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We introduce the notions of quasitriangular and factorizable Poisson bialgebras. A factorizable Poisson bialgebra induces a factorization of the underlying Poisson algebra. We prove that the Drinfeld classical double of a Poisson bialgebra naturally admits a factorizable Poisson bialgebra structure. Furthermore, we introduce the notion of quadratic Rota–Baxter Poisson algebras and show that a quadratic Rota–Baxter Poisson algebra of zero weight induces a triangular Poisson bialgebra. Moreover, we establish a one-to-one correspondence between factorizable Poisson bialgebras and quadratic Rota–Baxter Poisson algebras of nonzero weights. Finally, we establish the quasitriangular and factorizable theories for differential antisymmetric infinitesimal (ASI) bialgebras, and construct quasitriangular and factorizable Poisson bialgebras from quasitriangular and factorizable (commutative and cocommutative) differential ASI bialgebras respectively.

1. Introduction

Poisson algebras serve as fundamental structures in various areas of mathematics and mathematical physics, including Poisson geometry [28; 29], classical and quantum mechanics [3; 9; 22], algebraic geometry [13; 23], quantization theory [14; 17] and quantum groups [8; 12]. A Poisson algebra is both a Lie algebra and a commutative associative algebra which are compatible in a certain sense.

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Definition 1.1 [19; 29]. A *Poisson algebra* is a triple $(A, [\ ,\ ], \cdot)$, where $(A, [\ ,\ ])$ is a Lie algebra and $(A, \cdot)$ is a commutative associative algebra satisfying

$$[a, b \cdot c] = [a, b] \cdot c + b \cdot [a, c], \quad \forall a, b, c \in A.$$

Both Lie algebras and associative algebras possess well-developed theories of bialgebras, which have found extensive applications in various mathematical fields. In the Lie algebra setting, Lie bialgebras, introduced by Drinfeld [11], play a fundamental role as the infinitesimalization of Poisson Lie groups [8; 16]. On the side of associative algebras, the notion of infinitesimal bialgebras was introduced by Joni and Rota to provide an algebraic framework for the calculus of divided differences [15]. Variants of this notion, such as balanced infinitesimal bialgebras (also termed antisymmetric infinitesimal bialgebras or associative D-bialgebras in different contexts [2; 4; 30]), have been systematically studied by Aguiar [1; 2], who established their close analogy with Lie bialgebras. These structures have been found significant applications in combinatorial mathematics.

Building on these ideas, a unified bialgebra theory for Poisson algebras, called Poisson bialgebras, was later developed in [21], combining aspects of both Lie and infinitesimal bialgebras.

Meanwhile, In the realm of Lie bialgebras, quasitriangular Lie bialgebra structures have been pivotal objects in mathematical physics [12; 26]. Among these quasitriangular Lie bialgebra structures, factorizable Lie bialgebras constitute a particularly important subclass, linking classical r-matrices to factorization problems and playing a key role in integrable systems [5; 24; 25]. Recently, quasitriangular and factorizable theories has been extended to antisymmetric infinitesimal bialgebras with [27] introducing quasitriangular and factorizable antisymmetric infinitesimal bialgebras.

Naturally, we try to establish the quasitriangular and factorizable theories in the context of Poisson bialgebras, synthesizing concepts from both quasitriangular Lie bialgebras and quasitriangular antisymmetric infinitesimal bialgebras. More precisely, we introduce the notion of quasitriangular Poisson bialgebras based on the (ad, L)-invariant condition. In particular, if the symmetric part of the solution of the Poisson Yang–Baxter equation in a quasitriangular Poisson bialgebra is nondegenerate, then a factorizable Poisson bialgebra is obtained. We prove that every factorizable Poisson bialgebra induces a factorization of its underlying Poisson algebra. Furthermore, we establish that the Drinfeld classical double of a Poisson bialgebra is automatically endowed with a canonical factorizable Poisson bialgebra structure.

Recent results have shown that factorizable Lie bialgebras and factorizable antisymmetric infinitesimal bialgebras could be characterized by quadratic Rota–Baxter Lie algebras of nonzero weights and symmetric Rota–Baxter Frobenius algebras of nonzero weights respectively [18; 27]. This motivates our investigation of analogous Rota–Baxter characterizations of factorizable Poisson bialgebras. For this purpose, we introduce the notion of a quadratic Rota–Baxter Poisson algebra by equipping a quadratic Poisson algebra with a Rota–Baxter operator satisfying a compatibility condition. We show that a quadratic Rota–Baxter Poisson algebra of zero weight can give rise to a triangular Poisson bialgebra. Moreover, we establish a one-to-one correspondence between factorizable Poisson bialgebras and quadratic Rota–Baxter Poisson algebras of nonzero weights.

Building on the fact that a Poisson algebra can be obtained from a commutative differential algebra with two commuting derivations [7], Lin, Liu and Bai [20] extended such a connection to the context of bialgebras, utilizing the theory of differential antisymmetric infinitesimal (ASI) bialgebras to construct Poisson bialgebras from commutative and cocommutative differential ASI bialgebras. In this paper, we further investigate this relationship in greater depth. Specifically, we develop the theories of quasitriangular and factorizable differential ASI bialgebras, and apply them to the study of their Poisson bialgebra counterparts. We establish the constructions of quasitriangular and factorizable Poisson bialgebras from quasitriangular and factorizable (commutative and cocommutative) differential ASI bialgebras respectively.

Outline. Section 2 introduces the notion of quasitriangular (triangular) Poisson bialgebras as a special class of coboundary Poisson bialgebras. Section 3 presents the concept of factorizable Poisson bialgebras, which is a distinguished subclass of quasitriangular Poisson bialgebras. We demonstrate that a factorizable Poisson bialgebra induces a factorization of the underlying Poisson algebra. Furthermore, we prove that the Drinfeld classical double of a Poisson bialgebra is naturally endowed with a factorizable Poisson bialgebra structure. Section 4 establishes the Rota–Baxter characterization of factorizable Poisson bialgebras. We introduce the notion of quadratic Rota–Baxter Poisson algebras and establish a one-to-one correspondence between quadratic Rota–Baxter Poisson algebras of nonzero weights and factorizable Poisson bialgebras. Moreover, we show that a quadratic Rota–Baxter Poisson algebra of zero weight can give rise to a triangular Poisson bialgebra. In Section 5, we introduce the notions of quasitriangular and factorizable differential ASI bialgebras, and give Rota–Baxter characterization of factorizable differential ASI bialgebras. The constructions of quasitriangular and factorizable Poisson bialgebras from quasitriangular and factorizable (commutative and cocommutative) differential ASI bialgebras are illustrated respectively.

Throughout this paper, we work over a base field $\mathbb{K}$ of characteristic 0, and all vector spaces and algebras are assumed to be finite-dimensional. We adopt the following conventions and notations.

