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THE GENERIC EXTENSION MAP AND MODULAR STANDARD MODULES

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We study two classes of ℓ-modular standard modules of the general linear group. The first class is obtained by reducing existing standard modules over $\overline{\mathbb{Q}}_\ell$ to $\overline{\mathbb{F}}_\ell$ with respect to their natural integral structure. The second class is obtained by studying the generic extension map of the cyclical quiver, which was motivated by the construction of certain monomial bases of quantum algebras. In the latter case we also manage to prove a modular version of the Langlands classification, similar to the work of Langlands and Zelevinsky over C. We also compute the corresponding ℓ-modular Rankin–Selberg L-functions and check that they agree with the L-functions of their C-parameters constructed by Kurinczuk and Matringe.

1. Introduction

Let F be a nonarchimedean local field with ring of integers $\mathfrak{O}_\mathrm{F}$ and residue field k, whose cardinality we denote by q, and G be a reductive group over F. We also fix a prime ℓ not dividing q together with a field of coefficients $R \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell, \mathbb{C}\}$. The classification of irreducible smooth representations of $G(\mathrm{F})$ over R is of great importance in establishing the local Langlands correspondence. If $R = \mathbb{C}$, the classification was first achieved by Langlands in what is now known as the *Langlands classification*. His classification associates to each irreducible smooth representation π of $G(\mathrm{F})$ a parabolic subgroup $P \subseteq G$, a tempered representation σ of the F-points of the Levi component M of P and a character η of the center of M satisfying certain inequalities. Then π can be realized as the unique quotient of the parabolically induced representation $\mathcal{S}(\pi) = \mathrm{ind}_{P(\mathrm{F})}^{G(\mathrm{F})}(\sigma \otimes \eta)$, known as the standard module associated to π, and the triple (P, σ, η) is unique up to certain natural symmetries.

The analytical nature of the Langlands classification as sketched above makes it clear that it is a nontrivial task to extend it to fields of nonzero characteristic, as soon as one ventures beyond the banal setting of [22]. Therefore the construction of modular standard modules remained an open problem. The proposed definitions

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given in this paper avoid the above issue by considering standard modules which are, in the modular case, not necessarily induced representations, but rather certain subrepresentations of the space of Whittaker functions.

To motivate this approach we return for a moment to the problem of the classification of irreducible modular representations. We recall that in the case $G = \mathrm{GL}_n$ Bernstein and Zelevinsky gave in [32; 2; 3] a second classification of irreducible complex representation in terms of combinatorial objects called *multisegments*. To this date this complete second classification is a feature only present in the representation theory of GL_n and its inner forms, although certain extensions of these ideas to more general groups have been pursued [10]. Furthermore, the Langlands and Zelevinsky classifications are related by the Aubert–Zelevinsky involution, an involution on the set of isomorphism classes of the irreducible representations of $\mathrm{GL}_n(\mathrm{F})$; see [32; 1]. It thus fell upon the classification of Bernstein and Zelevinsky in the case $G = \mathrm{GL}_n$ to offer itself to generalization in the modular case.

In the works of Vignéras [30] and Mínguez and Sécherre [23], this feat has been accomplished by extending their ideas. In general, the study of irreducible representations of GL_n over $\overline{\mathbb{F}}_\ell$ shows many similarities to the study of complex representations, but with several key differences as we will see later. For example, no longer is every cuspidal representation supercuspidal.

Note that the usefulness of standard modules in the complex case goes much further than the simple classification of irreducible representations. A prominent example, and the main motivator of this paper, are the local *Rankin–Selberg L-factors* $L(s, \rho, \rho')$ of [11], which are defined for pairs of complex representations of Whittaker type (ρ, ρ') and they are rational functions in q^s. In particular, if $G = \mathrm{GL}_n$, every standard module $\mathcal{S}(\pi)$ is of Whittaker type, which in turn allows one to define the invariant $L(s, \mathcal{S}(\pi), \mathcal{S}(\pi'))$ for any pair of irreducible representations of GL_n. It comes as no surprise that the L-factor $L(s, \mathcal{S}(\pi), \mathcal{S}(\pi'))$ can be explicitly computed in terms of the Langlands parameters of π and π'.

Passing from the complex to the modular setting, the definition of local Rankin–Selberg L-factors has been extended in [14] to the more general fields R we are going to consider in this paper. The authors again associate to a pair of representations of Whittaker type an L-factor $L(X, \pi, \pi')$, which is a rational function in X (with $X = q^{-s}$ in the $\mathbb{C}$-case this recovers the original construction of [12]). The Rankin–Selberg L-factors so obtained satisfy a functional equation, which in turn gives rise to ϵ- and γ-factors associated to the pair (π, π'). In [16] the authors associate to each irreducible representation of $\mathrm{GL}_n(\mathrm{F})$ a so-called C-parameter $\mathrm{C}(\pi)$, which is a modular version of the Langlands parameter. Moreover, they equip C-parameters with a tensor product $\otimes_{ss}$ and define their L-, ϵ- and γ- factors.

In this paper we hope to initiate the investigation of possible candidates for standard modules in the modular setting and their properties. We do this by giving two natural possible candidates for standard modules, one obtained by reducing the ones defined over $\overline{\mathbb{Q}}_\ell$ to $\overline{\mathbb{F}}_\ell$, the other one by considering the so-called *generic extension map*, which is rooted in the analysis of certain (quantum)-algebras associated to certain Dynkin or affine Dynkin quivers. We will prove that the former always contain the latter and note that at the moment the evidence points towards them being equal, although we do not treat this question in this paper.

The properties discussed above give us natural constraints our potential standard modules should satisfy. Firstly, they should admit a modular formulation of the Langlands classification and secondly, they should allow us to define Rankin–Selberg L-functions, which should be explicitly computable using the C-parameters of the underlying irreducible representations.

The contents of this paper can thus be roughly divided into three parts: the construction of standard modules, establishing certain representation-theoretic properties and finally the computation of their Rankin–Selberg L-factors.

To state our results more precisely, we need to introduce some notation. Let $\psi : \mathrm{F} \to R$ be a smooth, additive character and recall the space of Whittaker-functions,

$$\mathcal{W}(\psi) := \left\{ f : G_n \to R : f\left(\begin{pmatrix} 1 & u_{1,2} & \dots & \dots \\ 0 & \ddots & \dots & \dots \\ 0 & \dots & 1 & u_{n-1,n} \\ 0 & \dots & 0 & 1 \end{pmatrix} g\right) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right) f(g),\ f \text{ locally constant} \right\}.$$

We also recall that a smooth representation π of G_n of finite length is called of Whittaker type if

$$\dim_R \mathrm{Hom}_{G_n}(\pi, \mathcal{W}(\psi)) = 1,$$

and in this case we denote its image by $\mathcal{W}(\pi, \psi)$. We let

$$\mathfrak{Rep}_{W,\psi,n} = \{\mathcal{W}(\pi, \psi) : \pi \text{ a representation of } G_n \text{ of Whittaker type}\}.$$

Let us recall that to a cuspidal representation of G_m we can associate $o(\rho) \in \mathbb{N} \cup \{\infty\}$, with $o(\rho) = \infty$ if and only if $\mathrm{char}(R) = 0$, and and that if $o(\rho)$ is finite, it is the minimal $k \in \mathbb{Z}_{>0}$ such that $\rho \cong \rho|-|^k$. We let moreover be $e(\rho) = \ell$ if $o(\rho) = 1$ and $e(\rho) = o(\rho)$ otherwise. To two integers $a \leq b$ and ρ cuspidal, we associate a segment (over R) $[a, b]_\rho = (\rho|-|^a, \dots, \rho|-|^b)$ and we identify $[a, b]_\rho$ with $[a + o(\rho), b + o(\rho)]_\rho$. We also write $[a, b]_\rho^\vee = [-b, -a]_{\rho^\vee}$. A finite formal sum of segments is called a multisegment and we denote the set of multisegments over R by $\mathcal{M}ult_R$, to which we extend the operation $(-)^\vee$. A multisegment is called aperiodic if it does not contain a submultisegment of the form

$$[a, b]_\rho + [a+1, b+1]_\rho + \dots + [a+e(\rho)-1, b+e(\rho)-1]_\rho,$$

and we denote the set of aperiodic multisegments by $\mathcal{M}ult_R^{\mathrm{ap}}$. We decompose any multisegment $\mathfrak{m}$ as $\mathfrak{m}_b + \mathfrak{m}_{nb}$, where $\mathfrak{m}_{nb}$ consists of those $[a, b]_\rho$ with $o(\rho) = 1$. We let $\mathcal{M}ult_{R,\square}^{\mathrm{ap}}$ be the set of aperiodic multisegments such that $\mathfrak{m} = \mathfrak{m}_b$. Note that if $\mathrm{char}(R) = 0$ all multisegments are aperiodic and $\mathcal{M}ult_{\overline{\mathbb{Q}}_\ell} = \mathcal{M}ult_{\overline{\mathbb{Q}}_\ell,\square}^{\mathrm{ap}}$. Then there exists a bijection

$$\langle - \rangle : \mathcal{M}ult_R^{\mathrm{ap}} \to \mathfrak{Irr}(R);$$

see [25] and [23]. The definition of $\mathcal{M}ult_{R,\square}^{\mathrm{ap}}$ is motivated by the following facts. If, as usual, we denote by $\times$ the normalized parabolic induction, then $\langle \mathfrak{m} \rangle \cong \langle \mathfrak{m}_b \rangle \times \langle \mathfrak{m}_{nb} \rangle$ and

$$L(X, \mathrm{C}(\langle \mathfrak{m} \rangle) \otimes_{ss} \mathrm{C}(\langle \mathfrak{n} \rangle)) = L(X, \mathrm{C}(\langle \mathfrak{m}_b \rangle) \otimes_{ss} \mathrm{C}(\langle \mathfrak{n}_b \rangle)).$$

In particular, representations with nontrivial L-functions are precisely those coming from $\mathcal{M}ult_{R,\square}^{\mathrm{ap}}$. The set $\mathcal{M}ult_R^{\mathrm{ap}}$ moreover admits an order, denoted by $\preceq$, which comes from the *degeneration order* on certain quiver-varieties, which we will discuss in a moment. Our goal is then to construct a map

$$\mathcal{S}_\psi : \mathcal{M}ult_R^{\mathrm{ap}} \to \mathfrak{Rep}_{W,\psi} = \bigcup_{n \in \mathbb{N}} \mathfrak{Rep}_{W,\psi,n}$$

satisfying, among others, the following properties.

(1) $L(X, \mathcal{S}_\psi(\mathfrak{m}) \otimes \mathcal{S}_{\psi^{-1}}(\mathfrak{n})) = L(X, \mathrm{C}(\langle \mathfrak{m} \rangle) \otimes \mathrm{C}(\langle \mathfrak{n} \rangle))$.

(2) $\mathcal{S}_\psi(\mathfrak{m})$ admits $\langle \mathfrak{m} \rangle$ as a quotient and $\langle \mathfrak{m} \rangle$ appears in the Jordan–Hölder decomposition of $\mathcal{S}_\psi(\mathfrak{m})$ with multiplicity one.

(3) $\mathcal{S}_\psi(\mathfrak{m})$ admits $\langle \mathfrak{m} \rangle$ as its unique irreducible quotient.

Properties (2) and (3) would imply that $\dim_R \mathrm{Hom}_{G_n}(\mathcal{S}_\psi(\mathfrak{m}), \mathcal{S}_{\psi^{-1}}(\mathfrak{m}^\vee)^\vee) = 1$ and every morphism in this space would factor through $\langle \mathfrak{m} \rangle$. The description of irreducible representations in the style of properties (2) and (3) is also known as the *Langlands classification*, and has not yet been achieved over $\overline{\mathbb{F}}_\ell$, unlike the case $\mathrm{char}(R) = 0$, where the relevant statements were first proved in [32] (see p. 2).

In this paper we construct such a map $\mathcal{S}_\psi$ satisfying (1) and (2), as well as a map

$$\mathcal{S}_{\mathrm{gen},\psi} : \mathcal{M}ult_{R,\square}^{\mathrm{ap}} \to \mathfrak{Rep}_{W,\psi}$$

satisfying the analogues of properties (2) and (3) for multisegments in $\mathcal{M}ult_{R,\square}^{\mathrm{ap}}$. We also show that $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) \subseteq \mathcal{S}_\psi(\mathfrak{m})$ for $\mathfrak{m} \in \mathcal{M}ult_{R,\square}$. If $\mathrm{char}(R) = 0$, these two constructions agree.

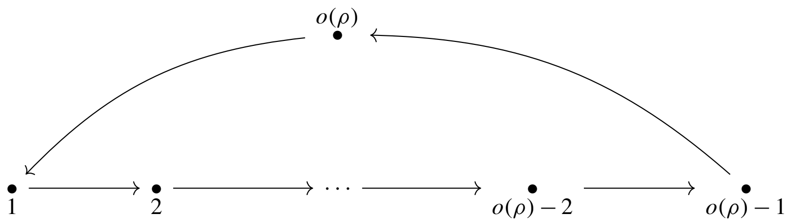
We start with a sketch of the construction of $\mathcal{S}_\psi$. If $\tilde{\rho}$ is a cuspidal representation over $\overline{\mathbb{Q}}_\ell$ admitting an integral structure, we denote its reduction mod ℓ by $\mathrm{r}_\ell(\tilde{\rho})$. If $\mathrm{r}_\ell(\tilde{\rho})$ is again a cuspidal representation ρ, we say $\tilde{\rho}$ is a lift of ρ. It is then straightforward to define a lift $\tilde{\mathfrak{m}} \in \mathcal{M}ult_{\overline{\mathbb{Q}}_\ell}$ of $\mathfrak{m} \in \mathcal{M}ult_{\overline{\mathbb{F}}_\ell}$ by first saying that a segment $[a', b']_{\tilde{\rho}}$ is a lift of $[a, b]_\rho$ if $\tilde{\rho}$ is a lift of ρ, $a = a' \bmod o(\rho)$ and $b - a = b' - a'$, and then extending this notion linearly to all multisegments. In this case one can equip $\mathcal{S}_\psi(\tilde{\mathfrak{m}})$ with its natural integral structure given by those Whittaker functions that take values in $\overline{\mathbb{Z}}_\ell$, and we denote the reduction mod ℓ of this specific integral structure by $\overline{\mathcal{S}_\psi(\tilde{\mathfrak{m}})}$. For $\mathfrak{m}$ an aperiodic multisegment over $\overline{\mathbb{F}}_\ell$, we write

$$\mathcal{S}_\psi(\mathfrak{m}) := \bigcap_{\tilde{\mathfrak{m}} \text{ lift of } \mathfrak{m}} \overline{\mathcal{S}_\psi(\tilde{\mathfrak{m}})},$$

where the intersection was taken in $\mathcal{W}(\psi)$.

Theorem 1. *The map $\mathcal{S}_\psi : \mathcal{M}ult^{\mathrm{ap}}_{\overline{\mathbb{F}}_\ell} \to \mathfrak{Rep}_{W,\psi}$ satisfies property* (1)*. If $\mathfrak{m} \in \mathcal{M}ult^{\mathrm{ap}}_{\overline{\mathbb{F}}_\ell,\square}$, then $\langle\mathfrak{m}\rangle$ is a quotient of $\mathcal{S}_\psi(\mathfrak{m})$ and appears with multiplicity* 1 *in it.*

The proof of this theorem uses our second construction of standard modules, denoted by $\mathcal{S}_{\mathrm{gen},\psi}$, to which we come now. We will focus on the case $R = \overline{\mathbb{F}}_\ell$, but the construction works similarly for $\mathrm{char}(R) = 0$. In the introduction, for the sake of clarity, we will focus on multisegments of the form $[a_1, b_1]_\rho + \cdots + [a_k, b_k]_\rho$ for a fixed cuspidal ρ with $o(\rho)>1$. We denote this set by $\mathcal{M}ult_{\overline{\mathbb{F}}_\ell}(\rho)$ and the aperiodic multisegments in it by $\mathcal{M}ult_{\overline{\mathbb{F}}_\ell}(\rho)^{\mathrm{ap}}$. Let Q be the cyclical quiver with $o(\rho)$ vertices and oriented counterclockwise.



Then the isomorphism classes of finite-dimensional, nilpotent $\mathbb{C}$-representations of Q, denoted by $[Q]$, are in bijection with $\mathcal{M}ult_{\overline{\mathbb{F}}_\ell}(\rho)$. The order $\preceq$ on multisegments mentioned above is nothing but the degeneration order on $[Q]$ transported to $\mathcal{M}ult_{\overline{\mathbb{F}}_\ell}(\rho)$. In [28; 4; 27] the authors investigate the so-called *generic extension map* $* : [Q] \times [Q] \to [Q]$, which sends the tuple $([M], [N])$ to $[X]$, where $X \in \mathrm{Ext}^1_Q(M, N)$ is such that $\dim_{\mathbb{C}} \mathrm{Hom}_Q(X, X)$ is minimal. The product so constructed,

$$* : \mathcal{M}ult_{\overline{\mathbb{F}}_\ell}(\rho) \times \mathcal{M}ult_{\overline{\mathbb{F}}_\ell}(\rho) \to \mathcal{M}ult_{\overline{\mathbb{F}}_\ell}(\rho),$$

is associative. The subset of $\mathcal{M}ult_{\overline{\mathbb{F}}_\ell}(\rho)$ generated by multisegments of the form $[i, i]_\rho$ is then precisely the set of aperiodic multisegments $\mathcal{M}ult_{\overline{\mathbb{F}}_\ell}(\rho)^{\mathrm{ap}}$.

The next result was motivated by a similar consideration of monomial bases of quantum and Hall algebras; see for example [7] or [28].

Theorem 2. *Let* $\mathfrak{m} = [i_1, i_1]_\rho * \cdots * [i_k, i_k]_\rho \in \mathcal{M}ult^{\mathrm{ap}}_{\overline{\mathbb{F}}_\ell}(\rho)$. *Then*

$$\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) := \mathcal{W}(\rho|-|^{i_1} \times \cdots \times \rho|-|^{i_k}, \psi)$$

only depends on $\mathfrak{m}$ *and satisfies properties* (1) *and* (2).

This construction allows us to define the desired map:

Theorem 3. *Let* $\mathfrak{m}, \mathfrak{m}' \in \mathcal{M}ult^{\mathrm{ap}}_{R,\square}$. *Then* $\mathcal{S}_{\mathrm{gen},\psi} : \mathcal{M}ult^{\mathrm{ap}}_{R,\square} \to \mathfrak{Rep}_{W,\psi}$ *is defined by*

$$\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m} * \mathfrak{m}') = \mathcal{W}(\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) \times \mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}'), \psi).$$

Moreover, if $\mathrm{char}(R) = 0$, $\mathcal{S}_\psi = \mathcal{S}_{\mathrm{gen},\psi}$ *and if* $R = \overline{\mathbb{F}}_\ell$, $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) \subseteq \mathcal{S}_\psi(\mathfrak{m})$.

Although the following seems at the moment out of reach, examples of small rank seem to suggest the equality $\mathcal{S}_\psi = \mathcal{S}_{\mathrm{gen},\psi}$ on $\mathcal{M}ult^{\mathrm{ap}}_{R,\square}$ as well as the equality $\mathcal{S}_\psi(\mathfrak{m}) = \overline{\mathcal{S}_\psi(\tilde{\mathfrak{m}})}$ for any lift $\tilde{\mathfrak{m}}$ of $\mathfrak{m}$.

2. Notation

Let F be a nonarchimedean local field with ring of integers $\mathfrak{O}_\mathrm{F}$ and residue field k, whose cardinality we denote by q. We also choose a uniformizer ϖ_F of the maximal ideal of $\mathfrak{O}_\mathrm{F}$ and fix a prime ℓ not dividing q. We let $|-|$ be the absolute value of F such that $|\varpi_\mathrm{F}| = q^{-1}$.

From now on R is one of the algebraically closed fields $\overline{\mathbb{F}}_\ell$ or $\overline{\mathbb{Q}}_\ell$. We let Λ be the maximal ideal of $\overline{\mathbb{Z}}_\ell \subseteq \overline{\mathbb{Q}}_\ell$ and fix a square root $q^{\frac{1}{2}}$ of q in $\overline{\mathbb{Z}}_\ell$. By abuse of notation, we are also going to denote its image in $\overline{\mathbb{F}}_\ell$ by $q^{\frac{1}{2}}$.

For $n \in \mathbb{Z}_{\geq 0}$ we let $G_n := \mathrm{GL}_n(F)$. We denote by 1_n the identity in G_n and by

$$w_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

the antidiagonal. For $m, n \in \mathbb{N}$, we denote by $w_{n,m}$ the block-diagonal embedding of $(1_m, w_n)$ into G_{n+m} and the space $M_{n,m}$ of $n \times m$ matrices with entries in F.

Inside G_n we consider for $k \in \mathbb{Z}$ the closed subsets

$$G_n^k := \{g \in G_n : \mathrm{val}_\mathrm{F}(\det(g)) = k\},$$

where val_F denotes the valuation of F with $\mathrm{val}_\mathrm{F}(\varpi) = 1$. We let B_n be the Borel subgroup of G_n of upper diagonal matrices and N_n its unipotent subgroup. More generally, if $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a partition of n, we denote by P_α the parabolic subgroup containing B_n with Levi component $G_\alpha := G_{\alpha_1} \times \cdots \times G_{\alpha_k}$. Its opposite parabolic subgroup $\overline{P}_\alpha$ is conjugate to $P_{(\alpha_k,\dots,\alpha_1)}$. We denote by

$$H_{n,m} := \left\{ \begin{pmatrix} g & x \\ 0 & n \end{pmatrix} : \, g \in G_n, \; x \in M_{n,m}, \; n \in N_m \right\} \subseteq G_{n+m}$$

and let $P_n := H_{n-1,1}$ be the mirabolic subgroup of G_n. Let $C_c^\infty(\mathrm{F}^n)$ be the space of R-valued locally constant and compactly supported functions on F^n and we set $\epsilon_n := (0, \dots, 0, 1) \in \mathrm{F}^n$.

2.1. ***Representations.*** In the next sections we recall the usual setup for ℓ-adic and ℓ-modular representation theory. For details, see [29]. Let $G \subseteq G_n$ be a closed subgroup. We denote by $\mathfrak{Rep}(G, R)$ the category of smooth, admissible representations of finite length of G. Whenever possible we will omit the field of coefficients R. Let $\alpha = (\alpha_1, \dots, \alpha_k)$ be a partition of n. We let $\mathfrak{Rep}_\alpha = \mathfrak{Rep}(G_\alpha)$, let $\mathfrak{Irr}_\alpha$ be the set of isomorphism classes of irreducible representations in $\mathfrak{Rep}_\alpha$ and set

$$\mathfrak{Rep} := \bigcup_{n\in\mathbb{N}} \mathfrak{Rep}_n, \quad \mathfrak{Irr} := \bigcup_{n\in\mathbb{N}} \mathfrak{Irr}_n.$$

If $G \subseteq H \subseteq G_n$ are closed subgroups, we denote the functors of normalized induction by Ind_H^G and their compactly supported version by ind_H^G. We recall the normalized Jacquet functor and parabolic induction corresponding to parabolic subgroups $P = MN$ of G_α, which give rise to the exact functors

$$r_P : \mathfrak{Rep}_\alpha \to \mathfrak{Rep}(M), \quad \mathrm{ind}_P^{G_\alpha} : \mathfrak{Rep}(M) \to \mathfrak{Rep}_\alpha.$$

We write $r_\alpha := r_{P_\alpha}$ and $\overline{r}_\alpha := r_{\overline{P}_\alpha}$. Recall that r_α and $\overline{r}_\alpha$ are the left and right adjoints, respectively, of $\mathrm{ind}_{P_\alpha}^{G_n}$, by Frobenius and Bernstein reciprocities (respectively). By abuse of notation we will also notate the maps they induce between the respective Grothendieck groups by the same letters. As is conventional, we will write

$$\pi_1 \times \cdots \times \pi_k := \mathrm{ind}_{P_\alpha}^{G_n}(\pi_1 \otimes \ldots \otimes \pi_k).$$

If χ is a smooth character of F and $\pi \in \mathfrak{Rep}$, we write $\chi\pi = \chi(\det)\pi$. If $\pi \in \mathfrak{Rep}_n$, we denote the corresponding element in the Grothendieck of $\mathfrak{Rep}_n$ by $[\pi]$ and define the length of π as the number of its irreducible subquotients counted with multiplicity. If π is a representation of G_n, we denote by $\pi^\mathfrak{c}$ the representation obtained by twisting π by $g \mapsto w_n(g^{-1})^t w_n$. We recall that if π is irreducible and $\ell > 2$, then $\pi^\mathfrak{c} \cong \pi^\vee$ (see for example [29] or [23, Remark 2.7]), and

$$(\pi_1 \times \pi_2)^\mathfrak{c} = {\pi_2}^\mathfrak{c} \times {\pi_1}^\mathfrak{c}.$$

Finally, we recall that parabolic induction on G_n is commutative on the level of Grothendieck groups, in particular for $\pi, \pi' \in \mathfrak{Irr}$ such that $\pi \times \pi'$ is irreducible,

$\pi \times \pi' \cong \pi' \times \pi$; see [29, 1.16] for $\ell > 2$, [32, Theorem 1.9] for the case $R = \mathbb{C}$ and [23, Proposition 2.6] in general.

2.1.1. *Cuspidal representations.* A representation $\rho \in \mathfrak{Irr}_n$ is called cuspidal if for all nontrivial partitions α of n, $r_\alpha(\rho) = 0$. It is called supercuspidal if there exists no nontrivial partition α and $\pi \in \mathfrak{Irr}_\alpha$ such that ρ appears as a subquotient of $\mathrm{ind}_{P_\alpha}^{G_n} \pi$. In general, supercuspidal implies cuspidal and if $R = \overline{\mathbb{Q}}_\ell$, cuspidal implies supercuspidal. We denote the subset of $\mathfrak{Irr}_n$ consisting of cuspidal respectively supercuspidal representations by $\mathfrak{C}_n$ respectively $\mathfrak{SC}_n$ and define

$$\mathfrak{C} := \bigcup_{n\in\mathbb{N}} \mathfrak{C}_n, \quad \mathfrak{SC} := \bigcup_{n\in\mathbb{N}} \mathfrak{SC}_n.$$

We also recall the notation of cuspidal support, meaning that if $\pi \in \mathfrak{Irr}$ there exist up to possible permutation and isomorphism unique $\rho_1, \ldots, \rho_k \in \mathfrak{C}$ such that $\pi \hookrightarrow \rho_1 \times \cdots \times \rho_k$. We denote by $\mathrm{cusp}(\pi) := [\rho_1] + \cdots + [\rho_k]$ the cuspidal support of π. Weakening in the above definition the condition of being a subrepresentation to being a subquotient and cuspidal representations to supercuspidal representations gives rise to the supercuspidal support $\mathrm{scusp}(\pi)$. If at any point the field of coefficients becomes important, we will add it in parentheses to the respective category or set, e.g., $\mathfrak{Irr}(\overline{\mathbb{F}}_\ell)$ versus $\mathfrak{Irr}(\overline{\mathbb{Q}}_\ell)$.

We recall the following notions; see for example [24, §3.4, §4.5]. Let $\rho \in \mathfrak{C}$ and recall that $\rho \times \chi\rho$ is reducible if and only if $\chi \cong |-|^{\pm}$. We recall the cuspidal line

$$\mathbb{Z}[\rho] := \{[\rho|-|^k] : k \in \mathbb{Z}\}$$

and denote the cardinality of $\mathbb{Z}[\rho]$ by $o(\rho)$. Note that $o(\rho)$ is finite if and only if $R = \overline{\mathbb{F}}_\ell$. Set

$$e(\rho) := \begin{cases} o(\rho) & \text{if } o(\rho) > 1, \\ \ell & \text{otherwise.} \end{cases}$$

One can associate to ρ an integer $f(\rho)$ via type theory. If $o(\rho) > 1$, $o(\rho)$ is the order of $q^{f(\rho)}$ in R. We let $\mathfrak{C}_n^{\square}$ be the subset of $\mathfrak{C}_n$ consisting of ρ such that $o(\rho) > 1$; i.e., those ρ which make $\rho \times \rho$ is irreducible. Finally, we say ρ and ρ' are in different cuspidal lines if $[\rho'] \notin \mathbb{Z}[\rho]$ and we set $\mathfrak{C}^{\square} := \bigcup_{n\in\mathbb{N}} \mathfrak{C}_n^{\square}$.

2.2. ***Multisegments.*** We now recall the combinatorics of multisegment (for details, see [32; 23]). Let $\rho \in \mathfrak{C}_m$ and $a \leq b \in \mathbb{Z}$. A segment is a sequence

$$[a, b]_\rho = ([\rho|-|^a], \ldots, [\rho|-|^b]).$$

Two segments $[a,b]_\rho$ and $[a',b']_{\rho'}$ are equal if and only if $\rho'|-|^a \cong \rho|-|^a$ and $b-a=b'-a'$. We also let $[a,b]_\rho^\vee=[-b,-a]_{\rho^\vee}$. Define

$$
\begin{aligned}
{}^-[a,b]_\rho &= [a+1,b]_\rho, \quad & [a,b]_\rho^- &= [a,b-1]_\rho,\\
{}^+[a,b]_\rho &= [a-1,b]_\rho, \quad & [a,b]_\rho^+ &= [a,b+1]_\rho,
\end{aligned}
$$

$$
a_\rho([a,b]_\rho)=a\in\begin{cases}\mathbb{Z}/(o(\rho)\mathbb{Z}) & \text{if } o(\rho)<\infty,\\ \mathbb{Z} & \text{if } o(\rho)=\infty,\end{cases}
$$

$$
b_\rho([a,b]_\rho)=b\in\begin{cases}\mathbb{Z}/(o(\rho)\mathbb{Z}) & \text{if } o(\rho)<\infty,\\ \mathbb{Z} & \text{if } o(\rho)=\infty.\end{cases}
$$

We define the length and degree of a segment as $l([a,b]_\rho):=b-a+1\in\mathbb{Z}$ and $\deg([a,b]_\rho)=(b-a+1)m\in\mathbb{Z}$. The cuspidal support of $[a,b]_\rho$ is defined as $\mathrm{cusp}([a,b]_\rho):=[\rho|-|^a]+\cdots+[\rho|-|]^b$. By abuse of notation we will often write ρ for the segment $[0,0]_\rho$.

A multisegment is a formal finite sum and we extend the length l, the degree deg, the notion of cuspidal support, and $(-)^\vee$ linearly. We let $\mathcal{M}ult_R$ be the set of multisegments and $\mathcal{M}ult_R(\rho)$ be the set of multisegments consisting only of segments of the form $[a,b]_\rho$. A multisegment is called aperiodic if it does not contain a submultisegment of the form

$$[a,b]_\rho+[a+1,b+1]_\rho+\cdots+[a+e(\rho)-1,b+e(\rho)-1]_\rho.$$

For any subset $\mathcal{N}\subseteq\mathcal{M}ult_R$, we denote by $\mathcal{N}^{\mathrm{ap}}$ the aperiodic multisegments in $\mathcal{N}$. For any $\mathfrak{m}\in\mathcal{M}ult_R$, we decompose $\mathfrak{m}=\mathfrak{m}_b+\mathfrak{m}_{nb}$ with $\mathfrak{m}_b$ consisting of all segments $[a,b]_\rho$ in $\mathfrak{m}$ with $\rho\in\mathfrak{C}^\square$. We let $\mathcal{M}ult_{R,\square}$ be the set of multisegments such that $\mathfrak{m}=\mathfrak{m}_b$.

Theorem 2.2.1 [23, §9; 32, §6]. *There exists a surjective map*

$$\mathrm{Z}:\mathcal{M}ult_R\to\mathfrak{Irr}$$

satisfying the following.

(1) $\mathrm{Z}(\mathfrak{m})\in\mathfrak{Irr}_{\deg(\mathfrak{m})}$.

(2) Z *restricted to aperiodic multisegments is a bijection.*

(3) $\mathrm{Z}(\mathfrak{m})^\vee\cong\mathrm{Z}(\mathfrak{m}^\vee)$.

(4) *For* $\mathfrak{m}\in\mathcal{M}ult_R^{\mathrm{ap}}$, $\mathrm{cusp}(\mathrm{Z}(\mathfrak{m}))=\mathrm{cusp}(\mathfrak{m})$. *If* $\mathfrak{m}\in\mathcal{M}ult_R(\rho)^{\mathrm{ap}}$ *with* $\rho\in\mathfrak{SC}$, *then* $\mathrm{scusp}(\mathrm{Z}(\mathfrak{m}))=\mathrm{cusp}(\mathrm{Z}(\mathfrak{m}))$.

(5) *If* $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2$ *then* $\mathrm{Z}(\mathfrak{m})$ *is a subquotient of* $\mathrm{Z}(\mathfrak{m}_1)\times\mathrm{Z}(\mathfrak{m}_2)$ *and appears with multiplicity one in its Jordan–Hölder decomposition.*

(6) $\mathrm{Z}([a,b]_\rho)$ *is a subrepresentation of* $\rho|-|^a\times\cdots\times\rho|-|^b$.

(7) *If* $\rho_1, \dots, \rho_l \in \mathfrak{C}$ *lie in pairwise different cuspidal lines and* $\mathfrak{m} = \mathfrak{m}_1 + \dots + \mathfrak{m}_k$ *with* $\mathfrak{m}_i \in \mathcal{M}ult_R(\rho_i)$,

$$\mathrm{Z}(\mathfrak{m}) \cong \mathrm{Z}(\mathfrak{m}_1) \times \dots \times \mathrm{Z}(\mathfrak{m}_k).$$

If $\mathfrak{m}$ consists only of segments of length 1, we call $\mathrm{Z}(\mathfrak{m})$ nondegenerate. Finally, we recall the Aubert–Zelevinsky involution $(-)^* : \mathfrak{Irr} \to \mathfrak{Irr}$ (see [1; 32; 25]), and denote

$$\langle - \rangle := (\mathrm{Z})^* : \mathcal{M}ult_R^{\mathrm{ap}} \to \mathfrak{Irr}.$$

This map preserves the cuspidal support and if $\mathfrak{m}, \mathfrak{m}_1, \dots, \mathfrak{m}_k \in \mathcal{M}ult_R^{\mathrm{ap}}$, the multiplicity of $\langle \mathfrak{m} \rangle$ in $\langle \mathfrak{m}_1 \rangle \times \dots \times \langle \mathfrak{m}_k \rangle$ is the multiplicity of $\mathrm{Z}(\mathfrak{m})$ in $\mathrm{Z}(\mathfrak{m}_1) \times \cdots \mathrm{Z}(\mathfrak{m}_k)$.

Lemma 2.2.1. *Let* $\rho, \rho' \in \mathfrak{C}^{\square}$ *be in different cuspidal lines. Moreover, let* π *be an irreducible subquotient of* $\rho_1 \times \dots \times \rho_k,\ \rho_i \in \mathbb{Z}[\rho]$, *and* π' *be an irreducible subquotient of* $\rho_1' \times \dots \times \rho_l',\ \rho_i' \in \mathbb{Z}[\rho]$. *Then* $\pi \times \pi'$ *is irreducible.*

Proof. By the classification of cuspidal representations in [21, §6], the cuspidal lines of the cuspidal supports of π and π' do not intersect. Indeed, if σ is a cuspidal representation appearing in the cuspidal support of π, its supercuspidal support consists of representations in $\mathbb{Z}[\rho]$; see [23, Theorem 9.36(3)]. By the uniqueness of the cuspidal and supercuspidal supports, it follows that the cuspidal representations appearing in π and π' lie in different cuspidal lines. The claim follows from property (7) in Theorem 2.2.1. □

Let $\mathfrak{m} \in \mathcal{M}ult_R$ and $\Delta = [a, b]_\rho,\ \Delta' = [a', b']_\rho$ be two segments in $\mathfrak{m}$ with

$$a + 1 \le a' \le b + 1 \le b'.$$

An *elementary operation* on $\mathfrak{m}$ refers to changing the segment in the following way:

$$\mathfrak{m} \mapsto \mathfrak{m} - \Delta - \Delta' + [a, b']_\rho + [a', b]_\rho.$$

We will say that the elementary operation is of the form $[a, b]_\rho + [a', b']_\rho \mapsto [a, b']_\rho + [a', b]_\rho$. We write $\mathfrak{n} \preceq \mathfrak{m}$ if $\mathfrak{n}$ can be obtained by repeated applications of elementary operations to $\mathfrak{m}$ and $\mathfrak{n} \prec \mathfrak{m}$ if $\mathfrak{n} \preceq \mathfrak{m}$ and $\mathfrak{m} \neq \mathfrak{n}$. We call two segments Δ, Δ' linked if there exists a presentation $\Delta = [a, b]_\rho,\ \Delta' = [a', b']_\rho$ such that one of the inequalities

$$a + 1 \le a' \le b + 1 \le b' \quad \text{or} \quad a' + 1 \le a \le b' + 1 \le b$$

is satisfied, i.e., there exists a multisegment $\mathfrak{n}$ with $\mathfrak{n} \prec \Delta + \Delta'$. The multisegment $\mathfrak{m} = \Delta_1 + \dots + \Delta_k$ is called unlinked, if for all $i \neq j \in \{1, \dots, k\}$ Δ_i and Δ_j are unlinked.

Lemma 2.2.2 [32, Theorem 7.1; 8, Theorem 6.4.1]. *Let* $\mathfrak{m}=\Delta_1+\cdots+\Delta_k\in\mathcal{M}ult_R$ *and* $\mathfrak{n}\in\mathcal{M}ult_R^{\mathrm{ap}}$. *Then* $\langle\Delta_1\rangle\times\cdots\times\langle\Delta_k\rangle$ *contains* $\langle\mathfrak{n}\rangle$ *as an irreducible subquotient if and only if* $\mathfrak{n}\preceq\mathfrak{m}$. *In particular,* $\langle\mathfrak{m}\rangle\cong\langle\Delta_1\rangle\times\cdots\times\langle\Delta_k\rangle$ *if and only if* $\mathfrak{m}$ *is unlinked.*

Proof. The case in $R=\overline{\mathbb{Q}}_\ell$ was treated in [32, Theorem 7.1] and the case $R=\overline{\mathbb{F}}_\ell$ in [8, Theorem 6.4.1]. (In [8], the claim was proven for $\mathrm{Z}(-)$ instead of $\langle-\rangle$, however, applying the Aubert–Zelevinsky yields the equivalence.)

To see the claim regarding the irreducibility, note that the induced representation is irreducible if and only if it contains no irreducible subquotient $\langle\mathfrak{n}'\rangle$ for $\mathfrak{m}\neq\mathfrak{n}'\in\mathcal{M}ult_R$. If $\mathfrak{n}'$ is not aperiodic, it is easy to construct $\mathfrak{n}''\prec\mathfrak{n}'$ via one elementary operation, and hence $\mathfrak{n}''\prec\mathfrak{n}'\preceq\mathfrak{m}$. Repeating this process, we arrive at $\mathfrak{n}\prec\mathfrak{n}'\preceq\mathfrak{m}$ with $\mathfrak{n}$ aperiodic, and hence by the above lemma $\langle\mathfrak{n}\rangle$ is a subquotient of the induced representation. Thus the induced representation is irreducible if and only if it contains no $\langle\mathfrak{n}\rangle$ with $\mathfrak{n}\in\mathcal{M}ult_R^{\mathrm{ap}}$, which by the above can only happen if and only if $\mathfrak{n}\preceq\mathfrak{m}$. □

Assume for a moment that $R=\overline{\mathbb{Q}}_\ell$ and let Δ, Δ' be two segments. We say Δ precedes Δ' if $\Delta=[a,b]_\rho,\ \Delta'=[a',b']_\rho$ and

$$a+1\leq a'\leq b+1\leq b'.$$

Thus, Δ and Δ' are unlinked if and only if Δ does not precede Δ' and vice versa. Let $\mathfrak{m}=\Delta_1+\cdots+\Delta_k\in\mathcal{M}ult_{\overline{\mathbb{Q}}_\ell}$. We say $(\Delta_1,\ldots,\Delta_k)$ is in *arranged form* if for all $i,j\in\{1,\ldots,k\}$, $i<j$ Δ_i does not precede Δ_j. Any multisegment over $\overline{\mathbb{Q}}_\ell$ admits an arranged form and any two arranged forms can be obtained from each other by repeatedly changing the order of two neighboring unlinked segments, i.e., replacing $(\ldots,\Delta_i,\Delta_{i+1},\ldots)\mapsto(\ldots,\Delta_{i+1},\Delta_i,\ldots)$ if Δ_i and Δ_{i+1} are unlinked.

2.3. ***Integral structures.*** We recall the following results of [29, I.9] on integral structures. Let α be a partition of $n\in\mathbb{N}$ and $(\tilde{\pi},V)\in\mathfrak{Rep}_\alpha(\overline{\mathbb{Q}}_\ell)$. We recall that an integral structure of $\tilde{\pi}$ is a $\overline{\mathbb{Z}}_\ell$-lattice $\mathfrak{l}$ in V which generates V, i.e., the natural map induces an isomorphism $\mathfrak{l}\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{Q}}_\ell\cong V$. If $\tilde{\pi}$ admits an integral structure it is called *integral*. For $\tilde{\pi}\in\mathfrak{Rep}_\alpha$ integral and $\mathfrak{l}$ an integral structure, $\mathfrak{l}\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{F}}_\ell$ is an object in $\mathfrak{Rep}_\alpha(\overline{\mathbb{F}}_\ell)$. We denote by

$$\mathrm{r}_\ell(\tilde{\pi}):=[\mathfrak{l}\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{F}}_\ell],$$

which is an element in the Grothendieck group of $\mathfrak{Rep}_\alpha(\overline{\mathbb{F}}_\ell)$. It is $\mathrm{r}_\ell(\tilde{\pi})$ which is independent of the chosen integral structure; the representation $\mathfrak{l}\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{F}}_\ell$ is highly dependent on the specific $\mathfrak{l}$ and in general not irreducible. Moreover, the representation $\mathrm{ind}_{P_\alpha}^{G_n}\tilde{\pi}$ is integral, with the natural integral structure $\mathrm{ind}_{P_\alpha}^{G_n}\mathfrak{l}$, i.e., the $\mathfrak{l}$-valued functions. Then

$$(\mathrm{ind}_{P_\alpha}^{G_n}\mathfrak{l})\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{F}}_\ell\cong\mathrm{ind}_{P_\alpha}^{G_n}(\mathfrak{l}\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{F}}_\ell).$$

Moreover, if $\tilde{\tau}$ is a subrepresentation (resp. quotient) of $\tilde{\pi}$, then $\mathfrak{l} \cap \tilde{\tau}$ (resp. the image of $\mathfrak{l}$ in $\tilde{\tau}$) is an integral structure of $\tilde{\tau}$.

Finally, the functors r_α and r_ℓ commute on the level of Grothendieck groups by [6, Proposition 1.4(i)], i.e.,

$$r_\alpha(\mathrm{r}_\ell(\tilde{\pi})) = \mathrm{r}_\ell(r_\alpha(\tilde{\pi})).$$

An integral representation $\tilde{\pi} \in \mathfrak{Irr}(\overline{\mathbb{Q}}_\ell)$ is called a *lift* of $\pi \in \mathfrak{Irr}(\overline{\mathbb{F}}_\ell)$ if $\mathrm{r}_\ell(\tilde{\pi}) = [\pi]$. If $\Delta = [a, b]_\rho$ is a segment over $\overline{\mathbb{F}}_\ell$, a segment $\tilde{\Delta} = [a', b']_{\tilde{\rho}}$ over $\overline{\mathbb{Q}}_\ell$ is called a lift of Δ if $\tilde{\rho}$ is a lift of ρ, $b' - a' = b - a$ and $a = a' \bmod o(\rho)$. Finally, we extend this notation of a lift to segments $\mathcal{M}ult_{\overline{\mathbb{F}}_\ell}$ linearly.

In [29, III] the Bushnell–Kutzko construction for ℓ-adic and modular representations was carried out. It was shown that every cuspidal representation over $\overline{\mathbb{F}}_\ell$ admits a cuspidal lift $\tilde{\rho}$ to $\overline{\mathbb{Q}}_\ell$, and that if $\tilde{\rho}$ is such a lift of ρ, $f(\rho) = f(\tilde{\rho})$.

Lemma 2.3.1 [23, Theorem 9.39]. *Let* $\mathfrak{m} \in \mathcal{M}ult^{\mathrm{ap}}_{\overline{\mathbb{F}}_\ell}$ *and let* $\tilde{\mathfrak{m}}$ *be a lift of* $\mathfrak{m}$. *Then* $\mathrm{Z}(\mathfrak{m})$ *appears with multiplicity* 1 *in* $\mathrm{r}_\ell(\mathrm{Z}(\tilde{\mathfrak{m}}))$ *and* $\langle\mathfrak{m}\rangle$ *appears with multiplicity* 1 *in* $\mathrm{r}_\ell(\langle\tilde{\mathfrak{m}}\rangle)$.

Proof. The claim for Z can be found in [23, Theorem 9.39]. The one for $\langle - \rangle$ is derived from the definition of $\langle - \rangle$ in [1] as follows. Recall the definition of $(-)^*$: For $[\pi]$ an element in the Grothendieck group of $\mathfrak{Rep}_n$, one first defines a nonzero element $\mathrm{D}([\pi])$ in the same Grothendieck group, which is represented by the cohomology of a complex obtained from $[\pi]$ by applying repeatedly the Jacquet functor and parabolic induction. In particular, if $[\pi] \leq [\tau]$, then $\mathrm{D}([\pi]) \leq \mathrm{D}([\tau])$. Moreover, if π is irreducible, there exists a unique irreducible summand $[\pi^*]$ in $\mathrm{D}(\pi)$ with the same cuspidal support as π and if $R = \overline{\mathbb{Q}}_\ell$, then $\mathrm{D}(\pi) = [\pi^*]$. The above construction implies that D commutes with r_ℓ. Thus, $\mathrm{r}_\ell(\langle\tilde{\mathfrak{m}}\rangle) = \mathrm{r}_\ell(\mathrm{D}(\mathrm{Z}(\tilde{\mathfrak{m}}))) = \mathrm{D}(\mathrm{r}_\ell(\mathrm{Z}(\tilde{\mathfrak{m}})))$, where the latter contains $\mathrm{D}(\mathrm{Z}(\mathfrak{m})) \geq [\langle\mathfrak{m}\rangle]$ by the first claim. □

2.4. ***Generic extensions.*** In this section we recall the composition algebra of the cyclic quiver; compare [28]. We fix $1 < n \in \mathbb{N} \cup \{\infty\}$ and consider the cyclic quiver Q with vertices

$$I = \begin{cases} \mathbb{Z}/(n\mathbb{Z}) & \text{if } n < \infty, \\ \mathbb{Z} & \text{if } n = \infty, \end{cases}$$

and an arrow from i to j if $j = i + 1 \bmod n$ if n is finite and $j = i + 1$ if $n = \infty$. We recall that a representation of Q is nothing but a finite-dimensional I-graded $\mathbb{C}$-vector V space together with a linear map $T : V \mapsto V$ of weight 1. We call the representation nilpotent if $T^N = 0$ for large enough $N \in \mathbb{N}$, and associate to V the dimension vector $\mathrm{grdim} V \in \mathbb{N}^I$, whose i-th entry equals $\dim_{\mathbb{C}} V_i$. Note that if $n = \infty$, for all but finitely many $i \in I$, the dimension of $\dim_{\mathbb{C}} V_i$ vanishes. We call a vector $\boldsymbol{d} \in \mathbb{N}^I$ a dimension vector if it is 0 for all but finitely many $i \in I$. We let $E_{\boldsymbol{d}} = \{(V_{\boldsymbol{d}}, T)\}$ be the set of finite-dimensional, nilpotent representations of Q over $\mathbb{C}$ with underlying graded vector space

$$V_{\boldsymbol{d}} := \bigoplus_{i \in I} \mathbb{C}^{\boldsymbol{d}_i},$$

where we write for $\mathbb{C}^0$ the trivial vector space. We let $G_{\boldsymbol{d}} := \prod_{i \in I} \mathrm{GL}_{d_i}(\mathbb{C})$, where $\mathrm{GL}_0(\mathbb{C})$ denotes the trivial group, and note that $G_{\boldsymbol{d}}$ acts on $E_{\boldsymbol{d}}$ by conjugation. The orbits of this action are naturally parametrized as follows. We let $\mathcal{M}ult(Q)$ be the set of multisegments, i.e., the formal finite sums of segments $[a, b]$, $a \leq b \in \mathbb{Z}$ up to the equivalence

$$[a, b] \sim [a', b'] \quad \text{if} \quad \begin{cases} a = a' \bmod n, \ b - a = b' - a' & \text{if } n < \infty, \\ a = a', \ b' = b & \text{if } n = \infty. \end{cases}$$

We write $[a, b]^\vee = [-b, -a]$, $l([a, b]) = b - a + 1$ and extend these operations linearly to $\mathcal{M}ult(Q)$. To $[a, b] \in \mathcal{M}ult(Q)$ we associate the indecomposable representation $\lambda([a, b])$ whose underlying vector space has as a basis $b - a + 1$ vectors $e_1, \dots, e_{b-a+1}$, with e_i in degree $i \bmod n$ if $n < \infty$ and in degree i if $n = \infty$ and $T(e_i) = e_{i+1}$ for $i \leq b - a$ and $T(e_{b-a+1}) = 0$. The dimension vector of this representation, denoted by $\mathrm{grdim}[a, b]$, has as its i-th entry

$$(\mathrm{grdim}[a, b])_i = \begin{cases} \#\{x \in \mathbb{Z} : a \leq x \leq b, x = i \bmod n\} & \text{if } n < \infty, \\ 1 & \text{if } n = \infty, a \leq i \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

More generally, we associate to a multisegment $\mathfrak{m} = [a_1, b_1] + \dots + [a_k, b_k]$ the representation $\lambda(\mathfrak{m}) := \lambda([a_1, b_1]) \oplus \dots \oplus \lambda([a_k, b_k])$. We denote its dimension vector $\mathrm{grdim}\,\mathfrak{m} = \mathrm{grdim}[a_1, b_1] + \dots + \mathrm{grdim}[a_k, b_k]$. We call a multisegment aperiodic if either $n = \infty$ or $n < \infty$ and it does not contain a multisegment of the form $[a, b] + \dots + [a + n - 1, b + n + 1]$. We denote the set of aperiodic multisegments by $\mathcal{M}ult(Q)^{\mathrm{ap}}$.

Via this construction, the $G_{\boldsymbol{d}}$-orbits $[E_{\boldsymbol{d}}]$ of $E_{\boldsymbol{d}}$ are then in bijection with multisegments $\mathfrak{m}$ such that $\boldsymbol{d} = \mathrm{grdim}\,\mathfrak{m}$. For M a representation in $E_{\boldsymbol{d}}$, we denote by $[M] = G_{\boldsymbol{d}} \cdot M$ its orbit.

We write $[M] \preceq [N]$ for two orbits in $E_{\boldsymbol{d}}$ if $[N]$ is in the closure of $[M]$ with respect to the analytic topology. This relation gives rise to the so-called *degeneration order*. We also recall the associative product

$$* : [E_{\boldsymbol{d}}] \times [E_{\boldsymbol{d}'}] \to [E_{\boldsymbol{d}+\boldsymbol{d}'}]$$

given by sending $([M], [N]) \mapsto [M] * [N]$ to the orbit of their *generic extension*; see [7, §3] for cyclic quivers and [27, §2] for Dynkin quivers. The generic extension of two representations M and N is defined as the set of $X \in \mathrm{Ext}^1_Q(M, N)$ for which the dimension of the complex algebraic variety $\overline{[X]}$ is maximal, or equivalently, for which $\dim_{\mathbb{C}} \mathrm{Hom}_Q(X, X)$ is minimal. Any X, X' in the generic extension of M and N lie in the same equivalence class, thus we can define $[M] * [N]$ as $[X]$ for some X in the generic extension of M and N.

Lemma 2.4.1 ([27, Proposition 2.4]; see also [7, Proposition 3.4]). *Let $M \in E_{\mathbf{d}}, N \in E_{\mathbf{d}'}, X \in E_{\mathbf{d}+\mathbf{d}'}$. Then $[M] * [N] \preceq [X]$ if and only if there exist $[M] \preceq [M']$, $[N] \preceq [N']$ such that there exists a short exact sequence $0 \to M' \to X \to N' \to 0$. In particular, if $[M] \preceq [M'], [N] \preceq [N']$ then $[M] * [N] \preceq [M'] * [N']$.*

For a word $w = i_1 \dots i_k$ of indices in I, we write

$$\mathfrak{m}_{\mathrm{gen}}(w) = [i_1, i_1] * \cdots * [i_k, i_k],$$

and we denote the set of words in I by Ω. We can describe $\mathfrak{m}_{\mathrm{gen}}(w)$ recursively as follows; see for example [7, p. 285 and Proposition 3.7].

For $\mathfrak{m} \in \mathcal{M}ult(Q)$ and $i \in I$ we let

$$i+\mathfrak{m} := \begin{cases} \mathfrak{m} + [i, i] & \text{if there does not exist a segment of the form } [i+1, b] \text{ in } \mathfrak{m}, \\ \mathfrak{m} + {}^{+}\Delta - \Delta & \text{where } \Delta \text{ is the longest segment of the form } [i+1, b] \text{ in } \mathfrak{m}. \end{cases}$$

Similarly, we let

$$\mathfrak{m}+i := \begin{cases} \mathfrak{m} + [i, i] & \text{if there does not exist a segment of the form } [a, i-1] \text{ in } \mathfrak{m}, \\ \mathfrak{m} + \Delta^{+} - \Delta & \text{where } \Delta \text{ is the longest segment of the form } [a, i-1] \text{ in } \mathfrak{m}. \end{cases}$$

Lemma 2.4.2 [7, Proposition 3.7]. *In this notation, we have $\mathfrak{m}_{\mathrm{gen}}(iw) = i + \mathfrak{m}_{\mathrm{gen}}(w)$ and $\mathfrak{m}_{\mathrm{gen}}(wi) = \mathfrak{m}_{\mathrm{gen}}(w) + i$.*

From now on we implicitly identify the isomorphism classes of representations of Q with $\mathcal{M}ult(Q)^{\mathrm{ap}}$.

Theorem 2.4.1. *The map $\mathfrak{m}_{\mathrm{gen}} : \Omega \to \mathcal{M}ult(Q)$ has image $\mathcal{M}ult(Q)^{\mathrm{ap}}$ and two words w and w' give rise to the same multisegment if and only if they are related by the degenerate Serre relations.*

(1) *$ij = ji$ if i and j are not neighbors.*

(2) *$i(i+1)i = ii(i+1)$ and $i(i+1)(i+1) = (i+1)i(i+1)$ if $n > 2$.*

(3) *$i(i+1)ii = ii(i+1)i$ if $n = 2$.*

Proof. For a proof see [28, Theorem A, Theorem B], where the author considers the algebras $H_t(Q)$ we consider below specialised to the case that t is a prime number. We sketch the argument for the sake of the reader; see also [27, Theorem 4.2]. Let t be an indeterminate and consider the $\mathbb{Q}[t]$-algebra $H_t(Q)$, which is generated by variables $x_1, \dots, x_n$ satisfying the following relations:

(1) $x_i x_j = x_j x_i$ if i, j are not neighbors in Q;

(2) $x_i^2 x_{i+1} - t x_{i+1} x_i^2 = (t+1) x_i x_{i+1} x_i$ and $x_i x_{i+1}^2 - t x_{i+1}^2 x_i = (t+1) x_{i+1} x_i x_{i+1}$ if $n > 2$;

(3) $t x_2 x_1^3 - (t^2+t+1)(x_1 x_2 x_1^2 + x_1^2 x_2 x_1) = t x_1^3$ and $t x_1 x_2^3 - (t^2+t+1)(x_2 x_1 x_2^2 + x_2^2 x_1 x_2) = t x_2^3$ if $n = 2$.

Given a dimension vector $\boldsymbol{d}$, we can ask for the rank of the free $\mathbb{Q}[t]$-submodule $H_t^{\boldsymbol{d}}(Q)$ of $H_t(Q)$ spanned by the monomials containing x_i with multiplicity $\boldsymbol{d}_i$. Specializing at $t = 1$, we have that $H_1(Q)$ is the universal enveloping algebra of a certain upper-triangular part of a Lie algebra, and obtain by [28, Proposition 7.2] that the rank equals the number of aperiodic elements in $\mathcal{M}ult(Q)$ with dimension vector $\boldsymbol{d}$. The author achieves this by constructing an explicit PBW basis of the latter space. But on the other hand, we obtain by specializing at $t = 0$ the associative algebra on Ω subject to the Serre relations. We have a morphism of algebras

$$H_0(Q) \to \mathbb{Q}[\mathcal{M}ult(Q)^{\mathrm{ap}}],$$

where the right side is equipped with the generic extension product and we send i to $[i, i]$. Indeed, one just needs to check that the expressions involved in the Serre relations give rise to the same multisegments. For example, if $n > 2$,

$$i(i+1)\,i = [i, i+1] + [i, i] = ii(i+1).$$

The other relations can be checked similarly. This map is surjective by [28, §4], and since for $\boldsymbol{d}$ a cuspidal support, $H_0^{\boldsymbol{d}}(Q)$ has image in the multisegments with cuspidal support $\boldsymbol{d}$, the claim follows, because the dimensions agree. □

Remark. There exists a second map $\mathfrak{m}_{\mathrm{crys}} : \Omega \to \mathcal{M}ult(Q)$ linked to ρ-derivatives and certain crystal bases of quantum groups; see for example [8] or [17]. Even though $\mathfrak{m}_{\mathrm{crys}}$ admits a similar, although slightly more involved, recursive description as $\mathfrak{m}_{\mathrm{gen}}$, these two maps differ in general.

If $w = i_1 \dots i_k \in \Omega$, we let $w^\vee = (-i_k) \dots (-i_1)$. It follows from the definitions that $(i + \mathfrak{m})^\vee = \mathfrak{m}^\vee + (-i)$ and hence

$$\mathfrak{m}_{\mathrm{gen}}(w^\vee) = \mathfrak{m}_{\mathrm{gen}}(w)^\vee.$$

2.4.1. *Degeneration order and Serre relations.* In this subsection we describe how, given two words $v, w \in \Omega$, one can decide whether $[\mathfrak{m}_{\mathrm{gen}}(v)] \preceq [\mathfrak{m}_{\mathrm{gen}}(w)]$. Elementary operations on $\mathcal{M}ult(Q)$ can be defined in a manner completely analogous to those on $\mathcal{M}ult_R$ and the first step in answering the above question is the following.

Proposition 2.1 [33, Theorem 2.2; 13; 26, Theorem 3.12]. *The degeneration order and the order by elementary operations are equivalent, i.e.,*

$$[\lambda(\mathfrak{m})] \preceq [\lambda(\mathfrak{n})]$$

if and only if $\mathfrak{m}$ *can be obtained by* $\mathfrak{n}$ *via finitely many elementary operations.*

Let $v, w \in \Omega$. We write $v \leq w$ if v can be obtained from w by applying a finite sequence of the following moves.

(1) $ij \mapsto ji$ if i and j are not neighbors in I.

(2) If $n > 2$, $(i+1)\,i \mapsto i(i+1)$ and $ii(i+1) \mapsto i(i+1)\,i$.

(3) If $n = 2$, $i(i+1)(i+1) \mapsto (i+1)\,i(i+1)$.

The Serre relations are invariant under these moves.

The goal of this section is to prove the following proposition.

Proposition 2.2. *Let $v, w \in \Omega$. If $v \geq w$ then $\mathfrak{m}_{\mathrm{gen}}(v) \leq \mathfrak{m}_{\mathrm{gen}}(w)$. If $n = \infty$, then $\mathfrak{m}_{\mathrm{gen}}(v) \leq \mathfrak{m}_{\mathrm{gen}}(w)$ implies $v \geq w$.*

It is possible that there exists a more straightforward proof than the direct, purely combinatorial proof we offer, namely by relating the questions to properties of PBW bases as in the proof of Theorem 2.4.1. Note that the second implication in the proposition is wrong if n is not infinite. For example, if $n = 3$, we can set $v = 002211102$ and $w = 022110021$. Then w is maximal with respect to the above order but

$$\begin{aligned}\mathfrak{m}_{\mathrm{gen}}(v) &= [0,2]+[0,1]+[2,3]+[2,2]+[1,1]\\ &> \mathfrak{m}_{\mathrm{gen}}(w) = [0,2]+[-1,1]+[2,3]+[1,1].\end{aligned}$$

Proof of Proposition 2.2. Assume first that $v \leq w$. Then it suffices by Lemma 2.4.1 to show that $[i,i]+[i+1,i+1] = (i+1)+i \succeq i+(i+1) = [i,i+1]$ and $i+i+(i+1) = i+(i+1)+i$ if $n > 2$. The first is seen to be a simple elementary operation whereas the second is a Serre relation. The case $n = 2$ can be checked analogously.

For the other direction (in the case n being infinite) write $\mathfrak{n} = \mathfrak{m}_{\mathrm{gen}}(v)$ and $\mathfrak{m} = \mathfrak{m}_{\mathrm{gen}}(w)$, $\mathfrak{n} \prec \mathfrak{m}$. Using the Serre relations, it suffices to find some words $v' \preceq w'$ with $\mathfrak{n} = \mathfrak{m}_{\mathrm{gen}}(v')$ and $\mathfrak{m} = \mathfrak{m}_{\mathrm{gen}}(w')$. We will thus construct for any aperiodic $\mathfrak{n} \prec \mathfrak{m}$ two such words $v' \leq w'$ via induction on the length of $\mathfrak{m}$.

It clearly suffices to treat the case where $\mathfrak{n}$ is obtained from $\mathfrak{m}$ via one elementary operation $[a,b]+[a',b'] \mapsto [a,b']+[a',b]$, $a \leq a'+1 \leq b \leq b'+1$ and $\mathfrak{n} \prec \mathfrak{m}$ is minimal, i.e., there exists no $\mathfrak{k}$ with $\mathfrak{n} \prec \mathfrak{k} \prec \mathfrak{m}$. This implies in particular the following. If there exists $[c,d] \in \mathfrak{m}$ with $a \leq c \leq a' \leq b \leq d \leq b'$, then $[c,d]$ is one of $[a,b]$, $[a',b']$, $[a,b']$ or $[a',b]$. Indeed, if there would exist a $[c,d]$ different from these four segments we could decompose the above elementary operation in the following way. Assume that $a < c < a'$; the other case follows analogously. Then we can first apply $[a',b']+[c,d] \mapsto [a',d]+[c,b']$ to $\mathfrak{m}$, then $[a,b]+[a',d] \mapsto [a',b]+[a,d]$ and finally $[a,d]+[c,b'] \mapsto [a,b']+[c,d]$ to obtain $\mathfrak{n}$, contradicting the minimality of the elementary operation.

Note that it makes no difference whether we prove the claim for $\mathfrak{m}$ or $\mathfrak{m}^\vee$; thus we can assume without loss of generality that $b-a \geq b'-a'$. We now let $[c,d]$

be the longest segment in $\mathfrak{m}$ with $a \le c$ such that $[c+1,d+1]$ does not appear in $\mathfrak{m}$. Set $\mathfrak{m}' = \mathfrak{m} - [c,d] + [c+1,d]$ and by construction $c + \mathfrak{m}' = \mathfrak{m}$. If $[c,d]$ is not the precise copy of $[a',b']$ involved in the elementary operation, it follows straightforwardly that the elementary operation descends to an elementary operation on $\mathfrak{m}'$ yielding a multisegment $\mathfrak{n}' \preceq \mathfrak{m}'$ with $c + \mathfrak{n}' = \mathfrak{n}$. In this case the claim follows by the induction hypothesis. On the other hand, if $[c,d] = [a',b']$, it follows by construction and the assumption $b-a \ge b'-a'$ that $b-a = b'-a'$ and for all $i \in \{0,\dots,a'-a\}$, $[a+i, b+i]$ appears in $\mathfrak{m}$. By the minimality of the elementary operation this implies that $[a',b'] = [a-1,b-1]$. Moreover, $[a,b]$ is a longest segment in $\mathfrak{m}$.

It follows that for $\mathfrak{m}'' = \mathfrak{m} - [a,b] - [a',b'] + [a+1,b] + [a'+1,b']$ we have $a + (a+1) + \mathfrak{m}' = \mathfrak{n}$ and $(a+1) + a + \mathfrak{m}' = \mathfrak{m}$, finishing the claim. □

Let $w = i_1 \dots i_l \in \Omega$ and $\mathfrak{m} \in \mathcal{M}ult(Q)$. We call w a *descendant* of $\mathfrak{m}$ if it is obtained from $\mathfrak{m}$ in the following, recursive, way. Write $\mathfrak{m} = [a_1,b_1] + \dots + [a_k,b_k]$ and choose $i \in \{1,\dots,k\}$. Then choose a descendant w' of $\mathfrak{m}' = [a_1,b_1] + \dots + [a_i+1,b_i] + \dots + [a_k,b_k]$. If $\mathfrak{m}'$ is empty, w' is the empty word. Finally, w is a descendant of $\mathfrak{m}$ if it is of the form $w = a_i w'$ for some a_i and w' as above.

Lemma 2.4.3. *Let $w \in \Omega$ and $\mathfrak{m} \in \mathcal{M}ult(Q)$. Then w is a descendant of $\mathfrak{m}$ if and only if $\mathfrak{m}_{\mathrm{gen}}(w) \preceq \mathfrak{m}$.*

Proof. We argue by induction on the length of $\mathfrak{m}$, the case of length 1 being trivially true, and we use the notation employed above the lemma. First assume that w is a descendant of $\mathfrak{m}$ and write $w = a_i w'$, w' a descendant of $\mathfrak{m}'$, as above the lemma. By the induction hypothesis we have that $\mathfrak{m}_{\mathrm{gen}}(w') \preceq \mathfrak{m}'$ and hence $\mathfrak{m}_{\mathrm{gen}}(w) \preceq a_i + \mathfrak{m}'$. It thus suffices to show that $a_i + \mathfrak{m}' \preceq \mathfrak{m}$, which follows quickly. If $[a_i+1, b_i]$ is the longest segment in $\mathfrak{m}'$ starting in $a_i + 1 \bmod n$, we have in fact equality. Otherwise, let $[a_i+1, d]$ be the respective longest segment. Then $a_i + \mathfrak{m}' = \mathfrak{m} - [a_i+1,d] + [a_i,d]$. Note that we can perform the elementary operation $[a_i,b_i] + [a_i+1,d] \mapsto [a_i,d] + [a_i+1,b_i]$ on $\mathfrak{m}$, yielding $a_i + \mathfrak{m}'$, and thus proving the claim.

For the other direction, assume that $\mathfrak{m}_{\mathrm{gen}}(w) \preceq \mathfrak{m}$ and write $w = cw'$, $c \in I$. Then it follows from [8, Lemma 6.3.3] that $c \in \{a_1,\dots,a_k\}$ and $\mathfrak{m}_{\mathrm{gen}}(w') \preceq \mathfrak{m}'$, where $\mathfrak{m}'$ is of the form $[a_1,b_1] + \dots + [a_j+1,b_j] + \dots + [a_k,b_k]$ for some j with $a_j = c \bmod n$. The claim follows immediately. □

3. Whittaker models

We follow the setup of [2] (see [14] for the case of nonzero characteristic). Let ψ_R be an additive character $\psi_R : \mathrm{F} \to R$. Moreover, we demand that $\psi_{\overline{\mathbb{Q}}_\ell} \otimes_{\overline{\mathbb{Z}}_\ell} \overline{\mathbb{F}}_\ell = \psi_{\overline{\mathbb{F}}_\ell}$. By abuse of notation, we will from now on write $\psi = \psi_R$. We extend ψ to N_n by defining

$$\psi(u) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right).$$

We define the space of Whittaker functions of G_n with respect to ψ as

$$\mathcal{W}(\psi) = \mathcal{W}_n(\psi) :== \operatorname{Ind}_{N_n}^{G_n}\psi = \big\{W : G_n \to R \text{ locally constant} : \\ W(ug) = \psi(u)W(g) \text{ for all } u \in N_n,\ g \in G_n\big\},$$

on which G_n acts by right-translation. We call $\pi \in \mathfrak{Rep}_n$ of Whittaker type if

$$\dim_R \operatorname{Hom}_{G_n}(\pi, \mathcal{W}(\psi)) = 1,$$

in which case we denote the image of π in $\mathcal{W}(\psi)$ by $\mathcal{W}(\pi, \psi)$. Thus, $\mathcal{W}(\pi, \psi)$ is socle-irreducible and its unique irreducible subrepresentation appears with multiplicity 1 in it. Moreover, if

$$\mathcal{W}(\pi, \psi) \to \mathcal{W}(\pi', \psi)$$

is a nonzero map for a second representation π' of Whittaker type, it is injective. If π is on top of that irreducible we call it *generic*.

Let W be a Whittaker function. We define the map $\widetilde{W}$ by

$$g \mapsto \widetilde{W}(g) := W(w_n(g^{-1})^t).$$

Finally, if π is of Whittaker type, we set

$$\mathcal{W}(\pi, \psi)^{\mathfrak{c}} = \{\widetilde{W}(g) :\ W \in \mathcal{W}(\pi, \psi)\} = \mathcal{W}(\pi^{\mathfrak{c}}, \psi^{-1}).$$

We recall that if π_1 and π_2 are of Whittaker type, so is $\pi_1 \times \pi_2$ (see [2; 3], and also [29, III.1.10]), and then $\mathcal{W}(\pi_1 \times \pi_2, \psi) = \mathcal{W}(\mathcal{W}(\pi_1, \psi) \times \mathcal{W}(\pi_2, \psi), \psi)$. We denote by $\mathfrak{Rep}_{W,\psi,n}$ the set of finite-length subrepresentations of $\mathcal{W}_n(\psi)$ which are of Whittaker type and we set

$$\mathfrak{Rep}_{W,\psi} := \bigcup_{n\in\mathbb{N}} \mathfrak{Rep}_{W,\psi,n}.$$

Then $\mathfrak{Rep}_{W,\psi}$ can be equipped with an associative product

$$* : \mathfrak{Rep}_{W,\psi} \times \mathfrak{Rep}_{W,\psi} \to \mathfrak{Rep}_{W,\psi}, \quad (\pi, \pi') \mapsto \mathcal{W}(\pi \times \pi', \psi).$$

Associativity is an easy consequence of the uniqueness of the equality $\mathcal{W}(\pi_1 \times \pi_2, \psi) = \mathcal{W}(\mathcal{W}(\pi_1, \psi) \times \mathcal{W}(\pi_2, \psi), \psi)$.

If $\Delta = [a, b]_\rho$ is a segment, the representation $\langle\Delta\rangle$ is of Whittaker type if $l(\Delta) < e(\rho)$, by [23, Remark 8.14]. Every cuspidal representation is of Whittaker type.

3.1. ***Derivatives.*** We recall the four exact functors in the definition of the Bernstein–Zelevinsky derivatives; see [2; 3], and compare [29, III.1]. Recall also the groups P_n from Section 2 and write the unipotent subgroups as

$$\left\{U_{n-1,1} := \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in G_n : x \in M_{n-1,1}\right\}.$$

The four functors are

(1) $\Psi^+ : \mathfrak{Rep}_{n-1} \to \mathfrak{Rep}(P_n)$, the extension by the trivial representation of $U_{n-1,1}$ and twisted by $|-|^{\frac{1}{2}}$;

(2) $\Psi^- : \mathfrak{Rep}(P_n) \to \mathfrak{Rep}_{n-1}$, the $U_{n-1,1}$-coinvariants twisted by $|-|^{-\frac{1}{2}}$;

(3) $\Phi^+ = \mathrm{ind}_{P_{n-1}U_{n-1,1}}^{P_n}(-\otimes\psi) : \mathfrak{Rep}(P_{n-1}) \to \mathfrak{Rep}(P_n)$;

(4) $\Phi^- : \mathfrak{Rep}(P_n) \to \mathfrak{Rep}(P_{n-1})$, the $(U_{n-1,1}, \psi)$-coinvariants twisted by $|-|^{-\frac{1}{2}}$.

Let $\tau \in \mathfrak{Rep}(P_n)$, $k \in \{1, \dots, n\}$ and set $\tau_{(k)} := (\Phi^-)^{k-1}(\tau)$ and $\tau^{(k)} := \Psi^-(\tau_{(k)})$. For $\pi \in \mathfrak{Rep}_n$ we set $\pi_{(k)} := (\pi|_{P_n})_{(k)}$, $\pi^{(k)} := (\pi|_{P_n})^{(k)}$ and $\pi^{(0)} := \pi$.

3.2. ***Derivatives and Whittaker models.*** We recall also the Kirillov model of a representation π of Whittaker type. It is defined as the P_n-representation given by

$$\mathcal{K}(\pi, \psi) := \{W|_{P_n} : W \in \mathcal{W}(\pi, \psi)\} \subseteq \mathrm{Ind}_{N_n}^{P_n} \psi.$$

Theorem 3.2.1 [9, 4.3; 20; 15]. *Let $\pi \in \mathfrak{Rep}_n$ be of Whittaker type. The map $W \mapsto W|_{P_n}$ is injective on $\mathcal{W}(\pi, \psi)$.*

We have the following description of $(\Phi^-)^k$.

Lemma 3.2.1 [5, Proposition 1.3]. *Let π be of Whittaker type. Then we can identify $(\Phi-)^k(\mathcal{K}(\pi, \psi))$ with the space*

$$\left\{p \mapsto |\det(p)|^{-\frac{k}{2}} W\begin{pmatrix} p & \\ & 1_k \end{pmatrix} : p \in P_{n-k},\ W \in \mathcal{W}(\pi, \psi)\right\}.$$

The authors only prove the claim for $R = \mathbb{C}$, but the same method works for arbitrary base fields.

The description of the space $\Psi^-(\Phi^-)^k(\mathcal{K}(\pi, \psi))$ is trickier. Let τ be a subrepresentation of $\mathcal{K}(\pi, \psi)^{(k)}$ with central character χ. Let σ be the inverse image of τ in $\mathcal{K}(\pi, \psi)$. For $W \in \sigma$ and $g \in G_{n-k}$ define the map

$$S(W)(g) = \lim_{z\to 0} |z|^{\frac{k-n}{2}} |\det(g)|^{\frac{k}{2}} \chi^{-1}(z) W\begin{pmatrix} zg & \\ & 1_k \end{pmatrix}. \tag{1}$$

Here $z \in \mathrm{F}^\times$ is seen as an element of Z_{n-k} and the limit becomes stationary for z small enough.

Proposition 3.1 [5, Proposition 1.7; 18, Corollary 2.1]. *The map $S : \sigma \mapsto \mathcal{W}(\psi)$ induces the nonzero, injective map*

$$\bar{S} : \tau \hookrightarrow \mathcal{W}(\psi).$$

Whittaker models are a useful place to look for integral structures, as the next theorem shows.

Theorem 3.2.2 [31, Theorem 2]. *For π an integral representation of Whittaker type over $\overline{\mathbb{Q}}_\ell$ of G_n, set*

$$\mathcal{W}^{\mathrm{en}}(\pi,\psi)=\left\{W\in\mathcal{W}(\pi,\psi):\ W(G_n)\subseteq\overline{\mathbb{Z}}_\ell\right\}.$$

If $\pi\in\mathfrak{Rep}$ is integral and of Whittaker type, then $\mathcal{W}^{\mathrm{en}}(\pi,\psi)$ is an integral structure of $\mathcal{W}(\pi,\psi)$.

In this case we denote

$$\overline{\mathcal{W}(\pi,\psi)}:=\mathcal{W}^{\mathrm{en}}(\pi,\psi)\otimes_{\overline{\mathbb{Z}}_\ell}\overline{\mathbb{F}}_\ell.$$

Lemma 3.2.2. *Let $\pi_1,\dots,\pi_k$ be integral representations of Whittaker type. Then the nonzero map*

$$\mathcal{W}(\pi_1,\psi)\times\cdots\times\mathcal{W}(\pi_n,\psi)\to\mathcal{W}(\pi_1\times\cdots\times\pi_k,\psi)$$

respects the integral structures and hence induces a nonzero map

$$\overline{\mathcal{W}(\pi_1,\psi)}\times\cdots\times\overline{\mathcal{W}(\pi_n,\psi)}\to\overline{\mathcal{W}(\pi_1\times\cdots\times\pi_k,\psi)}.$$

Proof. See for example the proof of [14, Theorem 2.26]. □

We recall some useful properties of Whittaker models.

Proposition 3.2 [15, Proposition 3.7]. *Let $\pi_1\in\mathfrak{Rep}_{n_1}$, $\pi_2\in\mathfrak{Rep}_{n_2}$*

$$\mathcal{W}(\pi_2,\psi)\subseteq\mathcal{W}(\pi_1\times\pi_2,\psi)^{(n_1)}.$$

Theorem 3.2.3 [15, Theorem 3.10]. *Let $n\geq 2$ and τ be an a submodule of $\mathrm{Ind}_{N_n}^{P_n}\psi$. If for $k\in\{1,\dots,n-1\}$, $\tau^{(k)}$ admits a central character, for any $W_0\in\mathcal{W}(\tau^{(k)},\psi)$ and $\phi\in C_c^\infty(\mathrm{F}^{n-k})$ there exists $W\in\tau$ such that for all $g\in G_{n-k}$*

$$W\begin{pmatrix} g & \\ & 1_k\end{pmatrix}=W_0(g)\phi(\epsilon_{n-k}g)|\det(g)|^{\frac{k-1}{2}}.$$

The proofs of these two results in [15] apply to $R=\mathbb{C}$, but the methods readily generalize to the more general settings presented here.

3.3. ***Rankin–Selberg L-factors.*** We now recall the construction of Rankin–Selberg L-factors as presented in [14]. For complex representations this goes back to the classical text [12]. Let $\pi\in\mathfrak{Rep}_n$ and $\pi'\in\mathfrak{Rep}_m$ be representations of Whittaker type.

Definition. Let $W\in\mathcal{W}(\pi,\psi)$, $W'\in\mathcal{W}(\pi',\psi^{-1})$ and $k\in\mathbb{Z}$.

(1) The case $n = m$: Let $\phi \in C_c^\infty(\mathrm{F}^n)$ and define

$$c_k(W, W', \phi) := \int_{N_n \backslash G_n^k} W(g)W'(g)\phi(\epsilon_k g)\,\mathrm{d}_l g,$$
$$I(X, W, W, \phi) = \sum_{k\in\mathbb{Z}} c_k(W, W', \phi)X^k \in R(X).$$

(2) The case $n < m$: Let $j \in \{0, \dots, n-m-1\}$ and define

$$c_k(W, W', j) := \int_{M_{j,m}} \int_{N_n \backslash G_n^k} W(g)W'\begin{pmatrix} g & & \\ x & 1_j & \\ & & 1_{n-m-j} \end{pmatrix} \mathrm{d}_l g\,\mathrm{d}_l x,$$
$$I(X, W, W, j) = \sum_{k\in\mathbb{Z}} c_k(W, W', j)X^k \in R(X).$$

(3) The case $n > m$: Analogous to the case $n < m$.

Having defined this, we can now recall the definition of the L-factors.

Theorem 3.3.1 [14, Theorem 3.5]. *Let $\pi \in \mathfrak{Rep}_n$ and $\pi' \in \mathfrak{Rep}_m$ be representations of Whittaker type.*

(1) *The case $n = m$. The ideal spanned by $I(X, W, W', \phi)$, where we vary over $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$ and $\phi \in C_c^\infty(\mathrm{F}^n)$ is fractional and admits a generator $L(X, \pi, \pi')$ with $L(X, \pi, \pi')^{-1} \in R[X]$, which is normalized by demanding $L(0, \pi, \pi') = 1$.*

(2) *The case $n \neq m$. Fix $j \in \{0, \dots, |n-m|-1\}$. The ideal spanned by $I(X, W, W', j)$, where we vary over $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi^{-1})$ is fractional, independent of j and admits a generator $L(X, \pi, \pi')$ with $L(X, \pi, \pi')^{-1} \in R[X]$, which is normalized by demanding $L(0, \pi, \pi') = 1$.*

In particular $L(X, \pi, \pi') = L(X, \pi', \pi)$. As usual, these L-factors satisfy a functional equation, giving rise to an ϵ-factor. For $\phi \in C_c^\infty(\mathrm{F}^n)$, we denote by $\hat{\phi}$ its Fourier transform with respect to the character ψ.

Theorem 3.3.2 [14, Corollary 3.11; 14, Lemma 3.12]. *Let $\pi \in \mathfrak{Rep}_n$, $\pi' \in \mathfrak{Rep}_m$ be representations of Whittaker type with central characters c_π and $c_{\pi'}$, and let $n \leq m$. Let $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$, $\phi \in C_c^\infty(\mathrm{F}^n)$ and $j \in \{0, \dots, m-n-1\}$. Then there exists $\epsilon(X, \pi, \pi', \psi) \in R[X, X^{-1}]^\times$ such that the following holds.*

(1) *The case $n = m$.*

$$\frac{I(q^{-1}X^{-1}, \tilde{W}, \tilde{W}', \hat{\phi})}{L(q^{-1}, \pi^{\mathfrak{c}}, \pi'^{\mathfrak{c}})} = c_{\pi'}(-1)^{n-1}\epsilon(X, \pi, \pi', \psi)\frac{I(X, W, W', \phi)}{L(X, \pi, \pi')}.$$

(2) *The case $n < m$.*

$$\frac{I(q^{-1}X^{-1}, \rho(w_{n,n-m})\tilde{W}, \tilde{W}', m-n-j-1)}{L(q^{-1}, \pi^{\mathfrak{c}}, \pi'^{\mathfrak{c}})} = c_{\pi'}(-1)^{n-1}\epsilon(X, \pi, \pi', \psi)\frac{I(X, W, W', j)}{L(X, \pi, \pi')}.$$

The last local factor, the γ-factor, is finally defined as follows.

Definition. Let $\pi, \pi' \in \mathfrak{Rep}$ be representations of Whittaker type. Then we define

$$\gamma(X, \pi, \pi', \psi) := \epsilon(X, \pi, \pi', \psi)\frac{L(q^{-1}X^{-1}, \pi^{\mathfrak{c}}, \pi'^{\mathfrak{c}})}{L(X, \pi, \pi')} \in R(X).$$

We now recall the most important properties of these local factors.

Lemma 3.3.1 [14, Theorem 3.13]. *Let $\pi, \pi', \pi'' \in \mathfrak{Rep}$ be representations of Whittaker type and τ a subrepresentation of π of Whittaker type. Then*

$$\gamma(X, \pi, \pi', \psi) = \gamma(X, \tau, \pi', \psi) \quad \textit{and} \quad L(X, \tau, \pi')^{-1} | L(X, \pi, \pi')^{-1}.$$

Moreover, we have the so-called inductivity relation

$$\gamma(X, \pi \times \pi'', \pi', \psi) = \gamma(X, \pi, \pi', \psi)\gamma(X, \pi'', \pi', \psi).$$

Finally, let us remark how these factors interact with respect to reduction mod ℓ.

Lemma 3.3.2 [14, Theorem 4.1, §4.1]. *Let $\pi, \pi' \in \mathfrak{Rep}$ be two integral representations of Whittaker type over $\overline{\mathbb{Q}}_\ell$. Then $L(X, \pi, \pi')$, $\epsilon(X, \pi, \pi', \psi)$, $\gamma(X, \pi, \pi', \psi)$ lie in $\overline{\mathbb{Z}}_\ell(X)$ and*

$$\gamma(X, \overline{\mathcal{W}(\pi, \psi)}, \overline{\mathcal{W}(\pi', \psi)}, \psi) = \mathrm{r}_\ell(\gamma(X, \pi, \pi', \psi)),$$
$$L(X, \overline{\mathcal{W}(\pi, \psi)}, \overline{\mathcal{W}(\pi', \psi)})^{-1} | \mathrm{r}_\ell(L(X, \pi, \pi')^{-1}).$$

Lemma 3.3.3. *Let $\pi \in \mathfrak{Rep}_n$, $\pi' \in \mathfrak{Rep}_m$ be two representation of Whittaker type and $k \in \{0, \dots, n\}$. Let τ be a subrepresentation of $\pi^{(k)}$ admitting a central character. Then*

$$L(X, \tau, \pi')^{-1} | L(X, \pi, \pi')^{-1}.$$

Proof. As explained in the proof of [14, Lemma 4.6, Proposition 4.7] (see also [12, Lemma 9.2]), the claim holds true if one shows the following. For any $W_0 \in \mathcal{W}(\tau, \psi)$ and $\phi \in C_c^\infty(\mathrm{F}^{n-k})$ there exists $W \in \mathcal{W}(\pi, \psi)$ such that for all $g \in G_{n-k}$

$$W\begin{pmatrix} g & \\ & 1_{n-k} \end{pmatrix} = W_0(g)\phi(\epsilon_{n-k}g)\,|\det(g)|^{\frac{k-1}{2}}.$$

But by assumption on $\tau^{(k)}$, $\mathcal{W}(\tau^{(k)}, \psi)$ admits a central character; hence this follows from Theorem 3.2.3. □

We finish by recalling the L-factors of C-parameters of [16]. For the moment we slightly extend this notation to aperiodic multisegments and to avoid confusion we will denote our parameters by C'.

Definition. Let $\rho, \rho' \in \mathfrak{C}$. We set

$$L(X, \mathrm{C}'(\rho), \mathrm{C}'(\rho')) := \begin{cases} (1 - (\chi(\varpi_{\mathrm{F}})X)^{f(\rho)})^{-1} & \text{if } \rho' \cong \chi\rho^{\vee},\ \rho \in \mathfrak{C}^{\square},\ \chi \text{ an unramified character,} \\ 1 & \text{otherwise.} \end{cases}$$

Let $\Delta = [a, b]_{\rho}$, $\Delta' = [a', b']'_{\rho}$ be two segments. Then we set

$$L(X, \mathrm{C}'(\Delta), \mathrm{C}'(\Delta')) := \begin{cases} \prod_{i=a}^{b} L(X, \mathrm{C}'(\rho|-|^{i}), \mathrm{C}'(\rho'|-|^{b})) & \text{if } l(\Delta) \leq l(\Delta'), \\ \prod_{i=a'}^{b'} L(X, \mathrm{C}'(\rho|-|^{b}), \mathrm{C}'(\rho'|-|^{i})) & \text{if } l(\Delta) \geq l(\Delta'). \end{cases}$$

Finally, if $\mathfrak{m}, \mathfrak{n} \in \mathcal{M}ult_R$ of the form $\mathfrak{m} = \Delta_1 + \cdots + \Delta_k$, $\mathfrak{n} = \Delta'_1 + \cdots + \Delta'_l$ we set

$$L(X, \mathrm{C}'(\mathfrak{m}), \mathrm{C}'(\mathfrak{n})) := \prod_{\substack{1 \leq i \leq k \\ 1 \leq j \leq l}} L(X, \mathrm{C}'(\Delta_i), \mathrm{C}'(\Delta'_j)).$$

3.4. ***Associative products and the map $\mathcal{S}_{\mathbf{gen},\psi}$.*** For the rest of this section, we fix an additive character ψ of F. We also fix $\rho \in \mathfrak{C}^{\square}$ and let Q be the quiver of Section 2.4 with $n = o(\rho) > 1$ and set of indices I. Fix a cuspidal support

$$\boldsymbol{d} = \sum_{i \in I} d_i [\rho|-|^{i}]$$

and note that $d_i = 0$ for all but finitely many $i \in I$ and hence $\boldsymbol{d}$ gives rise to a dimension vector denoted by the same letter $\boldsymbol{d}$ for Q. We let $\mathcal{M}ult_R(\rho)_{\boldsymbol{d}}$ be the set of multisegments with support $\boldsymbol{d}$. We then recall the natural bijection

$$\mathcal{M}ult_R(\rho)_{\boldsymbol{d}} \leftrightarrow \{G_{\boldsymbol{d}}\text{-orbits in } E_{\boldsymbol{d}}\},$$

which respects the orders on each side, by Proposition 2.1. This bijection allows us to import the generic extension product to $\mathcal{M}ult_R(\rho)$, which by abuse of notation we also denote by $*$. This product has the following representation-theoretic interpretation. By Theorem 2.4.1, there exists for $\mathfrak{m} \in \mathcal{M}ult_R^{\mathrm{ap}}(\rho)$ representations $\rho_1, \ldots, \rho_k$ in $\mathbb{Z}[\rho]$ such that $\mathfrak{m} = \rho_1 * \cdots * \rho_k$. We then set

$$\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) := \mathcal{W}(\rho_1 \times \cdots \times \rho_k, \psi).$$

This representation is independent of our choices; to prove this, we first note the following analogue of the degenerate Serre relations.

Lemma 3.4.1. *Let $\rho' \in \mathbb{Z}[\rho]$ be such that $\rho' \not\cong \rho|-|^{\pm 1}$. Then*

$$\mathcal{W}(\rho \times \rho', \psi) = \mathcal{W}(\rho' \times \rho, \psi).$$

If $o(\rho) > 2$,

$$\mathcal{W}(\rho \times \rho \times \rho|-|, \psi) = \mathcal{W}(\rho \times \rho|-|\times\rho, \psi),$$
$$\mathcal{W}(\rho \times \rho|-|\times\rho|-|, \psi) = \mathcal{W}(\rho|-|\times\rho \times \rho|-|, \psi).$$

If $o(\rho) = 2$,

$$\mathcal{W}(\rho \times \rho|-|\times\rho \times \rho, \psi) = \mathcal{W}(\rho \times \rho \times \rho|-|\times\rho, \psi).$$

Proof. Under the hypothesis, $\rho \times \rho'$ is irreducible, and

$$\mathcal{W}(\rho \times \rho', \psi) \cong \rho \times \rho' \cong \rho' \times \rho \cong \mathcal{W}(\rho' \times \rho, \psi).$$

Hence they are equal. We only show the remaining claims for $o(\rho) > 2$; the case $o(\rho) = 2$ follows analogously. We start with the equality $\mathcal{W}(\rho \times \rho \times \rho|-|, \psi) = \mathcal{W}(\rho \times \rho|-|\times\rho, \psi)$. By [23, Proposition 7.17],

$$\rho \times \rho \times \rho|-| \twoheadrightarrow \rho \times \langle [0, 1]_\rho \rangle,$$

which is irreducible by Lemma 2.2.2 and of Whittaker type. Thus the uniqueness of the Whittaker model forces

$$\mathcal{W}(\rho \times \rho \times \rho|-|, \psi) \cong \rho \times \langle [0,1]_\rho \rangle.$$

Similarly,

$$\mathcal{W}(\rho \times \rho|-|\times\rho, \psi) \cong \langle [0,1]_\rho \rangle \times \rho.$$

By the commutativity of parabolic induction on the Grothendieck group we have $\langle [0,1]_\rho \rangle \times \rho \cong \rho \times \langle [0,1]_\rho \rangle$; the claim follows. The equality $\mathcal{W}(\rho\times\rho|-|\times\rho|-|, \psi) = \mathcal{W}(\rho|-|\times\rho \times \rho|-|, \psi)$ follows by an analogous argument. □

From the lemma and Theorem 2.4.1 we obtain:

Corollary 3.4.1. *The representation $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m})$ is independent of the sequence $\rho_1, \dots, \rho_k$ with $\rho_1 * \cdots * \rho_k = \mathfrak{m}$. Moreover, the map $\mathcal{S}_{\mathrm{gen},\psi} : \mathcal{M}ult_R(\rho)^{\mathrm{ap}} \to \mathfrak{Rep}_W$ respects the respective products, i.e., for $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathcal{M}ult_R(\rho)^{\mathrm{ap}}$, we have*

$$\mathcal{W}(\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}_1) \times \mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}_2), \psi) = \mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}_1 * \mathfrak{m}_2)$$

Next we define the map $\mathcal{S}^{\cup}_{\mathrm{gen},\psi}$ by setting for $\mathfrak{m} \in \mathcal{M}ult_R(\rho)$ not necessarily aperiodic

$$\mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m}) := \bigcup_{\substack{\mathfrak{n} \preceq \mathfrak{m}, \\ \mathfrak{n} \text{ aperiodic}}} \mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{n}).$$

Here the union is taken in the space of Whittaker functions.

Proposition 3.3. *Let* $\mathfrak{n}, \mathfrak{m} \in \mathcal{M}ult_R(\rho)$ *with* $\mathfrak{n} \preceq \mathfrak{m}$. *Then*

$$\mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{n}) \subseteq \mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m}).$$

Moreover, let $\mathfrak{m}' \in \mathcal{M}ult_R(\rho)$ *such that* $\mathfrak{m}' * \rho = \mathfrak{m}$. *Then*

$$\mathcal{W}(\mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m}') \times \rho, \psi) \subseteq \mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m}).$$

Similarly, if $\mathfrak{m}' \in \mathcal{M}ult_R(\rho)$ *such that* $\rho * \mathfrak{m}' = \mathfrak{m}$, *then*

$$\mathcal{W}(\rho \times \mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m}'), \psi) \subseteq \mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m}).$$

Proof. The first claim follows from the definition. For the second claim we argue as follows. We only treat the case $\mathfrak{m}' * \rho = \mathfrak{m}$, as the other one follows similarly. Then for $\mathfrak{n}' \preceq \mathfrak{m}'$ we have by Lemma 2.4.1 that $\mathfrak{n}' * \rho \preceq \mathfrak{m}$ and hence $\mathcal{W}(\mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m}') \times \rho, \psi) \subseteq \mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m})$ by the first claim. □

Finally, for $\mathfrak{m} \in \mathcal{M}ult_{R,\square}$, write $\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_k$ with $\mathfrak{m}_i \in \mathcal{M}ult_{R,\square}(\rho_i)$, where the $\rho_i \in \mathfrak{C}^{\square}$ are in pairwise different cuspidal lines. Then we set

$$\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) := \mathcal{W}(\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}_1) \times \cdots \times \mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}_k), \psi), \quad \mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m}) := \bigcup_{\substack{\mathfrak{n} \preceq \mathfrak{m}, \\ \mathfrak{n} \text{ aperiodic}}} \mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{n}),$$

and if $\mathfrak{m}' = \mathfrak{m}'_1 + \cdots + \mathfrak{m}'_k \in \mathcal{M}ult_{R,\square}$, $\mathfrak{m}'_i \in \mathcal{M}ult_R(\rho_i)$ is a second multisegment, we define

$$\mathfrak{m} * \mathfrak{m}' := \mathfrak{m}_1 * \mathfrak{m}'_1 + \cdots + \mathfrak{m}_k * \mathfrak{m}'_k,$$

extending the product $*$ to $* : \mathcal{M}ult_{R,\square} \times \mathcal{M}ult_{R,\square} \to \mathcal{M}ult_{R,\square}$.

Lemma 3.4.2. *Let* $\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_k \in \mathcal{M}ult^{\mathrm{ap}}_{R,\square}$ *as above. Then*

$$\mathcal{W}(\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}), \psi) \cong \mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}_1) \times \cdots \times \mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}_k)$$

and hence for $\mathfrak{m}' \in \mathcal{M}ult^{\mathrm{ap}}_{R,\square}$ *a second multisegment we have*

$$\mathcal{W}(\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) \times \mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}'), \psi) = \mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m} * \mathfrak{m}').$$

An analogous claim holds for $\mathcal{S}^{\cup}_{\mathrm{gen},\psi}$ *and if* $\mathfrak{n} \preceq \mathfrak{m}$ *then* $\mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{n}) \subseteq \mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m})$.

Proof. There exists a nonzero map

$$\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}_1) \times \cdots \times \mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}_k) \to \mathcal{W}(\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}), \psi).$$

Let π be the unique irreducible subrepresentation of the latter representation. It is then enough to show that π is the unique subrepresentation of the former representation. To see this note that by Theorem 2.2.1 and Lemma 2.2.1, the representation $\pi_1 \times \cdots \times \pi_k$ is irreducible, where π_i is the unique irreducible subrepresentation of

$\mathcal{S}_{\text{gen},\psi}(\mathfrak{m}_i)$. Since $\pi_1 \times \cdots \times \pi_k$ is generic, it has to be isomorphic to π. Thus π is a subrepresentation of

$$\mathcal{S}_{\text{gen},\psi}(\mathfrak{m}_1) \times \cdots \times \mathcal{S}_{\text{gen},\psi}(\mathfrak{m}_k).$$

For the second point, note that if $\mathfrak{m}$ and $\mathfrak{m}'$ are in different cuspidal lines, then we just proved that $\mathcal{W}(\mathcal{S}_{\text{gen},\psi}(\mathfrak{m}) \times \mathcal{S}_{\text{gen},\psi}(\mathfrak{m}'), \psi) = \mathcal{W}(\mathcal{S}_{\text{gen},\psi}(\mathfrak{m}+\mathfrak{m}'), \psi) = \mathcal{W}(\mathcal{S}_{\text{gen},\psi}(\mathfrak{m}'+\mathfrak{m}), \psi) = \mathcal{W}(\mathcal{S}_{\text{gen},\psi}(\mathfrak{m}') \times \mathcal{S}_{\text{gen},\psi}(\mathfrak{m}), \psi)$. From this the second claim follows. Finally, if $\mathfrak{n} \preceq \mathfrak{m}$ with $\mathfrak{m}_i, \mathfrak{n}_i \in \mathcal{M}ult_R(\rho_i)$ as above, we have by definition $\mathfrak{n} \preceq \mathfrak{m}$ if and only if $\mathfrak{n}_i \preceq \mathfrak{m}_i$ for all i. From the first point and Proposition 3.3 it then follows that there exists an injection

$$\mathcal{S}^{\cup}_{\text{gen},\psi}(\mathfrak{n}) \hookrightarrow \mathcal{S}^{\cup}_{\text{gen},\psi}(\mathfrak{m}),$$

which by the uniqueness of the Whittaker model must be an inclusion.

The claim for $\mathcal{S}^{\cup}_{\text{gen},\psi}$ follows the exact same pattern. □

4. Standard modules

For the rest of this section, we fix an additive character ψ.

Definition. Two maps $\mathcal{T}_\psi, \mathcal{T}_{\psi^{-1}} : \mathcal{M}ult_R \to \mathfrak{Rep}$ are called of Whittaker type if they satisfy the following.

(1) For each $\mathfrak{m} \in \mathcal{M}ult_R$, $\mathcal{T}_\psi(\mathfrak{m})$ is a representation of $G_{\deg(\mathfrak{m})}$ and is contained in $\mathfrak{Rep}_{W,\psi}$.
(2) $\mathcal{T}_\psi(\mathfrak{m})^{\mathfrak{c}} = \mathcal{T}_{\psi^{-1}}(\mathfrak{m}^\vee)$.
(3) If $\mathfrak{n} \preceq \mathfrak{m}$ then $\mathcal{T}_\psi(\mathfrak{n}) \subseteq \mathcal{T}_\psi(\mathfrak{m})$.

Since by (2), $\mathcal{T}_{\psi^{-1}}$ is determined by $\mathcal{T}_\psi$, we will usually omit $\mathcal{T}_{\psi^{-1}}$.

The map $\mathcal{T}_\psi$ is called *L-standard* if it is of Whittaker type and for all $\mathfrak{n}, \mathfrak{m} \in \mathcal{M}ult_R^{\text{ap}}$

$$L(X, \mathcal{T}_\psi(\mathfrak{n}), \mathcal{T}_{\psi^{-1}}(\mathfrak{m})) = L(X, \mathrm{C}'(\mathfrak{n}), \mathrm{C}'(\mathfrak{m})).$$

Finally, an L-standard map $\mathcal{T}$ is called *standard* if it moreover satisfies the following.

(1) For each $\mathfrak{m} \in \mathcal{M}ult_R^{\text{ap}}$ there exists a nonzero map $\mathcal{T}_\psi(\mathfrak{m}) \twoheadrightarrow \langle\mathfrak{m}\rangle$.
(2) For each $\mathfrak{m} \in \mathcal{M}ult_R^{\text{ap}}$, $\dim_R \operatorname{Hom}_{\Delta G_{\deg(\mathfrak{m})}}(\mathcal{T}_\psi(\mathfrak{m}) \otimes \mathcal{T}_{\psi^{-1}}(\mathfrak{m}^\vee), R) = 1$.
(3) The multiplicity of $\langle\mathfrak{m}\rangle$ in $\mathcal{T}_\psi(\mathfrak{m})$ is 1.

Replacing $\mathcal{M}ult_R$ by $\mathcal{M}ult_{R,\square}$ in the above definition, we obtain the notions of $\square$-Whittaker type, $\square$-L-standard and $\square$-standard.

It is easy to see that if $\mathcal{T}_\psi$ is standard and $\ell \neq 2$, the nonzero map $\mathcal{T}_\psi(\mathfrak{m}) \to \mathcal{T}_{\psi^{-1}}(\mathfrak{m}^\vee)^\vee$, which is unique up to a scalar, factors through $\langle\mathfrak{m}\rangle$. Indeed, since $\ell \neq 2$, we have that $\langle\mathfrak{m}\rangle^{\mathfrak{c}} \cong \langle\mathfrak{m}\rangle^{\vee}$ is a quotient of $\mathcal{T}_{\psi^{-1}}(\mathfrak{m}^{\vee})$, and since Aubert duality commutes with taking duals, we have a nonzero map

$$\mathcal{T}_{\psi}(\mathfrak{m}) \twoheadrightarrow \langle\mathfrak{m}\rangle \hookrightarrow \mathcal{T}_{\psi^{-1}}(\mathfrak{m}^{\vee})^{\vee}.$$

For $\mathcal{T}$ a standard map, the representations $\mathcal{T}_{\psi^{-1}}(\mathfrak{m})$ with $\mathfrak{m}$ aperiodic are called *standard modules*.

4.1. *Standard modules over $\overline{\mathbb{Q}}_{\ell}$.* For this section we set $R = \overline{\mathbb{Q}}_{\ell}$. For an arranged form $(\Delta_1, \dots, \Delta_k)$ of a multisegment $\mathfrak{m}$ we denote

$$T_{\psi}(\mathfrak{m}) := \mathcal{W}(\langle\Delta_1\rangle) \times \cdots \times \mathcal{W}(\langle\Delta_k\rangle, \psi),$$

whose isomorphism type is independent of the arranged order; see, for example, [32, Theorem 6.1]. If $\mathfrak{m}$ is moreover integral we write

$$T_{\psi}^{\mathrm{en}}(\mathfrak{m}) := \mathcal{W}^{\mathrm{en}}(\langle\Delta_1\rangle, \psi) \times \cdots \times \mathcal{W}^{\mathrm{en}}(\langle\Delta_k\rangle, \psi).$$

Theorem 4.1.1. *The map $\mathcal{S}_{\psi} : \mathcal{M}ult_{\overline{\mathbb{Q}}_{\ell}} \to \mathfrak{Rep}$ given by $\mathfrak{m} \mapsto \mathcal{W}(T(\mathfrak{m}), \psi)$ is standard.*

Proof. For the computation of L-factors see [12, Theorem 8.2]. The other properties are proved in [32, §6, §7, §9]. □

We let $\mathcal{S}_{\psi}(\mathfrak{m})^{\mathrm{en}}$ be the subspace of $\overline{\mathbb{Z}}_{\ell}$-valued function in $\mathcal{S}_{\psi}(\mathfrak{m})$.

Lemma 4.1.1. *For all $\mathfrak{m} \in \mathcal{M}ult_{\overline{\mathbb{Q}}_{\ell}}$, $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) = \mathcal{S}_{\psi}(\mathfrak{m})$.*

Proof. We argue by induction on $\deg(\mathfrak{m})$. We first treat the case where $\mathfrak{m}$ is one segment $[0, b]_{\rho}$. Then by [32, Proposition 9.5] and Frobenius reciprocity we have $\rho \times \langle[1, b]_{\rho}\rangle \twoheadrightarrow \langle[0, b]_{\rho}\rangle$, which is a generic representation by the uniqueness of the Whittaker model. Thus $\mathcal{W}(\rho \times \langle[1, b]_{\rho}\rangle, \psi) = \mathcal{W}(\langle[0, b]_{\rho}\rangle, \psi)$. By induction on b the left-hand side equals $\mathcal{S}_{\mathrm{gen},\psi}([0, b]_{\rho})$ by Proposition 3.3.

We come to the general case. Write $\mathfrak{m} = \rho * \mathfrak{m}'$ as follows. Let $[0, b]_{\rho}$, $\rho \in \mathfrak{C}$, be a longest segment in $\mathfrak{m}$ such all segments Δ in $\mathfrak{m}$ with $a_{\rho}(\Delta) = 1$ satisfy $l(\Delta) < b$. Taking $\mathfrak{m}' = \mathfrak{m} - [0, b] + [1, b]_{\rho}$ we obtain $\rho * \mathfrak{m}' = \mathfrak{m}$. One can choose an arranged form $(\Delta_1, \dots, \Delta_k)$ of $\mathfrak{m}'$ such that for $j < i$, where i is the minimal i such that $\Delta_i = [1, b]_{\rho}$, either Δ_j has cuspidal support not intersecting $\mathbb{Z}[\rho]$ or $a_{\rho}(\Delta_j)$ is not equal to 1. In any case, $[0, 0] + \Delta_j$ is unlinked for $j < i$. We recall that by the uniqueness of the Whittaker model, we have for three representations π_1, π_2, π_3 of Whittaker type

$$\mathcal{W}(\pi_1 \times \pi_2 \times \pi_3, \psi) = \mathcal{W}(\pi_1 \times \mathcal{W}(\pi_2, \psi) \times \pi_3, \psi).$$

Thus by the case of a single segment

$$\begin{aligned}\mathcal{W}(\rho\times\langle\Delta_1\rangle\times\cdots\times\langle\Delta_k\rangle,\psi) &= \mathcal{W}(\langle\Delta_1\rangle\times\cdots\times\rho\times\langle\Delta_i\rangle\times\cdots\times\langle\Delta_k\rangle,\psi)\\ &= \mathcal{W}(\langle\Delta_1\rangle\times\cdots\times\mathcal{W}(\rho\times\langle\Delta_i\rangle,\psi)\times\cdots\times\langle\Delta_k\rangle,\psi)\\ &= \mathcal{W}(\langle\Delta_1\rangle\times\cdots\times\langle{}^{+}\Delta_i\rangle\times\cdots\times\langle\Delta_k\rangle,\psi).\end{aligned}$$

It is easy to see that $(\Delta_1,\dots,{}^{+}\Delta_i,\dots,\Delta_k)$ is an arranged form of $\mathfrak{m}$; hence the right-hand side is equal to $\mathcal{S}_\psi(\mathfrak{m})$. By induction and Proposition 3.3 the left-hand side is equal to $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m})$. □

4.2. *Standard modules over $\overline{\mathbb{F}}_\ell$.* We come to our definition of standard modules over $\overline{\mathbb{F}}_\ell$. We saw in the introduction that if $\tilde{\mathfrak{m}}$ is a lift of $\mathfrak{m}\in\mathcal{M}ult_{\overline{\mathbb{F}}_\ell}$ (Section 2.3), Theorem 3.2.2 says that $\mathcal{S}_\psi(\tilde{\mathfrak{m}})$ can be equipped with a natural integral structure, whose reduction mod ℓ is again of Whittaker type and is denoted by $\overline{\mathcal{S}_\psi(\tilde{\mathfrak{m}})}$. For $\mathfrak{m}\in\mathcal{M}ult_{\overline{\mathbb{F}}_\ell}$ we then defined the intersection

$$\mathcal{S}_\psi(\mathfrak{m}) := \bigcap_{\tilde{\mathfrak{m}}\text{ lift of }\mathfrak{m}} \overline{\mathcal{S}_\psi(\tilde{\mathfrak{m}})}$$

in the space of Whittaker functions. The representation is nonzero if $\mathfrak{m}\in\mathcal{M}ult_{R,\square}$: indeed, each $\overline{\mathcal{S}_\psi(\tilde{\mathfrak{m}})}$ contains with multiplicity one and as a unique subrepresentation the degenerate representation $\mathrm{Z}(\mathfrak{s})$, where $\mathfrak{s}=\mathrm{cusp}(\mathfrak{m})$. Then $\mathrm{Z}(\mathfrak{s})$ is a subrepresentation of $\mathcal{S}_\psi(\mathfrak{m})$ since $\dim_R\mathrm{Hom}_{G_n}(\mathrm{Z}(\mathfrak{s}),\mathcal{W}(\psi))=1$.

