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SPUN NORMAL SURFACES IN 3-MANIFOLDS II: THE TOROIDAL CASE

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Spun normal surfaces are a useful way of representing proper essential surfaces using ideal triangulations for 3-manifolds with tori boundaries. Here we consider spinning surfaces in the case of a 3-manifold with a nontrivial JSJ decomposition, where each of the JSJ components is hyperbolic. We prove that a proper essential surface Σ can be spun, so long as none of the JSJ components are bundles with fiber a subsurface of Σ and the ideal triangulation satisfies similar properties to a taut structure.

1. Introduction

Definition 1. A *crushing* of a space X is an epimorphism $f : X \to Y$ that is cell-like, i.e., $f^{-1}(y)$ is a cell for each $y \in Y$.

It is well-known that such a map is homotopic to a homeomorphism; see [Sie]. We will be interested in crushings associated to triangulations and normal surfaces. In particular, if S is a closed 2-sided normal surface in a triangulation $\mathcal{T}$ we will be concerned with the process of cutting M open along S and then crushing each component of S to a point. If we delete the image points of S after crushing, we obtain a new triangulation which is ideal. We want to crush the cell structure of the cut open triangulation into a new triangulation. This crushing process is detailed in [JR1] in the case of 2-spheres and also will be described in Section 2 for our case of tori.

In [KR2], it is shown that a properly embedded nonfibered incompressible and ∂-incompressible 2-sided surface in M can be spin normalized along boundary tori of M, if M is a suitable 3-manifold with 1-efficient ideal triangulation.

Definition 2. An ideal triangulation $\Im$ of a 3-manifold M is 1-*efficient* if it satisfies the following conditions:

- There are no embedded normal 2-spheres, projective planes, or Klein bottles.
- Every embedded normal torus is peripheral.

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Equivalently, there are no normal surfaces with nonnegative Euler characteristic, except for peripheral tori and Klein bottles.

[KR2, **Theorem 8**]. *Let M be an anannular, atoroidal, irreducible and P^2-irreducible 3-manifold with tori boundary components and $\mathfrak{T}$ be a 1-efficient ideal triangulation of M. If F is a properly embedded, incompressible and ∂-incompressible 2-sided surface in M which is not a fiber, then F can be spin normalized in $(M, \mathfrak{T})$ with 2^r choices of spinning direction, where r is the number of boundary components of M containing a curve of ∂F.*

We generalize this result by allowing JSJ-normal tori in M. To deal with such tori, we need to restrict how they can be realized as normal surfaces.

Definition 3. An ideal triangulation $\mathfrak{T}$ of an irreducible and P^2-irreducible 3-manifold M is *T-efficient* if it satisfies the following conditions:

- There are no embedded normal 2-spheres or projective planes.
- Every embedded normal torus or Klein bottle is essential, i.e., π_1-injective.
- For each isotopy class of essential tori or Klein bottles in M, there is exactly one embedded normal torus or Klein bottle representing that class.

Our main theorem is the following.

Theorem 6. *Suppose that M is compact, irreducible, and P^2-irreducible and has essential tori boundary components. Assume that each of the JSJ components M_i of M is hyperbolic for $1 \leq i \leq k$. Let Σ be any properly embedded 2-sided essential surface in M with the property that no subsurface of Σ is a fiber of a bundle structure for one of the JSJ components.*

Assume that M has an ideal triangulation $\mathfrak{T}$ which is T-efficient. Then Σ can be spin normalized with boundary curves spinning in all possible combinations of directions.

To prove Theorem 6, we carry out the crushing process along JSJ-normal tori T_i, cutting M open along those T_i and then crushing and deleting the boundaries to have an ideal triangulation for the interior of each JSJ component of M. This new ideal triangulation of each of the JSJ components is 1-efficient, so that we can apply the results of [KR2]. Also, in Section 5, we will show that there always exists an ideal triangulation that satisfies all the conditions described in Theorem 6, so it is reasonable to impose the above restrictions.

Another generalization would be to replace "1-efficiency" with "0-efficiency" for the ideal triangulation of the JSJ components we are considering. However, in this case, it is difficult to avoid inessential normal tori in spinning. Whether a properly embedded surface can spin along an inessential normal torus is not clear to us.

Spun normal surfaces and generalized spun normal surfaces play a central role in the 3d index of Dimofte, Gaiotto, and Gukov [DGG]. The relationships between spun and generalized spun normal surfaces and the 3d index are discussed in [GHRH]. To define the 3d index, a 1-efficient ideal triangulation is required. In [GHRS] it is shown that the 3d index is independent of the choice of triangulation, so long as a suitable class of triangulations are used, which come from the hyperbolic structure on the 3-manifold. It is conjectured that the 3d index counts surfaces in 3-manifolds without reference to a triangulation. Moreover, an important problem is to understand if the 3d index can be defined for toroidal manifolds. Exploring the behaviour of spun normal surfaces in toroidal manifolds may give some insight into this issue.

2. Crushing JSJ tori in an ideal triangulation

When a triangulation is cut open along a normal surface, we get a cell decomposition. The cells are either truncated tetrahedra, truncated (triangular) prisms or normal prisms (quadrilateral normal prisms and triangular normal prisms) with base a quadrilateral or triangular disk included in the normal surface. As the normal surface consists of quadrilateral and triangular disks, when these are crushed, we need to extend the crushing to these cells to produce a new triangulation. In this section, we will discuss such crushing. A detailed background and discussion is given in [JR1].

Most of the arguments in this section will be developed under the conditions given in the assumptions of the main theorem (Theorem 6) in Section 1. Although there are some arguments that hold true in more general situations, this article will focus on normal spinning a properly embedded incompressible and ∂-incompressible surface, rather than on crushing a more general triangulation.

Let M be a compact, irreducible, and P^2-irreducible 3-manifold with essential tori boundary components, and $\mathfrak{I}$ be an ideal T-efficient triangulation of $\mathring{M}$. Assume that M is toroidal and each of the JSJ components M_i of M is hyperbolic, for $1 \leq i \leq k$. We ultimately want to find a 1-efficient ideal triangulation for each JSJ component M_i. To make further arguments easier, we will truncate $(\mathring{M}, \mathfrak{I})$ by removing an open regular neighborhood of ideal vertices whose boundaries are peripheral normal tori, and denote the resulting manifold by $\hat{M}$.

Let S be the union of all the JSJ tori that are in (unique) normal form for $(\mathring{M}, \mathfrak{I})$ and all the boundary components of $\hat{M}$. We assume that each JSJ component $\hat{M}_i$ is obtained by cutting $\hat{M}$ open along the JSJ-normal tori in S. Then all the boundary components of $\hat{M}_i$ are in normal form. Since we truncated $(\mathring{M}, \mathfrak{I})$, $\hat{M}_i$ has no ideal vertices and has the induced cell decomposition C_i from the ideal triangulation $\mathfrak{I}$, which consists of four types of cells: truncated tetrahedra, truncated

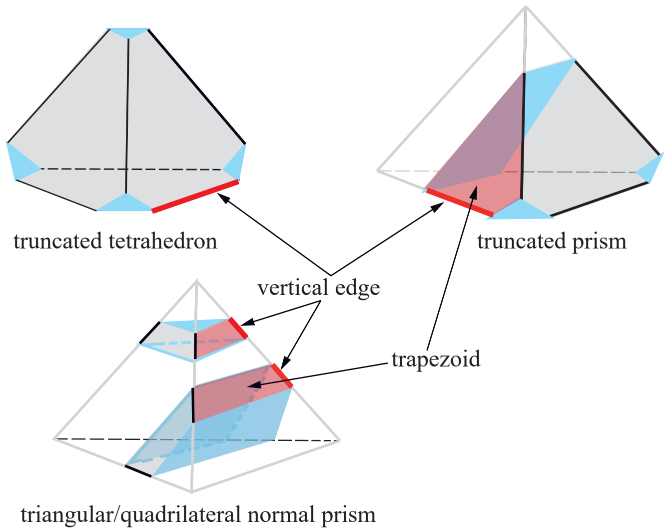

Figure 1. Four types of cells in C_i.

prisms, triangular normal prisms, and quadrilateral normal prisms (see Figure 1).

We will define a nice crushing $f_i : \hat{M}_i \to \overline{M}_i$, in Theorem 1, which satisfies the following conditions; f_i maps each component of $\partial \hat{M}_i$ to a point, $\overline{M}_i^*$ obtained by deleting vertices from $\overline{M}_i$ is homeomorphic to $\mathring{M}_i$, and the induced cell decomposition of $\overline{M}_i$ is an ideal triangulation of $\mathring{M}_i$ after deleting the vertices. We will show that the new induced ideal triangulation of $\overline{M}_i^*$, or equivalently $\mathring{M}_i$, is 1-efficient (Theorem 5).