(1) Let $(A, \diamond)$ be a vector space equipped with a bilinear operation $\diamond : A \otimes A \to A$. Let $L_\diamond(a)$ and $R_\diamond(a)$ denote the left and right multiplication operators, that is,

$$L_\diamond(a)b = R_\diamond(b)a = a \diamond b, \quad \forall a, b \in A.$$

We also simply denote them by $L(a)$ and $R(a)$ without confusion. If $(A, [\ ,\ ])$ is a Lie algebra, we let $\mathrm{ad}_{[,]}(a) = \mathrm{ad}(a)$ denote the adjoint operator, that is,

$$\mathrm{ad}_{[,]}(a)b = \mathrm{ad}(a)b = [a, b], \quad \forall a, b \in A.$$

(2) Let V be a vector space. Denote the flip operator by $\tau : V \otimes V \to V \otimes V$:

$$\tau(u \otimes v) = v \otimes u, \quad \forall u, v \in V.$$

(3) Let $(A, \diamond)$ be a vector space equipped with a bilinear operation $\diamond : A \otimes A \to A$. Let $r = \sum_i a_i \otimes b_i \in A \otimes A$. Set

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad r_{23} = \sum_i 1 \otimes a_i \otimes b_i,$$

where 1 is the unit if $(A, \diamond)$ is unital or a symbol playing a similar role as the unit for the nonunital cases. Further define compound symbols such as $r_{12} \diamond r_{13}$ by

$$r_{12} \diamond r_{13} = \sum_{i,j} a_i \diamond a_j \otimes b_i \otimes b_j, \quad r_{13} \diamond r_{23} = \sum_{i,j} a_i \otimes a_j \otimes b_i \diamond b_j,$$
$$r_{23} \diamond r_{12} = \sum_{i,j} a_i \otimes a_j \diamond b_i \otimes b_j, \quad r_{12} \diamond r_{23} = \sum_{i,j} a_i \otimes b_i \diamond a_j \otimes b_j.$$

(4) Denote the standard pairing between the dual space V^* and V by $\langle\ ,\ \rangle$, so that

$$\langle f, v\rangle := f(v) =: \langle v, f\rangle, \quad \forall f \in V^*,\ v \in V.$$

(5) Let V, W be two vector spaces and $T : V \to W$ be a linear map. Denote the dual map by $T^* : W^* \to V^*$:

$$\langle v, T^*(w^*)\rangle = \langle T(v), w^*\rangle, \quad \forall v \in V, w^* \in W^*.$$

(6) Let A, V be vector spaces. For a linear map $\mu : A \to \mathrm{End}(V)$, define a linear map $\mu^* : A \to \mathrm{End}(V^*)$ by $\mu^*(a) = (\mu(a))^*$, or, more explicitly,

$$\langle \mu^*(a)v^*, u\rangle = \langle v^*, \mu(a)u\rangle, \quad \forall a \in A, u \in V, v^* \in V^*,$$

(7) Let $\Pi_1 = \{\alpha_k : V_1 \to V_1\}_{k=1}^m$ and $\Pi_2 = \{\beta_k : V_2 \to V_2\}_{k=1}^m$ be two m-tuples of commuting linear maps. Then obviously $\{\alpha_k + \beta_k\}_{k=1}^m$ is still an m-tuple of commuting linear maps, which is denoted by $\Pi_1 + \Pi_2$.

2. Quasitriangular Poisson bialgebras

In this section, we recall the notion of coboundary Poisson bialgebras and introduce the notion of quasitriangular Poisson bialgebras as a special case of the coboundary Poisson bialgebras, based on the notion of (ad, L)-invariance. The notion of a quadratic Poisson algebra is also introduced, which gives rise to an (ad, L)-invariant 2-tensor and serves as the foundation for the subsequent notion of a quadratic Rota–Baxter Poisson algebra.

Definition 2.1 [21]. Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra, V a vector space and $\rho, \mu : A \to \mathrm{End}(V)$ two linear maps. The triple (V, ρ, μ) is called a *representation* of the Poisson algebra $(A, [\ ,\ ]_A, \cdot_A)$ if the following conditions hold:

(1) (V, ρ) is a representation of the Lie algebra $(A, [\ ,\ ]_A)$, that is, $\rho([a, b]_A) = \rho(a)\rho(b) - \rho(b)\rho(a)$ for all $a, b \in A$.

(2) (V, μ) is a representation of $(A, \cdot_A)$, that is, $\mu(a \cdot_A b) = \mu(a)\mu(b)$ for all $a, b \in A$.

(3) The following equations hold:

$$\rho(a \cdot_A b) = \mu(b)\rho(a) + \mu(a)\rho(b),$$
$$\mu([a, b]_A) = \rho(a)\mu(b) - \mu(b)\rho(a), \quad \forall a, b \in A.$$

Proposition 2.2 [21]. *Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra, V a vector space and $\rho, \mu : A \to \mathrm{End}(V)$ two linear maps. Then (V, ρ, μ) is a representation of $(A, [\ ,\ ]_A, \cdot_A)$ if and only if there is a Poisson algebra structure on $A \oplus V$ with the bilinear operations $[\ ,\ ]$ and $\cdot$ defined as follows, for all $a_1, a_2 \in A$ and $v_1, v_2 \in V$:*

$$[a_1 + v_1, a_2 + v_2] = [a_1, a_2]_A + \rho(a_1)v_2 - \rho(a_2)v_1,$$
$$(a_1 + v_1) \cdot (a_2 + v_2) = a_1 \cdot_A a_2 + \mu(a_1)v_2 + \mu(a_2)v_1,$$

The resulting Poisson algebra structure on $A \oplus V$ is denoted by $(A \ltimes_{\rho,\mu} V, [\ ,\ ], \cdot)$ and called the ***semidirect product Poisson algebra*** *by $(A, [\ ,\ ]_A, \cdot_A)$ and (V, ρ, μ).*

Example 2.3 [21]. Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra. Then (A, ad, L) is a representation of $(A, [\ ,\ ]_A, \cdot_A)$, called the *adjoint representation*, and $(A^*, -\mathrm{ad}^*, L^*)$ is also a representation of $(A, [\ ,\ ]_A, \cdot_A)$, called the *coadjoint representation*.

A *Lie bialgebra* is a pair of Lie algebras $(A, [\ ,\ ]_A)$ and $(A^*, [\ ,\ ]_{A^*})$ such that

$$\delta([a, b]_A) = (\mathrm{ad}(a) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(a))\delta(b) - (\mathrm{ad}(b) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(b))\delta(a), \quad \forall a, b \in A,$$

where $\delta : A \to \mathrm{Alt}^2(A)$ is defined by $\langle \delta(a), x^* \otimes y^* \rangle = \langle a, [x^*, y^*]_{A^*} \rangle$. A Lie bialgebra is denoted by $\big((A, [\ ,\ ]_A), (A^*, [\ ,\ ]_{A^*})\big)$ or $(A, [\ ,\ ]_A, \delta)$.

An *infinitesimal bialgebra* is a pair of associative algebras $(A, \cdot_A)$ and $(A^*, \cdot_{A^*})$ such that

$$\Delta(a \cdot_A b) = (L(a) \otimes \mathrm{id})\Delta(b) + (\mathrm{id} \otimes R(b))\Delta(a), \quad \forall a, b \in A,$$

where $\Delta : A \to A \otimes A$ is defined by $\langle \Delta(a), x^* \otimes y^* \rangle = \langle a, x^* \cdot_{A^*} y^* \rangle$. An infinitesimal bialgebra is denoted by $\big((A, \cdot_A), (A^*, \cdot_{A^*})\big)$ or $(A, \cdot_A, \Delta)$.