Definition. Let $\mathcal{T}_\psi$ a map of Whittaker type. We call $\mathcal{T}$ *extending* if for all $\mathfrak{m}\in\mathcal{M}ult_{R,\square}$ and $\rho\in\mathfrak{C}^\square$

$$\mathcal{W}(\rho\times\mathcal{T}_\psi(\mathfrak{m}),\psi)\subseteq\mathcal{T}_\psi(\rho*\mathfrak{m}),\ \mathcal{W}(\mathcal{T}_\psi(\mathfrak{m})\times\rho,\psi)\subseteq\mathcal{T}_\psi(\mathfrak{m}*\rho)$$

Lemma 4.2.1. *The maps $\mathcal{S}_\psi$ and $\mathcal{S}^{\cup}_{\mathrm{gen},\psi}$ are extending.*

Proof. For $\mathcal{S}^{\cup}_{\mathrm{gen},\psi}$ the claim is a consequence of Proposition 3.3. The claim for $\mathcal{S}_\psi$ in the case $R=\overline{\mathbb{F}}_\ell$ follows by noting that on the one hand, for every lift $\tilde{\mathfrak{m}}'$ of $\rho*\mathfrak{m}$, one can find lifts $\tilde{\rho}$ and $\tilde{\mathfrak{m}}$ of ρ and $\mathfrak{m}$ such that $\tilde{\rho}*\tilde{\mathfrak{m}}=\tilde{\mathfrak{m}}'$ and vice versa. By Lemma 3.2.2 the surjective, and hence nonzero, map $\mathcal{W}(\tilde{\rho},\psi)\times\mathcal{S}_\psi(\tilde{\mathfrak{m}})\twoheadrightarrow\mathcal{W}(\rho\times\mathcal{S}_\psi(\tilde{\mathfrak{m}}),\psi)=\mathcal{W}(\mathcal{S}_\psi(\tilde{\rho}*\tilde{\mathfrak{m}}),\psi)$ reduces to a nonzero map

$$\overline{\mathcal{W}(\tilde{\rho},\psi)}\times\overline{\mathcal{S}_\psi(\tilde{\mathfrak{m}})}\to\overline{\mathcal{W}(\mathcal{S}_\psi(\tilde{\rho}*\tilde{\mathfrak{m}}),\psi)}.$$

Taking the intersection in the space of Whittaker models we obtain a nonzero map

$$\rho\times\mathcal{S}_\psi(\mathfrak{m})\to\mathcal{S}_\psi(\rho*\mathfrak{m}).$$

To see that the map on the intersection is nonzero, it suffices to note that the map

$$\overline{\mathcal{W}(\tilde{\rho},\psi)}\times\overline{\mathcal{S}_\psi(\tilde{\mathfrak{m}})}\to\overline{\mathcal{W}(\mathcal{S}_\psi(\tilde{\rho}*\tilde{\mathfrak{m}}),\psi)}$$

does not vanish on the subrepresentation $\overline{\mathcal{W}(\tilde{\rho}, \psi)} \times \mathrm{Z}(\mathrm{cusp}(\mathfrak{m}))$, where $\mathrm{Z}(\mathrm{cusp}(\mathfrak{m}))$ is the unique irreducible subrepresentation of $\overline{\mathcal{S}_\psi(\tilde{\mathfrak{m}})}$ that is independent of the lift $\tilde{\mathfrak{m}}$.

By the uniqueness of the Whittaker model we thus have that $\mathcal{W}(\rho \times \mathcal{S}_\psi(\tilde{\mathfrak{m}}), \psi) \subseteq \mathcal{S}_\psi(\rho \times \mathfrak{m})$. The other inclusion follows by the same argument. □

Corollary 4.2.1. *The map* $\mathcal{S}^{\cup}_{\mathrm{gen},\psi} : \mathcal{M}ult_{\overline{\mathbb{F}}_\ell,\square} \to \mathfrak{Rep}$ *is of* $\square$*-Whittaker type. For* $\mathfrak{m} \in \mathcal{M}ult_{\overline{\mathbb{F}}_\ell}$, $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) \subseteq \mathcal{S}_\psi(\mathfrak{m})$ *and* $\mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m}) \subseteq \mathcal{S}_\psi(\mathfrak{m})$.

Proof. We start with proving properties (1) and (2) of being a $\square$-Whittaker type for $\mathcal{S}_{\mathrm{gen},\psi}$, which proves them for $\mathcal{S}^{\cup}_{\mathrm{gen},\psi}$. The first claim is true by construction. The second claim follows from the following observation. Let $\mathfrak{m} \in \mathcal{M}ult_{\overline{\mathbb{F}}_\ell,\square}$ and $\rho \in \mathfrak{C}^{\square}$. Then it follows from the construction of $\mathfrak{m}_{\mathrm{gen}}$ that $(\rho * \mathfrak{m})^\vee = \mathfrak{m}^\vee * \rho^\vee$. In particular, if $\mathfrak{m}$ is of the form $\rho_1 * \cdots * \rho_k$ for suitable $\rho_i \in \mathfrak{C}^{\square}$, then $\mathfrak{m}^\vee = \rho_k^\vee * \cdots * \rho_1^\vee$. Moreover, $\mathcal{M}ult_{\overline{\mathbb{F}}_\ell,\square}$ is nonzero only if $\ell \neq 2$ and hence $\rho_i^{\mathfrak{c}} \cong \rho_i^\vee$. We thus have that

$$\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m})^{\mathfrak{c}} = \mathcal{W}(\rho_1 \times \cdots \times \rho_k, \psi)^{\mathfrak{c}} = \mathcal{W}(\rho_k^{\mathfrak{c}} \times \cdots \times \rho_1^{\mathfrak{c}}, \psi^{-1}) = \mathcal{S}_{\mathrm{gen},\psi^{-1}}(\mathfrak{m}^\vee)$$

and the second claim follows. The third claim for $\mathcal{S}_{\mathrm{gen},\psi}$ follows from Lemma 2.4.1.

Finally, to prove $\mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m}) \subseteq \mathcal{S}_\psi(\mathfrak{m})$ it suffices to prove that for $\mathfrak{n}$ aperiodic with $\mathfrak{n} \preceq \mathfrak{m}$ we have $\mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{n}) \subseteq \mathcal{S}_\psi(\mathfrak{m})$. Moreover, by Lemma 3.4.2 it suffices to assume that $\mathfrak{m} \in \mathcal{M}ult_R(\rho)$. Write $\mathfrak{n} = \rho_1 * \cdots * \rho_k$. By the uniqueness of the Whittaker model it suffices to give a nonzero map $\rho_1 \times \cdots \times \rho_k \to \mathcal{S}_\psi(\mathfrak{m})$. By Lemma 2.4.3 and the Geometric Lemma, we have that $\rho_1 \otimes \ldots \otimes \rho_k$ appears in $r_{\overline{P}_\alpha}(\overline{\mathcal{S}_\psi(\tilde{\mathfrak{m}})})$, for any lift $\tilde{\mathfrak{m}}$ of $\mathfrak{m}$, as a subquotient. Since $\mathfrak{m} \in \mathcal{M}ult_R(\rho)$, considering the central characters implies that it appears as a subrepresentation, which in turn implies that there exists a nonzero map $\rho_1 \times \cdots \times \rho_k \to \overline{\mathcal{S}_\psi(\tilde{\mathfrak{m}})}$. Since $\rho_1 \times \cdots \times \rho_k$ is of Whittaker type, $\mathcal{W}(\rho_1 \times \cdots \times \rho_k, \psi)$ is contained in the intersection of the $\overline{\mathcal{S}_\psi(\tilde{\mathfrak{m}})}$, implying the claim. □

Remark. If $\mathfrak{m}$ is a banal multisegment (see [22]), i.e., if for each cuspidal representation ρ, the cuspidal support of $\mathfrak{m}$ does not contain $\rho|-|^k$ for some k, the same argument as in Lemma 4.1.1 shows that $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m})$ and $\mathcal{S}_\psi(\mathfrak{m})$ agree.

Lemma 4.2.2. *Assume* $\mathcal{T}_\psi$ *is an extending map of Whittaker type. Let* $\mathfrak{m} \in \mathcal{M}ult_{R,\square}$ *and let* $\Delta = [a, b]_\rho$, $\rho \in \mathfrak{C}^{\square}_m$, *be a segment in* $\mathfrak{m}$. *Then* $\mathcal{T}_\psi(\mathfrak{m} - \Delta + {}^{-}\Delta) \subseteq \mathcal{T}_\psi(\mathfrak{m})^{(m)}$.

Proof. We show that $\rho|-|^a * (\mathfrak{m} - \Delta + {}^{-}\Delta) \preceq \mathfrak{m}$. Let $\Gamma = [a-1, b']_\rho$ be the longest segment in $\mathfrak{m} - \Delta + {}^{-}\Delta$ with $a_\rho(\Gamma) = a-1$. If $\Gamma = {}^{-}\Delta$ we have that the left side equals the right side. Otherwise, we have $b' > b$ and that the left side equals

$$\mathfrak{m} - \Delta + {}^{-}\Delta - \Gamma + {}^{+}\Gamma,$$

which can be obtained from $\mathfrak{m}$ via the elementary operation $\Gamma + \Delta \mapsto {}^{+}\Gamma + {}^{-}\Delta$. By the above observation and the properties of $\mathcal{T}_\psi$, we obtain that

$$\mathcal{W}(\rho|-|^a \times \mathcal{T}_\psi(\mathfrak{m} - \Delta + {}^{-}\Delta), \psi) \subseteq \mathcal{T}_\psi(\mathfrak{m}).$$

The claim then follows from Proposition 3.2. □

4.3. *L-factors of standard modules.* Fix an additive character ψ. The goal of this section is to show that $\mathcal{S}_\psi$ and $\mathcal{S}_{\mathrm{gen},\psi}$ are L-standard.

Theorem 4.3.1. *Let $\mathcal{T}_\psi$ be an extending map of Whittaker type such that $\mathcal{T}_\psi(\mathfrak{m}) \subseteq \mathcal{S}_\psi(\mathfrak{m})$ for all $\mathfrak{m} \in \mathcal{M}ult_R$. Then $\mathcal{T}_\psi$ is L-standard, i.e., for $\mathfrak{n}, \mathfrak{m} \in \mathcal{M}ult^{\mathrm{ap}}_{R,\square}$*

$$L(X, \mathcal{T}_\psi(\mathfrak{m}), \mathcal{T}_{\psi^{-1}}(\mathfrak{n})) = L(X, \mathrm{C}'(\mathfrak{m}), \mathrm{C}'(\mathfrak{n})).$$

This has been achieved in [16, Theorem 4.22] for generic representations if R is arbitrary. In the case $R = \overline{\mathbb{Q}}_\ell$, this follows from the work of [12] as mentioned in Section 4.1, yielding the following corollary of Lemma 3.3.2 and Lemma 3.3.3.

Corollary 4.3.1. *Let $\mathfrak{n}, \mathfrak{m} \in \mathcal{M}ult_R$. Then*

$$L(X, \mathcal{T}_\psi(\mathfrak{m}), \mathcal{T}_{\psi^{-1}}(\mathfrak{n}))^{-1} | L(X, \mathrm{C}'(\mathfrak{m}), \mathrm{C}'(\mathfrak{n}))^{-1}.$$

Before we come to the proof of Theorem 4.3.1, we note the following useful lemmas, all of which follow easily from the definitions.

Lemma 4.3.1. *Let $\mathfrak{m} \in \mathcal{M}ult_R$ containing segments $\Delta = [a, b]_\rho$, $\Delta' = [a+1, b+1]_\rho$ for some $\rho \in \mathfrak{C}^\square$. Let $\mathfrak{n} \in \mathcal{M}ult_R$ and let $\mathfrak{n}'$ be the submultisegment of $\mathfrak{n}$ consisting of segments of the form $[c, d]_{\rho'}$ with $\rho' \cong \chi\rho^\vee$ for some unramified character χ and $c - d = b - a$. Finally, set $\mathfrak{m}' = \mathfrak{m} - \Delta - \Delta' + [a, b+1]_\rho + [a+1, b]_\rho$. Then*

$$\frac{L(X, \mathrm{C}'(\mathfrak{m}), \mathrm{C}'(\mathfrak{n}))}{L(X, \mathrm{C}'(\mathfrak{m}'), \mathrm{C}'(\mathfrak{n}))} = \prod_{[c,d]_{\chi\rho^\vee} \in \mathfrak{n}'} (1 - \chi(\varpi_{\mathrm{F}}) q^{-a-d} X)^{-f(\rho)}.$$

Lemma 4.3.2. *Let $\mathfrak{m}, \mathfrak{n} \in \mathcal{M}ult_R$ and $\Delta = [a, b]_\rho$ a segment of maximal length in $\mathfrak{m}$, where we assume $\rho \in \mathfrak{C}^\square$. Let $\mathfrak{n}'$ be the submultisegment of $\mathfrak{n}$ consisting of segments $[c, d]_{\rho'}$ with $\rho' \cong \chi\rho^\vee$ for some unramified character χ and $c - d \geq b - a$. Finally set $\mathfrak{m}' = \mathfrak{m} - \Delta + {}^-\Delta$. Then*

$$\frac{L(X, \mathrm{C}'(\mathfrak{m}), \mathrm{C}'(\mathfrak{n}))}{L(X, \mathrm{C}'(\mathfrak{m}'), \mathrm{C}'(\mathfrak{n}))} = \prod_{[c,d]_{\chi\rho^\vee} \in \mathfrak{n}'} (1 - \chi(\varpi_{\mathrm{F}}) q^{-a-d} X)^{-f(\rho)}.$$

Lemma 4.3.3. *Let $\mathfrak{m}, \mathfrak{n} \in \mathcal{M}ult_R$ and $\mathcal{T}_\psi$ a map of Whittaker type such that $\mathcal{T}_\psi(\mathfrak{n}) \subseteq \mathcal{S}_\psi(\mathfrak{n})$, $\mathcal{T}_\psi(\mathfrak{m}) \subseteq \mathcal{S}_\psi(\mathfrak{m})$. Then*

$$\frac{L(X, \mathcal{T}_\psi(\mathfrak{m}), \mathcal{T}_{\psi^{-1}}(\mathfrak{n})) L(q^{-1}X^{-1}, \mathrm{C}'(\mathfrak{m}^\vee), \mathrm{C}'(\mathfrak{n}^\vee))}{L(q^{-1}X^{-1}, \mathcal{T}_\psi(\mathfrak{m}^\vee), \mathcal{T}_{\psi^{-1}}(\mathfrak{n}^\vee)) L(X, \mathrm{C}'(\mathfrak{m}), \mathrm{C}'(\mathfrak{n}))}$$

is a unit in $R[X, X^{-1}]$.

Proof. Combining Lemmas 3.3.1 and 3.3.2 shows that the above fraction equals

$$\mathrm{r}_\ell(\epsilon(X,\mathcal{S}_\psi(\tilde{\mathfrak{m}}),\mathcal{S}_{\psi^{-1}}(\tilde{\mathfrak{n}}),\psi))\epsilon(X,\mathcal{T}_\psi(\mathfrak{m}),\mathcal{T}_{\psi^{-1}}(\mathfrak{n}),\psi)^{-1}$$

for suitable lifts $\tilde{\mathfrak{m}}$ and $\tilde{\mathfrak{n}}$ of $\mathfrak{m}$ and $\mathfrak{n}$. Since ϵ-factors are units in $R[X,X^{-1}]$, the claim follows. □

Lemma 4.3.4. *Let $\alpha,\beta\in R$ such that $\alpha^f\neq\beta^f$. Then*

$$\gcd(1-(\alpha X)^f,1-(\beta X)^f)=1.$$

Proof of Theorem 4.3.1. In fact we will show the slightly stronger statement that if $\mathfrak{m},\mathfrak{n}\in\mathcal{M}ult_R$ with at least one of them aperiodic, then

$$L(X,\mathcal{T}_\psi(\mathfrak{m}),\mathcal{T}_{\psi^{-1}}(\mathfrak{n}))=L(X,\mathrm{C}'(\mathfrak{m}),\mathrm{C}'(\mathfrak{n})).$$

We argue firstly by induction on $\deg(\mathfrak{m})$ and $\deg(\mathfrak{n})$ and for fixed $\deg(\mathfrak{m})$ and $\deg(\mathfrak{n})$ we argue moreover by induction on the order $\preceq$ on the set of multisegments. The base case is [16, Theorem 4.22].

By Corollary 4.3.1 we know that there exists $P\in\overline{\mathbb{F}}_\ell[X]$ such that

$$L(X,\mathcal{T}_\psi(\mathfrak{m}),\mathcal{T}_{\psi^{-1}}(\mathfrak{n}))^{-1}P(X)=L(X,\mathrm{C}'(\mathfrak{m}),\mathrm{C}'(\mathfrak{n}))^{-1}.$$

Moreover, let $\mathfrak{m}_b$ and $\mathfrak{n}_b$ be the banal parts of $\mathfrak{m}$ and $\mathfrak{n}$. By definition

$$L(X,\mathrm{C}'(\mathfrak{m}),\mathrm{C}'(\mathfrak{n}))=L(X,\mathrm{C}'(\mathfrak{m}_b),\mathrm{C}'(\mathfrak{n}_b))$$

and by Lemma 3.3.1 and Proposition 3.2 and the base case,

$$L(X,\mathcal{T}_\psi(\mathfrak{m}_b),\mathcal{T}_{\psi^{-1}}(\mathfrak{n}_b))=L(X,\mathcal{T}_\psi(\mathfrak{m}),\mathcal{T}_{\psi^{-1}}(\mathfrak{n})),$$

hence it is enough to treat the case where $\mathfrak{m}=\mathfrak{m}_b,\ \mathfrak{n}=\mathfrak{n}_b$.

We first assume without loss of generality that $\mathfrak{m}$ is not aperiodic, i.e., it contains a multisegment

$$[a,b]_\rho+\cdots+[a+o(\rho)-1,b+o(\rho)-1]_\rho.$$

Choose $i\in\{0,\dots,o(\rho)-1\}$ and using the notation of Lemma 4.3.1, we set $\Delta=[a+i,b+i]_\rho,\ \Delta'=[a+i+1,b+i+1]_\rho$. By assumption on, $\mathcal{T}_\psi,\mathcal{T}_\psi(\mathfrak{m}')\subseteq\mathcal{T}_\psi(\mathfrak{m})$ and hence by induction on $\preceq$ we obtain by Lemma 4.3.1 that

$$P(X)\,\Big|\prod_{\substack{[c,d]_{\chi\rho^\vee}\in\mathfrak{n}\\ d-c=b-a}}(1-\chi(\varpi_{\mathrm{F}})q^{-a-i-d}X)^{f(\rho)}.$$

Let $I_i=\{\chi(\varpi_{\mathrm{F}})q^{-a-i-d}:[c,d]_{\chi\rho^\vee}\in\mathfrak{n}:\ d-c=b-a\}$. Assume that $P(X)$ is nonzero and let $(1-\alpha X)$ be one of its nonzero factors with

$$(1-\alpha X)\,\Big|1-\chi(\varpi_{\mathrm{F}})q^{-a-i-d}X)^{f(\rho)}$$

for some i and $[c,d]_{\chi\rho^\vee} \in \mathfrak{n} : d-c = b-a$. Let $j \in \{1,\dots,o(\rho)\}$. Since

$$(1-\alpha X) \mid \prod_{\substack{[c,d]_{\chi\rho^\vee}\in\mathfrak{n}\\ d-c=b-a}} (1-\chi(\varpi_{\mathrm{F}})q^{-a-j-d}X)^{f(\rho)},$$

it follows that $(1-\alpha X) \mid (1-\chi'(\varpi_{\mathrm{F}})q^{-a-j-d'}X)^{f(\rho)}$ for some $[c',d']_{\chi'\rho^\vee} \in \mathfrak{n}$. Therefore, by Lemma 4.3.4,

$$\chi'(\varpi_{\mathrm{F}})^{f(\rho)}q^{f(\rho)(-a-j-d')} = \chi(\varpi_{\mathrm{F}})^{f(\rho)}q^{f(\rho)(-a-i-d)}.$$

It follows straightforwardly that this implies that for each j, $[c+j,d+j]_{\chi\rho^\vee}$ appears in $\mathfrak{m}$, contradicting the assumption on its being aperiodic.

Secondly, we assume that both $\mathfrak{m}$ and $\mathfrak{n}$ are aperiodic and, without loss of generality, that the length of the longest segment in $\mathfrak{m}$ is greater than or equal than the length of the longest segment in $\mathfrak{n}$. We will now use the notation of Lemma 4.3.2, e.g., $\Delta = [a,b]_\rho$ is a longest segment in $\mathfrak{m}$. Now Lemma 4.2.2 and the induction hypothesis in combination with Proposition 3.2 and Lemma 3.3.3 give

$$P(X)| \prod_{[c,d]_{\chi\rho^\vee}\in\mathfrak{n}'} (1-\chi(\varpi_{\mathrm{F}})q^{-a-d}X)^{f(\rho)}.$$

If $\mathfrak{n}'$ is empty, we are done; hence we can assume that the longest segment in $\mathfrak{m}$ and the longest segment in $\mathfrak{n}$ have the same length.

Replacing $\mathfrak{m}$ and $\mathfrak{n}$ by $\mathfrak{m}^\vee$ and $\mathfrak{n}^\vee$ we obtain a polynomial $P^\vee(X)$ such that by Lemma 4.3.3

$$\frac{P(X)}{P^\vee(q^{-1}X^{-1})} = rX^k,\ r \in \overline{\mathbb{F}}_\ell,\ k \in \mathbb{Z}.$$

Applying the same reasoning as above to $P^\vee(X)$ we obtain

$$P^\vee(X) \mid \prod_{[c,d]_{\chi\rho^\vee}\in\mathfrak{n}'} (1-\chi(\varpi_{\mathrm{F}})^{-1}q^{b+c}X)^{f(\rho)}.$$

We now assume that $P(X)$ is not a constant, and has a zero at $(\chi(\varpi_{\mathrm{F}})q^{-a-d})^{-1}$ with $[c,d]_{\chi\rho^\vee} \in \mathfrak{n}'$. Then $P^\vee(q^{-1}X^{-1})$ has to have a zero also at $\chi(\varpi_{\mathrm{F}})q^{-a-d}$, implying that there exists $[c',d']_{\chi'\rho^\vee} \in \mathfrak{n}'$ such that

$$\chi'(\varpi_{\mathrm{F}})q^{-b-c'+1} = \chi(\varpi_{\mathrm{F}})q^{-a-d}.$$

Thus $\chi' = \chi|-|^{-c'+1+c}$ and hence $[c+1,d+1]_{\chi\rho^\vee} \in \mathfrak{n}'$.

But now we can switch the roles of $\mathfrak{m}$ and $\mathfrak{n}$ as

$$L(X,\mathcal{T}_\psi(\mathfrak{m}),\mathcal{T}_{\psi^{-1}}(\mathfrak{n})) = L(X,\mathcal{T}_\psi(\mathfrak{n}),\mathcal{T}_{\psi^{-1}}(\mathfrak{m}))$$

and apply Lemma 4.3.2 with the longest segment $[c+1,d+1]_{\chi\rho^\vee}$, yielding that $[a-1,b-1]_\rho$ has to be a segment of $\mathfrak{m}$, since $(\chi(\varpi_{\mathrm{F}})q^{-a-d})^{-1}$ is a zero of $P(X)$.

Repeating this process we obtain that for all $i \in \mathbb{Z}_{\geq 0}$ the segment $[a-i, b-i]_\rho$ is contained in $\mathfrak{m}$, a contradiction to the assumption that $\mathfrak{m}$ is aperiodic. □

Recall now the C-parameters of [16], i.e., the image of the injective map constructed in [16]

$$\mathrm{C} : \mathfrak{Irr}_n \to \{\text{semisimple Deligne } R-\text{representations of length } n\}.$$

For the precise definition we refer to [16]. The right hand side of the above map is equipped with a tensor product denoted by $\otimes_{ss}$ and one can associate to two C-parameters $\mathrm{C}(\pi)$ and $\mathrm{C}(\pi')$ the three local factors

$$L(X, \mathrm{C}(\pi) \otimes_{ss} \mathrm{C}(\pi')),\ \epsilon(X, \mathrm{C}(\pi) \otimes_{ss} \mathrm{C}(\pi'), \psi),\ \gamma(X, \mathrm{C}(\pi) \otimes_{ss} \mathrm{C}(\pi'), \psi).$$

As a corollary to Theorem 4.3.1, one can prove exactly as in [16, §6.4] the following.

Corollary 4.3.2. *Let* $\pi = \langle \mathfrak{m} \rangle,\ \pi' = \langle \mathfrak{m}' \rangle \in \mathfrak{Irr}$. *Then*

$$L(X, \mathcal{S}_\psi(\mathfrak{m}), \mathcal{S}_{\psi^{-1}}(\mathfrak{n})) = L(X, \mathrm{C}(\pi) \otimes_{ss} \mathrm{C}(\pi')),$$
$$\epsilon(X, \mathcal{S}_\psi(\mathfrak{m}), \mathcal{S}_{\psi^{-1}}(\mathfrak{n}), \psi) = \epsilon(X, \mathrm{C}(\pi) \otimes_{ss} \mathrm{C}(\pi'), \psi),$$
$$\gamma(X, \mathcal{S}_\psi(\mathfrak{m}), \mathcal{S}_{\psi^{-1}}(\mathfrak{n}), \psi) = \gamma(X, \mathrm{C}(\pi) \otimes_{ss} \mathrm{C}(\pi'), \psi).$$

If $\mathfrak{m} \in \mathcal{M}ult_{R,\square}$, *the same is true if one replaces* $\mathcal{S}$ *by* $\mathcal{S}^{\cup}_{\mathrm{gen},\psi}$.

4.4. ***Quotients of standard modules.*** Let $\mathfrak{m} \in \mathcal{M}ult_{R,\square}$. We can define

$$J_{\mathfrak{m}} : \mathcal{S}_\psi(\mathfrak{m}) \otimes \mathcal{S}_{\psi^{-1}}(\mathfrak{m}^\vee) \otimes C_c^\infty(\mathrm{F}^n) \to R$$

by

$$(W, W', \phi) \mapsto L(X, \mathcal{S}_\psi(\mathfrak{m}), \mathcal{S}_{\psi^{-1}}(\mathfrak{m}^\vee))^{-1} I(X, W, W', \phi)|_{X=1}.$$

Let $C_{c,0}^\infty(\mathrm{F}^n)$ be the subspace of $C_c^\infty(\mathrm{F}^n)$ consisting of all function vanishing at 0.

Proposition 4.1. *The map* $J_{\mathfrak{m}}$ *vanishes for all* $\phi \in C_{c,0}^\infty(\mathrm{F}^n)$. *In particular, we obtain a nonzero map*

$$J_{\mathfrak{m}} : \mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) \otimes \mathcal{S}_{\mathrm{gen},\psi^{-1}}(\mathfrak{m}^\vee) \hookrightarrow \mathcal{S}_\psi(\mathfrak{m}) \otimes \mathcal{S}_{\psi^{-1}}(\mathfrak{m}^\vee) \to R$$

given by

$$W \otimes W' \mapsto J_{\mathfrak{m}}(W, W', \phi),$$

where ϕ *is some fixed element in* $C_c^\infty(\mathrm{F}^n)$ *such that* $\phi(0) \neq 0$.

Moreover, if $\tilde{\mathfrak{m}}$ *is a lift of* $\mathfrak{m}$ *then* $J_{\tilde{\mathfrak{m}}}$ *restricts to a map*

$$J_{\tilde{\mathfrak{m}}} : \mathcal{S}_\psi(\tilde{\mathfrak{m}})^{\mathrm{en}} \otimes \mathcal{S}_{\psi^{-1}}(\tilde{\mathfrak{m}}^\vee)^{\mathrm{en}} \to \overline{\mathbb{Z}}_\ell$$

whose reduction mod ℓ *equals* $J_{\mathfrak{m}}$.

Proof. We start with the case $R = \overline{\mathbb{Q}}_\ell$. By [19, Proposition 4.6], if $\tilde{\mathfrak{m}}$ is totally unlinked the map

$$J_{\tilde{\mathfrak{m}}} : \mathcal{S}_\psi(\tilde{\mathfrak{m}}) \otimes \mathcal{S}_{\psi^{-1}}(\tilde{\mathfrak{m}}^\vee) \otimes C^\infty_{c,0}(\mathrm{F}^n) \to \overline{\mathbb{Q}}_\ell$$

vanishes. More generally, let $\tilde{\mathfrak{m}} = \Delta_1 + \cdots + \Delta_k$ with the segments in an arranged order, choose $s \in \mathbb{Z}^k$, and set

$$\mathcal{S}_{\psi,s}(\tilde{\mathfrak{m}}) = \mathcal{W}\big(\mathcal{W}(\langle \Delta_1 \rangle, \psi)|-|^{s_1} \times \cdots \times \mathcal{W}(\langle \Delta_k \rangle, \psi)|-|^{s_k}, \psi\big),$$
$$\mathcal{S}_{\psi^{-1},s}(\tilde{\mathfrak{m}}^\vee) = \mathcal{W}\big(\mathcal{W}(\langle \Delta_k^\vee \rangle, \psi^{-1})|-|^{-s_k} \times \cdots \times \mathcal{W}(\langle \Delta_1^\vee \rangle|-|^{-s_1}, \psi^{-1}), \psi^{-1}\big).$$

Fix flat sections

$$f_s \in \mathcal{S}_{\psi,s}(\tilde{\mathfrak{m}}),\ f_s' \in \mathcal{S}_{\psi^{-1},s}(\tilde{\mathfrak{m}}^\vee)$$

in the sense of [5, §3]. We obtain for fixed $\phi \in C^\infty_{c,0}(\mathrm{F}^n)$ by [12, Proposition 3.3] a rational function over $\overline{\mathbb{Q}}_\ell$ such that $P(X_1, \ldots, X_k)$ such that $J_{\mathfrak{m}_s}(f_s, f_s', \phi) = P(q^{s_1}, \ldots, q^{s_k})$. As observed above, for all but finitely many s, $P(q^{s_1}, \ldots, q^{s_k}) = 0$ and hence it vanishes everywhere.

The case $R = \overline{\mathbb{F}}_\ell$ follows readily from the case $R = \overline{\mathbb{Q}}_\ell$. Let $\tilde{\mathfrak{m}}$ be any lift of $\mathfrak{m}$ to $\overline{\mathbb{Q}}_\ell$. Then the map

$$J_{\tilde{\mathfrak{m}}} : \mathcal{S}_\psi(\tilde{\mathfrak{m}}) \otimes \mathcal{S}_{\psi^{-1}}(\tilde{\mathfrak{m}}^\vee) \otimes C^\infty_{c,0}(\mathrm{F}^n) \to \overline{\mathbb{Q}}_\ell$$

vanishes. Since $J_{\tilde{\mathfrak{m}}}$ obviously respects the integral structures, it reduces to the map

$$J_{\mathfrak{m}} : \mathcal{S}_\psi(\mathfrak{m}) \otimes \mathcal{S}_{\psi^{-1}}(\mathfrak{m}^\vee) \otimes C^\infty_{c,0}(\mathrm{F}^n) \to \overline{\mathbb{F}}_\ell,$$

which therefore also vanishes. Moreover, if $\tilde{\mathfrak{m}}$ is any multisegment over $\overline{\mathbb{Q}}_\ell$, the map $\mathcal{S}_\psi(\tilde{\mathfrak{m}}) \otimes \mathcal{S}_{\psi^{-1}}(\tilde{\mathfrak{m}}^\vee) \to \overline{\mathbb{Q}}_\ell$ factors through $\langle \tilde{\mathfrak{m}} \rangle \otimes \langle \tilde{\mathfrak{m}}^\vee \rangle$ and in particular vanishes on $\mathcal{S}_\psi(\tilde{\mathfrak{n}}) \otimes \mathcal{S}_{\psi^{-1}}(\tilde{\mathfrak{n}}^\vee)$ for any $\tilde{\mathfrak{n}} \prec \tilde{\mathfrak{m}}$. Hence $J_{\tilde{\mathfrak{m}}}$ vanishes on

$$\mathcal{S}_\psi(\tilde{\mathfrak{n}}) \otimes \mathcal{S}_{\psi^{-1}}(\tilde{\mathfrak{m}}^\vee) \otimes C^\infty_c(\mathrm{F}^n).$$

We now argue that $J_{\mathfrak{m}}$ does not vanish on the restriction to

$$\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) \otimes \mathcal{S}_{\mathrm{gen},\psi^{-1}}(\mathfrak{m}^\vee) \otimes C^\infty_c(\mathrm{F}^n);$$

this will prove the claim. It suffices to show that for $\mathfrak{n} \prec \mathfrak{m}$ the map $J_{\mathfrak{m}}$ vanishes on $\mathcal{S}^\cup_{\mathrm{gen},\psi}(\mathfrak{n}) \otimes \mathcal{S}^\cup_{\mathrm{gen},\psi^{-1}}(\mathfrak{m}^\vee) \otimes C^\infty_c(\mathrm{F}^n)$. It suffices to show the claim for $\mathfrak{n}$ maximal, i.e., that it is obtained from $\mathfrak{m}$ via one elementary operation. But then we can choose a lift $\tilde{\mathfrak{m}}$ of $\mathfrak{m}$ and $\tilde{\mathfrak{n}}$ of $\mathfrak{n}$ with $\tilde{\mathfrak{n}} \prec \tilde{\mathfrak{m}}$ and the claim follows from the observation above and the fact that $\mathcal{S}^\cup_{\mathrm{gen},\psi}(\mathfrak{n}) \subseteq \mathcal{S}_\psi(\mathfrak{n}) \subseteq \overline{\mathcal{S}_\psi(\tilde{\mathfrak{n}})}$; see Corollary 4.2.1. By an analogous argument we can also show that $J_{\mathfrak{m}}$ vanishes on $\mathcal{S}^\cup_{\mathrm{gen},\psi}(\tilde{\mathfrak{m}}) \otimes \mathcal{S}^\cup_{\mathrm{gen},\psi^{-1}}(\tilde{\mathfrak{n}}^\vee) \otimes C^\infty_c(\mathrm{F}^n)$ and hence it cannot vanish on $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) \otimes \mathcal{S}_{\mathrm{gen},\psi^{-1}}(\mathfrak{m}^\vee) \otimes C^\infty_c(\mathrm{F}^n)$. By the previous arguments we know that $J_{\mathfrak{m}}$ vanishes on $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) \otimes \mathcal{S}_{\mathrm{gen},\psi^{-1}}(\mathfrak{m}^\vee) \otimes C^\infty_{c,0}(\mathrm{F}^n)$, which finishes the argument. □

Theorem 4.4.1. *Let* $\mathfrak{m} \in \mathcal{M}ult^{\mathrm{ap}}_{R,\square}$*. Then* $\langle\mathfrak{m}\rangle$ *is the unique irreducible quotient of* $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m})$.

Proof. Let $\mathfrak{m} \in \mathcal{M}ult^{\mathrm{ap}}_{R,\square}$. We argue by induction on $\deg(\mathfrak{m})$ and $\preceq$. The base case is trivial. Let $\pi = \langle\mathfrak{n}\rangle$ be a quotient of $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m})$. Note that $\mathrm{cusp}(\pi) = \mathrm{cusp}(\mathfrak{m})$. Since π has to be an irreducible subquotient of $\mathcal{S}_\psi(\mathfrak{m})$, we have by Lemma 2.2.2 that $\mathfrak{n} \preceq \mathfrak{m}$.

We write $\mathfrak{m} = \mathfrak{m}' * \rho$, $\rho \in \mathfrak{C}^{\square}_m$ and $\mathfrak{m}' \in \mathcal{M}ult^{\mathrm{ap}}_{R,\square}$ and hence $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}') \times \rho \twoheadrightarrow \pi$ by Lemma 2.4.1. In particular $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}') \otimes \rho \hookrightarrow \overline{r_{(\deg(\mathfrak{m})-m,m)}}(\pi)$. By induction and Proposition 3.3 we obtain that $\langle\mathfrak{m}'\rangle \otimes \rho$ appears in $\overline{r_{(\deg(\mathfrak{m})-m,m)}}(\mathcal{S}_\psi(\mathfrak{n}))$. By the geometric lemma of Bernstein and Zelevinsky [2, Theorem 5.2; 32, Theorem 1.1], it follows that $\mathfrak{m}' \preceq \mathfrak{n}'$, where $\mathfrak{n}'$ is of the form $\mathfrak{n}' = \mathfrak{n} - [a, 0]_\rho + [a, -1]_\rho$ for a suitable segment $[a, 0]_\rho$. By the same argument as in Lemma 4.2.2 we have $\mathfrak{n}' * \rho \preceq \mathfrak{n}$. On the other hand by Lemma 2.4.1 we have $\mathfrak{m} = \mathfrak{m}' * \rho \preceq \mathfrak{n}' * \rho \preceq \mathfrak{n}$ and hence $\mathfrak{n} = \mathfrak{m}$. Since $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) \subseteq \mathcal{S}_\psi(\mathfrak{m})$, and $\langle\mathfrak{m}\rangle$ appears in the latter with multiplicity one by Lemma 2.3.1, this is also true for $\mathcal{S}_{\mathrm{gen},\psi}$. □

Let us state two corollaries to this result. By abuse of notation we will also write $J_\mathfrak{m} : \mathcal{S}_\psi(\mathfrak{m}) \to \mathcal{S}_{\psi^{-1}}(\mathfrak{m}^\vee)^\vee$ for the map obtained from Proposition 4.1. As a consequence of this proposition, it restricts to a map $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) \to \mathcal{S}_{\mathrm{gen},\psi^{-1}}(\mathfrak{m}^\vee)^\vee$, which factors through $\langle\mathfrak{m}\rangle$.

Corollary 4.4.1. *Let* $\mathfrak{m} \in \mathcal{M}ult_{R,\square}$*. Then* $\langle\mathfrak{m}\rangle$ *appears in the image of* $J_\mathfrak{m}$ *as a quotient.*

Proof. Let $I(\mathfrak{m})$ denote the image of $J_\mathfrak{m} : \mathcal{S}_\psi(\mathfrak{m}) \to \mathcal{S}_{\psi^{-1}}(\mathfrak{m}^\vee)^\vee$. Let $\Sigma(\mathfrak{m})$ be the kernel of the map $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m}) \to \langle\mathfrak{m}\rangle$. Then $\Sigma(\mathfrak{m}^\vee) = \Sigma(\mathfrak{m})^\mathfrak{c}$. Since $\mathcal{S}_{\mathrm{gen},\psi}$ is $\square$-standard, the kernel of the map $J_\mathfrak{m}$ restricted to $\mathcal{S}_{\mathrm{gen},\psi}(\mathfrak{m})$ is $\Sigma(\mathfrak{m})$; thus $I(\mathfrak{m})$ is a quotient of $\Pi(\mathfrak{m}) := \Sigma(\mathfrak{m})\backslash\mathcal{S}_\psi(\mathfrak{m})$. we know that both $I(\mathfrak{m})$ and $\Pi(\mathfrak{m})$ contain $\langle\mathfrak{m}\rangle$ with multiplicity 1 by Theorem 4.4.1 and $\Pi(\mathfrak{m})$ contains it moreover as a subrepresentation. Since $\Pi(\mathfrak{m})^\mathfrak{c} \cong \Pi(\mathfrak{m}^\vee)$, it also follows that $\Pi(\mathfrak{m})$ admits $\langle\mathfrak{m}\rangle$ as a quotient, and hence as a direct summand since it appears only with multiplicity 1. Thus $\langle\mathfrak{m}\rangle$ is also a quotient of $I(\mathfrak{m})$. □

The second corollary follows readily from Theorem 4.4.1, Proposition 3.3, Lemma 2.2.2 and Corollary 4.2.1.

Corollary 4.4.2. *For* $\mathfrak{n}, \mathfrak{m} \in \mathcal{M}ult_{R,\square}$ *we have* $\mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{n}) \subseteq \mathcal{S}^{\cup}_{\mathrm{gen},\psi}(\mathfrak{m})$ *if and only if* $\mathfrak{n} \preceq \mathfrak{m}$*. We thus have an order-preserving injection*

$$\mathcal{S}^{\cup}_{\mathrm{gen},\psi} : \mathcal{M}ult_{R,\square} \hookrightarrow \mathfrak{Rep}_{W,\psi},$$

respecting the products.

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TRANSVERSE MINIMAL FOLIATIONS ON UNIT TANGENT BUNDLES AND APPLICATIONS

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We show that given two transverse minimal foliations on the unit tangent bundle of a surface of genus ≥ 2, their intersection either is an Anosov foliation or contains a Reeb surface. The existence of a Reeb surface is incompatible with partially hyperbolic foliations, so we deduce from this that certain partially hyperbolic diffeomorphisms in unit tangent bundles are collapsed Anosov flows. We also conclude that every volume preserving partially hyperbolic diffeomorphism of a unit tangent bundle is ergodic.

1. Introduction

This article studies geometric and dynamical properties of one-dimensional subfoliations obtained as the intersection of two transverse minimal foliations on unit tangent bundles of higher genus surfaces. We prove some strong geometric properties that imply that under certain conditions, the foliation must be homeomorphic to the orbit foliation of the geodesic flow for a hyperbolic metric on the surface.

One big motivating example for us comes from partially hyperbolic dynamics in dimension 3: under very general orientability conditions there is a pair of two-dimensional branching foliations, which are approximated by regular foliations. The pair of foliations are transverse to each other, yielding a one-dimensional subfoliation of both. Suppose that one proves that the subfoliation is the flow foliation of a topological Anosov flow. Then the partially hyperbolic diffeomorphism is what is called a collapsed Anosov flow [5; 25]. This has some important consequences, such as accessibility and ergodicity in the volume preserving case [23; 24].

This naturally leads to the following very general question, which was the initial goal of this project:

Question. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be transverse minimal foliations in a closed 3-manifold.*

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Are there simple conditions that guarantee that the intersection foliation is homeomorphic to an Anosov foliation?

Here, by an *Anosov foliation* we mean the orbit foliation of a (topological) Anosov flow.[1] The scope of the question is quite general, and the conditions are somewhat more restrictive than studying vector fields tangent to a foliation (for instance, the horocycle flow is a subfoliation of the weak stable foliation of an Anosov flow but it is rarely obtained as the intersection of two foliations). Having a vector field tangent to a foliation already imposes some obstructions on the foliation and the manifold (see for instance [11; 30]), so we expect here to have even more. This very general question can be extended to the problem of understanding in general the one-dimensional foliations induced by intersecting two general transverse foliations in a closed 3-manifold; we have included minimality to simplify certain formulations[2] but it makes sense to ask this question in general. (In fact, a similar question for three transverse taut foliations is suggested in [39, § 7.1].)

We stress that the question of analyzing general transverse intersections of foliations in 3-manifolds is very natural, interesting in itself, and has appeared in other contexts, such as Anosov and pseudo-Anosov flows transverse to foliations [19; 20; 21; 39].

In this article we start the general study of geometric properties of one-dimensional subfoliations of a pair of transverse foliations in 3-manifolds. In this generality the problem is at this point complex (see Section 1.2 for recent progress). Here we restrict to a class of 3-manifolds: unit tangent bundles of surfaces of negative Euler characteristic. In this case we have strong rigidity for single foliations and this substantially helps study this problem. It is also relevant for us since many people working in partially hyperbolic dynamics are more familiar with this family of 3-manifolds. On the other hand many of the techniques introduced in this paper should be useful for the general problem. In fact, since this paper was released, much progress has been made in the problem, which we survey in Section 1.2, showing the impact it has had.

Although minimal foliations in unit tangent bundles are homeomorphic to the weak stable foliation of an Anosov flow (by a result of [34]; see also [29]), T. Barbot pointed out to us the paper [35], which gives a beautiful example showing that even in these manifolds, there may be obstructions to the intersection being an Anosov foliation. (In fact, [35] contains a triple of pairwise transverse minimal foliations.)

[1]A short definition of a topological Anosov flow is an expansive flow preserving a foliation; see [2] for a nice introduction. In this paper we work in unit tangent bundles, where every topological Anosov flow is orbit equivalent to the geodesic flow of some constant curvature metric (which is a smooth Anosov flow) so we will not differentiate between them.

[2]One can always blow up one of the foliations and the intersection will no longer be an Anosov foliation. Other phenomena can also arise; see for instance the examples constructed in [10].

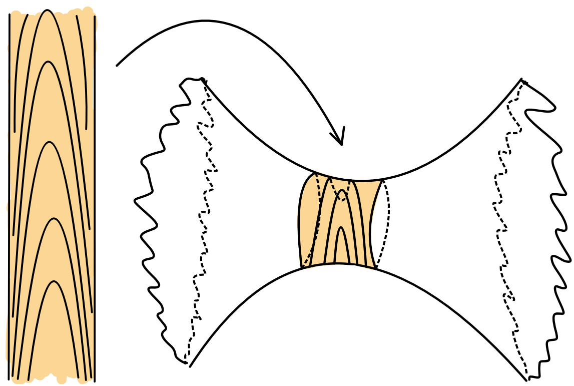

Figure 1. A Reeb surface and its lift to the universal cover.

We show that, in unit tangent bundles, the conditions that obstruct the intersection to be homeomorphic to an Anosov foliation, are similar to those appearing in the example of [35]. In Section 7 we describe the example of [35] from the point of view of this paper as well as discuss some possible extensions, and in Section 9 we prove that the obstruction to intersect in a foliation homeomorphic to an Anosov foliation can be explained by behavior identical to the ones discussed in Section 7.

The two-dimensional foliations $\mathcal{F}_1, \mathcal{F}_2$ we deal with have Gromov hyperbolic leaves. The strategy employed here to prove the Anosov behavior of the intersection foliation $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$ in certain cases is the following: show geometric properties of the subfoliations inside the leaves of the two-dimensional foliations. More specifically we try to show that the subfoliations are by uniform quasigeodesics inside these two-dimensional leaves. We obtain a structure that is enough to identify the obstruction for this to happen: *Reeb surfaces*. These are surfaces in some leaf of $\mathcal{F}_1$ or $\mathcal{F}_2$ which are finitely covered by a two-dimensional annulus whose boundary circles are leaves of the foliation $\mathcal{G}$ and the leaves of $\mathcal{G}$ in the interior of the annulus spiral towards the boundary components in opposite directions (see Figure 1).

Our main result is the following:

Theorem A. *Let S be a closed orientable hyperbolic surface and let $M = T^1S$. Let $\mathcal{F}_1, \mathcal{F}_2$ be minimal, two-dimensional foliations in M which are transverse to each other. Let $\mathcal{G}$ be the intersection of $\mathcal{F}_1$ with $\mathcal{F}_2$. Then either $\mathcal{G}$ is homeomorphic to the orbit foliation of the geodesic flow of a hyperbolic metric on S or $\mathcal{G}$ contains a Reeb surface.*

We remark that minimality is necessary as one can easily obtain counterexamples by blowing up weak stable and weak unstable foliations of Anosov flows. We note that we get a description of $\mathcal{G}$ when there are Reeb surfaces: we refer the reader to Section 9 for precise formulations (see in particular, Corollary 9.20). The description says that if there are Reeb surfaces, then some structure very similar to the Matsumoto–Tsuboi example has to occur. This is a rigidity result in the sense that counterexamples to Anosov behavior for $\mathcal{G}$ have very few possibilities.

An important point is that in Theorem A there is no (a priori) dynamical hypothesis, but there is a strong dynamical consequence. On the two-dimensional level, there is an a priori result which is the starting point of our study here and which explains the choice to work with unit tangent bundles rather than general Seifert fibered manifolds: as mentioned earlier, Matsumoto [34] proved that each $\mathcal{F}_i$ individually is topologically equivalent to the weak stable foliation[3] of the geodesic flow in M. While many arguments in our proof hold in more generality, we have chosen to avoid using too much terminology and background on foliations, so that someone willing to believe Theorem 2.1 could in principle follow the proof of the main theorem. It will also be helpful to convey the main strategy and ideas that we are trying to communicate. See Section 1.2 for updates on recent results.

As a consequence we prove the following result for partially hyperbolic diffeomorphisms (see Section 10 for precise definitions and more general results).

Corollary B. *Let $f : M \to M$ be a volume preserving partially hyperbolic diffeomorphism in $M = T^1 S$ with S a closed orientable surface of genus $g \geq 2$. Then, f is a collapsed Anosov flow. Hence f is accessible and if f is C^2, then f is ergodic.*

1.1. ***Outline of the strategy and organization of the paper.*** A way to detect that a one-dimensional foliation is an Anosov foliation is to show that when seen as the orbits of a flow, this flow is expansive. This implies that the flow is pseudo-Anosov [31; 37], and since our one-dimensional foliation preserves a two-dimensional foliation, being pseudo-Anosov is enough to show it is an Anosov foliation [5, § 5].

In this article we will instead consider a more geometric point of view. As in [25; 5], the main strategy to show that the flow generated by the foliation is a topological Anosov flow, is to show that its flow foliation subfoliates the two-dimensional leaves by quasigeodesics. We say that the foliation is *leafwise quasigeodesic* when in each leaf of say $\widetilde{\mathcal{F}}_1$ the leaves of the intersected foliation $\widetilde{\mathcal{G}}$ are quasigeodesics (note that we do not ask the one-dimensional leaves to be quasigeodesics of $\widetilde{M}$).

The idea is to try extend arguments that work in a compact surface (see [32] or [30, Appendix A]). Note however that in closed surfaces, compactness gives many deck transformations that preserve the universal cover, while here, leaves of the foliation in the universal cover may typically have small stabilizer. Here,

[3] In contrast, for general Seifert manifolds, it was already known that minimal foliations come from representations of surface groups into Homeo(S^1) by [38; 13] (see also [16]), but what we use here is the rigidity result of Matsumoto that gives conjugacy to a Fuchsian representation. This is also based on previous work of Ghys [29].

the key point is to use minimality and compactness of M to bring leaves together. Transversality of the foliations allows to push some behavior to nearby leaves, and then, arguments like in [22] can help partially reproducing the arguments in the case of closed surfaces. However, there are some subtleties when trying to push behavior to nearby leaves associated with the transverse geometry of leaves in the universal cover. The fact that our two-dimensional foliations are *Anosov* (in the sense that they are homeomorphic to the weak stable foliation of an Anosov flow when $M = T^1S$) is extremely useful. This allows for a very strong and precise form of "pushing" at most points in the leaf approaching the boundary at infinity (see Proposition 3.4). By this, we mean that if two leaves are close by in the leafspace, then, we can sort of copy the intersected foliation $\widetilde{\mathcal{G}}$ in one leaf to nearby leaves by following the intersection with leaves of the other foliation. Note that pushing requires having two transverse foliations and this is crucial in our arguments. In fact, for flows tangent to a foliation, more diverse behavior is possible as it is shown in [25] for the strong stable foliation of the examples constructed in [9].

The route taken for showing that the foliation $\widetilde{\mathcal{G}}$ is leafwise quasigeodesic follows the rough outline that was used in [25] for foliations that come from some special dynamical systems. Not all steps hold in full generality, as examples show, but we still describe here the main steps and explain under which assumptions they work.

Landing. In each leaf $L \in \widetilde{\mathcal{F}}_1$ we look at the restriction of $\widetilde{\mathcal{G}}$ to the leaf L. Then every ray of any given leaf of $\widetilde{\mathcal{G}}$ (which is always properly embedded in L) has a well defined unique limit point in the compactification $L \cup S^1(L)$ by the Gromov boundary of L. In Theorem 4.1 we show that in our setting, this holds in full generality.

Small visual measure. There are many ways a properly embedded ray in a hyperbolic plane can be extended to the boundary, in particular, we wish to rule out the possibility that it lands like horocycles do. For this, a technical property that we call small visual measure is relevant. An important consequence of small visual measure is that geodesic rays starting at a point of a ray r of a leaf $c \in \widetilde{\mathcal{G}}$ and landing at the same point as the ray r must be contained in a uniform neighborhood of r. This is also something that holds always in our setting as we prove in Section 5. We point out here that the fact that $\mathcal{G}$ is obtained as the intersection of two transverse foliations is crucial, as the horocyclic flow of an Anosov flow subfoliates a minimal foliation of T^1S but its leaves do not satisfy the small visual measure property.

No bubble leaves. To get a quasigeodesic foliation we need to rule out the existence of leaves c of $\widetilde{\mathcal{G}}$ such that both rays of c land in the same point of $S^1(L)$ (where L is the two-dimensional leaf containing c). We call such a leaf c a *bubble leaf*. We note that the example in [35] contains bubble leaves, so we cannot expect to prove the nonexistence of bubble leaves in general. But the existence of some nonbubble leaves is important in our analysis of the small visual measure and the construction of Reeb surfaces.

Hausdorff leaf space. Another consequence of a one-dimensional foliation subfoliating a two-dimensional foliation by Gromov hyperbolic leaves and being leafwise quasigeodesic is that the leaf space of the foliation $\widetilde{\mathcal{G}}$ in each leaf $L \in \widetilde{\mathcal{F}}_i$ must be Hausdorff. A priori, the nonexistence of bubble leaves is not enough to rule out non-Hausdorffness, so more analysis is needed. This is the content of Section 8 where we show that non-Hausdorffness of the leaf space leads to Reeb surfaces.

Getting the quasigeodesic property. Having Hausdorff leaf space is not enough to deduce the leafwise quasigeodesic property as the horocycle flow shows. In our context, we can show that it is enough and we do so in Section 6.

Outline. In Section 2 we study general properties of minimal foliations of T^1S and derive the consequences of [34] that we will use. In Section 3 we show some properties of pairs of transverse foliations, defining and describing the landing property and showing that in some settings it is possible to *push* behavior to nearby leaves. In Sections 4 and 5 we address landing and the small visual measure property. When the leaf space of $\widetilde{\mathcal{G}}$ is leafwise Hausdorff we show in Section 6 that the foliation $\mathcal{G}$ must be an Anosov foliation. In Section 7 we revisit the examples from [35] and describe their properties from the point of view of this paper as well as some possible extensions. In Section 8 we complete the proof of Theorem A. In Section 9 we explore further properties that in some sense show that the examples of [35] are the only possible way to introduce Reeb surfaces. In Section 10 we study the applications of our result to the classification and ergodicity of partially hyperbolic dynamics. We note that the applications to partial hyperbolicity do not use Section 7 and Section 9, which can be skipped by the reader interested only in the applications to partial hyperbolicity.

1.2. ***Recent developments.*** Since this paper was released in early 2023, several new developments have been obtained that we explain here, trying to emphasize the influence of this particular paper. Let us comment on the papers [1; 27; 26].

In [1] we extended part of the results of this paper to a more general setting, in particular, we showed that if $\mathcal{F}_1, \mathcal{F}_2$ are transverse $\mathbb{R}$-covered Anosov foliations in a 3-manifold which are uniformly equivalent, then, they either intersect in the orbit foliation of an Anosov flow, or they contain a Reeb surface. The goal of [1] was to present a completely different approach and also present results in higher dimensions (and some results in dimension 3 that are important in [26]), and while it reproves Theorem A of this paper, the proof is completely different and the techniques do not allow to obtain the more precise results that we obtain here in Section 9, and which we consider to be one of the main contributions of this paper (and also, that up to now do not have any counterpart in any context).

The paper [27], which was released later than this paper, also has some overlap with it, though they are largely independent. In [27] we reversed the strategy presented in Section 1.1 by assuming from the start a very strong property on the intersected foliation: that the leaf space of the intersected foliation is Hausdorff. Under that assumption, we were able to work out the full outline of Section 1.1 by showing landing, small visual measure (assuming that the manifold has fundamental group which is not virtually solvable) and finally the quasigeodesic property of leaves. While [27] works in much wider generality than this paper (no assumptions on the topology of M), its results would not materially shorten the current paper; they would only serve in reducing Section 6, by allowing us to apply directly the results in [27] (but that would be less natural, as our proof here is more direct).

Finally, in the recent [26], we proved a general statement needed for the classification of partially hyperbolic dynamics. Though it owes much to the ideas developed here, [26] does not use or depend on this paper. Its main result is about transverse foliations with Gromov hyperbolic leaves and says that the only obstruction to the intersected foliation being leafwise quasigeodesic is the presence of what we have called *generalized Reeb surfaces*, which extend the concept of Reeb surfaces used here. We note that [26] gives hope in progressing in the understanding of general transverse foliations, and in our opinion makes Section 9 of this paper even more relevant, since no analogue of this has been shown, and the existence of Reeb surfaces in some setting could be combined with the techniques in [26] to see if one can produce some incompressible torus in M. This could for instance be relevant in addressing the following question, which we believe may well have a positive answer (see also the question on page 39):

Question. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two transverse minimal foliations in a closed hyperbolic 3-manifold. Then, they intersect in the orbit foliation of a (topological) Anosov flow.*

2. Minimal foliations on unit tangent bundles

We consider $M = T^1S$, the unit tangent bundle of a closed orientable surface S of genus $g \geq 2$, together with a minimal foliation $\mathcal{F}$ on M. Such foliations have been completely classified by Matsumoto [34]: each is homeomorphic to the weak stable foliation of the geodesic flow on S for a hyperbolic metric. We will expand on this as well as on previous results in [38; 13] to describe the foliations in a way that is useful for our purposes.

We will assume throughout the article that the foliations we consider are $C^{0,1+}$. This means that the leaves are C^1 surfaces; see for instance [17]. This assumption is mostly for convenience, as having smooth leaves simplifies the definition of distances and lengths inside leaves. In [15] it is shown that for each foliation there is a smooth structure in M which makes it $C^{0,\infty+}$, but it is unclear that this can be done simultaneously for both foliations. To avoid discretizations of distances and local problems, we will keep this assumption throughout.

2.1. ***Unit tangent bundles.*** Consider the universal covering map $\pi : \tilde{S} \to S$ in which one can identify $\tilde{S} \cong \mathbb{H}^2$ as follows: fix a discrete subgroup Γ of $\mathrm{Isom}_+(\mathbb{H}^2) \cong \mathrm{PSL}(2, \mathbb{R})$ (acting on the right) where $\Gamma \cong \pi_1(S)$. We can identify π with the quotient map from $\mathbb{H}^2 \to \mathbb{H}^2/_\Gamma$.

Since $\mathrm{PSL}(2, \mathbb{R})$ naturally identifies with $T^1\mathbb{H}^2$ we obtain that T^1S is identified by this action with $T^1\mathbb{H}^2/_\Gamma$. We parametrize $T^1\mathbb{H}^2$ by $\mathbb{H}^2 \times \partial\mathbb{H}^2$ by identifying a unit vector $v \in T_x\mathbb{H}^2$ with the pair $(x, v_+) \in \{x\} \times \partial\mathbb{H}^2$, where v_+ is the limit point of the geodesic in $\mathbb{H}^2$ starting at x with speed v. The action of $\mathrm{PSL}(2, \mathbb{R})$ on $\partial\mathbb{H}^2 \cong S^1$ is given by extending the action of isometries on geodesic rays (if one uses the upper half model of $\mathbb{H}^2$ this action corresponds to the standard action by rational transformations on $\mathbb{R} \cup \{\infty\}$).

We will denote by $M = T^1S$, and by $\widehat{M} = T^1\tilde{S}$ its intermediate cover with deck transformations identified with the action of Γ in the coordinates given by the identification of $T^1\tilde{S} \cong \mathbb{H}^2 \times \partial\mathbb{H}^2$. We will denote by $\widetilde{M}$ the universal cover of M which covers $\widehat{M}$ with deck transformation group associated with the center of $\pi_1(M)$ (which corresponds to the deck transformation associated to the circle fiber of the circle bundle over S).

2.2. ***Horizontal foliations.*** Consider the foliation $\widehat{\mathcal{F}}_{ws}$ of $T^1\tilde{S} \cong \mathbb{H}^2 \times \partial\mathbb{H}^2$ given by $\widehat{\mathcal{F}}_{ws} = \{\mathbb{H}^2 \times \{\xi\}\}_{\xi \in \partial\mathbb{H}^2}$. This foliation is Γ-invariant and since the Γ action on $\partial\mathbb{H}^2$ is minimal it descends to a minimal foliation $\mathcal{F}_{ws}$ of M which is exactly the weak stable foliation for the geodesic flow associated to the metric on S induced by the choice of $\Gamma \subset \mathrm{Isom}_+(\mathbb{H}^2)$.

The following result from [34] will be very important in our study and says that $\mathcal{F}_{ws}$ is the unique minimal foliation of M up to homeomorphisms isotopic to the identity on the base:

Theorem 2.1 (Matsumoto [34]). *Let $\mathcal{F}$ be a minimal foliation of M, then, there exists a homeomorphism $h : M \to M$ inducing the identity on the base such that $h(\mathcal{F}) = \mathcal{F}_{ws}$.*

We note that the result of [34] is stated for C^2 foliations without compact leaves, but all that is needed is that $\mathcal{F}$ is *horizontal*, see also [29; 38].

To get the horizontal property for $\mathcal{F}$ we use the fact that $\mathcal{F}$ is minimal and apply Brittenham's theorem [13]. There is a slightly technical issue in Brittenham's result: in [13] a lamination is one that is carried by a branched surface, so technically a foliation is not a lamination and must first be split along a finite set of leaves to produce an essential lamination $\mathcal{L}$ which, due to [13], has a minimal sublamination which is either horizontal or vertical, which since $\mathcal{F}$ is minimal this is $\mathcal{L}$ itself. But $\mathcal{F}$ cannot be vertical.[4] It follows that $\mathcal{F}$ is horizontal and then one can apply Matsumoto's result. See also [16] for generalities on foliations on circle bundles.

That h induces the identity on the base means that if $p : T^1S \to S$ is the projection, then the induced actions on the fundamental group satisfy $p_* \circ h_* = p_*$. Note however, that h may not be homotopic to identity as a map of M and therefore two minimal foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ may not be *uniformly equivalent* in the sense that leaves in the universal cover may not be a bounded distance away from a leaf of the other foliation. (See [39] for discussion on this notion, which is different from the notion, also used sometimes, of having homotopic plane fields.)

We will use some other properties of minimal foliations on M. Some of these hold more generally for *Reebless foliations* (due to Novikov's theorem; see [16]). We state the properties we need in the setting we will use where the proofs are a direct consequence of the corresponding properties for $\mathcal{F}_{ws}$ and Theorem 2.1 (the last point also uses [13] for smoothness):

Corollary 2.2. *Let $\mathcal{F}$ be a minimal foliation on $M = T^1S$ and denote by $\widehat{\mathcal{F}}$ and $\widetilde{\mathcal{F}}$ the lifts to the covers $\widehat{M}$ and $\widetilde{M}$. Then*:

(i) *If τ is a curve transverse to $\widetilde{\mathcal{F}}$ then it intersects each leaf at most once.*

(ii) *The leaf space $\widehat{\mathcal{L}} = \widehat{M}/_{\widehat{\mathcal{F}}}$ of $\widehat{\mathcal{F}}$ is homeomorphic to S^1 and the leaf space $\widetilde{\mathcal{L}}$ of $\widetilde{\mathcal{F}}$ is homeomorphic to $\mathbb{R}$.*

(iii) *There is a (smooth) isotopy of $\mathcal{F}$ that makes every leaf transverse to the circle fibers of T^1S.*

Note that a transversal to $\widetilde{\mathcal{F}}$ or $\mathcal{F}$ is a continuous curve $\tau : I \to M$ where I is some interval such that for every $t \in I$, there is $\varepsilon > 0$ such that the curve $\tau|_{(t-\varepsilon,t+\varepsilon)}$ is monotone in the leaf space of a foliation chart of $\widetilde{\mathcal{F}}$ or $\mathcal{F}$ (which is an interval) around $\tau(t)$. We will many times abuse notation and denote by τ the image of a transversal.

2.3. ***Universal circle.*** One consequence of Matsumoto's result (Theorem 2.1) is that every minimal foliation in M is homeomorphic to our model and so we can compare the geometry of leaves with that of hyperbolic disks simultaneously. This allows to make natural projections into $\mathbb{H}^2$ of lifts of leaves of a minimal foliation $\mathcal{F}$ in M to the intermediate cover $\widehat{M}$ or universal cover $\widetilde{M}$.

We consider $p : M \to S$ to be the projection of the fiber bundle $S^1 \to M = T^1S \to S$. By our coordinate choices, the map p lifts to a projection $\hat{p} : \widehat{M} \to \mathbb{H}^2$,

[4] If it were, it would induce a nonsingular foliation in the base surface which has nonzero Euler characteristic.

that, given the identification $\widehat{M} \cong \mathbb{H}^2 \times \partial\mathbb{H}^2$ corresponds to the map $\hat{p}(x, \xi) = x$. We also denote by $\tilde{p} : \widetilde{M} \to \mathbb{H}^2$ the lift to the universal cover.

For a minimal foliation $\mathcal{F}$ on M we will denote by $\widehat{\mathcal{F}}$ and $\widetilde{\mathcal{F}}$ the lifts to $\widehat{M}$ and $\widetilde{M}$. For concreteness we will consider the following metric on $\widehat{M} \cong \mathbb{H}^2 \times \partial\mathbb{H}^2$, given by the fact that the metric on $\mathbb{H}^2 \times \{\xi\}$ is the one that makes projection in the first coordinate an isometry, makes the sets $\mathbb{H}^2 \times \{\xi\}$ and $\{x\} \times \partial\mathbb{H}^2$ orthogonal and measures distances in $\{x\} \times \partial\mathbb{H}^2$ via the visual metric; namely, we define the length of an interval $[\xi_1, \xi_2] \subset \partial\mathbb{H}^2$ as as the angle between the geodesic rays starting from x and landing on ξ_1 and ξ_2 respectively, and measured in the direction on which rays land in the interior of the interval. This metric is invariant under the action of Γ and thus produces a Riemannian metric on the quotient M which makes the projection $p : M \to S$ to be a Riemannian submersion on S when given the metric induced by the action of Γ on $\mathbb{H}^2$. In coordinates $\widehat{M} \cong \mathbb{H}^2 \times \partial\mathbb{H}^2$ the Riemannian metric is

$$\langle v, w\rangle_{(x,\xi)} = \left(\langle v_{\mathbb{H}^2}, w_{\mathbb{H}^2}\rangle^2_{T_x\mathbb{H}^2} + \langle v_{\partial\mathbb{H}^2}, w_{\partial\mathbb{H}^2}\rangle^2_{(x,\xi)}\right)^{1/2}. \tag{2-1}$$

where $v_{\mathbb{H}^2}, w_{\mathbb{H}^2}, v_{\partial\mathbb{H}^2}, w_{\partial\mathbb{H}^2}$ are the projections of the vectors on the first and second coordinates respectively, the inner product $\langle \cdot\,, \cdot\rangle_{T_x\mathbb{H}^2}$ is the standard inner product in the hyperbolic plane and $\langle \cdot\,, \cdot\rangle_{(x,\xi)}$ is an inner product on $T_{(x,\xi)}(\{x\} \times \partial\mathbb{H}^2)$ given by the identification of the landing of geodesic rays of $T^1_x\mathbb{H}^2$ with $\partial\mathbb{H}^2$ at the point $\xi \in \partial\mathbb{H}^2$.

The metric on $\widetilde{M}$ will be the path metric associated with the pullback of the Riemannian metric above by the universal cover projection. This metric in $\widetilde{M}$ will be denoted by $d : \widetilde{M} \times \widetilde{M} \to \mathbb{R}_{\geq 0}$.

Given a leaf L of a foliation $\mathcal{F}$ we consider the path distance in L induced by the restriction of the ambient Riemannian metric on L. The choices we have made are not important if we consider objects up to *quasi-isometry*: if one chooses another metric, one obtains a distance that is quasi-isometric to the first one. Recall that a (not necessarily continuous) map $q : (X_1, d_1) \to (X_2, d_2)$ is a Q-quasi-isometry if for every $x, y \in X_1$ one has

$$\frac{1}{Q} d_1(x, y) - Q \leq d_2(q(x), q(y)) \leq Q d_1(x, y) + Q \tag{2-2}$$

and the image of q is Q-dense in X_2. Being quasi-isometric (meaning there exists a quasi-isometry between the spaces) is an equivalence relation between metric spaces which is particularly relevant for Gromov hyperbolic spaces such as $\mathbb{H}^2$. We will use several basic properties of Gromov hyperbolic spaces and refer to [12] for proofs of those facts.

Theorem 2.1 implies the following which in particular shows that leaves of a minimal foliation are Gromov hyperbolic:

Proposition 2.3. *There is a uniform constant $Q_0 := Q_0(\mathcal{F})$ such that for every $L \in \widehat{\mathcal{F}}$ (or $\in \widetilde{\mathcal{F}}$) it follows that the restriction $\hat{p}|_L : L \to \mathbb{H}^2$ (resp. $\tilde{p}|_L : L \to \mathbb{H}^2$) is a Q_0-quasi-isometry.*

Proof. After an isotopy one can assume the foliation is horizontal (Corollary 2.2(iii)), and thus, by choosing an appropriate metric, we can ensure that the projection is an isometry. Since M is compact it follows that the lift of either $\hat{p}$ or $\tilde{p}$ restricted to any leaf is a uniform quasi-isometry. Since the statement (up to changing the constants) is invariant under change of metric on M, we conclude. □

Remark 2.4. A far-reaching generalization is Candel's uniformization theorem (see [16, Chapter 7]). It can be used to show that every minimal foliation on a 3-manifold with fundamental group of exponential growth admits a metric which makes every leaf of negative curvature (see [23, § 5] and [5, Appendix A]).

We can identify the Gromov boundary of L or, equivalently, the *circle at infinity* $S^1(L)$ of each leaf $L \in \widehat{\mathcal{F}}$ (or $\widetilde{\mathcal{F}}$) with $\partial\mathbb{H}^2$ in a canonical way. Notice that the *universal circle* of $\mathcal{F}$ as defined in [39] is also canonically identified with $\partial\mathbb{H}^2$ in this case.

2.4. ***Nonmarker points.*** Here we introduce the notion of marker and nonmarker points for the foliations we are interested in. More general definitions can be found in [16]. We start by analyzing $\mathcal{F}_{ws}$ using the metric given by (2-1) and then in the next subsection we show similar properties for every minimal foliation (because by Theorem 2.1, they are all homeomorphic to $\mathcal{F}_{ws}$).

An orientation will be fixed on $\mathbb{H}^2$. Then given an oriented geodesic $\ell \in \mathbb{H}^2$ we denote $H_+(\ell)$ and $H_-(\ell)$ the half spaces determined by ℓ (that is, the closure in $\mathbb{H}^2$ of the connected components of $\mathbb{H}^2 \setminus \ell$) with respect to the chosen orientations. Specifically $H_+(\ell)$ is the half space to the left of ℓ and $H_-(\ell)$ is the half space to the right of ℓ, with respect to the orientation in $\mathbb{H}^2$. For $X \subset \mathbb{H}^2$ and $C > 0$ denote $B_C(X)$ to be the set of points $x \in \mathbb{H}^2$ whose distance to X is less than or equal to C.

Proposition 2.5. *Given $\varepsilon \in (0, \pi)$ there exists $C := C(\varepsilon)$ such that for every interval $[\eta, \xi] \in \partial\mathbb{H}^2$ the set of points $x \in \mathbb{H}^2$ such that the visual length from x of the interval $[\eta, \xi]$ is less than ε is $H_-(\ell) \setminus B_C(\ell)$ where ℓ is a geodesic joining η, ξ oriented so that the interval $[\eta, \xi]$ is contained in the closure of $H_+(\ell)$ in the compactification $\mathbb{H}^2 \cup \partial\mathbb{H}^2$.*

Proof. For every $x \in H_+(\ell)$ the visual length from x of $[\eta, \xi]$ is greater than or equal to $\pi > \varepsilon$, so the set we wish to describe does not intersect $H_+(\ell)$.

Now, fix a geodesic ray $r : [0, \infty) \to \mathbb{H}^2$, parametrized by unit speed, starting at some point $r(0)$ in ℓ, orthogonal to it and contained in $H_-(\ell)$. For $t > t'$ the ideal geodesic triangle joining $r(t), \xi, \eta$ contains the triangle joining $r(t'), \xi, \eta$ so

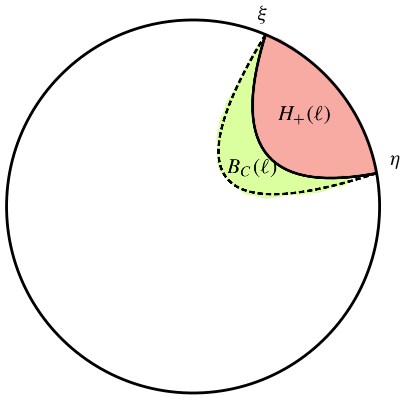

Figure 2. The set $D_\varepsilon(I)$ for $I = [\xi, \eta]$ is the complement of the union of $H_+(\ell)$ and $B_C(\ell)$. If ε is very small, then C is very large.

the inner angle at the point $r(t)$ is smaller than the one at the point $r(t')$ by Gauss–Bonnet formula. As it varies continuously, and since the angle is π at $r(0)$ and tends to 0 at $r(\infty)$, it follows that every such geodesic ray has a unique point $r(t_0)$ on which the angle is exactly ε and it follows also that $d(r(t_0), \ell) = t_0 := C(\varepsilon)$. To show that this does not depend on $r(0)$ consider the one parameter family of isometries of $\mathbb{H}^2$ fixing ξ, η; these isometries map $r(t_0)$ transitively along the boundary of $B_{t_0}(\ell)$ preserving angles. Hence t_0 does not depend on $r(0)$, and depends only on ε. This proves the proposition. □

We can parametrize leaves of $\widehat{\mathcal{F}}_{ws}$ by $\partial\mathbb{H}^2$ as these are of the form $\mathbb{H}^2 \times \{\xi\}$ with $\xi \in \partial\mathbb{H}^2$. We denote by $L_\xi = \mathbb{H}^2 \times \{\xi\}$ and call ξ the *nonmarker* point of L_ξ. This will be denoted as $\alpha(L_\xi) = \xi$. The key point is that the choices of coordinates make this point in $\partial\mathbb{H}^2$ special with respect to the leaf L_ξ as it will now be explained.

Given $\varepsilon > 0$ and an interval $I \subset \partial\mathbb{H}^2$ we define the set (see Figure 2)

$$D_\varepsilon(I) = \{x \in \mathbb{H}^2 : d_{\widehat{M}}((x, \xi), L_\eta) < \varepsilon \text{ for all } \xi, \eta \in I\}, \tag{2-3}$$

where $d_{\widehat{M}}$ is the distance in $\widehat{M}$. The point (x, ξ) belongs to L_ξ by definition. We will always assume that I is not $\partial\mathbb{H}^2$ nor a single point. For any interval $I = [\xi_-, \xi_+]$, the points in $D_\varepsilon(I)$ form a subset of $\mathbb{H}^2$ and the points in $\partial\mathbb{H}^2$ for which the biggest distance between the corresponding leaves is achieved is when $\{\eta, \xi\} = \{\xi_-, \xi_+\}$. So it is enough to look at the distance of points of the form (x, ξ_-) to L_{ξ_+} (or $(x, \xi_+) \in L_{\xi_-}$). From the previous proposition we deduce:

Corollary 2.6. *Given $I \subset \partial\mathbb{H}^2$ an interval and a constant $\varepsilon \in (0, \pi)$, there exists $C := C(I, \varepsilon)$ such that $D_\varepsilon(I)$ is equal to the set $H_-(\ell) \setminus B_C(\ell)$. Here ℓ is the geodesic joining the endpoints of I oriented so that I is contained in the closure of $H_+(\ell)$ in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$.*

We will denote the lift of the points of $D_\varepsilon(I)$ in L_ξ by

$$D_\varepsilon(\xi, I) = \{(x, \xi) \in \mathbb{H}^2 \times \{\xi\} : x \in D_\varepsilon(I)\} \subset L_\xi. \tag{2-4}$$

2.5. ***Minimal foliations in $M = T^1 S$.*** Now, let $\mathcal{F}$ be an arbitrary minimal foliation on M. By Theorem 2.1 there is a homeomorphism $h : M \to M$ mapping $\mathcal{F}$ to $\mathcal{F}_{ws}$.

We will produce maps Φ_L for L in $\widetilde{\mathcal{F}}$ which are uniform quasi-isometries (Proposition 2.3). First fix an isotopy of M, which restricted to each individual leaf of $\mathcal{F}$ is as regular as the leaves of $\mathcal{F}$ are, to get to a foliation $\mathcal{F}'$ which is transverse to the standard Seifert fibration of $M = T^1 S$ (Corollary 2.2(iii)). This is the Seifert fibration whose fibers are the unit vectors over a given point in S. The lift of this isotopy to $\widetilde{M}$ is denoted by ν. For any leaf of $\widetilde{\mathcal{F}}'$ project to $\mathbb{H}^2$ using the lift of this Seifert fibration, this projection is $\tilde{p}$. Since the angle between $\mathcal{F}'$ and the Seifert fibration is bounded below, this projection is a uniform quasi-isometry. The composition of the initial isotopy (lifted to $\widetilde{M}$) and the projection is well defined up to a bounded distortion (depending on $\mathcal{F}$). We denote this composition by Φ_L. The conclusion is that we obtain an equivariant collection of uniform quasi-isometries Φ_L from leaves $L \in \widetilde{\mathcal{F}}$ to $\mathbb{H}^2$. Also, if $\tilde{p} : \widetilde{M} \to \mathbb{H}^2$ is the standard projection, there is a uniform constant $C_0 > 0$ such that $d_{\mathbb{H}^2}(\Phi_L(x), \tilde{p}(x)) < C_0$. Note that $\Phi_L = \tilde{p} \circ \nu$ restricted to L.

This allows us to:

- Associate to each $L \in \widetilde{\mathcal{F}}$ a point, which we will denote throughout the article by $\alpha(L) \in \partial\mathbb{H}^2$, and that we will call the *nonmarker point* of L. It also identifies the leaf space of $\widehat{\mathcal{F}}$ with $\partial\mathbb{H}^2 \cong S^1$. This uses both h and the quasi-isometries above. First the map h shows that for any leaf F of $\mathcal{F}$ there is one ideal direction which is transversely noncontracting (the nonmarker direction), and all other directions are contracting. In particular, by lifting to appropriate covers, this is also true for leaves in $\widetilde{\mathcal{F}}$ or $\widehat{\mathcal{F}}$. The uniform quasi-isometries then allow for $L \in \widetilde{\mathcal{F}}$ to obtain the unique nonmarker point $\alpha(L)$ in $\partial\mathbb{H}^2$. Finally using h again one can show that the map α is a homeomorphism when considered as a map from the leaf space of $\widehat{\mathcal{F}}$ and $\partial\mathbb{H}^2 \cong S^1$.

- For each L in $\widetilde{\mathcal{F}}$ (or $\widehat{\mathcal{F}}$) Identify the (Gromov) boundary at infinity $S^1(L)$ of L with $\partial\mathbb{H}^2$ using the fact that Φ_L is a quasi-isometry. Importantly, the induced map $\Phi_L : S^1(L) \to \partial\mathbb{H}^2$ is independent on the choice of the isotopy of $\mathcal{F}$ to a horizontal foliation.

- We have maps

$$\Phi_L : L \cup S^1(L) \to \mathbb{H}^2 \cup \partial\mathbb{H}^2, \tag{2-5}$$

As explained before, each such map, when restricted to L is a diffeomorphism, for each L; and the diffeomorphisms have uniformly bounded derivatives (thus, it is a uniform quasi-isometry, independent on L).

Note that $\alpha(L)$ belongs to $\partial\mathbb{H}^2$ but we can identify it canonically with a point in $S^1(L)$ via Φ_L and sometimes we will go back and forth with these identifications. Also, these facts hold for leaves $L \in \widehat{\mathcal{F}}$ and we will abuse notation and denote by $\Phi_L : L \cup S^1(L) \to \mathbb{H}^2 \cup \partial\mathbb{H}^2$ all such maps.

2.6. *Nearby sets in distinct leaves.* For $\varepsilon > 0$, a leaf $L \in \widetilde{\mathcal{F}}$ (or $\widehat{\mathcal{F}}$ using the metric $d_{\widehat{M}}$) and an interval I of the leaf space of $\widetilde{\mathcal{F}}$ (or $\widehat{\mathcal{F}}$) we define

$$\widehat{D}_\varepsilon(L, I) = \{y \in L : \forall E \in I \ , \ d(y, E) < \varepsilon\} \tag{2-6}$$

Corollary 2.6 leads to the next result, thanks to the map Φ_L and the fact that $\mathcal{F}$ is homeomorphic to $\mathcal{F}_{ws}$:

Proposition 2.7. *There is a constant $C > 0$ independent of L such that the set $\widehat{D}_\varepsilon(L, I)$ it is at Hausdorff distance less than C from $\Phi_L^{-1}(D_\varepsilon(I))$. In particular, since Φ_L is a quasi-isometry, there is another constant $C' > 0$ such that the Hausdorff distance between $\Phi_L(\widehat{D}_\varepsilon(L, I))$ and $D_\varepsilon(I)$ is less than C'. Moreover, if $L_1 \in I$ and $x_n \in \widehat{D}_\varepsilon(L, I)$ is a sequence of points converging to some point $\xi \in S^1(L)$ which is not $\alpha(L')$ for some $L' \in I$ then we have that $d(x_n, L_1) \to 0$.*

Proof. Recall that we start with an isotopy from $\mathcal{F}$ to a horizontal foliation, then project using the Seifert fibration. These are the maps Φ_L. Choose ε' depending on ε and such that if points are within ε' then after undoing the isotopy the points are at most ε from each other. This shows that

$$\Phi_L^{-1}(D_{\varepsilon'}(I)) \ \subset \ \hat{D}_\varepsilon(L, I).$$

Since $D_{\varepsilon'}(I)$ and $D_\varepsilon(I)$ are a bounded Hausdorff distance from each other, there is C_1 with $B_{C_1}(\Phi_L^{-1}(D_\varepsilon(I)) \subset \hat{D}_\varepsilon(L, I)$.

Conversely, given ε, since the isotopy is the lift of a compact isotopy there is $\varepsilon_1 = \varepsilon_1(\varepsilon)$ such that $\Phi_L(\hat{D}_\varepsilon(L, I)) \subset D_{\varepsilon_1}(I)$. But $D_{\varepsilon_1}(I)$ is a bounded Hausdorff distance from $D_\varepsilon(I)$, so there is $C_2 > 0$ such that

$$\Phi_L(\hat{D}_\varepsilon(L, I)) \ \subset \ B_{C_2}(D_\varepsilon(I)).$$

Taking Φ_L^{-1}, and noticing that it is a quasi-isometry, produces $C_3 > 0$ such that $\hat{D}_\varepsilon(L, I) \subset B_{C_3}(\Phi_L^{-1}(D_\varepsilon(I)))$. This finishes the proof of the proposition. □

This proposition will combine well with Corollary 2.6 to control the geometry of the sets $D_\varepsilon(L, I)$.

2.7. ***Minimality of the action in the universal circle.*** In the next proposition we collect some facts about the action of the fundamental group of S in the boundary $\partial\mathbb{H}^2$. Recall that we have chosen a fixed hyperbolic metric on S which is induced by a subgroup $\Gamma \cong \pi_1(S)$ of isometries of $\mathbb{H}^2$ that induces an action on $\partial\mathbb{H}^2$ (recall that for such groups every non identity element acts as a hyperbolic isometry, so it has exactly two fixed points in $\partial\mathbb{H}^2$, one attracting and one repelling).

The fundamental group $\pi_1(M)$ of M is a central extension of $\pi_1(S)$ (that is, there is a surjective morphism $\pi_1(M) \to \pi_1(S)$ such that the preimage of the identity is the center of the group $\pi_1(M)$ and is generated by the homotopy class of the fibers) and its action on $\partial\mathbb{H}^2$ is induced by the action of its projection on $\pi_1(S)$.

Proposition 2.8. *The action of $\Gamma \cong \pi_1(S)$ in $\partial\mathbb{H}^2$ is minimal* (*i.e.*, *every orbit is dense*). *Given open sets U, V in $\partial\mathbb{H}^2$ there exists an element $\gamma \in \Gamma$ such that*

- $\gamma(U) \cap V \neq \varnothing$, *and*
- *the fixed points of γ are one contained in U and one contained in V.*

This result is classical; a proof can be found for instance in [22] where we prove an extension to general uniform $\mathbb{R}$-covered foliations ([22, Proposition 5.3] applied to the foliation by compact surfaces in $S \times S^1$ gives the previous result). Note that since we can identify the leaf space of $\widehat{\mathcal{F}}$ with $\partial\mathbb{H}^2$ this also gives information about the action of Γ acting on the leaf space of $\widehat{\mathcal{F}}$ on $T^1\mathbb{H}^2$ which is a circle identified with $\partial\mathbb{H}^2$ via the map α sending a leaf into its nonmarker point. If we go to the universal cover, then, the central extension of Γ that is $\pi_1(M)$ provides all lifts of the action of Γ in $\partial\mathbb{H}^2$ to the universal cover (in particular, there are always lifts with fixed points and the action is minimal).

2.8. ***Some plane topology.*** We will use the following standard consequence of the classical Schoenflies theorem (see [36, § 9], for instance). We will always be using piecewise smooth curves, so the proof is simpler.

Proposition 2.9. *Let c be a properly embedded curve in the plane. Then, the complement of c is the union of two topological open disks whose boundary in the plane is exactly c.*

Since we will always be working on $\mathbb{H}^2$, where we have a natural compactification $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ homeomorphic to a disk, we want to understand the complements of properly embedded curves in this compactification. For a set $K \subset \mathbb{H}^2$ the *limit set* of K is the closure of K in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ intersected with $\partial\mathbb{H}^2$. We consider a circular order in $\partial\mathbb{H}^2$. Let c be a properly embedded curve in $\mathbb{H}^2$ and denote by I_1 and I_2 the limit sets of the two rays of c (i.e., consider $x \in c$ and denote by c_1, c_2 the connected components of $c \setminus \{x\}$, then I_i is limit set of c_i). Notice that both I_1, I_2 are connected subsets of $\partial\mathbb{H}^2$ and are either $\partial\mathbb{H}^2$ or an interval. In the case that they (or one of them) are not $\partial\mathbb{H}^2$ we denote them by: $I_1 := [a,b]$, $I_2 := [c,d]$ in the circular order of $\partial\mathbb{H}^2$.[5]

Corollary 2.10. *Let c be a properly embedded curve in $\mathbb{H}^2$ as above. Let D^+ and D^- denote the connected components of $\mathbb{H}^2 \setminus c$, and let J^+, J^- be their respective limit sets. Then*

- *If $I_1 \cup I_2 = \partial\mathbb{H}^2$ then both J^+ and J^- coincide with $\partial\mathbb{H}^2$ (this includes the case where one of the intervals I_1 or I_2 coincides with $\partial\mathbb{H}^2$).*
- *Assume that $[a,b] \cap [c,d] = \varnothing$, and suppose that a, b, c, d are circularly ordered in $\partial\mathbb{H}^2$. Then one of J^+, J^- is $[a,d]$ and the other is $[c,b]$.*
- *Finally suppose $[a,b]$, $[c,d]$ intersect (and their union is not $\partial\mathbb{H}^2$). Then one of J^+, J^- is $\partial\mathbb{H}^2$ and the other is $[a,b] \cup [c,d]$.*

Proof. The limit set $J^\pm$ of $D^\pm$ is a compact connected subset of the boundary $\partial\mathbb{H}^2$. Since c is the boundary in $\mathbb{H}^2$ of both D^+ and D^- it follows that $I_1 \cup I_2$ is contained in both J^+ and J^-. This already proves the first point.

For the second and third items, it is implicitly assumed that none of I_1, I_2 are $\partial\mathbb{H}^2$. Since $D^+ \cup D^-$ must accumulate in all of $\partial\mathbb{H}^2$, it follows that if one considers a point $\xi \in \partial\mathbb{H}^2 \setminus (I_1 \cup I_2)$ then it has a neighborhood N in the compactification $\mathbb{H}^2 \cup \partial\mathbb{H}^2$, so that $N \cap \mathbb{H}^2$ which is contained in either D^+ or D^-. To finish we must show that (b,c) is contained in one of J^+, J^- and (d,a) is contained in the other. To do that consider a geodesic μ in L with one ideal point in (b,c) and the other in (d,a). The ideal points are disjoint from $I_1 \cup I_2$ so μ has rays contained in $D_1 \cup D_2$. If both rays are contained in say D_1 it follows that both rays of c are also contained in the same complementary component of μ. This contradicts that c limits on both $[a,b]$ and $[c,d]$. Hence only one ideal point is in J^+ and the other is in J^-, and consequently one of (b,c), (d,a) is contained in J^+ and the other in J^-. This proves the second statement.

For the third statement: since I_1 intersects I_2, then the above fact implies that $\partial\mathbb{H}^2 \setminus (I_1 \cup I_2)$ is contained in one and only one of J^+ or J^-. Since $I_1 \cup I_2 \subset J^+ \cap J^-$, the third statement follows. □

We will also need the following consequence of Proposition 2.9

Proposition 2.11. *Let r_1, r_2 be two disjoint properly embedded rays in $\mathbb{H}^2$ which limit in intervals I_1 and I_2 respectively. Assume that $I_1 \neq \partial\mathbb{H}^2$, then I_2 cannot be contained in the interior of I_1.*

Proof. Consider a point $o \in \mathbb{H}^2$ and two geodesic rays g_1, g_2 from o landing at points ξ_1, ξ_2 in the interior of I_1 and a ray g_3 landing at a point $\xi_3 \notin I_1$. We claim that there are arcs v_n of r_1 joining g_1, g_2 which converge in the topology of $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ to the

[5] We allow $a = b$ or $c = d$, in which case the interval is a point.

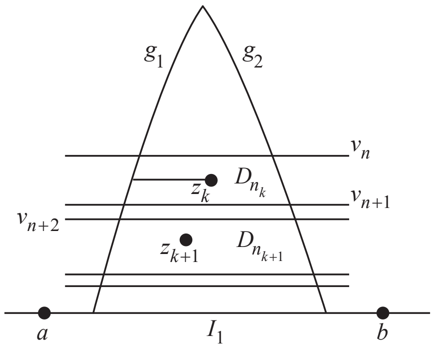

Figure 3. The limit set of r_1 is the interval I_1 with endpoints a, b. g_1, g_2 are geodesic rays with ideal points in the interior of I_1. The segments v_n are segments in r_1 which limit to $[a, b]$. The points z_k are in r_2 and have to connect outside the v_n to one of g_1 or g_2 in D_{n_k}. So the intersections with (say) g_1 limit to an arbitrary point in the interior of I_1.

interval $[\xi_1, \xi_2]$ contained in I_1. This is because given $R > 0$ we have that outside the disk $B_R(o)$ of radius R centered in o there are sequences of points x_n, y_n of r_1 converging to points in $I_1 \setminus [\xi_1, \xi_2]$ and such that the arc J_n joining x_n to y_n is completely outside $B_R(o)$ and cannot intersect g_3. Thus, there must be an arc inside J_n that we call v_n joining g_1 and g_2 and which is outside $B_R(o)$ and contained in the region bounded by $\partial B_R(o) \cup g_1 \cup g_2$ which accumulates in $[\xi_1, \xi_2]$. This proves our claim.

We refer to Figure 3 for the situation in this Lemma.

Let g_i' be the subray of g_i starting in $v_n \cap g_i$. If we consider the regions D_n obtained as the closure in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ bounded by $v_n \cup g_1 \cup g_2$ and accumulating in $[\xi_1, \xi_2]$ we get that after subsequence we can assume it is a nested sequence $D_{n+1} \subset D_n$ of disks such that $\bigcap D_n = [\xi_1, \xi_2]$. If r_2 is a properly embedded ray disjoint from r_1 which accumulates in a point $\xi \in (\xi_1, \xi_2)$ we can see that there is a sequence of points $z_k \in r_2$ converging to ξ and thus must be contained in some sets of the form $D_{n_k} \setminus D_{n_{k+1}}$. Since r_2 is connected, to join z_k with z_{k+1} it must exit D_{n_k} without intersecting $D_{n_{k+1}}$ thus there must be an arc t_k of r_2 that joins z_k to some point w_k in either g_1 or g_2. Up to subsequence, we can assume without loss of generality that w_k is in g_1 and we deduce that r_2 must limit in $[\xi_1, \xi]$. Since ξ_1, ξ_2 were arbitrary, we deduce that r_2 cannot limit in an interval completely contained in the interior of I_1 as desired. □

Remark 2.12. In Proposition 2.11, if $I_1 = \partial\mathbb{H}^2$, we cannot consider the ray g_3 which forces the curves v_n to join g_1, g_2 on the "correct" side. It is possible however to restrict the possible landing regions of disjoint properly embedded rays in this setting. This will be used later, but we will not give a precise general statement.

3. Pairs of transverse foliations

In this section we will consider two transverse minimal foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ in $M = T^1S$. We will call $\mathcal{G} := \mathcal{F}_1 \cap \mathcal{F}_2$ the one-dimensional foliation by the connected components of intersections of leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$. Note that since S is orientable it induces orientations on M, $\mathcal{F}_1$ and $\mathcal{F}_2$ and thus also on $\mathcal{G}$, we fix one such orientation and lift it to $\widetilde{M}$.

We consider the maps Φ^1_L and Φ^2_E from the compactification of leaves $L \in \widehat{\mathcal{F}}_1$ (or $\widetilde{\mathcal{F}}_1$) and $E \in \widehat{\mathcal{F}}_2$ (or $\widetilde{\mathcal{F}}_2$) as defined in (2-5). Since the foliation will be implicit from the leaf, we will usually omit the superscript, writing Φ_L for Φ^1_L when it is clear that L lies in $\widetilde{\mathcal{F}}_1$ or $\widehat{\mathcal{F}}_1$.

3.1. ***Some general properties.*** Given a leaf $c \in \mathcal{G}$ and $x \in c$ we will denote by

$$c^+_x \cup c^-_x = c \setminus \{x\} \tag{3-1}$$

the two rays of c, where c^+_x is the one in the positive orientation starting at x. We will denote by c^+ and c^- the *ray class*, meaning the equivalence class of rays of c such that any two equivalent rays coincide in a subray.

Given a curve $c \in \widehat{\mathcal{G}}$ (or $\widetilde{\mathcal{G}}$) we know that c is a connected component of $L \cap E$ where $L \in \widehat{\mathcal{F}}_1$ and $E \in \widehat{\mathcal{F}}_2$ (or in the corresponding lifts to the universal cover).[6] The following is a direct consequence of the quasi-isometric properties of Φ^1_L and Φ^2_E:

Lemma 3.1. *The curves $\Phi^1_L(c)$ and $\Phi^2_E(c)$ are proper and a bounded distance away from each other in $\mathbb{H}^2$.*

In particular, if we denote by $\bar{c}_L$ the closure of c in $L \cup S^1(L)$ and by $\bar{c}_E$ the respective closure in $E \cup S^1(E)$ we get the following important property:

Corollary 3.2. *The sets $\Phi_L(\bar{c}_L \setminus c)$ and $\Phi_E(\bar{c}_E \setminus c)$ are contained in $\partial\mathbb{H}^2$ and coincide.*

We will then write, for $c \in \widehat{\mathcal{G}}$,

$$\partial\Phi(c) := \Phi_L(\bar{c}_L \setminus c) = \Phi_E(\bar{c}_E \setminus c) \subset \partial\mathbb{H}^2. \tag{3-2}$$

Since c is properly embedded we can write $\partial\Phi(c) = \partial^+\Phi(c) \cup \partial^-\Phi(c)$ where each denotes the accumulation points of the positive and negative rays of c once a point is removed (it is easy to see that this is independent on the removed point and so it is a property of the ray class). Note that each of $\partial^+\Phi(c)$ and $\partial^-\Phi(c)$ are compact connected and nonempty subsets of $\partial\mathbb{H}^2$ (see Figure 4). Now we define a fundamental property to be analyzed in this article:

[6] For the remainder of this section we will omit saying that things hold both in the cover $\widehat{M}$ and the universal cover $\widetilde{M}$.

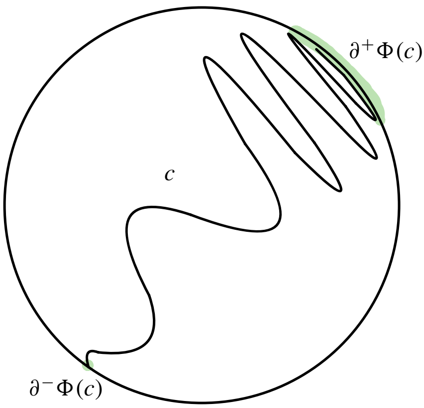

Figure 4. An example where the negative ray of c lands and the positive ray of c accumulates in an interval which is not a point.

Definition 3.3. Given $c \in \widehat{\mathcal{G}}$ we say that the *positive ray lands* (resp. the negative ray lands) if $\partial^+\Phi(c)$ is a point (resp. $\partial^-\Phi(c)$ is a point).

For notational simplicity, we also denote by $\partial\Phi(r)$ the accumulating set of a ray when r is a ray of a leaf $c \in \mathcal{G}_L$. In addition if r lands, we say that $\partial\Phi(r)$ is the landing point of r. If r in L lands, then r also limits in a single point b in $S^1(L)$ (in particular $\Phi_L(b) = \partial\Phi(r)$). We also say that r lands in b.

3.2. ***Pushing to nearby leaves.*** If ε is sufficiently small it is smaller than charts on which $\mathcal{F}_1$ is horizontal and $\mathcal{F}_2$ vertical and this means that the whole structure of $\mathcal{G}_L$ that one sees in a leaf L of (say) $\widetilde{\mathcal{F}}_1$ can be pushed isotopically to nearby leaves L' of $\widetilde{\mathcal{F}}_1$ in the set $\widehat{D}_\varepsilon(L, I)$. The roles of the foliations can be reversed.

Proposition 3.4. *There exists $\varepsilon > 0$ and $C > 0$ such that if L, E are leaves of $\widetilde{\mathcal{F}}_1$ which are contained in an interval I of the leaf space of $\widetilde{\mathcal{F}}_1$ and r is a segment of a leaf of $\widetilde{\mathcal{G}}$ which is contained in $\widehat{D}_\varepsilon(L, I)$, as in equation (2-6), such that $r \subset F \cap L$ where $F \in \widetilde{\mathcal{F}}_2$, then, there exists a segment r' of a leaf in $\widetilde{\mathcal{G}}$ contained in $F \cap E$ such that $\Phi^1_L(r)$ and $\Phi^1_E(r')$ are at distance less than C. In particular, if r is a ray, then the landing sets $\partial\Phi(r)$, $\partial\Phi(r')$ of r and r' coincide.*

Proof. This is a direct consequence of transversality of the foliations. If ε is small enough, then as long as points remain at distance less than ε the intersection of F with L and E will happen in a closeby set. Since the maps Φ^1_L and Φ^1_E are quasi-isometries and a bounded distance away from the projection to $\mathbb{H}^2$, the uniform bound C is obtained. □

3.3. *Nonseparated leaves.* Given a leaf $L \in \widetilde{\mathcal{F}}_i$ $(i = 1, 2)$ we defined the foliation $\mathcal{G}_L = \widetilde{\mathcal{G}}|_L$. Denote by $\mathcal{L}_L = L/_{\mathcal{G}_L}$ its leaf space which is a one-dimensional (possibly) non-Hausdorff manifold.

When $\mathcal{L}_L$ is non-Hausdorff one has *nonseparated leaves*, that is, distinct leaves $c_1, c_2 \in \mathcal{G}_L$ which are accumulated by a sequence d_n of leaves of $\mathcal{G}_L$ (since these are foliations, this is equivalent to having sequences $x_n, y_n \in d_n$ such that $x_n \to x_\infty \in c_1$ and $y_n \to y_\infty \in c_2$).

Remark 3.5. Two leaves $c_1, c_2 \in \mathcal{G}_L$ may be separated while there is no transversal to $\mathcal{G}_L$ which intersects both, in that case, there must be leaves e_1, e_2 (one of which could coincide with c_1 or c_2) which are nonseparated and separate c_1 from c_2 in the sense that they lie in different connected components of $L \setminus \{e_i\}$.

In principle, there is no a priori relation between $\mathcal{L}_L$ and $\mathcal{L}_E$ for different leaves L, E. However:

Proposition 3.6. *Assume that there is some $L \in \widetilde{\mathcal{F}}_i$ $(i = 1, 2)$ such that the leaf space $\mathcal{L}_L$ of $\mathcal{G}_L$ is Hausdorff. Then, for every $E \in \widetilde{\mathcal{F}}_1$ and $F \in \widetilde{\mathcal{F}}_2$ the leaf spaces $\mathcal{L}_E$ and $\mathcal{L}_F$ are Hausdorff.*

To show this, we first need the following useful property (see Figure 5):

Lemma 3.7. *Fix $L \in \widetilde{\mathcal{F}}_1$. The leaf space $\mathcal{L}_L$ of $\mathcal{G}_L$ is non-Hausdorff if and only if there is $E \in \widetilde{\mathcal{F}}_2$ such that $L \cap E$ is not connected. Moreover, if two distinct leaves $c_1, c_2 \in \mathcal{G}_L$ are nonseparated then, there is $E \in \widetilde{\mathcal{F}}_2$ such that $c_1 \cup c_2 \subset L \cap E$.*

Proof. Assume first that $L \cap E$ is not connected, therefore, there are leaves $c_1, c_2 \in \mathcal{G}_L$ which belong to E. Assume that the leaf space $\mathcal{L}_L$ of $\mathcal{G}_L$ is Hausdorff, this means

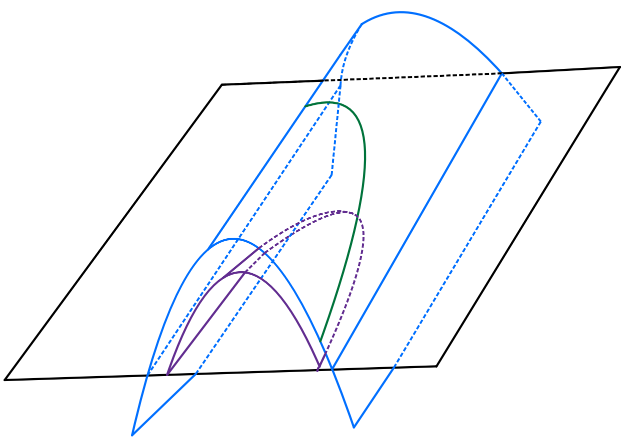

Figure 5. Intersection in more than one connected component forces non-Hausdorff leaf space of the induced one-dimensional foliation.

that there exists a transversal to $\mathcal{G}_L$ joining c_1 and c_2. This transversal is a transversal to $\widetilde{\mathcal{F}}_2$ intersecting E twice, in contradiction with Corollary 2.2.

Now, assume that c_1 and c_2 are nonseparated leaves in $\mathcal{G}_L$. Let $d_n \in \mathcal{G}_L$ be a sequence of leaves converging to both c_1 and c_2. It follows that $d_n \in E_n \cap L$ where E_n is a sequence of leaves of $\widetilde{\mathcal{F}}_2$. Consider $x_n, y_n \in d_n$ such that $x_n \to x_\infty \in c_1$ and $y_n \to y_\infty \in c_2$. It follows that E_n converges to both the leaf through x_∞ and the leaf through y_∞. Since the leaf space of $\widetilde{\mathcal{F}}_2$ is Hausdorff (see Corollary 2.2), these leaves must coincide, and thus $c_1 \cup c_2 \subset E \cap L$ where $E \in \widetilde{\mathcal{F}}_2$ is the leaf containing x_∞ (and y_∞). □

Proof of Proposition 3.6. By the previous lemma, if there is a leaf $L \in \widetilde{\mathcal{F}}_1$ for which $\mathcal{L}_L$ is non-Hausdorff then the same holds for some leaf $E \in \widetilde{\mathcal{F}}_2$ and vice versa.

This implies that it is enough to show that if there is a leaf $L \in \widetilde{\mathcal{F}}_1$ for which $\mathcal{L}_L$ is non-Hausdorff then the same holds for every $L' \in \widetilde{\mathcal{F}}_1$.

Since $\mathcal{F}_1$ is minimal, given an open interval I in the leaf space $\mathcal{L}_1 = \widetilde{M}/_{\widetilde{\mathcal{F}}_1}$ and a leaf $L' \in \widetilde{\mathcal{F}}_1$ there is some deck transformation $\gamma \in \pi_1(M)$ such that $\gamma L' \in I$. Thus, since having non-Hausdorff leaf space is invariant under deck transformations, it is enough to show that there is an interval in the leaf space where every leaf there has non-Hausdorff leaf space.

Take $L \in \widetilde{\mathcal{F}}_1$ and $E \in \widetilde{\mathcal{F}}_2$ such that $L \cap E$ is not connected and let $c_1, c_2 \in \mathcal{G}_L$ be two different connected components. Note that c_1, c_2 also belong to $\mathcal{G}_E$. Consider a transversal $\tau : (-\varepsilon, \varepsilon) \to E$ to $\mathcal{G}_E$ through some point $x \in c_1$ (i.e., $\tau(0) = x$) and denote by c_t the leaf of $\mathcal{G}_E$ through the point $\tau(t)$. Up to changing orientation in τ, we can assume that for $t > 0$ the leaves c_t and c_2 belong to different connected components of $E \setminus c_1$. This is because c_2 is in one component of $E \setminus c_1$. Denote L_t the leaf of $\widetilde{\mathcal{F}}_1$ through $\tau(t)$. Given a point $y \in c_2$ it follows that for small enough $t \in (-\varepsilon, \varepsilon)$ the leaf L_t intersects E close to y, thus, in the same connected component of $E \setminus c_1$ as c_2. It follows that for small $t > 0$ the leaf L_t intersects E in two connected components and thus $\mathcal{G}_{L_t}$ has non-Hausdorff leaf space as we wanted to show. □

3.4. ***Nonseparated rays and landing.*** If $c \in \mathcal{G}_L$, recall that a ray of c is a connected component of $c \setminus \{x\}$ for some $x \in c$ and a ray class is one equivalence class of rays by the relation of one of them being contained in the other; there are exactly two ray classes for each $c \in \mathcal{G}_L$.

