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We describe the structure and indices of the members of the derived series of a GGS group G defined over the p-adic tree. The values $|G : G^{(n)}|$ exhibit a very limited dependence on the defining vector of the group. Furthermore, we establish that the derived and Frattini series of a GGS group defined by a nonconstant vector are identical.

1. Introduction and statement of results

Groups of automorphisms of regular rooted trees provide examples with intriguing asymptotic and structural properties. One particularly well-studied case is the family of Grigorchuk–Gupta–Sidki groups (usually abbreviated as GGS groups), generalising the second Grigorchuk group and the Gupta–Sidki p-groups. This family encompasses at least one group of intermediate word growth, as shown in [6], and numerous finitely generated infinite periodic groups, as demonstrated in [10]. The GGS groups acting on the p-adic tree, where p denotes an odd prime, are best understood; hence, in the following, by a GGS group we shall mean specifically a GGS group acting on the p-adic tree.

GGS groups are defined by a nonzero element $\boldsymbol{e}$ of $\mathbb{F}_p^{p-1}$ as “input data”. It is fortunate that many properties of interest are satisfied by the group defined by $\boldsymbol{e}$ if and only if $\boldsymbol{e}$ satisfies certain linear conditions. For instance, a GGS group is periodic and just-infinite if and only if the sum of the entries of $\boldsymbol{e}$ is zero [9; 17]. Similarly, it is a branch group (with the congruence subgroup property) if and only if not all entries of $\boldsymbol{e}$ are equal [7; 8]. Its Hausdorff dimension is governed by a function depending only on certain linear invariants of $\boldsymbol{e}$ [7]. Some of these results naturally extend to larger classes of groups [1; 2; 11; 14], but have been established first for GGS groups, making the class of GGS groups a fertile soil for establishing new techniques.

In this article, we provide a description of the derived series — i.e., the subgroups defined by $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for all $n \in \mathbb{N}$ — for every GGS

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group G. Conventionally, we write G' and G'' for $G^{(1)}$ and $G^{(2)}$, respectively. Our description is formulated in terms of linear conditions on the defining vector $\boldsymbol{e}$.

A description of the derived series was obtained by Vieira [16] for the special case of the Gupta–Sidki 3-group, along with some results on its lower central series. However, the proof in [16] does not readily generalise to general GGS groups. We approach the problem by adopting methods developed by Fernández-Alcober and Zugadi-Reizabal in [7]. We note that the sequence of indices of the members of the derived series for the (first) Grigorchuk group was computed by Grigorchuk in [9].

Our main result is the following.

Theorem 1.1. *Let p be an odd prime and let G be a GGS group acting on the p-adic tree defined by a nonconstant vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$. Let $\boldsymbol{e}'$ be the tuple of differences between the entries of $\boldsymbol{e}$ and let $\boldsymbol{e}''$ be the tuple of differences of $\boldsymbol{e}'$. Then, for $n \geq 2$,*

$$\log_p |G : G^{(n)}| = p^{n-2}(p + \operatorname{con}(\boldsymbol{e}') + \operatorname{sym}(\boldsymbol{e}'')) - \frac{p^{n-1}-1}{p-1} \cdot \operatorname{sym}(\boldsymbol{e}) + 1.$$

Here, a vector $\boldsymbol{d}$ is called symmetric if its entries are the same when read from left to right or right to left, and $\operatorname{sym}(\boldsymbol{d})$ equals 1 if the vector $\boldsymbol{d}$ is symmetric and 0 otherwise. Similarly, $\operatorname{con}(\boldsymbol{d})$ equals 1 if the vector $\boldsymbol{d}$ is constant, i.e., if all its entries are equal, and 0 otherwise. In [8, Proposition 3.4], Fernández-Alcober, Garrido and Uria-Albizuri show that GGS groups with constant defining vector admit an infinite metabelian quotient, i.e., that $|G : G''| = \infty$. It is an elementary fact that $|G : G'| = p^2$, so Theorem 1.1 completes the description of all indices $|G : G^{(n)}|$ for all GGS groups G and all integers $n \in \mathbb{N}$.

To arrive at Theorem 1.1, we need a description of the derived subgroups:

Theorem 1.2. *Let p be an odd prime and let G be a GGS group acting on the p-adic tree defined by a nonconstant vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$. Let $n \geq 3$. Then*

$$\psi(G^{(n)}) = \mathsf{X}_p\, G^{(n-1)}.$$

If $\operatorname{con}(\boldsymbol{e}') + \operatorname{sym}(\boldsymbol{e}'') = 0$, *the same holds for $n = 2$.*

Together with a description of the second derived subgroup, which we present in Proposition 3.2, this theorem yields a good account of the structure of the derived subgroups of G. One notable consequence is that the quotients $G^{(n)}/G^{(n+1)}$ are elementary abelian p-groups, which immediately yields the following corollary.

Corollary 1.3. *Let G be a GGS group defined by a nonconstant vector. The Frattini series of G coincides with the derived series of G.*

Finally, we investigate the GGS groups defined by nonconstant vectors with the maximal possible indices $|G : G^{(n)}|$. We find that there are precisely two isomorphism classes of GGS groups with $|G : G^{(n)}|$ maximal among GGS groups defined by nonconstant vectors (acting on a fixed p-adic tree); see Proposition 3.5.

2. On Grigorchuk–Gupta–Sidki-groups

Notation. The letter p refers to a fixed odd prime. Given integers m and n subject to $m \leq n$, the set $\{m, m+1, \dots, n-1, n\}$ is denoted by $[m, n]$. The symbol X refers to the set $[0, p-1]$ underlying the field $\mathbb{F}_p$. For a group G, the p-fold direct product $G \times \cdots \times G$ is denoted by $\mathsf{X}_p\, G$. For $k \in \mathbb{N}$ distinct elements $i_0, \dots, i_{k-1} \in X$ and p elements $g_0, \dots, g_{p-1}$ of a group G, the expression

$$(i_0 : g_0,\ \dots,\ i_{k-1} : g_{k-1},\ \diamond : g_\diamond) \in \mathsf{X}_p\, G$$

denotes the tuple indexed by X, with g_j at position i_j for $j \in [0, k-1]$, and with $g_\diamond$ (possibly varying with $\diamond$) at every other position $\diamond \in X \smallsetminus \{i_j \mid j \in [0, k-1]\}$. The symbol $\diamond$ is reserved for this use.

Given a group G and two of its elements g and h, we use the following conventions for conjugation and the commutator:

$$g^h = h^{-1}gh \quad \text{and} \quad [g, h] = g^{-1}h^{-1}gh = g^{-1}g^h.$$

Automorphisms of rooted trees. The Cayley graph X^* of the free monoid on X is the rooted p-adic tree, i.e., a loop-free graph with a distinguished vertex (the “root”) $\varnothing$ of valency p and all other vertices of valency $p+1$. We write X^n for the set of all vertices of a given (geodesic) distance n to $\varnothing$. This set is called the *n-th level of X^**.

Any (graph) automorphism $g \in \mathrm{Aut}(X^*)$ necessarily fixes $\varnothing$ for its unique valency, and must consequently leave all levels X^n invariant. We write $\mathrm{St}(n)$ for the (pointwise) stabiliser of X^n, and $\mathrm{St}_G(n)$ for its intersection with a subgroup $G \leq \mathrm{Aut}(X^*)$.

Let u and v be vertices of X^*. We write u^g for the image of u under g. Since every level is invariant under g, the equation

$$(uv)^g = u^g v^{g|_u}$$

uniquely defines a map $|_u \colon \mathrm{Aut}(X^*) \to \mathrm{Aut}(X^*)$, called the *section map at u*. The image of g is called *the section of g at u*. For $u \in X^*$, the restriction of $|_u$ to the stabiliser $\mathrm{st}(u)$ of u is a group homomorphism, indeed, the map

$$\psi \colon \mathrm{St}(1) \to \mathsf{X}_p\, \mathrm{Aut}(X^*), \qquad g \mapsto (\diamond : g|_\diamond)$$

is an isomorphism. We record some equations for sections. Let u and v be vertices of X^* and g and h be automorphisms, then

$$(g|_u)|_v = g|_{uv}, \quad (gh)|_u = g|_u h|_{u^g}, \quad g^{-1}|_u = (g|_{u^{g^{-1}}})^{-1}.$$

The action of an automorphism g on the first level $X = X^1$ is denoted $g|^{\varnothing} \in \mathrm{Sym}(X)$. An element such that $g|_u = \mathrm{id}$ for all $u \neq \varnothing$ is called *rooted*. We identify rooted elements with the their images under $|^{\varnothing}$. Let $u \in X^n$ and $g \in \operatorname{Aut}(X^*)$. The unique element $h \in \operatorname{St}(n)$ satisfying $h|_u = g$ and $h|_v = \mathrm{id}$ for all $v \in X^n \smallsetminus \{u\}$ is denoted $\operatorname{ins}_u(g)$ and called the *insertion of g at u*.

