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ÁRPÁD BARICZ, DHIVYA PRABHU K,
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Á. Baricz dedicates this paper to Mourad E. H. Ismail on the occasion of his 80th birthday

We study certain continuous univariate probability distributions supported on $[0, \infty)$ — the McKay distribution and its generalizations, the generalized inverse Gaussian distribution and the K-distribution — , all of which are related to modified Bessel functions of the first and second kinds. In most cases we show that they belong to the class of infinitely divisible distributions, self-decomposable distributions, generalized gamma convolutions and hyperbolically completely monotone densities. Some of the results are known, but new proofs are provided using special functions techniques: Integral representations of quotients of Tricomi hypergeometric functions, Gaussian hypergeometric functions, and modified Bessel functions of the second kind, play an important role in our study. In addition, by using a different approach based on asymptotic properties of modified Bessel functions, we rediscover a Stieltjes transform representation due to Hermann Hankel for the product of modified Bessel functions of the first and second kinds and we deduce a series of new Stieltjes transform representations for products, quotients and their reciprocals concerning modified Bessel functions of the first and second kinds. By using these results we obtain new infinitely divisible modified Bessel distributions with Laplace transforms related to modified Bessel functions of the first and second kind. We show that the new Stieltjes transform representations have some interesting applications and we list some open problems that may be of interest for further research. In addition, we present a new proof, using the Pick function characterization theorem, for the infinite divisibility of the ratio of two gamma random variables and some new Stieltjes transform representations of quotients of Tricomi hypergeometric functions.

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1. Introduction

1.1. ***Preliminaries on infinite divisibility.*** A probability distribution is *infinitely divisible* (see Steutel and Van Harn [SH03]) if it can be expressed as the probability distribution of the sum of an arbitrary number of independent and identically distributed random variables. The concept of infinite divisibility of probability distributions was introduced in 1929 by Bruno de Finetti and the most basic results were discovered by Andrey Kolmogorov, Paul Lévy and Aleksandr Khinchin. This type of decomposition of a distribution is used in probability and statistics to find families of probability distributions that might be natural choices for certain models or applications. Infinitely divisible distributions also play an important role in the context of limit theorems. The characteristic function of any infinitely divisible distribution is called an infinitely divisible characteristic function and such a function may be represented, for any value of n, as the n-th power of some other characteristic function. More precisely, a probability distribution ν on the half-line $[0, \infty)$ is infinitely divisible if for any $n \in \mathbb{N}$ there exists a probability distribution ν_n on $(0, \infty)$ such that

$$\int_0^\infty e^{-xt}d\nu = \Big(\int_0^\infty e^{-xt}d\nu_n\Big)^n.$$

A function $f : (0, \infty) \to \mathbb{R}$ is *completely monotone* (or completely monotonic) if it has derivatives of all orders and $(-1)^n f^{(n)}(x) > 0$ for all $x > 0$ and $n \in \{0, 1, 2, \dots\}$. The classes of completely monotonic functions and infinitely divisible distributions are related by the following well-known result (see [Fe66, p. 425]).

Lemma 1. *The function $\omega : (0, \infty) \to (0, \infty)$ is the Laplace transform of an infinitely divisible distribution if and only if $\omega(x) = e^{-\varphi(x)}$, where $\varphi(0^+) = 0$ and $\varphi : (0, \infty) \to (0, \infty)$ is a Bernstein function, that is, φ' is completely monotonic.*

Every continuous-time Lévy process has infinitely divisible distributions, and conversely every infinitely divisible distribution generates uniquely a Lévy process (see for example Steutel and Van Harn [SH03]).

In various real life situations — for instance in biology, physics, economics and actuarial science — certain models postulate a random effect that is the sum of several independent random components with the same distribution. A convenient way to enforce this condition is to suppose the infinite divisibility of the distribution of these random effects.

The concept of *self-decomposability of probability measures* is due to Paul Lévy and goes back to 1937. A random variable X, distributed according to a law, is called *self-decomposable* if for every $c \in (0, 1)$ there exists a random variable X_c, independent of X, such that X and $X_c + cX$ are equal in law. A distribution is self-decomposable if and only if it is a weak limit of partial normed centered sums of a simple sequence of independent random variables. More precisely, a probability distribution is self-decomposable if it is the limit of

$$(X_1 + \cdots + X_n - b_n)/a_n,$$

where the X_i are independent random variables and $\{a_n\}$ and $\{b_n\}$ are sequences of constants with $a_n \to \infty$ and $a_{n+1}/a_n \to 1$. Every self-decomposable distribution is infinitely divisible, and the class of self-decomposable distributions is closed under the convolution and the weak convergence. This class contains stable distributions and generalized gamma convolutions.

The next auxiliary result (see [Fe66, p. 589]) characterizes self-decomposable distributions with support $[0, \infty)$.

Lemma 2. *A random variable X with support* $[0, \infty)$ *having the Laplace transform* $\omega : (0, \infty) \to (0, \infty)$ *is self-decomposable if and only if for every α, where $\alpha \in (0, 1)$, the function $\omega(x)/\omega(\alpha x)$ is a Laplace transform.*

Now, let us focus on two other subclasses of infinitely distributions. A function $f : (0, \infty) \to (0, \infty)$ is *hyperbolically completely monotone* if for each $u > 0$ the function $f(uv)f(u/v)$ is completely monotone as a function of $w = v + 1/v$, where $v > 0$. A distribution is said to be hyperbolically completely monotone if its probability density function is hyperbolically completely monotone. This class of lifetime distributions was discovered by Lennart Bondesson (see [Bo92]) and it is strongly connected to the class of *generalized gamma convolutions*, which was introduced by Olof Thorin [Th77] and constitutes the smallest class of distributions on $(0, \infty)$ that contains all gamma distributions and is closed under convolution and weak convergence. A positive continuous random variable X belongs to the class of generalized gamma convolutions if its Laplace transform is of the form

$$L(s) = \exp\Big(-as + \int_0^\infty \ln \frac{t}{s+t}\, d\mu(t)\Big),$$

where $s \geq 0$, $a \geq 0$ and $d\mu(t)$ is a nonnegative measure. Since the gamma distribution is infinitely divisible and self-decomposable, so is every generalized gamma convolution. An interesting result of Bondesson (see [Bo92]) states that a probability density that is hyperbolically completely monotone is the density of a generalized gamma convolution.

The next result is a *Pick function characterization theorem* of generalized gamma convolutions.

Lemma 3 [Bo92, Theorem 3.1.2]. *The probability distribution of a continuous random variable X on* $[0, \infty)$ *is a generalized gamma convolution if and only if its moment generating function ψ is analytic and zero-free in* $\mathbb{C} \setminus [0, \infty)$, *and satisfies* $\operatorname{Im}\big(\psi'(s)/\psi(s)\big) \geq 0$ *for* $\operatorname{Im} s > 0$.

Another simple characterization of generalized gamma convolutions is due to Bondesson:

Lemma 4 [Bo92, Theorem 6.1.1]. *A function ϕ on* $[0, \infty)$ *is the Laplace transform of a generalized gamma convolution if and only if* $\phi(0) = 1$ *and* ϕ *is hyperbolically completely monotone.*

Thorin's class of generalized gamma convolutions is closed with respect to rescaling, weak limits, and addition of independent random variables. Bondesson [Bo15, Theorem 1] has shown that the generalized gamma convolution class also has the remarkable property of being closed with respect to multiplication of independent random variables.

Proving or disproving infinite divisibility of a certain distribution is sometimes a complicated task and it may need a specialized approach; see for example the papers of John Kent [Ke78] and those of Pierre Bosch and Thomas Simon [BS15; BS16]. The Laplace transforms of probability measures are usually transcendental special functions, which has led authors to study the complete monotonicity of various quotients of special functions (such as modified Bessel functions, Tricomi hypergeometric functions, parabolic cylinder functions) and of the logarithmic derivatives of solutions of differential and difference equations; see for example the papers of Philip Hartman [Ha78; Ha79].

The present study is motivated by the *special function technique* approach of Mourad Ismail and coauthors (see [Is77a; Is77b; IK79; IM79; IM82; Is90]); we extend and complement the results from these papers by deducing a series of new Stieltjes transform representations for products and quotients of modified Bessel functions of the first and second kinds and by obtaining new infinitely divisible modified Bessel distributions.

Some of our main results are proved using ideas from the works just cited, but we also work with the theory of generalized gamma convolutions and hyperbolically completely monotone densities.

Outline. In the remainder of this section we recall some basic lemmas on Stieltjes transforms. In Section 2 we present a series of results on infinite divisibility of McKay distributions and its generalizations, the K-distribution and generalized inverse Gaussian distribution. In Section 3 we obtain a series of new Stieltjes transform representations for products and quotients of modified Bessel functions of the first and second kinds and using these results we obtain new infinitely divisible modified Bessel distributions. These new distributions have Laplace transforms related to modified Bessel functions of the first and second kinds. Section 4 is devoted to remarks and open problems concerning some distributions related to modified Bessel functions and Tricomi hypergeometric functions, while Section 5 contains the proofs of all the main results of this paper.

1.2. ***Preliminaries on Stieltjes transforms.*** Before presenting our results on infinitely divisible distributions whose probability density function or Laplace transform involves modified Bessel functions, we recall some basic facts concerning Stieltjes transforms. The first two lemmas of this subsection are variants of the *representation theorem for Stieltjes transforms*. In some applications we use Lemma 5 and in others Lemma 6. The ensuing Lemma 7 is called the *inversion theorem for Stieltjes transforms* or *Perron–Stieltjes inversion formula* and it is also a key ingredient in our proofs.

For proofs of Lemmas 5 and 6 we refer to [HW55] (pp. 235 and 210, respectively), while Lemma 7 can be found in [St32]. (Alternatively, Lemmas 5–7 can be found in [Is77a, Lemma 2.1], [IK79, Theorem 1.2] and [Is77a, Lemma 2.2], respectively.)

Lemma 5 (first representation theorem for Stieltjes transforms). *A complex function $F(z)$ admits a Stieltjes transform representation*

$$F(z)=\int_0^\infty \frac{d\mu(t)}{z+t}, \quad \text{with} \int_0^\infty |d\mu(t)|<\infty, \tag{1-1}$$

if and only if the following conditions hold true:

a. *$F(z)$ is analytic for $|\arg z|<\pi$.*
b. *$F(z)=o(1)$ as $|z|\to\infty$ and $F(z)=o(|z|^{-1})$ as $|z|\to 0$, uniformly in every sector $|\arg z|\le \pi-\varepsilon$ for $\varepsilon>0$.*

Lemma 6 (second representation theorem for Stieltjes transforms). *If*

a. *$F(z)$ is analytic for $|\arg z|<\pi/\theta$ for some θ such that $0<\theta<1$, and*
b. *$F(z)=o(1)$ as $z\to\infty$ and $F(z)=o(|z|^{-1})$ as $z\to 0$, uniformly in every sector $|\arg z|\le\pi/\eta$ with $\theta<\eta<1$,*

then the Stieltjes transform representation

$$F(x)=\frac{1}{\pi}\int_0^\infty \frac{dt}{x+t}\frac{1}{2\pi \mathrm{i}}\int_{\mathcal{C}} \frac{ze^{\frac{z}{2}}F(te^z)}{z^2+\pi^2}dz \tag{1-2}$$

is valid for all $x>0$, where $\mathcal{C}$ is a rectifiable closed curve going around $[-\mathrm{i}\pi,\mathrm{i}\pi]$ in the positive direction and lying in the strip $|\operatorname{Im} z|<\pi/\theta$.

Lemma 7 (inversion theorem for Stieltjes transforms). *If F has the representation* (1-1), *then*

$$\mu(t_2)-\mu(t_1)=\lim_{\eta\to 0^+}\frac{1}{2\pi \mathrm{i}}\int_{t_1}^{t_2}[F(-t-\mathrm{i}\eta)-F(-t+\mathrm{i}\eta)]\,dt, \tag{1-3}$$

where $\mu(t)$ is normalized by $\mu(0)=\mu(0^+)=0$ and $2\mu(t)=\mu(t^+)+\mu(t^-)$ for $t>0$.

2. Distributions whose probability density function involves modified Bessel functions

In this section we consider some known distributions whose probability density function involves the modified Bessel function of the first or second kind and study whether these distributions belong to the class of infinitely divisible distributions or to one of its subclasses: self-decomposable distributions, generalized gamma convolutions and hyperbolically completely monotone densities.

2.1. ***The McKay distribution of type I.*** The McKay distribution of type I involves the modified Bessel function of the first kind, I_μ. Its probability density function is given by

$$\varphi_{\mu,a,b}(x) = \frac{\sqrt{\pi}(b^2-a^2)^{\mu+\frac{1}{2}}}{(2a)^\mu \Gamma\big(\mu+\frac{1}{2}\big)} x^\mu e^{-bx} I_\mu(ax), \tag{2-1}$$

where $b > a > 0$, $\mu > -\frac{1}{2}$, and its support is $(0, \infty)$. In view of the asymptotic relation

$$I_\mu(x) \sim \frac{x^\mu}{2^\mu \Gamma(\mu+1)}$$

as $x \to 0$ and by using the Legendre duplication formula for the Euler gamma function

$$\Gamma(2x) = \frac{1}{\sqrt{\pi}} 2^{2x-1} \Gamma(x) \Gamma\big(x+\tfrac{1}{2}\big)$$

we obtain

$$\lim_{a\to 0} \varphi_{\mu,a,b}(x) = \frac{b^{2\mu+1}}{\Gamma(2\mu+1)} e^{-bx} x^{2\mu},$$

which shows that the McKay distribution of type I for $a \to 0$ reduces to a gamma distribution with shape parameter $2\mu + 1$ and inverse scale parameter b (also known as the rate parameter). Since the gamma distribution is infinitely divisible and self-decomposable, and obviously belongs to the class of generalized gamma convolutions, it is natural to ask whether the McKay distribution of type I belongs to those same classes of distributions. It is known (see [Ba10, p. 580]) that $x \mapsto e^{-x} x^\mu I_\mu(x)$ is completely monotonic on $(0, \infty)$ for all $\mu \in \big[-\frac{1}{2}, 0\big]$, which in turn implies that $x \mapsto e^{-ax}(ax)^\mu I_\mu(ax)$ is completely monotonic on $(0, \infty)$ for all $\mu \in \big[-\frac{1}{2}, 0\big]$ and $a > 0$. On the other hand, the function $x \mapsto e^{(a-b)x}$ is also completely monotonic on $(0, \infty)$ for all $b > a$, and since the product of two completely monotonic functions is also completely monotonic, we conclude:

Fact. *The probability density function $\varphi_{\mu,a,b}$ is completely monotonic on $(0, \infty)$ for all $\mu \in \big(-\frac{1}{2}, 0\big]$ and $b > a > 0$, and according to the Goldie–Steutel law the McKay distribution of type I is infinitely divisible for all $\mu \in \big(-\frac{1}{2}, 0\big]$ and $b > a > 0$.*

With a more sophisticated analysis it is possible to show the next theorem, to the effect that the McKay distribution of type I is infinitely divisible for all $\mu > -\frac{1}{2}$ and $b > a > 0$. The proof may be found in Section 5.

Theorem 1. *If* $\mu > -\frac{1}{2}$ *and* $b > a > 0$, *the McKay distribution, whose probability density function is defined by* (2-1), *belongs to the class of infinitely divisible distributions, self-decomposable distributions and generalized gamma convolutions.*

2.2. ***Another McKay-type distribution.*** Another distribution similar to the McKay distribution of type I also involves the modified Bessel function of the first kind I_μ. Its support is $[0, \infty)$ and its probability density function is given by

$$\psi_{\mu,a,b}(x) = \frac{\sqrt{\pi}(b^2-a^2)^{\mu+\frac{3}{2}}}{2b(2a)^\mu \Gamma\left(\mu+\frac{3}{2}\right)} x^{\mu+1} e^{-bx} I_\mu(ax), \tag{2-2}$$

where $b > a > 0$, $\mu > -1$.

Theorem 2. *If* $\mu > -1$ *and* $b > a > 0$, *the distribution with probability density function* (2-2) *belongs to the class of infinitely divisible distributions.*

2.3. ***Generalization of the McKay distribution of type I.*** We turn to a generalization of McKay-type distributions. Its support is also $(0, \infty)$ and its probability density function is given by

$$\varphi_{\mu,\nu,a,b}(x) = \frac{1}{c_{\mu,\nu,a,b}} x^{\nu-1} e^{-bx} I_\mu(ax), \tag{2-3}$$

where $\mu+1 > 0$, $\mu+\nu > 0$, $b > a > 0$, and

$$c_{\mu,\nu,a,b} = \frac{(a/2)^\mu}{b^{\mu+\nu}} \frac{\Gamma(\mu+\nu)}{\Gamma(\mu+1)} \cdot {}_2F_1\left(\tfrac{\mu+\nu}{2}, \tfrac{\mu+\nu+1}{2}, \mu+1, a^2/b^2\right)$$

(here ${}_2F_1(a, b, c, x)$ is the Gaussian hypergeometric function). Observe that

$$c_{\mu,\mu+1,a,b} = \frac{(2a)^\mu \Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{\pi}\, b^{2\mu+1}} \, {}_1F_0\left(\mu+\tfrac{1}{2}, a^2/b^2\right) = \frac{(2a)^\mu \Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{\pi}(b^2-a^2)^{\mu+\frac{1}{2}}}$$

and

$$c_{\mu,\mu+2,a,b} = \frac{2(2a)^\mu \Gamma\left(\mu+\frac{3}{2}\right)}{\sqrt{\pi}\, b^{2\mu+2}} \, {}_1F_0\left(\mu+\tfrac{3}{2}, a^2/b^2\right) = \frac{(2b)(2a)^\mu \Gamma\left(\mu+\frac{3}{2}\right)}{\sqrt{\pi}(b^2-a^2)^{\mu+\frac{3}{2}}},$$

and consequently for all $x > 0$, $\mu > -\frac{1}{2}$ and $b > a > 0$ we have

$$\varphi_{\mu,\mu+1,a,b}(x) = \varphi_{\mu,a,b}(x) = \frac{\sqrt{\pi}(b^2-a^2)^{\mu+\frac{1}{2}}}{(2a)^\mu \Gamma\left(\mu+\frac{1}{2}\right)} x^\mu e^{-bx} I_\mu(ax)$$

and for all $x > 0$, $\mu > -1$ and $b > a > 0$ we have

$$\varphi_{\mu,\mu+2,a,b}(x) = \psi_{\mu,a,b}(x) = \frac{\sqrt{\pi}(b^2-a^2)^{\mu+\frac{3}{2}}}{2b(2a)^{\mu}\Gamma\big(\mu+\frac{3}{2}\big)} x^{\mu+1} e^{-bx} I_{\mu}(ax).$$

From [Ba10, p. 578] we know that the function $x \mapsto e^{-x}x^{-\mu}I_{\mu}(x)$ is completely monotonic on $(0,\infty)$ for all $\mu \geq -\frac{1}{2}$, which in turn implies that $x \mapsto e^{-ax}(ax)^{-\mu}I_{\mu}(ax)$ is completely monotonic on $(0,\infty)$ for all $\mu \geq -\frac{1}{2}$ and $a > 0$. On the other hand, the function $x \mapsto e^{(a-b)x}$ is also completely monotonic on $(0,\infty)$ for all $b > a$. As the product of completely monotonic functions, therefore, the probability density function $\varphi_{\mu,\nu,a,b}$ is completely monotonic on $(0,\infty)$ for all $\mu \geq -\frac{1}{2}$, $\mu+\nu \leq 1$ and $b > a > 0$; thus, according to the Goldie–Steutel law, *the generalization of the McKay distribution of type I is infinitely divisible for such values of* μ, ν, b, a.

Theorem 3. *If* $\mu+\nu > 0$, $\mu+1 \geq \nu$ *and* $b > a > 0$, *the distribution with probability density function defined by* (2-3) *is infinitely divisible.*

With the previous paragraph, this gives two domains in the (μ, ν)-plane where our distribution is infinitely divisible (with $b > a > 0$). Neither domain contains the other. It would of interest to find the largest domain of μ and ν with this property.

2.4. ***One more McKay-type distribution.*** We turn to another distribution similar to the above generalization of the McKay distribution of type I. Its support is also $(0,\infty)$ and its probability density function is given by

$$\xi_{\mu,a,b}(x) = \frac{1}{c_{\mu,a,b}} x^{2\mu} e^{-bx} \left(I_{\mu}(ax)\right)^2, \tag{2-4}$$

where $\mu > -\frac{1}{4}$, $b > 2a > 0$ and

$$c_{\mu,a,b} = \frac{2^{4\mu}a^{2\mu}}{\pi b^{4\mu+1}} \frac{\Gamma\big(\mu+\frac{1}{2}\big)\Gamma\big(2\mu+\frac{1}{2}\big)}{\Gamma(\mu+1)} \cdot {}_2F_1\big(\mu+\tfrac{1}{2}, 2\mu+\tfrac{1}{2}, \mu+1, 4a^2/b^2\big).$$

From [Ba10, p. 580] we know that the function $x \mapsto e^{-x}x^{\mu}I_{\mu}(x)$ is completely monotonic on $(0,\infty)$ for all $\mu \in \big[-\frac{1}{2}, 0\big]$, and thus so are $x \mapsto e^{-ax}(ax)^{\mu}I_{\mu}(ax)$ and $x \mapsto e^{-2ax}(ax)^{2\mu}(I_{\mu}(ax))^2$ for all $a > 0$. Taking the product with $x \mapsto e^{(2a-b)x}$, which is completely monotonic on $(0,\infty)$ for $b > 2a$, we see that $\xi_{\mu,a,b}$ is completely monotonic on $(0,\infty)$ for all $\mu \in \big(-\frac{1}{4}, 0\big]$ and $b > 2a > 0$. Again by Goldie–Steutel, then, *the above McKay type distribution is infinitely divisible for all* $\mu \in \big(-\frac{1}{4}, 0\big]$ *and* $b > 2a > 0$. With a finer analysis the range of μ can be extended:

Theorem 4. *If* $\mu \in \big(-\frac{1}{4}, \frac{1}{2}\big]$ *and* $b > 2a > 0$, *the distribution with probability density function* (2-4) *is infinitely divisible.*

2.5. ***The K-distribution or gamma-gamma distribution.*** Suppose that a random variable X has gamma distribution with mean σ and shape parameter α, where σ is a random variable also having a gamma distribution, with mean μ and shape parameter β, where $\alpha, \beta, \mu > 0$. The result (see [JP78] and [Re99]) is that X has the following probability density function, supported on $(0, \infty)$:

$$\omega_{\alpha,\beta,\mu}(x) = \frac{2}{\Gamma(\alpha)\Gamma(\beta)}(\alpha\beta/\mu)^{\frac{\alpha+\beta}{2}} x^{\frac{\alpha+\beta}{2}-1} K_{\alpha-\beta}(2\sqrt{\alpha\beta x/\mu}). \tag{2-5}$$

The corresponding probability distribution, known as the K-distribution, besides being a compound distribution, it is also a product distribution: namely, the distribution of the product of two independent gamma-distributed random variables, one with mean 1 and shape parameter $\alpha > 0$, the other with mean $\mu > 0$ and shape parameter $\beta > 0$.

Since gamma densities are hyperbolically completely monotone and the product of hyperbolically completely monotone functions belongs to the same class (see [Bo92] or [Bo15, Proposition 4]), we conclude that the K-distribution is a hyperbolically completely monotone distribution. This proves the next theorem, but we will later give an alternative proof of it using special function techniques.