Let us define a crushing $f_i : \hat{M}_i \to \overline{M}_i$ by collapsing each cell of C_i as shown in Figure 2, and call it a *canonical crushing* on $\hat{M}_i$.

We will investigate if f_i is cell-like (see Figures 2 and 3). We denote the induced cell decomposition of $\overline{M}_i$ by $\Im_i$. Note that each boundary component of $\hat{M}_i$ crushes to a vertex of $\Im_i$. But there is no guarantee that the new manifold $\overline{M}_i^*$ given by deleting vertices from $\overline{M}_i$ is homeomorphic to $\mathring{M}_i$, or that the induced cell structure of $\overline{M}_i^*$ is a well-defined ideal triangulation. If f_i is cell-like, the first assertion follows from [Sie]. We will find that this is the case under our assumptions.

To investigate this, we need to look at the cycles of truncated prisms or trapezoids. Here a *trapezoid* is a rectangular face of C_i bounded by two parallel (i.e., normal isotopic) normal arcs and two vertical edges, where a *vertical edge* is an edge of C_i whose interior is inside $\hat{M}_i$ and both of whose boundary points are in $\partial \hat{M}_i$, i.e., in S (see Figure 1). A *trapezoidal cycle* is an annulus or a Möbius band properly embedded in M, formed by gluing trapezoids together along their vertical edges.

Note that a truncated prism crushes to a triangle and a trapezoid to an edge by the canonical crushing f_i. To achieve that $\overline{M}_i^*$ is homeomorphic to $\mathring{M}_i$, we will verify

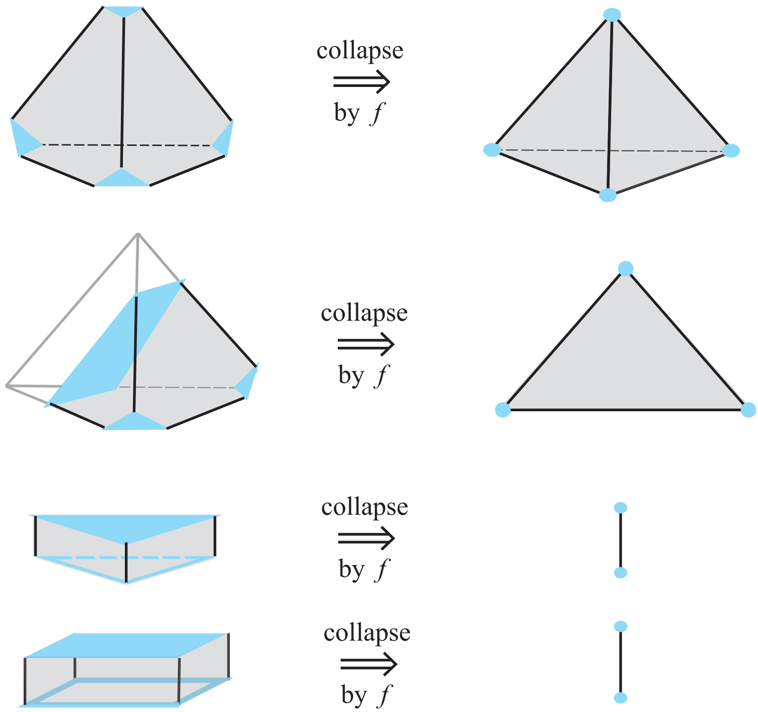

Figure 2. Canonical crushing $f = f_i$.

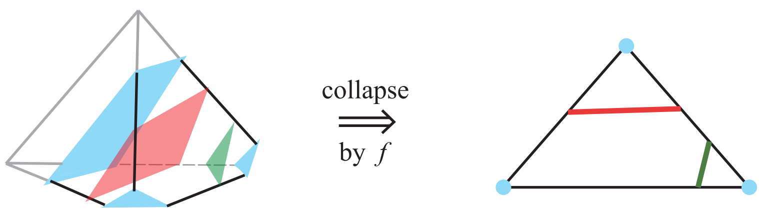

Figure 3. Crushings of quadrilateral and triangular disks by $f = f_i$.

that the regions surrounded by the cycles of trapezoids are simply connected, so that such regions can be crushed into the edges. By modifying f_i in this way, we are able to achieve that the modified f_i is cell-like and so the topology of $\hat{M}_i$ is that of M_i (Lemma 2). We also need to verify that there are no cycles of truncated prisms (Lemma 3) which collapse to a triangle by f_i, to finish the verification that the modified f_i is cell-like and that the induced cell structure of $\overline{M}_i^*$ is a well-defined ideal triangulation. This is the main topic of discussion in this section.

Let $P(C_i)$ be the collection of all cells which crush to edges of $\mathfrak{I}_i$ by the canonical crushing f_i. Then $P(C_i) = \{\text{all normal prisms of } C_i\} \cup \{\text{all trapezoids of } C_i\} \cup \{\text{all vertical edges of truncated tetrahedra and truncated prisms of } C_i\}$. We call $P(C_i)$

the *combinatorial product region* of C_i. Each component of $P(C_i)$ crushes to an edge of $\Im_i$ by the canonical crushing $f_i : \hat{M}_i \to \overline{M}_i$, and for $\overline{M}_i^*$ to be topologically identical to $\mathring{M}_i$, the inverse image of an edge must be simply connected. If each component of $P(C_i)$ is simply connected, the canonical crushing f_i is what we need for spinning. If not, we will modify f_i by further crushing as follows: each region enclosed by the cycles of trapezoids (together with normal disks in $\partial \hat{M}_i$) crushes into an edge (Figure 4, right, shows an example of such a region — in this case, a truncated prism that is crushed to an edge under modified crushing), so that the enlarged crushing region is simply connected. We denote the union of the new enlarged regions by $P(\hat{M}_i)$ and call it the *induced product region*, and the modified crushing is called a *combinatorial crushing*. Under suitable conditions, the desired $P(\hat{M}_i)$ can be constructed with properties required to achieve spinning. In our case (assuming the hypotheses of the main theorem (Theorem 6), $P(C_i)$ has simply connected components so that $P(C_i) = P(\hat{M}_i)$ (Lemma 4) and furthermore the canonical crushing f_i itself is a combinatorial crushing and does not change the topology of the original manifold. Under what general circumstances $P(C_i)$ is simply connected, and under what conditions $P(\hat{M}_i)$ exists, is a subject for further study.

The following theorem states that the canonical crushing in our case induces a suitable cell decomposition of the resulting manifold.

Theorem 1. *Let M be compact, irreducible, and P^2-irreducible with essential tori boundary components. Assume that each of the JSJ components M_i of M is hyperbolic for $1 \le i \le k$ and the ideal triangulation $\Im$ of $\mathring{M}$ is T-efficient. Then for each JSJ component M_i, $1 \le i \le k$, the canonical crushing $f_i : \hat{M}_i \to \overline{M}_i$ gives that $\mathring{M}_i$ is homeomorphic to $\overline{M}_i^*$, with vertices deleted from $\overline{M}_i$. Moreover the induced cell decomposition $\Im_i^*$ of $\overline{M}_i^*$ is an ideal triangulation of $\mathring{M}_i$.*

We will prove this by using Lemmas 2, 3 and 4 below and [JR1, Theorem 4.1].

Lemma 2. *With the same hypotheses as Theorem 1, any trapezoidal cycle in $\hat{M}_i$ is an annulus and together with two disks in S bounds a 3-cell, where $\hat{M}_i$ is a truncated JSJ component obtained by cutting $\hat{M}$ open along the normal representatives of the JSJ tori.*

Note. The hypothesis of "unique normal representative of each JSJ torus" in Theorem 1 could be replaced by "least-weight normal representative" in this lemma.

Proof. Let T be a trapezoidal cycle in $\hat{M}_i$. Then ∂T is on S, which is the union of all normal JSJ tori of M and the normal boundary components of $\hat{M}$ and T is a properly embedded annulus or Möbius band in $\hat{M}_i$. By the assumptions of Theorem 1, $\hat{M}_i$ is hyperbolic and so there are no properly embedded essential surfaces with nonnegative Euler characteristic. If T is an inessential Möbius band, then T is either ∂-parallel or has a boundary compression to S. But since $\hat{M}_i$ is hyperbolic and has tori boundary, neither can occur. Therefore T is an inessential annulus with both boundary curves on S. Here since S is incompressible, either both boundary curves of T are essential in S or both are not.