Definition 2.4 [21]. A *Poisson bialgebra* is a pair of Poisson algebras $(A, [\ ,\ ]_A, \cdot_A)$ and $(A^*, [\ ,\ ]_{A^*}, \cdot_{A^*})$ such that

(1) $\big((A, [\ ,\ ]_A), (A^*, [\ ,\ ]_{A^*})\big)$ is a Lie bialgebra,

(2) $\big((A, \cdot_A), (A^*, \cdot_{A^*})\big)$ is an infinitesimal bialgebra, and

(3) δ and Δ are compatible in the sense that, for all $a, b \in A$,

$$\delta(a \cdot_A b) \\ = (L(a) \otimes \mathrm{id})\delta(b) + (L(b) \otimes \mathrm{id})\delta(a) + (\mathrm{id} \otimes \mathrm{ad}(a))\Delta(b) + (\mathrm{id} \otimes \mathrm{ad}(b))\Delta(a),$$

and

$$\Delta([a, b]_A) = (\mathrm{ad}(a) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(a))\Delta(b) + (L(b) \otimes \mathrm{id} - \mathrm{id} \otimes L(b))\delta(a),$$

where δ, Δ are the linear duals of $[\ ,\ ]_{A^*}$ and $\cdot_{A^*}$ respectively.

A Poisson bialgebra is denoted by $\big((A, [\ ,\ ]_A, \cdot_A),\ (A^*, [\ ,\ ]_{A^*}, \cdot_{A^*})\big)$ or, in full, $(A, [\ ,\ ]_A, \cdot_A, \delta, \Delta)$.

Poisson bialgebras can be equivalently characterized by Manin triples of Poisson algebras [21]. Notably, for a Poisson bialgebra $\big((A, [\ ,\ ]_A, \cdot_A), (A^*, [\ ,\ ]_{A^*}, \cdot_{A^*})\big)$, the pair $\big((A^*, [\ ,\ ]_{A^*}, \cdot_{A^*}), (A, [\ ,\ ]_A, \cdot_A)\big)$ forms a Poisson bialgebra as well.

Definition 2.5. Let $(A, [\ ,\ ]_A, \cdot_A, \delta_A, \Delta_A)$ and $(B, [\ ,\ ]_B, \cdot_B, \delta_B, \Delta_B)$ be Poisson bialgebras. A *homomorphism* of Poisson bialgebras is a homomorphism of Poisson algebras $\varphi : A \to B$ such that

$$(\varphi \otimes \varphi)\delta_A = \delta_B \circ \varphi, \quad (\varphi \otimes \varphi)\Delta_A = \Delta_B \circ \varphi.$$

If $\varphi : A \to B$ is a linear isomorphism of vector spaces, then $\varphi : A \to B$ is called an *isomorphism* of Poisson bialgebras.

Proposition 2.6. *Let* $((A, [\ ,\ ]_A, \cdot_A), (A^*, [\ ,\ ]_{A^*}, \cdot_{A^*}))$ *be a Poisson bialgebra and* B *be a vector space. Suppose that* $\varphi : A \to B$ *is a linear isomorphism of vector spaces. Define brackets* $[\ ,\ ]_B : B \otimes B \to B$ *and* $[\ ,\ ]_{B^*} : B^* \otimes B^* \to B^*$ *by*

$$[a, b]_B = \varphi([\varphi^{-1}(a), \varphi^{-1}(b)]_A), \quad [x^*, y^*]_{B^*} = (\varphi^*)^{-1}([\varphi^*(x^*), \varphi^*(y^*)]_{A^*}),$$

and multiplications $\cdot_B : B \otimes B \to B$ *and* $\cdot_{B^*} : B^* \otimes B^* \to B^*$ *by*

$$a \cdot_B b = \varphi(\varphi^{-1}(a) \cdot_A \varphi^{-1}(b)), \quad x^* \cdot_{B^*} y^* = (\varphi^*)^{-1}(\varphi^*(x^*) \cdot_{A^*} \varphi^*(y^*)),$$

the equalities holding, as the case may be, for all $a, b \in B$ *and all* $x^*, y^* \in B^*$*. Then* $\big((B, [\ ,\ ]_B, \cdot_B), (B^*, [\ ,\ ]_{B^*}, \cdot_{B^*})\big)$ *is a Poisson bialgebra, and* φ *is an isomorphism of Poisson bialgebras.*

Proof. This follows from a straightforward verification. □

Definition 2.7 [21]. A Poisson bialgebra $(A, [\ ,\ ]_A, \cdot_A, \delta, \Delta)$ is called *coboundary* if there exists $r \in A \otimes A$ such that, for all $a \in A$,

$$\delta(a) = (\mathrm{id} \otimes \mathrm{ad}(a) + \mathrm{ad}(a) \otimes \mathrm{id})(r), \tag{2-1}$$

$$\Delta(a) = (\mathrm{id} \otimes L(a) - L(a) \otimes \mathrm{id})(r). \tag{2-2}$$

A coboundary Poisson bialgebra is denoted by $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$.

Theorem 2.8 [21, Theorem 2]. *Let* $(A, [\ ,\ ]_A, \cdot_A)$ *be a Poisson algebra and* $r \in A \otimes A$*. Write* r *as* $r = S + \Lambda$ *with* $S \in \mathrm{Sym}^2(A)$ *and* $\Lambda \in \mathrm{Alt}^2(A)$*. Let* $\delta : A \to A \otimes A$ *and* $\Delta : A \to A \otimes A$ *be the linear maps defined by* (2-1) *and* (2-2)*. Then* $(A^*, \delta^*, \Delta^*)$ *is a Poisson algebra such that* $(A, [\ ,\ ]_A, \cdot_A, \delta, \Delta)$ *is a Poisson bialgebra if and only if the following conditions are satisfied, for all* $a \in A$:

(1) $(\mathrm{ad}(a) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(a))S = 0.$

(2) $(L(a) \otimes \mathrm{id} - \mathrm{id} \otimes L(a))S = 0.$

(3) $(\mathrm{ad}(a) \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(a) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ad}(a))\boldsymbol{C}(r) = 0.$

(4) $(L(a) \otimes \mathrm{id} \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id} \otimes L(a))\boldsymbol{A}(r) = 0.$

(5) $(\mathrm{ad}(a) \otimes \mathrm{id} \otimes \mathrm{id})\boldsymbol{A}(r) - (\mathrm{id} \otimes L(a) \otimes \mathrm{id} - \mathrm{id} \otimes \mathrm{id} \otimes L(a))\boldsymbol{C}(r) = 0.$

Here

$$\boldsymbol{C}(r) := [r_{12}, r_{13}]_A + [r_{13}, r_{23}]_A + [r_{12}, r_{23}]_A,$$
$$\boldsymbol{A}(r) := r_{12} \cdot_A r_{13} + r_{13} \cdot_A r_{23} - r_{23} \cdot_A r_{12}.$$

Definition 2.9 [21]. Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra and $r \in A \otimes A$. Then r is called a solution of the *Poisson Yang–Baxter equation* in $(A, [\ ,\ ]_A, \cdot_A)$ if $\boldsymbol{C}(r) = \boldsymbol{A}(r) = 0$.