A group $G \leq \operatorname{Aut}(X^*)$ is called *self-similar*, if all sections $g|_u$ are contained in G for all $u \in X^*$ and $g \in G$. A group $G \leq \operatorname{Aut}(X^*)$ is called *fractal* if $\operatorname{st}_G(x)|_x = G$ for every $x \in X$. A group $G \leq \operatorname{Aut}(X^*)$ is called *spherically transitive* if it acts transitively on every level X^n. A self-similar group $G \leq \operatorname{Aut}(X^*)$ is called a *regular branch group* if it is spherically transitive and if there is a finite-index subgroup $K \leq \operatorname{St}_G(1)$ such that $\mathsf{X}_p K \leq \psi(K)$. A standard technique to establish that a group is regular branch is the following:

Proposition 2.1 [7, Proposition 2.18]. *Let $G \leq \operatorname{Aut}(X^*)$ be a spherically transitive fractal group, let $H \leq G$ be a subgroup and let $S \subseteq G$ be a subset. If $\operatorname{ins}_0(S) \subseteq H$, then $\mathsf{X}_p \langle S \rangle^G \leq \psi(H^G)$.*

GGS groups and their defining vectors. Let $\boldsymbol{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ be a nonzero vector. The group G generated by the rooted automorphism $a = (0\ 1\ \ldots\ p{-}1)$ induced by the addition of 1 in $\mathbb{F}_p$, and the automorphism defined by

$$b = \psi^{-1}(0 : b,\ \diamond : a^{e_\diamond})$$

is called the *GGS group defined by $\boldsymbol{e}$*. The vector $\boldsymbol{e}$ is the *defining vector of G*.

More generally, GGS groups acting on the (not necessarily prime) m-adic tree are defined in the same way, but using elements of $(\mathbb{Z}/m\mathbb{Z})^{m-1}$ whose entries have no common divisor other than 1. In general, the structure of these groups is much less understood than the case considered here. Even for prime powers $m = p^n$, the situation is much more involved, see for example [5], where the branching structures for these groups have been evaluated. Various further generalisations of GGS groups have been studied, e.g. in [1; 3; 12; 14].

Distinct defining vectors may give rise to identical or isomorphic GGS groups. Nonzero multiples of a given vector $\boldsymbol{e}$ define the same subgroup of $\operatorname{Aut}(X^*)$. Apart from that, certain reorderings of a given defining vector give rise to isomorphic (but not necessarily identical) GGS groups. We make use of the following characterisation. The group $\mathbb{F}_p^\times \times \mathbb{F}_p^\times \cong \mathrm{C}_{p-1}^2$ acts on the set $\mathbb{F}_p^{p-1} \smallsetminus \{\boldsymbol{0}\}$ of defining vectors by

$$(e_1, \ldots, e_{p-1}) * (\lambda, \mu) = (\lambda \cdot e_\mu, \lambda \cdot e_{2\cdot\mu}, \ldots, \lambda \cdot e_{(p-2)\cdot\mu}, \lambda \cdot e_{(p-1)\cdot\mu}). \tag{2-1}$$

Theorem 2.2 [13, Corollary 4.5]. *Two GGS groups G and $\tilde{G}$ defined by $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$, respectively, are isomorphic if and only if $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$ share the same orbit under the action of $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$.*

This allows us to choose defining vectors with desirable properties, as also done in [7, Theorem 2.16].

Corollary 2.3. *Let G be the GGS group defined by $\boldsymbol{e}$ and let $\alpha \in \mathbb{F}_p^\times$.*

(i) *There exists a defining vector $\tilde{\boldsymbol{e}}$ with $\tilde{e}_1 = \alpha$ such that the GGS group defined by $\tilde{\boldsymbol{e}}$ is isomorphic to G.*

(ii) *If $\boldsymbol{e}$ is not constant, there exists a defining vector $\tilde{\boldsymbol{e}}$ with $e_i - e_{i+1} = \alpha$, for some $i \in \{1, \dots, p-2\}$, such that the GGS group defined by $\tilde{\boldsymbol{e}}$ is isomorphic to G.*

Difference vectors. Let $\boldsymbol{d} = (d_1, \dots, d_n) \in \mathbb{F}_p^n$ be a vector. The *difference vector of $\boldsymbol{d}$* is the vector

$$\boldsymbol{d}' = (d'_1, d'_2, \dots, d'_{n-1}) = (d_1 - d_2, d_2 - d_3, \dots, d_{n-1} - d_n) \in \mathbb{F}_p^{n-1}.$$

A vector $\boldsymbol{d}$ is called *constant* if $\boldsymbol{d}' = \boldsymbol{0}$, i.e., if all entries of $\boldsymbol{d}$ are equal. We put $\operatorname{con}(\boldsymbol{d}) = 1$ if $\boldsymbol{d}$ is constant, and $\operatorname{con}(\boldsymbol{d}) = 0$ otherwise.

The structure of the derived subgroups of the GGS group defined by $\boldsymbol{e}$ is influenced by the difference vector $\boldsymbol{e}'$ and by its difference vector $\boldsymbol{e}''$, called, for convenience, the *second difference vector* of $\boldsymbol{e}$. The significance of $\boldsymbol{e}'$ is suggested by the following computation. Every GGS group G is two-generated and its derived subgroup is normally generated by the commutator $c = [a^{-1}, b]$, whose section decomposition is closely related to $\boldsymbol{e}'$,

$$\begin{aligned} \psi(c) = \psi([a^{-1}, b]) &= \psi(b^{a^{-1}})^{-1}\psi(b) \\ &= (\diamond : a^{-e_{\diamond+1}},\ p{-}1 : b^{-1})(0 : b,\ \diamond : a^{e_\diamond}) \\ &= (0 : a^{-e_1}b,\ \diamond : a^{e'_\diamond},\ p{-}1 : b^{-1}a^{e_{p-1}}). \end{aligned} \tag{2-2}$$

The entries of $\boldsymbol{e}''$ appear in a similar way in the section decomposition of $[a^{-1}, c]$.

Perhaps surprisingly, the higher difference vectors of $\boldsymbol{e}$ do not affect the structure of the derived series.

Symmetric vectors. Let $n \in \mathbb{N}$ be even, let K be a field not of characteristic 2, and let $\boldsymbol{d} = (d_1, \dots, d_n) \in K^n$ be a vector. It is called *symmetric* if

$$d_i = d_{n-i+1}$$

for all $i \in [1, n/2]$. We put $\operatorname{sym}(\boldsymbol{d}) = 1$ if $\boldsymbol{d}$ is symmetric, and $\operatorname{sym}(\boldsymbol{d}) = 0$ otherwise. Evidently, the set of all symmetric vectors constitutes a subspace S of K^n, being subject to the conditions $d_i - d_{n-i+1} = 0$ for $i \in [1, n/2]$. If $\boldsymbol{d}$ is symmetric, the second difference vector $\boldsymbol{d}''$ is also symmetric; see Table 1 for an overview of the possible configurations of the values $\operatorname{con}(\boldsymbol{e}), \operatorname{sym}(\boldsymbol{e}), \operatorname{con}(\boldsymbol{e}')$ and $\operatorname{sym}(\boldsymbol{e}'')$ for a defining vector. More precisely, the second difference vector is symmetric if and only if

$$d_i - 2d_{i+1} + d_{i+2} = d''_i = d''_{n-i-1} = d_{n-1-i} - 2d_{n-i} + d_{n-i+1},$$

i.e., if and only if $d''_i - d''_{n-i-1} = 0$ for all $i \in [1, n/2 - 1]$. It is apparent that

con($\boldsymbol{e}$)	sym($\boldsymbol{e}$)	con($\boldsymbol{e}'$)	sym($\boldsymbol{e}''$)
0	0	0	0
0	0	0	1
0	0	1	1
0	1	0	1
1	1	1	1

Table 1. Possible configurations of values for the invariants of $\boldsymbol{e}$ influencing the indices of the members of the derived series of the corresponding GGS group. For $p = 3$, the first, second and fourth rows are nonexistent.

the set of all vectors such that $\boldsymbol{d}''$ is symmetric is a subspace containing the space of symmetric vectors as a subspace of codimension 1. We need the following computational lemma.

