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SEVERI VARIETIES ON RULED SURFACES OVER ELLIPTIC CURVES

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We prove that the general members of Severi varieties on an Atiyah ruled surface over a general elliptic curve have nodes and ordinary triple points as singularities.

1. Introduction

Severi varieties of projective surfaces are roughly parameter (moduli) spaces of curves of fixed geometric genus, in a given linear system or homology class on the projective surface. The natural expectation is that a *general* curve in any irreducible component of a Severi variety will have only nodes as singularities. This expectation often turns out to be true under the natural assumption that the surface has general moduli, although there are notable known exceptions, for instance, in the case of abelian surfaces [DS17, Example (4.17)]. This problem has been investigated for many classes of surfaces beyond the classical case of $\mathbb{P}^2$, including K3 surfaces [Che02; Che19], abelian surfaces [KLM19; KL22], Enriques surfaces [CDGK23b], surfaces in $\mathbb{P}^3$ [CC99], and some ruled surfaces [CDGK23a].

In this paper, we consider Severi varieties on a ruled surface over a smooth elliptic curve. We will see that the unexpected phenomenon of general curves with worse than nodal singularities occurs again on this ruled surface.

Let E be a smooth elliptic curve, let $\mathscr{E}$ be a rank 2 vector bundle on E given by a nonzero vector in $\operatorname{Ext}(\mathcal{O}_E, \mathcal{O}_E)$ and let $R = \mathbb{P}\mathscr{E}$. Such surfaces arise naturally when we study the degeneration of abelian and K3 surfaces [Zah22]. We call such a ruled surface the *Atiyah ruled surface* over E.

The main purpose of this note is to prove the following:

Theorem 1.1. *Let E be a smooth elliptic curve, let $\mathscr{E}$ be a rank 2 vector bundle on E given by a nonzero vector in* $\operatorname{Ext}(\mathcal{O}_E, \mathcal{O}_E)$ *and let $R = \mathbb{P}\mathscr{E}$. For a line bundle L on R, let $V_{R,L,g} \subset |L|$ be the locus of integral curves $C \in |L|$ of geometric genus g. Then when E is general, L is ample and $g \geq 1$, for a general member $[C] \in V_{R,L,g}$,*

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- *if $L.D \geq 2$, C is nodal, and*
- *if $L.D = 1$, C has only nodes and/or ordinary triple points as singularities,*

where D is the unique section of R over E with self-intersection $D^2 = 0$.

Theorem 1.2 below shows that triple points do occur in some cases.

The Picard group $\mathrm{Pic}(R)$ of R is generated by D and $\pi^* \mathrm{Pic}(E)$, where $\pi : R \to E$ is the projection.

For an integral curve $C \subset R$ of geometric genus g with normalization $f : \widehat{C} \to R$, we have

$$\deg(\omega_{\widehat{C}} \otimes f^* \omega_R^\vee) = 2g - 2 - K_R.C = 2(g - 1 + C.D),$$

since $-K_R = 2D$. Based on [DS17, Corollary 2.11] (see also [HM98, Section B, pp. 108–111] and [AC81, Lemma 1.4, p. 345]), we know the following.

- If the degree of $\omega_{\widehat{C}} \otimes f^* \omega_R^\vee$ is at least $2g$, or equivalently,

$$C.D \geq 1, \tag{1-1}$$

  then a general deformation of f is immersive.
- If the degree of $\omega_{\widehat{C}} \otimes f^* \omega_R^\vee$ is at least $2g + 2$, or equivalently,

$$C.D \geq 2, \tag{1-2}$$

  then a general deformation of f has nodal image.

Consequently, our main theorem holds for every $L = mD + \pi^* M$ if $m > 0$ and $\deg M \geq 2$. Therefore, the only remaining case for Theorem 1.1 is $m > 0$ and $\deg M = 1$. Furthermore, we will show that the case $g \geq 2$ can be reduced to $g = 1$ by a degeneration argument. That is, it suffices to prove the theorem for $L = mD + R_p$ and $g = 1$, where $R_p = \pi^* p$ is the fiber of R over a point $p \in E$. Indeed, we have a more precise statement for this case:

Theorem 1.2. *Let E be a smooth elliptic curve, let $\mathscr{E}$ be a rank 2 vector bundle on E given by a nonzero vector in $\mathrm{Ext}(\mathcal{O}_E, \mathcal{O}_E)$ and let $R = \mathbb{P}\mathscr{E}$. When E is general, for $L = mD + R_p$ and every $[C] \in V_{R,L,1}$,*

- *if $4 \nmid m$, C is nodal, and*
- *if $4 \mid m$, C has only nodes and/or ordinary triple points as singularities,*

where D is the unique section of R over E with self-intersection $D^2 = 0$ and R_p is the fiber of R over $p \in E$.

In addition, if $4 \mid m$, then there exists at least one irreducible component V of $V_{R,L,1}$ such that the general curve $[C] \in V$ has at least one triple point.

Such elliptic curves were also studied by E. Sernesi in [Ser23].

Conventions. We work exclusively over $\mathbb{C}$.

2. The Atiyah ruled surface $\mathbb{P}\mathscr{E}$

We start with some basic facts about the Atiyah ruled surface $\mathbb{P}\mathscr{E}$.

Proposition 2.1. *Let E be a smooth elliptic curve, let $\mathscr{E}$ be a rank 2 vector bundle on E given by a nonzero vector in $\operatorname{Ext}(\mathcal{O}_E, \mathcal{O}_E)$, let $R = \mathbb{P}\mathscr{E}$ and let $D \subset R$ be the section of R/E with $D^2 = 0$.*

(1) *For every point $p \in E$, $|D + R_p|$ is a pencil such that every curve $C \neq D \cup R_p \in |D + R_p|$ is a smooth elliptic curve and any pair $C_1 \neq C_2 \in |D + R_p|$ of curves meet only at p (viewed as a point of D) with multiplicity 2, where R_p is the fiber of R over $p \in E$.*

(2) *For every point $p \in E$, $R \setminus (D \cup R_p) \cong (E \setminus \{p\}) \times \mathbb{A}^1$.*

(3) *For every pair of points $p \neq q \in E$, $R \setminus D$ is isomorphic to the gluing of $(E \setminus \{p\}) \times \mathbb{A}^1$ and $(E \setminus \{q\}) \times \mathbb{A}^1$ via an automorphism*

$$(E \setminus \{p, q\}) \times \mathbb{A}^1 \xrightarrow{\ \eta\ } (E \setminus \{p, q\}) \times \mathbb{A}^1$$

given by

$$\eta(z, s) = (z, s + h(z)),$$

where $h(z)$ is a meromorphic function on E with simple poles at p and q.

(4) *There is an exact sequence of group schemes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{G}_a & \longrightarrow & \operatorname{Aut}(R)_0 & \longrightarrow & \operatorname{Aut}(D)_0 & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \operatorname{Aut}(E)_0, & & \end{array} \tag{2-1}$$

where $\mathbf{G}_a$ is the additive group of $\mathbb{C}$ and $\operatorname{Aut}(R)_0$ and $\operatorname{Aut}(E)_0$ are the connected components of $\operatorname{Aut}(R)$ and $\operatorname{Aut}(E)$, respectively, containing the identity. Every $\phi \in \operatorname{Aut}(R)_0$ is given by

$$\begin{aligned} \phi(z, s) &= (z + \tau, s + b_1(z)) \quad \text{on } (E \setminus \{p, p - \tau\}) \times \mathbb{A}^1, \\ \phi(z, s) &= (z + \tau, s + b_2(z)) \quad \text{on } (E \setminus \{q, q - \tau\}) \times \mathbb{A}^1, \end{aligned} \tag{2-2}$$

where $\tau \in \operatorname{Pic}^0(E) = J(E)$, p and q are two distinct points on E satisfying $p - q \neq \pm\tau$, $b_1(z)$ is a meromorphic function on E with simple poles at p and $p - \tau$, $b_2(z)$ is a meromorphic function on E with simple poles at q and $q - \tau$, and $b_1(z)$ and $b_2(z)$ satisfy

$$b_1(z) + h(z) = b_2(z) + h(z + \tau) \tag{2-3}$$

on $E \setminus \{p, p - \tau, q, q - \tau\}$ with $h(z)$ given in (3).

Proof. By the exact sequence

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathscr{E} \longrightarrow \mathcal{O}_E \longrightarrow 0$$

we obtain

$$h^0(\mathscr{E}^\vee \otimes \mathcal{O}_E(p)) = h^0(\mathcal{O}_E(p)) + h^0(\mathcal{O}_E(p)) = 2$$

and hence $|D + R_p|$ is a pencil. Since

$$\mathcal{O}_R(D + R_p)\big|_D = \mathcal{O}_E(p)$$

every $C \in |D + R_p|$ passes through p. If C is reducible, C must contain a section of R/E and hence it must contain D. Consequently, the only reducible member of $|D + R_p|$ is $D \cup R_p$. Every other member of $|D + R_p|$ is a section of R/E. For $C_1 \neq C_2 \in |D + R_p|$, one of C_1 and C_2 must be integral. Let us assume that C_1 is a section of R/E. Then

$$\mathcal{O}_{C_1}(C_2) = \mathcal{O}_{C_1}(D + R_p) = \mathcal{O}_{C_1}(2p).$$

We know that both C_1 and C_2 pass through p and they have intersection number 2. So $C_1.C_2 = p + p'$. Then $p + p' \sim_{\text{rat}} 2p$ on C_1 and hence $p' = p$. That is, C_1 and C_2 meet at p with multiplicity 2 and they do not have any other intersections. This proves (1).

Let $\alpha_p : R \dashrightarrow \mathbb{P}^1$ be the rational map given by the pencil $|D + R_p|$. By (1), the map

$$R \setminus (D \cup R_p) \xrightarrow[\cong]{\pi \times \alpha_p} (E \setminus \{p\}) \times \mathbb{A}^1$$

is an isomorphism, where $\pi : R \to E$ is the projection. This proves (2).