More precisely, $\boldsymbol{C}(r) = 0$ is called the *classical Yang–Baxter equation* in the Lie algebra $(A, [\ ,\ ]_A)$ and $\boldsymbol{A}(r) = 0$ is called the *associative Yang–Baxter equation* in the associative algebra $(A, \cdot_A)$.

Let A be a vector space. Any $r \in A \otimes A$ can be identified with the pair of maps $r_+, r_- : A^* \to A$ defined by

$$\langle r_+(x^*), y^* \rangle = -\langle x^*, r_-(y^*) \rangle = \langle r, x^* \otimes y^* \rangle, \quad \forall x^*, y^* \in A^*.$$

Note that $(\tau(r))_+ = -r_-$ and $(\tau(r))_- = -r_+$. The bracket and multiplication on A^*

defined by (2-1)–(2-2) (as the duals) are given by, for all $x^*, y^* \in A^*$,

$$[x^*, y^*]_r = -\mathrm{ad}^*(r_+(x^*))y^* + \mathrm{ad}^*(r_-(y^*))x^*, \tag{2-3}$$

$$x^* \cdot_r y^* = L^*(r_+(x^*))y^* + L^*(r_-(y^*))x^*. \tag{2-4}$$

Lemma 2.10. *Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra and $r \in A \otimes A$. Then the following statements are equivalent*:

(1) *r is a solution of the Poisson Yang–Baxter equation in $(A, [\ ,\ ]_A, \cdot_A)$.*

(2) *$\tau(r)$ is a solution of the Poisson Yang–Baxter equation in $(A, [\ ,\ ]_A, \cdot_A)$.*

(3) *The following equations hold, for all $x^*, y^* \in A^*$:*

$$[r_+(x^*), r_+(y^*)]_A = r_+\left(-\mathrm{ad}^*(r_+(x^*))y^* + \mathrm{ad}^*(r_-(y^*))x^*\right), \tag{2-5}$$

$$r_+(x^*) \cdot_A r_+(y^*) = r_+\left(L^*(r_+(x^*))y^* + L^*(r_-(y^*))x^*\right). \tag{2-6}$$

(4) *The following equations hold, for all $x^*, y^* \in A^*$:*

$$[r_-(x^*), r_-(y^*)]_A = r_-\left(-\mathrm{ad}^*(r_-(x^*))y^* + \mathrm{ad}^*(r_+(y^*))x^*\right), \tag{2-7}$$

$$r_-(x^*) \cdot_A r_-(y^*) = r_-\left(L^*(r_-(x^*))y^* + L^*(r_+(y^*))x^*\right). \tag{2-8}$$

Proof. (1) $\Longleftrightarrow$ (2): Let $r = \sum_i a_i \otimes b_i \in A \otimes A$. Then we have

$$\begin{aligned}
\boldsymbol{C}(\tau(r)) &= \sum_{i,j} [b_i, b_j]_A \otimes a_i \otimes a_j + b_i \otimes [a_i, b_j]_A \otimes a_j + b_i \otimes b_j \otimes [a_i, a_j]_A \\
&= \sigma_{13}\big(\sum_{i,j} a_j \otimes a_i \otimes [b_i, b_j]_A + a_j \otimes [a_i, b_j]_A \otimes b_i + [a_i, a_j]_A \otimes b_j \otimes b_i\big) \\
&= -\sigma_{13}(\boldsymbol{C}(r)),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{A}(\tau(r)) &= \sum_{i,j} b_i \cdot_A b_j \otimes a_i \otimes a_j + b_i \otimes b_j \otimes a_i \cdot_A a_j - b_i \otimes b_j \cdot_A a_i \otimes a_j \\
&= \sigma_{13}\big(\sum_{i,j} a_j \otimes a_i \otimes b_i \cdot_A b_j + a_i \cdot_A a_j \otimes b_j \otimes b_i - a_j \otimes b_j \cdot_A a_i \otimes b_i\big) \\
&= \sigma_{13}\big(\sum_{i,j} a_j \otimes a_i \otimes b_j \cdot_A b_i + a_j \cdot_A a_i \otimes b_j \otimes b_i - a_j \otimes a_i \cdot_A b_j \otimes b_i\big) \\
&= \sigma_{13}(\boldsymbol{A}(r)),
\end{aligned}$$

where $\sigma_{13} \in \mathrm{End}(A \otimes A \otimes A)$ is defined by $\sigma_{13}(a \otimes b \otimes c) = c \otimes b \otimes a$ for all $a, b, c \in A$. Thus, r is a solution of the Poisson Yang–Baxter equation if and only if $\tau(r)$ is a solution of the Poisson Yang–Baxter equation.

(1) $\Longleftrightarrow$ (3): This follows from [5, Proposition 3.8] and [6, Theorem 3.5].

(2) $\Longleftrightarrow$ (4): This is similar to the proof of (1) $\Longleftrightarrow$ (3). □

We now turn to the definition of quasitriangular Poisson bialgebras as a special case of the coboundary Poisson bialgebras. The notion of (ad, L)-invariance of a 2-tensor in $A \otimes A$ is the main ingredient employed, and is motivated by Theorem 2.8.

Definition 2.11. Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra. An element $r \in A \otimes A$ is called (ad, L)-*invariant* if, for all $a \in A$, we have

$$(\mathrm{ad}(a) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(a))(r) = 0, \tag{2-9}$$

$$(L(a) \otimes \mathrm{id} - \mathrm{id} \otimes L(a))(r) = 0. \tag{2-10}$$

Obviously, if r is (ad, L)-invariant, then so is $\tau(r)$. Denote by I_r the operator

$$I_r = r_+ - r_- : A^* \to A. \tag{2-11}$$

Note that $r_- = -r_+^*$ and hence $I_r^* = I_r$.

Letting S be the symmetric part of r, we have $S_+ = \frac{1}{2} I_r = \frac{1}{2} I_{\tau(r)}$. In particular, if r is antisymmetric, then $I_r = 0$.

Definition 2.12. Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra. If r is a solution of the Poisson Yang–Baxter equation in $(A, [\ ,\ ]_A, \cdot_A)$ and the symmetric part of $r \in A \otimes A$ is (ad, L)-invariant, then the coboundary Poisson bialgebra $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ induced by r is called a *quasitriangular Poisson bialgebra*. If r is also antisymmetric, then $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ is called a *triangular Poisson bialgebra*.

Proposition 2.13. *Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra and $r \in A \otimes A$. Then $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ is a quasitriangular Poisson bialgebra if and only if*

$$(A, [\ ,\ ]_A, \cdot_A, \delta_{\tau(r)}, \Delta_{\tau(r)})$$

is a quasitriangular Poisson bialgebra.