Lemma 3.8. *Let $c_1, c_2 \in \mathcal{G}_L$ be two nonseparated leaves, then, there is exactly one ray class r of c_1 and transversals $\tau_1, \tau_2 : [0, \varepsilon) \to L$ with $\tau_i(0) \in c_i$ and an increasing homeomorphism $\varphi : (0, t_1] \to (0, t_2]$ for some small t_1, t_2 with the property that if $0 < t \leq t_1$ and e_t is the ray class of the leaf of $\mathcal{G}_L$ through $\tau_1(t)$ oriented as r then e_t intersects τ_2 at the point $\tau_2(\varphi(t))$.*

Proof. The existence of such segments is because being nonseparated implies that there are leaves $e_n \in \mathcal{G}_L$ accumulating in both, so if we fix two transversals τ and η to c_1 and c_2 we have segments of d_n of e_n intersecting τ and η in points x_n and y_n and such that $x_n \to c_1$ and $y_n \to c_2$. The segments d_n have length going to infinity.

Assume first that there are segments d_n and d_m with opposite orientation or which cut τ or η in different connected components of $\tau \setminus c_1$ or $\eta \setminus c_2$. We get a Jordan curve $\mathcal{J}$ made by $d_n \cup d_m$ and two segments of τ and η joining the points x_n and x_m and y_n and y_m. Assume without loss of generality that x_n is closer to c_1 than x_m. Consider the ray r_m of e_m starting from x_m which does not contain d_m. By construction, r_m enters the interior of the curve $\mathcal{J}$ and since it cannot intersect d_n or d_m it must exit J intersecting either τ or η in some point different from x_m or y_m which is a contradiction with Corollary 2.2 (in this case, it is just the Poincaré–Bendixson theorem that is being used since we are arguing inside L).

Now, the same argument implies that between two segments d_n and d_m every leaf intersecting the transversal τ must exit through η thus completing the proof of the lemma by choosing convenient parametrizations τ_1 and τ_2 of τ and η respectively. □

Define a *nonseparated ray class* of a leaf $c \in \mathcal{G}_L$ to be one which is nonseparated from some other leaf in $\mathcal{G}_L$ as in the previous Lemma. Now we can prove:

Proposition 3.9. *Let $c_1, c_2 \in \mathcal{G}_L$ be nonseparated leaves of $\mathcal{G}_L$ such that $c_1 \cup c_2 \subset L \cap E$ with $L \in \widetilde{\mathcal{F}}_1$ and $E \in \widetilde{\mathcal{F}}_2$. Then, if c_1^+ is the nonseparated ray class of c_1 with c_2 then one has that the positive ray lands and $\partial^+\Phi(c_1) = \alpha(E)$.*

For the following proof, a crucial fact[7] will be that if $\xi \neq \alpha(L)$ and $\varepsilon > 0$ is small, then there is a neighborhood I of L in the leaf space and a neighborhood J of ξ in $S^1(L)$ such that in $L \cup S^1(L)$ the set $\widehat{D}_\varepsilon(L, I)$ contains $N \cap L$, where N is a neighborhood of J in $L \cup S^1(L)$ (see Proposition 2.7).

If we assume that $\alpha(E) \notin \partial^+\Phi(c_1)$ then it is easier to obtain a contradiction, since we can fix a transversal τ to c_1^+ sufficiently close to some $\xi \in \partial^+\Phi(c_1)$ so that the full ray from this point is contained in $\widehat{D}_\varepsilon(E, I)$ and thus, we can show that if E' is a leaf nearby to E intersecting τ in the direction of c_2, then the $E' \cap L$ must follow closely the curve c_1 all along the ray, but on the other hand, since c_1 and c_2 are nonseparated the curve must joint a point close to τ to some point in a transversal to c_2, and since leaves of $\widetilde{\mathcal{F}}_2$ cannot intersect a transversal twice, one gets a contradiction. The main difficulty in what follows is that c_1^+ could approach $\alpha(E)$ and then we need to argue more carefully to get control on the nearby intersection.

Proof of Proposition 3.9. Assume that $\partial^+\Phi(c_1) \neq \{\alpha(E)\}$; this means that it contains some point $\xi \neq \alpha(E)$ in $\partial\mathbb{H}^2$.

[7]This property is true for general foliations homeomorphic to weak stable foliations of Anosov flows, but not true for general minimal foliations.

Fix a sequence $x_k \in c_1^+$ such that $\Phi_L(x_k) \to \xi$ which also implies that $\Phi_E(x_k) \to \xi$ by Lemma 3.1. We can assume by taking some subsequence that for some small ε smaller than the one given by Proposition 3.4 and some small intervals of leaves I of $\widetilde{\mathcal{F}}_2$ containing E in the boundary we have that $x_k \in \widehat{D}_\varepsilon(E, I)$.

We fix a sequence of transversals $\tau_k : [0, \varepsilon) \to L$ to $\mathcal{G}_L$ in L, through c_1 with $\tau_k(0) = x_k$ of length ε for ε as above. We assume that the transversals intersect the component of $L \setminus c_1$ which contains c_2. We also fix a transversal $\eta : [0, \varepsilon) \to L$ such that $\eta(0) \in c_2$ as in Lemma 3.8.

Since ξ is a marker point for $\widetilde{\mathcal{F}}_2$ (i.e., $\xi \neq \alpha(E)$) by our choice of I we know that if a leaf $E' \in I$, it intersects $\tau_1([0, \delta))$ for some small δ, and by Proposition 3.4 it contains a curve which remains at distance less than $\varepsilon/2$ from E and which limits in ξ (in other words, whose image under $\Phi_{E'}$ limits in ξ). Note that this curve has nothing to do with the ray c_1^+.

In particular, we know that E' will intersect τ_k for all sufficiently large k (in fact, Proposition 3.4 also implies that E' will intersect τ_k in points arbitrarily close to x_k). Note however that it could be that the intersection of E' with τ_k happens in a different connected component of $E' \cap L$ since we do not know that c_1^+ stays far from $\alpha(E)$. Our goal in what follows is to show that this “splitting” does not take place. n

Since c_1^+ is nonseparated from c_2 it follows from Lemma 3.8 that, up to reducing δ, for every $t \in (0, \delta]$ the leaf e_t of $\mathcal{G}_L$ through $\tau_1(t)$ intersects η in some point defining a segment d_t which joins both transversals.

We claim that for every large k, every point in $\tau_k((0, \varepsilon))$ must belong to some d_t. To see this, first take the segment d_{t_1} and consider k so large that $\tau_k((0, \varepsilon))$ is completely contained in the region of L bounded by c_1^+, the arc of the transversals τ_1 joining $\tau_1(0)$ with $\tau_1(t_1)$, d_{t_1}, the arc of η joining $\eta(0)$ with the intersection point with d_{t_1} with η and the corresponding ray of c_2 (this bounds a region by Proposition 2.9 and by the choice of the transversals τ_k their image is contained in this region). Now, fix some point $\tau_k(s)$ with $s \in (0, \varepsilon)$. By continuity of $\mathcal{G}_L$ there is $t \in (0, \delta)$ such that e_t, the curve of $\mathcal{G}_L$ through $\tau_1(t)$ intersects τ_k in some $\tau_k(s')$ with $s' < s$. Now consider the region $\mathcal{J}$ bounded by the Jordan curve formed by d_t, d_{t_1} and the arcs of η and τ_1 joining their endpoints. Then we get that the curve $e \in \mathcal{G}_L$ passing through $\tau_m(s)$ must escape $\mathcal{J}$ intersecting once the transversal τ_1 and once the transversal η, and the intersection point with τ_1 is in a point $\tau_1(t')$ with $t < t' < t_1$, which shows the claim.

For different values of t the segments d_t belong to distinct leaves of $\widetilde{\mathcal{F}}_2$, since each leaf intersects a transversal at most once (see Corollary 2.2). On the other hand, given some t, since d_t is compact and the points x_k escape to infinity, it follows that d_t cannot intersect τ_k for very large m because c_1 is properly embedded. This contradicts the fact that the leaf $E' \in \widetilde{\mathcal{F}}_2$ through some point $\tau_1(t)$ described before must intersect τ_k for all k. This contradiction then implies that c_1 cannot limit on $\xi \neq \alpha(E)$, or in other words $\partial^+\Phi(c_1) = \{\alpha(E)\}$. □

Remark 3.10. If $L \in \widetilde{\mathcal{F}}_1$ and two curves $c_1, c_2 \in \mathcal{G}_L$ are nonseparated, then, as proved in Lemma 3.7, they belong to $L \cap E$ with some $E \in \widetilde{\mathcal{F}}_2$. However, it should be noted that it does not follow from Lemma 3.7 that c_1 and c_2 must be nonseparated in the leaf space of $\mathcal{G}_E$ (only that there is no transversal in E from c_1 to c_2, recall Remark 3.5) and so it does not follow that $\alpha(L) = \alpha(E)$. See Section 7 for concrete examples.

4. Landing of rays

In this section we will show:

Theorem 4.1. *Let $\mathcal{F}_1, \mathcal{F}_2$ be two transverse foliations on $M = T^1S$ and let $\mathcal{G}$ denote the foliation obtained by intersection. Then, every ray of a leaf of $\widetilde{\mathcal{G}}$ lands* (cf. Definition 3.3).

The strategy of the proof is as follows. We show that if enough rays land, then every ray should land. In Proposition 3.9 we have shown that if rays are nonseparated then they should land in the nonmarker point of one of the foliations. We will use this to reduce to the case where leaves have Hausdorff leaf space, where landing can also be shown rather directly by "pushing" to close-by leaves.

4.1. *Some rays land then every ray lands.*

We will first prove the following:

Proposition 4.2. *Assume that there is a leaf $c \in \mathcal{G}_L$ for some $L \in \widetilde{\mathcal{F}}_i$ ($i = 1$ or 2) having one ray that lands in a point which is not $\alpha(L)$ (i.e., $\partial^+\Phi(c) = \{\xi\} \neq \{\alpha(L)\}$ or $\partial^-\Phi(c) = \{\xi\} \neq \{\alpha(L)\}$). Then, every ray of any leaf of $\widetilde{\mathcal{G}}$ lands in both foliations.*

We will use minimality of the foliation to show that there is a dense set of points in $S^1(L)$ which are landing points of leaves of $\mathcal{G}_L$.

Lemma 4.3. *Under the assumptions of Proposition 4.2, for every leaf $E \in \widetilde{\mathcal{F}}_i$ and every $I \subset \partial\mathbb{H}^2$ nontrivial interval, there is some leaf $e \in \mathcal{G}_E$ such that either $\partial^+\Phi(e) = \{\eta\}$ or $\partial^-\Phi(e) = \{\eta\}$ for some $\eta \in I$.*

Proof. Let us work in $\widehat{M}$, else, we can apply some deck transformations in the center of $\pi_1(M)$ so that everything makes sense. Let $\Gamma = \pi_1(M)/\mathbb{Z}$ where $\mathbb{Z}$ is the center, notice that $\Gamma \cong \pi_1(S)$ (cf. Section 2.1).

Up to considering a subinterval of I we can assume that the closure of I is disjoint from $\alpha(E)$. Consider a deck transformation $\gamma \in \Gamma$ of $\widehat{M} \to M$ such that $\gamma\xi \in I$ (see Proposition 2.8). Again, up to reducing I we can assume that $\gamma\alpha(L) \notin I$.

Fix some open interval J, such that the closures of J, I are disjoint, and in addition such that $\alpha(E), \gamma\alpha(L)$ are in the interior of J.

Fix $\varepsilon > 0$ given by Proposition 3.4. There is a subray $r \subset c$ landing in ξ such that the projection of γr is completely contained in $\widehat{D}_\varepsilon(E, J)$. This is because γr lands in $\gamma\xi \in I \subset J^c$ (see (2-6)) and $\alpha(E), \gamma\alpha(L)$ are in J. Here $\gamma r \subset \gamma L$ and the pushed through ray is in E and has ideal point in $\gamma\xi \in I$. This finishes the proof. $\square$

Proof of Proposition 4.2. Let $E \in \widetilde{\mathcal{F}}_i$. Suppose that some ray c in $\mathcal{G}_E$ does not land, and let I be its limit set, which is then not a point.

Let J be a closed nondegenerate interval whose closure is strictly contained in the interior of I (possibly $I = \partial\mathbb{H}^2$).

By Lemma 4.3 one can find in E at least two rays e_1, e_2 of $\mathcal{G}_E$ landing in different points $\xi_1 \neq \xi_2$ in the interior of J. We can further assume that e_1, e_2 are disjoint. Fix a curve joining the starting points of e_1 and e_2 and this defines two regions D^+ and D^- whose limit sets in $\partial\mathbb{H}^2$ are the two closed intervals $[\xi_1, \xi_2]$ and $[\xi_2, \xi_1]$ in $\partial\mathbb{H}^2$ with respect to the circular order (cf. Corollary 2.10). Since every ray of $\mathcal{G}_E$ is properly embedded and rays are disjoint or contained in one another, it follows that up to removing a compact piece, then the ray c must belong to either D^+ or D^-, and so can only accumulate on $[\xi_1, \xi_2]$ or $[\xi_2, \xi_1]$. This contradicts the fact that c accumulates on all of I, and I is an interval intersecting the interior of both $[\xi_1, \xi_2]$ and $[\xi_2, \xi_1]$. This proves the proposition. □

The following stronger statement will be used later:

Lemma 4.4. *Let $e \subset L \cap E$ be such that $\partial^+\Phi(e) = I$ where I is a not a point (in other words, the positive ray of e does not land). Then $\alpha(L) \in I$. In addition if $I \neq \partial\mathbb{H}^2$, then for every ξ in the interior of I there is no ray of a leaf of $\mathcal{G}_E$ which lands in ξ. The same holds for $\partial^-\Phi(e)$.*

Proof. In this lemma we also work in $\widehat{M}$, but the arguments can be easily adapted to do the same in $\widetilde{M}$. All leaves in this lemma are in the same foliation. First notice that if $I = \partial\mathbb{H}^2$ then there is nothing to prove.

First assume that $\alpha(L) \notin I$. Consider $\gamma \in \pi_1(M)$ such that γ acting on $\partial\mathbb{H}^2$ has a repelling fixed point $\gamma^- \notin I$ (and close to $\alpha(L)$) and an attracting fixed point $\gamma^+ \in I$ (Proposition 2.8).

Now, consider the leaf E' associated to the repelling fixed point, that is, such that $\alpha(E') = \gamma^-$. It follows that this leaf is invariant under γ. Consider a ray r of e, such that $\partial\Phi(r) = I$ which is contained in $\widehat{D}_\varepsilon(E, J)$ where J is some interval of the leaf space containing E' and E and $\varepsilon > 0$ given by Proposition 3.4. Using Proposition 3.4 we deduce that there is a ray r' of a leaf of $\mathcal{G}_{E'}$ such that $\partial\Phi(r) = I$. Moreover, $\gamma r'$ is a ray in $\mathcal{G}_{E'}$ with $\partial\Phi(\gamma(r)) = \gamma(I)$, which is strictly contained in I. This contradicts Proposition 2.11.

The second statement follows immediately from Proposition 2.11. □

4.2. *Finding landing rays.*

In this section we will show the following:

Proposition 4.5. *Let $L \in \widetilde{\mathcal{F}}_1$ and $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ such that $\gamma L = L$. Then, there is some $c \in \mathcal{G}_L$ which has at least one ray which lands.*

We consider a closed curve in the annulus $A = L/_{\langle\gamma\rangle}$ not homotopically trivial (for instance take a closed geodesic in A with some metric in L induced by M) and denote by ℓ a lift to L of this closed curve, which is thus γ invariant, that is $\gamma\ell = \ell$. For $K > 0$ we denote as $B_K(\ell)$ the closed tubular neighborhood of radius K around ℓ for some γ-invariant metric on L; note that the quotient $B_K(\ell)/_{\langle\gamma\rangle}$ is compact.

Lemma 4.6. *Assume that γ does not fix any leaf of $\mathcal{G}_L$. For every $K > 0$ there is $K' > 0$ such that if $c \in \mathcal{G}_L$ then any connected component of $c \cap B_K(\ell)$ has length less than K'.*

Proof. Assume, toward a contradiction, that there are sequences of arcs d_n of leaves $c_n \in \mathcal{G}_L$ of length going to infinity such that $d_n \subset B_K(\ell)$. There is a sequence γ^{k_n} of iterates of γ with $k_n \in \mathbb{Z}$ such that $\gamma^{k_n} d_n$ has its midpoint in a compact fundamental domain of $B_K(\ell)$ which is γ-invariant by choice. It follows that in the limit, $\gamma^{k_n} d_n$ converges to at least one leaf $c_\infty \in \mathcal{G}_L$ which is completely contained in $B_K(\ell)$.

Let $\pi_\gamma : L \to L/_{\langle\gamma\rangle}$ be the projection. Then $\pi_\gamma(c_\infty)$ is contained in a compact annulus. It follows that each ray of c_∞ either projects to a closed curve or to a curve asymptotic to a closed curve. This is a contradiction to hypothesis. □

Note that if there is a leaf $c \in \mathcal{G}_L$ which is fixed by γ then immediately we have that the landing points of c are $\partial^{\pm}\Phi_L(c) = \{\gamma^+, \gamma^-\}$ the attractor and repeller of γ acting at infinity.

Now we show that under the assumptions of the previous lemma, rays must keep intersecting $B_K(\ell)$ indefinitely.

Lemma 4.7. *Under the assumptions of Lemma 4.6 we have that there is $K > 0$ such that if $c \in \mathcal{G}_L$ and r is a ray of c that does not land, then r must intersect $B_K(\ell)$.*

We stress that K is independent of c and r. Note that since r cannot be contained in $B_K(\ell)$ by the previous lemma, we deduce that every ray needs to enter and leave $B_K(\ell)$ infinitely many times.

Proof. Fix $\varepsilon > 0$ given by Proposition 3.4. Now fix some large $K > 0$ so that if I is a closed interval of the leaf space of $\widetilde{\mathcal{F}}_1$ containing L as an endpoint, then $\widehat{D}_\varepsilon(L, I)$ contains the connected component of the complement of $B_K(\ell)$ which does not accumulate in I (see Proposition 2.7). Note that this value of K is independent of the interval I as long as L is an endpoint and I is sufficiently small (so that it is contained in the closure of one of the components of the complement of ℓ).

Assume for a contradiction that r is completely contained in the complement of $B_K(\ell)$ which then must be contained in a unique connected component because r is connected. Therefore, the limit set $I = \partial\Phi_L(r)$ is a proper interval contained in $\partial\mathbb{H}^2$ which does not intersect the interior of the limit set of the image by Φ_L of the connected component of $L \setminus B_K(\ell)$ which does not contain r. Now pick an interval J of the leaf space with L as an endpoint so that $\alpha(E) \in J$ only if $E = L$. For any $E \neq L$ in I we can then push the ray r to E applying Proposition 3.4 and we deduce that there is a ray r' in E which lands in an interval which is disjoint from $\alpha(E)$ contradicting Lemma 4.4. □

Proof of Proposition 4.5. As noted, if γ fixes some leaf of $\mathcal{G}_L$ then this leaf must land in the attractor and repeller of γ so the proposition holds. Therefore, we can assume we are under the assumptions of Lemma 4.6.

Let c be a leaf of $\mathcal{G}_L$ and r a ray of c. If r lands, the Proposition is proved, so assume that r does not land. Then we know that r must keep intersecting $B_K(\ell)$ for K as in Lemma 4.7. Consider a sequence d_n of segments of r completely contained in the complement of $B_K(\ell)$ except for the endpoints which belong to $\partial B_K(\ell)$. Using Lemma 4.6 we can choose the segments d_n with arbitrarily large length, since otherwise we would get arbitrarily large segments of r contained in $B_{K'}(\ell)$ for some larger K'.

Let x_n, y_n be the endpoints of d_n. We claim first that $d(x_n, y_n)$ must be bounded: if not, then up to considering an inverse we can assume that γx_n is contained in the segment J of $\partial B_K(\ell)$ joining x_n and y_n. Since $d_n \cup J$ is a Jordan curve and γy_n does not belong to J we deduce that γd_n intersects d_n which is not possible since they belong to different leaves of $\mathcal{G}_L$.

There is a sequence $k_n \in \mathbb{Z}$ such that $\gamma^{k_n} x_n$ belongs to a compact set. Thus, up to taking subsequences both $\gamma^{k_n} x_n$ and $\gamma^{k_n} y_n$ converge to points x_∞ and y_∞ in $\partial B_K(\ell)$. Note that x_∞ and y_∞ cannot be in the same leaf of $\mathcal{G}_L$ and being accumulated by $\gamma^{k_n} d_n$ their leaves must be nonseparated. Therefore the proposition follows from Proposition 3.9. □

4.3. *Proof of Theorem 4.1.* To prove Theorem 4.1 we want to apply Proposition 4.2. For this, it will be useful to use both foliations. We first show that if there is a leaf of $\mathcal{G}_L$ which has both rays landing in the same point, then after going to a closeby leaf of the other foliation, we will be able to apply Proposition 4.2 and we will get landing for all leaves in both foliations:

Lemma 4.8. *Let $L \in \widetilde{\mathcal{F}}_1$ be such that there is $c \in \mathcal{G}_L$ both of whose rays land in the same point, i.e., $\partial^+ \Phi^1_L(c) = \partial^- \Phi^1_L(c) = \{\xi\}$. Then, there is some leaf $E \in \widetilde{\mathcal{F}}_2$ with $\alpha(E) \neq \xi$ such that it contains a leaf $c' \in \mathcal{G}_E$ with $\partial^+ \Phi^1_E(c') = \partial^- \Phi^1_E(c') = \{\xi\}$.*

Proof. Consider the Jordan curve obtained by $c \cup \{\xi\}$ in $L \cup S^1(L)$ and denote by D the disk bounded by it.

Every curve $\hat{c} \in \mathcal{G}_L$ intersecting D must be contained in D and thus has both rays landing on ξ. Given a transversal $\tau : [0, t_0) \to L$ to $\mathcal{G}_L$ with $\tau(0) \in c$ and $\tau(t) \in D$ for $t > 0$ we denote $E_t \in \widetilde{\mathcal{F}}_2$ to the leaf through $\tau(t)$.

If we denote by c_t the curve of $\mathcal{G}_L$ containing $\tau(t)$ we get $c_t \subset L \cap E_t$. Since in L both rays of c_t land and the landing point is ξ, we get by Corollary 3.2 that for every small t, then c_t lands in E_t and the landing point is ξ. Since the nonmarker point $\alpha(E_t)$ varies monotonically we can choose t so that $\alpha(E_t) \neq \xi$. □

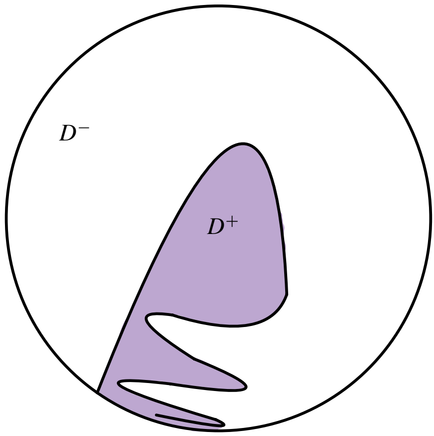

Figure 6. A depiction of the objects in the proof of Lemma 4.9.

The argument can be slightly extended to get:

Lemma 4.9. *Let $c \in \mathcal{G}_L$ satisfy $\partial^+\Phi_L(c) = \alpha(L)$ and $\partial^-\Phi_L(c) \neq \partial\mathbb{H}^2$. Then all rays of $\widetilde{\mathcal{G}}$ land.*

Proof. Let $I = \partial^-\Phi_L(c)$. Suppose first that $I = \alpha(L)$, then both rays of c land in the same point $\alpha(L)$. The previous lemma implies that there is some ray of $\widetilde{\mathcal{G}}$ in some leaf E of $\widetilde{\mathcal{F}}_2$ landing in a point different from $\alpha(E)$. Then Proposition 4.2 implies that all rays of $\widetilde{\mathcal{G}}$ land and the lemma is proved.

If I is a single point $\xi \neq \alpha(L)$, then again Proposition 4.2 also implies that all rays land, and the lemma is also proved.

Finally from now on assume that I is not a single point. Since $I \neq \partial\mathbb{H}^2$, Lemma 4.4 implies that $\alpha(L) \in I$.

The curve c separates L in two connected components, each diffeomorphic to a disk (see Proposition 2.9). Denote these components by D^+ and D^-. It follows that up to relabeling we can assume that the closure of $\Phi_L(D^+)$ in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ is equal to $\Phi_L(c \cup D^+) \cup I$ while the closure of $\Phi_L(D^-)$ in $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ is equal to $\Phi_L(c \cup D^-) \cup \partial\mathbb{H}^2$. See Figure 6.

Consider a transversal $\tau : (-\varepsilon, \varepsilon) \to L$ to $\mathcal{G}_L$ with $\tau(0) \in c$. Denote by c_t the leaf of $\mathcal{G}_L$ through the point $\tau(t)$. We can assume that if $t > 0$ then $\tau(t) \in D^+$, thus, the limit set $\partial^{\pm}\Phi_L(c_t) = I_t$ of c_t must be contained in I. Again as in the beginning of the proof we either finish the proof of the lemma or we are in the case that, the interval I_t is nondegenerate for any $t > 0$. Hence I_t contains $\alpha(L)$ by Lemma 4.4. In addition I_t is weakly monotonically decreasing, meaning that if $t > t'$ then $I_t \subset I_{t'}$ (they could coincide).

Denote by E_t the leaf of $\widetilde{\mathcal{F}}_2$ such that $c_t \subset L \cap E_t$. Then $\alpha(E_0) = \alpha(L)$; otherwise again Proposition 4.2 would imply that all rays of $\widetilde{\mathcal{G}}$ land in their appropriate leaves, but at this point we are assuming that the negative ray of c does not land. This contradiction shows that $\alpha(E_0) = \alpha(L)$.

In addition Lemma 4.4, applied to $\mathcal{F}_2$, implies that $\alpha(E_t)$ is contained in I_t. For small $t > 0$, $\alpha(E_t)$ is contained in the interior of I. Notice that $\alpha(E_t)$ varies continuously with t. We fix some small $t_0 > 0$. Now fix a t_1 with $0 < t_1 < t_0$ and such that E_{t_1} is invariant under some deck transformation γ which has a fixed point outside I (such leaves are dense, so there must be one). The other fixed point of γ is necessarily $\alpha(E_{t_1})$. We can arrange that this point is contained in the interior of $I_{t_0} \subset I_{t_1} \subset I$. This is because by choosing t_1 small we can ensure that $\alpha(E_{t_1})$ is in the interior of I_{t_0}, and since $I_{t_0} \subset I_{t_1}$, the claim follows.

We get that c_{t_1} has a ray converging to an interval I_{t_1} which contains only one of the fixed points of γ. We proved before that $\alpha(L) = \alpha(E_0) \in I_t$ for any t. In addition since $I_{t_0} \subset I_{t_1}$ we get that $\alpha(E_{t_0})$ is also in I_{t_1}, so the interval $[\alpha(E_0), \alpha(E_{t_0})]$ contained in I is contained in I_{t_1}. We conclude that $\alpha(E_{t_1})$ is an interior point of I_{t_1}, and I_{t_1} does not contain the other fixed point of γ. Applying γ to E_{t_1} we obtain two rays whose landing sets are proper intervals one contained in the interior of the other (I_{t_1} and $\gamma(I_{t_1})$), contradicting Proposition 2.11 and completing the proof. □

Proof of Theorem 4.1. Using Proposition 4.5 we know that there is at least one $L \in \widetilde{\mathcal{F}}_1$ and one $c \in \mathcal{G}_L$ having one landing ray. Up to orientation we can assume that $\partial^+\Phi_L(c) = \xi$. Using Proposition 4.2, either the result holds or $\xi = \alpha(L)$. Using Lemma 4.9 we can further assume that $\partial^-\Phi_L(c) = \partial\mathbb{H}^2$ (it is the only possibility of that lemma that does not yield that all rays land), in particular this assumption means that there is a ray of $\widetilde{\mathcal{G}}$ that does not land.

Note that all these results apply to both $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ so we will now work with $\widetilde{\mathcal{F}}_2$. Showing landing in both foliations is equivalent because of Lemma 3.1.

Let $\tau : (-\varepsilon, \varepsilon) \to L$ be a transversal to $\mathcal{G}_L$ with $\tau(0) \in c$. Call c_t the leaf of $\mathcal{G}_L$ passing through $\tau(t)$ and $E_t \in \widetilde{\mathcal{F}}_2$ such that $c_t \subset L \cap E_t$. Notice that the $\{E_t\}$ are pairwise distinct leaves.

We know that $\alpha(E_0) = \alpha(L)$ because $\partial^+\Phi_{E_0}(c) = \partial^+\Phi_L(c) = \alpha(L)$ (else we can apply Proposition 4.2 to $\widetilde{\mathcal{F}}_2$). We choose the transversal τ so small that $\alpha(E_t)$ is injective in $(-\varepsilon, \varepsilon)$ and $\alpha(E_t)$ is very close to $\alpha(L)$. Let I_t be the short closed interval in $\partial\mathbb{H}^2$ from $\alpha(L)$ to $\alpha(E_t)$ and J_t the closure of the complementary interval in $\partial\mathbb{H}^2$. Note that for $t > 0$ since $c_t \in L \cap E_t$, if it has a landing ray, then Proposition 4.2 applied to both foliations implies it must land $\alpha(L)$ and in $\alpha(E_t)$ which are different, so the limit set of both rays is a nontrivial connected set. Lemma 4.4 applied to both foliations implies that the limit set of each ray of c_t must contain both $\alpha(L)$ and $\alpha(E_t)$ thus it is some nontrivial connected set containing I_t or J_t.

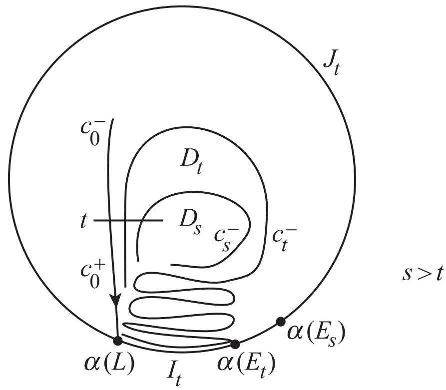

Figure 7. A depiction of some parts of the argument of the proof of Theorem 4.1: inclusion of sets $D_s \subset D_t$ and the "pairing" configuration for rays.

For any E_t which is a deck translate of E_0 there is a ray of $\mathcal{G}_{E_t}$ which lands (i.e., the image of the landing ray of c_0). By Proposition 4.2 again, it follows that this ray must land in $\alpha(E_t)$. Let r_t be one such ray. Consider

$$\mathcal{B} = \big\{t \in (-\varepsilon, \varepsilon) : E_t = \gamma E_0 \text{ for some } \gamma \in \pi_1(M)\big\}.$$

By minimality $\mathcal{B}$ is dense in $(-\varepsilon, \varepsilon)$. We will consider $\mathcal{B}^+ = \mathcal{B} \cap (0, \varepsilon)$.

Consider $t \in \mathcal{B}$ and let g_t be a ray of c_t. Denote by I_g its limit set (that is $\partial\Phi_{E_t}(g_t)$) in $\partial\mathbb{H}^2$. Recall that I_g must contain I_t or J_t (in particular, it not a point). The ray g_t is disjoint from r_t, because r_t lands and no ray of c_t lands. If $I_g \neq \partial\mathbb{H}^2$, then applying Lemma 4.4 (first to L and c_0^+ and then to E_t and r_t) we deduce that I_g coincides with either I_t or J_t because there cannot be a ray landing in its interior. If $I_g = \partial\mathbb{H}^2$ let us show a contradiction with a similar argument: in L the ray g_t has to be disjoint from the ray of c_0 which lands in $\alpha(L)$, in particular there cannot be a sequence of segments in g_t which limit to an interval in $S^1(L)$ with $\alpha(L)$ in its interior. When seen in E_t then g_t has to be disjoint from r_t, so again it cannot have a sequence of segments with a limit which is an interval in $S^1(E_t)$ with $\alpha(E_t)$ in its interior. It follows that I_g could not be all of $\partial\mathbb{H}^2$ and thus I_g is either I_t or J_t.

Recall that c_0^+ is a ray which lands in $\alpha(L)$. Let D_t be the component of $L \setminus c_t$ which does not contain $c_0 = c$, in particular D_t is an open set in L which has boundary (in L) equal to c_t. Notice that if $s > t > 0$, then $D_s \subset D_t$, and in particular that c_s is contained in D_t. For each $t > 0$ let c_t^+, c_t^- be the subrays of c_t determined by $\tau(t)$ and oriented coherently with c_0.

Consider $t \in \mathcal{B}^+$. We know that $\partial\Phi_L(c_t^-)$ can only be I_t or J_t for each such t. There are also two possibilities depending on where $\alpha(E_t)$ is (so in total, four possibilities). We say that $\alpha(E_t)$ and $\tau(t)$ *cross* if we consider a ray r from $\tau(0)$ to

$\xi \in J_t$ avoiding $c_0^+ \cup \tau((0, t])$ and there for every curve in $L \cup S^1(L)$ joining $\tau(t)$ with $\alpha(E_t)$ the curve intersects $c_0^+ \cup r$. Note that this is independent on the choice of r, and it is also independent on $t > 0$. We say they *pair* otherwise (i.e., there is a curve joining $\tau(t)$ with $\alpha(E_t)$ and avoiding $c_0^+ \cup r$).

We suppose first that $\tau(t), \alpha(E_t)$ pair, see this in Figure 7. Suppose first that for some $t \in \mathcal{B}^+$ we have that $\partial\Phi_L(c_t^-) = I_t$. In this case, note that the limit of $\Phi_L(D_t)$ is also I_t. For any $s > t$ we have $D_s \subset D_t$, so we now obtain that $\partial\Phi_L(c_s) \subset I_t$. But for such s the limit set of c_s^- has to contain $\alpha(E_s)$ which is not contained in I_t a contradiction. See Figure 7.

Now assume that $\tau(t), \alpha(E_t)$ pair but $\partial\Phi_L(c_t^-) = J_t$. Then c_t^- has points which limit to $\alpha(E_t)$ but c_t^- does not limit in any point in the interior of I_t. For any $s > t$ in $\mathcal{B}^+$ we cannot have that $\partial\Phi_L(c_s^-)$ is J_s. This is because the ray c_t^- limiting to $\alpha(E_t)$ forces the ray c_s^- to limit to some point in the interior of I_s. Thus, we can apply the previous analysis to s to get a contradiction. This finishes the analysis in the case where the transversal τ pairs.

We now consider the case where $\tau(t), \alpha(E_t)$ cross. Suppose first that for some $t \in \mathcal{B}^+$ the ray c_t^- limits to I_t. We depict this in Figure 8, left. Then for $s < 0$, it follows that the limit sets of both c_s^+, c_s^- have to be contained in I_t. This property is impossible, because for $s < 0$ contained in $\mathcal{B}$ the limit sets would have to be either I_s or J_s and neither is contained in I_t.

The remaining case is that for any $t \in \mathcal{B}^+$ we have that $\tau(t), \alpha(E_t)$ cross and $\partial\Phi_L(c_t^-) = J_t$. Fix some $t \in \mathcal{B}^+$. We will need to consider a different disk Z_t in L whose boundary is made up of c_t^-, the arc $\tau([0, t])$ and c_0^+ and chosen so that it contains c_t^+. Notice that this disk limits only on J_t (see Figure 8, right). Now consider s with $0 < s < t$, and $s \in \mathcal{B}^+$. Then c_s^+ is contained in Z_t, so its limit set

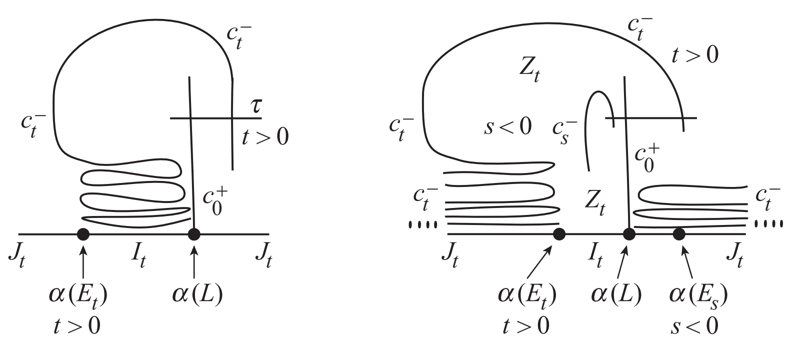

Figure 8. A depiction of the cases that $\tau(t), \alpha(t)$ cross. Left: the case that the limit set of c_t^- is I_t. Right: the case that the limit set is J_t. For simplicity of viewing we depict $\partial\mathbb{H}^2$ in a line segment, concentrating on what is happening near $\alpha(L)$.

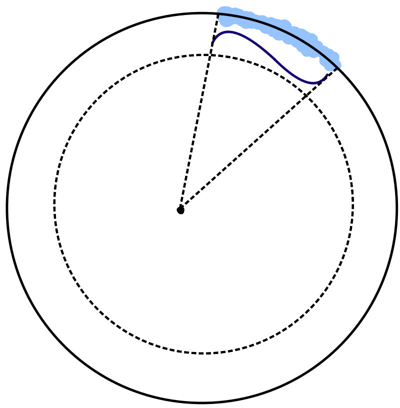

Figure 9. The shadow of a set from a point $x \in L$ is the set of points in $S^1(L)$ blocked by this set (when seen in $\mathbb{H}^2$). To have small visual measure means that if an arc of $\mathcal{G}_L$ is very far from x then its shadow has to be very small (independently of its length).

has to be also contained in J_t. On the other hand it has to be either J_s or I_s but neither is contained in J_t, which is a contradiction.

This finishes the proof of Theorem 4.1. □

5. Small visual measure

Let $\mathcal{F}_1, \mathcal{F}_2$ be two transverse minimal foliations on $M = T^1S$ and $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$ the intersection foliation.

We define the *shadow* of a subset $X \subset L$ seen from $x \in L$ as the set of points $\xi \in \partial\mathbb{H}^2$ for which there is a geodesic ray in $\mathbb{H}^2$ starting at $\Phi_L(x)$ and passing through some point $y \in \Phi_L(X)$ with ideal point ξ.

As in [25, § 4.3] we will say that $\mathcal{G}$ has the *small visual measure* property if for every $\varepsilon > 0$ there exists $R > 0$ such that if $L \in \widetilde{\mathcal{F}}_i$, $x \in L$ and ℓ a segment of some leaf of $\mathcal{G}_L$ such that $\ell \cap B_R(x) = \emptyset$, then the shadow of ℓ seen from x has length smaller than ε (meaning that the angle of the interval of vectors such that geodesic rays in $\mathbb{H}^2$ from $\Phi_L(x)$ to the shadow of ℓ is less than ε).

In this section we will show:

Proposition 5.1. *Let $\mathcal{F}_1, \mathcal{F}_2$ be two transverse minimal foliations on $M = T^1S$ and let $\mathcal{G}$ denote the foliation obtained by intersection. Then, $\mathcal{G}$ has the small visual measure property.*

Before we proceed with the proof let us explore some of its consequences. In [25, Lemma 5.9] the following is proved.

Proposition 5.2. *Assume that $\mathcal{G}$ has the small visual measure property, then, there exists $a_0 > 0$ such that for every segment ℓ of leaf of $\mathcal{G}_L$ we have that the geodesic segment joining the endpoints of ℓ is contained in $B_{a_0}(\ell)$.*

We stress that $B_{a_0}(\ell)$ is the neighborhood of ℓ of size a_0 in L, rather than in $\widetilde{M}$.

Note that from this we get that small visual measure allows to control endpoints of rays of leaves of $\mathcal{G}_L$. That is, if one considers a point $x \in L$ which belongs to some leaf $c \in \mathcal{G}_L$ and considers points y_n in c_x^+ going to infinity. It follows that the geodesic segments joining x and y_n are all contained in the a_0 neighborhood of c_x^+, thus, it follows that the geodesic ray joining x with $\partial^+\Phi(c)$ is contained in the a_0-neighborhood of c_x^+.

5.1. *Bubble leaves.*

Definition 5.3. (bubble leaves) We say that $c \in \widetilde{\mathcal{G}}$ is a *bubble leaf* if $\partial^+\Phi(c) = \partial^-\Phi(c) = \{\xi\}$ where ξ is some point in $\partial\mathbb{H}^2$.

We have dealt with such leaves in Lemma 4.8 showing that nearby leaves of the other foliation must also have bubble leaves with the same endpoint. We will perform a similar argument now, but trying to control the size of the interval on the leaf space where this holds.

The goal of this subsection is to show the following:

Proposition 5.4. *There exists a leaf $c \in \widetilde{\mathcal{G}}$ such that $\partial^+\Phi(c) \neq \partial^-\Phi(c)$* (see (3-2)).

We will need to control the place where the bubble leaves land. For this, we will separate the leaf L in *bubble regions*. To introduce this, let us first make some definitions. Notice first that by Corollary 2.10 we know that if $c \in \mathcal{G}_L$ is a bubble leaf with $\partial^\pm\Phi(c) = \{\xi\}$ we can define D_c to be the disk in the complement of c such that the accumulation of $\Phi_L(D_c)$ in $\partial\mathbb{H}^2$ is exactly ξ.

Definition 5.5. (bubble region of c) Given $L \in \widetilde{\mathcal{F}}_i$ and $c \in \mathcal{G}_L$ a bubble leaf with $\partial^\pm\Phi(c) = \{\xi\}$ we denote by $\mathcal{B}(c, L)$ the *bubble region of* c in L as the union of all leaves $c' \in \mathcal{G}_L$ such that there is some $c'' \in \mathcal{G}_L$ which is a bubble leaf and is such that $D_c \cup D_{c'} \subset D_{c''}$. We call ξ the *landing point* of the bubble region.

We also remark that in general the bubble region of c is not the union of all bubble leaves c' in $\mathcal{G}_{L_i}$ such that $\partial\Phi^\pm(c') = \{\xi\}$. For example it could be that c, c' bound maximal disjoint disks $D_c, D_{c'}$. By maximal for c we mean there is no leaf c'' of $\mathcal{G}_{L_i}$ distinct from c, and such that $D_{c''} \supset D_c$. In this case $D_c, D_{c'}$ are two disjoint bubble regions with the same ideal point ξ.

Lemma 5.6. *Every bubble region is either open or closed. Each leaf contains at most countably many distinct bubble regions. If every leaf in $\mathcal{G}_L$ is a bubble leaf, then there is a unique open bubble region $\mathcal{B}$ such that $\Phi_L(\mathcal{B})$ accumulates in all of $\partial\mathbb{H}^2$.*

Figure 10. Depiction of a leaf L such that every leaf of $\mathcal{G}_L$ is a bubble leaf. The painted regions are the closed bubble regions and their complement is the open bubble region.

In Figure 10 a depiction of the conclusion of this lemma is presented.

Proof. Each bubble region is a set saturated by $\mathcal{G}_L$. There is a natural partial order between the leaves in a bubble region given by $c_1 \prec c_2$ if $D_{c_1} \subset D_{c_2}$. It follows that if there is a maximal element in the bubble region, then it is unique (by definition of bubble region, every pair of elements is smaller or equal than some element in the region) and the bubble region is closed, while if there is no maximal element then the bubble region is open. As bubble regions have nonempty interior we know that there are at most countably many bubble regions.

Assume now that every leaf of $\mathcal{G}_L$ belongs to some bubble region. If there is a unique bubble region, then it is an open bubble region since there cannot be a maximal element. We next prove that not every bubble is closed: Let B be a closed bubble with boundary leaf c and let τ be a small transversal to $\mathcal{G}$ in L with an endpoint in c and not intersecting the interior of B. Parametrize τ as x_t, $0 \leq t \leq 1$ with x_0 in c. Suppose a bubble region B' intersects τ not in x_0. Then its boundary intersects τ (as x_0 is not in B'). In addition its boundary intersects τ in a single point, because if it intersected τ in two points, one would produce a bubble leaf intersecting τ in two points, contradiction. It follows that there is a single bubble region B' containing $\tau \setminus \{x_0\}$ and B' is an open bubble.

Finally, given an open bubble region $\mathcal{B}$ we can consider $\widehat{\mathcal{B}}$ to be $\mathcal{B}$ together with all the closed bubbles whose boundary intersects the boundary of $\mathcal{B}$. It follows that

$\widehat{\mathcal{B}}$ is an open set and that if $\mathcal{B}'$ is another open bubble region, then $\widehat{\mathcal{B}} \cap \widehat{\mathcal{B}}' = \varnothing$, thus, we get there is a unique open bubble region.

To conclude, using Corollary 2.10 we know that the complement of a closed bubble region accumulates in all of $S^1(L)$, so the same holds for the complement of countably many closed bubble regions, thus the open bubble region accumulates everywhere, also using the properties derived above. □

We remark that the uniqueness of open bubble regions in a leaf L of $\widetilde{\mathcal{F}}_i$ does not necessarily work if not every leaf in $\mathcal{G}_L$ is a bubble leaf.

Since open bubble regions are special, it is natural to expect that their landing point will also be special:

Lemma 5.7. *If every leaf in $\widetilde{\mathcal{G}}$ is a bubble leaf then in each $L \in \widetilde{\mathcal{F}}_i$ the unique open bubble region of $\mathcal{G}_L$ has $\alpha(L)$ as its landing point.*

Proof. The proof is by contradiction. Assume that there is a leaf $L \in \widetilde{\mathcal{F}}_i$ where the unique open bubble region $\mathcal{O}$ has landing point $\xi \neq \alpha(L)$. We fix a closed interval I in the leaf space of $\widetilde{\mathcal{F}}_i$ containing L in its interior and such that for every $L' \in I$ we have that $\alpha(L') \neq \xi$. Denote by $J \subset \partial\mathbb{H}^2$ the interval $J = \{\alpha(L') : L' \in I\}$. We consider $\varepsilon > 0$ from Proposition 3.4 and fix the region $\widehat{D}_\varepsilon(L, I)$.

We first notice that Lemma 4.3 shows that the existence of a leaf whose landing point is $\xi \neq \alpha(L)$ implies that in each leaf L the set of landing points of rays in the leaf is dense in $\partial\mathbb{H}^2$. Thus, for a given leaf L there must be rays converging to this dense set of points and thus also closed bubble regions in L the union of whose landing points is a dense in $\partial\mathbb{H}^2$. Pick one closed bubble region $\mathcal{B}$ whose ideal point is neither $\alpha(L)$ nor ξ.

Up to shrinking I to a smaller interval we assume that $\mathcal{B}$ completely contained in $\widehat{D}_\varepsilon(L, I)$. Moreover, if we call c the maximal element of $\mathcal{B}$, it is a leaf of $\mathcal{G}_L$ whose endpoints are $\partial^\pm\Phi(c) = \{\eta\}$ with $\eta \notin J$. We can assume that every closed bubbles leaf whose landing point is in the interval K joining ξ and η and not intersecting J is contained in $\widehat{D}_\varepsilon(L, I)$. Indeed any such bubble region $\mathcal{B}'$ not contained in $\hat{D}_\varepsilon(L, I)$ has points limiting to points in the interval K as well as points a bounded distance from the geodesic g in L with ideal points the endpoints of I. Suppose there are infinitely many of these. They cannot accumulate to some point in L, hence segments of the boundary of these limit on a nondegenerate open interval of $S^1(L)$. This contradicts that the set of ideal points of leaves of $\mathcal{G}_L$ is dense in $S^1(L)$. It follows that there are only finitely many bubbles in L satisfying this property and we take the last one satisfying the property.

Since $\mathcal{O}$ is an open bubble, we can find a sequence of nested curves e_n in $\mathcal{O}$ which accumulates on c. More precisely, if D_{e_n} are the disks defined by e_n and contained in $\mathcal{O}$ it follows that $D_{e_n} \subset D_{e_{n+1}}$ and $\mathcal{O} = \bigcup_n D_{e_n}$. It follows that there are rays r_n of e_n contained in $\widehat{D}_\varepsilon(L, I)$ and that accumulate on c (note that the full curve e_n must enter $\widehat{D}_\varepsilon(L, I)$, but it cannot happen that for large n both rays are contained in $\widehat{D}_\varepsilon(L, I)$, since the union of the disks D_{e_n} is all of $\mathcal{O}$).

Now we fix a leaf $E \in I$ that is invariant under some $\gamma \in \pi_1(M)\setminus\{\text{id}\}$ (cf. Section 2.7) and such that the fixed points of γ are different from ξ (note that one must be contained in J). If we apply Proposition 3.4 we deduce that the leaf c pushes to a leaf c' of $\mathcal{G}_E$ whose endpoints are η and the rays r_n push to rays r_n' which have one endpoint in ξ and accumulate on a ray of c'. If e_n' are the leaves of $\mathcal{G}_E$ containing the rays r_n' we deduce that they must belong to an open bubble region since every leaf of $\mathcal{G}_E$ is a bubble leaf and the leaves e_n' are all landing in ξ. It follows that E has an open bubble region with limit point ξ. Applying γ we deduce that there is more than one open bubble region in E contradicting the previous lemma. □

To prove Proposition 5.4 we will proceed by contradiction and therefore assume that all leaves of $\widetilde{\mathcal{G}}$ are bubble leaves. This will allow us to use the previous results. To be able to get a contradiction we will need to construct uncountably many bubble leaves with different landing points in a leaf of the other foliation.

Proof of Proposition 5.4. Assume for a contradiction that every leaf of $\widetilde{\mathcal{G}}$ is a bubble leaf.

We claim that for each $L \in \widetilde{\mathcal{F}}_1$ there is a nontrivial interval of the leaf space of $\widetilde{\mathcal{F}}_2$ which contains bubble leaves whose limit points are $\alpha(L)$. This is just because Lemma 5.7 states that the open bubble region $\mathcal{O}_L$ of L, which exists due to Lemma 5.6, must land in $\alpha(L)$. Considering a transversal τ to $\mathcal{G}_L$ inside $\mathcal{O}_L$ gives a non trivial interval I_L of the leaf space of $\widetilde{\mathcal{F}}_2$ (i.e., those leaves which intersect τ) containing bubble leaves whose landing point is $\alpha(L)$. The interval I_L also depends on τ, but we are omitting this dependence. Notice that we are not claiming that the elements of $\{I_L, L \in \widetilde{\mathcal{F}}_1\}$ are pairwise distinct.

Pick x_n dense and countable in the leaf space of $\widetilde{\mathcal{F}}_2$ (which is homeomorphic to $\mathbb{R}$). If for every x_n there are only countably many $L \in \widetilde{\mathcal{F}}_1$ such that $x_n \in I_L$, then the set of those $L \in \widetilde{\mathcal{F}}_1$ for which I_L does not contain any x_n is still uncountable. Hence this set is nonempty, contradicting the density of $\{x_n\}$, since any I_L is an open set in the leaf space of $\widetilde{\mathcal{F}}_2$.

We deduce that there is a leaf $E \in \widetilde{\mathcal{F}}_2$ for which there are uncountably many $L \in \widetilde{\mathcal{F}}_1$ such that $E \in I_L$ with pairwise distinct $\alpha(L)$. It follow that E contains uncountably many bubble leaves which land in pairwise different points. This contradicts the fact that a leaf can contain at most countably many distinct bubble regions (see Lemma 5.6). This contradiction completes the proof. □

5.2. ***Proof of Proposition 5.1.*** The proof will be by contradiction. Using what we have proved we will have a ray landing at a marker point. Then, assuming that the small visual property fails, we can use the following lemma and minimality to get a contradiction.

Lemma 5.8. *For every $\delta > 0$ there exists $K := K(\delta) > 0$ such that if*

- *I is a segment in $\partial\mathbb{H}^2$,*
- *$x \in \mathbb{H}^2$, and*
- *r is a geodesic ray starting from x and landing in some point ξ of I separating I in two intervals of visual measure $> \delta/2$ seen from x,*

then if $y \in r$ satisfies $d_{\mathbb{H}^2}(x, y) > K$, it follows that the visual measure of I seen from y is larger than $2\pi - \delta$. Here $d_{\mathbb{H}^2}$ is the hyperbolic metric in $\mathbb{H}^2$.

Proof. Since isometries are transitive in $T^1\mathbb{H}^2$ we can assume in the upper half space model of $\mathbb{H}^2$, that $x = i$, the geodesic ray r is the ray $\{ai : a \geq 1\}$ and the interval I contains, for some large t depending on δ a set of the form $A_t = [t, +\infty) \cup \{\infty\} \cup (-\infty, -t]$. The lemma then reduces to computing the visual measure of A_t from a point of the form ai which goes to 2π with $a \to \infty$. □

Proof of Proposition 5.1. Consider $c \in \widetilde{\mathcal{G}}$ given by Proposition 5.4 and let $L \in \widetilde{\mathcal{F}}_1$, $E \in \widetilde{\mathcal{F}}_2$ be such that $c \subset L \cap E$. We will argue for L (the case for E is symmetric).

Let r be a ray of c with landing point $\partial\Phi(r) = \xi \neq \alpha(L)$. Let I be a closed interval of the leaf space of $\widetilde{\mathcal{F}}_1$ containing L in its interior and such that if $L' \in I$ then $\alpha(L') \neq \xi$. Up to reducing the ray r we can assume that $r \subset \widehat{D}_\varepsilon(L, I)$ for ε as given in Proposition 3.4.

Now assume for a contradiction that there is $\delta > 0$ and a sequence ℓ_n of segments of $\mathcal{G}_{L_n}$ with $L_n \in \widetilde{\mathcal{F}}_1$ and points $x_n \in L_n$ such that $d_{L_n}(x_n, g_n) > 5n$ and ℓ_n has shadow in $S^1(L_n)$ of length $> 10\delta$ when seen from x_n. Up to cutting the segments ℓ_n we can assume that the shadow has length $> 5\delta$ and is disjoint from $\alpha(L_n)$.

Consider for each n a geodesic ray from x_n to the middle point of the shadow of ℓ_n and a point y_n in the geodesic ray and at distance $= 2n$ from x_n. By minimality of $\mathcal{F}_1$ and up to changing slightly the geodesic ray, we can assume that there is γ_n sending y_n to a point very close to L and in a compact set of $\widetilde{M}$. Using Lemma 5.8 we get that, up to a subsequence,

- $d_{L_n}(y_n, g_n) > 2n$,
- the visual measure of ℓ_n seen from y_n is larger than $2\pi - a_n$ with $a_n \to 0$ (this is after identifying $S^1(L)$ with $\partial\mathbb{H}^2$), and
- $\gamma_n y_n \to y_\infty \in L$.

We chose ℓ_n so that $\alpha(L_n)$ does not belong to the shadow of ℓ_n from x_n and thus we get that the shadow from $\gamma_n y_n$ of $\gamma_n g_n$ contains ξ in its interior if n is big enough. This is because $\gamma_n L_n \to L$ and $\alpha(\gamma_n L_n) \to \alpha(L)$. Since for large n the leaf $\gamma_n L_n$ is close to L we can assume that $\gamma_n L_n$ is in I and we can push arcs ℓ'_n of $\gamma_n g_n$ to L so that $\Phi_L(\ell'_n)$ accumulates in an interval containing ξ in its interior. This forces ℓ'_n to eventually intersect r which is a contradiction. □

6. Leafwise Hausdorff leaf space implies quasigeodesic

Here we will show the following:

Theorem 6.1. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be transverse minimal foliations of $M = T^1S$ and let $\mathcal{G}$ denote the foliation obtained by intersection. Assume that in some leaf $L \in \widetilde{\mathcal{F}}_1$ the leaf space of $\mathcal{G}_L$ is Hausdorff. Then, the foliation $\mathcal{G}$ is an Anosov foliation.*

This also follows from [27] which gives an alternative proof. Here we present a proof in our restricted setting, since it is vastly simpler and can show some of the ideas in a more transparent way.

Recall that an *Anosov foliation* is a one-dimensional foliation in M which is homeomorphic to the orbit foliation of a topological Anosov flow. As follows from [5, § 6], due to minimality, it is enough to show that the one-dimensional foliation is *quasigeodesic*, that is, for every leaf $L \in \widetilde{\mathcal{F}}_i$ the leaves of $\mathcal{G}_L$ are quasigeodesics in L.

Here we are working with $M = T^1S$ so it makes sense to compare our one-dimensional foliation with the one of the geodesic flow in negative curvature: this foliation, when restricted to a leaf L of the weak-stable or weak-unstable foliation has the following properties that we will try to produce to show Theorem 6.1:

- One of the landing points of every leaf of $\mathcal{G}_L$ is $\alpha(L)$.
- Given any point $\xi \in \partial\mathbb{H}^2 \setminus \{\alpha(L)\}$ there is a (unique) leaf whose landing points are ξ and $\alpha(L)$.

Thanks to some classical results, showing these properties will be enough to establish that the foliation is an Anosov foliation (see [28]). This implies that it is quasigeodesic (see [18]). We refer the reader to [5, § 5] for details.

6.1. ***Bubble leaves and landing.*** First we need to show that bubble leaves (recall Section 5.1) produce non-Hausdorff behavior. For this, the small visual measure property is crucial, in particular Proposition 5.2.

We recall here that the assumptions of Theorem 6.1 imply that in *every* leaf of $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ the restriction of $\widetilde{\mathcal{G}}$ has Hausdorff leaf space because of Proposition 3.6.

Lemma 6.2. *If $\mathcal{G}_L$ has a bubble leaf for some $L \in \widetilde{\mathcal{F}}_i$, then there is a leaf $L' \in \widetilde{\mathcal{F}}_i$ such that $\mathcal{G}_{L'}$ has non-Hausdorff leaf space.*

Proof. Consider a bubble leaf $c \in \mathcal{G}_L$ (assume $L \in \widetilde{\mathcal{F}}_1$) and fix a geodesic ray r starting at some point in c which has the same landing point $\xi \in \mathbb{H}^2$ as c (i.e., $\partial\Phi(r) = \partial^{\pm}\Phi(c) = \{\xi\}$). Let x_0 be the starting point of r in c and consider the rays $c^+_{x_0}$ and $c^-_{x_0}$. Passing to the limit in Proposition 5.2 we know that there is some $a_0 > 0$ such that r is contained in the a_0-neighborhood of $c^+_{x_0}$ and $c^-_{x_0}$.

Fix a sequence $y_n \in r$ such that $\Phi_L(y_n) \to \xi$ and consider points $p_n \in c^+_{x_0}$ and $q_n \in c^-_{x_0}$ at distance less than a_0 from y_n in L. It follows that

- $d_L(p_n, q_n) \leq 2a_0$,
- $\Phi_L(p_n) \to \xi$ and $\Phi_L(q_n) \to \xi$, and
- the length of the segment of c joining p_n and q_n goes to infinity.

Now consider deck transformations $\gamma_n \in \pi_1(M)$ which map p_n to a compact fundamental domain $K \subset \widetilde{M}$. Since γ_n are isometries and $d_L(p_n, q_n)$ is bounded, the points $\gamma_n q_n$ also belong to a compact set. Up to subsequence we can assume that both $\gamma_n p_n$ and $\gamma_n q_n$ converge to points p_∞ and q_∞. Note that since the distance in L from p_n and q_n is bounded, both points p_∞ and q_∞ belong to the same leaf $L' \in \widetilde{\mathcal{F}}_i$.

Since the length from p_n to q_n along c goes to infinity it follows that p_∞ and q_∞ cannot belong to the same leaf of $\mathcal{G}_{L'}$. On the other hand p_n, q_n belong to the same leaf of $\widetilde{\mathcal{G}}$ and thus the same leaf E_n of $\widetilde{\mathcal{F}}_2$ which since p_n, q_n are close to L' it follows that either $E_n \cap L'$ is not connected (in which case, the leaf space of $\mathcal{G}_{L'}$ is not Hausdorff) or $E_n \cap L'$ is a sequence of leaves of $\mathcal{G}_{L'}$ converging to both the leaf through p_∞ and q_∞ and again we get that $\mathcal{G}_{L'}$ does not have Hausdorff leaf space.

Since p_∞, q_∞ do not belong to the same leaf of $\mathcal{G}_{L'}$ but p_n, q_n belong to the same leaf of $\widetilde{\mathcal{F}}_2$ thus, $\mathcal{G}_{L'}$ does not have Hausdorff leaf space as we wanted to show. □

Now we show that at least one of the endpoints of each leaf in $\mathcal{G}_L$ must be the nonmarker point. Fix an orientation in L and consider, for a given $c \in \mathcal{G}_L$ which we know separates L in two open disks $D_c^\pm$, each one such that the limit set of $\Phi_L(D_c^\pm)$ in $\partial\mathbb{H}^2$ is one of the intervals joining the landing points of c (see Corollary 2.10).

Lemma 6.3. *Let $L \in \widetilde{\mathcal{F}}_i$ and $c \in \mathcal{G}_L$ then either $\partial^+\Phi(c) = \{\alpha(L)\}$ or $\partial^-\Phi(c) = \{\alpha(L)\}$.*

Proof. Let $L \in \widetilde{\mathcal{F}}_i$ and $c \in \mathcal{G}_L$ and assume that $\alpha(L) \notin \partial^+\Phi(c) \cup \partial^-\Phi(c)$.

Proposition 3.4 lets us assume that the leaf L is fixed by some $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$, and such that both points in $\partial^+\Phi(c) \cup \partial^-\Phi(c)$ belong to the same connected component of $\partial\mathbb{H}^2$ minus the fixed points of γ. It is enough to consider an interval I of the leaf space of $\widetilde{\mathcal{F}}_i$ containing L in the interior so that the nonmarker points of leaves in I are never in $\partial^+\Phi(c) \cup \partial^-\Phi(c)$: hence given an even smaller interval we can push the entire leaf c to nearby leaves of $\widetilde{\mathcal{F}}_i$ along the leaf of the other foliation with Proposition 3.4. Then since leaves with nontrivial stabilizer are dense (see Section 2.7), we can push to one of them in I to get the same property in such a leaf. To get that the fixed points of γ do not link with $\partial^+\Phi(c) \cup \partial^-\Phi(c)$ we can use for instance that the set of closed geodesics in S is dense in S (Section 2.7).

We now choose orientations so that D_c^+ is the complementary component of c which does not limit on γ^+, γ^-. If there is a transversal to $\mathcal{G}_L$ in L from c to γc, then the image of this under γ is a transversal from γc to $\gamma^2 c$. Notice

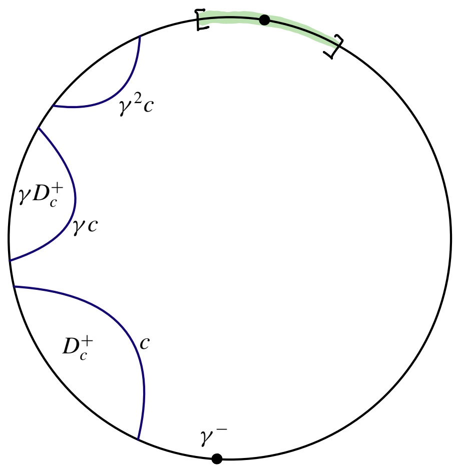

Figure 11. If no ray lands in $\alpha(L)$ the leaf space cannot be Hausdorff.

that $D_c^+, \gamma(D_c^+), \gamma^2(D_c^+)$ are pairwise disjoint. It now follows that $c, \gamma^2 c$ cannot intersect a common transversal and the leaf space cannot be Hausdorff. (See Figure 11.) □

6.2. ***Construction of the Anosov foliation.*** To complete the proof that the foliation $\mathcal{G}$ is an Anosov foliation we need to show that for every $L \in \widetilde{\mathcal{F}}_i$ and $\xi \in \partial\mathbb{H}^2 \setminus \{\alpha(L)\}$ there is a leaf $c \in \mathcal{G}_L$ such that $\partial\Phi(c) = \{\xi, \alpha(L)\}$ and that such a leaf is unique; this will produce an equivariant equivalence with the leaf space of the orbit flow of the geodesic flow in negative curvature [28] which is known to be enough to show that the foliation has the desired properties.

We divide this into some steps:

Lemma 6.4. *Let $L \in \widetilde{\mathcal{F}}_i$ and $\xi \in \partial\mathbb{H}^2$, then, there exists $c \in \mathcal{G}_L$ with a ray landing in ξ (i.e., such that $\xi \in \partial^{\pm}\Phi(c)$).*

Proof. Recall from Lemma 4.3 that the set of points in $\partial\mathbb{H}^2$ for which there is a ray in L which lands in that point is dense.

Consider a sequence c_n of leaves of $\mathcal{G}_L$ converging to some leaf $c \in \mathcal{G}_L$ (which is unique because the leaf space of $\mathcal{G}_L$ is Hausdorff). Let ξ_n be the landing point of c_n different from $\alpha(L)$ and let ξ and $\alpha(L)$ be the landing points of c. Assume that (up to taking a subsequence) $\xi_n \to \eta \neq \xi$. One can then consider a ray which lands in a point which belongs to the interval between ξ and η not containing $\alpha(L)$. This ray belongs to some leaf $\ell \in \mathcal{G}_L$. By the previous lemma, the leaf ℓ has one ideal point in $\alpha(L)$. It follows that ℓ must then separate the curves c_n with large n from c, contradicting the fact that $c_n \to c$.

This continuity plus the fact that the landing points are dense implies that every point different from $\alpha(L)$ is a landing point as we wanted to show. □

Now we show uniqueness as a consequence of minimality of the foliations, and derive from Theorem 2.1 that the nonmarker points of leaves vary monotonically (Section 2.4).

Lemma 6.5. *Let $L \in \widetilde{\mathcal{F}}_i$ and $\xi \in \partial\mathbb{H}^2 \setminus \{\alpha(L)\}$, then, the leaf $c \in \mathcal{G}_L$ such that $\partial^{\pm}\Phi(c) = \{\xi, \alpha(L)\}$ is unique.*

Proof. Assume that in a leaf $L \in \widetilde{\mathcal{F}}_1$ (the case where $L \in \widetilde{\mathcal{F}}_2$ is symmetric) there are two leaves of $\mathcal{G}_L$ landing in ξ and $\alpha(L)$. Since the leaf space of $\mathcal{G}_L$ is Hausdorff and one of the landing points of every $c \in \mathcal{G}_L$ is $\alpha(L)$ we deduce that there is an interval I in the leaf space of $\widetilde{\mathcal{F}}_2$ consisting on leaves E such that $E \cap L$ is a leaf $c \in \mathcal{G}_L$ such that $\partial^{\pm}\Phi_L(c) = \{\xi, \alpha(L)\}$. From the previous lemma applied to $\widetilde{\mathcal{F}}_2$ we know that if $E \in I$ and if $c' \in \mathcal{G}_E$ then $\alpha(E) \in \partial^{\pm}\Phi_E(c')$. Considering $c \in E \cap L$ we deduce that for every $E \in I$ we have that $\alpha(E) \in \{\xi, \alpha(L)\}$. Since the point $\alpha(E)$ varies continuously with E we deduce that there is an interval of $\widetilde{\mathcal{F}}_2$ which has the same nonmarker point while the nonmarker point varies monotonically. □

Now we are ready to prove the main result of this section.

Proof of Theorem 6.1. From our previous results we have a bijection between the space of leaves of $\widehat{\mathcal{G}}$ (the lift of the foliation $\mathcal{G}$ to $\widehat{M}$ the intermediate cover of M) and the set of pairs of distinct points of $\partial\mathbb{H}^2$. This bijection is continuous and equivariant under deck transformations which is enough to show that the foliation is homeomorphic to the orbit foliation of the geodesic flow of a hyperbolic metric on S. (See [28] or [5] for details.) □

7. The Matsumoto–Tsuboi example revisited

Here we revisit the example from [35] from the point of view of our results. In [35] an example of a pair of transverse foliations[8] of $\mathbb{T}^2 \times [-1, 1]$ is constructed in such a way that the boundaries match with the weak stable and weak unstable foliations seen in the lift to T^1S of a simple closed geodesic. That is, if ℓ is a closed geodesic in S, it lifts to a torus $T \subset T^1S$ which contains two periodic orbits of the geodesic flow (associated to the orbits associated to γ with both orientations). Note that the weak stable and weak unstable foliations are horizontal, so they are transverse to T. Figure 12 depicts the intersection of the foliations with the torus in its universal cover (which become tangent at the two periodic orbits that are common to both weak foliations).

This way one can cut T^1S along such a torus and glue this new foliation of $\mathbb{T}^2 \times [-1, 1]$ to obtain a new pair of transverse foliations of T^1S. From the way this pair of foliations is constructed, it is clear that the new foliations, called $\mathcal{F}_1$, $\mathcal{F}_2$,

[8]In [35] triples of transverse foliations are considered, but for the interests of our work we will ignore the third foliation.

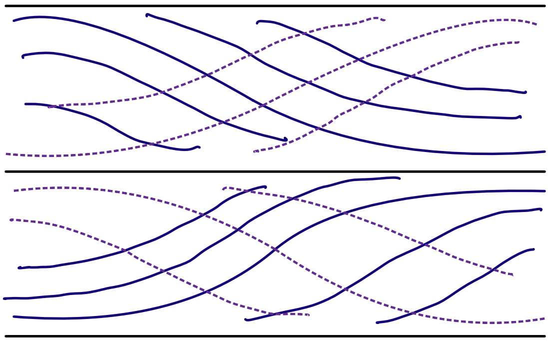

Figure 12. The intersection of the transverse tori with the weak stable and weak unstable of the geodesic flow.

will continue to be minimal and their intersection foliation $\mathcal{G}$ will be such that for every leaf $L \in \widetilde{\mathcal{F}}_i$ the foliation $\mathcal{G}_L$ will not have Hausdorff leaf space. In this paper we will not use this construction, but we thought it could be relevant to spend some time explaining the example from our point of view which was relevant for us to formulate the right statements in the next two sections. We plan to study this example as well as other examples of transverse foliations with Reeb surfaces in a future work.

7.1. ***The construction and its possible variants.*** Let S be a closed surface with a hyperbolic metric and let $\varphi_t : T^1S \to T^1S$ be the geodesic flow. The metric induces an action of $\Gamma = \pi_1(S)$ on $\mathbb{H}^2$ and on $T^1\mathbb{H}^2 = \widehat{M}$ such that the weak stable foliation of φ_t is the foliation $\mathcal{F}_{ws}$ defined in Section 2 by identifying $T^1\mathbb{H}^2$ with $\mathbb{H}^2 \times \partial\mathbb{H}^2$. The weak unstable foliation will be called $\mathcal{F}_{wu}$ and is obtained in a similar process, only that the identification of $T^1\mathbb{H}^2$ with $\mathbb{H}^2 \times \partial\mathbb{H}^2$ is made by identifying the point at infinity as the limit of the geodesic in the backwards direction. Both $\mathcal{F}_{ws}$ and $\mathcal{F}_{wu}$ are transverse to the fibers.

Fix any simple closed geodesic ℓ in $M = T^1S$ for the metric g and let T_ℓ be the torus obtained by looking at the unit vectors tangent at points of ℓ. This torus intersects both $\mathcal{F}_{ws}$ and $\mathcal{F}_{wu}$ transversally and the flow is transverse to T_ℓ except at the two orbits of φ_t contained in T_ℓ. See Figure 12.

In [35] a construction of two transverse foliations $\mathcal{S}$, $\mathcal{U}$ of $T_\ell \times [0, 1]$ are given intersecting the boundary tori with the following properties:

- Each leaf $S \in \mathcal{S}$ is either a cylinder (i.e., homeomorphic to $S^1 \times [0, 1]$) or a band (i.e., homeomorphic to $\mathbb{R} \times [0, 1]$), and the intersection of S with each boundary torus $T_\ell \times \{0\}$ and $T_\ell \times \{1\}$ are the same (in the trivialization $T_\ell \times [0, 1]$) and coincides with the intersection of some leaf of $\mathcal{F}_{ws}$ with T_ℓ.

- Each leaf $U \in \mathcal{U}$ is either a cylinder or a band, and the intersection with each boundary torus $T_\ell \times \{0\}$ and $T_\ell \times \{1\}$ is the same and coincides with the intersection of some leaf of $\mathcal{F}_{wu}$ with T_ℓ.
- There are exactly two cylinder leaves S_1, S_2 of $\mathcal{S}$ and U_1, U_2 of $\mathcal{U}$ corresponding to the periodic orbits of the flow in T_ℓ.
- Each leaf $S \in \mathcal{S} \setminus \{S_1, S_2\}$ intersects either U_1 or U_2 in two connected components that are infinite lines and bound a Reeb band.
- Each leaf $U \in \mathcal{S} \setminus \{U_1, U_2\}$ intersects either S_1 or S_2 in two connected components that are infinite lines and bound a Reeb band.
- S_1 intersects U_1 in at least three circles,
- S_2 intersects U_2 in at least three circles and U_1 in at least four circles.

We depict one possibility in Figures 13 and 14.

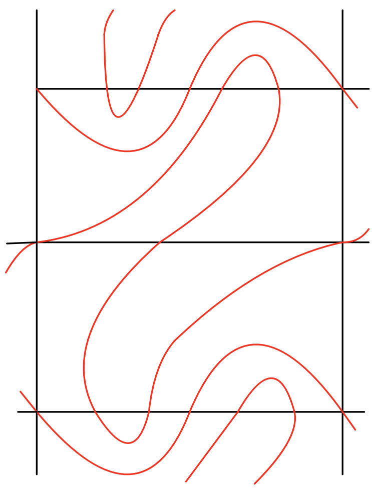

Figure 13. The horizontal lines represent the cylinders U_1 and U_2 (and their translations up by deck transformations) and the red curves represent the cylinders S_1 and S_2 (and their translations up by deck transformations). The figure depicts how the leaves intersect and each intersection corresponds to a circle in the leaf. The rest of the leaves of the $\mathcal{U}$ foliation are also horizontal leaves, but when going around the holonomy of the compact leaves, they intersect in a different height (in particular, the corresponding leaf is an infinite band whose intersection with the torus is an infinite family of horizontal segments which accumulate in U_1 and U_2 when going forward and backward respectively). The other leaves of the $\mathcal{S}$ foliation interpolate the traces of S_1 and S_2 and also change as going around the torus in the flow direction.

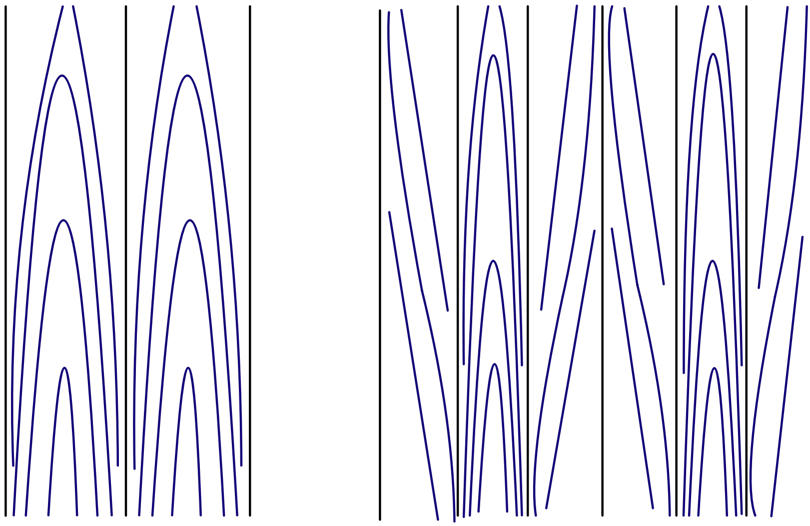

Figure 14. Left: the lift of one of the cylinders (U_2) and the intersected foliation lifted to this cover. Right: the same for the other cylinder (U_1). These intersections correspond to intersecting the leaves as one moves in the direction of the flow (which makes leaves of the $\mathcal{S}$ foliation approach in the forward or backward direction to the leaves S_1 and S_2). The straight lines corresponds to the intersection with the leaves S_1 and S_2 which are invariant under moving one step up.

We now consider the foliations $\mathcal{F}_1, \mathcal{F}_2$ in M_0 obtained by first cutting T^1S foliated with $\mathcal{F}_{ws}$ and $\mathcal{F}_{wu}$ along T_γ and gluing a copy of $T_\ell \times [0, 1]$ foliated by $\mathcal{S}$ and $\mathcal{U}$ and gluing the foliation $\mathcal{F}_{ws}$ with $\mathcal{S}$ and $\mathcal{F}_{wu}$ with $\mathcal{U}$. The manifold M_0 is still diffeomorphic to T^1S (we have changed T_γ for $T_\gamma \times [0, 1]$ with the trivial identification of $T_\gamma \times \{0\}$ and $T_\gamma \times \{1\}$ with the two copies of T_γ one gets after "cutting" T^1S) and the foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are everywhere transverse. We note that it is possible to choose the foliations $\mathcal{S}$ and $\mathcal{U}$ in order that the resulting foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ are *not* uniformly equivalent, that is, the homeomorphisms h_1 and h_2 given by Theorem 2.1 sending $\mathcal{F}_1$ and $\mathcal{F}_2$ to $\mathcal{F}_{ws}$ cannot be homotopic to each other (see Figures 15 and 16). The example in [35] provides examples for which both foliations are uniformly equivalent.

Remark 7.1. One can glue more than one block $T_\ell \times [0, 1]$, concatenating consecutive blocks. It is also possible to do the cut and paste process in several disjoint simple closed geodesic to obtain variants of these examples. More generally, take any Anosov flow in a closed 3-manifold with an embedded Birkhoff torus with only two Birkhoff annuli (that is, when the intersection of the weak stable and weak unstable foliations of the Anosov flow intersect the tori exactly as in this

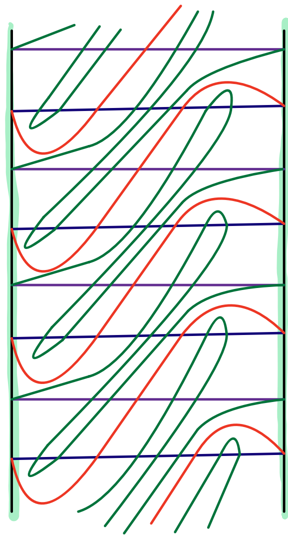

Figure 15. How the cylinders of each foliation can intersect so as to yield a pair of non-uniformly equivalent foliations intersecting transversally.

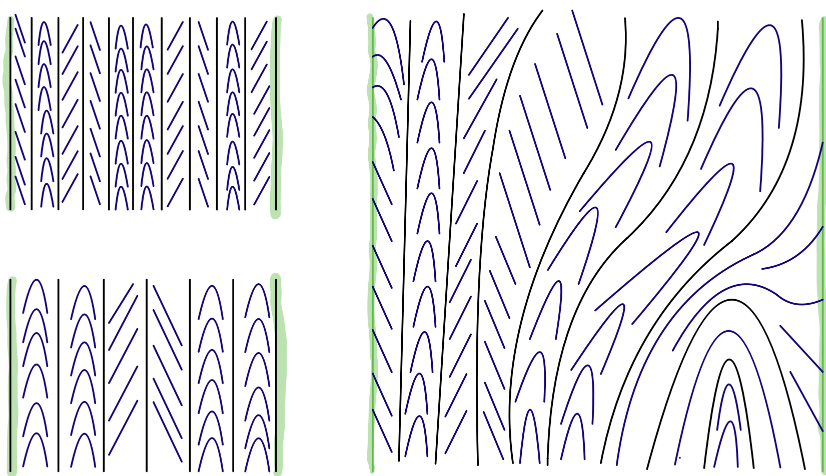

Figure 16. Left: A diagram of the foliations in the cylinder leaves when crossing the tori. Right: the foliation interpolating both foliations of a leaf that accumulates in both cylinders in the different directions.

case). Then, one can glue some of the foliations of $\mathbb{T}^2 \times [0, 1]$ and produce similar examples. We leave to a forthcoming work the (similar) analysis of this case. One can also do this with embedded Birkhoff tori with more Birkhoff annuli, but then the building blocks have to be accordingly adjusted: one needs at least the same number of Birkhoff annuli, which can be achieved by finite covers of $\mathbb{T}^2 \times [0, 1]$. This will also be pursued in the future.

7.2. ***Properties of the intersected foliation.*** From the point of view of this paper, it is relevant for us to understand the properties of the foliation $\mathcal{G}$ obtained by intersecting $\mathcal{F}_1$ and $\mathcal{F}_2$. The main observations are the following:

- Given $L \in \widetilde{\mathcal{F}}_1$ we have that the foliation $\mathcal{G}_L$ coincides with the foliation by geodesics as long as L does not intersect the lift to the universal cover of $T_\ell \times [0, 1]$. This lift has the property that it intersects every leaf in some regions which are bounded by two curves at bounded distance and landing at given points of $\partial\mathbb{H}^2$ independent of the leaf L.
- Inside each of the regions of intersection of the lift of $T_\ell \times [0, 1]$ to $\widetilde{M}$ the foliation $\mathcal{G}_L$ has some design that depends on the specific foliation (see for instance Figure 14 for a depiction of the leaves in some specific examples). However, it is always the case that in these regions there is a pair of nonseparated leaves at bounded distance of each other and landing at the two points at infinity of the region. Every leaf that enters the region must land in one of the endpoints of the region. In particular each ray of a leaf that enters this region does not leave this region.
- Given two nonseparated leaves $c_1, c_2 \in \mathcal{G}_L$ they must accumulate in one of the regions mentioned in the previous point. Note that if the nonseparated leaves are not contained in the region modified between the two tori, the nonseparated leaves may be at unbounded distance inside their leaf.

These properties will become more apparent in Section 9 which actually shows that behavior like this is the only possible.

8. A dichotomy Hausdorff vs Reeb surfaces

Let $\mathcal{F}_1, \mathcal{F}_2$ be two transverse minimal foliations on $M = T^1S$ and let $\mathcal{G}$ be the foliation obtained by intersecting them.

We define a *Reeb surface* to be a compact surface with boundary R satisfying the following properties:

- R is contained in a leaf S of $\mathcal{F}_i$ (with $i = 1, 2$),
- the boundary components of R are closed leaves of $\mathcal{G}$,
- if L is a lift of S to the universal cover, then, the lift of R to L is a surface B whose boundary consists of exactly two leaves c_1, c_2 of $\widetilde{\mathcal{G}}$ at bounded distance which are nonseparated in $\mathcal{G}_L$.

In particular, it is easy to see that the surface must be either an annulus or a Möbius band.

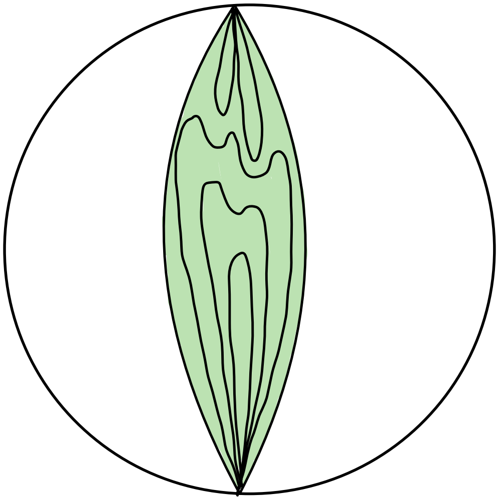

Figure 17. The interior of a non-Hausdorff bigon may not be trivially foliated if it is not γ-invariant for some $\gamma \in \pi_1(M)$.

We will define for a leaf $L \in \widetilde{\mathcal{F}}_i$ a *non-Hausdorff bigon* to be a pair of nonseparated leaves $c_1, c_2 \in \mathcal{G}_L$ such that $\partial^+\Phi(c_1) \neq \partial^-\Phi(c_1)$ and $\partial^\pm\Phi(c_1) = \partial^\pm\Phi(c_2)$. It does not follow that the leaf space of $\mathcal{G}_L$ inside the bigon is the reals (see Figure 17). For example one can have bubble leaves c, c' in the bigon so that the disks $D_c, D_{c'}$ are disjoint. On the other hand it is easy to see that if the boundaries of the bigon are invariant by a nontrivial deck transformation, then the leaf space of $\mathcal{G}_L$ inside the bigon is the reals. In particular, the existence of a Reeb surface is equivalent to having a non-Hausdorff bigon invariant under some deck transformation $\gamma \neq \mathrm{id}$.

Here we show:

Theorem 8.1. *Either there is a leaf $L \in \widetilde{\mathcal{F}}_1$ such that the leaf space of $\mathcal{G}_L$ is Hausdorff, or there exists a Reeb surface.*

Theorem 8.1 together with Theorem 6.1 is enough to conclude the proof of Theorem A. The purpose of this section is to prove Theorem 8.1.

8.1. *Some analysis on nonseparated leaves.* We first find a useful criterion to obtain Reeb surfaces:

Proposition 8.2. *Let $L \in \widetilde{\mathcal{F}}_i$ containing two nonseparated leaves $c_1, c_2 \in \mathcal{G}_L$ such that there is $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ which fixes L (i.e., $\gamma L = L$). Assume that the nonseparated rays r_1, r_2 of c_1, c_2* (cf. Section 3.4) *satisfy $\partial\Phi(r_1) = \partial\Phi(r_2) = \xi$ with $\gamma\xi = \xi$. Then, $\mathcal{F}_i$ has a Reeb surface in the projection of L to M.*

Note in particular that if $\xi = \alpha(L)$ then the hypothesis that $\gamma\xi = \xi$ holds automatically.

Proof. Let $\ell \subset L$ be a γ-invariant curve joining ξ with the other fixed point of γ in $S^1(L)$, without loss of generality we can assume that ℓ intersects r_1 and r_2. The small visual measure property (Proposition 5.2) implies that there is a_0 such that every ray r of $\mathcal{G}_L$ which lands in ξ and intersects ℓ satisfies the condition that the a_0-neighborhood of r contains a ray of ℓ. This applies to r_1, r_2 but also their iterates $\gamma^k r_i$ for $i = 1, 2$ and $k \in \mathbb{Z}$. Since $\gamma^j r_1$ is nonseparated with $\gamma^j r_2$ it follows that in a given fundamental domain, one can have at most finitely many such rays. This implies that there is $k \in \mathbb{Z} \setminus \{0\}$ such that $\gamma^k r_i \cap r_i \neq \varnothing$ and thus $\gamma^k c_i = c_i$ for $i = 1, 2$.

This implies that c_1 and c_2 join ξ with the other fixed point of γ and project to a (simple) closed curve in M. If γ preserves orientation, it follows that both c_1 and c_2 are γ-invariant, else, it can be that γ permutes them and γ^2 preserves them. In both cases we deduce that they bound a Reeb surface. □

8.2. *Proof of Theorem 8.1.* The following proposition will be proved in the next subsection.

Proposition 8.3. *Let $L \in \widetilde{\mathcal{F}}_i$ and $c_1, c_2 \in \mathcal{G}_L$ be two nonseparated leaves such that their nonseparated rays r_1, r_2 satisfy $\partial\Phi(r_1) = \partial\Phi(r_2) = \xi \neq \alpha(L)$. Then, there is $L' \in \widetilde{\mathcal{F}}_i$ and a deck transformation $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ such that*

- *$\gamma\xi = \xi$ and $\gamma L' = L'$, and*
- *there is a γ-invariant non-Hausdorff bigon in L' (which has ξ as one of its endpoints).*

We explain how Theorem 8.1, and thus Theorem A, follow from this proposition. As shown in Proposition 3.6, if some leaf $L \in \widetilde{\mathcal{F}}_i$ has the property that the leaf space of $\mathcal{G}_L$ is non-Hausdorff then the same holds for every leaf of both foliations. By Proposition 8.2, if there is a leaf $L \in \widetilde{\mathcal{F}}_i$ with nontrivial stabilizer for which the nonseparated rays of a pair of non separated leaves land in $\alpha(L)$ then we can conclude the existence of a Reeb surface. So, we can assume that there is a leaf $L \in \mathcal{F}_1$ which has nonseparated leaves c_1, c_2 such that their nonseparated rays r_1, r_2 satisfy $\partial\Phi(r_1) = \partial\Phi(r_2) \neq \alpha(L)$. Thus, we can apply Proposition 8.3 to deduce the existence of a γ-invariant non-Hausdorff bigon in some leaf $L' \in \mathcal{F}_1$ which implies the existence of a Reeb surface by Proposition 8.2. This completes the proof of Theorem 8.1.

For the purposes of the study we will make in Section 9, it is useful to obtain the following additional information:

Addendum 8.4. *Under the assumptions of Proposition 8.3 one can choose L' so that the rays r_1, r_2 are asymptotic to L' in $\widetilde{M}$.*

Here, by "asymptotic" to L' in M we mean that if we parametrize r_i in M by arc length as $r_i(t)$, then, as t goes to infinity, $d(r_i(t), L')$ converges to zero. In other words the projection $\pi(r_i)$ of r_i to M is asymptotic to a closed curve of $\mathcal{G}$ in $\pi(L')$.

8.3. ***Proof of Proposition 8.3.*** The proof of this proposition has two main steps: we first analyze some impossible configurations of returns of the curves r_i when projected to M and then we use the small visual measure to determine that the rays $\pi(r_1)$ and $\pi(r_2)$ need to accumulate as desired.

Let r_1, r_2 be nonseparated rays of $\mathcal{G}_L$ converging to $\xi \in S^1(L)$. Let c_i be the leaves of $\mathcal{G}_L$ containing them. Let $\sigma : [0, 1] \to L$ be a properly embedded segment satisfying the following conditions:

(1) $\sigma(0) \in r_1$, $\sigma(1) \in r_2$.

(2) σ intersects $c_1 \cup c_2$ only in the endpoints and σ transverse to $\mathcal{G}_L$ except at one interior point.

(3) There is a small transverse arc β to $\mathcal{G}_L$ starting in $\alpha(0)$ and going in the direction of c_2 so that if a leaf of $\mathcal{G}_L$ intersects the interior of α then it intersects β.

Let $\mathcal{C} = r_1 \cup r_2 \cup \sigma([0, 1])$ which is a Jordan curve and let $\mathcal{B} = \mathcal{B}(r_1, r_2, \sigma)$ be the closure of the connected component of the complement of $\mathcal{C}$ which accumulates only on ξ in $S^1(L)$. We say that $\mathcal{B}$ is a *good half-band* if $\mathcal{B}$ is the closure of the union of the arcs of leaves of $\mathcal{G}_L$ contained in $\mathcal{B}$ and joining points of $\sigma = \sigma([0, 1])$. Equivalently, we say that $\mathcal{B}$ is a good half-band if the boundary contains two nonseparated rays and it does not contain any complete leaf in its interior.

Lemma 8.5. *Let $c_1', c_2' \in \mathcal{G}_L$ be two nonseparated leaves in $L \in \widetilde{\mathcal{F}}_i$ such that their nonseparated rays r_1', r_2' satisfy $\partial\Phi(r_1') = \partial\Phi(r_2') = \xi$. Then, there exist nonseparated leaves $c_1, c_2 \in \mathcal{G}_L$ with nonseparated rays r_1, r_2 satisfying $\partial\Phi(r_1) = \partial\Phi(r_2) = \xi$ and moreover it bounds a good half-band.*

Proof. Applying the consequence of the small visual measure property given in Proposition 5.2 we get that for any geodesic ray ℓ_0 landing in $\partial\Phi(r_1') = \partial\Phi(r_2') = \xi$ one has that ℓ_0 has a subray contained in the $a_0 + a_1$-neighborhood of every ray β of $\mathcal{G}_L$ that lands in ξ: this is because there is a geodesic ray with starting point the starting point of β, ideal point ξ and contained in the a_0 neighborhood of β. Then by Gromov hyperbolicity, there is a global constant $a_1 > 0$ such that any two geodesic rays with ideal point ξ, have subrays which are $< a_1$ Hausdorff distant from each other in L.

A priori there could be some leaves of $\mathcal{G}_L$ nonseparated from c_1' and c_2' and between c_1', c_2' (i.e., if we pick a curve σ' as above and define $\mathcal{B}' = \mathcal{B}(c_1', c_2', \sigma')$, there may be curves nonseparated from c_1' and c_2' which have both endpoints in ξ and intersect $\mathcal{B}'$ – and hence contained in $\mathcal{B}'$). But since all of these keep intersecting the $a_0 + a_1$ neighborhood of ℓ_0, it follows that there are only finitely many of them.

Therefore up to replacing c_2' if necessary by the leaf nonseparated from c_1', c_2' and closest to c_1' we assume there are no nonseparated leaves from c_1', c_2' and between them. This way, up to replacing the curves and choosing the rays conveniently one can produce σ so that it produces a good half-band as above. □

We can then start the analysis with a good half-band $\mathcal{B}_L = \mathcal{B}(r_1, r_2, \sigma)$. Denote by $E = E(r_1, r_2)$ the leaf of $\widetilde{\mathcal{F}}_2$ such that $r_1 \cup r_2 \subset L \cap E$. Choosing α in a similar way as σ but contained in E and such that the curve β in the last condition is small, this implies that every curve of $\mathcal{G}$ intersecting the interior of the band intersects $\mathcal{B}_L$ in a compact set. It also implies that every leaf of $\widetilde{\mathcal{F}}_2$ that intersects the interior of $\mathcal{B}_L$ must intersect a transversal to E, in particular, E cannot intersect the interior of $\mathcal{B}_L$, unless $\gamma E = E$, in which case we get the conclusion of Proposition 8.3.

We consider an arc τ in E joining the endpoints of σ. Note that there is a uniform a_1 such that we can choose τ to have length less than a_1 because leaves of $\widetilde{\mathcal{F}}_2$ are uniformly properly embedded.[9] The r_i are contained in leaves c_i of $\widetilde{\mathcal{G}}$. We also choose τ to that it does not intersect $c_1 \cup c_2$ in its interior. We denote by $\mathcal{B}_E = \widehat{\mathcal{B}}(r_1, r_2, \tau)$ the half-band in E whose boundary in the compactification of E is $r_1 \cup r_2 \cup \tau \cup \xi$. Note that this may not be a "good half-band" as we defined before, because there is no reason for r_1 and r_2 to be nonseparated in $\mathcal{G}_E$.

Lemma 8.6. *There is an embedded disk D in $\widetilde{M}$ whose boundary is $\sigma \cup \tau$ intersecting $\mathcal{B}_L \cup \mathcal{B}_E$ only in the boundary.*

Proof. The bands $\mathcal{B}_L$ and $\mathcal{B}_E$ only intersect at their boundaries $r_1 \cup r_2$, and thus $\mathcal{B}_L \cup \mathcal{B}_E$ is a properly (and tamely) embedded copy of $S^1 \times [0, \infty)$ (in fact, it is piecewise C^1). Taking the one point compactification of $\widetilde{M}$ to S^3 we get that $\hat{D} = \mathcal{B}_L \cup \mathcal{B}_E \cup \{\infty\}$ is a tamely embedded disk whose boundary is $\tau \cup \sigma$ and therefore, there is a homeomorphism of S^3 sending $\hat{D}$ to the standard disk in a hemisphere (see [36, § 17]). Thus one can homotope the disk $\hat{D}$ rel $\partial \hat{D}$ away from ∞ and so that it does not intersects $\hat{D}$ in the interior and get an embedded disk D in $\widetilde{M}$ with the desired properties. □

Let $\mathcal{V} := \mathcal{V}(r_1, r_2, \sigma, \tau)$ be the region defined in $\widetilde{M}$ bounded by $D \cup \mathcal{B}_L \cup \mathcal{B}_E$ such that in each leaf inside, it limits only on ξ. This is a topological ball.[10] We call such a region a *good half-region*.

We analyze a good half-region $\mathcal{V} = \mathcal{V}(r_1, r_2, \sigma, \tau)$ with boundary $D \cup \mathcal{B}_L \cup \mathcal{B}_E$ where $\mathcal{B}_L$ is a good half-band in $L \in \widetilde{\mathcal{F}}_1$ and $\mathcal{B}_E$ is a half-band in E. Let $p_i \in \sigma \cap \tau \cap r_i$ the corner points of D and small foliation boxes B_1^i and B_2^i of $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ respectively

[9] This follows from Theorem 2.1 in our case, but is also a general fact for $\mathbb{R}$-covered foliations; see, e.g., [16, Lemma 4.48].

[10] Again, this is a consequence of Schoenflies theorem in dimension 3, see [36, § 17]. Note that the surfaces we are constructing are all tamely embedded by construction (they are piecewise leaves of foliations, and D can be chosen smooth).

around p_i (for $i = 1, 2$). We consider I_1^i, I_2^i small nondegenerate closed intervals in the leaf spaces of $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_2$ consisting on the leaves intersecting B_1^i and B_2^i respectively. We can assume that the boxes are small enough that (for $i = 1, 2$ and $j = 1, 2$),

- every leaf of $\widetilde{\mathcal{F}}_1$ through B_1^i intersects E,
- every leaf of $\widetilde{\mathcal{F}}_2$ through B_2^i intersects L, and
- the set $\widehat{D}_\varepsilon(L, I_1^i)$ contains[11] $\mathcal{B}_L$ (see Proposition 2.7).

By slightly changing B_1^i and B_2^i we can and will assume that $I_1^1 = I_1^2 = I_1$ and $I_2^1 = I_2^2 = I_2$. These conditions on the sets B_i^j and the intervals I_1, I_2 will be assumed in what follows. We can also consider I_1, I_2 small enough that they are disjoint from their images by the deck transformation associated to the center of $\pi_1(M)$ (i.e., they correspond to an interval in the leaf space $\hat{\mathcal{L}}$ in $\widehat{M}$; see Corollary 2.2). This implies that every $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ can have at most two fixed points in I_i and if it has two, one must be attracting and one repelling.

We are going to show that there are some restrictions on how $\mathcal{V}$ can intersect with its translates under deck transformations. See Figure 18.

Lemma 8.7. *Assume that $\gamma \in \pi_1(M)$ is such that*

- *there is a point $z_1 \in r_1$ such that $\gamma z_1 \in B_1^1 \cap B_2^1$,*
- *there is a point $z_2 \in r_2$ such that the distance in L from z_1 to z_2 is less than $2a_0 + 1$ and such that $\gamma z_2 \in B_1^2 \cap B_2^2$,*
- $\gamma(I_1) \subset \mathrm{Int}(I_1)$, *and*
- $I_2 \subset \mathrm{Int}(\gamma(I_2))$.

Then, γL cannot intersect the interior of $\mathcal{V}$, unless $\gamma E = E$, in which case we achieve the conclusion of Proposition 8.3.

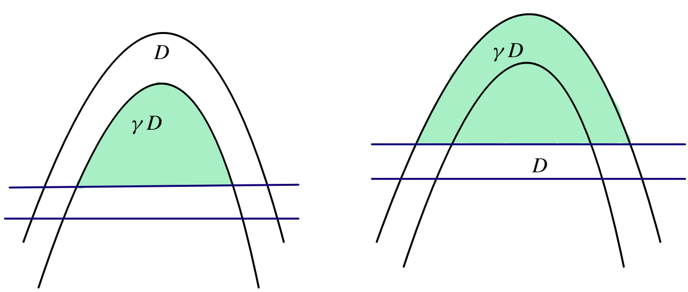

Figure 18. Forbidden returns of a good half-region according to Lemma 8.7.

[11] This uses the fact that $\xi \neq \alpha(L)$.

Proof. We assume for a contradiction that γL intersects the interior of $\mathcal{V}$. From how we chose the intervals it follows that there is a (unique) fixed point of γ in I_1 (which is attracting) and one in I_2 (which is repelling). Let L' be the fixed point by γ in I_1 and similarly E' the fixed point of γ in I_2. Note that since γ is attracting on I_1 then either $L' = L$ or L' belongs to the same connected component of $I_1 \setminus \{L\}$ as γL and since γ is expanding on I_2, either $E' = E$ or E' belongs to the connected component of $I_2 \setminus \{E\}$ not containing γE (recall that the intervals where chosen small enough that the action of γ has a unique fixed point in I_1 and I_2). In particular, both E' and L' intersect B_i^j for $i = 1, 2$; $j = 1, 2$ and using Corollary 2.2 we know that they intersect each such box in a unique plaque.

Since we have assumed that γL intersects the interior of $\mathcal{V}$ then $\gamma L \neq L$, and so by our choice of I_1, it follows that L' also intersects the interior of $\mathcal{V}$, since it belongs to the connected component of $I_1 \setminus L$ containing γL (the set of leaves of $\widetilde{\mathcal{F}}_1$ which intersect the interior of $\mathcal{V}$). Since $L' \in I_1$ and $\mathcal{B}_L \subset \widehat{D}_\varepsilon(L, I_1)$ we have that L and L' are asymptotic in $\mathcal{B}_L$ (i.e., $d(L \cap \mathcal{B}_L, L') < \varepsilon$, meaning every point in $\mathcal{B}_L$ is $< \varepsilon$ from L') and we can apply Proposition 3.4. Denote by $\mathcal{B}_{L'}$ the band in L' obtained by pushing from $\mathcal{B}_L$ to L' in the sense of Proposition 3.4 (i.e., the band $\mathcal{B}_{L'}$ is the one bounded by the intersections of E with L' which are close to r_1 and r_2). Note that this implies that $\mathcal{B}_{L'}$ is a good half-band, and its boundary rays that we denote as r_1', r_2' (and are contained in $L' \cap E$) land in ξ and are nonseparated.

Let us first assume that γE intersects $\mathcal{V}$. In this case, since $\gamma z_1 \in B_1^1 \cap B_2^1$ and $\gamma z_2 \in B_1^2 \cap B_2^2$ it follows that $\gamma \mathcal{B}_L$ and $\gamma \mathcal{B}_E$ must intersect $\mathcal{V}$. We obtain that $\mathcal{V}$ projects to a solid torus in $M_\gamma = \widetilde{M}/_{\langle \gamma \rangle}$. Then that $\gamma \mathcal{B}_L$, whose boundaries are γr_1 and γr_2, has a definite width in γL, meaning that the distance from one curve to the other is bounded below by some uniform constant due to the fact that they belong to the same leaf of $\widetilde{\mathcal{F}}_2$. This implies that the band cannot disappear and is completely contained in $\mathcal{V}$. It must thus be asymptotic to an infinite strip strictly inside $\mathcal{B}_{L'}$ which must be invariant under γ and such that the boundaries are curves which limit on ξ and separate r_1' from r_2' contradicting the fact that $\mathcal{B}_{L'}$ is a good half-band.

Suppose now that $\gamma E = E$. The argument in the previous case still applies so the band $\mathcal{B}_L$ is asymptotic to a band in L' which is invariant under γ. This produces a bigon in L' and $\mathcal{B}_L$ is asymptotic to this γ-invariant bigon. This achieves the conclusion of Proposition 8.3 in this case.

Finally we treat the case that γE does not intersect $\mathcal{V}$. This implies that E', the fixed point of γ in I_2 belongs to the connected component of $I_2 \setminus \{E\}$ not containing γE. Thus, E' must intersect $\mathcal{V}$. Since $\mathcal{B}_{L'}$ is a good half-band, each connected component of $E' \cap \mathcal{B}_{L'}$ is a piece of a leaf of $\mathcal{G}_{L'}$ and must be a compact interval by definition of good half-band. The same holds for $E' \cap \mathcal{B}_L$ due to Proposition 3.4. However, since γ fixes both L' and E' and because E' intersects B_1^2 in a unique plaque we get that γ must fix the connected component of the intersection $L' \cap E'$ intersecting B_1^2 and thus projects to a closed curve which lifts to a curve separating r_1' from r_2' we have shown had to be nonseparated. This is a contradiction and completes the proof. $\square$

Now, we will show that it is possible to find deck transformations with the properties required by Lemma 8.7.

Consider a geodesic ray ℓ_0 in L that limits in ξ. The small visual measure property implies that there is some $a_0 > 0$ so that for each x in ℓ_0 there is z in r_i with $d_L(x, z) < a_0$ (see Proposition 5.2).

Lemma 8.8. *Let $\mathcal{V} = \mathcal{V}(r_1, r_2, \sigma, \tau)$ be a good half-region and $p_1 = \sigma \cap r_1 \cap \tau$. Consider B_1^1, B_2^1 and I_1, I_2 as above and assume that there exists a sequence of points z_n in r_1, with $d_L(z_n, x_n) < a_0$ where the x_n are in a geodesic ray ℓ_0 as above and $x_n \to \xi$. Assume moreover that there are deck transformations γ_n such that $\gamma_n z_n \in B_1^1 \cap B_2^1$. Then, for large enough n we have that $\gamma_n I_1 \subset \mathrm{Int}(I_1)$ and $\gamma_n^{-1} I_2 \subset \mathrm{Int}(I_2)$.*

Proof. This will use that $\xi \neq \alpha(L)$, but $\xi = \alpha(E)$ by Proposition 3.9. Up to subsequence assume that $\gamma_n z_n$ and $\gamma_n x_n$ converge.

Fix a transversal ζ to $\mathcal{F}_1$ through p_1 intersecting exactly the leaves of $\widetilde{\mathcal{F}}_1$ in I_1. Let L_1, L_2 be the leaves of $\widetilde{\mathcal{F}}_1$ through the endpoints of ζ. Recall that the ideal point of r_1, r_2 and hence that of ℓ_0 is $\xi \neq \alpha(L)$ so it is a contracting direction for $\widetilde{\mathcal{F}}_1$. So for big enough n there is a transversal to $\widetilde{\mathcal{F}}_1$ of arbitrarily short length, through x_n and connecting the leaves L_1, L_2. By assumption $d_L(x_n, z_n)$ is bounded by a_0 independently of n, so using the local product structure of the foliation $\mathcal{F}_1$ we obtain that there is also a transversal β_n to $\widetilde{\mathcal{F}}_1$ through z_n of very small length and connecting L_1 to L_2. Then $\gamma_n \beta_n$ is a transversal of very small length passing through $\gamma_n z_n$ which is very close to p. In particular for n sufficiently $\gamma_n \beta_n$ intersects a set of leaves of $\widetilde{\mathcal{F}}_1$ which is strictly contained in I_1. This implies that $\gamma_n I_1 \subset \mathrm{Int}(I_1)$ for n big enough.

Next consider I_2 and $\widetilde{\mathcal{F}}_2$. We first have to verify that in E the points z_n are a bounded distance from a geodesic ray in E. First recall that Proposition 2.3 proves that there is a constant $Q_0 > 0$ such that for any F a leaf of $\widetilde{\mathcal{F}}_i$ then $\Phi_F^i : F \to \mathbb{H}^2$ is a Q_0 quasi-isometry.

The z_n are a bounded distance from the geodesic ray ℓ_0 in L. By Proposition 2.3 the image $\Phi_L^1(\ell_0)$ is a quasigeodesic and hence the points $\Phi_L^1(z_n)$ are a bounded distance from a geodesic ray in $\mathbb{H}^2$ which we denote by ℓ_1. Lemma 3.1 implies that there is $a_1 > 0$ such that for any point x in $\widetilde{M}$, if $x \in L \in \widetilde{\mathcal{F}}_1$ and $x \in E \in \widetilde{\mathcal{F}}_2$, then $d_{\mathbb{H}^2}(\Phi_L^1(x), \Phi_E^2(x)) < a_1$. Hence $\Phi_E^2(z_n)$ are a bounded distance from ℓ_1. Applying $(\Phi_E^2)^{-1}$ shows that z_n are a bounded distance from $(\Phi_E^2)^{-1}(\ell_1)$. The last curve is a quasigeodesic in E (again Proposition 2.3) and we conclude that z_n are a bounded distance from a geodesic ray in E as claimed.

