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JUAN LI, HONGWEI XU AND ENTAO ZHAO

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DIFFERENTIABLE SPHERE THEOREMS FOR COMPACT SUBMANIFOLDS

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We investigate the differentiable structure on compact simply connected submanifolds in Riemannian manifolds under curvature pinching conditions. We prove a sharp differentiable sphere theorem that an n-dimensional compact simply connected submanifold M^n ($n \geq 5$, $n \neq 7, 8$) in the sphere $\mathbb{S}^N(1/\sqrt{c})$ ($c > 0$) with the second fundamental form A and the mean curvature vector H satisfying $|A|^2 \leq 4c + \frac{|H|^2}{n-2}$ is diffeomorphic to the standard sphere. The similar differentiable sphere theorem also holds for compact simply connected submanifolds in the space form $\mathbb{F}^N(c)$ with $c \leq 0$.

1. Introduction

The sphere theorems characterize the geometric and topological properties of compact Riemannian manifolds through curvature pinching conditions, representing a forefront topic in global Riemannian geometry. Rauch [30] first introduced the concept of curvature pinching for Riemannian manifolds. A Riemannian manifold M^n is δ-pinched (globally) for $\delta > 0$ if the sectional curvature K_M of M satisfies $\delta < K_M \leq 1$. Rauch [30] proved a topological sphere theorem for compact simply connected δ-pinched Riemannian manifolds with $\delta \approx 3/4$. Berger [3] and Klingenberg [17] provided a topological sphere theorem under the $1/4$-curvature pinching condition. Subsequently, Brendle and Schoen [6] proved the following differentiable sphere theorem by using Ricci flow techniques.

Theorem A [6]. *Let M be an n-dimensional ($n \geq 4$) complete and simply connected Riemannian manifold. If $1/4 < K_M \leq 1$, then M is diffeomorphic to the standard sphere $\mathbb{S}^n$.*

In fact, Brendle and Schoen proved the differentiable sphere theorem for pointwise $1/4$-pinched Riemannian manifolds. They proved in [5] that a compact simply connected weakly pointwise $1/4$-pinched Riemannian manifold is diffeomorphic to the standard sphere $\mathbb{S}^n$ or isometric to a compact rank one symmetric space (CROSS).

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Since any Riemannian manifold can be regarded as a zero-codimensional submanifold of itself, it is an interesting problem whether it is possible to extend the Brendle–Schoen differentiable sphere theorem to the case of submanifolds of arbitrary codimension in a general Riemannian manifold.

Let M^n be an n-dimensional submanifold in an N-dimensional Riemannian manifold $\overline{M}^N$. Denote by A and H the second fundamental form and the mean curvature vector of M, respectively. Lawson and Simons [19] proved a topological sphere theorem for compact submanifolds in the unit sphere by combining the homology vanishing theorem for compact submanifolds with Smale's [33] proof of the generalized Poincaré conjecture in dimensions $n \geq 5$.

Theorem B [19]. *Let M^n be an n-dimensional oriented compact submanifold in the unit sphere $\mathbb{S}^N$.*

(i) *If $n \neq 3, 4$ and $|A|^2 < 2\sqrt{n-1}$, then M is homeomorphic to a sphere.*

(ii) *If $n = 3, 4$ and $|A|^2 < n-1$, then M is a homotopy sphere.*

Inspired by the rigidity theorems for submanifolds with parallel mean curvature (see [32; 36; 37; 38]), Shiohama and Xu [31] improved and extended the theorem of Lawson and Simons [19]. They proved the following optimal topological sphere theorem for complete submanifolds in space forms.

Theorem C [31]. *Let M^n be an n-dimensional oriented complete submanifold in a simply connected space form $\mathbb{F}^N(c)$ with nonnegative constant curvature c. Assume*

$$\sup_M \big(|A|^2 - \alpha(n, |H|, c)\big) < 0,$$

where

$$\alpha(n, |H|, c) = nc + \frac{n}{2(n-1)}|H|^2 - \frac{n-2}{2(n-1)}\sqrt{|H|^4 + 4(n-1)c|H|^2}.$$

(i) *If $n \neq 3$, then M is homeomorphic to an n-dimensional sphere.*

(ii) *If $n = 3$, then M is diffeomorphic to a 3-dimensional spherical space form.*

For compact submanifolds in the hyperbolic space, a similar topological sphere theorem was proved by Fu and Xu [11].

Xu and Zhao [40] were the first to apply the Ricci flow to prove differentiable sphere theorems for compact submanifolds in general Riemannian manifolds. In particular, they proved the following theorem.

Theorem D [40]. *Let M^n be an n-dimensional ($n \geq 4$) oriented complete submanifold in the unit sphere $\mathbb{S}^N$.*

(i) *If $n = 4, 5, 6$ and $\sup_M |A|^2 < 2\sqrt{n-1}$, then M is diffeomorphic to $\mathbb{S}^n$.*

(ii) *If $n \geq 7$ and $|A|^2 < 2\sqrt{2}$, then M is diffeomorphic to $\mathbb{S}^n$.*

Xu and Zhao [40] also obtained a topological sphere theorem for compact simply connected submanifolds in a Riemannian manifold under the assumption that

$$|A|^2 < \tfrac{16}{3}\left(\overline{K}_{\min} - \tfrac{1}{4}\overline{K}_{\max}\right) + \frac{|H|^2}{n-2}.$$

Here $\overline{K}_{\min}$ and $\overline{K}_{\max}$ are the minimum and the maximum of the sectional curvatures of $\overline{M}$ at a point.

Xu and Gu [39] further advanced the field by proving an optimal differential sphere theorem for complete submanifolds in space forms.

Theorem E [39]. *Let M^n be an n-dimensional $(n \geq 2)$ oriented complete submanifold in a simply connected space form $\mathbb{F}^N(c)$ with nonnegative constant curvature c. If*

$$\sup_M \left(|A|^2 - \frac{|H|^2}{n-1} - 2c \right) < 0,$$

then M is diffeomorphic to $\mathbb{S}^n$.

Many other differentiable sphere theorems for compact submanifolds were obtained by using Ricci flow and mean curvature flow techniques [1; 2; 12; 13; 22; 21; 23; 24; 26]. For instance, Lei and Xu [22; 21; 23] proved several optimal or sharp smooth convergence theorems for the mean curvature flow of submanifolds in space forms, which imply optimal or sharp differentiable sphere theorems for submanifolds in space forms.

Inspired by these developments, we prove the following differentiable sphere theorem for compact submanifolds in Riemannian manifolds.

Theorem 1.1. *Let M^n $(n \geq 4, n \neq 7, 8)$ be an n-dimensional compact simply connected submanifold in an N-dimensional Riemannian manifold $\overline{M}^N$. If*

$$|A|^2 < \frac{16}{3}\left(\overline{K}_{\min} - \frac{1}{4}\overline{K}_{\max}\right) + \frac{|H|^2}{n-2},$$

then M is diffeomorphic to $\mathbb{S}^n$.

Theorem 1.1 refines Xu and Zhao's topological sphere theorem in [40] to a differentiable sphere theorem. More differentiable sphere theorems for compact submanifolds in Riemannian manifolds will be proved in Section 6.

If the ambient space is a space form of constant sectional curvature c, then the condition in Theorem 1.1 simplifies to $|A|^2 < 4c + \frac{|H|^2}{n-2}$. With the aid of the homology vanishing theorem for compact submanifolds and the mean curvature flow of arbitrary codimension, we prove the following differentiable sphere theorem for compact submanifolds in space forms under a weak pinching condition.

Theorem 1.2. *Let M^n be an n-dimensional compact simply connected submanifold in a simply connected space form $\mathbb{F}^N(c)$ of constant sectional curvature c.*

(i) *If* $c \geq 0, n \geq 5, n \neq 7, 8, |H| > 0$ *for* $c = 0$, *and*

$$|A|^2 \leq 4c + \frac{|H|^2}{n-2},$$

then M *is diffeomorphic to* $\mathbb{S}^n$.

(ii) *If* $c < 0, n \geq 9$, *and*

$$|A|^2 \leq 4c + \frac{|H|^2}{n-2},$$

then M *is diffeomorphic to* $\mathbb{S}^n$.

For $c > 0$ and $n \geq 5$, consider

$$M^n(\varepsilon) := \mathbb{S}^2(\varepsilon) \times \mathbb{S}^{n-2}\big(\sqrt{1/c - \varepsilon^2}\big) \subset \mathbb{S}^{n+1}\big(1/\sqrt{c}\big),$$

with $0 < \varepsilon < 1/\sqrt{c}$. Intuitively, $M^n(\varepsilon)$ is simply connected and not homeomorphic to the sphere. By a direct computation, we have

$$|A|^2 - \frac{|H|^2}{n-2} = \frac{2(n-4)}{n-2}\frac{1}{\varepsilon^2} + \frac{2n}{n-2}c.$$

For $n \geq 5$, one has

$$\frac{2(n-4)}{n-2}\frac{1}{\varepsilon^2} + \frac{2n}{n-2}c \to +\infty \quad \text{as } \varepsilon \to 0,$$

$$\frac{2(n-4)}{n-2}\frac{1}{\varepsilon^2} + \frac{2n}{n-2}c \to 4c \quad \text{as } \varepsilon \to \frac{1}{\sqrt{c}}.$$

Therefore, for any constant $C > 4c$, there is $0 < \varepsilon < 1/\sqrt{c}$ such that $M^n(\varepsilon)$ satisfies $|A|^2 \leq C + \frac{|H|^2}{n-2}$. So for $c > 0$, Theorem 1.2 is sharp in the sense that the constant $4c$ in the pinching condition is the largest constant such that the differentiable sphere theorem holds. For $c > 0$ and $n = 7, 8$, M is homeomorphic to the sphere under the same pinching condition (see Theorem 4.1 in Section 4).