If both curves of ∂T are essential, then T must be boundary parallel, since $\hat{M}_i$ is hyperbolic. Let S_1 be the component of S containing ∂T and A the annulus in S_1 which is parallel to T. Note that T is 0-weight (meaning that its interior does not intersect any edges of the triangulation). The new torus $T \cup (S_1 \setminus A)$ is a barrier (see [JR1]) so normalizing produces a topologically boundary parallel normal torus with less weight than S_1. This contradicts the fact that there is a unique normal representation of each JSJ torus in S. Therefore the only possibility is that the curves of ∂T are inessential and each bounds a disk in S_1, say D_1 and D_2 respectively. If say $D_1 \subset D_2$ then T must be ∂-parallel (to the annulus $D_2 \setminus \mathring{D}_1$). But then we get a contradiction to the unique normal representation of JSJ tori again by a similar argument to the previous paragraph. Hence $D_1 \cap D_2 = \varnothing$ and D_1 and D_2 together with T form a 2-sphere. Therefore T together with two disks in S bounds a 3-cell in $\hat{M}_i$ since M is irreducible, and the proof is complete. □

The following remark comes from the proofs of Lemmas 3 and 2.

Remark 1. (1) No 0-weight annulus properly embedded in $\hat{M}_i$ can be ∂-parallel to an annulus with positive weight in S.

(2) Any non-∂-parallel 0-weight annulus properly embedded in $\hat{M}_i$ has inessential ∂-curves in S and together with two disks in S bounds a 3-cell in $\hat{M}_i$.

Since any trapezoidal cycle together with two disks in S bounds a 3-cell by Lemma 2, we can define $P(\hat{M}_i)$ by adding to $P(C_i)$ these 3-cells enclosed by trapezoidal cycles of $P(C_i)$. Note that $P(\hat{M}_i)$ is obtained by gluing a collection of project regions of the form $D \times [0, 1]$, where $D \times 0$ and $D \times 1$ are disks in S, to $P(C_i)$, and so the components of $P(\hat{M}_i)$ are simply connected.

The following lemma says that the components of $P(C_i)$ itself are simply connected with our hypotheses and so $P(C_i) = P(\hat{M}_i)$ and the canonical crushing f_i is a combinatorial crushing. This plays an important role in proving Theorem 1, and also in Theorem 5, which asserts the 1-efficiency of the induced ideal triangulation $\Im_i^*$ of $\mathring{M}_i$.

Lemma 3. *With the same hypotheses as Theorem 1, there is no cycle of truncated prisms in* C_i*, the induced cell decomposition of* $\hat{M}_i$*.*

Proof. We mostly follow the ideas of Theorem 5.5 in [JR1]. But in the details, the argument is simpler because of our strong hypotheses.

Let X be a cycle of truncated prisms of C_i. We can exclude the case that the cycle X is formed along a single edge, i.e., the hexagonal faces of all truncated prisms of X are glued along a single edge. For in that case, there is a properly embedded

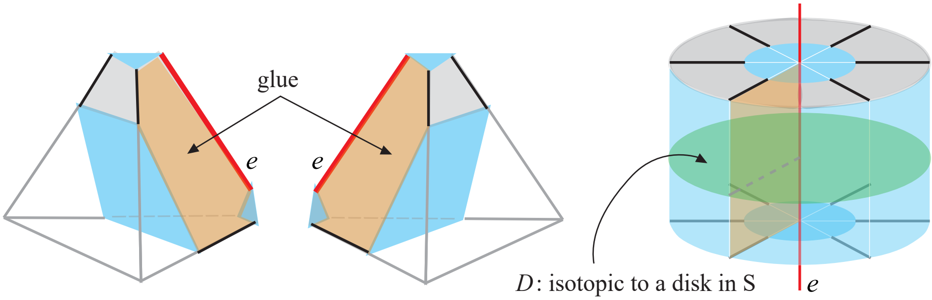

Figure 4. A cycle of truncated prisms glued along a single edge.

disk D in M_i which is isotopic to a disk in S (due to the incompressibility of S) and meeting edges of C_i in precisely one point (see Figure 4). By replacing the disk in S with D and normalizing, we obtain a new normal representative of a JSJ torus, contradicting the uniqueness assumption.

Hence we may assume that X is a cycle about more than one edge and so is a solid torus with boundary identifications along some trapezoids (if any). Figure 5 describes X without any identifications along trapezoids.

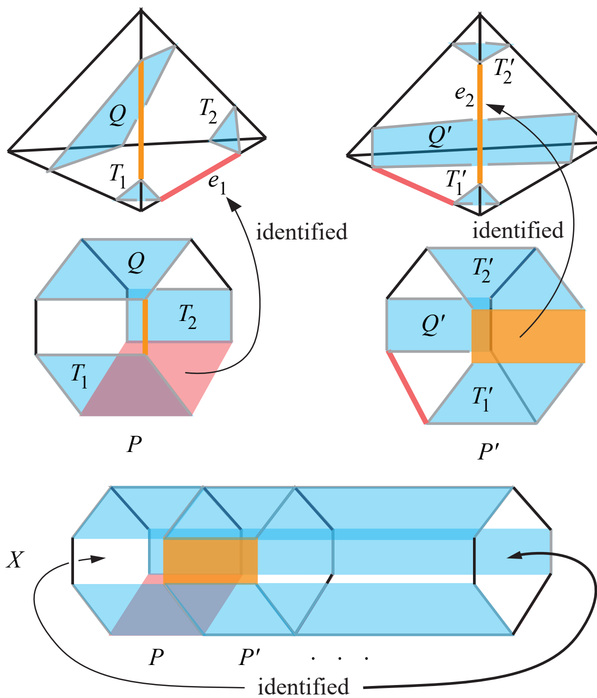

Figure 5. A cycle of truncated prisms without any trapezoidal identification.

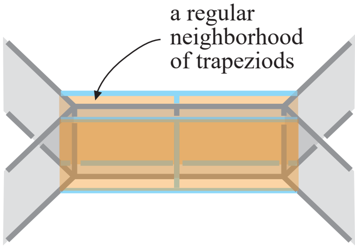

Figure 6. Truncating prisms along a trapezoidal identification.

The first case is of six strips in ∂X, three from S and three from trapezoidal cycles, and the second case is of two strips in ∂X, one from S and the other from trapezoidal cycles. If there are identifications along some trapezoids, we will truncate an open regular neighborhood of the trapezoidal cycles at X (see Figure 6) so that the new cycle of truncated prisms, again denoted by X, is a solid torus without any boundary identifications. In this case, the boundary of the solid torus is covered by either six strips (three from S and three 0-weight annuli parallel to the original trapezoidal cycles in ∂X) or two strips (one from S and one 0-weight annulus parallel to the original trapezoidal cycles in ∂X).

Case 1. Assume that the boundary of X has three strips, say A_i ($i = 1, 2, 3$), from S and three 0-weight strips, say T_i ($i = 1, 2, 3$), from trapezoidal cycles. Figure 7 describes this case.

By Lemma 2 and Remark 1, T_i with two disks D_i and D_i' in S bounds a 3-cell Δ_i, and each boundary curve of T_i (and so of A_i) bounds a disk in S. Fix an annulus A_i, say A_1, which has boundary curves, say C_1 and C_2, and let D_1 and D_2 be disks in S bounded by C_1 and C_2 respectively. Since D_1 and D_2 are glued along ∂A_1,

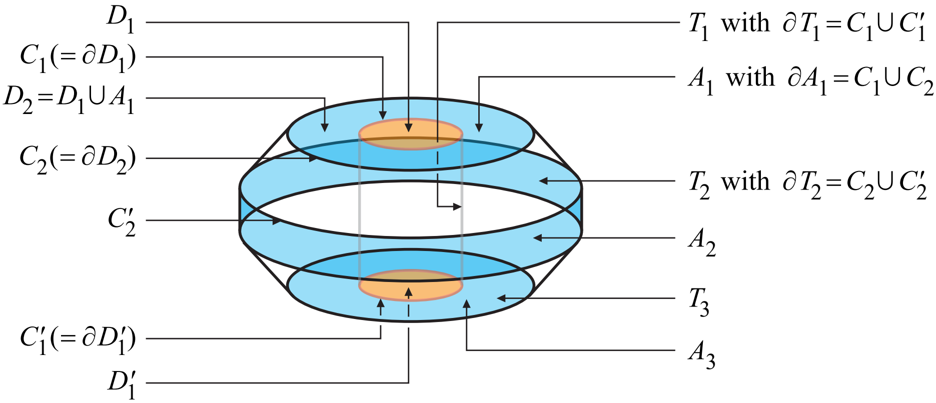

Figure 7. A cycle X of truncated prisms with six strips in its boundary.

these are all contained in the same component of S, and so either $D_2 = D_1 \cup A_1$ or $D_1 = D_2 \cup A_1$, say $D_2 = D_1 \cup A_1$ which means that $A_1 \subset D_2$. (Otherwise, $D_1 \cup D_2 \cup A_1$ is a 2-sphere.)