We have

$$R \setminus D = (R \setminus (D \cup R_p)) \cup (R \setminus (D \cup R_q))$$

with $(R \setminus (D \cup R_p))$ and $(R \setminus (D \cup R_q))$ isomorphic to $(E \setminus \{p\}) \times \mathbb{A}^1$ and $(E \setminus \{q\}) \times \mathbb{A}^1$ via $\pi \times \alpha_p$ and $\pi \times \alpha_q$, respectively. So $R \setminus D$ is the gluing of $(E \setminus \{p\}) \times \mathbb{A}^1$ and $(E \setminus \{q\}) \times \mathbb{A}^1$ via an automorphism $\eta \in \operatorname{Aut}(U \times \mathbb{A}^1/U)$

$$U \times \mathbb{A}^1 \xrightarrow{\eta} U \times \mathbb{A}^1$$

for $U = E \setminus \{p, q\}$. Such an automorphism is given by

$$\eta(z, s) = (z, h(z)s + f(z)),$$

where $h(z)$ and $f(z)$ are meromorphic functions on E such that they are holomorphic on U and $h(z) \neq 0$ on U. So $h(z)$ has zeros and poles only at p and q and $f(z)$ has poles only at p and q.

A member of the pencil $|D + R_p|$ other than $D \cup R_p$ is given by

$$(\pi \times \alpha_p)^{-1}\{s = a\}$$

for $a \in \mathbb{C}$. Similarly, a member of the pencil $|D + R_q|$ other than $D \cup R_q$ is given by

$$(\pi \times \alpha_q)^{-1}\{s = b\}$$

for $b \in \mathbb{C}$. These two curves meet at two points lying in $R \backslash (D \cup R_p \cup R_q)$. Therefore,

$$\{s = a\} \cap \eta^{-1}\{s = b\}$$

has two intersections (counted with multiplicity) in $U \times \mathbb{A}^1$ for all $a, b \in \mathbb{C}$. That is, the function

$$ah(z) + f(z) - b$$

has exactly two zeros over U for all a, b. It follows that $h(z)$ is a nonzero constant and $f(z)$ has simple poles at p and q. We may choose $h(z) \equiv 1$. This proves (3).

Clearly, every automorphism of R preserves the section D. Let $\phi : R \to R$ be an automorphism of R in the kernel of $\mathrm{Aut}(R) \to \mathrm{Aut}(D)$ and let ϕ_1 and ϕ_2 be the restriction of ϕ to $(E \setminus \{p\}) \times \mathbb{A}^1$ and $(E \setminus \{q\}) \times \mathbb{A}^1$, respectively. Suppose that ϕ_1 and ϕ_2 are given by

$$\phi_1(z, s) = (z, a_1(z)s + b_1(z)),$$
$$\phi_2(z, s) = (z, a_2(z)s + b_2(z)),$$

where $a_1(z)$ and $b_1(z)$ are meromorphic functions on E with poles at p, $a_2(z)$ and $b_2(z)$ are meromorphic functions on E with poles at q, $a_1(z) \neq 0$ on $E \setminus \{p\}$ and $a_2(z) \neq 0$ on $E \setminus \{q\}$. Clearly, $a_1(z) \equiv a_1$ and $a_2(z) \equiv a_2$ must be constants. In addition, since $\phi_1 \circ \eta = \eta \circ \phi_2$, we have

$$a_1(s + h(z)) + b_1(z) = a_2 s + b_2(z) + h(z)$$

on $(E \setminus \{p, q\}) \times \mathbb{A}^1$. Obviously, $a_1 = a_2 = a$ and hence

$$b_1(z) - b_2(z) = (1 - a)h(z).$$

Since $h(z)$ has simple poles at p and q, $b_1(z)$ has a single pole at p and $b_2(z)$ has a single pole at q, $b_1(z)$ and $b_2(z)$ must have simple poles at p and q, respectively, and hence they must be constant. It follows that $a = 1$ and $b_1(z) \equiv b_2(z) \equiv b$. This proves that

$$\mathbf{G}_a = \ker(\mathrm{Aut}(R) \to \mathrm{Aut}(D)).$$

To complete the proof of (2-1), it remains to prove that the map

$$\mathrm{Aut}(R)_0 \twoheadrightarrow \mathrm{Aut}(D)_0$$

is surjective.

Every automorphism $\lambda \in \mathrm{Aut}(E)_0$ is given by a translation $\lambda(p) = p + \tau$ for some $\tau \in \mathrm{Pic}^0(E) = J(E)$.

For a given $\tau \in J(E)$, if there exists a pair of meromorphic functions $b_1(z)$ and $b_2(z)$ satisfying (2-3), then $\phi \in \mathrm{Aut}(R)_0$ given by (2-2) maps to $\lambda \in \mathrm{Aut}(E)_0$ with $\lambda(p) = p+\tau$. So it suffices to prove the existence of $b_1(z)$ and $b_2(z)$ satisfying (2-3).

If $\tau = 0$, we can simply take $b_1(z) \equiv b_2(z) \equiv b$ to be a constant.

Suppose that $\tau \neq 0$. We lift (2-3) from $E \cong \mathbb{C}/\Lambda$ to $\mathbb{C}$. Then $b_1(z)$, $b_2(z)$ and $h(z)$ are doubly periodic meromorphic functions on $\mathbb{C}$. We choose $b_1(z)$ such that

$$\mathrm{Res}_p\, b_1(z) = -\,\mathrm{Res}_p\, h(z).$$

Since

$$\mathrm{Res}_p\, b_1(z) + \mathrm{Res}_{p-\tau}\, b_1(z) = 0$$

we have

$$\mathrm{Res}_{p-\tau}\, b_1(z) = \mathrm{Res}_p\, h(z) = \mathrm{Res}_{p-\tau}\, h(z+\tau).$$

So $b_2(z) = b_1(z) + h(z) - h(z+\tau)$ is analytic at p and $p-\tau$. This proves the existence of $b_1(z)$ and $b_2(z)$ satisfying (2-3) and hence (4). □

Let $C \in |mD + R_p|$ be a (possibly singular) elliptic curve on R and let $\nu : \mathscr{C} \to R$ be the normalization of C. We let

$$S = \mathscr{C} \times_E R = \mathbb{P}(\pi \circ \nu)^*\mathscr{E}$$

via the maps $\pi \circ \nu : \mathscr{C} \to E$ and $\pi : R \to E$. Clearly, $(\pi \circ \nu)^*\mathscr{E}$ is a rank 2 vector bundle on $\mathscr{C}$ given by a nonzero vector in $\mathrm{Ext}(\mathcal{O}_{\mathscr{C}}, \mathcal{O}_{\mathscr{C}})$.

The map $g : S \to R$ is induced by $\pi \circ \nu : \mathscr{C} \to E$ and is hence étale. Let us consider the preimage

$$g^{-1}(C) = \mathscr{C} \times_E C$$

of C. It contains the curve $G = \{(s, \nu(s)) : s \in \mathscr{C}\} \cong \mathscr{C}$. It is not hard to see that $G \in |\mathcal{O}_S(\mathscr{D} + S_q)|$, where $\mathscr{D} = g^*D$ is the unique section of $S/\mathscr{C}$ with self-intersection 0, $q \in (\pi \circ \nu)^{-1}(p)$ and S_q is the fiber of $S/\mathscr{C}$ over q.

Since $g : S \to R$ is Galois,

$$g^*C = \sum_{\sigma \in \mathrm{Aut}(S/R)} \sigma(G).$$

The map $g : g^*C \to C$ is étale. So C is nodal if and only if g^*C is, i.e., it has normal crossings.

Since $h = \pi \circ \nu : \mathscr{C} \to E$ is an isogeny, the dual isogeny $h^\vee : E \to \mathscr{C}$ has the property that $h^\vee \circ h : \mathscr{C} \to \mathscr{C}$ is a multiplication map given by $x \to p + n(x-p)$ for some integer n. So the Galois group $\mathrm{Aut}(\mathscr{C}/E)$ is a subgroup of $\mathrm{Aut}(h^\vee \circ h)$. Hence $\mathrm{Aut}(\mathscr{C}/E)$ is given by a finite subgroup of $J(\mathscr{C}) = \mathrm{Pic}^0(\mathscr{C})$. That is, every $\sigma \in \mathrm{Aut}(\mathscr{C}/E)$ is given by a translation $\sigma(x) = x + \tau$ for some torsion element $\tau \in J(\mathscr{C})$.

To prove Theorem 1.1, it suffices to prove the following:

Proposition 2.2. *Let E be a smooth elliptic curve, let $\mathscr{E}$ be a rank 2 vector bundle on E given by a nonzero vector in $\mathrm{Ext}(\mathcal{O}_E, \mathcal{O}_E)$, let $R = \mathbb{P}\mathscr{E}$, let $D \subset R$ be the section of R/E with $D^2 = 0$ and let $A \subset \mathrm{Aut}(R)_0$ be a finite subgroup of $\mathrm{Aut}(R)_0$ acting freely on R. Then when E is general, for every point $p \in E$ and every smooth curve $G \in |D + R_p|$,*

$$\sum_{\sigma \in A} \sigma(G)$$

has normal crossings if A does not contain the subgroup

$$J(E)_2 = \left\{\tau \in J(E) : 2\tau = 0\right\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

and has only nodes and ordinary triple points as singularities otherwise.

When $C \in |mD + E|$, the Galois group $\mathrm{Aut}(\mathscr{C}/E)$ has order m. If $4 \nmid m$, $\mathrm{Aut}(\mathscr{C}/E)$ does not contain a subgroup of order 4 and hence C is nodal by the above proposition.

Here we let

$$J(E)_n = \{\tau \in J(E) : n\tau = 0\} \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \quad \text{and} \quad J(E)_{\mathrm{tors}} = \bigcup_{n=1}^{\infty} J(E)_n$$

be the torsion subgroups of $J(E)$. For every $\tau \in J(E)_{\mathrm{tors}}$, we define the order $\mathrm{ord}(\tau)$ of τ to be the smallest positive integer n such that $n\tau = 0$ and let $\mathrm{ord}(\tau) = \infty$ if $\tau \notin J(E)_{\mathrm{tors}}$.