Lemma 2.4. *Let* $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$ *be such that* $\operatorname{sym}(\boldsymbol{e}'') = 1$. *Then*

$$2(e_{p-1} - e_1) + (e_2 - e_{p-2}) = 0.$$

Proof. Write $s_i = e_i - e_{p-i}$ for $i \in [1, p-1]$. Note that

$$s_i' = s_i - s_{i+1} = e_i - e_{p-i} - e_{i+1} + e_{p-i-1} = e_i' - e_{p-i-1}',$$

and in the same way, $s_i'' = e_i'' - e_{p-i-2}''$. Moreover, $s_i = -s_{p-i}$, whence $s_{(p-1)/2}' = s_{(p-1)/2} - s_{(p+1)/2} = 2 \cdot s_{(p-1)/2}$. Two simple telescope sum computations yield

$$\sum_{i=1}^{(p-3)/2} s_i'' = \sum_{i=1}^{(p-3)/2} (s_i' - s_{i+1}') = s_1' - s_{(p-1)/2}' = s_1 - s_2 - 2 \cdot s_{(p-1)/2}$$

and

$$\begin{aligned}\sum_{i=1}^{(p-3)/2} i \cdot s_i'' = \sum_{i=1}^{(p-3)/2} i \cdot (s_i' - s_{i+1}') &= \sum_{i=1}^{(p-3)/2} s_i' - ((p-3)/2) \cdot s_{(p-1)/2}' \\ &= s_1 - s_{(p-1)/2} + 3 \cdot s_{(p-1)/2} \\ &= s_1 + 2 \cdot s_{(p-1)/2}.\end{aligned}$$

Combining them, one finds that

$$-\sum_{i=1}^{(p-3)/2} (i+1) \cdot s_i'' = -2s_1 + s_2.$$

Since $\boldsymbol{e}''$ is symmetric, $s_i'' = 0$ for all $i \in [1, p-3]$. Thus the left-hand side of the equality above is zero, while the right-hand side evaluates to the desired expression $2(e_{p-1} - e_1) + (e_2 - e_{p-2})$. □

Circulant spaces. Let K be a field and let $\boldsymbol{d} = (d_0, \dots, d_{\ell-1}) \in K^\ell$ be a vector. The *circulant matrix* $\mathrm{Circ}(\boldsymbol{d})$ associated to $\boldsymbol{d}$ is the matrix whose rows are the cyclic shifts of $\boldsymbol{d}$, i.e.,

$$\begin{pmatrix} d_0 & d_1 & \dots & d_{\ell-2} & d_{\ell-1} \\ d_{\ell-1} & d_0 & \dots & d_{\ell-3} & d_{\ell-2} \\ \vdots & \vdots & & \vdots & \vdots \\ d_1 & d_2 & \dots & d_{\ell-1} & d_0 \end{pmatrix}.$$

The *semicirculant matrix* $\mathrm{SCirc}(\boldsymbol{d})$ associated to $\boldsymbol{d}$ is the upper triangular matrix given by

$$\begin{pmatrix} d_0 & d_1 & \dots & d_{\ell-2} & d_{\ell-1} \\ 0 & d_0 & \dots & d_{\ell-3} & d_{\ell-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & d_0 \end{pmatrix}.$$

More generally, a *circulant subspace of* K^ℓ is a subspace that is invariant under the automorphism induced by the cyclic permutation of the standard basis elements of K^ℓ. Given a subset $M \subseteq K^\ell$, we write $\mathrm{Circ}(M)$ for the minimal circulant subspace containing M. In case of a singleton set, $\mathrm{Circ}(\{\boldsymbol{d}\})$ is spanned by the rows of the circulant matrix $\mathrm{Circ}(\boldsymbol{d})$.

Ranks of circulant matrices, their computation and interpretation have long been studied; see for example [4] for the situation over the field of complex numbers. In positive characteristic p, the special case $\ell = p^n$ allows an easy description of the circulant subspaces of K^ℓ. Let $\Pi \in K^{\ell \times \ell}$ be the Pascal matrix with entries $\binom{i}{j}$ for $i, j \in [0, \ell-1]$, using the convention that $\binom{i}{j} = 0$ for $i < j$, and write Π_i for the ℓ-by-i matrix consisting of the first i columns of Π. Note that Π is a lower unitriangular matrix; in particular, it is invertible.

Proposition 2.5. *Let K be a field of characteristic $p > 0$ and let $\ell = p^n$ be a power of the characteristic. There exists a complete flag*

$$\{\mathbf{0}\} = \mathrm{Circ}_0 \subset \mathrm{Circ}_1 \subset \cdots \subset \mathrm{Circ}_\ell = K^\ell$$

containing all circulant subspaces of K^ℓ, which are given by

$$\mathrm{Circ}_i = \ker \Pi_{\ell-i} = \{\boldsymbol{d} \in K^\ell \mid \operatorname{rank} \mathrm{Circ}(\boldsymbol{d}) \le i\}.$$

Proof. Fix a vector $\boldsymbol{d} = (d_0, \dots, d_{\ell-1}) \in K^\ell$. Put

$$f_{\boldsymbol{d}}(X) = d_0 + d_1 X + d_2 X^2 + \cdots + d_{\ell-1} X^{\ell-1}$$

and

$$P = \begin{pmatrix} 0_{\ell-1,1} & \mathrm{I}_{\ell-1} \\ 1 & 0_{1,\ell-1} \end{pmatrix},$$

where $0_{n,m}$ and I_n stand for the zero and unit matrices, respectively, of the indicated formats. The matrix P is the permutation matrix associated to the cyclic shift of the basis elements, and

$$\mathrm{Circ}(\boldsymbol{d}) = d_0 I + d_1 P + d_2 P^2 + \cdots + d_{\ell-1} P^{\ell-1} = f_{\boldsymbol{d}}(P).$$

The characteristic polynomial of P is $X^\ell - 1 = (X-1)^\ell$ and splits over K; here we use that ℓ is a power of p. The geometric multiplicity of its unique eigenvalue 1 is 1, whence P is conjugate to the matrix $J_\ell(1)$ consisting of a single Jordan block of eigenvalue 1. Consequently, $\mathrm{Circ}(\boldsymbol{d})$ is conjugate to $f_{\boldsymbol{d}}(J_\ell(1))$. For $k \in \mathbb{N}$, we have $J_\ell(1)^k = \big(\binom{k}{j-i}\big)_{i,j\in[1,\ell]}$, i.e., $J_\ell(1)^k = \mathrm{SCirc}\big(\big(\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{\ell-1}\big)\big)$. Thus $f_{\boldsymbol{d}}(J_\ell(1))$ is the semicirculant matrix with respect to the vector $\boldsymbol{b} = (b_0, \dots, b_{\ell-1})$, with

$$b_i = \sum_{j=0}^{\ell-1} d_j \binom{j}{i}.$$

Then $\boldsymbol{b} = \boldsymbol{d}\Pi$. Since the rank of $\mathrm{SCirc}(\boldsymbol{b})$ is equal to $\ell - \min\{j \in [0, \ell-1] \mid b_j \neq 0\}$, we see that $\operatorname{rank}\mathrm{Circ}(\boldsymbol{d}) \le i$ if and only if $\boldsymbol{d} \in \ker \Pi_{\ell-i}$, with equality precisely if $\boldsymbol{d} \notin \ker \Pi_{\ell-i+1}$. Since Π is invertible, we see that $\operatorname{rank}\Pi_{\ell-i} = i$. Thus Circ_i is indeed a circulant space, as $\mathrm{Circ}_i = \mathrm{Circ}(\boldsymbol{d})$ for any $\boldsymbol{d} \in \mathrm{Circ}_i \smallsetminus \mathrm{Circ}_{i-1}$. Evidently $\mathrm{Circ}_i \subset \mathrm{Circ}_{i+1}$. It remains to show that there are no further circulant spaces. Let C be a circulant subspace of dimension i. For every $\boldsymbol{d} \in C$, we necessarily find $\operatorname{rank}\mathrm{Circ}(\boldsymbol{d}) \le i$. If there exists $\boldsymbol{d}$ such that the rank of $\mathrm{Circ}(\boldsymbol{d})$ is i, naturally $C = \mathrm{Circ}(\boldsymbol{d}) = \mathrm{Circ}_i$. Thus assume that $\mathrm{Circ}(\boldsymbol{d}) < i$ for all $\boldsymbol{d} \in C$. But then $C \subseteq \mathrm{Circ}_{i-1}$, an $(i-1)$-dimensional subspace, which is absurd. □