Proof. This follows from Lemma 2.10. □

Lemma 2.14. *Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra and $r \in A \otimes A$. Let S be the symmetric part of r. Then the following conditions are equivalent*:

(1) *S is* (ad, L)*-invariant.*

(2) *The following equations hold, for all $a \in A$ and $x^* \in A^*$*:

$$S_+(\mathrm{ad}^*(a)x^*) + [a, S_+(x^*)]_A = 0, \tag{2-12}$$

$$S_+(L^*(a)x^*) - a \cdot_A S_+(x^*) = 0. \tag{2-13}$$

(3) *The following equations hold, for all $x^*, y^* \in A^*$*:

$$\mathrm{ad}^*(S_+(x^*))y^* + \mathrm{ad}^*(S_+(y^*))x^* = 0, \tag{2-14}$$

$$L^*(S_+(x^*))y^* - L^*(S_+(y^*))x^* = 0. \tag{2-15}$$

Proof. (1) $\Longleftrightarrow$ (2): For all $a \in A$ and $x^*, y^* \in A^*$, we have

$$\begin{aligned}\langle S_+(\mathrm{ad}^*(a)x^*) + [a, S_+(x^*)], y^*\rangle &= \langle S, \mathrm{ad}^*(a)x^* \otimes y^* + x^* \otimes \mathrm{ad}^*(a)y^*\rangle \\ &= \langle (\mathrm{ad}(a) \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{ad}(a))S, x^* \otimes y^*\rangle, \\ \langle S_+(L^*(a)x^*) - a \cdot S_+(x^*), y^*\rangle &= \langle S, L^*(a)x^* \otimes y^* - x^* \otimes L^*(a)y^*\rangle \\ &= \langle (L(a) \otimes \mathrm{id} - \mathrm{id} \otimes L(a))S, x^* \otimes y^*\rangle,\end{aligned}$$

which shows that S is (ad, L)-invariant if and only if equations (2-12)–(2-13) hold.

(2) $\Longleftrightarrow$ (3): For all $a \in A$ and $x^*, y^* \in A^*$, we have

$$\begin{aligned}\langle \mathrm{ad}^*(S_+(x^*))y^* + \mathrm{ad}^*(S_+(y^*))x^*, a\rangle &= \langle y^*, [S_+(x^*), a]_A\rangle + \langle x^*, [S_+(y^*), a]_A\rangle \\ &= -\langle y^*, [a, S_+(x^*)]_A + S_+(\mathrm{ad}^*(a)(x^*))\rangle, \\ \langle L^*(S_+(x^*))y^* - L^*(S_+(y^*))x^*, a\rangle &= \langle y^*, S_+(x^*) \cdot_A a\rangle - \langle x^*, S_+(y^*) \cdot_A a\rangle \\ &= -\langle y^*, -a \cdot_A S_+(x^*) + S_+(L^*(a)x^*)\rangle,\end{aligned}$$

which shows that equations (2-12)–(2-13) hold if and only if (2-14)–(2-15) do. □

Theorem 2.15. *Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra and $r = S + \Lambda \in A \otimes A$, with $S \in \mathrm{Sym}^2(A)$ and $\Lambda \in \mathrm{Alt}^2(A)$. Suppose that S is (ad, L)-invariant. Then r is a solution of the Poisson Yang–Baxter equation in $(A, [\ ,\ ]_A, \cdot_A)$ if and only if $(A^*, [\ ,\ ]_r, \cdot_r)$ is a Poisson algebra and the linear maps $r_+, r_- : (A^*, [\ ,\ ]_r, \cdot_r) \to (A, [\ ,\ ]_A, \cdot_A)$ are both Poisson algebra homomorphisms, where $[\ ,\ ]_r : A^* \otimes A^* \to A^*$ and $\cdot_r : A^* \otimes A^* \to A^*$ are defined by equations* (2-3) *and* (2-4).

Proof. ($\Longrightarrow$) By Definition 2.12, $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ is a quasitriangular Poisson bialgebra where δ_r, Δ_r are defined by (2-1) and (2-2). Thus, $(A^*, [\ ,\ ]_r, \cdot_r)$ is a Poisson algebra where $[\ ,\ ]_r$ and $\cdot_r$ are given by (2-3) and (2-4). By Lemmas 2.10 and 2.14, for all $x^*, y^* \in A^*$, we have

$$\begin{aligned}r_+([x^*, y^*]_r) &= r_+(-\mathrm{ad}^*(r_+(x^*))y^* + \mathrm{ad}^*(r_-(y^*))x^*) = [r_+(x^*), r_+(y^*)]_A, \\ r_+(x^* \cdot_r y^*) &= r_+(L^*(r_+(x^*))y^* + L^*(r_-(y^*))x^*) = r_+(x^*) \cdot_A r_+(y^*), \\ r_-([x^*, y^*]_r) &= r_-(-\mathrm{ad}^*(r_+(x^*))y^* + \mathrm{ad}^*(r_-(y^*))x^*) \\ &= r_-\big(-2\mathrm{ad}^*(S_+(x^*))y^* - \mathrm{ad}^*(r_-(x^*))y^* + \mathrm{ad}^*(r_+(y^*))x^* \\ &\qquad - 2\mathrm{ad}^*(S_+(y^*))x^*\big) \\ &= r_-\big(-\mathrm{ad}^*(r_-(x^*))y^* + \mathrm{ad}^*(r_+(y^*))x^*\big) \\ &= [r_-(x^*), r_-(y^*)]_A, \\ r_-(x^* \cdot_r y^*) &= r_-\big(L^*(r_+(x^*))y^* + L^*(r_-(y^*))x^*\big) \\ &= r_-\big(2L^*(S_+(x^*))y^* + L^*(r_-(x^*))y^* + L^*(r_+(y^*))x^* \\ &\qquad - 2L^*(S_+(y^*))x^*\big) \\ &= r_-\big(L^*(r_-(x^*))y^* + L^*(r_+(y^*))x^*\big) \\ &= r_-(x^*) \cdot_A r_-(y^*),\end{aligned}$$

which shows that r_+, r_- are both Poisson algebra homomorphisms.

($\Longleftarrow$) This follows from Lemma 2.10. □

Recall that a bilinear form $\mathfrak{B} \in \otimes^2 A^*$ on a Poisson algebra $(A, [\ ,\ ]_A, \cdot_A)$ is called *invariant* if

$$\mathfrak{B}([a, b]_A, c) = \mathfrak{B}(a, [b, c]_A), \quad \mathfrak{B}(a \cdot_A b, c) = \mathfrak{B}(a, b \cdot_A c), \quad \forall a, b, c \in A.$$

A *quadratic Poisson algebra* is a quadruple

$$(A, [\ ,\ ]_A, \cdot_A, \mathfrak{B}),$$

where $(A, [\ ,\ ]_A, \cdot_A)$ is a Poisson algebra and $\mathfrak{B} \in \otimes^2 A^*$ is a nondegenerate symmetric invariant bilinear form on $(A, [\ ,\ ]_A, \cdot_A)$.