Let $\phi \in \mathrm{Aut}(R)_0$ be an automorphism of order n. By (2-2), ϕ is given by a meromorphic function $b_1(z)$ on E with simple poles at p and $p - \tau$ satisfying

$$b_1(z) + b_1(z+\tau) + \cdots + b_1(z + (n-1)\tau) = 0, \tag{2-4}$$

where $\tau \in J(E)_{\mathrm{tors}}$ has order $\mathrm{ord}(\tau) = n$.

To prove that G and $\phi(G)$ intersect transversely, it suffices to prove that $b_1(z)$ does not have a zero of multiplicity 2, i.e.,

$$b_1(p - \eta) \neq 0 \quad \text{for } \tau = 2\eta \tag{2-5}$$

when E is a general elliptic curve.

Let $\phi_1 \neq \phi_2 \in \mathrm{Aut}(R)_0$ be two automorphisms of finite order. Similarly, ϕ_1 and ϕ_2 are given by two meromorphic functions $b_1(z)$ and $b_2(z)$ on E with simple poles at $\{p, p - \tau_1\}$ and $\{p, p - \tau_2\}$, respectively, satisfying

$$b_i(z) + b_i(z + \tau_i) + \cdots + b_i(z + (n_i - 1)\tau_i) = 0 \tag{2-6}$$

for $i = 1, 2$, where $\tau_i \in J(E)_{\mathrm{tors}}$ has order n_i and $\tau_1 \neq \tau_2$. To show that G, $\phi_1(G)$ and $\phi_2(G)$ do not meet at one point, it suffices to show that

$$\{b_1(z) = 0\} \cap \{b_2(z) = 0\} = \varnothing, \tag{2-7}$$

where E is a general elliptic curve. So it remains to prove (2-5) and (2-7).

Let us start with the observation that the meromorphic functions $b_i(z)$ satisfying (2-6) are unique up to a scalar, depending only on p and τ_i.

Proposition 2.3. *Let E be an elliptic curve and let p be a point of E. For every $\tau \in J(E)_{\text{tors}}$ of order n and every meromorphic function $b(z)$ on E with simple poles at p and $p-\tau$ and no other poles,*

$$\sum_{k=0}^{n-1} b(z+k\tau)$$

is constant.

In addition, there is a unique meromorphic function $b(z) = b_{\tau,p}(z)$ on E, up to a scalar, with simple poles at p and $p-\tau$ and no other poles such that

$$\sum_{k=0}^{n-1} b(z+k\tau) = 0. \tag{2-8}$$

Furthermore, for all positive integers m with $n \mid m$ and every meromorphic function $b(z)$ on E with simple poles at p and $p-\tau$ and no other poles,

$$\sum_{\lambda \in J(E)_m} b(z+\lambda) = \frac{m^2}{n} \sum_{k=0}^{n-1} b(z+k\tau). \tag{2-9}$$

Consequently, (2-8) holds if and only if

$$\sum_{\lambda \in J(E)_m} b(z+\lambda) = 0 \tag{2-10}$$

for some positive integer m with $n \mid m$.

Proof. Let $\omega \in H^0(\Omega_E)$ be a nonzero holomorphic 1-form on E. Then $b(z)\omega$ is a meromorphic 1-form on E with simple poles at p and $p-\tau$. So

$$\operatorname{Res}_p b(z)\omega + \operatorname{Res}_{p-\tau} b(z)\omega = 0.$$

It follows that

$$\sum_{k=0}^{n-1} b(z+k\tau)\omega$$

is a holomorphic 1-form on E and hence

$$\sum_{k=0}^{n-1} b(z+k\tau)$$

is constant on E.

Let $V = H^0(\mathcal{O}_E(p_1 + p_2)) \cong \mathbb{C}^2$ be the vector space of meromorphic functions on E with at worst simple poles at $p_1 = p$ and $p_2 = p - \tau$ and let $L : V \to \mathbb{C}$ be the map given by

$$L(b(z)) = \sum_{k=0}^{n-1} b(z + k\tau).$$

Clearly, L is linear. When $b(z) \equiv c$ is constant, $L(b(z)) = nc$ and hence L is surjective. Thus, $\ker(L)$ is a one-dimensional subspace of V. So there exists a unique $b(z) \in V$, up to a scalar, such that

$$\sum_{k=0}^{n-1} b(z + k\tau) = 0.$$

Obviously, $G = \{k\tau : k \in \mathbb{Z}\}$ is a subgroup of $J(E)_m$ for $n \mid m$. So

$$J(E)_m = \bigsqcup_{i=1}^{d} (\lambda_i + G)$$

for some $\lambda_1, \lambda_2, \ldots, \lambda_d \in J(E)_m$ and $d = m^2/n$. Then

$$\sum_{\lambda \in J(E)_m} b(z + \lambda) = \sum_{i=1}^{d} \sum_{\lambda \in G} b(z + \lambda_i + \lambda)$$

We have proved that $\sum_{\lambda \in G} b(z + \lambda)$ is constant. Therefore,

$$\sum_{\lambda \in G} b(z + \lambda) \equiv \sum_{\lambda \in G} b(z + \lambda_i + \lambda)$$

for all i and hence

$$\sum_{\lambda \in J(E)_m} b(z + \lambda) = \sum_{i=1}^{d} \sum_{\lambda \in G} b(z + \lambda_i + \lambda) = d \sum_{\lambda \in G} b(z + \lambda).$$

This proves (2-9). □

We formally state the context around (2-5) and (2-7):

Proposition 2.4. *For a general elliptic curve E, every point $p \in E$, every $\tau \in J(E)_{\text{tors}}$ of order $n \geq 2$ and every $\eta \in J(E)_{\text{tors}}$ satisfying $2\eta = \tau$, we have*

$$b_{\tau,p}(p - \eta) \neq 0,$$

where $b_{\tau,p}(z)$ is the meromorphic function on E given in Proposition 2.3.

Proposition 2.5. *Let E be an elliptic curve, let $p \in E$ be a point on E and let $b_{\tau,p}$ be the meromorphic function on E given in Proposition 2.3 for a nonzero torsion point $\tau \in J(E)_{\text{tors}}$.*

For E general and any two torsion points $\tau_1 \neq \tau_2 \in J(E)_{\text{tors}}$ of orders $n_1 \geq 2$ and $n_2 \geq 2$, respectively, one of the following holds:

$$\{b_{\tau_1,p}(z) = 0\} \cap \{b_{\tau_2,p}(z) = 0\} = \varnothing \tag{2-11}$$

or

$$(n_1, n_2) = (2, 2) \tag{2-12}$$

or

$$(n_1, n_2) = (6, 6),\ \langle \tau_1, \tau_2 \rangle = 3 \text{ in } J(E)_6 \quad \text{and} \quad \operatorname{ord}(\tau_1 - \tau_2) = 6. \tag{2-13}$$

In addition, when $(n_1, n_2) = (2, 2)$,

$$\{b_{\tau_1,p}(z) = 0\} \cap \{b_{\tau_2,p}(z) = 0\} = \{p - \tau_3\}, \tag{2-14}$$

where $\tau_3 \in J(E)_{\text{tors}}$ is a torsion point of order 2 different from τ_1 and τ_2.

For E general and any three distinct nonzero torsion points $\tau_1, \tau_2, \tau_3 \in J(E)_{\text{tors}}$,

$$\{b_{\tau_1,p}(z) = 0\} \cap \{b_{\tau_2,p}(z) = 0\} \cap \{b_{\tau_3,p}(z) = 0\} = \varnothing. \tag{2-15}$$

The intersection pairing $\langle \cdot, \cdot \rangle$ on $J(E)_n$ will be defined in the next section.

Let us explain how Propositions 2.4 and 2.5 imply Proposition 2.2. Proposition 2.4 implies that any pair of curves among $\{\sigma(G) : \sigma \in A\}$ meet transversely and thus $\sum \sigma(G)$ has only ordinary singularities, i.e., singularities whose local branches are smooth and meet transversely pairwise. Then Proposition 2.4 says that no three curves among $\{\sigma(G) : \sigma \in A\}$ meet at one point with the exceptions (2-12) and (2-13), in which cases no more than three curves among $\{\sigma(G) : \sigma \in A\}$ meet at one point by (2-15). In case (2-12), τ_1 and τ_2 generate $J(E)_2 \subset A$. In case (2-13), τ_1 and τ_2 generate a subgroup of $J(E)_6$ of order 12 contained in A; such a subgroup clearly contains $J(E)_2$.

Finally, let us explicitly illustrate how the considerations above lead to curves with triple points in the case $4 \mid m$. Let us first consider the case $m = 4$. Let $[2] : E \to E$ be the multiplication by 2 map on E relative to the choice of a point on E. It is clear that $[2]^*\mathscr{E} \cong \mathscr{E}$ hence $R \cong E \times_{[2],E,\pi} R$. The group of deck transformations of the projection to the second factor $g : R \to R$ is $\{\mathrm{id}_R, \phi_{\tau_1}, \phi_{\tau_2}, \phi_{\tau_3}\}$, with $\phi_{\tau_i} : R \to R$ lying above $z \mapsto z + \tau_i$, where $J(E)_2 = \{0, \tau_1, \tau_2, \tau_3\}$. If $G \in |D + R_p|$, then (2-14) implies that

$$G \cap \phi_{\tau_1}(G) \cap \phi_{\tau_2}(G) \neq \varnothing$$

by the same reasoning as above. Therefore, the curve $g(G) \in |4D + R_p|$ has a triple point. In general, if $4 \mid m$, consider an isogeny $\mathscr{C} \to E$ of degree $m/4$. As above, let $S = \mathscr{C} \times_E R$, which is the Atiyah surface associated to $\mathscr{C}$, $\mathscr{D} \subset S$ the section of $S/\mathscr{C}$ of self-intersection 0, and $q \in \mathscr{C}$. By the case $m = 4$ discussed above, $|4\mathscr{D} + S_q|$ contains genus 1 curves with triple points. Then, their images in R by $S \to R$

are curves $C \in |mD + R_p|$ with triple points. Furthermore, if $f : \widehat{C} \to R$ is the normalization, then f is immersive and $\deg N_f = 2$, so

$$h^0(\widehat{C}, N_f) = 2 = \dim \operatorname{Aut}(R)_0,$$

which implies that all equigeneric deformations of $C \subset R$ come from automorphisms of R, and thus have triple points as well. (For ease of language, we have included the deformations which change the linear system.) Hence, if $4 \mid m$, there exists at least one irreducible component of the Severi variety $V_{R,mD+R_p,1}$ in which the general curve has triple points, as claimed in Theorem 1.2.