This description extends [7, Lemma 2.7]. Note that the crucial point is that the polynomial $X^\ell - 1$ splits over K. Let $\boldsymbol{e}$ be the defining vector of the GGS group G and put $\bar{\boldsymbol{e}} = (0, e_1, \dots, e_{p-1})$. The structure of G is heavily influenced by the *cyclic rank* $\mathrm{cr}(\boldsymbol{e})$ *of* $\boldsymbol{e}$, which is given by $\mathrm{cr}(\boldsymbol{e}) = \dim \mathrm{Circ}(\bar{\boldsymbol{e}})$. By Proposition 2.5, $\mathrm{cr}(\boldsymbol{e}) = p$ if and only if the sum of entries of $\boldsymbol{e}$ is nonzero, whence by [9, Example 9.1], the group G is periodic if and only if $\mathrm{cr}(\boldsymbol{e}) < p$. The Hausdorff dimension of G solely depends on whether $\boldsymbol{e}$ is symmetric or constant and on the value of $\mathrm{cr}(\boldsymbol{e})$, as demonstrated by Fernández-Alcober and Zugadi-Reizabal in [7, Theorem B].

Corollary 2.6. *Let K be a field of characteristic $p > 0$, let $i \in [0, p]$ and let* Circ_i *be the members of the flag of circulant subspaces of K^p. A basis for* Circ_i *is given by the first i columns of the Pascal matrix $\Pi \in K^{p\times p}$.*

Proof. In view of Proposition 2.5 and the fact that the columns of Π are linearly independent, we have to show that

$$\sum_{n=j}^{p-1} \binom{n}{m}\binom{n}{j} \equiv_p 0$$

for every $m \in [0, i-1]$ and $j \in [0, p-i-1]$. Put $\phi(k, m, j) = \sum_{n=j}^{p-1} n^k \binom{n}{m}\binom{n}{j}$ and compute

$$\phi(k,m,j) = \sum_{n=j}^{p-1} n^k \binom{n}{m}\binom{n}{j-1}\frac{n-j+1}{j} = \frac{1}{j}\phi(k+1,m,j-1) + \frac{1-j}{j}\phi(k,m,j-1);$$

analogously,

$$\phi(k,m,j) = \frac{1}{m}\phi(k+1,m-1,j) + \frac{1-m}{m}\phi(k,m-1,j).$$

By iteration we find that $\phi(0,m,j)$ is a linear combination of the values of $\phi(k,0,0)$ for $k \in [0, m+j]$. But

$$\phi(k,0,0) = \sum_{n=0}^{p-1} n^k = \sum_{n=0}^{p-1} n \equiv_p 0$$

for every k that is not a multiple of $p-1$. Since $m+j < p-1$, we obtain the desired congruence. □

Properties and structure of GGS groups. Recall that, for a given GGS group G, the rooted generator is denoted a, the directed generator is denoted b, and we write c for the commutator $[a^{-1}, b]$, whose sections are given in (2-2). We shall use the following shorthand notation for the conjugates of c:

$$c_i = c^{a^i} = [a^{-1}, b^{a^i}].$$

In particular, $c_0 = c$. We collect some facts about GGS groups.

Lemma 2.7. *Every GGS group is fractal and self-similar.*

For a proof, see e.g. [8, Section 2] or [13, Section 2.3]. Next, we record some information on certain small quotients of GGS groups.

Lemma 2.8. *Let G be a GGS group. Then*

(i) $\log_p |G : G'| = 2$ *and* G/G' *is elementary abelian,*

(ii) $\log_p |G' : \gamma_3(G)| = 1$ *and* $G/\gamma_3(G)$ *is of exponent* p, *and*

(iii) $\log_p |G' : \mathrm{St}_G(1)'| = p-1$.

For statements (i) and (ii), see Theorem 2.1(iii) of [7] and note that $a^p = b^p = \mathrm{id}$ and $c^p \equiv_{\gamma_3(G)} \mathrm{id}$. Statement (iii) is proven as part of Theorem 2.14 of [7].

It is a result of Fernández-Alcober and Zugadi-Reizabal that every GGS group defined by a nonconstant vector is a regular branch group (see below). The same is not true for GGS groups defined by constant tuples, explaining their divergent behaviour.

Theorem 2.9. *Let G be a GGS group with nonconstant defining vector $\boldsymbol{e}$. Then*:

(i) $\psi(\gamma_3(\mathrm{St}_G(1))) = \mathsf{X}_p\, \gamma_3(G)$.

(ii) $\psi(\mathrm{St}_G(1)') \le \mathsf{X}_p\, G'$ *and* $\log_p |\mathsf{X}_p\, G' : \psi(\mathrm{St}_G(1)')| = \mathrm{sym}(\boldsymbol{e})$.

In particular, *the group G is regular branch over $\gamma_3(G)$*, *and it is regular branch over G' if $\boldsymbol{e}$ is not symmetric.*

These statements appear as Lemmas 3.2 and 3.4 of [7].

We need another fact concerning the subgroups of GGS groups.

Proposition 2.10. *Let G be a GGS group with a nonconstant defining vector. Then*

$$[\mathrm{St}_G(1)', G'] = \gamma_3(\mathrm{St}_G(1)).$$

In particular, $\gamma_3(\mathrm{St}_G(1)) \le G''$.

Proof. The inclusion $[\mathrm{St}_G(1)', G'] \le \gamma_3(\mathrm{St}_G(1))$ is a straightforward consequence of $G' \le \mathrm{St}_G(1)$, which itself follows from $G/\mathrm{St}_G(1)$ being cyclic. We have to establish the other inclusion. In view of Proposition 2.1, it is enough to prove that $\mathrm{ins}_0([c, b])$ and $\mathrm{ins}_0([c, a])$ are contained in $[\mathrm{St}_G(1)', G']$, as $\gamma_3(G)$ is normally generated by $[c, b]$ and $[c, a]$. We distinguish two cases.

<u>Case $\mathrm{sym}(\boldsymbol{e}) = 0$</u>: By Theorem 2.9(ii), the element $\mathrm{ins}_0(c)$ is contained in $\mathrm{St}_G(1)'$. By Corollary 2.3(ii) we may assume $e'_i = 1$ for some $i \in \{1, \dots, p-2\}$. By (2-2) $c|_i = a$, hence

$$c_{p-i}|_0 = c|_i = a, \qquad (c_{p-i}^{e_1} c)|_0 = a^{e_1} a^{-e_1} b = b.$$

Since c_{p-i} and c are elements of G', we obtain

$$[\mathrm{ins}_0(c), c_{p-i}] = \mathrm{ins}_0([c, a]) \quad \text{and} \quad [\mathrm{ins}_0(c), c_{p-i}^{e_1} c] = \mathrm{ins}_0([c, b])$$

are contained in $[\mathrm{St}_G(1)', G']$.