Theorem 5. *If* $\alpha, \beta, \mu > 0$, *the* K*-distribution*, *with probability density function defined by* (2-5), *belongs to the class of infinitely divisible distributions*, *self-decomposable distributions*, *generalized gamma convolutions and hyperbolically completely monotone distributions.*

The K-distribution (also known as the gamma-gamma distribution, since it arises from the product of two independent gamma random variables) is well-known in engineering, having been used for example in modeling land and sea radar clutter, as well as the combined effects of fading and shadowing encountered in mobile communications channels. It has been applied also to optical wireless systems, involving the transmission of optical signals through the atmosphere (see [BSMKR06; CK11]), and in modeling microwave sea echo (see [JP78]).

In [BPV11, Theorem 5] the authors proved that the probability density function of the gamma-gamma distribution $x \mapsto \omega_{\alpha,\beta,\mu}(x)$ is *geometrically concave* on $(0, \infty)$ for all $\alpha, \beta, \mu > 0$, that is, the function $x \mapsto x\omega'_{\alpha,\beta,\mu}(x)/\omega_{\alpha,\beta,\mu}(x)$ is decreasing on $(0, \infty)$ for all $\alpha, \beta, \mu > 0$. It can be shown easily (see for example [Bo92, p. 102]) that this is equivalent to the probability density function of the K-distribution being hyperbolically monotone (of order 1), that is, the function $w \mapsto \omega_{\alpha,\beta,\mu}(uv)\omega_{\alpha,\beta,\mu}(u/v)$ is decreasing on $(0, \infty)$ for all $u, v, \alpha, \beta, \mu > 0$, where $w = v + 1/v$. Clearly Theorem 5 states much more than this: the function $w \mapsto \omega_{\alpha,\beta,\mu}(uv)\omega_{\alpha,\beta,\mu}(u/v)$ is not only decreasing on $(0, \infty)$ for all $u, v, \alpha, \beta, \mu > 0$, it is even completely monotonic.

2.6. ***The generalized inverse Gaussian distribution.*** The generalized inverse Gaussian distribution is a three-parameter family of continuous probability distributions with probability density function

$$\pi_{\mu,a,b}(x) = \frac{(a/b)^{\mu/2}}{2K_\mu(\sqrt{ab})} x^{\mu-1} e^{-\frac{1}{2}(ax+b/x)} \tag{2-6}$$

and support $(0, \infty)$, where $a, b > 0$ and μ is a real parameter. Barndorff-Nielsen and Halgreen [BH77] have proved that the generalized Gaussian distribution is infinitely divisible. Barndorff-Nielsen et al. [BBH78] have shown that the generalized inverse Gaussian distribution is a first hitting time for certain time-homogeneous diffusion processes provided the parameter μ is negative, and in this case infinite divisibility follows from the general central limit theorem. Halgreen [Ha79] and Bondesson [Bo79] have shown that the generalized inverse Gaussian distribution is a generalized gamma convolution, and according to [Bo92, p. 74] the generalized inverse Gaussian distribution belongs to the class of hyperbolically completely monotone densities and hence to the class of generalized gamma convolutions and self-decomposable distributions. Thus the next theorem is previously known, but we will provide an alternative proof, based on the special function approach, for the fact that the generalized inverse Gaussian distribution is a generalized gamma convolution.

Theorem 6. *If $\mu \in \mathbb{R}$ and $a, b > 0$, the generalized inverse Gaussian distribution, with probability density function* (2-6), *belongs to the class of generalized gamma convolutions and hence to the class of self-decomposable distributions and infinitely divisible distributions.*

3. Distributions whose Laplace transform involves modified Bessel functions

At the end of his book [Bo92] Bondesson wrote: "Since the class of infinitely divisible distributions is extremely large, it is not a very interesting class. In fact, as the research during the last two decades has shown, infinitely divisibility seems to be more a rule than an exception. This is not surprising if one considers that the class of (univariate) distributions which are infinitely divisible with respect to the maximum operation contains all distributions. On the other hand, to investigate whether or not infinitely divisibility holds for a particular distribution may lead to a deep insight into the structure of that and other distributions and also to a lot of by-products. (Cf. Riemann's hypothesis in mathematics, for example.) This work would certainly not have been written had not Steutel (1973) asked whether the lognormal distribution is infinitely divisible and had not Thorin (1977) attacked and solved this problem."

In this section our aim is to consider some new lifetime distributions whose Laplace transform contains the modified Bessel function of the first and/or second kind and to study whether these distributions belong to the class of infinitely divisible distributions or to one of its subclasses such as self-decomposable distributions or generalized gamma convolutions. It is an interesting problem to find out whether or not particular distributions are infinitely divisible; in this connection, the first part of this section was motivated by the following open conjecture of Ismail and Miller [IM82, p. 234]:

Conjecture. *If $\nu > \mu \geq 0$ and $b > a > 0$, is the function*

$$x \mapsto \left(\frac{b}{a}\right)^{\mu-\nu} \frac{I_\mu(a\sqrt{x})I_\nu(b\sqrt{x})}{I_\mu(b\sqrt{x})I_\nu(a\sqrt{x})}$$

the Laplace transform of an infinitely divisible probability distribution?

Theorems 8 and 9 show that it is possible to generate the Laplace transform of an infinitely divisible probability distribution from the above quotient of modified Bessel functions of the first kind, with some slight modifications.

3.1. ***Quotients of modified Bessel functions of the first kind.*** Due to Ismail and Kelker [IK79, Theorem 1.10], we know that the function

$$x \mapsto \rho_{\mu,a}(x) = \frac{1}{2^\mu \Gamma(\mu+1)} \frac{(a\sqrt{x})^\mu}{I_\mu(a\sqrt{x})} \tag{3-1}$$

is the Laplace transform of an infinitely divisible distribution on $[0, \infty)$, when $\mu > 0$ and $a > 0$. The same distribution is also self-decomposable; the proof follows naturally from Lemma 2.

The next result shows that in fact the above function is the Laplace transform of a generalized gamma convolution.

Theorem 7. *If $\mu > -1$ and $a > 0$, then $x \mapsto \rho_{\mu,a}(x)$ is the Laplace transform of a generalized gamma convolution and therefore it is also the Laplace transform of a self-decomposable distribution.*

Theorem 8. *If $\mu > -1$, $\nu > \sigma > -1$ and $b > a > 0$, then*

$$\Omega_{\mu,\nu,\sigma,a,b}(x) = \left(\frac{b}{a}\right)^{\mu-\nu} \frac{I_\mu(a\sqrt{x})I_\nu(b\sqrt{x})}{I_\mu(b\sqrt{x})I_\nu(a\sqrt{x})} \cdot \rho_{\sigma,b}(x)$$

is the Laplace transform of an infinitely divisible distribution with support $[0, \infty)$.

Theorem 9. *If $\mu > -1$, $\nu > \frac{1}{2}$ and $b > a > 0$, the function*

$$\Omega_{\mu,\nu,a,b}(x) = \left(\frac{b}{a}\right)^{\mu-\nu} \frac{I_\mu(a\sqrt{x})I_\nu(b\sqrt{x})}{I_\mu(b\sqrt{x})I_\nu(a\sqrt{x})} \cdot e^{-b\sqrt{x}}$$

is the Laplace transform of an infinitely divisible distribution with support $[0, \infty)$.

3.2. ***Stieltjes transform representations and infinite divisibility.*** We now list results related to Stieltjes transforms of modified Bessel functions of the first and second kinds, their products and quotients. We show that these new Stieltjes transform representations are closely related to some new infinitely divisible modified Bessel distributions. The proofs of the results are based on representation and inversion theorems for Stieltjes transforms. Some of the proofs related to infinitely divisibility were inspired from the papers of Mourad Ismail, but in each case we will also show the integral representation of the probability density functions in question.

Theorem 10. *If $a>0$, $\mu>-\frac{1}{2}$ and $|\arg z|<\pi$, the following Stieltjes transform representation is valid*:

$$e^{-a\sqrt{z}}z^{-\frac{\mu}{2}}I_{\mu}(a\sqrt{z})=\frac{1}{\pi}\int_0^{\infty}\frac{t^{-\frac{\mu}{2}}}{z+t}J_{\mu}(a\sqrt{t})\sin(a\sqrt{t})\,dt.$$

It can be rewritten as the two-fold Laplace transform

$$e^{-a\sqrt{z}}z^{-\frac{\mu}{2}}I_{\mu}(a\sqrt{z})=\frac{1}{\pi}\int_0^{\infty}e^{-zs}\Big(\int_0^{\infty}e^{-st}t^{-\frac{\mu}{2}}J_{\mu}(a\sqrt{t})\sin(a\sqrt{t})\,dt\Big)\,ds.$$

We know from [Is90, Theorem 2] that if $\mu>\frac{1}{2}$, the function

$$x\mapsto 2^{\mu}\Gamma(\mu+1)x^{-\mu/2}I_{\mu}(\sqrt{x})e^{-\sqrt{x}}$$

is the Laplace transform of an infinitely divisible distribution that is not a generalized gamma convolution. This implies that the function

$$x\mapsto\frac{2^{\mu}\Gamma(\mu+1)}{a^{\mu}}x^{-\frac{\mu}{2}}e^{-a\sqrt{x}}I_{\mu}(a\sqrt{x})$$

is also the Laplace transform of an infinitely divisible distribution that is not a generalized gamma convolution. In view of Theorem 10, the function

$$s\mapsto\frac{1}{\pi}\frac{2^{\mu}\Gamma(\mu+1)}{a^{\mu}}\int_0^{\infty}e^{-st}t^{-\frac{\mu}{2}}J_{\mu}(a\sqrt{t})\sin(a\sqrt{t})\,dt$$

is the corresponding probability density function for the above distribution when $\mu>\frac{1}{2}$. This complements [Is90, Theorem 2]. Also, from [EMOT54, eq. (19), p. 226] we know that

$$z^{-\frac{\mu}{2}}e^{-a\sqrt{z}}I_{\mu}(b\sqrt{z})=\frac{1}{\pi}\int_0^{\infty}\frac{t^{-\frac{\mu}{2}}}{z+t}J_{\mu}(b\sqrt{t})\sin(a\sqrt{t})\,dt,$$

where $\operatorname{Re}\mu>-\frac{1}{2}$, $a>b>0$.

Theorem 10 can be extended to the case when $a>0$, $\operatorname{Re}\mu>-\frac{1}{2}$ and $|\arg z|<\pi$, which clearly complements the above formula for the case $a=b$.

The next theorem is a similar result for the product of modified Bessel functions.

Theorem 11. *Let $a \le b$, $\mu, \nu > -1$, $\nu - \mu < 1$ and $|\arg z| < \pi$. Then*

$$z^{\frac{\nu-\mu}{2}} I_\mu(a\sqrt{z}) K_\nu(b\sqrt{z}) = \frac{1}{2}\int_0^\infty \frac{t^{\frac{\nu-\mu}{2}}}{z+t} J_\mu(a\sqrt{t}) J_\nu(b\sqrt{t})\, dt. \tag{3-2}$$

This Stieltjes transform representation can be rewritten as

$$z^{\frac{\nu-\mu}{2}} I_\mu(a\sqrt{z}) K_\nu(b\sqrt{z}) = \frac{1}{2}\int_0^\infty e^{-zs}\Big(\int_0^\infty e^{-st} t^{\frac{\nu-\mu}{2}} J_\mu(a\sqrt{t}) J_\nu(b\sqrt{t})\, dt\Big)\, ds.$$

Theorem 11 also holds for modified Bessel functions whose orders μ and ν are complex numbers: specifically, under the conditions $a \le b$, $\operatorname{Re}\mu > -1$, $\operatorname{Re}\nu > -1$, $\operatorname{Re}(\nu - \mu) < 1$ and $|\arg z| < \pi$.

During our work we found out that the integral representation in (3-2) existed in the literature (see [MOS66, p. 96]) in the form

$$z^{\nu-\mu} I_\mu(az) K_\nu(bz) = \int_0^\infty \frac{t^{\nu-\mu+1}}{z^2+t^2} J_\mu(at) J_\nu(bt)\, dt \tag{3-3}$$

with the better conditions $a \le b$, $\operatorname{Re}\nu > -1$, $\operatorname{Re}(\nu-\mu) < 2$ and $\operatorname{Re} z > 0$. By replacing z by $\sqrt{z}$ and introducing a suitable transformation, equation (3-3) becomes (3-2) with the conditions $a \le b$, $\operatorname{Re}\nu > -1$, $\operatorname{Re}(\nu-\mu) < 2$ and $|\arg z| < \pi$. The representation (3-2) is also given in [EMOT54, eq. (24), p. 227] with conditions $0 < a < b$, $2 + \operatorname{Re}\mu > \operatorname{Re}\nu > -1$. By using the asymptotic relations 10.30.4 and 10.25.3 in [NIST10], we have, as $z \to \infty$,

$$z^{\frac{\nu-\mu}{2}} I_\mu(a\sqrt{z}) K_\nu(b\sqrt{z}) \sim z^{\frac{\nu-\mu}{2}} \frac{e^{a\sqrt{z}}}{\sqrt{2\pi a\sqrt{z}}} \sqrt{\frac{\pi}{2b\sqrt{z}}}\, e^{-b\sqrt{z}} = \frac{1}{2\sqrt{ab}}\, e^{(a-b)\sqrt{z}} z^{\frac{\nu-\mu-1}{2}}.$$

The factor $e^{(a-b)\sqrt{z}}$ does not goes to zero as $|z| \to \infty$ uniformly in every sector $|\arg z| \le \pi - \epsilon$, for $\epsilon > 0$. Consequently, condition (b) in Lemma 5 is not valid when $\operatorname{Re}(\nu-\mu) \ge 1$. Thus, when $\operatorname{Re}(\nu-\mu) \ge 1$ the Stieltjes measure presented in [MOS66, p. 96] and [EMOT54, eq. (24), p. 227] is not absolutely integrable.

For integer $\mu = \nu = n$, equation (3-3) appears in [Mi36] with a reference to von Hermann Hankel [Ha75], while a more general form of (3-3) for integer orders has been considered in [Ha75]. Namely, the extension of (3-3) in the case when $k \in \mathbb{N}$ and $n + m + q$ is an even integer has the form

$$-\frac{1}{2\Gamma(k+1)}\Big(\frac{\partial}{\partial r^2}\Big)^k r^q J_n(ar)\big(Y_n(br) - \pi \mathrm{i} J_n(br)\big) = \int_0^\infty t^{q+1} \frac{J_n(at) J_m(bt)}{(t^2 - r^2)^{k+1}} dt.$$

The following result provides an immediate application of (3-2).

Theorem 12. *If $\mu > -1$ and $|\arg z| < \pi$, the following representations hold:*

$$2 I_\mu(\sqrt{z}) K_\mu(\sqrt{z}) = \int_0^\infty \frac{J_\mu^2(\sqrt{t})}{z+t} dt = \int_0^\infty e^{-zs}\Big(\int_0^\infty e^{-st} J_\mu^2(\sqrt{t})\, dt\Big)\, ds. \tag{3-4}$$

and consequently for $\mu > 0$ the function $x \mapsto 2\mu I_\mu(\sqrt{x})K_\mu(\sqrt{x})$ is the Laplace transform of an infinitely divisible probability distribution with support $[0, \infty)$, *and its corresponding probability density function is*

$$\varsigma_\mu(x) = \mu \int_0^\infty e^{-tx} J_\mu^2(\sqrt{t})\, dt = \frac{\mu}{x} e^{-\frac{1}{2x}} I_\mu\Big(\frac{1}{2x}\Big).$$

A Stieltjes transform can be viewed, at least formally, as a two-fold Laplace transform:

$$\int_0^\infty \frac{d\mu(t)}{z+t} = \int_0^\infty e^{-zs} \int_0^\infty e^{-st} d\mu(t)\, ds.$$

This relation immediately implies the second equation in Theorems 10 and 11, as well as the second equality in (3-4). The formula (3-4) preexisted in the literature as [EMOT54, eq. (22), p. 226]) with condition $\mu \in \mathbb{C}$. But for the case when $\operatorname{Re}\mu \le -1$, $\mu \ne -1, -2, \dots$, by using the asymptotic relations [NIST10, 10.30.1, 10.30.2 and 10.27.3], we observe that as $z \to 0$

$$I_\mu(\sqrt{z})K_\mu(\sqrt{z}) = I_\mu(\sqrt{z})K_{-\mu}(\sqrt{z}) \sim \frac{\Gamma(-\mu)}{2^{2\mu+1}\Gamma(\mu+1)} z^\mu.$$

It is clear that $|z|\, I_\mu(\sqrt{z})K_\mu(\sqrt{z})$ does not goes to zero as $z \to 0$. This violates one of the conditions in Lemma 5. Thus, when $\operatorname{Re}\mu \le -1$, $\mu \ne -1, -2, \dots$ the Stieltjes measure presented in [EMOT54, eq. (22), p. 226] is not absolutely integrable.

The next corollary contains some immediate applications of the formula in (3-4) concerning the product of modified Bessel functions of the first and second kinds. The first part of the next corollary is well-known and was proved using nontrivial arguments by Penfold et al. [PVG07] for $\mu \ge 0$, by Baricz [Ba09] for $\mu \ge -\frac{1}{2}$ and by Segura [Se21] for $\mu \ge -1$.

Corollary 1.

a. *The function $x \mapsto I_\mu(x)K_\mu(x)$ is decreasing on* $(0, \infty)$ *for all $\mu > -1$.*
b. *The function $x \mapsto I_\mu(\sqrt{x})K_\mu(\sqrt{x})$ is completely monotonic on* $(0, \infty)$ *for all $\mu > -1$.*
c. *The function $x \mapsto x I_\mu(\sqrt{x})K_\mu(\sqrt{x})$ is a Bernstein function on* $(0, \infty)$ *for all $\mu > -1$.*
d. *The function $x \mapsto \big(x I_\mu(\sqrt{x})K_\mu(\sqrt{x})\big)^{-1}$ is completely monotonic on* $(0, \infty)$ *for all $\mu \ge 0$.*
e. *If $\mu > 0$ and $x > 0$, then $I_\mu(x)K_\mu(x) \le \dfrac{\pi c_L^2}{\sqrt{3}x^{\frac{2}{3}}}$, where $c_L = \sup_{t\in\mathbb{R}_+} \sqrt[3]{t}\, J_0(t) \simeq$* 0.7857468704 . . . .

In [BMPS16, Theorem 1] the authors obtained a more general bound for $\mu \ge \nu$

and $x>0$:

$$I_\mu(x)K_\nu(x)\le\frac{2\pi^{\frac{3}{2}}c_L}{\sqrt{3}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)(2x)^{\frac{1}{3}}}.$$

Since

$$c_L<\frac{2^{\frac{2}{3}}\sqrt{\pi}}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)}\simeq 1.8407427466\ldots,$$

the upper bound in Corollary 1**e** is sharper than the above bound from [BMPS16] in the case when $\mu=\nu$.

Another infinitely divisible distribution related to the product of $I_\mu(a\sqrt{x})K_\nu(b\sqrt{x})$ can be found in the next theorem.

Theorem 13. *If $\mu,\nu>-1$, $\nu-\mu<1$, $a,b>0$ and $|\arg z|<\pi$, then*

$$z^{\frac{\nu-\mu}{2}}e^{-a\sqrt{z}}I_\mu(a\sqrt{z})K_\nu(b\sqrt{z})=\frac{1}{2}\int_0^\infty\frac{t^{\frac{\nu-\mu}{2}}J_\mu(a\sqrt{t})}{z+t}\big(J_\nu(b\sqrt{t})\cos(a\sqrt{t})-Y_\nu(b\sqrt{t})\sin(a\sqrt{t})\big)\,dt.$$

This Stieltjes transform representation can be rewritten as

$$\frac{I_\mu(a\sqrt{z})K_\nu(b\sqrt{z})}{z^{\frac{\mu-\nu}{2}}e^{a\sqrt{z}}}=\frac{1}{2}\int_0^\infty e^{-zs}\Big(\int_0^\infty e^{-st}t^{\frac{\nu-\mu}{2}}J_\mu(a\sqrt{t})\big(J_\nu(b\sqrt{t})\cos(a\sqrt{t})-Y_\nu(b\sqrt{t})\sin(a\sqrt{t})\big)\,dt\Big)\,ds.$$

The next theorem shows that the function

$$x\mapsto\frac{2^{\mu-\nu+1}\Gamma(\mu+1)b^\nu}{a^\mu\Gamma(\nu)}e^{-a\sqrt{x}}x^{\frac{\nu-\mu}{2}}I_\mu(a\sqrt{x})K_\nu(b\sqrt{x})$$

is the Laplace transform of an infinitely divisible distribution on $(0,\infty)$ with the conditions $a,b,\nu>0$ and $\mu>\frac{1}{2}$. Based on Theorem 13 we can express the probability density function of the corresponding distribution as

$$s\mapsto\frac{2^{\mu-\nu}\Gamma(\mu+1)b^\nu}{a^\mu\Gamma(\nu)}\times\int_0^\infty e^{-st}t^{\frac{\nu-\mu}{2}}J_\mu(a\sqrt{t})\big(J_\nu(b\sqrt{t})\cos(a\sqrt{t})-Y_\nu(b\sqrt{t})\sin(a\sqrt{t})\big)\,dt$$

whenever $a,b,\nu>0$, $\mu>\frac{1}{2}$ and $\nu-\mu<1$.

Theorem 14. *If $a,b,\nu>0$ and $\mu>\frac{1}{2}$, the function*

$$\chi_{\mu,\nu,a,b}(x)=\frac{2^{\mu-\nu+1}\Gamma(\mu+1)b^\nu}{a^\mu\Gamma(\nu)}e^{-a\sqrt{x}}x^{\frac{\nu-\mu}{2}}I_\mu(a\sqrt{x})K_\nu(b\sqrt{x})$$

is the Laplace transform of an infinitely divisible distribution on $(0,\infty)$.

The next result is also related to an infinitely divisible distribution, but involves only the modified Bessel functions of the second kind.

Theorem 15. *If $a, b, \mu, \nu > 0$, $\nu+\mu < 1$ and $|\arg z| < \pi$, then*

$$z^{\frac{\mu+\nu}{2}} K_\mu(a\sqrt{z})K_\nu(b\sqrt{z}) = -\frac{\pi}{4}\int_0^\infty \frac{t^{\frac{\mu+\nu}{2}}}{z+t}\big(J_\mu(a\sqrt{t})Y_\nu(b\sqrt{t}) + J_\nu(b\sqrt{t})Y_\mu(a\sqrt{t})\big)\,dt. \tag{3-5}$$

This representation can be rewritten as

$$z^{\frac{\mu+\nu}{2}} K_\mu(a\sqrt{z})K_\nu(b\sqrt{z}) = \int_0^\infty e^{-zs}\Big(-\frac{\pi}{4}\int_0^\infty e^{-st}t^{\frac{\mu+\nu}{2}}\big(J_\mu(a\sqrt{t})Y_\nu(b\sqrt{t}) + J_\nu(b\sqrt{t})Y_\mu(a\sqrt{t})\big)\,dt\Big)\,ds.$$

The following result shows that the above product of modified Bessel functions naturally generates a generalized gamma convolution. On the other hand, if we let a, b, μ, ν and s be strictly positive real numbers, and $\nu+\mu < 1$, then the function

$$s \mapsto \frac{-\pi a^\mu b^\nu}{2^{\mu+\nu}\Gamma(\mu)\Gamma(\nu)}\int_0^\infty e^{-st}t^{\frac{\mu+\nu}{2}}\big(J_\mu(a\sqrt{t})Y_\nu(b\sqrt{t}) + J_\nu(b\sqrt{t})Y_\mu(a\sqrt{t})\big)\,dt$$

is in fact a probability density function and

$$\int_0^\infty\Big(\int_0^\infty e^{-st}t^{\frac{\mu+\nu}{2}}\big(J_\mu(a\sqrt{t})Y_\nu(b\sqrt{t}) + J_\nu(b\sqrt{t})Y_\mu(a\sqrt{t})\big)\,dt\Big)\,ds = \frac{-2^{\mu+\nu}\Gamma(\mu)\Gamma(\nu)}{\pi a^\mu b^\nu}.$$

This follows from Theorem 15 and the fact that function in (3-6) is the Laplace transform of an infinitely divisible distribution. In the case when $a = b = 1$, in view of the integral representation

$$J_\mu(x)Y_\nu(x) + J_\nu(x)Y_\mu(x) = -\frac{4}{\pi}\int_0^\infty J_{\mu+\nu}(2x\cosh s)\cosh((\mu-\nu)s)\,F ds$$

(see [EMOT53b, eq. (65), p. 97]), we obtain that (3-5) reduces to

$$z^{\frac{\mu+\nu}{2}} K_\mu(\sqrt{z})K_\nu(\sqrt{z}) = \int_0^\infty\Big(\int_0^\infty \frac{t^{\frac{\mu+\nu}{2}}}{z+t}J_{\mu+\nu}(2\sqrt{t}\cosh s)\cosh((\mu-\nu)s)\,ds\Big)\,dt,$$

where $\mu, \nu > 0$, $\nu+\mu < 1$ and $|\arg z| < \pi$, as before.

