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Andrés Larraín-Hubach, Yulong Li,
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RESOLVENT BOUNDS FOR REPULSIVE POTENTIALS

ANDRÉS LARRAÍN-HUBACH, YULONG LI,
JACOB SHAPIRO AND JOSEPH TILLER

We prove limiting absorption resolvent bounds for the semiclassical Schrödinger operator with a repulsive potential in dimensions $n \geq 3$, which may have a singularity at the origin. As an application, we obtain time decay for the weighted energy of the solution to the associated wave equation with a short range repulsive potential and compactly supported initial data.

1. Introduction and statement of results

We establish limiting absorption resolvent bounds for the semiclassical Schrödinger operator with a repulsive potential in dimensions three and higher. The one-dimensional case was studied in [ChDa21, Section 2]. As an application, we obtain time decay of a weighted energy for the solution to the associated wave equation with a short range repulsive potential and compactly supported initial data.

Let $\Delta := \sum_{j=1}^{n} \partial_j^2 \leq 0$ be the Laplacian on $\mathbb{R}^n$. We use $(r, \theta) = (|x|, x/|x|) \in (0, \infty) \times \mathbb{S}^{n-1}$ for polar coordinates on $\mathbb{R}^n \setminus \{0\}$. Throughout, we equip $\mathbb{S}^{n-1}$ with the standard Borel measure $d\theta$ such that the product measure $r^{n-1}dr \times d\theta$ gives Lebesgue measure on $(0, \infty) \times \mathbb{S}^{n-1}$. Put $B(0, r_0) := \{x \in \mathbb{R}^n : |x| < r_0\}$. For a function u defined on a subset of $\mathbb{R}^n$, we write $u(r, \theta) := u(r\theta)$, and denote the radial derivative by $u' := \partial_r u$. If $E \subseteq \mathbb{R}^n$ is a Borel set, $\mathbf{1}_E$ stands for its indicator function.

Our Schrödinger operator takes the form

$$P(h) := -h^2\Delta + V(x) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad x \in \mathbb{R}^n, \tag{1}$$

where $h > 0$ is the semiclassical parameter. The conditions we place on the Borel measurable potential $V : \mathbb{R}^n \to \mathbb{R}$ are as follows. We suppose $V = V^+ - V^-$ with $V^+ = \max(V, 0)$ and $V^- = -\min(V, 0)$; moreover

$$V^- \in L^\infty(\mathbb{R}^n), \tag{2}$$

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$$r^{\rho(n)} V(x) \text{ is bounded for } r \le 1, \tag{3}$$

$$V(x) \text{ is bounded for } r \ge 1, \tag{4}$$

$$\text{for each } \theta \in \mathbb{S}^{n-1},\ (0,\infty) \ni r \mapsto V(r,\theta) := V(r\theta) \text{ has bounded variation.} \tag{5}$$

Here, $\rho(n) > 0$ depends on the dimension n:

$$\rho(n) < \begin{cases} \frac{3}{2} & \text{if } n = 3, \\ 2 & \text{if } n \ge 4. \end{cases} \tag{6}$$

Recall that a function f of locally bounded variation on an interval $I \subseteq \mathbb{R}$ has distributional derivative equal to a locally finite signed Borel measure, which we denote by df. In particular

$$\int \varphi \, df = -\int f \varphi' dx, \quad \varphi \in C_0^\infty(I), \tag{7}$$

where the dx on the right side denotes Lebesgue measure on I; df further satisfies

$$\int_{(a,b]} df = f^R(b) - f^R(a), \tag{8}$$

for any interval $(a,b] \subseteq \mathring{I}$, where $f^R(x) := \lim_{\delta \to 0^+} f(x+\delta)$.

The last condition we impose on V is that there exists $C_V > 0$ such that for all $\theta \in \mathbb{S}^{n-1}$ and every bounded Borel set $E \subseteq (0,\infty)$,

$$\int_E dV(r,\theta) \le -C_V \int_E (r+1)^{-1} V^+(r,\theta) dr. \tag{9}$$

We emphasize that since the inequality (9) is one sided, each measure $dV(\cdot,\theta)$ is allowed to have negative point masses. Moreover, because only the positive part V^+ appears on the right side, the potential is allowed to approach a negative constant as $r \to \infty$. A prototype potential satisfying the above conditions is

$$V(r,\theta) = g(\theta)\big(\mathbf{1}_{B(0,1)} r^{-\rho(n)} - 2^{-1} \mathbf{1}_{\mathbb{R}^n \setminus B(0,1)} (r^{-\delta} - 2^{-1})\big)$$

for some $\delta > 0$ and $g \ge 0$ a bounded measurable function on $\mathbb{S}^{n-1}$. Note that for $V \in C^1(\mathbb{R}^n; [0,\infty))$, (9) implies each $V(\cdot,\theta)$ is repulsive in sense of classical mechanics, i.e., that $V(r,\theta) > 0$ implies $V'(r,\theta) < 0$. The local bound (3) allows for mild singularities at the origin, most notably the repulsive Coulomb behavior.

For a real-valued potential $V \in L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ with $p \ge 2$, $p > n/2$, $P(h)$ is self-adjoint if one takes the Sobolev space $H^2(\mathbb{R}^n)$ as the domain [Ne64, Theorem 8]. The conditions (2), (3), and (4) imply that $V \in L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ for some such p.

Our main results are the following weighted limiting absorption resolvent bounds for $P(h)$.

Theorem 1.1. *Suppose $n \ge 3$ and V satisfies (2)–(5) and (9). Define $P(h)$ by (1) and equip it with the domain $H^2(\mathbb{R}^n)$. For all $s > \frac{1}{2}$ and $z = E \pm i\varepsilon$ with $E > 0$*

fixed, there is a constant $\mathfrak{C}(E, s, \varepsilon, \|V^-\|_{L^\infty}) > 0$ *defined in* (40) *such that*

$$\|(r+1)^{-s}(P(h)-z)^{-1}(r+1)^{-s}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le \frac{\mathfrak{C}(E, s, \varepsilon, \|V^-\|_{L^\infty})}{h}. \tag{10}$$

When $V^- = 0$, we prove stronger estimates.

Theorem 1.2. *Suppose* $n \ge 3$ *and* V *satisfies* (2)–(5) *and* (9) *with* $V^- = 0$. *Define* $P(h)$ *by* (1) *and equip it with the domain* $H^2(\mathbb{R}^n)$. *For all* s, s_1, $s_2 > \frac{1}{2}$, *with* $s_1 + s_2 > 2$, *there is* $C > 0$ *such that for all* $z \in \mathbb{C} \setminus [0, \infty)$ *and* $h > 0$,

$$\left\|(r+1)^{-s}(P(h)-z)^{-1}(r+1)^{-s}\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le \frac{C}{h|z|^{1/2}}, \tag{11}$$

$$\left\|(r+1)^{-s_1}(P(h)-z)^{-1}(r+1)^{-s_2}\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le \frac{C}{h^2}. \tag{12}$$

Remark 1.3. From (42), we see how the constant C in (11) depends on s and C_V. The dependence of C in (12) on s_1, s_2 and C_V can be deduced from (49).

Section 2 is devoted to the proof of Theorem 1.2, which builds on [ChDa21, Theorem 1.2], where the authors obtained (11) and (12) for bounded, repulsive potentials, nonnegative potentials on the half-line. The novelty of our paper is that it extends these bounds to dimensions $n \ge 3$ for repulsive potentials that may be singular as $r \to \infty$. The corresponding problem in dimension two appears to be delicate, and to our knowledge remains open; we comment below on a possible approach.

Remark 1.4. In Appendix D, we recall how for the case $V = 0$ and $n = 3$, the conditions on s, s_1 and s_2 in Theorem 1.2, as well as the h- and z-dependencies of the right sides of (11) and (12), are nearly optimal in a suitable sense.

Remark 1.5. The weighted estimates underlying (10), (11), and (12) hold under a condition weaker than (3), namely that $|r^{n-1}V(r, \theta)| \to 0$ as $r \to 0$; see (32). (The factor r^{n-1} reflects the volume element of Lebesgue measure in polar coordinates.) These estimates are derived for test functions in $C_0^\infty(\mathbb{R}^n)$ and are transferred to resolvent bounds via the density argument in Appendix C. This step uses that the operator domain is $H^2(\mathbb{R}^n)$ and that $C_0^\infty(\mathbb{R}^n)$ is dense in $H^2(\mathbb{R}^n)$, which is why we impose the stronger hypothesis (3). We note, however, that self-adjoint realizations of $-\Delta + V$ exist under weaker local assumptions on V. For example, if $V \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $V \ge 0$, then $-\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ [ReSi75, Theorem X.28]; essential self-adjointness on $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ holds under inverse square lower bounds [ReSi75, Theorem X.30]. It would be interesting to formulate the weighted estimates in such frameworks, which would allow more singular potentials. However, for the wave decay results discussed below, a restriction comes from the need to control rV near the origin. We thus adopt the more streamlined approach here.

We prove Theorems 1.1 and 1.2 using the *spherical energy method*, a widely used technique for deriving weighted estimates for Schrödinger operators. The approach is based on separation of variables and the classical identity

$$r^{\frac{n-1}{2}}(-\Delta)r^{-\frac{n-1}{2}} = -\partial_r^2 + r^{-2}\Lambda, \tag{13}$$

where

$$\Lambda := -\Delta_{\mathbb{S}^{n-1}} + \tfrac{1}{4}(n-1)(n-3), \tag{14}$$

and $\Delta_{\mathbb{S}^{n-1}}$ denotes the negative Laplace–Beltrami operator on $\mathbb{S}^{n-1}$. The repulsivity condition is sufficiently advantageous to allow the use of a relatively simple weight — specifically, the same weight employed in [ChDa21] — to obtain (35) and (46). For more general potentials, it is usually necessary to instead conjugate the Laplacian by $e^{\varphi/h}r^{(n-1)/2}$ (see, e.g., [CaVo02; Da14; GaSh22]) for a suitable phase φ. This results in a Carleman estimate with exponential losses as $h \to 0^+$.

We use in a crucial way that $\Lambda \geq 0$ on $L^2(\mathbb{S}^{n-1})$; see (30). This is why our approach does not cover the case $n = 2$ where the effective potential $-1/(4r^2)$ has a strong negative singularity as $r \to 0$. We expect that repulsive potentials in dimension two can be treated by adapting the Mellin transform methods used in [DGS23; Ob24].

As an application of (12), we prove weighted energy decay for the solution to the wave equation

$$\begin{cases} (\partial_t^2 - \Delta + V(x))u(t,x) = 0 & \text{for } (t,x) \in \mathbb{R}\times\mathbb{R}^n,\ n \geq 3, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x) & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{15}$$

where $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ have compact support. The potential obeys $V \geq 0$, (3), (5) with $\rho = \rho(n) = 1$, (9), and the extra short range condition

$$\mathbf{1}_{\mathbb{R}^n \setminus B(0,1)} V \leq C(r+1)^{-\delta(n)}, \tag{16}$$

for some $C > 0$ and $\delta(n) > 0$ such that

$$\delta(n) > \begin{cases} \frac{1}{2} + \frac{n+3}{4} & \text{if } n \neq 8, \\ \frac{1}{2} + 3 & \text{if } n = 8. \end{cases} \tag{17}$$

Since $P := P(1) = -\Delta + V$ is self-adjoint (and nonnegative) under such conditions, we may use the spectral theorem for self-adjoint operators to represent the solution to (15) by

$$u(t,\cdot) = \cos(t\sqrt{P})u_0 + \frac{\sin(t\sqrt{P})}{\sqrt{P}}u_1. \tag{18}$$

For $s > 0$ fixed, define the weighted energy of the solution u to (15) to be

$$E_s[u](t) = E_s(t) := \int_{\mathbb{R}^n} \langle x\rangle^{-2s}(|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 + |u(t,x)|^2)\,dx.$$

Set also

$$E(0) := \|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2.$$

Theorem 1.6. *Suppose $V \geq 0$ satisfies (3), (5) with $\rho = 1$, (9), as well as (16). Let $u_0 \in H^1(\mathbb{R}^n)$ and $u_1 \in L^2(\mathbb{R}^n)$ have compact support. For each s such that*

$$s > \begin{cases} \frac{n+3}{4} & \text{if } n \neq 8, \\ 3 & \text{if } n = 8, \end{cases} \tag{19}$$

there exists $C_s > 0$ depending on s but independent of t, u_0, and u_1 so that

$$E_s(t) \leq C_s \langle t \rangle^{-2} E(0), \tag{20}$$

where $\langle t \rangle := (1 + |t|^2)^{1/2}$.

Remark 1.7. Since $u(-t, \cdot) = \cos(t\sqrt{P})u_0 + (\sin(t\sqrt{P})/\sqrt{P})(-u_1)$, to prove (20) it suffices to suppose $t \geq 0$.

Remark 1.8. We expect that, by adapting the arguments of [Vo04b, Section 3], the assumption of compact support on the initial data can be relaxed. Specifically, the decay should continue to hold for initial conditions lying in an appropriate weighted Sobolev spaces. However, to keep the presentation technically streamlined, we suppose u_0 and u_1 have compact support.

Remark 1.9. The condition (17) has a quirk in dimension eight compared to other dimensions. This appears to be an artifact of our approach, which analyzes the low-frequency behavior of the resolvent kernel in terms of Hilbert–Schmidt norms (Appendix E).

For smooth, nonnegative potentials of compact support, the local energy

$$E_{r_0}(t) := \int_{B(0,r_0)} |\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 + |u(t,x)|^2 dx, \quad r_0 > 0,$$

obeys

$$E_{r_0}(t) = \begin{cases} O(e^{-ct}) \text{ for some } c > 0 & \text{if } n \geq 3 \text{ is odd}, \\ O(t^{-2n}) & \text{if } n \geq 4 \text{ is even}. \end{cases} \tag{21}$$

Indeed, Vainberg [Va75] showed (21) for compactly supported perturbations of the Laplacian satisfying the generalized Huygens principle (as defined in [Vo04c]). A result of Melrose and Sjöstrand on the propagation of singularities [MeSj78; MeSj82] implies that this principle holds for a broad class of nontrapping perturbations of the Laplacian, which includes smooth, nonnegative, compactly supported potentials. The study of energy decay for nontrapping perturbations has a long history, going back to the works of Lax, Morawetz, and Phillips [LMP63; Mo66; Mo75].