The paper is organized as follows. In Section 2, we introduce the relevant concepts and some curvature inequalities for Riemannian manifolds. In Section 3, we prove Theorem 1.1 with the aid of the classification theorem for Riemannian manifolds proved by using Ricci flow techniques. In Section 4, we prove a topological sphere theorem for compact submanifolds in space forms $\mathbb{F}^N(c)$ with $c \geq 0$, utilizing the homology vanishing theorem for compact submanifolds. In Section 5, we prove Theorem 1.2 by applying the mean curvature flow techniques. In Section 6, more differentiable sphere theorems for compact submanifolds under different curvature pinching conditions are proved.

2. Notation and formulas

Let (M^n, g) be an n-dimensional Riemannian submanifold in a Riemannian manifold $\overline{M}^N$ of dimension N with metric $\overline{g}$. We shall make use of the following convention on the range of indices:

$$1 \le i, j, k, \ldots \le n, \quad 1 \le A, B, C, \ldots \le N, \quad \text{and} \quad n+1 \le \alpha, \beta, \gamma, \ldots \le N.$$

We choose a local orthonormal frame $\{e_i\}$ for the tangent bundle and a local orthonormal frame $\{e_\alpha\}$ for the normal bundle. Let $\{\omega_A\}$ be the dual frame field corresponding to $\{e_A\}$. Denote by Rm and $\overline{Rm}$ the Riemannian curvature tensors of M and $\overline{M}$. Then we have

$$Rm = \sum_{i,j,k,l} R_{ijkl}\omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l,$$
$$\overline{Rm} = \sum_{A,B,C,D} \bar{R}_{ABCD}\omega_A \otimes \omega_B \otimes \omega_C \otimes \omega_D.$$

Let A and H be the second fundamental form and the mean curvature vector of M, given by

$$A = \sum_{i,j,\alpha} A^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \quad H = \sum_\alpha H^\alpha e_\alpha, \quad H^\alpha = \sum_i A^\alpha_{ii}.$$

We have the Gauss equation

$$R_{ijkl} = \bar{R}_{ijkl} + \sum_\alpha \big(A^\alpha_{ik}A^\alpha_{jl} - A^\alpha_{il}A^\alpha_{jk}\big).$$

The trace-free second fundamental form $\mathring{A}$ is defined by $\mathring{A} = A - \frac{1}{n} g \otimes H$. Then

$$\mathring{A} = \sum_{i,j,\alpha} \mathring{A}^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \quad \mathring{A}^\alpha_{ij} = A^\alpha_{ij} - \frac{1}{n}H^\alpha\delta_{ij}.$$

The sectional curvature, Ricci curvature, and scalar curvature on M and $\overline{M}$ are defined as K, $\overline{K}$, Ric, $\overline{\text{Ric}}$, R, and $\bar{R}$, respectively. Thus, we have

$$\text{Ric}(e_i) = \sum_j R_{ijij}, \quad \overline{\text{Ric}}(e_A) = \sum_B \bar{R}_{ABAB}, \quad R = \sum_{i,j} R_{ijij}, \quad \bar{R} = \sum_{A,B} \bar{R}_{ABAB}.$$

The normalized scalar curvature $\bar{R}_0$ on $\overline{M}$ is defined as

$$\bar{R}_0 = \frac{\bar{R}}{N(N-1)}.$$

Set

$$\overline{K}_{\min}(x) = \min_{\pi \subset T_x\overline{M}} \overline{K}(\pi), \quad \overline{\text{Ric}}_{\min}(x) = \min_{u \in U_x\overline{M}} \overline{\text{Ric}}(u),$$
$$\overline{K}_{\max}(x) = \max_{\pi \subset T_x\overline{M}} \overline{K}(\pi), \quad \overline{\text{Ric}}_{\max}(x) = \max_{u \in U_x\overline{M}} \overline{\text{Ric}}(u).$$

We have Berger's inequalities:

$$|\bar{R}_{ACBC}| \le \tfrac{1}{2}(\bar{K}_{\max} - \bar{K}_{\min}) \quad \text{for all distinct indices } A, B, C,$$
$$|\bar{R}_{ABCD}| \le \tfrac{2}{3}(\bar{K}_{\max} - \bar{K}_{\min}) \quad \text{for all distinct indices } A, B, C, D.$$

For any unit tangent vector $u \in U_x\bar{M}$ at point $x \in \bar{M}$, let V_x^k be a k-dimensional subspace of $T_x\bar{M}$ satisfying $u \perp V_x^k$. Choose an orthonormal basis $\{e_A\}$ in $T_x\bar{M}$ such that for distinct indices $1 \le A_0, A_1, \dots, A_k \le N$, we have

$$e_{A_0} = u, \quad \operatorname{span}\{e_{A_1}, \dots, e_{A_k}\} = V_x^k.$$

The k-th Ricci curvature on $\bar{M}$ is defined as

$$\overline{\mathrm{Ric}}^{(k)}\left(u; V_x^k\right) = \overline{\mathrm{Ric}}^{(k)}\left([e_{A_0}, e_{A_1}, \dots, e_{A_k}]\right) = \sum_{p=1}^{k} \bar{R}_{A_0A_pA_0A_p}.$$

Set

$$\overline{\mathrm{Ric}}^{(k)}_{\min}(x) = \min_{u \in U_x\bar{M}} \min_{u \perp V_x^k \subset T_x\bar{M}} \overline{\mathrm{Ric}}^{(k)}(u; V_x^k),$$
$$\overline{\mathrm{Ric}}^{(k)}_{\max}(x) = \max_{u \in U_x\bar{M}} \max_{u \perp V_x^k \subset T_x\bar{M}} \overline{\mathrm{Ric}}^{(k)}(u; V_x^k).$$

Extend the orthonormal s-frame $\{e_{A_0}, \dots, e_{A_{s-1}}\}$ on $T_x\bar{M}$ to an orthonormal $(k+1)$-frame $\{e_{A_0}, \dots, e_{A_k}\}$ for $1 \le s \le k+1 \le N$. The (k, s)-curvature on $\bar{M}$ is defined as

$$\bar{R}^{(k,s)}([e_{A_0}, e_{A_1}, \dots, e_{A_k}]) = \sum_{p=0}^{s-1} \sum_{q=0}^{k} \bar{R}_{A_pA_qA_pA_q}.$$

Set

$$\bar{R}^{(k,s)}_{\min}(x) = \min_{\{e_{A_0}, e_{A_1}, \dots, e_{A_k}\} \subset T_x\bar{M}} \bar{R}^{(k,s)}([e_{A_0}, e_{A_1}, \dots, e_{A_k}]),$$
$$\bar{R}^{(k,s)}_{\max}(x) = \max_{\{e_{A_0}, e_{A_1}, \dots, e_{A_k}\} \subset T_x\bar{M}} \bar{R}^{(k,s)}([e_{A_0}, e_{A_1}, \dots, e_{A_k}]).$$

The s-th weak Ricci curvature on $\bar{M}$ is defined as

$$\begin{aligned}\overline{\mathrm{Ric}}^{[s]}([e_{A_0}, e_{A_1}, \dots, e_{A_{N-1}}]) &= \bar{R}^{(N-1,s)}([e_{A_0}, e_{A_1}, \dots, e_{A_{N-1}}])\\ &= \sum_{p=0}^{s-1} \sum_{q=0}^{N-1} \bar{R}_{A_pA_qA_pA_q}.\end{aligned}$$

Set

$$\overline{\mathrm{Ric}}^{[s]}_{\min}(x) = \min_{\{e_{A_0}, e_{A_1}, \dots, e_{A_{N-1}}\} \subset T_x\bar{M}} \overline{\mathrm{Ric}}^{[s]}([e_{A_0}, e_{A_1}, \dots, e_{A_{N-1}}]),$$
$$\overline{\mathrm{Ric}}^{[s]}_{\max}(x) = \max_{\{e_{A_0}, e_{A_1}, \dots, e_{A_{N-1}}\} \subset T_x\bar{M}} \overline{\mathrm{Ric}}^{[s]}([e_{A_0}, e_{A_1}, \dots, e_{A_{N-1}}]).$$

The k-th scalar curvature on $\overline{M}$ is defined as

$$\bar{R}^{(k)}([e_{A_0}, e_{A_1}, \ldots, e_{A_k}]) = \bar{R}^{(k,k+1)}([e_{A_0}, e_{A_1}, \ldots, e_{A_k}]) = \sum_{p=0}^{k} \sum_{q=0}^{k} \bar{R}_{A_p A_q A_p A_q}.$$

Set

$$\bar{R}^{(k)}_{\min} = \min_{\{e_{A_0}, e_{A_1}, \ldots, e_{A_k}\} \subset T_x \overline{M}} \bar{R}^{(k)}([e_{A_0}, e_{A_1}, \ldots, e_{A_k}]),$$

$$\bar{R}^{(k)}_{\max} = \max_{\{e_{A_0}, e_{A_1}, \ldots, e_{A_k}\} \subset T_x \overline{M}} \bar{R}^{(k)}([e_{A_0}, e_{A_1}, \ldots, e_{A_k}]).$$

Remark 2.1. Based on the definitions, the Ricci curvature of $\overline{M}$ is equivalent to its $(N-1, 1)$-curvature, its $(N-1)$-th Ricci curvature, and its 1-st weak Ricci curvature. The scalar curvature of $\overline{M}$ is equivalent to its $(N-1, N)$-curvature, its N-th weak Ricci curvature, and its $(N-1)$-th scalar curvature.

Without loss of generality, all manifolds and submanifolds in this paper are assumed to be connected and without boundary.