Proposition 2.16. *Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra. Then*

$$\big(A \ltimes_{-\mathrm{ad}^*, L^*} A^*, [\ ,\ ], \cdot, \mathfrak{B}_d\big)$$

is a quadratic Poisson algebra, where $\mathfrak{B}_d$ is defined by

$$\mathfrak{B}_d\left(a + x^*, b + y^*\right) = \langle a, y^* \rangle + \langle b, x^* \rangle, \quad \forall a, b \in A,\ x^*, y^* \in A^*. \tag{2-16}$$

Proof. This follows from a straightforward computation. □

Let A be a vector space and $\mathfrak{B}$ be a nondegenerate bilinear form. Denote by $I_{\mathfrak{B}} : A^* \to A$ the induced linear isomorphism defined by

$$\langle I_{\mathfrak{B}}^{-1}(a), b \rangle := \mathfrak{B}(a, b), \quad \forall a, b \in A. \tag{2-17}$$

Denote by $r_{\mathfrak{B}} \in A \otimes A$ the 2-tensor form of $I_{\mathfrak{B}}$, that is,

$$\langle r_{\mathfrak{B}}, x^* \otimes y^* \rangle := \langle I_{\mathfrak{B}}(x^*), y^* \rangle, \quad \forall x^*, y^* \in A^*. \tag{2-18}$$

Proposition 2.17. *Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra and $\mathfrak{B}$ a nondegenerate bilinear form. Let $I_{\mathfrak{B}} : A^* \to A$ be the linear isomorphism induced by $\mathfrak{B}$ and $r_{\mathfrak{B}} \in A \otimes A$ be the 2-tensor form of $I_{\mathfrak{B}}$ given by* (2-18). *Then $(A, [\ ,\ ]_A, \cdot_A, \mathfrak{B})$ is a quadratic Poisson algebra if and only if $r_{\mathfrak{B}} \in A \otimes A$ is symmetric and* (ad, L)-*invariant.*

Proof. Suppose that $(A, [\ ,\ ]_A, \cdot_A, \mathfrak{B})$ is a quadratic Poisson algebra. For all $a, b, c \in A$, there exist $x^*, y^*, z^* \in A^*$ such that $I_{\mathfrak{B}}(x^*) = a$, $I_{\mathfrak{B}}(y^*) = b$ and $I_{\mathfrak{B}}(z^*) = c$. Then we have

$$\begin{aligned} \langle r_{\mathfrak{B}} - \tau(r_{\mathfrak{B}}), x^* \otimes y^* \rangle &= \langle I_{\mathfrak{B}}(x^*), I_{\mathfrak{B}}^{-1}(b) \rangle - \langle I_{\mathfrak{B}}(y^*), I_{\mathfrak{B}}^{-1}(a) \rangle \\ &= \mathfrak{B}(b, a) - \mathfrak{B}(a, b) = 0. \end{aligned}$$

Thus, $r_{\mathfrak{B}}$ is symmetric. Moreover, we have

$$\begin{aligned}\langle (\mathrm{ad}(c)\otimes \mathrm{id}+\mathrm{id}\otimes \mathrm{ad}(c))(r_{\mathfrak{B}}), x^*\otimes y^*\rangle &= \langle r_{\mathfrak{B}}, \mathrm{ad}^*(c)(x^*)\otimes y^*\rangle + \langle r_{\mathfrak{B}}, x^*\otimes \mathrm{ad}^*(c)(y^*)\rangle \\ &= \langle I_{\mathfrak{B}}(y^*), \mathrm{ad}^*(c)(x^*)\rangle + \langle I_{\mathfrak{B}}(x^*), \mathrm{ad}^*(c)(y^*)\rangle \\ &= \mathfrak{B}([c,b]_A, a)+\mathfrak{B}([c,a]_A, b)=0,\end{aligned}$$

$$\begin{aligned}\langle (L(c)\otimes \mathrm{id}-\mathrm{id}\otimes L(c))(r_{\mathfrak{B}}), x^*\otimes y^*\rangle &= \langle r_{\mathfrak{B}}, L^*(c)(x^*)\otimes y^*\rangle - \langle r_{\mathfrak{B}}, x^*\otimes L^*(c)(y^*)\rangle \\ &= \langle I_{\mathfrak{B}}(y^*), L^*(c)(x^*)\rangle - \langle I_{\mathfrak{B}}(x^*), L^*(c)(y^*)\rangle \\ &= \mathfrak{B}(c\cdot_A b, a)-\mathfrak{B}(c\cdot_A a, b)=0,\end{aligned}$$

which shows that $r_{\mathfrak{B}}\in A\otimes A$ is (ad, L)-invariant. The converse statement can be proved by reversing the argument. □

The next theorem demonstrates that every quadratic Poisson algebra naturally induces an isomorphism between the adjoint and coadjoint representations of the corresponding Poisson algebra.

Definition 2.18. Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra, and (V_1, ρ_1, μ_1) and (V_2, ρ_2, μ_2) be two representations of $(A, [\ ,\ ]_A, \cdot_A)$. A *homomorphism of representations* from (V_1, ρ_1, μ_1) to (V_2, ρ_2, μ_2) is a linear map $\varphi : V_1 \to V_2$ such that

$$\varphi\circ\rho_1(a)=\rho_2(a)\circ\varphi,\quad \varphi\circ\mu_1(a)=\mu_2(a)\circ\varphi,\quad \forall a\in A.$$

When $\varphi : V_1\to V_2$ is a vector space isomorphism satisfying this same condition, φ is called an *isomorphism of representations* from (V_1, ρ_1, μ_1) to (V_2, ρ_2, μ_2).

Theorem 2.19. *Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra. If there is a bilinear form $\mathfrak{B}$ such that $(A, [\ ,\ ]_A, \cdot_A, \mathfrak{B})$ is a quadratic Poisson algebra, then the linear map $I_{\mathfrak{B}}^{-1} : A\to A^*$ defined by $\langle I_{\mathfrak{B}}^{-1}(a), b\rangle = \mathfrak{B}(a,b)$ is an isomorphism from the adjoint representation (A, ad, L) to the coadjoint representation $(A^*, -\mathrm{ad}^*, L^*)$.*

Conversely, if $I^{-1} : A\to A^*$ is an isomorphism from the adjoint representation (A, ad, L) to the coadjoint representation $(A^*, -\mathrm{ad}^*, L^*)$, then the bilinear form $\mathfrak{B}$ defined by $\mathfrak{B}(a,b)=\langle I^{-1}(a), b\rangle$ is nondegenerate invariant on $(A, [\ ,\ ]_A, \cdot_A)$.

Proof. Suppose that $(A, [\ ,\ ]_A, \cdot_A, \mathfrak{B})$ is a quadratic Poisson algebra. For all $a, b, c\in A$, we have

$$\langle I_{\mathfrak{B}}^{-1}\mathrm{ad}(a)b, c\rangle = -\mathfrak{B}([b,a]_A, c) = -\mathfrak{B}(b, [a,c]_A) = \langle -\mathrm{ad}^*(a)I_{\mathfrak{B}}^{-1}(b), c\rangle,$$

$$\langle I_{\mathfrak{B}}^{-1}L(a)b, c\rangle = \mathfrak{B}(a\cdot_A b, c) = \mathfrak{B}(b\cdot_A a, c) = \mathfrak{B}(b, a\cdot_A c) = \langle L^*(a)I_{\mathfrak{B}}^{-1}(b), c\rangle,$$

that is,

$$I_{\mathfrak{B}}^{-1}\mathrm{ad}(a) = -\mathrm{ad}^*(a)I_{\mathfrak{B}}^{-1},\quad I_{\mathfrak{B}}^{-1}L(a)=L^*(a)I_{\mathfrak{B}}^{-1}.$$

Hence, $I_{\mathfrak{B}}^{-1} : A\to A^*$ is an isomorphism from the adjoint representation (A, ad, L)

to the coadjoint representation $(A^*, -\mathrm{ad}^*, L^*)$. The converse can be proved by a similar argument. □

We conclude this section with a proposition that will be valuable in Section 4, in obtaining Rota–Baxter characterizations of factorizable Poisson bialgebras.