3. Torsion points on generic elliptic curves

We will prove Proposition 2.4 and 2.5 by letting E vary in a complete family of elliptic curves X/B with a unique section P. There are many choices of such X. Let us choose X to be a K3 surface with Picard lattice $\left[\begin{smallmatrix} -2 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. We call such X a *Bryan–Leung K*3 [BL00]. Such X admits an elliptic fibration $\pi : X \to B = \mathbb{P}^1$. For X general, it has 24 nodal fibers over $S \subset B$. The (-2)-curve $P \subset X$ is the only section of π. For each positive integer n, let us consider

$$\Sigma_n = \overline{\left\{q \in X_b : b \notin S,\ \operatorname{ord}(p-q) = n \text{ for } p = P_b = P \cap X_b\right\}} \tag{3-1}$$

Clearly, Σ_n is a multisection of X/B of degree

$$n^2 \prod_{\substack{p \text{ prime} \\ p \mid n}} \left(1 - \frac{1}{p^2}\right).$$

We claim that Σ_n is irreducible. This is proved by studying the monodromy action of $\pi_1(B \setminus S)$ on Σ_n. Actually, the monodromy action of $\pi_1(B \setminus S)$ on Σ_n is induced by its action on $H^1(X_b, \mathbb{Z})$.

Fix a smooth fiber $E = X_b$ of X over $b \in B^\circ = B \setminus S$ and let us consider the monodromy action of $\pi_1(B^\circ)$ on $J(E)_{\text{tors}}$ and $H^1(E, \mathbb{Z})$. From the exponential sequence, we have the diagram

$$\begin{array}{ccccccccc}
 & & & & & & J(E)_n & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & H^1(E,\mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}_E) & \longrightarrow & J(E) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \times n} & & \|{\scriptstyle \times n} & & \downarrow{\scriptstyle \times n} & & \\
0 & \longrightarrow & H^1(E,\mathbb{Z}) & \longrightarrow & H^1(\mathcal{O}_E) & \longrightarrow & J(E) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & H^1(E,\mathbb{Z})/nH^1(E,\mathbb{Z}) & & & & & &
\end{array}$$

Thus, we have

$$J(E)_n \cong H^1(E,\mathbb{Z})/nH^1(E,\mathbb{Z})$$

and the action of $\pi_1(B^\circ)$ on $J(E)_{\text{tors}}$ is induced by its action of $H^1(E,\mathbb{Z})$.

The action $\pi_1(B^\circ)$ on $H^1(E,\mathbb{Z})$ preserves the intersection product of $H^1(E,\mathbb{Z})$. Thus, it is given by a group homomorphism

$$\pi_1(B^\circ) \longrightarrow \operatorname{Aut}(H^1(E,\mathbb{Z})) \cong \mathrm{SL}_2(\mathbb{Z}),$$

where $\operatorname{Aut}(H^1(E,\mathbb{Z}))$ is the automorphism group of $H^1(E,\mathbb{Z})$ as a lattice. Thus, the induced action of $\pi_1(B^\circ)$ on Σ_n is given by the group homomorphism

$$\begin{array}{ccc} \pi_1(B^\circ) & \longrightarrow & \mathrm{SL}_2(\mathbb{Z}) \\ & \searrow & \downarrow \\ & & \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) \end{array}$$

Proposition 3.1. *Let $\pi : X \to B = \mathbb{P}^1$ be a Bryan–Leung* K3 *surface with* 24 *nodal fibers. Then the monodromy action $\pi_1(B^\circ) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is surjective and Σ_n is irreducible for all $n \in \mathbb{Z}^+$ with $\Sigma_n \subset X$ defined by* (3-1).

The action of $\pi_1(B^\circ)$ on $H^1(E,\mathbb{Z})$ is well understood. At each $b_i \in \{b_1, b_2, \dots, b_{24}\}$, the loop around b_i acts on $H^1(E,\mathbb{Z})$ by a Lefschetz–Picard transform [Lew99]:

$$T_{\delta_i}(\lambda) = \lambda + \langle \lambda, \delta_i \rangle \delta_i,$$

where $\delta_i \in H^1(E,\mathbb{Z})$ is called the *vanishing cycle* at the node of X_{b_i} for $i = 1, 2, \dots, 24$ and $\langle \cdot, \cdot \rangle$ is the intersection pairing on $H^1(E,\mathbb{Z})$. The monodromy action of $\pi_1(B^\circ)$ on $H^1(E,\mathbb{Z})$ is the subgroup of $\operatorname{Aut}(H^1(E,\mathbb{Z}))$ generated by $T_{\delta_1}, T_{\delta_2}, \dots, T_{\delta_{24}}$. Clearly, T_{δ_i} lift to actions on $H^1(E,\mathbb{Z})/nH^1(E,\mathbb{Z})$. We start with a simple observation:

Lemma 3.2. *Let $\delta_1, \delta_2, \dots, \delta_{24} \in H^1(X_b,\mathbb{Z})$ be the vanishing cycles associated to a Bryan–Leung* K3 *surface $\pi : X \to B = \mathbb{P}^1$ with* 24 *nodal fibers. Then*:

(1) *The δ_i are indivisible, i.e., there do not exist $\eta \in H^1(X_b,\mathbb{Z})$ and an integer $m \ge 2$ such that $\delta_i = m\eta$.*

(2) *For every indivisible $\lambda \in H^1(X_b,\mathbb{Z})$,*

$$\gcd(\langle \lambda, \delta_1 \rangle, \langle \lambda, \delta_2 \rangle, \dots, \langle \lambda, \delta_{24} \rangle) = 1.$$

Proof. It is well known that the δ_i are indivisible, as a consequence of the smoothness of X. (See [Lew99, Example 6.6, p. 72], for instance.) Here we give another argument based on torsion points.

Suppose that $\delta/m \in H^1(E,\mathbb{Z})$ for some $\delta = \delta_i$ and $m \ge 2$. For simplicity, let us assume that m is prime. Then $H^1(E,\mathbb{Z})/mH^1(E,\mathbb{Z})$ is fixed by T_δ so Σ_m is the union $Q_1 \cup Q_2 \cup \cdots \cup Q_{m^2-1}$ of $m^2 - 1$ local sections over a disk $U \subset B$ around the point $s = b_i \in S$. Since X is smooth, each Q_j meets X_s at a point away from the node x of X_s. Let $f : X \dashrightarrow X$ be the rational map given by $f(q) = p + m(q - p)$ for $q \in X_b$, $b \in B^\circ$ and $p = P \cap X_b$. Then f can be extended to a regular, quasifinite and unramified morphism

$$X \setminus \{x_1, x_2, \ldots, x_{24}\} \xrightarrow{f} X,$$

where $x_1, x_2, \ldots, x_{24}$ are the nodes of the 24 fibers $X_S = \pi^{-1}(S)$. Then

$$X_U \cap f^{-1}(P) = P \cup Q_1 \cup Q_2 \cup \cdots \cup Q_{m^2-1}$$

for $X_U = \pi^{-1}(U)$. Since f is unramified, $P, Q_1, Q_2, \ldots, Q_{m^2-1}$ are disjoint. Therefore, $p = P \cap X_s$ and $q_j = Q_j \cap X_s$ are m^2 distinct points on $X_s \setminus \{x\}$. But there are only m distinct points q on $X_s \setminus \{x\}$ such that $m(q - p) = 0$ in $\mathrm{Pic}^0(X_s) \cong \mathbb{C}^*$, which is a contradiction.

For (2), if

$$\gcd(\langle \lambda, \delta_1 \rangle, \langle \lambda, \delta_2 \rangle, \ldots, \langle \lambda, \delta_{24} \rangle) = m \geq 2,$$

then $\lambda \in H^1(E, \mathbb{Z})/mH^1(E, \mathbb{Z})$ is fixed by T_{δ_i} for all i. Therefore, Σ_m contains a section. But P is the only section of X/B, which is a contradiction. □

Proof of Proposition 3.1. If $n = n_1 n_2$ for two coprime integers n_1 and n_2, then the surjectivity of $\pi_1(B^\circ) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ follows from those of $\pi_1(B^\circ) \to \mathrm{SL}_2(\mathbb{Z}/n_i\mathbb{Z})$ for $i = 1, 2$ via the group isomorphism

$$\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Z}/n_1\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}/n_2\mathbb{Z})$$

So by induction on the number of prime divisors of n, it suffices to prove the proposition for $n = p^d$ with p prime.

For simplicity, suppose that $\delta_1 = e_1$, where $\{e_1, e_2\}$ is the standard basis of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. By Lemma 3.2,

$$\gcd(\langle \delta_1, \delta_2 \rangle, \langle \delta_1, \delta_3 \rangle, \ldots, \langle \delta_1, \delta_{24} \rangle) = 1$$

So there exists $2 \leq i \leq 24$ such that $p \nmid \langle \delta_1, \delta_i \rangle$. We may assume that $p \nmid \langle \delta_1, \delta_2 \rangle$. Then $\delta_2 = ae_1 + be_2$ for some $p \nmid b$. Let m be an integer such that $bm \equiv 1 \pmod{n}$. By changing the basis from $\{e_1, e_2\}$ to $\{e_1, ame_1 + e_2\}$, we may assume that $\delta_2 = be_2$.

Clearly,

$$T_{\delta_1}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad (T_{\delta_2})^m \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

for $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = xe_1 + ye_2$, and hence $\pi_1(B^\circ) \to \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is surjective. □

Let us consider the degeneration of the function $b_{\tau,p}(z)$ when X_t degenerates to X_0 for some $0 \in S$.