3. Differentiable sphere theorem for compact submanifolds in Riemannian manifolds

To prove the sphere theorem, we need the classification theorem for compact Riemannian manifolds with positive isotropic curvature. For an n-dimensional $(n \geq 4)$ Riemannian manifold M^n, if the Riemannian curvature tensor R of M^n satisfies $R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} > 0$ for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$, then we say that M has positive isotropic curvature. Recently, Brendle [4] utilized curvature pinching estimates under Ricci flow to demonstrate that compact Riemannian manifolds with positive isotropic curvature in dimensions $n \geq 12$ maintain the pinching condition under Ricci flow. This led to the development of a classification theorem for compact Riemannian manifolds with positive isotropic curvature, thereby strengthening the intrinsic connection between curvature conditions and differential structure, and extending the applicability of classical differentiable sphere theorems. Chen [7] further extended the theorem to dimensions $n \geq 9$.

Theorem F [4; 7]. *Let M^n be an n-dimensional $(n \geq 9)$ compact manifold with positive isotropic curvature. If M does not contain any essential incompressible $(n-1)$-dimensional space forms, then M is diffeomorphic to a connected sum of finitely many spaces, each of which is a quotient of $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{R}$ by standard isometries.*

In the above theorem, an incompressible space form N^{n-1} in M^n is an $(n-1)$-dimensional submanifold diffeomorphic to $\mathbb{S}^{n-1}/\Gamma$ such that the fundamental group

$\pi_1(N)$ injects into $\pi_1(M)$. The space form is said to be essential unless Γ is trivial, or $\Gamma = \mathbb{Z}_2$ and the normal bundle is nonorientable.

Proposition 3.1. *Let M be an n-dimensional $(n \geq 4)$ compact simply connected Riemannian manifold. Then there is no essential $(n-1)$-dimensional incompressible space form in M.*

Proof. Let N be an $(n-1)$-dimensional submanifold that is diffeomorphic to $\mathbb{S}^{n-1}/\Gamma$ in the Riemannian manifold M. By applying the Killing–Hopf theorem [15; 16], we deduce that $\pi_1(N) = \pi_1(\mathbb{S}^{n-1}/\Gamma) \approx \Gamma$ as $\mathbb{S}^{n-1}$ is simply connected. Clearly, if Γ is nontrivial, then there is no injective homomorphism from $\pi_1(N)$ to $\pi_1(M)$, since Γ contains more than one element, while $\pi_1(M)$ is trivial. Therefore, there is no essential $(n-1)$-dimensional incompressible space form in M. □

Using the above proposition, we present the proof of Theorem 1.1.

Proof of Theorem 1.1. From the definition of the second fundamental form, we have

$$|A|^2 = \sum_{i,j=1}^{n} (A^\alpha_{ij})^2 = \sum_{i=1}^{n} (A^\alpha_{ii})^2 + \sum_{i \neq j} (A^\alpha_{ij})^2.$$

For all distinct indices p, q, k, l, applying the Cauchy inequality yields

$$\begin{aligned}\left(\sum_{i=1}^{n} A^\alpha_{ii}\right)^2 &\leq (n-2)\left[(A^\alpha_{pp} + A^\alpha_{qq})^2 + (A^\alpha_{kk} + A^\alpha_{ll})^2 + \sum_{i \neq p,q,k,l} (A^\alpha_{ii})^2\right] \\ &= (n-2)\left(\sum_{i=1}^{n} (A^\alpha_{ii})^2 + 2A^\alpha_{pp}A^\alpha_{qq} + 2A^\alpha_{kk}A^\alpha_{ll}\right).\end{aligned}$$

This inequality implies

$$\begin{aligned}2A^\alpha_{pp}A^\alpha_{qq} + 2A^\alpha_{kk}A^\alpha_{ll} &\geq \frac{\left(\sum_{i=1}^{n} A^\alpha_{ii}\right)^2}{n-2} - \sum_{i=1}^{n} (A^\alpha_{ii})^2 \\ &= \frac{\left(\sum_{i=1}^{n} A^\alpha_{ii}\right)^2}{n-2} + \sum_{i \neq j} (A^\alpha_{ij})^2 - \sum_{i,j=1}^{n} (A^\alpha_{ij})^2.\end{aligned} \tag{3-1}$$

Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame. By the Gauss equation, we have

$$\begin{aligned}&R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\ &\quad = \bar{R}_{1313} + \bar{R}_{1414} + \bar{R}_{2323} + \bar{R}_{2424} - 2\bar{R}_{1234} \\ &\qquad + \sum_\alpha \big[A^\alpha_{11}A^\alpha_{33} + A^\alpha_{11}A^\alpha_{44} + A^\alpha_{22}A^\alpha_{33} + A^\alpha_{22}A^\alpha_{44} - (A^\alpha_{13})^2 - (A^\alpha_{14})^2 \\ &\qquad\qquad - (A^\alpha_{23})^2 - (A^\alpha_{24})^2 - 2(A^\alpha_{13}A^\alpha_{24} - A^\alpha_{14}A^\alpha_{23})\big].\end{aligned} \tag{3-2}$$

Applying Berger's inequality yields

$$\bar{R}_{1234} \le \tfrac{2}{3}(\bar{K}_{\max} - \bar{K}_{\min}).$$

Hence,

$$\bar{R}_{1313} + \bar{R}_{1414} + \bar{R}_{2323} + \bar{R}_{2424} - 2\bar{R}_{1234} \ge 4\bar{K}_{\min} - \tfrac{4}{3}(\bar{K}_{\max} - \bar{K}_{\min}). \tag{3-3}$$

By (3-1), we obtain the following estimate for the first four terms in brackets on the right-hand side of (3-2):

$$\begin{aligned} A^{\alpha}_{11}A^{\alpha}_{33} + A^{\alpha}_{11}A^{\alpha}_{44} + A^{\alpha}_{22}A^{\alpha}_{33} + A^{\alpha}_{22}A^{\alpha}_{44} \\ \ge \frac{\left(\sum_{i=1}^{n} A^{\alpha}_{ii}\right)^2}{n-2} + \sum_{i\ne j}(A^{\alpha}_{ij})^2 - \sum_{i,j=1}^{n}(A^{\alpha}_{ij})^2. \end{aligned} \tag{3-4}$$

Combining (3-2)–(3-4), we obtain

$$\begin{aligned} R_{1313} &+ R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\ &\ge \tfrac{16}{3}(\bar{K}_{\min} - \tfrac{1}{4}\bar{K}_{\max}) \\ &\quad + \sum_{\alpha}\Bigg[\frac{\left(\sum_{i=1}^{n} A^{\alpha}_{ii}\right)^2}{n-2} + \sum_{i\ne j}(A^{\alpha}_{ij})^2 - \sum_{i,j=1}^{n}(A^{\alpha}_{ij})^2 \\ &\qquad - 2(A^{\alpha}_{13})^2 - 2(A^{\alpha}_{14})^2 - 2(A^{\alpha}_{23})^2 - 2(A^{\alpha}_{24})^2\Bigg] \\ &\ge \tfrac{16}{3}\left(\bar{K}_{\min} - \tfrac{1}{4}\bar{K}_{\max}\right) + \frac{|H|^2}{n-2} - |A|^2. \end{aligned} \tag{3-5}$$

Therefore, under the assumption of Theorem 1.1, M has positive isotropic curvature.

When $n = 4$, by [8; 9; 14], M is diffeomorphic to the standard sphere.

When $n = 5, 6$, it is known from [27] that M is homeomorphic to a sphere. As the differentiable structure on the topological sphere of dimension 5 or 6 is unique, M is diffeomorphic to the standard sphere.

When $n \ge 9$, Proposition 3.1 implies that M contains no essential $(n-1)$-dimensional incompressible spatial form. By Theorem F, M is diffeomorphic to the standard sphere. □

4. Topological sphere theorem for compact submanifolds in space forms

In this section, we prove a topological sphere theorem for compact submanifolds in space forms of nonnegative sectional curvature.

Theorem 4.1. *Let M^n be an n-dimensional $(n \geq 5)$ compact simply connected submanifold in a simply connected space $\mathbb{F}^N(c)$ with $c \geq 0$. Assume that $|H| > 0$ for $c = 0$. If*

$$|A|^2 \leq 4c + \frac{|H|^2}{n-2},$$

then M is homeomorphic to $\mathbb{S}^n$.

Theorem 4.1 provides a proof of the case $n = 5, 6$ of (i) in Theorem 1.2.

We need the following homology vanishing theorem for compact submanifolds in space forms to prove Theorem 4.1. Lawson and Simons [19] initially established the homology vanishing theorem for compact submanifolds in spheres under a strict pointwise pinching condition. Subsequently, Xin [35] generalized this theorem to compact submanifolds in the Euclidean space. It was observed by Elworthy and Rosenberg [10] that the Lawson–Simons theorem still holds under a weak pinching condition that is strict at some point. This generalization is also true for compact submanifolds in the Euclidean space.

Theorem 4.2 [10; 19; 35]. *Let M^n be an n-dimensional $(n \geq 5)$ compact submanifold in a simply connected space form $\mathbb{F}^N(c)$ with nonnegative constant curvature c. Assume that, for an integer $0 < q < n$,*

$$\sum_{k=q+1}^{n} \sum_{i=1}^{q} \big(2|A(e_i, e_k)|^2 - \langle A(e_i, e_i), A(e_k, e_k)\rangle\big) \leq q(n-q)c \tag{4-1}$$

holds for any orthonormal basis $\{e_i\}$ of T_xM at any point $x \in M$. If there is a point such that (4-1) is strict for any orthonormal basis $\{e_i\}$ at that point, then there do not exist any stable q-currents, and

$$H_q(M; \mathbb{Z}) = H_{n-q}(M; \mathbb{Z}) = 0,$$

where $H_i(M; \mathbb{Z})$ is the i-th homology group of M with integer coefficients.