On the other hand, two 0-weight annuli, say T_1 and T_2, from trapezoidal cycles are adjacent to A_1 along the boundary curves C_1 and C_2. Let C_1' and C_2' be the remaining boundary curves of T_1 and T_2. By Remark 1, C_1' and C_2' also bound disks in S, say D_1' and D_2', and $T_1 \cup D_1 \cup D_1'$ and $T_2 \cup D_2 \cup D_2'$ bound 3-cells Δ_1 and Δ_2. Then the entirety of the cycle X is contained in Δ_2 and so in $P(\hat{M}_i)$ since $A_1 \subset D_2 \subset \Delta_2$ and so truncated prisms along A_1 are contained in Δ_2. Furthermore $\Delta_1 \subset \Delta_2$ (because $\partial \Delta_1 \subset \Delta_2$) and $A_2, A_3 \subset D_2'$ (because $\partial \Delta_2 = D_2 \cup T_2 \cup D_2'$ and $A_2, A_3 \subset \partial \Delta_2$).

Therefore, the remaining trapezoidal cycle T_3 adjacent to A_2 and A_3 must be parallel to an annulus in D_2'. This contradicts Remark 1(1).

Case 2. Assume that the boundary of the cycle X has two strips: one from S, say A, and the other from a trapezoidal cycle, say T. Let C_1 and C_2 be the curves in ∂X bounding A and T. Since T is a 0-weight annulus properly embedded in $\hat{M}_i$, by Remark 1(2), T with two disks D_1 and D_2 in S bounds a 3-cell Δ, where $\partial D_1 = C_1$ and $\partial D_2 = C_2$. Then $A \cup D_1 \cup D_2$ is a 2-sphere in S, which gives a contradiction. □

Lemma 4. *With the same hypotheses as Theorem 1, there are no truncated prisms or truncated tetrahedra in $P(\hat{M}_i)$, i.e., $P(\hat{M}_i) = P(C_i)$ and so the components of $P(C_i)$ are simply connected.*

Proof. Suppose that there is a truncated prism or truncated tetrahedron Δ in $P(\hat{M}_i)$. The closure of each component of $P(\hat{M}_i) \setminus P(C_i)$ is a product $D \times [0, 1]$, where $D \times 0$ and $D \times 1$ are disks in S. Since Δ is contained in such a closure of a component of $P(\hat{M}_i) \setminus P(C_i)$, at least two normal disks in $\partial \Delta$ belong to the same $D \times \epsilon$ ($\epsilon =$ either 0 or 1) and so the vertical arc α connecting these two normal disks is isotopic to an arc β of S along a disk in $D \times [0, 1] \subset P(\hat{M}_i)$ (see Figure 8).

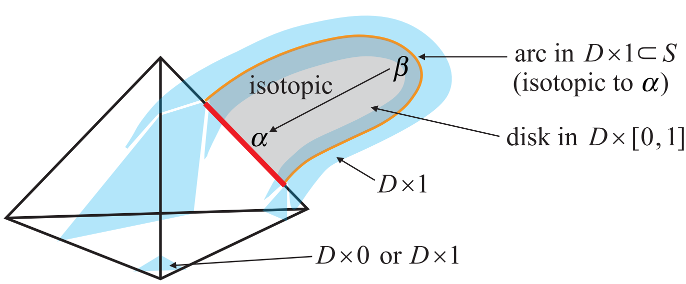

Figure 8. A vertical edge isotopic to an arc in S.

Now we can isotope S so that it doesn't meet the vertical edge and obtain a new normal representative of S which contradicts the uniqueness of normal representatives of each JSJ torus. (The new normal representation here may not be least-weight.)

This completes the claim that $P(\hat{M}_i)$ has no truncated prisms or tetrahedra and so $P(\hat{M}_i) = P(C_i)$. Thus the components of $P(C_i)$ are simply connected. □

Remark 2. From the arguments of the above lemmas, we can say that Lemma 2 (as already observed) and Lemma 3 hold even if the "unique normal representation of each JSJ torus" hypothesis is replaced by "unique least-weight normal representation". On the other hand, the proof of Lemma 4 required the assumption that there is a unique normal representation of each JSJ torus.

Now we are ready to prove that $\overline{M}_i^*$ is homeomorphic to $\mathring{M}_i$ and that $\Im_i^*$ is an ideal triangulation of $\mathring{M}_i$. In the general case of $P(\hat{M}_i) \neq P(C_i)$, an additional crushing is required and the 3 cells enclosed by the trapezoidal cycles of $P(\hat{M}_i)$ are further crushed to the edges. This modified crushing $\bar{f}_i : \hat{M}_i \to \overline{M}_i$ will be a combinatorial crushing. But in our situation described in Theorem 1, since the components of $P(C_i)$ are simply connected, no further crushing to edges is required and the canonical crushing will be a combinatorial crushing.

The following theorem from [JR1] provides sufficient conditions for $P(\hat{M}_i)$ to ensure that the manifold $\overline{M}_i^*$ obtained by a combinatorial crushing map is homeomorphic to $\mathring{M}_i$ and that the induced cell-structure $\Im_i^*$ is an ideal triangulation of $\mathring{M}_i$. In our case of Theorem 1, it is enough to show that $P(\hat{M}_i)$ satisfies the three conditions given by the theorem.

[JR1, **Theorem 4.1**]. *Suppose $\Im$ is a triangulation of a closed, orientable 3-manifold or an ideal triangulation of the interior of a compact, orientable 3-manifold M. Suppose S is a normal surface embedded in M, X is the closure of a component of the complement of S and X does not contain any vertices of $\Im$. Let $P(X)$ be the induced product region for X. Suppose the following conditions are met*:

- *$X \neq P(X)$.*
- *$P(X)$ is a trivial product region for X, i.e., it has simply connected components.*
- *There are no cycles of truncated prisms in X that are not in $P(X)$.*

Then the triangulation $\Im$ can be crushed along S into an ideal triangulation $\Im^*$ of $\mathring{X}$.

Proof of Theorem 1. We will show that the three properties described in Theorem 4.1 of [JR1] hold for $X = \hat{M}_i$.

Since $P(\hat{M}_i) = P(C_i)$ and $P(C_i)$ itself has simply connected components, we can work with the canonical crushing map $f_i : \hat{M}_i \to \overline{M}_i$, which crushes each cell of C_i, as shown in Figure 2. Here C_i is the cut-open cell decomposition of $\hat{M}_i$ induced from $\Im$.

- If $\hat{M}_i = P(\hat{M}_i)$, by Lemma 4, $\hat{M}_i$ is simply connected. But this contradicts that M_i is a JSJ component.
- Since $P(C_i)$ has simply connected components, $P(\hat{M}_i)$ $(= P(C_i))$ is a trivial product region.
- There are no cycles of truncated prisms outside $P(\hat{M}_i)$ by Lemma 3. □

Remark 3. (1) In [JR1], the induced product region $P(M_i)$ is defined in a different way; each component $D \times [0, 1]$ of $P(M_i)$ does not have to be simply connected, but the inclusion homomorphism $\pi_1(D \times \epsilon)$ into $\pi_1(S)$ must be injective for $\epsilon = 0, 1$. But in our case, we are able to satisfy the [JR1] requirements above and deduce Theorem 1. Therefore, Theorem 4.1 of [JR1] is also valid in our case.

(2) In [JR2] and [BJR], any crushing satisfying the three conditions above is called a "*combinatorial crushing*". In our case, the canonical crushing f_i satisfies those conditions, so it is also a combinatorial crushing in the sense of [JR2] and [BJR].

Now we are ready to prove that the constructed ideal triangulation $\Im_i^*$ of $\overline{M}_i^*$ is 1-efficient.

Theorem 5. *Let M be compact, irreducible, and P^2-irreducible with essential tori boundary components. Assume that the JSJ components M_i of M are hyperbolic for $1 \le i \le k$. If the ideal triangulation $\Im$ of $\mathring{M}$ is T-efficient, then for each JSJ component M_i, $1 \le i \le k$, the ideal triangulation $\Im_i^*$ of $\mathring{M}_i$ induced by the canonical crushing $f_i : \hat{M}_i \to \overline{M}_i$ is 1-efficient.*

Proof. By Theorem 1, the cell decomposition $\Im_i^*$ induced by the canonical crushing $f_i : \hat{M}_i \to \overline{M}_i$ is an ideal triangulation of $\mathring{M}_i$. To prove the 1-efficiency of $(\mathring{M}_i, \Im_i^*)$ (which is hyperbolic), we need to show that there is no normal 2-sphere or non-vertex-linking normal torus in $(\overline{M}_i^*, \Im_i^*)$. Suppose that there is one, denoted by T^*. It is enough to show that the inverse image of T^* by f_i, denoted by T, is a normal surface homeomorphic to T^* in (M_i, C_i) and so in $(\mathring{M}, \Im)$.