In Theorem 15 it is possible to assume that the orders μ and ν are complex numbers with the conditions such that $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$ and $\operatorname{Re}(\nu+\mu) < 1$. The integral representation in (3-5) preexisted in the literature (see [MOS66, p. 96]) in two forms:

$$(-1)^{l+1}\frac{2}{\pi}z^{\mu+\nu+2l}K_\nu(az)K_\mu(bz) = \int_0^\infty \frac{t^{\mu+\nu+2l+1}}{z^2+t^2}\big(J_\nu(at)Y_\mu(bt)+J_\mu(bt)Y_\nu(at)\big)\,dt,$$

where $l \in \{0, \pm 1, \pm 2, \dots\}$, $\operatorname{Re}(\nu+l) > -1$, $\operatorname{Re}(\mu+l) > -1$, $l-1 < \operatorname{Re}(\mu+\nu+2l) < l$, $\operatorname{Re} z > 0$, and

$$(-1)^{l+1}\frac{2}{\pi}z^{\mu+\nu+2l-1}K_\nu(az)K_\mu(bz) = \int_0^\infty \frac{t^{\mu+\nu+2l}}{z^2+t^2}\big(J_\nu(at)J_\mu(bt) - Y_\nu(at)Y_\mu(bt)\big)\,dt,$$

where $l \in \{0, \pm 1, \pm 2, \dots\}$, $\operatorname{Re}(\nu+l) > -\frac{1}{2}$, $\operatorname{Re}(\mu+l) > -\frac{1}{2}$, $l-\frac{1}{2} < \operatorname{Re}(\mu+\nu+2l) < l$, $\operatorname{Re} z > 0$. Clearly our product representation (3-5) corresponds to case when $z^{\mu+\nu+2l}$ reduces to $z^{\mu+\nu}$ and in this case the conditions will be $\operatorname{Re}\nu > -1$, $\operatorname{Re}\mu > -1$, $-1 < \operatorname{Re}(\nu+\mu) < 0$ and $\operatorname{Re} z > 0$, which complements our conditions $\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$, $\operatorname{Re}(\nu+\mu) < 1$ and $|\arg z| < \pi$.

Theorem 16. *If a, b, μ, and ν are strictly positive real numbers, the function*

$$\vartheta_{\mu,\nu,a,b}(x) = \frac{a^\mu b^\nu}{2^{\mu+\nu-2}\Gamma(\mu)\Gamma(\nu)}x^{\frac{\mu+\nu}{2}}K_\mu(a\sqrt{x})K_\nu(b\sqrt{x}) \tag{3-6}$$

is the Laplace transform of an infinitely divisible distribution on $(0,\infty)$ *that is self-decomposable and a generalized gamma convolution.*

Ismail and Kelker [IK79, Theorem 1.8] proved that if $\mu > \nu > -1$, the function $x \mapsto (\sqrt{x})^{\nu-\mu}K_\nu(\sqrt{x})/K_\mu(\sqrt{x})$ is the Laplace transform of an infinitely divisible distribution with support $(0,\infty)$ and consequently taking into account that $K_\mu(x) \sim 2^{\mu-1}x^{-\mu}\Gamma(\mu)$ as $\mu \to \infty$ for $x > 0$ fixed, the function $x \mapsto (\sqrt{x})^\nu K_\nu(\sqrt{x})/\big[2^{\nu-1}\Gamma(\nu)\big]$ is also the Laplace transform of an infinitely divisible distribution with support $(0,\infty)$ whenever $\nu > -1$. However, after verifying the proof of [IK79, Theorem 1.8] we reached the conclusion that the above results are only true when $\mu > \nu > 0$, and $\nu > 0$, respectively.

The next result is analogous to Theorem 15.

Theorem 17. *If $a, b > 0$, $\mu, \nu > -1$, $\mu+\nu > -1$ and $|\arg z| < \pi$, then*

$$e^{-(a+b)\sqrt{z}}z^{-\frac{\mu+\nu}{2}}I_\mu(a\sqrt{z})I_\nu(b\sqrt{z}) = \frac{1}{\pi}\int_0^\infty \frac{t^{-\frac{\mu+\nu}{2}}}{z+t}\big(J_\mu(a\sqrt{t})J_\nu(b\sqrt{t})\sin((a+b)\sqrt{t})\big)\,dt.$$

This Stieltjes transform representation can be rewritten as

$$e^{-(a+b)\sqrt{z}}z^{-\frac{\mu+\nu}{2}}I_\mu(a\sqrt{z})I_\nu(b\sqrt{z}) = \frac{1}{\pi}\int_0^\infty e^{-zs}\Big(\int_0^\infty e^{-st}t^{-\frac{\mu+\nu}{2}}\big(J_\mu(a\sqrt{t})J_\nu(b\sqrt{t})\sin((a+b)\sqrt{t})\big)\,dt\Big)\,ds.$$

The product of two modified Bessel functions of the first kind also generates an infinitely divisible distribution, which however will not be a generalized gamma convolution:

Theorem 18. *If $a, b > 0$ and $\mu, \nu > \frac{1}{2}$, the function*

$$\zeta_{\mu,\nu,a,b}(x) = \frac{2^{\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)}{a^\mu b^\nu} e^{-(a+b)\sqrt{x}} x^{-\frac{\mu+\nu}{2}} I_\mu(a\sqrt{x}) I_\nu(b\sqrt{x})$$

is a Laplace transform of an infinitely divisible distribution on $(0, \infty)$, but is not a generalized gamma convolution.

From Theorems 17 and 18 for $a, b > 0$ and $\mu, \nu > \frac{1}{2}$ we conclude that

$$s \mapsto \frac{2^{\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)}{a^\mu b^\nu \pi} \int_0^\infty e^{-st} t^{-\frac{\mu+\nu}{2}} \left(J_\mu(a\sqrt{t}) J_\nu(b\sqrt{t}) \sin(a+b)\sqrt{t}\right) dt$$

is a probability density function and

$$\int_0^\infty \left(\int_0^\infty e^{-st} t^{-\frac{\mu+\nu}{2}} \left(J_\mu(a\sqrt{t}) J_\nu(b\sqrt{t}) \sin(a+b)\sqrt{t}\right) dt\right) ds = \frac{\pi a^\mu b^\nu}{2^{\mu+\nu}\Gamma(\mu+1)\Gamma(\nu+1)}.$$

The next result is the counterpart of Theorem 15.

Theorem 19. *If $a, b > 0$, $\mu, \nu \in \mathbb{R}$, $\mu+\nu > 1$ and $|\arg z| < \pi$, then*

$$\frac{e^{-(a+b)\sqrt{z}}}{z^{\frac{\mu+\nu}{2}} K_\mu(a\sqrt{z}) K_\nu(b\sqrt{z})} = \frac{4}{\pi^3} \int_0^\infty \frac{t^{-\frac{\mu+\nu}{2}}}{z+t} \Gamma_{\mu,\nu,a,b}(t)\, dt.$$

This representation is equivalent to

$$\frac{e^{-(a+b)\sqrt{z}}}{z^{\frac{\mu+\nu}{2}} K_\mu(a\sqrt{z}) K_\nu(b\sqrt{z})} = \frac{4}{\pi^3} \int_0^\infty e^{-zs} \left(\int_0^\infty e^{-st} t^{-\frac{\mu+\nu}{2}} \Gamma_{\mu,\nu,a,b}(t)\, dt\right) ds,$$

where

$$\Gamma_{\mu,\nu,a,b}(t) = \frac{{}_1T_{\mu,\nu,a,b}(t) \cos((a+b)\sqrt{t}) - {}_2T_{\mu,\nu,a,b}(t) \sin((a+b)\sqrt{t})}{\left(J_\mu^2(a\sqrt{t}) + Y_\mu^2(a\sqrt{t})\right)\left(J_\nu^2(b\sqrt{t}) + Y_\nu^2(b\sqrt{t})\right)}$$

with

$${}_1T_{\mu,\nu,a,b}(t) = J_\mu(a\sqrt{t}) Y_\nu(b\sqrt{t}) + J_\nu(b\sqrt{t}) Y_\mu(a\sqrt{t}),$$
$${}_2T_{\mu,\nu,a,b}(t) = J_\mu(a\sqrt{t}) J_\nu(b\sqrt{t}) - Y_\mu(a\sqrt{t}) Y_\nu(b\sqrt{t}).$$

The next theorem shows that the reciprocal of the product of two modified Bessel functions of the second kind also generates an infinitely divisible distribution.

Theorem 20. *If $a, b > 0$, $\mu, \nu > \frac{1}{2}$, the function*

$$\kappa_{\mu,\nu,a,b}(x) = \frac{2^{\mu+\nu-2}\Gamma(\mu)\Gamma(\nu)}{a^\mu b^\nu} \cdot \frac{e^{-(a+b)\sqrt{x}}}{x^{\frac{\mu+\nu}{2}} K_\mu(a\sqrt{x}) K_\nu(b\sqrt{x})}$$

is the Laplace transform of an infinitely divisible distribution on $(0, \infty)$, which is a generalized gamma convolution.

In view of Theorems 19 and 20, the function

$$s \mapsto \frac{2^{\mu+\nu}\Gamma(\mu)\Gamma(\nu)}{a^{\mu}b^{\nu}\pi^{3}} \int_0^{\infty} e^{-st} t^{-\frac{\mu+\nu}{2}} \Gamma_{\mu,\nu,a,b}(t)\, dt$$

is a probability density function on $(0, \infty)$ whenever $a, b > 0$, $\mu, \nu > \frac{1}{2}$, and moreover

$$\int_0^{\infty}\Big(\int_0^{\infty} e^{-st} t^{-\frac{\mu+\nu}{2}} \Gamma_{\mu,\nu,a,b}(t)\, dt\Big)\, ds = \frac{a^{\mu}b^{\nu}\pi^{3}}{2^{\mu+\nu}\Gamma(\mu)\Gamma(\nu)}.$$

Now, we focus on the quotient of modified Bessel functions of the first and second kinds.

Theorem 21. *If* $a, b > 0$, $\mu > -1$, $\mu+\nu > 0$ *and* $|\arg z| < \pi$, *then*

$$\frac{z^{-\frac{\mu+\nu}{2}}}{e^{(a+b)\sqrt{z}}} \frac{I_{\mu}(a\sqrt{z})}{K_{\nu}(b\sqrt{z})} = -\frac{2}{\pi^2}\int_0^{\infty} \frac{t^{-\frac{\mu+\nu}{2}}}{z+t} J_{\mu}(a\sqrt{t})\gamma_{\nu,a,b}(t)\, dt.$$

This representation can be rewritten as

$$\frac{z^{-\frac{\mu+\nu}{2}}}{e^{(a+b)\sqrt{z}}} \frac{I_{\mu}(a\sqrt{z})}{K_{\nu}(b\sqrt{z})} = -\frac{2}{\pi^2}\int_0^{\infty} e^{-zs}\Big(\int_0^{\infty} \frac{e^{-st} J_{\mu}(a\sqrt{t})}{t^{\frac{\mu+\nu}{2}}}\gamma_{\nu,a,b}(t)\, dt\Big)\, ds,$$

where

$$\gamma_{\nu,a,b}(t) = \frac{J_{\nu}(b\sqrt{t})\cos((a+b)\sqrt{t}) + Y_{\nu}(b\sqrt{t})\sin((a+b)\sqrt{t})}{J_{\nu}^2(b\sqrt{t}) + Y_{\nu}^2(b\sqrt{t})}.$$

In particular, when $a \to 0$, we obtain the following result.

Corollary 2. *If* $b > 0$, $\nu > \frac{1}{2}$ *and* $|\arg z| < \pi$, *then*

$$z^{-\frac{\nu}{2}} e^{-b\sqrt{z}} \frac{1}{K_{\nu}(b\sqrt{z})} = -\frac{2}{\pi^2}\int_0^{\infty} \frac{t^{-\frac{\nu}{2}}}{z+t}\gamma_{\nu,0,b}(t)\, dt.$$

Finally, we show that the above quotient of modified Bessel functions of the first and second kinds also generates an infinitely divisible distribution.

Theorem 22. *If* $a, b > 0$, $\nu, \mu > \frac{1}{2}$, *then the function*

$$\varepsilon_{\mu,\nu,a,b}(x) = \frac{2^{\mu+\nu-1}}{a^{\mu}b^{\nu}}\Gamma(\nu)\Gamma(\mu+1)e^{-(a+b)\sqrt{x}} x^{-\frac{\mu+\nu}{2}} \frac{I_{\mu}(a\sqrt{x})}{K_{\nu}(b\sqrt{x})}$$

is the Laplace transform of an infinitely divisible distribution on $(0, \infty)$. *If* $a, b > 0$, $\nu > 0$ *and* $\mu > -1$, *then the reciprocal of* $e^{(a+b)\sqrt{x}}\varepsilon_{\mu,\nu,a,b}(x)$, *that is,*

$$\frac{e^{-(a+b)\sqrt{x}}}{\varepsilon_{\mu,\nu,a,b}(x)} = \frac{a^{\mu}b^{\nu}}{2^{\mu+\nu-1}\Gamma(\nu)\Gamma(\mu+1)} x^{\frac{\mu+\nu}{2}} \frac{K_{\nu}(b\sqrt{x})}{I_{\mu}(a\sqrt{x})},$$

is also a Laplace transform of an infinitely divisible distribution on $(0, \infty)$.

If $a, b > 0$ and $\mu, \nu > \frac{1}{2}$, Theorems 21 and 22 imply that the function

$$s \mapsto \frac{-2^{\mu+\nu}\Gamma(\nu)\Gamma(\mu+1)}{a^{\mu}b^{\nu}\pi^2}\int_0^{\infty} e^{-st}t^{-\frac{\mu+\nu}{2}} J_{\mu}(a\sqrt{t})\gamma_{\nu,a,b}(t)\,dt$$

is a probability density function and

$$\int_0^{\infty}\Big(\int_0^{\infty} e^{-st}t^{-\frac{\mu+\nu}{2}} J_{\mu}(a\sqrt{t})\gamma_{\nu,a,b}(t)\,dt\Big)\,ds = -\frac{a^{\mu}b^{\nu}\pi^2}{2^{\mu+\nu}\Gamma(\nu)\Gamma(\mu+1)}.$$

4. Remarks, open problems and challenges for future direction

4.1. *Remarks on quotients of Tricomi hypergeometric functions.* We now state some facts related to quotients of Tricomi hypergeometric functions. Goovaerts et al. [GHP78] proved that the distribution of the ratio of two independent gamma-distributed random variables is infinitely divisible; this solved a problem posed by Steutel [St73] in a survey of the theory of infinite divisibility. More precisely, based on a relation between the Laplace transform of the density of the quotient of two gamma random variables and the Tricomi confluent hypergeometric function, Goovaerts et al. proved that the distribution of the ratio of two independent gamma-distributed random variables is a generalized gamma convolution. In their proof the logarithmic derivative of the Laplace transform is obtained by the argument principle and contour integration. At the same time, Ismail and Kelker [IK79, Theorem 1.5] proved with the Stieltjes transform technique that the distribution of the two gamma random variables is self-decomposable, hence is infinitely divisible. In this subsection we provide another way to show that the distribution of the ratio of two gamma random variables is a generalized gamma convolution.

Theorem 23. *The distribution of the quotient of two gamma random variables is a generalized gamma convolution.*

Ismail and Kelker's proof of [IK79, Theorem 1.5] was based on the integral representation

$$\frac{\psi(a+1,c+1,z)}{\psi(a,c,z)} = \int_0^{\infty}\frac{t^{-c}e^{-t}|\psi(a,c,te^{\mathrm{i}\pi})|^{-2}}{(z+t)\Gamma(a+1)\Gamma(a-c+1)}dt \tag{4-1}$$

(see [IK79, p. 885]), where $a > 0$, $c < 1$ and $|\arg z| < \pi$. Our proof of Theorem 23 is also based on (4-1), bur our approach is based on the Pick function characterization theorem, Lemma 3. Moreover, we show that it is possible to obtain similar Stieltjes transform representations for quotients of Tricomi hypergeometric functions. These results complement [IK79, Theorem 1.4].

Theorem 24. *If* $a > 0$, $c < 1$ *and* $|\arg z| < \pi$, *then the following representations* *are valid*:

(4-2) $$\frac{\psi(a,c-1,z)}{\psi(a,c,z)}=\frac{1-c}{a-c+1}+\int_0^\infty\frac{z}{z+t}\frac{t^{-c}e^{-t}|\psi(a,c,e^{\mathrm{i}\pi}t)|^{-2}}{\Gamma(a)\Gamma(a-c+2)}dt,$$

$$\frac{\psi(a+1,c,z)}{\psi(a,c,z)}=\frac{1}{a-c+1}-\int_0^\infty\frac{z}{z+t}\frac{(a-c+1)t^{-c}e^{-t}}{\Gamma(a+1)\Gamma(a-c+2)}|\psi(a,c,e^{\mathrm{i}\pi}t)|^{-2}dt,$$

(4-3) $$\frac{\psi(a,c+1,z)}{\psi(a,c,z)}=1+\int_0^\infty\frac{t^{-c}e^{-t}}{z+t}\frac{|\psi(a,c,e^{\mathrm{i}\pi}t)|^{-2}}{\Gamma(a)\Gamma(a-c+1)}dt,$$

$$\frac{\psi(a-1,c,z)}{\psi(a,c,z)}=z-c+a+\int_0^\infty\frac{z}{z+t}\frac{t^{-c}e^{-t}|\psi(a,c,e^{\mathrm{i}\pi}t)|^{-2}}{\Gamma(a)\Gamma(a-c+1)}dt.$$

This follows naturally from (4-1), but for (4-3) we give a more detailed proof via the Stieltjes representation and inversion theorems. We mention that by using the Stieltjes transform technique Ismail and Kelker [IK79, eq. (1.5)] proved that

$$\frac{\psi(a,c-1,z)}{\psi(a,c,z)}=\int_0^\infty\frac{z}{z+t}\frac{t^{-c}e^{-t}|\psi(a,c,e^{\mathrm{i}\pi}t)|^{-2}}{\Gamma(a)\Gamma(a-c+2)}dt$$

for $a>0,\ 1<c<a+1$ and $|\arg z|<\pi$, and our result (4-2) complements this naturally.

Bondesson [Bo92, Example 4.3.1] remarked that the distribution of the quotient of two gamma random variables belongs also to the class of hyperbolically completely monotone densities. More precisely, if X and Y are independent gamma distributed random variables with parameters (α,β) and (α_0,β_0), then the probability density function of their quotient $Z=X/Y$ is given by

$$f(x)=\frac{\Gamma(\alpha+\alpha_0)}{\Gamma(\alpha)\Gamma(\alpha_0)}\Big(\frac{\beta_0}{\beta}\Big)^{\alpha}x^{\alpha-1}\Big(1+\frac{\beta_0}{\beta}x\Big)^{-(\alpha+\alpha_0)},$$

where $x>0$; see [IK79, p. 889]. Now, if $u,v>0$ and $w=v+1/v$, we have, by [Bo92, Example 4.3.1],

$$\begin{aligned}f(uv)f\Big(\frac{u}{v}\Big)&=\bigg(\frac{\Gamma(\alpha+\alpha_0)}{\Gamma(\alpha)\Gamma(\alpha_0)}\Big(\frac{\beta_0}{\beta}\Big)^{\alpha}\bigg)^2u^{2\alpha-2}\Big((1+\frac{\beta_0}{\beta}uv)\Big(1+\frac{\beta_0}{\beta}\frac{u}{v}\Big)\Big)^{-(\alpha+\alpha_0)}\\&=\bigg(\frac{\Gamma(\alpha+\alpha_0)}{\Gamma(\alpha)\Gamma(\alpha_0)}\Big(\frac{\beta_0}{\beta}\Big)^{\alpha}\bigg)^2u^{2\alpha-2}\Big(1+\Big(\frac{u\beta_0}{\beta}\Big)^2+\frac{\beta_0}{\beta}uw\Big)^{-(\alpha+\alpha_0)},\end{aligned}$$

and thus $w\mapsto f(uv)f(u/v)$ is completely monotonic on $(0,\infty)$ for all $\alpha,\alpha_0,\beta,\beta_0>0$.

As discussed in [AB23], the integral representation (4-1) was important in the study of the infinite divisibility of the Whittaker distribution. It is possible to obtain similar results on ratios of Tricomi hypergeometric functions where the difference between the parameters is not necessarily an integer; see [FS23, Section 2] for details.

4.2. ***The noncentral chi square distribution.*** The noncentral chi square distribution is a noncentral generalization of the chi-squared distribution and has the probability density function

$$\chi_{\mu,\lambda}(x) = \tfrac{1}{2}e^{-\frac{(x+\lambda)}{2}}(x/\lambda)^{\frac{\mu}{4}-\frac{1}{2}}I_{\frac{\mu}{2}-1}(\sqrt{\lambda x}), ,$$

where $\lambda > 0$ is the noncentral parameter and $\mu > 0$ is the degree of freedom. From [IK79, Theorem 1.6] we know that the noncentral chi square distribution is infinitely divisible for all degrees of freedom, including fractional ones. From [Bo92, Example 9.2.2] we also know that the noncentral chi square distribution belongs to the so-called class of generalized convolutions of mixtures of exponential distributions, introduced in [Bo81, p. 43], and which is in fact the smallest class of distributions that is closed under convolution and weak convergence and contains all mixtures of exponential distributions. We know that the class of generalized convolutions of mixtures of exponential distributions is a subclass of the infinitely divisible distributions, but contains all generalized gamma convolutions and hence hyperbolically completely monotone densities. Thus, it is very natural to ask whether the noncentral chi square distribution belongs to the class of generalized gamma convolutions or to the class of hyperbolically completely monotone densities. The next result suggests that under some conditions on the parameters μ and λ the noncentral chi square distribution belongs to the class of hyperbolically completely monotone densities, and hence to the class of generalized gamma convolutions. The ***first open problem*** is to find the optimal range for the parameters μ and λ such that the noncentral chi square distribution belongs to the class of hyperbolically completely monotone densities.

Theorem 25. *Let $\mu > 1$, $u, v > 0$ and $w = v + 1/v$. If $\lambda \leq \mu$, the function $w \mapsto \chi_{\mu,\lambda}(uv)\chi_{\mu,\lambda}(u/v)$ is strictly decreasing on $(2,\infty)$ and if $2\lambda \leq \mu$, then $w \mapsto \chi_{\mu,\lambda}(uv)\chi_{\mu,\lambda}(u/v)$ is strictly convex on $(2,\infty)$.*

4.3. ***Hyperbolically complete monotonicity of McKay distributions.*** In Theorems 1–4 of Section 2 we studied the infinite divisibility and self-decomposability of McKay-type distributions and checked their membership in the class of generalized gamma convolutions. We were not able to check whether these distributions belong to the class of hyperbolically completely monotone densities. (In Theorem 5, Macdonald's integral representation of the K-distribution was crucial; but to our knowledge there is no similar result for modified Bessel functions of the first kind.)