Proof of Theorem 4.1. We follow the computation in [31]. For any orthonormal basis $\{e_\alpha\}$ of the normal space at a point,

$$\begin{aligned} &\sum_{k=q+1}^{n} \sum_{i=1}^{q} \big(2|A(e_i, e_k)|^2 - \langle A(e_i, e_i), A(e_k, e_k)\rangle\big) \\ &\qquad = 2\sum_{\alpha} \sum_{k=q+1}^{n} \sum_{i=1}^{q} (A_{ik}^{\alpha})^2 - \sum_{\alpha} \sum_{k=q+1}^{n} \sum_{i=1}^{q} A_{ii}^{\alpha} A_{kk}^{\alpha} \\ &\qquad = \sum_{\alpha} \bigg[2 \sum_{k=q+1}^{n} \sum_{i=1}^{q} (A_{ik}^{\alpha})^2 - \Big(\sum_{i=1}^{q} A_{ii}^{\alpha}\Big)\Big(H^{\alpha} - \sum_{i=1}^{q} A_{ii}^{\alpha}\Big)\bigg]. \end{aligned} \tag{4-2}$$

We take $r = n - q$ and set

$$Z_\alpha := -\Big(\sum_{i=1}^{q} A_{ii}^\alpha\Big)\Big(H^\alpha - \sum_{i=1}^{q} A_{ii}^\alpha\Big), \quad S_\alpha := \sum_{i,j=1}^{n} (A_{ij}^\alpha)^2, \quad \tilde{S}_\alpha := \sum_{i=1}^{n} (A_{ii}^\alpha)^2.$$

Applying the Cauchy inequality, we have

$$\begin{aligned} qr\tilde{S}_\alpha &= qr\sum_{i=1}^{q}(A_{ii}^\alpha)^2 + qr\sum_{k=q+1}^{n}(A_{kk}^\alpha)^2 \\ &\geq r\Big(\sum_{i=1}^{q} A_{ii}^\alpha\Big)^2 + q\Big(\sum_{k=q+1}^{n} A_{kk}^\alpha\Big)^2 \\ &= (r+q)\Big(\sum_{i=1}^{q} A_{ii}^\alpha\Big)^2 - 2qH^\alpha\Big(\sum_{i=1}^{q} A_{ii}^\alpha\Big) + q(H^\alpha)^2. \end{aligned} \tag{4-3}$$

This inequality implies

$$nZ_\alpha + (r-q)H^\alpha\Big(\sum_{i=1}^{q} A_{ii}^\alpha\Big) + q(H^\alpha)^2 - qr\tilde{S}_\alpha \leq 0. \tag{4-4}$$

By the definition and applying the Cauchy inequality, we have

$$\begin{aligned} \tilde{S}_\alpha - \frac{1}{n}(H^\alpha)^2 &= \sum_{i=1}^{q}(\mathring{A}_{ii}^\alpha)^2 + \sum_{k=q+1}^{n}(\mathring{A}_{kk}^\alpha)^2 \geq \frac{1}{q}\Big(\sum_{i=1}^{q}\mathring{A}_{ii}^\alpha\Big)^2 + \frac{1}{r}\Big(\sum_{k=q+1}^{n}\mathring{A}_{kk}^\alpha\Big)^2 \\ &= \Big(\frac{1}{q} + \frac{1}{r}\Big)\Big(\sum_{i=1}^{q}\mathring{A}_{ii}^\alpha\Big)^2. \end{aligned}$$

Hence,

$$\Big|\sum_{i=1}^{q}\mathring{A}_{ii}^\alpha\Big| \leq \sqrt{\tfrac{qr}{n}\big(\tilde{S}_\alpha - \tfrac{1}{n}(H^\alpha)^2\big)}.$$

Combining this inequality with (4-4), we get

$$Z_\alpha \leq \frac{qr}{n}\tilde{S}_\alpha - \Big(\frac{q(r-q)}{n^2} + \frac{q}{n}\Big)(H^\alpha)^2 + \frac{|r-q|}{n}|H^\alpha|\sqrt{\tfrac{qr}{n}\big(\tilde{S}_\alpha - \tfrac{1}{n}(H^\alpha)^2\big)}.$$

Substituting this inequality into (4-2) yields

$$\begin{aligned} &\sum_{k=q+1}^{n}\sum_{i=1}^{q}\big(2|A(e_i,e_k)|^2 - \langle A(e_i,e_i), A(e_k,e_k)\rangle\big) \\ &\qquad \leq \sum_\alpha\Big\{2\sum_{k=q+1}^{n}\sum_{i=1}^{q}(A_{ik}^\alpha)^2 + \frac{qr}{n}\tilde{S}_\alpha - \Big(\frac{q(r-q)}{n^2} + \frac{q}{n}\Big)(H^\alpha)^2 \\ &\qquad\qquad + \frac{|r-q|}{n}|H^\alpha|\sqrt{\tfrac{qr}{n}\big(\tilde{S}_\alpha - \tfrac{1}{n}(H^\alpha)^2\big)}\Big\}. \end{aligned}$$

For $n \geq 5$ and $2 \leq q \leq n-2$, one has $qr > n$. So for any $n-1 \leq \alpha \leq N$,

$$
\begin{aligned}
\text{(4-5)}\quad & 2\sum_{k=q+1}^{n}\sum_{i=1}^{q}(A_{ik}^{\alpha})^2 + \frac{qr}{n}\sum_{i=1}^{n}(A_{ii}^{\alpha})^2 \\
&= \frac{qr}{n}\sum_{i,j=1}^{n}(A_{ij}^{\alpha})^2 - \Big(\frac{qr}{n}-1\Big)\sum_{i\neq j}(A_{ij}^{\alpha})^2 - \sum_{\substack{i,j=1\\ i\neq j}}^{q}(A_{ij}^{\alpha})^2 - \sum_{\substack{i,j=q+1\\ i\neq j}}^{n}(A_{ij}^{\alpha})^2 \\
&\leq \frac{qr}{n}\sum_{i,j=1}^{n}(A_{ij}^{\alpha})^2.
\end{aligned}
$$

Applying the Cauchy inequality, we have

$$
\text{(4-6)}\qquad \sum_{\alpha}|H^{\alpha}|\sqrt{\tilde{S}_{\alpha} - \frac{1}{n}(H^{\alpha})^2} \leq \sqrt{\sum_{\alpha}(H^{\alpha})^2\cdot\sum_{\alpha}\Big(\tilde{S}_{\alpha} - \frac{1}{n}(H^{\alpha})^2\Big)}.
$$

Hence,

$$
\begin{aligned}
\text{(4-7)}\quad & \sum_{k=q+1}^{n}\sum_{i=1}^{q}(2|A(e_i,e_k)|^2 - \langle A(e_i,e_i), A(e_k,e_k)\rangle) \\
&\leq \sum_{\alpha}\Big(\frac{qr}{n}S_{\alpha} - \frac{2qr}{n^2}(H^{\alpha})^2\Big) + \frac{|r-q|}{n}\sqrt{\frac{qr}{n}\sum_{\alpha}(H^{\alpha})^2\cdot\sum_{\alpha}\Big(\tilde{S}_{\alpha} - \frac{1}{n}(H^{\alpha})^2\Big)} \\
&\leq \frac{qr}{n}\Big(|A|^2 - \frac{2}{n}|H|^2 + \frac{|r-q|}{\sqrt{nqr}}\,|H|\sqrt{|A|^2 - \tfrac{1}{n}|H|^2}\Big) \\
&\leq \frac{qr}{n}\Big(|A|^2 - \frac{2}{n}|H|^2 - nc + \frac{n-4}{\sqrt{2n(n-2)}}\,|H|\sqrt{|A|^2 - \tfrac{1}{n}|H|^2}\Big) + qrc.
\end{aligned}
$$

Now we take the case $c > 0$. By a direct computation, we have, for $2 \leq q \leq n-2$ and $n \geq 5$,

$$
|A|^2 - \frac{2}{n}|H|^2 - nc + \frac{n-4}{\sqrt{2n(n-2)}}\,|H|\sqrt{|A|^2 - \tfrac{1}{n}|H|^2} < 0,
$$

provided $|A|^2 \leq 4c + \dfrac{|H|^2}{n-2}$. Therefore,

$$
\sum_{k=q+1}^{n}\sum_{i=1}^{q}\big(2|A(e_i,e_k)|^2 - \langle A(e_i,e_i), A(e_k,e_k)\rangle\big) < qrc
$$

for $2 \leq q \leq n-2$. By Theorem 4.2, $H_q(M;\mathbb{Z}) = H_{n-q}(M;\mathbb{Z}) = 0$ for $2 \leq q \leq n-2$.

Next, we consider the case $c = 0$. As $n \geq 5$ and $2 \leq q \leq n-2$, by a direct computation we have

$$\text{(4-8)} \quad \sum_{k=q+1}^{n} \sum_{i=1}^{q} \big(2|A(e_i, e_k)|^2 - \langle A(e_i, e_i), A(e_k, e_k)\rangle\big) \\ \leq \frac{qr}{n}\Big(|A|^2 - \frac{2}{n}|H|^2 + \frac{n-4}{\sqrt{2n(n-2)}}|H|\sqrt{|A|^2 - \tfrac{1}{n}|H|^2}\Big) \leq 0,$$

provided $|A|^2 \leq |H|^2/(n-2)$. Moreover, the second inequality becomes equality at a point if and only if $|A|^2 = |H|^2/(n-2)$ at this point.

If the second inequality in (4-8) is strict at a point for any orthonormal basis $\{e_i\}$ at that point, then by Theorem 4.2, $H_q(M; \mathbb{Z}) = H_{n-q}(M; \mathbb{Z}) = 0$ for $2 \leq q \leq n-2$.