Bounds similar to (20) were obtained in previous works for various classes of short range potentials. In [Za04], Zappacosta considered potentials $V \in C^1(\mathbb{R}^3; (0,\infty))$

with $\partial_x^\alpha V = O(\langle x\rangle^{-\delta-|\alpha|})$ for all $0 \le |\alpha| \le 1$ and some $\delta > 2$. For each $\chi \in C_0^\infty(\mathbb{R}^3)$, the bound

$$\|\chi\sqrt{V}(\sin(t\sqrt{P})/\sqrt{P})\sqrt{V}\chi\|^2_{L^2\to L^2} = O(t^{-2})$$

was proved. In [Vo04a], Vodev showed $E_{r_0}(t) = O(t^{-2})$ in dimension $n \ge 3$, where $V \in C^1(\mathbb{R}^n; [0,\infty))$ obeys

$$V = O(\langle x\rangle^{-\delta_0}) \quad \text{for some } \delta_0 > 2, \text{ and} \tag{22}$$

$$2V + r\partial_r V \le C\langle x\rangle^{-\delta} \quad \text{for some } C > 0 \text{ and some } \delta > 1. \tag{23}$$

Additionally, it is assumed that V has no resonance at zero energy, a condition closely related to the validity of a bound such as (12) for $|z| \ll 1$. Vodev also obtained weighted energy decay for a class of long-range, nontrapping perturbations of the Laplacian that includes perturbations by a nonnegative long-range potential, provided the initial conditions are spectrally localized away from $[0, a]$ for $a > 0$ sufficiently large [Vo04b]. (In our approach, the positivity of the potential appears crucial for obtaining (12). Indeed, the constant $\mathfrak{C}(E, s, \varepsilon, \|V^-\|_{L^\infty})$ in (40) blows up as $E \to 0^+$ unless $\|V^-\|_{L^\infty} = 0$.)

We note the connection between (23) and our repulsiveness condition. If $V \in C^1(\mathbb{R}^n; (0,\infty))$ satisfies (9), then $\partial_r(\log V(\,\cdot\,,\theta)) \le -C_V(r+1)^{-1}$ uniformly in θ. Integration in r yields $0 \le V(r,\theta) \le C(r+1)^{-C_V}$, so $2V + r\partial_r V \le 2V \le C(r+1)^{-C_V}$, which is (23) provided $C_V > 1$.

In the absence of a bound like (12), or a condition excluding a resonance or eigenvalue at zero, decay of wave equation solutions generally cannot be expected. Thus, in the present work the positivity of the potential serves as a sufficient condition to rule out threshold obstructions. Other sufficient conditions have been obtained in previous works; for example, smallness of the potential in the Rollnik and global Kato norms [RoSc04]. For related discussions and further references, see [JeNe01; ChDa25; CDY25].

The proof of Theorem 1.6 follows the strategy of [Vo04a, Section 3], with modifications to account for the possible singularity of V at the origin. The key step is to establish

$$t^2 E_s(t) \le Ct^2 \int_t^\infty E_s(\tau)\,d\tau \le CE(0), \quad t \ge 1. \tag{24}$$

Using Duhamel's formula and the Fourier transform (t dual to λ), the Fourier transform of u can be expressed in terms of $(P-\lambda^2)^{-1}$; see (64). Because the initial data are compactly supported, finite speed of propagation allows insertion of a cutoff function η, and Plancherel's theorem then reduces control of $E_s(t)$ to bounds on $\langle x\rangle^{-s}(P-\lambda^2)^{-1}\eta$, which comes from (12); see also Lemma 3.1.

However, the factor t^2 in (24) corresponds to differentiation with respect to λ, so it is also necessary to control

$$\langle x\rangle^{-s}\frac{d}{d\lambda}(P-\lambda^2)^{-1}\eta.$$

This is achieved under the stronger assumptions (3) with $\rho = 1$ (compare with (6)) and (16). In particular, the restriction $\rho = 1$ arises from the need for a uniform bound on $\lambda V(-\Delta-\lambda^2)^{-1}\langle x\rangle^{-s}$ for $\operatorname{Im}\lambda > 0$, see (58). It would be interesting to determine whether more singular potentials could be accommodated by controlling the λ-derivative in a less perturbative manner.

We expect that for certain potentials with mild spatial decay depending on the dimension, t^{-2} is the optimal decay for the local energy. A sharper description should depend on a low-frequency expansion of the resolvent around $\lambda = 0$ (see, e.g., [JeNe01]), rather than a bound alone.

Another energy studied is the quantity

$$E_K^{(1)}[u](t) := \int_K |\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2 + V(x)|\,u(t,x)|^2\,dx,$$

where $K \subseteq \mathbb{R}^n$ is a region of interest. In [Vo04a, Theorem 1.1], Vodev studied the case $K = B(0,\gamma_0 t) \subseteq \mathbb{R}^n$ for $n \geq 3$ and suitable $0 < \gamma_0 < 1$. Under the assumption of no resonance at zero, along with (22) for some $\delta_0 > 1$ and (23) for constants $C > 0$ and $\delta > 1$, he proved that $E_K^{(1)}[u](t) = O(t^{-1})$. In [Ik23, Theorems 1.1 and 1.2], Ikehata considered exterior subdomains Ω of $\mathbb{R}^n$, $n \geq 2$, excluding the origin. For compact subsets $K \subseteq \Omega$, he established the same decay rate assuming V is nonnegative, C^1, and obeys $x\cdot\nabla V + 2V \leq 0$. The $O(t^{-1})$-decay was first showed by Morawetz [Mo61] for $V = 0$ in the exterior of a three-dimensional star-shaped obstacle (later improved to exponential decay in [LMP63]).

There is an extensive body of literature on wave decay for higher-order perturbations. For general second-order perturbations that may exhibit trapping, logarithmic decay — rather than polynomial decay — is more typical. For historical background and related developments, see [Bu98; Vo99; Bu02; Bo11; CaVo04; Sh18; ChIk20].

Outline. In Section 2, we prove Theorem 1.2. In Section 3, we apply Theorem 1.2 to prove norm bounds for $\langle x\rangle^{-s}(P-\lambda^2)^{-1}\langle x\rangle^{-s}$ and its λ-derivative. In Section 4 we prove Theorem 1.6. Finally, we include several appendices of technical results that assist with the proofs of earlier sections.

2. Proof of Theorem 1.2

Throughout this section, we take $P(h)$ as in (1) and assume the potential V satisfies (2)–(5) and(9).

By (13),

$$P^{\pm}(h) := r^{\frac{n-1}{2}} (P(h) - E \pm i\varepsilon) r^{-\frac{n-1}{2}} = -h^2\partial_r^2 + h^2 r^{-2}\Lambda + V - E \pm i\varepsilon, \tag{25}$$

where we let E and ε vary in $[0, \infty)$. Let $u \in r^{(n-1)/2} C_0^{\infty}(\mathbb{R}^n)$. Define a spherical energy functional $F[u](r)$,

$$F(r) = F[u](r) := \|hu'(r, \cdot)\|^2 - \langle (h^2 r^{-2}\Lambda + V^R(r, \cdot) - E)u(r, \cdot), u(r, \cdot)\rangle, \tag{26}$$

where $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the norm and inner product on $L^2(\mathbb{S}_\theta^{n-1})$. We take complex conjugation to occur in the first argument of $\langle \cdot, \cdot \rangle$. Here, $V^R(r, \theta)$ is the measurable function defined by $V^R(r, \theta) := \lim_{k\to\infty} V((r + (1/k))\theta)$ for $k \in \mathbb{N}$. The limit exists for each r and θ since each $V(\cdot, \theta)$ has bounded variation. Each $V^R(\cdot, \theta)$ is decreasing thanks to (2), (8) and (9).

For a weight $w(r)$ which is absolutely continuous, nonnegative, and increasing, we compute the derivative of wF in the sense of distributions on $(0, \infty)$. For this we need the following technical lemma, whose proof we give in Appendix A.

Lemma 2.1. *The function $r \mapsto \int_{\mathbb{S}^{n-1}} V^R(r, \theta)|u(r, \theta)|^2 \, d\theta$ has locally bounded variation and its derivative in the sense of distributions on $(0, \infty)$ is the element of $C_0^{\infty}(0, \infty)$ given by*

$$\varphi \mapsto \int_0^{\infty} \varphi(r) \int_{\mathbb{S}^{n-1}} V(r, \theta)\, 2\,\mathrm{Re}(\bar{u}u')\, d\theta\, dr + \int_{\mathbb{S}^{n-1}} \int_0^{\infty} \varphi(r)|u(r, \theta)|^2 \, dV(r, \theta)\, d\theta. \tag{27}$$

Note that, aside from the term in $F(r)$ involving $V^R(r, \theta)$, the remaining terms are functions of r with locally bounded variation on $(0, \infty)$. Thus by Lemma 2.1, $F(r)$ itself is of locally bounded variation on $(0, \infty)$.

Using (27) we have, in the sense of distributions on $(0, \infty)$:

$$\begin{aligned} (wF)' &= wF' + w'F \\ &= w(-2\,\mathrm{Re}\langle(-h^2\partial_r^2 + h^2 r^{-2}\Lambda + V - E)u, u'\rangle \\ &\qquad + 2h^2 r^{-3}\langle \Lambda u, u\rangle - \textstyle\int_{\mathbb{S}^{n-1}} |u(r, \theta)|^2 \, dV(r, \theta)\, d\theta) \\ &\qquad + w'(\|hu'\|^2 - \langle h^2 r^{-2}\Lambda u, u\rangle + \langle (E - V)u, u\rangle \\ &= -2w\,\mathrm{Re}\langle P^{\pm}(h)u, u'\rangle \pm 2\varepsilon w\,\mathrm{Im}\langle u, u'\rangle + w'\|hu'\|^2 \\ &\qquad + (2wr^{-1} - w')\langle h^2 r^{-2}\Lambda u, u\rangle + Ew'\|u\|^2 \\ &\qquad - \textstyle\int_{\mathbb{S}^{n-1}} |u(r, \theta)|^2 (w(r)\, dV(r, \theta) + w'(r)V(r, \theta))\, d\theta. \end{aligned} \tag{28}$$

First we show (11). Since increasing s decreases the left side of (11), without loss of generality we may take $0 < \delta := 2s - 1 < 1$. We will show that the last line of (28) can be made to have a suitable lower bound, using

$$w(r) := 1 - \frac{C_V}{C_V + \delta}(1+r)^{-\delta}. \tag{29}$$

For such w, we clearly have

$$w'(r) = \frac{\delta C_V}{C_V + \delta}(r+1)^{-1-\delta}.$$

Therefore, on the one hand,

$$2wr^{-1} - w' = 2r^{-1}(r+1)^{-\delta}\left((r+1)^{\delta} - \frac{C_V}{C_V+\delta}\left(1 + \frac{\delta r}{2(r+1)}\right)\right) \geq 0,$$

where we used

$$(r+1)^{\delta} - \frac{C_V}{C_V+\delta} \geq \frac{\delta C_V}{C_V+\delta}\int_1^{r+1} s^{\delta-1}\,ds \geq \frac{\delta C_V r}{(C_V+\delta)(r+1)},$$

since $\delta < 1$. On the other hand, using (9), we have, in the sense of measures on bounded Borel subsets of $(0, \infty)$,

$$w\,dV + w'V = \frac{\delta C_V V}{(C_V+\delta)(r+1)^{1+\delta}} + w dV \leq \frac{C_V V^+}{1+r}((r+1)^{-\delta} - 1) \leq 0.$$

Thus, the last two estimates and (28) imply, for any $\varphi \in C_0^\infty((0,\infty);[0,\infty))$,

$$\begin{aligned}\int \varphi\, d(wF) &= -\int (wF)\varphi' dr \\ &\geq \int \left(-2w\,\mathrm{Re}\langle P^{\pm}(h)u, u'\rangle \pm 2\varepsilon w\,\mathrm{Im}\langle u, u'\rangle + w'\|hu'\|^2 + Ew'\|u\|^2\right)\varphi\, dr,\end{aligned} \tag{30}$$

where in the first line of (30) we applied (7). Now, take a sequence of $\varphi_k \in C_0^\infty((0,\infty);[0,1])$ that converges pointwise to the indicator function $\mathbf{1}_{(r_0,r_1]}$ with $0 < r_0 \ll 1$ and r_1 large enough so that $u(r,\theta) = 0$ for near $[r_1,\infty) \times \mathbb{S}^{n-1}$. Substituting $\varphi = \varphi_k$ in (30), sending $k \to \infty$, and applying the dominated convergence theorem and (8) gives

$$\begin{aligned}\int_{r_0}^{\infty} Ew'\|u\|^2 + w'\|hu'\|^2\,dr + w(r_0)F^R(r_0) \\ \leq \int_{r_0}^{\infty} 2w\,\mathrm{Re}\langle P^{\pm}(h)u, u'\rangle \mp 2\varepsilon w\,\mathrm{Im}\langle u, u'\rangle\,dr.\end{aligned} \tag{31}$$

Since $u = r^{(n-1)/2}v$ for some $v \in C_0^\infty(\mathbb{R}^n)$, we recognize that

$$\begin{aligned}F^R(r_0) = \|hu'(r_0,\cdot)\|^2 &+ r_0^{n-3}\langle h^2\Delta_{\mathbb{S}^{n-1}}v(r_0,\cdot), v(r_0,\cdot)\rangle \\ &+ \left(Er_0^{n-1} - h^2 4^{-1}(n-1)(n-3)r_0^{n-3}\right)\|v(r_0,\cdot)\|^2 \\ &+ r_0^{n-1}\langle V^R(r_0,\cdot)v(r_0,\cdot), v(r_0,\cdot)\rangle.\end{aligned} \tag{32}$$