Proposition 3.3. *Let $\pi : X \to \Delta$ be a flat projective family of curves over the unit disk Δ such that X is smooth, X_t is a smooth elliptic curve for $t \neq 0$ and X_0 is a rational curve with a node, where X_t is the fiber of X over $t \in \Delta$. Let P and Q be two sections of X/Δ such that $P_t - Q_t$ is a torsion class in $J(X_t)$ of order $n \geq 2$ for $t \neq 0$. Then there exists an integral curve $Z \subset X$ flat of degree 2 over Δ such that Z_0 is supported on the node of X_0 and*

$$\{b_{\tau,p}(z) = 0\} = Z_t \tag{3-2}$$

for $t \neq 0$, where $b_{\tau,p}(z)$ is the meromorphic function on X_t given in Proposition 2.3 with $\tau = P_t - Q_t$ and $p = P_t$.

Proof. Since P and Q are sections of X/Δ and X is smooth, P and Q meet X_0 at smooth points P_0 and Q_0 of X_0. By the argument in the proof of Lemma 3.2, $P_0 - Q_0$ is a torsion class in $\mathrm{Pic}^0(X_0) \cong \mathbb{C}^*$ of order n.

Let us consider $\pi_* \mathcal{O}_X(P + Q)$. This is a rank 2 vector bundle over Δ since $h^0(\mathcal{O}_{X_t}(P + Q)) = 2$ for all t. Therefore,

$$H^0(\pi_* \mathcal{O}_X(P + Q)) = H^0(\mathcal{O}_X(P + Q))$$

is a rank 2 free module over $\mathbb{C}[\![t]\!]$.

Let o be the node of X_0. Then $X_0 \setminus \{o\} \cong \mathbb{C}^*$. We may assume that $P_0 = 1$ and $Q_0 = \eta = \exp(2\pi i/n)$. Then $H^0(\mathcal{O}_{X_0}(P_0 + Q_0))$ is spanned by the constant function 1 and

$$s_0(z) = \frac{z}{(z-1)(z-\eta)}$$

over $\mathbb{C}$. We can choose $s \in H^0(\mathcal{O}_X(P + Q))$ such that s_0 is the restriction of s to X_0, i.e., $s_0(z) = s(0, z)$, where we consider $s = s(t, z)$ as a meromorphic function on X with simple poles along P and Q. Then $H^0(\mathcal{O}_X(P + Q))$ is generated by 1 and s over $\mathbb{C}[\![t]\!]$.

Let $\phi : X \setminus \{o\} \to X \setminus \{o\}$ be the automorphism given by $\phi(z) = z + (p - q)$ for $z \in X_t$, $p = P_t$ and $q = Q_t$. Then

$$\sum_{k=0}^{n-1} s(t, \phi^k(z))$$

is constant for each fixed $t \neq 0$ by Proposition 2.3. For $t = 0$, we have

$$\sum_{k=0}^{n-1} s(0, \phi^k(z)) = \sum_{k=0}^{n-1} \frac{\eta^k z}{(\eta^k z - 1)(\eta^k z - \eta)} = 0.$$

Therefore,

$$f(t) = \sum_{k=0}^{n-1} s(t, \phi^k(z))$$

for some $f(t) \in \mathbb{C}[[t]]$ with $f(0) = 0$. Then $ns(t, z) - f(t)$ is a section of $\mathcal{O}_X(P+Q)$ whose restriction to X_t is exactly the function $b_{\tau,p}(z)$.

Let

$$Z = \{ns(t, z) - f(t) = 0\} \tag{3-3}$$

be the vanishing locus of $ns(t, z) - f(t)$. Then (3-2) follows from our choice of $f(t)$. In addition, since $ns(0, z) - f(0) = ns_0(z)$ and s_0 only vanishes at the node o of X_0, we see that Z_0 is supported at o.

We know that Z is a closed subscheme of X of pure dimension one and flat of degree 2 over Δ. So we are in one of the following cases:

- Z is supported on a section of X/Δ with multiplicity 2.
- Z is a union of two distinct sections of X/Δ.
- Z is an irreducible multisection of degree 2 over Δ.

Since Z_0 is supported on the node o of X_0 and X is smooth, Z cannot contain any section of X/Δ. Thus, Z must be an integral curve flat of degree 2 over Δ. □

Proposition 2.4 follows immediately from the above proposition.

Proof of Proposition 2.4. Suppose that $b_{\tau,p}(p - \eta) = 0$ on a general elliptic curve E for some torsion class $\tau \in J(E)$ of order $n \geq 2$ and $2\eta = \tau$. Then by Proposition 3.1, this holds for every torsion class τ of order n.

Let $\pi : X \to B = \mathbb{P}^1$ be a Bryan–Leung K3 surface with 24 nodal fibers over $S \subset B$. We choose a point $s \in S$ and let $U \subset B$ be an open disk about s. Then there exists a section Q of $X_U = \pi^{-1}(U)$ over U such that $P_t - Q_t$ is a torsion class of order n for all $t \in U$. It follows from Proposition 3.3 that $b_{\tau,p}(z)$ has two distinct zeros on X_t for $\tau = P_t - Q_t$ and $p = P_t$, which is a contradiction. □

4. Proof of Proposition 2.5

In this section, we will prove Proposition 2.5. Combined with Proposition 2.4, we obtain Proposition 2.2. Then Theorem 1.2 follows.

We will prove the following two statements in sequence:

Proposition 4.1. *For a general elliptic curve E, a point $p \in E$ and a pair $\tau_1 \neq \tau_2 \in J(E)_{\text{tors}}$ of torsion points of orders $n_1 \geq 2$ and $n_2 \geq 2$, respectively, if*

$$\{b_{\tau_1,p}(z) = 0\} \cap \{b_{\tau_2,p}(z) = 0\} \neq \varnothing$$

then either

$$\{p - q : b_{\tau_i,p}(q) = 0\} \subset J(E)_{\text{tors}} \tag{4-1}$$

for $i = 1, 2$ or

$$n_1 = n_2 = 6, \quad \langle \tau_1, \tau_2 \rangle = 3 \text{ in } J(E)_6 \quad \text{and} \quad \operatorname{ord}(\tau_1 - \tau_2) = 6. \tag{4-2}$$

Proposition 4.2. *For a general elliptic curve E, a point $p \in E$ and a pair $\tau_1 \neq \tau_2 \in J(E)_{\text{tors}}$ of nonzero torsion points, if*

$$\{b_{\tau_1,p}(z)=0\} \cap \{b_{\tau_2,p}(z)=0\} \neq \varnothing \tag{4-3}$$

and

$$\{p-q : b_{\tau_i,p}(q)=0\} \subset J(E)_{\text{tors}} \quad \textit{for } i=1,2,$$

then $\operatorname{ord}(\tau_1)=\operatorname{ord}(\tau_2)=2$.

Our main tool is the monodromy action of $\pi_1(B^\circ)$ on $J(E)_{\text{tors}}$. We fix a Bryan–Leung K3 surface $X \to B = \mathbb{P}^1$ with 24 nodal fibers over $S \subset B$ and a general fiber $E = X_t$ of X/B. We extend the monodromy action on $J(E)_{\text{tors}}$ to the triples (τ, q_1, q_2) with $\tau \in J(E)_{\text{tors}}$ and $\{b_{\tau,p}(z)=0\} = \{q_1, q_2\}$.

Define a curve in $\operatorname{Pic}^0(X/B) \times_B X \times_B X$ by

$$\big\{(\tau, q_1, q_2) : \tau \in J(X_t)_n,\ t \in B \setminus S,\ q_1, q_2 \in X_t,\ \text{and} \\ \{b_{\tau,p}(z)=0\} = \{q_1,q_2\} \text{ for } p = P_t\big\}. \tag{4-4}$$

By Proposition 2.4, for each fixed $n \geq 2$, there exists a finite set $S_n \subset B$ such that for every $t \notin S \cup S_n$, $b_{\tau,p}(z)$ has no double zeros on X_t. So the curve defined by (4-4) is unramified over $B \setminus (S \cup S_n)$ and we have a well-defined monodromy action of $\pi_1(B \setminus (S \cup S_n))$ on such triples (τ, q_1, q_2) on a general fiber $E = X_t$. Let us use the notation $\lambda(\tau)$ and $\lambda(\tau, q_1, q_2)$ to denote the action of $\lambda \in \pi_1(B \setminus (S \cup S_n))$ on $\tau \in J(E)_{\text{tors}}$ and (τ, q_1, q_2).

Lemma 4.3. *Let $X \to B = \mathbb{P}^1$ be a Bryan–Leung* K3 *surface with* 24 *nodal fibers and let $E = X_t$ be a general fiber of X/B. Let $\tau \in J(E)_{\text{tors}}$ be a torsion class of order $n \geq 2$ and let $q_1, q_2 \in E$ be two points given by*

$$\{b_{d\tau,p}(z)=0\} = \{q_1, q_2\}$$

for some integer d with $d\tau \neq 0$. If $\lambda \in \pi_1(B \setminus (S \cup S_n))$ acts on $J(E)_n$ by

$$\lambda(\eta) = \eta + \langle \eta, \tau \rangle \tau$$

for all $\eta \in J(E)_n$, then

$$\lambda(d\tau, q_1, q_2) = (d\tau, q_2, q_1).$$

Proof. Fix a point $0 \in S$ and let δ be the vanishing cycle associated to the nodal fiber X_0. If $\tau = \delta$ in $J(E)_n$, then we must have $\lambda = T_\delta$ in $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$, where T_δ is the Picard–Lefschetz transform associated to δ. Since

$$T_\delta(d\tau) = d\tau,$$

there is a local section $Q \subset X_U = X \times_B U$ over a simply connected open neighborhood U of 0 such that $P_t - Q_t = d\tau$. Then the lemma follows from Proposition 3.3.