<u>Case $\mathrm{sym}(\boldsymbol{e}) = 1$</u>: Note that, since $\boldsymbol{e}$ is by assumption not constant, the prime p is necessarily greater than 3. By Corollary 2.3(i), we may assume $e_1 = e_{p-1} = -1$. Observe that

$$\begin{aligned} \psi([b, b^a]) = [\psi(b), \psi(b^a)] &= (0 : [b, a^{-1}],\ 1 : [a^{-1}, b],\ \diamond : \mathrm{id}) \\ &= (0 : c^{-1},\ 1 : c,\ \diamond : \mathrm{id}). \end{aligned}$$

We first show that there exists $j \in \mathbb{F}_p^\times \smallsetminus \{1, p-1\}$ such that $e_{j-1} \ne e_{p-j-1}$. Assume the converse for contradiction. Using that $\boldsymbol{e}$ is symmetric, we find

$$e_j = e_{(1+j)-1} = e_{p-(1+j)-1} = e_{j+2}.$$

Thus $e_2 = e_4 = \cdots = e_{p-1}$ and $e_3 = e_5 = \cdots = e_{p-2}$, using that $p > 3$. Since $\boldsymbol{e}$ is symmetric, $e_2 = e_{p-2}$ and $e_1 = e_{p-1}$, whence $\boldsymbol{e}$ is constant, which is excluded.

Now let j be an element as described above. Compute

$$\psi([a^{2j}, b]) = (0 : a^{-e_{p-2j}} b,\ 2j : b^{-1} a^{e_{2j}},\ \diamond : a^{e_\diamond - e_{\diamond - 2j}}),$$

so $[a^{2j}, b]|_j = a^{e_j - e_{p-j}} = \mathrm{id}$, since $\boldsymbol{e}$ is symmetric. Put $g = [a^{2j}, b]^{a^{1-j}}$ to find

$$g|_1 = [a^{2j}, b]|_j = \mathrm{id} \quad \text{and} \quad g|_0 = [a^{2j},\ b]|_{j-1} = a^{e_{j-1} - e_{p-j-1}},$$

since $j \notin \{1, p-1\}$ forbids $j - 1 \in \{0, 2j\}$. Let $i = (e_{j-1} - e_{p-j-1})^{-1}$, using that $e_{j-1} \neq e_{p-j-1}$. Observe that

$$\begin{aligned}[\mathrm{St}_G(1)', G'] \ni [[b^a, b], g^i] &= [\psi^{-1}(0 : c,\ 1 : c^{-1},\ \diamond : \mathrm{id}), \psi^{-1}(0 : a,\ 1 : \mathrm{id},\ \diamond : g^i|_\diamond)] \\ &= \mathrm{ins}_0([c, a]).\end{aligned}$$

It remains to show $\mathrm{ins}_0([c, b]) \in [\mathrm{St}_G(1)', G']$. Consider $h = g^{i e_{p-2}}[a^2, b]$, which fulfils

$$h|_0 = a^{e_{p-2}} a^{-e_{p-2}} b = b \quad \text{and} \quad h|_1 = a^{e_1 - e_{p-1}} = \mathrm{id}\,.$$

Then

$$[\mathrm{St}_G(1)', G'] \ni [[b^a, b], h] = \mathrm{ins}_0([c, b]).$$ ☐

Using Proposition 2.10, we derive the following adjunct to Theorem 2.9.

Proposition 2.11. *Let G be a GGS group with nonconstant defining vector. Then G is branch over G''.*

Proof. Using Theorem 2.9(i) and Proposition 2.10 we find

$$\mathsf{X}_p\, G'' \le \mathsf{X}_p\, \gamma_3(G) = \psi(\gamma_3(\mathrm{St}_G(1))) \le \psi(G'').$$ ☐

3. The derived series of GGS groups

The second derived subgroup. By Lemma 2.8(ii), the quotient $G'/\gamma_3(G)$ is a cyclic group of order p, generated by the element $\bar{c} = c \cdot \gamma_3(G)$. Thus the group $V := \mathsf{X}_p\, (G'/\gamma_3(G))$ is an elementary abelian p-group of rank p, which we write additively and treat as an $\mathbb{F}_p$-vector space with the natural basis

$$\{(\bar{c}, 0, \dots, 0), \dots, (0, \dots, 0, \bar{c})\}.$$

Write $\phi\colon\ \mathrm{St}_G(1)' \to V$ for the concatenation of ψ and the natural epimorphism $\mathsf{X}_p\, G' \to V$.

Lemma 3.1. *Let G be a GGS group with nonconstant defining vector and let $N \le G$ be a subgroup satisfying*

$$\mathsf{X}_p\, \gamma_3(G) \le \psi(N) \le \mathsf{X}_p\, G'.$$

Then N is normal in G if and only if $\phi(N)$ is a circulant subspace of V, and as a consequence, there exist precisely $p + 1 - \mathrm{sym}(\boldsymbol{e})$ such normal subgroups.

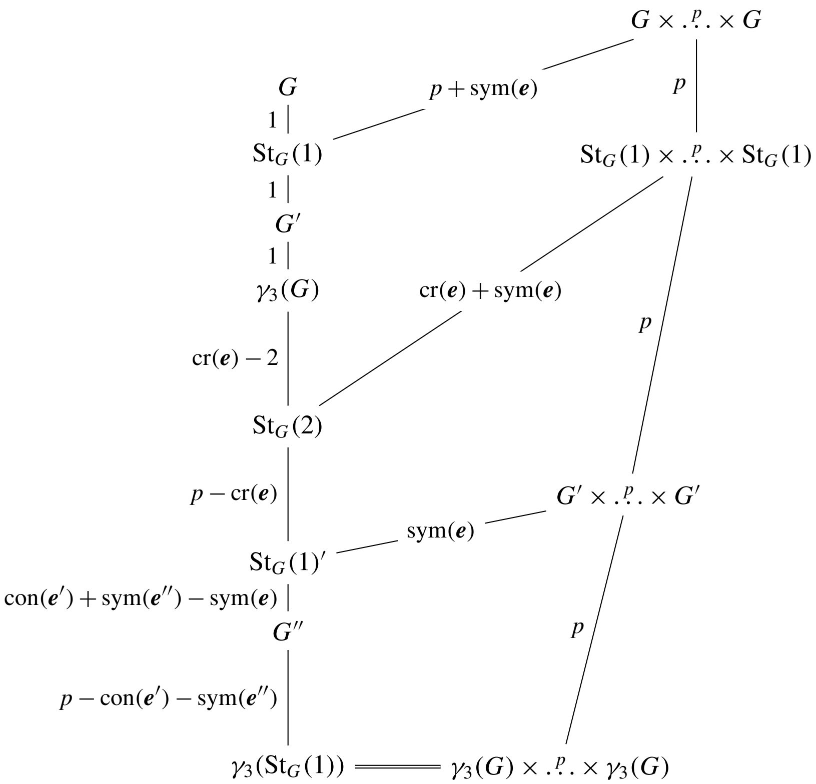

Figure 1. Part of the top of the subgroup lattice of a GGS group, with some supergroups added. Passage from the left to the right side signifies the application of ψ. All indices are logarithmic. See [7] for the computation of the index of $\mathrm{St}_G(2)$.

Proof. The conjugation action of a corresponds to a cyclic shift on $\psi(N)$, whence N is invariant under conjugation by a if and only if $\phi(N)$ is circulant. It remains to notice that N is automatically invariant under conjugation by b: for every $g \in \psi^{-1}(\mathsf{X}_p\, G') \cap G$ and every $h \in \mathrm{St}_G(1)$

$$\psi(g^h) = (\diamond : g|_\diamond^{h|_\diamond}) \equiv_{\mathsf{X}_p\, \gamma_3(G)} (\diamond : g|_\diamond) = g.$$

The last statement is a direct consequence of Proposition 2.5 and Theorem 2.9(ii). □

Proposition 3.2. *Let G be a GGS group with nonconstant defining vector. Then $G'' = \phi^{-1}(\mathrm{Circ}_{p-\mathrm{con}(\boldsymbol{e}')-\mathrm{sym}(\boldsymbol{e}'')}(V))$. In particular,*

$$\log_p |\mathsf{X}_p\, G' : \psi(G'')| = \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'').$$