We know that ℓ_1 converges to $\alpha(E)$, so holonomy of $\widetilde{\mathcal{F}}_2$ along ℓ_1 is expanding. We have chosen B_2^1 sufficiently small that the transversal through z_n intersecting the leaves corresponding to B_2^1 will have length much bigger than length of a transversal through p_1. Then one gets that $\mathrm{Int}(\gamma_n I_2) \supset I_2$, or equivalently $\gamma_n^{-1} I_2 \subset \mathrm{Int}(I_2)$ as claimed. □

Before we show that an accumulation point must be fixed by some deck transformation we will show some general property about geodesic rays in leaves of $\widetilde{\mathcal{F}}_i$.

Lemma 8.9. *Let ℓ be a geodesic ray in some leaf $L \in \widetilde{\mathcal{F}}_i$ with $\partial\Phi(\ell) = \xi \in \partial\mathbb{H}^2$ then one of the following holds*:

(1) *there is $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ and a leaf $L' \in \widetilde{\mathcal{F}}_i$ such that $\gamma L' = L'$ and $\gamma\xi = \xi$ and the projection of ℓ spirals towards the projection of L', or,*

(2) *for every $\varepsilon_1 > 0$ there is a foliated box U in $\widetilde{M}$ of diameter less than ε_1, a sequence of points $y_n \in \ell$ going to infinity and deck transformations $\eta_n \in \pi_1(M)$ for which $\eta_n y_n \in U$ are in different leaves of $\widetilde{\mathcal{F}}_i$ and such that $\eta_n L$ accumulate in infinitely many distinct leaves of $\widetilde{\mathcal{F}}_i$.*

Proof. Up to changing ℓ by a uniformly bounded amount which does not affect the result, one can assume that $\mathcal{F}_i$ is the weak stable foliation of the geodesic flow on $M = T^1S$ for a hyperbolic metric on S (cf. Theorem 2.1). Note that since this is the case, the accumulation set of ℓ when projected to M is the same as the accumulation of the orbit of the geodesic flow from $\alpha(L)$ to ξ inside L traversed in the direction opposite to the flow. In particular it consists of a compact connected set saturated by orbits that we shall denote by Λ.

First, note that unless $L = \gamma L$ and $\gamma\xi = \xi$ for some $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ (in which case Λ is the unique closed geodesic in L and we are in the first option) we have that all returns of the projection of ℓ to M to a foliation box must happen in distinct plaques of $\mathcal{F}_1$. This is the same for any backward orbit of the geodesic flow, if it limits in a point x and it is not periodic, then it must intersect a transversal to x in infinitely many distinct plaques of $\mathcal{F}_1$ (because $\mathcal{F}_1$ is the weak stable foliation of the geodesic flow by construction).

Next, we assume that there is some $\varepsilon_1 > 0$ such that for every $x \in \Lambda$ the two-dimensional transversal of size ε_1 to the geodesic flow through the point x does not intersect Λ except perhaps at x itself. In this case, we get that Λ is a closed geodesic which belongs to some leaf $\mathcal{F}_1$ and considering some lift L' of this leaf to $\widetilde{M}$ gives us the first option of the lemma.

Finally, assume that Λ is not a periodic orbit. Then, there is $x \in \Lambda$ such that every transversal to the flow through x intersects Λ in infinitely many distinct leaves. For every foliated neighborhood U of x we get that the returns need to have accumulation points every point of $\Lambda \cap U$ which has infinitely many different leaves obtaining the second option of the lemma. □

Using the small visual measure property, one can extend this to rays of $\mathcal{G}_L$ for every L.

Corollary 8.10. *Let r be a ray in some leaf of $\mathcal{G}_L$ for $L \in \widetilde{\mathcal{F}}_i$ with ideal point ξ and let ℓ_0 be a geodesic ray in L with ideal point ξ. Suppose that ℓ_0 does not spiral towards a leaf $L' \in \widetilde{\mathcal{F}}_i$ invariant under some deck transformation. Then, for every $\varepsilon_2 > 0$ there is a sequence of points $x_n \in r$ going to infinity and deck transformations $\eta_n \in \pi_1(M)$ such that for every $n_0 > 0$ there are $n_1 < n_2 < n_3$ such that $\eta_{n_1}x_{n_1}, \eta_{n_2}x_{n_2}, \eta_{n_3}x_{n_3}$ are ε_2 distance apart and $\eta_{n_2}L$ is between $\eta_{n_2}L$ and $\eta_{n_3}L$.*

Proof. Fix a_0 given by the small visual measure so that for every $y \in \ell_0$ there is a point $x \in r$ such that $d(x, y) < a_0$. For $y \in \widetilde{M}$ denote by D_y to the disk of $\widetilde{\mathcal{F}}_i$ of radius a_0 centered in y.

We fix ε_2 and choose ε_1 sufficiently small that if two points z, w are ε_1-close, then, the disks D_z and D_w are at Hausdorff distance less than ε_2. Now, choose k sufficiently large that if one chooses k-points $z_1, \dots, z_k$ in a transversal to $\widetilde{\mathcal{F}}_i$ of length ε_1 and picks one point w_i in each D_{z_i} then it holds that at least 3 of them are ε_2 apart.

Note that the assumption on r implies that ℓ_0 satisfies the second alternative in the previous lemma, implying that for every $\varepsilon_1 > 0$ there is a sequence of points $y_n \in \ell_0$ going to infinity in ℓ_0 and deck transformations η_n such that all points $\eta_n y_n$ belong to an open set U of diameter less than ε_1 and that the leaves $\eta_n L$ have infinitely many accumulation points. In particular, we can assume that $\eta_n y_n$ accumulate in k-points $z_1, \dots, z_k \in U$ belonging to different leaves of $\widetilde{\mathcal{F}}_i$.

Now, given n_0, we can choose $n_1 < n_2 < n_3$ and points y_{n_1}, y_{n_2} and y_{n_3} such that the $\eta_{n_i} y_{n_i}$ are in U, the leaf $\eta_{n_2}L$ is between the leaves $\eta_{n_1}L$ and $\eta_{n_3}L$, and the corresponding points x_{n_i} in r are ε_2 close. This completes the proof. □

The next lemma completes the proof of Proposition 8.3 and Addendum 8.4.

Lemma 8.11. *Let $c_1, c_2 \in \mathcal{G}_L$ be two nonseparated leaves in $L \in \widetilde{\mathcal{F}}_i$ whose non-separated rays r_1, r_2 satisfy $\partial\Phi(r_1) = \partial\Phi(r_2) = \xi \neq \alpha(L)$. Then there exists some $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ and $L' \in \widetilde{\mathcal{F}}_i$ such that $\gamma L' = L'$ and $\gamma\xi = \xi$. The rays r_i, when projected to M, spiral towards the projection of L', and there exists a γ-invariant bigon in L'.*

Proof. Assume without loss of generality (cf. Lemma 8.5) that r_1 and r_2 are boundaries of a good half-band $\mathcal{B}_L$ in L.

Let $\mathcal{V} := \mathcal{V}(r_1, r_2, \sigma, \tau)$ a good half-region with boundary $D \cup \mathcal{B}_L \cup \mathcal{B}_E$ constructed as in Lemma 8.6. We can choose an ordering of the leaf space of $\widetilde{\mathcal{F}}_1$ so that leaves that intersect $\mathcal{V}$ are above L.

By Proposition 8.2 we can assume that r_1 satisfies the assumptions of Corollary 8.10. Orient r_1 and given $x \in r_1$ consider $r_1(x)$ to be the ray starting at x. We can also define $\mathcal{V}_x := \mathcal{V}(r_1(x), r_2, \sigma_x, \tau_x)$ by changing the arcs to intersect in x. Now, using Corollary 8.10 given ε_2 we can find points x, y in r_1 such that y is in $r_1(x)$ and a deck transformation η such that x and ηy and are very close and that ηL is above L. This contradicts Lemma 8.7.

It follows that the point ξ is invariant under some γ and that the ray r_1 converges to a closed a leaf L' of $\widetilde{\mathcal{F}}_1$ invariant under γ.

In addition, the region between r_1 and r_2 in L is asymptotic to a region in L'. This region in L' has ideal point $\xi \neq \alpha(L')$. The rays r_1 and r_2 are thus converging to some limit rays e_1 and e_2 in L'. These rays are nonseparated in L' and they have ideal point ξ such that $\gamma\xi = \xi$ and $\gamma L' = L'$. Proposition 8.2 implies that e_1, e_2 are the boundaries of a γ-invariant bigon. This completes the proof of Lemma 8.11. □

Remark 8.12. Note that in the proof we also obtain that the endpoint ξ is not equal to $\alpha(L')$ for the leaf L' we have found.

9. Consequences of the existence of Reeb surfaces

The next result gives more structure and improves Theorem A. In a certain sense, it says that if the intersected foliation $\mathcal{G}$ is not an Anosov foliation, then the foliations should look very much like the ones studied in Section 7.

Theorem 9.1. *Assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ have a Reeb surface and let B be its lift to $L \in \widetilde{\mathcal{F}}_1$ (a non-Hausdorff bigon). Denote by ξ, η its limit points in $\partial\mathbb{H}^2$. Then, for every $L' \in \widetilde{\mathcal{F}}_i$ there is a pair of nonseparated leaves $c_1, c_2 \in \mathcal{G}_{L'}$ such that $\partial^{\pm}\Phi(c_i) = \{\xi, \eta\}$ for $i = 1, 2$.*

To prove this, we first provide some stability properties of non-Hausdorff bigons in nearby leaves. Let B be a non-Hausdorff bigon in a leaf $L \in \widetilde{\mathcal{F}}_i$ with boundaries $c_1, c_2 \in \mathcal{G}_L$, we denote by ξ_B^+ and ξ_B^- the points in $\partial\mathbb{H}^2$ given by $\partial^{\pm}\Phi(c_i) = \{\xi_B^+, \xi_B^-\}$ for $i = 1, 2$ and such that $\{\xi_B^+\}$ is the ideal point of the rays of c_1, c_2 which are nonseparated from each other. See Figure 19.

9.1. ***Persistence and extension to the closure.*** We next consider a Reeb surface lifting to a non-Hausdorff bigon B in some leaf $L \in \widetilde{\mathcal{F}}_i$ which is γ-invariant, and let $I_\gamma^{\pm}$ be the connected components of $\partial\mathbb{H}^2 \setminus \{\gamma^{\pm}\}$ where $\gamma^{\pm}$ are the fixed points of γ. We will show that for one of the intervals $I_\gamma^{\pm}$, say I_γ^+, it holds that for every $L' \in \widetilde{\mathcal{F}}_i$ such that $\alpha(L') \in I_\gamma^+$ the leaf L' has a non-Hausdorff bigon B' sharing the same endpoints as B. To do this, we will need to prove several stability properties,

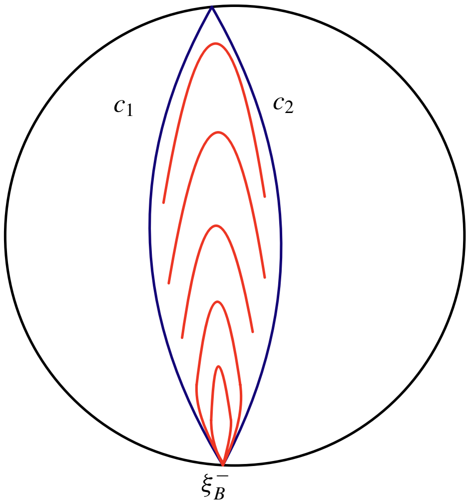

Figure 19. The ideal points of a non-Hausdorff bigon B.

and along the way we will obtain some useful information that will help us to prove Theorem 9.1.

We first show the easy consequence of Proposition 3.4 that bigons persist in nearby leaves when both endpoints are marker points.

Lemma 9.2. *Let B be a non-Hausdorff bigon in a leaf $L \in \widetilde{\mathcal{F}}_i$ with boundaries $c_1, c_2 \in \mathcal{G}_L$ such that $\alpha(L) \notin \{\xi_B^+, \xi_B^-\}$. Then, there is a neighborhood I of L in the leaf space of $\widetilde{\mathcal{F}}_i$ such that for every $L' \in I$ there is a non-Hausdorff bigon B' in L' which shares both endpoints with B.*

Proof. This is just an application of Proposition 3.4 since the non-Hausdorff bigon is contained in $\widehat{D}_\varepsilon(L, I)$ (see Proposition 2.7) for some ε for which Proposition 3.4 applies and some small neighborhood I of L in the leaf space of $\widetilde{\mathcal{F}}_i$. □

Such bigons extend to the boundary γ-invariant leaves:

Lemma 9.3. *Let B be a non-Hausdorff bigon in a leaf $L \in \widetilde{\mathcal{F}}_i$ with boundaries $c_1, c_2 \in \mathcal{G}_L$ such that $\alpha(L) \notin \{\xi_B^+, \xi_B^-\}$. Assume that there is $\gamma \in \pi_1(M)$ whose fixed point at $\partial\mathbb{H}^2$ are exactly ξ_B^+ and ξ_B^-. Then, for every $L' \in \widetilde{\mathcal{F}}_i$ such that $\alpha(L')$ is in the same connected component as $\alpha(L)$ in $\partial\mathbb{H}^2 \setminus \{\xi_B^+, \xi_B^-\}$ the leaf L' has a non-Hausdorff bigon with the same endpoints as B. Moreover, this property passes to the closure of that set of leaves and thus extends to the boundary leaves L_0, L_1 such that $\alpha(L_0) = \xi_B^+$ and $\alpha(L_1) = \xi_B^-$.*

We note that up to composing γ with a power of the deck transformation associated to fibers we can assume that the leaves L_0 and L_1 in the conclusion are γ-invariant.

Proof. We assume that $L \in \widetilde{\mathcal{F}}_1$. The set of leaves for which $\alpha(L')$ belongs to the same connected component of $\alpha(L)$ in $\partial\mathbb{H}^2 \setminus \{\xi_B^+, \xi_B^-\}$ is a countable union of open intervals. For any pair of such intervals, there is power of the deck transformation associated to the fiber (which induces the identity in $\partial\mathbb{H}^2$) which maps one interval to the other, so it is enough to show that one such open interval I_0 (the one containing L) satisfies the desired property. We can also assume that γ fixes I_0.

Let $E \in \widetilde{\mathcal{F}}_2$ be the leaf such that $E \cap L$ contains the boundaries of the non-Hausdorff bigon and consider the set of leaves $I_1 \subset I_0$ consisting of leaves L' such that $L' \cap E$ contains curves that bound a non-Hausdorff bigon joining $\{\xi_B^+, \xi_B^-\}$. Note that since ξ_B^+ is the point where the nonseparation happens, it follows from Proposition 3.9 that $\alpha(E) = \xi_B^+$ and is thus γ-invariant. We claim that this implies that $\gamma E = E$: note that since $\gamma\alpha(E) = \alpha(E)$ then either E is fixed, or it is mapped to a leaf E' obtained by acting on E with a power of the deck transformation associated to the fiber and therefore γ acts freely on the leaf space of $\widetilde{\mathcal{F}}_2$ (recall that we are assuming that γ is the element of $\pi_1(M) \setminus \{\mathrm{id}\}$ which also fixes I_0, which is an interval in the leaf space of $\widetilde{\mathcal{F}}_1$). To show that $\gamma E = E$ we therefore note that $\gamma^n(c_1), \gamma^n(c_2)$ are boundaries in bigons in $\gamma^n(L)$ and since γ fixes I_0 which is a bounded interval in the leaf space of $\widetilde{\mathcal{F}}_1$ the iterates $\gamma^n(L)$ converges to some leaf $L' \in \widetilde{\mathcal{F}}_1$. By the small visual measure property it follows that $\gamma^n(c_1)$ must remain intersecting a compact set in $\widetilde{M}$ and hence $\gamma^n(E)$ which contains $\gamma^n(c_1)$ must also intersect a compact set in $\widetilde{M}$ for all n. This shows that γ cannot act freely on the leaf space of $\widetilde{\mathcal{F}}_2$ and since $\alpha(E)$ is γ-invariant we deduce $\gamma E = E$.

Applying the argument in the previous lemma and using the fact that as long as $\alpha(L') \notin \{\xi_B^+, \xi_B^-\}$ we can push the non-Hausdorff bigon to nearby leaves we deduce that I_1 is open in I_0.

To complete the proof we will show that I_1 is also closed in I_0.

Consider $L_n \to L' \in I_0$ with L_n in an open interval I contained in I_1. The leaf L_n contains a non-Hausdorff bigon B_n bounded by leaves r_1^n, r_2^n of $\mathcal{G}_{L_n}$ contained in L_n but also in E, because pushing preserves the $\widetilde{\mathcal{F}}_2$ leaves they are in. Denote by D_n the region in E bounded by r_1^n and r_2^n. We can assume that the sequence D_n is monotonic. If the region D_n decreases with n (i.e., if $D_{n+1} \subset D_n$) then since D_n must contain a non-Hausdorff bigon of E we get that in the limit the curves r_1^n and r_2^n converge to curves joining ξ_B^+ and ξ_B^-, as we want to show.

We assume then that the sets D_n increase with n and consider the set $D_\infty = \overline{\bigcup_n D_n}$ whose closure in $E \cup S^1(L)$ is compact, connected, and cannot be the whole $E \cup S^1(E)$ (because of the small visual measure property). Let $R_1' = \{\ell_1, \dots, g_k, \dots\}$ be the (possibly finite, but countable) collection of all limits of the curves r_1^n in $\mathcal{G}_E$. Similarly, denote by $R_2' = \{\ell_1^2, \dots, \ell_m^2, \dots\}$ the limits of r_2^n. Since the r_1^n, r_2^n are in $L_n \cap E$, the limits are in $E \cap L'$. Hence they belong to $\mathcal{G}_E$ and $\mathcal{G}_{L'}$. By Proposition 3.9 the limit points of the leaves ℓ_i^j can be either the ones of r_j^n (that is,

$\xi_B^+ = \alpha(E)$ or ξ_B^-), or $\alpha(L')$ if there is more than one limit curve.

The set $R_1' \cup R_2'$ in $E \cup S^1(E)$ bounds the set D_∞ which is $\mathcal{G}_E$-saturated and has nonempty interior since the region between r_1^n and r_2^n must contain a non-Hausdorff bigon which belongs to all the D_n. Since the possible limit points of any leaf in the boundary are only $\xi_B^+, \xi_B^-, \alpha(L')$, It follows that R_1' or R_2' must contain at least one leaf with an ideal point in ξ_B^+ and one with an ideal point in ξ_B^- (they could be the same leaf).

Therefore, if L' does not contain a non-Hausdorff bigon joining ξ_B^+ and ξ_B^- bounded by leaves in $R_1' \cup R_2' \subset L' \cap E$, there must be a region whose limit in $\partial\mathbb{H}^2$ is either exactly $\{\xi_B^+, \xi_B^-, \alpha(L')\}$; or two regions, one with ideal points $\{\xi_B^+, \alpha(L')\}$, and one with ideal points $\{\xi_B^-, \alpha(L')\}$. We analyze the first possibility, the second one is similar. Since E is γ-invariant and since γ does not fix $\alpha(L')$, then applying γ or γ^{-1} to a curve with ideal points $\xi_B^-, \alpha(L')$ we obtain a curve with one ideal point ξ_B^- and another in the open interval $(\xi_B^+, \alpha(L'))$ which does not contain ξ_B^-. This gives a contradiction since the curves in $\mathcal{G}_E$ cannot cross.

Note that the boundary leaves bound a non-Hausdorff bigon in L'. Indeed these boundary leaves have ideal points ξ_B^-, ξ_B^+, and Proposition 3.4 implies that the same local picture has to be seen in L' as for L_n for large n. In other words the bigons in L_n push through to L'.

Note that in the case of $L_n \to L_i$ with $i = 0, 1$ we get the same conclusion (even simpler, since we do not have the possibility to have three limit points), only that when $\alpha(L_i) = \alpha(E)$ we cannot ensure that the region bounded by the leaves is a non-Hausdorff bigon. However, since there is no transversal intersecting both boundary leaves, it is easy to see that there must be one non-Hausdorff bigon in between. □

When the endpoints of the bigon contain the nonmarker point of L, stability is harder to establish. There are two cases depending on whether $\xi_B^+ = \alpha(L)$ or $\xi_B^- = \alpha(L)$.

9.2. *Half interval stability: the case where $\alpha(L) = \xi_B^+$.* The goal of this subsection is to give a proof of the following

Proposition 9.4. *Let B be a non-Hausdorff bigon in a leaf $L \in \widetilde{\mathcal{F}}_i$ with boundaries $c_1, c_2 \in \mathcal{G}_L$ such that $\alpha(L) = \xi_B^+$. Assume moreover that for some $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ we have that $\gamma B = B$. Then, there is a half neighborhood I of L in the leaf space of $\widetilde{\mathcal{F}}_i$ (i.e., I is a connected component of $J \setminus \{L\}$ where J is a neighborhood of L in the leaf space) such that for every $L' \in I$ there is a non-Hausdorff bigon B' in L' which shares both endpoints with B.*

Let us assume that $L \in \widetilde{\mathcal{F}}_1$. Since c_1 and c_2 are nonseparated in $\mathcal{G}_L$, we know that there is $E \in \widetilde{\mathcal{F}}_2$ such that $c_1 \cup c_2 \subset E \cap L$.

Let B_E denote the region in E bounded by $c_1 \cup c_2$ which is an infinite band with bounded width, limiting on $\xi_B^\pm$ (cf. Lemma 3.1). This set B_E is not necessarily a non-Hausdorff bigon since c_1 and c_2 could be separated in $\mathcal{G}_E$, but there must exist some non-Hausdorff bigon contained in B_E and limiting in the same points. By assumption $\alpha(L) = \xi_B^+$ and using Proposition 3.9 we have that $\alpha(E) = \xi_B^+ = \alpha(L)$. Up to considering the inverse, we can assume that γ acts as an expansion on ξ_B^+. (Since $\alpha(E) = \alpha(L)$ we have $E = \gamma E$.)

Fix small transversals τ_1 and τ_2 to $\mathcal{G}_E$ in E parametrized in such a way that the leaf $L_t \in \widetilde{\mathcal{F}}_1$ through $\tau_1(t)$ also passes through $\tau_2(t)$ and that $\tau_i(0)$ belongs to c_i. We also assume these are parametrized so that for $t > 0$ we have that $\tau_1(t)$ belongs to B_E.

For $t < 0$ denote by c_1^t and c_2^t the leaves of $\mathcal{G}_E$ through $\tau_1(t)$ and $\tau_2(t)$ respectively, which belong to $L_t \cap E$. Note that these cannot coincide because they are separated by c_1 (and c_2). Note also that the rays of c_1^t and c_2^t in the direction of ξ_B^- must approximate c_1 and c_2 for t small because a neighborhood of ξ_B^- in L is contained in the set where Proposition 3.4 applies.

The leaves c_1^t and c_2^t cannot be connected by a transversal in L_t (since they are both in E). Between them (maybe coinciding with one of them), there is a pair of leaves e_1^t, e_2^t which is nonseparated and such that e_1^t, e_2^t both separate c_1^t from c_2^t (unless they coincide with them). It follows that both have ξ_B^- as an ideal point.

We call $E_t \in \widetilde{\mathcal{F}}_2$ the leaf such that $e_1^t \cup e_2^t \subset L_t \cap E_t$. We will show that the other landing point of e_1^t, e_2^t is in ξ_B^+ and thus there is a non-Hausdorff bigon in L_t with ideal points ξ_B^-, ξ_B^+ as desired. Notice that a priori E_t does not vary continuously with t, even though we will show in the next lemma that E_t is continuous with t when $t = 0$.

The other rays of e_1^t and e_2^t must land in $\alpha(E_t)$ (because the rays we showed limit in ξ_B^- are separated by Proposition 3.4 and thus the other rays are nonseparated; therefore Proposition 3.9 applies). In particular, the curves e_1^t and e_2^t bound a non-Hausdorff bigon B_t in L_t which limits in $\xi_t = \alpha(E_t)$ and in ξ_B^-. Let $L_0 = L$ and $\xi_0 = \alpha(L_0) = \alpha(E)$.

Lemma 9.5. *The function ξ_t of t is continuous at zero. More precisely, for every J neighborhood of ξ_B^+ there is $\delta > 0$ such that for every $t \in (\delta, 0]$ we have that $\xi_t \in J$.*

Proof. To see this fix some small neighborhood U of ξ_0 in $\partial\mathbb{H}^2$. Now choose a transversal η to $\mathcal{G}_L$ starting at a point in c_1 and entering B.

We choose η small enough such that if a leaf ℓ of $\mathcal{G}_{L_0}$ intersects η, then the leaf ℓ is contained in a leaf $E' \in \widetilde{\mathcal{F}}_2$ such that $\alpha(E') \in U$. Denote by ℓ_0 the leaf of $\mathcal{G}_L$ through the endpoint of η which is not in c_1. In particular ℓ_0 is contained in the interior of B. Now choose a neighborhood I of $\alpha(L)$ such that $\ell_0 \cup \eta$ is contained in $\widehat{D}_\varepsilon(L, I)$ (this set is defined as in (2-6), where ε is chosen so that Proposition 3.4

holds). Recall that $L = L_0$. Let ℓ_t be the push through to L_t of the leaf $\ell_0 \subset L$. In other words ℓ_t is the component of $\widetilde{\mathcal{F}}_2(\ell_0) \cap L_t$ which is ε-close to ℓ_0. In the same way η can be pushed to a transversal η_t to $\mathcal{G}_{L_t}$ in L_t starting in c_1^t and ending in ℓ_t. For each t, ℓ_t is contained in between c_1^t, c_2^t in L_t. In addition there is a transversal η_t to $\mathcal{G}_{L_t}$ from c_1^t to ℓ_t. Therefore ℓ_t is in between e_1^t, e_2^t in L_t. In particular this implies that e_1^t is in between c_1^t and ℓ_t in L_t. We have that $c_1^t \subset E$, $e_1^t \subset E_t$ and $\ell_t \subset \widetilde{\mathcal{F}}_2(\ell_0)$, with $\alpha(\widetilde{\mathcal{F}}_2(\ell_0))$ in U. Since $\alpha(E)$ is in U, it now follows that $\alpha(E_t)$ is in U all well, as long as $\alpha(L_t)$ is in I. Since $\xi_t = \alpha(E_t)$ this completes the proof. □

Note that we have also shown that there is a transversal to $\mathcal{G}_{L_t}$ from e_1^t to c_1^t which ε pushes to $L = L_0$ and similarly there is a transversal from e_2^t to c_2^t which also pushes to L. It now makes sense to talk about monotonicity of ξ_t, we can indeed show:

Lemma 9.6. *The point ξ_t varies in a weakly monotonic way, that is, for small $t, t' \in (-\delta, 0]$ we have that if $t' < t$ then $\xi_{t'} \leq \xi_t$ for the orientation of J making ξ_B^+ the maximal point.*

Proof. For small $\delta' > 0$, consider $\eta : (-\delta', 0] \to L$ such that η is transverse to $\mathcal{G}_L$ and such that $\eta(0) \in c_1$ and $\eta(s) \in B$ for all $s \in (-\delta', 0)$. Note that as we have shown in Proposition 3.9, that if δ is small enough, we know that for every $t \in (-\delta, 0)$ we have that e_1^t belongs to the same leaf of $\widetilde{\mathcal{F}}_2$ as $\eta(s)$ for some $s \in (-\delta', 0]$. This identification will be recorded by a function $\rho : (-\delta, 0] \to (-\delta', 0]$ such that $\rho(t) = s$.

If δ is small enough, then the image of η is contained in $\widehat{D}_\varepsilon(L, I_\delta)$ where I_δ is the interval of $\partial\mathbb{H}^2$ made by $\alpha(L_t)$ with $t \in (-\delta, 0]$, so, applying Proposition 3.4 we find transversals $\eta^t : (-\delta', 0] \to L_t$ intersecting the same $\widetilde{\mathcal{F}}_2$ leaves. Denote by $\ell_s \in \mathcal{G}_L$ the leaf of $\mathcal{G}_L$ through the point $\eta(s)$ and by $E'_s \in \widetilde{\mathcal{F}}_2$ the leaf that contains ℓ_s. We note that all ℓ_s with $s \in (-\delta', 0)$ are bubble leaves with endpoint in ξ_B^-.

For the function ρ defined above, we have $E_t = E'_{\rho(t)}$. Consider $e_{t'}^t$ to be the leaf of $\mathcal{G}_{L_t}$ containing $\eta^t(\rho(t'))$. We get that $e_t^t = e_1^t$ by definition.

We can thus restate the claim stating that whenever $t' < t < 0$ one has that $\rho(t') \leq \rho(t)$. We assume by contradiction that this does not hold, that is, for a pair $t' < t < 0$ we have $\rho(t') > \rho(t)$. Then we get that the leaf $e_t^{t'}$ is a bubble leaf with endpoint ξ_B^-. On the other hand $e_t^t = e_1^t$ is a leaf in L_t with one ideal point ξ_B^-. It follows that the leaf $E_t = E'_{\rho(t)}$ must intersect $L_{t'}$ and L_0 in bubble leaves while it intersects L_t in at least $e_1^t \cup e_2^t$. We will show that this is impossible: Denote by c the corresponding bubble intersection of E_t with L. By the remark after the proof of Lemma 9.5, we know that there is a small transversal β to $\mathcal{G}_{L_{t'}}$ in $L_{t'}$ from $e_t^{t'}$ to $c_2^{t'}$ and this pushes to L_0, through $\widetilde{\mathcal{F}}_2(e_t^{t'}) = E_t$. Then E_t intersects L_t near the push through of β. The same happens for e_2^t. Hence the local leaf of E_t passes through e_2^t and also c. In other words there is a small transversal ν_2 to $\mathcal{G}_{E_t}$ in E_t intersecting

$e^{t'}_t$, e^t_2 and c in turn. In the same way there is a small transversal ν_1 to $\mathcal{G}_{E_t}$ in E_t from $e^{t'}_t$ to e^t_1 to c. Consider the closed curve in E_t which is the concatenation of ν_1, a segment in c, ν_2 and a segment in $e^{t'}_t$. This closed curve does not bound a disk in E_t because e^t_1, e^t_2 intersect this curve in a single point. This would show that E_t is not a plane. This contradiction completes the proof. □

Note that Lemma 9.2 applies directly as soon as $\xi_t \neq \alpha(L_t)$, so, we get:

Lemma 9.7. *For every $t \in (-\delta, 0]$ either ξ_s is locally constant near t or $\xi_t = \alpha(L_t)$.*

Finally, we show:

Lemma 9.8. *If $\xi_t = \alpha(L_t)$ for $t \neq 0$ then, one cannot have that L_t is invariant under some $\gamma_t \in \pi_1(M) \setminus \{\mathrm{id}\}$.*

Proof. First notice that if this is the case then also one has that $\gamma_t \xi_t = \xi_t$ since $\gamma_t L_t = L_t$ implies that $\alpha(L_t)$ is γ_t-invariant. On the other hand, by definition, L_t contains a non-Hausdorff bigon B_t whose endpoints are ξ_t and ξ^-_B. Since $t \neq 0$ and $\xi_t = \alpha(L_t)$ then $\xi_t \neq \alpha(L_0)$. But as $\xi^+_B = \alpha(L_0)$, then $\xi_t \neq \xi^+_B$. Also we know that ξ^+_B and ξ^-_B are the fixed points of $\gamma = \gamma_0$ (the deck transformation leaving $L = L_0$ invariant) then $\gamma_t \xi^-_B \neq \xi^-_B$. Let B_t be a bigon in L_t with ideal points ξ_t and ξ^-_B.

Now we argue as in Proposition 8.2: Due to the small visual measure property there is a uniform bound on the number of distinct non-Hausdorff bigons that share an endpoint in a given leaf of $\widetilde{\mathcal{F}}_i$. On the other hand, applying γ^n_t to B_t we obtain infinitely many disjoint non-Hausdorff bigons in L_t sharing one of the endpoints, namely ξ_t. This gives a contradiction and proves the lemma. □

Proof of Proposition 9.4. We show that ξ_t must be constant and equal to ξ^+_B, as desired. Assume first that we have that $\xi_t = \alpha(L_t)$ in an open interval $I \subset (-\delta, 0)$. Since leaves with nontrivial stabilizer are dense, it follows that for some t close to 0 we have that L_t is γ_t invariant for some $\gamma_t \in \pi_1(M) \setminus \{\mathrm{id}\}$ contradicting Lemma 9.8.

Now, assume that for some $t \in (-\delta, 0)$ we have that $\xi_t \neq \alpha(L_t)$. Consider $A = \{s \in (-\delta, 0) : \xi_s = \xi_t\}$. Note that Lemma 9.5 implies ξ_t is continuous at $t = 0$ we know that A avoids a neighborhood of 0. Therefore, if $s_0 < 0$ is the supremum of A, it follows from Lemma 9.7 that $\xi_{s_0} = \alpha(L_{s_0})$. In L_{s_0} we have a bigon B_{s_0} with $\xi^+_{B_{s_0}} = \xi_{s_0}$. Applying Proposition 8.3 to L_s with s in the interior of the interval (so $s \neq s_0$) we deduce that ξ_{s_0} is fixed by some deck transformation $\hat{\gamma}_s \in \pi_1(M)$ and not acting trivially on $\partial \mathbb{H}^2$. Since $\alpha(L_{s_0}) = \xi_{s_0}$ then up to changing $\hat{\gamma}_s \in \pi_1(M)$ by some power of the deck transformation generated by the fiber we get some $\gamma_s \in \pi_1(M) \setminus \{\mathrm{id}\}$ which fixes L_{s_0} again contradicting Lemma 9.8, unless $s_0 = 0$. In other words the interval where ξ_s is constant has an endpoint in 0.

This completes the proof that ξ_t must be constant equal to ξ^+_B for $t \in (-\delta, 0)$ and thus completes the proof of Proposition 9.4. □

Remark 9.9. We have in fact showed the following: If B is a γ-invariant bigon in a leaf $L \in \widetilde{\mathcal{F}}_1$, and $E \in \widetilde{\mathcal{F}}_2$ is the leaf which intersects B in the boundary $c_1, c_2 \in \mathcal{G}_L$, is such that the nonseparated rays of c_1, c_2 in L land *in* $\alpha(L)$, then for every leaf $L' \in \widetilde{\mathcal{F}}_1$ close to L intersecting E *outside* the region B_E bounded by $c_1 \cup c_2$, the leaf L' contains a bigon B' whose endpoints coincide with those of B and its boundaries correspond to the intersection with E.

9.3. ***Half interval stability: the nonseparated side is a marker point.*** This means that $\xi_B^+ \neq \alpha(L)$, but $\xi_B^- = \alpha(L)$. When the endpoint of the bigon is a marker point for L, we can also push it to one side, but the argument is different:

Proposition 9.10. *Let B be a non-Hausdorff bigon in a leaf $L \in \widetilde{\mathcal{F}}_i$ with boundaries $c_1, c_2 \in \mathcal{G}_L$ such that $\alpha(L) = \xi_B^-$. Assume moreover that for some $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ we have that $\gamma B = B$. Then, there is a half neighborhood I of L in the leaf space of $\widetilde{\mathcal{F}}_i$ such that for every $L' \in I$ there is a non-Hausdorff bigon B' in L' which shares both endpoints with B.*

Proof. Assume for concreteness that $L \in \widetilde{\mathcal{F}}_1$. Let E be the leaf in $\widetilde{\mathcal{F}}_2$ such that $c_1 \cup c_2 \subset L \cap E$. Then $\alpha(L) = \xi_B^-$ by assumption and $\alpha(E) = \xi_B^+$ by Proposition 3.9.

Denote by B_E the region in E bounded between c_1 and c_2. Suppose that B' is a non-Hausdorff bigon contained in B_E. We claim that B' must have its nonseparated rays land in ξ_B^+ (and in particular, this shows that c_1 and c_2 cannot be nonseparated in E because they belong to L and $\alpha(L) = \xi_B^-$ and Proposition 3.9 would imply that the nonseparated rays land there). To prove the claim, consider first some E' very close to E in such a way that it intersects B in a curve which is very close to $c_1 \cup c_2$. If C is a non-Hausdorff bigon in B_E limiting in ξ_B^-, first let L'' be the $\widetilde{\mathcal{F}}_1$ leaf containing ∂C, and then can take a leaf L' of $\widetilde{\mathcal{F}}_1$ very close to L'' and in such a way that the intersection of L' with E is a bubble leaf with points very close to E'. It follows that the intersection of L' and E' in the region between B and B_E must be compact, which is a contradiction because it would give a circle leaf in $\mathcal{G}_{L'}$ (and $\mathcal{G}_{E'}$). This proves the claim.

The same argument shows that any bigon in B_E cannot have a boundary leaf in c_1 or c_2. This implies that inside B_E both c_1, c_2 have a neighborhood that does not intersect a bigon. Starting from c_1, consider leaves e of $\mathcal{G}_E$ inside B_E and near c_1. There is a small interval of leaves where γ acts as a contraction or expansion in this interval. Similarly for c_2. Hence there is a maximal interval $[c_1, e_1]$ where γ fixes the endpoints and no other leaf of $\mathcal{G}_E$ in between. Similarly there is a maximal interval $[c_2, e_2]$ with the same properties. The leaves e_1, e_2 are in the same leaf L' of $\widetilde{\mathcal{F}}_1$ and it is invariant under γ. By construction $\alpha(L') = \xi_B^+$. Finally any leaf L'' of $\widetilde{\mathcal{F}}_1$ between L and L' has a bigon $B_{L''}$ which is asymptotic to B in one direction and to a bigon $B_{L'}$ in the other direction.

This provides the interval required by the proposition. □

Remark 9.11. As a consequence of the proof we have the following property: Let B a γ-invariant non-Hausdorff bigon in a leaf $L \in \widetilde{\mathcal{F}}_1$ bounded by curves c_1, c_2 such that $c_1 \cup c_2 \subset L \cap E$ for some $E \in \widetilde{\mathcal{F}}_2$. Let $\{\xi_B^+, \xi_B^-\}$ the endpoints of the bigon, where as above, ξ_B^+ denotes the nonseparated point of the curves c_1, c_2. Then, we have that all non-Hausdorff bigons in E (note that there is at least one since there is no transversal in E from c_1 to c_2) that are contained between c_1 and c_2 have its nonseparated point in ξ_B^+.

In contrast with Remark 9.9 we notice the following:

Remark 9.12. If B is a γ-invariant bigon in a leaf $L \in \widetilde{\mathcal{F}}_1$ and $E \in \widetilde{\mathcal{F}}_2$ is the leaf which intersects B in the boundary $c_1, c_2 \in \mathcal{G}_L$ is such that the nonseparated rays of c_1, c_2 in L land in a point *different from* $\alpha(L)$, then, for every leaf $L' \in \widetilde{\mathcal{F}}_1$ close to L intersecting E *inside* the region B_E bounded by $c_1 \cup c_2$ contains a bigon B' whose endpoints coincide with those of B and its boundaries correspond to the intersection with E.

9.4. ***Putting all stability together.*** Let B be a non-Hausdorff bigon in $L \in \widetilde{\mathcal{F}}_1$ with boundaries $c_1, c_2 \in \mathcal{G}_L$ and endpoints ξ_B^+ and ξ_B^- (recall that ξ_B^+ denotes the endpoint which is the limit of the nonseparated rays of c_1 and c_2). Consider a transversal $\tau : [0, \delta) \to L$ to $\mathcal{G}_L$ such that $\tau(0) \in c_1$, so that $\tau((0, \delta))$ is contained in B. Let then $E_t \in \widetilde{\mathcal{F}}_2$ be the leaf through the point $\tau(t)$. We will denote by $\mathcal{I}_B$ the closed interval in $\partial\mathbb{H}^2$ which is the closure of the connected component of $\partial\mathbb{H}^2 \setminus \{\xi_B^+, \xi_B^-\}$ containing $\alpha(E_t)$, $t > 0$; that is,

$$\mathcal{I}_B = \overline{\mathrm{cc}_{\alpha(E_t)}(\partial\mathbb{H}^2 \setminus \{\xi_B^+, \xi_B^-\})}. \tag{9-1}$$

We note that the definition of $\mathcal{I}_B$ is independent on the choice of τ and $t > 0$ (and also works if $\tau(0) \in c_2$ instead of c_1).

Lemma 9.13. *Let B be a γ-invariant non-Hausdorff bigon in a leaf $L \in \widetilde{\mathcal{F}}_1$ with boundaries $c_1, c_2 \in \mathcal{G}_L$ such that $\alpha(L) = \xi_B^-$ and $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$. Then, for every $L' \in \widetilde{\mathcal{F}}_1$ such that $\alpha(L') \in \mathcal{I}_B$ (see (9-1)) we have that L' contains a non-Hausdorff bigon B' with endpoints ξ_B^-, ξ_B^+. In the same way if $\alpha(L) = \xi_B^+$ we also get that for every leaf $L' \in \widetilde{\mathcal{F}}_1$ such that $\alpha(L') \in \mathcal{I}_B$ we have that L' contains a non-Hausdorff bigon B' with endpoints ξ_B^-, ξ_B^+.*

Proof. Both $\mathcal{F}_1$ and $\mathcal{F}_2$ are transversally orientable. We choose an orientation such that if $\mu : (-\varepsilon, \varepsilon) \to \widetilde{M}$ is a positively oriented path transverse to *both* foliations $\widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2$. Then, if $L_t \in \widetilde{\mathcal{F}}_1$ and $E_t \in \widetilde{\mathcal{F}}_2$ are the leaves through $\mu(t)$ then $\alpha(L_t)$ and $\alpha(E_t)$ move both clockwise. To see that this is possible, recall that when lifted to $\widetilde{M}$ it follows that the transverse orientation induce a direction on which the point $\alpha(L_t)$ varies as L_t varies in the leaf space of $\widetilde{\mathcal{F}}_i$. Note that since both foliations inherit the orientation of the base (because they are horizontal) the transverse orientations match (recall that from Theorem 2.1 we know that there are homeomorphisms inducing the identity in the base which maps the foliations $\mathcal{F}_i$ to $\mathcal{F}_{ws}$), and so we can speak of moving in the clockwise or counterclockwise direction in $\partial\mathbb{H}^2$.

Let E be the $\widetilde{\mathcal{F}}_2$ leaf containing $c_1 \cup c_2$, and let B_E be the region in E bounded by $c_1 \cup c_2$. Let τ be a transversal to $\mathcal{G}_L$ starting at c_1 and entering B.

The following happens: moving E_t (the leaf through $\tau(t)$, so $E_0 = E$) moves $\alpha(E_t)$ in one direction of $\partial\mathbb{H}^2$, starting at ξ_B^+, by definition $\alpha(E_t)$ moves into $\mathcal{I}_B$. Without loss of generality, we can assume that $\alpha(E_t)$ moves clockwise, equivalently the leaves of $\widetilde{\mathcal{F}}_2$ intersecting B are *above* E_0. In other words B is above E_0. Next consider a curve $\eta : (-\delta, \delta) \to E_0$ transverse to $\mathcal{G}_{E_0}$, such that $\eta(0) \in c_1$, and parametrized clockwise; i.e., if L_s is the leaf of $\widetilde{\mathcal{F}}_1$ through the point $\eta(s)$ then $\alpha(L_s)$ moves clockwise as s increases.

Now if $\alpha(L_0) = \xi_B^-$, the previous lemma tells us that the leaves L_s which have bigons bounded by the curves in $E \cap L_s$ are the leaves intersecting the region B_E (see Remark 9.12). We saw above that the region B is above E_0 and therefore B_E is *below* L_0.

Therefore the leaves L_t which have these bigons, satisfy that the point $\alpha(L_t)$ is locally counterclockwise to ξ_B^- and thus in the same connected component as $\alpha(E_s)$ (for small s) in $\partial\mathbb{H}^2 \setminus \{\xi_B^+, \xi_B^-\}$.

Applying Lemma 9.3 we get that the full closed interval between ξ_B^+ and ξ_B^-, has a bigon with ideal points ξ_B^-, ξ_B^+. If on the other hand we assume that $\alpha(E_t)$ moves counterclockwise when t increases, we get that B is *below* E_0, and as above it will follows that B_E is above L_0. One obtains the same result as above.

Finally we consider the case that $\alpha(L) = \xi_B^+$. In this case the important fact to note is that in the proof of Proposition 9.4 we obtain the half neighborhood by moving in the opposite direction (see Remark 9.9), where we explain that the L_s near L_0 which intersect E_0 forming a bigon in L_s intersect E_0 *outside* B_E (as opposed to inside B_E in the previous case). Therefore with the conventions as in the previous case we have the following: if $\alpha(E_t) > \alpha(E_0)$ (for $t > 0$), we produce bigons in leaves L_s above L_0, so $\alpha(L_s) > \alpha(L_0)$. Therefore $\alpha(E_t)$ moves clockwise and $\alpha(L_t)$ moves clockwise. But $\alpha(E_t)$ moving clockwise when t increases means that $\alpha(E_t)$ moves inside $\mathcal{I}_B$ for $t > 0$, and henceforth $\alpha(L_s)$ moves inside $\mathcal{I}_B$ for $t > 0$. Thus we obtain the second statement of the lemma. □

Putting together what we have shown, we can deduce:

Proposition 9.14. *Let $\mathcal{F}_1, \mathcal{F}_2$ be two transverse minimal foliations of $M = T^1S$ and let $\mathcal{G}$ be their intersection. Then if $\mathcal{G}$ is not homeomorphic to the foliation given by the geodesic flow of a hyperbolic metric, it follows that there are finitely many disjoint simple closed curves $s_1, \dots, s_k$ in S such that for every periodic non-Hausdorff bigon B in a leaf $L \in \widetilde{\mathcal{F}}_i$ we have that $\partial\mathcal{I}_B$ (equation (9-1)) corresponds* *to the endpoints of a lift of one of the curves s_j to $\mathbb{H}^2$. In particular:*

- *If two periodic non-Hausdorff bigons share one endpoint, then they must share both endpoints.*
- *The endpoints of two different periodic non-Hausdorff bigons cannot be linked.*
- *Up to deck transformations, there are finitely many periodic non-Hausdorff bigons. Equivalently there are finitely many Reeb surfaces of $\mathcal{G}$ in $\mathcal{F}_i$.*

Proof. Assume that a leaf $L \in \widetilde{\mathcal{F}}_1$ has a γ-periodic non-Hausdorff bigon B. Due to Propositions 9.4 or 9.10 we know that we can push such a bigon to nearby leaves. Then, thanks to Lemma 9.3 we know that every leaf $L' \in \widetilde{\mathcal{F}}_1$ with $\alpha(L') \in \mathcal{I}_B$ has a bigon joining the endpoints ξ_B^+ and ξ_B^- of B.

First, assume that $\mathcal{I}_B$ and $\mathcal{I}_{B'}$ share an endpoint, then, since γ and γ' must fix those endpoints, we deduce that γ' and γ belong to the same cyclic group of $\pi_1(M)$. Thus, we deduce that both endpoints must coincide.

Now assume that there are two distinct non-Hausdorff bigons B, B' such that B is γ-periodic and B' is γ'-periodic and that the ideal points of B, B' are linked.

Again using Propositions 9.4 and 9.10, we can find bigons with same ideal points as B in all leaves L'' in $\widetilde{\mathcal{F}}_1$ in the interval defined by $\alpha(L'')$ in the closure of one complementary component of ξ_B^+, ξ_B^-. The set of such $\alpha(L'')$ produces an interval I_B of $\partial\mathbb{H}$. The same holds for B', with corresponding interval $I_{B'}$. So if the ideal points link, it follows that the interiors of I_B, $I_{B'}$ intersect, and we can find a leaf L' which has both a bigon with same endpoints as B and one which has both endpoints as B'. This is a contradiction since bigons cannot cross (since they are bounded by leaves of $\mathcal{G}_{L'}$ which is a foliation).

Finally, note that since a γ-invariant non-Hausdorff bigon cannot cross with its translates by other deck transformations, each such non-Hausdorff bigon corresponds to a simple closed curve in S. Similarly, distinct periodic non-Hausdorff bigons correspond to disjoint curves, and at most finitely many such disjoint curves can exist in S.

To prove the final property: we may assume periodic bigons B_1, B_2 are associated with same simple closed curve s_j of S, and are both bigons in say $\widetilde{\mathcal{F}}_1$. There is a unique element γ of $\pi_1(M)$ which projects to s_j in S and acts with fixed points in $\widetilde{\mathcal{F}}_i$. This γ has a discrete set of fixed points in the leaf space of $\widetilde{\mathcal{F}}_1$. So we may assume up to fiber translates, that B_1, B_2 are in the same leaf of $\widetilde{\mathcal{F}}_1$. But then there are finitely many such. This proves finiteness of Reeb surfaces in M. □

9.5. *Creating new bigons.* As in the examples of [35], it is possible that some bigons do not have continuations beyond one half-interval of the leaf space. Thus, to show that there are bigons in every leaf, we need to construct new bigons (i.e., which do not come from varying continuously from the original one) in order to

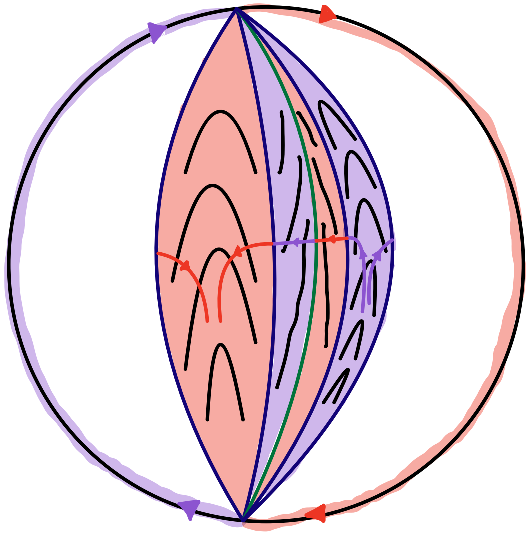

Figure 20. How the nonmarker points move on the other foliation as one considers the leaf associated to a transversal of $\mathcal{G}_L$.

produce bigons in the other half interval. This requires a careful analysis of the orientation of the foliations.

We first show that the existence of some periodic bigon forces landing points of rays to be rather restricted. We state this in somewhat more generality.

Lemma 9.15. *Assume that there is a leaf $L \in \widetilde{\mathcal{F}}_i$ that contains a leaf $e \in \mathcal{G}_L$ such that $\alpha(L) \notin \{\partial^+\Phi(e) \cup \partial^-\Phi(e)\}$. Assume also that $\partial^+\Phi(e) \neq \partial^-\Phi(e)$. Let J be a nontrivial interval in the leaf space of $\widetilde{\mathcal{F}}_i$. Then there is a leaf L' in J fixed by a nontrivial deck transformation β such that no leaf of $\mathcal{G}_{L'}$ has one ideal point fixed by β.*

Proof. Decrease J if necessary so that $\alpha(L')$ never attains either of $\partial^+\Phi(e)$ or $\partial^-\Phi(e)$ for $L' \in J$. In particular, α is injective in J. Let $a_1 = \partial^+\Phi(e),\ a_2 = \partial^-\Phi(e)$.

Let I be an open interval in $\partial\mathbb{H}^2$ contained in the connected component of $\partial\mathbb{H}^2 \setminus \{a_1, a_2\}$ which does not contain $\alpha(L)$.

Choose a nontrivial deck transformation β with one fixed point in I and one in $\alpha(J)$. Let L' be a leaf of $\widetilde{\mathcal{F}}_i$ fixed by β, and we can assume that L' is in J. This leaf satisfies the conclusion. Suppose that $\mathcal{G}_{L'}$ has a leaf c which is not a bubble leaf and has one ideal point fixed by β. Without loss of generality assume that c has an ideal point in I which is fixed by β. Then iterating by β or β^{-1} we eventually obtain a leaf c' of $\mathcal{G}_{L'}$ with one ideal point in I and one ideal point in $\alpha(J)$. But a_1, a_2 link with $I, \alpha(J)$. This contradicts that c' is a leaf of $\mathcal{G}_{L'}$ with ideal points a_1, a_2, which would cause crossing of different leaves of $\mathcal{G}_{L'}$.

If c is a bubble leaf landing in a point fixed by β, one can iterate and produce a leaf which is not a bubble leaf and has the same characteristics. □

Remark 9.16. If the leaf space of $\widetilde{\mathcal{G}}$ is not Hausdorff, it follows from Theorem 8.1 that there are some non-Hausdorff bigons in some leaf, and by Proposition 9.14 that an open set of leaves contains non-Hausdorff bigons. Using minimality, we deduce that every leaf $L \in \widetilde{\mathcal{F}}_i$ has infinitely many non-Hausdorff bigons with distinct endpoints in $\partial\mathbb{H}^2$ and therefore we are in the hypothesis of the previous lemma for every $L \in \widetilde{\mathcal{F}}_i$.

To produce new non-Hausdorff bigons we need to push to the other side of the bigons which is more delicate. We first show the following:

Lemma 9.17. *Let $L \in \widetilde{\mathcal{F}}_1$ and let $E \in \widetilde{\mathcal{F}}_2$ with $\alpha(L) \neq \alpha(E)$ be such that there is $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ fixing L and E. Let $\ell_0 \in E \cap L$ be a connected component of the intersection such that $\{\partial^+\Phi(\ell_0), \partial^-\Phi(\ell_0)\} = \{\alpha(L), \xi_0\}$ for some $\xi_0 \neq \alpha(L)$. Let $\tau : [0, \delta) \to L$ be a transversal to $\mathcal{G}_L$, with $\tau(0) \in \ell_0$, and such that if E_t denotes the leaf through $\tau(t)$ for $t > 0$ we have that $\alpha(E_t)$ and $\tau(t)$ are in different connected components of $L \setminus \ell_0$. Then, for small $t > 0$, if ℓ_t denotes the curve of $\mathcal{G}_L$ through $\tau(t)$ it follows that $\{\partial^+\Phi(\ell_t), \partial^-\Phi(\ell_t)\} \subset \{\alpha(L), \xi_0\}$ and at least one of the points is $\alpha(L)$.*

Proof. Consider I_t the interval of the leaf space of $\widetilde{\mathcal{F}}_2$ made of the leaves E_s with $s \in (0, t)$ and fix a small ε such that Proposition 3.4 applies for $\widetilde{\mathcal{F}}_2$. For t small we identify I_t with $\bigcup\{\alpha(E_s), s \in (0, t)\}$. Consider then the set $\widehat{D}_\varepsilon(E, I_t)$ for this foliation. It follows (see Proposition 2.7) that the set $\widehat{D}_t$ which is the projection of $\widehat{D}_\varepsilon(E, I_t)$ to L contains a complementary component of a neighborhood of uniform size around the geodesic joining the points $\alpha(E) = \alpha(E_0)$ and $\alpha(E_t)$. This complementary component has $\alpha(L)$ in its closure.

In particular, one can choose t so small that $\widehat{D}_t$ is disjoint from the image of τ. In particular, one gets that one of the rays of ℓ_t must converge to $\alpha(L)$. We now want to understand the other ray. If $\alpha(E) \neq \xi_0$ then, the same argument shows that the other ray of ℓ_t converges to ξ_0, so we will assume in what follows that $\alpha(E) = \xi_0$.

We assume by contradiction that the other ray of ℓ_t starting at $\tau(t)$, that we call r_t, lands in some point $\xi \notin \{\alpha(L), \alpha(E)\}$. It cannot land in any point of the interior of I_t since it would need to intersect ℓ_0.

We can thus apply Proposition 3.4 to the leaves E_s for $s \in (t, 0)$ to obtain a family of rays r_s in L that start at $\tau(s)$ and always land in ξ. To see this, note that while r_s intersects $\widehat{D}_t$ then the ray varies continuously with s, but if stops intersecting $\widehat{D}_t$ it could in principle split into more than one ray. However, due to Proposition 3.9 the landing should occur in $\alpha(E_s)$ which is impossible since it would force the ray to intersect ℓ_0. Finally, note that when $s \to 0$ there could be splitting, but in this case we obtain that there is a curve in $E_0 \cap L$ (note that $E_0 = E$) which goes from $\alpha(E)$ to ξ.

In conclusion, we have shown that if the result does not hold, then $E \cap L$ must have a curve ℓ' from one of the fixed points of γ to ξ. Since E and L are γ-invariant we can iterate this intersected curve ℓ' to obtain a sequence of distinct intersections of $E \cap L$ landing in the same point (and since they do not admit a common transversal, they must have some distance in between). This contradicts the small visual measure property and completes the proof of the lemma. □

There is a similar phenomenon when the splitting goes in the opposite direction:

Lemma 9.18. *Assume that $\widetilde{\mathcal{G}}$ is not Hausdorff.*[12] *Let $L \in \widetilde{\mathcal{F}}_1$ and $E \in \widetilde{\mathcal{F}}_2$ be such that $\alpha(E) = \alpha(L)$ so that there is $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ fixing both L and E. Let $\ell_0 \in E \cap L$ be a connected component of the intersection such that $\{\partial^+\Phi(\ell_0), \partial^-\Phi(\ell_0)\} = \{\alpha(L), \xi_0\}$ for some $\xi_0 \neq \alpha(E) = \alpha(L)$. Let $\tau : [0, \delta) \to L$ be a transversal to $\mathcal{G}_L$, with $\tau(0) \in \ell_0$. Then, for small $t > 0$, if ℓ_t denotes the leaf of $\mathcal{G}_L$ through $\tau(t)$ it follows that $\{\partial^+\Phi(\ell_0), \partial^-\Phi(\ell_0)\} \subset \{\alpha(L), \xi_0\}$ and at least one of the points is ξ_0.*

Proof. Since E and L are γ-invariant then, one must have that ℓ_0 also is. Otherwise ℓ_0 is not γ periodic, so we would get infinitely many distinct connected components of $E \cap L$, all of them sharing the endpoint $\alpha(L) = \alpha(E)$. But this contradicts small visual measure since no two of such components can intersect a common transversal. It follows that $\xi_0, \alpha(L)$ are the two fixed points of γ.

Consider I_δ the interval of the leaf space of $\widetilde{\mathcal{F}}_2$ which consists of the leaves E_t through $\tau(t)$ with $t \in (0, \delta)$. The curve $\ell_t \subset E_t \cap L$ is the curve of $\widetilde{\mathcal{G}}$ passing through $\tau(t)$. Since $\xi_0 \neq \alpha(E)$ we can apply Proposition 3.4 to deduce that one of the rays of ℓ_t converges to ξ_0. We now want to understand the other ray.

Consider for $t \in [0, \delta)$ the other ray of ℓ_t starting at $\tau(t)$, which we call r_t, and let $\xi_t \in \partial\mathbb{H}^2$ be the landing point of ξ_t. As in the proof of Proposition 9.4 we claim that the point ξ_t is weakly monotonic as we move t, that it is continuous at $t = 0$ and that either it is $\xi_s = \alpha(E_t)$ or ξ_t is locally constant. Weak monotonicity is simpler in this case, since if $t' > t$ we have that ℓ_t separates $\ell_{t'}$ from ℓ_0. Using the pushing argument of Proposition 3.4 we get that if ξ_t is not $\alpha(E_t)$ then ξ_t must be locally constant. Finally for continuity at 0: we assume that $\xi_t \neq \xi_0$ for some small t (else we get the result if $\xi_t = \xi_0$ for all t sufficiently small). Assume that $\alpha(L)$ is the attracting fixed point of γ. Then $\gamma^n(\ell_t)$ converges to $\alpha(L)$ when $n \to \infty$. Since $\gamma^n(\ell_t) = \ell_{t_n}$ with $t_n \to 0$, continuity of ξ_t at $t = 0$ follows.

Now suppose that $\xi_t = \alpha(E_t)$ for a nontrivial interval J in $(0, t)$. We identify J with an interval in the leaf space of $\widetilde{\mathcal{F}}_2$, which we can identify with an interval in $\partial\mathbb{H}^2$ as well. As noted in Remark 9.16 we satisfy the hypothesis of Lemma 9.15. Hence

[12] This is a standing assumption in this section, but we emphasize it here because the proof makes crucial use of Lemma 9.15.

there is a leaf E_t in J such that E_t is fixed by a nontrivial deck transformation β and no leaf of $\mathcal{G}_{E_t}$ has one ideal point fixed by β. However we know that $\xi_t = \alpha(E_t)$ and $\alpha(E_t)$ is fixed by β. This is a contradiction since ℓ_t has one ideal point ξ_t.

Exactly as in Proposition 9.4, we deduce that if ξ_s is not constant and equal to $\alpha(E) = \alpha(L)$ then either it is constant and equal to ξ_0 or the function ξ_s is locally constant with jumps in a discrete set of $(0, \delta)$ in points $a_n \to 0$ such that for each n we have ℓ_{a_n} lands in $\xi_n := \alpha(E_{a_n})$. This implies that the intersection of $E_{a_n} \cap L$ contains, besides ℓ_{a_n} a curve joining ξ_{n-1} and ξ_n. These curves are accumulated by the curves ℓ_t with $t \in (a_{n-1}, a_n)$ and therefore, due to Proposition 8.3 we know that there is some $\gamma_n \in \pi_1(M)\setminus\{\mathrm{id}\}$ fixing E_{a_n}. This implies that E_{a_n} has a non-Hausdorff bigon joining ξ_n with some point in the interval from ξ_0 to ξ_{n-1} not containing ξ_n. This contradicts Proposition 9.14 because it produces bigons whose endpoints are linked or have a unique common endpoint. This completes the proof. □

We can now prove the main result of this section:

Proposition 9.19. *Let B be a γ-periodic non-Hausdorff bigon in a leaf $L \in \widetilde{\mathcal{F}}_i$ for some $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$. Then, one of the following options hold*:

(1) *there are γ-invariant bigons B_1 and B_2 in L such that $\xi^+_{B_1} = \xi^+_{B_2}$ and such that $\mathcal{I}_{B_1} \cup \mathcal{I}_{B_2} = \partial\mathbb{H}^2$, or*

(2) *there is an even number of ordered[13] γ-invariant bigons $B_1, B_2, \dots B_{2k}$ in L such that consecutive ones have different nonseparated points. Moreover, if the order is chosen so that B_1 is the bigon such that $\xi^-_{B_1} = \alpha(L)$ then one has that $\mathcal{I}_{B_1}$ is the interval which is separated from B_1 by B_2.*

Proof. Due to small visual measure, there are finitely many γ-invariant non-Hausdorff bigons in L, which we denote by $B_1, \dots, B_k$ in order (i.e., B_j separates B_{j-1} from B_{j+1} in L). Each such γ-periodic non-Hausdorff bigon $B_j \in L$ corresponds to the intersection of L with some $E_j \in \widetilde{\mathcal{F}}_2$ such that $\alpha(E_j) \in \{\xi, \eta\}$, which are the fixed points of γ (and also the separated and nonseparated points of B). Since L is γ-invariant we know that $\alpha(L) \in \{\xi, \eta\}$. Moreover, if $\alpha(E_j) = \alpha(L)$ then it follows that $\xi^+_{B_j} = \alpha(L)$ while if $\alpha(E_j) \neq \alpha(L)$ then $\xi^+_{B_j} \neq \alpha(L)$.

There are also a finite number of γ-invariant leaves of $\mathcal{G}_L$, say $\ell_0, \dots, g_m$, corresponding to intersections with leaves $F_i \in \widetilde{\mathcal{F}}_2$ such that $\alpha(F_i) \in \{\xi, \eta\}$. (Indeed, these leaves are at some minimal distance away from each other and each one must be uniformly close to the geodesic joining ξ and η). We also consider these leaves ordered in the same direction. Some of them are contained in the boundaries of the bigons B_j. We know that $m \geq 1$ since there is at least one γ-periodic non-Hausdorff bigon.

[13]This means that in L the bigon B_j separates B_{j-1} from B_{j+1} for all $j = 2 \dots k-1$.

Let D be the closure of the region in L bounded by ℓ_0, g_m. Since $m \geq 1$ then D is nonempty, with nonempty interior. Fix a transversal orientation to $\mathcal{F}_2$.

Consider, for every $i \in \{0, \ldots, m-1\}$, a transversal $\tau_i : [0, \delta) \to L$ such that $\tau_i(0) \in \ell_i$ and that $\tau_i(t)$ is between ℓ_i and ℓ_{i+1}. Denote by $\mathcal{C}_i$ the (closed) interval from ξ to η and containing $\alpha(H_t)$ where $H_t \in \widetilde{\mathcal{F}}_2$ is the leaf such that $\tau_i(t) \in H_t$. This interval is well defined and is independent of the choice of τ_i and t.

Claim 1. $\mathcal{C}_i \cup \mathcal{C}_{i+1} = \partial\mathbb{H}^2$. *The region between ℓ_i and ℓ_{i+1} is a bigon if and only if $\alpha(F_i) = \alpha(F_{i+1})$.*

Proof. The interior of $\mathcal{C}_i$ is exactly the set of points ν in $\partial\mathbb{H}^2$ such that there is a leaf $E \in \widetilde{\mathcal{F}}_2$ that intersects the region between ℓ_i and ℓ_{i+1} in L and such that $\alpha(E) = \nu$. This is because both sets are γ invariant and contain small neighborhoods of the endpoints. Hence if one considers a transversal to $\mathcal{G}_L$ intersecting in the interior the leaf ℓ_{i+1} (which is a connected component of $L \cap F_{i+1}$), one sees that $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ correspond to distinct intervals and thus $\mathcal{C}_i \cup \mathcal{C}_{i+1} = \partial\mathbb{H}^2$ as desired. □

To see the last property, notice that if the region is a bigon then $F_i = F_{i+1}$ so the α's coincide. Otherwise F_i, F_{i+1} are distinct, fixed by γ, and there is no other γ invariant leaf between F_i and F_{i+1}. Hence when acting on the leaf space of $\widetilde{\mathcal{F}}_2$, it follows that up to inverse, γ is attracting in F_i and repelling in F_{i+1}. This implies that $\alpha(F_i) \neq \alpha(F_{i+1})$.

Claim 2. *Consider B_j, B_{j+1} consecutive non-Hausdorff bigons in L. Then if $\xi^+_{B_j} = \xi^+_{B_{j+1}}$ we have $\mathcal{I}_{B_j} \cup \mathcal{I}_{B_{j+1}} = \partial\mathbb{H}^2$, and if $\xi^+_{B_j} = \xi^-_{B_{j+1}}$ then $\mathcal{I}_{B_j} = \mathcal{I}_{B_{j+1}}$.*

Proof. In the first case one has between the boundaries of B_j, B_{j+1} an even (possibly zero, if $B_j \cap B_{j+1}$ intersect in some leaf ℓ_j) number of regions between consecutive curves ℓ_i. Since each intersection makes a half turn in $\partial\mathbb{H}^2$ without changing orientation (because this only happens when one crosses a non-Hausdorff bigon) we obtain that the leaves of $\widetilde{\mathcal{F}}_2$ intersecting the interior of B_j and B_{j+1} correspond to different intervals in $\partial\mathbb{H}^2$ whose boundaries are $\{\xi, \eta\}$ thus $\mathcal{I}_{B_j} \cup \mathcal{I}_{B_{j+1}} = \partial\mathbb{H}^2$. The other case is similar, since one has an odd number of such regions in between. □

Conclusion. Claim 2 implies that the only way that all non-Hausdorff bigons with the same nonseparated point have the same associated interval is that adjacent non-Hausdorff bigons must have distinct nonseparated points. Note that we still could have that there is a unique such non-Hausdorff bigon.

To obtain that there are an even number of such bigons, we will use Lemmas 9.17 and 9.18. Using Lemma 9.18 we will deduce that ℓ_0 and ℓ_m satisfy $\alpha(F_0) = \alpha(F_m) \neq \alpha(L)$. Indeed, suppose to the contrary that $\alpha(L) = \alpha(F_0)$. Then we can apply Lemma 9.18 to the curve ℓ_0 contained in $L \cap F_0$. It has ideal points ξ, η, which are distinct; one of them is $\alpha(L) = \alpha(F_0)$, the other we call ξ_0. Now consider a transversal τ starting in ℓ_0 and exiting D and ℓ' a curve of $\mathcal{G}_L$ intersecting τ

outside D and near $\tau(0)$. Apply Lemma 9.18 to ℓ_0: it implies that the ideal points of ℓ' are contained in $\alpha(L)$, ξ_0. But that is impossible given the choice of ℓ_0: no curve outside D has ideal points contained in $\{\xi, \eta\}$. This shows that $\alpha(F_0) \neq \alpha(L)$. This also applies to ℓ_m, showing that $\alpha(L) \neq \alpha(F_m)$. Hence $\alpha(F_m) = \alpha(F_0) \neq \alpha(L)$.

Moreover Lemma 9.17 implies that $\mathcal{C}_0$ is the interval of $\partial\mathbb{H}^2 \setminus \{\xi, \eta\}$ which is in the opposite connected component of $\partial\mathbb{H}^2 \setminus \{\eta, \xi\}$ of ℓ_m in $L \setminus \ell_0$. Similarly, $\mathcal{C}_{m-1}$ is the opposite interval, that is, the interval of $\partial\mathbb{H}^2 \setminus \{\xi, \eta\}$ which is in the opposite connected component to ℓ_0 in $L \setminus \ell_m$. In particular, the two intervals $\mathcal{C}_0$, $\mathcal{C}_{m-1}$ are different.

We will now check the progress along leaves $\widetilde{\mathcal{F}}_2$ when we cross the region D starting from ℓ_0 all the way through ℓ_m. Notice that crossing each periodic non-Hausdorff bigon changes the orientation in which one makes progress in the leaves of $\widetilde{\mathcal{F}}_2$. Lemma 9.17 implies that in both ℓ_0 and ℓ_m when one crosses into D, then the α's of the $\widetilde{\mathcal{F}}_2$ leaves move in the opposite direction (e.g., when crossing ℓ_0 it is in the complementary component of ℓ_0 which does not contain D). In particular, crossing through ℓ_0 into D and through ℓ_m outside D, one needs to cross $\widetilde{\mathcal{F}}_2$ leaves with the same orientation. Thus there must be an even number of non-Hausdorff bigons in between.

Finally, if we choose the order so that B_1 is the one such that $\xi^-_{B_1} = \alpha(L)$ we have that one needs to intersect an even number of curves ℓ_i after ℓ_0 to get to B_1 (possibly ℓ_0 is the boundary of B_1, in which case we do not intersect any). We get that the boundary of B_1 is of the form $\{\ell_{2i} \cup \ell_{2i+1}\}$ for some $i \geq 0$. This implies that $\mathcal{C}_{2_i} = \mathcal{C}_0$, because if G is the leaf of $\widetilde{\mathcal{F}}_2$ containing ∂B_1, then $\alpha(G) = \xi^+_{B_1}$. This completes the proof. □

9.6. *Proof of Theorem 9.1 and applications.*

Proof of Theorem 9.1. By assumption we know that there is at least one leaf L with a non-Hausdorff bigon B and some $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ for which we have $\gamma B = B$ and if we denote the endpoints as $\{\xi, \eta\} \subset \partial\mathbb{H}^2$ these are also γ-invariant. Without loss of generality we assume that $L \in \widetilde{\mathcal{F}}_1$. In particular, $\alpha(L) \in \{\xi, \eta\} = \{\xi^+_B, \xi^-_B\}$. Let $E \in \widetilde{\mathcal{F}}_2$ be such that the boundary of B is contained in $E \cap L$.

Denote by $\hat{L}$ a leaf in $\widetilde{\mathcal{F}}_1$ such that $\alpha(\hat{L}) \in \{\xi, \eta\}$ and that $\alpha(\hat{L}) \neq \alpha(L)$. The leaf $\hat{L}$ must also be γ-invariant. It follows from applying either Proposition 9.4 or Proposition 9.10 that $\hat{L}$ contains at least one non-Hausdorff bigon which up to changing $\hat{L}$ by a deck transformation associated to the fiber we can assume is bounded by leaves of $E \cap \hat{L}$.

We now apply Proposition 9.19. If the first condition of the proposition happens for either L or $\hat{L}$, we get that there are γ-periodic non-Hausdorff bigons B_i, B_j in L (or $\hat{L}$) with the same nonseparated point and such that $\mathcal{I}_{B_i} \cup \mathcal{I}_{B_j} = \partial\mathbb{H}^2$. Applying

Lemma 9.13 we deduce that for every $L' \in \widetilde{\mathcal{F}}_1$ there must be a non-Hausdorff bigon joining ξ and η in L'. This completes this case.

Thus, we can assume that both for L and $\hat{L}$ we have that the second option of Proposition 9.19 holds. We need to show:

Claim 3. *If we consider $B_1, \ldots, B_{2k}$ the γ-invariant non-Hausdorff bigons in L with a chosen order and use the same order in $\hat{L}$ and get γ-invariant non-Hausdorff bigons $\hat{B}_1, \ldots, \hat{B}_{2m}$ we have that $k = m$ and that the nonseparated point of B_1 and $\hat{B}_1$ coincide.*

Proof. Using Lemma 9.13 we know that there is a closed interval $\mathcal{I} \subset \partial\mathbb{H}^2$ (with boundary $= \{\xi, \eta\}$) such that each bigon B_i in L pushes to a bigon in every L' with $\alpha(L') \in \mathcal{I}$. Moreover, if E_i is the leaf of $\widetilde{\mathcal{F}}_2$ such that E_i intersects L in the boundary of B_i, then we have that every leaf L' as above has the corresponding bigon contained in between the intersection of L' and E_i. Note that Remark 9.11 implies that the nonseparated point of the corresponding bigons are in the same direction.

Since the intersection of E_i with the leaves L' from L to $\hat{L}$ separates the region between L and $\hat{L}$, the order of the bigons cannot be reversed. We deduce that in $\hat{L}$ we have bigons in the same order and the same directions. It remains to show that there cannot be bigons in $\hat{L}$ that do not come from pushing those in L, but this just follows by a symmetric argument. □

Conclusion. Assume that B_1 satisfies $\xi^-_{B_1} = \alpha(L)$, the other case is symmetric. In this case we deduce from Claim 1 that $\xi^+_{\hat{B}_1} = \alpha(\hat{L})$, and hence $\xi^-_{\hat{B}_{2k}} = \alpha(\hat{L})$. By Proposition 9.19 it follows that $\mathcal{I}_{B_1} \cup \mathcal{I}_{\hat{B}_{2k}} = \partial\mathbb{H}^2$; to see this, first apply it to L so $\mathcal{I}_{B_1}$ is the interval opposite to $B_i, i > 1$. When seen in $\hat{L} \cup S^1(\hat{L})$ we have that $\mathcal{I}_{B_1}$ is the interval in the same complementary component of $\hat{L} \setminus \hat{B}_{2k}$ that contains $\hat{B}_1$. Now apply it to $\hat{L}$: then $\mathcal{I}_{\hat{B}_{2k}}$ is the component opposite from $\hat{B}_1$ when seen from $\hat{B}_{2k}$. Hence $\mathcal{I}_{B_1} \cup \mathcal{I}_{\hat{B}_{2k}} = \partial\mathbb{H}^2$.

Finally, applying Lemma 9.13, we deduce that for every $L' \in \widetilde{\mathcal{F}}_1$ there must be a non-Hausdorff bigon joining ξ and η in L'. □

Corollary 9.20. *Assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are two transverse minimal foliations of $M = T^1S$ such that the foliation $\mathcal{G}$ obtained as their intersection is not homeomorphic to the orbit foliation of the geodesic flow for a hyperbolic metric on S. Then, there are finitely many disjoint simple closed curves $s_1, \ldots, s_k$ in S such that for every lift $\tilde{s}_j$ of some of these curves, if ξ, η denote the endpoints of $\tilde{s}_j$ in $\partial\mathbb{H}^2$, they satisfy the following: for every leaf $L \in \widetilde{\mathcal{F}}_i$ we have that L has a non-Hausdorff bigon joining ξ and η.*

Even if we fix the homotopy classes of the curves s_i which contain Reeb surfaces, it could be that the intersected foliations are not equivalent since there are possible variants to the Matsumoto–Tsuboi construction (see Remark 7.1). Thus we can add several copies of $\mathbb{T}^2 \times I$ associated with the same s_i and they will not produce equivalent foliations. This statement gives a combinatorial way to describe the possible intersected foliations.

Proof. It follows from Theorem 9.1 that once a leaf L contains a non-Hausdorff bigon invariant by some $\gamma \in \pi_1(M) \setminus \{\mathrm{id}\}$ then every leaf contains a non-Hausdorff bigon joining the same endpoints. Note that since everything is equivariant under the action of $\pi_1(M)$ we get that every leaf must contain a non-Hausdorff bigon joining the fixed points of all elements in $\pi_1(M)$ conjugated to γ. Since non-Hausdorff bigons cannot intersect, these form a lamination in $\partial\mathbb{H}^2$ (by this we mean a collection of pairs of distinct points in $\partial\mathbb{H}^2$ which are pairwise not linked) and thus corresponds to a simple closed curve in S (see also Proposition 9.14). Moreover, if there is another γ'-invariant non-Hausdorff bigon, it must be also disjoint from the first one, so we get that it is associated to a disjoint simple closed curve in S. Since there are only finitely many homotopy classes of disjoint simple closed curves in a surface S of genus $g \geq 2$ we obtain the result. □

This section motivates the following:

Question. *Let $\mathcal{F}_1, \mathcal{F}_2$ be two transverse minimal foliations in a closed 3-manifold M. Assume that the the lift $\tilde{\mathcal{G}}$ of the intersection foliation $\mathcal{G} = \mathcal{F}_1 \cap \mathcal{F}_2$ to the universal cover does not have Hausdorff leaf space. Is it true that M contains a π_1-injective torus T such that every leaf $L \in \widetilde{\mathcal{F}}_i$ contains non-Hausdorff bigons joining the endpoints of the intersections of lifts of T with L? In particular, is it true that if M is atoroidal then two transverse foliations intersect with Hausdorff leaf space in the universal cover?*

10. Application to partially hyperbolic diffeomorphisms

A diffeomorphism $f : M \to M$ will be said *partially hyperbolic* if it admits a Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that there is some $n > 0$ such that for $x \in M$ and unit vectors $v^s \in E^s(x)$, $v^c \in E^c(x)$ and $v^u \in E^u(x)$ we have

$$\|Df^n v^s\| < \tfrac{1}{2}\min\{1, \|Df^n v^c\|\}, \quad \|Df^n v^u\| > 2\max\{1, \|Df^n v^c\|\}. \tag{10-1}$$

The problem of classification of partially hyperbolic diffeomorphisms in 3-manifolds introduced in [7; 14] has seen a lot of activity in the last few years. In particular, we point to [6; 4; 25] for the analysis when the 3-manifold is hyperbolic, and more generally for diffeomorphisms homotopic to the identity in general 3-manifolds. In [5] we have introduced a class of systems called *collapsed Anosov flows* which would provide a natural and useful notion of classification of such systems. In [25] we have shown that this class contains all partially hyperbolic diffeomorphisms in hyperbolic 3-manifolds. The proof involves a careful study of pairwise transverse foliations, but also dynamics is introduced at several points in a crucial way. As mentioned in the introduction, the goal of this paper is to see to which extent we can extract dynamical information by using only the geometric properties of transverse foliations.

We note that besides the classical examples of time one maps of geodesic flows (or more generally, *discretized Anosov flows* ([6; 33]), unit tangent bundles admit many other classes of examples (see [8; 9]), some of which have been studied in [3; 25].

The goal of this section is to prove Corollary B which states that every conservative partially hyperbolic diffeomorphism in $M = T^1S$ is a collapsed Anosov flow up to finite cover and it is thus accessible (see [24]). This section will assume some familiarity with standard results and notions of partial hyperbolicity; all of them can be found in [5].

10.1. ***Preliminary results and precise statement.*** Recall from [14] that when a partially hyperbolic diffeomorphism has orientable bundles whose orientation is preserved by f, then it preserves transverse *branching foliations* $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent respectively to $E^s \oplus E^c$ and $E^c \oplus E^u$. In [30, Theorem 3.1] some conditions are obtained which imply that these foliations do not have vertical leaves (for instance, being volume preserving is one of such conditions). Vertical means the leaf can be homotoped to be a union of Seifert fibers in $M = T^1S$. If no vertical leaves exist, then, arguments like in [6; 4] (see in particular [25, Proposition 8.3]) will allow us to prove the following:

Theorem 10.1. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism on $M = T^1S$ where S is a surface of genus $g \geq 2$. Assume moreover that f preserves branching foliations which do not have vertical leaves. Then, f is a collapsed Anosov flow.*

Here, being a collapsed Anosov flow means that there is a semiconjugacy between f and a self-orbit equivalence of an Anosov flow of M (which by a classical result of Ghys must be orbit equivalent to a geodesic flow, see [28]). The semiconjugacy is required to have some technical properties relating the flow and the center direction. We obtain the strongest such condition, called *strong collapsed Anosov flow* in [5]. Since we will deduce a property of the leaves that implies such condition by [5, Theorem D] we will refer the reader to [5] for the actual definition of a collapsed Anosov flow.

These are the two results we shall use to be able to apply Theorem A to partially hyperbolic diffeomorphisms:

Theorem 10.2 (Burago and Ivanov). *Let $f : M \to M$ be a partially hyperbolic diffeomorphism preserving an orientation of its invariant bundles. Then, f admits* *branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ which are approximated by foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$. Moreover, if the branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ are minimal, then one can choose $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ to also be minimal.*

We refer the reader to [5, § 3] for discussion of this result, in particular the final property. The approximation is such that there is a *collapsing* map $h : M \to M$ which sends leaves of $\mathcal{F}^{cs}$ to leaves of $\mathcal{W}^{cs}$ (same for cu), is homotopic to the identify, is C^1 along leaves, is ε-close to the identity, and has derivatives along the leaves which is C^1 close to the identity along the leaves, and has further properties. Many properties transfer between the two foliations, in particular the topological types of leaves of both $\mathcal{W}^{cs}$ and $\mathcal{F}^{cs}$ is the same.

To obtain Theorem 10.1 we will then apply the following criterion given by [5, Theorem D]:

Theorem 10.3. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism preserving an orientation of its invariant bundles. If leaves of the foliation obtained by intersecting $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ are quasigeodesics in the universal cover of their respective leaves, then f is a collapsed Anosov flow.*

Accessibility and ergodicity then follow from [24, Theorem A]. We note that Corollary B follows since the branching foliations of volume preserving partially hyperbolic diffeomorphisms are what we call f-minimal, so they cannot have vertical leaves, and therefore satisfy the assumptions of Theorem 10.1. We note in fact that f-minimality is studied in [30] in many situations, including the case where f is *chain-recurrent* (something weaker than volume preserving or transitive) or belongs to certain isotopy classes. It is also shown in [5, Proposition 4.8] that being f-minimal is an open and closed condition on partially hyperbolic diffeomorphisms, so that if some f is known to be isotopic to a chain recurrent one along partially hyperbolic diffeomorphisms, then it will be in the hypothesis of Theorem 10.1.

10.2. ***Partially hyperbolic foliations do not admit Reeb surfaces.*** We now explain why existence of a Reeb surface in the approximating foliations is not possible if the branching foliations come from a partially hyperbolic diffeomorphism. This will reduce the proof of Theorem 10.1 to showing that under its assumptions the branching foliations exist and are minimal.

Proposition 10.4. *Let $f : M \to M$ be a partially hyperbolic diffeomorphism preserving branching foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ which are approximated in the sense of Theorem 10.2 by true transverse foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$. Then, the approximating foliations do not have Reeb surfaces.*

Proof. A Reeb surface of $\mathcal{F}^{cs}$ is finitely covered by an annulus whose boundary components are leaves of $\mathcal{F}^{cs} \cap \mathcal{F}^{cu}$ and whose interior is made of infinitely many curves spiraling to the boundary. When collapsing to $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$, these annuli collapse to “branched” annuli S contained in some leaf of $\mathcal{W}^{cs}$ or $\mathcal{W}^{cu}$ (say $\mathcal{W}^{cs}$) which is an annulus, the boundaries are leaves of the one-dimensional branching foliation induced by the intersection and no transversal from one side intersects the other. Using the Poincaré–Bendixson theorem we deduce that every flow transverse to the boundaries of S must have a periodic orbit. Applying this to the flow generated by a unit vector field tangent to E^s we deduce the existence of a closed curve tangent to E^s which is a contradiction with partial hyperbolicity (because f would contract the curve until you find a circle tangent to E^s in an arbitrarily small ball). $\square$

10.3. *Proof of Theorem 10.1.* Using Theorem 8.1, Theorem 6.1, and Theorem 10.3 it is enough to show that the approximating foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ given by Theorem 10.2 are minimal. For this, it is enough to show that this is true for the branching foliations under the assumption that there are no vertical leaves for such foliations.

Theorem 10.2 shows that the collapsing map can be chosen to be a bijection between the sets of leaves of $\mathcal{F}^{cs}$ and $\mathcal{F}_{ws}$ (say), and the collapsing map preserves their homotopic properties. Therefore $\mathcal{F}^{cs}, \mathcal{F}^{cu}$ do not have vertical leaves by assumption.

Since $\mathcal{F}^{cs}$ is horizontal, the leaf space of $\widetilde{\mathcal{F}}^{cs}$ is homeomorphic to the reals $\mathbb{R}$, and $\mathcal{F}^{cs}$ blows down to a minimal foliation (see [20]). By Theorem 2.1 this foliation has only planes and annuli leaves. Hence $\mathcal{W}^{cs}$ has only planes and annuli leaves (here, the topology of the leaf is, by definition, the topology of the quotient of the leaf in the universal cover by the deck transformations that fix the given leaf).

Since every leaf of $\mathcal{W}^{cs}$ is a cylinder or a plane and the foliation is $\mathbb{R}$-covered we can argue exactly as in [25, Proposition 8.3] to get a contradiction with partial hyperbolicity (the quasigeodesic property is used in the proof of [25, Proposition 8.3] only to show that leaves are cylinders or planes). This completes the proof of minimality.

10.4. *Proof of Corollary B.* Take a regular finite cover M_1 and iterate in order to have orientability of the bundles as well as their preservation. As explained in [30, § 7], once we take a finite cover M_1, since the foliations will not have vertical leaves they need to be horizontal. To show that there are no vertical leaves and that the branching foliations are minimal in the cover M_1, we use the volume preservation assumption as in [30]. In [30, Lemma 7.1] and [30, Subsection 6.4] it is proved that deck translations associated with the cover $M_1 \to M$ preserve the orientations of all the bundles. This implies that the original bundles were horizontal and the orientability conditions were satisfied in M.

Hence the partially hyperbolic diffeomorphism f is a collapsed Anosov flow. It follows that f is accessible, and if f is C^2 (and volume preserving) then f is ergodic (see [24]). This proves Corollary B.

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REFINED BOUNDS FOR THE EIGENVALUES OF THE STOKES OPERATOR

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We analyze bounds for the sums of eigenvalues of the Stokes operator restricted to a bounded domain $\Omega \subset \mathbb{R}^d$ with $d \geq 2$. We improve upon existing lower bound estimates due to Ilyin as well as those by the second author and S. Yıldırım Yolcu, while preserving sharpness in the sense of Weyl asymptotics.

1. Introduction

Let Ω be an open bounded set in $\mathbb{R}^d$, $d \geq 2$. We derive sharper estimates for the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ of Stokes problem defined by

$$(1)\qquad \begin{cases} -\Delta \boldsymbol{u}_k + \nabla p_k = \lambda_k \boldsymbol{u}_k & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}_k = 0 & \text{in } \Omega, \\ \boldsymbol{u}_k = 0 & \text{on } \partial\Omega. \end{cases}$$

The eigenvalues of the Stokes problem in (1) are important due to their numerous applications in fluid mechanics. They can be interpreted as the eigenvalues associated with linear (small or infinitesimal) self-oscillations of the fluid within the domain Ω [4]. When the eigenfunctions are required to meet specific conditions that are crucial from a physical perspective in fluid dynamics, exact eigenvalues for the Stokes problem are not obtainable. Consequently, both theoretical and practical aspects necessitate the close identification of the eigenvalues. The literature concerning the eigenvalues of the Stokes problem is vast, and recent studies aimed at deriving estimates for these eigenvalues are documented in [6; 9; 10; 12; 24].

Before presenting the results, we first review some basic facts about the theory of the Stokes operator. Let $\mathcal{U}$ denote the set of smooth, divergence-free vector functions with compact supports. Specifically,

$$\mathcal{U} = \{\boldsymbol{u} : \Omega \to \mathbb{R}^d,\ \boldsymbol{u} \in \boldsymbol{C}_0^{\infty}(\Omega),\ \operatorname{div} \boldsymbol{u} = 0\}$$

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and let the closure of $\mathcal{U}$ in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$ be denoted by L and U, respectively. In particular,

$$U \subseteq \{\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega),\ \operatorname{div} \boldsymbol{u} = 0\}$$

and if Ω is open, bounded and locally Lipschitz, we have the equality [3]:

$$U = \{\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega),\ \operatorname{div} \boldsymbol{u} = 0\}.$$

We note that $\boldsymbol{L}^2(\Omega)$ can be written as $\boldsymbol{L}^2(\Omega) = L \oplus L^{\perp}$, where

$$L^{\perp} = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{u} = \nabla p,\ p \in L^2_{\mathrm{loc}}(\Omega)\}$$

(see [3; 4; 19]). The Stokes operator is formally expressed as

$$A\boldsymbol{u} = -P_L(\Delta \boldsymbol{u}),$$

where the linear operator $P_L : \boldsymbol{L}^2(\Omega) \to L$, defined by

$$P_L(\boldsymbol{v}) = \boldsymbol{v} - \nabla \Delta^{-1}(\operatorname{div} \boldsymbol{v}),$$

is called the Leray projection [4], i.e., $P_L(P_L(\boldsymbol{v})) = P_L(\boldsymbol{v})$. Notice that P_L becomes the identity operator for the divergence-free vector fields because $P_L(\boldsymbol{v}) = \boldsymbol{v}$ for $\operatorname{div} \boldsymbol{v} = 0$. Furthermore, P_L can be understood as the projection onto divergence-free vector fields. This projection is particularly employed to remove some terms and components in the Stokes and Navier–Stokes equations. Going back to the Stokes problem in (1), one can also see that

$$A\boldsymbol{u} = -\Delta \boldsymbol{u} + \nabla p, \quad p = \Delta^{-1}(\operatorname{div} \Delta \boldsymbol{u}),$$

and that for all $\boldsymbol{u}, \boldsymbol{w}$ in U, A is defined by

$$(A\boldsymbol{u}, \boldsymbol{w}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{w}),$$

i.e.,

$$\int_{\Omega} A\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{w}(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\Omega} \sum_{i=1}^{d} \frac{\partial \boldsymbol{u}}{\partial x_i} \frac{\partial \boldsymbol{w}}{\partial x_i}\, d\boldsymbol{x}.$$

The Stokes operator A is an unbounded, linear, self-adjoint, positive definite operator in L. In addition to these nice properties, its inverse A^{-1} is a compact, self-adjoint operator [3; 4]. Thus, there exists an orthonormal basis $\{\boldsymbol{u}_k\}_{k=1}^{\infty}$ in L and a set of positive eigenvalues $\{\delta_k\}_{k=1}^{\infty}$ accumulating at zero such that

$$A^{-1} \boldsymbol{u}_k = \delta_k\, \boldsymbol{u}_k,$$

for $k = 1, 2, 3, \ldots$. Therefore, $\{\boldsymbol{u}_k\}_{k=1}^{\infty} \in U$ with corresponding eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ with $\lambda_k = 1/\delta_k$ are such that

$$A\boldsymbol{u}_k = \lambda_k \boldsymbol{u}_k. \tag{2}$$

The eigenvalues (including multiplicities) satisfy

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \to \infty \quad \text{as } k \to \infty. \tag{3}$$

Observe that by taking the scalar product with orthonormal $\boldsymbol{u}_k$ we have

$$\lambda_k = \|\nabla \boldsymbol{u}_k\|^2 = \int_\Omega \sum_{i=1}^{d} \frac{\partial \boldsymbol{u}_k}{\partial x_i} \cdot \frac{\partial \boldsymbol{u}_k}{\partial x_i}\, d\boldsymbol{x}. \tag{4}$$

The eigenvalues also satisfy the Weyl asymptotic formulas [4; 8; 17] for $d \ge 2$:

$$\lambda_k \sim 4\pi \left(\frac{\Gamma\left(1 + \frac{d}{2}\right)}{(d-1)|\Omega|} \right)^{\frac{2}{d}} k^{\frac{2}{d}}, \tag{5}$$

where $\Gamma(x)$ denotes the gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$ for $x > 0$ and $|\Omega|$ denotes the Lebesgue measure of Ω.

The eigenvalue bounds that will be established in this paper closely resemble those associated with the Dirichlet Laplacian. A substantial body of literature exists regarding inequalities for the eigenvalues of the Dirichlet Laplacian; notable references include the survey articles cited in [1; 2; 7]. In his renowned paper [18], G. Pólya demonstrated that for small values of k, the ratio $4\pi k/|\Omega|$ serves as a lower bound for the eigenvalues μ_k of the Dirichlet Laplacian in tiling domains in $\mathbb{R}^2$. Pólya conjectured that this result could be extended to any bounded domain in $\mathbb{R}^d$. This conjecture remains unresolved. A significant advance in establishing a lower bound was achieved by P. Li and S.-T. Yau [14], who demonstrated the following inequality related to the sums of the eigenvalues μ_j of the Dirichlet Laplacian on the domain Ω:

$$\sum_{j=1}^{k} \mu_j \ge 4\pi \frac{d}{d+2} \left(\frac{\Gamma\left(1 + \frac{d}{2}\right)}{|\Omega|} \right)^{\frac{2}{d}} k^{1+\frac{2}{d}}, \tag{6}$$

Recent research has focused significantly on these types of bounds, along with their extensions and enhancements, particularly in relation to other operators. For instance, A. Melas [16] provided an improvement on (6) that incorporates moments of inertia. In two dimensions, H. Kovařík, S. Vugalter, and T. Weidl [13] further advanced Melas's findings by introducing a positive correction term that involves the size of the boundary. T. Weidl [21] achieved an enhancement of the sharp Berezin-type bounds concerning the Riesz means $\sum_k (z - \mu_k)_+^\sigma$ of the eigenvalues associated with the Dirichlet Laplacian operator within a specified domain for $\sigma \ge 3/2$. Regarding other operators, Harrell and Yıldırım Yolcu [5] along with Yolcu [29] proved inequalities of Berezin–Li–Yau type applicable to the eigenvalues of Klein–Gordon operators. Yıldırım Yolcu and Yolcu [25; 28; 27] established

Melas-type improvements for the eigenvalues of fractional Laplacian operators $(-\Delta)^{\alpha/2}, \alpha \in (0, 2]$.

It is noteworthy that the coefficient of $k^{1+\frac{2}{d}}$ in (6) is slightly larger for the tiling domains [18]. See [23] for more extensive research and a generalized conjecture.

Lower bounds on the sums of eigenvalues are important for the theoretical framework of attractors associated with the Navier–Stokes equations [9; 10]. The eigenvalue bounds of the Stokes operator have garnered attention following the work by A. A. Ilyin [9], who proved an inequality of Berezin–Li–Yau type for the Stokes operator:

$$\sum_{j=1}^{k} \lambda_j \geq 4\pi \frac{d}{d+2}\left(\frac{\Gamma\left(1+\frac{d}{2}\right)}{(d-1)\,|\Omega|}\right)^{\frac{2}{d}} k^{1+\frac{2}{d}}, \tag{7}$$

where $k \geq 1$. In this article, we first build upon (7) by establishing upper bounds for the sums of negative powers and lower bounds for the sums of positive powers of the eigenvalues of the Stokes operator as follows:

Theorem 1. *For $k \geq 1$, $0 < b < d/2$, and $d \geq 2$, the sums of negative powers of eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the Stokes operator on Ω in (1) satisfy*

$$\sum_{j=1}^{k} \lambda_j^{-b} \leq (4\pi)^{-b} \frac{d}{d-2b}\left(\frac{(d-1)|\Omega|}{\Gamma\left(1+\frac{d}{2}\right)}\right)^{\frac{2b}{d}} k^{1-\frac{2b}{d}}. \tag{8}$$

Theorem 2. *For $k \geq 1$, $0 < a \leq 1$, and $d \geq 2$, the sums of positive powers of eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the Stokes operator on Ω in (1) satisfy*

$$\sum_{j=1}^{k} \lambda_j^{a} \geq (4\pi)^{a} \frac{d}{d+2a}\left(\frac{\Gamma\left(1+\frac{d}{2}\right)}{(d-1)|\Omega|}\right)^{\frac{2a}{d}} k^{1+\frac{2a}{d}}. \tag{9}$$

By appropriately translating the open set Ω if needed, we can assume that the second moment of inertia $I(\Omega)$ is defined as

$$I(\Omega) = \int_{\Omega} |\boldsymbol{x}|^2 \, d\boldsymbol{x}.$$

Ilyin [10] was able to improve the Berezin–Li–Yau inequality in (7) for the Stokes operator by adding a lower-order term involving the moment of inertia $I(\Omega)$ in dimensions 2, 3, and 4 as follows:

$$\sum_{j=1}^{k} \lambda_j \geq 4\pi \frac{d}{d+2}\left(\frac{\Gamma\left(1+\frac{d}{2}\right)}{(d-1)\,|\Omega|}\right)^{\frac{2}{d}} k^{1+\frac{2}{d}} + \frac{d-1}{48}\beta_d^{\mathrm{S}} \frac{|\Omega|}{I(\Omega)} k, \tag{10}$$

where in the two-dimensional case $\beta_2^{\mathrm{S}} = \frac{239}{240}$, while for $d = 3, 4$ it suffices to take $\beta_3^{\mathrm{S}} = 0.986$ and $\beta_4^{\mathrm{S}} = 0.978$.

Let

$$B_R(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}^d : |\boldsymbol{y} - \boldsymbol{x}| \le R\}$$

be the ball of radius R centered at $\boldsymbol{x}$ in $\mathbb{R}^d$. Let

$$\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma\big(1 + \frac{d}{2}\big)} \tag{11}$$

be the volume of the unit ball $B_1(\boldsymbol{x})$ in $\mathbb{R}^d$.

Building on the improvements in (10), Yıldırım Yolcu and Yolcu [24] introduced additional correction terms beyond those dependent on $k^{1+\frac{2}{d}}$ and $k \ge 1$, along with their explicit coefficients, applicable for any $d \ge 2$:

$$\begin{aligned}\sum_{j=1}^{k} \lambda_j \ge 4\pi \frac{d}{d+2}\left(\frac{\Gamma\big(1+\frac{d}{2}\big)}{(d-1)\,|\Omega|}\right)^{\frac{2}{d}} k^{1+\frac{2}{d}} + \frac{d-1}{24(d^2+2d)} \frac{|\Omega|}{I(\Omega)}\, k \\ + \frac{((d-1)\,|\Omega|)^{\frac{3d+2}{2d}}}{144\sqrt{\pi}\, d^{\frac{3}{2}}\, (d+2)\, I(\Omega)^{\frac{3}{2}}\, \Gamma\big(1+\frac{d}{2}\big)^{\frac{1}{d}}}\, k^{1-\frac{1}{d}}.\end{aligned} \tag{12}$$

In this paper we establish several effective lower bounds for the sums of the eigenvalues of the Stokes problem, including the following improved estimates:

Theorem 3 (refinement of the Berezin–Li–Yau inequality for the Stokes operator). *Let $\Omega \subset \mathbb{R}^d$. For any $d \ge m+1 \ge 2$, the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the Stokes operator in (1) satisfy*

$$\begin{aligned}\sum_{i=1}^{k} \lambda_i \ge 4\pi \left(\frac{\Gamma\big(1+\frac{d}{2}\big)}{(d-1)\,|\Omega|}\right)^{\frac{2}{d}} k^{1+\frac{2}{d}} - \frac{2\omega_d^{\frac{m-1}{d}} A_{m+2}\varrho^{\frac{(m+1)d+m-1}{d}}}{(d+2)M^{m+1}} k^{1-\frac{m-1}{d}} \\ + c\,\frac{2\omega_d^{\frac{m}{d}}(m+1)A_{m+3}\varrho^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)M^{m+2}} k^{1-\frac{m}{d}},\end{aligned}$$

where $k \ge 1$, a is defined by the equality in (39),

$$\varrho = \frac{d-1}{(2\pi)^d}|\Omega|, \quad m_s = \frac{2\sqrt{(d^2-d)|\Omega| I(\Omega)}}{(2\pi)^d}, \quad M = \frac{\sqrt{d^2-d}}{(2\pi)^d}|\Omega|^{\frac{d+1}{d}}\omega_d^{-\frac{1}{d}}, \tag{13}$$

and the remaining notation is defined by

$$A_r = (a+1)^r - a^r, \qquad c = \min\big\{1, \max\{a_1, a_2\}\big\},$$

$$a_1 = \frac{(d+2)(m+3)}{(m+1)\big((m+2)d+m\big)} \frac{(2\pi)^{m+2}}{A_{m+3}} d^{\frac{m+2}{2}} (d-1)^{-\frac{m(d+2)+2m}{2d}},$$

and

$$a_2 = \frac{\sqrt{2} d^{\frac{1}{d}}}{2(d-1)^{\frac{d+2}{2d}}} \frac{[(m+1)d+m-1](m+3)}{[(m+2)d+m](m+1)} \frac{A_{m+2}}{A_{m+3}}.$$

Theorem 3 yields no improvement when $d = 2$, but it quickly yields the following enhanced lower bound for any dimension $d \geq 3$:

Corollary 1. *For any bounded domain $\Omega \subseteq \mathbb{R}^d$, with $d \geq 3$, and any $k \geq 1$, the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the Stokes operator in (1) satisfy*

$$\sum_{i=1}^{k} \lambda_i \geq \frac{4\pi d}{d+2} \left(\frac{\Gamma\left(1+\frac{d}{2}\right)}{(d-1)\,|\Omega|} \right)^{\frac{2}{d}} k^{1+\frac{2}{d}} + \frac{\varrho^2}{6(d+2)m_s^2} k + \frac{\omega_d^{-\frac{1}{d}} \varrho^{\frac{3d+1}{d}}}{9(d+2)m_s^3} k^{\frac{d-1}{d}} + \frac{3\omega_d^{-\frac{1}{d}} \varrho^{\frac{4d+2}{d}}}{80(n+2)m_s^4} k^{\frac{d-2}{d}},$$

where ϱ and m_s are as in (13).

When $d = 3$, we have an improvement:

Theorem 4. *Let $\Omega \subset \mathbb{R}^d$ with $d = 3$. Then the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the Stokes operator in (1) satisfy*

$$\sum_{i=1}^{k} \lambda_i \geq \frac{2\pi \varrho^6}{15 m_s^5} \left((t+1)^6 - t^6 \right), \tag{14}$$

where m_s and ϱ are defined in (13) and

$$t = \frac{1}{2}\left(\left(T + \sqrt{T^2 + \tfrac{1}{27}} \right)^{\frac{1}{3}} - \left(-T + \sqrt{T^2 + \tfrac{1}{27}} \right)^{\frac{1}{3}} - 1 \right), \quad T = \frac{(d+1)m_s^d}{\omega_d \varrho^{d+1}} k.$$

Furthermore,

$$\sum_{i=1}^{k} \lambda_i \geq \frac{3}{5} \left(\frac{2\pi^3}{2\omega_3 |\Omega|} \right)^{\frac{2}{3}} k^{\frac{5}{3}} + \frac{1}{24} \frac{|\Omega|}{I(\Omega)} k - \frac{11\pi \omega_3^{-\frac{1}{3}}}{180} \frac{\varrho^{\frac{14}{3}}}{m_s^4} k^{\frac{1}{3}} + \frac{659\pi \omega_3^{\frac{1}{3}}}{75000 \cdot 4^{\frac{1}{3}}} \frac{\varrho^{\frac{22}{3}}}{m_s^6} k^{-\frac{1}{3}}.$$

Inspired by Melas's research [16], we adopted a similar methodology using fundamental techniques from prior studies [10; 22; 24; 27; 28; 29; 26]. While preserving the core strategy of these works, we introduced significant modifications to achieve stronger lower bounds.

Outline. Section 2 presents results needed for proving the theorems discussed in this work. The main content is found in Sections 3–6, which respectively establish Theorems 1, 2, 3 and 4, proving intermediate results as needed.

2. Preliminaries

The set of eigenfunctions $\{\boldsymbol{u}_j\}_{j=1}^{\infty}$ of the Stokes operator is orthonormal in $L^2(\Omega)$, and so the set of Fourier transforms $\{\hat{\boldsymbol{u}}_j\}_{j=1}^{\infty}$ also forms an orthonormal set in $L^2(\mathbb{R}^d)$, by Plancherel's theorem. Let us set

$$U_k(\xi) = \sum_{j=1}^{k} |\hat{\boldsymbol{u}}_j(\xi)|^2 = \sum_{j=1}^{k} \left| \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\Omega} e^{-i\boldsymbol{x}\cdot\xi} \boldsymbol{u}_j(\boldsymbol{x})\, d\boldsymbol{x} \right|^2 \geq 0. \tag{15}$$

The integral is taken over Ω instead of $\mathbb{R}^d$ because the support of $\boldsymbol{u}_j$ is Ω.

The following result, proven by A. A. Ilyin [10, Section 2], is essential to our calculations.

Lemma 1 [10]. *The functions U_k defined by (15) satisfy*

$$\int_{\mathbb{R}^d} U_k(\xi)\, d\xi = k, \tag{16}$$

$$U_k(\xi) \leq \frac{d-1}{(2\pi)^d} |\Omega|, \quad \xi \in \mathbb{R}^d, \tag{17}$$

$$\int_{\mathbb{R}^d} |\xi|^2\, U_k(\xi)\, d\xi = \sum_{j=1}^{k} \lambda_j. \tag{18}$$

Note that equations (16)–(18) bear a striking resemblance to their Laplacian counterparts. However, the proof of Lemma 1 employs suborthonormal functions [10].

We highlight some important properties of the function U. Let $U^*(\xi)$ designate the radial decreasing rearrangement of $U(\xi)$. By approximating U if needed, we may assume that there exists a real-valued absolutely continuous function

$$\zeta : [0, \infty) \to \left[0, \frac{d-1}{(2\pi)^d} |\Omega|\right]$$

such that $U^*(\xi) = \zeta(|\xi|)$. Then

$$0 \leq -\zeta' \leq m_s, \tag{19}$$

where m_s depends only on d and Ω and is explicitly defined in (13). To see (19), let μ be the distribution function defined by

$$\mu(s) = |\{U(\xi) > s\}| = |\{U^*(\xi) > s\}|.$$

Then, we observe that

$$\mu(\zeta(t)) = \left|\{U^*(\xi) > \zeta(t)\}\right| = \left|\{\xi : |\xi| < t\}\right| = |B_t(0)| = w_d t^d.$$

Invoking the coarea formula in view of (17), we have

$$\mu(s) = \int_s^\infty \int_{\{U^{-1}(t)\}} \frac{1}{|\nabla U|} d\mathcal{H}\, dt = \int_s^{\frac{d-1}{(2\pi)^d}|\Omega|} \int_{\{U=t\}} \frac{1}{|\nabla U|} d\mathcal{H}\, dt,$$

where $\mathcal{H}$ is the $(d-1)$-dimensional Hausdorff measure. Let us consider $t > 0$ values such that $\zeta'(t) < 0$.