(i) If T is a 2-sphere then 0-efficiency is contradicted.

(ii) If T is a torus then, by uniqueness of normal tori, T must be vertex-linking in $\Im$ or boundary-linking normally parallel to a boundary component of M_i that crushes onto a vertex-linking normal torus in $(\overline{M}_i^*, \Im_i^*)$. This contradicts that T^* is not vertex-linking.

We will follow the proof ideas of Lemma 3.4 and Theorem 3.5 in [BJR]. (In [BJR], the cell decomposition of $\hat{M}_i$ must be an inflation of $\Im_i^*$ which is a minimal-vertex triangulation of $\hat{M}_i$ (in the sense of [BJR]) crushing to $\Im_i^*$. So direct application of the proof technique is problematic. However, the proof ideas are valid in our case using the cell decomposition C_i of $\hat{M}_i$.)

Note that the canonical crushing $f_i : \hat{M}_i \to \overline{M}_i$ is a combinatorial crushing and T crushes onto T^* by f_i. Since f_i is a combinatorial crushing and $P(M_i) = P(C_i)$, there is a 1-1 correspondence between the tetrahedra of $\Im_i^*$ and the truncated tetrahedra of C_i, and a 1-1 correspondence between the normal disks in a tetrahedron of $\Im_i^*$ and the normal disks in the corresponding truncated tetrahedron of C_i. Therefore, when comparing T and T^*, we need only consider the inverse image of T^* intersecting with 2-simplices or 1-simplices of $\Im_i^*$.

We first show that T and T^* are homeomorphic. In the proof of [BJR, Lemma 3.4], which uses the approximation theorem in [Sie], it is only necessary to prove that the canonical crushing f_i gives a proper cell-like map from T to T^*, i.e., for each point y of T^* the inverse image $f_i^{-1}(y)$ is compact and contractible. Let y be a point of T^* contained in a tetrahedron Δ of $\Im_i^*$. If y is placed in the interior of Δ, the inverse image $f_i^{-1}(y)$ is a point in a truncated tetrahedron of C_i. Assume that y is in the interior of a face of Δ. The inverse image of a face of a tetrahedron via f_i is either a face between two truncated tetrahedra or a chain of truncated prisms that is not a cycle. So the inverse image $f_i^{-1}(y)$ in this case is either a point or a long arc inside the chain of truncated prisms. Finally, assume that y is in an edge of Δ. The inverse image of an edge of $\Im_i^*$ by f_i is a product component $D \times [0, 1]$ of $P(C_i)$, so the inverse image $f_i^{-1}(y)$ is a horizontal cross section $D \times t$ which is simply connected. Therefore, $f_i^{-1}(y)$ is contractible for all cases and by the approximation theorem in [Sie], T and T^* are homeomorphic. (For the crushing f_i, see Figures 2 and 3.)

Next, we will prove that T, the inverse image of T^* by f_i, is a normal surface. We will prove that the inverse image of each normal disk, which is a quadrilateral or a triangle, must be in normal form, i.e., a union of normal disks. Let A^* be a normal disk of T^*. Since there is a 1-1 correspondence between the tetrahedra of $\Im_i^*$ and the truncated tetrahedra of C_i, there must be a unique normal disk A, which crushes onto A^*, in a truncated tetrahedron of C_i.

We need now consider inverse images of edges and vertices of A^*. Let α be an edge of A^*; then α is an arc inside a face of a tetrahedron in $\Im_i^*$. As we described above, the inverse image of a face of a tetrahedron via f_i is either a face between two truncated tetrahedra or a chain of truncated prisms that is not a cycle. Hence the inverse image of α must be an arc in a face of truncated tetrahedra (so it is an edge of A in this case) or a strip inside the chain and further normally parallel to a horizontal boundary strip of the chain (so in this case it is a union of normal disks parallel to normal disks in the boundary of truncated prisms). Now we will look at the inverse image of a vertex v of A^*. Since v is a point in an edge of $\Im_i^*$, the inverse image of v is a horizontal cross section of a product component $D \times [0, 1]$ of $P(C_i)$, so it is either a vertex of A or a disk that is a union of normal disks parallel to those in D. This completes the proof. □

Remark 4. From the reasoning in this proof, we can derive the following results.

(1) The inverse image under the combinatorial crushing f_i of a normal surface T in $(\overline{M}_i^*, \Im_i^*)$, or equivalently $(\mathring{M}_i, \Im_i^*)$, is a normal surface T^* in $(\hat{M}_i, C_i)$, and further T and T^* are homeomorphic.

(2) The inverse image under the combinatorial crushing f_i of a spun normal surface in $(\overline{M}_i^*, \Im_i^*)$ is a surface that spins in normal form along some JSJ tori and some boundary tori of M. We call such a surface a *pseudo-spun normal surface*. Such a surface can spin along nonperipheral essential normal tori embedded in the interior of M, and thus may contain infinitely many quadrilateral normal disks. However there are only finitely many quadrilateral normal disks which are not contained in the infinite annuli spinning around nonperipheral essential normal tori.

(3) The combinatorial crushing $f_i : \hat{M}_i \to \overline{M}_i$ is a proper cell-like map, i.e., for each point y of $\overline{M}_i$, the inverse image $f_i^{-1}(y)$ is compact and contractible.

3. Review of the construction of a spun normal surface in the case of 1-efficient ideal triangulations

The way that normal surfaces with nonnegative Euler characteristic can be an obstruction in normal spinning of a properly embedded incompressible and ∂-incompressible surfaces has been described in [KR2]. We will briefly review the construction of a spun normal surface in the absence of normal surfaces with nonnegative Euler characteristic.

Definition 4. Let M be a 3-manifold with an ideal triangulation $\Im$ and let $\hat{M}$ be the compact 3-manifold obtained by deleting regular neighborhoods of ideal vertices of $\Im$ from M with the induced truncated triangulation $\hat{\Im}$.

(1) A normal surface embedded in M is a closed surface that intersects each tetrahedron in finitely many elementary disks.

(2) A spun normal surface S in M is an embedded surface formed from elementary disks in the tetrahedra satisfying the following conditions:

- S consists of finitely many quadrilaterals and infinitely many triangular disks.
- S has a finite collection of disjoint infinite cylinders that spiral around ideal vertices.
- Each such cylinder consists of an infinite subset of the triangular disks in S.
- The remainder of S outside these cylinders is compact and consists of finitely many quadrilateral and triangular disks.

(3) Suppose that F is a properly embedded incompressible and ∂-incompressible surface in $\hat{M}$. We say that F spin normalizes (or normally spins) if there is a spun normal surface S in M such that $S \cap \hat{M}$ is isotopic to F.

[KR2, **Theorem 8]**. *Let M be an anannular, atoroidal, irreducible and P^2-irreducible 3-manifold with tori boundary components and $\Im$ be a 1-efficient ideal triangulation of M. If F is a properly embedded, incompressible and ∂-incompressible 2-sided surface in M which is not a fiber, then F can be spin normalized in $(M, \Im)$ with 2^r choices of spinning direction, where r is the number of boundary components of M containing a curve of ∂F.*

According to this theorem, if the manifold M we are considering is equipped with a 1-efficient ideal triangulation then we can always normally spin a properly embedded 2-sided essential surface F if F is not a fiber. To prove this, we truncated M at an open regular neighborhood of the ideal vertices, to work on a compact manifold with a truncated triangulation, and applied normal surface theory adapted to the truncated triangulation.

Here is a rough sketch of the proof of [KR2, Theorem 8]. Each step will be discussed in some detail later.

Step 1. Find a sequence $< F_k >$ of topological spinnings of F, which contains an infinite number of isotopy classes with a fixed boundary.

Step 2. Normalize each F_k to obtain a sequence $\langle \hat{F}_k \rangle$ of normal surfaces, which also has an infinite number of normal isotopy classes.

Step 3. Find a fixed common core surface S of an infinite number of normal surfaces $\hat{F}_k$, which means that the $\hat{F}_k$ differ only by annuli (with only triangular normal disks) parallel to boundary tori or Klein bottles of M.

Remark 5. (1) A sequence of "topological spinnings" in Step 1 requires that there are infinitely many different isotopy classes in the sequence. These are compact properly embedded surfaces with the same boundary curves and isotopies must keep the boundaries fixed.