We rewrite the term in (32) involving $\Delta_{\mathbb{S}^{n-1}}$ using the formula for the Laplacian in spherical coordinates:

$$r^{-2}\Delta_{\mathbb{S}^{n-1}} = \Delta - \partial_r^2 - (n-1)r^{-1}\partial_r.$$

This leads to

$$\begin{aligned}(33)\quad & r_0^{n-3}\langle h^2\Delta_{\mathbb{S}^{n-1}} v(r_0,\cdot), v(r_0,\cdot)\rangle \\ &= h^2 r_0^{n-1}\langle(\Delta v)(r_0,\cdot), v(r_0,\cdot)\rangle - h^2 r_0^{n-1}\langle(\partial_r^2 v)(r_0,\cdot), v(r_0,\cdot)\rangle \\ &\qquad - h^2(n-1)r_0^{n-2}\langle(\partial_r v)(r_0,\cdot), v(r_0,\cdot)\rangle.\end{aligned}$$

We can express the differential operators ∂_r and ∂_r^2 with respect to the Euclidean coordinate system as

$$(34)\qquad \partial_r = r^{-1}\sum_{j=1}^n x_j\partial_{x_j}, \quad \partial_r^2 = r^{-2}\sum_{k=1}^n x_k \sum_{j=1}^n x_j \partial_{x_k}\partial_{x_j}.$$

By (3), (33) and (34), all terms in (32) tend to zero as $r_0 \to 0$, except possibly for $\|hu'(r,\cdot)\|^2$ in dimension three, which in that case tends to $|v(0)|^2\int_{\mathbb{S}^{n-1}} d\theta$. We conclude that

$$\lim_{r_0\to 0} w(r_0)F(r_0) = w(0)F(0) = \begin{cases}\omega_{n-1}w(0)|v(0)|^2 & \text{if } n=3,\\ 0 & \text{if } n\ge 4,\end{cases}$$

where ω_{n-1} is the $(n-1)$-dimensional volume of $\mathbb{S}^{n-1}$.

Thus, in view of (31) and $0 < w \le 1$,

$$\begin{aligned}(35)\quad & \int_0^\infty Ew'\|u\|^2 + w'\|hu'\|^2\,dr \\ &\le 2\Big(\int_0^\infty \frac{1}{h^2 w'}\|P^\pm(h)u\|^2\,dr\Big)^{1/2}\Big(\int_0^\infty w'\|hu'\|^2\,dr\Big)^{1/2} \\ &\qquad + \frac{2\varepsilon}{h}\Big(\int_0^\infty \|u\|^2\,dr\Big)^{1/2}\Big(\int_0^\infty \|hu'\|^2\,dr\Big)^{1/2}.\end{aligned}$$

We now estimate

$$\begin{aligned}&\int_0^\infty \|hu'\|^2\,dr \\ &= \operatorname{Re}\int_0^\infty \langle u, -h^2u''\rangle\,dr \\ &= \operatorname{Re}\Big(\int_0^\infty \langle u, P^\pm(h)u\rangle\,dr + \int_0^\infty \langle u, (E-V-h^2r^{-2}\Lambda)u\rangle\,dr \mp i\varepsilon\int_0^\infty \|u\|^2\,dr\Big) \\ &= \operatorname{Re}\int_0^\infty \langle u, P^\pm(h)u\rangle\,dr + \int_0^\infty \langle u, (E-V-h^2r^{-2}\Lambda)u\rangle\,dr \\ &\le \Big(\int_0^\infty \frac{1}{w'}\|P^\pm(h)u\|^2\,dr\Big)^{1/2}\Big(\int_0^\infty w'\|u\|^2\,dr\Big)^{1/2} + (E+\|V^-\|_{L^\infty})\int_0^\infty \|u\|^2\,dr\end{aligned}$$

and

$$\begin{aligned}\varepsilon\int_0^\infty \|u\|^2\,dr &= \varepsilon\|v\|_{L^2}^2\\ &= \left|\operatorname{Im}\langle (P(h)-E\pm i\varepsilon)v, v\rangle_{L^2}\right| = \left|\operatorname{Im}\int_0^\infty \langle P^\pm(h)u,u\rangle\,dr\right|\\ &\le \Big(\int_0^\infty \frac{1}{w'}\|P^\pm(h)u\|^2\,dr\Big)^{1/2}\Big(\int_0^\infty w'\|u\|^2\,dr\Big)^{1/2}.\end{aligned}$$

Combining these estimates gives

$$\begin{aligned}&\frac{\varepsilon^2}{h^2}\int_0^\infty \|u\|^2\,dr\int_0^\infty \|hu'\|^2\,dr\\ &\qquad\le (E+\varepsilon+\|V^-\|_{L^\infty})\int_0^\infty \frac{1}{h^2 w'}\|P^\pm(h)u\|^2\,dr\cdot\int_0^\infty w'\|u\|^2\,dr.\end{aligned}$$

Plugging this into (35) yields

$$\begin{aligned}&\int_0^\infty Ew'\|u\|^2+w'\|hu'\|^2\,dr\\ &\quad\le 2\Big(\int_0^\infty \frac{1}{h^2w'}\|P^\pm(h)u\|^2\,dr\Big)^{1/2}\\ &\quad\cdot\Big(\Big(\int_0^\infty w'\|hu'\|^2\,dr\Big)^{1/2}+\big(E+\varepsilon+\|V^-\|_{L^\infty}\big)^{1/2}\Big(\int_0^\infty w'\|u\|^2\,dr\Big)^{1/2}\Big).\end{aligned}\tag{36}$$

Now restrict to $E>0$ and complete the square in (36) to find

$$\begin{aligned}&\Big(\sqrt{E}\Big(\int_0^\infty w'\|u\|^2\,dr\Big)^{\frac12}-\frac{\sqrt{E+\varepsilon+\|V^-\|_{L^\infty}}}{\sqrt{E}}\Big(\int_0^\infty \frac{1}{h^2w'}\|P^\pm(h)u\|^2\,dr\Big)^{\frac12}\Big)^2\\ &\quad+\Big(\Big(\int_0^\infty w'\|hu'\|^2\,dr\Big)^{\frac12}-\Big(\int_0^\infty \frac{1}{h^2w'}\|P^\pm(h)u\|^2\,dr\Big)^{\frac12}\Big)^2\\ &\qquad\le \frac{2E+\varepsilon+\|V^-\|_{L^\infty}}{E}\int_0^\infty \frac{1}{h^2w'}\|P^\pm(h)u\|^2\,dr.\end{aligned}\tag{37}$$

Dropping the second term on the left side of (37) implies, for all $E>0$ and $\varepsilon\ge 0$,

$$\begin{aligned}&\sqrt{E}\Big(\int_0^\infty w'\|u\|^2\,dr\Big)^{\frac12}\\ &\quad\le\Big(\frac{\sqrt{E+\varepsilon+\|V^-\|_{L^\infty}}}{\sqrt{E}}+\frac{\sqrt{2E+\varepsilon+\|V^-\|_{L^\infty}}}{\sqrt{E}}\Big)\Big(\int_0^\infty \frac{1}{h^2w'}\|P^\pm(h)u\|^2\,dr\Big)^{\frac12}.\end{aligned}\tag{38}$$

Recall that $w'=C_V\delta(C_V+\delta)^{-1}(r+1)^{-1-\delta}$ and $\delta=2s-1$. From (38) and the density argument in Appendix C we get, rewriting $z=E\pm i\varepsilon$, that

$$\|(1+r)^{-2s}(P-z)^{-1}(1+r)^{-2s}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}\le\frac{\mathfrak{C}(E,s,\varepsilon,\|V^-\|_{L^\infty})}{h},\tag{39}$$

where

$$\text{(40)}\quad \mathfrak{C}(E, s, \varepsilon, \|V^-\|_{L^\infty}) = \frac{C_V+\delta}{E\delta C_V}\left((E+\varepsilon+\|V^-\|_{L^\infty})^{1/2}+(2E+\varepsilon+\|V^-\|_{L^\infty})^{1/2}\right).$$

From this point on, we assume $V^- = 0$. Fix $0 < \alpha < 1$. From (38), we get for all $h > 0$, all $E \pm i\varepsilon$ in the sector $\{z \in \mathbb{C} : |\operatorname{Im} z| < \alpha \operatorname{Re} z\}$, and $u \in r^{n-1/2}C_0^\infty(\mathbb{R}^n)$,

$$\text{(41)}\quad \left|(E\pm i\varepsilon)^{1/2}\right|\left(\int_0^\infty (r+1)^{-2s}\|u\|^2\,dr\right)^{1/2} \le h^{-1}(1+\alpha^2)^{1/4}\left(\frac{1}{\delta}+\frac{1}{C_V}\right)\left((1+\alpha)^{1/2}+(2+\alpha)^{1/2}\right) \cdot\left(\int_0^\infty (r+1)^{2s}\|P^\pm(h)u\|^2\,dr\right)^{1/2}.$$

Here, our branch of the complex square root is chosen so that $\operatorname{Im}(E\pm i\varepsilon)^{1/2} > 0$, and we used that $|(E\pm i\varepsilon)|^{1/2} = (E^2+\varepsilon^2)^{1/4} \le E^{1/2}(1+\alpha^2)^{1/4}$ because $E \pm i\varepsilon \in \{z \in \mathbb{C} : |\operatorname{Im} z| < \alpha \operatorname{Re} z\}$. Since $u \in r^{(n-1)/2}C_0^\infty(\mathbb{R}^n)$, a standard density argument, which we review in Appendix C, shows that (41) implies

$$\text{(42)}\quad \left\|z^{1/2}(r+1)^{-s}(P(h)-z)^{-1}(r+1)^{-s}\right\|_{L^2\to L^2} \le h^{-1}(1+\alpha^2)^{1/4}\left(\frac{1}{\delta}+\frac{1}{C_V}\right)\left((1+\alpha)^{1/2}+(2+\alpha)^{1/2}\right),$$

on $\{z \in \mathbb{C} : |\operatorname{Im} z| < \alpha \operatorname{Re} z\}$ and for any $0 < \alpha < 1$. To extend this bound to all $z \in \mathbb{C}\setminus[0,\infty)$, we will use the Phragmén–Lindelöf method [EM] in the following way. For $u, v \in L^2(\mathbb{R}^n)$, put

$$U(z) := z^{1/2}\left\langle (r+1)^{-s}(P(h)-z)^{-1}(r+1)^{-s}u, v\right\rangle_{L^2}.$$

Then $U(z)$ in analytic in $\Omega_\alpha := \{z \in \mathbb{C} : \alpha \operatorname{Re} z < |\operatorname{Im} z|\}$. By (42), on $\partial\Omega_\alpha \setminus \{0\}$ we have

$$\text{(43)}\quad |U(z)| \le h^{-1}(1+\alpha^2)^{1/4}\left(\frac{1}{\delta}+\frac{1}{C_V}\right)((1+\alpha)^{1/2}+(2+\alpha)^{1/2})\|u\|_{L^2}\|v\|_{L^2}.$$

On the other hand, in Ω_α, we have the standard bound

$$\text{(44)}\quad |U(z)| \le \frac{|z|^{1/2}\|u\|_{L^2}\|v\|_{L^2}}{\operatorname{dist}(z,[0,\infty))} = \begin{cases} \dfrac{\|u\|_{L^2}\|v\|_{L^2}}{|z|^{1/2}} & \text{if } \operatorname{Re} z < 0, \\[2ex] \dfrac{|z|^{1/2}\|u\|_{L^2}\|v\|_{L^2}}{|\operatorname{Im} z|} & \text{if } \operatorname{Re} z \ge 0,\ z \in \Omega_\alpha, \end{cases}$$

where we used

$$\frac{1}{\operatorname{dist}(z,[0,\infty))} = \frac{1}{\inf_{r\ge 0}((\operatorname{Re} z - r)^2 + (\operatorname{Im} z)^2)^{1/2}} = \begin{cases} |z|^{-1} & \text{if } \operatorname{Re} z < 0, \\ |\operatorname{Im} z|^{-1} & \text{if } \operatorname{Re} z \ge 0,\ z \in \Omega_\alpha. \end{cases}$$

Finally, define, $g(z) = e^{i(z^{-1})^{1/2}}$, where our branch of the square root is as above. In Ω_α, $|g(z)| \le e^{-c_\alpha |z|^{-1/2}}$ for some $0 < c_\alpha < 1$ depending on α; this is because, from the definition of Ω_α, there exists $\theta_\alpha \in (0, \pi/4)$ such that any $z \in \Omega_\alpha$ takes the form $|z|e^{i\theta}$ with $\theta_\alpha < \theta < 2\pi - \theta_\alpha$. Hence $\operatorname{Re}(i(z^{-1})^{1/2}) = -|z|^{-1/2}\sin(\theta/2) \le -|z|^{-1/2}\sin(\theta_\alpha/2)$. Combining with (44) gives

$$\limsup_{z\to 0,\, z\in\Omega_\alpha} |g(z)|^\sigma |U(z)| = 0, \quad \sigma > 0. \tag{45}$$

Therefore, from (43) and (45), the Phragmén–Lindelöf theorem (Theorem B.1 in Appendix B) implies that (42) holds for all $z \in \Omega_\alpha$ too. Sending $\alpha \to 0^+$ completes the proof of (11).