Now we consider the case that the equality holds for the second inequality in (4-8) at any point for a certain orthonormal basis $\{e_i\}$ at that point. Then at any point, all inequalities become equalities. In particular, from (4-8), there holds $|A|^2 - |H|^2/(n-2) \equiv 0$.

At any point, given a corresponding $\{e_i\}$ and for any $\{e_\alpha\}$, the equality case of (4-3) implies that for any $n+1 \leq \alpha \leq N$,

$$A^\alpha_{11} = \cdots = A^\alpha_{qq}, \quad A^\alpha_{q+1q+1} = \cdots = A^\alpha_{nn}.$$

Moreover, the equality case of (4-5) gives that for any $n+1 \leq \alpha \leq N$,

$$A^\alpha_{ij} = 0, \quad \text{for any } i \neq j.$$

This implies $S_\alpha = \tilde{S}_\alpha$ for any $n+1 \leq \alpha \leq N$.

From the equality case of (4-6) we conclude that there exists a constant ρ for this point such that

$$\tilde{S}_\alpha = \rho \cdot (H^\alpha)^2$$

for any $n+1 \leq \alpha \leq N$. Since $S_\alpha = \tilde{S}_\alpha$ and $|A|^2 - \dfrac{|H|^2}{n-2} \equiv 0$, one has $\rho = \dfrac{1}{n-2}$.

As $|H| > 0$ and $|A|^2 - |H|^2/n > 0$, the equality case of (4-7) implies

$$\frac{|r-q|}{\sqrt{nqr}} = \frac{n-4}{\sqrt{2n(n-2)}}.$$

Hence, $q = 2$ or $q = n-2$. Without loss of generality, we choose $q = 2$.

As $|H| > 0$ on M, at any point we can choose $\{e_\alpha\}$ such that $e_{n+1} = H/|H|$. Then $H^{n+1} = |H|$ and $H^\alpha = 0$ for any $\alpha = n+2, \ldots, N$. Since $A^\alpha_{ij} = 0$ for any α and $i \neq j$, and $\tilde{S}_\alpha = \frac{1}{n-2}(H^\alpha)^2$ for any α, we conclude that $A^\alpha = 0$ for $\alpha = n+2, \ldots, N$. Thus,

$$S_{n+1} = \sum_{i,j}(A^{n+1}_{ij})^2 = |A|^2 = \frac{|H|^2}{n-2}.$$

This implies $A^{n+1}_{ij} \leq |A| = |H|/\sqrt{n-2} < |H|$. Therefore, by Proposition 2.5 in [28], M lies in an $(n+1)$-dimensional affine subspace of $\mathbb{R}^N$. Since M is a hypersurface in $\mathbb{R}^{n+1}$, its second fundamental form can be considered as a symmetric 2-tensor $A = \{A_{ij}\}$ and the mean curvature H of M is essentially a scalar function. We have $|A|^2 - H^2/(n-2) \equiv 0$, and at any point there is an orthonormal basis $\{e_i\}$ such that

$$A_{11} = A_{22}, \qquad A_{33} = \cdots = A_{nn}, \qquad A_{ij} = 0 \quad \text{for all } i \neq j.$$

Therefore, M is a compact hypersurface in $\mathbb{R}^{n+1}$ with two distinct principal curvatures of multiplicities 2 and $n-2$. We denote the two principal curvatures by λ and μ, respectively. Then λ and μ are smooth functions on M. We may assume $H > 0$. Since

$$2\lambda + (n-2)\mu = H \quad \text{and} \quad 2\lambda^2 + (n-2)\mu^2 = \frac{H^2}{n-2},$$

we have, by a direct computation,

$$\lambda = 0 \quad \text{and} \quad \mu = \frac{H}{n-2},$$

or

$$\lambda = \frac{2}{n}H \quad \text{and} \quad \mu = \frac{n-4}{n(n-2)}H.$$

In the first subcase, one has $\lambda\mu = 0$. By the Gauss equation, at any point of M there exists a tangent plane with zero sectional curvature. However, since M is a compact hypersurface in $\mathbb{R}^{n+1}$, there exists a point of M at which all the sectional curvatures are positive. Therefore, this subcase is impossible.

For the second subcase, we will prove that M is isoparametric. By Theorem 2 in [29] and its corollary, the distribution of the space of the principal vectors corresponding to λ (resp., μ) is completely integrable, and λ (resp., μ) is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors. At any point of M, there exists a corresponding $\{e_i\}$ such that e_1, e_2 are principal vectors of λ and $e_3, \ldots, e_n$ are principal vectors of μ. Putting

$$d\lambda = \sum_{k=1}^{n} \lambda_{,k}\omega_k, \quad d\mu = \sum_{k=1}^{n} \mu_{,k}\omega_k,$$

one has at this point that

$$\lambda_{,1} = \lambda_{,2} = 0, \quad \mu_{,3} = \cdots = \mu_{,n} = 0.$$

Putting

$$dH = \sum_{k=1}^{n} H_k\omega_k,$$

we have

$$H_i = \frac{n}{2}\lambda_{,i} = 0, \quad \text{for } i = 1, 2,$$

and

$$H_j = \frac{n(n-2)}{n-4}\mu_{,j} = 0, \quad \text{for } j = 3, \ldots, n.$$

Consequently, $dH = 0$ at this point. Since the point is arbitrary, one gets $dH = 0$ on M. This implies that the mean curvature is constant on M and the principal curvatures λ and μ are both constant. Therefore, M is an isoparametric hypersurface in $\mathbb{R}^{n+1}$ with two distinct principal curvatures λ and μ. By Cartan's formula, $\lambda\mu = 0$. This leads to a contradiction.

In summary, we conclude that there always exists a point of M such that the second inequality in (4-8) is strict at this point for any orthonormal basis $\{e_i\}$. Therefore, $H_q(M; \mathbb{Z}) = H_{n-q}(M; \mathbb{Z}) = 0$ for $2 \le q \le n-2$.

Finally, we give the proof of Theorem 4.1. Since M is simply connected, it is oriented (see Theorem 15.43 in [20] for a proof). By the Hurewicz theorem (see Theorem 5 in Chapter 7 of [34]), the homotopy groups of M satisfy $\pi_i(M) = 0$ for $i = 1, \ldots, n-1$. So M is a homotopy sphere. By the proof of the generalized Poincaré conjecture [33], M is a topological sphere. □

5. Differentiable sphere theorem for compact submanifolds in space forms

We will use the mean curvature flow as a tool to prove Theorem 1.2.

Let M be an n-dimensional compact submanifold in a simply connected space form $\mathbb{F}^N(c)$ of constant sectional curvature c. Denote by F the immersion. We deform M by the mean curvature flow $F : M^n \times [0, T) \to \mathbb{F}^N(c)$ that satisfies

$$\frac{\partial}{\partial t} F(x,t) = H(x,t),$$

where $H(x,t)$ is the mean curvature vector of the submanifold $M_t = F(M,t)$.

Lemma 5.1. *Let $F : M^n \times [0, T) \to \mathbb{F}^N(c)$ be a mean curvature flow of compact submanifolds of dimension $n \ge 8$ in a simply connected space form $\mathbb{F}^N(c)$. Assume M_0 satisfies $|A|^2 \le 4c + \frac{|H|^2}{n-2}$, and $|H| > 0$ for $c = 0$. Then M_t satisfies $|A|^2 < 4c + \frac{|H|^2}{n-2}$ for $t > 0$.*

Proof. As in [1; 2; 26], we set $Q = |A|^2 - a|H|^2 - bc$. Then

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q = -2(|\nabla A|^2 - a|\nabla H|^2) + 2R_1 - 2aR_2 - 2nc|\mathring{A}|^2 - 2n\left(a - \frac{1}{n}\right)c|H|^2, \tag{5-1}$$

where

$$R_1 = \sum_{\alpha,\beta}\left(\sum_{i,j} A^\alpha_{ij} A^\beta_{ij}\right)^2 + |R^\perp|^2 \quad \text{and} \quad R_2 = \sum_{i,j}\left(\sum_\alpha H^\alpha A^\alpha_{ij}\right)^2,$$

with

$$|R^\perp|^2 = \sum_{i,j,\alpha,\beta}\left(\sum_k (A^\alpha_{ik} A^\beta_{jk} - A^\beta_{ik} A^\alpha_{jk})\right)^2 = \sum_{\alpha,\beta} N(A^\alpha A^\beta - A^\beta A^\alpha).$$

Here $A^\alpha = (A^\alpha_{ij})$ and $N(\cdot)$ denotes the squared Frobenius norm of a matrix.