(2) The statement of Step 3 means that F is spin normalized. The reason is that if a surface $\hat{F}_k$ spins once along a boundary torus or Klein bottle then we can attach an infinite normal annulus along the surface without introducing any new folds, and obtain a spun normal surface where the core surface S is isotopic to F.

(3) Step 3 is the only place that requires 1-efficiency in the proof of the theorem.

(4) All steps are performed on the truncated manifold and triangulation $(\hat{M}, \hat{\Im})$.

To apply this process to the proof of our main theorem in the next section, we need additional information on how to construct the surfaces mentioned in each step. Without loss of generality, we can assume that F is properly embedded in the truncated manifold $(\hat{M}, \hat{\Im})$ and its boundary curves are normal curves on the boundary of $\hat{M}$ and have the least intersection with the 1-simplices of $\partial \hat{M}$ in their isotopy classes.

<u>Step</u> 1: *Constructing F_k, the k-th spinning of F.* The *k-th spinning F_k* of F is a surface obtained by spinning F k times for each boundary curve, along the boundary tori or Klein bottles of M. That is, we attach a boundary parallel annulus of length k along each boundary curve of F so that the boundary slope of F_k is exactly the same as that of F.

<u>Step</u> 2: *Normalizing F_k with the boundary fixed.* Here, we perform the normalizing process on the truncated triangulation of M. Since F, and so F_k, are essential, we can always normalize F_k keeping the boundary fixed, so that the boundary curve of the resulting normal surface $\hat{F}_k$ is identical to the boundary curve of F and $\hat{F}_k$ is in normal form.

<u>Step</u> 3: *Finding a core surface S.* There is a finite number of fundamental surfaces from which any closed normal surface can be represented by geometric sums. Since the original ideal triangulation is 1-efficient, there are no closed normal surfaces with nonnegative Euler characteristic, so all fundamental surfaces except the peripheral tori or Klein bottles have negative Euler characteristic. Therefore, for a fixed Euler characteristic $\chi(F) = \chi(\hat{F}_k)$, only a finite number of possibilities for A and C arise when we write $\hat{F}_k = A + B + C$, where A is a proper normal surface with the same boundary as $\hat{F}_k$, B is a multiple of a peripheral normal surface and C is a closed non-peripheral normal surface. (Here we assume that A cannot be written as a sum $A = A' + B$.) This allows us to choose an infinite number of $\hat{F}_k$'s with a fixed core, say S:

$$\hat{F}_k = S + \sum_{j=1}^{r} l_{k_i,j} T_j$$

where the $l_{k,j}$ are nonnegative integers, and the T_j are boundary components of $\hat{M}$ containing some curves of ∂F. Furthermore, the boundary slope of S is the same as the boundary slope of F.

4. Spinning for suitable ideal triangulations in the toroidal case

Now we prove the main theorem, which guarantees the existence of a spun normal surface for the case with only hyperbolic JSJ components. As mentioned in the introduction, we first crush the JSJ tori to points by an appropriate cell-like crushing map, yielding a 1-efficient ideal triangulation of each JSJ component, and then apply the result of the 1-efficient case in [KR2].

Theorem 6. *Suppose that M is compact, irreducible and P^2-irreducible and has essential tori boundary components. Assume that each of the JSJ components M_i of M is hyperbolic, for $1 \leq i \leq k$. Let Σ be any properly embedded 2-sided essential surface in M with the property that no subsurface of Σ is a fiber of a bundle structure for one of the JSJ components.*

Assume that M has a $\mathcal{T}$-efficient ideal triangulation $\mathcal{T}$. Then Σ can be spin normalized with boundary curves spinning in all possible combinations of directions.

Proof. Let $\mathcal{C}$ be the collection of all JSJ tori which are the only normal surfaces that have nonnegative Euler characteristic, except for boundary parallel normal tori, and let S be the union of tori in the collection $\mathcal{C}$. Since S is a collection of closed 2-sided essential normal surfaces, crushing each of these tori to a new ideal vertex produces a 1-efficient ideal triangulation for each JSJ piece M_i of M, for $1 \leq i \leq k$, by Theorem 5 in Section 2.

Now, given an essential proper surface Σ embedded in M, we can isotope Σ until it meets each of the tori of $\mathcal{C}$ in a minimal set of essential loops. (We may assume that Σ meets all JSJ tori in $\mathcal{C}$. The reason is that if there are JSJ tori and JSJ components that Σ does not meet, we can cut and remove those JSJ components along the JSJ tori and then work on the remaining M.) In particular, it then follows that $\Sigma \cap M_i = \Sigma_i$ is a properly embedded essential surface in M_i. We can further isotope Σ to have minimal intersection with the edges of tori in $\mathcal{C}$, which is required in the spin normalizing process of each piece Σ_i. This immediately implies that Σ intersects tori of $\mathcal{C}$ in normal curves. So when we crush all the tori of $\mathcal{C}$ to ideal vertices, Σ_i becomes an essential surface Σ_i^* with the same boundary slope as Σ_i at each vertex-linking torus of $\Im_i^*$ and properly embedded in $(\overline{M}_i^*, \Im_i^*)$, where $\overline{M}_i^*$ is obtained by deleting vertices from the result of the canonical crushing $f_i : \hat{M}_i \to \overline{M}_i$ and $\Im_i^*$ is the induced ideal triangulation.

Since, by Theorem 5, $\Im_i^*$ is 1-efficient, Σ_i^* can be spin normalized by [KR2] with any combination of directions of spinning for the boundary curves. We will use these spun normal surfaces, each representing Σ_i^* in $\overline{M}_i^*$ $(1 \leq i \leq k)$, as barriers when normalizing a spinning of Σ, and do this in a covering space of M. Here is some notation for the construction.

- Σ^s (resp. Σ_i^s): the spinning of Σ (resp. Σ_i) obtained by attaching infinite annuli parallel to the boundary tori of M (resp. M_i) along $\partial\Sigma$ (resp. $\partial\Sigma_i$).
- Σ^{ns} (resp. Σ_i^{ns}): the normal spinning of Σ (resp. Σ_i) obtained by normalizing Σ^s (resp. Σ_i^s) in $(\mathring{M}, \Im^*)$ (resp. $(\overline{M}_i^*, \Im_i^*)$). (Here, Σ_i^{ns} is a pseudo-spun normal surface in M.)
- $\tilde{M}$: the covering space of M corresponding to $\pi_1(\Sigma)$.
- $\tilde{M}_i$: the lift of M_i to $\tilde{M}$, which is the covering space of M_i corresponding to $\pi_1(\Sigma_i)$.
- $\tilde{\Sigma}$: the lift of Σ to $\tilde{M}$.
- $\tilde{\Sigma}'$ and $\tilde{\Sigma}''$: parallel copies of $\tilde{\Sigma}$ placed on opposite sides of $\tilde{\Sigma}$.

- $\tilde{\Sigma}_i = \tilde{\Sigma} \cap \tilde{M}_i$: the lift of Σ_i to $\tilde{M}_i$.
- $\tilde{\Sigma}^s$ (resp. $\tilde{\Sigma}_i^s$): the spinning of $\tilde{\Sigma}$ (resp. $\tilde{\Sigma}_i$) obtained by attaching infinite annuli parallel to the boundary infinite annuli of $\tilde{M}$ (resp. $\tilde{M}_i$) along $\partial\tilde{\Sigma}$ (resp. $\partial\tilde{\Sigma}_i$).

Let $\tilde{M}$ be the covering space of M corresponding to $\pi_1(\Sigma)$ and $\tilde{\Sigma}$ the lift of Σ to $\tilde{M}$. It is obvious that the covering space $\tilde{M}_i$ of M_i corresponding to $\pi_1(\Sigma_i)$ is embedded in $\tilde{M}$, and each piece $\tilde{\Sigma}_i = \tilde{\Sigma} \cap \tilde{M}_i$ is an essential surface properly embedded in $\tilde{M}_i$. Since Σ is 2-sided, we can take a parallel copy of $\tilde{\Sigma}$ on each side and denote the copies by $\tilde{\Sigma}'$ and $\tilde{\Sigma}''$. All three copies are parallel, i.e., normally isotopic in $\tilde{M}$ with the induced ideal triangulation. (Here, the ideal triangulation of $\tilde{M}$ contains an infinite number of ideal tetrahedra.) Let $\tilde{\Sigma}_i' = \tilde{\Sigma}' \cap \tilde{M}_i$ and $\tilde{\Sigma}_i'' = \tilde{\Sigma}'' \cap \tilde{M}_i$. Now we will spin all these $\tilde{\Sigma}_i'$ and $\tilde{\Sigma}_i''$ surfaces by attaching infinite annuli parallel to $\partial\tilde{M}_i$ along the boundary curves of $\tilde{\Sigma}_i'$ and $\tilde{\Sigma}_i''$ (see Figure 9).