More generally, by Proposition 3.1, there exists $\alpha \in \pi_1(B \setminus (S \cup S_n))$ such that $\alpha(\delta) = \tau$. Then $T_\delta = \alpha^{-1} \circ \lambda \circ \alpha$ since

$$\begin{aligned}\alpha^{-1} \circ \lambda \circ \alpha(\eta) &= \alpha^{-1}\big(\alpha(\eta) + \langle \alpha(\eta), \alpha(\delta)\rangle \alpha(\delta)\big)\\ &= \alpha^{-1}\big(\alpha(\eta) + \langle \eta, \delta\rangle \alpha(\delta)\big)\\ &= \alpha^{-1} \circ \alpha(\eta + \langle \eta, \delta\rangle \delta) = T_\delta(\eta).\end{aligned}$$

Thus, the lemma follows. □

Lemma 4.4. *Let $X \to B = \mathbb{P}^1$ be a Bryan–Leung* K3 *surface with* 24 *nodal fibers and let $E = X_t$ be a general fiber of X/B. Let τ_1 and $\tau_2 \in J(E)_{tors}$ be two torsion classes of the same order $n \geq 2$ with $m = \langle \tau_1, \tau_2\rangle$ in $J(E)_n$, let n_1, n_2 be two integers such that $n \nmid n_i$ and let*

$$\{b_{n_1\tau_1,p}(z) = 0\} = \{q_1, q_2\}.$$

If $b_{n_2\tau_2,p}(q_1) = 0$, then

$$\begin{aligned}&b_{n_2(\tau_2+km\tau_1),p}(q_1) = 0 \quad \text{if } 2 \mid k,\\ &b_{n_2(\tau_2+km\tau_1),p}(q_2) = 0 \quad \text{if } 2 \nmid k.\end{aligned} \tag{4-5}$$

If, in addition, $(2\gcd(mn_2, n)) \nmid n$, then $n_1\tau_1 = n_2\tau_2$.

Proof. By Proposition 3.1, we can find $\lambda \in \pi_1(B \setminus (S \cup S_n))$ such that

$$\lambda(\alpha) = \alpha + \langle \alpha, \tau_1\rangle \tau_1$$

for all $\alpha \in J(E)_n$. Then $\lambda(\tau_1) = \tau_1$. Hence, by Lemma 4.3, we have

$$\begin{aligned}&\lambda^k(n_1\tau_1, q_1, q_2) = (n_1\tau_1, q_1, q_2) \quad \text{if } 2 \mid k,\\ &\lambda^k(n_1\tau_1, q_1, q_2) = (n_1\tau_1, q_2, q_1) \quad \text{if } 2 \nmid k.\end{aligned} \tag{4-6}$$

Obviously,

$$\lambda^k(\tau_2) = \tau_2 - km\tau_1 \tag{4-7}$$

for all integers k. Combining (4-6) and (4-7), we obtain (4-5).

If $(2\gcd(mn_2, n)) \nmid n$, then $k_0 = n/\gcd(mn_2, n)$ is odd. Setting $k = k_0$ in (4-5), we obtain

$$b_{n_2\tau_2,p}(q_2) = b_{n_2(\tau_2+k_0m\tau_1),p}(q_2) = 0.$$

On the other hand, we assume that $b_{n_2\tau_2,p}(q_1) = 0$. So

$$\{b_{n_i\tau_i,p}(z) = 0\} = \{q_1, q_2\}$$

for $i = 1, 2$. This implies

$$n_1\tau_1 = (p - q_1) + (p - q_2) = n_2\tau_2.$$

□

Lemma 4.5. *Let E be an elliptic curve, let p be a point on E and let $\tau \in J(E)_{\text{tors}}$ be a torsion point of order 2. Then*

$$\{b_{\tau,p}(z) = 0\} = \{q_1, q_2\},$$

where q_1, q_2 are such that τ, $p - q_1$ and $p - q_2$ are the three distinct 2-torsion points.

Proof. Let τ, τ_1 and τ_2 be the three distinct 2-torsion points. Clearly,

$$\tau = \tau_1 + \tau_2.$$

So there exist a rational function $b(z)$ on E with simple poles at p and $p - \tau$ and simple zeros at $p - \tau_1$ and $p - \tau_2$. Note that $b(z + \tau)$ also has simple poles at p and $p - \tau$ and simple zeros at $p - \tau_1$ and $p - \tau_2$. Therefore,

$$b(z + \tau) \equiv cb(z)$$

for a constant c. And since $b(z) + b(z + \tau)$ is a constant by Proposition 2.3, we must have $c = -1$ and

$$b(z) + b(z + \tau) \equiv 0.$$

Therefore, $b_{\tau,p}(z) \equiv \lambda b(z)$ for a constant $\lambda \neq 0$ by the uniqueness of $b_{\tau,p}(z)$ and the lemma follows. □

Lemma 4.6. *Let E be an elliptic curve, let p be a point on E and let $\tau_1 \neq \tau_2 \in J(E)_{\text{tors}}$ be two distinct nonzero torsion classes. If*

$$\{b_{\tau_1,p}(z) = 0\} = \{q_1, q_2\} \quad \textit{and} \quad \{b_{\tau_2,p}(z) = 0\} = \{q_1, q_3\},$$

then

$$b_{\tau_1-\tau_2,p}(q_2) = 0.$$

Proof. From $\tau_1 = (p - q_1) + (p - q_2)$ and $\tau_2 = (p - q_1) + (p - q_3)$ we have $q_2 = q_3 - (\tau_1 - \tau_2)$. Let us consider the meromorphic function $b_{\tau_2,p}(z + (\tau_1 - \tau_2))$. It has simple poles at $p - (\tau_1 - \tau_2)$ and $(p - \tau_2) - (\tau_1 - \tau_2) = p - \tau_1$ and a zero at

$$q_3 - (\tau_1 - \tau_2) = q_2.$$

Therefore,

$$b(z) = b_{\tau_1,p}(z) + cb_{\tau_2,p}(z + (\tau_1 - \tau_2))$$

has simple poles at p and $p - (\tau_1 - \tau_2)$ and a zero at q_2 for the constant c given by

$$c = -\frac{\operatorname{Res}_{p-\tau_1} b_{\tau_1,p}(z)\omega}{\operatorname{Res}_{p-\tau_1} b_{\tau_2,p}(z + (\tau_1 - \tau_2))\omega},$$

where ω is a nonvanishing holomorphic 1-form on E.

Let n be a positive integer such that $\tau_1, \tau_2 \in J(E)_n$. Then

$$\sum_{\lambda \in J(E)_n} b(z+\lambda) = \sum_{\lambda \in J(E)_n} b_{\tau_1,p}(z+\lambda) + c \sum_{\lambda \in J(E)_n} b_{\tau_2,p}(z+(\tau_1-\tau_2)+\lambda)$$
$$= \sum_{\lambda \in J(E)_n} b_{\tau_1,p}(z+\lambda) + c \sum_{\lambda \in J(E)_n} b_{\tau_2,p}(z+\lambda) \equiv 0$$

by Proposition 2.3. Then by the uniqueness of $b_{\tau_1-\tau_2,p}(z)$, we must have

$$b_{\tau_1-\tau_2,p}(z) \equiv ab(z)$$

for some constant $a \neq 0$ and the lemma follows. □

Lemma 4.7. *Let E be an elliptic curve, let n be a positive integer satisfying $4 \mid n$ and $8 \nmid n$ and let $\alpha_1 \neq \alpha_2 \in J(E)_{\text{tors}}$ be two torsion classes of order n. If*

$$\langle \alpha_1, \alpha_2 \rangle = n/2 \ \textit{in } J(E)_n \quad \textit{and} \quad 4(d_1\alpha_1 - d_2\alpha_2) = 0$$

for some odd integers d_1 and d_2, then

$$\operatorname{ord}(d_1\alpha_1 - d_2\alpha_2) = 2.$$

Proof. Let $m = n/2$. We may assume that $\alpha_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\alpha_2 = \left[\begin{smallmatrix}a\\m\end{smallmatrix}\right]$ in $J(E)_n \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, where $\gcd(a, m) = 1$ and hence a is odd. Then

$$d_1\alpha_1 - d_2\alpha_2 = \begin{bmatrix} d_1 - ad_2 \\ -d_2 m \end{bmatrix}$$

and $2m \mid 4(d_1 - ad_2)$. Since $d_1 - ad_2$ is even and $4 \nmid m$, we see that $2m \mid 2(d_1 - ad_2)$ and hence $d_1\alpha_1 - d_2\alpha_2$ has order 2. □

Proof of Proposition 4.1. Suppose that E is a general fiber of a Bryan–Leung K3 surface $\pi : X \to B = \mathbb{P}^1$ with 24 nodal fibers. Let

$$n = \operatorname{lcm}(n_1, n_2), \quad d_1 = \frac{n}{n_1} \quad \text{and} \quad d_2 = \frac{n}{n_2}.$$

Suppose that

$$\{b_{\tau_1,p}(z) = 0\} = \{q_1, q_2\} \quad \text{and} \quad \{b_{\tau_2,p}(z) = 0\} = \{q_1, q_3\}.$$

It suffices to prove that one of $p - q_1$, $p - q_2$ and $p - q_3$ is torsion.

Since $\operatorname{ord}(\tau_i) = n_i$, we have $\tau_i = d_i\alpha_i$ for $i = 1, 2$ and some $\alpha_i \in J(E)_{\text{tors}}$ of order n. Let $m = \langle \alpha_1, \alpha_2 \rangle \in \mathbb{Z}/n\mathbb{Z}$.

By Lemma 4.4,

$$b_{\tau_2+kd_2m\alpha_1,p}(q_1) = 0 \quad \text{if } 2 \mid k,$$
$$b_{\tau_2+kd_2m\alpha_1,p}(q_2) = 0 \quad \text{if } 2 \nmid k.$$

If $k_0 = n/\gcd(d_2m, n)$ is odd, then $\tau_1 = \tau_2$ by Lemma 4.4, which is a contradiction. Therefore, k_0 and n are even. If $k_0 \neq 2$, we have two cases:

Suppose that $4 \mid k_0$. We have

$$b_{\tau_2,p}(q_1) = b_{\tau_2+(k_0/2)d_2m\alpha_1,p}(q_1) = 0.$$

Let

$$\tau_1' = \tau_2 + (k_0/2)d_2m\alpha_1 \quad \text{and} \quad \tau_2' = \tau_2.$$

Suppose that

$$\{b_{\tau_1',p}(z) = 0\} = \{q_1, q_2'\} \quad \text{and} \quad \{b_{\tau_2',p}(z) = 0\} = \{q_1, q_3'\}.$$

By Lemma 4.6,

$$b_{\tau_1'-\tau_2',p}(q_2') = 0.$$

Obviously, $\operatorname{ord}(\tau_1' - \tau_2') = 2$. Therefore, $p - q_2' \in J(E)_{\text{tors}}$ by Lemma 4.5. It follows that $p - q_1 \in J(E)_{\text{tors}}$ and we are done.