At the point where $|H| \neq 0$, we choose $\{e_\alpha\}$ such that $e_{n+1} = H/|H|$ at this point. Let $A_H = \sum_{i,j} A_{ij}^{n+1} \omega^i \otimes \omega^j$, $A_I = A - A_H$, $\mathring{A}_H = A_H - \frac{1}{n} H \otimes g$, $\mathring{A}_I = \mathring{A} - \mathring{A}_H$. From the definitions, it follows that

$$|A_H|^2 = |A^{n+1}|^2, \quad |A_I|^2 = \sum_{\alpha > n+1} |A^\alpha|^2 = |A|^2 - |A_H|^2,$$
$$|\mathring{A}_H|^2 = |\mathring{A}^{n+1}|^2, \quad |\mathring{A}_I|^2 = \sum_{\alpha > n+1} |A^\alpha|^2 = |\mathring{A}|^2 - |\mathring{A}_H|^2.$$

Notice that $|A_H|^2 = |\mathring{A}_H|^2 + \frac{1}{n}|H|^2$ and $|\mathring{A}_I|^2 = |A_I|^2$. By the calculations in [1], one has

$$\begin{aligned}(5\text{-}2)\quad 2R_1 - 2aR_2 = {} & 2|\mathring{A}_H|^4 - 2\Big(a - \frac{2}{n}\Big)|\mathring{A}_H|^2|H|^2 - \frac{2}{n}\Big(a - \frac{1}{n}\Big)|H|^4 \\ & + 4\sum_{\alpha > n+1}\Big(\sum_{i,j} \mathring{A}_{ij}^{n+1}\mathring{A}_{ij}^\alpha\Big)^2 + 4\sum_{\alpha > n+1} N(A^{n+1}\mathring{A}^\alpha - \mathring{A}^\alpha A^{n+1}) \\ & + 2\sum_{\alpha,\beta > n+1}\Big(\sum_{i,j}\mathring{A}_{ij}^\alpha \mathring{A}_{ij}^\beta\Big)^2 + 2\sum_{\alpha,\beta > n+1} N(\mathring{A}^\alpha\mathring{A}^\beta - \mathring{A}^\beta\mathring{A}^\alpha).\end{aligned}$$

For the first line of the right-hand side of (5-2), we replace $|H|^2$ with

$$\frac{|\mathring{A}_H|^2 + |\mathring{A}_I|^2 - bc - Q}{a - \frac{1}{n}}$$

and get

$$\begin{aligned}(5\text{-}3)\quad & 2|\mathring{A}_H|^4 - 2\Big(a - \frac{2}{n}\Big)|\mathring{A}_H|^2|H|^2 - \frac{2}{n}\Big(a - \frac{1}{n}\Big)|H|^4 \\ & = -\bigg(2 + \frac{2}{n\big(a - \frac{1}{n}\big)}\bigg)|\mathring{A}_H|^2|\mathring{A}_I|^2 + \bigg(2 + \frac{2}{n\big(a - \frac{1}{n}\big)}\bigg)|\mathring{A}_H|^2 Q \\ & \quad + \bigg(2b + \frac{2b}{n\big(a - \frac{1}{n}\big)}\bigg)c|\mathring{A}_H|^2 - \frac{2}{n\big(a - \frac{1}{n}\big)}|\mathring{A}_I|^4 + \frac{4}{n\big(a - \frac{1}{n}\big)}|\mathring{A}_I|^2 Q \\ & \quad + \frac{4bc}{n\big(a - \frac{1}{n}\big)}|\mathring{A}_I|^2 - \frac{4bc}{n\big(a - \frac{1}{n}\big)}Q - \frac{2b^2c^2}{n\big(a - \frac{1}{n}\big)} - \frac{2}{n\big(a - \frac{1}{n}\big)}Q^2.\end{aligned}$$

For the last two lines of the right side of (5-2), the computations in [1] give

$$(5\text{-}4)\quad \sum_{\alpha > n+1}\Big(\sum_{i,j}\mathring{A}_{ij}^{n+1}\mathring{A}_{ij}^\alpha\Big)^2 + \sum_{\alpha > n+1} N(A^{n+1}\mathring{A}^\alpha - \mathring{A}^\alpha A^{n+1}) \le 2|\mathring{A}_H|^2|\mathring{A}_I|^2$$

and

$$2\sum_{\alpha,\beta>n+1}\Big(\sum_{i,j}\mathring{A}^{\alpha}_{ij}\mathring{A}^{\beta}_{ij}\Big)^2+2\sum_{\alpha,\beta>n+1}N(\mathring{A}^{\alpha}\mathring{A}^{\beta}-\mathring{A}^{\beta}\mathring{A}^{\alpha})\le 3|\mathring{A}_I|^4. \tag{5-5}$$

Combining (5-2)–(5-5), we have

$$\begin{aligned}2R_1-2aR_2\le{}&\Big(6-\frac{2}{n\big(a-\frac{1}{n}\big)}\Big)|\mathring{A}_H|^2|\mathring{A}_I|^2+\Big(3-\frac{2}{n\big(a-\frac{1}{n}\big)}\Big)|\mathring{A}_I|^4\\&+\Big(2+\frac{2}{n\big(a-\frac{1}{n}\big)}\Big)|\mathring{A}_H|^2Q+\frac{4}{n\big(a-\frac{1}{n}\big)}|\mathring{A}_I|^2Q\\&+\Big(2b+\frac{2b}{n\big(a-\frac{1}{n}\big)}\Big)c|\mathring{A}_H|^2+\frac{4bc}{n\big(a-\frac{1}{n}\big)}|\mathring{A}_I|^2\\&-\frac{4bc}{n\big(a-\frac{1}{n}\big)}Q-\frac{2b^2c^2}{n\big(a-\frac{1}{n}\big)}-\frac{2}{n\big(a-\frac{1}{n}\big)}Q^2.\end{aligned}\tag{5-6}$$

We also calculate that

$$\begin{aligned}-2nc|\mathring{A}|^2-2n\Big(a-\frac{1}{n}\Big)c|H|^2&=-2nc|\mathring{A}|^2-2nc(|\mathring{A}|^2-bc-Q)\\&=-4nc|\mathring{A}_H|^2-4nc|\mathring{A}_I|^2+2nc^2b+2ncQ.\end{aligned}\tag{5-7}$$

Substituting (5-6) and (5-7) into (5-1) yields

$$\begin{aligned}\Big(\frac{\partial}{\partial t}-\Delta\Big)Q&\\ \le -2\big(|\nabla A|^2-a|\nabla H|^2\big)&\\&+\Big(6-\frac{2}{n\big(a-\frac{1}{n}\big)}\Big)|\mathring{A}_H|^2|\mathring{A}_I|^2+\Big(3-\frac{2}{n\big(a-\frac{1}{n}\big)}\Big)|\mathring{A}_I|^4\\&+\Big(2b+\frac{2b}{n\big(a-\frac{1}{n}\big)}-4n\Big)c|\mathring{A}_H|^2+\Big(\frac{4b}{n\big(a-\frac{1}{n}\big)}-4n\Big)c|\mathring{A}_I|^2\\&+\Big(\frac{2\big(a-\frac{2}{n}\big)}{a-\frac{1}{n}}|\mathring{A}_H|^2+\frac{4}{n\big(a-\frac{1}{n}\big)}(|\mathring{A}|^2-bc)+2nc\Big)Q\\&+\Big(2nb-\frac{2b^2}{n\big(a-\frac{1}{n}\big)}\Big)c^2-\frac{2}{n\big(a-\frac{1}{n}\big)}Q^2.\end{aligned}\tag{5-8}$$

For $n\ge 8$, we choose $a=\frac{1}{n-2}$ and $b=4$, satisfying the condition $a<\frac{3}{n+2}$. We also have the following gradient inequality for submanifolds in $\mathbb{F}^N(c)$:

$$|\nabla A|^2\ge\frac{3}{n+2}|\nabla H|^2. \tag{5-9}$$

From this inequality, we derive $-2\big(|\nabla A|^2-\frac{1}{n-2}|\nabla H|^2\big)\le 0$. Consequently, we discard this term in (5-8).

We will analyze the following three cases separately: $c=0$, $c<0$ and $c>0$ for the constant curvature c of the space form $\mathbb{F}^N(c)$.

Case $c=0$: Since $|H|>0$ at $t=0$, there is a $t_0>0$ such that $|H|>0$ for $t\in[0,t_0]$. At any point in M_t for $t\in[0,t_0]$, (5-8) implies

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)Q\le\bigg(6-\frac{2}{n\big(a-\frac{1}{n}\big)}\bigg)|\mathring{A}_H|^2|\mathring{A}_I|^2+\bigg(3-\frac{2}{n\big(a-\frac{1}{n}\big)}\bigg)|\mathring{A}_I|^4 -\frac{2}{n\big(a-\frac{1}{n}\big)}Q^2+\bigg(\frac{2\big(a-\frac{2}{n}\big)}{a-\frac{1}{n}}|\mathring{A}_H|^2+\frac{4}{n\big(a-\frac{1}{n}\big)}|\mathring{A}|^2\bigg)Q.$$

Since $a=\frac{1}{n-2}$, one has

$$6-\frac{2}{n\big(a-\frac{1}{n}\big)}=-(n-8)\le 0,\qquad -\frac{2}{n\big(a-\frac{1}{n}\big)}=-(n-2)<0.$$

Therefore,

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)Q\le-(n-5)|\mathring{A}_I|^4+\big(-(n-4)|\mathring{A}_H|^2+2(n-2)|\mathring{A}|^2\big)Q. \tag{5-10}$$

By the maximum principle, (5-10) implies that $Q\le 0$ is preserved on $[0,t_0]$.

By (5-10) we have, on $[0,t_0]$,

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)Q\le-(n-5)|\mathring{A}_I|^4-(n-4)|\mathring{A}_H|^2Q. \tag{5-11}$$

Since $-(n-5)|\mathring{A}_I|^4\le 0$, by the strong maximum principle, either $Q<0$ for all $t\in(0,t_0]$, or $Q\equiv 0$ for all $t\in[0,t_0]$.

Recall that we discarded the nonpositive term $-2\big(|\nabla A|^2-\frac{1}{n-2}|\nabla H|^2\big)$ in the above computations. If we retain this item, then (5-11) becomes

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)Q\le-2\Big(|\nabla A|^2-\frac{1}{n-2}|\nabla H|^2\Big)-(n-5)|\mathring{A}_I|^4-(n-4)|\mathring{A}_H|^2Q.$$

If $Q\equiv 0$ for all $t\in[0,t_0]$, this inequality implies $|\nabla A|^2-\frac{1}{n-2}|\nabla H|^2\equiv 0$ and $\mathring{A}_I\equiv 0$. As $\frac{1}{n-2}<\frac{3}{n+2}$ for $n\ge 8$, it follows from (5-9) that $\nabla A\equiv 0$, i.e., the second fundamental form is parallel. Therefore, M lies in an $(n+1)$-dimensional affine subspace of $\mathbb{R}^N$, and the second fundamental form of the hypersurface M in $\mathbb{R}^{n+1}$ is parallel. Then by [18], $M=\mathbb{S}^k(r)\times\mathbb{R}^{n-k}$ for $k=0,1,2,3,\dots,n$. Since M is compact, we conclude that $k=n$ and $M=\mathbb{S}^n(r)$. This implies $|A|^2\equiv\frac{1}{n}|H|^2$, which is a contradiction to that $Q\equiv 0$. Therefore, $Q<0$ for all $t\in(0,t_0]$. By the maximum principle, $Q<0$ for all $t>0$.