A function $f : (0,\infty) \to \mathbb{R}$ is *absolutely monotone* (or absolutely monotonic) if it has derivatives of all orders and $f^{(n)}(x) > 0$ for all $x > 0$ and $n \in \{0, 1, 2, \dots\}$. The next result shows that $w \mapsto I_\mu(uv)I_\mu(u/v)$ is absolutely monotonic on $(2,\infty)$; thus a more sophisticated analysis is needed to verify whether McKay type distributions belong to the class of hyperbolically completely monotone densities.

Theorem 26. *If* $\mu > -\frac{1}{2}$, $u, v > 0$ *and* $w = v + 1/v$, *then the function* $w \to I_\mu(uv) I_\mu(u/v)$ *is absolutely monotonic on* $(2, \infty)$.

The ***second open problem*** is to verify under what conditions the distributions considered in Theorems 1–4 belong to the class of hyperbolically completely monotone densities, and under what conditions the distributions in Theorems 2–4 belong to the class of self-decomposable distributions as well as of generalized gamma convolutions. One can also ask under what conditions the probability density function in Theorem 12 is hyperbolically completely monotonic.

4.4. ***Self-decomposability of modified Bessel distributions.*** Let $w(x)$ be the Laplace transform of a probability distribution on $[0, \infty)$. It is known that if $-dw(x)/dx$ is the Stieltjes transform of a positive measure, then the original distribution is self-decomposable, and hence is infinitely divisible. In other words, a nonnegative random variable is self-decomposable if its Laplace transform satisfies

$$\int_0^\infty e^{-xt} d\varpi(t) = e^{-h(x)}, \quad h'(x) = \int_0^\infty \frac{d\varpi(t)}{x+t}, \quad d\varpi(t) \ge 0. \tag{4-4}$$

By Lemma 1, a probability measure $d\omega$ supported on $[0, \infty)$ is infinitely divisible if and only if its Laplace transform satisfies

$$\int_0^\infty e^{-xt} d\omega(t) = e^{-h(x)}, \quad h(0) = 0, \quad \text{and } h'(x) \text{ is completely monotonic.} \tag{4-5}$$

Recall that self-decomposable functions are infinitely divisible and a probability distribution satisfying (4-4) and (4-5) is called a generalized gamma convolution. In Theorems 14, 18 and 22 we have infinitely divisible modified Bessel distributions whose Laplace transform can be written as Stieltjes transforms, but these Stieltjes transforms do not have positive kernels. Thus, the ***third open problem*** is to verify under which conditions the distributions considered in Theorems 14, 18 and 22 are self-decomposable.

5. Proofs of the main results

Proof of Theorem 1. From Prudnikov et al. [PBM88, eq. 2.15.3.2], we have

$$\int_0^\infty x^\mu e^{-bx} I_\mu(ax)\, dx = \frac{(2a)^\mu \Gamma\left(\mu + \frac{1}{2}\right)}{\sqrt{\pi}(b^2 - a^2)^{\mu + \frac{1}{2}}},$$

which in turn implies that the Laplace transform

$$L[\varphi_{\mu,a,b}(x)] = \int_0^\infty e^{-xt} \varphi_{\mu,a,b}(t)\, dt$$

of the probability density function $\varphi_{\mu,a,b}$ is given by

$$L[\varphi_{\mu,a,b}(x)]=\frac{\sqrt{\pi}(b^2-a^2)^{\mu+\frac{1}{2}}}{(2a)^{\mu}\Gamma\left(\mu+\frac{1}{2}\right)}\int_0^{\infty}t^{\mu}e^{-(b+x)t}I_{\mu}(at)\,dt=\left(\frac{b^2-a^2}{(x+b)^2-a^2}\right)^{\mu+\frac{1}{2}}.$$

First we show that the McKay distribution with probability density function (2-1) is infinitely divisible. In view of Lemma 1, this is equivalent to

$$x\mapsto -\frac{d\ln L[\varphi_{\mu,a,b}(x)]}{dx}=\left(\mu+\tfrac{1}{2}\right)\left(\frac{1}{x+b-a}+\frac{1}{x+b+a}\right)$$

being complete monotone on $(0,\infty)$ when $\mu>-\frac{1}{2}$ and $b>a>0$, which is true.

We show that the same distribution is self-decomposable. Write $\omega_{\mu,a,b}(x)=L[\varphi_{\mu,a,b}(x)]/L[\varphi_{\mu,a,b}(\alpha x)]$. By Lemma 2, self-decomposability is equivalent to the complete monotonicity on $(0,\infty)$ of

$$x\mapsto -\frac{d\ln\omega_{\mu,a,b}(x)}{dx}=\left(\mu+\tfrac{1}{2}\right)\left(\frac{(1-\alpha)(b-a)}{(x+b-a)(\alpha x+b-a)}+\frac{(1-\alpha)(b+a)}{(x+b+a)(\alpha x+b+a)}\right)$$

when $\alpha\in(0,1)$, $\mu>-\frac{1}{2}$ and $b>a>0$, which is also true.

We show that our distribution belongs to the class of generalized gamma convolutions. The moment generating function of the distribution is

$$\psi(z):=M_X(z)=\int_0^{\infty}e^{tz}\varphi_{\mu,a,b}(t)\,dt=L[\varphi_{\mu,a,b}(-z)]=\left(\frac{b^2-a^2}{(z-b)^2-a^2}\right)^{\mu+\frac{1}{2}}.$$

This function is analytic and zero-free in $\mathbb{C}\setminus[0,\infty)$, so to apply Lemma 3 we just need to verify that $\operatorname{Im}\left(\psi'(s)/\psi(s)\right)\geq 0$ for $\operatorname{Im}s>0$. We have

$$\frac{\psi'(s)}{\psi(s)}=-\left(\mu+\tfrac{1}{2}\right)\left(\frac{1}{s+a-b}+\frac{1}{s-a-b}\right),$$

which implies that for $s=x+\mathrm{i}y$ and $y=\operatorname{Im}s>0$ we have, as needed,

$$\operatorname{Im}\frac{\psi'(s)}{\psi(s)}=\left(\mu+\tfrac{1}{2}\right)\left(\frac{y}{(x+a-b)^2+y^2}+\frac{y}{(x-a-b)^2+y^2}\right)>0.$$

(Another proof that the distribution belongs to the class of generalized gamma convolutions uses Lemma 4: we just observe that if $\phi(x)=L[\varphi_{\mu,a,b}(x)]$, then $\phi(uv)\phi(u/v)$ can be written as

$$\phi(uv)\phi(u/v)=\frac{(b^2-a^2)^{\mu+\frac{1}{2}}u^{-2\mu-1}}{\left(w+\dfrac{u^2+(b-a)^2}{(b-a)u}\right)^{\mu+\frac{1}{2}}\left(w+\dfrac{u^2+(b+a)^2}{(b+a)u}\right)^{\mu+\frac{1}{2}}},$$

which is completely monotonic in $w=v+1/v>0$ for all $\mu>-\frac{1}{2}$ and $b>a>0$.)

□

Proof of Theorem 2. By [PBM88, equation 2.15.3.2] we have

$$\int_0^\infty x^{\mu+1}e^{-bx}I_\mu(ax)\,dx = \frac{2b(2a)^\mu\Gamma\left(\mu+\frac{3}{2}\right)}{\sqrt{\pi}(b^2-a^2)^{\mu+\frac{3}{2}}},$$

which implies that the Laplace transform $L[\psi_{\mu,a,b}(x)] = \int_0^\infty e^{-xt}\psi_{\mu,a,b}(t)\,dt$ of the probability density function $\psi_{\mu,a,b}$ is given by

$$\frac{\sqrt{\pi}(b^2-a^2)^{\mu+\frac{3}{2}}}{2b(2a)^\mu\Gamma\left(\mu+\frac{3}{2}\right)}\int_0^\infty t^{\mu+1}e^{-(b+x)t}I_\mu(at)\,dt = \left(1+\frac{x}{b}\right)\left(\frac{b^2-a^2}{(x+b)^2-a^2}\right)^{\mu+\frac{3}{2}}.$$

By Lemma 1, showing that the distribution with probability density function (2-2) is infinitely divisible is equivalent to showing that

$$x \mapsto -\frac{d\ln L[\psi_{\mu,a,b}(x)]}{dx} = \frac{\mu+1}{x+b-a} + \frac{\mu+1}{x+b+a} + \frac{a^2}{(x+b)(x+b-a)(x+b+a)}$$

is completely monotone on $(0,\infty)$ when $\mu > -1$ and $b > a > 0$, which is true. □

Proof of Theorem 3. Equation 2.15.3.2 of [PBM88] reads $\int_0^\infty x^{\nu-1}e^{-bx}I_\mu(ax)\,dx = c_{\mu,\nu,a,b}$, so the Laplace transform

$$L[\varphi_{\mu,\nu,a,b}(x)] = \int_0^\infty e^{-xt}\varphi_{\mu,\nu,a,b}(t)\,dt$$

of the probability density function $\varphi_{\mu,\nu,a,b}$ is given by

$$L[\varphi_{\mu,\nu,a,b}(x)] = \left(\frac{b}{x+b}\right)^{\mu+\nu}\cdot\frac{{}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}, \mu+1, \frac{a^2}{(x+b)^2}\right)}{{}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}, \mu+1, \frac{a^2}{b^2}\right)}.$$

Recall that the Gaussian hypergeometric function satisfies

$$\frac{d}{dx}\left({}_2F_1(a,b,c,x)\right) = \frac{ab}{c}\cdot{}_2F_1(a+1,b+1,c+1,x)$$

and for a, b and c such that $-1 \le a \le c$ and $0 < b \le c$ we have the Küstner integral representation [Kü02]

$$\frac{z\cdot{}_2F_1(a+1,b+1,c+1,z)}{{}_2F_1(a,b,c,z)} = \int_0^1 \frac{z}{1-tz}dq_{a,b,c}(t), \tag{5-1}$$

where $z \in \mathbb{C}\setminus[1,\infty)$, $q_{a,b,c}(1) - q_{a,b,c}(0) = 1$ and $q_{a,b,c}$ is a nondecreasing self-mapping of $[0,1]$. To prove infinite divisibility, in view of Lemma 1, we just need to observe that the function

$$x \mapsto -\frac{d\ln L[\varphi_{\mu,\nu,a,b}(x)]}{dx}$$

$$= \frac{\mu+\nu}{x+b} + \frac{a^2(\mu+\nu)(\mu+\nu+1)}{2(\mu+1)(x+b)^3} \cdot \frac{{}_2F_1\left(\frac{\mu+\nu+2}{2}, \frac{\mu+\nu+3}{2}, \mu+2, \frac{a^2}{(x+b)^2}\right)}{{}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}, \mu+1, \frac{a^2}{(x+b)^2}\right)}$$

$$= \frac{\mu+\nu}{x+b}\left(1 + \frac{a^2(\mu+\nu+1)}{2(\mu+1)} \int_0^1 \frac{dq_{\mu,\nu}(t)}{(x+b-a\sqrt{t})(x+b+a\sqrt{t})}\right)$$

is completely monotonic on $(0, \infty)$ for all $\mu+\nu > 0$, $\nu \le \mu+1$ and $b > a > 0$, where in view of (5-1) we have $q_{\mu,\nu}(1) - q_{\mu,\nu}(0) = 1$ and $q_{\mu,\nu}$ is a nondecreasing self-mapping of [0, 1]. □

Proof of Theorem 4. Equation 2.15.20.5 of [PBM88] reads $\int_0^\infty x^{2\mu} e^{-bx} (I_\mu(ax))^2 dx = c_{\mu,a,b}$, so the Laplace transform

$$L[\xi_{\mu,a,b}(x)] = \int_0^\infty e^{-xt} \xi_{\mu,a,b}(t)\, dt$$

of the probability density function $\xi_{\mu,a,b}$ is given by

$$L[\xi_{\mu,a,b}(x)] = \left(\frac{b}{x+b}\right)^{4\mu+1} \cdot \frac{{}_2F_1\left(\mu+\frac{1}{2}, 2\mu+\frac{1}{2}, \mu+1, 4a^2/(x+b)^2\right)}{{}_2F_1\left(\mu+\frac{1}{2}, 2\mu+\frac{1}{2}, \mu+1, 4a^2/b^2\right)}.$$

To prove infinite divisibility, in view of Lemma 1, we just need to observe that

$$x \mapsto -\frac{d \ln L[\xi_{\mu,a,b}(x)]}{dx}$$

$$= \frac{4\mu+1}{x+b} + \frac{8a^2(\mu+\frac{1}{2})(2\mu+\frac{1}{2})}{(\mu+1)(x+b)^3} \frac{{}_2F_1\left(\mu+\frac{3}{2}, 2\mu+\frac{3}{2}, \mu+2, 4a^2/(x+b)^2\right)}{{}_2F_1\left(\mu+\frac{1}{2}, 2\mu+\frac{1}{2}, \mu+1, 4a^2/(x+b)^2\right)}$$

$$= \frac{4\mu+1}{x+b}\left(1 + \frac{2a^2(2\mu+1)}{\mu+1} \int_0^1 \frac{dq_\mu(t)}{(x+b-2a\sqrt{t})(x+b+2a\sqrt{t})}\right)$$

is completely monotonic on $(0, \infty)$ for all $-1 \le \mu+\frac{1}{2} \le \mu+1$, $0 < 2\mu+\frac{1}{2} \le \mu+1$ and $b > 2a > 0$, where in view of (5-1) we have $q_\mu(1) - q_\mu(0) = 1$ and q_μ is a nondecreasing self-mapping of [0, 1]. □

Proof of Theorem 5. We will consider only the case when $\alpha \ne \beta$ for the proof of the infinite divisibility, self-decomposability as well as when we prove that the K-distribution belongs to the class of generalized gamma convolutions. However, in the case of hyperbolically complete monotonicity we will consider also the case when $\alpha = \beta$.

By using Lemma 1, first we show that the K-distribution is infinitely divisible. Equation 2.16.8.4 of [PBM88] reads

$$\int_0^\infty t^{q-1/2} e^{-rt} K_{2\nu}(2s\sqrt{t})\, dt = \frac{\Gamma\left(q+\nu+\frac{1}{2}\right)\Gamma\left(q-\nu+\frac{1}{2}\right)}{2sr^q e^{-s^2/2r}} W_{-q,\nu}\left(\frac{s^2}{r}\right),$$

where $W_{\kappa,\mu}$ stands for the Whittaker function of the second kind. Hence

$$L[\omega_{\alpha,\beta,\mu}(x)] = \int_0^\infty e^{-xt}\omega_{\alpha,\beta,\mu}(t)\,dt = \Big(\frac{\alpha\beta}{\mu x}\Big)^{\frac{\alpha+\beta-1}{2}} e^{\frac{\alpha\beta}{2\mu x}} W_{-\frac{\alpha+\beta-1}{2},\frac{\alpha-\beta}{2}}\Big(\frac{\alpha\beta}{\mu x}\Big).$$

$W_{\kappa,\mu}$ is related to the Tricomi hypergeometric function by

$$W_{\kappa,\mu}(x) = e^{-\frac{x}{2}}x^{\mu+\frac{1}{2}}\cdot\psi\big(\mu-\kappa+\tfrac{1}{2}, 1+2\mu, x\big),$$

so the Laplace transform of the K-distribution is

$$L[\omega_{\alpha,\beta,\mu}(x)] = \Big(\frac{\alpha\beta}{\mu x}\Big)^{\alpha}\psi\Big(\alpha, 1+\alpha-\beta, \frac{\alpha\beta}{\mu x}\Big).$$

Now, recall the recurrence relation [NIST10, eq. 13.3.22]

$$\psi'(a, c, x) = -a\psi(a+1, c+1, x)$$

and the integral representation (4-1), which is valid for $|\arg z| < \pi$, $a > 0$ and $c < 1$. In view of Lemma 1 and the above relations, to show that the K-distribution is infinitely divisible we just need to show that for $\theta(x) = L[\omega_{\alpha,\beta,\mu}(x)]$ we have

$$-\frac{d}{dx}\ln\theta(x) = \frac{\alpha}{x} + \frac{\alpha\beta}{\mu x^2}\frac{\psi'\big(\alpha, 1+\alpha-\beta, \frac{\alpha\beta}{\mu x}\big)}{\psi\big(\alpha, 1+\alpha-\beta, \frac{\alpha\beta}{\mu x}\big)} = \frac{\alpha}{x} - \frac{\alpha^2\beta}{\mu x^2}\frac{\psi\big(\alpha+1, 2+\alpha-\beta, \frac{\alpha\beta}{\mu x}\big)}{\psi\big(\alpha, 1+\alpha-\beta, \frac{\alpha\beta}{\mu x}\big)},$$

that is,

$$-\frac{d}{dx}\ln\theta(x) = \frac{\alpha}{x}\Big(1 - \frac{\alpha\beta}{\mu x}\int_0^\infty \frac{\omega_{\alpha,\beta}(t)}{\frac{\alpha\beta}{\mu x}+t}\,dt\Big),$$

where

$$\omega_{\alpha,\beta}(t) = \frac{t^{\beta-\alpha-1}e^{-t}}{\Gamma(\alpha+1)\Gamma(\beta)}\big|\psi(\alpha, 1+\alpha-\beta, te^{\mathrm{i}\pi})\big|^{-2},$$

is completely monotonic on $(0, \infty)$. To show this, observe that if in

$$\frac{\alpha\beta}{\mu x}\frac{\psi\big(\alpha+1, 2+\alpha-\beta, \frac{\alpha\beta}{\mu x}\big)}{\psi\big(\alpha, 1+\alpha-\beta, \frac{\alpha\beta}{\mu x}\big)} = \int_0^\infty \frac{\frac{\alpha\beta}{\mu x}\omega_{\alpha,\beta}(t)}{\frac{\alpha\beta}{\mu x}+t}\,dt \tag{5-2}$$

we make the change of variable $\frac{\alpha\beta}{\mu x} = s$ and take s to infinity, then in view of the asymptotic expansion [NIST10, eq. 13.7.3]

$$\psi(a, c, x) \sim x^{-a}\Big(1 + a(c-a-1)\frac{1}{x} + \frac{1}{2}a(a+1)(a+1-c)(a+2-c)\frac{1}{x^2} + \dots\Big),$$

which is valid for large real x and fixed a and c, we obtain

$$\int_0^\infty \omega_{\alpha,\beta}(t)\,dt = 1. \tag{5-3}$$

This implies that

$$-\frac{d}{dx}\ln\theta(x)=\frac{\alpha}{x}\Big(\int_0^\infty \omega_{\alpha,\beta}(t)\,dt-\frac{\alpha\beta}{\mu x}\int_0^\infty\frac{\omega_{\alpha,\beta}(t)}{\frac{\alpha\beta}{\mu x}+t}dt\Big)=\int_0^\infty\frac{\alpha\omega_{\alpha,\beta}(t)}{x+\frac{\alpha\beta}{\mu t}}dt,$$

and this is completely monotonic in x on $(0,\infty)$ for all $\alpha,\beta,\mu>0$ such that $\alpha<\beta$. All we need is to observe that the Whittaker function of the first kind is symmetric in the second parameter, that is, $W_{\kappa,\mu}=W_{\kappa,-\mu}$, and this shows that for every $\alpha,\beta,\mu>0$ the Laplace transform of the K-distribution $L[\omega_{\alpha,\beta,\mu}(x)]$ is the Laplace transform of an infinitely divisible distribution. Another way to show the complete monotonicity of $-d\ln\theta(x)/dx$ is to use the Kummer transformation [NIST10, eq. 13.2.40]

$$\psi(a,c,x)=x^{1-c}\psi(a-c+1,2-c,x)$$

and then apply the previous approach to the Laplace transform

$$L[\omega_{\alpha,\beta,\mu}(x)]=\Big(\frac{\alpha\beta}{\mu x}\Big)^{\beta}\psi\Big(\beta,1-\alpha+\beta,\frac{\alpha\beta}{\mu x}\Big), \tag{5-4}$$

to check that it is the Laplace transform of an infinite divisible distribution for all $\alpha,\beta,\mu>0$ such that $\beta<\alpha$.

Now set $\theta(x)=L[\omega_{\alpha,\beta,\mu}(x)]$. By Lemma 2, the K-distribution will be self-decomposable if, for every $a\in(0,1)$, the function $\eta(x)=\theta(x)/\theta(ax)$ is a Laplace transform. Lemma 1 reduces the proof of the last condition to checking that $-d\ln\eta(x)/dx$ is completely monotonic on $(0,\infty)$. Thus, by (5-2) we have

$$\begin{aligned}-\frac{d}{dx}[\ln\eta(x)]&=-\frac{\alpha\beta}{a\mu x^2}\frac{\psi'\big(\alpha,1+\alpha-\beta,\frac{\alpha\beta}{a\mu x}\big)}{\psi\big(\alpha,1+\alpha-\beta,\frac{\alpha\beta}{a\mu x}\big)}+\frac{\alpha\beta}{\mu x^2}\frac{\psi'\big(\alpha,1+\alpha-\beta,\frac{\alpha\beta}{\mu x}\big)}{\psi\big(\alpha,1+\alpha-\beta,\frac{\alpha\beta}{\mu x}\big)}\\
&=\frac{\alpha^2\beta}{a\mu x^2}\frac{\psi\big(\alpha+1,2+\alpha-\beta,\frac{\alpha\beta}{a\mu x}\big)}{\psi\big(\alpha,1+\alpha-\beta,\frac{\alpha\beta}{a\mu x}\big)}-\frac{\alpha^2\beta}{\mu x^2}\frac{\psi\big(\alpha+1,2+\alpha-\beta,\frac{\alpha\beta}{\mu x}\big)}{\psi\big(\alpha,1+\alpha-\beta,\frac{\alpha\beta}{\mu x}\big)}\\
&=\frac{\alpha}{x}\bigg(\int_0^\infty\frac{\frac{\alpha\beta}{a\mu x}\omega_{\alpha,\beta}(t)}{\frac{\alpha\beta}{a\mu x}+t}\,dt-\int_0^\infty\frac{\frac{\alpha\beta}{\mu x}\omega_{\alpha,\beta}(t)}{\frac{\alpha\beta}{\mu x}+t}\,dt\bigg)\\
&=\int_0^\infty\frac{\alpha^2\beta(1-a)\omega_{\alpha,\beta}(t)}{a\mu t\big(x+\frac{\alpha\beta}{t\mu}\big)\big(x+\frac{\alpha\beta}{at\mu}\big)}\,dt,\end{aligned}$$

which is completely monotonic as a function of x on $(0,\infty)$, $a\in(0,1)$, for all $\alpha,\beta,\mu>0$ and $\alpha<\beta$. Thus the K-distribution is self-decomposable if $\alpha<\beta$. The case $\beta<\alpha$ is handled by repeating the process for the other version of the Laplace transform, (5-4).