Let $\overline{K}$ denote the closure of $K \subset \mathbb{R}^d$ and ∂K the boundary of K. The isoperimetric inequality,

$$\mathcal{H}(\partial K) \geq d w_d^{\frac{1}{d}} |\overline{K}|^{\frac{d-1}{d}},$$

enables us to deduce

$$\begin{aligned}\frac{d w_d t^{d-1}}{\zeta'(t)} &= \mu'(\zeta(t)) = -\int_{\{U=\zeta(t)\}} \frac{1}{|\nabla U|} d\mathcal{H} \leq -\frac{1}{m_S}\mathcal{H}(\{U = \zeta(t)\}) \\ &\leq -\frac{1}{m_S} d w_d^{\frac{1}{d}} \mu(\zeta(t))^{\frac{d-1}{d}} = -\frac{1}{m_S} d w_d t^{d-1}.\end{aligned}$$

This inequality together with $\zeta' \leq 0$ easily yields (19). Note that (19) essentially states that if the gradient vector of the original function is bounded, then the gradient of the rearrangement retains the same bound. (An alternative derivation is possible, using the Pólya–Szegő inequality.)

Now, let r represent the real number such that $|\Omega| = w_d r^d$. We can get a lower bound for $I(\Omega)$:

$$I(\Omega) \geq \int_{B_r(0)} |\boldsymbol{x}|^2\, d\boldsymbol{x} = \frac{d w_d}{d+2} r^{d+2} = \frac{d}{d+2} w_d^{-\frac{2}{d}} |\Omega|^{\frac{d+2}{d}},$$

which quickly results in a lower bound for m_s as follows:

$$m_s = \frac{2\sqrt{(d^2-d)|\Omega| I(\Omega)}}{(2\pi)^d} \geq \frac{\sqrt{d^2-d}}{(2\pi)^d} |\Omega|^{\frac{d+1}{d}} \omega_d^{-\frac{1}{d}} =: M. \tag{20}$$

3. Proof of Theorem 1

The following proof is inspired from the proof of the Berezin–Li–Yau inequality in [5; 14; 28; 29]. An analogous proof is also given in [20] by means of the bathtub principle [15].

Proof of Theorem 1. Assume the properties (16)–(18). Since $|\hat{\boldsymbol{u}}_j(\xi)|^2\, d\xi$ is a probability measure on $\mathbb{R}^d$ and $t \mapsto t^{-b}$ is convex for $t > 0$ and $b > 0$, employing Jensen's inequality and (15), we obtain

$$\sum_{j=1}^k \lambda_j^{-b} = \sum_{j=1}^k \left(\int_{\mathbb{R}^d} |\xi|^2\, |\hat{\boldsymbol{u}}_j(\xi)|^2\, d\xi \right)^{-b} \leq \int_{\mathbb{R}^d} \frac{U_k(\xi)}{|\xi|^{2b}}\, d\xi. \tag{21}$$

Let $\mathbf{1}_\Omega(t)$ denote the characteristic function of the set Ω. Define

$$\Psi_k(\xi) = \frac{d-1}{(2\pi)^d}|\Omega|\mathbf{1}_{B_{S_k}(0)}(\xi) \quad \text{and} \tag{22}$$
$$S_k = \left(\frac{(d-2b)(2\pi)^d\left(\sum_{j=1}^k \lambda_j^{-b}\right)}{d\,(d-1)\,\omega_d\,|\Omega|}\right)^{\frac{1}{d-2b}},$$

so that

$$\int_{\mathbb{R}^d} \frac{\Psi_k(\xi)}{|\xi|^{2b}}\,d\xi = \frac{d-1}{(2\pi)^d}|\Omega|\,d\,\omega_d \int_0^{S_k} \frac{1}{t^{2b}}\,t^{d-1}\,dt \tag{23}$$
$$= \frac{d-1}{(2\pi)^d}|\Omega|\,\frac{d\,\omega_d}{d-2b} S_k^{d-2b} = \sum_{j=1}^k \lambda_j^{-b}.$$

Notice that

$$\left(\frac{1}{|\xi|^{2b}} - \frac{1}{S_k^{2b}}\right)(U_k(\xi) - \Psi_k(\xi)) \le 0. \tag{24}$$

Integrating (24) on $\mathbb{R}^d$ and using (23) we arrive at

$$\frac{1}{S_k^{2b}} \int_{\mathbb{R}^d} (U_k(\xi) - \Psi_k(\xi))\,d\xi \ge \int_{\mathbb{R}^d} \frac{U_k(\xi) - \Psi_k(\xi)}{|\xi|^{2b}}\,d\xi \ge 0 \tag{25}$$

from which it follows that

$$\int_{\mathbb{R}^d} U_k(\mu)\,dmu \ge \int_{\mathbb{R}^d} \Psi_k(\mu)\,d\mu.$$

Thus, by (16), we obtain

$$k \ge \int_{\mathbb{R}^d} \Psi_k(\mu)\,d\mu = \frac{d-1}{(2\pi)^d}|\Omega|\,d\omega_d\left(\frac{S_k^d}{d}\right). \tag{26}$$

Substituting ω_d given by (11) and S_k given by (22) into (26) and rearranging the terms, we obtain (8). □

4. Proof of Theorem 2

We sketch a direct proof of (9) for the sake of completeness.

Proof of Theorem 2. Recall that U_k satisfies (16)–(18). Since $|\hat{\boldsymbol{u}}_j(\xi)|^2\,d\xi$ is a probability measure on $\mathbb{R}^d$ and $r \mapsto r^a$ is concave for $r > 0$ and $0 < a \le 1$, we can use Jensen's inequality to derive that

$$\sum_{j=1}^k \lambda_j^a = \sum_{j=1}^k \left(\int_{\mathbb{R}^d} |\xi|^2\,|\hat{\boldsymbol{u}}_j(\xi)|^2\,d\xi\right)^a \ge \int_{\mathbb{R}^d} |\xi|^{2a}\,U_k(\xi)\,d\xi. \tag{27}$$

Setting

$$\Theta_k(\xi) = \frac{d-1}{(2\pi)^d}|\Omega|\,\mathbf{1}_{B_{T_k}(0)}(\xi), \quad T_k = \left(\frac{(d+2a)(2\pi)^d\left(\sum_{j=1}^k \lambda_j^a\right)}{d\,(d-1)\,\omega_d\,|\Omega|}\right)^{\frac{1}{d+2a}}, \tag{28}$$

we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\xi|^{2a}\,\Theta_k(\xi)\,d\xi &= \frac{d-1}{(2\pi)^d}|\Omega|\,d\,\omega_d \int_0^{T_k} r^{2a} r^{d-1}\,dr \\ &= \frac{d-1}{(2\pi)^d}|\Omega|\,\frac{d\,\omega_d}{d+2a}T_k^{d+2a} = \sum_{j=1}^k \lambda_j^a.\end{aligned} \tag{29}$$

Now observe that

$$(|\xi|^{2a} - T_k^{2a})(U_k(\xi) - \Theta_k(\xi)) \ge 0. \tag{30}$$

Integrating (30) on $\mathbb{R}^d$ and using (27) and (29) we conclude that

$$T_k^{2a}\int_{\mathbb{R}^d}(U_k(\xi) - \Theta_k(\xi))\,d\xi \le \int_{\mathbb{R}^d} |\xi|^{2a}\,(U_k(\xi) - \Theta_k(\xi))\,d\xi \le 0, \tag{31}$$

which yields

$$\int_{\mathbb{R}^d} U_k(\xi)\,d\xi \le \int_{\mathbb{R}^d} \Theta_k(\xi)\,d\xi.$$

Now we use (16) to obtain

$$k \le \int_{\mathbb{R}^d} \Theta_k(\xi)\,d\xi = \frac{d-1}{(2\pi)^d}|\Omega|\,d\,\omega_d\left(\frac{T_k^d}{d}\right). \tag{32}$$

Substituting the values of ω_d and T_k given in (11) and (28) into (32) and simplifying, we deduce (9). □

5. Proof of Theorem 3

Before the proof of Theorem 3, we give some elementary results from [11; 24]. To connect two key integrals in our proof, we introduce an important quantity, a, using the idea in [24]:

Lemma 2. *For any function* $\alpha : [0,\infty) \to [0,1]$ *and integer* $d \ge 1$ *with*

$$\int_0^\infty \phi(t)\,dt = 1, \quad \int_0^\infty t^d\phi(t)\,dt < \infty, \quad \int_0^\infty t^{d+2}\phi(t)\,dt \le \infty,$$

there exists an $a \ge 0$ *such that*

$$\int_a^{a+1} t^d\,dt = \int_0^\infty t^d\phi(t)\,dt \quad \textit{and} \quad \int_a^{a+1} t^{d+2}dt \le \int_0^\infty t^{d+2}\phi(t)\,dt.$$

The following sharp inequality is key in estimating the lower bounds for $\sum_{i=1}^{k}\lambda_i$.

Lemma 3 [11]. *For an integer $d \geq m+1 \geq 2$ and positive real numbers t and τ we have*

$$dt^{d+2}-(d+2)\tau^2t^d+2\tau^{d+2}-\sum_{k=1}^{m+1}2kt^{k-1}\tau^{n-k+1}(\tau-t)^2 \geq 0. \tag{33}$$

Considering all of these findings, we arrive at the following intermediate estimate.

Proposition 1. *If $d \geq m+1 \geq 2$, we have*

$$\sum_{i=1}^{k}\lambda_i \geq \omega_d\,(dA)^{\frac{d+2}{d}}(\zeta(0))^{-\frac{2}{d}}-\frac{2\omega_d A_{m+2}(dA)^{\frac{d-m+1}{d}}(\zeta(0))^{\frac{(m+1)d+m-1}{d}}}{(d+2)m_s^{m+1}} + \frac{2\omega_d(m+1)A_{m+3}(dA)^{\frac{d-m}{n}}(\zeta(0))^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)m_s^{m+2}}, \tag{34}$$

where m_s is defined in (13) and

$$A_r := (a+1)^r - a^r \geq 1, \quad A := \frac{1}{d}\int_a^{a+1} t^d\,dt, \quad 0 \leq \zeta(0) \leq \frac{d-1}{(2\pi)^d}|\Omega|,$$

the quantity a being defined by

$$\int_a^{a+1} t^d\,dt = \frac{k}{\omega_d}.$$

Proof. We use a strategy similar to that of [11; 24]. Since the map $\xi \mapsto |\xi|^2$ is radial and increasing, Lemma 1 implies that

$$k = \int_{\mathbb{R}^d} U(\xi)\,d\xi = \int_{\mathbb{R}^d} U^*(\xi)\,d\xi = d\omega_d\int_0^\infty t^{d-1}\zeta(t)\,dt, \tag{35}$$

$$\sum_{i=1}^{k}\lambda_i = \int_{\mathbb{R}^d}|\xi|^2U(\xi)\,d\xi \geq \int_{\mathbb{R}^d}|\xi|^2U^*(\xi)\,d\xi = d\omega_d\int_0^\infty t^{d+1}\zeta(t)\,dt. \tag{36}$$

For convenience, we rescale ζ as

$$S(t) = \frac{1}{\zeta(0)}\zeta\left(\frac{\zeta(0)}{m_s}t\right).$$

Obviously, $S(t)$ is a positive function with $S(0)=1$ and $0 \leq -S'(t) \leq 1$. Let $\phi(t) := -S'(t)$ and define

$$A = \int_0^\infty t^{d-1}S(t)\,dt, \quad B = \int_0^\infty t^{d+1}S(t)\,dt. \tag{37}$$

Integrating by parts, we get

$$\int_0^\infty t^d \phi(t)\,dt = dA, \quad \int_0^\infty t^{d+2}\phi(t)\,dt = (d+2)B. \tag{38}$$

According to (38) and Lemma 2, there exists $a \geq 0$ such that

$$\int_a^{a+1} t^d dt = dA \quad \text{and} \quad \int_a^{a+1} t^{d+2} dt \leq (d+2)B. \tag{39}$$

For $i \geq 0$, $\tau \geq \frac{1}{2}$ and $a \geq 0$, we have

$$\int_a^{a+1} t^i (\tau - t)^2 dt = \frac{A_{i+3}}{i+3} - \frac{2A_{i+2}}{i+2}\tau + \frac{A_{i+1}}{i+1}\tau^2, \tag{40}$$

where

$$A_r = (a+1)^r - a^r \geq 1.$$

Taking into account Lemma 3 and (40), we may integrate (33) in t from a to $a+1$ to get

$$\begin{aligned} d(d+2)B \geq d(d+2)A\tau^2 - 2\tau^{d+2} \\ + \sum_{i=1}^{m+1} 2i\tau^{d-i+1}\left(\frac{A_i}{i}\tau^2 - \frac{2A_{i+1}}{i+1}\tau + \frac{A_{i+2}}{i+2}\right). \end{aligned} \tag{41}$$

The summation on the second row of (41) can be rewritten as

$$\begin{aligned} &= 2\tau^{d+2} + 2\sum_{i=1}^{m} A_{i+1}\tau^{d-i+2} - 2\sum_{i=1}^{m+1} \frac{2iA_{i+1}}{i+1}\tau^{d-i+2} + 2\sum_{i=1}^{m+1} \frac{iA_{i+2}}{i+2}\tau^{d-i+1} \\ &= 2\tau^{d+2} + 2A_2\tau^{d+1} + \frac{2mA_{m+2}}{m+2}\tau^{d-m+1} + \frac{2(m+1)A_{m+3}}{m+3}\tau^{d-m} \\ &\quad -2A_2\tau^{d+1} - \frac{4(m+1)A_{m+2}}{m+2}\tau^{d-m+1} + 2\sum_{i=2}^{m}\left(1 + \frac{i-1}{i+1} - \frac{2i}{i+1}\right)A_{i+1}\tau^{d-i+2}. \end{aligned}$$

The parenthetical factor in this last summation vanishes, so the right-hand side of the preceding display boils down to

$$\frac{2(m+1)A_{m+3}}{m+3}\tau^{d-m} + 2\tau^{d+2} - 2A_{m+2}\tau^{d-m+1},$$

and we deduce that

$$d(d+2)B - (d+2)\tau^2 dA \geq \frac{2(m+1)A_{m+3}}{m+3}\tau^{d-m} - 2A_{m+2}\tau^{d-m+1}. \tag{42}$$

Using Jensen's inequality, we obtain $(dA)^{\frac{1}{d}} \geq \int_\delta^{\delta+1} t\,dt \geq \frac{1}{2}$ for any $\delta \geq 0$. Hence, putting $\tau = (dA)^{\frac{1}{d}}$ in (42), we arrive at

$$B \geq \frac{(dA)^{\frac{d+2}{d}}}{d} - \frac{2A_{m+2}(dA)^{\frac{d-m+1}{n}}}{d(d+2)} + \frac{2(m+1)A_{m+3}(nA)^{\frac{d-m}{d}}}{d(d+2)(m+3)}. \tag{43}$$

Taking the rescaling of $\zeta(t)$ into consideration, we obtain from (43) the integral inequality

$$\int_0^\infty t^{s+1}\zeta(t)\,dt \geq \frac{(dA)^{\frac{d+2}{n}}(\zeta(0))^{-\frac{2}{d}}}{d} - \frac{2S_{m+2}(dA)^{\frac{d-m+1}{d}}(\zeta(0))^{\frac{(m+1)d+m-1}{d}}}{d(d+2)m_s^{m+1}} + \frac{2(m+1)S_{m+3}(dA)^{\frac{d-m}{d}}(\zeta(0))^{\frac{(m+2)n+m}{d}}}{d(d+2)(m+3)m_s^{m+2}}.$$

Notice that $\inf_t \zeta(t) \leq \zeta(0) \leq \sup_t \zeta(t)$. Due to (36), the estimate of $\int_0^\infty t^{s+1}\zeta(t)\,dt$ provides a lower bound

$$\sum_{i=1}^k \lambda_i \geq \omega_d\,(dA)^{\frac{d+2}{d}}(\zeta(0))^{-\frac{2}{d}} - \frac{2\omega_d A_{m+2}(dA)^{\frac{d-m+1}{d}}(\zeta(0))^{\frac{(m+1)d+m-1}{d}}}{(d+2)m_s^{m+1}} + \frac{2\omega_d(m+1)A_{m+3}(dA)^{\frac{d-m}{n}}(\zeta(0))^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)m_s^{m+2}}.$$

This is what we required in Proposition 1. □

The right-hand side of the intermediate lower bound (34) depends on $\zeta(0)$. Our goal is to derive a final eigenvalue inequality that depends solely on d and k. To achieve this, we adopt a strategy of minimizing (34) with respect to $\zeta(0)$ over the interval $[0,\varrho]$, where $\varrho = \frac{d-1}{(2\pi)^d}|\Omega|$ as defined in (13).

Proof of Theorem 3. Using (35) and (38) in combination with integration by parts, we obtain

$$\frac{k}{\omega_n} = d\int_0^\infty t^{d-1}\zeta(t)\,dt = \int_0^\infty -t^d\zeta'(t)\,dt = \int_0^\infty t^d\phi(t)\,dt = dA.$$

Using the fact that $dA = k/\omega_d$, we define $G_1(x)$, $G_2(x)$ and $G(x)$ for $x \in [0,\varrho]$ by

$$G_1(x) = \omega_d^{-\frac{2}{d}}x^{-\frac{2}{d}}k^{\frac{d+2}{d}} + c_2\frac{2\omega_d^{\frac{m}{d}}(m+1)A_{m+3}x^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)m_s^{m+2}}k^{\frac{d-m}{d}},$$

$$G_2(x) = -\frac{2\omega_d^{\frac{m-1}{d}}A_{m+2}x^{\frac{(m+1)d+m-1}{d}}}{(d+2)m_s^{m+1}}k^{\frac{d-m+1}{d}},$$

$$G(x) = G_1(x) - G_2(x).$$

Here $0 < c_2 \le 1$ is a positive number to be determined. Since $m \ge 1$, we conclude that $G_2(x)$ is decreasing on $[0, \varrho]$. By direct calculation, we have

$$\frac{d}{2}\frac{\omega_d^{\frac{2}{d}} G_1'(x)}{k^{\frac{d-m}{d}}} = -k^{\frac{2+d}{d}} x^{-\frac{d+2}{d}} + c_2 \omega_d^{\frac{m+2}{d}} \frac{(m+1)\big((m+2)d+m\big)}{(d+2)(m+3)} \frac{A_{m+3}}{m_s^{m+2}} x^{\frac{(m+1)d+m}{d}}.$$

Therefore, when

$$c_2 \le \frac{k^{\frac{d+2}{d}}}{\omega_d^{\frac{m+2}{d}}} \frac{(d+2)(m+3)}{(m+1)[(m+2)d+m]} \frac{m_s^{m+2}}{A_{m+3}} \varrho^{-\frac{(m+2)(d+1)}{d}},$$

we conclude that $G_1'(x) \le 0$ on $[0, \varrho]$. Since

$$m_s \ge M = \frac{\sqrt{d^2-d}}{(2\pi)^d \omega_d^{\frac{1}{d}}} |\Omega|^{\frac{d+1}{d}}, \quad \varrho = \frac{d-1}{(2\pi)^d} |\Omega|, \tag{44}$$

we can choose $c_1 = \min\{1, a_1\}$, where a_1 is defined by

$$a_1 = \frac{(d+2)(m+3)}{(m+1)\big[(m+2)d+m\big]} \frac{(2\pi)^{m+2}}{A_{m+3}} \frac{d^{\frac{m+2}{2}}}{(d-1)^{\frac{m(d+2)+2m}{2d}}}, \tag{45}$$

to obtain

$$\sum_{i=1}^{k} \lambda_i \ge \omega_d^{-\frac{2}{d}} \varrho^{-\frac{2}{d}} k^{\frac{d+2}{d}} - \frac{2\omega_d^{\frac{m-1}{d}} A_{m+2} \varrho^{\frac{(m+1)d+m-1}{d}}}{(d+2)M^{m+1}} k^{\frac{d-m+1}{d}} + c_1 \frac{2\omega_d^{\frac{m}{d}} (m+1) A_{m+3} \varrho^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)M^{m+2}} k^{\frac{d-m}{d}}, \tag{46}$$

where ϱ and M are defined in (13). Similarly, we can split $G(x)$ as a sum of two functions:

$$G(x) = g_1(x) + g_2(x), \quad x \in [0, \varrho],$$

where g_1 and g_2 are defined by $g_1(x) = \omega_d^{-\frac{2}{d}} x^{-\frac{2}{d}} k^{\frac{d+2}{d}}$ and

$$g_2(x) = c_2 \frac{2\omega_d^{\frac{m}{d}} (m+1) A_{m+3} x^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3) m_s^{m+2}} k^{\frac{d-m}{d}} - \frac{2\omega_d^{\frac{m-1}{d}} A_{m+2} x^{\frac{(m+1)d+m-1}{d}}}{(d+2) m_s^{m+1}} k^{\frac{d-m+1}{d}}.$$

Obviously, $g_1(x)$ is decreasing on $[0, \varrho]$. Rewriting $g_2(x)$ as

$$\frac{d(d+2)}{2} \frac{m_s}{\omega_d^{\frac{m}{d}} k^{\frac{d-m}{d}}} g_2(x) = c_2 \frac{[(m+2)d+m](m+1)A_{m+3}}{(m+3)m_s} x^{\frac{(m+1)d+m}{d}} - ((m+1)d+m-1)\omega_d^{-\frac{1}{d}} A_{m+2} k^{\frac{1}{d}} x^{\frac{(d+1)m-1}{d}}$$

allows us to observe that if

$$c_2 \leq \frac{[(m+1)d+m-1](m+3)}{[(m+2)d+m](m+1)} \frac{A_{m+2}}{A_{m+3}} \frac{m_s}{\omega_d^{1/d}} \varrho^{-\frac{d+1}{d}},$$

then $g_2'(x) \leq 0$ on $[0, \varrho]$. Using (44) again, we choose $c_3 = \min\{1, a_2\}$, where a_2 is defined by

$$a_2 = \frac{\sqrt{2} d^{\frac{1}{d}}}{2(d-1)^{\frac{d+2}{2d}}} \frac{[(m+1)d+m-1](m+3)}{[(m+2)d+m](m+1)} \frac{A_{m+2}}{A_{m+3}}, \tag{47}$$

to derive that

$$\sum_{i=1}^{k} \lambda_i \geq \omega_d^{-\frac{2}{d}} \varrho^{-\frac{2}{d}} k^{\frac{d+2}{d}} - \frac{2\omega_d^{\frac{m-1}{d}} A_{m+2} \varrho^{\frac{(m+1)d+m-1}{d}}}{(d+2)M^{m+1}} k^{\frac{d-m+1}{d}} + c_3 \frac{2\omega_d^{\frac{m}{d}} (m+1) A_{m+3} \varrho^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)M^{m+2}} k^{\frac{d-m}{d}}. \tag{48}$$

In light of equations (46) and (48), we can use $c = \min\{1, \max\{a_1, a_2\}\}$, where a_1 and a_2 are defined in (45) and (47), to get

$$\sum_{i=1}^{k} \lambda_i \geq \omega_d^{-\frac{2}{d}} \varrho^{-\frac{2}{d}} k^{\frac{d+2}{d}} - \frac{2\omega_d^{\frac{m-1}{d}} A_{m+2} \varrho^{\frac{(m+1)d+m-1}{d}}}{(d+2)M^{m+1}} k^{\frac{d-m+1}{d}} + c \frac{2\omega_d^{\frac{m}{d}} (m+1) A_{m+3} \varrho^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)M^{m+2}} k^{\frac{d-m}{d}}. \tag{49}$$

Finally, we use

$$\omega_d^{-\frac{2}{d}} \varrho^{-\frac{2}{d}} k^{\frac{d+2}{d}} = 4\pi \left(\frac{\Gamma\left(1+\frac{d}{2}\right)}{(d-1)\,|\Omega|} \right)^{\frac{2}{d}} k^{1+\frac{2}{d}} \tag{50}$$

to complete the proof of Theorem 3. □

Proof of Corollary 1. An argument similar to that of [11] gives

$$d(d+2)B - d(d+2)A\tau^2 + 2\tau^{d+2} \geq \tfrac{1}{6}(\tau^d) + \tfrac{1}{9}(\tau^{d-1}) + \tfrac{3}{80}\tau^{d-2}.$$

Hence, we get

$$\sum_{i=1}^{k} \lambda_i \geq \frac{d\omega_d^{-\frac{2}{d}} \zeta(0)^{-\frac{2}{d}}}{d+2} k^{\frac{d+2}{d}} + \frac{\zeta(0)^2}{6(d+2)m_s^2} k + \frac{c_4 \omega_d^{-\frac{1}{d}} \zeta(0)^{\frac{3d+1}{d}}}{(d+2)m_s^3} k^{\frac{d-1}{d}} + \frac{c_5 \omega_d^{-\frac{1}{d}} \zeta(0)^{\frac{4d+2}{d}}}{(n+2)m_s^4} k^{\frac{d-2}{d}},$$

where $0 < c_4 \le \frac{1}{9}$ and $0 < c_5 \le \frac{3}{80}$ are two constants to be chosen. Following an approach analogous to the discussion in the proof of Theorem 3, we can choose $c_4 = \frac{1}{9}$ and $c_5 = \frac{3}{80}$. Using the equality in (50) completes the proof. □

6. Proof of Theorem 4

The following lemma from [10] plays a pivotal role in establishing our eigenvalue inequality and makes use of the piecewise continuous function

$$\Phi_s(r) = \begin{cases} m_s & \text{if } 0 \le r \le s, \\ m_s - \varrho(r-s) & \text{if } s \le r \le s + m_s/\varrho, \\ 0 & \text{if } s + m_s/\varrho \le r, \end{cases}$$

where m_s and ϱ are defined in (13).

Lemma 4 [10]. *Let $\alpha > 0$ be a real number and $\Psi(r)$ be a decreasing and absolutely continuous function such that*

$$\int_0^\infty r^\alpha\, \Psi(r)\, dr = \int_0^\infty r^\alpha\, \Phi_s(r)\, dr, \quad 0 \le \Psi \le m_s, \quad -L < \Psi' < 0.$$

Then, for any $\beta \ge \alpha$, the following integral inequality holds:

$$\int_0^\infty r^\beta\, \Psi(r)\, dr \ge \int_0^\infty r^\beta\, \Phi_s(r)\, dr.$$

Moreover, for any $\gamma \ge 0$, we have

$$\int_0^\infty r^\gamma\, \Phi_s(r)\, dr = \frac{\varrho^{\gamma+2}}{(\gamma+1)(\gamma+2) m_s^{\gamma+1}} \big((t+1)^{\gamma+2} - t^{\gamma+2}\big), \quad s = \frac{t m_s}{\varrho}.$$

Proof of Theorem 4. Let m_s be as in (13). Applying Lemma 4 to $\Psi(x) = \zeta(x)$ and setting $\alpha = d-1$, we obtain

$$\int_0^\infty t^{d-1} \zeta(t)\, dt = \frac{k}{d\omega_d} = \frac{\varrho^{d+1}}{d(d+1) m_s^d} \big((t+1)^{d+1} - t^{d+1}\big),$$

which in turn implies that t is the unique root of the equation

$$(t+1)^{d+1} - t^{d+1} = T := \frac{(d+1) m_s^d}{\omega_d \varrho^{d+1}} k.$$

Then Lemma 4 implies

$$\int_0^\infty t^{d+1} \zeta(t)\, dt \ge \frac{\varrho^{d+3}}{(d+2)(d+3) m_s^{d+2}} \big((t+1)^{d+3} - t^{d+3}\big). \tag{51}$$

One can easily check that T satisfies

$$T \geq 1, \quad T \geq \frac{d+1}{d-1} \frac{(4\pi)^d}{\omega_d^2} \left(\frac{d^2}{(d-1)(d+2)} \right)^{\frac{d}{2}}.$$

In particular, when $d = 3$, we consider the equation

$$(t+1)^4 - t^4 = T = \frac{4\varrho^3}{\omega_3 m_s^4} k,$$

whose positive root is

$$t = \frac{1}{2} \left(\left(T + \sqrt{T^2 + \tfrac{1}{27}} \right)^{\frac{1}{3}} - \left(-T + \sqrt{T^2 + \tfrac{1}{27}} \right)^{\frac{1}{3}} - 1 \right).$$

From (36), we obtain our desired result as

$$\sum_{i=1}^{k} \lambda_i \geq \frac{2\pi \varrho^6}{15 m_s^5} \left((t+1)^6 - t^6 \right). \tag{52}$$

Considering the Taylor series of t, we arrive at

$$t \geq -\tfrac{1}{2} + 4^{-\frac{1}{3}} T^{\frac{1}{3}} - \tfrac{1}{6} 2^{-\frac{1}{3}} T^{-\frac{1}{3}} + \tfrac{1}{324} 4^{-\frac{1}{3}} T^{-\frac{5}{3}} + \tfrac{1}{1944} 2^{-\frac{1}{3}} T^{-\frac{7}{3}} - \tfrac{1}{26244} 4^{-\frac{1}{3}} T^{-\frac{11}{3}} - \tfrac{1}{629856} 2^{-\frac{1}{3}} T^{-\frac{11}{3}},$$

where we used the fact that $T \geq 1$. Hence, we obtain

$$(t+1)^6 - t^6 \geq \tfrac{3}{4} 2^{-\frac{1}{3}} T^{\frac{5}{3}} + \tfrac{5}{8} T - \tfrac{11}{24} 2^{\frac{2}{3}} T^{\frac{1}{3}} + \tfrac{659}{10000} T^{-\frac{1}{3}}.$$

Putting the lower bound of $(t+1)^6 - t^6$ into (52), we get

$$\sum_{i=1}^{k} \lambda_i \geq \frac{3}{5} \left(\frac{2\pi^3}{2\omega_3 |\Omega|} \right)^{\frac{2}{3}} k^{\frac{5}{3}} + \frac{1}{24} \frac{|\Omega|}{I(\Omega)} k - \frac{11\pi}{180} \frac{\varrho^{\frac{14}{3}}}{\omega_3^{\frac{1}{3}} m_s^4} k^{\frac{1}{3}} + \frac{659\pi \omega_3^{\frac{1}{3}}}{75000 \cdot 4^{\frac{1}{3}}} \frac{\varrho^{\frac{22}{3}}}{m_s^6} k^{-\frac{1}{3}}.$$

This concludes the proof of Theorem 4. □

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THE CLASS Q AND MIXTURE DISTRIBUTIONS WITH DOMINATED CONTINUOUS SINGULAR PARTS

ALEXEY A. KHARTOV

We consider a new class Q of distribution functions F that have the property of rational-infinite divisibility: there exist some infinitely divisible distribution functions F_1 and F_2 such that $F_1 = F * F_2$. A distribution function of class Q is quasi-infinitely divisible in the sense that its characteristic function admits the Lévy-type representation with a "signed spectral measure". This class is a wide natural extension of the fundamental class of infinitely divisible distribution functions and it is being actively studied now. We are interested in conditions for a distribution function F to belong to class Q for the unexplored case, where F may have a continuous singular part. We propose a criterion under the assumption that the continuous singular part of F is dominated by the discrete part in a certain sense. The criterion generalizes the previous results by Alexeev and Khartov for discrete probability laws and the results by Berger and Kutlu for the mixtures of discrete and absolutely continuous laws. In addition, we describe the characteristic triplet of the corresponding Lévy-type representation, which may contain a continuous singular part. We also show that the assumption of the dominated continuous singular part cannot be omitted or even slightly extended (without some special assumptions). We apply the general criterion to some interesting particular examples. We also positively solve the decomposition problem stated by Lindner, Pan and Sato within the case being considered.

1. Introduction

This paper is devoted to the study of a new class of probability laws that naturally extends the fundamental class of infinitely divisible distributions.

Let F be a distribution function on the real line $\mathbb{R}$. Recall that F and the corresponding probability law are called *infinitely divisible* if for every positive integer n there exists a distribution function $F_{1/n}$ such that $F = (F_{1/n})^{*n}$, where

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"$*$" denotes the convolution operation, i.e., F is the n-fold convolution power of $F_{1/n}$. Let f be the characteristic function of F, i.e.,

$$f(t) := \int_{\mathbb{R}} e^{itx}\, dF(x), \quad t \in \mathbb{R}.$$

It is well-known (see [12; 31; 37]) that F is infinitely divisible if and only if f admits the *Lévy representation*

$$f(t) = \exp\left\{it\gamma - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \big(e^{itx} - 1 - it\sin(x)\big)\, dL(x)\right\}, \quad t \in \mathbb{R}, \tag{1}$$

with a *shift parameter* $\gamma \in \mathbb{R}$, a *the Gaussian variance* $\sigma^2 \geqslant 0$, and a *Lévy spectral function* $L : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, which is nondecreasing on each of the intervals $(-\infty, 0)$ and $(0, +\infty)$, and satisfies

$$\lim_{x\to-\infty} L(x) = \lim_{x\to+\infty} L(x) = 0, \tag{2}$$

and also

$$\int_{O_\delta} x^2\, dL(x) < \infty \quad \text{for any} \quad \delta > 0,$$

where $O_\delta := (-\delta, 0) \cup (0, \delta)$. The function L is assumed to be right-continuous at every point of the real line. Importantly, the *characteristic triplet* (γ, σ^2, L) is uniquely determined by f and hence by F. Due to this representation, the class of infinitely divisible probability laws has found a lot of applications through Lévy processes (see [31]), the stochastic calculus (see [4]), teletraffic models (see [25]), and actuarial mathematics (see [32]).

Let $\boldsymbol{I}$ denote the class of all infinitely divisible distribution functions on the real line. This class is naturally extended in the following way. We call a distribution function F (and the corresponding probability law) *rational-infinitely divisible* if there exist some infinitely divisible distribution functions F_1 and F_2 such that $F_1 = F * F_2$. In terms of characteristic functions, this definition is equivalent to the formula $f(t) = f_1(t)/f_2(t)$, $t \in \mathbb{R}$, for the characteristic function f of F, where f_1 and f_2 are the characteristic functions of some infinitely divisible distribution functions F_1 and F_2. We denote by $\boldsymbol{Q}$ the class of all rational-infinitely divisible distribution functions. Since F_2 may be chosen as degenerate at some point a (i.e., $f_2(t) = e^{ita}$, $t \in \mathbb{R}$), it is clear that, indeed, $\boldsymbol{I} \subset \boldsymbol{Q}$. Moreover, from the definition, it is seen that the characteristic function f of any $F \in \boldsymbol{Q}$ admits a *Lévy-type representation*. Namely, if F_1 and F_2 have characteristic triplets $(\gamma_1, \sigma_1^2, L_1)$ and $(\gamma_2, \sigma_2^2, L_2)$, then formula (1) holds with *the shift parameter* $\gamma = \gamma_1 - \gamma_2 \in \mathbb{R}$, *the Gaussian variance* $\sigma^2 = \sigma_1^2 - \sigma_2^2$, and *the spectral function* $L = L_1 - L_2$. In that case, L has a bounded total variation on $\mathbb{R} \setminus O_\delta$ for every $\delta > 0$, and, in general, it is nonmonotonic on the intervals $(-\infty, 0)$ and $(0, +\infty)$. The function L also inherits from L_1 and L_2 right-continuity on $\mathbb{R}$ and property (2). Moreover,

$$\int_{O_\delta} x^2\, d|L|(x) < \infty \quad \text{for any} \quad \delta > 0,$$

where we integrate over the variation of L. We now suppose that, conversely, f admits representation (1) with some $\gamma \in \mathbb{R}$, $\sigma^2 \geqslant 0$, and L satisfying the above conditions. Following Lindner and Sato [26], the corresponding distribution function F and probability law are called *quasi-infinitely divisible*. Let us fix any real γ_1 and γ_2 such that $\gamma = \gamma_1 - \gamma_2$. Let us fix any nonnegative σ_1^2 and σ_2^2 such that $\sigma_1^2 \geqslant \sigma_2^2$ and $\sigma^2 = \sigma_1^2 - \sigma_2^2$. Due to the Hahn–Jordan decomposition, it is not difficult to show that there exist some canonical Lévy spectral functions L_1 and L_2 satisfying the usual conditions with monotonicity such that $L = L_1 - L_2$. Then $f(t) = f_1(t)/f_2(t)$, $t \in \mathbb{R}$, where f_1 and f_2 are represented by the canonical Lévy formula (1) with characteristic triplets $(\gamma_1, \sigma_1^2, L_1)$ and $(\gamma_2, \sigma_2^2, L_2)$. So f_1 and f_2 are the characteristic functions for some infinitely divisible distribution functions and hence the distribution function F corresponding to such an f is rational-infinitely divisible. Thus $F \in \boldsymbol{Q}$ if and only if f admits representation (1) with some (γ, σ^2, L) satisfying the above conditions. Additionally, *the characteristic triplet* (γ, σ^2, L) is uniquely determined by f and hence by F as for infinitely divisible laws (this can be concluded from the assertion in [12, p. 80]). It is also clear that for any rational-infinitely divisible F its characteristic function f has no zeroes on the real line, i.e., $f(t) \neq 0$, $t \in \mathbb{R}$.

The class $\boldsymbol{Q}$ and its multivariate analog are objects of active study (see [2; 8; 26]) and they find interesting applications in probability limit and compactness theorems (see Sections 4 and 8 in [26], Section 3 in [3], the paper [19], and also [1; 17]), and in other areas (see, for instance, [10; 29; 30]). But, actually, nondegenerate representatives of $\boldsymbol{Q} \setminus \boldsymbol{I}$ appeared even earlier in the theory of decompositions of probability laws as components of certain infinitely divisible distribution functions (see [12, pp. 81–83; 27, p. 165]).

The class $\boldsymbol{Q}$ is seen to be rather wide. For instance, it contains the distribution function of every probability law that has a mass $> 1/2$ at some point. Hence the class $\boldsymbol{Q}$ contains nondegenerate distribution functions of some probability laws with bounded supports (see examples in [26]), which are “far” from the infinite divisibility property in a known sense (see [5]). So it is interesting and important to obtain criteria for belonging to the class $\boldsymbol{Q}$. The existing results in this direction usually have simple and nice formulations in terms of characteristic functions. The first quite general result of such type was obtained by Lindner, Pan, and Sato in [26] (see Theorem 8.1, p. 30). It states that a lattice distribution function F belongs to the class $\boldsymbol{Q}$ if and only if its characteristic function f does not have zeroes on the real line, i.e., $f(t) \neq 0$ for any $t \in \mathbb{R}$.

In [3] and [18], this result was generalized to the class of arbitrary discrete probability laws. Namely, a discrete distribution function F belongs to $\boldsymbol{Q}$ if and only if its characteristic function f is separated from zero, i.e., $|f(t)| \geqslant \mu$ for any $t \in \mathbb{R}$ and for some constant $\mu > 0$. Moreover, in that case, the components of the characteristic triplet are fully described. This result is a generalization of the previous one for discrete lattice distributions, because the absolute value $|f(\,\cdot\,)|$ of the characteristic function f of a discrete lattice distribution is a periodic continuous function on $\mathbb{R}$. Therefore such f is zero-free on the period segment (and hence on $\mathbb{R}$) if any only if it is separated from zero.

In [6], Berger considered mixtures of a degenerate law (with a nonzero coefficient) and absolutely continuous distributions. According to his result, a distribution function F of such type belongs to $\boldsymbol{Q}$ if and only if $f(t) \neq 0$, $t \in \mathbb{R}$. Moreover, the result describes the structure of the components of the characteristic triplet in that case. The author also formulated more general criterion for the case, when the degenerate law from the previous one is replaced by a discrete lattice distribution with characteristic function, which has no zeroes on the real line. At present, however, the most general criterion (among those that use assumptions about the type of distribution) is the following result for the mixtures of discrete and absolutely continuous probability laws, which was obtained by Berger and Kutlu in the paper [7]. Let us formulate it with more details here. Namely, assume that $F(x) = c_d F_d(x) + c_a F_a(x)$, $x \in \mathbb{R}$, where F_d is a discrete distribution function, F_a is an absolutely continuous distribution function, $c_d > 0$, $c_a \geqslant 0$, and $c_d + c_a = 1$. We write the characteristic function f in the corresponding form: $f(t) = c_d f_d(t) + c_a f_a(t)$, $t \in \mathbb{R}$. Then $F \in \boldsymbol{Q}$ if and only if $f(t) \neq 0$ and $|f_d(t)| \geqslant \mu$ for any $t \in \mathbb{R}$ with some constant $\mu > 0$. It is equivalent to the condition $|f(t)| \geqslant \mu'$ for any $t \in \mathbb{R}$ and for some constant $\mu' > 0$. Moreover, Berger and Kutlu showed the existence of some discrete part in the spectral function and they fully described its absolutely continuous part for this case. It should be noted that we are not aware of any similar results for purely absolutely continuous distribution functions F. However, for some cases the problem of membership in class $\boldsymbol{Q}$ for a given distribution function of such type is not difficult to solve by the general criteria proposed in [20] with some additional analysis.

This article is devoted to generalizing and complementing all the mentioned results (except [20]) for the case, when F may have a continuous singular part. Namely, we propose a criterion for a distribution function F to belong to class $\boldsymbol{Q}$ under the assumption that its continuous singular part is dominated by its discrete part in a certain sense. In fact, we show that the conditions on f from the results [6] and [7] are carried over to this case. In addition, we describe the characteristic triplet of the corresponding Lévy-type representation, which may contain some continuous singular part.

We next show that the assumption of a dominated continuous singular part cannot be omitted or even slightly extended without some additional assumptions. In addition, for any $F \in \boldsymbol{Q}$ we solve the decomposition problem, which was stated by Lindner, Pan, Sato in [26] (see Open Question 8.4), within the considered case. Here we obtain a positive solution generalizing similar results from [6] and [7].

The article has the following structure. Section 2 contains necessary preliminaries, more detailed statements of some preexisting results mentioned above and the formulations of the new results of the paper. In Section 3, we formulate some important known theorems and useful lemmata, which will be auxiliary tools needed for the proofs of our results. In Section 4, we first prove a key auxiliary lemma and we next propose the proofs of the main results of the article.

Throughout the paper, we use the following notation. We denote by $\mathbb{N}$ the set of positive integers and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbols $\mathbb{Z}$ and $\mathbb{Q}$ denote as usual the sets of all integers and rational numbers, respectively. Next, $\mathbb{C}$ is the set of all complex numbers. For any $z \in \mathbb{C}$ we denote by $\mathrm{Im}\{z\}$ and $\arg(z)$ its imaginary part and the principal value of the argument of z, respectively. If ψ is a complex-valued continuous function on $\mathbb{R}$ satisfying $\psi(0) = c \in \mathbb{C} \setminus \{0\}$ and $\psi(t) \neq 0$ for any $t \in \mathbb{R}$, then *the distinguished logarithm* $\mathrm{Ln}\,\psi$ is defined by the formula $\mathrm{Ln}\,\psi(t) := \ln|\psi(t)| + i\,\mathrm{Arg}\,\psi(t)$, $t \in \mathbb{R}$, where $\mathrm{Arg}\,\psi(t)$ is the argument of $\psi(t)$ uniquely defined on $\mathbb{R}$ by the continuity with the condition $\mathrm{Arg}\,\psi(0) = \arg(c) \in (-\pi, \pi]$. The symbol $\mathbb{1}_a$ with fixed $a \in \mathbb{R}$ denotes the distribution function of the degenerate law concentrated at the point a, i.e., $\mathbb{1}_a(x) = 1$ for $x \geqslant a$ and $\mathbb{1}_a(x) = 0$ for $x < a$. For any set A we denote by $\mathbb{I}_A$ the indicator function of A, i.e., $\mathbb{I}_A(x) = 1$ for any $x \in A$ and $\mathbb{I}_A(x) = 0$ for any $x \notin A$. The signum function is denoted by $\mathrm{sgn}(\,\cdot\,)$, i.e., $\mathrm{sgn}(x) = +1$ for $x > 0$, $\mathrm{sgn}(x) = -1$ for $x < 0$, and $\mathrm{sgn}(0) = 0$. For any finite set A the symbol $|A|$ denotes the number of elements of A. We always set $\sum_{k \in A} a_k = 0$ and $\prod_{k \in A} a_k = 1$ in the case $A = \varnothing$. For any two vectors x and y from $\mathbb{R}^n$ the standard scalar product is denoted by $\langle x, y\rangle$.

For any function G defined on $\mathbb{R}$ the limits $\lim_{x \to \pm\infty} G(x)$ are denoted by $G(\pm\infty)$, respectively, if these limits exist. The class of all functions $G : \mathbb{R} \to \mathbb{R}$ of bounded total variation on $\mathbb{R}$ (nonmonotonic in general), which are right-continuous at every point and $G(-\infty) = 0$, is denoted by $\boldsymbol{V}$. We denote by $\boldsymbol{V}_{\mathbb{C}}$ the set of all functions $G : \mathbb{R} \to \mathbb{C}$ of the form $G(x) = G_1(x) + iG_2(x)$, $x \in \mathbb{R}$, where G_1 and G_2 are from $\boldsymbol{V}$. For every $G \in \boldsymbol{V}$ (or $\boldsymbol{V}_{\mathbb{C}}$) its total variation on $\mathbb{R}$ will be denoted by $\|G\|$ and the total variation on $(-\infty, x]$ by $|G|(x)$, $x \in \mathbb{R}$. So we have $|G(x)| \leqslant |G|(x) \leqslant \|G\|$, $x \in \mathbb{R}$, and $|G|(+\infty) = \|G\|$. Next, we adopt the following convention. Let G be a function from $\boldsymbol{V}$ with the Fourier–Stieltjes transform g, i.e., $g(t) = \int_{\mathbb{R}} e^{itx}\, dG(x)$, $t \in \mathbb{R}$. In view of the uniqueness theorem for functions from $\boldsymbol{V}$, we set $\|g\| := \|G\|$. So $\|g\| = 0$ if and only if $g(t) = 0$, $t \in \mathbb{R}$, and $\|c \cdot g\| = |c| \cdot \|g\|$ for any $c \in \mathbb{R}$. Let G_1 and G_2 be functions from $\boldsymbol{V}$ with the

Fourier–Stieltjes transforms g_1 and g_2, correspondingly. The known inequalities $\|G_1 + G_2\| \leqslant \|G_1\| + \|G_2\|$ and $\|G_1 * G_2\| \leqslant \|G_1\| \cdot \|G_2\|$ are correspondingly written as $\|g_1 + g_2\| \leqslant \|g_1\| + \|g_2\|$ and $\|g_1 \cdot g_2\| \leqslant \|g_1\| \cdot \|g_2\|$. We recall that both $\boldsymbol{V}$ and the corresponding space of functions g with norm $\|\cdot\|$ will be complete normed spaces (see [11, p. 165]).

2. Criteria for belonging to class $\boldsymbol{Q}$

Let F be an arbitrary distribution function on the real line. According to the Lebesgue decomposition theorem, F admits the representation

$$F(x) = c_d F_d(x) + c_a F_a(x) + c_s F_s(x), \quad x \in \mathbb{R}, \tag{3}$$

where F_d, F_a, and F_s are discrete, absolutely continuous and continuous singular distribution functions, respectively. Here the coefficients c_d, c_a, and c_s are nonnegative constants such that $c_d + c_a + c_s = 1$. Let f be the characteristic function of F. It is represented in a similar way:

$$f(t) = c_d f_d(t) + c_a f_a(t) + c_s f_s(t), \quad t \in \mathbb{R}, \tag{4}$$

where f_d, f_a, and f_s are the characteristic functions corresponding to F_d, F_a, and F_s, respectively. It is well known that the summands in (3) or (4) are uniquely determined. So if any of the terms is not identically zero, then the corresponding coefficient, distribution function, and characteristic function are uniquely determined.

We will consider only the case that F has nonzero discrete part, i.e., $c_d > 0$ in (3). We write the distribution function F_d in the form

$$F_d(x) = \sum_{\substack{k \in \mathbb{N}_0 \\ x_k \leqslant x}} p_k, \quad x \in \mathbb{R}, \tag{5}$$

where x_k are distinct reals associated with weights $p_k \geqslant 0$, $k \in \mathbb{N}_0$, $\sum_{k=0}^{\infty} p_k = 1$. Hence f_d has the form

$$f_d(t) = \sum_{k \in \mathbb{N}_0} p_k e^{itx_k}, \quad t \in \mathbb{R}. \tag{6}$$

We define the carrier of the distribution corresponding to F_d:

$$\mathcal{X} := \{x_k : p_k > 0, k \in \mathbb{N}_0\}.$$

Obviously, $\mathcal{X} \neq \varnothing$. We also need the set of all finite $\mathbb{Z}$-linear combinations of elements from the set $\mathcal{X}$:

$$\langle \mathcal{X} \rangle := \left\{ \sum_{k=1}^{m} a_k z_k : a_k \in \mathbb{Z},\ z_k \in \mathcal{X},\ m \in \mathbb{N} \right\}.$$

So $\langle\mathcal{X}\rangle$ is the module over the ring $\mathbb{Z}$ with the generating set $\mathcal{X}$. It easily seen that, in particular, $\mathcal{X}\subset\langle\mathcal{X}\rangle$ and $0\in\langle\mathcal{X}\rangle$. If $\mathcal{X}\neq\{0\}$, then $\langle\mathcal{X}\rangle$ is an infinite countable set.

Now we are ready to formulate in detail the most general existing results on criteria for belonging to class $\boldsymbol{Q}$. We start with the result obtained in [3] and [18] by Alexeev and Khartov for the case of discrete F.

Theorem 1. *Suppose that F is a discrete distribution function, $c_d=1$ and $c_a=c_s=0$ in (3) (F and F_d are identical and hence f and f_d are too; F has the form (5), f is represented by (6)). Then $F\in\boldsymbol{Q}$ if and only if $\inf_{t\in\mathbb{R}}|f(t)|>0$. In that case, f admits the representation*

$$f(t)=\exp\Big\{it\gamma_0+\sum_{u\in\langle\mathcal{X}\rangle\setminus\{0\}}\lambda_u\big(e^{itu}-1\big)\Big\},\quad t\in\mathbb{R}, \tag{7}$$

with some $\gamma_0\in\langle\mathcal{X}\rangle$ and $\lambda_u\in\mathbb{R}$ for all $u\in\langle\mathcal{X}\rangle\setminus\{0\}$, and $\sum_{u\in\langle\mathcal{X}\rangle\setminus\{0\}}|\lambda_u|<\infty$.

It is not difficult to rewrite the representation (7) in integral form (1) with some $\gamma\in\mathbb{R}$, $\sigma^2=0$, and some discrete L, which will satisfy all the conditions for a spectral function for the quasi-infinitely divisibility. We will do this below for a more general case. Hence, if the characteristic function f is represented by (7), then $F\in\boldsymbol{Q}$.

It is clear that if $F\in\boldsymbol{Q}$ and $\lambda_u\geqslant 0$ for all $u\in\langle\mathcal{X}\rangle\setminus\{0\}$ in representation (7), then F is infinitely divisible, i.e., $F\in\boldsymbol{I}$. If there exists $\lambda_v<0$ with some $v\in\langle\mathcal{X}\rangle\setminus\{0\}$, then $F\in\boldsymbol{Q}\setminus\boldsymbol{I}$, because the (uniquely defined) function L will be decreasing in the neighborhood of v. For examples of the latter case see [26, p. 10] and [27, p. 165].

We now formulate Berger and Kutlu's result in [7] for the important case $c_a\geqslant 0$. As mentioned, at present, this is the most general criterion using information about the type of the distribution function F.

Theorem 2. *Suppose that F has the decomposition (3) with some $c_d>0$, $c_a\geqslant 0$, and $c_s=0$. Then the following statements are equivalent:*

(i) $F\in\boldsymbol{Q}$.

(ii) $\inf_{t\in\mathbb{R}}|f(t)|>0$.

(iii) $f(t)\neq 0$ *for any* $t\in\mathbb{R}$, *and* $\inf_{t\in\mathbb{R}}|f_d(t)|>0$.

If one (hence all) of the conditions is satisfied, then f admits the representation

$$f(t)=\exp\Big\{it\gamma_0+\sum_{u\in\mathcal{U}}\lambda_u\big(e^{itu}-1\big)+\int_{\mathbb{R}}(e^{itx}-1)\Big(v(x)+\operatorname{sgn}(x)\,\frac{\mathfrak{m}\cdot e^{-|x|}}{|x|}\Big)\,dx\Big\},\quad t\in\mathbb{R}, \tag{8}$$

where $\gamma_0\in\mathbb{R}$, $\mathfrak{m}\in\mathbb{Z}$, $\mathcal{U}$ is a discrete set, $\lambda_u\in\mathbb{R}$ for all $u\in\mathcal{U}$, with $\sum_{u\in\mathcal{U}}|\lambda_u|<\infty$, and v is a real-valued function from $L_1(\mathbb{R})$.

In addition to this theorem, the paper [7] contains a number of interesting assertions concerning decomposition of F and characterization of the existence of the H-moment of F for certain positive functions H. Additionally, there is a rather general theorem similar to Theorem 2 for the function f, but without the assumption that it is the characteristic function of some probability law (Theorem 2.1 in [7]).

Theorem 2 generalizes the results by Berger ([6, Theorems 4.5 and 4.12]) mentioned in the introduction, where F_d was assumed to be a discrete lattice. It is also seen that, in fact, Theorem 2 is a strengthening of Theorem 1. However, we note that γ_0 and the discrete part in (7) are described in greater detail than in formula (8).

We now propose a generalization of the previous criteria for the case when F may have a continuous singular part, i.e., when $c_s \geqslant 0$ in (3). The following theorem is the main result of this article.

For convenience, we preliminarily select the following property of distributions. Let us define $\mu_d := \inf_{t\in\mathbb{R}} |f_d(t)|$. We say that a distribution function F has a *dominated continuous singular part* if $c_s < c_d\mu_d$ for the case $\mu_d > 0$ and if $c_s = 0$ for the case $\mu_d = 0$.

Theorem 3. *Suppose that F has a decomposition* (3) *with $c_d > 0$, $c_a \geqslant 0$, $c_s \geqslant 0$, and that F has a dominated continuous singular part. Then the following statements are equivalent*:

(i) $F \in \boldsymbol{Q}$.

(ii) $\inf_{t\in\mathbb{R}} |f(t)| > 0$.

(iii) $f(t) \neq 0$ *for any* $t \in \mathbb{R}$, *and* $\inf_{t\in\mathbb{R}} |f_d(t)| > 0$.

If one (*hence all*) *of the conditions is satisfied*, f *admits the representation*

$$(9) \quad f(t) = \exp\Big\{ it\gamma_0 + \sum_{u\in\langle\mathcal{X}\rangle\setminus\{0\}} \lambda_u\big(e^{itu}-1\big) + \int_{\mathbb{R}\setminus\{0\}} (e^{itx}-1)\left(v_a(x) + \operatorname{sgn}(x)\,\frac{\mathfrak{m}_a\cdot e^{-|x|}}{|x|}\right)dx + \int_{\mathbb{R}\setminus\{0\}} (e^{itx}-1)\,dW(x)\Big\}, \quad t\in\mathbb{R}.$$

Here $\gamma_0 \in \langle\mathcal{X}\rangle$, $\lambda_u \in \mathbb{R}$ for all $u \in \langle\mathcal{X}\rangle \setminus \{0\}$, and $\sum_{u\in\langle\mathcal{X}\rangle\setminus\{0\}} |\lambda_u| < \infty$. Next, the function $v_a : \mathbb{R} \mapsto \mathbb{R}$ satisfies $\int_{\mathbb{R}} |v_a(x)|\,dx < \infty$, and, in the case $c_a = 0$, v_a is identically 0; *the constant $\mathfrak{m}_a \in \mathbb{Z}$ is defined by the formula*

$$(10) \qquad \mathfrak{m}_a := \frac{1}{2\pi}\Big(\lim_{t\to\infty} \operatorname{Arg} R_a(t) - \lim_{t\to-\infty} \operatorname{Arg} R_a(t)\Big),$$

with

$$R_a(t) := 1 + \frac{c_a f_a(t)}{c_d f_d(t) + c_s f_s(t)}, \quad t\in\mathbb{R},$$

where, in particular, $\mathfrak{m}_a = 0$ *for the case* $c_a = 0$. *Next, the function* $W : \mathbb{R} \to \mathbb{R}$ *belongs to* $\boldsymbol{V}$ *and it is always continuous on* $\mathbb{R}$. *If* $c_s = 0$ *then* W *is identically* 0. *If* $c_s \neq 0$ *then* W *is not absolutely continuous on* $\mathbb{R}$, *i.e., it always contains some continuous singular part. In addition, if all the functions* F_s^{*k}, $k \in \mathbb{N}$, *are continuous singular, then the function* W *is (purely) continuous singular.*

Note that the discrete part in the exponent in (9) depends only on discrete part of f, i.e., on f_d. More precisely, we have the following remark, which will be seen from the proof of Theorem 3.

Remark 1. In the representation (9),

$$\operatorname{Ln} f_d(t) = it\gamma_0 + \sum_{u \in \langle \mathcal{X} \rangle \setminus \{0\}} \lambda_u (e^{itu} - 1), \quad t \in \mathbb{R}.$$

It is easily seen that if we set $c_a := 0$ and $c_s := 0$ in the assumptions of this theorem, then we come to Theorem 1. If we set only $c_s := 0$, then we conclude the statement of Theorem 2, but with a full description of the discrete part in (8). In the unexplored case $c_s > 0$, the new function W appears and it always has some continuous singular part.

Using Theorem 3, it is easy to construct a lot of particular examples of $F \in \boldsymbol{Q}$ with nonzero continuous singular parts.

Example 1. Suppose that

$$F(x) := c_d \mathbb{1}_0(x) + c_a F_a(x) + c_s F_s(x), \quad x \in \mathbb{R},$$

with $c_d > c_s > 0$, $c_a \geqslant 0$, and $c_d + c_a + c_s = 1$. Let F_s be an arbitrary continuous singular function, but F_a be an absolutely continuous distribution function, whose characteristic function f_a is real and nonnegative (for instance, f_a is a Pólya-type characteristic function, or f_a corresponds to a symmetric continuous stable distribution). Then $F \in \boldsymbol{Q}$. Indeed, F has dominated continuous singular part, because f_d is identically 1 here, i.e., $\mu_d = 1$, and hence $c_d \mu_d = c_d > c_s$. Let us check (ii) from Theorem 3. For this, we consider the characteristic function of F,

$$f(t) = c_d + c_a f_a(t) + c_s f_s(t), \quad t \in \mathbb{R}.$$

Under the assumptions, we observe that

$$|f(t)| \geqslant |c_d + c_a f_a(t)| - c_s |f_s(t)| = c_d + c_a f_a(t) - c_s |f_s(t)|, \quad t \in \mathbb{R}.$$

Since $f_a(t) \geqslant 0$ and $|f_s(t)| \leqslant 1$ for any $t \in \mathbb{R}$, we have $|f(t)| \geqslant c_d - c_s > 0$ for any $t \in \mathbb{R}$, i.e., (ii) holds. Thus $F \in \boldsymbol{Q}$ by Theorem 3.

In the general case, of course, checking (ii) or (iii) of Theorem 3 may require more subtle analysis. We illustrate this with the following special example, which is interesting also because there is a (purely) continuous singular function W in (9).

Example 2. Let us consider the mixture of the degenerate law, the uniform distribution on [0, 1] and the classical Cantor distribution. Namely, we set

$$F(x) := c_d \mathbb{1}_0(x) + c_a U(x) + c_s S(x), \quad x \in \mathbb{R},$$

where U is an absolutely continuous distribution function with density $\mathbb{I}_{[0,1]}$, and S is the cumulative function of the classical Cantor distribution supported on the Cantor set $\mathcal{C} \subset [0, 1]$; $c_d > c_s > 0$, $c_a \geqslant 0$, and $c_d + c_a + c_s = 1$.

We first observe that f_d is identically 1 here and, as in Example 1, F has dominated continuous singular part. We next check condition (iii) of Theorem 3. Since $\inf_{t\in\mathbb{R}} |f_d(t)| = 1$, it remains to show that $f(t) \neq 0$ for any $t \in \mathbb{R}$. The function f is expressed by the formula

$$f(t) = c_d + c_a \frac{e^{it} - 1}{it} + c_s e^{it/2} \prod_{k=1}^{\infty} \cos(t/3^k), \quad t \in \mathbb{R}.$$

We set

$$\hat{f}(t) := f(t)e^{-it/2} = c_d e^{-it/2} + c_a \frac{\sin(t/2)}{t/2} + c_s \prod_{k=1}^{\infty} \cos(t/3^k), \quad t \in \mathbb{R}.$$

The functions f and $\hat{f}$ have the same set of zeroes on $\mathbb{R}$, a subset of the set of zeroes of the function $\mathrm{Im}\{\hat{f}(t)\} = -c_d \sin(t/2)$, $t \in \mathbb{R}$. The last set is exactly $\{2\pi m : m \in \mathbb{Z}\}$. Therefore it is sufficient to show that $f(2\pi m) \neq 0$ for any $m \in \mathbb{Z}$. Obviously, we can exclude $m = 0$. For any $m \in \mathbb{Z} \setminus \{0\}$ we write

$$\begin{aligned} |f(2\pi m)| = |\hat{f}(2\pi m)| &= \left| c_d \cos(\pi m) + c_a \frac{\sin(\pi m)}{\pi m} + c_s \prod_{k=1}^{\infty} \cos(2\pi m/3^k) \right| \\ &= \left| c_d (-1)^m + c_s \prod_{k=1}^{\infty} \cos(2\pi m/3^k) \right|. \end{aligned}$$

It follows that

$$|f(2\pi m)| \geqslant c_d - c_s \prod_{k=1}^{\infty} \left| \cos(2\pi m/3^k) \right| \geqslant c_d - c_s > 0.$$

Thus $f(t) \neq 0$ for any $t \in \mathbb{R}$ and condition (iii) of Theorem 3 is satisfied. So, by the theorem, $F \in \boldsymbol{Q}$.

Let us consider the representation (9) for f. Since $f_d(t) = 1$ for any $t \in \mathbb{R}$, the sum

$$it\gamma_0 + \sum_u \lambda_u (e^{itu} - 1)$$

is identically 0 according to Remark 1. There are the function $v_a \in L_1(\mathbb{R})$ and the constant $\mathfrak{m}_a \in \mathbb{Z}$, but we will not determine these here. Next, we turn to the function

W. It is known that all convolution powers S^{*n}, $n \in \mathbb{N}$, of the Cantor distribution function S are continuous singular; this follows from the famous Jessen–Wintner theorem (see [16], Theorem 35) and the representation of $S\big(x+\frac{1}{2}\big)$, $x \in \mathbb{R}$, as the infinite symmetric Bernoulli convolution, and it was also explicitly shown in [36] (see p. 520). According to Theorem 3, this implies that W is purely continuous singular. Moreover, it will be seen from the proof of Theorem 3 that W is the limit of the sums

$$W_n(x) := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k S^{*k}(x), \quad x \in \mathbb{R}, \quad n \in \mathbb{N},$$

as $n \to \infty$ with respect to convergence in total variation on $\mathbb{R}$.

So Theorem 3 yields a sufficient condition for W to be (purely) continuous singular. Is it true that W is always a (purely) continuous singular function in the case $c_s \neq 0$?

Proposition 1. *Suppose that $F \in \boldsymbol{Q}$ satisfies the assumptions of Theorem 3 with $c_s > 0$ and the characteristic function f is represented by (9) with some W. Suppose there is an integer $n_a \geqslant 2$ such that the function $F_s^{*(n_a-1)}$ is (purely) continuous singular, but the function $F_s^{*n_a}$ is not, i.e., $F_s^{*n_a}(x) = \alpha H_a(x) + (1-\alpha) H_s(x)$, $x \in \mathbb{R}$, where α is a number in $(0, 1]$, H_a is an absolutely continuous distribution function, and H_s is a continuous singular distribution function. If*

$$\alpha \geqslant \frac{n_a}{n_a+1} \cdot \frac{c_s}{c_d}$$

then the function W is not (purely) continuous singular. In particular, this is always true for the case $\alpha = 1$.

It should be recalled here that there exist examples of continuous singular distribution functions (say of F_s), whose convolution squares (say F_s^{*2}) are absolutely continuous (see [14; 15; 36]). Therefore we answer the question asked before Proposition 1 in the negative.

The following remark yields the characteristic triplet for $F \in \boldsymbol{Q}$ in the explicit form under the assumptions of Theorem 3.

Remark 2. The representation (9) can be written in the form

$$\begin{aligned} f(t) &= \exp\left\{it\gamma_0 + \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\, dL(x)\right\} \\ &= \exp\left\{it\gamma + \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1 - it\sin(x))\; dL(x)\right\}, \quad t \in \mathbb{R}, \end{aligned}$$

where

$$\gamma := \gamma_0 + \int_{\mathbb{R}\setminus\{0\}} \sin(x)\; dL(x)$$

and $L(x) := L_d(x) + L_a(x) + L_s(x)$ for any $x \in \mathbb{R} \setminus \{0\}$ with

$$L_d(x) := \begin{cases} \sum\limits_{\substack{u \in \langle \mathcal{X} \rangle \setminus \{0\} \\ u \leqslant x}} \lambda_u & \text{if } x < 0, \\ -\sum\limits_{\substack{u \in \langle \mathcal{X} \rangle \setminus \{0\} \\ u > x}} \lambda_u & \text{if } x > 0, \end{cases} \tag{11}$$

$$L_a(x) := \begin{cases} \int\limits_{u \leqslant x} \Big(v_a(u) + \operatorname{sgn}(u) \, \dfrac{\mathfrak{m}_a \cdot e^{-|u|}}{|u|} \Big) \, du & \text{if } x < 0, \\ -\int\limits_{u > x} \Big(v_a(u) + \operatorname{sgn}(u) \, \dfrac{\mathfrak{m}_a \cdot e^{-|u|}}{|u|} \Big) \, du & \text{if } x > 0, \end{cases} \tag{12}$$

$$L_s(x) := \begin{cases} W(x) & \text{if } x < 0, \\ W(x) - W(+\infty) & \text{if } x > 0. \end{cases} \tag{13}$$

It is seen that L satisfies all admissible conditions for a spectral function in the Lévy type representation. Thus, under the assumptions of Theorem 3, $(\gamma, 0, L)$ is the characteristic triplet for F satisfying one of the conditions (i)–(iii). On the other hand, if we know that (9) represents the characteristic function of some probability law, then its distribution function F is quasi-infinitely divisible by the definition and hence $F \in \boldsymbol{Q}$.

The following remark shows that the property of dominated singular part is not a necessary condition for belonging to class $\boldsymbol{Q}$.

Remark 3. Let F satisfy the assumptions of Theorem 3 with $c_s > 0$. Suppose that $F \in \boldsymbol{Q}$. Then $F^{*n} \in \boldsymbol{Q}$ for any $n \in \mathbb{N}$, but F^{*n} does not have dominated singular part for all sufficiently large n.

Indeed, it is seen from the definition of the rational-infinite divisibility that the convolution of two any distribution functions from $\boldsymbol{Q}$ belongs to $\boldsymbol{Q}$. Therefore $F^{*n} \in \boldsymbol{Q}$ for any $n \in \mathbb{N}$. Next, we consider the characteristic function f of F. It admits the decomposition (4) with $c_d > 0$, $c_a \geqslant 0$, $c_s > 0$, and $c_s < c_d \mu_d$, where $\mu_d = \inf_{t \in \mathbb{R}} |f_d(t)|$ as before. Then every F^{*n} has the characteristic function

$$f(t)^n = \big(c_d f_d(t) + c_a f_a(t) + c_s f_s(t) \big)^n, \quad t \in \mathbb{R}.$$

From this it is easily seen that $t \mapsto c_d^n f_d(t)^n$, $t \in \mathbb{R}$, is the Fourier–Stieltjes transform of the discrete part of F^{*n}. Hence c_d^n is the weight of this part and $\inf_{t \in \mathbb{R}} |f_d(t)^n| = \mu_d^n$. Next, there are n terms $c_s c_d^{n-1} f_s(t) f_d(t)^{n-1}$ that appear in the Fourier–Stieltjes transform of the continuous singular part of F^{*n}. This is because a convolution of continuous singular function with a discrete function is continuous singular (see [28, p. 190] or [34, p. 319]). So the weight of the continuous singular part of F^{*n} is not less than $n c_s c_d^{n-1}$. For any integer $n \geqslant c_d / c_s$ we have $n c_s c_d^{n-1} \geqslant c_d^n \geqslant c_d^n \mu_d^n$. Therefore the exact weight of the continuous singular part of F^{*n} is not less than

$c_d^n \mu_d^n$ too; i.e., the condition of dominated singular part doesn't hold for F^{*n} for $n \geqslant c_d/c_s$.

The following interesting example shows that the condition of a dominated singular part cannot be simply omitted, or even extended to the case $c_s = c_d \mu_d$ with $\mu_d > 0$ without certain additional assumptions.

Example 3. Let B denote the distribution function of the Bernoulli law on the points ± 1 with equal probabilities, i.e., $B(x) = \frac{1}{2}\,\mathbb{1}_{-1}(x) + \frac{1}{2}\,\mathbb{1}_1(x)$, $x \in \mathbb{R}$. We set

$$F_*(x) := \tfrac{1}{2}\,\mathbb{1}_0(x) + \tfrac{1}{2}\,F_s(x), \qquad x \in \mathbb{R}, \tag{14}$$

where F_s is the following infinite symmetric Bernoulli convolution

$$F_s = B_1 * B_2 * \cdots * B_n * \cdots$$

with $B_k(x) := B(k!\,x)$, $x \in \mathbb{R}$, $k \in \mathbb{N}$. Let f_* and f_s denote the characteristic functions of F_* and F_s. Then $f_*(t) = \frac{1}{2} + \frac{1}{2}\,f_s(t)$, $t \in \mathbb{R}$, and

$$f_s(t) = \prod_{k=1}^{\infty} \cos(t/k!), \qquad t \in \mathbb{R}. \tag{15}$$

It is known (see [28, pp. 20, 67]) that the function F_s is continuous singular. Let us consider Lebesgue's decomposition (4) for $f = f_*$. We have $c_d = \frac{1}{2}$, $c_a = 0$, and $c_s = \frac{1}{2}$. Observe that the component f_d is identically 1 for f_* and hence $\mu_d = \inf_{t\in\mathbb{R}} |f_d(t)| = 1$. So $c_d \mu_d = c_s$, i.e., the condition of dominated continuous singular part, doesn't hold. Next, $f_*(t) \neq 0$ for any $t \in \mathbb{R}$, since otherwise $f_s(t_0) = -1$ for some $t_0 \neq 0$ and hence $|f_s(t_0)| = 1$, which would mean that F_s is a discrete lattice distribution function (see [28, Theorem 2.1.4]), a contradiction. Thus F_* satisfies condition (iii) of Theorem 3 and the assumption $c_d > 0$, but F_* doesn't have dominated continuous singular part.

Proposition 2. *The function F_* doesn't belong to $\boldsymbol{Q}$.*

Thus, in general, conditions (i) and (iii) from Theorem 3 are not equivalent even if $c_s = c_d \mu_d > 0$.

By the way, we recall that distribution functions of discrete laws with a point mass $\frac{1}{2}$, which have characteristic functions without zeroes on the real line, don't always belong to class $\boldsymbol{Q}$ (see [22] for more details). However, if the distribution is an atom of mass $\frac{1}{2}$ plus a continuous part, which is not purely singular, then the answer is definite.

Example 4. Suppose that

$$F(x) := \tfrac{1}{2}\,\mathbb{1}_{\gamma_0}(x) + c_a F_a(x) + c_s F_s(x), \qquad x \in \mathbb{R},$$

where F_a is an absolutely continuous distribution function, F_s is a continuous singular distribution function, $\gamma_0 \in \mathbb{R}$, $c_a > 0$, $c_s \geqslant 0$, and $c_a + c_s = \frac{1}{2}$. Such a function F always belongs to class $\boldsymbol{Q}$. Let us check it. We consider the characteristic function of F:

$$f(t) = \tfrac{1}{2}\, e^{it\gamma_0} + c_a f_a(t) + c_s f_s(t), \quad t \in \mathbb{R}.$$

Here $c_d = \frac{1}{2}$ and $f_d(t) = e^{it\gamma_0}$, $t \in \mathbb{R}$. Hence $\mu_d = \inf_{t\in\mathbb{R}} |f_d(t)| = 1$ and $c_s = \frac{1}{2} - c_a < \frac{1}{2} = c_d\mu_d$, i.e., F has dominated continuous singular part. Next, $f(t) \neq 0$ for any $t \in \mathbb{R}$. For otherwise $2c_a f_a(t_0) + 2c_s f_s(t_0) = -e^{it_0\gamma_0}$ for some $t_0 \neq 0$ and so $|2c_a f_a(t_0) + 2c_s f_s(t_0)| = 1$. This equality implies that the distribution function with the characteristic function $2c_a f_a(\,\cdot\,) + 2c_s f_s(\,\cdot\,)$ is discrete lattice, but this is actually continuous by the assumptions, a contradiction.

So F satisfies condition (iii) and the other assumptions of Theorems 3. It follows that $F \in \boldsymbol{Q}$.

The next assertion solves the decomposition problem for any distribution function $F \in \boldsymbol{Q}$ satisfying the assumptions of Theorem 3.

Proposition 3. *Let F be a distribution function with decomposition (3) with some $c_d > 0$, $c_a \geqslant 0$, $c_s \geqslant 0$, and a dominated continuous singular part. Suppose that $F \in \boldsymbol{Q}$ and $F = F_1 * F_2$ with some distribution functions F_1 and F_2. Then F_1 and F_2 belong to $\boldsymbol{Q}$.*

This proposition generalizes a similar result from [7] (see Corollary 2.3 to Theorem 2.2), which solves the problem for $F \in \boldsymbol{Q}$ satisfying the assumptions of Theorem 2. We note that there is a general decomposition problem for arbitrary $F \in \boldsymbol{Q}$, which was stated by Lindner, Pan and Sato in [26] (see Open Question 8.4): *Is it true that if $F \in \boldsymbol{Q}$ and $F = F_1 * F_2$ (F_1 and F_2 being distribution functions on $\mathbb{R}$), then $F_1 \in \boldsymbol{Q}$ and $F_2 \in \boldsymbol{Q}$?* So Proposition 3 answers this question in the affirmative for any $F \in \boldsymbol{Q}$ satisfying the assumptions of Theorem 3. However, there is a result in [21] that asserts that the general answer is negative.

3. Auxiliary theorems and lemmata

The proof of the main result (Theorem 3) of the article essentially uses the following Wiener–Pitt theorem [35] (see also [33; 11, p. 191]).

Theorem 4. *Let H be a function in $\boldsymbol{V}_{\mathbb{C}}$ with Lebesgue decomposition*

$$H(x) = H_d(x) + H_a(x) + H_s(x), \quad x \in \mathbb{R},$$

where H_d, H_a, H_s are the discrete, absolutely continuous and singular parts of H, each of which belongs to $\boldsymbol{V}_{\mathbb{C}}$. Let

$$h(t) := \int_{\mathbb{R}} e^{itx}\, dH(x), \quad \textit{and} \quad h_d(t) := \int_{\mathbb{R}} e^{itx}\, dH_d(x), \quad t \in \mathbb{R}.$$

Suppose that $\inf_{t\in\mathbb{R}} |h(t)| > 0$ *and* $\|H_s\| < \inf_{t\in\mathbb{R}} |h_d(t)|$. *Then there exists a function* K *in* $\boldsymbol{V}_{\mathbb{C}}$ *such that*

$$\frac{1}{h(t)} = \int_{\mathbb{R}} e^{itx}\, dK(x), \qquad t \in \mathbb{R}.$$

The following theorem was proposed by Berger [6]. In fact, it is a modification of one of Krein's results (see [23, Theorem L, p. 15]).

Theorem 5. *Suppose that a function* $h : \mathbb{R} \to \mathbb{C}$ *is defined by the formula*

$$h(t) := c + \int_{\mathbb{R}} e^{itx} u(x)\, dx, \qquad t \in \mathbb{R}, \tag{16}$$

where $c \in \mathbb{C} \setminus \{0\}$ *is a constant,* $u : \mathbb{R} \to \mathbb{C}$ *is a function satisfying the condition* $\int_{\mathbb{R}} |u(x)|\, dx < \infty$. *Assume that* $h(0) = 1$ *and* $h(t) \neq 0$ *for any* $t \in \mathbb{R}$. *Then*

$$h(t) = \exp\left\{ \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\left(v(x) + \operatorname{sgn}(x)\, \frac{\mathfrak{m} \cdot e^{-|x|}}{|x|} \right) dx \right\}, \qquad t \in \mathbb{R},$$

where $\mathfrak{m}$ *is a constant defined by the formula*

$$\mathfrak{m} := \frac{1}{2\pi}\big(\lim_{t\to\infty} \operatorname{Arg} h(t) - \lim_{t\to-\infty} \operatorname{Arg} h(t) \big),$$

and the function $v : \mathbb{R} \to \mathbb{C}$ *satisfies the condition* $\int_{\mathbb{R}} |v(x)|\, dx < \infty$.

Let us consider the quantity $\mathfrak{m}$ from the theorem. It is the *index* of h (see [23]). By the Riemann–Lebesgue lemma, for the function h defined by (16) we have $h(t) \to c$ as $t \to \pm\infty$. Therefore it is not difficult to conclude the following fact.

Remark 4. The quantity $\mathfrak{m}$ from Theorem 5 is well-defined and it is an integer.

We next formulate a very useful result obtained by Berger [6].

Theorem 6. *Let* F *be a distribution function on* $\mathbb{R}$ *with the characteristic function* f. *Suppose that* f *admits the representation*

$$f(t) = \exp\left\{ it\gamma_1 - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \big(e^{itx} - 1 - it\, \mathbb{1}_{[-1,1]}(x) \big)\, dL(x) \right\}, \qquad t \in \mathbb{R},$$

with some $\gamma_1 \in \mathbb{C}$, $\sigma^2 \in \mathbb{C}$, *and the function* $L(x) = L_1(x) + iL_2(x)$, $x \in \mathbb{R} \setminus \{0\}$. *Here for every* $j \in \{1, 2\}$ *the function* L_j *has bounded variation on* $\mathbb{R} \setminus O_\delta$ *for any* $\delta > 0$ (*in general,* L_j *may be nonmonotonic on the intervals* $(-\infty, 0)$ *and* $(0, +\infty)$), L_j *is right-continuous at every point of* $\mathbb{R}$, $L_j(-\infty) = L_j(+\infty) = 0$, *and*

$$\int_{O_\delta} x^2\, d|L_j|(x) < \infty \quad \text{for any} \quad \delta > 0,$$

where $O_\delta := (-\delta, 0) \cup (0, \delta)$. *Then* $\gamma_1 \in \mathbb{R}$, $\sigma^2 \geqslant 0$, *and* $\operatorname{Im}\{L(x)\} = L_2(x) = 0$ *for any* $x \in \mathbb{R} \setminus \{0\}$. *In addition,* $F \in \boldsymbol{Q}$.

The next theorem yields one special property of characteristic functions of rational-infinitely divisible laws. Its proof can be found in [18, p. 3] or [19, p. 360].

Theorem 7. *Let F be a distribution function on $\mathbb{R}$ with the characteristic function f. If $F \in \boldsymbol{Q}$ then for any $\tau > 0$ there exists $C_\tau > 0$ such that*

$$\sup_{t\in\mathbb{R}} \left| \frac{f(t-\tau) f(t+\tau)}{f(t)^2} \right| \leqslant C_\tau .$$

Lemma 1 (Bochner [9]). *Let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{V}$ such that $\sup_{n\in\mathbb{N}} \|W_n\| = B < \infty$. Let*

$$w_n(t) := \int_{\mathbb{R}} e^{itx}\, dW_n(x), \quad t \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Suppose that $w_n(t) \to w(t)$ as $n \to \infty$ uniformly on every bounded interval. Then there exists $W \in \boldsymbol{V}$ with $\|W\| \leqslant B$ such that

$$w(t) = \int_{\mathbb{R}} e^{itx}\, dW(x), \quad t \in \mathbb{R}.$$

We will need the following multivariate version of the Leibniz rule for partial derivatives of a product of two functions (see [13, p. 10]).

Lemma 2. *Suppose that the functions $f : \mathbb{R}^d \to \mathbb{C}$ and $g : \mathbb{R}^d \to \mathbb{C}$ have continuous mixed partial derivatives up to and including order d in some open set $T \subset \mathbb{R}^d$. Then, for any $t = (t_1, \dots, t_d) \in T$,*

$$\frac{\partial^d (f(t) \cdot g(t))}{\partial t_1 \cdots \partial t_d} = \sum_{\mathfrak{u}\subset D} \left(\frac{\partial^{|\mathfrak{u}|} f(t)}{\prod_{j\in\mathfrak{u}} \partial t_j} \cdot \frac{\partial^{d-|\mathfrak{u}|} g(t)}{\prod_{j\in D\setminus\mathfrak{u}} \partial t_j} \right),$$

where $D = \{1, \dots, d\}$, and the parameter $\mathfrak{u}$ of the sum ranges over all the subsets of D.

We will also need the multivariate form of Faà di Bruno's formula for the partial derivatives of a composition of two functions (see [13, p. 4]).

Lemma 3. *Suppose that a function $g : \mathbb{R}^d \to \mathbb{C}$ has continuous mixed partial derivatives up to and including order d in some open set $T \subset \mathbb{R}^d$. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function in some domain in $\mathbb{C}$ that which include the values of $g(t)$, $t \in T$. Suppose that $D := \{1, \dots, d\}$, $\mathfrak{u}$ is nonempty subset of D, $\mathcal{P}(\mathfrak{u})$ is the set of all partitions of the set $\mathfrak{u}$. Then, for any $t = (t_1, \dots, t_d) \in T$,*

$$\frac{\partial^{|\mathfrak{u}|} f(g(t))}{\prod_{j\in\mathfrak{u}} \partial t_j} = \sum_{P\in\mathcal{P}(\mathfrak{u})} \left(\frac{d^{|P|} f(z)}{dz^{|P|}}\bigg|_{z=g(t)} \cdot \prod_{s\in P} \frac{\partial^{|s|} g(t)}{\prod_{j\in s} \partial t_j} \right).$$

We now formulate a lemma that will be a key tool in proving of Theorem 3. Its proof is given at the start of the next section.

Lemma 4. *Suppose a function $\varphi : \mathbb{R}^d \to \mathbb{C}$, where $d \in \mathbb{N}$, admits the representation*

$$\varphi(t) = \sum_{m=1}^{N} q_m e^{i\langle t, c_m\rangle}, \quad t = (t_1, \dots, t_d) \in \mathbb{R}^d,$$

with $N \in \mathbb{N}$, $q_m \in \mathbb{C}$, and $c_m = (c_{m,1}, \dots, c_{m,d}) \in \mathbb{Z}^d$, $m = 1, \dots, N$. Then, for any $k \in \mathbb{N}$,

$$\|\varphi^k\| \leqslant \left(\frac{\pi}{\sqrt{6}}\right)^d \sum_{\mathfrak{u} \subset D} \left(\prod_{j \in D \setminus \mathfrak{u}} |k\alpha_j - 1| \cdot \sum_{\substack{P \in \mathcal{P}(\mathfrak{u}) \\ |P| \leqslant k}} \left(\prod_{l=1}^{|P|} (k - l + 1) \cdot S_\varphi^{k-|P|} \cdot R_\varphi(P) \right)\right), \tag{17}$$

where $D = \{1, \dots, d\}$,

$$\begin{aligned} &\alpha_j := \min\nolimits_{m=1,\dots,N} c_{m,j}, \quad j = 1, \dots, d, \\ &S_\varphi := \sup_{t \in [-\pi,\pi]^d} |\varphi(t)|, \quad R_\varphi(P) := \sup_{t \in [-\pi,\pi]^d} \prod_{s \in P} \left| \frac{\partial^{|s|}\varphi(t)}{\prod_{j \in s} \partial t_j} \right|. \end{aligned} \tag{18}$$

In particular,

$$\|\varphi^k\| \leqslant A_\varphi k^d S_\varphi^k \quad \textit{for any } k \in \mathbb{N}, \tag{19}$$

with

$$A_\varphi := \left(\frac{\pi}{\sqrt{6}}\right)^d \sum_{\mathfrak{u} \subset D} \left(\prod_{j \in D \setminus \mathfrak{u}} \big(|\alpha_j| + 1\big) \cdot \sum_{P \in \mathcal{P}(\mathfrak{u})} S_\varphi^{-|P|} R_\varphi(P) \right), \tag{20}$$

which is independent of k.

4. Proofs

Proof of Lemma 4. We will first prove inequality (17) for $k = 1$. We define a vector $a := (a_1, \dots, a_d)$ with $a_j := 1 - \alpha_j$, $j = 1, \dots, d$, and we introduce the function

$$\hat{\varphi}_a(t) := \varphi(t) e^{i\langle t, a\rangle} = \sum_{m=1}^{N} q_m e^{i\langle t, c_m + a\rangle}, \quad t \in \mathbb{R}^d.$$

Note that $c_m + a \in \mathbb{N}^d$ for any $m = 1, \dots, N$. Without loss of generality we can assume that c_m are distinct if $N > 1$. Then we have

$$\begin{aligned} \|\varphi\| &= \sum_{m=1}^{N} |q_m| = \sum_{m=1}^{N} \left(|q_m| \cdot \prod_{j=1}^{d} (c_{m,j} + a_j) \cdot \prod_{j=1}^{d} \frac{1}{c_{m,j} + a_j} \right) \\ &\leqslant \left(\sum_{m=1}^{N} \left(|q_m|^2 \cdot \prod_{j=1}^{d} (c_{m,j} + a_j)^2 \right) \right)^{1/2} \cdot \left(\sum_{m=1}^{N} \prod_{j=1}^{d} \frac{1}{(c_{m,j} + a_j)^2} \right)^{1/2}. \end{aligned} \tag{21}$$

Since $c_m + a = (c_{m,1} + a_1, \ldots, c_{m,d} + a_d) \in \mathbb{N}^d$ are distinct vectors for different m, the following estimate holds:

$$\sum_{m=1}^{N} \prod_{j=1}^{d} \frac{1}{(c_{m,j} + a_j)^2} \leqslant \prod_{j=1}^{d} \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \cdots\right) = \left(\frac{\pi^2}{6}\right)^d.$$

We next write the function $\hat{\varphi}_a$ in expanded form as

$$\hat{\varphi}_a(t) = \sum_{m=1}^{N} q_m \exp\left\{i \sum_{j=1}^{d} t_j (c_{m,j} + a_j)\right\}, \quad t = (t_1, \ldots, t_d) \in \mathbb{R}^d.$$

So it is easily seen that

$$\frac{\partial^d \hat{\varphi}_a(t)}{\partial t_1 \cdots \partial t_d} = \sum_{m=1}^{N} \left(q_m \cdot i^d \cdot \prod_{j=1}^{d} (c_{m,j} + a_j) \cdot \exp\left\{i \sum_{j=1}^{d} t_j (c_{m,j} + a_j)\right\}\right), \quad t = (t_1, \ldots, t_d) \in \mathbb{R}^d.$$

We consider the written exponential functions (additionally normed by $(2\pi)^d$) as a rank-N orthonormal system of $L_2\big([-\pi, \pi]^d\big)$ and, by Parseval's identity, we get

$$\sum_{m=1}^{N} \left(|q_m|^2 \cdot \prod_{j=1}^{d} (c_{m,j} + a_j)^2\right) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \left|\frac{\partial^d \hat{\varphi}_a(t)}{\partial t_1 \cdots \partial t_d}\right|^2 dt_1 \cdots dt_d.$$

Hence

$$\sum_{m=1}^{N} \left(|q_m|^2 \cdot \prod_{j=1}^{d} (c_{m,j} + a_j)^2\right) \leqslant \sup_{t \in [-\pi,\pi]^d} \left|\frac{\partial^d \hat{\varphi}_a(t)}{\partial t_1 \cdots \partial t_d}\right|^2.$$

Using these estimates in (21), we come to the inequality

$$\|\varphi\| \leqslant \left(\frac{\pi}{\sqrt{6}}\right)^d \sup_{t \in [-\pi,\pi]^d} \left|\frac{\partial^d \hat{\varphi}_a(t)}{\partial t_1 \cdots \partial t_d}\right|.$$

We next find an upper estimate for the latter supremum in terms of φ and a. We apply Lemma 2 for the function $\hat{\varphi}_a(t) = \varphi(t) e^{i\langle t, a\rangle}$ with $f(t) := \varphi(t)$ and $g(t) := e^{i\langle t, a\rangle}$, $t \in \mathbb{R}^d$:

$$\frac{\partial^d \hat{\varphi}_a(t)}{\partial t_1 \cdots \partial t_d} = \sum_{\mathfrak{u} \subset D} \left(\frac{\partial^{|\mathfrak{u}|} \varphi(t)}{\prod_{j \in \mathfrak{u}} \partial t_j} \cdot \frac{\partial^{d-|\mathfrak{u}|} e^{i\langle t, a\rangle}}{\prod_{j \in D \setminus \mathfrak{u}} \partial t_j}\right) = \sum_{\mathfrak{u} \subset D} \left(\frac{\partial^{|\mathfrak{u}|} \varphi(t)}{\prod_{j \in \mathfrak{u}} \partial t_j} e^{i\langle t, a\rangle} i^{d-|\mathfrak{u}|} \prod_{j \in D \setminus \mathfrak{u}} a_j\right), \quad t \in \mathbb{R}^d.$$

Here $D := \{1, \ldots, d\}$ and the index $\mathfrak{u}$ of the sum ranges over all subsets of D.

Hence we get the estimate

$$\sup_{t\in[-\pi,\pi]^d}\left|\frac{\partial^d \hat{\varphi}_a(t)}{\partial t_1\cdots\partial t_d}\right| \leqslant \sum_{\mathfrak{u}\subset D}\left(\prod_{j\in D\setminus\mathfrak{u}}|a_j|\cdot\sup_{t\in[-\pi,\pi]^d}\left|\frac{\partial^{|\mathfrak{u}|}\varphi(t)}{\prod_{j\in\mathfrak{u}}\partial t_j}\right|\right).$$

Thus (on account of the equalities $a_j = 1-\alpha_j$, $j=1,\dots,d$) we have

$$\|\varphi\| \leqslant \left(\frac{\pi}{\sqrt{6}}\right)^d \sum_{\mathfrak{u}\subset D}\left(\prod_{j\in D\setminus\mathfrak{u}}|\alpha_j-1|\cdot\sup_{t\in[-\pi,\pi]^d}\left|\frac{\partial^{|\mathfrak{u}|}\varphi(t)}{\prod_{j\in\mathfrak{u}}\partial t_j}\right|\right). \tag{22}$$

The right-hand side of this inequality coincides with that of (17) for $k=1$. Indeed, in the inner sum in (17), the index P can be equal to $\{\mathfrak{u}\}$, i.e., $|P|=1$, and, in the product in $R_\varphi(P)$, the index s assumes only the value $\mathfrak{u}$. Therefore $\prod_{l=1}^{|P|}(k-l+1)=1$, $S_\varphi^{k-|P|}=S_\varphi^0=1$, and

$$R_\varphi(P) = \sup_{t\in[-\pi,\pi]^d}\left|\frac{\partial^{|\mathfrak{u}|}\varphi(t)}{\prod_{j\in\mathfrak{u}}\partial t_j}\right|.$$

Substituting these values in (17), we see the match with (22).

We now prove that (17) is true for any $k\in\mathbb{N}$. Let us fix an arbitrary $k\in\mathbb{N}$. We consider the function φ^k given by

$$\varphi^k(t) = \left(\sum_{m=1}^N q_m e^{i\langle t,c_m\rangle}\right)^k, \quad t\in\mathbb{R}^d.$$

Clearly, it can be written in the form

$$\varphi^k(t) = \sum_{m=1}^M Q_m e^{i\langle t,C_m\rangle}, \quad t\in\mathbb{R}^d,$$

with some $M\in\mathbb{N}$, $Q_m\in\mathbb{C}$, and distinct $C_m=(C_{m,1},\dots,C_{m,d})\in\mathbb{Z}^d$, $m=1,\dots,M$. Observe that

$$C_m \in \left\{\sum_{j=1}^k c_{m_j} : m_1,\dots,m_k\in\{1,\dots,N\}\right\}, \quad m=1,\dots,M.$$

Therefore

$$\min_{m=1,\dots,M} C_{m,j} = k\min_{m=1,\dots,N} c_{m,j} = k\alpha_j, \quad j=1,\dots,d.$$

Taking this into account, we use inequality (22) for the function φ^k:

$$\|\varphi^k\| \leqslant \left(\frac{\pi}{\sqrt{6}}\right)^d \sum_{\mathfrak{u}\subset D}\left(\prod_{j\in D\setminus\mathfrak{u}}|k\alpha_j-1|\cdot\sup_{t\in[-\pi,\pi]^d}\left|\frac{\partial^{|\mathfrak{u}|}\varphi^k(t)}{\prod_{j\in\mathfrak{u}}\partial t_j}\right|\right).$$

We now write expressions for the mixed partial derivatives from the right-hand side. We apply Lemma 3 with $g(t) := \varphi(t)$, $t \in \mathbb{R}^d$, and $f(z) := z^k$, $z \in \mathbb{C}$:

$$\frac{\partial^{|\mathfrak{u}|}\varphi^k(t)}{\prod_{j\in\mathfrak{u}}\partial t_j} = \sum_{P\in\mathcal{P}(\mathfrak{u})}\left(\frac{d^{|P|}z^k}{dz^{|P|}}\bigg|_{z=\varphi(t)} \cdot \prod_{s\in P}\frac{\partial^{|s|}\varphi(t)}{\prod_{j\in s}\partial t_j}\right)$$

$$= \sum_{\substack{P\in\mathcal{P}(\mathfrak{u})\\ |P|\leqslant k}}\left(\prod_{l=1}^{|P|}(k-l+1)\cdot\varphi(t)^{k-|P|}\cdot\prod_{s\in P}\frac{\partial^{|s|}\varphi(t)}{\prod_{j\in s}\partial t_j}\right).$$

This formula holds even in the case $\mathfrak{u} = \varnothing$, where P must be $\varnothing$, i.e., $|P| = 0$ and $\mathcal{P}(\mathfrak{u}) = \{\varnothing\}$. Indeed, there is only one summand in the latter sum, in which the written products are equal to 1 (because the indexes formally belong to the empty set). Hence the right-hand side equals $\varphi(t)^k$, as does the left-hand side, since $|\mathfrak{u}| = 0$.

We have the estimate

$$\sup_{t\in[-\pi,\pi]^d}\left|\frac{\partial^{|\mathfrak{u}|}\varphi^k(t)}{\prod_{j\in\mathfrak{u}}\partial t_j}\right| \leqslant \sum_{\substack{P\in\mathcal{P}(\mathfrak{u})\\ |P|\leqslant k}}\left(\prod_{l=1}^{|P|}(k-l+1)\cdot S_\varphi^{k-|P|}\cdot R_\varphi(P)\right),$$

where S_φ and $R_\varphi(P)$ are defined by (18). Thus we obtain the required inequality (17).

We next show that (19) holds. Let us obtain an upper estimate for the right-hand side of inequality (17). We fix $k \geqslant 1$. Observe that

$$\prod_{j\in D\setminus\mathfrak{u}}|k\alpha_j - 1| \leqslant \prod_{j\in D\setminus\mathfrak{u}}\big(k|\alpha_j|+1\big) \leqslant \prod_{j\in D\setminus\mathfrak{u}}\big(k|\alpha_j|+k\big) \leqslant k^{d-|\mathfrak{u}|}\cdot\prod_{j\in D\setminus\mathfrak{u}}\big(|\alpha_j|+1\big).$$

Next, since $|P| \leqslant |\mathfrak{u}|$ for any $P \in \mathcal{P}(\mathfrak{u})$, we have

$$\prod_{l=1}^{|P|}(k-l+1) \leqslant \prod_{l=1}^{|P|}(k-1+1) = k^{|P|} \leqslant k^{|\mathfrak{u}|}.$$

We apply these inequalities to (17), obtaining

$$\|\varphi^k\| \leqslant \left(\frac{\pi}{\sqrt{6}}\right)^d \sum_{\mathfrak{u}\subset D}\left(k^{d-|\mathfrak{u}|}\prod_{j\in D\setminus\mathfrak{u}}(|\alpha_j|+1)\cdot\sum_{\substack{P\in\mathcal{P}(\mathfrak{u})\\ |P|\leqslant k}}k^{|\mathfrak{u}|}\cdot S_\varphi^{k-|P|}\cdot R_\varphi(P)\right)$$

$$= k^d S_\varphi^k\cdot\left(\frac{\pi}{\sqrt{6}}\right)^d \sum_{\mathfrak{u}\subset D}\left(\prod_{j\in D\setminus\mathfrak{u}}\big(|\alpha_j|+1\big)\cdot\sum_{\substack{P\in\mathcal{P}(\mathfrak{u})\\ |P|\leqslant k}}S_\varphi^{-|P|}R_\varphi(P)\right).$$

Since S_φ and $R_\varphi(P)$ are always nonnegative, we come to the needed inequality

$$\|\varphi^k\| \leqslant k^d S_\varphi^k\left(\frac{\pi}{\sqrt{6}}\right)^d \sum_{\mathfrak{u}\subset D}\left(\prod_{j\in D\setminus\mathfrak{u}}\big(|\alpha_j|+1\big)\cdot\sum_{P\in\mathcal{P}(\mathfrak{u})}S_\varphi^{-|P|}R_\varphi(P)\right) = A_\varphi k^d S_\varphi^k. \quad \square$$

Proof of Theorem 3. We assume that F admits the decomposition (3) with some $c_d \in (0, 1]$, and that F has a dominated singular part. We set $\mu_d := \inf_{t\in\mathbb{R}} |f_d(t)|$.

(i) $\Rightarrow$ (ii). Suppose that $F \in \boldsymbol{Q}$. If $\mu_d = 0$ then $c_s = 0$ by the assumption of a dominated singular part, and hence we can apply Theorem 2. According to that theorem, (i) implies (ii). Next, if $\mu_d > 0$ then we know that $c_s < c_d\mu_d$, i.e., $c_d\mu_d - c_s > 0$. Observe that, for any $t \in \mathbb{R}$,

$$\begin{aligned}|f(t)| = \left|c_d f_d(t) + c_s f_s(t) + c_a f_a(t)\right| &\geqslant c_d |f_d(t)| - c_s \left|f_s(t)| - c_a|f_a(t)\right| \\ &\geqslant c_d\mu_d - c_s - |f_a(t)|.\end{aligned}$$

Since $f_a(t) \to 0$ as $t \to \pm\infty$, there exists $T > 0$ such that $|f_a(t)| < \frac{1}{2}(c_d\mu_d - c_s)$ as $|t| > T$. Hence $|f(t)| > \frac{1}{2}(c_d\mu_d - c_s) > 0$ as $|t| > T$. Next, let us consider the function $t \mapsto |f(t)|$ on the segment $[-T, T]$. Since it is continuous, there exists $t_{\min} \in [-T, T]$ such that $|f(t)| \geqslant |f(t_{\min})|$ for any $t \in [-T, T]$. Since $F \in \boldsymbol{Q}$, we know that $f(t) \neq 0$ for any $t \in \mathbb{R}$ (see the introduction) and thus $C_T := |f(t_{\min})| > 0$. So we get $|f(t)| \geqslant C_T > 0$ for any $t \in [-T, T]$. Thus, for any $t \in \mathbb{R}$,

$$|f(t)| \geqslant \min\left\{\tfrac{1}{2}(c_d\mu_d - c_s), C_T\right\} > 0,$$

i.e., $\inf_{t\in\mathbb{R}} |f(t)| > 0$. So we come to (ii).

(ii) $\Rightarrow$ (iii). Obviously, (ii) yields that $f(t) \neq 0$ for any $t \in \mathbb{R}$. To obtain a contradiction, suppose that $\mu_d = \inf_{t\in\mathbb{R}} |f_d(t)| = 0$. Then $c_s = 0$ by the condition of dominated singular part. According to Theorem 2, (iii) follows from (ii), i.e., in particular, we have $\mu_d > 0$, a contradiction. Thus we conclude that $\inf_{t\in\mathbb{R}} |f_d(t)| > 0$ and (iii) holds.

(iii) $\Rightarrow$ (i). Let us consider f represented by formula (4) with $c_d > 0$. Here we assume that $f(t) \neq 0$ for any $t \in \mathbb{R}$, that

$$\mu_d = \inf_{t\in\mathbb{R}} |f_d(t)| > 0, \tag{23}$$

and that $c_s < c_d\mu_d$. Due to these assumptions, for any $t \in \mathbb{R}$ we have

$$\left|c_d f_d(t) + c_s f_s(t)\right| \geqslant c_d|f_d(t)| - c_s|f_s(t)| \geqslant c_d\mu_d - c_s > 0,$$

i.e.,

$$\inf_{t\in\mathbb{R}} |c_d f_d(t) + c_s f_s(t)| > 0. \tag{24}$$

Then for any $t \in \mathbb{R}$ we can write

$$\begin{aligned}f(t) &= c_d f_d(t) + c_s f_s(t) + c_a f_a(t) \\ &= f_d(t) \cdot \frac{c_d f_d(t) + c_s f_s(t) + c_a f_a(t)}{c_d f_d(t) + c_s f_s(t)} \cdot \frac{c_d f_d(t) + c_s f_s(t)}{f_d(t)} \\ &= f_d(t) \cdot \left(1 + \frac{c_a f_a(t)}{c_d f_d(t) + c_s f_s(t)}\right) \cdot \frac{c_d f_d(t) + c_s f_s(t)}{f_d(t)}.\end{aligned}$$

So it is convenient to represent f as

$$f(t) = f_d(t) f_{a,ds}(t) f_{s,d}(t), \quad t \in \mathbb{R}, \tag{25}$$

where

$$f_{a,ds}(t) := c_d + c_s + \frac{c_a(c_d+c_s) f_a(t)}{c_d f_d(t) + c_s f_s(t)}, \quad f_{s,d}(t) := \frac{c_d f_d(t) + c_s f_s(t)}{(c_d+c_s) f_d(t)}, \quad t \in \mathbb{R}.$$

Let us consider f_d represented by (6). Due to (23), by Theorem 1, it admits the representation

$$f_d(t) = \exp\Bigg\{ it\gamma_0 + \sum_{u \in \langle \mathcal{X} \rangle \setminus \{0\}} \lambda_u \big(e^{itu} - 1\big) \Bigg\}, \quad t \in \mathbb{R}, \tag{26}$$

with some $\gamma_0 \in \langle \mathcal{X} \rangle$ and $\lambda_u \in \mathbb{R}$ for all $u \in \langle \mathcal{X} \rangle \setminus \{0\}$, and $\sum_{u \in \langle \mathcal{X} \rangle \setminus \{0\}} |\lambda_u| < \infty$.

We now consider the function $f_{a,ds}$. If $c_a = 0$ then $f_{a,ds}(t) = c_d + c_s = 1 - c_a = 1$. We next suppose that $c_a > 0$. The function $t \mapsto c_d f_d(t) + c_s f_s(t)$, $t \in \mathbb{R}$, is separated from 0 according to (24) and for its continuous singular part we have

$$\|c_s f_s\| = c_s < c_d \mu_d = c_d \inf_{t \in \mathbb{R}} |f_d(t)|.$$

By Theorem 4, there exists a function $I_{ds} : \mathbb{R} \to \mathbb{C}$ in $\boldsymbol{V}_{\mathbb{C}}$ such that

$$\frac{1}{c_d f_d(t) + c_s f_s(t)} = \int_{\mathbb{R}} e^{itx}\, dI_{ds}(x), \quad t \in \mathbb{R}.$$

We next observe that

$$\frac{f_a(t)}{c_d f_d(t)+c_s f_s(t)} = \int_{\mathbb{R}} e^{itx}\, dF_a(x) \cdot \int_{\mathbb{R}} e^{itx}\, dI_{ds}(x) = \int_{\mathbb{R}} e^{itx}\, d(F_a * I_{ds})(x), \quad t \in \mathbb{R},$$

and we write

$$\frac{c_a(c_d+c_s) f_a(t)}{c_d f_d(t) + c_s f_s(t)} = \int_{\mathbb{R}} e^{itx}\, d\widetilde{F}_a(x), \quad t \in \mathbb{R},$$

where $\widetilde{F}_a(x) := c_a(c_d+c_s)(F_a * I_{ds})(x)$, $x \in \mathbb{R}$. The function $\widetilde{F}_a \in \boldsymbol{V}_{\mathbb{C}}$ inherits absolute continuity from F_a, i.e., there exists a function $\tilde{p}_a : \mathbb{R} \to \mathbb{C}$ such that $\int_{\mathbb{R}} |\tilde{p}_a(x)|\, dx < \infty$ and $\widetilde{F}_a(x) = \int_{y \leqslant x} \tilde{p}_a(y)\, dy$, $x \in \mathbb{R}$. Hence

$$\int_{\mathbb{R}} e^{itx}\, d\widetilde{F}_a(x) = \int_{\mathbb{R}} e^{itx} \tilde{p}_a(x)\, dx, \quad t \in \mathbb{R},$$

and we have

$$f_{a,ds}(t) = c_d + c_s + \frac{c_a(c_d+c_s) f_a(t)}{c_d f_d(t) + c_s f_s(t)} = c_d + c_s + \int_{\mathbb{R}} e^{itx} \tilde{p}_a(x)\, dx, \quad t \in \mathbb{R},$$

where $c_d + c_s \geqslant c_d > 0$. Note that $f_{a,ds}(0) = 1$. Since $f(t) \neq 0$ for any $t \in \mathbb{R}$ and the function $t \mapsto c_d f_d(t) + c_s f_s(t)$, $t \in \mathbb{R}$, is bounded, we conclude that $f_{a,ds}(t) \neq 0$

for any $t \in \mathbb{R}$. By Theorem 5, $f_{a,ds}$ admits the representation

$$f_{a,ds}(t) = \exp\left\{\int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\left(v_a(x) + \operatorname{sgn}(x)\,\frac{\mathfrak{m}_a \cdot e^{-|x|}}{|x|}\right) dx\right\}, \quad t \in \mathbb{R}, \tag{27}$$

where $v_a : \mathbb{R} \to \mathbb{C}$ is a function satisfying the condition $\int_{\mathbb{R}} |v_a(x)|\, dx < \infty$, $\mathfrak{m}_a$ is a constant defined by the formula

$$\mathfrak{m}_a := \frac{1}{2\pi}\big(\lim_{t\to\infty} \operatorname{Arg} f_{a,ds}(t) - \lim_{t\to-\infty} \operatorname{Arg} f_{a,ds}(t)\big).$$

Since $c_d + c_s$ is positive, formula (10) gives an equivalent definition of $\mathfrak{m}_a$. By the way, it is easily seen that $\mathfrak{m}_a = 0$ if $c_a = 0$. In general, $\mathfrak{m}_a \in \mathbb{Z}$ (see Remark 4). Therefore, in the case $c_a = 0$, we set $v_a(x) := 0$ for every $x \in \mathbb{R}$ and the formula (27) remains valid.