Proof. Since $G' \le \mathrm{St}_G(1)$, the second derived subgroup G'' is contained in $\mathrm{St}_G(1)'$, which, by Theorem 2.9(ii), is in turn contained in $\psi^{-1}(\mathsf{X}_p\, G')$. On the other hand, using Proposition 2.10 and Theorem 2.9(i), we find

$$\mathsf{X}_p\,\gamma_3(G) = \psi(\gamma_3(\mathrm{St}_G(1))) \le \psi(G'').$$

Thus Lemma 3.1 applies and it remains to compute the dimension of $\phi(G'')$. To achieve this, we pick a subset S of G'' normally generating G'' and use Proposition 2.5 to compute the dimension of the circulant space generated by $\phi(S)$. Since the group G is 2-generated, a natural choice for S is

$$\{[c^g, c] \mid g \in G\}.$$

The kernel of ϕ is $\gamma_3(\mathrm{St}_G(1)) = \psi^{-1}(\mathsf{X}_p\,\gamma_3(G))$, whence $\phi([c^g, c]) = \phi([c_i, c])$ for $a^i \equiv_{\mathrm{St}_G(1)} g$. Notice that

$$[c_i, c]^{-1} = [c, c_i] = [c, c^{a^i}] = [c_{p-i}, c]^{a^i}.$$

Thus the set $\phi(\{[c_i, c] \mid i \in [1, (p-1)/2]\})$ generates $\phi(G'')$ as a circulant space. The sections of the elements $[c_i, c]$ are given by (compare (2-2))

$$\begin{aligned}\psi([c_1, c]) &= (0 : [b^{-1}a^{e_{p-1}}, a^{-e_1}b],\ 1 : [a^{-e_1}b, a^{e'_1}],\ \diamond : \mathrm{id},\ p-1 : [a^{e'_{p-2}}, b^{-1}a^{e_{p-1}}]) \\ &\equiv (0 : c^{e_1 - e_{p-1}},\ 1 : c^{e'_1},\ \diamond : \mathrm{id},\ p-1 : c^{e'_{p-2}}) \pmod{\mathsf{X}_p\,\gamma_3(G)}\end{aligned}$$

and

$$\begin{aligned}\psi([c_i, c]) &= \left(\begin{array}{rlrl} 0 : & [a^{e'_{p-i}}, a^{-e_1}b], & p-1 : & [a^{e'_{p-i-1}}, b^{-1}a^{e_{p-1}}] \\ i-1 : & [b^{-1}a^{e_{p-1}}, a^{e'_{i-1}}], & i : & [a^{-e_1}b, a^{e'_i}] \\ \diamond : & \mathrm{id} & & \end{array}\right) \\ &\equiv \left(\begin{array}{rlrl} 0 : & c^{-e'_{p-i}}, & p-1 : & c^{e'_{p-i-1}}, \\ i-1 : & c^{-e'_{i-1}}, & i : & c^{e'_i}, \\ \diamond : & \mathrm{id} & & \end{array}\right) \pmod{\mathsf{X}_p\,\gamma_3(G)}\end{aligned}$$

for $i \in [2, (p-1)/2]$. Thus the images $\boldsymbol{d}_i = \phi([c_i, c])$ in V are given by

$$\boldsymbol{d}_1 = (e_1 - e_{p-1}, e'_1, 0, \dots, 0, e'_{p-2}),$$

and by

$$\boldsymbol{d}_i = (-e'_{p-i}, \underbrace{0, \dots, 0}_{i-2}, -e'_{i-1}, e'_i, \underbrace{0, \dots, 0}_{p-i-2}, e'_{p-i-1}),$$

for $i \in [2, (p-1)/2]$. By Proposition 2.5, the dimension of $\phi(G'')$ is equal to the maximum value of $\dim \mathrm{Circ}(\boldsymbol{d}_i)$ for $i \in [1, (p-1)/2]$. To determine the dimension of the latter spaces, recall that for any $\boldsymbol{f} = (f_0, \dots, f_{p-1})$ we find $\boldsymbol{f}\,\Pi_3 = (\boldsymbol{f}\,\Xi_p, \boldsymbol{f}\,\Xi_{p-1}, \boldsymbol{f}\,\Xi_{p-2})$,

$$\boldsymbol{f}\,\Xi_p = \sum_{i=0}^{p-1} f_i, \quad \boldsymbol{f}\,\Xi_{p-1} = \sum_{i=0}^{p-1} i f_i, \quad \text{and} \quad \boldsymbol{f}\,\Xi_{p-2} = \sum_{i=0}^{p-1} \binom{i}{2} f_i$$

and compute

$$\boldsymbol{d}_1\,\Xi_p = 2(e_1 - e_{p-1}) + e_{p-2} - e_2, \tag{3-1}$$

$$\boldsymbol{d}_1\,\Xi_{p-1} = e'_1 - e'_{p-2}. \tag{3-2}$$

$$\boldsymbol{d}_1\,\Xi_{p-2} = e'_{p-2}, \tag{3-3}$$

using $\binom{p-1}{2} \equiv_p 1$ in the last line. Then, for $i \in [2, (p-1)/2]$, we compute

$$\boldsymbol{d}_i\,\Xi_p = -e'_{p-i} - e'_{i-1} + e'_i + e'_{p-i-1} = e''_{p-i-1} - e''_{i-1}, \tag{3-4}$$

and for $i \in [2, (p-1)/2]$ we find

$$\begin{aligned} \boldsymbol{d}_i\,\Xi_{p-1} &= -0 \cdot e'_{p-i} - (i-1)e'_{i-1} + i e'_i + (p-1)e'_{p-i-1} \\ &= -i \cdot e''_{i-1} + e'_{i-1} - e'_{p-i-1} \\ &= -i \cdot e''_{i-1} + \sum_{j=i-1}^{p-i-2} e''_j. \end{aligned} \tag{3-5}$$

Note that for $i = (p-1)/2$, this is equal to

$$\boldsymbol{d}_{(p-1)/2}\,\Xi_{p-1} = (p+3)/2 \cdot e''_{(p-3)/2}. \tag{3-6}$$

Case $\operatorname{sym}(\boldsymbol{e}'') = 0$. This implies $\operatorname{sym}(\boldsymbol{e}) = 0$, and $V = \phi(\operatorname{St}_G(1)')$ by Theorem 2.9(ii). By definition, there exists $i \in [1, (p-3)/2]$ such that $e''_i \neq e''_{p-2-i}$, hence $\boldsymbol{d}_{i+1}\,\Xi_p \neq 0$ by (3-4) and $\dim \operatorname{Circ}(\boldsymbol{d}_{i+1}) = p$ by Proposition 2.5. Thus $\phi(G'')$ is equal to V, i.e., $G'' = \operatorname{St}_G(1)'$. Note that $\operatorname{sym}(\boldsymbol{e}'') = 0$ implies $\operatorname{con}(\boldsymbol{e}') = 0$, since the difference vector of a constant vector is zero and in particular symmetric. Thus we find

$$\dim \phi(G'') = p = p - (\operatorname{sym}(\boldsymbol{e}'') + \operatorname{con}(\boldsymbol{e}')).$$

Case $\operatorname{sym}(\boldsymbol{e}'') = 1$. By (3-4), $\boldsymbol{d}_i\,\Xi_p = 0$ for all $i \in [2, (p-1)/2]$, and, by Lemma 2.4 and (3-1), also $\boldsymbol{d}_1\,\Xi_p = 0$; hence Proposition 2.5 implies $\dim \phi(G'') \leq p-1$. Using (3-5), we see that

$$\boldsymbol{d}_i\,\Xi_{p-1} = -i \cdot e''_{i-1} + \sum_{j=i-1}^{p-i-2} e''_j = (1-i) \cdot e''_{i-1} + 2 \sum_{j=i}^{(p-3)/2} e''_j \tag{3-7}$$

for $i \in [2, (p-3)/2]$.

If $\operatorname{con}(\boldsymbol{e}') = 0$, then $p \neq 3$ and $\boldsymbol{e}'' \neq 0$. By symmetry, there exists $i \in [1, (p-3)/2]$ such that $e''_i \neq 0$. If $e''_{(p-3)/2} \neq 0$, we find $\boldsymbol{d}_{(p-1)/2}\,\Xi_{p-1} \neq 0$ by (3-6). Otherwise, the prime p is at least 7. Let $i \in [1, (p-5)/2]$ be maximal such that $e''_i \neq 0$. Then $\boldsymbol{d}_{i+1}\,\Xi_{p-1} \neq 0$ by (3-7). Thus, by Proposition 2.5, $\dim \phi(G'') = p-1 = p - (\operatorname{sym}(\boldsymbol{e}'') + \operatorname{con}(\boldsymbol{e}'))$.