To prove (12), start again at (37) and drop the first term on the left-hand side. Still working on $\{z \in \mathbb{C} : |\operatorname{Im} z| < \alpha \operatorname{Re} z\}$, some manipulations give

$$\Big(\int_0^\infty w' \|hu'\|^2\, dr\Big)^{1/2} \le (1 + \sqrt{2+\alpha})\Big(\int_0^\infty \frac{1}{h^2 w'} \|P^\pm(h)u\|^2\, dr\Big)^{1/2}. \tag{46}$$

By integration by parts, we have

$$\begin{aligned}
&\int_0^\infty (r+1)^{-3-\delta}\|u\|^2\, dr \\
&\qquad = \frac{2}{2+\delta}\int_0^\infty (r+1)^{-2-\delta}\operatorname{Re}\langle u, u'\rangle\, dr \\
&\qquad \le h^{-1}\Big(\int_0^\infty (r+1)^{-1-\delta}\|hu'\|^2\, dr\Big)^{1/2}\Big(\int_0^\infty (r+1)^{-3-\delta}\|u\|^2\, dr\Big)^{1/2},
\end{aligned}$$

which implies

$$\Big(\int_0^\infty (r+1)^{-3-\delta}\|u\|^2\, dr\Big)^{1/2} \le h^{-1}\Big(\int_0^\infty (r+1)^{-1-\delta}\|hu'\|^2\, dr\Big)^{1/2}. \tag{47}$$

From (46), (47) and $w' = C_V\delta(C_V+\delta)^{-1}(r+1)^{-1-\delta}$, we obtain

$$\begin{aligned}
&\Big(\int_0^\infty (r+1)^{-3-\delta}\|u\|^2\, dr\Big)^{1/2} \\
&\qquad \le h^{-2}\Big(\frac{1}{\delta} + \frac{1}{C_V}\Big)(1+\sqrt{2+\alpha})\Big(\int_0^\infty (r+1)^{1+\delta}\|P^\pm(h)u\|^2\, dr\Big)^{1/2}.
\end{aligned}$$

Using again the density argument in Appendix C, for $0 < \delta < 1$, we get

$$\big\|(1+r)^{-\frac{3+\delta}{2}}(P(h)-z)^{-1}(1+r)^{-\frac{1+\delta}{2}}\big\|_{L^2\to L^2} \le h^{-2}\big(\delta^{-1} + C_V^{-1}\big)(1+\sqrt{2+\alpha}) \tag{48}$$

on $\{z \in \mathbb{C} : |\operatorname{Im} z| < \alpha \operatorname{Re} z\}$. Then, as above, we can use (44) and the Phragmén–Lindelöf theorem, and take the limit $\alpha \to 0^+$, to obtain

$$\big\|(1+r)^{-\frac{3+\delta}{2}}(P(h)-z)^{-1}(1+r)^{-\frac{1+\delta}{2}}\big\|_{L^2\to L^2} \le h^{-2}\big(\delta^{-1} + C_V^{-1}\big)(1+\sqrt{2}),$$

for $z \in \mathbb{C} \setminus [0, \infty)$. Since the norm of an operator coincides with that of its adjoint, we reach

$$\big\| (1+r)^{-\frac{1+\delta}{2}} (P(h)-z)^{-1} (1+r)^{-\frac{3+\delta}{2}} \big\|_{L^2 \to L^2} \le h^{-2} \big(\delta^{-1} + C_V^{-1}\big)(1+\sqrt{2}),$$

for $z \in \mathbb{C} \setminus [0, \infty)$. The three lines lemma then says that for fixed $z \in \mathbb{C} \setminus [0, \infty)$, the analytic mapping

$$\lambda \mapsto (1+r)^{-\frac{3+\delta}{2}+\lambda} (P(h)-z)^{-1} (1+r)^{-\frac{1+\delta}{2}-\lambda}, \quad 0 < \operatorname{Re} \lambda < 1,$$

with values in the space of bounded operators $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, obeys

$$\big\| (1+r)^{-\frac{3+\delta}{2}+\theta} (P(h)-z)^{-1} (1+r)^{-\frac{1+\delta}{2}-\theta} \big\|_{L^2 \to L^2} \le h^{-2} \big(\delta^{-1} + C_V^{-1}\big)(1+\sqrt{2}), \quad \theta \in [0, 1]. \tag{49}$$

Having established (49), to finish, we need to see that we can choose δ and θ appropriately to arrive at (12). That is, we need to attain the more general weights characterized by $s_1, s_2 > \frac{1}{2}$ and $s_1 + s_2 > 2$. However, because we have the restrictions $\delta \in (0, 1)$ and $\theta \in [0, 1]$, we first need to make reductions as follows. Since decreasing s_1 or s_2 in (12) increases the left side, it suffices to suppose $s_1,\ s_2 > \frac{1}{2}$ and $2 < s_1 + s_2 < 3$. Furthermore, by taking the adjoint, it is no restriction to have $s_1 \le s_2$. If we write $s_1 = (1 + 2\delta_2)/2$ for some $\delta_2 > 0$, then we may replace s_2 by $\min(s_2, (3+\delta_2)/2)$. Having made these reductions, (12) follows from (49) by setting $\delta = s_1 + s_2 - 2 < 1$ and

$$\theta = \tfrac{1}{2}(s_2 - s_1 + 1) \le \tfrac{1}{4}(4 - \delta_2) < 1.$$

Remark 2.2. The bound (38) with $\varepsilon = 0$ rules out $P(h)$ having an eigenvalue $E > 0$. When $V^- = 0$, a zero eigenvalue is ruled out by combining (36) (with $E = \varepsilon = 0$) with (47).

3. Resolvent bounds for wave decay

In this section, we consider the operator $P := P(1) = -\Delta + V$, with $P(h)$ as in (1) and V obeying (2)–(5) and (9). As a consequence of Theorem 1.2, we prove several more resolvent bounds for P, which enable us in Section 4 to establish weighted energy decay for the solution to the wave equation (15). Throughout this section, C denotes a positive constant whose precise value may change, but is always independent of λ, which plays the role of our spectral parameter.

Lemma 3.1. *Fix $s_1,\ s_2 > \frac{1}{2}$ with $s_1 + s_2 > 2$. There exist $C > 0$ such that for all $\lambda \in \mathbb{C}$ with $0 < |\operatorname{Im} \lambda| \le 1$ and for all multiindices $\alpha_1,\ \alpha_2$ with $|\alpha_1| + |\alpha_2| \le 2$,*

$$\big\| \langle x \rangle^{-s_1} \partial_x^{\alpha_1} (P - \lambda^2)^{-1} \partial_x^{\alpha_2} \langle x \rangle^{-s_2} \big\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \le C (1 + |\operatorname{Re} \lambda|)^{|\alpha_1| + |\alpha_2| - 1}. \tag{50}$$

Proof. Since $((P-\lambda^2)^{-1})^* = (P-\bar{\lambda}^2)^{-1}$, to prove (50) is suffices to assume $\operatorname{Im}\lambda > 0$.

First, we treat the case $|\alpha_2| = 0$. Using (11) if $|\operatorname{Re}\lambda| > 1$ or (12) if $|\operatorname{Re}\lambda| \le 1$, we get

$$\big\| \langle x\rangle^{-s_1}(P-\lambda^2)^{-1}\langle x\rangle^{-s_2}\big\|_{L^2\to L^2} \le C(1+|\operatorname{Re}\lambda|)^{-1}, \quad 0<\operatorname{Im}\lambda\le 1, \tag{51}$$

Recall from standard elliptic theory that for all $f\in H^2(\mathbb{R}^n)$ and all $\gamma>0$,

$$\begin{aligned} &\|f\|_{H^2} \le C(\|f\|_{L^2}+\|\Delta f\|_{L^2}),\\ &\|f\|_{H^1}^2 \le C\|f\|_{L^2}\|f\|_{H^2} \le C(\gamma^{-1}\|f\|_{L^2}^2+\gamma\|\Delta f\|_{L^2}^2). \end{aligned} \tag{52}$$

Therefore, for any $f\in L^2(\mathbb{R}^n)$,

$$\begin{aligned} &\|\langle x\rangle^{-s_1}(P-\lambda^2)^{-1}\langle x\rangle^{-s_2} f\|_{H^2(\mathbb{R}^n)}\\ &\le C(\|\langle x\rangle^{-s_1}(P-\lambda^2)^{-1}\langle x\rangle^{-s_2} f\|_{L^2(\mathbb{R}^n)} + \|(-\Delta)\langle x\rangle^{-s_1}(P-\lambda^2)^{-1}\langle x\rangle^{-s_2} f\|_{L^2(\mathbb{R}^n)})\\ &\le C(\|\langle x\rangle^{-s_1}(P-\lambda^2)^{-1}\langle x\rangle^{-s_2} f\|_{H^1(\mathbb{R}^n)} + \|\langle x\rangle^{-s_1}(-\Delta)(P-\lambda^2)^{-1}\langle x\rangle^{-s_2} f\|_{L^2(\mathbb{R}^n)})\\ &\le C(\gamma^{-1}+|\operatorname{Re}\lambda|^2)\|\langle x\rangle^{-s_1}(P-\lambda^2)^{-1}\langle x\rangle^{-s_2} f\|_{L^2(\mathbb{R}^n)}\\ &\qquad + C\gamma\|\Delta\langle x\rangle^{-s_1}(P-\lambda^2)^{-1}\langle x\rangle^{-s_2} f\|_{L^2(\mathbb{R}^n)} + C\|f\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Selecting γ sufficiently small depending on C and applying (51) yields

$$\big\| \langle x\rangle^{-s_1}(P-\lambda^2)^{-1}\langle x\rangle^{-s_2} f\big\|_{H^2(\mathbb{R}^n)} \le C(1+|\operatorname{Re}\lambda|)\|f\|_{L^2(\mathbb{R}^n)}, \tag{53}$$

as desired. This confirms (50) for $|\alpha_1| = 2$. For $|\alpha_1| = 1$ (still with $|\alpha_2| = 0$), combine (51) and (53) via the second line of (52).

If $|\alpha_2| > 0$, let $f\in C_0^\infty(\mathbb{R}^n)$, and put

$$u = \langle x\rangle^{-s_1}(P-\lambda^2)^{-1}\langle x\rangle^{-s_2}\partial_x^{\alpha_2} f.$$

We need to show

$$\|u\|_{H^{|\alpha_1|}} \le C(1+|\operatorname{Re}\lambda|)^{|\alpha_1|+|\alpha_2|-1}\|f\|_{L^2}, \quad H^0 := L^2(\mathbb{R}^n).$$

If $|\alpha_1| = 0$, we use self-adjointness and the already proved estimate

$$\|\langle x\rangle^{-s_2}(P-\lambda^2)^{-1}\langle x\rangle^{-s_1} f\|_{H^j} \le C(1+|\operatorname{Re}\lambda|)^{j-1}\|f\|_{L^2}, \quad j\in\{0,1,2\},$$

to get

$$\begin{aligned} \|u\|_{L^2}^2 &= \langle u, \langle x\rangle^{-s_1}(P-\lambda^2)^{-1}\langle x\rangle^{-s_2}\partial_x^{\alpha_2} f\rangle_{L^2}\\ &\le \|\partial_x^{\alpha_2}\langle x\rangle^{-s_2}(P-\bar{\lambda}^2)^{-1}\langle x\rangle^{-s_1} u\|_{L^2}\|f\|_{L^2}\\ &\le C(1+|\operatorname{Re}\lambda|)^{|\alpha_2|-1}\|u\|_{L^2}\|f\|_{L^2}. \end{aligned}$$

If $|\alpha_1| = 1$, we recognize that

$$(P-\lambda^2)u = \langle x\rangle^{-s_1-s_2}\partial_x^{\alpha_2} f + [-\Delta, \langle x\rangle^{-s_1}]\langle x\rangle^{s_1} u.$$

Then multiply by $\bar{u}$, integrate over $\mathbb{R}^n$, and integrate by parts as appropriate:

$$\begin{aligned}\|\nabla u\|_{L^2}^2 &= \int (\lambda^2 - V)|u|^2 - \int \partial_x^{\alpha_2}(\langle x\rangle^{-s_1-s_2}\bar{u}) f + \int \bar{u}[-\Delta, \langle x\rangle^{-s_1}]\langle x\rangle^{s_1} u \\ &\leq \lambda^2 \int |u|^2 - \int \partial_x^{\alpha_2}(\langle x\rangle^{-s_1-s_2}\bar{u}) f + \int \bar{u}[-\Delta, \langle x\rangle^{-s_1}]\langle x\rangle^{s_1} u.\end{aligned}$$

Because both $\partial_x^{\alpha_2}\langle x\rangle^{-s_1-s_2}$ and

$$[-\Delta, \langle x\rangle^{-s_1}]\langle x\rangle^{s_1} = (-\Delta\langle x\rangle^{-s_1})\langle x\rangle^{s_1} - 2(\nabla\langle x\rangle^{-s_1})\cdot\nabla\langle x\rangle^{s_1}$$

are first-order differential operators with bounded coefficients, we conclude, for all $\gamma > 0$,

$$\begin{aligned}\|\nabla u\|_{L^2}^2 &\leq C_\gamma((1+|\operatorname{Re}\lambda|)^2\|u\|_{L^2}^2 + \|f\|_{L^2}^2) + \gamma\|\nabla u\|_{L^2}^2 \\ &\leq C_\gamma(1+|\operatorname{Re}\lambda|)^2\|f\|_{L^2}^2 + \gamma\|\nabla u\|_{L^2}^2,\end{aligned}$$

for some $C_\gamma > 0$ depending on γ.

Fixing γ small enough, we absorb the second term on the right side into the left side, confirming (50) when $|\alpha_1| = |\alpha_2| = 1$. □

Next, we prove an estimate for the λ-derivative of the weighted resolvent, which requires the extra short range conditions (16) and (17) on the potential. As input we need the following bound for the weighted square of the free resolvent, which we prove in Appendix E.