We now turn to the function

$$f_{s,d}(t) = \frac{c_d f_d(t) + c_s f_s(t)}{(c_d + c_s) f_d(t)} = \left(1 + \frac{c_s f_s(t)}{c_d f_d(t)}\right) \cdot \frac{c_d}{c_d + c_s}, \quad t \in \mathbb{R}.$$

From (24) we know that $f_{s,d}(t) \neq 0$ for any $t \in \mathbb{R}$ and we see that $f_{s,d}(0) = 1$. Hence the distinguished logarithm $\operatorname{Ln} f_{s,d}$ is uniquely defined on $\mathbb{R}$ with condition $\operatorname{Ln} f_{s,d}(0) = 0$. Due to the assumptions,

$$\left|\frac{c_s f_s(t)}{c_d f_d(t)}\right| \leqslant \frac{c_s}{c_d \mu_d} < 1 \quad \text{for any} \quad t \in \mathbb{R}, \tag{28}$$

and we can write

$$\operatorname{Ln} f_{s,d}(t) = \ln\left(1 + \frac{c_s f_s(t)}{c_d f_d(t)}\right) - \ln\left(1 + \frac{c_s}{c_d}\right), \quad t \in \mathbb{R}, \tag{29}$$

where $\ln(\,\cdot\,)$ returns the principal value of the logarithm. Let us consider the first logarithm in the right-hand side. Due to (28), we have the expansion

$$\begin{aligned} &\ln\left(1 + \frac{c_s f_s(t)}{c_d f_d(t)}\right) \\ &\quad = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{c_s f_s(t)}{c_d f_d(t)}\right)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{c_s}{c_d}\right)^k f_s(t)^k g_d(t)^k, \quad t \in \mathbb{R}, \end{aligned} \tag{30}$$

where we have set $g_d(t) := 1/f_d(t)$, $t \in \mathbb{R}$. The series converges uniformly in $\mathbb{R}$ by the Weierstrass M-test, because, due to (28), for any $t \in \mathbb{R}$ the absolute values of its terms are majorized by the terms of the convergent numerical series

$$\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{c_s}{c_d \mu_d}\right)^k.$$

Let us consider the function g_d and show that it is actually a Fourier–Stieltjes transform of some function from class $\boldsymbol{V}$. Indeed, according to (26),

$$g_d(t) = e^{-it\gamma_0} \cdot e^{-\lambda_0} \cdot \exp\Big\{-\sum_{u\in\langle\mathcal{X}\rangle\setminus\{0\}} \lambda_u e^{itu}\Big\}, \quad t \in \mathbb{R},$$

where we have set

$$\lambda_0 := -\sum_{u\in\langle\mathcal{X}\rangle\setminus\{0\}} \lambda_u.$$

There is an absolutely convergent Fourier series with real coefficients in the last exponential function. Hence this function itself expands into an absolutely convergent Fourier series with coefficients in $\mathbb{R}$ and exponents in $\langle\mathcal{X}\rangle$. This remains true after multiplying this series by $e^{-\lambda_0}$ and $e^{-it\gamma_0}$ (which leads to g_d), because $\lambda_0 \in \mathbb{R}$ and $\gamma_0 \in \langle\mathcal{X}\rangle$. Thus we have

$$g_d(t) = \sum_{y\in\langle\mathcal{X}\rangle} q_y e^{ity}, \quad t \in \mathbb{R}, \tag{31}$$

where $q_y \in \mathbb{R}$ for every $y \in \langle\mathcal{X}\rangle$ and $\sum_{y\in\langle\mathcal{X}\rangle} |q_y| < \infty$. Obviously, the series (31) can be written as a Fourier–Stieltjes transform,

$$g_d(t) = \int_{\mathbb{R}} e^{itx}\, dI_d(x), \quad t \in \mathbb{R},$$

with the discrete function $I_d \in \boldsymbol{V}$ given by

$$I_d(x) := \sum_{\substack{y\in\langle\mathcal{X}\rangle \\ y\leqslant x}} q_y, \quad x \in \mathbb{R}. \tag{32}$$

Let us return to the formula (30). We observe that for any $k \in \mathbb{N}$ the functions f_s^k, g_d^k, $f_s^k \cdot g^k$ are the Fourier–Stieltjes transforms of functions from $\boldsymbol{V}$:

$$\begin{gathered} f_s(t)^k = \int_{\mathbb{R}} e^{itx}\, dF_s^{*k}(x), \quad g_d(t)^k = \int_{\mathbb{R}} e^{itx}\, dI_d^{*k}(x), \\ f_s(t)^k g_d(t)^k = \int_{\mathbb{R}} e^{itx}\, d(F_s^{*k} * I_d^{*k})(x), \quad t \in \mathbb{R}. \end{gathered} \tag{33}$$

Consequently, the partial sums of the series (30) are the Fourier–Stieltjes transforms of the functions

$$W_n(x) := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k \big(F_s^{*k} * I_d^{*k}\big)(x), \quad x \in \mathbb{R},\ n \in \mathbb{N}. \tag{34}$$

Let us show that the sum of the whole series (30) admits the representation

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k f_s(t)^k g_d(t)^k = \int_{\mathbb{R}} e^{itx}\, dW(x), \quad t \in \mathbb{R}, \tag{35}$$

with some function $W \in \boldsymbol{V}$. According to Lemma 1, for this it is sufficient that (1) the partial sums of (30) uniformly converge to the infinite sum on every bounded interval and (2) $\sup_{n\in\mathbb{N}} \|W_n\| < \infty$. The uniform convergence of the partial sums (even on whole real line) was shown above, i.e., (1) holds. Let us prove (2). For any $n \in \mathbb{N}$ we have the estimate

$$\|W_n\| \leqslant \sum_{k=1}^{n} \frac{1}{k}\left(\frac{c_s}{c_d}\right)^k \|F_s^{*k} * I_d^{*k}\| \leqslant \sum_{k=1}^{n}\left\{\frac{1}{k}\left(\frac{c_s}{c_d}\right)^k \|F_s^{*k}\| \cdot \|I_d^{*k}\|\right\}.$$

Here $\left\|F_s^{*k}\right\| = 1$ for every $k \in \mathbb{N}$, because all the F_s^{*k} are distribution functions. Also, it is convenient for us to set $\|g_d^k\| = \|I_d^{*k}\|$, $k \in \mathbb{N}$. Thus we come to the inequalities

$$\|W_n\| \leqslant \sum_{k=1}^{n} \frac{1}{k}\left(\frac{c_s}{c_d}\right)^k \|g_d^k\|, \quad n \in \mathbb{N}. \tag{36}$$

We now estimate $\|g_d^k\|$ for every $k \in \mathbb{N}$. Let us return to decomposition (31) for the function g_d. The set $\langle\mathcal{X}\rangle$ is countable, and $\sum_{y\in\langle\mathcal{X}\rangle} |q_y| < \infty$. So we fix an arbitrary $\varepsilon \in (0, 1)$ and choose $N_\varepsilon \in \mathbb{N}$ and distinct points $y_1, \dots, y_{N_\varepsilon} \in \langle\mathcal{X}\rangle$ such that

$$\sum_{y\in\langle\mathcal{X}\rangle\setminus\{y_1,\dots,y_{N_\varepsilon}\}} |q_y| < \varepsilon.$$

We introduce the polynomial

$$\tilde{g}_\varepsilon(t) := \sum_{m=1}^{N_\varepsilon} q_{y_m} e^{ity_m}, \quad t \in \mathbb{R}. \tag{37}$$

So we have $\|g_d - \tilde{g}_\varepsilon\| < \varepsilon$. We also observe that

$$\left\|\frac{g_d - \tilde{g}_\varepsilon}{g_d}\right\| \leqslant \|g_d - \tilde{g}_\varepsilon\| \cdot \left\|\frac{1}{g_d}\right\| = \|g_d - \tilde{g}_\varepsilon\| \cdot \|f_d\| = \|g_d - \tilde{g}_\varepsilon\| < \varepsilon. \tag{38}$$

Let us fix $k \in \mathbb{N}$ and write

$$\|g_d^k\| = \left\|\tilde{g}_\varepsilon^k \cdot \left(\frac{g_d}{\tilde{g}_\varepsilon}\right)^k\right\| \leqslant \|\tilde{g}_\varepsilon^k\| \cdot \left\|\frac{g_d}{\tilde{g}_\varepsilon}\right\|^k.$$

In order to get a convenient representation for $g_d/\tilde{g}_\varepsilon$, we observe that for any $n \in \mathbb{N}$

$$1 - \left(\frac{g_d(t) - \tilde{g}_\varepsilon(t)}{g_d(t)}\right)^n = \left(1 - \frac{g_d(t) - \tilde{g}_\varepsilon(t)}{g_d(t)}\right) \cdot \sum_{j=0}^{n-1}\left(\frac{g_d(t) - \tilde{g}_\varepsilon(t)}{g_d(t)}\right)^j$$
$$= \frac{\tilde{g}_\varepsilon(t)}{g_d(t)} \cdot \sum_{j=0}^{n-1}\left(\frac{g_d(t) - \tilde{g}_\varepsilon(t)}{g_d(t)}\right)^j, \quad t \in \mathbb{R}.$$

So we have

$$\frac{g_d(t)}{\tilde{g}_\varepsilon(t)} = \sum_{j=0}^{n-1}\left(\frac{g_d(t)-\tilde{g}_\varepsilon(t)}{g(t)}\right)^j + \left(\frac{g_d(t)-\tilde{g}_\varepsilon(t)}{g_d(t)}\right)^n \cdot \frac{g_d(t)}{\tilde{g}_\varepsilon(t)}, \quad t\in\mathbb{R}, \quad n\in\mathbb{N}.$$

Hence, due to (38), we get

$$\left\|\frac{g_d}{\tilde{g}_\varepsilon}\right\| \leqslant \sum_{j=0}^{n-1}\left\|\frac{g_d-\tilde{g}_\varepsilon}{g_d}\right\|^j + \left\|\frac{g_d-\tilde{g}_\varepsilon}{g_d}\right\|^n \cdot \left\|\frac{g_d}{\tilde{g}_\varepsilon}\right\| < \sum_{j=0}^{n-1}\varepsilon^j + \varepsilon^n\cdot\left\|\frac{g_d}{\tilde{g}_\varepsilon}\right\|, \quad n\in\mathbb{N}.$$

Since $\varepsilon\in(0,1)$, letting $n\to\infty$ yields

$$\left\|\frac{g_d}{\tilde{g}_\varepsilon}\right\| \leqslant \sum_{j=0}^{\infty}\varepsilon^j = \frac{1}{1-\varepsilon}.$$

Thus we have

$$\|g_d^k\| \leqslant \frac{\|\tilde{g}_\varepsilon^k\|}{(1-\varepsilon)^k}. \tag{39}$$

So the estimation of $\|I_d^{*k}\| = \|g_d^k\|$ is reduced to finding an upper bound for $\|\tilde{g}_\varepsilon^k\|$.

Let us consider $\tilde{g}_\varepsilon$ defined by formula (37). Let $Y_\varepsilon := \{y_1, y_2, \dots, y_{N_\varepsilon}\}$. Suppose that $Y_\varepsilon = \{0\}$, i.e., $\sum_{y\in\langle\mathcal{X}\rangle\setminus\{0\}} |q_y| < \varepsilon$ and $\tilde{g}_\varepsilon(t) = q_0$ for any $t\in\mathbb{R}$. Then $\|\tilde{g}_\varepsilon^k\| = |q_0|^k$ for any $k\in\mathbb{N}$. Observe that

$$|q_0| = |\tilde{g}_\varepsilon(0)| \leqslant |g_d(0)| + |\tilde{g}_\varepsilon(0) - g_d(0)|,$$

where $|g_d(0)| = |1/f_d(0)| = 1$ and

$$|\tilde{g}_\varepsilon(0) - g_d(0)| = \left|\sum_{y\in\langle\mathcal{X}\rangle\setminus\{0\}} q_y\right| \leqslant \sum_{y\in\langle\mathcal{X}\rangle\setminus\{0\}} |q_y| < \varepsilon.$$

Thus $|q_0| < 1+\varepsilon$ and hence $\|\tilde{g}_\varepsilon^k\| < (1+\varepsilon)^k$.

We next assume that $Y_\varepsilon \neq \{0\}$, i.e., Y_ε contains nonzero elements. We select a basis over $\mathbb{Q}$ in the set Y_ε (see [24, p. 67–68]), i.e., we choose nonzero elements $\beta_1, \dots, \beta_d \in Y_\varepsilon$ with some $d\in\{1,\dots,N_\varepsilon\}$, which are linearly independent over $\mathbb{Q}$ and for any $m\in\{1,\dots,N_\varepsilon\}$ there exist some $r_{m,1},\dots,r_{m,d}\in\mathbb{Q}$ such that $y_m = \sum_{l=1}^d r_{m,l}\beta_l$. Linear independence over $\mathbb{Q}$ means that with $r_1,\dots,r_d\in\mathbb{Q}$, vanishes only in the case $r_1 = r_2 = \cdots = r_d = 0$. It is clear that the coefficients $r_{m,l}$ of the decomposition of y_m are uniquely determined. Let $\varkappa$ be the minimal positive integer such that $\bar{r}_{m,l} := \varkappa\cdot r_{m,l}\in\mathbb{Z}$ for any admissible m and l. We set $\bar{\beta}_l := \beta_l/\varkappa$ for every $l\in\{1,\dots,d\}$. Then we have $y_m = \sum_{l=1}^d \bar{r}_{m,l}\bar{\beta}_l$ for any $m\in\{1,\dots,N_\varepsilon\}$. Here the coefficients $\bar{r}_{m,l}$ are uniquely determined by y_m too. We define the vectors $\bar{r}_m := (\bar{r}_{m,1},\dots,\bar{r}_{m,d})\in\mathbb{Z}^d$, $m = 1,\dots,N_\varepsilon$. Since $y_1,\dots,y_{N_\varepsilon}$ are assumed to be distinct, all $\bar{r}_m$ are distinct too. Let us introduce the function

$$\varphi_\varepsilon(t_1,\dots,t_d) := \sum_{m=1}^{N_\varepsilon} q_{y_m}\exp\left\{i\sum_{l=1}^d \bar{r}_{m,l}t_l\right\}, \quad t_1,\dots,t_d\in\mathbb{R}.$$

It is easily seen that this function is continuous and 2π-periodic over every variable. We will also use a shorthand for it:

$$\varphi_\varepsilon(t) = \sum_{m=1}^{N_\varepsilon} q_{y_m} e^{i\langle t,\bar{r}_m\rangle}, \qquad t = (t_1, \dots, t_d) \in \mathbb{R}^d.$$

The functions φ_ε and $\tilde{g}_\varepsilon$ are related:

$$\varphi_\varepsilon\big(\bar{\beta}_1 t, \dots, \bar{\beta}_d t\big) = \sum_{m=1}^{N_\varepsilon} q_{y_m} \exp\left\{it\sum_{l=1}^{d} \bar{r}_{m,l}\bar{\beta}_l\right\} = \sum_{m=1}^{N_\varepsilon} q_{y_m} e^{ity_m} = \tilde{g}_\varepsilon(t), \qquad t \in \mathbb{R},$$

i.e., $\tilde{g}_\varepsilon$ is diagonal for the function $(t_1, \dots, t_d) \mapsto \varphi_\varepsilon(\bar{\beta}_1 t_1, \dots, \bar{\beta}_d t_d)$. Hence the image of the first function is dense in the image of the second one (see [24, p. 116, Theorem 2.4.1]) and, consequently, in the image of φ_ε. This yields

$$\sup_{t\in\mathbb{R}} |\tilde{g}_\varepsilon(t)| = \sup_{t\in[-\pi,\pi]^d} |\varphi_\varepsilon(t)|. \tag{40}$$

Let us show that

$$\|\tilde{g}_\varepsilon^k\| = \|\varphi_\varepsilon^k\| \quad \text{for any } k \in \mathbb{N}. \tag{41}$$

We fix $k \in \mathbb{N}$ and write

$$\tilde{g}_\varepsilon^k(t) = \left(\sum_{m=1}^{N_\varepsilon} q_{y_m} e^{ity_m}\right)^k = \sum_{m_1=1}^{N_\varepsilon} \cdots \sum_{m_k=1}^{N_\varepsilon} \big(q_{y_{m_1}} \cdot \dots \cdot q_{y_{m_k}} e^{it(y_{m_1}+\dots+y_{m_k})}\big), \qquad t \in \mathbb{R},$$

$$\varphi_\varepsilon^k(t) = \left(\sum_{m=1}^{N_\varepsilon} q_{y_m} e^{i\langle t,\bar{r}_m\rangle}\right)^k = \sum_{m_1=1}^{N_\varepsilon} \cdots \sum_{m_k=1}^{N_\varepsilon} \big(q_{y_{m_1}} \cdot \dots \cdot q_{y_{m_k}} e^{i\langle t,\bar{r}_{m_1}+\dots+\bar{r}_{m_k}\rangle}\big), \ t \in \mathbb{R}^d.$$

We next have the representation

$$\tilde{g}_\varepsilon^k(t) = \sum_{z\in\mathcal{Y}_\varepsilon^{(k)}} \Bigg(\sum_{\substack{1\leqslant m_1,\dots,m_k\leqslant N_e \\ y_{m_1}+\dots+y_{m_k}=z}} q_{y_{m_1}} \cdot \dots \cdot q_{y_{m_k}}\Bigg) e^{itz}, \qquad t \in \mathbb{R},$$

$$\varphi_\varepsilon^k(t) = \sum_{s\in\mathcal{R}_\varepsilon^{(k)}} \Bigg(\sum_{\substack{1\leqslant m_1,\dots,m_k\leqslant N_e \\ \bar{r}_{m_1}+\dots+\bar{r}_{m_k}=s}} q_{\bar{r}_{m_1}} \cdot \dots \cdot q_{\bar{r}_{m_k}}\Bigg) e^{i\langle t,s\rangle}, \quad t \in \mathbb{R}^d,$$

where

$$\mathcal{Y}_\varepsilon^{(k)} := \big\{y_{m_1} + \dots + y_{m_k} : m_1, \dots, m_k \in \{1, \dots, N_\varepsilon\}\big\},$$
$$\mathcal{R}_\varepsilon^{(k)} := \big\{\bar{r}_{m_1} + \dots + \bar{r}_{m_k} : m_1, \dots, m_k \in \{1, \dots, N_\varepsilon\}\big\}.$$

Thus we have

$$\|\tilde{g}_\varepsilon^k\| = \sum_{z\in\mathcal{Y}_\varepsilon^{(k)}} \Bigg| \sum_{\substack{m_1=1,\dots,N_\varepsilon,\\ \dots\\ m_k=1,\dots,N_\varepsilon\\ y_{m_1}+\dots+y_{m_k}=z}} q_{y_{m_1}}\cdot\dots\cdot q_{y_{m_k}}\Bigg|, \tag{42}$$

$$\|\varphi_\varepsilon^k\| = \sum_{s\in\mathcal{R}_\varepsilon^{(k)}} \Bigg| \sum_{\substack{m_1=1,\dots,N_\varepsilon,\\ \dots\\ m_k=1,\dots,N_\varepsilon\\ \bar{r}_{m_1}+\dots+\bar{r}_{m_k}=s}} q_{y_{m_1}}\cdot\dots\cdot q_{y_{m_k}}\Bigg|. \tag{43}$$

There is a natural map between the two finite sets $\mathcal{Y}_\varepsilon^{(k)}$ and $\mathcal{R}_\varepsilon^{(k)}$: every number $z=\sum_{j=1}^k y_{m_j}$ from the first set is paired with a vector $s=\sum_{j=1}^k \bar{r}_{m_j}$ from the second set. Here s is the vector of coefficients of the decomposition for z with the basis $\bar{\beta}_1,\dots,\bar{\beta}_d$. This map is actually a bijection. Indeed, it is injective, because distinct numbers from $\mathcal{Y}_\varepsilon^{(k)}$ have the distinct decompositions, i.e., they are paired with the distinct vectors from the set $\mathcal{R}_\varepsilon^{(k)}$. The map is surjective, because any vector $v\in\mathcal{R}_\varepsilon^{(k)}$ is a sum $\bar{r}_{m_1'}+\dots+\bar{r}_{m_k'}$ with some indices $m_1',\dots,m_d'\in\{1,\dots,N_\varepsilon\}$, and this sum corresponds to the number $z=y_{m_1'}+\dots+y_{m_k'}$ from the set $\mathcal{Y}_\varepsilon^{(k)}$ by construction. Next, it is clear from the basis decompositions that for any fixed pair of corresponding elements $z\in\mathcal{Y}_\varepsilon^{(k)}$ and $s\in\mathcal{R}_\varepsilon^{(k)}$ the equalities $y_{m_1}+\dots+y_{m_k}=z$ and $\bar{r}_{m_1}+\dots+\bar{r}_{m_k}=s$ are equivalent for varying index vectors $(m_1,\dots,m_k)$. Since the inner sums in (42) and (43) add the same weight $q_{y_{m_1}}\cdot\dots\cdot q_{y_{m_k}}$ for every index vector $(m_1,\dots,m_k)$, we conclude that these inner sums are equal. Thus we come to the equality $\|\tilde{g}_\varepsilon^k\|=\|\varphi_\varepsilon^k\|$.

We now apply Lemma 4 to estimate $\|\varphi_\varepsilon^k\|$ for any $k\in\mathbb{N}$ (we set $\varphi:=\varphi_\varepsilon$, $N:=N_\varepsilon$, $q_m:=q_{y_m}$ and $c_m:=\bar{r}_m$ for every $m=1,\dots,N_\varepsilon$). Using inequality (19), we get

$$\|\varphi_\varepsilon^k\| \leqslant A_{\varphi_\varepsilon} k^d \sup_{t\in[-\pi,\pi]^d} |\varphi_\varepsilon(t)|^k, \quad k\in\mathbb{N},$$

where A_{φ_ε} is a constant defined by (20), which doesn't depend on k. Applying (40) and (41), we come to the inequality

$$\|\tilde{g}_\varepsilon^k\| \leqslant A_{\varphi_\varepsilon} k^d \sup_{t\in\mathbb{R}} |\tilde{g}_\varepsilon(t)|^k, \quad k\in\mathbb{N}.$$

Observe that

$$\sup_{t\in\mathbb{R}} |\tilde{g}_\varepsilon(t)| \leqslant \sup_{t\in\mathbb{R}} |g_d(t)| + \sup_{t\in\mathbb{R}} |g_d(t)-\tilde{g}_\varepsilon(t)|.$$

On account of (38), we have

$$\frac{\sup_{t\in\mathbb{R}} |g_d(t)-\tilde{g}_\varepsilon(t)|}{\sup_{t\in\mathbb{R}} |g_d(t)|} \leqslant \sup_{t\in\mathbb{R}} \left|\frac{g_d(t)-\tilde{g}_\varepsilon(t)}{g_d(t)}\right| \leqslant \left\|\frac{g_d-\tilde{g}_\varepsilon}{g_d}\right\| < \varepsilon,$$

i.e., $\sup_{t\in\mathbb{R}} |g_d(t)-\tilde{g}_\varepsilon(t)| < \varepsilon \sup_{t\in\mathbb{R}} |g_d(t)|$. Therefore

$$\sup_{t\in\mathbb{R}} |\tilde{g}_\varepsilon(t)| \leqslant \sup_{t\in\mathbb{R}} |g_d(t)| + \varepsilon \sup_{t\in\mathbb{R}} |g_d(t)| = (1+\varepsilon) \sup_{t\in\mathbb{R}} |g_d(t)|,$$

where

$$\sup_{t\in\mathbb{R}}|g_d(t)| = \sup_{t\in\mathbb{R}}\left|\frac{1}{f_d(t)}\right| = \frac{1}{\inf_{t\in\mathbb{R}}|f_d(t)|} = \frac{1}{\mu_d}.$$

So we obtain the estimate

$$\|\tilde{g}_\varepsilon^k\| \leqslant A_{\varphi_\varepsilon} k^d \cdot \frac{(1+\varepsilon)^k}{\mu_d^k}, \quad k\in\mathbb{N}.$$

We note that here $k^d \geqslant 1$ and $1/\mu_d^k \geqslant 1$ for any $k\in\mathbb{N}$. Therefore the estimates of $\|\tilde{g}_\varepsilon^k\|$ in the cases $Y_\varepsilon = \{0\}$ and $Y_\varepsilon \neq \{0\}$ can be unified as

$$\|\tilde{g}_\varepsilon^k\| \leqslant C_\varepsilon k^d \cdot \frac{(1+\varepsilon)^k}{\mu_d^k} \quad \text{for any } k\in\mathbb{N},$$

with some constant $C_\varepsilon > 0$. We use this in (39), obtaining:

$$\|g_d^k\| \leqslant \frac{\|\tilde{g}_\varepsilon^k\|}{(1-\varepsilon)^k} \leqslant C_\varepsilon k^d \cdot \frac{1}{\mu_d^k}\cdot\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^k, \quad k\in\mathbb{N}.$$

Let us return to (36). Due to the last estimate, for any $\varepsilon\in(0,1)$ and $n\in\mathbb{N}$ we have

$$\|W_n\| \leqslant \sum_{k=1}^n\left(\frac{1}{k}\left(\frac{c_s}{c_d}\right)^k C_\varepsilon k^d\cdot\frac{1}{\mu_d^k}\cdot\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^k\right) = C_\varepsilon\sum_{k=1}^n k^{d-1}\left(\frac{c_s}{c_d\mu_d}\cdot\frac{1+\varepsilon}{1-\varepsilon}\right)^k. \tag{44}$$

Since $c_s < c_d\mu_d$ by assumption, the fixed number $\varepsilon\in(0,1)$ can be specified by

$$0 < \frac{c_s}{c_d\mu_d}\cdot\frac{1+\varepsilon}{1-\varepsilon} < 1.$$

Then we obtain

$$\sup_{n\in\mathbb{N}}\|W_n\| \leqslant C_\varepsilon\sum_{k=1}^\infty k^{d-1}\left(\frac{c_s}{c_d\mu_d}\cdot\frac{1+\varepsilon}{1-\varepsilon}\right)^k < \infty,$$

i.e., we showed that the sufficient condition (2) for (35) is satisfied (condition (1) was proved above). So, according to (30) and (35), we have

$$\ln\left(1+\frac{c_s f_s(t)}{c_d f_d(t)}\right) = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k}\left(\frac{c_s}{c_d}\right)^k f_s(t)^k g_d(t)^k = \int_{\mathbb{R}} e^{itx}\,dW(x), \quad t\in\mathbb{R}, \tag{45}$$

with some function $W\in\boldsymbol{V}$. Hence (29) takes the form

$$\operatorname{Ln} f_{s,d}(t) = \int_{\mathbb{R}} e^{itx}\,dW(x) - \ln\left(1+\frac{c_s}{c_d}\right), \quad t\in\mathbb{R}.$$

The equality $\operatorname{Ln} f_{s,d}(0) = 0$, which was mentioned above, implies

$$\ln\left(1+\frac{c_s}{c_d}\right) = \int_{\mathbb{R}} dW(x).$$

Then

$$\operatorname{Ln} f_{s,d}(t) = \int_{\mathbb{R}} (e^{itx} - 1)\ dW(x), \quad t \in \mathbb{R},$$

i.e., we come to the representation

$$(46) \qquad f_{s,d}(t) = \exp\left\{\int_{\mathbb{R}} (e^{itx} - 1)\ dW(x)\right\} = \exp\left\{\int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\ dW(x)\right\}, \quad t \in \mathbb{R}.$$

Let us investigate the function W. If $c_s = 0$, then W is identically 0, which is seen from (45). We next focus on the case $c_s \neq 0$. We will prove that W is continuous on $\mathbb{R}$. For this we first observe that the W_n, $n \in \mathbb{N}$, are continuous functions on $\mathbb{R}$. Indeed, according to (34), W_n is a finite linear combination of the functions $F_s^{*k} * I_d^{*k}$, $k \in \mathbb{N}$, which are continuous on $\mathbb{R}$, being convolutions with continuous F_s. Next, we observe that, similarly to (44), we have the estimate

$$\|W_{n_2} - W_{n_1}\| \leqslant C_\varepsilon \sum_{k=n_1+1}^{n_2} k^{d-1}\left(\frac{c_s}{c_d \mu_d} \cdot \frac{1+\varepsilon}{1-\varepsilon}\right)^k,$$

with the same ε, C_ε and for any positive integers n_1 and n_2 such that $n_1 \leqslant n_2$. The sum on the right can be made arbitrarily small for all sufficiently large n_1 and n_2, because its terms (for $k \in \mathbb{N}$) form a convergent series. This means that $(W_n)_{n\in\mathbb{N}}$ is a fundamental sequence in the space $\boldsymbol{V}$. Since $\boldsymbol{V}$ is a complete norm space (see the comments at the end of the introduction), there exists $W_* \in \boldsymbol{V}$ such that $\|W_n - W_*\| \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} \int_{\mathbb{R}} e^{itx}\, dW_n(x) = \int_{\mathbb{R}} e^{itx}\, dW_*(x), \quad t \in \mathbb{R}.$$

On the other hand, we know that

$$\lim_{n\to\infty} \int_{\mathbb{R}} e^{itx}\, dW_n(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\left(\frac{c_s}{c_d}\right)^k f_s(t)^k g_d(t)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{c_s}{c_d}\right)^k f_s(t)^k g_d(t)^k = \int_{\mathbb{R}} e^{itx}\, dW(x), \quad t \in \mathbb{R}.$$

Then $\int_{\mathbb{R}} e^{itx}\, dW_*(x) = \int_{\mathbb{R}} e^{itx}\, dW(x)$ for any $t \in \mathbb{N}$, and we conclude that $W_* = W$. So we have proved that

$$(47) \qquad \|W_n - W\| \to 0 \quad \text{as } n \to \infty.$$

In general, from the well-known relation $\sup_{x\in\mathbb{R}} |U(x)| \leqslant \|U\|$ for any $U \in \boldsymbol{V}$, we have uniform convergence:

$$\sup_{x\in\mathbb{R}} \bigl|W_n(x) - W(x)\bigr| \to 0 \quad \text{as } n \to \infty.$$

Due to the already proved continuity of W_n, $n \in \mathbb{N}$, we conclude that W is continuous on $\mathbb{R}$ too.

We now show that W cannot be purely absolutely continuous (in the case $c_s \neq 0$). For suppose that W is absolutely continuous. Let

$$w(t) := \int_{\mathbb{R}} e^{itx}\, dW(x), \quad t \in \mathbb{R}.$$

Then, according to (45), we have

$$1 + \frac{c_s f_s(t)}{c_d f_d(t)} = \exp\left\{\int_{\mathbb{R}} e^{itx}\, dW(x)\right\} = \exp\{w(t)\} = 1 + \sum_{k=1}^{\infty} \frac{w(t)^k}{k!}, \quad t \in \mathbb{R}.$$

Hence, on the one hand,

$$\begin{aligned} \frac{c_s f_s(t)}{c_d f_d(t)} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{w(t)^k}{k!} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!} \int_{\mathbb{R}} e^{itx}\, dW^{*k}(x) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} e^{itx}\, d\left(\sum_{k=1}^{n} \frac{1}{k!}\, W^{*k}(x)\right), \quad t \in \mathbb{R}. \end{aligned}$$

On the other hand, on account of (33), we have

$$\frac{c_s f_s(t)}{c_d f_d(t)} = \frac{c_s}{c_d} \cdot f_s(t) \cdot g_d(t) = \frac{c_s}{c_d} \cdot \int_{\mathbb{R}} e^{itx}\, d(F_s * I_d)(x) = \int_{\mathbb{R}} e^{itx}\, d\left(\frac{c_s}{c_d} \cdot (F_s * I_d)(x)\right),$$

for $t \in \mathbb{R}$. Since the variance

$$\left\| \sum_{k=n_1+1}^{n_2} \frac{1}{k!}\, W^{*k} \right\| \leqslant \sum_{k=n_1+1}^{n_2} \frac{\|W^{*k}\|}{k!} \leqslant \sum_{k=n_1+1}^{n_2} \frac{\|W\|^k}{k!}$$

can be made arbitrarily small for all sufficiently large n_1 and n_2, the sequence of sums $\sum_{k=1}^{n} W^{*k}/k!$, $n \in \mathbb{N}$, is fundamental in the space $\boldsymbol{V}$. Due to the completeness of $\boldsymbol{V}$, these sums converge in variation to some function from $\boldsymbol{V}$, namely, to $(c_s/c_d) \cdot (F_s * I_d)$ by the uniqueness of the Fourier–Stieltjes transform:

$$\left\| \sum_{k=1}^{n} \frac{1}{k!}\, W^{*k} - \frac{c_s}{c_d} \cdot (F_s * I_d) \right\| \to 0 \quad \text{as } n \to \infty.$$

But this is impossible, because $\sum_{k=1}^{n} W^{*k}/k!$, $n \in \mathbb{N}$, are absolutely continuous as linear combinations of convolution powers of absolutely continuous W by the assumption, but $(c_s/c_d) \cdot (F_s * I_d)$ is continuous singular as the convolution of the continuous singular function F_s and the discrete function I_d (see the comments after Remark 3). Thus W is not a (purely) absolutely continuous function from $\boldsymbol{V}$, i.e., it always has some continuous singular part.

Let us return to (47) and prove the sufficient condition (from the statement of the theorem) for W to be (purely) continuous singular. Suppose that all the functions F_s^{*k}, $k \in \mathbb{N}$, are continuous singular. Hence all W_n, $n \in \mathbb{N}$ are too. Let W_a and W_s

be the absolutely continuous part and the continuous singular part of the Lebesgue decomposition for W: $W = W_a + W_s$. Then

$$\|W_n - W\| = \|W_n - W_a - W_s\| = \|W_n - W_s\| + \|W_a\| \geqslant \|W_a\| \geqslant 0.$$

Due to (47), we conclude that $\|W_a\| = 0$, i.e., $W_a(x) = 0$ for any $x \in \mathbb{R}$. Thus $W = W_s$.

We now combine the representations (26), (27) and (46) with formula (25):

$$\begin{aligned} f(t) = \exp\Big\{ it\gamma_0 + \sum_{u \in \langle \mathcal{X} \rangle \setminus \{0\}} \lambda_u \big(e^{itu} - 1\big) \\ + \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\Big(v_a(x) + \operatorname{sgn}(x)\,\frac{\mathfrak{m}_a e^{-|x|}}{|x|}\Big)\,dx + \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\; dW(x)\Big\} \\ = \exp\Big\{ it\gamma_0 + \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\; dL(x)\Big\}, \quad t \in \mathbb{R}, \end{aligned}$$

with $L(x) := L_d(x) + L_a(x) + L_s(x)$, $x \in \mathbb{R} \setminus \{0\}$, where L_d, L_a, and L_s are defined by formulas (11), (12), and (13). It is not difficult to check that L_d, L_a, L_s, and, consequently, L satisfy all conditions the for a spectral function of the Lévy type representation (see introduction or Theorem 6). However, we recall that the function v_a is potentially complex-valued and hence L is too. Next, it is seen that

$$\int_{S_1} |x|\; d|L_*|(x) < \infty$$

with $S_1 := [-1, 1] \setminus \{0\}$ and for any $L_* \in \{L_d, L_a, L_s\}$. Hence it is also true for $L_* = L$, and we can write

$$f(t) = \exp\Big\{ it\gamma_1 + \int_{\mathbb{R}\setminus\{0\}} \big(e^{itx} - 1 - itx\, \mathbb{I}_{[-1,1]}(x)\big)\; dL(x)\Big\}, \quad t \in \mathbb{R},$$

with $\gamma_1 := \gamma_0 + \int_{\mathbb{R}\setminus\{0\}} x\, \mathbb{I}_{[-1,1]}(x)\; dL(x)$, which is potentially complex.

We now apply Theorem 6 to derive that L is actually real-valued, which means that v_a is a real-valued function. Moreover, we have $F \in \boldsymbol{Q}$, i.e., (i) is proved. □

Proof of Proposition 1. To obtain a contradiction, we assume that W is continuous singular. For convenience, we introduce the operator $[\,\cdot\,]_s$, which acts from $\boldsymbol{V}$ to $\boldsymbol{V}$, and for any $U \in \boldsymbol{V}$ it returns the continuous singular part of U as $[U]_s$ (which can be identically zero). We start with (47) from the proof of Theorem 3, which holds under the assumptions of the proposition. For every $n \in \mathbb{N}$ we write

$$\|W_n - W\| = \big\|(W_n - [W_n]_s) + ([W_n]_s - W)\big\| = \big\|W_n - [W_n]_s\big\| + \big\|[W_n]_s - W\big\|,$$

where the latter equality is valid because $W_n - [W_n]_s$ is absolutely continuous (or identically zero) and $[W_n]_s - W$ is continuous singular by assumption. Therefore $\|W_n - W\| \geqslant \big\|[W_n]_s - W\big\| \geqslant 0$ and, due to (47), we conclude that

$$\big\|[W_n]_s - W\big\| \to 0 \quad \text{as} \quad n \to \infty. \tag{48}$$

Next, according to (34), we have

$$[W_n]_s = \left(\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\left(\frac{c_s}{c_d}\right)^k F_s^{*k} * I_d^{*k}\right)_s = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\left(\frac{c_s}{c_d}\right)^k \left[F_s^{*k} * I_d^{*k}\right]_s$$

for $n \in \mathbb{N}$. Since I_d is discrete (see (32)), the functions I_d^{*k} are discrete and hence $\left[F_s^{*k} * I_d^{*k}\right]_s = \left[F_s^{*k}\right]_s * I_d^{*k}$ for any $k \in \mathbb{N}$. Thus we come to the equalities

$$[W_n]_s = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\left(\frac{c_s}{c_d}\right)^k \left[F_s^{*k}\right]_s * I_d^{*k}, \quad n \in \mathbb{N}. \tag{49}$$

Let us consider the functions $\left[F_s^{*k}\right]_s$ and numbers $\left[F_s^{*k}\right]_s(\infty)$. By the definition of n_a, for any $k < n_a$ we have $\left[F_s^{*k}\right]_s = F_s^{*k}$ and, in particular, $\left[F_s^{*k}\right]_s(\infty) = 1$. For the case $k = n_a$, by the assumption, we have $\left[F_s^{*n_a}\right]_s(x) = (1-\alpha)H_s(x)$, $x \in \mathbb{R}$, and hence $\left[F_s^{*n_a}\right]_s(\infty) = 1-\alpha$. Next, for any integer $k \geqslant n_a$ we observe that

$$\left[F_s^{*(k+1)}\right]_s(x) = \left[F_s^{*k} * F_s\right]_s(x) = \left[[F_s^{*k}]_s * F_s\right]_s(x) \leqslant \left([F_s^{*k}]_s * F_s\right)(x), \quad x \in \mathbb{R}.$$

By the Lebesgue dominated convergence theorem, we conclude that

$$\left([F_s^{*k}]_s * F_s\right)(\infty) = \lim_{z\to\infty} \int_{\mathbb{R}} F_s(z-x)\; d[F_s^{*k}]_s(x) = \int_{\mathbb{R}} d[F_s^{*k}]_s(x) = \left[F_s^{*k}\right]_s(\infty).$$

Thus $\left[F_s^{*(k+1)}\right]_s(\infty) \leqslant \left[F_s^{*k}\right]_s(\infty)$ for any integer $k \geqslant n_a$. Let us introduce the sequence $A_k := \left[F_s^{*k}\right]_s(\infty)$, $k \in \mathbb{N}$. So we have

$$\begin{gathered} A_1 = \cdots = A_{n_a-1} = 1, \qquad A_{n_a} = 1-\alpha, \\ A_k \geqslant A_{k+1} \geqslant 0 \quad \text{for any } \; k \geqslant n_a. \end{gathered} \tag{50}$$

Due to (48), we next conclude that

$$\int_{\mathbb{R}} e^{itx}\, dW(x) = \lim_{n\to\infty} \int_{\mathbb{R}} e^{itx}\, d[W_n]_s(x), \quad t \in \mathbb{R}.$$

Then, according to (45) and (49), we have

$$\ln\left(1 + \frac{c_s f_s(t)}{c_d f_d(t)}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\left(\frac{c_s}{c_d}\right)^k f_{k,s}(t) g_d(t)^k \quad \text{for any} \quad t \in \mathbb{R},$$

where

$$f_{k,s}(t) := \int_{\mathbb{R}} e^{itx}\; d\left[F_s^{*k}\right]_s(x), \quad t \in \mathbb{R}.$$

In particular, we write

$$\ln\left(1 + \frac{c_s f_s(0)}{c_d f_d(0)}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{c_s}{c_d}\right)^k f_{k,s}(0) g_d(0)^k.$$

Recall that $f_s(0) = f_d(0) = g_d(0) = 1$ and $f_{k,s}(0) = [F_s^{*k}]_s(\infty) = A_k$ for any $k \in \mathbb{N}$. Then, on the one hand,

$$\ln\left(1 + \frac{c_s}{c_d}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k A_k.$$

On the other hand, since $c_s < c_d \mu_d \leqslant c_d$, we have the expansion

$$\ln\left(1 + \frac{c_s}{c_d}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k.$$

Thus we come to the equality

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k A_k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k.$$

Due to (50), it is reduced to

$$\sum_{k=n_a}^{\infty} \frac{(-1)^{k-n_a}}{k} \left(\frac{c_s}{c_d}\right)^k A_k = \sum_{k=n_a}^{\infty} \frac{(-1)^{k-n_a}}{k} \left(\frac{c_s}{c_d}\right)^k. \tag{51}$$

The series on the left is alternating with nonincreasing absolute values of the terms. Therefore

$$\sum_{k=n_a}^{\infty} \frac{(-1)^{k-n_a}}{k} \left(\frac{c_s}{c_d}\right)^k A_k \leqslant \frac{1}{n_a} \left(\frac{c_s}{c_d}\right)^{n_a} A_{n_a} = \frac{1}{n_a} \left(\frac{c_s}{c_d}\right)^{n_a} (1 - \alpha).$$

Since the series on the right in (51) is alternating with strictly decreasing absolute values, the following inequality holds:

$$\sum_{k=n_a}^{\infty} \frac{(-1)^{k-n_a}}{k} \left(\frac{c_s}{c_d}\right)^k > \frac{1}{n_a} \left(\frac{c_s}{c_d}\right)^{n_a} - \frac{1}{n_a+1} \left(\frac{c_s}{c_d}\right)^{n_a+1} = \frac{1}{n_a} \left(\frac{c_s}{c_d}\right)^{n_a} \left(1 - \frac{n_a}{n_a+1} \cdot \frac{c_s}{c_d}\right).$$

We see that the assumption $\alpha \geqslant \frac{n_a}{n_a+1} \cdot \frac{c_s}{c_d}$ implies the inequality

$$\sum_{k=n_a}^{\infty} \frac{(-1)^{k-n_a}}{k} \left(\frac{c_s}{c_d}\right)^k A_k < \sum_{k=n_a}^{\infty} \frac{(-1)^{k-n_a}}{k} \left(\frac{c_s}{c_d}\right)^k.$$

Thus we come to a contradiction with (51). So the assumption that W is continuous singular is false.

Due to the condition of dominated continuous singular part, we have $c_s < c_d$. Since $\frac{n_a}{n_a+1} < 1$, the inequality $\alpha \geqslant \frac{n_a}{n_a+1} \cdot \frac{c_s}{c_d}$ always holds in the case $\alpha = 1$. □

Proof of Proposition 2. Let us consider the function f_s defined by formula (15). We first find the limit of the sequence $f_s(t_n)$, $n \in \mathbb{N}$, with $t_n := \pi(2n)!$ as $n \to \infty$. We write $f_s(t_n) = M_n \cdot R_n$, $n \in \mathbb{N}$, where

$$M_n := \prod_{k=1}^{2n} \cos(t_n/k!) \quad \text{and} \quad R_n := \prod_{k=2n+1}^{\infty} \cos(t_n/k!), \quad n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$

$$\begin{aligned} M_n &= \cos\left(\frac{\pi(2n)!}{(2n)!}\right) \cdot \cos\left(\frac{\pi(2n)!}{(2n-1)!}\right) \cdot \cos\left(\frac{\pi(2n)!}{(2n-2)!}\right) \cdot \cdots \cdot \cos\left(\frac{\pi(2n)!}{1!}\right) \\ &= \cos(\pi) \cdot \cos(\pi \cdot 2n) \cdot \cos\big(\pi \cdot 2n(2n-1)\big) \cdot \cdots \cdot \cos\big(\pi \cdot 2n(2n-1) \cdot \cdots \cdot 1\big) \\ &= (-1) \cdot 1 \cdot 1 \cdot \cdots \cdot 1 = -1. \end{aligned}$$

Next, for every $n \in \mathbb{N}$ and for every integer $k \geqslant 2n+1$ we have

$$0 < \frac{t_n}{k!} = \frac{\pi(2n)!}{k \cdot (k-1)!} \leqslant \frac{\pi}{k} \leqslant \frac{\pi}{3},$$

and, by the well-known inequality $\cos(x) \geqslant 1 - x^2/2$, $x \in \mathbb{R}$, we get

$$\cos(t_n/k!) \geqslant 1 - \frac{(t_n/k!)^2}{2} \geqslant 1 - \frac{\pi^2}{2k^2} \geqslant 1 - \frac{\pi^2}{18} > 0.$$

Therefore, on the one hand, it is clear that for any $n \in \mathbb{N}$ we have $\cos(t_n/k!) < 1$ for any $k \geqslant 2n+1$ and hence $R_n < 1$. On the other hand,

$$R_n \geqslant \prod_{k=2n+1}^{\infty} \left(1 - \frac{\pi^2}{2k^2}\right) = \exp\left\{\sum_{k=2n+1}^{\infty} \ln\left(1 - \frac{\pi^2}{2k^2}\right)\right\}, \quad n \in \mathbb{N},$$

where the sum in the exponent tends to 0 as $n \to \infty$. So we conclude that $R_n \to 1$ as $n \to \infty$. Thus

$$f_s(t_n) = M_n \cdot R_n = -R_n \to -1 \quad \text{as} \quad n \to \infty,$$

and we know that $f_s(t_n) > -1$ for any $n \in \mathbb{N}$.

We next observe that

$$f_s(t_n \pm \pi) = \prod_{k=1}^{\infty} \cos\left(\frac{t_n \pm \pi}{k!}\right) = 0.$$

Indeed,

$$\cos\left(\frac{t_n \pm \pi}{k!}\right)\bigg|_{k=2} = \cos\left(\frac{\pi(2n)! \pm \pi}{2!}\right) = \cos\left(\pi \cdot 3 \cdot \cdots \cdot (2n) \pm \frac{\pi}{2}\right) = 0.$$

We now return to the characteristic function $f_*(t) = \frac{1}{2} + \frac{1}{2} f_s(t)$, $t \in \mathbb{R}$, and we consider the quantities

$$\frac{f_*(t_n - \pi) f_*(t_n + \pi)}{f_*(t_n)^2} = \frac{\left(\frac{1}{2} + \frac{1}{2} f_s(t_n - \pi)\right)\left(\frac{1}{2} + \frac{1}{2} f_s(t_n + \pi)\right)}{\left(\frac{1}{2} + \frac{1}{2} f_s(t_n)\right)^2}, \quad n \in \mathbb{N}.$$

By the above,

$$\frac{f_*(t_n - \pi) f_*(t_n + \pi)}{f_*(t_n)^2} = \frac{1}{(1 + f_s(t_n))^2} \to \infty \quad \text{as} \quad n \to \infty.$$

Therefore, by Theorem 7, f_* cannot be the characteristic function of a distribution function from class $\boldsymbol{Q}$. Thus $F_* \notin \boldsymbol{Q}$. □

Proof of Proposition 3. We have decompositions (3) and (4) for the distribution function F and its characteristic function f, respectively. Since $F \in \boldsymbol{Q}$, we know that $f(t) \neq 0$ for any $t \in \mathbb{R}$ and $\mu_d = \inf_{t\in\mathbb{R}} |f_d(t)| > 0$ according to equation (9). Due to the assumption of dominated singular part and $c_d > 0$, we have $c_s < c_d \mu_d$.

Let f_1 and f_2 denote the characteristic functions of F_1 and F_2. The assumed decomposition $F = F_1 * F_2$ means that $f(t) = f_1(t) f_2(t)$, $t \in \mathbb{R}$. This means that $f_1(t) \neq 0$ and $f_2(t) \neq 0$ for all $t \in \mathbb{R}$.

Let us write the Lebesgue decompositions for F_1 and F_2:

$$F_j(x) = c_{j,d} F_{j,d}(x) + c_{j,a} F_{j,a}(x) + c_{j,s} F_{j,s}(x), \quad x \in \mathbb{R}, \quad j \in \{1, 2\}.$$

We also write the corresponding decompositions for f_1 and f_2:

$$f_j(t) = c_{j,d} f_{j,d}(t) + c_{j,a} f_{j,a}(t) + c_{j,s} f_{j,s}(t), \quad t \in \mathbb{R}, \quad j \in \{1, 2\}.$$

Here $c_{j,d}$, $c_{j,a}$, $c_{j,s}$ are nonnegative and $c_{j,d} + c_{j,a} + c_{j,s} = 1$ for $j = 1$ and $j = 2$. For clarity, we write the equality $F = F_1 * F_2$ in the expanded form:

$$\begin{aligned}(52)\quad & c_d F_d + c_a F_a + c_s F_s \\ &\qquad = \left(c_{1,d} F_{1,d} + c_{1,a} F_{1,a} + c_{1,s} F_{1,s}\right) * \left(c_{2,d} F_{2,d} + c_{2,a} F_{2,a} + c_{2,s} F_{2,s}\right).\end{aligned}$$

Since F has nonzero discrete part ($c_d > 0$), the functions F_1 and F_2 have nonzero discrete parts too, i.e., $c_{1,d} > 0$ and $c_{2,d} > 0$. Since a convolution of any two distribution functions is discrete if and only if these functions are discrete, we conclude that $c_d F_d(x) = c_{1,d} c_{2,d} (F_{1,d} * F_{2,d})(x)$, $x \in \mathbb{R}$, i.e., $c_d = c_{1,d} c_{2,d}$ and $F_d = F_{1,d} * F_{2,d}$. Thus we have $f_d(t) = f_{1,d}(t) f_{2,d}(t)$, $t \in \mathbb{R}$. Since $|f_{1,d}(t)| \leqslant 1$ and $|f_{2,d}(t)| \leqslant 1$ for any $t \in \mathbb{R}$, we conclude that

$$\mu_{1,d} := \inf_{t\in\mathbb{R}} |f_{1,d}(t)| \geqslant \inf_{t\in\mathbb{R}} |f_d(t)| = \mu_d > 0,$$

and, analogously,

$$\mu_{2,d} := \inf_{t\in\mathbb{R}} |f_{2,d}(t)| \geqslant \inf_{t\in\mathbb{R}} |f_d(t)| = \mu_d > 0.$$

We next observe that $F_{1,s} * F_{2,d}$ and $F_{2,s} * F_{1,d}$ are continuous singular. Therefore the corresponding summands from (52) are included in the continuous part of F, i.e., $c_s F_s(x) \geqslant c_{1,s} c_{2,d} (F_{1,s} * F_{2,d})(x) + c_{2,s} c_{1,d} (F_{2,s} * F_{1,d})(x)$, $x \in \mathbb{R}$. Consequently,

$$c_s = c_s F_s(\infty) \geqslant c_{1,s} c_{2,d} (F_{1,s} * F_{2,d})(\infty) + c_{2,s} c_{1,d} (F_{2,s} * F_{1,d})(\infty) \\ = c_{1,s} c_{2,d} + c_{2,s} c_{1,d}.$$

Then we get

$$c_{1,s} c_{2,d} \leqslant c_s < c_d \mu_d = c_{1,d} c_{2,d} \mu_d \leqslant c_{1,d} c_{2,d} \mu_{1,d},$$

i.e., $c_{1,s} < c_{1,d} \mu_{1,d}$. Analogously,

$$c_{2,s} c_{1,d} \leqslant c_s < c_d \mu_d = c_{1,d} c_{2,d} \mu_d \leqslant c_{1,d} c_{2,d} \mu_{2,d},$$

i.e., $c_{2,s} < c_{2,d} \mu_{2,d}$. Thus F_1 and F_2 have dominated continuous singular parts.

We have shown that F_1 and F_2 satisfy the assumptions of Theorem 3 and condition (iii) from it. Thus, according to that theorem, F_1 and F_2 belong to class $\boldsymbol{Q}$. □

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DATA FOR SHIMURA VARIETIES INTERSECTING THE TORELLI LOCUS

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For infinitely many Hurwitz spaces parametrizing cyclic covers of the projective line, we provide a method to determine the integral PEL datum of the Shimura variety that contains the image of the Hurwitz space under the Torelli morphism.

1. Introduction

1.1. ***Overview.*** In [15], Shimura studied unitary groups associated with Hermitian spaces over algebraic number fields and their maximal lattices. In [14], he developed this theory to study isomorphism classes of polarized abelian varieties and Riemann forms. Using this, in [16], Shimura determined the lattice and Hermitian matrix associated with each of six families of cyclic covers of the projective line $\mathbb{P}^1$. The lattice and Hermitian matrix determine *the integral PEL datum* of the family, as defined in Section 2.5.

For m an odd prime such that $\mathbb{Q}(\zeta_m)$ has class number one, we provide in Theorem 5.5 a method to determine the lattice and Hermitian matrix, and thus the integral PEL datum, for all positive-dimensional families of degree m cyclic covers of $\mathbb{P}^1$.

Our results compute the integral PEL datum of certain unitary Shimura varieties that contain Hurwitz spaces of cyclic covers of the projective line.

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1.2. *Results.* Consider a family H_γ of cyclic covers of the projective line $\mathbb{P}^1$, indexed by the monodromy datum $\gamma = (m, N, a)$, where m is the degree, N is the number of branch points, and a is the inertia type (Section 2.1). Let $g = g_\gamma$ be the genus of the curves in this family; see (2-1).

For $g \geq 1$, let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g. Let Z_γ be the image of the family H_γ in $\mathcal{A}_g$ under the Torelli morphism. In [12, Section 3], Moonen describes the smallest PEL type Shimura substack S_γ of $\mathcal{A}_g$ containing Z_γ. Our goal is to compute the Shimura datum of S_γ or, more precisely, the *integral PEL datum* of S_γ as defined in Definition 2.2. We use [17, Proposition 4.1], which shows that the integral PEL datum of S_γ can be computed at any point P of Z_γ.

Suppose m is an odd prime such that $\mathbb{Q}(\zeta_m)$ has class number 1. In other words, $m \in \{3, 5, 7, 11, 13, 17, 19\}$. In Theorem 5.5, for every monodromy datum γ for degree m covers, we provide a method that determines the integral PEL datum of S_γ. Our approach is to find a point $P \in Z_\gamma$ for which we can analyze the Jacobian J_P represented by P and the lattice $\Lambda = H^1(J_P, \mathbb{Z})$. From this, we can

(1) express $V = \mathbb{Q}^{2g}$ as a vector space over $F = \mathbb{Q}(\zeta_m)$, find the $\mathcal{O}_F$-lattice $\Lambda \subset V$, and

(2) explicitly find the Hermitian form $\langle \cdot, \cdot \rangle$ on V, which takes integral values on Λ.

Our technique also applies to infinitely many monodromy data γ when m is a composite number such that $\mathbb{Q}(\zeta_m)$ has class number 1; see Section 7.

Specifically, we choose P to be a *distinguished point* of Z_γ, meaning that the Jacobian J_P represented by P is a product of principally polarized abelian varieties each having complex multiplication, see Definition 5.1. For every monodromy datum γ with m an odd prime, we prove in Proposition 5.2 that Z_γ has a distinguished point P such that each of the abelian varieties in the product has complex multiplication by $\mathbb{Z}[\zeta_m]$. In fact, J_P is the Jacobian of a singular curve in the boundary of the Hurwitz family.

1.3. *Comparison of methods.* Our approach to this problem is different from Shimura's and may be more accessible to people with background in the areas of algebraic number theory and moduli spaces of curves (Hurwitz spaces).

One advantage of our approach is that it provides a straightforward method to determine the lattice and the Hermitian form explicitly. We provide many examples of this in Section 6 (for m prime) and Section 7 (for m composite). The reason is that the decomposition of J_P provides a basis for the lattice as a free $\mathbb{Z}[\zeta_m]$-module. The Hermitian form is diagonal with respect to that basis and is determined by the CM-types of the factors of J_P, which can be explicitly computed from γ. In Remark 6.6 we give more careful attention to the subtleties caused by the choice of primitive m-th root of unity.

With Shimura's approach, it is necessary to find a Witt decomposition of the signature and lattice; this is straightforward for Shimura's six examples, five of which are of the form $y^m = f(x)$ for m an odd prime and $f(x)$ a separable polynomial, but could cause subtleties for more complicated families.

One drawback to our approach is that it only works when the CM-types produced in the method are simple. The simple property guarantees that the principal polarization on each of the abelian varieties is unique up to isomorphism; this guarantees that the Hermitian form we compute is correct. We use several methods to address this issue.

For composite m, there are several complications with both approaches.

1.4. ***Outline.*** Section 2 contains information about families of cyclic covers of $\mathbb{P}^1$ and information about Shimura varieties and Shimura data.

Section 3 contains some results in algebraic number theory about the narrow class group and units of independent signs that we need for this paper and later papers.

In Section 4, we prove a result about uniqueness of principal polarizations on abelian varieties with complex multiplication (Propositions 4.4 and 4.5).

Section 5 contains the main result of the paper, Theorem 5.5, which determines the integral PEL data of the unitary Shimura varieties S_γ for all families S_γ when m is an odd prime such that $\mathbb{Q}(\zeta_m)$ has class number 1.

Section 6 contains examples of Theorem 5.5: for all γ when $m = 3$; and for all γ with $N = 4$ branch points when $m = 5$ and $m = 7$. In Section 6.4, we apply the technique of Theorem 5.5 to determine the integral PEL datum using a different kind of distinguished point that represents the Jacobian of a curve with extra automorphisms.

In Section 7, we provide some examples when $m = 4, 6, 10$ to illustrate that the technique of Theorem 5.5 sometimes works when m is composite.

Remark 1.1. In [12, Theorem 3.6], Moonen proved that there are exactly 20 equivalence classes of γ for which Z_γ is open and dense in S_γ; in this situation, the family is called *special*. The six families in Shimura's paper are $M[n]$ when $n = 6, 10, 8, 11, 16, 17$, where $M[n]$ denotes the n-th row in [12, Table 1]. As an application of Theorem 5.5, we determine the integral PEL datum for 12 of these 20 families, including the 6 families from [16]. We emphasize that the special property is not necessary for either approach.

We would like to thank the referee for this observation: when $\dim(Z_\gamma) < \dim(S_\gamma)$, the Torelli locus gives a cycle on S_γ which is not obviously associated to a Shimura subvariety.

2. Hurwitz families and Shimura varieties

2.1. ***Families of cyclic covers of the projective line.*** This section is a shortened version of [8, Section 2.2]. Consider the universal family of μ_m-covers $\psi : \mathcal{C} \to \mathbb{P}^1$, branched at N points, with *inertia type* $a = (a(1), \dots, a(N))$; the data $\gamma = (m, N, a)$ is called a *monodromy datum*. Over $\mathbb{Q}(\zeta_m)$, such a cover ψ has an equation of the form $y^m = \prod_{i=1}^{N} (x - t(i))^{a(i)}$, and a chosen automorphism h of order m given by $(x, y) \mapsto (x, \zeta_m y)$. By the Riemann–Hurwitz formula, the genus of the fibers of $\mathcal{C}$ is

$$g = g_\gamma = 1 + \frac{1}{2}\Big((N-2)m - \sum_{i=1}^{N} \gcd(a(i), m)\Big) \tag{2-1}$$

For $0 \le n < m$, let $\tau_n : \mathbb{Q}[\mu_m] \to \mathbb{C}$ be given by $\tau_n(\zeta_m) = e^{2\pi I n/m}$. We identify $\mathbb{Z}/m\mathbb{Z} = \operatorname{Hom}(\mathbb{Q}[\mu_m], \mathbb{C})$ by $n \mapsto \tau_n$. The *signature type* of γ is a function $\mathfrak{f} : \operatorname{Hom}(\mathbb{Q}[\mu_m], \mathbb{C}) \to \mathbb{Z}_{\ge 0}$ which we denote by $\mathfrak{f} = (\mathfrak{f}(\tau_1), \dots, \mathfrak{f}(\tau_{m-1}))$, where $\mathfrak{f}(\tau_n)$ is the dimension of the eigenspace of $H^0(\mathcal{C}(\mathbb{C}), \Omega^1)$ on which h acts by multiplication by ζ_m^n.

For any $x \in \mathbb{Q}$, let $\langle x \rangle$ denote the fractional part of x. By [12, Lemma 2.7, §3.2],

$$\mathfrak{f}(\tau_n) = \begin{cases} -1 + \sum_{i=1}^{N} \left\langle \dfrac{-na(i)}{m} \right\rangle & \text{if } n \not\equiv 0 \bmod m, \\ 0 & \text{if } n \equiv 0 \bmod m. \end{cases} \tag{2-2}$$

We describe how the inertia type a and signature $\mathfrak{f}$ change under the action of $\operatorname{Aut}(\mu_m) \simeq (\mathbb{Z}/m\mathbb{Z})^*$. Let $\sigma_i \in \operatorname{Aut}(\mu_m)$ denote the automorphism such that $\sigma_i(\zeta_m) = \zeta_m^i$.

Lemma 2.1. *The automorphism $\sigma_i \in \operatorname{Aut}(\mu_m)$ takes the inertia type*

$$a = (a_1, a_2, \dots, a_N) \quad \textit{to} \quad a' = i^{-1} \cdot a = (i^{-1} \cdot a_1, i^{-1} \cdot a_2, \dots, i^{-1} \cdot a_N)$$

and the signature

$$\mathfrak{f} \quad \textit{to} \quad \mathfrak{f}', \qquad \textit{where } \mathfrak{f}'(\tau_n) = \mathfrak{f}(\tau_{ni^{-1}}).$$

Proof. The j-th entry a_j of the inertia type signifies that the canonical generator of inertia above the j-th branch point is the a_j-th power of the generator ζ_m of μ_m. The canonical generator of inertia is the $i^{-1}a_j$-th power of the new generator ζ_m^i of μ_m, so the j-th entry of the inertia type a' is $i^{-1}a_j$.

The automorphism σ_i permutes the eigenspaces for the action of h on $H^0(\mathcal{C}(\mathbb{C}), \Omega^1)$, taking the eigenspace indexed by n to the one indexed by ni^{-1}. This yields the formula $\mathfrak{f}'(\tau_n) = \mathfrak{f}(\tau_{ni^{-1}})$, which can also be deduced from the formula for a' and (2-2). □

Consider the Hurwitz family of μ_m-covers of $\mathbb{P}^1$ with monodromy datum $\gamma = (m, N, a)$. As in [8, Definition 2.1], let $Z^0 = Z^0_\gamma$ denote the image of this family in $\mathcal{A}_g$. Let $Z = Z_\gamma$ denote the closure of Z^0_γ in $\mathcal{A}_g$.

2.2. ***Complex abelian varieties.*** Let $g \geq 1$ and let $V = \mathbb{R}^{2g}$. Suppose V has a complex structure and $\Lambda \subset V$ is a lattice of rank $2g$. A Riemann form on a pair (V, Λ) is an alternating pairing $\Psi : \Lambda \times \Lambda \to \mathbb{Z}$ such that the pairing $\Psi_{\mathbb{R}}(\,\cdot\,, \sqrt{-1}\,\cdot\,) : V \times V \to \mathbb{R}$ is symmetric and positive definite. Every complex abelian variety of dimension g is isomorphic to V/Λ for some pair (V, Λ) that admits a Riemann form.

By Riemann's theorem, the category of polarized abelian varieties of dimension g over $\mathbb{C}$ is equivalent to the category of pairs (V, Λ), where V is a nontrivial $\mathbb{C}$-vector space of dimension g and Λ is a lattice in V together with a Riemann form integral on Λ.

2.3. ***Siegel modular varieties.*** For $g \geq 1$, let $V = \mathbb{Q}^{2g}$ together with the standard lattice $\Lambda = \mathbb{Z}^{2g}$. Let $\Psi : V \times V \to \mathbb{Q}$ be the standard symplectic form, which is integral on Λ. The points of the Siegel upper half space $\mathcal{H}_g$ are complex structures J on $V_{\mathbb{R}}$ such that $\Psi_J = \Psi_{\mathbb{R}}(\,\cdot\,, J\,\cdot\,)$ is a Riemann form. These points parametrize principally polarized complex abelian varieties A of dimension g, equipped with a trivialization $\Lambda \simeq H^1(A, \mathbb{Z})$. The Shimura datum for $\mathcal{A}_g$ arises from the symplectic $\mathbb{Q}$-vector space (V, Ψ); its integral datum given by the self-dual $\mathbb{Z}$-lattice Λ. The associated algebraic group over $\mathbb{Q}$ is $G_g := \mathrm{GSp}(V, \Psi)$, the algebraic group of symplectic similitudes; see [11, Example 11.12].

2.4. ***Shimura data of PEL type.*** A *Shimura datum of PEL type* is a Shimura datum arising from a *PEL datum* $(B, *, V, \langle\,,\rangle, h)$, where (see [5, Section 4])

- B is a semisimple finite dimensional $\mathbb{Q}$-algebra;
- $*$ is a positive involution on B;
- V is a finitely generated B-module;
- $\langle\,\cdot\,,\cdot\,\rangle$ is a nondegenerate skew-symmetric $*$-Hermitian form on V; and
- $h : \mathbb{C} \to \mathrm{End}_{B\otimes\mathbb{R}}(V_{\mathbb{R}})$ is an $\mathbb{R}$-algebra homomorphism satisfying $\langle h(z)\,\cdot\,, \cdot\,\rangle = \langle\,\cdot\,, h(\bar{z})\,\cdot\,\rangle$, for all $z \in \mathbb{C}$; and such that the symmetric bilinear form $\langle\,\cdot\,, h(i)\cdot\,\rangle$ on V is positive definite.

The associated algebraic group H over $\mathbb{Q}$ is $H = \mathrm{GL}_B(V) \cap \mathrm{GSp}(V, \langle\,,\rangle)$. By definition, $H \subset G_g$ for $g = (\dim_{\mathbb{Q}} V)/2$.

Given a PEL datum $(B, *, V, \langle\,,\rangle, h)$, there is an associated PEL moduli space $\mathrm{Sh} = \mathrm{Sh}(B, *, V, \langle\,,\rangle, h)$, which is a subspace of $\mathcal{A}_g$. The space Sh is defined over a finite extension of $\mathbb{Q}$, called the *reflex field* [5, Section 5]. The manifold $\mathrm{Sh}(\mathbb{C})^{\mathrm{an}}$ is the disjoint union of finitely many connected Shimura varieties [5, Sections 7–8].

2.5. ***Integral PEL data.*** Let $(B, *, V, \langle\, ,\rangle, h)$ be a PEL datum.

Definition 2.2. An *integral PEL datum* of the PEL datum $(B, *, V, \langle\, ,\rangle, h)$ consists of a tuple $(\mathcal{O}_B, *, \Lambda, \langle\, ,\rangle, h)$ where

- $\mathcal{O}_B$ is an order of B that is $*$-stable, and
- Λ is a lattice of V that is $\mathcal{O}_B$-stable, such that the $*$-Hermitian form $\langle\cdot,\cdot\rangle$ is integral on Λ.

A rational prime p is *good* for an integral PEL datum if $\mathcal{O}_B$ is maximal at p and Λ is self-dual at p. All but finitely many primes are good, [6, Definition 1.4.1.1].

To an integral PEL datum $(\mathcal{O}_B, *, \Lambda, \langle\, ,\rangle, h)$, there is an associated (canonical) integral model $\mathrm{Sh}(\mathcal{O}_B, *, \Lambda, \langle\, ,\rangle, h)$ of $\mathrm{Sh}(B, *, V, \langle\, ,\rangle, h)$, defined over the ring of integers of the reflex field localized at good primes, [6, Theorem 1.4.1.11]. If p is a good prime for the integral PEL datum, then $\mathrm{Sh}(\mathcal{O}_B, *, \Lambda, \langle\, ,\rangle, h)$ has good reduction at p.

2.6. ***Shimura varieties given by the action of the covering group.*** Let $\gamma = (m, N, a)$ be a monodromy datum, and let Z_γ be as in Section 2.1. Then Z_γ is contained in a subspace of PEL type S_γ of $\mathcal{A}_g$ [12, §3]. The PEL datum $(B, *, V, \langle\, ,\rangle, h)$ of S_γ (defined as in Section 2.4) satisfies the following conditions:

- $B = \mathbb{Q}[\mu_m]$ is the group algebra of μ_m over $\mathbb{Q}$.
- $*$ is the involution on $\mathbb{Q}[\mu_m]$ induced by the inverse map on μ_m.
- $V = \mathbb{Q}^{2g}$ for $g = g_\gamma$ as given in (2-1).

The structure of V as a B-module, and the complex structure on $V_{\mathbb{R}}$ (namely, h) are uniquely determined by the signature type $\mathfrak{f} = \mathfrak{f}_\gamma$ given in (2-2).

Our goal is to compute the integral PEL datum $(\mathcal{O}_B, *, \Lambda, \langle\, ,\rangle, h)$ of S_γ.

More precisely, we set $\mathcal{O}_B = \mathbb{Z}[\mu_m]$, which is a $*$-stable order of B, and compute an $\mathcal{O}_B$-stable lattice Λ of V and a skew Hermitian form $\langle\cdot,\cdot\rangle$ on V which is integral on Λ. By construction, the order $\mathcal{O}_B$ is maximal away from m and the lattice Λ is self-dual away from m; hence S_γ has good reduction at every prime p not dividing m.

2.7. ***Complex multiplication.*** Let L be a CM-field and let L_0 be the maximal totally real subfield of L. Let $n = [L_0 : \mathbb{Q}]$; hence, $[L : \mathbb{Q}] = 2n$. A CM-type of L is an ordered set Φ of distinct embeddings $\phi_i : L \hookrightarrow \mathbb{C}$, for $1 \le i \le n$, no two of which are complex conjugate. A CM-type of L is called *simple* if it is not induced from the CM-type of a proper CM-subfield of L.

Let A be a complex torus such that $L \subset \mathrm{End}(A) \otimes \mathbb{Q}$. We say that A is of CM-type (L, Φ) if $\dim(A) = n$ and the complex representation of L on $\mathrm{Lie}(A)$ is isomorphic to $\sum_{\phi\in\Phi} \phi$. If, in addition, $\mathrm{End}(A) \simeq \mathcal{O}_L$, we say A has type $(\mathcal{O}_L, \Phi)$.

The following lemma is well-known. See [9, Lemma 3.1] for some explanation.

Lemma 2.3. *Suppose $\psi : \mathcal{C} \to \mathbb{P}^1$ is a μ_m-cover, branched at $N = 3$ points, with inertia type $a = (a_1, a_2, a_3)$ and signature type $\mathfrak{f}$. Then the Jacobian of $\mathcal{C}$ has complex multiplication by $\prod_d \mathbb{Q}(\zeta_d)$, where d satisfies $1 < d \mid m$, and $d \nmid a(i)$ for $1 \le i \le 3$. Its CM-type Φ satisfies*

$$\phi \in \Phi \iff \mathfrak{f}(\phi) > 0. \tag{2-3}$$

The maximal order in $\prod_d \mathbb{Q}(\zeta_d)$ is $\prod_d \mathbb{Z}[\zeta_d]$. The image of the group ring $\mathbb{Z}[\mu_m]$ in $\prod_d \mathbb{Q}(\zeta_d)$ has finite index inside the maximal order. This index is 1 if and only if there is a unique d satisfying $1 < d \mid m$, and $d \nmid a(i)$ for $1 \le i \le 3$. This is true if m is prime.

The next result is immediate from Lemma 2.1 and (2-3).

Lemma 2.4. *If $N = 3$, then $\sigma_i \in \operatorname{Aut}(\mu_m)$ takes the CM-type Φ to $\Phi' = i \cdot \Phi = \{\tau_{ni} \mid \tau_n \in \Phi\}$.*

For any CM-type (L, Φ), there exists a unique 0-dimensional PEL type moduli space $\mathrm{Sh} = \mathrm{Sh}(L, \Phi)$ whose points represent abelian varieties with complex multiplication of type $(\mathcal{O}_L, \Phi)$. (When $L = \mathbb{Q}(\zeta_m)$, the points of Sh represent the Jacobians of curves for which there is a μ_m-cover of $\mathbb{P}^1$ branched at $N = 3$ points with CM-type Φ.) The signature type $\mathfrak{f}$ of Sh is given by

$$\text{for each } \phi : L \to \mathbb{C}, \quad \mathfrak{f}(\phi) = \begin{cases} 1 & \text{if } \phi \in \Phi, \\ 0 & \text{if } \phi \notin \Phi. \end{cases} \tag{2-4}$$

For any signature type $\mathfrak{f}$ on L taking values in $\{0, 1\}$, there is a unique CM-type Φ of L compatible with $\mathfrak{f}$.

3. Some background from algebraic number theory

3.1. ***Class group and narrow class group.*** For a number field L: let $\mathcal{O}_L$ denote its ring of integers; let $\mathcal{U}_L$ denote the units in $\mathcal{O}_L$; and let $D_{L/\mathbb{Q}}$ denote the different of L over $\mathbb{Q}$.

Let cl_L be the class group of L. Recall that $\mathrm{cl}_L = I_L/P_L$, where I_L is the group of nonzero fractional ideals and P_L is the subgroup of nonzero principal fractional ideals. Let $h_L = |\mathrm{cl}_L|$ be the class number of L.

An element $\alpha \in L$ is totally positive if it is positive under every real embedding of L. Let $\mathcal{U}_L^+ \subset \mathcal{U}_L$ be the subgroup of totally positive units. Let $P_L^+ \subset P_L$ be the subgroup of principal ideals generated by a totally positive element. The narrow class group of L is $\mathrm{cl}_L^+ = I_L/P_L^+$ and the narrow class number is $h_L^+ = |\mathrm{cl}_L^+|$. There is a surjection

$$\nu_L : \mathrm{cl}_L^+ \to \mathrm{cl}_L.$$

3.2. ***Units of independent signs.*** Let L be a CM-field and L_0 its maximal totally real subfield. Let $n = [L_0 : \mathbb{Q}]$; hence, $[L : \mathbb{Q}] = 2n$. By [19, Theorem 4.10], h_{L_0} divides h_L.

We fix an ordering $\tau_1, \dots, \tau_n$ of the n real embeddings $L_0 \hookrightarrow \mathbb{R}$. If $L_0/\mathbb{Q}$ is Galois, we identify these with the elements of $\mathrm{Gal}(L_0/\mathbb{Q})$. Consider the group homomorphism

$$\rho_{L_0} : \mathcal{U}_{L_0} \to \{\pm 1\}^n, \quad \rho_{L_0}(u) = (\tau_i(u)/|\tau_i(u)|)_{1 \le i \le n} \text{ for } u \in \mathcal{U}_{L_0}. \tag{3-1}$$

We say that L_0 has *units of independent signs* if, for each real embedding τ, there is a unit which is negative under τ but positive under all other real embeddings [2, Definition 12.1]. This is equivalent to saying that ρ_{L_0} is surjective or that $\nu_{L_0} : \mathrm{cl}^+_{L_0} \to \mathrm{cl}_{L_0}$ is an isomorphism [2, Lemma 11.2].

We say that L_0 has *units of almost independent signs* if every unit in $\mathcal{U}_{L_0}$ is negative under an even number of real embeddings and, for every pair of real embeddings, there is a unit which is negative under exactly the two embeddings in that pair [2, Definition 12.13]. This condition is equivalent to $|\ker(\nu_{L_0})| = 2$ or $|\mathrm{coker}(\rho_{L_0})| = 2$, by [2, Lemma 11.2].

3.3. ***Norms and the Hasse unit index.*** Consider the norm map $N : \mathcal{U}_L \to \mathcal{U}_{L_0}$ given by $N(y) = N_{L/L_0}(y)$ for $y \in \mathcal{U}_L$. By Dirichlet's unit theorem, $[\mathcal{U}_{L_0} : \mathcal{U}^2_{L_0}] = 2^n$.

Lemma 3.1. *Suppose L_0 is a totally real field, and L/L_0 is a CM-extension. Then*

$$\mathcal{U}^2_{L_0} \subseteq N(\mathcal{U}_L) \subseteq \mathcal{U}^+_{L_0} \subseteq \mathcal{U}_{L_0}. \tag{3-2}$$

Proof. By definition, $\mathcal{U}^+_{L_0} \subseteq \mathcal{U}_{L_0}$. All elements in $N(\mathcal{U}_L)$ are totally positive units because L is a CM-field, quadratic over its totally real subfield L_0. So $N(\mathcal{U}_L) \subseteq \mathcal{U}^+_{L_0}$. Also, $\mathcal{U}^2_{L_0} = N(\mathcal{U}_{L_0})$, hence $\mathcal{U}^2_{L_0} \subset N(\mathcal{U}_L)$. □

Let μ_L be the group of roots of unity of L.

Definition 3.2. The *Hasse unit index* of a CM-extension L/L_0 is

$$Q(L) = [\mathcal{U}_L : \mu_L \mathcal{U}_{L_0}].$$

By [19, Theorem 4.12], $Q(L) = 1$ or $Q(L) = 2$. Since $\mathrm{Ker}(N) = \mu_L$, it follows that $\mathcal{U}_L = \mu_L \mathcal{U}_{L_0}$ if and only if $N(\mathcal{U}_L) = N(\mathcal{U}_{L_0})$. Also $N(\mathcal{U}_{L_0}) = \mathcal{U}^2_{L_0}$. Thus

$$Q(L) = [N(\mathcal{U}_L) : \mathcal{U}^2_{L_0}]. \tag{3-3}$$

Let t be the number of finite primes ramified in L/L_0.

Lemma 3.3. *Suppose L has odd class number. Let L_0 be a totally real field, and L/L_0 a CM-extension. Then $Q(L) = 1$ if and only if $t = 1$; and $Q(L) = 2$ if and only if $t = 0$.*

Proof. The material in [2, Chapter 13] is stated in terms of the type of the CM-extension L/L_0. By a theorem of Kummer [2, Theorem 13.4] L/L_0 has type I (resp. II) if and only if $Q(L) = 1$ (resp. $Q(L) = 2$). The result is then immediate from [2, page 73]. □

We summarize the result from this section that we need in this and later papers.

Lemma 3.4. *Suppose L is a CM-field with maximal totally real subfield L_0.*

(1) *L_0 has units of independent signs if and only if $Q(L)[\mathcal{U}^+_{L_0} : N(\mathcal{U}_L)] = 1$.*

(2) *L_0 has units of almost independent signs if and only if $Q(L)[\mathcal{U}^+_{L_0} : N(\mathcal{U}_L)] = 2$.*

Proof. (1) By [2, Lemma 12.2], L_0 has units of independent signs if and only if every element of $\mathcal{U}^+_{L_0}$ is a square. The result then follows from Lemma 3.1.

(2) It follows from the definitions that L_0 has units of almost independent signs if and only if $[\mathcal{U}_{L_0} : \mathcal{U}^+_{L_0}] = 2^{n-1}$. Since $[\mathcal{U}_{L_0} : \mathcal{U}^2_{L_0}] = 2^n$, this is equivalent to $[\mathcal{U}^+_{L_0} : \mathcal{U}^2_{L_0}] = 2$. □

3.4. *Cyclotomic fields.* Let m be a positive integer and let $\zeta_m = e^{2\pi I/m} \in \mathbb{C}$. Let $\mu_m \subset \mathbb{C}$ be the group of m-th roots of unity. Let $\mathbb{Q}[\mu_m]$ be the group algebra of μ_m over $\mathbb{Q}$.

The cyclotomic field $F = \mathbb{Q}(\zeta_m)$ is a CM-field over $\mathbb{Q}$ with maximal totally real subfield $F_0 = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$. The degree of F_0 over $\mathbb{Q}$ is $n = \phi(m)/2$.

Notation 3.5. The choice of ζ_m fixes an embedding $\sigma_1 : F \hookrightarrow \mathbb{C}$. For $1 \leq i < m$, with $\gcd(i, m) = 1$, let σ_i be the embedding $F \hookrightarrow \mathbb{C}$ (or automorphism in $\mathrm{Gal}(F/\mathbb{Q})$) determined by $\sigma_i(\zeta_m) = \zeta_m^i$. For $x \in F$, let $\bar{x} = \sigma_{m-1}(x)$ denote its complex conjugate.

If $m = p^r$ is a prime power, then F/F_0 is ramified at the unique prime of F_0 above p and at the n infinite primes of F_0 and is unramified at all other primes. See [19, Proposition 2.15].

Lemma 3.6. *Let $F = \mathbb{Q}(\zeta_m)$ and $F_0 = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$. Suppose F has class number* 1.

(1) *If m is a prime power* (*or twice a prime power*), *then F_0 has narrow class number* 1 *and thus has units of independent signs. The complete list of these m is*

$m = 1, 2, 4, 8, 16, 32$;
$m = 3, 5, 7, 9, 11, 13, 17, 19, 25, 27$; *or twice a value of m on this row.*

(2) *If m is not a prime power* (*or twice a prime power*), *then F_0 has narrow class number* 2 *and thus has units of almost independent signs. The complete list of these m is*

$m = 12, 16, 20, 24, 28, 36, 40, 44, 48, 60, 84$;
$m = 15, 21, 33, 35, 45$; *or twice a value of m on this row.*

Proof. The complete list of m such that $h_F = 1$ is well-known; it is the union of the lists in parts (1) and (2). For these m, since $h_F = 1$, also $h_{F_0} = 1$.

(1) By a result of Hasse (see [2, Corollary 3.9]), since h_F and h_{F_0} are odd, F_0 has units of independent signs if and only if F/F_0 is ramified at exactly one finite prime. This happens if and only if $m = p^r$ or $m = 2 \cdot p^r$, for some prime p.

(2) Since h_F and h_{F_0} are odd, then F_0 has units of almost independent signs if and only if F/F_0 is not ramified at any finite prime [2, Corollary 13.10]. This happens if m is not a prime power (or twice a prime power). □

3.5. ***A generator for the different of cyclotomic fields.***

Lemma 3.7. *Let $F = \mathbb{Q}(\zeta_m)$. The element β_0 below generates $D_{F/\mathbb{Q}}$ and $\beta_0 = -\bar{\beta}_0$.*

(1) *If m is an odd prime, then*

$$\beta_0 = \frac{m}{\zeta_m^{(m+1)/2} - \zeta_m^{(m-1)/2}}. \tag{3-4}$$

(2) *If $m = 2^k$ with $k \geq 2$, then $\beta_0 = -2^{k-1} i$.*

(3) *If $m = 3^k$, then $\beta_0 = -3^{k-1}\sqrt{3} i$.*

(4) *If $m = pq$ with p, q distinct odd primes, then*

$$\beta_0 = m \frac{\zeta_m^{(m+1)/2} - \zeta_m^{(m-1)/2}}{\left(\zeta_m^{q(p+1)/2} - \zeta_m^{q(p-1)/2}\right)\left(\zeta_m^{p(q+1)/2} - \zeta_m^{p(q-1)/2}\right)}. \tag{3-5}$$

Proof. In each case, β_0 is on the imaginary axis, so $\beta_0 = -\bar{\beta}_0$.

Let $c_m(x)$ denote the m-th cyclotomic polynomial. The different $D_{F/\mathbb{Q}}$ is generated by $\langle c'_m(\zeta_m)\rangle$ by [13, III, Proposition 2.4]. To show that β_0 generates $D_{F/\mathbb{Q}}$, it suffices to show that it is an associate of $\langle c'_m(\zeta_m)\rangle$ in $\mathcal{O}_F$.

(1) When m is an odd prime, then $c_m(x) = (x^m - 1)/(x - 1)$. Then

$$c'_m(x) = \frac{(x-1)mx^{m-1} - (x^m - 1)}{(x-1)^2}.$$

Hence $c'_m(\zeta_m) = m\zeta_m^{-1}/(\zeta_m - 1)$, and β_0 from (3-4) is an associate of $c'_m(\zeta_m)$ in $\mathcal{O}_F$.

(2) If $m = 2^k$, then $c_m(x) = x^{2^{k-1}} + 1$. Thus $c'_m(\zeta_m) = 2^{k-1}\zeta_m^{2^{k-1}-1} = -2^{k-1}/\zeta_m$, and the corresponding β_0 is an associate of $c'_m(\zeta_m)$ in $\mathcal{O}_F$.

(3) If $m = 3^k$, then $c_m(x) = x^{2\cdot 3^{k-1}} + x^{3^{k-1}} + 1$. Thus $c'_m(\zeta_m) = 3^{k-1}\zeta_m^{3^{k-1}-1}\sqrt{-3}$. and this β_0 is an associate of $c'_m(\zeta_m)$ in $\mathcal{O}_F$.

(4) In this case, $c_m(x) = \dfrac{(x^m-1)(x-1)}{(x^p-1)(x^q-1)}$, and β_0 in (3-5) is an associate of $c'_m(\zeta_m) = \dfrac{m(\zeta_m - 1)}{\zeta_m(\zeta_m^p - 1)(\zeta_m^q - 1)}$. □

3.6. ***Simple types.*** Let $F = \mathbb{Q}(\zeta_m)$ and recall that $\mathrm{Gal}(F/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^*$. A CM-type Φ of F is simple, if it is not induced from any CM-field $F' \subset F$.

Let $H \subset \mathrm{Gal}(F/\mathbb{Q})$ and let F^H be the fixed field of F under H. If F^H is a CM-field, then Φ is induced from F^H if and only if it is a union of cosets of H.

Lemma 3.8. *Suppose $m = 4$ or m is a Fermat prime or twice a Fermat prime. Then Φ is simple.*

We will use Lemma 3.8 for $m = 4$ and $m = 3, 5, 17$ and $m = 6, 10$.

Proof. For these values of m, the field F contains no proper nontrivial CM-fields. □

Lemma 3.9. *Let $m > 3$ be prime. Let $\gamma = (m, 3, a)$ be a monodromy datum with inertia type $a = (1, a_2, a_3)$. Let Φ be the CM-type of F corresponding to a. Then Φ is simple if and only if $a \neq (1, x, x^2)$ for some $x \in (\mathbb{Z}/m\mathbb{Z})^*$ with order 3.*

Proof. This follows from [4, Theorem 2]. See also [7, Theorem 6.2]. □

For example, when $m = 7$ and $a = (1, a_2, a_3)$, Φ is simple unless $a = (1, 2, 4)$.

By a direct computation, one can check that Φ is always simple when $m = 25$ or $m = 27$.

4. Principal polarizations of CM abelian varieties

Let L be a CM-field and L_0 its maximal totally real subfield. We will now study principal polarizations on abelian varieties having complex multiplication by L.

Section 4.1 contains background about complex tori with complex multiplication. In Section 4.2, we study principal polarizations on CM-abelian varieties, following work of van Wamelen. In Section 4.3, we prove a result about existence and uniqueness of principal polarizations for abelian varieties with simple CM-type when L has class number 1. In Section 4.4, we specialize to the case that L is a cyclotomic field with class number 1.

4.1. ***Complex tori.*** We review some complex multiplication theory, following Lang in [7, Chapter 1]. Let A be a complex torus such that $L \subset \mathrm{End}(A) \otimes \mathbb{Q}$. We say that A is of type (L, Φ) if the complex representation of $\mathrm{End}(A) \otimes \mathbb{Q}$ is isomorphic to $\sum_{\phi \in \Phi} \phi$. If, in addition, $\mathrm{End}(A) \simeq \mathcal{O}_L$, we say A has type $(\mathcal{O}_L, \Phi)$.

Theorem 4.1. [7, Chapter 1: Theorems 3.3, 3.5, 4.1, 4.2]

(1) *If $\mathfrak{a}$ is a lattice in L and Φ is a CM-type of L, then $\mathbb{C}^n/\Phi(\mathfrak{a})$ is a complex torus of type (L, Φ).*

(2) *If A is a complex torus of type (L, Φ), then there exists a lattice $\mathfrak{a}$ of L such that $A \simeq \mathbb{C}^n/\Phi(\mathfrak{a})$.*

(3) *If Φ is a simple type and $\mathfrak{a}$ is a fractional ideal of L, then $\mathrm{End}(\mathbb{C}^n/\Phi(\mathfrak{a})) \simeq \mathcal{O}_L$.*

(4) *If* Φ *is a simple type and* $\mathfrak{a}, \mathfrak{b}$ *are fractional ideals of* L, *then* $\mathbb{C}^n/\Phi(\mathfrak{a}) \simeq \mathbb{C}^n/\Phi(\mathfrak{b})$ *if and only if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are in the same ideal class.*

In particular, if (L, Φ) is a simple CM-type, then the set of isomorphism classes of complex tori of type $(\mathcal{O}_L, \Phi)$ is in bijection with the class group of L.

Furthermore, by [7, Chapter 1, Theorem 4.5], every (admissible, nondegenerate) Riemann form on $\mathbb{C}^n/\Phi(\mathfrak{a})$ is given by

$$\mathbb{E}(\Phi(x), \Phi(y)) = \mathrm{tr}_{L/\mathbb{Q}}(\xi x \bar{y}), \text{ for } x, y \in L, \tag{4-1}$$

for some ξ such that $L = L_0(\xi)$, $\xi^2 \in L_0$ is totally negative, and $\mathrm{Im}(\phi(\xi)) > 0$ for $\phi \in \Phi$.

4.2. ***Principal polarizations on CM abelian varieties.*** In [18, p. 310], van Wamelen developed an algorithm to produce isomorphism classes of principally polarized abelian varieties of type $(\mathcal{O}_L, \Phi)$ based on the following result.

Theorem 4.2 (van Wamelen [18]). *Let* (L, Φ) *be a CM-type.*

(1) (Theorem 4)[1] *Writing* $L = L_0(\sqrt{-c})$ *for some* $c \in \mathcal{O}_{L_0}$, *there exist a fractional ideal* $\mathfrak{a} \subset L$ *and an element* $b \in \mathcal{O}_{L_0}$ *such that* $D_{L/\mathbb{Q}} \cdot \mathfrak{a}\bar{\mathfrak{a}} = \langle b\sqrt{-c} \rangle$.

(2) (Theorem 3) *Let* $\xi \in L$ *be such that* $L = L_0(\xi)$, $\xi^2 \in L_0$, *and* $D_{L/\mathbb{Q}} \cdot \mathfrak{a}\bar{\mathfrak{a}} = \langle \xi^{-1} \rangle$, *for some fractional ideal* $\mathfrak{a}$ *of* L (*for example, with the notation of part* (1), $\xi = (b\sqrt{-c})^{-1}$). *Define a Riemann form* $\mathbb{E} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ *by*

$$\mathbb{E}(z, w) = \sum_{i=1}^{n} \phi_i(\xi)(\bar{z}_i w_i - z_i \bar{w}_i),$$

for $z, w \in \mathbb{C}^n$. *If* $\mathrm{Im}(\phi(\xi)) > 0$ *for* $\phi \in \Phi$, *then* $\mathbb{E}$ *defines a principal polarization on* $\mathbb{C}^n/\Phi(\mathfrak{a})$ *of type* $(\mathcal{O}_L, \Phi)$. *Furthermore, if* (L, Φ) *is a simple CM-type, then all principal polarizations on* $\mathbb{C}^n/\Phi(\mathfrak{a})$ *of type* $(\mathcal{O}_L, \Phi)$ *are given by such a* ξ.[2]

(3) (Corollary 1) *Two principal polarizations of the same CM-type on* $\mathbb{C}^n/\Phi(\mathfrak{a})$ *arising from* ξ_1 *and* ξ_2 *give isomorphic principally polarized abelian varieties if and only if there exists a unit* $u \in \mathcal{O}_L^*$ *such that* $\xi_1 = u\bar{u}\xi_2$.

Corollary 4.3. *Let* Φ *be a CM-type of* L. *An element* $\xi = \beta^{-1}$ *for some* $\beta \in \mathcal{O}_L$ *defines a principal polarization on* $A_\Phi = \mathbb{C}^n/\Phi(\mathcal{O}_L)$ *of CM-type* $(\mathcal{O}_L, \Phi)$ *if and only if*

(1) β *generates the different* $D_{L/\mathbb{Q}}$,

(2) $\beta = -\bar{\beta}$, *and*

(3) $\mathrm{Im}(\phi(\beta)) < 0$, *for each* $\phi \in \Phi$.

[1] We have made a small adjustment to the notation to be consistent with other parts of this theorem.

[2] Note that type (L, Φ) and type $(\mathcal{O}_L, \Phi)$ are equivalent in the last two statements.

Two elements β, β' satisfying the above conditions yield isomorphic principally polarized abelian varieties if and only if there exists a unit $u \in \mathcal{U}_L$ such that $\beta = u\bar{u}\beta'$. If the CM-type Φ is simple, then all principal polarizations of A_Φ of CM-type $(\mathcal{O}_L, \Phi)$ arise this way.

Proof. This follows from Theorem 4.2, replacing ξ, ξ_1, ξ_2 with $\beta^{-1}, \beta^{-1}, \beta'^{-1}$. □

In Lemma 3.7, we determined an element $\beta_0 \in \mathcal{O}_F$ satisfying conditions (1) and (2) in Corollary 4.3 when $F = \mathbb{Q}(\zeta_m)$ for many values of m.

4.3. ***Existence and uniqueness of principal polarizations.*** In this section, we study principal polarizations on CM-abelian varieties. Under conditions on the class group and unit group of the field, we show such principal polarizations exist and are uniquely determined. Recall that L is a CM-field with maximal totally real subfield L_0 and $n = [L_0 : \mathbb{Q}]$.

Proposition 4.4. *Suppose L_0 has units of independent signs and (L, Φ) is a CM-type.*

(1) *There exists a principally polarized CM-abelian variety of type $(\mathcal{O}_L, \Phi)$.*

(2) *If (L, Φ) is simple, then any CM-abelian variety of type $(\mathcal{O}_L, \Phi)$ has at most one principal polarization, up to isomorphism.*

(3) *If (L, Φ) is simple and L has class number 1, then there exists a unique principally polarized CM-abelian variety of type $(\mathcal{O}_L, \Phi)$, up to isomorphism.*

Proof. By Theorem 4.2(1), there exists a fractional ideal $\mathfrak{a}$ in L and an element $\beta_0 \in \mathcal{O}_L$ such that (1) β_0 generates $D_{L/\mathbb{Q}}\mathfrak{a}\bar{\mathfrak{a}}$ and (2) $\beta_0 = -\overline{\beta}_0$. An element $\beta' \in \mathcal{O}_L$ satisfies conditions (1) and (2) if and only if $\beta' = u_0\beta_0$ for a totally real unit $u_0 \in \mathcal{U}_{L_0}$.

By Theorem 4.2(2), $\beta \in \mathcal{O}_L$ defines a principal polarization of type $(\mathcal{O}_L, \Phi)$ on $\mathbb{C}^n/\Phi(\mathfrak{a})$ if and only if it satisfies conditions (1) and (2), and also (3) $\mathrm{Im}(\phi(\beta)) < 0$, for each $\phi \in \Phi$.

Hence, to finish the first claim, it suffices to check that there exists $u_0 \in \mathcal{U}_{L_0}$ such that $\beta = u_0\beta_0$ satisfies condition (3), for any CM-type Φ of L. If L_0 has units of independent signs, then ρ_{L_0} is surjective; this shows that the unit u_0 described above exists.

By Theorem 4.2(2), if (L, Φ) is simple, all principal polarizations of type Φ on $\mathbb{C}^n/\Phi(\mathfrak{a})$ arise from some $\beta \in \mathcal{O}_L$ satisfying conditions (1)–(3). By Theorem 4.2(3), $\beta, \beta' \in \mathcal{O}_L$ satisfying conditions (1)–(3) define isomorphic principally polarized abelian varieties if and only if $\beta' = N(u)\beta$ for some unit $u \in \mathcal{U}_L$.

By definition, given a fractional ideal $\mathfrak{a}$ of L, if $\beta \in \mathcal{O}_L$ satisfying conditions (1)–(3) exists, then $\beta' \in \mathcal{O}_L$ also satisfies conditions (1)–(3) if and only if $\beta' = u^+\beta$ for some totally positive unit $u^+ \in \mathcal{U}_{L_0}^+$.

Hence, to finish the second claim, it suffices to check that for any $u^+ \in \mathcal{U}_{L_0}^+$ there is $u \in \mathcal{U}_L$ such that $u^+ = N(u)$. Since L_0 has units of independent signs, this follows from Lemma 3.4. Finally, if L has class number 1, then by Theorem 4.2 and [18, Theorem 5], any CM-abelian variety of type $(\mathcal{O}_L, \Phi)$ is isomorphic to $A_\Phi = \mathbb{C}^n/\Phi(\mathcal{O}_L)$. □

Proposition 4.5. *Suppose L_0 has units of almost independent signs and $Q(L) = 2$, where $Q(L)$ is defined in Definition 3.2. Suppose (L, Φ) is a simple CM-type.*

(1) *The number of isomorphism classes of principal polarizations on a CM-abelian variety of type $(\mathcal{O}_L, \Phi)$ is at most one.*

(2) *Suppose in addition that L has class number* 1. *If there is a principally polarized CM-abelian variety of type $(\mathcal{O}_L, \Phi)$, then it is unique up to isomorphism.*

Proof. The arguments in the proof of Proposition 4.4 still apply, after observing that $U_{L_0}^+ = N(\mathcal{U}_L)$ when L_0 has units of almost independent signs and $Q(L) = 2$ by Lemma 3.4. □

4.4. ***CM-types for cyclotomic fields.*** We consider the case when L is the cyclotomic field $F = \mathbb{Q}(\zeta_m)$. By Section 3.2, the next result is a special case of Propositions 4.4 and 4.5.

Corollary 4.6. *Let m such that $F = \mathbb{Q}(\zeta_m)$ has class number* 1. *Let Φ be a CM-type of F.*

(1) *If F_0 has narrow class number* 1, *then there exists a principally polarized CM-abelian variety of type $(\mathcal{O}_F, \Phi)$. Furthermore, if (F, Φ) is simple, then it is unique up to isomorphism.*

(2) *Suppose F_0 has narrow class number* 2 *and Φ is simple. If there exists a principally polarized CM-abelian variety of type $(\mathcal{O}_F, \Phi)$, then it is unique up to isomorphism.*

We will now explicitly describe the data of the principal polarization.

Corollary 4.7. *Let m, $F = \mathbb{Q}(\zeta_m)$, and β_0 be as in Lemma 3.7. Let Φ be a CM-type of F. Consider conditions* (1)–(3) *in Corollary 4.3.*

(1) *If $\Phi' = i\Phi$ and β satisfies conditions* (1)–(3) *for Φ, then $\beta' = \sigma_{i^{-1}}(\beta)$ satisfies conditions* (1)–(3) *for Φ'.*

(2) *Suppose F_0 has units of independent signs. Then, for any CM-type Φ of F, there exists a unit $u_0 \in \mathcal{U}_{F_0}$ such that $\beta = u_0\beta_0$ satisfies conditions* (1)–(3) *for Φ. Furthermore, β is the unique element satisfying conditions* (1)–(3) *for Φ, up to multiplication by an element of $N_{F/F_0}(\mathcal{U}_F)$.*

The condition that F_0 has units of independent signs is satisfied if the narrow class number of F_0 is odd [2, bottom of page 58].

Proof. (1) This follows from Lemma 2.1 and Lemma 2.4.

(2) By Lemma 3.7, β_0 satisfies conditions (1) and (2) of Corollary 4.3. Since F_0 has units of independent signs, there exists $u_0 \in \mathcal{U}_{F_0}$ such that $\beta = u_0\beta_0$ satisfies condition (3). Since $u_0 \in \mathcal{U}_{F_0}$ is a real unit, β also satisfies conditions (1) and (2) of Corollary 4.3. Note that β is unique up to multiplication by a totally positive unit of F_0. By Lemma 3.4, $\mathcal{U}_{F_0}^{+} = N(\mathcal{U}_F)$, proving the uniqueness statement. □

See Examples 6.3 and 6.7 for illustrations of Corollary 4.7 when $m = 5$ and $m = 7$, respectively. As another example, if $m = 8$, then $\beta_0 = -4i$ from Lemma 3.7: if $\mathfrak{f} = (0, 1, 0, 1)$, set $u_0 = -1$ and $\beta = 4i$; if $\mathfrak{f} = (1, 1, 0, 0)$, set $u_0 = \sqrt{2} - 1$ and $\beta = 4(1 - \sqrt{2})i$.

5. Shimura data

Consider a monodromy datum $\gamma = (m, N, a)$ as in Section 2.1. Recall that H_γ is the Hurwitz space of μ_m-cyclic covers of $\mathbb{P}^1$ with monodromy datum γ. Recall also that Z_γ is the closure in $\mathcal{A}_g$ of the image of H_γ under the Torelli morphism and S_γ is the smallest PEL type Shimura substack of $\mathcal{A}_g$ containing Z_γ. In this section, under certain assumptions on m, we provide a method to determine the integral PEL datum of S_γ.

A point P in Z_γ represents the Jacobian of a stable curve C_P of compact type, where compact type means that the dual graph of C_P is a tree. Furthermore, the curve C_P is a μ_m-cover of a tree of projective lines. The main idea of the proof is to find a point P in Z_γ such that the Jacobian of every irreducible component of C_P has complex multiplication by a CM-field.

5.1. ***Admissible covers.*** We briefly review some information about the boundary of H_γ, see [20, Chapter 4] or [3, Section 1]. The generic point of Z_γ represents the Jacobian of a smooth curve. More generally, a point P of Z_γ represents the Jacobian J_P of a stable curve C_P of compact type. There is a μ_m-cover $\psi : C_P \to T$ where T is a tree of projective lines. The nodes of C_P and T are ordinary double points.

Since P is in Z_γ, the closure of Z_γ^0, the cover ψ is *admissible* as defined below. If η is a node of C_P, consider the restrictions ϕ_1 and ϕ_2 of ϕ to the two irreducible components of C_P that intersect at η. When m is prime, the compact type condition implies that ϕ_1 and ϕ_2 are each totally ramified at η. For $i = 1, 2$, let α_i be the canonical generator of inertia of ϕ_i at η. The admissible condition is that α_1 and α_2 are inverses in μ_m.

5.2. ***Distinguished points.*** We prove that there exists a distinguished point P in Z_γ such that we can compute the lattice and Hermitian form for J_P.

Definition 5.1. A point P in Z_γ is a *distinguished point* if the Jacobian J_P is a principally polarized abelian variety with complex multiplication by a maximal order in a CM-field, or the direct sum of such together with the product polarization.

Proposition 5.2. *Let m be an odd prime. Let $\gamma = (m, N, a)$ be a monodromy datum. Then Z_γ has a distinguished point P. More specifically, for $r = N - 2$:*

(1) *In the family H_γ of μ_m-covers of a genus 0 curve with monodromy datum γ, there is a point which represents an admissible μ_m-cover $\psi : C_P \to T$, where T is a tree of r projective lines and C_P is a curve of compact type, with r irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_r$, each of which is a curve of genus $(m-1)/2$ admitting a μ_m-cover of $\mathbb{P}^1$ branched at 3 points.*

(2) *The Jacobian J_P of C_P is of the form $A \simeq \bigoplus_{j=1}^r A_j$, where each A_j is a principally polarized abelian variety of dimension $(m-1)/2$ having complex multiplication by $\mathcal{O}_F = \mathbb{Z}[\zeta_m]$, together with the product polarization.*

Proof. The fact that Z_γ has a distinguished point P is immediate from part (2), which we will show follows from part (1).

(1) Let $a = (a(1), \ldots, a(N))$ be the inertia type. If $N \geq 4$ and m is an odd prime, then a has the following property: there is a pair (i, j) with $1 \leq i < j \leq N$, such that $a(i) + a(j) \not\equiv 0 \bmod m$. Without loss of generality, we can suppose $i = 1$ and $j = 2$. Let $\alpha_2 = a(1) + a(2)$ and $\alpha_1 = -\alpha_2$.

When $N \geq 4$, then Z_γ^0 is affine. Consider a family of μ_m-covers with monodromy datum γ, where the first branch point b_1 approaches the second branch point b_2. When $b_1 = b_2$, the curve is singular and its normalization is a μ_m-cover ϕ of a tree of two projective lines. Let ϕ_1 (resp. ϕ_2) be the restriction of ϕ over the first (resp. second) of these.

The cover ϕ_1 (resp. ϕ_2) is ramified at the specializations of 2 (resp. $N-2$) ramification points. For each of these, the canonical generator of inertia remains the same at the specialization. The values in the inertia type of ϕ_1 (or ϕ_2) sum to 0 modulo m. Also, ϕ is admissible and ramified at the ordinary double point. Thus ϕ_1 is a μ_m-cover of $\mathbb{P}^1$ branched at 3 points with inertia type $(a(1), a(2), \alpha_1)$, and ϕ_2 is a μ_m-cover of $\mathbb{P}^1$ branched at $N-1$ points with inertia type $(\alpha_2, a(3), \ldots, a(N))$. The cover ϕ is called *a degeneration of compact type* in [8, Remark 5.2].

By induction on N, we see that H_γ contains a family of μ_m-covers which degenerates completely to an admissible μ_m-cover $\psi : C_P \to T$, where T is a tree of r projective lines and the restriction of ψ above each component of T is branched at 3 points. By (2-1), each irreducible component of C_P has genus $(m-1)/2$.

(2) Choose a labeling $C_1, \ldots, C_r$ of the irreducible components of C_P. For $1 \leq j \leq r$, let $A_j = \mathrm{Jac}(C_j)$. Note that A_j is a principally polarized abelian variety of

dimension $(m-1)/2$. By [1, Section 9.2, Example 8, page 246], $J_P \simeq \bigoplus_{j=1}^{r} A_j$, and the principal polarization on J_P decomposes as the product polarization.

Consider the CM-field $F = \mathbb{Q}(\zeta_m)$. Then $\mathcal{O}_F = \mathbb{Z}[\zeta_m] \subset \mathrm{End}(A_j)$ since μ_m acts on C_j for $1 \le j \le r$. Since $\deg(F/\mathbb{Q}) = 2 \cdot \dim(A_j)$, the abelian variety A_j has complex multiplication by $\mathcal{O}_F$ for $1 \le j \le r$. □

5.3. ***Shimura datum for Hurwitz spaces.*** We turn to the main result of the paper. Recall that $\gamma = (m, N, a)$ and $r = N - 2$.

Definition 5.3. Let $P \in Z_\gamma$ be a distinguished point as described in Proposition 5.2. Suppose $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$ has units of independent signs. (This is guaranteed by Lemma 3.6 when m is an odd prime such that $F = \mathbb{Q}(\zeta_m)$ has class number 1.)

For $1 \le j \le r$, define β_j and $\xi_j = \beta_j^{-1}$ as follows. Recall that C_j is an irreducible component of C_P. The inertia type of the μ_m-cover $C_j \to \mathbb{P}^1$ determines its signature $\mathfrak{f}_j$ by (2-2), which determines its CM-type Φ_j as in (2-4). Let β_0 be as in Lemma 3.7. By Corollary 4.7, there exists $u_j \in \mathcal{U}_{F_0}$ such that $\beta_j := u_j \beta_0$ satisfies conditions (1)–(3) of Corollary 4.3 for Φ_j. Furthermore, β_j is the unique element satisfying conditions (1)–(3) for Φ_j, up to multiplication by an element of $N_{F/F_0}(\mathcal{U}_F)$.