Suppose that $4 \nmid k_0$ and $k_0 > 2$. We have

$$b_{\tau_2,p}(q_1) = b_{\tau_2+2d_2m\alpha_1,p}(q_1) = 0.$$

Let

$$\tau_1' = \tau_2 + 2d_2m\alpha_1 \quad \text{and} \quad \tau_2' = \tau_2.$$

We see that $\tau_1' \neq \tau_2'$, $\operatorname{ord}(\tau_1') \mid n_2 = \operatorname{ord}(\tau_2')$ and

$$\langle \tau_1', \tau_2' \rangle = m' = 2(d_2m)^2$$

with $n_2/\gcd(m', n_2)$ odd. Then $\tau_1' = \tau_2'$ by Lemma 4.4, which is a contradiction.

So we have $k_0 = 2$. That is,

$$n = 2\gcd(d_2m, n).$$

Similarly, we have

$$n = 2\gcd(d_1m, n).$$

So we have

$$d_2m \equiv d_1m \equiv \frac{n}{2} \pmod{n}.$$

And since $\gcd(d_1, d_2) = 1$, we conclude that

$$m \equiv \frac{n}{2} \pmod{n}$$

and d_1 and d_2 are both odd. That is, we have reduced the proposition to the case that

$$2 \mid n,\ 2 \nmid d_1d_2 \quad \text{and} \quad m = \frac{n}{2}. \tag{4-8}$$

Note that under these assumptions,

$$m\tau_j = d_i m\alpha_j = m\alpha_j$$

for all $i, j = 1, 2$.

If one of the τ_i is 2-torsion, then it follows immediately from Lemma 4.5 that $p - q_1 \in J(E)_{\text{tors}}$ and we are done. So we may assume that $n_i \ge 3$ for $i = 1, 2$.

By Lemma 4.6,

$$b_{\tau_1-\tau_2, p}(q_2) = 0.$$

If $\tau_1 - \tau_2$ is a 2-torsion class, then $p - q_2 \in J(E)_{\text{tors}}$ by Lemma 4.5. We are again done. So we may assume that none of τ_1, τ_2 and $\tau_1 - \tau_2$ are 2-torsion classes. That is, we may assume that

$$n_1 \ge 3, \quad n_2 \ge 3 \quad \text{and} \quad \operatorname{ord}(\tau_1 - \tau_2) \ge 3, \tag{4-9}$$

in addition to (4-8).

Repeatedly applying Lemma 4.4, we obtain

$$\begin{aligned}
\{b_{\tau_1, p}(z) = 0\} &= \{q_1, q_2\},\\
\{b_{\tau_2, p}(z) = 0\} &= \{q_1, q_3\},\\
\{b_{\tau_2+m\alpha_1, p}(z) = 0\} &= \{q_2, q_4\},\\
\{b_{\tau_1+m\alpha_2, p}(z) = 0\} &= \{q_3, q_5\}.
\end{aligned}$$

Continuing this process, we obtain

$$b_{\tau_1+m(\alpha_2+m\alpha_1), p}(q_4) = 0.$$

Suppose that $4 \mid n$, i.e., $2 \mid m$. Then $m(\alpha_2 + m\alpha_1) = m\alpha_2$ and hence

$$b_{\tau_1+m\alpha_2, p}(q_4) = 0.$$

Since $\{b_{\tau_1+m\alpha_2, p}(z) = 0\} = \{q_3, q_5\}$, we have either $q_3 = q_4$ or $q_4 = q_5$.

- If $q_3 = q_4$, then

$$\begin{aligned}
\{b_{\tau_1, p}(z) = 0\} &= \{q_1, q_2\},\\
\{b_{\tau_2, p}(z) = 0\} &= \{q_1, q_3\},\\
\{b_{\tau_2+m\alpha_1, p}(z) = 0\} &= \{q_2, q_3\},
\end{aligned}$$

and hence

$$\begin{aligned}
(p - q_1) + (p - q_2) &= \tau_1 \in J(E)_{\text{tors}},\\
(p - q_1) + (p - q_3) &= \tau_2 \in J(E)_{\text{tors}},\\
(p - q_2) + (p - q_3) &= \tau_2 + m\alpha_1 \in J(E)_{\text{tors}}.
\end{aligned}$$

It follows that $p - q_1, p - q_2, p - q_3 \in J(E)_{\text{tors}}$. We are done.

- If $q_4 = q_5$, then

$$\begin{aligned}
\{b_{\tau_1,p}(z) = 0\} &= \{q_1, q_2\},\\
\{b_{\tau_2,p}(z) = 0\} &= \{q_1, q_3\},\\
\{b_{\tau_2+m\alpha_1,p}(z) = 0\} &= \{q_2, q_4\},\\
\{b_{\tau_1+m\alpha_2,p}(z) = 0\} &= \{q_3, q_4\},
\end{aligned}$$

and hence

$$\begin{aligned}
(p-q_1)+(p-q_2) &= \tau_1,\\
(p-q_1)+(p-q_3) &= \tau_2,\\
(p-q_2)+(p-q_4) &= \tau_2+m\alpha_1 = \tau_2+m\tau_1,\\
(p-q_3)+(p-q_4) &= \tau_1+m\alpha_2 = \tau_1+m\tau_2.
\end{aligned}$$

It follows that

$$(m-2)(\tau_1-\tau_2) = 0 \Rightarrow \gcd(m-2,n)(\tau_1-\tau_2) = 0.$$

Since $\gcd(m-2,n) = \gcd(m-2,2m)$ is either 2 or 4, the order of $\tau_1-\tau_2$ is either 2 or 4. By our hypothesis (4-9), $\operatorname{ord}(\tau_1-\tau_2) \neq 2$. So $\operatorname{ord}(\tau_1-\tau_2) = 4$. Then $\gcd(m-2,2m) = 4$ and $4 \nmid m$. This contradicts Lemma 4.7.

So far we have proved the proposition when m is even. Suppose that $2 \nmid m$. Then $m(\alpha_2+m\alpha_1) = m(\alpha_1+\alpha_2)$ and hence

$$b_{\tau_1+m(\alpha_1+\alpha_2),p}(q_4) = 0.$$

Continuing with the use of Lemma 4.4, we obtain

$$\begin{aligned}
\{b_{\tau_1,p}(z) = 0\} &= \{q_1, q_2\},\\
\{b_{\tau_2,p}(z) = 0\} &= \{q_1, q_3\},\\
\{b_{\tau_2+m\alpha_1,p}(z) = 0\} &= \{q_2, q_4\},\\
\{b_{\tau_1+m\alpha_2,p}(z) = 0\} &= \{q_3, q_5\},\\
\{b_{\tau_1+m(\alpha_1+\alpha_2),p}(z) = 0\} &= \{q_4, q_6\},\\
\{b_{\tau_2+m(\alpha_1+\alpha_2),p}(z) = 0\} &= \{q_5, q_7\}.
\end{aligned}$$

Applying Lemma 4.4 to $(\tau_1+m(\alpha_1+\alpha_2), \tau_2+m\alpha_1)$, we obtain

$$b_{\tau_2+m(\alpha_1+\alpha_2),p}(q_6) = 0.$$

Similarly,

$$b_{\tau_1+m(\alpha_1+\alpha_2),p}(q_7) = 0.$$

That is, $q_6 \in \{q_5, q_7\}$ and $q_7 \in \{q_4, q_6\}$. Since $\{q_5, q_7\} \neq \{q_4, q_6\}$, we must have $q_6 = q_7$. Then from

$$\begin{aligned}
\{b_{\tau_1,p}(z) = 0\} &= \{q_1, q_2\},\\
\{b_{\tau_2,p}(z) = 0\} &= \{q_1, q_3\},
\end{aligned}$$

$$\{b_{\tau_2+m\alpha_1,p}(z)=0\}=\{q_2,q_4\},$$
$$\{b_{\tau_1+m\alpha_2,p}(z)=0\}=\{q_3,q_5\},$$
$$\{b_{\tau_1+m(\alpha_1+\alpha_2),p}(z)=0\}=\{q_4,q_6\},$$
$$\{b_{\tau_2+m(\alpha_1+\alpha_2),p}(z)=0\}=\{q_5,q_6\},$$

we obtain

$$3(\tau_1-\tau_2)=m(\alpha_1-\alpha_2).$$

Hence $\tau_1-\tau_2$ has order 2 or 6.

By our hypothesis (4-9), $\operatorname{ord}(\tau_1-\tau_2)\neq 2$. So $\tau_1-\tau_2$ has order 6. Hence $6\,|\,n$, $3\,|\,m$ and $3\,|\,n_1n_2$.

Since d_1 and d_2 are odd, $n_1=n/d_1$ and $n_2=n/d_2$ are even. So at least one of n_1 and n_2 is divisible by 6. Without loss of generality, let us assume that $6\,|\,n_1$. Then

$$n_1(\tau_1-\tau_2)=0 \implies n_1\tau_2=0 \implies n_2\,|\,n_1 \implies n=n_1.$$

Let

$$\tau_1'=\tau_1 \quad\text{and}\quad \tau_2'=\tau_1-\tau_2.$$

By Lemma 4.6,

$$b_{\tau_1',p}(q_2)=b_{\tau_2',p}(q_2)=0.$$

Applying the whole argument to (τ_1',τ_2'), we again arrive at

$$\operatorname{ord}(\tau_1'-\tau_2')=6.$$

That is, $n_2=\operatorname{ord}(\tau_2)=6$. Then this implies that $\tau_1=\tau_2+(\tau_1-\tau_2)$ also has order 6. So we have (4-2). □

Proof of Proposition 4.2. Let

$$\{b_{\tau_1,p}(z)=0\}=\{q_1,q_2\} \quad\text{and}\quad \{b_{\tau_2,p}(z)=0\}=\{q_1,q_3\},$$

where $\eta_i=p-q_i$ are torsion for $i=1,2,3$.