Lemma 3.2. *Let $n \geq 3$. Suppose s satisfies (19). There exists $C > 0$ such that for all $\lambda \in \mathbb{C}$ with $0 < |\operatorname{Im}\lambda| \leq 1$, and any multiindex α such that $|\alpha| \leq 1$,*

$$\left\|\lambda\langle x\rangle^{-s}\partial_x^\alpha(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s}\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C(1+|\lambda|)^{|\alpha|-1}. \tag{54}$$

Remark 3.3. In [Vo04a], the estimate (54) is stated to hold in any dimension $n \geq 3$ provided $s > \frac{3}{2}$. However, our proof of Lemma 3.2 in dimension $n \geq 4$ needs s larger if (54) is to hold uniformly as $|\lambda| \to 0$. In our approach, we use the integral kernel of $\lambda\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s}$ to assess L^2-boundedness as $|\lambda| \to 0$. The kernel is given in terms of the Macdonald function [DLMF, 10.27.4, 10.27.5] of order $(n/2) - 2$, along with other factors. We are able to conclude boundedness on $L^2(\mathbb{R}^n)$ for s as in (19).

Lemma 3.4. *Let $n \geq 3$ and suppose s is as in (19). Assume V obeys (2)–(5) and (9), as well as (16) and (17). There exists $C > 0$ such that for all $\lambda \in \mathbb{C}$ with $0 < |\operatorname{Im}\lambda| \leq 1$, and any $j \in \{0, 1\}$ and multiindex α such that $j + |\alpha| \leq 1$,*

$$\left\|\frac{d}{d\lambda}\langle x\rangle^{-s}\lambda^j\partial_x^\alpha(P-\lambda^2)^{-1}\langle x\rangle^{-s}\right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq C. \tag{55}$$

Proof. Without loss of generality, we take s sufficiently close to (but larger than) $\frac{1}{4}(n+3)$ when $n \neq 8$, or sufficiently close to (but larger than) 3 when $n = 8$, so that by (17) we may fix $s' > \frac{1}{2}$ such that $s + s' < \delta$. With these choices we have

$s' + s > 2$, so we are permitted to apply (50) as the need arises.

We begin from the resolvent identity

(56) $$(P - \lambda^2)^{-1} \langle x \rangle^{-s} (I + K(\lambda)) = R_0(\lambda) \langle x \rangle^{-s},$$

where

$$K(\lambda) := V(x) \langle x \rangle^{s+s'} \langle x \rangle^{-s'} R_0(\lambda) \langle x \rangle^{-s} \quad \text{and} \quad R_0(\lambda) := (-\Delta - \lambda^2)^{-1}.$$

We know that $\langle x \rangle^{-s'} R_0(\lambda) \langle x \rangle^{-s} : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ has a continuous extension from either half-plane ($\pm \operatorname{Im} \lambda > 0$) to $\mathbb{R}$ [GiMo74, Proposition 2.4]. Let us denote this extension by $R^{\pm}_{0,s',s}(\lambda)$ and put $K^{\pm}(\lambda) = V(x) \langle x \rangle^{s+s'} R^{\pm}_{0,s',s}(\lambda)$.

We now show that $K^{\pm}(\lambda)$ is a compact operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. To see this, write $K^{\pm}(\lambda)$ as the sum

$$K^{\pm}(\lambda) = (\chi \langle x \rangle^{s+s'} V) R^{\pm}_{0,s',s}(\lambda) + ((1 - \chi) V \langle x \rangle^{\delta}) \langle x \rangle^{s+s'-\delta} R^{\pm}_{0,s',s}(\lambda).$$

where $\chi \in C_0^\infty(\mathbb{R}^n; [0,1])$ is supported in $B(0,1)$ and $\chi \equiv 1$ near the origin in $\mathbb{R}^n$. The second operator on the right side of the display is compact by [DyZw19, Theorem B.4]). The first operator on the right is compact, as follows: It is the composition of bounded $R^{\pm}_{0,s',s}(\lambda) : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ with multiplication by $\chi \langle x \rangle^{s+s'} V$. Due to (3) and Lemma F.1, we have $\|\chi \langle x \rangle^{s+s'} V u\|_{L^2(\mathbb{R}^n)} \le C \|u\|_{H^1(B(0,1))}$ for some $C > 0$ and all $u \in H^2(\mathbb{R}^n)$. By the Kondrachov embedding theorem the inclusion $H^2(B(0,1)) \to H^1(B(0,1))$ is compact. So compactness of $(\chi \langle x \rangle^{s+s'} V) R^{\pm}_{0,s',s}(\lambda)$ holds as desired.

We claim further that $I + K^{\pm}(\lambda) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is invertible for all λ with $\pm \operatorname{Im} \lambda \ge 0$. By compactness of $K^{\pm}(\lambda)$ and the Fredholm alternative [ReSi80, Theorem VI.14], $I + K^{\pm}(\lambda)$ will be invertible if we can show that for $g \in L^2(\mathbb{R}^n)$, $(I + K^{\pm}(\lambda))g = 0$ implies $g = 0$. To this end, put $u := \langle x \rangle^{s'} R^{\pm}_{0,s',s}(\lambda) g$, which belongs to $\langle x \rangle^{s'} H^2(\mathbb{R}^n)$. If we can show $u = 0$, then in fact $g = 0$. This is because $(-\Delta - \lambda^2) u = \langle x \rangle^{-s} g$ in the distributional sense.

Now let us show $u = 0$. If $\lambda^2 \in \mathbb{C} \setminus [0, \infty)$ (so that $K^{\pm}(\lambda) = K(\lambda)$), this follows immediately from $(P - \lambda^2) u = \langle x \rangle^{-s} g + V R_0(\lambda) \langle x \rangle^{-s} g = \langle x \rangle^{-s} (I + K(\lambda)) g = 0$. If $\lambda^2 \in [0, \infty)$, the idea is the same, but we incorporate a limiting step. Set $u_{\pm,\varepsilon} = (-\Delta - \lambda^2 \pm i\varepsilon)^{-1} \langle x \rangle^{-s} g$. Proposition 2.4 of [GiMo74] implies that $\langle x \rangle^{-s'} u_{\pm,\varepsilon}$ converges to $\langle x \rangle^{-s'} u$ in $H^2(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. We also have

$$\begin{aligned} u_{\pm,\varepsilon} &= (-\Delta - \lambda^2 \pm i\varepsilon)^{-1} \langle x \rangle^{-s} g \\ &= (P - \lambda^2 \pm i\varepsilon)^{-1} (P - \lambda^2 \pm i\varepsilon)(-\Delta - \lambda^2 \pm i\varepsilon)^{-1} \langle x \rangle^{-s} g \\ &= (P - \lambda^2 \pm i\varepsilon)^{-1} \langle x \rangle^{-s} (I + V \langle x \rangle^{s} (-\Delta - \lambda^2 \pm i\varepsilon)^{-1} \langle x \rangle^{-s}) g. \end{aligned}$$

Therefore, by (12),

$$\begin{aligned}\|\langle x\rangle^{-s'}u\|_{L^2} &= \lim_{\varepsilon\to 0^+}\|\langle x\rangle^{-s'}u_{\pm,\varepsilon}\|_{L^2}\\ &\le C\lim_{\varepsilon\to 0^+}\|(I+V\langle x\rangle^{s}(-\Delta-\lambda^2\pm i\varepsilon)^{-1}\langle x\rangle^{-s})g\|_{L^2}\\ &= \|(I+K^{\pm}(\lambda))g\|_{L^2}=0.\end{aligned}$$

Thus we have demonstrated that $I+K^{\pm}(\lambda)$ is invertible for $\pm\operatorname{Im}\lambda\ge 0$. As $\lambda\to\infty$, $\|K(\lambda)\|_{L^2\to L^2}\to 0$ thanks to (50); hence we can compute $(I+K^{\pm}(\lambda))^{-1}$ by a Neumann series, thanks to (50). Therefore

(57) $$\|(I+K^{\pm}(\lambda))^{-1}\|_{L^2\to L^2}\le C.$$

Now for $0<|\operatorname{Im}\lambda|\le 1$ the λ-derivative of (56) is

(58) $$\begin{aligned}&\Big(\frac{d}{d\lambda}\langle x\rangle^{-s}\lambda^{j}\partial_x^{\alpha}(P-\lambda^2)^{-1}\langle x\rangle^{-s}\Big)(I+K(\lambda))\\ &\quad=\frac{d}{d\lambda}\langle x\rangle^{-s}\lambda^{j}\partial_x^{\alpha}R_0(\lambda)\langle x\rangle^{-s}\\ &\qquad-2\langle x\rangle^{-s}\lambda^{j}\partial_x^{\alpha}(P-\lambda^2)^{-1}\langle x\rangle^{-s'}V\langle x\rangle^{s+s'}\lambda\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s},\end{aligned}$$

where we used

(59) $$\frac{d}{d\lambda}\langle x\rangle^{-s'}\partial_x^{\alpha}R_0(\lambda)\langle x\rangle^{-s}=2\lambda\langle x\rangle^{-s'}\partial_x^{\alpha}(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s},\qquad j\in\{0,1\}.$$

The operator norm $L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ of the term in the second line of (58) is bounded above by a constant, due to (54) and (59). As for the third line, $\big\|\langle x\rangle^{-s}\lambda^{j}\partial_x^{\alpha}(P-\lambda^2)^{-1}\langle x\rangle^{-s'}\big\|_{L^2\to L^2}\le C$ by (50). Moreover,

$$\big\|V\langle x\rangle^{s+s'}\lambda\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s}\big\|_{L^2\to L^2}\le C\big\|\lambda\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s}\big\|_{H^1\to L^2},$$

since multiplication by $V\langle x\rangle^{s+s'}$ is a bounded operator $H^1(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ (see (3) and Lemma F.1). Finally, because $\big\|\lambda\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s}\big\|_{H^1\to L^2}\le C$ by (54), the proof of (55) is complete. □

4. Proof of Theorem 1.6

In this section we prove Theorem 1.6 by combining the resolvent bounds of the previous section with an argument from [Vo04a, Section 3]. As before we set $P=-\Delta+V:L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$, $n\ge 3$, where V obeys (2)–(9), (16) and (17).

In several steps below, we use that for all $0\le\alpha\le 1$ there exists $C>0$ such that, for any $f\in H^1(\mathbb{R}^n)$,

(60) $$\|V^{\alpha}f\|_{L^2}^2\le C(\|\nabla f\|_{L^2}^2+\|f\|_{L^2}^2)\le C\|\nabla f\|_{L^2}^2.$$

The first inequality follows from (3), (16) and Lemma F.1, while the second follows from the Poincaré inequality (as we work in dimension $n\ge 3$).

Given $s>0$ and u as in (18) solving the wave equation (15), with compactly supported initial conditions $u(0,x)=u_0(x)\in H^1(\mathbb{R}^n)$, $\partial_t u(0,x)=u_1(x)\in L^2(\mathbb{R}^n)$, define

$$E_s(t) := \int_{\mathbb{R}^n} \langle x\rangle^{-2s}(|\partial_t u(t,x)|^2+|\nabla u(t,x)|^2+|u(t,x)|^2)\,dx,$$
$$E(0) := \|\nabla u_0\|_{L^2}^2+\|u\|_{L^2}^2.$$

Lemma 4.1. *If $s>\frac{1}{2}$ and V satisfies (2)–(9), there exists $C>0$ such that*

(61) $$\int_0^\infty E_s(\tau)\,d\tau \le CE(0).$$

If in addition s satisfies (19) and V (16) and (17), there exists $C>0$ such that, for $t\ge 1$,

(62) $$\int_t^\infty E_s(\tau)\,d\tau \le Ct^{-2}E(0).$$

Proof. Choose $\phi\in C^\infty(\mathbb{R})$ with $\phi\ge 0$, $\phi(t)=0$ near $\left(-\infty,\frac{1}{2}\right]$, and $\phi(t)=1$ near $[1,\infty)$. Since $(\partial_t^2+P)u=0$, where $P=-\Delta+V$, we have

(63) $$(\partial_t^2+P)\phi u=(\phi''+2\phi'\partial_t)u := v(t).$$

By Duhamel's formula for the solution to an inhomogeneous wave equation with zero initial conditions,

$$\phi u(t)=\int_0^t \frac{\sin(t-\tau)\sqrt{P}}{\sqrt{P}}v(\tau)\,d\tau.$$

On the other hand,

$$(P-(\lambda-i\varepsilon)^2)^{-1}=\int_0^\infty e^{-it(\lambda-i\varepsilon)}\frac{\sin(t\sqrt{P})}{\sqrt{P}}dt,\qquad \varepsilon>0.$$

It follows from the last two identities that the Fourier transform $\widehat{\phi u}$ of ϕu satisfies

(64) $$\widehat{\phi u}(\lambda-i\varepsilon) := \int_{-\infty}^\infty e^{-it(\lambda-i\varepsilon)}\phi(t)u(\,\cdot\,,t)\,dt=(P-(\lambda-i\varepsilon)^2)^{-1}\hat{v}(\lambda-i\varepsilon).$$

By finite propagation speed for the wave equation, and because $v(t)$ is compactly supported in t, $\operatorname{supp}_x v(t)$, and thus also $\operatorname{supp}_x \hat{v}(\lambda)$, is contained in some compact subset of $\mathbb{R}^n$ independent of t. Choose $\eta\in C_0^\infty(\mathbb{R}^n)$ such that $\eta=1$ near $\operatorname{supp}_x v(t)$ for all $t\in\mathbb{R}$. By (64),

$$\langle x\rangle^{-s}\widehat{\phi u}(\lambda-i\varepsilon)=\langle x\rangle^{-s}(P-(\lambda-i\varepsilon)^2)^{-1}\eta\hat{v}(\lambda-i\varepsilon),$$
$$\langle x\rangle^{-s}\widehat{\partial_t(\phi u)}(\lambda-i\varepsilon)=\langle x\rangle^{-s}(\lambda-i\varepsilon)(P-(\lambda-i\varepsilon)^2)^{-1}\eta\hat{v}(\lambda-i\varepsilon),$$
$$\langle x\rangle^{-s}\nabla\widehat{\phi u}(\lambda-i\varepsilon)=\langle x\rangle^{-s}\nabla(P-(\lambda-i\varepsilon)^2)^{-1}\eta\hat{v}(\lambda-i\varepsilon).$$

Therefore, by (50), for $s>\frac{1}{2}$ and V obeying (2)–(9), there is $C>0$ independent of λ and ε and such that for all $\lambda\in\mathbb{R}$, $0<\varepsilon\le 1$, we have

$$\left\| \frac{d^k}{d\lambda^k} \langle x\rangle^{-s} \widehat{\partial_t(\phi u)}(\lambda - i\varepsilon) \right\|_{L^2} + \left\| \frac{d^k}{d\lambda^k} \langle x\rangle^{-s} \nabla \widehat{\phi u}(\lambda - i\varepsilon) \right\|_{L^2} + \left\| \frac{d^k}{d\lambda^k} \langle x\rangle^{-s} \widehat{\phi u}(\lambda - i\varepsilon) \right\|_{L^2} \le C \|\hat{v}(\lambda - i\varepsilon)\|_{L^2} + Ck \|\widehat{tv}(\lambda - i\varepsilon)\|_{L^2} \tag{65}$$

for $k = 0$. If in addition we suppose s satisfies (19) and V satisfies (16) and (17), then by (55), (65) holds for $k \in \{0, 1\}$. Here when $k = 1$ we used the product rule and the identity $d\hat{v}(\lambda - i\varepsilon)/d\lambda = -i\widehat{tv}(\lambda - i\varepsilon)$.