On the other hand, if $\operatorname{con}(\boldsymbol{e}') = 1$, we immediately find $\boldsymbol{d}_1\,\Xi_{p-1}$ by (3-2). Furthermore, $\boldsymbol{e}'' = \boldsymbol{0}$, whence we also find $\boldsymbol{d}_i\,\Xi_{p-1} = 0$ for $i \in [2, (p-1)/2]$, using (3-5). But by (3-3), $\boldsymbol{d}_1 \Xi_{p-2} = e'_{p-2} \neq 0$; otherwise, $\boldsymbol{e}' = \boldsymbol{0}$ since it is constant, which implies $\operatorname{con}(\boldsymbol{e}) = 1$, which was excluded. Thus $\dim \phi(G'') = p-2 = p-(\operatorname{sym}(\boldsymbol{e}'')+\operatorname{con}(\boldsymbol{e}'))$. □

Lemma 3.3. *Let G be a GGS group defined by the nonconstant vector $\boldsymbol{e}$. Then*

$$\log_p |G' : G''| = p + \operatorname{con}(\boldsymbol{e}') + \operatorname{sym}(\boldsymbol{e}'') - \operatorname{sym}(\boldsymbol{e}) - 1.$$

Proof. This is an immediate consequence of Lemma 2.8(iii), Proposition 3.2 and Theorem 2.9(ii):

$$\begin{aligned}\log_p |G' : G''| &= \log_p |G' : \operatorname{St}_G(1)'| + \log_p |\mathsf{X}_p\, G' : \psi(G'')| - \log_p |\mathsf{X}_p\, G' : \psi(\operatorname{St}_G(1)')| \\ &= p + \operatorname{con}(\boldsymbol{e}') + \operatorname{sym}(\boldsymbol{e}'') - \operatorname{sym}(\boldsymbol{e}) - 1.\end{aligned}$$ □

Proofs of the main results. We are now in the position to prove our theorems, which we state again for the convenience of the reader.

Theorem 1.2. *Let p be an odd prime and let G be a GGS group acting on the p-adic tree defined by a nonconstant vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$. Let $n \geq 3$. Then*

$$\psi(G^{(n)}) = \mathsf{X}_p\, G^{(n-1)}.$$

If $\operatorname{con}(\boldsymbol{e}') + \operatorname{sym}(\boldsymbol{e}'') = 0$, *the same holds for $n = 2$.*

Proof. Assume that the given equation holds true for some $n \geq 1$. Then we find

$$\psi(G^{(n+1)}) = \psi(G^{(n)})' = (\mathsf{X}_p\, G^{(n-1)})' = \mathsf{X}_p\, G^{(n)},$$

since $G^{(n)} \leq \operatorname{St}_G(1)$. Thus it is enough to consider the case $n = 3$, or the case $n = 2$, respectively.

First assume that $\operatorname{con}(\boldsymbol{e}') = \operatorname{sym}(\boldsymbol{e}'') = 0$, which also implies $\operatorname{sym}(\boldsymbol{e}) = 0$. By Lemma 3.3, $\mathsf{X}_p\, G' = \psi(G'')$. Thus the equation holds for $n = 2$.

We forgo the above assumptions on the defining vector and prove the desired equation for $n = 3$. In view of Proposition 2.1, it is sufficient to prove that $\operatorname{ins}_0([c, c^g]) \in G^{(3)}$ for any $g \in G$, since G'' is normally generated by the elements $\{[c, c^g] \mid g \in G\}$. By Proposition 2.10, the group G'' contains $\psi^{-1}(\mathsf{X}_p\, \gamma_3(G))$, in particular $\operatorname{ins}_0([c, g]) \leq G'$ for all $g \in G$. By Proposition 3.2 we find $h \in G''$ with

$$\psi(h) = (0 : c,\ 1 : c^{-2},\ 2 : c,\ \diamond : \mathrm{id}).$$

Let $g \in G$. Then

$$[h, \operatorname{ins}_0([c, g])] = \operatorname{ins}_0([h|_0, [c, g]]) = \operatorname{ins}_0([c, [c, g]]) = \operatorname{ins}_0([c, c^g]) \in G^{(3)}.$$ □

Before we prove Theorem 1.1, we use Theorem 1.2 to derive some more results on the structure of G.

Corollary 3.4. *Let G be a GGS group defined by a nonconstant vector and let $n \in \mathbb{N}$. Then the quotient $G^{(n)}/G^{(n+1)}$ is an elementary abelian p-group.*

Proof. It is sufficient to show that all p-th powers in $G^{(n)}$ are contained in $G^{(n+1)}$ for all $n \in \mathbb{N}$. For $n = 0$, this is the statement of Lemma 2.8(i). Since G is self-similar, we find $\psi(G') \leq \mathsf{X}_p\, G$, and, as a consequence, $\psi(G^{(n)}) \leq \mathsf{X}_p\, G^{(n-1)}$ for all $n \geq 1$. Thus for $n = 1$, notice that

$$\psi((G')^p) \leq (\mathsf{X}_p\, G)^p \leq \mathsf{X}_p\, \gamma_3(G) = \psi(\gamma_3(\mathrm{St}_G(1))) \leq \psi(G''),$$

using Lemma 2.8(ii), Theorem 2.9(i) and Proposition 2.10. For general $n > 1$, using induction and Theorem 1.2, we see that

$$\psi(G^{(n)})^p \leq (\mathsf{X}_p\, G^{(n-1)})^p = \mathsf{X}_p\, (G^{(n-1)})^p \leq \mathsf{X}_p\, G^{(n)} = \psi(G^{(n+1)}).$$ ☐

Corollary 1.3 follows immediately. It remains to prove Theorem 1.1.

Theorem 1.1. *Let p be an odd prime and let G be a GGS group acting on the p-adic tree defined by a nonconstant vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$. Let $\boldsymbol{e}'$ be the tuple of differences between the entries of $\boldsymbol{e}$ and let $\boldsymbol{e}''$ be the tuple of differences of $\boldsymbol{e}'$. Then for $n \geq 2$,*

$$\log_p |G : G^{(n)}| = p^{n-2}(p + \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'')) - \frac{p^{n-1}-1}{p-1} \cdot \mathrm{sym}(\boldsymbol{e}) + 1.$$

Proof. Using Theorem 1.2, we find for $n \geq 3$

$$\log_p |G^{(n)} : G^{(n+1)}| = \log_p |\mathsf{X}_p\, G^{(n-1)} : \mathsf{X}_p\, G^{(n)}| = p \cdot \log_p |G^{(n-1)} : G^{(n)}|,$$

and consequently

$$\log_p |G'' : G^{(n)}| = \sum_{i=0}^{n-3} p^i \log_p |G'' : G^{(3)}| = \frac{p^{n-2}-1}{p-1} \cdot \log_p |G'' : G^{(3)}|.$$

Employing our previous results we find

$$\begin{aligned}
\log_p |G'':G^{(3)}| &\overset{\text{Thm. 1.2}}{=} \log_p |\mathsf{X}_p\, G':\mathsf{X}_p\, G''| - \log_p |\mathsf{X}_p\, G':\psi(G'')| \\
&\overset{\text{Prop. 3.2}}{=} p \cdot \log_p |G':G''| - (\mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'')) \\
&\overset{\text{Lem. 3.3}}{=} p(p-1+\mathrm{con}(\boldsymbol{e}')+\mathrm{sym}(\boldsymbol{e}'')-\mathrm{sym}(\boldsymbol{e})) - (\mathrm{con}(\boldsymbol{e}')+\mathrm{sym}(\boldsymbol{e}'')) \\
&= (p-1)(p+\mathrm{con}(\boldsymbol{e}')+\mathrm{sym}(\boldsymbol{e}'')) - p \cdot \mathrm{sym}(\boldsymbol{e}).
\end{aligned}$$

Putting everything together (using Lemma 2.8(i) and, again, Lemma 3.3), we find

$$\begin{aligned}
\log_p |G : G^{(n)}| &= (p^{n-2}-1)(p + \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'')) \\
&\qquad - \frac{p^{n-1}-p}{p-1} \cdot \mathrm{sym}(\boldsymbol{e}) + \log_p |G' : G''| + \log_p |G : G'| \\
&= p^{n-2}(p + \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'')) - \frac{p^{n-1}-1}{p-1} \cdot \mathrm{sym}(\boldsymbol{e}) + 1.
\end{aligned}$$ ☐

GGS groups with differentially constant defining vector. A vector $\boldsymbol{e}$ is called *differentially constant* if it satisfies $\mathrm{con}(\boldsymbol{e}') = 1$ (and thus also $\mathrm{sym}(\boldsymbol{e}'') = 1$). In view of Theorem 1.1, GGS groups with differentially constant defining vector display the largest indices $|G : G^{(n)}|$ among GGS groups with nonconstant defining vector, as $\mathrm{con}(\boldsymbol{e}') = 1$ implies $\mathrm{sym}(\boldsymbol{e}) = 0$. The condition $\mathrm{con}(\boldsymbol{e}') = 1$ is a strong restriction on the defining vector, making it possible to determine the isomorphism classes of differentially constant GGS groups.