Suppose that $n=\operatorname{lcm}(\operatorname{ord}(\tau_1),\operatorname{ord}(\eta_1),\operatorname{ord}(\eta_2))$ and that $\tau_1=d\alpha_1$ for some $\alpha_1\in J(E)_{\text{tors}}$ of order n. Let E be a general fiber of a Bryan–Leung K3 surface $X\to B=\mathbb{P}^1$ with 24 nodal fibers. Clearly, each $\lambda\in\pi_1(B\setminus(S\cup S_n))$ acts on (τ_1,q_1,q_2) by

$$\lambda(\tau_1,q_1,q_2)=(\lambda(\tau_1),\,p-\lambda(\eta_1),\,p-\lambda(\eta_2)).$$

On the other hand, for $\lambda(\eta)=\eta+\langle\eta,\alpha_1\rangle\alpha_1$,

$$\lambda(\tau_1,q_1,q_2)=(\lambda(\tau_1),q_2,q_1)$$

by Lemma 4.3. Therefore,

$$\eta_2=\lambda(\eta_1)=\eta_1+m\alpha_1$$

for $m = \langle \eta_1, \alpha_1 \rangle$. And since $\tau_1 = \eta_1 + \eta_2$, we have

$$\tau_1 = 2\eta_1 + m\alpha_1 \implies \langle (d-m)\alpha_1, \alpha_1 \rangle = \langle 2\eta_1, \alpha_1 \rangle \implies 2m = 0$$

in $\mathbb{Z}/n\mathbb{Z}$. If $m = 0$, then $\eta_1 = \eta_2$, which contradicts Proposition 2.4. So n is even and $m = n/2$. Therefore, we have

$$\operatorname{ord}(\eta_1 - \eta_2) = \operatorname{ord}(\tau_1 - 2\eta_1) = 2. \tag{4-10}$$

Similarly,

$$\operatorname{ord}(\eta_1 - \eta_3) = \operatorname{ord}(\tau_2 - 2\eta_1) = 2. \tag{4-11}$$

It follows that $\tau_1 - \tau_2$ is a 2-torsion class as well. By Lemma 4.6,

$$b_{\tau_1 - \tau_2, p}(q_2) = 0.$$

Hence η_2 is a 2-torsion class by Lemma 4.7. Together with (4-10) and (4-11), we see that all of $\tau_1, \tau_2, \eta_1, \eta_2, \eta_3$ are 2-torsion classes. □

To finish the proof of Proposition 2.5, it remains to justify (2-15).

Proof of Proposition 2.5. We have proved (2-11) with two exceptions outlined in the proposition.

If (2-15) fails, we must have one of the following:

A. τ_1, τ_2, τ_3 are three distinct 2-torsion points.

B. τ_1, τ_2, τ_3 are three distinct 6-torsion points satisfying that $\langle \tau_i, \tau_j \rangle = 3$ and $\operatorname{ord}(\tau_i - \tau_j) = 6$ for all $1 \le i < j \le 3$.

In case A, by (2-14),

$$\{b_{\tau_1,p}(z) = 0\} \cap \{b_{\tau_2,p}(z) = 0\} = \{p - \tau_3\},$$
$$\{b_{\tau_2,p}(z) = 0\} \cap \{b_{\tau_3,p}(z) = 0\} = \{p - \tau_1\},$$

and (2-15) follows.

In case B, we may assume that

$$\tau_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in J(E)_6 \cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$$

Since $\langle \tau_i, \tau_j \rangle = 3$ for $i \ne j$, we must have $\tau_2 = \begin{bmatrix} a \\ 3 \end{bmatrix}$ and $\tau_3 = \begin{bmatrix} b \\ 3 \end{bmatrix}$ for some $a, b \in \mathbb{Z}$ satisfying $3 \nmid ab$ and $2 \nmid (a-b)$.

Since $\operatorname{ord}(\tau_1 - \tau_2) = \operatorname{ord}(\tau_1 - \tau_3) = 6$, $3 \nmid (a-1)(b-1)$. Together with $3 \nmid ab$, we must have

$$a \equiv b \equiv 2 \pmod 3.$$

Then $\operatorname{ord}(\tau_2 - \tau_3) = 2$. Therefore, there are no such triples (τ_1, τ_2, τ_3). □

5. Proof of Theorem 1.1 for $g \geq 2$

It remains to prove Theorem 1.1 for $g \geq 2$. As mentioned in Section 1, we will reduce it to the case $g = 1$ by a degeneration argument.

Let E be a smooth elliptic curve. We first construct a smooth projective family X of surfaces over $\Delta = \mathbb{A}^1$ such that $X_0 \cong E \times \mathbb{P}^1$ and $X_t \cong \mathbb{P}\mathscr{E}$ for $t \neq 0$, where $\mathscr{E}$ is the rank 2 vector bundle on E given by a nonzero vector in $\operatorname{Ext}(\mathcal{O}_E, \mathcal{O}_E)$.

Let $\mathcal{V}$ be a rank 2 vector bundle over $E \times \Delta$ given by

$$t \in \operatorname{Ext}(\mathcal{O}_{E\times\Delta}, \mathcal{O}_{E\times\Delta}) = H^1(\mathcal{O}_{E\times\Delta}) = \mathbb{C}[t]$$

and let $X = \mathbb{P}\mathcal{V}$. Clearly, X is such a family.

There is an effective divisor $D \subset X$, flat over Δ, such that D_t is the section of X_t/E with $D_t^2 = 0$. Fix a point $p \in E$ and let $L = mD + \pi^* p$, where π is the projection $X \to E$.

For $t \neq 0$, the Severi variety $V_{X_t,L,g}$ has expected dimension g. If we fix g general points on X_t, there exist finitely many $[C] \in V_{X_t,L,g}$ such that C passes through these points. Let us fix g general sections $P_1, P_2, \ldots, P_g \subset X$ of X/Δ. Then after a base change, there exists a flat projective family $C \subset X$ of curves over Δ such that C_t is an integral curve in $|L|$ on X_t passing through $P_i \cap X_t$ for $i = 1, 2, \ldots, g$ and $t \neq 0$. Here we replace Δ by an analytic disk or a smooth affine curve finite over $\mathbb{A}^1$.

We may choose the base change in such a way that there exists a family of stable maps $\varphi : \mathscr{C} \to X$ over Δ such that φ maps $\mathscr{C}$ birationally onto C.

On X_0, the linear system $|L|$ is completely reducible in the sense that

$$H^0(\mathcal{O}_{X_0}(L)) = \operatorname{Sym}^m H^0(\mathcal{O}_{X_0}(D)) \otimes H^0(\mathcal{O}_{X_0}(\pi^* p)).$$

Therefore,

$$C_0 = m_1 D_1 + m_2 D_2 + \cdots + m_g D_g + F,$$

where D_i are the sections of X_0/E passing through $P_i \cap X_0$ for $i = 1, 2, \ldots, g$, F is the fiber of $\pi : X_0 \to E$ over p and the m_i are positive integers such that $\sum m_i = m$.

Clearly, C_t only has singularities in open neighborhoods of D_i. So it suffices to show that C_t has only nodes and ordinary triple points as singularities in an analytic neighborhood of each D_i for $i = 1, 2, \ldots, g$, if E is general.

Since $\mathscr{C}_t$ is a smooth projective curve of genus g for $t \neq 0$, there are exactly g irreducible components $\Gamma_1, \Gamma_2, \ldots, \Gamma_g$ of $\mathscr{C}_0$ such that each Γ_i is a smooth elliptic curve dominating D_i for $i = 1, 2, \ldots, g$.

Let us fix i. If $m_i = 1$, there is nothing to do. Otherwise, suppose that $m_i \geq 2$. Let $\psi : \widehat{X} \to X$ be the blowup of X along D_i. Then the central fiber $\widehat{X}_0 = S \cup R$ is a union of two smooth projective surfaces S and R, where S is the proper transform of X_0, R is the exceptional divisor of ψ and S and R meet transversely along a curve over D_i, which we still denote by D_i. Let $\widehat{C}$ be the proper transform of C under ψ.

The rational map $\psi^{-1} \circ \varphi : \mathscr{C} \dashrightarrow \widehat{X}$ is regular at a general point of Γ_i. We claim that

$$\psi^{-1} \circ \varphi(\Gamma_i) \not\subset D_i = S \cap R.$$

Otherwise, we choose a local section Q of $\mathscr{C}/\Delta$ passing through a general point of Γ_i. Then $\varphi(Q)$ is a local section of $\widehat{X}/\Delta$ meeting $D_i = S \cap R$, which is impossible since $\widehat{X}$ is smooth. So $\psi^{-1} \circ \varphi$ maps Γ_i to an irreducible curve on R other than D_i. That is, $\widehat{C}_0$ does not contain D_i.

We have either $R \cong \mathbb{P}\mathscr{E}$ or $R \cong E \times \mathbb{P}^1$.

A. If $R \cong \mathbb{P}\mathscr{E}$, then $\widehat{C} \cap R$ must be an integral curve in $|m_i \widehat{D} + \hat{\pi}^* p|$ of geometric genus 1, where $\widehat{D}$ is the proper transform of D and $\hat{\pi} = \pi \circ \psi$ is the projection $\widehat{X} \to E$. Then by Theorem 1.2, $\widehat{C} \cap R$ has only nodes and ordinary triple points as singularities and the same holds for C_t in an open neighborhood of D_i.

B. If $R \cong E \times \mathbb{P}^1$, then $\widehat{C} \cap R = m_i \widehat{D}_i + \widehat{F}$, where $\widehat{D}_i$ is the section R/E passing through the point $\widehat{P}_i \cap R$ with $\widehat{P}_i$ being the proper transform of P_i under ψ and $\widehat{F}$ is the fiber of R over $p \in E$. So we continue to blow up $\widehat{X}$ along $\widehat{D}_i$. By embedded resolution of singularities, there exists a sequence of blowups over D_i, say $f : X' \to X$, such that the proper transform C' of C is smooth over a general point of D_i. Then by Zariski's main theorem, the map $f^{-1} \circ \varphi : \mathscr{C} \dashrightarrow X'$ has connected fiber over $f^{-1}(D_i)$. This means that C'_0 is smooth over a general point of D_i. So we will eventually end up in case A after a sequence of blowups over D_i.

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