Next, by (65) and Plancherel's theorem, there exist $C_1, C_2, C_3, C > 0$ independent of ε and such that

$$\begin{aligned} &\int_{-\infty}^{\infty} (\|\langle x\rangle^{-s} \partial_t(\phi u)\|_{L^2}^2 + \|\langle x\rangle^{-s} \nabla(\phi u)\|_{L^2}^2 + \|\langle x\rangle^{-s} \phi u\|_{L^2}^2) e^{-2\varepsilon t} dt \\ &= C_1 \int_{-\infty}^{\infty} (\|\langle x\rangle^{-s} \widehat{\partial_t(\phi u)}(\lambda - i\varepsilon)\|_{L^2}^2 + \|\langle x\rangle^{-s} \nabla \widehat{\phi u}(\lambda - i\varepsilon)\|_{L^2}^2 \\ &\qquad + \|\langle x\rangle^{-s} \widehat{\phi u}(\lambda - i\varepsilon)\|_{L^2}^2)\, d\lambda \\ &\le C_2 \int_{-\infty}^{\infty} \|\hat{v}(\lambda - i\varepsilon)\|_{L^2}^2 d\lambda = C_3 \int_{-\infty}^{\infty} \|v(t)\|_{L^2}^2 e^{-2\varepsilon t} dt \le C \sup_{t\in\mathbb{R}} \|v(t)\|^2. \end{aligned} \tag{66}$$

The last constant C is independent of ε because $v(t)$ has compact support in t; see (63). The proof of (61) is completed by sending $\varepsilon \to 0$ in (66) and observing that

$$\begin{aligned} \|v(t)\|_{L^2} &\le C(\|u_0\|_{L^2} + \|\sqrt{P} u_0\|_{L^2} + \|u_1\|_{L^2}) \\ &\le C(\|\nabla u_0\|_{L^2} + \|u_1\|_{L^2}) = C\sqrt{E(0)}. \end{aligned} \tag{67}$$

In the last inequality we used that for any $f \in H^2(\mathbb{R}^n)$ (and thus any $f \in H^1(\mathbb{R}^n)$, since $H^2(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$),

$$\|\sqrt{P} f\|_{L^2}^2 = \langle f, Pf\rangle_{L^2} = \|\nabla f\|_{L^2}^2 + \|\sqrt{V} f\|_{L^2}^2 \le C\|\nabla f\|_{L^2}^2,$$

with the inequality being due to (60).

To prove (62), we again use Plancherel's theorem with (65), so that for all $0 < \varepsilon \le 1$ and $T \ge 1$,

$$\begin{aligned} &T^2 \int_T^{\infty} \big(\|\langle x\rangle^{-s} \partial_t(\phi u)\|_{L^2}^2 + \|\langle x\rangle^{-s} \nabla(\phi u)\|_{L^2}^2 + \|\langle x\rangle^{-s} \phi u\|_{L^2}^2\big) e^{-2\varepsilon t}\, dt \\ &\le \int_{-\infty}^{\infty} \big(\|\langle x\rangle^{-s} t\partial_t(\phi u)\|_{L^2}^2 + \|\langle x\rangle^{-s} t\nabla(\phi u)\|_{L^2}^2 + \|\langle x\rangle^{-s} t\phi u\|_{L^2}^2\big) e^{-2\varepsilon t}\, dt \\ &= C_1 \int_{-\infty}^{\infty} \Big(\Big\| \frac{d}{d\lambda} \langle x\rangle^{-s} \widehat{\partial_t(\phi u)}(\lambda - i\varepsilon) \Big\|_{L^2}^2 + \Big\| \frac{d}{d\lambda} \langle x\rangle^{-s} \nabla \widehat{\phi u}(\lambda - i\varepsilon) \Big\|_{L^2}^2 \\ &\qquad + \Big\| \frac{d}{d\lambda} \langle x\rangle^{-s} \widehat{\phi u}(\lambda - i\varepsilon) \Big\|_{L^2}^2 \Big)\, d\lambda \\ &\le C_2 \int_{-\infty}^{\infty} \big(\|\hat{v}(\lambda - i\varepsilon)\|_{L^2}^2 + \|\widehat{tv}(\lambda - i\varepsilon)\|_{L^2}^2 \big)\, d\lambda \\ &= C_3 \int_{-\infty}^{\infty} \big(\|v(t)\|_{L^2}^2 + \|tv(t)\|_{L^2}^2 \big) e^{-2\varepsilon t}\, dt \le C \sup_{t\in\mathbb{R}} \|v(t)\|^2 \le CE(0). \end{aligned} \tag{68}$$

Once again sending $\varepsilon \to 0^+$ concludes the proof of (62). □

The local energy decay (20) follows from (62) and the following:

Lemma 4.2. *If $s > 0$ and $V \geq 0$ satisfies (3) and (4), there exists $C > 0$ such that, for all $t \geq 1$,*

$$E_s(t) \leq C \int_t^\infty E_s(\tau)\, d\tau. \tag{69}$$

Proof. The strategy is the same as that of [Vo04a, Lemma 3.2]. Computing $\frac{d}{dt}E_s(t)$, one finds

$$\begin{aligned}\frac{d}{dt}E_s(t) = -2\,\mathrm{Re}\int_{\mathbb{R}^n} \partial_r u(t,x)\overline{\partial_t u(t,x)}\partial_r \langle x\rangle^{-2s}\, dx \\ + 2\,\mathrm{Re}\int_{\mathbb{R}^n}(-Vu(t,x)\overline{\partial_t u(t,x)} + u(t,x)\overline{\partial_t u(t,x)})\langle x\rangle^{-2s}\, dx.\end{aligned} \tag{70}$$

By (60),

$$\begin{aligned}\|V\langle x\rangle^{-s}u(t,x)\|_{L^2} &\leq C\|\nabla\langle x\rangle^{-s}u(t,x)\|_{L^2} \\ &\leq C\,\|\langle x\rangle^{-s}\nabla u(t,x)\|_{L^2} + C\,\|\langle x\rangle^{-s}u(t,x)\|_{L^2}.\end{aligned}$$

for $C > 0$ independent of t, and whose precise value may change between lines. Thus we can bound the right side of (70) from above by Cauchy–Schwarz:

$$\begin{aligned}&\frac{d}{dt}E_s(t) \\ &\leq C\left\|\langle x\rangle^{-s}\partial_r u(t,x)\right\|_{L^2}\left\|\langle x\rangle^{-s}\partial_t u(t,x)\right\|_{L^2} + C\left\|V\langle x\rangle^{-s}u(t,x)\right\|_{L^2}\left\|\langle x\rangle^{-s}\partial_t u(t,x)\right\|_{L^2} \\ &\qquad + C\left\|\langle x\rangle^{-s}u(t,x)\right\|_{L^2}\left\|\langle x\rangle^{-s}\partial_t u(t,x)\right\|_{L^2} \\ &\leq CE_s(t).\end{aligned}$$

We then have, for all $T > t \geq 1$,

$$E_s(t) \leq E_s(T) + C_s \int_t^T E_s(\tau)\, d\tau. \tag{71}$$

From (61), we also have a sequence $T_j \to \infty$ such that $\lim_{T_j \to \infty} E_s(T_j) = 0$. So setting $T = T_j$ in (71) and sending $T_j \to \infty$ completes the proof of (69). □

Appendix A. Proof of Lemma 2.1

First we check that

$$C_0^\infty(0,\infty) \ni \varphi \mapsto \int_{\mathbb{S}^{n-1}}\int_0^\infty \varphi(r)|u(r,\theta)|^2\, dV(r,\theta)\, d\theta \tag{72}$$

(part of the expression in (27)) is well defined as a distribution on $(0,\infty)$. Indeed, by (7), for each $\theta \in \mathbb{S}^{n-1}$,

$$\int_0^\infty \varphi(r)|u(r,\theta)|^2\, dV(r,\theta) = -\int_0^\infty V(r,\theta)(|u(r,\theta)|^2\varphi(r))' dr.$$

By Fubini's theorem the expression on the right side belongs to $L^1(\mathbb{S}^{n-1})$. Hence the quantity in (72) is well defined as a distribution on $(0,\infty)$.

Next we demonstrate that the function $r \mapsto \int_{\mathbb{S}^{n-1}} V^R(r,\theta)\,|u(r,\theta)|^2\,d\theta$ has locally bounded variation. Suppose $[a,b] \subseteq (0,\infty)$ and $a = r_0 < r_1 < \cdots < r_N = b$. We have

$$\begin{aligned}\sum_{k=1}^{N} \Big| \int_{\mathbb{S}^{n-1}} V^R(r_k,\theta)\,|u(r_k,\theta)|^2\,d\theta - \int_{\mathbb{S}^{n-1}} V^R(r_{k-1},\theta)\,|u(r_{k-1},\theta)|^2\,d\theta \Big| \\ \le \int_{\mathbb{S}^{n-1}} \sum_{k=1}^{N} |u(r_k,\theta)|^2\,|V^R(r_k,\theta) - V^R(r_{k-1},\theta)|\,d\theta \\ + \int_{\mathbb{S}^{n-1}} \sum_{k=1}^{N} |V^R(r_{k-1},\theta)|\,\big||u(r_k,\theta)|^2 - |u(r_{k-1},\theta)|^2\big|\,d\theta.\end{aligned}$$

For each θ, the function $r \mapsto V^R(r,\theta)$ is decreasing, so

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \sum_{k=1}^{N} |u(r_k,\theta)|^2 |V^R(r_k,\theta) - V^R(r_{k-1},\theta)|\,d\theta \\ \le \|u\|^2_{L^\infty([a,b]\times\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \sum_{k=1}^{N} |V^R(r_k,\theta) - V^R(r_{k-1},\theta)|\,d\theta \\ \le \|u\|^2_{L^\infty([a,b]\times\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} (V^R(a,\theta) - V^R(b,\theta))\,d\theta \\ \le \omega_{n-1} \|u\|^2_{L^\infty([a,b]\times\mathbb{S}^{n-1})} \|V\|_{L^\infty([a,b]\times\mathbb{S}^{n-1})},\end{aligned}$$

where ω_n is the $(n-1)$-dimensional volume of $\mathbb{S}^{n-1}$. On the other hand,

$$\begin{aligned}\sum_{k=1}^{N} \int_{\mathbb{S}^{n-1}} |V^R(r_{k-1},\theta)|\,\big||u(r_k,\theta)|^2 - |u(r_{k-1},\theta)|^2\big|\,d\theta \\ \le \|V\|_{L^\infty([a,b]\times\mathbb{S}^{n-1})} \sum_{k=1}^{N} \int_{\mathbb{S}^{n-1}} \int_{r_{k-1}}^{r_k} |\partial_r |u(r,\theta)|^2|\,dr\,d\theta \\ \le \omega_{n-1}(b-a) \|V\|_{L^\infty([a,b]\times\mathbb{S}^{n-1})} \|\partial_r |u|^2\|_{L^\infty([a,b]\times\mathbb{S}^{n-1})}.\end{aligned}$$

Taken together, the previous two estimates show that $r \mapsto \int_{\mathbb{S}^{n-1}} V^R(r,\theta)|u(r,\theta)|^2\,d\theta$ has locally bounded variation.

We finish by confirming (27). Let $\varphi \in C_0^\infty(0,\infty)$. In the following chain of equalities, the first uses Fubini's theorem, the second arises because for each $\theta \in \mathbb{S}^{n-1}$, $V^R(\,\cdot\,,\theta) = V(\,\cdot\,,\theta)$ almost everywhere with respect to the measure dr, and the last uses (7):

$$\begin{aligned}&-\int_0^\infty \varphi'(r) \int_{\mathbb{S}^{n-1}} V^R(r,\theta)|u(r,\theta)|^2\,d\theta\,dr \\ &\quad= -\int_{\mathbb{S}^{n-1}} \int_0^\infty \varphi'(r) V^R(r,\theta)|u(r,\theta)|^2\,dr\,d\theta \\ &\quad= -\int_{\mathbb{S}^{n-1}} \int_0^\infty \varphi'(r) V(r,\theta)|u(r,\theta)|^2\,dr\,d\theta \\ &\quad= -\int_{\mathbb{S}^{n-1}} \int_0^\infty \big((\varphi(r)\,|u(r,\theta)|^2)' - \varphi(r)\,2\,\mathrm{Re}(\bar{u}(r,\theta)u'(r,\theta))\big)\,V(r,\theta)\,dr\,d\theta\end{aligned}$$

$$= \int_{\mathbb{S}^{n-1}} \int_0^\infty \varphi'(r) |u(r,\theta)|^2 \, dV(r,\theta) \, d\theta$$
$$+ 2 \int_{\mathbb{S}^{n-1}} \int_0^\infty \varphi(r) \operatorname{Re}(\bar{u}(r,\theta) u'(r,\theta))) V(r,\theta) \, dr \, d\theta.$$

Appendix B. The Phragmén–Lindelöf theorem

Let $f(z)$ be a holomorphic function in a domain D of the complex plane with boundary Γ. We say that $f(z)$ does not exceed a number $M \geq 0$ in modulus at a boundary point $\zeta \in \Gamma$ if $\limsup_{z \to \zeta,\, z \in D} |f(z)| \leq M$.