Lemma 3 is used to prove that the K-distribution belongs to the class of generalized gamma convolutions. The moment generating function

$$M_X(s) = L[\omega_{\alpha,\beta,\mu}(-s)] = \Big(-\frac{\alpha\beta}{\mu s}\Big)^{\alpha}\psi\Big(\alpha,\,1+\alpha-\beta,\,-\frac{\alpha\beta}{\mu s}\Big)$$

of the K-distribution is analytic and zero-free in $\mathbb{C}\setminus[0,\infty)$; we show that it satisfies $\operatorname{Im}\big(\psi'(s)/\psi(s)\big)\geq 0$ for $\operatorname{Im} s > 0$. In view of (5-2) and (5-3) for $\psi(s) = M_X(s)$ and $\alpha < \beta$ we obtain

$$\begin{aligned}\frac{\psi'(s)}{\psi(s)} &= -\frac{\alpha}{s} - \frac{\alpha^2\beta}{\mu s^2}\,\frac{\psi\big(\alpha+1,\,2+\alpha-\beta,\,-\frac{\alpha\beta}{\mu s}\big)}{\psi\big(\alpha,\,1+\alpha-\beta,\,-\frac{\alpha\beta}{\mu s}\big)} = -\frac{\alpha}{s} + \frac{\alpha}{s}\int_0^\infty \frac{-\frac{\alpha\beta}{\mu s}\omega_{\alpha,\beta}(t)}{-\frac{\alpha\beta}{\mu s}+t}\,dt\\ &= -\alpha\int_0^\infty \frac{\omega_{\alpha,\beta}(t)}{\big(s-\frac{\alpha\beta}{\mu t}\big)}\,dt,\end{aligned}$$

so for $s = x+\mathrm{i}y$ such that $y > 0$ we have

$$\operatorname{Im}\frac{\psi'(s)}{\psi(s)} = \int_0^\infty \frac{\alpha y\,\omega_{\alpha,\beta}(t)}{\big(\big(x-\frac{\alpha\beta}{\mu t}\big)^2+y^2\big)}\,dt > 0.$$

This concludes the proof in the case $\alpha < \beta$. The case $\alpha > \beta$ is handled by applying the same process to the other version of the Laplace transform, (5-4).

Finally, we show that the probability density function of the K-distribution is hyperbolically completely monotone. By using Macdonald's integral representation

$$K_\mu(x)K_\mu(y) = \frac{1}{2}\int_0^\infty \exp\Big(-\frac{t}{2}-\frac{x^2+y^2}{2t}\Big)K_\mu\Big(\frac{xy}{t}\Big)\frac{dt}{t} \tag{5-5}$$

(see [Wa44, p. 439]), where $x, y > 0$ and $\mu\in\mathbb{R}$, we obtain

$$\begin{aligned}&\omega_{\alpha,\beta,\mu}(uv)\,\omega_{\alpha,\beta,\mu}(u/v)\\ &\qquad= \frac{2u^{\alpha+\beta-2}}{\Gamma^2(\alpha)\Gamma^2(\beta)}\Big(\frac{\alpha\beta}{\mu}\Big)^{\alpha+\beta}\int_0^\infty \exp\Big(-\frac{t}{2}-\frac{w}{\mu t}(2\alpha\beta u)\Big)K_{\alpha-\beta}\Big(\frac{4\alpha\beta u}{\mu t}\Big)\frac{dt}{t},\end{aligned}$$

which is completely monotonic on $(0,\infty)$ as a function of $w = v+1/v$ for all $\alpha, \beta, \mu, u > 0$. □

Proof of Theorem 6. We apply Lemma 4. Observe that the Laplace transform of the generalized inverse Gaussian distribution is given by

$$\phi(x) := L[\pi_{\mu,a,b}(x)] = \Big(\frac{a}{2x+a}\Big)^{\frac{\mu}{2}}\cdot\frac{K_\mu\sqrt{b(2x+a)}}{K_\mu(\sqrt{ab})}.$$

Macdonald's integral representation (5-5) then gives

$$\phi(uv)\,\phi(u/v) = \frac{1}{2}\,\frac{a^\mu\big(K_\mu(\sqrt{ab})\big)^{-2}}{(\alpha w+\beta)^{\mu/2}}\int_0^\infty e^{-\frac{t}{2}-\frac{b}{t}(uw+a)}K_\mu(b\sqrt{\alpha w+\beta})\frac{dt}{t},$$

where $\alpha = 2au > 0$ and $\beta = 4u^2+a^2 > 0$. On the other hand, by [Ba10, p. 589], the function $x\mapsto x^{\mu/2}K_\mu(\sqrt{x})$ is completely monotonic on $(0,\infty)$ for all $\mu\in\mathbb{R}$. This implies that the function $w \mapsto (b\sqrt{\alpha w+\beta})^{\mu} K_{\mu}(b\sqrt{\alpha w+\beta})$ is also completely monotonic on $(0,\infty)$ for all $\mu \in \mathbb{R}$ and $b,\alpha,\beta>0$. Since $w \mapsto (\alpha w+\beta)^{-\mu}$ is completely monotonic on $(0,\infty)$ for all $\mu \geq 0$ and $\alpha,\beta>0$, and $w \mapsto e^{-\frac{b}{t}(uw+a)}$ is completely monotonic on $(0,\infty)$ for all $a,b,u,t>0$, we conclude that indeed the function $w \mapsto \phi(uv)\phi(u/v)$ is completely monotonic on $(0,\infty)$ for all $\mu \geq 0$ and $a,b>0$. For the case when $\mu<0$ we just need to observe that $K_{\mu}(x)=K_{-\mu}(x)$ and in view of the above mentioned results $x \mapsto x^{-\mu/2}K_{\mu}(\sqrt{x})$ is completely monotonic on $(0,\infty)$ for all $\mu<0$, which implies that the function $w \mapsto (b\sqrt{\alpha w+\beta})^{-\mu} K_{\mu}(b\sqrt{\alpha w+\beta})$ is also completely monotonic on $(0,\infty)$ for all $\mu<0$ and $b,\alpha,\beta>0$. Thus, we conclude that the function $w \mapsto \phi(uv)\phi(u/v)$ is completely monotonic on $(0,\infty)$ also for all $\mu<0$ and $a,b>0$. Now, applying Lemma 4, the proof is complete. □

Proof of Theorem 7. We will use Lemma 3. From [IK79, Theorem 1.9] we know that $x \mapsto \rho_{\mu,a}(x)$ is the Laplace transform of a probability distribution, and so the moment generating function $s \mapsto \phi_{\mu,a}(s)$ is

$$\phi_{\mu,a}(s)=\rho_{\mu,a}(-s)=\left(\frac{a}{2}\right)^{\mu}\frac{(-s)^{\frac{\mu}{2}}}{\Gamma(\mu+1)}\frac{1}{I_{\mu}(a\sqrt{-s})}=\left(\frac{a}{2}\right)^{\mu}\frac{s^{\frac{\mu}{2}}}{\Gamma(\mu+1)}\frac{1}{J_{\mu}(a\sqrt{s})},$$

in view of the relation $I_{\mu}(\mathrm{i}x)=\mathrm{i}^{\mu}J_{\mu}(x)$. Therefore $s \mapsto \phi_{\mu,a}(s)$ is analytic and zero-free in $\mathbb{C}\setminus[0,\infty)$. Taking the logarithmic derivative of both sides of the preceding display, we obtain

$$\frac{\phi'_{\mu,a}(s)}{\phi_{\mu,a}(s)}=\frac{\mu}{2s}-\frac{a}{2\sqrt{s}}\frac{J'_{\mu}(a\sqrt{s})}{J_{\mu}(a\sqrt{s})}. \tag{5-6}$$

Taking the logarithmic derivative of both sides of the infinite product representation

$$J_{\mu}(x)=\frac{\left(\frac{1}{2}x\right)^{\mu}}{\Gamma(\mu+1)}\prod_{n\geq 1}\left(1-\frac{x^2}{j_{\mu,n}^2}\right),$$

where $j_{\mu,n}$ stands for the n-th positive zero of $x \mapsto J_{\mu}(x)$, we obtain the classical Mittag-Leffler expansion

$$\frac{J'_{\mu}(x)}{J_{\mu}(x)}=\frac{\mu}{x}-\sum_{n\geq 1}\frac{2x}{j_{\mu,n}^2-x^2},$$

which implies that

$$\operatorname{Im}\frac{\phi'_{\mu,a}(s)}{\phi_{\mu,a}(s)}=\operatorname{Im}\left(\sum_{n\geq 1}\frac{a^2}{j_{\mu,n}^2-a^2s}\right)=\sum_{n\geq 1}\frac{a^4y}{(j_{\mu,n}^2-a^2x)^2+(a^2y)^2}>0,$$

whenever $x=\operatorname{Re}s\in\mathbb{R}$ and $y=\operatorname{Im}s>0$. Consequently, the conditions in Lemma 3 are satisfied and the proof is complete. □

Proof of Theorem 8. Since $2^{\mu}\Gamma(\mu+1)x^{-\mu}I_{\mu}(x)\to 1$ as $x\to 0$, it follows that $\Omega_{\mu,\nu,\sigma,a,b}(x)\to 1$ as $x\to 0$. Using the recurrence relation $I'_{\mu}(x)=I_{\mu+1}(x)+(\mu/x)I_{\mu}(x)$ (see [Wa44, p. 79]) and the Mittag-Leffler expansion

$$\frac{I_{\mu+1}(x)}{I_{\mu}(x)}=\sum_{n\geq 1}\frac{2x}{x^2+j_{\mu,n}^2},$$

we obtain for $-d\ln\Omega_{\mu,\nu,\sigma,a,b}(x)/dx$ the value

$$\begin{aligned}
&-\frac{a}{2\sqrt{x}}\frac{I'_{\mu}(a\sqrt{x})}{I_{\mu}(a\sqrt{x})}-\frac{b}{2\sqrt{x}}\frac{I'_{\nu}(b\sqrt{x})}{I_{\nu}(b\sqrt{x})}+\frac{b}{2\sqrt{x}}\frac{I'_{\mu}(b\sqrt{x})}{I_{\mu}(b\sqrt{x})}+\frac{a}{2\sqrt{x}}\frac{I'_{\nu}(a\sqrt{x})}{I_{\nu}(a\sqrt{x})}+\sum_{n\geq 1}\left(\frac{b^2}{b^2x+j_{\sigma,n}^2}\right)\\
&=\sum_{n\geq 1}\left(-\frac{1}{x+j_{\mu,n}^2a^{-2}}-\frac{1}{x+j_{\nu,n}^2b^{-2}}+\frac{1}{x+j_{\mu,n}^2b^{-2}}+\frac{1}{x+j_{\nu,n}^2a^{-2}}+\frac{1}{x+j_{\sigma,n}^2b^{-2}}\right)\\
&=\int_0^{\infty}e^{-xt}\left(\sum_{n\geq 1}\left(-e^{-j_{\mu,n}^2a^{-2}t}-e^{-j_{\nu,n}^2b^{-2}t}+e^{-j_{\mu,n}^2b^{-2}t}+e^{-j_{\nu,n}^2a^{-2}t}+e^{-j_{\sigma,n}^2b^{-2}t}\right)\right)dt\\
&=\int_0^{\infty}e^{-xt}\left(\sum_{n\geq 1}\left(\left(-e^{-j_{\mu,n}^2a^{-2}t}+e^{-j_{\mu,n}^2b^{-2}t}\right)+\left(-e^{-j_{\nu,n}^2b^{-2}t}+e^{-j_{\sigma,n}^2b^{-2}t}\right)+e^{-j_{\nu,n}^2a^{-2}t}\right)\right)dt.
\end{aligned}$$

Since $a<b$ and $-1<\sigma<\nu$, we have $e^{-j_{\mu,n}^2a^{-2}t}<e^{-j_{\mu,n}^2b^{-2}t}$ and $e^{-j_{\nu,n}^2b^{-2}t}<e^{-j_{\sigma,n}^2b^{-2}t}$ for every $n\in\mathbb{N}$ and $\mu>-1$, where we used the fact that $\mu\mapsto j_{\mu,n}$ is increasing on $(-1,\infty)$ for every $n\in\mathbb{N}$. Consequently, the last expression is positive, and this implies that $-d\ln\Omega_{\mu,\nu,\sigma,a,b}(x)/dx$ is completely monotonic in x on $(0,\infty)$. In view of Lemma 1 we conclude that $\Omega_{\mu,\nu,\sigma,a,b}(x)$ is indeed the Laplace transform of an infinitely divisible distribution. □

Proof of Theorem 9. The result will be established by verifying the conditions in Lemma 1 and by using the main idea of the proof of [Is90, Theorem 1]. Clearly $\Omega_{\mu,\nu,a,b}(x)\to 1$ as $x\to 0$, and by using the same approach as in the proof of Theorem 8, we obtain

$$\begin{aligned}
&-\frac{d\ln\Omega_{\mu,\nu,a,b}(x)}{dx}\\
&\quad=\frac{b}{2\sqrt{x}}+\sum_{n\geq 1}\left(-\frac{1}{x+j_{\mu,n}^2a^{-2}}-\frac{1}{x+j_{\nu,n}^2b^{-2}}+\frac{1}{x+j_{\mu,n}^2b^{-2}}+\frac{1}{x+j_{\nu,n}^2a^{-2}}\right)\\
&\quad=\frac{b}{2\sqrt{x}}-\sum_{n\geq 1}\frac{1}{x+j_{\nu,n}^2b^{-2}}+\int_0^{\infty}e^{-xt}\sum_{n\geq 1}\left(-e^{-j_{\mu,n}^2a^{-2}t}+e^{-j_{\mu,n}^2b^{-2}t}+e^{-j_{\nu,n}^2a^{-2}t}\right)dt.
\end{aligned}$$

Note that $e^{-j_{\mu,n}^2a^{-2}t}<e^{-j_{\mu,n}^2b^{-2}t}$ for all $\mu>-1$, $b>a$, $t>0$ and $n\in\mathbb{N}$. Consequently,

$$x\mapsto\int_0^{\infty}e^{-xt}\sum_{n\geq 1}\left(-e^{-j_{\mu,n}^2a^{-2}t}+e^{-j_{\mu,n}^2b^{-2}t}+e^{-j_{\nu,n}^2a^{-2}t}\right)dt$$

is a completely monotonic function on $(0,\infty)$ for all $\mu>-1,\ \nu>-1$, and $b>a$. Now, define

$$\eta_{\mu,b}(x)=\frac{b}{2\sqrt{x}}-\sum_{n\geq 1}\frac{1}{x+j_{\nu,n}^2 b^{-2}}.$$

Then $\dfrac{b}{2\sqrt{x}}=\dfrac{1}{\pi}\displaystyle\int_0^\infty\frac{dt}{x+b^{-2}t^2}$ and thus

$$\eta_{\mu,b}(x)=\frac{1}{\pi}\int_0^\infty\frac{dt}{x+b^{-2}t^2}-\sum_{n\geq 1}\frac{1}{x+j_{\nu,n}^2 b^{-2}}.$$

On the other hand, for every s such that $j_{\nu,m}\leq s<j_{\nu,m+1}$, we have

$$\begin{aligned}\int_0^s\frac{dt}{x+b^{-2}t^2}&\geq\int_{j_{\nu,0}}^{j_{\nu,m}}\frac{dt}{x+b^{-2}t^2}=\sum_{n=1}^m\int_{j_{\nu,n-1}}^{j_{\nu,n}}\frac{dt}{x+b^{-2}t^2}\\&\geq\sum_{n=1}^m\int_{j_{\nu,n-1}}^{j_{\nu,n}}\frac{dt}{x+b^{-2}j_{\nu,n}^2}=\sum_{n=1}^m\frac{j_{\nu,n}-j_{\nu,n-1}}{x+b^{-2}j_{\nu,n}^2},\end{aligned}$$

where we used the fact that $t\mapsto 1/(x+b^{-2}t^2)$ is a decreasing function on $(0,\infty)$ for all $b>0$ and $x>0$. We thus arrive at

$$\eta_{\mu,b}(x)\geq\frac{1}{\pi}\sum_{n\geq 1}\frac{j_{\nu,n}-j_{\nu,n-1}}{x+b^{-2}j_{\nu,n}^2}-\sum_{n\geq 1}\frac{1}{x+j_{\nu,n}^2 b^{-2}}.$$

Since $j_{\nu,n}-j_{\nu,n-1}>\pi$ for $\nu>\frac{1}{2}$ (see [Se01, Theorem 1.6], for example), we conclude that $\eta_{\mu,b}(x)>0$ for all $x>0$ and $b>0$. Moreover, from the previous results we obtain that

$$\begin{aligned}\frac{(-1)^k}{k!}\eta_{\mu,b}^{(k)}(x)&=\frac{1}{\pi}\int_0^\infty\frac{dt}{(x+b^{-2}t^2)^{k+1}}-\sum_{n\geq 1}\frac{1}{(x+j_{\nu,n}^2 b^{-2})^{k+1}}\\&\geq\frac{1}{\pi}\sum_{n\geq 1}\frac{j_{\nu,n}-j_{\nu,n-1}}{(x+j_{\nu,n}^2 b^{-2})^{k+1}}-\sum_{n\geq 1}\frac{1}{(x+j_{\nu,n}^2 b^{-2})^{k+1}}\end{aligned}$$

for all $x>0,\ b>0$ and $k\in\mathbb{N}$, where we have used that $t\mapsto 1/(x+b^{-2}t^2)^{k+1}$ is a decreasing function on $(0,\infty)$ for all $x>0,\ b>0$ and $k\in\mathbb{N}$. Consequently, $(-1)^k\eta_{\mu,b}^{(k)}(x)\geq 0$ for all $k\in\mathbb{N}$ and $x>0,\ b>0$. Therefore $\eta_{\mu,b}$ is also a completely monotonic function on $(0,\infty)$ and this implies that $x\mapsto -d\ln\Omega_{\mu,\nu,a,b}(x)/dx$ is the sum of two completely monotonic functions on $(0,\infty)$. □

Proof of Theorem 10. Let $F(z)=e^{-a\sqrt{z}}z^{-\frac{\mu}{2}}I_\mu(a\sqrt{z})$. Using eq. 10.30.1 of [NIST10], we have

$$F(z)\sim\frac{e^{-a\sqrt{z}}a^\mu}{2^\mu\Gamma(\mu+1)}\quad\text{as } z\to 0,$$

leading to $F(z) = o(|z|^{-1})$ as $|z| \to 0$. Using eq. 10.30.4 of [NIST10], we obtain

$$F(z) \sim \frac{1}{\sqrt{2\pi a}} z^{-\frac{(\mu+1/2)}{2}} \quad \text{as } |z| \to \infty.$$

These relations hold uniformly in every sector $|\arg z| \le \pi - \epsilon$, $\epsilon > 0$. Hence conditions **a** and **b** of Lemma 5 hold, and by using (5-8) we arrive at

$$F(-t - \mathrm{i}\eta) = F(e^{-\mathrm{i}\pi}(t + \mathrm{i}\eta)) = e^{a\mathrm{i}\sqrt{t+\mathrm{i}\eta}}(t + \mathrm{i}\eta)^{-\frac{\mu}{2}} J_\mu(a\sqrt{t+\mathrm{i}\eta}),$$
$$F(-t + \mathrm{i}\eta) = F(e^{\mathrm{i}\pi}(t - \mathrm{i}\eta)) = e^{-a\mathrm{i}\sqrt{t-\mathrm{i}\eta}}(t - \mathrm{i}\eta)^{-\frac{\mu}{2}} J_\mu(a\sqrt{t-\mathrm{i}\eta}).$$

By using the asymptotic relation

$$J_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)}, \quad \nu \ne -1, -2, \dots, \quad \text{as } z \to 0 \tag{5-7}$$

(see [NIST10, eq. 10.7.3]) we conclude that the function $F(-t - \mathrm{i}\eta) - F(-t + \mathrm{i}\eta)$ is continuous in every rectangle $[t_1, t_2] \times [0, \eta]$, where $t_1,\ t_2,\ \eta > 0$. Thus, the limit and the integral in (1-3) can be interchanged and the fundamental theorem of calculus yields

$$\alpha'(t) = \frac{1}{\pi} t^{-\frac{\mu}{2}} J_\mu(a\sqrt{t}) \sin(a\sqrt{t}),$$

so that $\alpha(t)$ is continuous and $\lim_{t\to 0^+} \alpha(t)$ exists. Letting $\tilde{\alpha}(t) = \alpha(t) - \alpha(0)$, we get the normalized measure $\tilde{\alpha}(t)$ with $\tilde{\alpha}'(t) = \alpha'(t)$. □

Proof of Theorem 11. Let us consider the function

$$F(z) = z^{\frac{\nu-\mu}{2}} I_\mu(a\sqrt{z}) K_\nu(b\sqrt{z}).$$

Using eqs. 10.30.1 and 10.30.2 of [NIST10] we get, for $\nu > 0$ and $\mu > -1$,

$$F(z) \sim \left(\tfrac{1}{2}\right)^{\mu-\nu+1} \frac{\Gamma(\nu)}{\Gamma(\mu+1)} \frac{a^\mu}{b^\nu} \quad \text{as } z \to 0.$$

And using eqs. 10.30.1 and 10.30.3 of [NIST10] we get, for $\nu = 0$ and $\mu > -1$,

$$F(z) \sim -\frac{a^\mu}{2^\mu \Gamma(\mu+1)} \ln(b\sqrt{z}) \quad \text{as } z \to 0.$$

Similarly, by using the relations $K_\nu(z) = K_{-\nu}(z)$ (see eqs. 10.27.3, 10.30.2 and 10.30.1 of [NIST10]) we have, for $-1 < \nu < 0$ and $\mu > -1$,

$$F(z) \sim \frac{a^\mu b^\nu}{2^{\mu+\nu+1}} \frac{\Gamma(-\nu)}{\Gamma(\mu+1)} z^{\nu+1} \quad \text{as } z \to 0.$$

We thus obtain $F(z) = o(|z|^{-1})$ as $z \to 0$ for all $\nu > -1$ and $\mu > -1$. Further using eqs. 10.30.4 and 10.25.3 of [NIST10], we see that

$$F(z) \sim z^{\frac{\nu-\mu-1}{2}} \frac{e^{-(b-a)\sqrt{z}}}{2} \quad \text{as } z \to \infty,$$

uniformly in every sector $|\arg z| \le \pi - \varepsilon$, $\varepsilon > 0$. This leads to $F(z) = o(1)$ as $|z| \to \infty$, whenever $b \ge a$ and $\nu - \mu < 1$. Thus, the conditions in Lemma 5 have been verified because $F(z)$ is analytic in $|\arg z| < \pi$.