Proposition 2.20. *Let $(A, [\ ,\ ]_A, \cdot_A, \mathfrak{B})$ be a quadratic Poisson algebra and $I_{\mathfrak{B}} : A^* \to A$ be the induced linear isomorphism by $\mathfrak{B}$. Suppose that $r \in A \otimes A$. Then r is a solution of the Poisson Yang–Baxter equation in $(A, [\ ,\ ]_A, \cdot_A)$ if and only if the linear map $P := r_+ \circ I_{\mathfrak{B}}^{-1} : A \to A$ satisfies the following equations, for all $a, b \in A$:*

$$[P(a), P(b)]_A = P\big([P(a), b]_A + [a, P(b)]_A - [a, (I_r \circ I_{\mathfrak{B}}^{-1})(b)]_A\big), \tag{2-19}$$

$$P(a) \cdot_A P(b) = P\big(P(a) \cdot_A b + a \cdot_A P(b) - a \cdot_A (I_r \circ I_{\mathfrak{B}}^{-1})(b)\big), \tag{2-20}$$

Proof. By Theorem 2.19, we have

$$I_{\mathfrak{B}}^{-1}\mathrm{ad}(a) = -\mathrm{ad}^*(a) I_{\mathfrak{B}}^{-1}, \quad I_{\mathfrak{B}}^{-1} L(a) = L^*(a) I_{\mathfrak{B}}^{-1}, \quad \forall a \in A.$$

For all $a, b \in A$, there exist $x^*, y^* \in A^*$ such that $I_{\mathfrak{B}}(x^*) = a$ and $I_{\mathfrak{B}}(y^*) = b$. Then we have

$$\begin{aligned}
[P(a), P(b)]_A &= [(r_+ \circ I_{\mathfrak{B}}^{-1})(a), (r_+ \circ I_{\mathfrak{B}}^{-1})(b)]_A = [r_+(x^*), r_+(y^*)]_A,\\
P([P(a), b]_A) &= (r_+ \circ I_{\mathfrak{B}}^{-1})([(r_+ \circ I_{\mathfrak{B}}^{-1})(a), b]_A)\\
&= (r_+ \circ I_{\mathfrak{B}}^{-1})(\mathrm{ad}(r_+(x^*)) I_{\mathfrak{B}}(y^*)) = -r_+(\mathrm{ad}^*(r_+(x^*)) y^*),\\
P([a, P(b)]_A) &= (r_+ \circ I_{\mathfrak{B}}^{-1})([a, (r_+ \circ I_{\mathfrak{B}}^{-1})(b)]_A)\\
&= -(r_+ \circ I_{\mathfrak{B}}^{-1})(\mathrm{ad}(r_+(y^*)) I_{\mathfrak{B}}(x^*)) = r_+(\mathrm{ad}^*(r_+(y^*)) x^*),\\
-P([a, (I_r \circ I_{\mathfrak{B}}^{-1})(b)]_A) &= -(r_+ \circ I_{\mathfrak{B}}^{-1})([a, (I_r \circ I_{\mathfrak{B}}^{-1})(b)]_A)\\
&= (r_+ \circ I_{\mathfrak{B}}^{-1})(\mathrm{ad}(I_r(y^*)) I_{\mathfrak{B}}(x^*)) = -r_+(\mathrm{ad}^*(I_r(y^*)) x^*),\\
P(a) \cdot_A P(b) &= (r_+ \circ I_{\mathfrak{B}}^{-1})(a) \cdot_A (r_+ \circ I_{\mathfrak{B}}^{-1})(b) = r_+(x^*) \cdot_A r_+(y^*),\\
P(P(a) \cdot_A b) &= (r_+ \circ I_{\mathfrak{B}}^{-1})((r_+ \circ I_{\mathfrak{B}}^{-1})(a) \cdot_A b)\\
&= (r_+ \circ I_{\mathfrak{B}}^{-1})(L(r_+(x^*)) I_{\mathfrak{B}}(y^*)) = r_+(L^*(r_+(x^*)) y^*),\\
P(a \cdot_A P(b)) &= (r_+ \circ I_{\mathfrak{B}}^{-1})(a \cdot_A (r_+ \circ I_{\mathfrak{B}}^{-1})(b))\\
&= (r_+ \circ I_{\mathfrak{B}}^{-1})(L(r_+(y^*)) I_{\mathfrak{B}}(x^*)) = r_+(L^*(r_+(y^*)) x^*),\\
-P(a \cdot_A (I_r \circ I_{\mathfrak{B}}^{-1})(b)) &= -(r_+ \circ I_{\mathfrak{B}}^{-1})(a \cdot_A (I_r \circ I_{\mathfrak{B}}^{-1})(b))\\
&= -(r_+ \circ I_{\mathfrak{B}}^{-1})(L(I_r(y^*)) I_{\mathfrak{B}}(x^*)) = -r_+(L^*(I_r(y^*)) x^*).
\end{aligned}$$

By (2-11) and Lemma 2.10, the conclusion follows. □

3. Factorizable Poisson bialgebras

A factorizable Poisson bialgebra is a special quasitriangular Poisson bialgebra such that the map $I_r : A^* \to A$ is a linear isomorphism of vector spaces. We will show that the Drinfeld classical double of a Poisson bialgebra is naturally a factorizable Poisson bialgebra.

Definition 3.1. A quasitriangular Poisson bialgebra $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ is called *factorizable* if the symmetric part S of r is nondegenerate, which means that the linear map $I_r : A^* \to A$ defined by (2-11) is a linear isomorphism of vector spaces.

Proposition 3.2. *Let* $(A, [\ ,\ ]_A, \cdot_A)$ *be a Poisson algebra and* $r \in A \otimes A$. *Then* $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ *is a factorizable Poisson bialgebra if and only if the same is true of* $(A, [\ ,\ ]_A, \cdot_A, \delta_{\tau(r)}, \Delta_{\tau(r)})$.

Proof. This follows from Proposition 2.13. □

Consider the map

$$A^* \xrightarrow{r_+ \oplus r_-} A \oplus A \xrightarrow{(a,b) \mapsto a-b} A.$$

The next result justifies the term "factorizable Poisson bialgebra".

Proposition 3.3. *Let* $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ *be a factorizable Poisson bialgebra. Then* $\operatorname{Im}(r_+ \oplus r_-)$ *is a Poisson subalgebra of the direct sum Poisson algebra* $A \oplus A$, *which is isomorphic to the Poisson algebra* $(A^*, [\ ,\ ]_r, \cdot_r)$, *where* $[\ ,\ ]_r$ *and* $\cdot_r$ *are defined by* (2-3) *and* (2-4). *Any* $a \in A$ *has a unique decomposition* $a = a_+ - a_-$ *with* $(a_+, a_-) \in \operatorname{Im}(r_+ \oplus r_-)$.