Theorem B.1 (Phragmén–Lindelöf [EM]). *Suppose that $E \subseteq \Gamma$ and that $f : D \to \mathbb{C}$ is analytic and does not exceed M in modulus at any point of $\Gamma \setminus E$. Suppose also there is a function $g(z)$ with the following properties*:

(1) *$g(z)$ is analytic in D.*

(2) *$|g(z)| < 1$ in D.*

(3) *$g(z) \neq 0$ in D.*

(4) *For every $\sigma > 0$, the function $|g(z)|^\sigma \, |f(z)|$ does not exceed M in modulus at any $\zeta \in E$.*

Then $|f(z)| \leq M$ everywhere in D.

Appendix C. Density argument: proof of (42) and (48)

Estimates (42) and (48) are consequences of the following fact:

Lemma C.1. *Fix h, $s_1 > 0$, $0 < s_2 < 1$, and $z \in \mathbb{C} \setminus [0, \infty)$. Let $P(h)$ be as in (1) with $V : \mathbb{R}^n \to \mathbb{R}$ obeying (3) and (4) (so that $P(h)$ is self-adjoint with respect to the domain $H^2(\mathbb{R}^n)$). Suppose there exists $C > 0$ so that for all $v \in C_0^\infty(\mathbb{R}^n)$,*

$$\|\langle x\rangle^{-s_1} v\|_{L^2}^2 \leq C \|\langle x\rangle^{s_2} (P(h) - z) v\|_{L^2}^2. \tag{73}$$

Then

$$\|\langle x\rangle^{-s_1} (P(h) - z)^{-1} \langle x\rangle^{-s_2}\|_{L^2 \to L^2} \leq C. \tag{74}$$

Proof. The operator

$$[P(h), \langle x\rangle^{s_2}]\langle x\rangle^{-s_2} = \left(-h^2(\Delta\langle x\rangle^{s_2}) - 2h^2(\nabla\langle x\rangle^{s_2}) \cdot \nabla\right) \langle x\rangle^{-s_2}$$

is bounded $H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. So, for $v \in H^2(\mathbb{R}^n)$ such that $\langle x\rangle^{s_2} v \in H^2(\mathbb{R}^n)$,

$$\begin{aligned} \|\langle x\rangle^{s_2}(P(h) - z)v\|_{L^2} &\leq \|(P(h) - z)\langle x\rangle^{s_2} v\|_{L^2} + \|[P(h), \langle x\rangle^{s_2}]\langle x\rangle^{-s_2}\langle x\rangle^{s_2} v\|_{L^2} \\ &\leq C_{z,h} \|\langle x\rangle^{s_2} v\|_{H^2}, \end{aligned} \tag{75}$$

for some constant $C_{z,h} > 0$ depending on z and h.

Given $f \in L^2(\mathbb{R}^n)$, the function $u = \langle x\rangle^{s_2}(P(h)-z)^{-1}\langle x\rangle^{-s_2} f \in H^2(\mathbb{R}^n)$ because

$$u = (P(h)-z)^{-1}(f+w), \quad w = [P(h), \langle x\rangle^{s_2}]u,$$

with $[P(h), \langle x\rangle^{s_2}]$ being bounded $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ since $s_2 < 1$.

Now, choose a sequence $v_k \in C_0^\infty$ such that $v_k \to \langle x\rangle^{s_2}(P(h)-z)^{-1}\langle x\rangle^{-s_2} f$ in $H^2(\mathbb{R}^n)$. Define $\tilde{v}_k := \langle x\rangle^{-s_2} v_k$. Then, as $k \to \infty$,

$$\left\| \langle x\rangle^{-s_1}\tilde{v}_k - \langle x\rangle^{-s_1}(P(h)-z)^{-1}\langle x\rangle^{-s_2} f \right\|_{L^2} \le \left\| v_k - \langle x\rangle^{s_2}(P(h)-z)^{-1}\langle x\rangle^{-s_2} f \right\|_{H^2},$$

which tends to 0. Also, applying (75),

$$\| \langle x\rangle^{s_2}(P(h)-z)\tilde{v}_k - f\|_{L^2} \le C_{z,h}\| v_k - \langle x\rangle^{s_2}(P(h)-E \pm i\varepsilon)^{-1}\langle x\rangle^{-s_2} f\|_{H^2} \to 0.$$

Thus (74) follows by replacing v by $\tilde{v}_k$ in (73) and sending $k \to \infty$. □

Appendix D. Justification of Remark 1.4

In the setting of Theorem 1.2, consider the case of $V = 0$ and $n = 3$. In that scenario the integral kernel of $(P(h)-z)^{-1} = (-h^2\Delta - z)^{-1}$ with $z \in \mathbb{C} \setminus [0, \infty)$ is given by

$$R_0(x, y, z) := h^{-2}\frac{e^{i\frac{\sqrt{z}}{h}|x-y|}}{4\pi|x-y|}, \quad \operatorname{Im}\sqrt{z} > 0.$$

Looking at the operator

$$\langle \cdot \rangle^{-s_1}(-h^2\Delta - z)^{-1}\langle \cdot \rangle^{-s_2} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \tag{76}$$

we recall why having a bound on it like (11) requires $s_1,\ s_2 > \frac{1}{2}$.

Since the norm of an operator and its adjoint coincide, it suffices to show that $s_1 > \frac{1}{2}$ is necessary. Use $\sqrt{z}$ of the form $\sqrt{z} = E + i\varepsilon$ for $E > 0$ fixed and $\varepsilon > 0$ tending to zero. As calculated in the proof of [DyZw19, Theorem 3.5], for $f \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} \langle x\rangle^{-s_1}\int_{\mathbb{R}^3} R_0(x, y, z) f(y)\, dy \\ = h^{-2}\frac{\langle x\rangle^{-s_1}}{4\pi|x|} e^{\frac{i}{h}(E+i\varepsilon)|x|}\Big(\hat{f}\Big(\frac{E}{h}\frac{x}{|x|}\Big) + o(1)\Big) + O(|x|^{-2}) \end{aligned} \tag{77}$$

as $\varepsilon \to 0^+$ and $|x| \to \infty$. If f is chosen so that $|\hat{f}| > c$ for some $c > 0$ on $\{|x| = E/h\}$, then (77) and $s_1 \le \frac{1}{2}$ imply $\|\langle x\rangle^{-s_1}\int_{\mathbb{R}^3} R_0(x, y, z) f(y)\, dy\|_{L^2} \to \infty$ as $\varepsilon \to 0^+$.

Next, supposing $s_1,\ s_2 > \frac{1}{2}$, we show why a bound like (12) on (76) requires additionally that $s_1 + s_2 \ge 2$, which is nearly the condition we impose for (12). Using $\sqrt{z} = i\varepsilon$ for $\varepsilon > 0$ tending to zero, and $f_\eta(y) = \langle y\rangle^{-\eta-\frac{3}{2}}$, $\eta > 0$, we see as in [BoHa10, proof of Remark 2] that

$$\begin{aligned}\langle x\rangle^{-s_1}\int_{\mathbb{R}^3} & R_0(x,y,z)\langle y\rangle^{-s_2} f(y)\\ &= h^{-2}\langle x\rangle^{-s_1}\int_{\mathbb{R}^3}\frac{e^{-\frac{\varepsilon}{h}|x-y|}}{4\pi|x-y|}\langle y\rangle^{-s_2} f(y)\,dy\\ &\gtrsim h^{-2}e^{-\frac{3\varepsilon}{2h}|x|}\langle x\rangle^{-s_1-1}\int_{|y|\le\frac{|x|}{2}}\langle y\rangle^{-s_2-\eta-\frac{3}{2}}\,dy \gtrsim h^{-2}e^{-\frac{3\varepsilon}{2h}|x|}\langle x\rangle^{-s_1-s_2-\eta+\frac{1}{2}},\end{aligned}$$

where the implicit constants indicated by $\gtrsim$ are independent of ε and η. First sending $\varepsilon\to 0^+$ gives $s_1+s_2\ge 2-\eta$, but since $\eta>0$ is arbitrary, we in turn get $s_1+s_2\ge 2$.

To see that the $O(|z|^{-\frac{1}{2}}h^{-1})$-dependence of the right side of (11) is optimal, consider the function $u=e^{i\frac{\sqrt{z}}{h}x_1}\chi$ for nontrivial $\chi\in C_0^\infty(\mathbb{R}^3;[0,1])$. We have

$$\langle x\rangle^s(-h^2\Delta-z)u=-i\sqrt{z}h\langle x\rangle^s\partial_{x_1}\chi-h^2e^{i\frac{\sqrt{z}}{h}x_1}\langle x\rangle^s\Delta\chi=:f,$$

whence $\langle\cdot\rangle^{-s}(-h^2\Delta-z)^{-1}\langle\cdot\rangle^{-s}f=\langle\cdot\rangle^{-s}u$ and thus, as $h\to 0$,

$$\frac{\|\langle\cdot\rangle^{-s}(-h^2\Delta-z)^{-1}\langle\cdot\rangle^{-s}f\|_{L^2}}{\|f\|_{L^2}}=\frac{\|\langle\cdot\rangle^{-s}u\|_{L^2}}{\|f\|_{L^2}}\gtrsim|z|^{-\frac{1}{2}}h^{-1}.$$

Finally, we argue why the $O(h^{-2})$-dependence of the right-hand side of (12) is sharp. As noted before, Proposition 2.4 or [GiMo74] gives that $R_{0,s_1,s_2}(\lambda)=\langle\cdot\rangle^{-s_1}(-h^2\Delta-\lambda^2)^{-1}\langle\cdot\rangle^{-s_2}$, with $s_1,\ s_2>\frac{1}{2}$ and $s_1+s_2>2$, extends continuously from $\operatorname{Im}\lambda>0$ to $\mathbb{R}$ in the space of bounded operators $\mathbb{R}^3\to\mathbb{R}^3$. In this case,

$$h^{-2}\Big\|\langle x\rangle^{-s_1}\int_{\mathbb{R}^3}\frac{1}{4\pi|x-y|}\langle y\rangle^{-s_2}dy\Big\|_{L^2\to L^2}=\lim_{\varepsilon\to 0}\|R_{0,s_1,s_2}(i\varepsilon)\|_{L^2\to L^2}.$$

Appendix E. Proof of Lemma 3.2

Lemma 3.2. *Let $n\ge 3$. Suppose s satisfies (19). There exists $C>0$ such that for all $\lambda\in\mathbb{C}$ with $0<|\operatorname{Im}\lambda|\le 1$, and any multiindex α such that $|\alpha|\le 1$,*

(78) $$\big\|\lambda\langle x\rangle^{-s}\partial_x^\alpha(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s}\big\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}\le C(1+|\lambda|)^{|\alpha|-1}.$$

Proof of Lemma 3.2. Initially we take $|\alpha|=0$ in (54), so may assume without loss of generality that $\operatorname{Im}\lambda>0$. We treat the $|\alpha|=1$ case at the end of the proof. Observe that

$$\frac{d}{d\lambda}\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-1}\langle x\rangle^{-s}=2\lambda\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s},$$

so we can bound the $L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ norm of either quantity.

We now consider two cases for $|\lambda|$.

Case $|\lambda| \geq 1$. If we assume $|\lambda| \geq 1$, then

$$(79)\quad -\lambda^2\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s} = -\tfrac{1}{2}\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-1}(-2\Delta)(-\Delta-\lambda^2)^{-1}\langle x\rangle^{-s}+\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-1}\langle x\rangle^{-s}.$$

By (50), the $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ norm of the second line of (79) is bounded by $C(1+|\operatorname{Re}\lambda|)^{-1}$. So it suffices to investigate the first summand on the right side of (79). For notational brevity, put $R_0(\lambda) := (-\Delta-\lambda^2)^{-1}$. We show that, for all $f \in C_0^\infty(\mathbb{R}^n)$,

$$(80)\quad \langle x\rangle^{-s}R_0(\lambda)(-2\Delta)R_0(\lambda)\langle x\rangle^{-s}f = -\langle x\rangle^{-s}R_0(\lambda)\partial_r(r\langle x\rangle^{-s}f)+\langle x\rangle^{-s}R_0(\lambda)\langle x\rangle^{-s}f+\langle x\rangle^{-s}r\partial_r R_0(\lambda)\langle x\rangle^{-s}f.$$

Since $s > \frac{3}{2}$ by (19), the $L^2(\mathbb{R}^n)$-norm of the right side of (80) is bounded by $C\|f\|_{L^2}$, thanks to (50). So it remains to show (80).