Now, in order to apply Lemma 7, we observe that

$$\begin{aligned}
F(-t-\mathrm{i}\eta) &= F(e^{-\mathrm{i}\pi}(t+\mathrm{i}\eta))\\
&= (e^{-\mathrm{i}\pi}(t+\mathrm{i}\eta))^{\frac{\nu-\mu}{2}} I_\mu(ae^{-\frac{\mathrm{i}\pi}{2}}\sqrt{t+\mathrm{i}\eta})K_\nu(be^{-\frac{\mathrm{i}\pi}{2}}\sqrt{t+\mathrm{i}\eta})\\
&= \tfrac{1}{2}\mathrm{i}\pi(t+\mathrm{i}\eta)^{\frac{\nu-\mu}{2}} J_\mu(a\sqrt{t+\mathrm{i}\eta})H_\nu^{(1)}(b\sqrt{t+\mathrm{i}\eta}),\\
F(-t+\mathrm{i}\eta) &= F(e^{\mathrm{i}\pi}(t-\mathrm{i}\eta))\\
&= (e^{\mathrm{i}\pi}(t-\mathrm{i}\eta))^{\frac{\nu-\mu}{2}} I_\mu(ae^{\frac{\mathrm{i}\pi}{2}}\sqrt{t-\mathrm{i}\eta})K_\nu(be^{\frac{\mathrm{i}\pi}{2}}\sqrt{t-\mathrm{i}\eta})\\
&= -\tfrac{1}{2}\mathrm{i}\pi(t-\mathrm{i}\eta)^{\frac{\nu-\mu}{2}} J_\mu(a\sqrt{t-\mathrm{i}\eta})H_\nu^{(2)}(b\sqrt{t-\mathrm{i}\eta}),
\end{aligned}$$

where we have used the following relations (see [IM79, p. 459; NIST10, 10.27]):

$$\text{(5-8)}\quad \begin{cases}
H_\mu^{(1)}(z) = J_\mu(z) + \mathrm{i}Y_\mu(z), & H_\mu^{(2)}(z) = J_\mu(z) - \mathrm{i}Y_\mu(z),\\
K_\mu(ze^{-\frac{1}{2}\mathrm{i}\pi}) = \frac{1}{2}\mathrm{i}\pi e^{\frac{1}{2}\mathrm{i}\pi\mu}H_\mu^{(1)}(z), & K_\mu(ze^{\frac{1}{2}\mathrm{i}\pi}) = -\frac{1}{2}\mathrm{i}\pi e^{-\frac{1}{2}\mathrm{i}\pi\mu}H_\mu^{(2)}(z),\\
I_\mu(ze^{\pm\frac{1}{2}\mathrm{i}\pi}) = e^{\pm\frac{1}{2}\mathrm{i}\mu\pi}J_\mu(z), & I_\mu(z) = e^{-\frac{1}{2}\mathrm{i}\mu\pi}J_\mu(ze^{\frac{1}{2}\mathrm{i}\pi}).
\end{cases}$$

We next justify interchanging the limit and the integral in (1-3), in three cases:

<u>(i) $\nu > 0$ and $\mu > -1$</u>: By using the asymptotic relation

$$\text{(5-9)}\qquad H_\nu^{(1)}(z) \sim -H_\nu^{(2)}(z) \sim -\frac{\mathrm{i}}{\pi}\Gamma(\nu)(z/2)^{-\nu}, \quad z \to 0, \quad \operatorname{Re}\nu > 0$$

(see [NIST10, eq. 10.7.7]) and (5-7), we observe that the function $F(-t-\mathrm{i}\eta) - F(-t+\mathrm{i}\eta)$ is continuous in every rectangle $[t_1, t_2] \times [0, \eta]$, where $t_1,\ t_2,\ \eta > 0$. Thus, in this case, we can interchange the limit and the integral in (1-3).

<u>(ii) $\nu = 0$ and $\mu > -1$</u>: By using the asymptotic relation

$$\text{(5-10)}\qquad H_0^{(1)}(z) \sim -H_0^{(2)}(z) \sim \frac{2\mathrm{i}}{\pi}\ln(z), \quad z \to 0$$

(see [NIST10, eq. 10.7.2]) and (5-7), we obtain

$$\begin{aligned}
F(-t-\mathrm{i}\eta) &\sim -\frac{a^\mu}{2^\mu\Gamma(\mu+1)}\ln(b\sqrt{t+\mathrm{i}\eta}), \quad t \to 0,\ \eta \to 0,\\
F(-t+\mathrm{i}\eta) &\sim -\frac{a^\mu}{2^\mu\Gamma(\mu+1)}\ln(b\sqrt{t-\mathrm{i}\eta}), \quad t \to 0,\ \eta \to 0.
\end{aligned}$$

By the definition of asymptotic convergence, this means there are constants c, α, $\beta > 0$ such that

$$\bigl|F(-t\mp\mathrm{i}\eta)\bigr| \le \frac{ca^\mu}{2^\mu\Gamma(\mu+1)}\bigl|\ln(b\sqrt{t\pm\mathrm{i}\eta})\bigr|, \quad \text{for } 0 < t < \alpha,\ 0 < \eta < \beta.$$

Because the right side of this (double) inequality is integrable over $[t_1, t_2]$, where $0 < t_1 < t_2 < \alpha$, we are allowed to interchange the limit and the integral in (1-3).

<u>(iii) $-1 < \nu < 0$ and $\mu > -1$</u>: Using the relations

$$H^{(1)}_{-\nu}(z) = e^{\nu\pi \mathrm{i}} H^{(1)}_{\nu}(z), \quad H^{(2)}_{-\nu}(z) = e^{-\nu\pi \mathrm{i}} H^{(2)}_{\nu}(z) \tag{5-11}$$

(see [NIST10, eq. 10.4.6]) and the asymptotic relations (5-7) and (5-9), we have

$$F(-t-\mathrm{i}\eta) \sim \frac{e^{-\nu\pi\mathrm{i}}\Gamma(-\nu)a^{\mu}b^{\nu}}{2^{\mu+\nu+1}\Gamma(\mu+1)}(t+\mathrm{i}\eta)^{\nu},$$
$$F(-t+\mathrm{i}\eta) \sim \frac{e^{\nu\pi\mathrm{i}}\Gamma(-\nu)a^{\mu}b^{\nu}}{2^{\mu+\nu+1}\Gamma(\mu+1)}(t-\mathrm{i}\eta)^{\nu},$$

as $t \to 0$ and $\eta \to 0$. Then the same reasoning as in the previous case justifies interchanging the limit and the integral in (1-3).

Altogether we get

$$\alpha'(t) = \frac{1}{2\pi\mathrm{i}} \lim_{\eta\to 0^+} \big(F(-t-\mathrm{i}\eta) - F(-t+\mathrm{i}\eta)\big) = \tfrac{1}{2} t^{\frac{\nu-\mu}{2}} J_{\mu}(a\sqrt{t}) J_{\nu}(b\sqrt{t}),$$

showing that $\alpha(t)$ is continuous and $\lim_{t\to 0} \alpha(t)$ exists. Then $\tilde{\alpha}(t) = \alpha(t) - \alpha(0)$ is a normalized measure with $\tilde{\alpha}'(t) = \alpha'(t)$. □

Proof of Theorem 12. A function $x \mapsto \phi(x)$ is the Laplace transform of a probability distribution on $(0, \infty)$ if and only if $\phi(0) = 1$ and $\phi(x)$ has the form

$$\phi(x) = \int_0^{\infty} e^{-xt}\, d\mu(t)$$

and is completely monotonic on $(0, \infty)$ (see [Bo92, p. 8]). In view of (3-4) this implies that $x \mapsto 2\mu I_{\mu}(\sqrt{x}) K_{\mu}(\sqrt{x})$ is the Laplace transform of a probability distribution with support $(0, \infty)$. Combining this with the second equality in (3-4) we see that this distribution's probability density function,

$$\varsigma_{\mu}(x) = \mu \int_0^{\infty} e^{-tx} J^2_{\mu}(\sqrt{t})\, dt,$$

is a completely monotonic function on $(0, \infty)$ for all $\mu > 0$, and by using the Goldie–Steutel law we obtain that indeed the function $x \mapsto 2\mu I_{\mu}(\sqrt{x}) K_{\mu}(\sqrt{x})$ is the Laplace transform of an infinitely divisible probability distribution with support $(0, \infty)$. Now using the formula $\int_0^{\infty} e^{-st} J^2_{\mu}(\sqrt{t})\, dt = \frac{1}{s} e^{-\frac{1}{2s}} I_{\mu}\big(\frac{1}{2s}\big)$, where $\mu > 0$ (see [OB12, p. 139]), we get

$$\varsigma_{\mu}(x) = \frac{\mu}{x} e^{-\frac{1}{2x}} I_{\mu}\Big(\frac{1}{2x}\Big),$$

which completes the derivation of the probability density function. □

Proof of Corollary 1. The assertions in parts **a**, **b** and **c** follow immediately from the integral representation (3-4). We prove part **d**. From [Se21, p. 8], we have

$$[xI_\mu(x)K_\mu(x)]^{-1}=\frac{I_{\mu-1}(x)}{I_\mu(x)}+\frac{K_{\mu-1}(x)}{K_\mu(x)}$$

and in view of the three-term recurrence relation

$$I_{\mu-1}(x)=I_{\mu+1}(x)+\frac{2\mu}{x}I_\mu(x),$$

we arrive at

$$\big(xI_\mu(x)K_\mu(x)\big)^{-1}=\frac{2\mu}{x}+\frac{I_{\mu+1}(x)}{I_\mu(x)}+\frac{K_{\mu-1}(x)}{K_\mu(x)}.$$

Now, replacing x by $\sqrt{x}$ and dividing both sides by $\sqrt{x}$, we obtain

$$\big(xI_\mu(\sqrt{x})K_\mu(\sqrt{x})\big)^{-1}=\frac{2\mu}{x}+\frac{1}{\sqrt{x}}\frac{I_{\mu+1}(\sqrt{x})}{I_\mu(\sqrt{x})}+\frac{1}{\sqrt{x}}\frac{K_{\mu-1}(\sqrt{x})}{K_\mu(\sqrt{x})}.$$

Using the integral representation

$$\frac{K_{\mu-1}(\sqrt{x})}{\sqrt{x}K_\mu(\sqrt{x})}=\frac{2}{\pi^2}\int_0^\infty\frac{t^{-1}}{x+t}\big(J_\mu^2(\sqrt{t})+Y_\mu^2(\sqrt{t})\big)^{-1}dt, \tag{5-12}$$

where $x>0$, $\mu\ge 0$ (see for example [Is77a, eq. (1.3)] or [IM82, eq. (2.4)]) and the series and integral representations for the ratios of modified Bessel functions of the first and second kinds, we obtain

$$\big(xI_\mu(\sqrt{x})K_\mu(\sqrt{x})\big)^{-1}=\frac{2\mu}{x}+\sum_{n\ge1}\frac{2}{x+j_{\mu,n}^2}+\frac{2}{\pi^2}\int_0^\infty\frac{t^{-1}}{x+t}\big(J_\mu^2(\sqrt{t})+Y_\mu^2(\sqrt{t})\big)^{-1}dt,$$

which is the sum of three completely monotonic functions on $(0,\infty)$ for all $\mu>0$.

For **e** we use the integral representation of $I_\mu(\sqrt{x})K_\mu(\sqrt{x})$ and the Landau bound

$$|J_\mu(t)|\le c_L|t|^{-\frac{1}{3}},\quad \mu>0,\quad t\in\mathbb{R},\quad c_L=0.7857468704\dots$$

(see [La00] or [BMPS16]). We obtain, as desired,

$$I_\mu(\sqrt{x})K_\mu(\sqrt{x})\le\frac{c_L^2}{2}\int_0^\infty\frac{t^{-\frac{2}{3}}}{x+t}dt=\frac{\pi c_L^2}{\sqrt{3}x^{\frac{2}{3}}}.$$

□

Proof of Theorem 13. Let

$$F(z)=z^{\frac{\nu-\mu}{2}}e^{-a\sqrt{z}}I_\mu(a\sqrt{z})K_\nu(b\sqrt{z}).$$

The steps on the first half-page of the proof of Theorem 11 apply unchanged until the use of eq. 10.30.4 of [NIST10], which is replaced by 10.30.5. We arrive at

$$F(z)\sim\frac{z^{\frac{\nu-\mu-1}{2}}e^{-b\sqrt{z}}}{2\sqrt{ab}}\quad\text{as } |z|\to\infty,$$

uniformly in every sector $|\arg z| \le \pi - \epsilon$, $\epsilon > 0$, and thus $F(z) = o(1)$ as $|z| \to \infty$. Hence conditions **a** and **b** in Lemma 5 have been verified, and this implies that the corresponding Stieltjes transform representation exists. The use of the inversion theorem to obtain the corresponding measure is also almost the same: from Lemma 7 and equations (5-8) we obtain

$$F(-t-\mathrm{i}\eta) = F(e^{-\mathrm{i}\pi}(t+\mathrm{i}\eta)) = \frac{\mathrm{i}\pi}{2}(t+\mathrm{i}\eta)^{\frac{\nu-\mu}{2}} e^{a\mathrm{i}\sqrt{t+\mathrm{i}\eta}} J_\mu(a\sqrt{t+\mathrm{i}\eta}) H_\nu^{(1)}(b\sqrt{t+\mathrm{i}\eta}),$$

$$F(-t+\mathrm{i}\eta) = F(e^{\mathrm{i}\pi}(t-\mathrm{i}\eta)) = -\frac{\mathrm{i}\pi}{2}(t-\mathrm{i}\eta)^{\frac{\nu-\mu}{2}} e^{-a\mathrm{i}\sqrt{t-\mathrm{i}\eta}} J_\mu(a\sqrt{t-\mathrm{i}\eta}) H_\nu^{(2)}(b\sqrt{t-\mathrm{i}\eta}).$$

The justification for interchanging the limit and the integral in (1-3) also goes as in the proof of Theorem 11, and we omit it. Finally, from the relations (5-8), we get

$$\alpha'(t) = \tfrac{1}{2} t^{\frac{\nu-\mu}{2}} J_\mu(a\sqrt{t})\big(J_\nu(b\sqrt{t})\cos(a\sqrt{t}) - Y_\nu(b\sqrt{t})\sin(a\sqrt{t})\big). \qquad \square$$

Proof of Theorem 14. From the recurrence relation $K_\mu'(x) = -K_{\mu-1}(x) - \frac{\mu}{x}K_\mu(x)$ (see [Wa44, p. 79]) and the equivalent form of the integral representation (5-12), namely,

$$\frac{K_{\mu-1}(\sqrt{x})}{\sqrt{x}K_\mu(\sqrt{x})} = \frac{4}{\pi^2}\int_0^\infty \frac{t^{-1}}{x+t^2}\big(J_\mu^2(t)+Y_\mu^2(t)\big)^{-1}dt,$$

where $x > 0$, $\mu \ge 0$, we obtain

$$\begin{aligned}
-\frac{d\ln\chi_{\mu,\nu,a,b}(x)}{dx} &= \frac{a}{2\sqrt{x}} + \frac{\mu-\nu}{2x} - \frac{a}{2\sqrt{x}}\frac{I_\mu'(a\sqrt{x})}{I_\mu(a\sqrt{x})} - \frac{b}{2\sqrt{x}}\frac{K_\nu'(b\sqrt{x})}{K_\nu(b\sqrt{x})} \\
&= \frac{a}{2\sqrt{x}} - \frac{a}{2\sqrt{x}}\frac{I_{\mu+1}(a\sqrt{x})}{I_\mu(a\sqrt{x})} + \frac{b}{2\sqrt{x}}\frac{K_{\nu-1}(b\sqrt{x})}{K_\nu(b\sqrt{x})} \\
&= \Theta(x) + \frac{2}{\pi^2}\int_0^\infty \frac{t^{-1}}{x+b^{-2}t^2}\big(J_\nu^2(t)+Y_\nu^2(t)\big)^{-1}dt,
\end{aligned}$$

where

$$\begin{aligned}
\Theta(x) &= \frac{1}{\pi}\int_0^\infty \frac{dt}{x+a^{-2}t^2} - \sum_{n\ge1}\frac{1}{x+a^{-2}j_{\mu,n}^2} \\
&\ge \frac{1}{\pi}\sum_{n\ge1}\frac{j_{\mu,n}-j_{\mu,n-1}}{x+a^{-2}j_{\mu,n}^2} - \sum_{n\ge1}\frac{1}{x+a^{-2}j_{\mu,n}^2} \ge 0.
\end{aligned}$$

Further,

$$\frac{(-1)^m}{m!}\frac{d^m\Theta(x)}{dx^m} \ge \frac{1}{\pi}\sum_{n\ge1}\frac{j_{\mu,n}-j_{\mu,n-1}}{(x+a^{-2}j_{\mu,n}^2)^{m+1}} - \sum_{n\ge1}\frac{1}{(x+a^{-2}j_{\mu,n}^2)^{m+1}} \ge 0,$$

since $j_{\mu,n} - j_{\mu,n-1} > \pi$ whenever $\mu > \frac{1}{2}$ and $n \in \mathbb{N}$. Thus, $x \mapsto -d\ln\chi_{\mu,\nu,a,b}(x)/dx$ is completely monotonic on $(0, \infty)$ for all a, b, $\nu > 0$ and $\mu > \frac{1}{2}$. In view of Lemma 1 the proof is complete. $\square$

Proof of Theorem 15. Let $F(z) = z^{\frac{\mu+\nu}{2}} K_\mu(a\sqrt{z})K_\nu(b\sqrt{z})$. Using the asymptotic eq. 10.30.2 from [NIST10] for $a, b, \mu, \nu > 0$, we have

$$z^{\frac{\mu+\nu}{2}} K_\mu(a\sqrt{z})K_\nu(b\sqrt{z}) \sim \tfrac{1}{4}\Gamma(\mu)\Gamma(\nu)\frac{2^{\mu+\nu}}{a^\mu b^\nu}$$

as $z \to 0$. Hence, $f(z) = o(|z|^{-1})$ as $|z| \to 0$. Using [NIST10, eq. 10.25.3], we have

$$z^{\frac{\mu+\nu}{2}} K_\mu(a\sqrt{z})K_\nu(b\sqrt{z}) \sim \frac{\pi}{2\sqrt{ab}}\sqrt{z}^{\mu+\nu-1}e^{-(a+b)\sqrt{z}}$$

as $|z| \to \infty$ and $|\arg z| < \frac{3}{2}\pi$. From this we see that $F(z) = o(1)$ as $|z| \to \infty$, uniformly in every sector $|\arg z| \le \pi - \varepsilon$, $\varepsilon > 0$. Therefore, the conditions of Lemma 5 hold true. At the same time, using (5-8) we obtain

$$F(-t-\mathrm{i}\eta) = F(e^{-\mathrm{i}\pi}(t+\mathrm{i}\eta)) = -\tfrac{1}{4}\pi^2(t+\mathrm{i}\eta)^{\frac{\mu+\nu}{2}} H_\mu^{(1)}(a\sqrt{t+\mathrm{i}\eta})H_\nu^{(1)}(b\sqrt{t+\mathrm{i}\eta}),$$
$$F(-t+\mathrm{i}\eta) = F(e^{\mathrm{i}\pi}(t-\mathrm{i}\eta)) = -\tfrac{1}{4}\pi^2(t-\mathrm{i}\eta)^{\frac{\mu+\nu}{2}} H_\mu^{(2)}(a\sqrt{t-\mathrm{i}\eta})H_\nu^{(2)}(b\sqrt{t-\mathrm{i}\eta}),$$

By using the asymptotic relation (5-9), we conclude that $F(-t-\mathrm{i}\eta) - F(-t+\mathrm{i}\eta)$ is continuous in every rectangle $[t_1, t_2] \times [0, \eta]$, $t_1, t_2, \eta > 0$. Consequently, we can interchange the limit and the integral in (1-3). In view of Lemma 7 we arrive at

$$\alpha'(t) = -14\pi t^{\frac{\mu+\nu}{2}}\big(J_\mu(a\sqrt{t})Y_\nu(b\sqrt{t}) + J_\nu(b\sqrt{t})Y_\mu(a\sqrt{t})\big).$$

Thus $\alpha(t)$ is continuous and $\lim_{t\to 0^+}\alpha(t)$ exists. Then $\tilde{\alpha}(t) = \alpha(t) - \alpha(0)$ is the normalized measure with $\tilde{\alpha}'(t) = \alpha'(t)$. □

Proof of Theorem 16. Infinite divisibility results from [IK79, Theorem 1.8]. Namely, we know that the function $x \mapsto (\sqrt{x})^\nu K_\nu(\sqrt{x})/(2^{\nu-1}\Gamma(\nu))$ is the Laplace transform of an infinitely divisible distribution with support $(0, \infty)$ whenever $\nu > 0$. Thus the Laplace transform $\vartheta_{\mu,\nu,a,b}(x)$ is the product of Laplace transforms of two infinitely divisible distributions, and hence the Laplace transform of an infinitely divisible distribution, by [SH03, Proposition 2.1]. More precisely, (5-12) gives

$$\begin{aligned}
-\frac{d\ln\vartheta_{\mu,\nu,a,b}(x)}{dx} &= -\frac{\mu+\nu}{2x} - \frac{a}{2\sqrt{x}}\frac{K_\mu'(a\sqrt{x})}{K_\mu(a\sqrt{x})} - \frac{b}{2\sqrt{x}}\frac{K_\nu'(b\sqrt{x})}{K_\nu(b\sqrt{x})}\\
&= \frac{a}{2\sqrt{x}}\frac{K_{\mu-1}(a\sqrt{x})}{K_\mu(a\sqrt{x})} + \frac{b}{2\sqrt{x}}\frac{K_{\nu-1}(b\sqrt{x})}{K_\nu(b\sqrt{x})}\\
&= \frac{1}{\pi^2}\bigg(\int_0^\infty \frac{t^{-1}}{x+ta^{-2}}\big(J_\mu^2(\sqrt{t}) + Y_\mu^2(\sqrt{t})\big)^{-1}dt\\
&\qquad + \int_0^\infty \frac{t^{-1}}{x+tb^{-2}}\big(J_\nu^2(\sqrt{t}) + Y_\nu^2(\sqrt{t})\big)^{-1}dt\bigg),
\end{aligned}$$

which as a function of x is completely monotonic on $(0, \infty)$ for all strictly positive real numbers a, b, μ and ν. Applying Lemma 1, this shows that $x \mapsto \vartheta_{\mu,\nu,a,b}(x)$ is the Laplace transform of an infinitely divisible distribution.

Set $\xi(x) = \vartheta_{\mu,\nu,a,b}(x)/\vartheta_{\mu,\nu,a,b}(\alpha x)$. Then

$$\begin{aligned}
&-\frac{d\ln\xi(x)}{dx}\\
&= -\frac{a}{2\sqrt{x}}\frac{K'_\mu(a\sqrt{x})}{K_\mu(a\sqrt{x})} - \frac{b}{2\sqrt{x}}\frac{K'_\nu(b\sqrt{x})}{K_\nu(b\sqrt{x})} + \frac{a\sqrt{\alpha}}{2\sqrt{x}}\frac{K'_\mu(a\sqrt{\alpha x})}{K_\mu(a\sqrt{\alpha x})} + \frac{b\sqrt{\alpha}}{2\sqrt{x}}\frac{K'_\nu(b\sqrt{\alpha x})}{K_\nu(b\sqrt{\alpha x})}\\
&= \frac{a}{2\sqrt{x}}\frac{K_{\mu-1}(a\sqrt{x})}{K_\mu(a\sqrt{x})} + \frac{b}{2\sqrt{x}}\frac{K_{\nu-1}(b\sqrt{x})}{K_\nu(b\sqrt{x})} - \frac{a\sqrt{\alpha}}{2\sqrt{x}}\frac{K_{\mu-1}(a\sqrt{\alpha x})}{K_\mu(a\sqrt{\alpha x})} - \frac{b\sqrt{\alpha}}{2\sqrt{x}}\frac{K_{\nu-1}(b\sqrt{\alpha x})}{K_\nu(b\sqrt{\alpha x})}\\
&= \frac{a^2}{\pi^2}\int_0^\infty \frac{(1-\alpha)}{(a^2\alpha x+t)(a^2x+t)}\big(J_\mu^2(\sqrt{t})+Y_\mu^2(\sqrt{t})\big)^{-1}dt\\
&\qquad\qquad + \frac{b^2}{\pi^2}\int_0^\infty \frac{(1-\alpha)}{(b^2\alpha x+t)(b^2x+t)}\big(J_\nu^2(\sqrt{t})+Y_\nu^2(\sqrt{t})\big)^{-1}dt
\end{aligned}$$

is completely monotonic with respect to x on $(0,\infty)$ for all $\alpha \in (0,1)$ and strictly positive real numbers a, b, μ and ν. Applying Lemmas 1 and 2 we conclude that ξ is the Laplace transform of an infinitely divisible distribution, and so $x \mapsto \vartheta_{\mu,\nu,a,b}(x)$ is the Laplace transform of a self-decomposable distribution.