Remark 5.4. The CM-type (F, Φ_j) of A_j may not be simple. For example, for the special family $M[17]$ with $m = 7$, we show in Section 6.3 that (F, Φ_2) is not simple.

Let $F = \mathbb{Q}(\zeta_m)$ and denote by $\mathcal{O}_F = \mathbb{Z}[\zeta_m]$ the ring of integers of F.

Theorem 5.5. *Let m be an odd prime. Let $\gamma = (m, N, a)$ be a monodromy datum with $N \ge 4$. Suppose $F = \mathbb{Q}(\zeta_m)$ has class number* 1. *Let $r = N - 2$.*

Let $P \in Z_\gamma$ be a distinguished point as described in Proposition 5.2. For $1 \le j \le r$, this determines an abelian variety A_j with CM-type $(\mathcal{O}_F, \Phi_j)$; let $\xi_j = \beta_j^{-1}$ be as in Definition 5.3.

Then the integral PEL datum for S_γ is given by

- *the F-vector space $V = F^r$, together with the standard $\mathcal{O}_F$-lattice $\Lambda = (\mathcal{O}_F)^r \subseteq V$;*
- *the Hermitian form $H_B = \langle \cdot, \cdot \rangle$ on V taking integral values on Λ and defined by*

$$\langle x, y \rangle = \mathrm{tr}_{F/\mathbb{Q}}(x B \bar{y}^T) \quad \textit{for } B = \mathrm{diag}[\xi_1, \dots \xi_r] \in \mathrm{GL}_r(F) = \mathrm{GL}(V). \tag{5-1}$$

It is straightforward to compute B from γ; we give many examples in Section 6.

Proof. The variety Z_γ is an irreducible algebraic subvariety of $\mathcal{A}_g$. By [17, Proposition 4.1], the lattice and Hermitian form for the integral PEL datum of S_γ can be computed at any point of Z_γ. Since m is an odd prime, there is a distinguished point $P \in Z_\gamma$ as described in Proposition 5.2. We compute the lattice and Hermitian form at P.

Let $V = H^1(C_P(\mathbb{C}), \mathbb{Q})$. Note that $r = N - 2 = 2g/(m-1)$. From (2-2), we deduce that there is an isomorphism of $\mathbb{Q}[\mu_m]$-modules $V \simeq F^r$, where multiplication of $\mathbb{Q}[\mu_m]$ on F^r is given by the natural homomorphism $\mathbb{Q}[\mu_m] \to F$. The complex structure on $V_\mathbb{R}$ is given as $V_\mathbb{R} \simeq (F \otimes_\mathbb{Q} \mathbb{R})^r \simeq \oplus_{n=1}^m \mathbb{C}^{\mathfrak{f}(\tau_n)}$. By (2-2), $\mathfrak{f}(\tau_n) + \mathfrak{f}(\tau_{-n}) = r$ for all $n \not\equiv 0 \bmod m$.

Let $\Lambda = H^1(C_P(\mathbb{C}), \mathbb{Z})$ be the first Betti cohomology of the curve C_P. When F has class number 1, we prove that there is an isomorphism of $\mathbb{Q}[\mu_m]$-modules $V \simeq F^r$ such that the $\mathbb{Z}[\mu_m]$-lattice $\Lambda \subset V$ maps isomorphically onto $(\mathcal{O}_F)^r \subset F^r$ and the Hermitian form ψ on V maps to the diagonal Hermitian form H_B on F^r in (5-1). Note $(\mathcal{O}_F)^r \subset F^r$ is a $\mathbb{Z}[\mu_m]$-lattice, and H_B is integral on $(\mathcal{O}_F)^r$.

By Lemma 3.6, if m is prime and F has class number 1, then $F_0 = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ has narrow class number 1 and has units of independent signs. For $1 \le j \le r$, by hypothesis, A_j has dimension $(m-1)/2$; it has complex multiplication by $\mathcal{O}_F$ and has CM-type $(\mathcal{O}_F, \Phi_j)$. Furthermore, β_j defines a principal polarization on A_j; the corresponding Hermitian form is given by $\mathbb{E}(\Phi(x), \Phi(y)) = \mathrm{tr}_{F/\mathbb{Q}}(\xi_j x \bar{y})$ for $x, y \in L$ by (4-1).

Since F has class number 1, there is a unique complex torus of CM-type $(\mathcal{O}_F, \Phi_j)$ up to isomorphism by Theorem 4.1; namely, $A_j \simeq \mathbb{C}^g/\Phi_j(\mathcal{O}_F)$. Since $J_P \simeq \oplus_{j=1}^r A_j$, we deduce that $\Lambda \simeq (\mathcal{O}_F)^r$. The product polarization on J_P defines an integral Hermitian form of the prescribed signature on Λ. By [16, Appendix, Proposition 8], there is a unique Hermitian form of given signature on Λ. Thus the Hermitian form on Λ is the one in (5-1). □

Remark 5.6. The values of m for which Theorem 5.5 applies are $3, 5, 7, 11, 13, 17, 19$.

For $m = 3, 5, 11, 17$, it is not necessary to refer to [16, Appendix, Proposition 8]; the reason is that (F, Φ_j) is automatically simple for $1 \le j \le r$ by Lemmas 3.8 and 3.9. Thus by Corollary 4.6(1), there is a unique principally polarized CM-abelian variety of type $(\mathcal{O}_F, \Phi_j)$ up to isomorphism. In Section 6.4, we give an alternative approach for $m = 7$ that relies on Corollary 4.6(2) and does not refer to [16, Appendix, Proposition 8].

The strategy of Theorem 5.5 may apply even when m is composite; see Section 7.

6. Applications for m prime

Theorem 5.5 gives a method to determine the integral PEL datum for S_γ for all monodromy data $\gamma = (m, N, a)$ when $m = 3, 5, 7, 11, 13, 17, 19$, so that $F = \mathbb{Q}(\zeta_m)$ has class number 1. In this section, we give examples of this. Specifically:

- In Section 6.1, when $m = 3$, we explicitly determine the Hermitian form for all γ.
- In Sections 6.2 and 6.3, when $m = 5, 7$, we give explicit examples for all equivalence classes of γ when $N = 4$.

- In Section 6.4, we follow an alternative approach for $m = 7$ which avoids the possible occurrence of nonsimple CM-types at the distinguished point.

In particular, we determine the integral PEL datum for the 6 special families for which the degree m is an odd prime; these are denoted by $M[n]$ as in [12, Table 1].

Given a signature $\mathfrak{f}$, let Φ be the CM-type of F determined by $\mathfrak{f}$, as in (2-4). Given the F-vector space $V = F^r$, together with the standard $\mathcal{O}_F$-lattice $\Lambda = (\mathcal{O}_F)^r \subseteq V$, let $H_B = \langle \cdot, \cdot \rangle$ denote the Hermitian form on V defined in (5-1).

6.1. *Integral PEL datum for all families when* $m = 3$.

Example 6.1. When $m = 3$, then $\beta_0 = 3/(\zeta_3^2 - \zeta_3) = \sqrt{-3}$ by Lemma 3.7.

For $\mathfrak{f}_1 = (1, 0)$, set $u_1 = -1$ and $b_1 = u_1\beta_0$ so that $b_1 = -\sqrt{-3}$. Then $\mathrm{Im}(\sigma_1(b_1)) < 0$.

For $\mathfrak{f}_2 = (0, 1)$, set $u_2 = 1$ and $b_2 = u_2\beta_0$ so that $b_2 = \sqrt{-3}$. Then $\mathrm{Im}(\sigma_2(b_2)) < 0$.

The CM-type Φ_i is simple by Lemma 3.8 and b_i satisfies Corollary 4.7 with respect to Φ_i.

For $m = 3$, there are simple formulas relating the signature $\mathfrak{f}$ and the inertia type a. If a has d_1 entries of 1 and d_2 entries of 2, then $\mathfrak{f} = (f_1, f_2)$ with $f_1 = (2d_1 + d_2 - 3)/3$ and $f_2 = (d_1 + 2d_2 - 3)/3$. Then $d_1 = 2f_1 - f_2 + 1$ and $d_2 = 2f_2 - f_1 + 1$. One can check that $0 \le \max(f_1, f_2) \le 2\min(f_1, f_2) + 1$.

Corollary 6.2. *Let $m = 3$ and $N \ge 4$. Let $\gamma = (3, N, a)$ be a monodromy datum with signature (f_1, f_2). Let $g = f_1 + f_2$. Let S_γ be the component of the Shimura variety containing Z_γ.*

Let $F = \mathbb{Q}(\zeta_3)$ and $\xi = -1/\sqrt{-3}$. Then the integral PEL datum of S_γ has lattice $(\mathcal{O}_F)^g$ with Hermitian form H_B where $B \in \mathrm{GL}_g(\mathcal{O}_F)$ is diagonal with f_1 entries of ξ and f_2 entries of $-\xi$.

Proof. Consider the admissible μ_3-cover $\varphi : \mathcal{C}_P \to T$ represented by the distinguished point from Proposition 5.2. Here T is a tree of $r = f_1 + f_2$ projective lines. Above f_1 (resp. f_2) components of T, the restriction of φ is a μ_3-cover branched at 3 points with signature $\mathfrak{f}_1 = (1, 0)$ (resp. $\mathfrak{f}_2 = (0, 1)$). The result then follows from Theorem 5.5. □

In particular, Corollary 6.2 includes the three special families $M[3]$, $M[6]$, $M[10]$:

M	(m, N, a)	B
$M[3]$	(3, 4, (1, 1, 2, 2))	$\mathrm{diag}[\xi, -\xi]$
$M[6]$	(3, 5, (1, 1, 1, 1, 2))	$\mathrm{diag}[\xi, \xi, -\xi]$
$M[10]$	(3, 6, (1, 1, 1, 1, 1, 1))	$\mathrm{diag}[\xi, \xi, \xi, -\xi]$

The cases $M[6] : \xi[1, 1, -1]$ and $M[10] : \xi[1, 1, 1, -1]$ match the table on [16, p. 1].

6.2. *Integral PEL datum when $m = 5$.* We illustrate Theorem 5.5 for several well-chosen examples when $m = 5$, including all families branched at $N = 4$ points up to equivalence and one family branched at $N = 5$ points. This includes the two special families when $m = 5$, namely $M[11]$ and $M[16]$. A similar result can be obtained for any monodromy datum γ with $N > 4$ by an inductive process.

Example 6.3. The following table summarizes the data for CM-types when $m = 5$.

a	$\mathfrak{f}$	Φ	β
$(4,3,3)$	$(0,1,0,1)$	$\{2,4\}$	$\beta_1 = 5/(\zeta_5^3 - \zeta_5^2)$
$(3,1,1)$	$(1,1,0,0)$	$\{1,2\}$	$\beta_2 = 5/(\zeta_5 - \zeta_5^4)$
$(1,2,2)$	$(1,0,1,0)$	$\{1,3\}$	$\beta_3 = -\beta_1$
$(2,4,4)$	$(0,0,1,1)$	$\{3,4\}$	$\beta_4 = -\beta_2$

In the i-th line of the table, the CM-type Φ_i is simple by Lemma 3.8 and β_i satisfies Corollary 4.7 with respect to Φ_i. The automorphism σ_3 permutes the rows via the cycle $(1, 2, 3, 4)$ and its inverse σ_2 permutes the fourth column via $\beta_1 \to \beta_2 \to -\beta_1 \to -\beta_2 \to \beta_1$.

Proof. When $m = 5$, then $\beta_0 = 5/(\zeta_5^3 - \zeta_5^2)$ by Lemma 3.7. For $\mathfrak{f}_1 = (0, 1, 0, 1)$, set $u_1 = 1$ and $\beta_1 = u_1\beta_0$. We compute that $\mathrm{Im}(\sigma_j(\beta_1)) < 0$ for $j = 2, 4$. For $\mathfrak{f}_2 = (1, 1, 0, 0)$, set $u_2 = (\zeta_5^3 - \zeta_5^2)/(\zeta_5 - \zeta_5^4)$ and $\beta_2 = u_2\beta_0$. We compute that $\mathrm{Im}(\sigma_j(\beta_2)) < 0$ for $j = 1, 2$. The signature $\mathfrak{f}_3 = (1, 0, 1, 0)$ (resp. $\mathfrak{f}_4 = (0, 0, 1, 1)$) is the complex conjugate of $\mathfrak{f}_1$ (resp. $\mathfrak{f}_2$), which negates the value of β. □

Corollary 6.4. *Let $m = 5$ and $F = \mathbb{Q}(\zeta_5)$. Every family of μ_5-covers of $\mathbb{P}^1$ with $N = 4$ is equivalent to either* (i), (ii) *or* $M[11]$ *in the table below. Recall $\beta_1, \beta_2 \in \mathcal{O}_F$ from Example 6.3 and let $\xi_i = \beta_i^{-1}$. For the monodromy data $\gamma = (5, N, a)$ and $r = N - 2$ as below, the integral PEL datum of S_γ has lattice $(\mathcal{O}_F)^r$ with Hermitian form H_B where $B \in \mathrm{GL}_r(\mathcal{O}_F)$ is as below.*

M	$(5,N,a)$	B
(i)	$(5,4,(1,1,4,4))$	$\mathrm{diag}[\xi_2, -\xi_2]$
(ii)	$(5,4,(1,2,3,4))$	$\mathrm{diag}[-\xi_1, \xi_1]$
$M[11]$	$(5,4,(1,3,3,3))$	$\mathrm{diag}[\xi_1, \xi_2]$
$M[16]$	$(5,5,(2,2,2,2,2))$	$\mathrm{diag}[-\xi_1, -\xi_2, -\xi_1]$

Proof. Suppose $m = 5$ and $N = 4$ and let a be the inertia type of γ. If three of the values of a are the same, the family is equivalent to the one with $a = (1, 3, 3, 3)$, which is $M[11]$. If two of the values of a are the same, the family is equivalent to (i). If all values of a are distinct, the family is equivalent to (ii).

By Theorem 5.5, it suffices to find the CM-type (F, Φ_i) for the abelian varieties A_i in the decomposition of the Jacobian of $\mathcal{C}_P$. We refer to [8, Remark 5.2, Lemma 6.4] for information about the admissible degeneration, given in shorthand by: (i) $(1,1,3)+(2,4,4)$; (ii) $(1,2,2)+(3,3,4)$; $M[11]$ $(1,3,1)+(4,3,3)$; and $M[16]$ $(2,2,1)+(4,2,4)+(1,2,2)$. Using the table in Example 6.3, we find the entries of the diagonal of B. □

Remark 6.5. The family (ii), with monodromy datum $\gamma=(5,4,(1,2,3,4))$, has a second degeneration of the form $(1,3,1)+(4,2,4)$, whose Hermitian form has matrix $B'=\mathrm{diag}[\xi_2,-\xi_2]$. We give two reasons why the Hermitian forms determined by B' and $B=\mathrm{diag}[-\xi_1,\xi_1]$ are isomorphic.

First, by Lemma 2.1, the automorphism σ_2 takes the inertia types in the first degeneration to those in the second by multiplying the entries by 3. By Corollary 4.7(1), the action on the entries of B is via $\sigma_{2^{-1}}=\sigma_3$ and

$$\sigma_3(B)=\mathrm{diag}[\sigma_3(-\xi_1),\sigma_3(\xi_1)]=\mathrm{diag}[\xi_2,-\xi_2]=B'.$$

Second, for family (ii), S_γ has signature type $(1,1,1,1)$; hence its reflex field is $\mathbb{Q}$ (which is smaller than $F_0\subset\mathbb{R}$). The matrices B and B' are conjugate under the action of $\sigma_3\in\mathrm{Gal}(F_0/\mathbb{Q})$ and correspond to the two choices of a $\mathbb{Q}$-linear embedding $F_0\hookrightarrow\mathbb{R}$.

Remark 6.6. To compare with Shimura's work, write $w=\zeta_5+\zeta_5^4$. Then we have $w^2+w-1=0$. So $w=(-1+\sqrt{5})/2$. Then $\xi_2/\xi_1=-w-1=-(1+\sqrt{5})/2$ so $\xi_1/\xi_2=(1-\sqrt{5})/2$.

Consider the family $\gamma'=(5,4,(1,1,1,1,1))$. A careful look at [16, Section 5] shows that Shimura replaced ζ_5 by ζ_5^3 in his computation for this family. This has the effect of switching to the family $M[16]$ with $\gamma=(5,4,(2,2,2,2,2))$; indeed, Shimura computes that the signature is $(2,0,3,1)$. By line 4 of the table in Corollary 6.4, the family γ has $B=-\xi_1[1,1,\xi_2/\xi_1]=-\xi_1[1,1,-(1+\sqrt{5})/2]$. This does not exactly match what is written in line 5 of the table on [16, page 1], namely $[1,1,\xi_1/\xi_2]=[1,1,(1-\sqrt{5})/2]$, but it has the same sign signature and thus yields an isomorphic Hermitian form.

Consider the family $\gamma'=(5,4,(2,1,1,1))$. The details for this family are not included in [16] but it appears that Shimura replaced ζ_5 by ζ_5^3 in his computation for this family also. This has the effect of switching to the family $M[11]$ with $\gamma=(5,4,(1,3,3,3))$. By line 3 of the table in Corollary 6.4, the family γ has $B=\xi_2[1,\xi_1/\xi_2]=\xi_2[1,(1-\sqrt{5})/2]$. This matches what is written in line 4 of the table on [16, page 1].

6.3. ***Integral PEL datum when $m=7$.*** We illustrate Theorem 5.5 for several well-chosen examples when $m=7$, including all families branched at $N=4$ points up to equivalence. This includes the special family $M[17]$. A similar result can be obtained for any monodromy datum γ with $N>4$ by an inductive process.

Example 6.7. The following table summarizes the cases when $m = 7$.

a	$\mathfrak{f}$	Φ	β
$(1,1,5)$	$(1,1,1,0,0,0)$	$\{1,2,3\}$	$\beta_1 = 7/(\zeta_7 - \zeta_7^6)$
$(3,3,1)$	$(1,0,1,0,1,0)$	$\{1,3,5\}$	$\beta_2 = 7/(\zeta_7^3 - \zeta_7^4)$
$(2,2,3)$	$(1,0,0,1,1,0)$	$\{1,4,5\}$	$\beta_3 = 7/(\zeta_7^2 - \zeta_7^5)$
$(6,6,2)$	$(0,0,0,1,1,1)$	$\{4,5,6\}$	$\beta_4 = -\beta_1$
$(4,4,6)$	$(0,1,0,1,0,1)$	$\{2,4,6\}$	$\beta_5 = -\beta_2$
$(5,5,4)$	$(0,1,1,0,0,1)$	$\{2,3,6\}$	$\beta_6 = -\beta_3$
$(1,2,4)$	$(1,1,0,1,0,0)$	$\{1,2,4\}$	$\beta = -\dfrac{7(\zeta_7^3 - \zeta_7^4)}{(\zeta_7 - \zeta_7^6)(\zeta_7^2 - \zeta_7^5)}$
$(3,1,5)$	$(0,0,1,0,1,1)$	$\{3,5,6\}$	$\beta' = -\beta$

Lemma 6.8. *In the i-th line of the table in Example 6.7, for $1 \le i \le 6$, the CM-type Φ_i is simple, and the element β_i satisfies Corollary 4.7 with respect to Φ_i. The generator σ_3 of the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$ permutes these via $\sigma_3^{i-1}(\beta_1) = \beta_i$, for $i = 1, \dots, 6$.*

In the last two lines of the table, the element β (resp. β') satisfies Corollary 4.7 with respect to the CM-type $\Phi = \{1, 2, 4\}$ (resp. $\Phi' = \{3, 5, 6\}$). The CM types Φ, Φ' are not simple.

Proof. When $m = 7$, then $\beta_0 = 7/(\zeta_7^4 - \zeta_7^3)$ by Lemma 3.7.

For $a_1 = (1, 1, 5)$ and $\mathfrak{f}_1 = (1, 1, 1, 0, 0, 0)$, set $u_1 = (\zeta_7^4 - \zeta_7^3)/(\zeta_7 - \zeta_7^6)$ and $\beta_1 = u_1\beta_0$. We compute that $\mathrm{Im}(\sigma_j(\beta_1)) < 0$ for $j = 1, 2, 3$.

Let $\sigma = \sigma_5$, which is a generator of $\mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$ and has inverse σ_3. Then, by Lemmas 2.1 and 2.4, for $1 \le i \le 6$, the action of σ_5^{i-1} changes the inertia type to $a_i := (3^{i-1}) \cdot a_1$ and the CM-type to $\Phi_i := 5^{i-1} \cdot \Phi$, which also determines the signature $\mathfrak{f}_i$. Also, the element $\beta_i = \sigma_3^{i-1}(\beta_1)$ satisfies Corollary 4.7 with respect to Φ_i.

For $a = (1, 2, 4)$ and $\mathfrak{f} = (1, 1, 0, 1, 0, 0)$, set

$$u = -\frac{(\zeta_7^4 - \zeta_7^3)^2}{(\zeta_7 - \zeta_7^6)(\zeta_7^2 - \zeta_7^5)} \quad \text{and} \quad \beta = u\beta_0 = -\frac{7(\zeta_7^3 - \zeta_7^4)}{(\zeta_7 - \zeta_7^6)(\zeta_7^2 - \zeta_7^5)}.$$

Then $\mathrm{Im}(\sigma_j(\beta_2)) < 0$ for $j = 1, 2, 4$. The last line follows from Lemma 2.1 by applying complex conjugation to the previous line.

By Lemma 3.9, Φ_i is simple unless it is $\{1, 2, 4\}$ or $\{3, 5, 6\}$. □

For $m = 7$ and $N = 4$, every family is equivalent to one in the next result.

Corollary 6.9. *Let $m = 7$ and $F = \mathbb{Q}(\zeta_7)$. Recall $\beta_1, \beta_2, \beta_3, \beta \in \mathcal{O}_F$ from Example 6.7. Let $\xi_i = \beta_i^{-1}$ for $i = 1, 2, 3$ and $\xi = \beta^{-1}$. For the monodromy* *data* $\gamma = (7, 4, a)$ *as below, the integral PEL datum of* S_γ *has lattice* $(\mathcal{O}_F)^2$ *with Hermitian form* H_B *where* $B \in \mathrm{GL}_2(\mathcal{O}_F)$ *is as below.*

M	a	B
(i)	$(1,1,2,3)$	$\mathrm{diag}[\xi_1, \xi_3]$
(ii)	$(1,1,6,6)$	$\mathrm{diag}[\xi_1, -\xi_1]$
(iii)	$(1,2,5,6)$	$\mathrm{diag}[\xi_1, -\xi_1]$
$M[17]$	$(2,4,4,4)$	$\mathrm{diag}[-\xi_2, \xi]$

Proof. We refer to [8, Remark 5.2, Lemma 6.4] for information about the admissible degeneration of each family, given in shorthand by (i) $(1, 1, 5) + (2, 2, 3)$; (ii) $(1, 1, 5) + (2, 6, 6)$; (iii) $(1, 5, 1) + (6, 2, 6)$; and $M[17]$ $(4, 4, 6) + (1, 4, 2)$. For each family, consider the CM-types (F, Φ_1) and (F, Φ_2) for the abelian varieties A_i in the decomposition of the Jacobian at the distinguished point arising from the degeneration. We compute the associated integral PEL data by Theorem 5.5, using the table in Example 6.7. □

Remark 6.10. In cases (i), (ii), and (iii), both (F, Φ_1) and (F, Φ_2) are simple. For $M[17]$, the CM-type (F, Φ_2) is not simple, but this is not a concern by [16, Appendix, Proposition 8] or Section 6.4. For $M[17]$, we note that $B = \xi[1, -\xi_2/\xi]$ where $-\xi_2/\xi = \zeta_7 + \zeta_7^6 - 1$. In line 6 of the table on [16, page 1], for the related family $\gamma = (7, 4, (4, 1, 1, 1))$, Shimura writes the Hermitian form as H_B with $B = \xi[1, -\sin(3\pi/7)/\sin(2\pi/7)]$.

6.4. ***Another approach for M[17].*** In this section, we compute the integral PEL datum for the family $M[17]$ using another approach. This approach utilizes a different kind of distinguished point Q in the family, namely one that represents a curve C_Q with extra automorphisms. In this section, we see that the Jacobian J_Q has complex multiplication by a larger field and that its CM-type is simple.

In fact, we proved that every positive-dimensional family of μ_7-covers of $\mathbb{P}^1$ has a distinguished point representing a product of principally polarized abelian varieties, each of which has complex multiplication with a CM-type that is simple. In this way, one can avoid the use of [16, Appendix, Proposition 8] when $m = 7$. We omit the details of this.

We start by finding another distinguished point in the family Z_γ, under certain restrictive conditions on γ. Similarly to Notation 3.5, for $0 \leq n \leq 3m$ with $\gcd(n, 3m) = 1$, let σ_n be the embedding $\mathbb{Q}(\zeta_{3m}) \hookrightarrow \mathbb{C}$ determined by $\sigma_n(\zeta_{3m}) = \zeta_{3m}^n$.

Proposition 6.11. *Let* $\gamma = (m, 4, a)$ *where* $m > 3$ *is prime and* $a = (1, 1, 1, m-3)$. *Then* Z_γ *has a distinguished point* Q. *More specifically: in the family of* μ_m-*covers with monodromy datum* γ, *there is a point which represents a* μ_m-*cover*

$\psi : C_Q \to \mathbb{P}^1$, *where* C_Q *is a curve of genus* $m-1$ *having an automorphism of order* 3.

The Jacobian of C_Q *has complex multiplication by* $(\mathbb{Z}[\zeta_{3m}], \Phi_Q)$, *where, for any* $0 < n < 3m$ *with* $(n, 3m) = 1$, *the embedding* σ_n *is in* Φ_Q *if and only if*

$$n \in [0, 2m/3] \cup [m, 5m/3] \cup [2m, 8m/3] \quad \textit{if } n \equiv 1 \bmod 3, \quad \textit{or}$$
$$n \in [0, m/3] \cup [m, 4m/3] \cup [2m, 7m/3] \quad \textit{if } n \equiv 2 \bmod 3.$$

It is possible to generalize Proposition 6.11, by replacing 3 by an odd prime ℓ relatively prime to m and letting $N = \ell + 1$ and $a = (1, \dots, 1, m - \ell)$. We omit this generalization.

Proof of Proposition 6.11. Let C_Q be the smooth projective curve with equation $y^m = x^3 - 1$. It admits a μ_m-cover to $\mathbb{P}^1$ branched at $1, \zeta_3, \zeta_3^2, \infty$ with inertia type $a = (1, 1, 1, m-3)$. The genus of C_Q is $m-1$ by (2-1). Also C_Q has an automorphism $(x, y) \mapsto (\zeta_3 x, y)$ of order 3.

The Jacobian $J_Q = \mathrm{Jac}(C_Q)$ is a principally polarized abelian variety having dimension $m-1$. Let $F = \mathbb{Q}(\zeta_m)$, and $L = \mathbb{Q}(\zeta_{3m})$. The field L is a CM-field of degree $2 \cdot \dim(J_Q)$. Then, the inclusion $\mathcal{O}_F \subset \mathrm{End}(J_Q)$ extends to an inclusion $\mathcal{O}_L = \mathcal{O}_F[\zeta_3] \subset \mathrm{End}(J_Q)$. Thus J_Q has complex multiplication by $\mathcal{O}_L = \mathbb{Z}[\zeta_{3m}]$.

Consider the morphism $\phi : C_Q \to \mathbb{P}^1$, taking $(x, y) \mapsto x^3$. Then ϕ is a μ_{3m}-cover of $\mathbb{P}^1$, branched at $0, 1, \infty$. We compute the CM-type of J_Q by finding the inertia type and signature type of ϕ. Let b_0 (resp. b_1, resp. b_∞) denote the element of $\mathbb{Z}/3m\mathbb{Z}$ that determines the canonical generator of inertia of ϕ above 0, (resp. 1, resp. ∞).

Note that $b_1 = 3$. This is because ϕ is branched at the 3 points $x = 1, \zeta_3, \zeta_3^2$ that lie above $x^3 = 1$; the canonical generator of inertia of ϕ at the points of C_Q above these is, by definition, the automorphism identified with $\zeta_7^1 = \zeta_{21}^3$.

Without loss of generality, $b_0 = m$. To see this, note there are m points of C_Q above $x^3 = 0$, so $\gcd(b_0, 3m) = m$; this implies that $b_0 = 2m$ or $b_0 = m$; possibly after replacing the order 3 automorphism with its inverse, we can suppose that $b_0 = m$. Third, $b_\infty = 3m - b_0 - b_1 = 2m - 3$, so $(b_0, b_1, b_\infty) = (m, 3, 2m-3)$.

We compute the signature $\mathfrak{f}$ of ϕ. Let $0 < n < 3m$ and $(n, 3m) = 1$. By (2-2):

$$\mathfrak{f}(\sigma_n) = -1 + \langle -n/3 \rangle + \langle -n/m \rangle + \langle (-2n)/3 + (n/m) \rangle;$$

that is,

$$\mathfrak{f}(\sigma_n) = \begin{cases} -1 + \frac{2}{3} + \langle -n/m \rangle + \langle -\frac{2}{3} + n/m \rangle & \text{if } n \equiv 1 \bmod 3, \\ -1 + \frac{1}{3} + \langle -n/m \rangle + \langle -\frac{1}{3} + n/m \rangle & \text{if } n \equiv 2 \bmod 3, \end{cases}$$

We deduce that $\mathfrak{f}(\sigma_n) = 1$ if $n \equiv 1 \bmod 3$ and $n \in [0, 2m/3] \cup [m, 5m/3] \cup [2m, 8m/3]$ or if $n \equiv 2 \bmod 3$ and $n \in [0, m/3] \cup [m, 4m/3] \cup [2m, 7m/3]$; otherwise $\mathfrak{f}(\sigma_n) = 0$.

Finally, by (2-3), the CM-type Φ_Q of J_Q is as given in the statement. $\square$

For $m = 7$, we verify that the CM-type Φ_Q of the Jacobian of C_Q is simple.

Lemma 6.12. *When $m = 7$, then the CM-type Φ_Q in Proposition 6.11 is simple.*

Proof. Let C_Q be the smooth projective curve with equation $y^7 = x^3 - 1$. From Proposition 6.11, its CM-type is $\Phi = \{\tau_n \mid n = 1, 2, 4, 8, 10, 16\}$. We check that Φ is simple by showing that it is not induced from any proper CM-field. Let $\sigma_i : \mathbb{Q}(\zeta_{3m}) \to \mathbb{C}$ be the embedding given by $\sigma_i(\zeta_{3m}) = \zeta_{3m}^i$, for $1 \le i \le 3m$ with $\gcd(i, 3m) = 1$. Every subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_{3m})/\mathbb{Q})$ not containing complex conjugation contains either σ_8 (order 2), σ_{13} (order 2) or σ_4 (order 3). It suffices to show that the subgroup generated by each of these has a coset not contained in Φ but having nontrivial intersection with Φ, so that Φ is not a union of cosets. For $\langle\sigma_8\rangle$, this is true because of the coset $\{\sigma_4, \sigma_{11}\}$. For $\langle\sigma_{13}\rangle$, this is true because of the coset $\{\sigma_1, \sigma_{13}\}$. For $\langle\sigma_4\rangle$, this is true because of the coset $\{\sigma_2, \sigma_8, \sigma_{11}\}$. □

The following lemma will be helpful.

Lemma 6.13. *In the field $L = \mathbb{Q}(\zeta_{21})$, write $\zeta = \zeta_{21}$. Let $\beta_3 = 7/(\zeta^6 - \zeta^{15})$. Let*

(6-1) $$\alpha = (\zeta^7 - \zeta^{14})(\zeta^2 - \zeta^{19}) = (\zeta^9 + \zeta^{12}) - (\zeta^5 + \zeta^{16}).$$

With respect to the CM-type $\Phi = \{1, 2, 4, 8, 10, 16\}$ of L, the following element z satisfies conditions (1)–(3) *in Corollary 4.3*:

(6-2) $$z = \beta_3\alpha = 21(\zeta^2 - \zeta^{19})/((\zeta^{14} - \zeta^7)(\zeta^6 - \zeta^{15})).$$

Proof. We verify the first claim by computation using that $\zeta^{14} - \zeta^7 = -\sqrt{-3}$. We verify the second claim from Lemma 3.7, part (4). Note that $z = -\sigma_4(\beta_0)$ for β_0 as in loc. cit. □

We now compute the integral PEL datum of $M[17]$ using Proposition 6.11.

Proposition 6.14. *Let $m = 7$ and $F = \mathbb{Q}(\zeta_7)$. Let $\xi_1 = \beta_1^{-1}$ where $\beta_1 \in \mathcal{O}_F$ is defined in Example 6.7. Consider the units $u_1 = \zeta_7^2 + \zeta_7^5$ and $v = (1 + \zeta_7^3 + \zeta_7^4)^{-1}$ of $\mathbb{Z}[\zeta_7]$. For the family $M[17]$, with monodromy datum $\gamma = (7, 4, (2, 4, 4, 4))$, the integral PEL datum of S_γ has lattice $\Lambda = (\mathcal{O}_F)^2$ with Hermitian form H_B where*

$$B = \xi_1 v \begin{bmatrix} u_1 & -\zeta_7^2 \\ -\zeta_7^5 & u_1 \end{bmatrix}.$$

Proof. Consider the family $Z = Z_\gamma$ with monodromy datum $\gamma = (7, 4, (1, 1, 1, 4))$. Because $(2, 4, 4, 4) = 4 \cdot (4, 1, 1, 1)$ and $4 \equiv 2^{-1} \bmod 7$, the action of σ_2 takes Z to the family $M[17]$.

Let

(6-3) $$A = \frac{\zeta_7^2 - \zeta_7^5}{7(1 + \zeta_7 + \zeta_7^6)} \begin{bmatrix} \zeta_7^3 + \zeta_7^4 & -\zeta_7^4 \\ -\zeta_7^3 & \zeta_7^3 + \zeta_7^4 \end{bmatrix}.$$

One can check that A is integral away from 7. We claim that the integral PEL datum of Z is given by the lattice $\Lambda = (\mathcal{O}_F)^2$ and the Hermitian form for the matrix A. This is sufficient to prove the proposition because, by the last statement of Corollary 4.3, the integral PEL datum of $M[17]$ is then given by the lattice $\Lambda = (\mathcal{O}_F)^2$ and the Hermitian form for the matrix $B = \sigma_2^{-1}(A)$.

We turn to computing the integral PEL datum of Z. Let $L = \mathbb{Q}(\zeta_{21})$. Consider the CM-type $\Phi = \{1, 2, 4, 8, 10, 16\}$ for L. Consider the curve C_Q of genus 6 given by the equation $y^7 = x^3 - 1$ and the associated μ_7-cover $C_Q \to \mathbb{P}^1$. By Proposition 6.11: this cover is represented by a point Q of Z; the Jacobian $\mathrm{Jac}(C_Q)$ is a principally polarized abelian variety with complex multiplication by $(\mathcal{O}_L, \Phi_Q)$; and this CM-type is simple by Lemma 6.12. In other words, Q is a simple distinguished point of Z.

Note that L has class number 1 and L_0 has narrow class number 2. By Corollary 4.6, there exists a unique principally polarized CM-abelian variety of type $(\mathcal{O}_L, \Phi)$ up to isomorphism. By Lemma 6.13, the element z from (6-2) satisfies conditions (1)–(3) in Corollary 4.3 with respect to Φ. Thus $\mathrm{Jac}(C_Q)$ is isomorphic to the torus $\mathbb{C}^6/\Phi(\mathcal{O}_L)$, with principal polarization given by $\xi = z^{-1}$. Furthermore, the integral PEL datum of Z is given by the lattice $\Lambda = \mathcal{O}_L \subset V = L$ and the Hermitian form $\langle x, y\rangle = \mathrm{tr}_{L/Q}(x\xi \bar{y}^T)$. Note that $\mathcal{O}_L = \mathbb{Z}[\zeta_{21}] = \mathcal{O}_F[\zeta_3]$. With respect to the ordered basis $1, \zeta_3$ for $\mathcal{O}_L$ over $\mathcal{O}_F$, the Hermitian form is given by a matrix in $\mathrm{GL}_2(\mathcal{O}_F)$. By Lemma 6.15, this matrix is A. □

Lemma 6.15. *With respect to the ordered basis* $1, \zeta_3$ *for* $\mathcal{O}_L$ *over* $\mathcal{O}_F$*, the Hermitian form* $\langle x, y\rangle = \mathrm{tr}_{L/Q}(x\xi \bar{y}^T)$ *is given by the matrix* $A \in \mathrm{GL}_2(\mathcal{O}_F)$*, where* A *is given in* (6-3).

Proof. Let $x, y \in \Lambda$. By Lemma 6.13, since $\beta_3 = 7/(\zeta_7^2 - \zeta_7^5)$ is in $\mathcal{O}_{F_0}$, we have

$$\langle x, y\rangle = \mathrm{tr}_{L/\mathbb{Q}}(xz\bar{y}) = \mathrm{tr}_{F/\mathbb{Q}}(\beta_3^{-1}\,\mathrm{tr}_{L/F}(x\alpha^{-1}\bar{y})),$$

where α is as in (6-1).

Let $\tau = \sigma_8 \in \mathrm{Gal}(L/\mathbb{Q})$. Then τ is the generator of $\mathrm{Gal}(L/F)$, and

$$\mathrm{tr}_{L/\mathbb{Q}}(x\alpha^{-1}\bar{y}) = \mathrm{tr}_{F/\mathbb{Q}}(x\alpha^{-1}\bar{y} + \tau(x)\tau(\alpha^{-1})\tau(\bar{y})).$$

Write $x = x_1 + x_2\zeta_3$ and $y = y_1 + y_2\zeta_3$, for $x_1, x_2, y_1, y_2 \in \mathcal{O}_F$. We compute

$$\mathrm{tr}_{L/\mathbb{Q}}(x\alpha^{-1}\bar{y}) = \mathrm{tr}_{F/\mathbb{Q}}\big((x_1+x_2\zeta_3)\alpha^{-1}(\bar{y}_1+\bar{y}_2\zeta_3^2)+(x_1+x_2\zeta_3^2)\tau(\alpha^{-1})(\bar{y}_1+\bar{y}_2\zeta_3)\big).$$

Write $a_{1,1} = \alpha^{-1} + \tau(\alpha^{-1})$, $a_{1,2} = \zeta_3^2\alpha^{-1} + \zeta_3\tau(\alpha^{-1})$, and $a_{2,1} = \zeta_3\alpha^{-1} + \zeta_3^2\tau(\alpha^{-1})$. Then $\mathrm{tr}_{L/\mathbb{Q}}(x\alpha^{-1}\bar{y}) = \mathrm{tr}_{F/\mathbb{Q}}(x_1\bar{y}_1a_{1,1} + x_1\bar{y}_2a_{1,2} + x_2\bar{y}_1a_{2,1} + x_2\bar{y}_2a_{1,1})$. We compute that $a_{1,1} = \alpha^{-1} + \tau(\alpha^{-1}) = (\zeta_7^4 + \zeta_7^3)/(1 + \zeta_7 + \zeta_7^6)$, $a_{1,2} = -\zeta_7^4/(1 + \zeta_7 + \zeta_7^6)$, and $a_{2,1} = -\zeta_7^3/(1 + \zeta_7 + \zeta_7^6)$. Thus, $\langle x, y\rangle = \mathrm{tr}_{F/\mathbb{Q}}((x_1, x_2)A(\bar{y}_1, \bar{y}_2)^T)$. □

7. Applications for composite *m*

We illustrate how to extend Theorem 5.5 by computing the integral PEL datum in some cases when m is composite. In particular, we determine the integral PEL datum for 6 of the Moonen special families where $m = 4, 6, 10$.

Remark 7.1. Our techniques do not apply well for symplectic Shimura varieties, so we exclude the modular curve $M[1]$ with $m = 2$, the Picard surface $M[2]$ with $m = 2$, the family $M[7]$ with $m = 4$, and $M[12]$ with $m = 6$. We exclude $M[15]$ with $m = 8$ and $M[20]$ with $m = 12$, since the CM-types for biquadratic CM-fields are never simple. Our techniques are not sufficient to handle $M[13]$ with $m = 6$ or $M[19]$ with $m = 9$.

7.1. ***Difficulties when m is composite.*** The key to Theorem 5.5 is the existence of a distinguished point P in Z_γ satisfying the properties in Proposition 5.2. When m is composite, this is not always possible for the following reasons.

Remark 7.2. One issue when m is composite is that the curve C_P might not have compact type. For example, the family $M[12]$, with $\gamma = (6, 4, (1, 1, 1, 3))$, has no admissible degenerations of compact type. The reason is that the two covers with inertia types $a_1 = (1, 1, 4)$ and $a_2 = (2, 1, 3)$ would need to be joined at two points, leading to a cycle in the dual graph of C_P.[3] The same is true for $M[7]$, with $\gamma = (4, 4, (1, 1, 1, 1))$.

Remark 7.3. Another issue when m is composite is that the integral group algebra $\mathbb{Z}[\mu_m]$ has nontrivial index in $\prod_{1\le d|m} \mathbb{Z}[\zeta_d]$. Hence, the Jacobian of a μ_m-cover of $\mathbb{P}^1$ branched at 3 points might not have complex multiplication by a maximal order. See Section 7.5.

7.2. ***Integral PEL data for two families with m = 4.*** We find the integral PEL datum for two of the three special families with $m = 4$: $M[4]$ and $M[8]$; we exclude $M[7]$ as it is not unitary. When $m = 4$, the hypotheses of Proposition 5.2 are not satisfied but we can find a distinguished point in the family by direct computation.

Example 7.4. If $m = 4$, then $\beta_0 = -2i$ from Lemma 3.7. Set $u_1 = -1$, so $\beta_1 = -\beta_0 = 2i$. If $\mathfrak{f}_1 = (0, 1)$, then β_1 satisfies the 3 conditions of Corollary 4.3 and all CM-types of $F = \mathbb{Q}(i)$ are simple.

Corollary 7.5. *Let $m = 4$ and $F = \mathbb{Q}(i)$. Let $\xi = \beta_1^{-1} = 1/2i$. For the family M with monodromy datum γ as in the next table* (top of page 206), *the integral PEL datum for S_γ has lattice $(\mathcal{O}_F)^r$ with Hermitian form H_B, where r and $B \in \mathrm{GL}_r(\mathcal{O}_F)$*

[3]There is a typo in this case in [8, Lemma 6.4].

are as follows:

M	$\gamma=(m,N,a)$	r	$B\in \mathrm{GL}_r(\mathcal{O}_F)$
$M[4]$	$(4,4,(1,2,2,3))$	2	$\mathrm{diag}[\xi,-\xi]$
$M[8]$	$(4,5,(1,1,2,2,2))$	3	$\mathrm{diag}[\xi,-\xi,\xi]$

Proof. When $m=4$, then $a=(3,2,3)$ has signature $\mathfrak{f}=(0,1)$ and $\sigma_2(\beta_1)<0$. Also $a=(1,2,1)$ has signature $\mathfrak{f}=(1,0)$ and $\sigma_1(-\beta_1)<0$. By [8, Lemma 6.4], there is a distinguished point in the family $M[4]$ and the family $M[8]$. The family $M[4]$ has an admissible degeneration, expressed in short as $(1,2,1)+(3,2,3)$. The family $M[8]$ has an admissible degeneration, expressed as $(1,2,1)+(3,2,3)+(1,2,1)$. The result then follows from Theorem 5.5. □

The case $M[8]:\xi[1,-1,1]$ matches line 3 of the table on [16, page 1].

7.3. *Remark when m = 9.* The techniques in this paper are not sufficient to find the integral PEL datum for the special family $M[19]$ when $m=9$. This family has inertia type $a=(3,5,5,5)$ and its only admissible degeneration can be expressed in short hand as $(3,5,1)+(8,5,5)$. For $a_2=(8,5,5)$, the Jacobian has complex multiplication by $\mathbb{Q}[\zeta_3]\times\mathbb{Q}[\zeta_9]$. As the action of $\mathbb{Z}[\mu_9]$ does not extend to the maximal order $\mathbb{Z}[\zeta_3]\times\mathbb{Z}[\zeta_9]$, we do not know how to compute its lattice.

We expect that our technique is sufficient to determine the integral PEL datum when $m=9$, $N=4$ and $a=(3,1,8,6)$. We leave the details to the reader to check.

7.4. *Generalities for twice a prime.*

Notation 7.6. Let $m=2m'$ where m' is an odd prime. Let $F=\mathbb{Q}(\zeta_m)$ and $F'=\mathbb{Q}(\zeta_{m'})$. We identify $F=F'$ and $\mathbb{Z}[\zeta_m]=\mathbb{Z}[\zeta_{m'}]$, by $\zeta_m=-\zeta_{m'}^{(m'+1)/2}$. There is a bijection between CM types Φ for F and CM types Φ' for F' where $\sigma_j\in\Phi$ for an odd integer $1\le j\le m$ if and only if $\sigma_{j'}\in\Phi'$, where j' is the reduction of j modulo m'.

If β satisfies Corollary 4.7 with respect to the CM type $(\mathbb{Z}[\zeta_{m'}],\Phi')$ then it also satisfies Corollary 4.7 with respect to the CM type $(\mathbb{Z}[\zeta_m],\Phi)$. This can be verified in general but we only need a few cases from Example 6.1 when $m'=3$ and Example 6.3 when $m'=5$.

Example 7.7.

m	β	$(\mathbb{Z}[\zeta_{m'}],\Phi')$	$(\mathbb{Z}[\zeta_m],\Phi)$
6	$b_2=\sqrt{-3}$	$(\mathbb{Z}[\zeta_3],\{2\})$	$(\mathbb{Z}[\zeta_6],\{5\})$
10	$\beta_1=5/(\zeta_5^3-\zeta_5^2)$	$(\mathbb{Z}[\zeta_5],\{2,4\})$	$(\mathbb{Z}[\zeta_{10}],\{7,9\})$
10	$\beta_2=5/(\zeta_5-\zeta_5^4)$	$(\mathbb{Z}[\zeta_5],\{1,2\})$	$(\mathbb{Z}[\zeta_{10}],\{1,7\})$

7.5. ***Integral PEL data for 4 families with m = 6, 10.*** We find the integral PEL datum for three special families with $m = 6$, namely $M[5]$, $M[9]$ and $M[14]$, and for the special family with $m = 10$, namely $M[18]$. For $m = 6$, we exclude $M[12]$ as it is not unitary and $M[13]$ (Remark 7.10). For m composite, the hypotheses of Proposition 5.2 are not satisfied. In each case, we find a distinguished point P in the family for which the issues in Remarks 7.2 and 7.3 do not occur.

Corollary 7.8. (Recall Notation 7.6.) *For the special family* $M[18]$ *with monodromy datum* $\gamma = (10, 4, (3, 5, 6, 6))$, *the integral PEL datum of* S_γ *has lattice*

$$\Lambda = (\mathcal{O}_{F'}) \oplus (\mathcal{O}_F)^2$$

with Hermitian form H_B, *where* $B \in \mathrm{GL}_1(\mathcal{O}_{F'}) \times \mathrm{GL}_2(\mathcal{O}_F)$ *is* $\mathrm{diag}[\xi_2] \oplus \mathrm{diag}[\xi_1, \xi_2]$ *with* $\xi_i = \beta_i^{-1}$.

Proof. Let $m = 10$ and $m' = 5$. Let $M = M[18]$ with $\gamma = (10, 4, (3, 5, 6, 6))$ and signature type $\mathfrak{f} = (1, 1, 0, 1, 0, 0, 2, 0, 1)$. The proof is similar to that of Theorem 5.5. We produce a simple distinguished point P in the family similar to that in Proposition 5.2(1); it represents an admissible μ_m-cover $\psi : C_P \to T$, where T is a tree of projective lines and C_P is a curve of compact type such that each of its irreducible components is a curve admitting a μ_m-cover of $\mathbb{P}^1$ branched at 3 points. We verify by direct computation that each irreducible component of C_P has complex multiplication by either $\mathbb{Z}[\zeta_m]$ or $\mathbb{Z}[\zeta_{m'}]$.

Consider the μ_5-cover $\psi_2 : C_2 \to \mathbb{P}^1$ branched at three points, with inertia type $a_2 = (4, 3, 3)$, and signature $\mathfrak{f}_2 = (0, 1, 0, 1)$. Then $A_2 = \mathrm{Jac}(C_2)$ has complex multiplication by $(\mathbb{Z}(\zeta_5), \{2, 4\})$. Consider the induced curve $\tilde{C}_2 = \mathrm{Ind}_5^{10}(C_2)$, which is the disconnected curve consisting of two copies of C_2, and the induced μ_{10}-cover $\Psi_2 : \tilde{C}_2 \to \mathbb{P}^1$. Then Ψ_2 is branched at three points and, somewhat imprecisely, we can say that it has inertia type $(8, 6, 6) = \mathrm{Ind}_5^{10}(4, 3, 3)$. Above the first branch point, there are two points η_2 and η_2' on $\tilde{C}_2$, and they are labeled by the two cosets of $\mu_5 \subset \mu_{10}$. Let $\mathcal{A}_2 = \mathrm{Jac}(\tilde{C}_2) \simeq A_2^2$.

As explained in [10, Section 3.1], the signature type of $\mathcal{A}_2$ is

$$\mathfrak{f}_2 = (0, 1, 0, 1, 0, 0, 1, 0, 1) = (0, 1, 0, 1, 0, 0, 0, 0, 0) + (0, 0, 0, 0, 0, 0, 1, 0, 1).$$

The action of $\mathbb{Z}[\mu_{10}]$ on $\mathcal{A}_2$ is the diagonal action of $\mathbb{Z}[\zeta_5] \times \mathbb{Z}[\zeta_{10}]$ on A^2. The first (resp. second) copy of A_2 has CM-type $(\mathbb{Z}[\zeta_5], \{2, 4\})$ (resp. $(\mathbb{Z}[\zeta_{10}], \{7, 9\})$); for this CM-type, the element β_1 (resp. β_1) satisfies Corollary 4.7, as seen in Example 7.7.

Consider the μ_{10}-cover $\psi_1 : C_1 \to \mathbb{P}^1$ branched at three points, with inertia type $a_1 = (3, 5, 2)$. Above the 3rd branch point, there are two points η_1 and η_1' on C_1 and they are labeled by the two cosets of $\mu_5 \subset \mu_{10}$. Let $A_1 = \mathrm{Jac}(C_1)$; it has signature type $\mathfrak{f}_1 = (1, 0, 0, 0, 0, 0, 1, 0, 0)$. By Lemma 2.3, A_1 has complex multiplication by $(\mathbb{Z}[\zeta_{10}], \{1, 7\})$. Since $\{1, 7\} = \{3 \cdot 7, 3 \cdot 9\}$ mod 10, the element β that satisfies Corollary 4.7 is $\sigma_7(\beta_1) = \beta_2$. All CM types of $\mathbb{Q}(\zeta_5) = \mathbb{Q}(\zeta_{10})$ are simple.

Let $\mathcal{C}$ denote the singular curve, whose components are C_1 and $\tilde{C}_2$, formed by identifying η_1 η_1' with η_2 and η_2', so that the cosets are matched correctly, in two ordinary double points. The curve $\mathcal{C}$ admits an admissible μ_{10}-cover Ψ to a chain of two projective lines. By construction, $\mathcal{C}$ has an action by $\mathcal{O}_{F'} \oplus \mathcal{O}_F^2$, with CM-type given by $\mathrm{diag}[\xi_2] \oplus \mathrm{diag}[\xi_1, \xi_2]$.

The last thing to check is that Ψ is represented by a point $P \in Z_\gamma$. Since Ψ is admissible, it can be deformed to a μ_{10}-cover of $\mathbb{P}^1$ with inertia type $(3, 5, 6, 6)$. This has signature type $\mathfrak{f} = (1, 1, 0, 1, 0, 0, 2, 0, 1)$. Hence, Ψ is represented by a point $P \in Z_\gamma$ and by the preceding paragraph P is a simple distinguished point. □

Corollary 7.9. (Recall Notation 7.6.) *Let $\xi = 1/\sqrt{-3}$. For the special families M with monodromy datum $\gamma = (6, N, a)$ as below, the integral PEL datum of S_γ has lattice $\Lambda = (\mathcal{O}_{F'})^{r'} \oplus (\mathcal{O}_F)^r$ with Hermitian form H_B where r', r, and B are as follows*:

M	$\gamma = (m, N, a)$	(r', r)	$B \in \mathrm{GL}_{r'}(\mathcal{O}_{F'}) \times \mathrm{GL}_r(\mathcal{O}_F)$
$M[5]$	$(6, 4, (2, 3, 3, 4))$	$(0, 2)$	$\mathrm{diag}[\xi, -\xi]$
$M[9]$	$(6, 4, (1, 3, 4, 4))$	$(1, 2)$	$\mathrm{diag}[-\xi] \oplus \mathrm{diag}[\xi, -\xi]$
$M[14]$	$(6, 5, (2, 2, 2, 3, 3))$	$(1, 3)$	$\mathrm{diag}[\xi] \oplus \mathrm{diag}[\xi, -\xi, \xi]$

Proof. The proof is very similar to that of Corollary 7.8 so we provide only a sketch.

(1) Let $M = M[5]$ with $\gamma = (6, 4, (2, 3, 3, 4))$ and $\mathfrak{f} = (1, 0, 0, 0, 1)$. Then C_P is the join of two μ_6-covers with inertia types $a_1 = (1, 2, 3)$ and $a_2 = (3, 4, 5)$. These have signatures $\mathfrak{f}_1 = (1, 0, 0, 0, 0)$ and $\mathfrak{f}_2 = (0, 0, 0, 0, 1)$. By Lemma 2.3, A_1 has CM by $\mathcal{O}_F$ of type $\Phi_1 = \{1\}$, and A_2 has CM by $\mathcal{O}_F$ of type $\Phi_2 = \{5\}$. Let $b_1 = -b_2 = -\sqrt{-3}$. For $i = 1, 2$, the CM-type Φ_i is simple, and the element $b_i \in \mathcal{O}_F$ satisfies Corollary 4.7 with respect to Φ_i.

(2) Let $M = M[9]$ with $\gamma = (6, 4, (1, 3, 4, 4))$ and $\mathfrak{f} = (1, 1, 0, 0, 1)$. Then C_P is the join of two covers with inertia types $a_1 = (1, 2, 3)$ and $a_2 = \mathrm{Ind}_3^6(2, 2, 2)$. These have signatures $\mathfrak{f}_1 = (1, 0, 0, 0, 0)$ and $\mathfrak{f}_2 = (0, 1, 0, 0, 1)$. By Lemma 2.3, $A_1 = \mathrm{Jac}(C_1)$ has CM-type $(\mathcal{O}_F, \{1\})$, which has $\beta = b_1 = -b_2$. Also $\mathcal{A}_2 \simeq A^2$ and the action of $\mathbb{Z}[\mu_6]$ on $\mathcal{A}_2$ is given by the diagonal action of $\mathbb{Z}[\zeta_3] \times \mathbb{Z}[\zeta_6]$ on A^2. The first copy of A has CM-type $(\mathbb{Z}[\zeta_3], \{2\})$ which has $\beta = b_2$ and the second copy of A has CM-type $(\mathbb{Z}[\zeta_6], \{5\})$ which has $\beta = b_2$.

(3) Let $M = M[14]$ with $\gamma = (6, 5, (2, 2, 2, 3, 3))$ and $\mathfrak{f} = (2, 0, 0, 1, 1)$. Then C_P is the join of three covers with inertia types $a_1 = (1, 2, 3)$, $a_2 = (3, 4, 5)$ and $a_3 = \mathrm{Ind}_3^6(1, 1, 1)$. These have signatures $\mathfrak{f}_1 = (1, 0, 0, 0, 0)$, $\mathfrak{f}_2 = (0, 0, 0, 0, 1)$, and $\mathfrak{f}_3 = (1, 0, 0, 1, 0)$. By Lemma 2.3, A_1 has CM-type $(\mathcal{O}_F, \{1\})$ which has $\beta = b_1$

and A_2 has CM-type $(\mathcal{O}_F, \{5\})$ which has $\beta = b_2$. Also $\mathcal{A}_3 \simeq A^2$ and the action of $\mathbb{Z}[\mu_6]$ on $\mathcal{A}_3$ is given by the diagonal action of $\mathbb{Z}[\zeta_3] \times \mathbb{Z}[\zeta_6]$ on A^2. The first copy of A has CM-type $(\mathbb{Z}[\zeta_3], \{1\})$ which has $\beta = b_1$ and the second copy of A has CM-type $(\mathbb{Z}[\zeta_6], \{4\})$ which has $\beta = b_1$. □

We checked Corollaries 7.8 and 7.9 independently, using the fact that the special families for $m = 6, 10$ are subspaces of those for $m' = 3, 5$, up to a Galois twist.

Remark 7.10. Let $M = M[13]$ be the special family with $\gamma = (6, 4, (1, 1, 2, 2))$ and $\mathfrak{f} = (2, 1, 0, 1, 0)$. This has an admissible degeneration to the join of two covers with inertia types $a_1 = (1, 1, 4)$ and $a_2 = \mathrm{Ind}_3^6(1, 1, 1)$. The first one has signature $\mathfrak{f}_1 = (1, 1, 0, 0, 0)$ and its Jacobian A_1 has CM by an order of finite index in $\mathcal{O}_{F'} \times \mathcal{O}_F$. The index is nontrivial since $\langle x^3 - 1, x^3 + 1\rangle = \langle 2\rangle$ in $\mathbb{Z}[x]/\langle x^6 - 1\rangle$, so our technique does not apply. The other degeneration to $a_1 = (1, 2, 3)$ and $a_2 = (3, 1, 2)$ does not have compact type.

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THE DERIVED SERIES OF GGS GROUPS

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We describe the structure and indices of the members of the derived series of a GGS group G defined over the p-adic tree. The values $|G : G^{(n)}|$ exhibit a very limited dependence on the defining vector of the group. Furthermore, we establish that the derived and Frattini series of a GGS group defined by a nonconstant vector are identical.

1. Introduction and statement of results

Groups of automorphisms of regular rooted trees provide examples with intriguing asymptotic and structural properties. One particularly well-studied case is the family of Grigorchuk–Gupta–Sidki groups (usually abbreviated as GGS groups), generalising the second Grigorchuk group and the Gupta–Sidki p-groups. This family encompasses at least one group of intermediate word growth, as shown in [6], and numerous finitely generated infinite periodic groups, as demonstrated in [10]. The GGS groups acting on the p-adic tree, where p denotes an odd prime, are best understood; hence, in the following, by a GGS group we shall mean specifically a GGS group acting on the p-adic tree.

GGS groups are defined by a nonzero element $\boldsymbol{e}$ of $\mathbb{F}_p^{p-1}$ as "input data". It is fortunate that many properties of interest are satisfied by the group defined by $\boldsymbol{e}$ if and only if $\boldsymbol{e}$ satisfies certain linear conditions. For instance, a GGS group is periodic and just-infinite if and only if the sum of the entries of $\boldsymbol{e}$ is zero [9; 17]. Similarly, it is a branch group (with the congruence subgroup property) if and only if not all entries of $\boldsymbol{e}$ are equal [7; 8]. Its Hausdorff dimension is governed by a function depending only on certain linear invariants of $\boldsymbol{e}$ [7]. Some of these results naturally extend to larger classes of groups [1; 2; 11; 14], but have been established first for GGS groups, making the class of GGS groups a fertile soil for establishing new techniques.

In this article, we provide a description of the derived series — i.e., the subgroups defined by $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for all $n \in \mathbb{N}$ — for every GGS

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group G. Conventionally, we write G' and G'' for $G^{(1)}$ and $G^{(2)}$, respectively. Our description is formulated in terms of linear conditions on the defining vector $\boldsymbol{e}$.

A description of the derived series was obtained by Vieira [16] for the special case of the Gupta–Sidki 3-group, along with some results on its lower central series. However, the proof in [16] does not readily generalise to general GGS groups. We approach the problem by adopting methods developed by Fernández-Alcober and Zugadi-Reizabal in [7]. We note that the sequence of indices of the members of the derived series for the (first) Grigorchuk group was computed by Grigorchuk in [9].

Our main result is the following.

Theorem 1.1. *Let p be an odd prime and let G be a GGS group acting on the p-adic tree defined by a nonconstant vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$. Let $\boldsymbol{e}'$ be the tuple of differences between the entries of $\boldsymbol{e}$ and let $\boldsymbol{e}''$ be the tuple of differences of $\boldsymbol{e}'$. Then, for $n \geq 2$,*

$$\log_p |G : G^{(n)}| = p^{n-2}(p + \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'')) - \frac{p^{n-1}-1}{p-1} \cdot \mathrm{sym}(\boldsymbol{e}) + 1.$$

Here, a vector $\boldsymbol{d}$ is called symmetric if its entries are the same when read from left to right or right to left, and $\mathrm{sym}(\boldsymbol{d})$ equals 1 if the vector $\boldsymbol{d}$ is symmetric and 0 otherwise. Similarly, $\mathrm{con}(\boldsymbol{d})$ equals 1 if the vector $\boldsymbol{d}$ is constant, i.e., if all its entries are equal, and 0 otherwise. In [8, Proposition 3.4], Fernández-Alcober, Garrido and Uria-Albizuri show that GGS groups with constant defining vector admit an infinite metabelian quotient, i.e., that $|G : G''| = \infty$. It is an elementary fact that $|G : G'| = p^2$, so Theorem 1.1 completes the description of all indices $|G : G^{(n)}|$ for all GGS groups G and all integers $n \in \mathbb{N}$.

To arrive at Theorem 1.1, we need a description of the derived subgroups:

Theorem 1.2. *Let p be an odd prime and let G be a GGS group acting on the p-adic tree defined by a nonconstant vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$. Let $n \geq 3$. Then*

$$\psi(G^{(n)}) = \mathsf{X}_p\, G^{(n-1)}.$$

If $\mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'') = 0$, *the same holds for* $n = 2$.

Together with a description of the second derived subgroup, which we present in Proposition 3.2, this theorem yields a good account of the structure of the derived subgroups of G. One notable consequence is that the quotients $G^{(n)}/G^{(n+1)}$ are elementary abelian p-groups, which immediately yields the following corollary.

Corollary 1.3. *Let G be a GGS group defined by a nonconstant vector. The Frattini series of G coincides with the derived series of G.*

Finally, we investigate the GGS groups defined by nonconstant vectors with the maximal possible indices $|G : G^{(n)}|$. We find that there are precisely two isomorphism classes of GGS groups with $|G : G^{(n)}|$ maximal among GGS groups defined by nonconstant vectors (acting on a fixed p-adic tree); see Proposition 3.5.

2. On Grigorchuk–Gupta–Sidki-groups

Notation. The letter p refers to a fixed odd prime. Given integers m and n subject to $m \le n$, the set $\{m, m+1, \dots, n-1, n\}$ is denoted by $[m, n]$. The symbol X refers to the set $[0, p-1]$ underlying the field $\mathbb{F}_p$. For a group G, the p-fold direct product $G \times \cdots \times G$ is denoted by $\mathsf{X}_p\, G$. For $k \in \mathbb{N}$ distinct elements $i_0, \dots, i_{k-1} \in X$ and p elements $g_0, \dots, g_{p-1}$ of a group G, the expression

$$(i_0 : g_0,\ \dots,\ i_{k-1} : g_{k-1},\ \diamond : g_\diamond) \in \mathsf{X}_p\, G$$

denotes the tuple indexed by X, with g_j at position i_j for $j \in [0, k-1]$, and with $g_\diamond$ (possibly varying with $\diamond$) at every other position $\diamond \in X \setminus \{i_j \mid j \in [0, k-1]\}$. The symbol $\diamond$ is reserved for this use.

Given a group G and two of its elements g and h, we use the following conventions for conjugation and the commutator:

$$g^h = h^{-1}gh \quad \text{and} \quad [g, h] = g^{-1}h^{-1}gh = g^{-1}g^h.$$

Automorphisms of rooted trees. The Cayley graph X^* of the free monoid on X is the rooted p-adic tree, i.e., a loop-free graph with a distinguished vertex (the "root") $\varnothing$ of valency p and all other vertices of valency $p+1$. We write X^n for the set of all vertices of a given (geodesic) distance n to $\varnothing$. This set is called the *n-th level of X^**.

Any (graph) automorphism $g \in \mathrm{Aut}(X^*)$ necessarily fixes $\varnothing$ for its unique valency, and must consequently leave all levels X^n invariant. We write $\mathrm{St}(n)$ for the (pointwise) stabiliser of X^n, and $\mathrm{St}_G(n)$ for its intersection with a subgroup $G \le \mathrm{Aut}(X^*)$.

Let u and v be vertices of X^*. We write u^g for the image of u under g. Since every level is invariant under g, the equation

$$(uv)^g = u^g v^{g|_u}$$

uniquely defines a map $|_u\colon \mathrm{Aut}(X^*) \to \mathrm{Aut}(X^*)$, called the *section map at u*. The image of g is called *the section of g at u*. For $u \in X^*$, the restriction of $|_u$ to the stabiliser $\mathrm{st}(u)$ of u is a group homomorphism, indeed, the map

$$\psi\colon\ \mathrm{St}(1) \to \mathsf{X}_p\, \mathrm{Aut}(X^*), \qquad g \mapsto (\diamond : g|_\diamond)$$

is an isomorphism. We record some equations for sections. Let u and v be vertices of X^* and g and h be automorphisms, then

$$(g|_u)|_v = g|_{uv}, \quad (gh)|_u = g|_u h|_{u^g}, \quad g^{-1}|_u = (g|_{u^{g^{-1}}})^{-1}.$$

The action of an automorphism g on the first level $X = X^1$ is denoted $g|^{\varnothing} \in \mathrm{Sym}(X)$. An element such that $g|_u = \mathrm{id}$ for all $u \neq \varnothing$ is called *rooted*. We identify rooted elements with the their images under $|^{\varnothing}$. Let $u \in X^n$ and $g \in \mathrm{Aut}(X^*)$. The unique element $h \in \mathrm{St}(n)$ satisfying $h|_u = g$ and $h|_v = \mathrm{id}$ for all $v \in X^n \smallsetminus \{u\}$ is denoted $\mathrm{ins}_u(g)$ and called the *insertion of g at u*.

A group $G \leq \mathrm{Aut}(X^*)$ is called *self-similar*, if all sections $g|_u$ are contained in G for all $u \in X^*$ and $g \in G$. A group $G \leq \mathrm{Aut}(X^*)$ is called *fractal* if $\mathrm{st}_G(x)|_x = G$ for every $x \in X$. A group $G \leq \mathrm{Aut}(X^*)$ is called *spherically transitive* if it acts transitively on every level X^n. A self-similar group $G \leq \mathrm{Aut}(X^*)$ is called a *regular branch group* if it is spherically transitive and if there is a finite-index subgroup $K \leq \mathrm{St}_G(1)$ such that $\mathsf{X}_p\, K \leq \psi(K)$. A standard technique to establish that a group is regular branch is the following:

Proposition 2.1 [7, Proposition 2.18]. *Let $G \leq \mathrm{Aut}(X^*)$ be a spherically transitive fractal group, let $H \leq G$ be a subgroup and let $S \subseteq G$ be a subset. If $\mathrm{ins}_0(S) \subseteq H$, then $\mathsf{X}_p\, \langle S\rangle^G \leq \psi(H^G)$.*

GGS groups and their defining vectors. Let $\boldsymbol{e} = (e_1, \dots, e_{p-1}) \in \mathbb{F}_p^{p-1}$ be a nonzero vector. The group G generated by the rooted automorphism $a = (0\ 1\ \dots\ p-1)$ induced by the addition of 1 in $\mathbb{F}_p$, and the automorphism defined by

$$b = \psi^{-1}(0 : b,\ \diamond : a^{e_\diamond})$$

is called the *GGS group defined by* $\boldsymbol{e}$. The vector $\boldsymbol{e}$ is the *defining vector of G*.

More generally, GGS groups acting on the (not necessarily prime) m-adic tree are defined in the same way, but using elements of $(\mathbb{Z}/m\mathbb{Z})^{m-1}$ whose entries have no common divisor other than 1. In general, the structure of these groups is much less understood than the case considered here. Even for prime powers $m = p^n$, the situation is much more involved, see for example [5], where the branching structures for these groups have been evaluated. Various further generalisations of GGS groups have been studied, e.g. in [1; 3; 12; 14].

Distinct defining vectors may give rise to identical or isomorphic GGS groups. Nonzero multiples of a given vector $\boldsymbol{e}$ define the same subgroup of $\mathrm{Aut}(X^*)$. Apart from that, certain reorderings of a given defining vector give rise to isomorphic (but not necessarily identical) GGS groups. We make use of the following characterisation. The group $\mathbb{F}_p^\times \times \mathbb{F}_p^\times \cong \mathrm{C}_{p-1}^2$ acts on the set $\mathbb{F}_p^{p-1} \smallsetminus \{\mathbf{0}\}$ of defining vectors by

$$(e_1, \dots, e_{p-1}) * (\lambda, \mu) = (\lambda \cdot e_\mu, \lambda \cdot e_{2\cdot\mu}, \dots, \lambda \cdot e_{(p-2)\cdot\mu}, \lambda \cdot e_{(p-1)\cdot\mu}). \tag{2-1}$$

Theorem 2.2 [13, Corollary 4.5]. *Two GGS groups G and $\tilde{G}$ defined by $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$, respectively, are isomorphic if and only if $\boldsymbol{e}$ and $\tilde{\boldsymbol{e}}$ share the same orbit under the action of $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$.*

This allows us to choose defining vectors with desirable properties, as also done in [7, Theorem 2.16].

Corollary 2.3. *Let G be the GGS group defined by $\boldsymbol{e}$ and let $\alpha \in \mathbb{F}_p^{\times}$.*

(i) *There exists a defining vector $\tilde{\boldsymbol{e}}$ with $\tilde{e}_1 = \alpha$ such that the GGS group defined by $\tilde{\boldsymbol{e}}$ is isomorphic to G.*

(ii) *If $\boldsymbol{e}$ is not constant, there exists a defining vector $\tilde{\boldsymbol{e}}$ with $e_i - e_{i+1} = \alpha$, for some $i \in \{1, \dots, p-2\}$, such that the GGS group defined by $\tilde{\boldsymbol{e}}$ is isomorphic to G.*

Difference vectors. Let $\boldsymbol{d} = (d_1, \dots, d_n) \in \mathbb{F}_p^n$ be a vector. The *difference vector of $\boldsymbol{d}$* is the vector

$$\boldsymbol{d}' = (d'_1, d'_2, \dots, d'_{n-1}) = (d_1 - d_2, d_2 - d_3, \dots, d_{n-1} - d_n) \in \mathbb{F}_p^{n-1}.$$

A vector $\boldsymbol{d}$ is called *constant* if $\boldsymbol{d}' = \mathbf{0}$, i.e., if all entries of $\boldsymbol{d}$ are equal. We put $\mathrm{con}(\boldsymbol{d}) = 1$ if $\boldsymbol{d}$ is constant, and $\mathrm{con}(\boldsymbol{d}) = 0$ otherwise.

The structure of the derived subgroups of the GGS group defined by $\boldsymbol{e}$ is influenced by the difference vector $\boldsymbol{e}'$ and by its difference vector $\boldsymbol{e}''$, called, for convenience, the *second difference vector* of $\boldsymbol{e}$. The significance of $\boldsymbol{e}'$ is suggested by the following computation. Every GGS group G is two-generated and its derived subgroup is normally generated by the commutator $c = [a^{-1}, b]$, whose section decomposition is closely related to $\boldsymbol{e}'$,

$$\begin{aligned} \text{(2-2)} \qquad \psi(c) = \psi([a^{-1}, b]) &= \psi(b^{a^{-1}})^{-1}\psi(b) \\ &= (\diamond : a^{-e_{\diamond+1}},\ p-1 : b^{-1})(0 : b,\ \diamond : a^{e_\diamond}) \\ &= (0 : a^{-e_1}b,\ \diamond : a^{e'_\diamond},\ p-1 : b^{-1}a^{e_{p-1}}). \end{aligned}$$

The entries of $\boldsymbol{e}''$ appear in a similar way in the section decomposition of $[a^{-1}, c]$.

Perhaps surprisingly, the higher difference vectors of $\boldsymbol{e}$ do not affect the structure of the derived series.

Symmetric vectors. Let $n \in \mathbb{N}$ be even, let K be a field not of characteristic 2, and let $\boldsymbol{d} = (d_1, \dots, d_n) \in K^n$ be a vector. It is called *symmetric* if

$$d_i = d_{n-i+1}$$

for all $i \in [1, n/2]$. We put $\mathrm{sym}(\boldsymbol{d}) = 1$ if $\boldsymbol{d}$ is symmetric, and $\mathrm{sym}(\boldsymbol{d}) = 0$ otherwise. Evidently, the set of all symmetric vectors constitutes a subspace S of K^n, being subject to the conditions $d_i - d_{n-i+1} = 0$ for $i \in [1, n/2]$. If $\boldsymbol{d}$ is symmetric, the second difference vector $\boldsymbol{d}''$ is also symmetric; see Table 1 for an overview of the possible configurations of the values $\mathrm{con}(\boldsymbol{e})$, $\mathrm{sym}(\boldsymbol{e})$, $\mathrm{con}(\boldsymbol{e}')$ and $\mathrm{sym}(\boldsymbol{e}'')$ for a defining vector. More precisely, the second difference vector is symmetric if and only if

$$d_i - 2d_{i+1} + d_{i+2} = d''_i = d''_{n-i-1} = d_{n-1-i} - 2d_{n-i} + d_{n-i+1},$$

i.e., if and only if $d''_i - d''_{n-i-1} = 0$ for all $i \in [1, n/2 - 1]$. It is apparent that

con($\boldsymbol{e}$)	sym($\boldsymbol{e}$)	con($\boldsymbol{e}'$)	sym($\boldsymbol{e}''$)
0	0	0	0
0	0	0	1
0	0	1	1
0	1	0	1
1	1	1	1

Table 1. Possible configurations of values for the invariants of $\boldsymbol{e}$ influencing the indices of the members of the derived series of the corresponding GGS group. For $p = 3$, the first, second and fourth rows are nonexistent.

the set of all vectors such that $\boldsymbol{d}''$ is symmetric is a subspace containing the space of symmetric vectors as a subspace of codimension 1. We need the following computational lemma.

Lemma 2.4. *Let* $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$ *be such that* $\operatorname{sym}(\boldsymbol{e}'') = 1$. *Then*

$$2(e_{p-1} - e_1) + (e_2 - e_{p-2}) = 0.$$

Proof. Write $s_i = e_i - e_{p-i}$ for $i \in [1, p-1]$. Note that

$$s'_i = s_i - s_{i+1} = e_i - e_{p-i} - e_{i+1} + e_{p-i-1} = e'_i - e'_{p-i-1},$$

and in the same way, $s''_i = e''_i - e''_{p-i-2}$. Moreover, $s_i = -s_{p-i}$, whence $s'_{(p-1)/2} = s_{(p-1)/2} - s_{(p+1)/2} = 2 \cdot s_{(p-1)/2}$. Two simple telescope sum computations yield

$$\sum_{i=1}^{(p-3)/2} s''_i = \sum_{i=1}^{(p-3)/2} (s'_i - s'_{i+1}) = s'_1 - s'_{(p-1)/2} = s_1 - s_2 - 2 \cdot s_{(p-1)/2}$$

and

$$\begin{aligned}\sum_{i=1}^{(p-3)/2} i \cdot s''_i = \sum_{i=1}^{(p-3)/2} i \cdot (s'_i - s'_{i+1}) &= \sum_{i=1}^{(p-3)/2} s'_i - ((p-3)/2) \cdot s'_{(p-1)/2} \\ &= s_1 - s_{(p-1)/2} + 3 \cdot s_{(p-1)/2} \\ &= s_1 + 2 \cdot s_{(p-1)/2}.\end{aligned}$$

Combining them, one finds that

$$-\sum_{i=1}^{(p-3)/2} (i+1) \cdot s''_i = -2s_1 + s_2.$$

Since $\boldsymbol{e}''$ is symmetric, $s''_i = 0$ for all $i \in [1, p-3]$. Thus the left-hand side of the equality above is zero, while the right-hand side evaluates to the desired expression $2(e_{p-1} - e_1) + (e_2 - e_{p-2})$. □

Circulant spaces. Let K be a field and let $\boldsymbol{d} = (d_0, \dots, d_{\ell-1}) \in K^\ell$ be a vector. The *circulant matrix* $\mathrm{Circ}(\boldsymbol{d})$ associated to $\boldsymbol{d}$ is the matrix whose rows are the cyclic shifts of $\boldsymbol{d}$, i.e.,

$$\begin{pmatrix} d_0 & d_1 & \dots & d_{\ell-2} & d_{\ell-1} \\ d_{\ell-1} & d_0 & \dots & d_{\ell-3} & d_{\ell-2} \\ \vdots & \vdots & & \vdots & \vdots \\ d_1 & d_2 & \dots & d_{\ell-1} & d_0 \end{pmatrix}.$$

The *semicirculant matrix* $\mathrm{SCirc}(\boldsymbol{d})$ associated to $\boldsymbol{d}$ is the upper triangular matrix given by

$$\begin{pmatrix} d_0 & d_1 & \dots & d_{\ell-2} & d_{\ell-1} \\ 0 & d_0 & \dots & d_{\ell-3} & d_{\ell-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & d_0 \end{pmatrix}.$$

More generally, a *circulant subspace of* K^ℓ is a subspace that is invariant under the automorphism induced by the cyclic permutation of the standard basis elements of K^ℓ. Given a subset $M \subseteq K^\ell$, we write $\mathrm{Circ}(M)$ for the minimal circulant subspace containing M. In case of a singleton set, $\mathrm{Circ}(\{\boldsymbol{d}\})$ is spanned by the rows of the circulant matrix $\mathrm{Circ}(\boldsymbol{d})$.

Ranks of circulant matrices, their computation and interpretation have long been studied; see for example [4] for the situation over the field of complex numbers. In positive characteristic p, the special case $\ell = p^n$ allows an easy description of the circulant subspaces of K^ℓ. Let $\Pi \in K^{\ell \times \ell}$ be the Pascal matrix with entries $\binom{i}{j}$ for $i, j \in [0, \ell-1]$, using the convention that $\binom{i}{j} = 0$ for $i < j$, and write Π_i for the ℓ-by-i matrix consisting of the first i columns of Π. Note that Π is a lower unitriangular matrix; in particular, it is invertible.

Proposition 2.5. *Let K be a field of characteristic $p > 0$ and let $\ell = p^n$ be a power of the characteristic. There exists a complete flag*

$$\{\mathbf{0}\} = \mathrm{Circ}_0 \subset \mathrm{Circ}_1 \subset \dots \subset \mathrm{Circ}_\ell = K^\ell$$

containing all circulant subspaces of K^ℓ, which are given by

$$\mathrm{Circ}_i = \ker \Pi_{\ell-i} = \{\boldsymbol{d} \in K^\ell \mid \operatorname{rank} \mathrm{Circ}(\boldsymbol{d}) \leq i\}.$$

Proof. Fix a vector $\boldsymbol{d} = (d_0, \dots, d_{\ell-1}) \in K^\ell$. Put

$$f_{\boldsymbol{d}}(X) = d_0 + d_1 X + d_2 X^2 + \dots + d_{\ell-1} X^{\ell-1}$$

and

$$P = \begin{pmatrix} 0_{\ell-1,1} & \mathrm{I}_{\ell-1} \\ 1 & 0_{1,\ell-1} \end{pmatrix},$$

where $0_{n,m}$ and I_n stand for the zero and unit matrices, respectively, of the indicated formats. The matrix P is the permutation matrix associated to the cyclic shift of the basis elements, and

$$\mathrm{Circ}(\boldsymbol{d}) = d_0 I + d_1 P + d_2 P^2 + \cdots + d_{\ell-1} P^{\ell-1} = f_{\boldsymbol{d}}(P).$$

The characteristic polynomial of P is $X^\ell - 1 = (X-1)^\ell$ and splits over K; here we use that ℓ is a power of p. The geometric multiplicity of its unique eigenvalue 1 is 1, whence P is conjugate to the matrix $J_\ell(1)$ consisting of a single Jordan block of eigenvalue 1. Consequently, $\mathrm{Circ}(\boldsymbol{d})$ is conjugate to $f_{\boldsymbol{d}}(J_\ell(1))$. For $k \in \mathbb{N}$, we have $J_\ell(1)^k = \left(\binom{k}{j-i}\right)_{i,j\in[1,\ell]}$, i.e., $J_\ell(1)^k = \mathrm{SCirc}\left(\left(\binom{k}{0}, \binom{k}{1}, \ldots, \binom{k}{\ell-1}\right)\right)$. Thus $f_{\boldsymbol{d}}(J_\ell(1))$ is the semicirculant matrix with respect to the vector $\boldsymbol{b} = (b_0, \ldots, b_{\ell-1})$, with

$$b_i = \sum_{j=0}^{\ell-1} d_j \binom{j}{i}.$$

Then $\boldsymbol{b} = \boldsymbol{d}\Pi$. Since the rank of $\mathrm{SCirc}(\boldsymbol{b})$ is equal to $\ell - \min\{j \in [0, \ell-1] \mid b_j \neq 0\}$, we see that $\operatorname{rank} \mathrm{Circ}(\boldsymbol{d}) \leq i$ if and only if $\boldsymbol{d} \in \ker \Pi_{\ell-i}$, with equality precisely if $\boldsymbol{d} \notin \ker \Pi_{\ell-i+1}$. Since Π is invertible, we see that $\operatorname{rank} \Pi_{\ell-i} = i$. Thus Circ_i is indeed a circulant space, as $\mathrm{Circ}_i = \mathrm{Circ}(\boldsymbol{d})$ for any $\boldsymbol{d} \in \mathrm{Circ}_i \smallsetminus \mathrm{Circ}_{i-1}$. Evidently $\mathrm{Circ}_i \subset \mathrm{Circ}_{i+1}$. It remains to show that there are no further circulant spaces. Let C be a circulant subspace of dimension i. For every $\boldsymbol{d} \in C$, we necessarily find $\operatorname{rank} \mathrm{Circ}(\boldsymbol{d}) \leq i$. If there exists $\boldsymbol{d}$ such that the rank of $\mathrm{Circ}(\boldsymbol{d})$ is i, naturally $C = \mathrm{Circ}(\boldsymbol{d}) = \mathrm{Circ}_i$. Thus assume that $\mathrm{Circ}(\boldsymbol{d}) < i$ for all $\boldsymbol{d} \in C$. But then $C \subseteq \mathrm{Circ}_{i-1}$, an $(i-1)$-dimensional subspace, which is absurd. □

This description extends [7, Lemma 2.7]. Note that the crucial point is that the polynomial $X^\ell - 1$ splits over K. Let $\boldsymbol{e}$ be the defining vector of the GGS group G and put $\bar{\boldsymbol{e}} = (0, e_1, \ldots, e_{p-1})$. The structure of G is heavily influenced by the *cyclic rank* $\mathrm{cr}(\boldsymbol{e})$ *of* $\boldsymbol{e}$, which is given by $\mathrm{cr}(\boldsymbol{e}) = \dim \mathrm{Circ}(\bar{\boldsymbol{e}})$. By Proposition 2.5, $\mathrm{cr}(\boldsymbol{e}) = p$ if and only if the sum of entries of $\boldsymbol{e}$ is nonzero, whence by [9, Example 9.1], the group G is periodic if and only if $\mathrm{cr}(\boldsymbol{e}) < p$. The Hausdorff dimension of G solely depends on whether $\boldsymbol{e}$ is symmetric or constant and on the value of $\mathrm{cr}(\boldsymbol{e})$, as demonstrated by Fernández-Alcober and Zugadi-Reizabal in [7, Theorem B].

Corollary 2.6. *Let K be a field of characteristic $p > 0$, let $i \in [0, p]$ and let* Circ_i *be the members of the flag of circulant subspaces of K^p. A basis for* Circ_i *is given by the first i columns of the Pascal matrix $\Pi \in K^{p\times p}$.*

Proof. In view of Proposition 2.5 and the fact that the columns of Π are linearly independent, we have to show that

$$\sum_{n=j}^{p-1} \binom{n}{m}\binom{n}{j} \equiv_p 0$$

for every $m \in [0, i-1]$ and $j \in [0, p-i-1]$. Put $\phi(k, m, j) = \sum_{n=j}^{p-1} n^k \binom{n}{m}\binom{n}{j}$ and compute

$$\phi(k,m,j) = \sum_{n=j}^{p-1} n^k \binom{n}{m}\binom{n}{j-1}\frac{n-j+1}{j} = \frac{1}{j}\phi(k+1,m,j-1) + \frac{1-j}{j}\phi(k,m,j-1);$$

analogously,

$$\phi(k,m,j) = \frac{1}{m}\phi(k+1,m-1,j) + \frac{1-m}{m}\phi(k,m-1,j).$$

By iteration we find that $\phi(0,m,j)$ is a linear combination of the values of $\phi(k,0,0)$ for $k \in [0, m+j]$. But

$$\phi(k,0,0) = \sum_{n=0}^{p-1} n^k = \sum_{n=0}^{p-1} n \equiv_p 0$$

for every k that is not a multiple of $p-1$. Since $m+j < p-1$, we obtain the desired congruence. □

Properties and structure of GGS groups. Recall that, for a given GGS group G, the rooted generator is denoted a, the directed generator is denoted b, and we write c for the commutator $[a^{-1}, b]$, whose sections are given in (2-2). We shall use the following shorthand notation for the conjugates of c:

$$c_i = c^{a^i} = [a^{-1}, b^{a^i}].$$

In particular, $c_0 = c$. We collect some facts about GGS groups.

Lemma 2.7. *Every GGS group is fractal and self-similar.*

For a proof, see e.g. [8, Section 2] or [13, Section 2.3]. Next, we record some information on certain small quotients of GGS groups.

Lemma 2.8. *Let G be a GGS group. Then*

(i) $\log_p |G : G'| = 2$ *and G/G' is elementary abelian,*

(ii) $\log_p |G' : \gamma_3(G)| = 1$ *and $G/\gamma_3(G)$ is of exponent p, and*

(iii) $\log_p |G' : \mathrm{St}_G(1)'| = p-1$.

For statements (i) and (ii), see Theorem 2.1(iii) of [7] and note that $a^p = b^p = \mathrm{id}$ and $c^p \equiv_{\gamma_3(G)} \mathrm{id}$. Statement (iii) is proven as part of Theorem 2.14 of [7].

It is a result of Fernández-Alcober and Zugadi-Reizabal that every GGS group defined by a nonconstant vector is a regular branch group (see below). The same is not true for GGS groups defined by constant tuples, explaining their divergent behaviour.

Theorem 2.9. *Let G be a GGS group with nonconstant defining vector $\boldsymbol{e}$. Then*:

(i) $\psi(\gamma_3(\mathrm{St}_G(1))) = \mathsf{X}_p\, \gamma_3(G)$.

(ii) $\psi(\mathrm{St}_G(1)') \le \mathsf{X}_p\, G'$ *and* $\log_p |\mathsf{X}_p\, G' : \psi(\mathrm{St}_G(1)')| = \mathrm{sym}(\boldsymbol{e})$.

In particular, the group G is regular branch over $\gamma_3(G)$, and it is regular branch over G' if $\boldsymbol{e}$ is not symmetric.

These statements appear as Lemmas 3.2 and 3.4 of [7].

We need another fact concerning the subgroups of GGS groups.

Proposition 2.10. *Let G be a GGS group with a nonconstant defining vector. Then*

$$[\mathrm{St}_G(1)', G'] = \gamma_3(\mathrm{St}_G(1)).$$

In particular, $\gamma_3(\mathrm{St}_G(1)) \le G''$.

Proof. The inclusion $[\mathrm{St}_G(1)', G'] \le \gamma_3(\mathrm{St}_G(1))$ is a straightforward consequence of $G' \le \mathrm{St}_G(1)$, which itself follows from $G/\mathrm{St}_G(1)$ being cyclic. We have to establish the other inclusion. In view of Proposition 2.1, it is enough to prove that $\mathrm{ins}_0([c, b])$ and $\mathrm{ins}_0([c, a])$ are contained in $[\mathrm{St}_G(1)', G']$, as $\gamma_3(G)$ is normally generated by $[c, b]$ and $[c, a]$. We distinguish two cases.

Case $\mathrm{sym}(\boldsymbol{e}) = 0$: By Theorem 2.9(ii), the element $\mathrm{ins}_0(c)$ is contained in $\mathrm{St}_G(1)'$. By Corollary 2.3(ii) we may assume $e'_i = 1$ for some $i \in \{1, \dots, p-2\}$. By (2-2) $c|_i = a$, hence

$$c_{p-i}|_0 = c|_i = a, \quad (c_{p-i}^{e_1} c)|_0 = a^{e_1} a^{-e_1} b = b.$$

Since c_{p-i} and c are elements of G', we obtain

$$[\mathrm{ins}_0(c), c_{p-i}] = \mathrm{ins}_0([c, a]) \quad \text{and} \quad [\mathrm{ins}_0(c), c_{p-i}^{e_1} c] = \mathrm{ins}_0([c, b])$$

are contained in $[\mathrm{St}_G(1)', G']$.

Case $\mathrm{sym}(\boldsymbol{e}) = 1$: Note that, since $\boldsymbol{e}$ is by assumption not constant, the prime p is necessarily greater than 3. By Corollary 2.3(i), we may assume $e_1 = e_{p-1} = -1$. Observe that

$$\begin{aligned} \psi([b, b^a]) = [\psi(b), \psi(b^a)] &= (0 : [b, a^{-1}],\ 1 : [a^{-1}, b],\ \diamond : \mathrm{id}) \\ &= (0 : c^{-1},\ 1 : c,\ \diamond : \mathrm{id}). \end{aligned}$$

We first show that there exists $j \in \mathbb{F}_p^\times \smallsetminus \{1, p-1\}$ such that $e_{j-1} \neq e_{p-j-1}$. Assume the converse for contradiction. Using that $\boldsymbol{e}$ is symmetric, we find

$$e_j = e_{(1+j)-1} = e_{p-(1+j)-1} = e_{j+2}.$$

Thus $e_2 = e_4 = \cdots = e_{p-1}$ and $e_3 = e_5 = \cdots = e_{p-2}$, using that $p > 3$. Since $\boldsymbol{e}$ is symmetric, $e_2 = e_{p-2}$ and $e_1 = e_{p-1}$, whence $\boldsymbol{e}$ is constant, which is excluded.

Now let j be an element as described above. Compute

$$\psi([a^{2j}, b]) = (0 : a^{-e_{p-2j}} b,\ 2j : b^{-1} a^{e_{2j}},\ \diamond : a^{e_\diamond - e_{\diamond-2j}}),$$

so $[a^{2j}, b]|_j = a^{e_j - e_{p-j}} = \mathrm{id}$, since $\boldsymbol{e}$ is symmetric. Put $g = [a^{2j}, b]^{a^{1-j}}$ to find

$$g|_1 = [a^{2j}, b]|_j = \mathrm{id} \quad \text{and} \quad g|_0 = [a^{2j},\ b]|_{j-1} = a^{e_{j-1} - e_{p-j-1}},$$

since $j \notin \{1, p-1\}$ forbids $j - 1 \in \{0, 2j\}$. Let $i = (e_{j-1} - e_{p-j-1})^{-1}$, using that $e_{j-1} \neq e_{p-j-1}$. Observe that

$$\begin{aligned}[\mathrm{St}_G(1)', G'] \ni [[b^a, b], g^i] &= [\psi^{-1}(0 : c,\ 1 : c^{-1},\ \diamond : \mathrm{id}), \psi^{-1}(0 : a,\ 1 : \mathrm{id},\ \diamond : g^i|_\diamond)] \\ &= \mathrm{ins}_0([c, a]).\end{aligned}$$

It remains to show $\mathrm{ins}_0([c, b]) \in [\mathrm{St}_G(1)', G']$. Consider $h = g^{ie_{p-2}}[a^2, b]$, which fulfils

$$h|_0 = a^{e_{p-2}} a^{-e_{p-2}} b = b \quad \text{and} \quad h|_1 = a^{e_1 - e_{p-1}} = \mathrm{id}\,.$$

Then

$$[\mathrm{St}_G(1)', G'] \ni [[b^a, b], h] = \mathrm{ins}_0([c, b]). \qquad \square$$

Using Proposition 2.10, we derive the following adjunct to Theorem 2.9.

Proposition 2.11. *Let G be a GGS group with nonconstant defining vector. Then G is branch over G''.*

Proof. Using Theorem 2.9(i) and Proposition 2.10 we find

$$\bigtimes_p G'' \leq \bigtimes_p \gamma_3(G) = \psi(\gamma_3(\mathrm{St}_G(1))) \leq \psi(G''). \qquad \square$$

3. The derived series of GGS groups

The second derived subgroup. By Lemma 2.8(ii), the quotient $G'/\gamma_3(G)$ is a cyclic group of order p, generated by the element $\bar{c} = c \cdot \gamma_3(G)$. Thus the group $V := \bigtimes_p (G'/\gamma_3(G))$ is an elementary abelian p-group of rank p, which we write additively and treat as an $\mathbb{F}_p$-vector space with the natural basis

$$\{(\bar{c}, 0, \dots, 0),\ \dots,\ (0, \dots, 0, \bar{c})\}.$$

Write $\phi\colon\ \mathrm{St}_G(1)' \to V$ for the concatenation of ψ and the natural epimorphism $\bigtimes_p G' \to V$.

Lemma 3.1. *Let G be a GGS group with nonconstant defining vector and let $N \leq G$ be a subgroup satisfying*

$$\bigtimes_p \gamma_3(G) \leq \psi(N) \leq \bigtimes_p G'.$$

Then N is normal in G if and only if $\phi(N)$ is a circulant subspace of V, and as a consequence, there exist precisely $p + 1 - \mathrm{sym}(\boldsymbol{e})$ such normal subgroups.

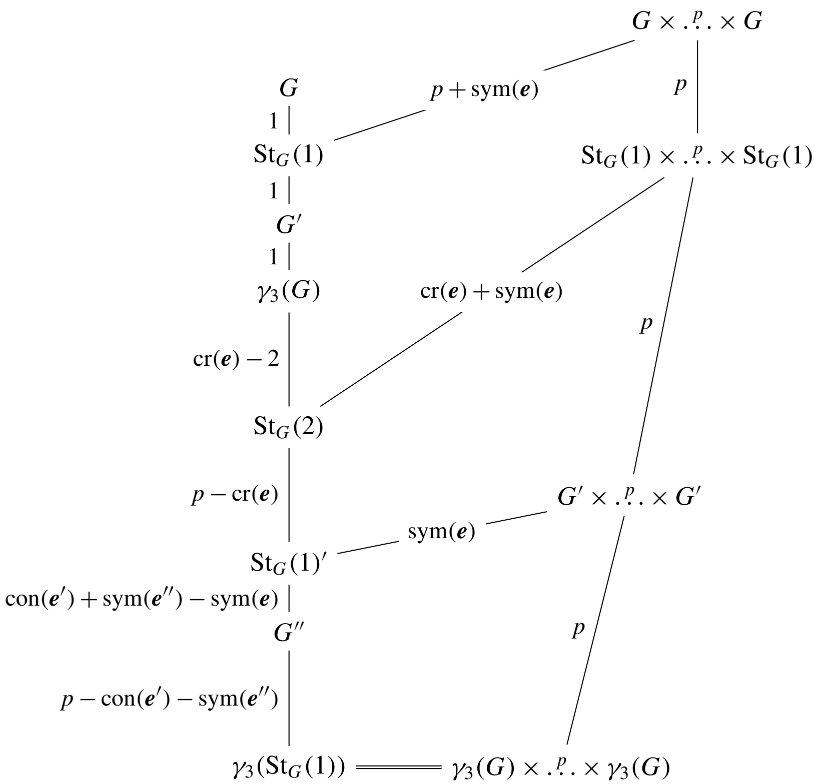

Figure 1. Part of the top of the subgroup lattice of a GGS group, with some supergroups added. Passage from the left to the right side signifies the application of ψ. All indices are logarithmic. See [7] for the computation of the index of $\mathrm{St}_G(2)$.

Proof. The conjugation action of a corresponds to a cyclic shift on $\psi(N)$, whence N is invariant under conjugation by a if and only if $\phi(N)$ is circulant. It remains to notice that N is automatically invariant under conjugation by b: for every $g \in \psi^{-1}(\times_p G') \cap G$ and every $h \in \mathrm{St}_G(1)$

$$\psi(g^h) = (\diamond : g|_\diamond^{h|_\diamond}) \equiv_{\times_p \gamma_3(G)} (\diamond : g|_\diamond) = g.$$

The last statement is a direct consequence of Proposition 2.5 and Theorem 2.9(ii). $\square$

Proposition 3.2. *Let G be a GGS group with nonconstant defining vector. Then $G'' = \phi^{-1}(\mathrm{Circ}_{p-\mathrm{con}(\boldsymbol{e}')-\mathrm{sym}(\boldsymbol{e}'')}(V))$. In particular,*

$$\log_p |\times_p G' : \psi(G'')| = \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'').$$

Proof. Since $G' \leq \mathrm{St}_G(1)$, the second derived subgroup G'' is contained in $\mathrm{St}_G(1)'$, which, by Theorem 2.9(ii), is in turn contained in $\psi^{-1}(\times_p G')$. On the other hand, using Proposition 2.10 and Theorem 2.9(i), we find

$$\mathsf{X}_p\,\gamma_3(G)=\psi(\gamma_3(\mathrm{St}_G(1)))\le\psi(G'').$$

Thus Lemma 3.1 applies and it remains to compute the dimension of $\phi(G'')$. To achieve this, we pick a subset S of G'' normally generating G'' and use Proposition 2.5 to compute the dimension of the circulant space generated by $\phi(S)$. Since the group G is 2-generated, a natural choice for S is

$$\{[c^g,c]\mid g\in G\}.$$

The kernel of ϕ is $\gamma_3(\mathrm{St}_G(1))=\psi^{-1}(\mathsf{X}_p\,\gamma_3(G))$, whence $\phi([c^g,c])=\phi([c_i,c])$ for $a^i\equiv_{\mathrm{St}_G(1)} g$. Notice that

$$[c_i,c]^{-1}=[c,c_i]=[c,c^{a^i}]=[c_{p-i},c]^{a^i}.$$

Thus the set $\phi(\{[c_i,c]\mid i\in[1,(p-1)/2]\})$ generates $\phi(G'')$ as a circulant space. The sections of the elements $[c_i,c]$ are given by (compare (2-2))

$$\begin{aligned}\psi([c_1,c])&=(0:[b^{-1}a^{e_{p-1}},a^{-e_1}b],\ 1:[a^{-e_1}b,a^{e'_1}],\ \diamond:\mathrm{id},\ p-1:[a^{e'_{p-2}},b^{-1}a^{e_{p-1}}])\\&\equiv(0:c^{e_1-e_{p-1}},\ 1:c^{e'_1},\ \diamond:\mathrm{id},\ p-1:c^{e'_{p-2}})\ (\mathrm{mod}\ \mathsf{X}_p\,\gamma_3(G))\end{aligned}$$

and

$$\begin{aligned}\psi([c_i,c])&=\begin{pmatrix}0:\ [a^{e'_{p-i}},a^{-e_1}b], & p-1:\ [a^{e'_{p-i-1}},b^{-1}a^{e_{p-1}}]\\ i-1:\ [b^{-1}a^{e_{p-1}},a^{e'_{i-1}}], & i:\ [a^{-e_1}b,a^{e'_i}]\\ \diamond:\ \mathrm{id} & \end{pmatrix}\\&\equiv\begin{pmatrix}0:\ c^{-e'_{p-i}}, & p-1:\ c^{e'_{p-i-1}},\\ i-1:\ c^{-e'_{i-1}}, & i:\ c^{e'_i},\\ \diamond:\ \mathrm{id} & \end{pmatrix}\ (\mathrm{mod}\ \mathsf{X}_p\,\gamma_3(G))\end{aligned}$$

for $i\in[2,(p-1)/2]$. Thus the images $\boldsymbol{d}_i=\phi([c_i,c])$ in V are given by

$$\boldsymbol{d}_1=(e_1-e_{p-1},e'_1,0,\dots,0,e'_{p-2}),$$

and by

$$\boldsymbol{d}_i=(-e'_{p-i},\underbrace{0,\dots,0}_{i-2},-e'_{i-1},e'_i,\underbrace{0,\dots,0}_{p-i-2},e'_{p-i-1}),$$

for $i\in[2,(p-1)/2]$. By Proposition 2.5, the dimension of $\phi(G'')$ is equal to the maximum value of $\dim\mathrm{Circ}(\boldsymbol{d}_i)$ for $i\in[1,(p-1)/2]$. To determine the dimension of the latter spaces, recall that for any $\boldsymbol{f}=(f_0,\dots,f_{p-1})$ we find $\boldsymbol{f}\Pi_3=(\boldsymbol{f}\Xi_p,\boldsymbol{f}\Xi_{p-1},\boldsymbol{f}\Xi_{p-2})$,

$$\boldsymbol{f}\Xi_p=\sum_{i=0}^{p-1}f_i,\quad \boldsymbol{f}\Xi_{p-1}=\sum_{i=0}^{p-1}if_i,\quad\text{and}\quad \boldsymbol{f}\Xi_{p-2}=\sum_{i=0}^{p-1}\binom{i}{2}f_i$$

and compute

$$\boldsymbol{d}_1 \Xi_p = 2(e_1 - e_{p-1}) + e_{p-2} - e_2, \tag{3-1}$$

$$\boldsymbol{d}_1 \Xi_{p-1} = e'_1 - e'_{p-2}. \tag{3-2}$$

$$\boldsymbol{d}_1 \Xi_{p-2} = e'_{p-2}, \tag{3-3}$$

using $\binom{p-1}{2} \equiv_p 1$ in the last line. Then, for $i \in [2, (p-1)/2]$, we compute

$$\boldsymbol{d}_i \Xi_p = -e'_{p-i} - e'_{i-1} + e'_i + e'_{p-i-1} = e''_{p-i-1} - e''_{i-1}, \tag{3-4}$$

and for $i \in [2, (p-1)/2]$ we find

$$\begin{aligned} \boldsymbol{d}_i \Xi_{p-1} &= -0 \cdot e'_{p-i} - (i-1)e'_{i-1} + i e'_i + (p-1)e'_{p-i-1} \\ &= -i \cdot e''_{i-1} + e'_{i-1} - e'_{p-i-1} \\ &= -i \cdot e''_{i-1} + \sum_{j=i-1}^{p-i-2} e''_j. \end{aligned} \tag{3-5}$$

Note that for $i = (p-1)/2$, this is equal to

$$\boldsymbol{d}_{(p-1)/2} \Xi_{p-1} = (p+3)/2 \cdot e''_{(p-3)/2}. \tag{3-6}$$

Case $\mathrm{sym}(\boldsymbol{e}'') = 0$. This implies $\mathrm{sym}(\boldsymbol{e}) = 0$, and $V = \phi(\mathrm{St}_G(1)')$ by Theorem 2.9(ii). By definition, there exists $i \in [1, (p-3)/2]$ such that $e''_i \neq e''_{p-2-i}$, hence $\boldsymbol{d}_{i+1} \Xi_p \neq 0$ by (3-4) and $\dim \mathrm{Circ}(\boldsymbol{d}_{i+1}) = p$ by Proposition 2.5. Thus $\phi(G'')$ is equal to V, i.e., $G'' = \mathrm{St}_G(1)'$. Note that $\mathrm{sym}(\boldsymbol{e}'') = 0$ implies $\mathrm{con}(\boldsymbol{e}') = 0$, since the difference vector of a constant vector is zero and in particular symmetric. Thus we find

$$\dim \phi(G'') = p = p - (\mathrm{sym}(\boldsymbol{e}'') + \mathrm{con}(\boldsymbol{e}')).$$

Case $\mathrm{sym}(\boldsymbol{e}'') = 1$. By (3-4), $\boldsymbol{d}_i \Xi_p = 0$ for all $i \in [2, (p-1)/2]$, and, by Lemma 2.4 and (3-1), also $\boldsymbol{d}_1 \Xi_p = 0$; hence Proposition 2.5 implies $\dim \phi(G'') \leq p-1$. Using (3-5), we see that

$$\boldsymbol{d}_i \Xi_{p-1} = -i \cdot e''_{i-1} + \sum_{j=i-1}^{p-i-2} e''_j = (1-i) \cdot e''_{i-1} + 2 \sum_{j=i}^{(p-3)/2} e''_j \tag{3-7}$$

for $i \in [2, (p-3)/2]$.

If $\mathrm{con}(\boldsymbol{e}') = 0$, then $p \neq 3$ and $\boldsymbol{e}'' \neq 0$. By symmetry, there exists $i \in [1, (p-3)/2]$ such that $e''_i \neq 0$. If $e''_{(p-3)/2} \neq 0$, we find $\boldsymbol{d}_{(p-1)/2} \Xi_{p-1} \neq 0$ by (3-6). Otherwise, the prime p is at least 7. Let $i \in [1, (p-5)/2]$ be maximal such that $e''_i \neq 0$. Then $\boldsymbol{d}_{i+1} \Xi_{p-1} \neq 0$ by (3-7). Thus, by Proposition 2.5, $\dim \phi(G'') = p-1 = p - (\mathrm{sym}(\boldsymbol{e}'') + \mathrm{con}(\boldsymbol{e}'))$.

On the other hand, if $\mathrm{con}(\boldsymbol{e}') = 1$, we immediately find $\boldsymbol{d}_1 \Xi_{p-1}$ by (3-2). Furthermore, $\boldsymbol{e}'' = \boldsymbol{0}$, whence we also find $\boldsymbol{d}_i \Xi_{p-1} = 0$ for $i \in [2, (p-1)/2]$, using (3-5). But by (3-3), $\boldsymbol{d}_1 \Xi_{p-2} = e'_{p-2} \neq 0$; otherwise, $\boldsymbol{e}' = \boldsymbol{0}$ since it is constant, which implies $\mathrm{con}(\boldsymbol{e}) = 1$, which was excluded. Thus $\dim \phi(G'') = p-2 = p-(\mathrm{sym}(\boldsymbol{e}'')+\mathrm{con}(\boldsymbol{e}'))$. □

Lemma 3.3. *Let G be a GGS group defined by the nonconstant vector $\boldsymbol{e}$. Then*

$$\log_p |G' : G''| = p + \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'') - \mathrm{sym}(\boldsymbol{e}) - 1.$$

Proof. This is an immediate consequence of Lemma 2.8(iii), Proposition 3.2 and Theorem 2.9(ii):

$$\begin{aligned}\log_p |G' : G''| &= \log_p |G' : \mathrm{St}_G(1)'| + \log_p |\mathsf{X}_p\, G' : \psi(G'')| - \log_p |\mathsf{X}_p\, G' : \psi(\mathrm{St}_G(1)')| \\ &= p + \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'') - \mathrm{sym}(\boldsymbol{e}) - 1.\end{aligned}$$ □

Proofs of the main results. We are now in the position to prove our theorems, which we state again for the convenience of the reader.