Resolving the definition, one finds that $\mathrm{con}(\boldsymbol{e}') = 1$ implies that there exist $k, m \in \mathbb{F}_p$ such that $\boldsymbol{e} = (k+m, k+2m, \dots, k+(p-1)m)$. We introduce the shorthand notation $\mathrm{dc}(k, m)$ for the vector given above. If $m = 0$, evidently $\mathrm{con}(\boldsymbol{e}) = 1$.

Proposition 3.5. *For any given odd prime p there are precisely three isomorphism classes of differentially constant GGS groups acting on the p-adic tree*:

(i) *one consisting of the constant GGS group*,

(ii) *one consisting of a single periodic GGS group*,

(iii) *one containing precisely $p-1$ nonperiodic GGS groups.*

Proof. Recall the isomorphism class preserving action $*$ of $(\mathbb{F}_p^\times)^2$ on the set of defining vectors given by (2-1). Let G be the GGS group defined by $\mathrm{dc}(k, m)$ and let $\lambda, \mu \in \mathbb{F}_p^\times$. Then

$$\mathrm{dc}(k, m) * (\lambda, \mu) = \mathrm{dc}(\lambda k, \lambda m) * (1, \mu) = \mathrm{dc}(\lambda k, \lambda\mu m).$$

If $m = 0$, the vector $\mathrm{dc}(k, 0)$ is constant, and we are in case (i). If $m \neq 0$, we find

$$\mathrm{dc}(0, m) * (1, m^{-1}) = \mathrm{dc}(0, 1) \quad \text{and} \quad \mathrm{dc}(k, m) * (k^{-1}, km^{-1}) = \mathrm{dc}(1, 1)$$

for $k \neq 0$. At the same time, $\mathrm{dc}(0, m) * (\lambda, \mu) \neq \mathrm{dc}(1, 1)$ for all $\lambda, \mu \in \mathbb{F}_p^\times$, whence $\mathrm{dc}(0, 1)$ and $\mathrm{dc}(1, 1)$ represent distinct isomorphism classes. The $(\mathbb{F}_p^\times)^2$-orbit of $\mathrm{dc}(0, 1)$ consists of the multiples of $\mathrm{dc}(0, 1)$ and all the associated GGS groups are identical. By [9, Example 9.1], the GGS group defined by $\boldsymbol{e}$ is periodic if and only if the sum of the entries of $\boldsymbol{e}$ vanishes, i.e., if $\dim \mathrm{Circ}(\bar{f}e) \leq p-1$. For $\mathrm{dc}(0, 1) = (1, \dots, p-1)$ this is the case, but not for $\mathrm{dc}(1, 1) = (2, \dots, p-1, 0)$. As seen above, $(\mathbb{F}_p^\times)^2$ acts transitively on $\{\mathrm{dc}(k, m) \mid k, m \in \mathbb{F}_p^\times\}$. Since only proportional vectors yield identical GGS groups, there are $p-1$ distinct GGS groups isomorphic to the group defined by $\mathrm{dc}(1, 1)$. □

In case of the prime $p = 3$, all GGS groups are differentially constant and the unique periodic GGS group is the Gupta–Sidki 3-group. It is interesting to see that other invariants take extremal values for the groups G_p defined by $(1, 2, \dots, p-1)$ (for arbitrary p): By [7, Theorem B], the Hausdorff dimension of the GGS group defined by $\boldsymbol{e}$ is given by

$$\frac{(p-1)\,\mathrm{cr}(\boldsymbol{e})}{p^2}-\frac{\mathrm{sym}(\boldsymbol{e})}{p^2}-\frac{\mathrm{con}(\boldsymbol{e})}{(p-1)p^2}.$$

By Corollary 2.6, $\mathrm{cr}(\boldsymbol{e})=2$ if and only if $\boldsymbol{e}$ is a nonzero multiple of $\mathrm{dc}(0,1)$. In particular, a symmetric defining vector $\boldsymbol{e}$ fulfils $\mathrm{cr}(\boldsymbol{e})>2$. Furthermore, a constant defining vector has $\mathrm{cr}(\boldsymbol{e})=p$. Thus the group G_p is the unique GGS group with the minimal (among GGS groups acting on a fixed tree) possible Hausdorff dimension $2(p-1)/p^2$, while the group defined by $\mathrm{dc}(1,1)$ is among those with maximal Hausdorff dimension $(p-1)/p$.

The automorphism group of G_p is as large as possible; cf. [15, Example 6.2].

Comparison with the congruence subgroups. The members of the derived series of a GGS group share certain properties with the level stabilisers. They both form filtrations of the group, with $\mathrm{St}_G(n)\geq G^{(n)}$ for all $n\in\mathbb{N}$; for sufficiently high values of n, they satisfy $\psi(G_n)=\mathsf{X}_p\,G_{n-1}$ by Theorem 1.2 and [7, Lemma 3.3], respectively; furthermore, the quotients of respective consecutive members are elementary abelian p-groups. Using the formula for the index of the n-th level stabiliser provided by Fernández-Alcober and Zugadi-Reizabal in [7, Theorem A] one finds

$$\log_p|G:\mathrm{St}_G(2)|=\mathrm{cr}(\boldsymbol{e})+1,$$

hence, using the consequence $\log_p|G:\mathrm{St}_G(1)'|=p+1$ of Lemma 2.8, we find that $\log_p\bigl|\mathrm{St}_G(2):\mathrm{St}_G(1)'\bigr|=p-\mathrm{cr}(\boldsymbol{e})$. A comparison with $\log_p|G:G''|=p+\mathrm{con}(\boldsymbol{e}')+\mathrm{sym}(\boldsymbol{e}'')-\mathrm{sym}(\boldsymbol{e})+1$ makes it apparent that $G''=\mathrm{St}_G(2)$ if and only if $\boldsymbol{e}'$ is nonconstant, $\mathrm{cr}(\boldsymbol{e})=p$, and $\mathrm{sym}(\boldsymbol{e})=\mathrm{sym}(\boldsymbol{e}'')$. Since $\psi(G_n)=\mathsf{X}_p\,G_{n-1}$ holds for both the derived series and the series of level stabilisers from $n=3$ onwards, the series only differ in their first term in this case, and otherwise have no equal terms at all. The largest difference is attained for the periodic groups G_p with the differentially constant defining vector $\mathrm{dc}(0,1)$; recall that $\mathrm{cr}(\mathrm{dc}(0,1))=2$, and thus

$$\log_p\,|\mathrm{St}_{G_p}(n):G_p^{(n)}|=p^{n-1}.$$

By Proposition 2.5, $\mathrm{cr}(\boldsymbol{e})=2$ is the minimum possible value; however one finds that $\mathrm{cr}(\boldsymbol{e})$ may take any value in $[2,p]$. Therefore the number of distinct sequences $(|G:\mathrm{St}_G(n)|)_{n\in\mathbb{N}}$ obtained by any GGS group is between p and $2p-1$ (note that symmetric defining vectors do not admit all values $[2,p]$ under cr); in particular, the number of classes of GGS groups separated by the sequence of indices of their level stabilisers grows with p. In contrast, the sequences $(|G:G^{(n)}|)_{n\in\mathbb{N}}$ yield a partition into five subsets for $p\neq 3$; for $p=3$, one obtains 2 classes.

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