The spinning direction for each boundary curve will be chosen so that the spinning does not meet $\tilde{\Sigma}$. Note that this choice of spinning direction causes $\tilde{\Sigma}_i'$ and $\tilde{\Sigma}_i''$ to spin in opposite directions in $\tilde{M}_i$ ($1 \le i \le k$). Now using the methods of [KR2], we claim that the spinnings of $\tilde{\Sigma}_i'$ and $\tilde{\Sigma}_i''$ can be pseudo-spin normalized as follows: Let Σ_i' and Σ_i'' be the image of $\tilde{\Sigma}_i'$ and $\tilde{\Sigma}_i''$ into M, and $(\Sigma_i')^*$ and $(\Sigma_i'')^*$ be the corresponding essential surfaces properly embedded in $\overline{M}_i^*$. As mentioned above, the induced cell-structure $\Im_i^*$ of $\overline{M}_i^*$ is a 1-efficient ideal triangulation by Theorem 5, so any spinnings of $(\Sigma_i')^*$ and $(\Sigma_i'')^*$ can be spin normalized by [KR2] and the inverse images of the resulting pseudo-spun normal surfaces are also pseudo-spun normal surfaces (by Remark 4) in M_i. Finally the latter surfaces can be lifted to pseudo-spun normal surfaces, say $(\tilde{\Sigma}_i')^{ns}$ and $(\tilde{\Sigma}_i'')^{ns}$, representing $\tilde{\Sigma}_i'$ and $\tilde{\Sigma}_i''$ in $\tilde{M}_i$. This establishes the claim that the spinnings of $\tilde{\Sigma}_i'$ and $\tilde{\Sigma}_i''$ can be pseudo-spin normalized.

Now we will use all the surfaces $(\tilde{\Sigma}_i')^{ns}$ and $(\tilde{\Sigma}_i'')^{ns}$, which are lifts of spun normal surfaces representing Σ_i' and Σ_i'', as a barrier when we normalize a spinning of $\tilde{\Sigma}$ in $\tilde{M}$ (see Figure 10).

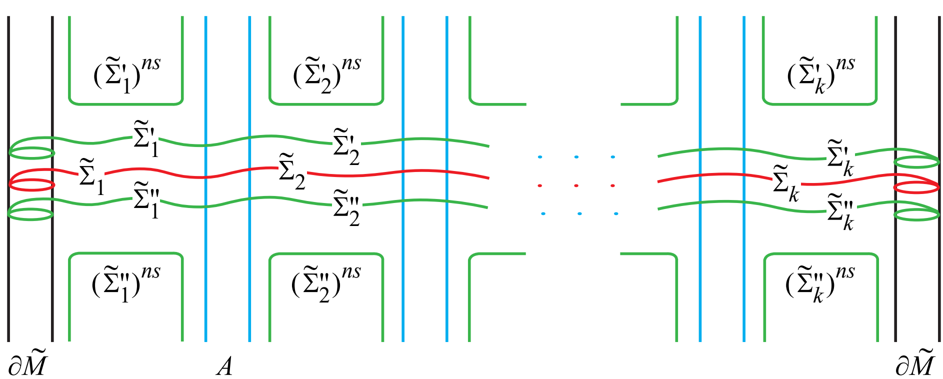

Figure 9. Both way spinnings of $\tilde{\Sigma}_i$ in $\tilde{M}$.

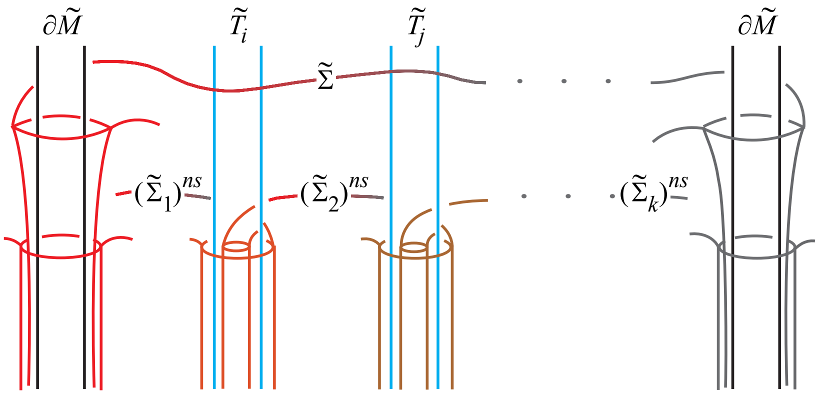

Figure 10. A barrier for spun normalizing $\tilde{\Sigma}$.

Note that the pseudo-spun normal surfaces $(\tilde{\Sigma}_i')^{ns}$ and $(\tilde{\Sigma}_i'')^{ns}$ are inverse images of spun normal surfaces obtained by normalizing the corresponding spinnings of $\tilde{\Sigma}_i'$ and $\tilde{\Sigma}_i''$, and the direction of spinnings of $\tilde{\Sigma}_i'$ and $\tilde{\Sigma}_i''$ were chosen to not meet $\tilde{\Sigma}$. This implies that $\tilde{\Sigma}$ does not meet any of $(\tilde{\Sigma}_i')^{ns}$ and $(\tilde{\Sigma}_i'')^{ns}$ for $1 \le i \le k$, and any combination of spinning directions for the boundary curves of $\tilde{\Sigma}$ can be chosen so that the spinnings with arbitrarily long annuli attached are all disjoint from the barriers $(\tilde{\Sigma}_i')^{ns}$ and $(\tilde{\Sigma}_i'')^{ns}$ for $1 \le i \le k$.

The normalizing process does not introduce any new intersections with the 1-skeleton of the triangulation; it just removes them by isotopies. With this observation, the union of pseudo-spun normal surfaces $(\tilde{\Sigma}_i')^{ns}$ and $(\tilde{\Sigma}_i'')^{ns}$ acts as a *barrier*, so that the normalization of a spinning of $\tilde{\Sigma}$ in the complement of this union does not touch the barrier (see Section 3 of [JR1] for details on barriers) and $\tilde{\Sigma}$ must normally spin in $\tilde{M}$ exactly as in the argument in [KR2]. What we need to additionally check here, unlike [KR2], is that the normal spinning of $\tilde{\Sigma}$ does not spin along the infinite annuli lifted from JSJ tori of M. Suppose that $\tilde{\Sigma}$ normally spins along the infinite annulus A lifted from a JSJ torus T bounding two JSJ components M_i and M_j. Here, there are two possible ways of spinning $\tilde{\Sigma}$ along A; spinning between $(\tilde{\Sigma}_i')^{ns}$ and $(\tilde{\Sigma}_j')^{ns}$, or spinning between $(\tilde{\Sigma}_i'')^{ns}$ and $(\tilde{\Sigma}_j'')^{ns}$ (see Figure 11).

However, we obtain the normal spinning $(\tilde{\Sigma})^{ns}$ of $\tilde{\Sigma}$ by normalizing a spinning $(\tilde{\Sigma})^{s}$, which is obtained by attaching infinitely long annuli along $\partial\tilde{\Sigma}$ lying on the annuli of $\partial\tilde{M}$ (note that none of these were lifted from JSJ tori), without touching the barriers. Therefore, neither $(\tilde{\Sigma})^{s}$ nor $(\tilde{\Sigma})^{ns}$ can meet the 1-skeleton located between the long annuli of $(\tilde{\Sigma}_i')^{ns}$ (resp. $(\tilde{\Sigma}_i'')^{ns}$) and $(\tilde{\Sigma}_j')^{ns}$ (resp. $(\tilde{\Sigma}_j'')^{ns}$). This establishes the claim that the spun normal surface $(\tilde{\Sigma})^{ns}$ does not spin along any lifts of JSJ tori of M.

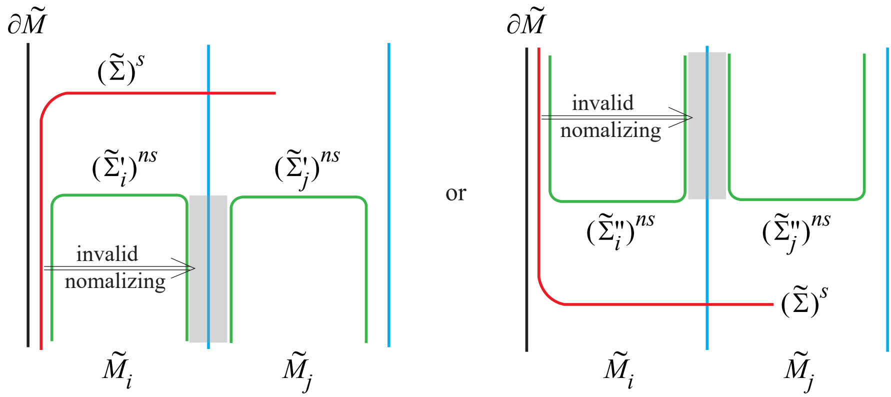

Figure 11. Spinnings of $\tilde{\Sigma}$ along a lift of a JSJ torus in $\tilde{M}$.