Finally, let $w = v + 1/v$. For a, b, μ, ν arbitrary positive real numbers set

$$\alpha_{\mu,\nu,a,b} = \frac{a^\mu b^\nu}{2^{\mu+\nu-2}\Gamma(\mu)\Gamma(\nu)}.$$

Using Macdonald's integral representation (5-5) for modified Bessel functions of the second kind we can write

$$\begin{aligned}
&\vartheta_{\mu,\nu,a,b}(uv)\vartheta_{\mu,\nu,a,b}(u/v)\\
&\quad = \alpha^2_{\mu,\nu,a,b}\cdot u^{\mu+\nu}K_\mu(a\sqrt{uv})K_\mu(a\sqrt{u/v})K_\nu(b\sqrt{uv})K_\nu(b\sqrt{u/v})\\
&\quad = \tfrac{1}{4}\alpha^2_{\mu,\nu,a,b}\,u^{\mu+\nu}\\
&\qquad \times\int_0^\infty\!\!\int_0^\infty \exp\Big(-wu\Big(\frac{b^2t+a^2s}{2ts}\Big)\Big)\exp\Big(-\frac{t+s}{2}\Big)K_\mu\Big(\frac{au^2}{t}\Big)K_\nu\Big(\frac{bu^2}{s}\Big)\frac{dt}{t}\frac{ds}{s}.
\end{aligned}$$

The factor containing w makes the function $w \mapsto \vartheta_{\mu,\nu,a,b}(uv)\vartheta_{\mu,\nu,a,b}(u/v)$ completely monotonic with respect to w on $(0,\infty)$. Then Lemma 4 gives that $x \mapsto \vartheta_{\mu,\nu,a,b}(x)$ is the Laplace transform of a generalized gamma convolution. □

Proof of Theorem 17. Let $F(z) = e^{-(a+b)\sqrt{z}}z^{-\frac{\mu+\nu}{2}}I_\mu(a\sqrt{z})I_\nu(b\sqrt{z})$. As $z \to 0$,

$$z^{-\frac{\mu+\nu}{2}}I_\mu(a\sqrt{z})I_\nu(b\sqrt{z}) \sim \big(\tfrac{1}{2}\big)^{\mu+\nu}\frac{a^\mu b^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)}$$

and hence $F(z)=o(|z|^{-1})$ as $|z|\to 0$. Using [NIST10, eq. 10.25.3] we obtain

$$F(z)\sim \frac{1}{2\pi\sqrt{ab}}z^{-\frac{\mu+\nu+1}{2}}$$

as $|z|\to\infty$, uniformly in every sector $|\arg z|\le \pi-\varepsilon$, $\varepsilon>0$. Thus, conditions **a** and **b** of Lemma 5 are met. Now we use (5-8) to evaluate

$$F(-t-\mathrm{i}\eta)=F(e^{-\mathrm{i}\pi}(t+\mathrm{i}\eta))=e^{(a+b)\mathrm{i}\sqrt{t+\mathrm{i}\eta}}(t+\mathrm{i}\eta)^{-\frac{\mu+\nu}{2}}J_\mu(a\sqrt{t+\mathrm{i}\eta})J_\nu(b\sqrt{t+\mathrm{i}\eta}),$$
$$F(-t+\mathrm{i}\eta)=F(e^{\mathrm{i}\pi}(t-\mathrm{i}\eta))=e^{-(a+b)\mathrm{i}\sqrt{t-\mathrm{i}\eta}}(t-\mathrm{i}\eta)^{-\frac{\mu+\nu}{2}}J_\mu(a\sqrt{t-\mathrm{i}\eta})J_\nu(b\sqrt{t-\mathrm{i}\eta}).$$

By using the asymptotic relation (5-7), we can validate the interchange of the integral and limit in (1-3). Then Lemma 7 gives

$$\alpha'(t)=\frac{1}{\pi}t^{-\frac{\mu+\nu}{2}}J_\mu(a\sqrt{t})J_\nu(b\sqrt{t})\sin((a+b)\sqrt{t}),$$

and thus $\alpha(t)$ is continuous and $\lim_{t\to 0^+}\alpha(t)$ exists. Letting $\tilde{\alpha}(t)=\alpha(t)-\alpha(0)$ we get the normalized measure with $\tilde{\alpha}'(t)=\alpha'(t)$. □

Proof of Theorem 18. From [Is90, Theorem 2] we know that if $\nu>\frac{1}{2}$, the function $x\mapsto 2^\nu\Gamma(\nu+1)x^{-\nu/2}I_\nu(\sqrt{x})e^{-\sqrt{x}}$ is the Laplace transform of an infinitely divisible distribution that is not a generalized gamma convolution. Hence $\zeta_{\mu,\nu,a,b}(x)$ is the Laplace transform of an infinitely divisible distribution that is not a generalized gamma convolution, being a product of two such.

To prove infinite divisibility we start with the observation that $-\ln\zeta_{\mu,\nu,a,b}(x)\to 0$ as $x\to 0$ (see [NIST10, eq. 10.30.1]). Using the same idea as in the proof of Theorem 9 we obtain

$$\begin{aligned}
&-\frac{\ln\zeta_{\mu,\nu,a,b}(x)}{dx}(x)\\
&=\frac{a+b}{2\sqrt{x}}+\frac{\mu+\nu}{2x}-\frac{a}{2\sqrt{x}}\frac{I'_\mu(a\sqrt{x})}{I_\mu(a\sqrt{x})}-\frac{b}{2\sqrt{x}}\frac{I'_\nu(b\sqrt{x})}{I_\nu(b\sqrt{x})}\\
&=\frac{a}{2\sqrt{x}}-\sum_{n\ge 1}\frac{1}{x+j_{\mu,n}^2a^{-2}}+\frac{b}{2\sqrt{x}}-\sum_{n\ge 1}\frac{1}{x+j_{\nu,n}^2b^{-2}}\\
&=\left(\frac{1}{\pi}\int_0^\infty\frac{dt}{x+t^2a^{-2}}-\sum_{n\ge 1}\frac{1}{x+j_{\mu,n}^2a^{-2}}\right)+\left(\frac{1}{\pi}\int_0^\infty\frac{dt}{x+t^2b^{-2}}-\sum_{n\ge 1}\frac{1}{x+j_{\nu,n}^2b^{-2}}\right);
\end{aligned}$$

this is a sum of two completely monotonic functions on $(0,\infty)$ if $a,b>0$ and $\mu,\nu>\frac{1}{2}$.

To show that $\zeta_{\mu,\nu,a,b}(x)$ is not a Laplace transform of a generalized gamma convolution we use Pick's characterization (Lemma 3). If $\psi(s)=\zeta_{\mu,\nu,a,b}(-s)$ is the moment generating function, we have $\operatorname{Im}\big(\psi'(s)/\psi(s)\big) < 0$ whenever $\operatorname{Im} s > 0$, as follows: With $s = re^{\mathrm{i}\theta} = x + \mathrm{i}y$, we can write

$$\begin{aligned}\frac{\psi'(s)}{\psi(s)} &= -\frac{(a+b)\mathrm{i}}{2\sqrt{s}} - \frac{\mu+\nu}{2s} + \frac{a}{2\sqrt{s}}\frac{J'_\mu(a\sqrt{s})}{J_\mu(a\sqrt{s})} + \frac{b}{2\sqrt{s}}\frac{J'_\nu(b\sqrt{s})}{J_\nu(b\sqrt{s})} \\ &= -\frac{(a+b)\mathrm{i}}{2\sqrt{s}} + \sum_{n\ge1}\frac{a^2}{s-j^2_{\mu,n}} + \sum_{n\ge1}\frac{b^2}{s-j^2_{\nu,n}} \\ &= \frac{\big(-\sin\frac{\theta}{2} - \mathrm{i}\cos\frac{\theta}{2}\big)(a+b)}{2\sqrt{r}} + \sum_{n\ge1}\frac{a^2(x-j^2_{\mu,n}-\mathrm{i}y)}{(x-j^2_{\mu,n})^2+y^2} + \sum_{n\ge1}\frac{b^2(a-j^2_{\nu,n}-\mathrm{i}y)}{(x-j^2_{\nu,n})^2+y^2},\end{aligned}$$

which is negative when $\operatorname{Im} s > 0$, as desired. □

Proof of Theorem 19. Define

$$F(z) = \frac{e^{-(a+b)\sqrt{z}}}{z^{\frac{\mu+\nu}{2}}K_\mu(a\sqrt{z})K_\nu(b\sqrt{z})}.$$

First take $\mu > 0$ and $\nu > 0$ and use [NIST10, eq. 10.30.2] to get

$$F(z) \sim \frac{a^\mu b^\nu e^{-(a+b)\sqrt{z}}}{2^{\mu+\nu-2}\Gamma(\mu)\Gamma(\nu)} \quad \text{as } z \to 0.$$

Then take $\mu > 0$ and $\nu = 0$ and use [NIST10, eq. 10.30.3] to get

$$F(z) \sim -\frac{a^\mu e^{-(a+b)\sqrt{z}}}{2^{\mu-1}\Gamma(\mu)\ln(b\sqrt{z})} \quad \text{as } z \to 0.$$

When $\mu > 0$ and $\nu < 0$, we use [NIST10, eq. 10.30.2] and $K_\nu(z) = K_{-\nu}(z)$ to get

$$F(z) \sim \frac{e^{-(a+b)\sqrt{z}}a^\mu b^{-\nu}z^{-\nu}}{2^{\mu-\nu-2}\Gamma(\mu)\Gamma(-\nu)} \quad \text{as } z \to 0.$$

To sum up, $F(z) = o(|z|^{-1})$ as $|z| \to 0$ whenever $\mu > 0$ and $\nu \in \mathbb{R}$, and by interchanging ν and μ we can conclude that in fact $F(z) = o(|z|^{-1})$ as $|z| \to 0$ whenever $\mu \in \mathbb{R}$, $\nu \in \mathbb{R}$ and $\mu + \nu > 1$. Using [NIST10, eq. 10.25.3], we also have

$$F(z) \sim \frac{2\sqrt{ab}z^{\frac{1-(\mu+\nu)}{2}}}{\sqrt{\pi}} \quad \text{as } |z| \to \infty$$

for $\mu + \nu > 1$. Hence, $F(z) = o(1)$ as $|z| \to \infty$ in the region $|\arg z| < \frac{3}{2}\pi$, showing that condition **b** in Lemma 6 is met.

$K_\mu(z)$ has no zeros in $|\arg z| \le \frac{\pi}{2}$ and has finitely many zeros in $\mathbb{C} \setminus \Omega$, where $\Omega = \big\{z : |\arg z| < \frac{\pi}{2}\big\}$. Consequently, $K_\mu(a\sqrt{z})$ has no zeros in $|\arg z| \le \pi$ and has finitely many zeros in $\pi < |\arg z| < 2\pi$. Hence, we can find a $\theta \in \big(\frac{2}{3}, 1\big)$ such that the function $F(z)$ is analytic in the region $|\arg| < \pi/\theta$, showing that condition **a** of Lemma 6 is also met.

Using the residue theorem, we evaluate the contour integral in (1-2) to

$$\varrho(t) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{z e^{\frac{z}{2}}}{z^2+\pi^2} F(e^z t)\, dz = \frac{\mathrm{i}}{2}\big(F(te^{\mathrm{i}\pi}) - F(te^{-\mathrm{i}\pi})\big),$$

where C is a rectifiable closed curve going around $[-\mathrm{i}\pi, \mathrm{i}\pi]$ in the positive direction and lying in the strip $|\mathrm{Im}\, z| < \pi/\theta$. On the other hand, in view of (5-8), we have

$$F(te^{\mathrm{i}\pi}) = \frac{-4e^{-(a+b)\mathrm{i}\sqrt{t}}}{\pi^2 t^{\frac{\mu+\nu}{2}} H_\mu^{(2)}(a\sqrt{t}) H_\nu^{(2)}(b\sqrt{t})},$$

$$F(te^{-\mathrm{i}\pi}) = \frac{-4e^{(a+b)\mathrm{i}\sqrt{t}}}{\pi^2 t^{\frac{\mu+\nu}{2}} H_\mu^{(1)}(a\sqrt{t}) H_\nu^{(1)}(b\sqrt{t})}.$$

All this leads to

$$\varrho(t) = \frac{4}{t^{\frac{\mu+\nu}{2}}\pi^2} \cdot \frac{{}_1T_{\mu,\nu,a,b}(t)\cos((a+b)\sqrt{t}) - {}_2T_{\mu,\nu,a,b}(t)\sin((a+b)\sqrt{t})}{\big(J_\mu^2(a\sqrt{t}) + Y_\mu^2(a\sqrt{t})\big)\big(J_\nu^2(b\sqrt{t}) + Y_\nu^2(b\sqrt{t})\big)}. \quad \square$$

Proof of Theorem 20. From [Is90, Theorem 1] we know that if $\nu > \frac{1}{2}$, the function $x \mapsto 2^{\nu-1}\Gamma(\nu) x^{-\nu/2} e^{-\sqrt{x}} / K_\nu(\sqrt{x})$ is the Laplace transform of an infinitely divisible distribution that is a generalized gamma convolution. This implies that $\kappa_{\mu,\nu,a,b}(x)$ is the Laplace transform of an infinitely divisible distribution that is also a generalized gamma convolution, being a product of two such. (See [Bo15, Theorem 1] or [BB17, Proposition 7].)

For infinite divisibility we calculate

$$\begin{aligned}
-\frac{d \ln \kappa_{\mu,\nu,a,b}(x)}{dx}
&= \frac{a+b}{2\sqrt{x}} + \frac{\mu+\nu}{2x} + \frac{a}{2\sqrt{x}} \frac{K_\mu'(a\sqrt{x})}{K_\mu(a\sqrt{x})} + \frac{b}{2\sqrt{x}} \frac{K_\nu'(b\sqrt{x})}{K_\nu(b\sqrt{x})} \\
&= \frac{a}{2\sqrt{x}} - \frac{a}{2\sqrt{x}} \frac{K_{\mu-1}(a\sqrt{x})}{K_\mu(a\sqrt{x})} + \frac{b}{2\sqrt{x}} - \frac{b}{2\sqrt{x}} \frac{K_{\nu-1}(b\sqrt{x})}{K_\nu(b\sqrt{x})} \\
&= \frac{1}{\pi} \int_0^\infty \frac{1}{x + t^2 a^{-2}} \left(1 - \frac{2(\pi t)^{-1}}{J_\mu^2(t) + Y_\mu^2(t)}\right) dt \\
&\qquad + \frac{1}{\pi} \int_0^\infty \frac{1}{x + t^2 b^{-2}} \left(1 - \frac{2(\pi t)^{-1}}{J_\nu^2(t) + Y_\nu^2(t)}\right) dt.
\end{aligned}$$

which for $\mu > \frac{1}{2}$ implies, in view of the inequality $J_\mu^2(t) + Y_\mu^2(t) > 2/(\pi t)$ (see [Wa44, p. 447]), that $x \mapsto -d \ln \kappa_{\mu,\nu,a,b}(x)/dx$ is indeed completely monotonic on

$(0, \infty)$ for all $\mu, \nu > \frac{1}{2}$ and $a, b > 0$. Then Lemma 1 shows that $\kappa_{\mu,\nu,a,b}(x)$ is the Laplace transform of an infinitely divisible distribution. □

Proof of Theorem 21. Letting

$$F(z) = z^{-\frac{\mu+\nu}{2}} e^{-(a+b)\sqrt{z}} \frac{I_\mu(a\sqrt{z})}{K_\nu(b\sqrt{z})},$$

we go through the same manipulations as in the proof of Theorem 19 to obtain the asymptotics of F as $z \to 0$: when $\nu > 0$, use eqs. 10.30.1 and 10.30.2 of [NIST10]; when $\nu = 0$, use 10.30.1 and 10.30.3; when $\nu < 0$, use 10.30.1, 10.27.3 and 10.30.2. Finally, when $a, b > 0$ and $\mu + \nu > 0$ use 10.30.4 and 10.25.3 to obtain

$$F(z) \sim \sqrt{b/a}\frac{z^{-\frac{\mu+\nu}{2}}}{\pi} \quad \text{as } |z| \to \infty.$$

Thus, condition **b** of Lemma 5 holds true. $K_\nu(z)$ has no zeros in $|\arg z| \le \frac{\pi}{2}$, so neither does $K_\nu(b\sqrt{z})$ have zeros in $|\arg z| \le \pi$. Thus, condition **a** of Lemma 5 is also met. From (5-8) we then obtain

$$F(-t - \mathrm{i}\eta) = F(e^{-\mathrm{i}\pi}(t + \mathrm{i}\eta)) = -\frac{2\mathrm{i}(t + \mathrm{i}\eta)^{-\frac{\mu+\nu}{2}} e^{(a+b)\mathrm{i}\sqrt{t+\mathrm{i}\eta}}}{\pi} \frac{J_\mu(a\sqrt{t + \mathrm{i}\eta})}{H_\nu^{(1)}(b\sqrt{t + \mathrm{i}\eta})},$$

$$F(-t + \mathrm{i}\eta) = F(e^{\mathrm{i}\pi}(t - \mathrm{i}\eta)) = \frac{2\mathrm{i}(t - \mathrm{i}\eta)^{-\frac{\mu+\nu}{2}} e^{-(a+b)\mathrm{i}\sqrt{t-\mathrm{i}\eta}}}{\pi} \frac{J_\mu(a\sqrt{t - \mathrm{i}\eta})}{H_\nu^{(2)}(b\sqrt{t - \mathrm{i}\eta})}.$$

To validate the interchange of limit and integral in (1-3) we consider three cases as in the proof of Theorem 11:

<u>(i) $\nu > 0$ and $\mu > -1$</u>: From (5-7) and (5-9) we see that $F(-t - \mathrm{i}\eta) - F(-t + \mathrm{i}\eta)$ is continuous in every rectangle $[t_1, t_2] \times [0, \eta]$.

<u>(ii) $\nu = 0$ and $\mu > -1$</u>: From (5-7) and (5-10) we get, as $t \to 0$ and $\eta \to 0$,

$$F(-t - \mathrm{i}\eta) \sim -\frac{a^\mu}{2^\mu \Gamma(\mu + 1)} \frac{1}{\ln(b\sqrt{t + \mathrm{i}\eta})},$$

$$F(-t + \mathrm{i}\eta) \sim -\frac{a^\mu}{2^\mu \Gamma(\mu + 1)} \frac{1}{\ln(b\sqrt{t - \mathrm{i}\eta})}.$$

Thus, both functions are bounded in every rectangle $[t_1, t_2] \times [0, \eta]$.

<u>(iii) $\nu < 0$ and $\mu > -1$</u>: From (5-7), (5-9) and (5-11) we get, as $t \to 0$ and $\eta \to 0$,

$$F(-t - \mathrm{i}\eta) \sim \frac{e^{\nu\pi\mathrm{i}} a^\mu}{2^{\mu-\nu-1} b^\nu \Gamma(\mu + 1)\Gamma(-\nu)} (t + \mathrm{i}\eta)^{-\nu},$$

$$F(-t + \mathrm{i}\eta) \sim \frac{e^{-\nu\pi\mathrm{i}} a^\mu}{2^{\mu-\nu-1} b^\nu \Gamma(\mu + 1)\Gamma(-\nu)} (t - \mathrm{i}\eta)^{-\nu};$$

that is, both functions are bounded in every rectangle $[t_1, t_2] \times [0, \eta]$.

Finally, in view of Lemma 7, we obtain

$$\frac{d\alpha(t)}{dt} = -\frac{2t^{-\frac{\mu+\nu}{2}}}{\pi^2} J_\mu(a\sqrt{t})\, \frac{\cos((a+b)\sqrt{t})J_\nu(b\sqrt{t}) + \sin((a+b)\sqrt{t})Y_\nu(b\sqrt{t})}{J_\nu^2(b\sqrt{t}) + Y_\nu^2(b\sqrt{t})}. \qquad \square$$

Proof of Corollary 2. We can simply take $a \to 0$ in Theorem 21. However, we give another proof using the Stieltjes transform representation in Lemma 6. Let

$$F(z) = (\sqrt{z})^{-\nu} e^{-b\sqrt{z}} \frac{1}{K_\nu(b\sqrt{z})}.$$

Equation 10.30.2 in [NIST10] gives $F(z) \sim o(|z|^{-1})$ as $z \to 0$, while 10.25.3 gives $F(z) \sim (\sqrt{z})^{-\nu+\frac{1}{2}}\sqrt{2b/\pi}$ as $|z| \to \infty$. Hence condition **b** in Lemma 6 holds true.

Next, $K_\nu(z)$ has no zero in the region $|\arg z| \le \frac{\pi}{2}$ and it has finitely many zeros in the regions $\frac{\pi}{2} < \arg z < \pi$ and $-\pi < \arg z < -\frac{\pi}{2}$. Hence $F(z)$ is analytic in $|\arg z| \le \pi$. We can find $\epsilon \in \left(\frac{2}{3}, 1\right)$ such that $F(z)$ is analytic in the region $|\arg z| < \frac{\pi}{\epsilon}$, so condition **a** in Lemma 6 is also met. In view of Lemma 7 we obtain

$$\frac{d\alpha(t)}{dt} = \frac{1}{2\pi \mathrm{i}}\big(F(e^{-\mathrm{i}\pi}t) - F(e^{\mathrm{i}\pi}t)\big),$$

where

$$F(e^{-\mathrm{i}\pi}t) = e^{\mathrm{i}\frac{\pi}{2}\nu}(\sqrt{t})^{-\nu}e^{\mathrm{i}b\sqrt{t}}\frac{1}{K_\nu(e^{-\mathrm{i}\frac{\pi}{2}}b\sqrt{t})} = \frac{2(\sqrt{t})^{-\nu}}{\mathrm{i}\pi}e^{\mathrm{i}b\sqrt{t}}\frac{J_\nu(b\sqrt{t}) - \mathrm{i}Y_\nu(b\sqrt{t})}{J_\nu^2(b\sqrt{t}) + Y_\nu^2(b\sqrt{t})},$$

$$F(e^{\mathrm{i}\pi}t) = e^{-\mathrm{i}\frac{\pi}{2}\nu}(\sqrt{t})^{-\nu}e^{-\mathrm{i}b\sqrt{t}}\frac{1}{K_\nu(e^{\mathrm{i}\frac{\pi}{2}}b\sqrt{t})} = -\frac{2(\sqrt{t})^{-\nu}}{\mathrm{i}\pi}e^{-\mathrm{i}b\sqrt{t}}\frac{J_\nu(b\sqrt{t}) + \mathrm{i}Y_\nu(b\sqrt{t})}{J_\nu^2(b\sqrt{t} + Y_\nu^2(b\sqrt{t}))}.$$

This leads to

$$\frac{d\alpha(t)}{dt} = -\frac{2}{\pi^2}(\sqrt{t})^{-\nu}\frac{J_\nu(b\sqrt{t})\cos(b\sqrt{t}) + Y_\nu(b\sqrt{t})\sin(b\sqrt{t})}{J_\nu^2(b\sqrt{t}) + Y_\nu^2(b\sqrt{t})} \qquad \square$$

Proof of Theorem 22. Applying Theorems 1 and 2 of [Is90] in the same way as in the proof of Theorem 18 shows that $\varepsilon_{\mu,\nu,a,b}(x)$ is the Laplace transform of an infinitely divisible distribution. More precisely, we use that

$$\begin{aligned}
-\frac{d\ln\varepsilon_{\mu,\nu,a,b}(x)}{dx}(x) &= \frac{a+b}{2\sqrt{x}} + \frac{\mu+\nu}{2x} - \frac{a}{2\sqrt{x}}\frac{I'(a\sqrt{x})}{I_\mu(a\sqrt{x})} + \frac{b}{2\sqrt{x}}\frac{K_\nu'(b\sqrt{x})}{K_\nu(b\sqrt{x})} \\
&= \frac{a+b}{2\sqrt{x}} - \frac{a}{2\sqrt{x}}\frac{I_{\mu+1}(a\sqrt{x})}{I_\mu(a\sqrt{x})} - \frac{b}{2\sqrt{x}}\frac{K_{\nu-1}(b\sqrt{x})}{K_\nu(b\sqrt{x})} \\
&= \frac{1}{\pi}\int_0^\infty \frac{1}{x+b^{-2}t^2}\left(1 - \frac{2(\pi t)^{-1}}{J_\nu^2(t) + Y_\nu^2(t)}\right)dt \\
&\qquad + \frac{1}{\pi}\int_0^\infty \frac{dt}{x+a^{-2}t^2} - \sum_{n\ge1}\frac{1}{x+a^{-2}j_{\mu,n}^2}
\end{aligned}$$

is completely monotonic on $(0, \infty)$ for all $\mu, \nu > \frac{1}{2}$ and $a, b > 0$, being a sum of two completely monotonic functions: the expression on the penultimate line is completely monotonic because of the inequality $J_\nu^2(t) + Y_\nu^2(t) > 2/(\pi t)$ where $\nu > \frac{1}{2}$ (see [Wa44, p. 447]), while the expression on the last line satisfies

$$\begin{aligned}\Theta(x) &= \frac{1}{\pi}\int_0^\infty \frac{dt}{x + a^{-2}t^2} - \sum_{n\geq 1} \frac{1}{x + a^{-2} j_{\mu,n}^2} \\ &\geq \frac{1}{\pi}\sum_{n\geq 1} \frac{j_{\mu,n} - j_{\mu,n-1}}{x + a^{-2} j_{\mu,n}^2} - \sum_{n\geq 1} \frac{1}{x + a^{-2} j_{\mu,n}^2} \geq 0\end{aligned}$$

and

$$\frac{(-1)^m}{m!}\frac{d^m \Theta(x)}{dx^m} \geq \frac{1}{\pi}\sum_{n\geq 1} \frac{j_{\mu,n} - j_{\mu,n-1}}{(x + a^{-2} j_{\mu,n}^2)^{m+1}} - \sum_{n\geq 1} \frac{1}{(x + a^{-2} j_{\mu,n}^2)^{m+1}} \geq 0,$$

since $j_{\mu,n} - j_{\mu,n-1} > \pi$ whenever $\mu > \frac{1}{2}$ and $n \in \mathbb{N}$. In view of Lemma 1, this proves the first part of Theorem 22.