Case $c<0$: At $t=0$, the pinching condition implies that $|H|>0$. Then there is a $t_0>0$ such that $|H|>0$ for $t\in[0,t_0]$.

As $a=\frac{1}{n-2}$ and $b=4$, (5-8) implies that, at any point in M_t for $t\in[0,t_0]$,

$$\left(\frac{\partial}{\partial t}-\Delta\right)Q \le \left[-(n-4)|\mathring{A}_H|^2+2(n-2)(|\mathring{A}|^2-bc)+2nc\right]Q-8(n-4)c^2. \tag{5-12}$$

As $-8(n-4)c^2<0$, by the maximum principle, $Q\le 0$ is preserved on $[0,t_0]$. By the strong maximum principle, either $Q<0$ for all $t\in(0,t_0]$, or $Q\equiv 0$ for all $t\in[0,t_0]$. For the second case, (5-12) implies

$$0\le -8(n-4)c^2<0,$$

which is impossible. Therefore, $Q<0$ for all $t\in(0,t_0]$. By the maximum principle, $Q<0$ for all $t>0$.

Case $c>0$: We first consider the point $|H|\neq 0$. For $x,y\ge 0$, define the function

$$F(x,y)=\left(6-\frac{2}{n\left(a-\frac{1}{n}\right)}\right)xy+\left(3-\frac{2}{n\left(a-\frac{1}{n}\right)}\right)y^2 +\left(2b+\frac{2b}{n\left(a-\frac{1}{n}\right)}-4n\right)cx+\left(\frac{4b}{n\left(a-\frac{1}{n}\right)}-4n\right)cy+\left(2nb-\frac{2b^2}{n\left(a-\frac{1}{n}\right)}\right)c^2.$$

As $a=\frac{1}{n-2}$ and $b=4$, $F(x,y)$ can be expressed as

$$F(x,y)=-(n-8)xy-(n-5)y^2+4(n-4)cy-[8(n-4)-\epsilon]c^2-\epsilon c^2.$$

As $n\ge 8$, we have $-(n-8)\le 0$. Hence, we only need to consider the function

$$f(y)=-(n-5)y^2+4(n-4)cy-[8(n-4)-\epsilon]c^2.$$

The discriminant of this quadratic satisfies

$$\Delta=16(n-4)^2-4(n-5)[8(n-4)-\epsilon]<0$$

for a sufficiently small $\epsilon>0$. So $f(y)<0$ for this ϵ. Therefore,

$$F\left(|\mathring{A}_H|^2,|\mathring{A}_I|^2\right)<-\epsilon c^2.$$

Thus, (5-8) implies

$$\left(\frac{\partial}{\partial t}-\Delta\right)Q\le\left[-(n-4)|\mathring{A}_H|^2+2(n-2)(|\mathring{A}|^2-bc)+2nc\right]Q-\epsilon c^2. \tag{5-13}$$

If $|H|=0$ at a point, then $Q=|A|^2-bc$, which implies $|\mathring{A}|^2=|A|^2=Q+bc$. Therefore, by the Li–Li inequality [25],

$$\left(\frac{\partial}{\partial t}-\Delta\right)Q\le 3|\mathring{A}|^4-2nc|\mathring{A}|^2=3|\mathring{A}|^2(Q+bc)-2nc|\mathring{A}|^2 3|\mathring{A}|^2Q+(3b-2n)c|\mathring{A}|^2.$$

As $b=4$, $3b-2n=-2(n-6)<0$. Hence,

$$\left(\frac{\partial}{\partial t}-\Delta\right)Q\le\left(3|\mathring{A}|^2-2(n-6)c\right)Q-8(n-6)c^2. \tag{5-14}$$

Therefore, by the maximum principle, we see from (5-13) and (5-14) that $Q \le 0$ is preserved.

Moreover, by (5-13) and (5-14), one always has

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q \le -\gamma_1 (|\mathring{A}|^2 + c) Q - \gamma_2$$

for positive constants γ_1 and γ_2. Therefore, by the strong maximum principle, either $Q < 0$ for all $t > 0$, or $Q \equiv 0$ for all $t \ge 0$. However, for the second case, the above inequality implies

$$0 \le -\gamma_1 < 0,$$

which is impossible. Therefore, $Q < 0$ for all $t > 0$. □

Remark 5.2. For $4 \le n \le 7$, by a similar computation, the pinching condition $|A|^2 \le a|H|^2 + bc$ for certain $\frac{1}{n-1} < a < \frac{1}{n-2}$ and $2 < b < 4$ is also preserved.

Proof of Theorem 1.2.. When $n = 5$, 6, if $c > 0$, or $c = 0$ and $|H| > 0$, by Theorem 4.1, M is a topological sphere. Since the differentiable structure on a topological sphere of dimension 5 or 6, M is diffeomorphic to the standard sphere.

When $n \ge 9$, combining Theorem 1.1 and Lemma 5.1, we conclude that M is diffeomorphic to the standard sphere. □

6. More differentiable sphere theorems for compact submanifolds

We present two more differentiable sphere theorems for compact submanifolds in Riemannian manifolds.

Theorem 6.1. *Let M^n $(n \ge 4, n \ne 7, 8)$ be an n-dimensional compact simply connected submanifold in an N-dimensional Riemannian manifold $\overline{M}^N$. Suppose that one of the following conditions holds*:

(i) $|A|^2 < \frac{10}{3}\left(\overline{\mathrm{Ric}}_{\min} - \frac{5N-11}{5}\overline{K}_{\max}\right) + \frac{|H|^2}{n-2}$.

(ii) $|A|^2 < \frac{7N}{6s}\left(\overline{\mathrm{Ric}}^{[s]}_{\min} - \frac{s(7N^2-7N-24)}{7N}\overline{K}_{\max}\right) + \frac{|H|^2}{n-2}$ *for some* $1 \le s \le N$.

(iii) $|A|^2 < \frac{7N(N-1)}{6}\left(\overline{R}_0 - \frac{7N^2-7N-24}{7N(N-1)}\overline{K}_{\max}\right) + \frac{|H|^2}{n-2}$.

Then M is diffeomorphic to $\mathbb{S}^n$.

Proof. As in Theorem 1.1, we need only show that M has positive isotropic curvature.

(i) For any $x \in \overline{M}$, suppose $u, v \in U_x\overline{M}$ be two orthonormal vectors such that $\overline{K}(\pi) = \overline{K}_{\min}(x)$, where $\pi = \mathrm{span}\{u, v\}$. Let $V_x^k \subset T_x\overline{M}$ be a k-dimensional subspace such that $v \in V_x^k$ and $u \perp V_x^k$. Define $e_{A_0} = u,\ e_{A_1} = v$, and let $\{e_{A_1}, \ldots, e_{A_k}\}$

be an orthonormal basis of V_x^k. We have

$$\overline{\mathrm{Ric}}^{(k)}\left([e_{A_0}, e_{A_1}, \dots, e_{A_k}]\right) = \sum_{p=1}^{k} \bar{R}_{A_0A_pA_0A_p} = \bar{R}_{A_0A_1A_0A_1} + \sum_{p=2}^{k} \bar{R}_{A_0A_pA_0A_p} \le \bar{K}_{\min} + (k-1)\bar{K}_{\max}.$$

By the definition of $\overline{\mathrm{Ric}}^{(k)}_{\min}$, one has

$$\bar{K}_{\min} \ge \overline{\mathrm{Ric}}^{(k)}_{\min} - (k-1)\bar{K}_{\max}. \tag{6-1}$$

Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame. Extend $\{e_1, e_2, e_3, e_4\}$ to an orthonormal $(k+1)$-frame $\{e_1, \dots, e_{k+1}\}$. Then

$$\overline{\mathrm{Ric}}^{(k)}([e_1, e_2, \dots, e_{k+1}]) = \sum_{p=2}^{k+1} \bar{R}_{1p1p} = \bar{R}_{1313} + \bar{R}_{1414} + \left(\bar{R}_{1212} + \sum_{p=5}^{k+1} \bar{R}_{1p1p}\right) \le \bar{R}_{1313} + \bar{R}_{1414} + (k-2)\bar{K}_{\max}.$$

This implies

$$\bar{R}_{1313} + \bar{R}_{1414} \ge \overline{\mathrm{Ric}}^{(k)}_{\min} - (k-2)\bar{K}_{\max}. \tag{6-2}$$

Combining (6-1) and (6-2) with Berger's inequality, we obtain

$$\begin{aligned} \bar{R}_{1313} + \bar{R}_{1414} - \bar{R}_{1234} &\ge \overline{\mathrm{Ric}}^{(k)}_{\min} - (k-2)\bar{K}_{\max} - \tfrac{2}{3}\big(k\bar{K}_{\max} - \overline{\mathrm{Ric}}^{(k)}_{\min}\big) \\ &= \tfrac{5}{3}\big[\overline{\mathrm{Ric}}^{(k)}_{\min} - \big(k - \tfrac{6}{5}\big)\bar{K}_{\max}\big]. \end{aligned} \tag{6-3}$$

Similarly,

$$\bar{R}_{2323} + \bar{R}_{2424} - \bar{R}_{1234} \ge \tfrac{5}{3}\big[\overline{\mathrm{Ric}}^{(k)}_{\min} - \big(k - \tfrac{6}{5}\big)\bar{K}_{\max}\big]. \tag{6-4}$$

Combining (3-2), (6-3) and (6-4), we obtain

$$\begin{aligned} &R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \\ &\quad\ge \tfrac{10}{3}\big[\overline{\mathrm{Ric}}^{(k)}_{\min} - \big(k - \tfrac{6}{5}\big)\bar{K}_{\max}\big] + \sum_{\alpha}\bigg[\frac{\big(\sum_{i=1}^{n} A^{\alpha}_{ii}\big)^2}{n-2} + \sum_{i\ne j}(A^{\alpha}_{ij})^2 - \sum_{i,j=1}^{n}(A^{\alpha}_{ij})^2 \\ &\qquad\qquad - 2(A^{\alpha}_{13})^2 - 2(A^{\alpha}_{14})^2 - 2(A^{\alpha}_{23})^2 - 2(A^{\alpha}_{24})^2\bigg] \\ &\quad\ge \tfrac{10}{3}\left[\overline{\mathrm{Ric}}^{(k)}_{\min} - \big(k - \tfrac{6}{5}\big)\bar{K}_{\max}\right] + \frac{|H|^2}{n-2} - |A|^2. \end{aligned}$$

Choosing $k = N-1$ and combining the assumption, we know that M has positive isotropic curvature.