Recall the formula

$$\Delta = \partial_r^2 + (n-1)r^{-1}\partial_r + r^{-2}\Delta_{\mathbb{S}^{n-1}}$$

for the Laplacian in polar coordinates, which implies the commutator identity

$$(81)\quad [r\partial_r, \Delta] := r\partial_r(\Delta) - \Delta(r\partial_r) = -2\Delta.$$

Fix $f \in C_0^\infty(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n)$, and put

$$u := R_0(\lambda)\langle x\rangle^{-s}f \in H^2(\mathbb{R}^n).$$

Let $\{u_k\}_{k=1}^\infty \subseteq C_0^\infty(\mathbb{R}^n)$ be a sequence converging to u in $H^2(\mathbb{R}^n)$. Starting from the left side of (80) and applying (81),

$$(82)\quad \langle g, \langle x\rangle^{-s}R_0(\lambda)(-2\Delta)R_0(\lambda)\langle x\rangle^{-s}f\rangle_{L^2} = \lim_{k\to\infty}\langle g, \langle x\rangle^{-s}R_0(\lambda)[r\partial_r, \Delta]u_k\rangle_{L^2} = \langle g, \langle x\rangle^{-s}r\partial_r u\rangle_{L^2} - \lim_{k\to\infty}\big\langle g, \langle x\rangle^{-s}R_0(\lambda)r\partial_r(-\Delta-\lambda^2)u_k\big\rangle_{L^2}.$$

The purpose of the following calculations is to show that the limit on the last line of (82) equals $\langle g, \langle x\rangle^{-s}R_0(\lambda)r\partial_r\langle x\rangle^{-s}f\rangle_{L^2}$. First, for any $v \in L^2(\mathbb{R}^n)$, we have $rR_0(\lambda)\langle x\rangle^{-1}v \in H^1(\mathbb{R}^n)$, for the following reason. Setting $w := \langle x\rangle R_0(\lambda)\langle x\rangle^{-1}v$, then $rR_0(\lambda)\langle x\rangle^{-1}v = r\langle x\rangle^{-1}w$ and

$$(-\Delta-\lambda^2)w = [-\Delta, \langle x\rangle]R_0(\lambda)\langle x\rangle^{-1}v + v \implies w = R_0(\lambda)\big([-\Delta, \langle x\rangle]R_0(\lambda)\langle x\rangle^{-1}v+v\big) \in H^2(\mathbb{R}).$$

Furthermore, for any $w, v \in C_0^\infty(\mathbb{R}^n)$,

$$\langle w, \partial_r v\rangle_{L^2} = \langle \partial_r^* w, v\rangle_{L^2} := (1-n)\langle r^{-1}w, v\rangle_{L^2} - \langle \partial_r w, v\rangle_{L^2}.$$

Hence, by the density of $C_0^\infty(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n)$, and setting $\tilde{u}_k = (-\Delta - \lambda)u_k$, we get

$$\begin{aligned}(83)\qquad \lim_{k\to\infty}\big\langle g, \langle x\rangle^{-s} R_0(\lambda)(r\partial_r)\tilde{u}_k\big\rangle_{L^2} &= \lim_{k\to\infty}\langle(\partial_r)^* r R_0(\bar{\lambda})\langle x\rangle^{-s} g, \tilde{u}_k\rangle_{L^2}\\ &= \langle(\partial_r)^* r R_0(\bar{\lambda})\langle x\rangle^{-s} g, \langle x\rangle^{-s} f\rangle_{L^2}\\ &= \langle g, \langle x\rangle^{-s} R_0(\lambda) r\partial_r \langle x\rangle^{-s} f\rangle_{L^2}.\end{aligned}$$

as desired. Taken together, (82) and (83) confirm (80).

Case $|\lambda| \le 1$. Now we turn to the case $|\lambda| \le 1$, and utilize the integral kernel of the free resolvent, which is given by [JeNe01, Section 3],

$$(84)\quad (-\Delta-\lambda^2)^{-1}(|x-y|) = \frac{1}{2\pi}\left(\frac{-i\lambda}{2\pi|x-y|}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(-i\lambda|x-y|), \qquad \operatorname{Im}\lambda > 0,$$

where $K_\nu(z)$ is the Macdonald function of order ν [DLMF, 10.27.4, 10.27.5]. Now, if $n = 3$, the integral kernel of

$$\langle x\rangle^{-s}\frac{d}{d\lambda}(-\Delta-\lambda^2)^{-1}\langle x\rangle^{-s}$$

is given by $i(4\pi)^{-1}\langle x\rangle^{-s} e^{i\lambda|x-y|}\langle y\rangle^{-s}$, which has Hilbert–Schmidt norm bounded uniformly in $|\lambda| \le 1$ provided $s > \frac{3}{2}$. Moving on to $n \ge 4$, by [DLMF, 10.29.2],

$$\frac{d}{d\lambda}\left(\frac{-i\lambda}{2\pi|x-y|}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(-i\lambda|x-y|) = \frac{-(-i)^{\frac{n}{2}}\lambda^{\frac{n}{2}-1}}{(2\pi)^{\frac{n}{2}-1}|x-y|^{\frac{n}{2}-2}} K_{\frac{n}{2}-2}(-i\lambda|x-y|).$$

The Macdonald function satisfies [DLMF, 10.25.3, 10.30.2, 10.30.3]

$$(85)\qquad |K_\nu(z)| \le \begin{cases} C|z|^{-\nu} & \text{if } 0<|z|\le 1,\ \nu>0,\\ C\big|\ln|z|\big| & \text{if } 0<|z|\le 1,\ \nu=0,\\ C|z|^{-1/2} & \text{if } |z|\ge 1,\ \operatorname{Re} z\ge 0,\ \nu\ge 0,\end{cases}$$

for $C > 0$ a constant independent of z. Therefore, there is a constant $C > 0$ independent of λ such that, if $n = 4$,

$$\begin{aligned}(86)\quad &\left|\frac{\lambda^{\frac{n}{2}-1}\langle x\rangle^{-s}\langle y\rangle^{-s}}{|x-y|^{\frac{n}{2}-2}} K_{\frac{n}{2}-2}(-i\lambda|x-y|)\right|\\ &\le C|\lambda|\langle x\rangle^{-s}\langle y\rangle^{-s}\big|\ln(|\lambda||x-y|)\big|\,\mathbf{1}_{\{|\lambda||x-y|\le 1\}} + C\frac{|\lambda|^{\frac{n}{2}-\frac{3}{2}}\langle x\rangle^{-s}\langle y\rangle^{-s}}{|x-y|^{\frac{n}{2}-\frac{3}{2}}}\mathbf{1}_{\{|\lambda||x-y|>1\}}\end{aligned}$$

and, if $n > 4$,

$$\begin{aligned}(87)\quad &\left|\frac{\lambda^{\frac{n}{2}-1}\langle x\rangle^{-s}\langle y\rangle^{-s}}{|x-y|^{\frac{n}{2}-2}} K_{\frac{n}{2}-2}(-i\lambda|x-y|)\right|\\ &\qquad\le C\frac{|\lambda|\langle x\rangle^{-s}\langle y\rangle^{-s}}{|x-y|^{n-4}}\mathbf{1}_{\{|\lambda||x-y|\le 1\}} + C\frac{|\lambda|^{\frac{n}{2}-\frac{3}{2}}\langle x\rangle^{-s}\langle y\rangle^{-s}}{|x-y|^{\frac{n}{2}-\frac{3}{2}}}\mathbf{1}_{\{|\lambda||x-y|>1\}}.\end{aligned}$$

As preparation for the conclusions we draw in the next paragraph, we observe that the first summand on the right-hand side of (87) has the bound, for $|\lambda| \le 1$ and $0 < \alpha \le 1$,

$$\frac{|\lambda|\langle x\rangle^{-s}\langle y\rangle^{-s}}{|x-y|^{n-4}}\mathbf{1}_{\{|\lambda||x-y|\le 1\}} = \frac{|\lambda||x-y|^{\alpha}\langle x\rangle^{-s}\langle y\rangle^{-s}}{|x-y|^{n-4+\alpha}}\mathbf{1}_{\{|\lambda||x-y|\le 1\}} \le \frac{\langle x\rangle^{-s}\langle y\rangle^{-s}}{|x-y|^{n-4+\alpha}}\mathbf{1}_{\{|\lambda||x-y|\le 1\}}. \tag{88}$$

In what follows we make repeated use of Lemma F.3 to verify that a given kernel is Hilbert–Schmidt. In both (86) and (87), the last summand on the right side is uniformly bounded in Hilbert–Schmidt norm for $|\lambda| \le 1$, provided $s > \frac{1}{4}(n+3)$. This also holds for the first summand on the right side of (86) if $s > \frac{3}{2}$. In addition, to address the first summand on the right side of (87), we utilize (88) in combination with Lemma F.3. Taken together this means that $\langle x\rangle^{-s}\langle y\rangle^{-s}|x-y|^{-n+4-\alpha}$ is Hilbert–Schmidt

when $n = 5$, if $\alpha = 1$ and $s > \frac{3}{2}$;
when $n = 6$, if $0 < \alpha < 1$ and $s > 2 - \frac{1}{2}\alpha$; and
when $n = 7$, if $0 < \alpha < \frac{1}{2}$ and $s > 2 - \frac{1}{2}\alpha$.

Thus, when $n = 6$ it is enough to take $s > \frac{3}{2}$, while when $n = 7$, $s > \frac{7}{4}$ suffices.

Finally, if $n \ge 8$, the first summand on the right in (87) is uniformly bounded as an operator $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ for $|\lambda| \le 1$ provided $s > 3$. This is due to (88) with $\alpha = 1$ and the Schur test; see Lemma F.2.

We finish by resolving the $|\alpha| = 1$ case for (54). By (54) in the $|\alpha| = 0$ case, and by (52), we need to show that

$$\left\|\lambda\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s} f\right\|_{H^2} \le O(1+|\lambda|)\|f\|_{L^2}.$$

According to (89) below,

$$\begin{aligned}\left\|\lambda\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s} f\right\|_{H^2} &\le C\left\|\lambda\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s} f\right\|_{L^2} + C\left\|\lambda\langle x\rangle^{-s}(-\Delta)(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s} f\right\|_{L^2} \\ &= C\|f\|_{L^2} + C\left\|\lambda\langle x\rangle^{-s}(-\Delta)(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s} f\right\|_{L^2}.\end{aligned}$$

Then use

$$\langle x\rangle^{-s}(-\Delta)(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s} f = \langle x\rangle^{-s}(-\Delta-\lambda^2)^{-1}\langle x\rangle^{-s} f + \lambda^2\langle x\rangle^{-s}(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s} f,$$

which in combination with (50), as well as (54) in the $|\alpha| = 0$ case, yields

$$\left\|\lambda\langle x\rangle^{-s}(-\Delta)(-\Delta-\lambda^2)^{-2}\langle x\rangle^{-s} f\right\|_{L^2} \le C(1+|\lambda|)\|f_{L^2},$$

completing the proof. □

Appendix F. Useful lemmas

Lemma F.1 [Fa67, Proposition 6]. *Let $n \geq 3$. Then*

$$\|r^{-1}u\|_{L^2}^2 \leq \left(\frac{2}{n-2}\right)^2 \|\nabla u\|_{L^2}^2, \quad u \in H^1(\mathbb{R}^n).$$

Lemma F.2 (Schur's test [DyZw19, Section A.5]). *Suppose $K(x, y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies*

$$\sup_x \int |K(x, y)|\, dy \leq C \quad \textit{and} \quad \sup_y \int |K(x, y)|\, dx \leq C.$$

Then the linear operator

$$TF(x) = \int K(x, y) f(y)\, dy,$$

obeys the estimate

$$\|Tf\|_{L^2} \leq C\|f\|_{L^2}.$$

Lemma F.3 [Pe24]. *The necessary and sufficient conditions for*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \langle x\rangle^{-s} \langle y\rangle^{-t} |x - y|^{-p}\, dx\, dy < \infty,$$

are

$$s + p > n, \quad t + p > n, \quad s + p + t > 2n, \quad p < n.$$

Lemma F.4. *Suppose $T : L^2(\mathbb{R}^n) \to H^2(\mathbb{R}^n)$ is a bounded operator. For any $s > 0$, there exists $C > 0$ such that*

$$\|\langle x\rangle^{-s} T\|_{L^2 \to H^2} \leq C(\|\langle x\rangle^{-s} T\|_{L^2 \to L^2} + \|\langle x\rangle^{-s} \Delta T\|_{L^2 \to L^2}). \tag{89}$$

Proof. Let $f \in L^2(\mathbb{R}^n)$ and put $u = TF$. By the first line of (52), there exists $C > 0$, whose precise value may change from line to line, such that

$$\|\langle x\rangle^{-s} u\|_{H^2} \leq C\|\langle x\rangle^{-s} u\|_{L^2} + C\|\Delta\langle x\rangle^{-s} u\|_{L^2}, \quad \tilde{u} \in H^2(\mathbb{R}^n). \tag{90}$$

Then use the second line of (52):

$$\begin{aligned}
\|\Delta\langle x\rangle^{-s} u\|_{L^2} &\leq \|[\Delta, \langle x\rangle^{-s}]u\|_{L^2} + \|\langle x\rangle^{-s} \Delta u\|_{L^2} \\
&\leq C\|\langle x\rangle^{-s} u\|_{H^1} + \|\langle x\rangle^{-s} \Delta u\|_{L^2} \\
&\leq C\gamma^{-1}\|\langle x\rangle^{-s} u\|_{L^2} + C\gamma\|\Delta\langle x\rangle^{-s} u\|_{L^2}) + \|\langle x\rangle^{-s} \Delta u\|_{L^2}, \quad \gamma > 0.
\end{aligned}$$

Fixing γ small enough yields

$$\|\Delta\langle x\rangle^{-s} u\|_{L^2} \leq C(\|\langle x\rangle^{-s} u\|_{L^2} + \|\langle x\rangle^{-s} \Delta u\|_{L^2}),$$

which in combination with (90) implies (89). ☐

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ANDRÉS LARRAÍN-HUBACH
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DAYTON
DAYTON, OH
UNITED STATES
alarrainhubach1@udayton.edu

YULONG LI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DAYTON
DAYTON, OH
UNITED STATES
yli004@udayton.edu

JACOB SHAPIRO
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DAYTON
DAYTON, OH9-2316
UNITED STATES
jshapiro1@udayton.edu

JOSEPH TILLER
PEERLESS TECHNOLOGIES
BEAVERCREEK, OH
UNITED STATES
joseph.tiller@epeerless.us

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