Theorem 1.2. *Let p be an odd prime and let G be a GGS group acting on the p-adic tree defined by a nonconstant vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$. Let $n \geq 3$. Then*

$$\psi(G^{(n)}) = \mathsf{X}_p\, G^{(n-1)}.$$

If $\mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'') = 0$, *the same holds for $n = 2$.*

Proof. Assume that the given equation holds true for some $n \geq 1$. Then we find

$$\psi(G^{(n+1)}) = \psi(G^{(n)})' = (\mathsf{X}_p\, G^{(n-1)})' = \mathsf{X}_p\, G^{(n)},$$

since $G^{(n)} \leq \mathrm{St}_G(1)$. Thus it is enough to consider the case $n = 3$, or the case $n = 2$, respectively.

First assume that $\mathrm{con}(\boldsymbol{e}') = \mathrm{sym}(\boldsymbol{e}'') = 0$, which also implies $\mathrm{sym}(\boldsymbol{e}) = 0$. By Lemma 3.3, $\mathsf{X}_p\, G' = \psi(G'')$. Thus the equation holds for $n = 2$.

We forgo the above assumptions on the defining vector and prove the desired equation for $n = 3$. In view of Proposition 2.1, it is sufficient to prove that $\mathrm{ins}_0([c, c^g]) \in G^{(3)}$ for any $g \in G$, since G'' is normally generated by the elements $\{[c, c^g] \mid g \in G\}$. By Proposition 2.10, the group G'' contains $\psi^{-1}(\mathsf{X}_p\, \gamma_3(G))$, in particular $\mathrm{ins}_0([c, g]) \leq G'$ for all $g \in G$. By Proposition 3.2 we find $h \in G''$ with

$$\psi(h) = (0 : c,\ 1 : c^{-2},\ 2 : c,\ \diamond : \mathrm{id}).$$

Let $g \in G$. Then

$$[h, \mathrm{ins}_0([c, g])] = \mathrm{ins}_0([h|_0, [c, g]]) = \mathrm{ins}_0([c, [c, g]]) = \mathrm{ins}_0([c, c^g]) \in G^{(3)}.$$ □

Before we prove Theorem 1.1, we use Theorem 1.2 to derive some more results on the structure of G.

Corollary 3.4. *Let G be a GGS group defined by a nonconstant vector and let $n \in \mathbb{N}$. Then the quotient $G^{(n)}/G^{(n+1)}$ is an elementary abelian p-group.*

Proof. It is sufficient to show that all p-th powers in $G^{(n)}$ are contained in $G^{(n+1)}$ for all $n \in \mathbb{N}$. For $n = 0$, this is the statement of Lemma 2.8(i). Since G is self-similar, we find $\psi(G') \leq \mathsf{X}_p\, G$, and, as a consequence, $\psi(G^{(n)}) \leq \mathsf{X}_p\, G^{(n-1)}$ for all $n \geq 1$. Thus for $n = 1$, notice that

$$\psi((G')^p) \leq (\mathsf{X}_p\, G)^p \leq \mathsf{X}_p\, \gamma_3(G) = \psi(\gamma_3(\mathrm{St}_G(1))) \leq \psi(G''),$$

using Lemma 2.8(ii), Theorem 2.9(i) and Proposition 2.10. For general $n > 1$, using induction and Theorem 1.2, we see that

$$\psi(G^{(n)})^p \leq (\mathsf{X}_p\, G^{(n-1)})^p = \mathsf{X}_p\, (G^{(n-1)})^p \leq \mathsf{X}_p\, G^{(n)} = \psi(G^{(n+1)}). \qquad \square$$

Corollary 1.3 follows immediately. It remains to prove Theorem 1.1.

Theorem 1.1. *Let p be an odd prime and let G be a GGS group acting on the p-adic tree defined by a nonconstant vector $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$. Let $\boldsymbol{e}'$ be the tuple of differences between the entries of $\boldsymbol{e}$ and let $\boldsymbol{e}''$ be the tuple of differences of $\boldsymbol{e}'$. Then for $n \geq 2$,*

$$\log_p |G : G^{(n)}| = p^{n-2}(p + \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'')) - \frac{p^{n-1}-1}{p-1} \cdot \mathrm{sym}(\boldsymbol{e}) + 1.$$

Proof. Using Theorem 1.2, we find for $n \geq 3$

$$\log_p |G^{(n)} : G^{(n+1)}| = \log_p |\mathsf{X}_p\, G^{(n-1)} : \mathsf{X}_p\, G^{(n)}| = p \cdot \log_p |G^{(n-1)} : G^{(n)}|,$$

and consequently

$$\log_p |G'' : G^{(n)}| = \sum_{i=0}^{n-3} p^i \log_p |G'' : G^{(3)}| = \frac{p^{n-2}-1}{p-1} \cdot \log_p |G'' : G^{(3)}|.$$

Employing our previous results we find

$$\begin{aligned}
\log_p |G'' : G^{(3)}| &\overset{\text{Thm. 1.2}}{=} \log_p |\mathsf{X}_p\, G' : \mathsf{X}_p\, G''| - \log_p |\mathsf{X}_p\, G' : \psi(G'')| \\
&\overset{\text{Prop. 3.2}}{=} p \cdot \log_p |G' : G''| - (\mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'')) \\
&\overset{\text{Lem. 3.3}}{=} p(p - 1 + \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'') - \mathrm{sym}(\boldsymbol{e})) - (\mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'')) \\
&= (p-1)(p + \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'')) - p \cdot \mathrm{sym}(\boldsymbol{e}).
\end{aligned}$$

Putting everything together (using Lemma 2.8(i) and, again, Lemma 3.3), we find

$$\begin{aligned}
\log_p |G : G^{(n)}| &= (p^{n-2} - 1)(p + \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'')) \\
&\qquad - \frac{p^{n-1} - p}{p-1} \cdot \mathrm{sym}(\boldsymbol{e}) + \log_p |G' : G''| + \log_p |G : G'| \\
&= p^{n-2}(p + \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'')) - \frac{p^{n-1}-1}{p-1} \cdot \mathrm{sym}(\boldsymbol{e}) + 1. \qquad \square
\end{aligned}$$

GGS groups with differentially constant defining vector. A vector $\boldsymbol{e}$ is called *differentially constant* if it satisfies $\mathrm{con}(\boldsymbol{e}') = 1$ (and thus also $\mathrm{sym}(\boldsymbol{e}'') = 1$). In view of Theorem 1.1, GGS groups with differentially constant defining vector display the largest indices $|G : G^{(n)}|$ among GGS groups with nonconstant defining vector, as $\mathrm{con}(\boldsymbol{e}') = 1$ implies $\mathrm{sym}(\boldsymbol{e}) = 0$. The condition $\mathrm{con}(\boldsymbol{e}') = 1$ is a strong restriction on the defining vector, making it possible to determine the isomorphism classes of differentially constant GGS groups.

Resolving the definition, one finds that $\mathrm{con}(\boldsymbol{e}') = 1$ implies that there exist $k, m \in \mathbb{F}_p$ such that $\boldsymbol{e} = (k+m, k+2m, \dots, k+(p-1)m)$. We introduce the shorthand notation $\mathrm{dc}(k, m)$ for the vector given above. If $m = 0$, evidently $\mathrm{con}(\boldsymbol{e}) = 1$.

Proposition 3.5. *For any given odd prime p there are precisely three isomorphism classes of differentially constant GGS groups acting on the p-adic tree*:

(i) *one consisting of the constant GGS group*,

(ii) *one consisting of a single periodic GGS group*,

(iii) *one containing precisely $p-1$ nonperiodic GGS groups.*

Proof. Recall the isomorphism class preserving action $*$ of $(\mathbb{F}_p^\times)^2$ on the set of defining vectors given by (2-1). Let G be the GGS group defined by $\mathrm{dc}(k, m)$ and let $\lambda, \mu \in \mathbb{F}_p^\times$. Then

$$\mathrm{dc}(k, m) * (\lambda, \mu) = \mathrm{dc}(\lambda k, \lambda m) * (1, \mu) = \mathrm{dc}(\lambda k, \lambda\mu m).$$

If $m = 0$, the vector $\mathrm{dc}(k, 0)$ is constant, and we are in case (i). If $m \neq 0$, we find

$$\mathrm{dc}(0, m) * (1, m^{-1}) = \mathrm{dc}(0, 1) \quad \text{and} \quad \mathrm{dc}(k, m) * (k^{-1}, km^{-1}) = \mathrm{dc}(1, 1)$$

for $k \neq 0$. At the same time, $\mathrm{dc}(0, m) * (\lambda, \mu) \neq \mathrm{dc}(1, 1)$ for all $\lambda, \mu \in \mathbb{F}_p^\times$, whence $\mathrm{dc}(0, 1)$ and $\mathrm{dc}(1, 1)$ represent distinct isomorphism classes. The $(\mathbb{F}_p^\times)^2$-orbit of $\mathrm{dc}(0, 1)$ consists of the multiples of $\mathrm{dc}(0, 1)$ and all the associated GGS groups are identical. By [9, Example 9.1], the GGS group defined by $\boldsymbol{e}$ is periodic if and only if the sum of the entries of $\boldsymbol{e}$ vanishes, i.e., if $\dim \mathrm{Circ}(\bar{f}e) \leq p-1$. For $\mathrm{dc}(0, 1) = (1, \dots, p-1)$ this is the case, but not for $\mathrm{dc}(1, 1) = (2, \dots, p-1, 0)$. As seen above, $(\mathbb{F}_p^\times)^2$ acts transitively on $\{\mathrm{dc}(k, m) \mid k, m \in \mathbb{F}_p^\times\}$. Since only proportional vectors yield identical GGS groups, there are $p-1$ distinct GGS groups isomorphic to the group defined by $\mathrm{dc}(1, 1)$. □

In case of the prime $p = 3$, all GGS groups are differentially constant and the unique periodic GGS group is the Gupta–Sidki 3-group. It is interesting to see that other invariants take extremal values for the groups G_p defined by $(1, 2, \dots, p-1)$ (for arbitrary p): By [7, Theorem B], the Hausdorff dimension of the GGS group defined by $\boldsymbol{e}$ is given by

$$\frac{(p-1)\,\mathrm{cr}(\boldsymbol{e})}{p^2} - \frac{\mathrm{sym}(\boldsymbol{e})}{p^2} - \frac{\mathrm{con}(\boldsymbol{e})}{(p-1)\,p^2}.$$

By Corollary 2.6, $\mathrm{cr}(\boldsymbol{e}) = 2$ if and only if $\boldsymbol{e}$ is a nonzero multiple of $\mathrm{dc}(0, 1)$. In particular, a symmetric defining vector $\boldsymbol{e}$ fulfils $\mathrm{cr}(\boldsymbol{e}) > 2$. Furthermore, a constant defining vector has $\mathrm{cr}(\boldsymbol{e}) = p$. Thus the group G_p is the unique GGS group with the minimal (among GGS groups acting on a fixed tree) possible Hausdorff dimension $2(p-1)/p^2$, while the group defined by $\mathrm{dc}(1, 1)$ is among those with maximal Hausdorff dimension $(p-1)/p$.

The automorphism group of G_p is as large as possible; cf. [15, Example 6.2].

Comparison with the congruence subgroups. The members of the derived series of a GGS group share certain properties with the level stabilisers. They both form filtrations of the group, with $\mathrm{St}_G(n) \geq G^{(n)}$ for all $n \in \mathbb{N}$; for sufficiently high values of n, they satisfy $\psi(G_n) = \mathsf{X}_p\, G_{n-1}$ by Theorem 1.2 and [7, Lemma 3.3], respectively; furthermore, the quotients of respective consecutive members are elementary abelian p-groups. Using the formula for the index of the n-th level stabiliser provided by Fernández-Alcober and Zugadi-Reizabal in [7, Theorem A] one finds

$$\log_p |G : \mathrm{St}_G(2)| = \mathrm{cr}(\boldsymbol{e}) + 1,$$

hence, using the consequence $\log_p |G : \mathrm{St}_G(1)'| = p + 1$ of Lemma 2.8, we find that $\log_p \bigl|\mathrm{St}_G(2) : \mathrm{St}_G(1)'\bigr| = p - \mathrm{cr}(\boldsymbol{e})$. A comparison with $\log_p |G : G''| = p + \mathrm{con}(\boldsymbol{e}') + \mathrm{sym}(\boldsymbol{e}'') - \mathrm{sym}(\boldsymbol{e}) + 1$ makes it apparent that $G'' = \mathrm{St}_G(2)$ if and only if $\boldsymbol{e}'$ is nonconstant, $\mathrm{cr}(\boldsymbol{e}) = p$, and $\mathrm{sym}(\boldsymbol{e}) = \mathrm{sym}(\boldsymbol{e}'')$. Since $\psi(G_n) = \mathsf{X}_p\, G_{n-1}$ holds for both the derived series and the series of level stabilisers from $n = 3$ onwards, the series only differ in their first term in this case, and otherwise have no equal terms at all. The largest difference is attained for the periodic groups G_p with the differentially constant defining vector $\mathrm{dc}(0, 1)$; recall that $\mathrm{cr}(\mathrm{dc}(0, 1)) = 2$, and thus

$$\log_p\, |\,\mathrm{St}_{G_p}(n) : G_p^{(n)}| = p^{n-1}.$$

By Proposition 2.5, $\mathrm{cr}(\boldsymbol{e}) = 2$ is the minimum possible value; however one finds that $\mathrm{cr}(\boldsymbol{e})$ may take any value in $[2, p]$. Therefore the number of distinct sequences $(|G : \mathrm{St}_G(n)|)_{n \in \mathbb{N}}$ obtained by any GGS group is between p and $2p-1$ (note that symmetric defining vectors do not admit all values $[2, p]$ under cr); in particular, the number of classes of GGS groups separated by the sequence of indices of their level stabilisers grows with p. In contrast, the sequences $(|G : G^{(n)}|)_{n \in \mathbb{N}}$ yield a partition into five subsets for $p \neq 3$; for $p = 3$, one obtains 2 classes.

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PROPERTY QT OF RELATIVELY HIERARCHICALLY HYPERBOLIC GROUPS

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Using the projection complex machinery, a number of authors (Bestvina, Bromberg and Fujiwara; Hagen and Petyt; and Han, Nguyen and Yang) have proved that several classes of nonpositively curved groups admit equivariant quasi-isometric embeddings into finite products of quasitrees, i.e., having property QT. Here we unify and generalize those results by establishing a sufficient condition for relatively hierarchically hyperbolic groups to have property QT.

As applications, we show that a group has property QT if it is residually finite and belongs to one of the following classes of groups: admissible groups, hyperbolic-2-decomposable groups with no distorted elements, and Artin groups of large and hyperbolic type. We also introduce a slightly stronger version of property QT, called property QT_0, and show the invariance of property QT_0 under graph products.

1. Introduction

Group actions on quasitrees have been studied intensively in recent years. A *quasitree* is a geodesic space quasi-isometric to a simplicial tree. We say that a finitely generated group G has *property QT* if G acts on a finite product of quasitrees (equipped with the ℓ^1-metric) such that the orbit map is a quasi-isometric embedding. Such an embedding is called a *QT embedding* of G. Since a quasitree has asymptotic dimension at most 1, property QT is a stronger form of finite asymptotic dimension. Examples of groups with property QT include

- Coxeter groups [DJ99];
- residually finite hyperbolic groups [BBF21];
- mapping class groups of finite-type surfaces [BBF21];
- virtually colorable hierarchically hyperbolic groups whose associated hyperbolic spaces are all quasitrees [HP22] (including virtually compact special

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groups [BHS17b], the genus 2 handlebody group [Che22], fundamental groups of nongeometric graph manifolds [HRSS24]);

- fundamental groups of compact orientable 3-manifolds whose sphere-disk decomposition does not support either Sol or Nil geometry [HNY25];

along with their undistorted subgroups.

The last four examples were proved to have property QT with the help of the projection complex techniques developed in [BBF15; BBFS19]. For mapping class groups, property QT strengthens Theorem C of [BBF15], which says that mapping class groups equivariantly quasi-isometrically embed in a finite product of hyperbolic graphs of finite asymptotic dimension. Counterexamples of property QT include certain special linear groups [Man06; Man08], generalized Baumslag–Solitar groups with infinite monodromy [But25] and groups with Property hereditary (NL) [BFG24]. For some basic corollaries of property QT, see [HNY25, §§2.1, 2.2]. Recently, Vergara [Ver24] proved that any finitely generated group with property QT has a proper uniformly Lipschitz affine action on ℓ^1 with quasi-isometrically embedded orbits.

As a generalization of the Masur–Minsky machinery [MM99; MM00], *hierarchically hyperbolic groups* [BHS17b; BHS19], abbreviated HHGs, have become an important bridge between mapping class groups, cubical groups, and many other nonpositively curved groups. A list of papers in this field can be found in [HRSS24]. Coarsely speaking, an HHG is a finitely generated group G whose geometry can be recovered from G-equivariant projections to a specified (possibly infinite) collection of hyperbolic spaces. For background on HHGs and relative HHGs, see Section 2.3. As shown in [BHS17a], HHGs have finite asymptotic dimension. This leads to a natural question:

Question 1.1. *Which HHGs have property QT?*

In this paper, we provide a sufficient condition for relative HHGs to have property QT, which reproves those of [BBF21; HP22; HNY25]. We also give a sufficient condition for the existence of a *quasimedian* QT embedding (for this notion, see [HP22]). This stronger property can be used to prove the existence of globally stable cylinders (see [PSZ25]), which connects to a long-standing question of Rips and Sela [RS95] about canonical representatives of elements in hyperbolic groups. The following is a collection of applications from Section 7. These results are new except for mapping class groups.

Theorem 1.2. *The following groups have property QT*:

- *mapping class groups of finite-type surfaces*;
- *residually finite admissible graphs of groups*;
- *residually finite hyperbolic-*2*-decomposable groups with no distorted elements*;

- *residually finite Artin groups of large and hyperbolic type.*

The QT embeddings for all these groups are quasimedian.

When studying the invariance of property QT in some cases, we want the group action on the product space to be diagonal. We say that a finitely generated group G has *property* QT_0 if G has property QT and the G-action on the finite product of quasitrees is diagonal. By [HNY25, Theorem 1.5], if a residually finite group G is hyperbolic relative to a collection of groups with property QT_0, then G has property QT. Without ambiguity, we also say that a G-action on a metric space X has *property* QT_0 if X admits a G-equivariant quasi-isometric embedding into a finite product of quasitrees on which G acts diagonally. In particular, if X itself is a finite product of quasitrees, then any diagonal action on X has property QT_0. We prove the following invariance of property QT_0 under graph products.

Theorem 7.10. *Any graph product of groups whose every vertex group has property* QT_0 *still has property* QT_0.

Now we give the main definitions needed to state our main theorem.

Definition 1.3. Let $(G, \mathfrak{S})$ be a relative HHG. For any $U \in \mathfrak{S}$, we write $G_U < \mathrm{Aut}(\mathfrak{S}_U)$ to mean the image of $\mathrm{Stab}_G(U)$ under the restriction homomorphism.

(1) We say a domain $U \in \mathfrak{S}$ is of *type I* if it has the following properties:
  (*hyperbolicity*) $\mathcal{C}U$ is hyperbolic.
  (*acylindrical image*) G_U acts on $\mathcal{C}U$ acylindrically.
  (*cobounded nested region*) G_U acts on $\boldsymbol{F}_U$ coboundedly.
  (*separable quasi-axes*) For any element $g \in \mathrm{Stab}_G(U)$ that acts loxodromically on $\mathcal{C}U$, the elementary closure $EC(g)$ is separable in G.

(2) We say a domain $U \in \mathfrak{S}$ is of *type II* if the action $G_U \curvearrowright \mathcal{C}U$ has property QT_0.

For any $U \in \mathfrak{S}$ of type II, property QT_0 provides quasitrees T_U^i along with G_U-equivariant maps $\iota_U^i : \mathcal{C}U \to T_U^i$ for $i = 1, \ldots, n_U$ such that

$$\prod_{i=1}^{n_U} \iota_U^i : \mathcal{C}U \to \prod_{i=1}^{n_U} T_U^i$$

is a quasi-isometric embedding. Our main theorem is as follows.

Theorem 1.4. *Let* $(G, \mathfrak{S})$ *be a relative HHG that is virtually colorable. If every* $U \in \mathfrak{S}$ *is of type I or II, then G has property QT.*

If, for any $D \geq 1$, *there exists* $D' \geq 1$ *such that for every* $U \in \mathfrak{S}$ *of type II and each* $i = 1, \ldots, n_U$ *the map* $\iota_U^i : \mathcal{C}U \to T_U^i$ *sends* (D, D)*-quasigeodesics to unparametrized* (D', D') *quasigeodesics, then G is coarse median and the QT embedding of G is quasimedian.*

Sketch of proof. We roughly explain how to prove Theorem 1.4 in the case that $(G, \mathfrak{S})$ is an HHG with only type I domains, excluding the "moreover" part. This case contains most of the key ideas.

First, we introduce a class of *thick distances* on G each of which is defined using a class of *thick segments* of hierarchy paths on G. We prove in Section 3 a thick distance formula saying that the word metric of G can be recovered by summing up these thick distances. This is an analogue of the distance formula for HHGs.

Then we show that any class of thick segments is cofinite up to the group action. Furthermore, these thick segments can be extended to a cofinite collection of quasi-axes. Using projections to these quasi-axes, we can estimate the thick distance. This is done in Section 4.

Finally, we take a finite-index subgroup of G, say H, such that a collection of quasi-axes as above is divided into finitely many H-orbits. Each H-orbit satisfies the Bestvina–Bromberg–Fujiwara projection axioms, so it gives us a quasitree with an H-action. We prove that H equivariantly quasi-isometrically embeds in the product of these finitely many quasitrees in Section 5. Since property QT is commensurably invariant, G has property QT. □

As stated in the proof, property QT is commensurably invariant [BBF21, §2.2]. It follows that the conclusion of Theorem 1.4 also holds for any group that is virtually a relative HHG that satisfies our condition, even though such a group may not be a relative HHG itself [PS23].

For most examples of HHGs that emerged from the study, every domain satisfies the first three conditions of type I. Virtual colorability is also common in practice. Therefore, the biggest restriction of our theorem comes from the assumption of separable quasi-axes. We further discuss it in Section 6 and show how residual finiteness helps to give an easy-to-use criterion for having separable quasi-axes.

2. Background

2.1. ***Quasi-isometric embeddings and acylindricity.*** Given constants $\lambda \geq 1,\ c \geq 0$, we say that a coarse map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is a (λ, c)-*quasi-isometric embedding* if

$$\frac{1}{\lambda} d_X(x_1, x_2) - c \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + c$$

for all $x_1, x_2 \in X$. A (λ, c)-quasi-isometric embedding $\gamma : [0, l] \to X$ is called a (parametrized) (λ, c)-*quasigeodesic* in X. A coarse map $\gamma : [0, l] \to X$ is an *unparametrized* (λ, c)-quasigeodesic if there exists a strictly increasing function $f : [0, l] \to [0, l]$ with $f(0) = 0$, $f(l) = l$ such that $\gamma \circ f$ is a (λ, c)-quasigeodesic. We also use the term "quasigeodesic" to mean a quasi-isometric embedding of $\mathbb{R}$. We will not distinguish between a quasigeodesic and its image in X.

A geodesic metric space is called δ-*hyperbolic* (or simply *hyperbolic*) for $\delta \geq 0$ if for any geodesics α, β, γ that form a triangle, α is contained in the δ-neighborhood of $\beta \cup \gamma$ [Gro87]. For a δ-hyperbolic space X, an isometry $g : X \to X$ is called *loxodromic* if the g-orbit $n \mapsto g^n x$ is a quasigeodesic for some (equivalently, any) $x \in X$.

Let X be a hyperbolic space and G be a group acting by isometries on X with a loxodromic element g. Given constants $\lambda \geq 1$ and $c \geq 0$, a (λ, c)-quasigeodesic $\gamma \subset X$ is called a (λ, c)-*quasi-axis* for g if γ is g-invariant. The *elementary closure* of g in G, $EC_G(g)$, is the subgroup of G that stabilizes γ up to bounded Hausdorff distance. If there is no ambiguity in G, we often simplify the notation as $EC(g)$. Equivalently, it is the stabilizer of the set $\gamma(\pm\infty)$, the points at infinity of γ. Thus, the elementary closure does not depend on the choice of γ. Everything that commutes with g is contained in $EC(g)$ (including powers and roots), but there may be other elements.

A group action $G \curvearrowright X$ by isometries is called *acylindrical* [Bow08] if for any $r \geq 0$ there exist constants R, $N \geq 0$ such that, for any pair $a, b \in X$ with $d(a, b) \geq R$,

$$\#\{g \in G \mid d(ga, a) \leq r \text{ and } d(gb, b) \leq r\} \leq N.$$

Let X be a hyperbolic space and G be a group acting acylindrically on X with a loxodromic element g. Some basic properties of this kind of action can be found in [Osi16]. In this case, the elementary closure $EC(g)$ is the unique maximal virtually cyclic subgroup of G that contains g [DGO17, Lemma 6.5]. Moreover, $EC(g)$ has a subgroup of index at most 2 that is a centralizer of a large power of g in G [DGO17, Corollary 6.6].

In this paper, we will consider group actions with a large kernel, in which case the action cannot be acylindrical. As in [BBF21], an action $G \curvearrowright X$ is said to have *acylindrical image* if the image of G in the isometry group of X is acylindrical.

2.2. ***Projection axioms.*** In this section, we review the construction of a quasitree of spaces in [BBF15] with improvements from [BBFS19].

Let $\boldsymbol{Y}$ be a collection of geodesic metric spaces, and $\pi_Y(X) \subset Y$ be specified subsets whenever $X \neq Y$ are elements of $\boldsymbol{Y}$. Set $d_Y^\pi(X, Z) := \operatorname{diam}(\pi_Y(X) \cup \pi_Y(Z))$ for $X \neq Y \neq Z$. We say that $(\boldsymbol{Y}, \{\pi_Y\})$ is a *projection system* with *projection constant* $\xi \geq 0$ if it satisfies the following *projection axioms*:

(P0) (*bounded projection*) $\operatorname{diam}(\pi_Y(X)) \leq \xi$ when $X \neq Y$.

(P1) (*Behrstock inequality*) If X, Y, Z are distinct and $d_Y^\pi(X, Z) > \xi$, then $d_X^\pi(Y, Z) \leq \xi$.

(P2) (*finiteness*) For $X \neq Z$ the set

$$\{Y \in \boldsymbol{Y} \mid d_Y^\pi(X, Z) > \xi\}$$

is finite.

We say that $(\boldsymbol{Y}, \{\pi_Y\})$ is a *G-projection system* if a group G acts on the set $\boldsymbol{Y}$ in such a way that every $g \in G$ acts as an isometry from Y to gY and the projections π_Y are G-equivariant, that is, $\pi_{gY}(gX) = g\pi_Y(X)$.

We say that $(\boldsymbol{Y}, \{\pi_Y\})$ satisfies the *strong projection axioms* if (P0), (P2) and the following replacement for (P1) are satisfied:

(P1′) If X, Y, Z are distinct and $d^{\pi}_Y(X, Z) > \xi$, then $\pi_X(Y) = \pi_X(Z)$.

While there are many natural situations where the projection axioms hold, the strong projection axioms are not as natural. However, we can modify the projections so that they do hold.

Theorem 2.1 [BBFS19, Theorem 4.1]. *If $(\boldsymbol{Y}, \{\pi_Y\})$ is a projection system with constant ξ, then there are projections $\{\pi'_Y\}$ such that $(\boldsymbol{Y}, \{\pi'_Y\})$ satisfies the strong projections axioms with constant ξ', where $\pi'_Y(X)$ and $\pi_Y(X)$ are apart from each other within a uniform Hausdorff distance ϵ, and ϵ and ξ' only depend on ξ.*

If $(\boldsymbol{Y}, \{\pi_Y\})$ is a G-projection system, then $(\boldsymbol{Y}, \{\pi'_Y\})$ is still a G-projection system.

Let $\mathcal{C}_K\boldsymbol{Y}$ denote the space obtained from the disjoint union

$$\bigsqcup_{Y \in \boldsymbol{Y}} Y$$

by joining points in $\pi_X(Z)$ with points in $\pi_Z(X)$ by an edge of length one whenever $d_Y(X, Z) < K$ for all $Y \in \boldsymbol{Y} - \{X, Z\}$. When the spaces are graphs and projections are subgraphs, we can join just the vertices in these projections. If $\boldsymbol{Y}$ is a G-projection system, then G acts isometrically on $\mathcal{C}_K\boldsymbol{Y}$.

Theorem 2.2 [BBF15, §4]. *Let $(\boldsymbol{Y}, \{\pi_Y\})$ satisfy the strong projection axioms with constant ξ, and take $K > 2\xi$. Then:*

- *$\mathcal{C}_K\boldsymbol{Y}$ is hyperbolic if all $Y \in \boldsymbol{Y}$ are δ-hyperbolic.*
- *$\mathcal{C}_K\boldsymbol{Y}$ is a quasitree if all $Y \in \boldsymbol{Y}$ are quasitrees with uniform QI constants.*

There is a very useful distance formula in $\mathcal{C}_K\boldsymbol{Y}$. Let $X, Z \in \boldsymbol{Y}$ and $x \in X$, $z \in Z$. We define $\pi_Y(x) = \pi_Y(X)$ if $Y \neq X$ and define $\pi_X(x) = x$. Then define $d_Y(x, z) = \operatorname{diam}(\pi_Y(x) \cup \pi_Y(z))$.

Notation. Given $A, B \geq 0$, we define a threshold function by

$$\{\!\{A\}\!\}_B = \begin{cases} A & \text{if } A \geq B, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.3 [BBFS19, Theorem 6.3]. *Let $(\boldsymbol{Y}, \{\pi_Y\})$ satisfy the strong projection axioms with constant ξ. Let $x \in X$ and $z \in Z$ be two points of $\mathcal{C}_K(\boldsymbol{Y})$ with $X, Z \in \boldsymbol{Y}$. Then*

$$\frac{1}{4}\sum_{Y\in\boldsymbol{Y}}\{\!\{d_Y(x,z)\}\!\}_K \le d_{\mathcal{C}_K\boldsymbol{Y}}(x,z) \le 2\sum_{Y\in\boldsymbol{Y}}\{\!\{d_Y(x,z)\}\!\}_K + 3K$$

for all $K \geq 4\xi$.

Next we recall a theorem that allows us to pass projection axioms from a projection system to a collection of certain subspaces. Let $\boldsymbol{Y}$ be a collection of δ-hyperbolic spaces and $(\boldsymbol{Y}, \{\pi_Y\})$ be a projection system with constant ξ. For each $Y \in \boldsymbol{Y}$, let $\mathcal{A}_Y$ be a collection of quasigeodesics in Y. Let $\mathcal{A}$ be the disjoint union of all $\mathcal{A}_Y$'s. We also make the following assumptions.

- As a collection of quasigeodesics, $\mathcal{A}$ has uniform coarse constants.
- For $\alpha, \beta \in \mathcal{A}_Y$, we define $\pi_\alpha(\beta)$ to be the closest-point projection of β to α.
- For $\alpha \in \mathcal{A}_X$ and $\beta \in \mathcal{A}_Y$ where $X \neq Y$, we define $\pi_\alpha(\beta)$ to be the closest point projection of $\pi_X(Y)$ to α.

Theorem 2.4 [BBF21, Theorem 4.17]. *For any $\theta > 0$, there exists $\xi' > 0$, depending only on θ, δ, ξ and coarse constants of $\mathcal{A}$, such that, if* $\operatorname{diam}(\pi_\alpha(\beta)) \le \theta$ *whenever α and β are distinct elements in the same $\mathcal{A}_Y$, then $(\mathcal{A}, \{\pi_\alpha\})$ is a projection system with constant ξ'.*

2.3. (*Relatively*) *hierarchically hyperbolic spaces.* In this paper, we deal with (*relatively*) *hierarchically hyperbolic spaces* and (*relatively*) *hierarchically hyperbolic groups*. Coarsely speaking, a (relative) HHS is a pair $(\mathcal{X}, \mathfrak{S})$, where $\mathcal{X}$ is a quasigeodesic space and $\mathfrak{S}$ is an index set, with some extra structure. A full definition can be found in [BHS19, Definition 1.1,1.21]. Some important information from the definition is collected below.

- An element $U \in \mathfrak{S}$ is called a *domain* of $\mathcal{X}$. $\mathfrak{S}$ has a partial order $\sqsubseteq$, called *nesting*, and a symmetric relation $\perp$, called *orthogonality*. These two relations are required to be mutually exclusive. Any two elements that are neither comparable under the partial order nor mutually orthogonal are by definition mutually *transversal*; we denote this relation by $\pitchfork$. We denote by $\mathfrak{S}_U$ the set of domains nested in U, and by $\mathfrak{S}_U^\circ$ the set of domains properly nested in U.
- There is a unique $\sqsubseteq$-maximal element S in $\mathfrak{S}$ and a uniform bound on the length of $\sqsubseteq$-chains in $\mathfrak{S}$, called the *complexity* of $(\mathcal{X}, \mathfrak{S})$. The *level* $\ell(V)$ of $V \in \mathfrak{S}$ is defined inductively as follows. If V is $\sqsubseteq$-minimal then $\ell(V) = 1$. The element V has level $k+1$ if k is the maximal integer such that there exists $U \sqsubsetneq V$ with $\ell(U) = k$.
- For HHSes, there is a set $\{(\mathcal{C}U, d_U) : U \in \mathfrak{S}\}$ of uniformly hyperbolic spaces and a set of uniformly coarsely Lipschitz and coarsely surjective maps $\pi_U : \mathcal{X} \to \mathcal{C}U$

for all $U \in \mathfrak{S}$. For relative HHSes, the complexity is at least 2. If U is $\sqsubseteq$-minimal, $\mathcal{C}U$ is not required to be hyperbolic. This is the only difference between HHSes and relative HHSes in definition.

- For $U \sqsubsetneq V$ or $U \pitchfork V$, there is a uniformly bounded set $\rho^U_V \subset \mathcal{C}V$.
- For $U \sqsubsetneq V$, there is a coarse map $\rho^V_U : \mathcal{C}V \to \mathcal{C}U$.
- Whenever $V \sqsubseteq W$ and $W \perp U$, we require that $V \perp U$.
- (*orthogonal containers*) For each $T \in \mathfrak{S}$ and each $U \in \mathfrak{S}_T$ for which the set $\{V \in \mathfrak{S}_T \mid V \perp U\} \neq \varnothing$, there exists $W \in \mathfrak{S}^\circ_T$, so that whenever $V \perp U$ and $V \sqsubseteq T$, we have $V \sqsubseteq W$. We say that W is an *orthogonal container* of U in T if W is a $\sqsubseteq$-minimal element satisfying the above property. Let $\mathrm{cont}^\perp_T U$ denote the set of all orthogonal containers of U in T. If T is the maximal element of $\mathfrak{S}$, then we suppress it from the notation and write $\mathrm{cont}^\perp U$. We set $\mathfrak{S}^\perp_U = \{V \in \mathfrak{S} \mid V \perp U\} \cup \{A\}$, where A is an arbitrary element of $\mathrm{cont}^\perp U$.
- (*consistency*) For every $x \in X$, the tuple $(\pi_U(x))_{U \in \mathfrak{S}}$ is κ_0-consistent (defined below). If $U \sqsubseteq V$, then $d_W(\rho^U_W, \rho^V_W) \leq \kappa_0$ whenever $W \in \mathfrak{S}$ satisfies either $V \sqsubsetneq W$ or $V \pitchfork W$ and $U \not\perp W$.
- (*bounded geodesic image*) There exists $E > 0$ such that for all $W \in \mathfrak{S}$, all $V \in \mathfrak{S}^\circ_W$, and all $x, y \in \mathcal{X}$ such that some geodesic from $\pi_W(x)$ to $\pi_W(y)$ stays E-far from ρ^V_W, we have $d_V(\pi_V(x), \pi_V(y)) \leq E$. We will refer this property as BGI in this paper.

Definition 2.5 (κ-consistent tuple). For $\kappa \geq 0$, let $\vec{b} = (b_U)_{U \in \mathfrak{S}} \in \prod_{U \in \mathfrak{S}} 2^{\mathcal{C}U}$ be a tuple such that every set b_U has diameter at most κ. We say that $\vec{b}$ is κ-consistent if

$$\min\left\{d_U(b_U, \rho^V_U), d_V(b_V, \rho^U_V)\right\} \leq \kappa \quad \text{whenever } U \pitchfork V, \text{ and}$$
$$\min\left\{d_V(b_V, \rho^U_V), \mathrm{diam}_U(b_U \cup \rho^V_U(b_V))\right\} \leq \kappa \quad \text{whenever } U \sqsubsetneq V.$$

For convenience, we always take E to be the greatest constant in all coarseness from the above list (see [BHS19, Remark 1.6] for discussion on these constants). For the rest of this subsection, let $(\mathcal{X}, \mathfrak{S})$ be a relative HHS.

Notation. Given $x, y \in \mathcal{X}$, we write $d_U(x, y)$ to mean $d_U(\pi_U(x), \pi_U(y))$. If $U \pitchfork V$ or $U \sqsubsetneq V$, we write $d_V(x, \rho^U_V)$ to mean $d_V(\pi_V(x), \rho^U_V)$.

Notation. Given two functions $f, g : X \to \mathbb{R}$ and $A, B > 0$, we write $f \preceq_{(A,B)} g$ to mean $f(x) \leq Ag(x) + B$ for any $x \in X$. We write $f \asymp_{(A,B)} g$ to mean

$$\frac{1}{A} f(x) - B \leq g(x) \leq Af(x) + B$$

for any $x \in X$. Sometimes we omit the constants, meaning that the inequality holds for some constants.

The powerful Masur–Minsky distance formula [MM00] shows that the distance between points in a mapping class group is coarsely the sum of the distances between the projections of these points to the curve graphs of all subsurfaces. Like mapping class groups, relative HHSes also satisfy a Masur–Minsky-style distance formula.

Theorem 2.6 (distance formula [BHS19, Theorem 6.10]). *There exists s_0 such that for all $s \geq s_0$ there exists a constant $C > 0$ such that, for all $x, y \in \mathcal{X}$,*

$$d_{\mathcal{X}}(x, y) \asymp_{(C,C)} \sum_{W \in \mathfrak{S}} \{\!\{d_W(x, y)\}\!\}_s.$$

Closely related to the distance formula is the existence of *hierarchy paths*.

Definition 2.7 (hierarchy path). A (D, D)-quasigeodesic $\gamma \subset \mathcal{X}$ is a *D-hierarchy path* if $\pi_U(\gamma)$ is an unparametrized (D, D)-quasigeodesic for each $U \in \mathfrak{S}$.

Theorem 2.8 (existence of hierarchy paths [BHS19, Theorem 6.11]). *There exists D_0 such that any two points in $\mathcal{X}$ are joined by a D_0-hierarchy path.*

Remark 2.9. Let γ be a D-hierarchy path connecting x and y. By the construction of hierarchy paths in [BHS19], $\pi_U(\gamma)$ is contained in the D-neighborhood of a geodesic connecting $\pi_U(x)$ and $\pi_U(y)$. If $\mathcal{C}U$ is hyperbolic, this is easy to see from the Morse lemma. In the general case, it deserves its own mention.

There is an important class of subspaces in relative HHSes. We will consider them in Section 4.

Definition 2.10 (standard product region, standard nested region, standard orthogonal region). Fix $U \in \mathfrak{S}$ and $\kappa \geq \kappa_0$. Let $\boldsymbol{F}_U$ be the set of κ-consistent tuples in $\prod_{V \in \mathfrak{S}_U} 2^{\mathcal{C}V}$. Let $\boldsymbol{E}_U$ be the set of κ-consistent tuples in $\prod_{V \in \mathfrak{S}_U^{\perp} - \{A\}} 2^{\mathcal{C}V}$. Let $\boldsymbol{P}_U = \boldsymbol{F}_U \times \boldsymbol{E}_U$. We can define a coarse map $\phi_U : \boldsymbol{P}_U \to \mathcal{X}$ as follows.

For each $(\vec{a}, \vec{b}) \in \boldsymbol{F}_U \times \boldsymbol{E}_U$ and each $V \in \mathfrak{S}$, define the coordinate $(\phi_U(\vec{a}, \vec{b}))_V$ as follows. If $V \sqsubseteq U$, then $(\phi_U(\vec{a}, \vec{b}))_V = a_V$; if $V \perp U$, then $(\phi_U(\vec{a}, \vec{b}))_V = b_V$; if $V \pitchfork U$, then $(\phi_U(\vec{a}, \vec{b}))_V = \rho^U_V$; and if $U \sqsubsetneq V$, then $(\phi_U(\vec{a}, \vec{b}))_V = \rho^U_V$. We can check that the tuple $\phi_U(\vec{a}, \vec{b})$ is κ-consistent, and thus the realization theorem [BHS19, Theorem 3.1] supplies the map $\phi_U : \boldsymbol{P}_U \to \mathcal{X}$. (See [BHS19, §5B] for details.)

For convenience, we do not distinguish between $\boldsymbol{P}_U$ and its image in $\mathcal{X}$. We call $\boldsymbol{P}_U$ the *standard product region*. By choosing any copy of $\boldsymbol{F}_U$ in the direct product, ϕ_U restricts to a coarse map $\phi^{\sqsubseteq} : \boldsymbol{F}_U \to \mathcal{X}$. We also define $\phi^{\perp} : \boldsymbol{E}_U \to \mathcal{X}$ in the same way. We call $\boldsymbol{F}_U$ and $\boldsymbol{E}_U$ the *standard nested region* and the *standard orthogonal region*, respectively.

Remark 2.11. By definition, $\boldsymbol{F}_U$, $\boldsymbol{E}_U$ and $\boldsymbol{P}_U$ depend on the constant κ. *From now on, we fix some $\kappa \geq \kappa_0$ and do not mention it again.*

It is known that $(\boldsymbol{F}_U, \mathfrak{S}_U)$, $(\boldsymbol{E}_U, \mathfrak{S}_U^{\perp})$ are both relatively hierarchically hyperbolic. By definition of $\boldsymbol{F}_U$, $\boldsymbol{E}_U$ and $\boldsymbol{P}_U$, there are natural retractions from $\mathcal{X}$ to these subspaces. We call such a map a *gate map*. Take $\boldsymbol{F}_U$ for example. We denote the gate map to $\boldsymbol{F}_U$ by $\mathfrak{g}_{\boldsymbol{F}_U}$. For all $x \in \mathcal{X}$ and all $V \in \mathfrak{S}$ such that $\mathcal{C}V$ is hyperbolic, $\pi_V(\mathfrak{g}_{\boldsymbol{F}_U}(x))$ uniformly coarsely coincides with the closest point projection of $\pi_V(x)$ to the quasiconvex subset $\pi_V(\boldsymbol{F}_U)$. In fact, gate maps can be defined for all "hierarchically quasiconvex" subspaces, which form a larger class of subspaces of relative HHSes (see [BHS19, §5] for HHSes).

For any (relative) HHS $(\mathcal{X}, \mathfrak{S})$, an automorphism is roughly speaking a bijection from $(\mathcal{X}, \mathfrak{S})$ to itself that preserves its (relative) HHS structure [BHS19, §1G]. The automorphisms of $(\mathcal{X}, \mathfrak{S})$ form a group $\operatorname{Aut}(\mathfrak{S})$, which we call the *automorphism group* of $(\mathcal{X}, \mathfrak{S})$.

Definition 2.12 (hierarchically hyperbolic and relatively hierarchically hyperbolic groups). A finitely generated group G is (*relatively*) *hierarchically hyperbolic* if there exists a (relatively) hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$ and an action $G \to \operatorname{Aut}(\mathfrak{S})$ such that the action $G \curvearrowright X$ is metrically proper and cobounded, and such that the induced action on $\mathfrak{S}$ is cofinite.

Note that if G is (relatively) hierarchically hyperbolic by virtue of its action on the (relatively) hierarchically hyperbolic space $(\mathcal{X}, \mathfrak{S})$, then $(G, \mathfrak{S})$ is a (relatively) hierarchically hyperbolic structure with respect to any word metric on G.

Let $\operatorname{Aut}(\mathfrak{S}; V)$ be the group of automorphisms $g \in \operatorname{Aut}(\mathfrak{S})$ such that $g \cdot V = V$. Then there is a *restriction homomorphism* $\theta_V : \operatorname{Aut}(\mathfrak{S}; V) \to \operatorname{Aut}(\mathfrak{S}_V)$ defined as follows. Given $g \in \operatorname{Aut}(\mathfrak{S}; V)$, let $\theta_V(g)$ act like g on the substructure $\mathfrak{S}_V$. For a group $G < \operatorname{Aut}(\mathfrak{S})$, we write $\operatorname{Stab}_G(V)$ to mean $G \cap \operatorname{Aut}(\mathfrak{S}; V)$ and write G_V to mean the image of $\operatorname{Stab}_G(V)$ under θ_V.

For many HHGs (for example, the case of mapping class groups), every G_V acts acylindrically on $\mathcal{C}V$. However, not all HHGs have this property [DHS20].

Definition 2.13 (colorability). Let $(G, \mathfrak{S})$ be a relative HHG. Let $\mathfrak{S}' \subset \mathfrak{S}$ be a G-invariant subset. We say $\mathfrak{S}'$ is *colorable* if $\mathfrak{S}'$ admits a decomposition $\mathfrak{S}' = \bigsqcup_{i=1}^{\chi} \mathfrak{S}'_i$ into finitely many G-invariant families $\mathfrak{S}'_i$ such that any two domains in the same family are transverse. Such a decomposition is called a *coloring* of $\mathfrak{S}'$. We say a relative HHG $(G, \mathfrak{S})$ is *colorable* if $\mathfrak{S}$ is colorable.

The notion of colorability is formalized in [DMS23; HP22]. There are many classes of (virtually) colorable HHGs, as listed in those papers. In particular, a coloring is constructed for (a finite-index subgroup of) a mapping class group in [BBF15, §5], from which the notion comes. However, one cannot expect that all HHGs are virtually colorable [Hag23]. Nevertheless, Proposition 3.2 of [HP22] provides a sufficient condition for an HHG to be virtually colorable.

Remark 2.14. In this paper, we only treat unbounded domains, i.e., domains with unbounded associated hyperbolic spaces. Abusing terminology, we say a relative HHG $(G, \mathfrak{S})$ is colorable if the collection of unbounded domains is colorable.

3. Thick distance formula

In this section, we will prove a *thick distance formula* that is similar to [BBF21, Theorem 4.13]. This allows us to estimate the distance in a relative HHS by counting only "thick" segments of a hierarchy path instead of the whole hierarchy path. The reader should be aware that the definitions in this section are different from those in [BBF21]. In particular, we do not have tight geodesics in a general HHG.

Let $(\mathcal{X}, \mathfrak{S})$ be a relative HHS and fix $T > 100E + 10D_0$ (see Section 2.3 for constants associated with a relative HHS). As in [BHS19, §2B], we say a domain $U \in \mathfrak{S}$ is *T-relevant* for $x, y \in \mathcal{X}$ if $d_U(x, y) > T$. We write $\mathrm{Rel}_T(x, y)$ for the set of T-relevant domains for x, y, and define $\mathrm{Rel}_T(V; x, y) := \mathrm{Rel}_T(x, y) \cap \mathfrak{S}_V^\circ$. We write $\mathrm{Rel}_T^m(V; x, y)$ for the set of $\sqsubseteq$-maximal elements in $\mathrm{Rel}_T(V; x, y)$. When x and y are fixed, we often omit them from the notation.

Lemma 3.1. *Given $x, y \in \mathcal{X}$ and $U \in \mathrm{Rel}_T(x, y)$, there exist at most two domains $V_1, V_2 \in \mathrm{Rel}_T(x, y)$ such that $U \in \mathrm{Rel}_T^m(V_i; x, y)$ for $i = 1, 2$.*

Proof. Suppose there exist three such domains V_1, V_2, V_3. Since U is maximal in each $\mathrm{Rel}_T(V_i; x, y)$, we know that the V_i are not $\sqsubseteq$-comparable. No two of them can be orthogonal since $U \subsetneq V_i$. Thus, the V_i must be pairwise transverse.

By [BHS19, Proposition 2.8], any set of pairwise transverse elements in $\mathrm{Rel}_T(x, y)$ has a total order $<$, obtained by setting $U < V$ whenever $d_U(y, \rho_U^V) \le E$. We assume that $V_1 < V_2 < V_3$.

On the one hand, $d_{V_2}(\rho_{V_2}^{V_1}, \rho_{V_2}^{V_3}) \ge d_{V_2}(x, y) - 2E > T - 2E$ by the triangle inequality. On the other hand, $d_{V_2}(\rho_{V_2}^{V_1}, \rho_{V_2}^{U}) \le \kappa_0$ and $d_{V_2}(\rho_{V_2}^{V_3}, \rho_{V_2}^{U}) \le \kappa_0$ by consistency, which gives $d_{V_2}(\rho_{V_2}^{V_1}, \rho_{V_2}^{V_3}) \le 2\kappa_0 < T - 2E$. This gives a contradiction. $\square$

Definition 3.2 (T-thickness). Given $\mathfrak{S}' \subset \mathfrak{S}$, we say a pair of points $(x, y) \in \mathcal{X} \times \mathcal{X}$ is *T-thick for $\mathfrak{S}'$* if $\mathrm{diam}(\pi_U(x) \cup \pi_U(y)) \le T$ for all $U \in \mathfrak{S}'$. We define $\mathcal{P}_T(\mathfrak{S}')$ to be the set of all T-thick pairs of points for $\mathfrak{S}'$. If $\mathfrak{S}' = \{U\}$, we also say (x, y) is T-thick for U and write $(x, y) \in \mathcal{P}_T(U)$.

Note that $(x, y) \in \mathcal{P}_T(U)$ if and only if $U \notin \mathrm{Rel}_T(x, y)$, and that $(x, y) \in \mathcal{P}_T(\mathfrak{S}_V^\circ)$ if and only if $\mathrm{Rel}_T(V; x, y) = \varnothing$.

Lemma 3.3. *Let D_0 be the constant provided by Theorem 2.8. For any $x, y \in \mathcal{X}$, let γ be a D_0-hierarchy path between x, y. Given any $U \in \mathfrak{S}$ and any $x', y' \in \gamma$, then*

$$d_U(x', y') \le d_U(x, y) + 2D_0.$$

In particular, if $(x, y) \in \mathcal{P}_T(U)$, then $(x', y') \in \mathcal{P}_{T+2D_0}(U)$.

Proof. By Remark 2.9, $\pi_U(\gamma)$ lies in the D_0-neighborhood of a geodesic connecting $\pi_U(x)$ and $\pi_U(y)$. The conclusion then follows from the triangle inequality. □

Notation. *Throughout this paper, let* $\hat{T} = T + 2D_0$ *and* $\check{T} = T - 2D_0$ *for every constant* $T > 2D_0$.

Notation. Given two points x, y in a hyperbolic space, we write $[x, y]$ to mean a geodesic segment between x, y, which is coarsely unique. For an interval I or a path γ, we write I^-, I^+ or γ^-, γ^+ to mean their endpoints.

Definition 3.4 ((T, R)-thick distance). Fix sufficiently large constants T, R. Let γ be a D_0-hierarchy path between x and y. Let $\gamma_1, \dots, \gamma_n \subset \gamma$ be disjoint subpaths occurring in this order such that $(\gamma_i^-, \gamma_i^+) \in \mathcal{P}_T(\mathfrak{S}_V^\circ)$ for each i.

The (T, R)-*thick distance in* V is denoted by $d_V^{T,R}(x, y)$ and is defined to be the supremum of $\sum_{i=1}^n \{\!\{d_V(\gamma_i^-, \gamma_i^+)\}\!\}_R$ over all such choices for the γ_i, and for all D_0-hierarchy paths from x to y.

It is always true that $d_V^{T,R}(x, y) \le \{\!\{d_V(x, y)\}\!\}_R$. This becomes an equality if V is $\sqsubseteq$-minimal. For the opposite direction, we have the following estimate.

Lemma 3.5. *Fix constants* $T, R > 100E$. *For any* $x, y \in \mathcal{X}$ *and* $W \in \mathfrak{S}$, *we have*

$$\{\!\{d_W(x, y)\}\!\}_R \le d_W^{T,R}(x, y) + (6E + 2R)\left|\mathrm{Rel}_{\check{T}}^m(W; x, y)\right|.$$

Proof. If $\mathrm{Rel}_{\check{T}}^m(W) = \varnothing$, then $(x, y) \in \mathcal{P}_T(\mathfrak{S}_W^\circ)$. Thus, both sides of the above inequality are equal.

Now assume that $\mathrm{Rel}_{\check{T}}^m(W) \neq \varnothing$ and $d_W(x, y) \ge R$. Then $\mathcal{C}W$ is not $\sqsubseteq$-minimal, so it is hyperbolic. Let $\gamma : I \to \mathcal{X}$ be a D_0-hierarchy path realizing $d_W^{T,R}(x, y)$, where I is an interval of $\mathbb{R}$. For any $V \sqsubsetneq W$, we define

$$s_V^- := \inf\{s \in I \mid \exists U \sqsubseteq V \text{such that } d_W(\gamma(s), \rho_W^U) \le 2E\},$$
$$s_V^+ := \sup\{s \in I \mid \exists U \sqsubseteq V \text{such that } d_W(\gamma(s), \rho_W^U) \le 2E\}.$$

For any $U \sqsubseteq V \sqsubsetneq W$, we know that $d_W(\rho_W^U, \rho_W^V) \le \kappa_0$ by consistency. Thus,

$$d_W(\gamma(s_V^-), \rho_W^V) \le 2E + \kappa_0 \le 3E \quad \text{and} \quad d_W(\gamma(s_V^+), \rho_W^V) \le 2E + \kappa_0 \le 3E.$$

Therefore,

$$d_W(\gamma(s_V^-), \gamma(s_V^+)) \le d_W(\gamma(s_V^-), \rho_W^V) + d_W(\gamma(s_V^+), \rho_W^V) \le 6E.$$

Let $J_0, \dots, J_n$ be the collection of maximal intervals in $I - \bigcup_{V \in \mathrm{Rel}_{\check{T}}^m(W)} (s_V^-, s_V^+)$. Note that $n \le |\mathrm{Rel}_{\check{T}}^m(W)|$. We now prove that $(\gamma(J_i^-), \gamma(J_i^+)) \in \mathcal{P}_T(\mathfrak{S}_W^\circ)$.

On the one hand, $d_W(\rho_W^U, \gamma(J_i)) \ge 2E$ for any $U \in \mathrm{Rel}_{\check{T}}(W)$ by the definition of J_i. By the Morse lemma,

$$d_W(\rho_W^U, [\gamma(J_i^-), \gamma(J_i^+)]) \ge d_W(\rho_W^U, \gamma(J_i)) - E \ge E.$$

Therefore, $d_U(\gamma(J_i^-), \gamma(J_i^+)) \le E < T$ by BGI. On the other hand, (x, y) is $\check{T}$-thick for $\mathfrak{S}_W^\circ - \mathrm{Rel}_{\check{T}}(W)$ by definition. It follows from Lemma 3.3 that $(\gamma(J_i^-), \gamma(J_i^+))$ is T-thick for $\mathfrak{S}_W^\circ - \mathrm{Rel}_{\check{T}}(W)$. In sum, $(\gamma(J_i^-), \gamma(J_i^+)) \in \mathcal{P}_T(\mathfrak{S}_W^\circ)$.

Finally, we estimate

$$\begin{aligned} d_W(x, y) &\le \sum_{i=0}^{n} d_W(\gamma(J_i^+), \gamma(J_i^-)) + 6E|\mathrm{Rel}_{\check{T}}^m(W)| \\ &\le d_W^{T,R}(x, y) + R(n+1) + 6E\,|\mathrm{Rel}_{\check{T}}^m(W)| \\ &\le d_W^{T,R}(x, y) + (6E+2R)\,|\mathrm{Rel}_{\check{T}}^m(W)|. \end{aligned}$$ □

Let S denote the unique maximal domain in $\mathfrak{S}$. Recall that the level $\ell(S)$ of S is equal to the complexity of $(\mathcal{X}, \mathfrak{S})$.

Theorem 3.6. *Fix constants T, R with $\check{T} \ge R > 100E$. Let $x, y \in \mathcal{X}$. Then, for each n,*

$$\sum_{\ell(W)\le n} \{\!\{d_W(x, y)\}\!\}_R \le \sum_{\ell(W)=n} d_W^{T,R}(x, y) + 7 \sum_{\ell(W)<n} \{\!\{d_W(x, y)\}\!\}_R.$$

Note that each sum is has finitely many terms since there are only finitely many W such that $d_W(x, y) \ge R$ for given x, y, by the distance formula (Theorem 2.6).

Proof. If $\ell(W) = n$ then by Lemma 3.5,

$$\begin{aligned} \{\!\{d_W(x, y)\}\!\}_R &\le d_W^{T,R}(x, y) + (6E+2R)\,|\mathrm{Rel}_{\check{T}}^m(W)| \\ &\le d_W^{T,R}(x, y) + (6E+2R) \sum_{V\in\mathrm{Rel}_{\check{T}}^m(W)} \frac{\{\!\{d_V(x, y)\}\!\}_{\check{T}}}{\check{T}} \\ &\le d_W^{T,R}(x, y) + 3 \sum_{V\in\mathrm{Rel}_{\check{T}}^m(W)} \{\!\{d_V(x, y)\}\!\}_{\check{T}} \\ &\le d_W^{T,R}(x, y) + 3 \sum_{V\in\mathrm{Rel}_{\check{T}}^m(W)} \{\!\{d_V(x, y)\}\!\}_R. \end{aligned}$$

By Lemma 3.1, any V appears in at most two $\mathrm{Rel}_{\check{T}}^m(W)$. Therefore, if we sum up the left side over all W with $\ell(W) = n$, we have

$$\sum_{\ell(W)=n} \{\!\{d_W(x, y)\}\!\}_R \le \sum_{\ell(W)=n} d_W^{T,R}(x, y) + 6 \sum_{\ell(W)<n} \{\!\{d_W(x, y)\}\!\}_R.$$

Adding $\sum_{\ell(W)<n} \{\!\{d_W(x, y)\}\!\}_R$ to both sides gives the desired inequality. □

Corollary 3.7. *Fix constants T, R with $\check{T} \ge R > 100E$. Let $x, y \in \mathcal{X}$. Then*

$$\frac{1}{D_0} \sum_{W\in\mathfrak{S}} d_W^{T,R}(x, y) - D_0 \le \sum_{W\in\mathfrak{S}} \{\!\{d_W(x, y)\}\!\}_R \le 7^{\ell(S)-1} \sum_{W\in\mathfrak{S}} d_W^{T,R}(x, y).$$

Proof. The first inequality is trivial since $(1/D_0)d_W^{T,R}(x,y) - D_0 \le \{\!\{d_W(x,y)\}\!\}_R$ for all W. By inductively applying Theorem 3.6, with base case $n = \ell(S)$, we have

$$\sum_{W\in\mathfrak{S}} \{\!\{d_W(x,y)\}\!\}_R \le 7^{\ell(S)-n}\Bigg(\sum_{n\le\ell(W)\le\ell(S)} d_W^{T,R}(x,y) + 7\sum_{\ell(W)<n} \{\!\{d_W(x,y)\}\!\}_R\Bigg).$$

When $n = 1$, the last term on the right is zero, and the result follows. □

Combining the distance formula (Theorem 2.6) with Corollary 3.7, we obtain:

Theorem 3.8 (thick distance formula). *There exists R_0 such that, for all T, R with $\check{T} \ge R > R_0$, there exists a constant $L > 0$ such that, for all $x, y \in \mathcal{X}$,*

$$d_{\mathcal{X}}(x,y) \asymp_{(L,L)} \sum_{W\in\mathfrak{S}} d_W^{T,R}(x,y).$$

4. Estimation of thick distance via quasi-axes

Our proof in this section is inspired by [NY23]. The main technique in the proof of [NY23, Lemma 5.5] that is different from [BBF21] is the use of the extension lemma [Yan19, Lemma 2.14]. Lemma 4.1 below is a trimmed version for acylindrical actions on hyperbolic spaces. In the original statement of Lemma 4.1 in [NY23], the group action is required to be cobounded, but it is easy to see from the proof that this condition can be removed.

Lemma 4.1 (extension lemma [NY23, Lemma 4.13]). *Let H be a group acting nonelementarily and acylindrically on a δ-hyperbolic space Y. Fix a base point $o \in Y$. There exists a set $F \subset H$ of three loxodromic elements and constants $\lambda \ge 1$, and $c \ge 0$ with the following property.*

For any $h \in H$ there exists $f \in F$ such that hf is a loxodromic element and the bi-infinite path

$$\gamma_h = \bigcup_{i\in\mathbb{Z}} (hf)^i([o,ho][ho,hfo])$$

is a (λ, c)-quasigeodesic.

Let $(\mathcal{X}, \mathfrak{S})$ be a relative HHS with a coarse constant E, and let G be a relative HHG by virtue of its action on $\mathcal{X}$. Corresponding to Definition 1.3, we say that a domain $U \in \mathfrak{S}$ has

(1) *hyperbolicity* if $\mathcal{C}U$ is hyperbolic.

(2) *acylindrical image* if G_U acts on $\mathcal{C}U$ acylindrically.

(3) *cobounded nested region* if G_U acts on $\boldsymbol{F}_U$ coboundedly.

(4) *separable quasi-axes* if for any element $g \in \mathrm{Stab}_G(U)$ that acts loxodromically on $\mathcal{C}U$, the elementary closure $EC(g)$ is *separable* in G, i.e., $EC(g)$ equals the intersection of all finite-index subgroups of G that contain $EC(g)$.

Till the end of this section, let $V \in \mathfrak{S}$ be an unbounded domain that has hyperbolicity, cobounded nested region and acylindrical image.

The next statement could be compared with Theorem 4.19 of [BBF21] for mapping class groups.

Lemma 4.2 (extension of thick segments). *There exist constants $\lambda \geq 1$, $c \geq 0$, $B \geq 0$ such that the following holds. For any $T, R > 0$, there exists a G_V-finite collection $\mathcal{A}_V = \mathcal{A}_V^{T,R}$ of (λ, c)-quasi-axes in $\mathcal{C}V$ such that for any pair of points $(x, y) \in \mathcal{P}_{\max\{\hat{T},R\}}(\mathfrak{S}_V)$ there exists $\gamma \in \mathcal{A}_V$ such that $[\pi_V(x), \pi_V(y)] \subset \mathcal{N}_B(\gamma)$.*

Proof. Fix a base point $o \in \boldsymbol{F}_V$ and project it to a base point in $\mathcal{C}V$. If the action $G_V \curvearrowright \mathcal{C}V$ is elementary, $\mathcal{C}V$ itself is a quasi-axis of a loxodromic element, and thus satisfies the requirement. Now assume that the action $G_V \curvearrowright \mathcal{C}V$ is nonelementary. Lemma 4.1 provides a finite set $F \subset G_V$ and constants $\lambda \geq 1$, $c \geq 0$. Recall that $(\boldsymbol{F}_V, \mathfrak{S}_V)$ is a relative HHS. $\boldsymbol{F}_V$ is proper because $\mathcal{X}$ is proper.

Since G_V acts coboundedly on $\boldsymbol{F}_V$, there exists $\epsilon > 0$ such that $\boldsymbol{F}_V$ is covered by the G_V-translates of any ϵ-ball. Let $T' = \max\{\hat{T}, R\}$. By Theorem 2.6, there exists $r > 0$, depending only on E, T', ϵ, such that the distance between any pair of points in $\boldsymbol{F}_V$ that is $(T' + \epsilon)$-thick for $\mathfrak{S}_V$ is bounded above by r. Fix any base point $o \in \boldsymbol{F}_V$. Since $\boldsymbol{F}_V$ is proper, there exists a finite subset $S \subset G_V$ such that $\mathcal{N}_{r+\epsilon}(o)$ is covered by $\bigcup_{s \in S} s \cdot \mathcal{N}_\epsilon(o)$.

Lemma 4.1 tells us that for each $s \in S$, there exists $f \in F$ such that sf is a loxodromic element acting on $\mathcal{C}V$. Let $\mathcal{A}_V = \mathcal{A}_V^{T,R}$ be the collection of G_V-translates of the (λ, c)-quasi-axes provided by Lemma 4.1 for all loxodromic elements of the form sf.

Now we verify that $\mathcal{A}_V$ meets our requirements. Let $(x, y) \in \mathcal{P}_{T'}(\mathfrak{S}_V)$. We can choose $g \in G_V$ such that $d_{\boldsymbol{F}_V}(x, go) < \epsilon$. Then

$$d_{\boldsymbol{F}_V}(o, g^{-1}y) \leq d_{\boldsymbol{F}_V}(x, y) + d_{\boldsymbol{F}_V}(x, go) < r + \epsilon.$$

By our choice of S, there exists $s \in S$ such that $d_{\boldsymbol{F}_V}(g^{-1}y, so) < \epsilon$. Thus,

$$d_V(x, go) < E\epsilon + E, \quad d_V(y, gso) < E\epsilon + E,$$

because π_V is E-coarsely Lipschitz. Since $\mathcal{C}V$ is E-hyperbolic, we can find $B > 0$ by the fellow-traveler property such that $[\pi_V(x), \pi_V(y)] \subset \mathcal{N}_B([g \cdot \pi_V(o), gs \cdot \pi_V(o)])$. By construction, $[g \cdot \pi_V(o), gs \cdot \pi_V(o)]$ is contained in some $\gamma \in \mathcal{A}_V$, so we are done. □

Notation. Assume that γ is a quasigeodesic in a hyperbolic space Y. We write $\pi_\gamma : Y \to \gamma$ to mean the closest point projection. For $x, y \in Y$, we write $d_\gamma(x, y)$ to mean $\operatorname{diam}(\pi_\gamma(x) \cup \pi_\gamma(y))$.

Notation. Assume that γ is a quasigeodesic in $\mathcal{C}V$. We write $\pi_\gamma^{\mathcal{X}}$ to mean $\pi_\gamma \circ \pi_V$. For $x, y \in \mathcal{X}$, we write $d_\gamma^{\mathcal{X}}(x, y)$ to mean $\operatorname{diam}(\pi_\gamma^{\mathcal{X}}(x) \cup \pi_\gamma^{\mathcal{X}}(y))$.

The following lemma is a well-known corollary of the Morse lemma for quasi-geodesics in δ-hyperbolic spaces so we omit the proof.

Lemma 4.3. *Let γ and α be two (λ, c)-quasigeodesics in a δ-hyperbolic space. Then for any $B > 0$, there exists a constant $C = C(\lambda, c, B, \delta) > 0$ such that*

$$d_\gamma(\alpha^-, \alpha^+) \geq \operatorname{diam}(\alpha \cap \mathcal{N}_B(\gamma)) - C.$$

The main result of this section is the following estimate, which generalizes Proposition 4.18 of [BBF21].

Proposition 4.4. *For any $K > 0$, there exists $R > 0$ such that the following holds. Given any $T > 0$, let $\mathcal{A}_V = \mathcal{A}_V^{T,R}$ be the collection of (λ, c)-quasi-axes provided by Lemma 4.2. Then for any two points $x, y \in \mathcal{X}$,*

$$d_V^{T,R}(x, y) \leq 2(D_0 + 1) \sum_{\gamma \in \mathcal{A}_V} \{\!\{ d_\gamma^{\mathcal{X}}(x, y) \}\!\}_K,$$

where D_0 is the constant provided by Theorem 2.8.

Proof. Let $C = C(\lambda + D_0, c + D_0, B, E)$ be the constant provided by Lemma 4.3. Let $R > 2D_0(C + 1) + K$. We will show that projections to quasi-axes $\mathcal{A}_V = \mathcal{A}_V^{T,R}$ bound the (T, R)-thick distance in V from above.

For any two points $x, y \in \mathcal{X}$, let β be a (D_0, D_0)-hierarchy path connecting x and y realizing $d_V^{T,R}(x, y)$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be the collection of disjoint subpaths of β with $d_V(\alpha_i^-, \alpha_i^+) \geq R$ and $(\alpha_i^-, \alpha_i^+) \in \mathcal{P}_T(\mathfrak{S}_V^\circ)$ such that

$$d_V^{T,R}(x, y) = \sum_{i=1}^{n} d_V(\alpha_i^-, \alpha_i^+).$$

By the definition of gate maps, $\pi_V(z)$ is coarsely $\pi_V(\mathfrak{g}_{\boldsymbol{F}_V}(z))$ for any $z \in \mathcal{X}$. Thus, the difference between $d_V(\alpha_i^-, \alpha_i^+)$ and $d_V(\mathfrak{g}_{\boldsymbol{F}_V}(\alpha_i^-), \mathfrak{g}_{\boldsymbol{F}_V}(\alpha_i^+))$ is uniformly bounded. This enables us to replace α_i with $\mathfrak{g}_{\boldsymbol{F}_V}(\alpha_i)$ from now on. We divide each α_i into several consecutive subpaths $\{\tilde{\alpha}_{i,j} \mid 1 \leq j \leq m_i\}$ with $d_V(\tilde{\alpha}_{i,j}^-, \tilde{\alpha}_{i,j}^+) = R$ for $j = 1, \ldots, m_i - 1$ and $d_V(\tilde{\alpha}_{i,m_i}^-, \tilde{\alpha}_{i,m_i}^+) \leq R$. By Lemma 3.3, we already know that $(\tilde{\alpha}_{i,j}^-, \tilde{\alpha}_{i,j}^+) \in \mathcal{P}_{\hat{T}}(\mathfrak{S}_V^\circ)$ for every pair (i, j). Thus,

$$d_U(\tilde{\alpha}_{i,j}^-, \tilde{\alpha}_{i,j}^+) \leq \max\{\hat{T}, R\}$$

for all $U \in \mathfrak{S}_V$. By Lemma 4.2, there exists $\gamma_{i,j} \in \mathcal{A}_V$ such that $\pi_U(\tilde{\alpha}_{i,j}) \subset \mathcal{N}_B(\gamma_{i,j})$ (with an increased B by a uniform constant), which yields

$$\operatorname{diam}(\pi_V(\alpha_i) \cap \mathcal{N}_B(\gamma_{i,j})) \geq R.$$

Let $\mathcal{A}_V'$ be the collection of all distinct $\gamma_{i,j}$. We see that

$$\pi_V(\alpha_i \setminus \tilde{\alpha}_{i,m_i}) \subset \bigcup_{\gamma \in \mathcal{A}_V'} \pi_V(\alpha_i) \cap \mathcal{N}_B(\gamma).$$

Thus, we have

$$d_V(\alpha_i^-, \alpha_i^+) \le \sum_{\gamma \in \mathcal{A}'_V} \operatorname{diam}(\pi_V(\alpha_i) \cap \mathcal{N}_B(\gamma)) + R \le 2 \sum_{\gamma \in \mathcal{A}'_V} \operatorname{diam}(\pi_V(\alpha_i) \cap \mathcal{N}_B(\gamma)).$$

Summing up from $i = 1$ to n yields

$$d_V^{T,R}(x, y) \le 2 \sum_{\gamma \in \mathcal{A}'_V} (D_0 \operatorname{diam}(\pi_V(\beta) \cap \mathcal{N}_B(\gamma)) + D_0). \tag{1}$$

Note that $R > 2D_0(C+1) + K$. Thus, Lemma 4.3 tells us that

$$d_\gamma^{\mathcal{X}}(x, y) \ge \operatorname{diam}(\pi_V(\beta) \cap \mathcal{N}_B(\gamma)) - C \ge R - C > D_0(C+1) + K$$

for each $\gamma \in \mathcal{A}'_V$. We now estimate by Lemma 4.3 that

$$D_0 \operatorname{diam}(\pi_V(\beta) \cap \mathcal{N}_B(\gamma)) + D_0 \le D_0(d_\gamma^{\mathcal{X}}(x, y) + C) + D_0 < (D_0 + 1) d_\gamma^{\mathcal{X}}(x, y).$$

Combining this with (1), we obtain

$$\begin{aligned} d_V^{T,R}(x, y) &\le 2(D_0 + 1) \sum_{\gamma \in \mathcal{A}'_V} d_\gamma^{\mathcal{X}}(x, y) \\ &= 2(D_0 + 1) \sum_{\gamma \in \mathcal{A}'_V} \{\!\{ d_\gamma^{\mathcal{X}}(x, y) \}\!\}_K \le 2(D_0 + 1) \sum_{\gamma \in \mathcal{A}_V} \{\!\{ d_\gamma^{\mathcal{X}}(x, y) \}\!\}_K. \end{aligned}$$ □

5. Construction of quasitrees

This section is devoted to the proof of Theorem 1.4. Let $(G, \mathfrak{S})$ be a relative HHG that is virtually colorable and assume that every domain in $\mathfrak{S}$ is of type I or II. The index set $\mathfrak{S}$ admits a G-invariant decomposition $\mathfrak{S} = \mathfrak{S}^I \sqcup \mathfrak{S}^{II}$, where $\mathfrak{S}^I$ (respectively, $\mathfrak{S}^{II}$) only contains domains of type I (respectively, type II). Note that types I and II are not mutually exclusive, but for those domains of both types, we can simply put them in $\mathfrak{S}^{II}$.

Before starting the proof, we summarize the dependencies of some important constants that will be used in the proof:

$$(E, D_0, A) \xrightarrow{\text{Corollary 5.3}} \theta \xrightarrow{\text{Theorem 2.4}} \xi \xrightarrow{\text{Lemma 5.4}} K \xrightarrow{\text{Proposition 4.4}} R \xrightarrow{\text{Theorem 3.8}} T.$$

Here A stands for the acylindrical constants. We draw an arrow from a constant M to N if N depends on M. Remember that the dependency graph shown above is incomplete, but we hope it is helpful to the reader.

5.1. *Quasitrees from domains of type I.*

Proposition 5.1. *There exists a finite-index subgroup $H < G$ satisfying the following. For any sufficiently large constant R and any $T > 0$, there exist quasitrees $\mathcal{T}_1, \ldots, \mathcal{T}_n$*

such that H acts on $\prod_{j=1}^{n} \mathcal{T}_j$ diagonally and for any choice of base points $o_j \in \mathcal{T}_j$ we have

$$\sum_{V \in \mathfrak{S}^I} d_V^{T,R}(1, h) \preceq \sum_{j=1}^{n} d_{\mathcal{T}_j}(o_j, ho_j)$$

for any $h \in H$.

Before proving this, we recall an auxiliary result.

Proposition 5.2 [BBF21, Proposition 3.4]. *Let a group H act on a δ-hyperbolic space Y. Assume that the image of H in* $\mathrm{Isom}(Y)$ *is acylindrical. Consider a loxodromic element $g \in H$ and the collection $\mathbb{A}$ of all H-translates of a fixed (λ, c)-quasi-axis of g. Then there exists a constant $\theta > 0$ depending only on λ, c, δ and the acylindrical constants such that for any $\gamma \in \mathbb{A}$, the set*

$$\left\{h \in H \mid \mathrm{diam}(\pi_\gamma(h\gamma)) \geq \theta\right\}$$

is a finite union of double $EC(g)$-cosets.

Corollary 5.3. *Let $U \in \mathfrak{S}^I$. Consider a (λ, c)-quasi-axis $\gamma \subset \mathcal{C}U$ for some loxodromic element of the acylindrical action of G_U. Then there exists $\theta > 0$, only depending on λ, c, E and the acylindrical constants, and a finite-index subgroup $G_\gamma < G$ such that every translate of γ by an element of $G_\gamma \cap \mathrm{Stab}_G(U)$ either has finite Hausdorff distance with γ or has θ-bounded projection to γ.*

Proof. This is clear by Proposition 5.2 and separability of quasi-axes. □

By definition of relative HHGs, $\mathfrak{S}^I$ consists of finitely many G-orbits, so acylindrical constants for $U \in \mathfrak{S}^I$ can be chosen uniformly. Thus, Lemma 4.2 provides uniform constants $\lambda \geq 1, c \geq 0$ for every $U \in \mathfrak{S}^I$. This further gives a uniform constant $\theta > 0$ by Corollary 5.3.

Let $\mathcal{U}$ be a G-representative set of $\mathfrak{S}^I$ such that $1 \in \boldsymbol{P}_U$ for any $U \in \mathcal{U}$. Let $U \in \mathcal{U}$. Let $T > 0$ and let $K > 0$ be a sufficiently large constant that will be decided by Lemma 5.4. Lemma 4.2 provides a G_U-finite collection $\mathcal{A}_U = \mathcal{A}_U^{T,R}$ of (λ, c)-quasi-axes, where R is provided by Proposition 4.4. By Corollary 5.3, we can find a finite-index subgroup $H_U < G$ such that for any $\gamma \in \mathcal{A}_U$ and $h \in H_U \cap \mathrm{Stab}_G(U)$, either $d_{\mathrm{Haus}}(h\gamma, \gamma) < \infty$ or $\mathrm{diam}\,\pi_\gamma(h\gamma) < \theta$. For any $g \in G$, we define $\mathcal{A}_{gU} := \{g\gamma \mid \gamma \in \mathcal{A}_U\}$.

Let $\mathcal{A} := \bigsqcup_{U \in \mathfrak{S}} \mathcal{A}_U$ and $H := \bigcap_{U \in \mathcal{U}} H_U$. Since $\mathcal{U}$ is finite, H is of finite index in G. By adding finitely many domains to $\mathcal{U}$ so that there is one representative for each H-orbit on $\mathfrak{S}^I$, we obtain an H-representative set $\tilde{\mathcal{U}}$ of $\mathfrak{S}^I$. We still assume that $1 \in \boldsymbol{P}_U$ for any $U \in \tilde{\mathcal{U}}$. Let $\{\gamma_1, \ldots, \gamma_n\}$ be an H-representative set of $\mathcal{A}$. We assume that every representative γ_j is contained in $\mathcal{C}U$ for some $U \in \tilde{\mathcal{U}}$. Let $\mathcal{A}_j \subset \mathcal{A}$ be the H-orbit of γ_j.

Without loss of generality, we assume that H is colorable instead of virtually colorable. Thus, the H-orbit of any domain is pairwise transverse. By [HP22, Lemma 3.4], every H-orbit of $\mathfrak{S}^I$ is an H-projection system with constant s_0+4E, where s_0 is the constant provided by Theorem 2.6. Thus, every $\mathcal{A}_j$ is an H-projection system with a uniform projection constant $\xi = \xi(\theta, s_0, \lambda, c, E)$ by Theorem 2.4. The projections defined there will be denoted by Π_γ.

Using Theorem 2.1, we obtain modified projections Π'_γ such that $(\mathcal{A}_j, \{\Pi'_\gamma\})$ satisfies the strong projection axioms with constant $\xi' = \xi'(\xi)$ and that $\Pi_\gamma(\alpha)$ and $\Pi'_\gamma(\alpha)$ are apart from each other within a uniform Hausdorff distance $\epsilon = \epsilon(\xi)$. For any $K' \geq 4\xi'$, $\mathcal{C}_{K'}\mathcal{A}_j$ is a quasitree by Theorem 2.2. The following lemma is an estimate via the orbit map between the projections $\pi^H_\gamma = \pi_\gamma \circ \pi_V$ in the relative HHG structure and the projections Π'_γ in the quasitree $\mathcal{C}_{K'}\mathcal{A}_j$. Recall from Section 4 that $d^H_\gamma(g,h)$ is defined as $\operatorname{diam}(\pi^H_\gamma(g) \cup \pi^H_\gamma(h))$ for any $g, h \in H$.

Lemma 5.4. *Fix a base point $o_j \in \gamma_j$ for each $j = 1, \ldots, n$. There exists a sufficiently large constant $K' = K'(\xi, \lambda, c, E)$ and a constant $\Delta > 0$ such that if $K \geq 2K'$ then*

$$\sum_{\gamma \in \mathcal{A}_j} \{\!\{ d^H_\gamma(1,h) \}\!\}_K \leq 8 d_{\mathcal{C}_{K'}\mathcal{A}_j}(o_j, ho_j) + \Delta$$

for any $h \in H$ and any $j = 1, \ldots, n$.

The proof depends on another lemma.

Lemma 5.5. *For any constants $A, B \geq 0$ and constants $L, M > 0$,*

$$\frac{\{\!\{A+B\}\!\}_{L+M}}{L+M} \leq \frac{\{\!\{A\}\!\}_L}{L} + \frac{\{\!\{B\}\!\}_M}{M}.$$

Proof. Assume that $A+B \geq L+M$. First, if $A < L$ then $B > M$. Thus, $\{\!\{B\}\!\}_M = B \geq \frac{M}{L+M}(A+B)$. Next, if $B < M$ then $A > L$ and the same argument holds. Finally, if $A \geq L$ and $B \geq M$, then all thresholds are reached and the inequality is obviously true. □

Proof of Lemma 5.4. For simplicity, we use $|p-q|$ to mean the distance between two points p, q in the same space. Assume that $\gamma_j \subset \mathcal{C}U$ for some $U \in \tilde{\mathcal{U}}$. For any $g \in H - \operatorname{Stab}_H(U)$, since $1 \in \boldsymbol{P}_U$ and $U \pitchfork gU$, we have $|\pi_{gU}(1) - \rho^U_{gU}| \leq E$ by the definition of standard product regions.

Assume $\gamma = g\gamma_j$. By hyperbolicity, there exists a constant $F = F(E, \lambda, c)$ such that if $\gamma \subset \mathcal{C}gU \neq \mathcal{C}U$ then $|\pi^H_\gamma(1) - \Pi'_\gamma(o_j)| \leq |\pi_{gU}(1) - \rho^U_{gU}| + F + \epsilon \leq E + F + \epsilon$. Let $M > \xi' + E + F + \epsilon$ and define $\delta_\gamma(h) = |\pi^H_\gamma(h) - \Pi'_\gamma(ho_j)|$. We see that if $\delta_\gamma(h) = \delta_{h^{-1}\gamma}(1) \geq M$ then $\gamma \subset \mathcal{C}hU$. Thus, for a fixed $h \in H$, there are only finitely many $\gamma \in \mathcal{A}_j$ such that $\delta_\gamma(h) \geq M$ by projection axiom (P2).

Let $K' > 2M + 4\xi'$. Define $D_\gamma(1,h) = \left|\Pi'_\gamma(o_j) - \Pi'_\gamma(ho_j)\right|$. By the triangle inequality and Lemma 5.5, we obtain

$$\{\!\{d^H_\gamma(1,h)\}\!\}_{K'+2M} \le \frac{K'+2M}{K'}\{\!\{D_\gamma(1,h)\}\!\}_{K'} + \frac{K'+2M}{2M}\{\!\{\delta_\gamma(1)+\delta_\gamma(h)\}\!\}_{2M}.$$

Therefore,

$$\{\!\{d^H_\gamma(1,h)\}\!\}_{K} \le 2\{\!\{D_\gamma(1,h)\}\!\}_{K'} + \frac{K'+2M}{M}\{\!\{\delta_\gamma(1)\}\!\}_{M}.$$

Summing over $\gamma \in \mathcal{A}_j$, we obtain

$$\sum_{\gamma\in\mathcal{A}_j}\{\!\{d^H_\gamma(1,h)\}\!\}_{K} \le 2\sum_{\gamma\in\mathcal{A}_j}\{\!\{D_\gamma(1,h)\}\!\}_{K'} + \frac{K'+2M}{M}\sum_{\gamma\in\mathcal{A}_j}\{\!\{\delta_\gamma(1)\}\!\}_{M}.$$

By the preceding discussion,

$$\Delta_j = \frac{K'+2M}{M}\sum_{\gamma\in\mathcal{A}_j}\{\!\{\delta_\gamma(1)\}\!\}_{M}$$

is a finite constant. Let $\Delta = \max_{1\le j\le n}\Delta_j$. We conclude using Theorem 2.3. □

Proof of Proposition 5.1. Let K and K' be the constants provided by Lemma 5.4. Let $T > 0$. From the discussion before Lemma 5.4, we know that the collection $\mathcal{A}$ of quasi-axes provided by Proposition 4.4 forms n quasitrees $\mathcal{C}_{K'}\mathcal{A}_j$, $j = 1, \dots, n$. Moreover, Proposition 4.4 tells us that there exists $R > 0$ such that

$$\sum_{V\in\mathfrak{S}^I} d^{T,R}_V(1,h) \preceq \sum_{\gamma\in\mathcal{A}}\{\!\{d^H_\gamma(1,h)\}\!\}_K.$$

Finally, we conclude by Lemma 5.4. □

5.2. ***Quasitrees from domains of type II.*** We now prove an analogue of Proposition 5.1 for domains of type II.

Let $H < G$ be the subgroup provided by Proposition 5.1. Fix an H-representative set $\mathcal{V}$ of $\mathfrak{S}^{II}$ such that $1 \in \boldsymbol{P}_V$ for any $V \in \mathcal{V}$. For $V \in \mathcal{V}$ we write $[V]$ to mean its H-orbit. By colorability and [HP22, Lemma 3.4], every $[V]$ is an H-projection system with constant $s_0 + 4E$.

Fix any $V \in \mathcal{V}$. By property QT_0 of the action $\mathrm{Stab}_G(V) \curvearrowright \mathcal{C}V$, there exist quasitrees T^i_V along with $\mathrm{Stab}_H(V)$-equivariant maps $\iota^i_V : \mathcal{C}V \to T^i_V$ for $i = 1, \dots, n_V$ such that

$$\prod_{i=1}^{n_V}\iota^i_V : \mathcal{C}V \to \prod_{i=1}^{n_V} T^i_V$$

is a (λ', c')-quasi-isometric embedding. Thus ι^i_V is (λ', c')-coarsely Lipschitz. Fix $i \in \{1, \dots, n_V\}$. It is conventional to extend the map ι^i_V on $[V]$ in an H-equivariant way. This means that we can construct a collection of quasitrees $\boldsymbol{T}^i_{[V]} = \{T^i_U \mid U \in [V]\}$

with an H-action and a collection of coarsely Lipschitz maps $\{\iota_U^i : \mathcal{C}U \to T_U^i \mid U \in [V]\}$ such that the following diagram commutes for any $h \in H$ and $U \in [V]$:

$$\begin{array}{ccc} \mathcal{C}U & \xrightarrow{\;h\;} & \mathcal{C}hU \\ {\scriptstyle \iota_U^i}\big\downarrow & & \big\downarrow{\scriptstyle \iota_{hU}^i} \\ T_U^i & \xrightarrow{\;h\;} & T_{hU}^i \end{array}$$

Define $\Pi_{T_{hU}^i}(T_U^i) := \iota_{hU}^i(\rho_{hU}^U)$ for any $U \in [V]$ and $hU \neq U$. Clearly, these projections are H-equivariant and the projection axioms pass to the collection $(\boldsymbol{T}_{[V]}^i, \{\Pi_{T_U^i}\})$ under coarsely Lipschitz maps $\{\iota_U^i\}$. We modify the projections within an error ϵ such that $(\boldsymbol{T}_{[V]}^i, \{\Pi'_{T_U^i}\})$ satisfies the strong projection axioms with constant $\zeta = \zeta(s_0, \lambda', c', E)$. For any $K'' \geq 4\zeta$, $\mathcal{C}_{K''}\boldsymbol{T}_{[V]}^i$ is a quasitree by Theorem 2.2. Define $d_{T_U^i}(1, h) := \left|\iota_U^i(\pi_U(1)) - \iota_U^i(\pi_U(h))\right|$ for any $U \in [V]$. For any $V \in \mathcal{V}$ and $i \in \{1, \ldots, n_V\}$, fix a base point $o_V^i \in T_V^i$. The following proposition is an analogue of Proposition 5.1.

Proposition 5.6. *There is a constant $K'' = K''(\zeta, \lambda', c', E)$ such that $R \geq 2K''$ implies*

$$\sum_{U \in \mathfrak{S}^{II}} \{\!\{d_U(1, h)\}\!\}_R \preceq \sum_{V \in \mathcal{V}} \sum_{i=1}^{n_V} d_{\mathcal{C}_{K''}\boldsymbol{T}_{[V]}^i}(o_V^i, ho_V^i),$$

for any $h \in H$.

Proof. Fix $V \in \mathcal{V}$ and $i \in \{1, \ldots, n_V\}$. For any $U \in [V] - \{V\}$, we have

$$\left|\iota_U^i(\pi_U(1)) - \Pi'_{T_U^i}(o_V^i)\right| \leq \lambda' |\pi_U(1) - \rho_U^V| + c' + \epsilon \leq \lambda' E + c' + \epsilon.$$

Let $K'' > 4\zeta + 2(\lambda' E + c' + \epsilon)$. As in the proof of Lemma 5.4, we estimate that

$$\sum_{U \in [V]} \{\!\{d_{T_U^i}(1, h)\}\!\}_R \preceq \{\!\{|\Pi'_{T_U^i}(o_i) - \Pi'_{T_U^i}(ho_i)|\}\!\}_{K''} \preceq d_{\mathcal{C}_{K''}\boldsymbol{T}_{[V]}^i}(o_V^i, ho_V^i).$$

Here the first inequality follows from the triangle inequality and Lemma 5.5, and the second holds by Theorem 2.3. Since the map $\prod_{i=1}^{n_U} \iota_U^i : \mathcal{C}U \to \prod_{i=1}^{n_U} T_U^i$ is a quasi-isometric embedding for any $U \in \mathfrak{S}^{II}$, we conclude by summing the inequality over $1 \leq i \leq n_V$ for all $V \in \mathcal{V}$. □

5.3. *Proof of Theorem 1.4.*

Proof. Let $R > 0$ be large enough to satisfy Propositions 5.1 and 5.6 and Theorem 3.8. Let $T \geq R + 2D_0$. By Proposition 5.1 and Proposition 5.6, there exists quasitrees $\mathcal{T}_1, \ldots, \mathcal{T}_m$ such that H acts on $\prod_{k=1}^m \mathcal{T}_k$ diagonally and for any choice of base points $o_k \in \mathcal{T}_k$ and any $h \in H$,

$$\sum_{V \in \mathfrak{S}^I} d_V^{T,R}(1, h) + \sum_{U \in \mathfrak{S}^{II}} \{\!\{d_U(1, h)\}\!\}_R \preceq \sum_{k=1}^m d_{\mathcal{T}_k}(o_k, ho_k).$$

By definition of thick distance, $d_U^{T,R}(1,h) \le \{\!\{d_U(1,h)\}\!\}_R$. Thus, by Theorem 3.8,

$$\sum_{V\in\mathfrak{S}} d_V^{T,R}(1,h) \preceq \sum_{k=1}^{m} d_{\mathcal{T}_k}(o_k, ho_k)$$

for any $h \in H$.

On the other hand, the orbit map from H to $\prod_{k=1}^m \mathcal{T}_k$ is coarsely Lipschitz since H is finitely generated. Therefore, H embeds quasi-isometrically into $\prod_{k=1}^m \mathcal{T}_k$, which means that H has property QT_0. Finally, we know that G has property QT since property QT passes to any finite-index supergroup.

For the "moreover" part, first note that G is coarse median for the same reason as [BHS19, Theorem 7.3]. The rest of the proof is just a combination of [HP22; Pet21]. The proof of [HP22, Proposition 3.9] can be naturally generalized to quasitrees from domains of type II. The proof in [Pet21, §3] for mapping class groups can be repeated verbatim to deal with quasitrees from domains of type I. □

For relative HHGs with only type II domains, we obtain the following stronger theorem.

Theorem 5.7. *Let $(G, \mathfrak{S})$ be a relative HHG that is colorable. If every $U \in \mathfrak{S}$ is of type II, then G has property QT_0.*

Proof. The proof is a simplified version of that of Theorem 1.4. Since G is colorable and every domain is of type II, the finite-index subgroup H in the above proof can be replaced with G itself. This gives QT_0 rather than just QT. □

6. A criterion for having separable quasi-axes

In this section, we provide a criterion for a relative HHG to have separable quasi-axes that is easy to use in application. For an acylindrical action on a hyperbolic space, we have seen in Section 2.1 that the elementary closure of any loxodromic element is a *virtual centralizer*. Inspired by the discussion in [BBF21, §4.3], the following lemma generalizes this fact. In general, we cannot expect the elementary closure to be a centralizer as in [BBF21, §4.3], because it may contain a flip.

Lemma 6.1. *Let G be a group acting on a δ-hyperbolic space X with an acylindrical image. Let K be the kernel of this action. Assume that there is a subgroup $H < G$ such that $[H, K] = 1$ and H is mapped to a finite-index subgroup of G/K. Then for any loxodromic element $g \in G$, the elementary closure $EC_G(g)$ is a virtual centralizer in G of some loxodromic element in H.*

Proof. Let $\bar{G} = G/K$, and let $\theta : G \to \bar{G}$ be the quotient map. For any $g \in G$, denote the image $\theta(g) \in \bar{G}$ by $\bar{g}$.

Let $g \in G$ be a loxodromic element. Since $\theta(H)$ is a finite-index subgroup of $\bar{G}$, we can choose $h \in H$ and $n > 0$ such that $\bar{h} = \bar{g}^n$. Since $\bar{G}$ acts acylindrically,

$EC_{\bar{G}}(\bar{g}) = EC_{\bar{G}}(\bar{h})$, which contains the cyclic subgroup $\langle\bar{h}\rangle$ as a finite-index subgroup. By definition, $EC_G(g) = \theta^{-1}(EC_{\bar{G}}(\bar{g}))$. Thus, the preimage $\theta^{-1}(\langle\bar{h}\rangle) = K \cdot \langle h\rangle$ is a finite-index subgroup of $EC_G(g)$.

On the other hand, any element in G that commutes with h has an image in $\bar{G}$ that commutes with $\bar{h}$. Thus, $C_G(h) < EC_G(g)$. Note that $K \cdot \langle h\rangle < C_G(h)$ since $[H, K] = 1$. In conclusion, $C_G(h)$ is a finite-index subgroup of $EC_G(g)$. □

Definition 6.2. For a relative HHG $(G, \mathfrak{S})$, we say a domain $V \in \mathfrak{S}$ has *neat kernel* if there exists a subgroup $H_V < \mathrm{Stab}_G(V)$ such that $[H_V, \ker(\theta_V)] = 1$ and θ_V maps H_V to a finite-index subgroup of G_V.

Proposition 6.3. *Let $(G, \mathfrak{S})$ be a relative HHG that is residually finite. Let $V \in \mathfrak{S}$. If V has hyperbolicity, acylindrical image and neat kernel, then V has separable quasi-axes.*

Proof. Let $g \in \mathrm{Stab}_G(V)$ be a loxodromic element. Since V has neat kernel, $EC(g)$ is a virtual centralizer in $\mathrm{Stab}_G(V)$ of some loxodromic element $h \in H_V$ by Lemma 6.1. Every element that commutes with h stabilizes V. Therefore, $EC(g)$ is a virtual centralizer of h in G. Moreover, a centralizer in a residually finite group is separable (see [BBF21, Lemma 2.1] or the proof of [Lon87, Proposition]). It is also known that a finite-index supergroup of a separable subgroup is still separable (easy to see from the profinite topology). Therefore, V has separable quasi-axes. □

Corollary 6.4. *Let $(G, \mathfrak{S})$ be a relative HHG that is residually finite. Let $S \in \mathfrak{S}$ be the unique maximal domain. Then S is of type I.*

Proof. By [BHS17b, Theorem 14.3], G acts on $\mathcal{C}S$ acylindrically. Now $\ker(\theta_V)$ is trivial, so S has neat kernel. Thus, S has separable quasi-axes by Proposition 6.3. Moreover, S has cobounded nested region because G acts on $\mathcal{X}$ coboundedly. In conclusion, S is of type I. □

7. Applications

7.1. ***Mapping class groups.*** We now explain how Theorem 1.4 applies to mapping class groups to recover the following theorem.

Theorem 7.1 [BBF21, Theorem 1.2]. *Mapping class groups of finite-type surfaces have property QT.*

Proof. Let Σ be a finite-type surface, i.e., a closed oriented surface with finitely many marked points. Let $\mathcal{M}(\Sigma)$ be its marking complex [MM00]. Let $\mathfrak{S}$ be the collection of isotopy classes of essential non-pants-closed subsurfaces of Σ, where disconnected subsurfaces are also allowed. Given any $V \in \mathfrak{S}$, let $\hat{V}$ be the surface obtained by gluing a once-punctured disk to each boundary component of V. Let $\mathcal{C}V$ be the curve graph of $\hat{V}$. Here the curve graph of a disconnected surface is defined as the join of the curve graphs of its components, and thus is bounded. It is known that $(\mathcal{M}(\Sigma), \mathfrak{S})$ is an HHS. Moreover, the mapping class group $\mathrm{MCG}(\Sigma)$ is an HHG by virtue of its action on $(\mathcal{M}(\Sigma), \mathfrak{S})$ [BHS19, §11].

For any $V \in \mathfrak{S}^\circ$, let $V^\perp$ be the closure of $\Sigma - V$ in Σ, and let

$$\mathrm{MCG}(\Sigma; V) < \mathrm{MCG}(\Sigma)$$

be the stabilizer of V. Let $\eta_V : \mathrm{MCG}(V) \to \mathrm{MCG}(\Sigma)$ be the homomorphism induced by the inclusion $V \hookrightarrow \Sigma$. Denote the image of η_V by $\overline{\mathrm{MCG}}(V)$. By Theorem 3.18 of [FM12], $\overline{\mathrm{MCG}}(V^\perp)$ is exactly the kernel of the restriction homomorphism $\theta_V : \mathrm{MCG}(\Sigma; V) \to G_V$, where G_V is a finite-index subgroup of $\mathrm{MCG}(\hat{V})$. It is clear that $\overline{\mathrm{MCG}}(V)$ commutes with $\overline{\mathrm{MCG}}(V^\perp)$ and θ_V maps $\overline{\mathrm{MCG}}(V)$ to a finite-index subgroup of G_V. Therefore, V has neat kernel.

It is known that $\mathrm{MCG}(\Sigma)$ is virtually colorable [BBF15, §5] and residually finite [Gro74]. We only need to prove that every unbounded domain $V \in \mathfrak{S}$ is of type I. First, it is clear that $\boldsymbol{F}_V$ is coarsely $\mathcal{M}(V)$. Since $\mathrm{MCG}(\hat{V})$ acts coboundedly on $\mathcal{M}(V)$ and acts acylindrically on $\mathcal{C}\hat{V}$ [Bow08], V has cobounded nested region and acylindrical image. Furthermore, V has separable quasi-axes by the above discussion and Proposition 6.3. Therefore, V is of type I.

In conclusion, mapping class groups of finite-type surfaces have property QT by Theorem 1.4. □

Certain quotients of a mapping class group are again HHGs, as proved in [BHS17a] and [BHMS24]. In particular, the quotient by the normal closure of a suitable power of a pseudo-Anosov element or by the normal closure of suitable powers of all Dehn twists is again an HHG. It would be interesting to determine whether these quotient groups are still residually finite, thereby satisfying the assumption of Theorem 1.4.

7.2. *Admissible graphs of groups.* *Admissible groups* were introduced by Croke and Kleiner in [CK02]; they generalize the fundamental groups of nongeometric 3-dimensional graph manifolds.

Definition 7.2. Let $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\})$ be a graph of groups. We say $\mathcal{G}$ is *admissible* if the following hold:

(1) Γ is a finite graph with at least one edge.

(2) Each vertex group G_v has center $Z_v \cong \mathbb{Z}$, and $H_v := G_v/Z_v$ is a nonelementary hyperbolic group.

(3) Every edge group G_e is isomorphic to $\mathbb{Z}^2$.

(4) If e is an edge with $v = e^+$ and $w = e^-$, and $\tau_e, \tau_{\bar{e}}$ are the edge monomorphisms, then the subgroup $\langle \tau_e^{-1}(Z_v), \tau_{\bar{e}}^{-1}(Z_w)\rangle$ has finite index in G_e.

(5) Let e_1 and e_2 be distinct edges entering a vertex v, and let $K_i \subset G_v$ be the image of the edge homomorphism τ_{e_i} for $i = 1, 2$. Then
 - for every $g \in G_v$, gK_1g^{-1} is not commensurable with K_2;
 - for every $g \in G_v - K_i$, gK_ig^{-1} is not commensurable with K_i.

A group G is *admissible* if it is the fundamental group of an admissible graph of groups.

Every admissible group G has a (combinatorial) HHG structure by [HRSS24, Theorem 1.4]. According to the classification of simplices by [HRSS24, Lemma 6.2], if $\Delta \perp g\Delta$, where Δ corresponds to an unbounded hyperbolic space, then Δ is of type 8 and g exchanges two adjacent vertices in the Bass–Serre tree (see [BHMS24, Definition 1.11] for definition of orthogonality in a combinatorial HHS). Therefore, it is easy to see that G has a subgroup of index at most 2 that is colorable (see [NY23, Lemma 4.6] for example). This shows the virtual colorability of G. Thus every nongeometric graph manifold group has a virtually colorable HHG structure with all associated hyperbolic spaces being quasitrees. Thus, nongeometric graph manifold groups have property QT by [HP22, Theorem 3.1] or Theorem 1.4. However, in the HHG structure of an admissible group, associated hyperbolic spaces are not necessarily quasitrees. As an application of Theorem 1.4, we show that property QT still holds true in this case if we assume G to be residually finite.

Theorem 7.3. *Let $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\})$ be an admissible graph of groups, and let $G = \pi_1\mathcal{G}$. If G is residually finite, then G has property QT.*

Proof. According to the classification of simplices [HRSS24, Lemma 6.2], any simplex that is not of type 7 corresponds to a quasitree so it is a domain of type II. Thus, we only need to check that simplices of type 7 are of type I. The stabilizer of such a simplex Δ is exactly a vertex group G_v that acts on $\mathcal{C}(\Delta)$ with image $H_v = G_v/Z_v$. Now $\mathcal{C}(\Delta)$ is coarsely the hyperbolic space obtained by coning off H_v as a relatively hyperbolic group, and $\boldsymbol{F}_\Delta$ is coarsely H_v itself. Therefore, acylindrical image and cobounded nested region hold true (see [Osi16, Proposition 5.2] for acylindricity). Since G_v is a central extension of H_v by Z_v, Δ has neat kernel. Therefore, we conclude by Proposition 6.3 and Theorem 1.4. □

There is another approach to property QT of nongeometric graph manifold groups in [HNY25]. For graph manifolds with nonempty boundary, they actually prove in a more general setting. A *Croke–Kleiner admissible group* (abbreviated as CKA group) is an admissible group that admits a geometric action on a complete proper CAT(0) space. As a corollary of Theorem 7.3, we recover the following theorem.

Corollary 7.4 ([HNY25, Theorem 1.3]). *Let G be a CKA group where for every vertex v the central extension $1 \to Z_v \to G_v \to H_v \to 1$ has an omnipotent hyperbolic quotient group H_v. Then G has property QT.*

For definition of omnipotence, we refer the reader to [Wis00]. Note that if every hyperbolic group is residually finite, then every hyperbolic group is omnipotent by [Wis00, Remark 3.4]. Under the assumption of Corollary 7.4, the central extension associated with any vertex virtually splits by [BH99, Theorem II.7.1]. Therefore, Theorem 7.3 implies Corollary 7.4 due to the following lemma.

Lemma 7.5. *Let G be an admissible group where for every vertex v the central extension $1 \to Z_v \to G_v \to H_v \to 1$ virtually splits and the hyperbolic quotient group H_v is omnipotent. Then G is residually finite.*

We omit the proof of Lemma 7.5 since it is almost the same as the proof of residual finiteness for graph manifold groups by Hempel [Hem87]. The reader can also see [Ngu26] for an improved result.

7.3. *Hyperbolic-2-decomposable groups.* A group G is *hyperbolic-2-decomposable* if G splits as a graph of hyperbolic groups with 2-ended edge groups.

Theorem 7.6. *Let G be a residually finite hyperbolic-2-decomposable group. Then G has property QT if and only if G does not contain any distorted element.*

Proof. If G has property QT, then G does not contain any distorted element by [HNY25, Lemma 2.5]. Now assume that G does not contain any distorted element. Let $\mathfrak{S}$ be the HHG structure of G given by [RS20]. By construction, there is no orthogonality in $\mathfrak{S}$. Thus, G is colorable. Let $U \in \mathfrak{S}$. Then $\mathcal{C}U$ is either a quasitree or a hyperbolic space obtained by coning off a vertex group G_v as a relatively hyperbolic group. In the former case, U is of type II. Now we only need to consider the latter case. Similarly to Theorem 7.3, we have $\mathrm{Stab}_G(U) = G_v$ and $\boldsymbol{F}_U$ is coarsely G_v itself. It is easy to see that acylindrical image, cobounded nested region and neat kernel hold true. Therefore, we conclude by Proposition 6.3 and Theorem 1.4. □

Similarly to Lemma 7.5, if G is a hyperbolic-2-decomposable group without any distorted element such that every vertex group is omnipotent, then G is residually finite (see [Wis00, §4]).

7.4. *Artin groups and extensions of lattice Veech groups.* Let G be either

- an Artin group of large and hyperbolic type, or
- the $\pi_1(\Sigma)$-extension group of a lattice Veech group in the mapping class group $\mathrm{MCG}(\Sigma)$ of a closed surface Σ.

As shown in [HMS24] and [DDLS24] respectively, G is a virtually colorable HHG. Moreover, the associated hyperbolic spaces of G are all quasitrees except the maximal one. By Corollary 6.4 and Theorem 1.4, G has property QT if G is residually finite. Hence:

Theorem 7.7. *Every residually finite Artin group of large and hyperbolic type has property QT.*

It is proved in [Jan22] that any 3-generator Artin groups with labels ≥ 4 except for $(2m+1, 4, 4)$ for any $m \geq 2$ is residually finite. Therefore, any 3-generator Artin group with labels ≥ 4 except for $(2m+1, 4, 4)$ for any $m \geq 2$ has property QT. On the other hand, we ask

Question 7.8. *When is the $\pi_1(\Sigma)$-extension group of a lattice Veech group residually finite*?

7.5. *Graph products.*

Definition 7.9 (graph product). Let Γ be a finite simplicial graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. Each vertex $v \in V(\Gamma)$ is labeled by a group G_v. The *graph product* G_Γ is the group

$$G_\Gamma = \big(\ast_{v \in V(\Gamma)} G_v\big) \big/ \langle\langle [g_v, g_w] \mid g_v \in G_v,\ g_w \in G_w,\ \{v, w\} \in E(\Gamma) \rangle\rangle\,.$$

We call the G_v the *vertex groups* of the graph product G_Γ.

Theorem 7.10. *Any graph product of groups whose every vertex group has property QT_0 still has property QT_0.*

Proof. Any graph product G_Γ has a relative HHG structure $\mathfrak{S}_\Gamma$ by [BR22]. By definition of $\mathfrak{S}_\Gamma$, any G_Γ-orbit on $\mathfrak{S}_\Gamma$ corresponds to a unique subgraph of Γ and is pairwise transversal. Thus, G_Γ is colorable. By [BR22, Theorem 4.4], for each domain $[g\Lambda] \in \mathfrak{S}_\Gamma$, either $[g\Lambda]$ is $\sqsubseteq$-minimal or $\mathcal{C}g\Lambda$ is a quasitree. Since each $\sqsubseteq$-minimal domain corresponds to a vertex group, this means that every domain is of type II. By Theorem 5.7, G_Γ has property QT. □

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