(ii) For any $x \in \overline{M}$, let $u, v \in U_x\overline{M}$ be two orthonormal vectors such that $\overline{K}(\pi) = \overline{K}_{\min}(x)$ with $\pi = \operatorname{span}\{u, v\}$. Let $\{e_{A_0}, e_{A_1}, \dots, e_{A_k}\}$ be an orthonormal $(k+1)$-frame such that $e_{A_0} = u$, $e_{A_1} = v$. Then

$$\begin{aligned}\overline{R}^{(k)}\big([e_{A_0}, e_{A_1}, \dots, e_{A_k}]\big) &= \sum_{p=0}^{k}\sum_{q=0}^{k} \overline{R}_{A_pA_qA_pA_q}\\ &= \overline{R}_{A_0A_1A_0A_1} + \overline{R}_{A_1A_0A_1A_0}\\ &\qquad + \sum_{p=0}^{k}\sum_{q=0}^{k} \overline{R}_{A_pA_qA_pA_q} - (\overline{R}_{A_0A_1A_0A_1} + \overline{R}_{A_1A_0A_1A_0})\\ &\le 2\overline{K}_{\min} + [k(k+1)-2]\,\overline{K}_{\max}.\end{aligned}$$

This implies

$$\overline{K}_{\min} \ge \tfrac{1}{2}\big(\overline{R}^{(k)}_{\min} - [k(k+1)-2]\,\overline{K}_{\max}\big). \tag{6-5}$$

By the definition of $\overline{R}^{(k,s)}$, one has

$$\frac{\overline{R}^{(k)}_{\min}}{k(k+1)} \ge \frac{\overline{R}^{(k,s)}_{\min}}{ks}. \tag{6-6}$$

Combining (6-5) and (6-6), we have

$$\overline{K}_{\min} \ge \tfrac{1}{2}\big(\tfrac{k+1}{s}\overline{R}^{(k,s)}_{\min} - [k(k+1)-2]\,\overline{K}_{\max}\big). \tag{6-7}$$

Suppose $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame. Extend $\{e_1, e_2, e_3, e_4\}$ to a $(k+1)$-frame $\{e_1, \dots, e_{k+1}\}$. As $\overline{R}^{(k)} = \overline{R}^{(k,k+1)}$, we have

$$\begin{aligned}\overline{R}^{(k)}([e_1, \dots, e_{k+1}]) &= \sum_{i=1}^{k+1}\sum_{j=1}^{k+1} \overline{R}_{ijij}\\ &= 2\big(\overline{R}_{1313} + \overline{R}_{1414} + \overline{R}_{2323} + \overline{R}_{2424}\big)\\ &\qquad + \sum_{i=1}^{k+1}\sum_{j=1}^{k+1} \overline{R}_{ijij} - 2(\overline{R}_{1313} + \overline{R}_{1414} + \overline{R}_{2323} + \overline{R}_{2424})\\ &\le 2\big(\overline{R}_{1313} + \overline{R}_{1414} + \overline{R}_{2323} + \overline{R}_{2424}\big) + [k(k+1)-8]\,\overline{K}_{\max}.\end{aligned}$$

By (6-6), this implies that for $1 \le s \le N$, there holds

$$\overline{R}_{1313} + \overline{R}_{1414} + \overline{R}_{2323} + \overline{R}_{2424} \ge \tfrac{1}{2}\big(\tfrac{k+1}{s}\overline{R}^{(k,s)}_{\min} - [k(k+1)-8]\overline{K}_{\max}\big). \tag{6-8}$$

Combining (3-5), (6-7) and (6-8) with Berger's inequality, we obtain

$$\begin{aligned}
&R_{1313}+R_{1414}+R_{2323}+R_{2424}-2R_{1234}\\
&\geq \tfrac{1}{2}\Big\{\tfrac{k+1}{s}\overline{R}^{(k,s)}_{\min}-[k(k+1)-8]\,\overline{K}_{\max}\Big\}\\
&\quad -\tfrac{2}{3}\Big[k(k+1)\overline{K}_{\max}-\tfrac{(k+1)}{s}\overline{R}^{(k,s)}_{\min}\Big]+\tfrac{|H|^2}{n-2}-|A|^2\\
&\geq \tfrac{7}{6}\Big[\tfrac{k+1}{s}\overline{R}^{(k,s)}_{\min}-\big(k^2+k-\tfrac{24}{7}\big)\,\overline{K}_{\max}\Big]+\tfrac{|H|^2}{n-2}-|A|^2.
\end{aligned}$$

Choosing $k=N-1$ and combining the assumption, we know that M has positive isotropic curvature.

(iii) Substituting $s=N$ into (ii), we derive that M has positive isotropic curvature. □

Theorem 6.2. *Let M^n $(n\geq 4, n\neq 7,8)$ be an n-dimensional compact simply connected submanifold in an N-dimensional Riemannian manifold $\overline{M}^N$. Suppose that one of the following conditions holds*:

(i) $|A|^2<\frac{4(N+2)}{3}\big(\overline{K}_{\min}-\frac{1}{N+2}\overline{\mathrm{Ric}}_{\max}\big)+\frac{|H|^2}{n-2}$.

(ii) $|A|^2<\frac{2(sN-s+6)}{3}\big(\overline{K}_{\min}-\frac{1}{sN-s+6}\overline{\mathrm{Ric}}^{[s]}_{\max}\big)+\frac{|H|^2}{n-2}$ *for some* $2\leq s\leq N$.

(iii) $|A|^2<\frac{2(N^2-N+6)}{3}\big(\overline{K}_{\min}-\frac{N^2-N}{N^2-N+6}\overline{R}_0\big)+\frac{|H|^2}{n-2}$.

Then M is diffeomorphic to $\mathbb{S}^n$.

Proof. As in Theorem 1.1, we need only show that M has positive isotropic curvature.

(i) As in the proof Theorem 6.1(i), one has

$$\overline{K}_{\max}\leq\overline{\mathrm{Ric}}^{(k)}_{\max}-(k-1)\overline{K}_{\min}.$$

Suppose $\{e_1,e_2,e_3,e_4\}$ is an orthonormal four-frame. Combining (3-5) with the above inequality, we obtain

$$\begin{aligned}
R_{1313}+R_{1414}+R_{2323}+R_{2424}-2R_{1234}&\geq\tfrac{16}{3}\big(\overline{K}_{\min}-\tfrac{1}{4}\overline{K}_{\max}\big)+\tfrac{|H|^2}{n-2}-|A|^2\\
&\geq\tfrac{4}{3}\big((k+3)\overline{K}_{\min}-\overline{\mathrm{Ric}}^{(k)}_{\max}\big)+\tfrac{|H|^2}{n-2}-|A|^2.
\end{aligned}$$

Choosing $k=N-1$ and combining with the assumption, we see that M has positive isotropic curvature.

(ii) As in the proof of Theorem 6.1(ii), one has that for $2\leq s\leq k+1$,

$$\overline{K}_{\max}\leq\tfrac{1}{2}\big(\overline{R}^{(k,s)}_{\max}-(ks-2)\overline{K}_{\min}\big).$$

Combining this with (3-5), we obtain

$$\begin{aligned}
R_{1313}+R_{1414}+R_{2323}+R_{2424}-2R_{1234}&\geq\tfrac{16}{3}\big(\overline{K}_{\min}-\tfrac{1}{4}\overline{K}_{\max}\big)+\tfrac{|H|^2}{n-2}-|A|^2\\
&\geq\tfrac{2}{3}\big((ks+6)\overline{K}_{\min}-\overline{R}^{(k,s)}_{\max}\big)+\tfrac{|H|^2}{n-2}-|A|^2.
\end{aligned}$$

Choosing $k = N - 1$ and combining with the assumption, we know that M has positive isotropic curvature.

(iii) Substituting $s = N$ into (ii), we derive that M has positive isotropic curvature. □

Remark 6.3. From the proofs of Theorems 6.1 and 6.2, similar differentiable sphere theorems can be established for compact submanifolds under pinching conditions involving the k-th Ricci curvature or the (k, s)-curvature.

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JUAN LI
CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU
CHINA
juanli@zju.edu.cn

HONGWEI XU
CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU
CHINA
xuhw@zju.edu.cn

ENTAO ZHAO
CENTER OF MATHEMATICAL SCIENCES
ZHEJIANG UNIVERSITY
HANGZHOU
CHINA
zhaoet@zju.edu.cn

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