Now, $(\tilde{\Sigma})^{ns}$, which normally spins along the boundary annuli of $\tilde{M}$, can be projected to M to obtain a normal spinning of Σ in M. This completes the proof. □

5. Constructing suitable triangulations

Among the assumptions of Theorem 6, the condition of T-efficiency of the triangulation, that the boundary of the JSJ components are the only normal tori, seems rather strong. However, in this section we show that such an ideal triangulation can always be constructed if the manifold under consideration admits a taut ideal triangulation. Moreover, Lackenby showed in [Lac] that taut ideal triangulations are very common; it is sufficient to assume the manifold is irreducible, P^2-irreducible and anannular to achieve a taut triangulation. The assumption that all JSJ components are hyperbolic implies these conditions and hence the existence of a T-efficient taut triangulation.

Lemma 7. *Suppose that M is compact, irreducible and P^2-irreducible and has essential tori and Klein bottle boundary components. Further suppose that M is toroidal and admits a taut ideal triangulation $\mathcal{T}$. Then every essential torus or Klein bottle admits a unique normal representative in its isotopy class.*

Proof. The argument follows by the same method as in [DGR]. We first consider the case of 2-sided surfaces. Namely, if there were two normal essential 2-sided tori or 2-sided Klein bottles isotopic to each other [DGR, Theorem 5.5], then we can find two disjoint such normal surfaces bounding a product region [Wal, Lemma 5.3]. So the surfaces are topologically but not normally parallel. However, by standard sweepout theory from [Rub] and [Sto], there must be an almost normal torus or Klein bottle in this product region, and this contradicts the taut structure of $\mathcal{T}$. According to the Euler characteristic calculation using the angles induced by the taut structure (see Remark 6), almost normal surfaces with a nonnegative Euler characteristic are not allowed in such a structure.

To deal with the case of 1-sided surfaces, assume that is a normal essential 1-sided torus or Klein bottle. The boundary of a small regular neighborhood is then a 2-sided normal torus or Klein bottle. By the 2-sided case, this surface is unique in its isotopy class. but then it follows immediately that so is also the 1-sided surface. □

Remark 6. Let Σ be a normal or almost normal surface in M with an ideal triangulation $\mathcal{T}$.

(1) By Gauss–Bonnet,

$$\begin{aligned} 2\pi\chi(\Sigma) = &\sum_T (\text{vertex angle sum of a triangle } T - \pi) \\ &+ \sum_Q (\text{vertex angle sum of a quadrilateral } Q - 2\pi) \\ &+ \sum_O (\text{vertex angle sum of an octagon } O - 6\pi). \end{aligned}$$

(2) In a taut structure, the vertex angle sum is π for a triangle, 0 or 2π for a quadrilateral, and 2π or 4π for an octagon. For details, see [KR1].

Theorem 8. *Suppose that M is compact, irreducible and P^2-irreducible and has essential tori boundary components. Further suppose that M has a JSJ decomposition with only hyperbolic pieces. Then M has a taut ideal triangulation $\mathcal{T}$ which is T-efficient. Consequently, any surface Σ in M satisfying the same conditions as Theorem 6 can be spin normalized with boundary curves spinning in all possible combinations of directions.*

Proof. By Theorem 2.6 of [KR1], any taut ideal triangulation is 0-efficient; that is, there are no embedded normal 2-spheres, projective planes, or Klein bottles, and moreover any normal torus or Klein bottle is essential. By Lemma 7, every essential torus or Klein bottle admits a unique normal representative in its isotopy class. Therefore, it suffices to establish the existence of a taut ideal triangulation.

According to Theorem 1 of [Lac], a compact, irreducible, P^2-irreducible, and anannular 3-manifold with incompressible torus boundary admits a taut ideal triangulation. The assumption that the 3-manifold has a JSJ decomposition with all components hyperbolic implies that the manifold is anannular. Indeed, suppose there exists an essential annulus or Möbius band B. Then B can be isotoped to intersect the JSJ tori in parallel essential curves. An innermost annulus between two such curves would lie in a single JSJ component and must be inessential, since all JSJ components are hyperbolic. Consequently, B can be isotoped to remove all intersections with the JSJ tori. This contradicts the assumption that B is essential, as it would then be properly embedded in a hyperbolic component. Hence, all such 3-manifolds admit taut ideal triangulations. This completes the proof. □

Example 1. Suppose that M is a punctured surface bundle over a circle. Assume that the fiber is a union of subsurfaces glued along their boundary curves. Finally suppose that the monodromy is a collection of pseudo-Anosov maps on these subsurfaces and the boundary curves of the subsurfaces are fixed under the monodromy. If we take a layered ideal triangulation of M, this clearly has a taut structure satisfying the conditions of Theorem 6 and the JSJ decomposition is given by the surface bundles over a circle with fibers given by the subsurface system. In general such bundles have many essential surfaces which do not have subsurfaces in the fiber bundle structure. These surfaces can all be spin normalized by Theorem 8.

Remark 7. In the proof of Theorem 8, it is sufficient to assume that all JSJ components of M adjacent to the boundary components of M are hyperbolic in order to conclude that M is anannular and hence admits a taut triangulation. Therefore, it may be possible to extend the main results of this paper to this more general setting. However, for such 3-manifolds, in addition to the tori defining the JSJ decomposition, further essential tori are allowed inside JSJ components that are not adjacent to the boundary of M. This must be taken into account in the construction, and as a consequence, applying the methods of [KR2] becomes more difficult.

References

[BJR] B. Bryant, W. Jaco, and J. H. Rubinstein, "Efficient triangulations and boundary slopes", *Topology Appl.* **297** (2021), art. id. 107689, 18 pp. MR

[DGG] T. Dimofte, D. Gaiotto, and S. Gukov, "3-manifolds and 3d indices", *Adv. Theor. Math. Phys.* **17**:5 (2013), 975–1076. MR

[DGR] N. M. Dunfield, S. Garoufalidis, and J. H. Rubinstein, "Counting essential surfaces in 3-manifolds", *Invent. Math.* **228**:2 (2022), 717–775. MR

[GHRH] S. Garoufalidis, C. D. Hodgson, N. R. Hoffman, and J. H. Rubinstein, "The 3D-index and normal surfaces", *Illinois J. Math.* **60**:1 (2016), 289–352. MR

[GHRS] S. Garoufalidis, C. D. Hodgson, J. H. Rubinstein, and H. Segerman, "1-efficient triangulations and the index of a cusped hyperbolic 3-manifold", *Geom. Topol.* **19**:5 (2015), 2619–2689. MR

[JR1] W. Jaco and J. H. Rubinstein, "0-efficient triangulations of 3-manifolds", *J. Differential Geom.* **65**:1 (2003), 61–168. MR

[JR2] W. Jaco and J. H. Rubinstein, "Inflations of ideal triangulations", *Adv. Math.* **267** (2014), 176–224. MR

[KR1] E. Kang and J. H. Rubinstein, "Ideal triangulations of 3-manifolds, II: Taut and angle structures", *Algebr. Geom. Topol.* **5** (2005), 1505–1533. MR

[KR2] E. Kang and J. H. Rubinstein, "Spun normal surfaces in 3-manifolds, I: 1-efficient triangulations", *Algebr. Geom. Topol.* **22**:8 (2022), 3533–3576. MR

[Lac] M. Lackenby, "Taut ideal triangulations of 3-manifolds", *Geom. Topol.* **4** (2000), 369–395. MR

[Rub] J. H. Rubinstein, "Polyhedral minimal surfaces, Heegaard splittings and decision problems for 3-dimensional manifolds", pp. 1–20 in *Geometric topology* (Athens, GA, 1993), vol. 1, AMS/IP Stud. Adv. Math. **2.1**, Amer. Math. Soc., 1997. MR

[Sie] L. C. Siebenmann, "Approximating cellular maps by homeomorphisms", *Topology* **11** (1972), 271–294. MR

[Sto] M. Stocking, "Almost normal surfaces in 3-manifolds", *Trans. Amer. Math. Soc.* **352**:1 (2000), 171–207. MR

[Wal] F. Waldhausen, "On irreducible 3-manifolds which are sufficiently large", *Ann. of Math.* (2) **87** (1968), 56–88. MR

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