For the second part we observe, using a similar approach as before, that the corresponding expression

$$\begin{aligned}-\frac{d}{dx}\ln\frac{e^{-(a+b)\sqrt{x}}}{\varepsilon_{\mu,\nu,a,b}(x)} &= -\frac{\nu+\mu}{2x} - \frac{b}{2\sqrt{x}}\frac{K_\nu'(b\sqrt{x})}{K_\nu(b\sqrt{x})} + \frac{a}{2\sqrt{x}}\frac{I_\mu'(a\sqrt{x})}{I_\mu(a\sqrt{x})} \\ &= \frac{b}{2\sqrt{x}}\frac{K_{\nu-1}(b\sqrt{x})}{K_\nu(b\sqrt{x})} + \frac{a}{2\sqrt{x}}\frac{I_{\mu+1}(a\sqrt{x})}{I_\mu(a\sqrt{x})} \\ &= \frac{2}{\pi^2}\int_0^\infty \frac{1}{x + b^{-2}t^2}[J_\nu^2(t) + Y_\nu^2(t)]^{-1}dt + \sum_{n\geq 1}\frac{1}{x + a^{-2} j_{\mu,n}^2}\end{aligned}$$

is completely monotonic on $(0, \infty)$, being a sum of two completely monotonic functions whenever $a, b, \nu > 0$ and $\mu > -1$. $\square$

Proof of Theorem 23. Let X and Y be independent gamma variables with parameters (α, β) and (α_0, β_0). The probability density function of the quotient $Z = X/Y$ is

$$f(z) = \frac{\Gamma(\alpha + \alpha_0)}{\Gamma(\alpha)\Gamma(\alpha_0)}\left(\frac{\beta_0}{\beta}\right)^\alpha x^{\alpha-1}\left(1 + \frac{\beta_0}{\beta}x\right)^{-(\alpha+\alpha_0)},$$

where $x > 0$ (see for example [IK79, p. 889]). Let $a = \alpha$ and $c = 1 - \alpha_0$. The Laplace transform $L(s)$ of the quotient of two gamma random variables is

$$L(s) = \frac{\Gamma(a - c + 1)}{\Gamma(1 - c)}\psi(a, c, s)$$

(see [IK79, p.889]) and thus the moment generating function $\phi(s) = L(-s)$ is

$$\phi(s) = \frac{\Gamma(a - c + 1)}{\Gamma(1 - c)}\psi(a, c, -s). \tag{5-13}$$

Since $a > 0$, by using [NIST10, eq. 13.9.13], we observe that $\psi(a, c, -s)$ has no zeros in $\mathbb{C} \setminus [0, \infty)$. Thus the first condition of the Pick function characterization theorem (Lemma 3) is verified, and by taking the logarithmic derivative of both sides of (5-13), we obtain

$$\frac{\phi'(s)}{\phi(s)} = -\frac{\psi'(a, c, -s)}{\psi(a, c, -s)} = \frac{a\psi(a+1, c+1, -s)}{\psi(a, c, -s)}.$$

For $s = x + \mathrm{i}y$, by using the integral representation (4-1), which is valid for $|\arg z| < \pi$, $a > 0$ and $c < 1$, we arrive at

$$\begin{aligned}\frac{\phi'(s)}{\phi(s)} &= \int_0^\infty \frac{at^{-c}e^{-t}|\psi(a, c, te^{\mathrm{i}\pi})|^{-2}\,dt}{(-s+t)\Gamma(a+1)\Gamma(a-c+1)} \\ &= \int_0^\infty \frac{at^{-c}e^{-t}|\psi(a, c, te^{\mathrm{i}\pi})|^{-2}\,dt}{(-x-\mathrm{i}y+t)\Gamma(a+1)\Gamma(a-c+1)} \\ &= \int_0^\infty \frac{(t-x+\mathrm{i}y)at^{-c}e^{-t}|\psi(a, c, te^{\mathrm{i}\pi})|^{-2}\,dt}{((t-x)^2+y^2)\Gamma(a+1)\Gamma(a-c+1)}.\end{aligned}$$

This shows that $\operatorname{Im} \phi'(s)/\phi(s) \geq 0$ whenever $\operatorname{Im} s > 0$. □

Proof of Theorem 24. By using equations (12)–(15) of [EMOT53a, p. 258] and the equality $\psi'(a, c, z) = -a\psi(a+1, c+1, z)$, we obtain

$$\begin{aligned}\psi(a, c-1, z) &= \frac{1-c}{a-c+1}\psi(a, c, z) + \frac{az}{a-c+1}\psi(a+1, c+1, z), \\ \psi(a+1, c, z) &= \frac{1}{a-c+1}\psi(a, c, z) - \frac{z}{a-c+1}\psi(a+1, c+1, z), \\ \psi(a, c+1, z) &= \psi(a, c, z) + a\psi(a+1, c+1, z), \\ \psi(a-1, c, z) &= z\psi(a, c, z) - (c-a)\psi(a, c, z) + az\phi(a+1, c+1, z).\end{aligned}$$

Dividing both sides of these equations by $\psi(a, c, z)$ and using the integral representation (4-1) yields the required results.

Thus, the representations in Theorem 24 follow naturally from (4-1), but for (4-3) we give a more detailed proof via the Stieltjes representation and inversion theorems. For this let

$$F(z) = \frac{\psi(a, c+1, z)}{\psi(a, c, z)} - 1.$$

Using eq. 13.7.3 of [NIST10], we have

$$F(z) \sim \frac{\sum_{s\geq 1} \frac{(a)_s((a-c)_s - (a-c+1)_s)(-z)^{-s}}{s!}}{\sum_{s\geq 0} \frac{(a)_s(a-c+1)_s(-z)^{-s}}{s!}} \quad \text{as } z \to \infty,$$

and $|\arg z| < \frac{3}{2}\pi$, showing that $f(z) \to 0$ as $|z| \to \infty$. Next, for $c < -1$, using eq. 13.2.22, we have

$$F(z) \sim \frac{\Gamma(-c)/\Gamma(a-c)}{\Gamma(1-c)/\Gamma(a-c+1)} - 1 \quad \text{as } z \to 0;$$

for $c = -1$, eqs. 13.2.21 and 13.2.22 give

$$F(z) \sim \frac{1/\Gamma(a+1)}{\Gamma(1-c)/\Gamma(a-c+1)} - 1 \quad \text{as } z \to 0;$$

for $-1 < c < 0$, eqs. 13.2.20 and 13.2.22 yield

$$F(z) \sim \frac{\Gamma(-c)/\Gamma(a-c)}{\Gamma(1-c)/\Gamma(a-c+1)} - 1 \quad \text{as } z \to 0;$$

for $c = 0$, eqs. 13.2.19 and 13.2.21 lead to

$$F(z) \sim \frac{-1/\Gamma(a)(\ln(z)+d)}{1/\Gamma(a+1)} - 1 \quad \text{as } z \to 0,$$

(where d is the constant in eq. 13.2.19), and for $0 < c < 1$, eqs. 13.2.18 and 13.2.20 give

$$F(z) \sim \frac{\Gamma(c)/\Gamma(a)z^{-c} + \Gamma(-c)/\Gamma(a-c)}{\Gamma(1-c)/\Gamma(a-c+1)} \quad \text{as } z \to 0.$$

Thus $F(z) = o(|z|^{-1})$ as $z \to 0$, for $a > 0$ and $c < 1$. The conditions in Lemma 6 have been verified, because $\psi(a, c, z)$ has no zeros in $|\arg z| < \pi/\alpha$, where $\alpha \in \left(\frac{2}{3}, 1\right)$. By using the residue calculus we obtain

$$\begin{aligned}
\frac{d}{dt}\alpha(t) &= \frac{1}{2\pi \mathrm{i}}\big(F(te^{-\mathrm{i}\pi}) - F(te^{\mathrm{i}\pi})\big) \\
&= \frac{1}{2\pi \mathrm{i}} \lim_{\eta \to 0^+} \big(F(-t - \mathrm{i}\eta) - F(-t + \mathrm{i}\eta)\big) \\
&= \frac{1}{2\pi \mathrm{i}} \lim_{\eta \to 0^+} \frac{\begin{array}{c}\psi(a, c, e^{\mathrm{i}\pi}(t - \mathrm{i}\eta))\psi(a, c+1, e^{-\mathrm{i}\pi}(t + \mathrm{i}\eta)) \\ -\psi(a, c, e^{-\mathrm{i}\pi}(t + \mathrm{i}\eta))\psi(a, c+1, e^{\mathrm{i}\pi}(t - \mathrm{i}\eta))\end{array}}{\big|\psi(a, c, e^{\mathrm{i}\pi}(t - \mathrm{i}\eta))\big|^2}.
\end{aligned}$$

Now, by using [EMOT53a, eq. (14), p. 263], for $-a < \min\{0, 1-c\}$ we have

$$\lim_{\eta \to 0^+} \psi(a, c, e^{\pm \mathrm{i}\pi}(t \mp \mathrm{i}\eta)) = k_1 y_1(-t) - e^{\mp \mathrm{i}\pi c} k_2 y_2(-t), \tag{5-14}$$

where

$$y_1(x) = \Phi(a, c, x) \quad \text{and} \quad y_2(x) = x^{1-c}\Phi(a-c+1, 2-c, x),$$

$\Phi(a, c, x)$ being the Kummer confluent hypergeometric function and

$$k_1 = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}, \quad k_2 = (-1)^{1-c}\frac{\Gamma(c-1)}{\Gamma(a)}.$$

On the other hand, from [EMOT53a, eq. (9), p. 253] we have

$$W[y_1(x), y_2(x)] = y_1(x)y_2'(x) - y_2(x)y_1'(x) = (1-c)x^{-c}e^x \tag{5-15}$$

and by using the recurrence relation $\psi(a, c+1, x) = -\psi'(a, c, x) + \psi(a, c, x)$ (see [EMOT53a, eq. (14), p. 258]) and equation (5-14), we obtain

$$\begin{aligned}\lim_{\eta\to 0^+} &\psi(a, c+1, e^{\mp i\pi}(t \pm i\eta)) \\ &= k_1 y_1'(-t) - e^{\pm i\pi c}k_2 y_2'(-t) + k_1 y_1(-t) - e^{\pm i\pi c}k_2 y_2(-t).\end{aligned} \tag{5-16}$$

From (5-16), (5-15) and (5-14) it follows that

$$\frac{d\alpha(t)}{dt} = -\frac{\sin(\pi c)k_1k_2(y_1(-t)y_2'(-t) - y_2(-t)y_1'(-t))}{\pi|\psi(a, c, e^{i\pi}t)|^2}$$

and by using the formula $\sin(\pi c) = \dfrac{\pi}{\Gamma(c-1)\Gamma(2-c)}$ (see [IK79, p. 890]) we arrive at

$$\frac{d\alpha(t)}{dt} = \frac{t^{-c}e^{-t}\,|\psi(a, c, e^{i\pi}t)|^{-2}}{\Gamma(a)\Gamma(a-c+1)}.$$

This completes the proof of (4-3). □

Proof of Theorem 25. From [MOS66, p. 90] we have this integral representation for the product of modified Bessel functions of the first kind, where $\mu > -\frac{1}{2}$:

$$\begin{aligned} I_\mu(a)I_\mu(b) = \\ \frac{\left(\frac{1}{2}ab\right)^\mu}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+\mu\right)} \int_0^\pi (a^2+b^2-2ab\cos t)^{-\frac{1}{2}\mu} I_\mu\left((a^2+b^2-2ab\cos t)^{1/2}\right)\sin^{2\mu} t\,dt,\end{aligned} \tag{5-17}$$

Combining this with

$$\chi_{\mu,\lambda}(uv)\chi_{\mu,\lambda}(u/v) = \tfrac{1}{4}e^{-\lambda}(u/\lambda)^{\frac{\mu}{2}-1}e^{-\frac{u}{2}w}I_{\frac{\mu}{2}-1}(\sqrt{\lambda uv})I_{\frac{\mu}{2}-1}(\sqrt{\lambda u/v}).$$

and using the notation $T = \sqrt{\lambda u(w - 2\cos t)}$, we obtain

$$\chi_{\mu,\lambda}(uv)\chi_{\mu,\lambda}(u/v) = c_{\mu,\lambda}(u)e^{-\frac{u}{2}w}\int_0^\pi \frac{I_{\frac{\mu}{2}-1}(T)}{T^{\frac{\mu}{2}-1}}\sin^{\mu-2} t\,dt, \tag{5-18}$$

with

$$c_{\mu,\lambda}(u) = \frac{e^{-\lambda}u^{(\mu-2)}}{2^{\frac{\mu}{2}+1}\sqrt{\pi}\,\Gamma\left(\frac{\mu-1}{2}\right)}.$$

Since $2dT/dw = \lambda u/T$ and by the recurrence relation $(z^{-\mu}I_\mu(z))' = z^{-\mu}I_{\mu+1}(z)$ (see for example [Wa44, p. 79]), after differentiating both sides of (5-18) with respect to $w = v + 1/v$, we arrive at

$$\frac{d}{dw}\left(\chi_{\mu,\lambda}(uv)\chi_{\mu,\lambda}(u/v)\right)$$

$$= c_{\mu,\lambda}(u)\Big(-\frac{1}{2}ue^{-\frac{uw}{2}}\int_0^\pi \frac{I_{\frac{\mu}{2}-1}(T)}{T^{\frac{\mu}{2}-1}}\sin^{\mu-2}t\,dt + e^{-\frac{uw}{2}}\int_0^\pi \frac{\lambda u}{2}\frac{I_{\frac{\mu}{2}}(T)}{T^{\frac{\mu}{2}}}\sin^{\mu-2}t\,dt\Big)$$

$$= c_{\mu,\lambda}(u)\Big(\frac{u}{2}e^{-\frac{uw}{2}}\int_0^\pi \frac{\sin^{\mu-2}t}{T^{\frac{\mu}{2}-1}}\Big(-I_{\frac{\mu}{2}-1}(T)+\lambda\frac{I_{\frac{\mu}{2}}(T)}{T}\Big)\,dt\Big).$$

The recurrence relation $(2\mu/z)I_\mu(z) = I_{\mu-1}(z) - I_{\mu+1}(z)$ (see [Wa44, p. 79]) yields

$$\frac{\lambda}{T}I_{\frac{\mu}{2}}(T) = \frac{\lambda}{\mu}I_{\frac{\mu}{2}-1}(T) - \frac{\lambda}{\mu}I_{\frac{\mu}{2}+1}(T),$$

which in turn gives

$$\frac{d}{dw}\big(\chi_{\mu,\lambda}(uv)\chi_{\mu,\lambda}(u/v)\big) = \frac{\frac{1}{2}uc_{\mu,\lambda}(u)}{e^{\frac{u}{2}w}}\int_0^\pi \frac{\sin^{\mu-2}t}{T^{\frac{\mu}{2}-1}}\Big(\Big(\frac{\lambda}{\mu}-1\Big)I_{\frac{\mu}{2}-1}(T) - \frac{\lambda}{\mu}I_{\frac{\mu}{2}+1}(T)\Big)\,dt < 0$$

whenever $0 < \lambda \le \mu$ and $u > 0$. By using a similar approach we obtain

$$\begin{aligned}\frac{d^2}{dw^2}\big(\chi_{\mu,\lambda}(uv)\chi_{\mu,\lambda}(u/v)\big) &= c_{\mu,\lambda}(u)\Big[-\frac{1}{4}u^2e^{-\frac{u}{2}w}\int_0^\pi \sin^{\mu-2}t\Big(-\frac{I_{\frac{\mu}{2}-1}(T)}{T^{\frac{\mu}{2}-1}}+\lambda\frac{I_{\frac{\mu}{2}}(T)}{T^{\frac{\mu}{2}}}\Big)\,dt \\ &\qquad + \frac{u}{2}e^{-\frac{u}{2}w}\int_0^\pi \sin^{\mu-2}t\Big(-\frac{\lambda u}{2}\frac{I_{\frac{\mu}{2}}(T)}{T^{\frac{\mu}{2}}}+\frac{\lambda^2 u}{2}\frac{I_{\frac{\mu}{2}+1}(T)}{T^{\frac{\mu}{2}+1}}\Big)\,dt\Big] \\ &= c_{\mu,\lambda}(u)\cdot\frac{u^2}{4}e^{-\frac{u}{2}w}\int_0^\pi \frac{\sin^{\mu-2}t}{T^{\frac{\mu}{2}-1}}\Big(I_{\frac{\mu}{2}-1}(T)-\frac{2\lambda}{T}I_{\frac{\mu}{2}}(T)+\frac{\lambda^2}{T^2}I_{\frac{\mu}{2}+1}(T)\Big)\,dt,\end{aligned}$$

which can be rewritten as

$$\frac{u^2c_{\mu,\lambda}(u)}{4e^{\frac{u}{2}w}}\int_0^\pi \frac{\sin^{\mu-2}t}{T^{\frac{\mu}{2}-1}}\Big(\Big(1-\frac{2\lambda}{\mu}\Big)I_{\frac{\mu}{2}-1}(T)+\frac{2\lambda}{\mu}I_{\frac{\mu}{2}+1}(T)+\frac{\lambda^2}{\mu^2}I_{\frac{\mu}{2}+1}(T)\Big)\,dt$$

and this is strictly positive whenever $0 < \lambda \le \mu/2$ and $u > 0$. □

Proof of Theorem 26. If $t \in \big(0, \frac{\pi}{2}\big]$ and $a, b > 0$, then $a^2+b^2-2ab\cos t > (a-b)^2 > 0$; if $t \in \big(\frac{\pi}{2}, \pi\big)$ and $a, b > 0$, we likewise have $a^2+b^2-2ab\cos t > 0$. Inserting $a = uv$ and $b = u/v$ in the integral representation (5-17), we obtain

$$I_\mu(uv)I_\mu(u/v) = \frac{\big(\frac{1}{2}u^2\big)^\mu}{\sqrt{\pi}\,\Gamma\big(\frac{1}{2}+\mu\big)}\int_0^\pi f_\mu(S)\sin^{2\mu}t\,dt,$$

where $f_\mu(S) = S^{-\mu}I_\mu(S)$ and $S = \sqrt{u^2((w^2-2)-2\cos t)} > 0$. The recurrence relation $(z^{-\mu}I_\mu(z))' = z^{-\mu}I_{\mu+1}(z)$ (see [Wa44, p. 79]) implies that each of

$$\frac{d}{dw} f_\mu(S) = u^2 w f_{\mu+1}(S), \quad \frac{d^2}{dw^2} f_\mu(S) = u^2 f_{\mu+1}(S) + (u^2 w)^2 f_{\mu+2}(S),$$
$$\frac{d^3}{dw^3} f_\mu(S) = 3wu^4 f_{\mu+2}(S) + (u^2 w)^3 f_{\mu+3}(S),$$
$$\frac{d^4}{dw^4} f_\mu(S) = 3u^4 f_{\mu+2}(S) + 6u^6 w^2 f_{\mu+3}(S) + (u^2 w)^4 f_{\mu+4}(S)$$

is positive for all $a, b > 0$, $w > 2$ and $\mu > -\frac{1}{2}$. In view of these relations we make the induction hypothesis that the $(2n+1)$-st derivative is of the form

$$\frac{d^{2n+1}}{dw^{2n+1}} f_\mu(S) = (u^2 w)^{2n+1} f_{\mu+2n+1}(S) + \alpha_{2n}(u) w^{2n-1} f_{\mu+2n}(S) + \cdots + \alpha_2(u) w f_{\mu+k}(S)$$

and this expression is positive, where $k \le 2n$ and $\alpha_{2n}(u)$, $\alpha_{2n-2}(u)$, $\ldots$, $\alpha_2(u)$ are nonnegative constants. We also make the induction hypothesis that the $2n$-th derivative is of the form

$$\frac{d^{2n}}{dw^{2n}} f_\mu(S) = (u^2 w)^{2n} f_{\mu+2n}(S) + \beta_{2n}(u) w^{2n-1} f_{\mu+2n-1}(S) + \cdots + \beta_2(u) f_{\mu+k}(S)$$

and this expression is positive, where $k \le 2n-1$ and $\beta_{2n}(u), \beta_{2n-2}(u), \ldots, \beta_2(u)$ are nonnegative constants. By using the recurrence relation $df_\mu(S)/dw = u^2 w f_{\mu+1}(S)$ repeatedly we see that the derivatives $d^{2n+3} f_\mu(S)/dw^{2n+3}$ and $d^{2n+2} f_\mu(S)/dw^{2n+2}$ have a similar form as $d^{2n+1} f_\mu(S)/dw^{2n+1}$ and $d^{2n} f_\mu(S)/dw^{2n}$, respectively, and that each of these expressions is positive. Consequently, for all $\mu > -\frac{1}{2}$, $u, v > 0$, $w > 2$ and $n \in \mathbb{N}$ we have

$$\frac{d^n}{dw^n}\big(I_\mu(uv) I_\mu(u/v)\big) = \frac{\left(\frac{1}{2}u^2\right)^\mu}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}+\mu\right)} \int_0^\pi \frac{d^n}{dw^n} f_\mu(S) \sin^{2\mu} t \, dt \ge 0. \qquad \square$$

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ÁRPÁD BARICZ
DEPARTMENT OF ECONOMICS
BABEŞ-BOLYAI UNIVERSITY
CLUJ-NAPOCA 400591
ROMANIA

and

INSTITUTE OF APPLIED MATHEMATICS
ÓBUDA UNIVERSITY
BUDAPEST 1034
HUNGARY

bariczocsi@yahoo.com

DHIVYA PRABHU K
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY INDORE
INDORE 453552
INDIA

phd2001241004@iiti.ac.in

SANJEEV SINGH
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY INDORE
INDORE 453552
INDIA

snjvsngh@iiti.ac.in

ANTONY VIJESH V
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY INDORE
INDORE 453552
INDIA

vijesh@iiti.ac.in

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