Proof. By Theorem 2.15, both r_+ and r_- are Poisson algebra homomorphisms. Therefore, $\operatorname{Im}(r_+ \oplus r_-)$ is a Poisson subalgebra of the Poisson algebra $A \oplus A$. Since $I_r = r_+ - r_-$ is a linear isomorphism of vector spaces, it follows that the Poisson algebra $\operatorname{Im}(r_+ \oplus r_-)$ is isomorphic to the Poisson algebra $(A^*, [\ ,\ ]_r, \cdot_r)$. Moreover, we have

$$r_+ I_r^{-1}(a) - r_- I_r^{-1}(a) = (r_+ - r_-) I_r^{-1}(a) = a,$$

which shows that $a = a_+ - a_-$ with $a_+ = r_+ I_r^{-1}(a)$ and $a_- = r_- I_r^{-1}(a)$. Uniqueness also follows from the fact that $I_r : A^* \to A$ is a linear isomorphism of vector spaces. □

Let $\big((A, [\ ,\ ]_A, \cdot_A), (A^*, [\ ,\ ]_{A^*}, \cdot_{A^*})\big)$ be an arbitrary Poisson bialgebra. We endow $\mathfrak{A} = A \oplus A^*$ with a bracket

$$\begin{aligned}&[(a, x^*), (b, y^*)]_{\mathfrak{A}} \\ &\quad = \big([a, b]_A - \operatorname{ad}^*_{[,]_{A^*}}(x^*)b + \operatorname{ad}^*_{[,]_{A^*}}(y^*)a, [x^*, y^*]_{A^*} - \operatorname{ad}^*_{[,]_A}(a)y^* + \operatorname{ad}^*_{[,]_A}(b)x^*\big)\end{aligned}$$

and a multiplication

$$(a, x^*) \cdot_{\mathfrak{A}} (b, y^*) \\ = \big(a \cdot_A b + L^*_{\cdot_{A^*}}(x^*)b + L^*_{\cdot_{A^*}}(y^*)a,\ x^* \cdot_{A^*} y^* + L^*_{\cdot_A}(a)y^* + L^*_{\cdot_A}(b)x^*\big),$$

where $a, b \in A$, $x^*, y^* \in A^*$. Then $(\mathfrak{A}, [\ ,\ ]_{\mathfrak{A}}, \cdot_{\mathfrak{A}})$ is a Poisson algebra, called the *Drinfeld classical double* of the Poisson bialgebra.

Let $\{e_1, e_2, \dots, e_n\}$ be a basis of A and $\{e_1^*, e_2^*, \dots, e_n^*\}$ be the dual basis of A^*. Then

$$r = \sum_i e_i \otimes e_i^* \in A \otimes A^* \subset \mathfrak{A} \otimes \mathfrak{A}$$

induces a coboundary Poisson bialgebra structure $(\mathfrak{A}, [\ ,\ ]_{\mathfrak{A}}, \cdot_{\mathfrak{A}}, \delta_r, \Delta_r)$ (see [21, Theorem 3] for details).

Theorem 3.4. *The Poisson bialgebra $(\mathfrak{A}, [\ ,\ ]_{\mathfrak{A}}, \cdot_{\mathfrak{A}}, \delta_r, \Delta_r)$ is a factorizable quasitriangular Poisson bialgebra.*

Proof. First, we need to verify that the symmetric part $S = \frac{1}{2}(e_i \otimes e_i^* + e_i^* \otimes e_i)$ of r is $(\mathrm{ad}_{[,]_{\mathfrak{A}}}, L_{\cdot_{\mathfrak{A}}})$-invariant. For all $(x^*, a) \in \mathfrak{A}^*$, we have $S_+(x^*, a) = \frac{1}{2}(a, x^*) \in \mathfrak{A}$. A straightforward computation gives

$$[(a, x^*), S_+(y^*, b)]_{\mathfrak{A}} \\ = \tfrac{1}{2}\big([a, b]_A - \mathrm{ad}^*_{[,]_{A^*}}(x^*)b + \mathrm{ad}^*_{[,]_{A^*}}(y^*)a,\ [x^*, y^*]_{A^*} - \mathrm{ad}^*_{[,]_A}(a)y^* + \mathrm{ad}^*_{[,]_A}(b)x^*\big),$$

$$\mathrm{ad}^*_{[,]_{\mathfrak{A}}}(a, x^*)(y^*, b) \\ = -\big([x^*, y^*]_{A^*} - \mathrm{ad}^*_{[,]_A}(a)y^* + \mathrm{ad}^*_{[,]_A}(b)x^*,\ [a, b]_A - \mathrm{ad}^*_{[,]_{A^*}}(x^*)b + \mathrm{ad}^*_{[,]_{A^*}}(y^*)a\big),$$

$$(a, x^*) \cdot_{\mathfrak{A}} S_+(y^*, b) \\ = \tfrac{1}{2}\big(a \cdot_A b + L^*_{\cdot_{A^*}}(x^*)b + L^*_{\cdot_{A^*}}(y^*)a,\ x^* \cdot_{A^*} y^* + L^*_{\cdot_A}(a)y^* + L^*_{\cdot_A}(b)x^*\big),$$

$$L^*_{\cdot_{\mathfrak{A}}}(a, x^*)(y^*, b) \\ = \big(x^* \cdot_{A^*} y^* + L^*_{\cdot_A}(a)y^* + L^*_{\cdot_A}(b)x^*,\ a \cdot_A b + L^*_{\cdot_{A^*}}(x^*)b + L^*_{\cdot_{A^*}}(y^*)a\big).$$

Thus, we have

$$S_+(\mathrm{ad}^*_{[,]_{\mathfrak{A}}}(a, x^*)(y^*, b)) + [(a, x^*), S_+(y^*, b)]_{\mathfrak{A}} = 0,$$
$$S_+(L^*_{\cdot_{\mathfrak{A}}}(a, x^*)(y^*, b)) - (a, x^*) \cdot_{\mathfrak{A}} S_+(y^*, b) = 0.$$

Therefore, by Lemma 2.14, the symmetric part S of r is $(\mathrm{ad}_{[,]_{\mathfrak{A}}}, L_{\cdot_{\mathfrak{A}}})$-invariant. On the other hand, by [21, Theorem 3], r is a solution of the Poisson Yang–Baxter equation in $(\mathfrak{A}, [\ ,\ ]_{\mathfrak{A}}, \cdot_{\mathfrak{A}})$. Hence, the Poisson bialgebra $(\mathfrak{A}, [\ ,\ ]_{\mathfrak{A}}, \cdot_{\mathfrak{A}}, \delta_r, \Delta_r)$ is a quasitriangular Poisson bialgebra.

That $I_r : \mathfrak{A}^* \to \mathfrak{A}$ is a linear isomorphism of vector spaces follows from the equality $I_r(x^*, a) = 2S_+(x^*, a) = (a, x^*)$. Therefore, the Poisson bialgebra $(\mathfrak{A}, [\ ,\ ]_{\mathfrak{A}}, \cdot_{\mathfrak{A}}, \delta_r, \Delta_r)$ is factorizable. □

4. Quadratic Rota–Baxter Poisson algebras

A linear map $P : A \to A$ is called a *Rota–Baxter operator of weight* λ on a Poisson algebra $(A, [\ ,\ ]_A, \cdot_A)$ if, for all $a, b \in A$, we have

$$[P(a), P(b)]_A = P([P(a), b]_A + [a, P(b)]_A + \lambda[a, b]_A),$$
$$P(a) \cdot_A P(b) = P(P(a) \cdot_A b + a \cdot_A P(b) + \lambda a \cdot_A b).$$

A *Rota–Baxter Poisson algebra* $(A, [\ ,\ ]_A, \cdot_A, P)$ *of weight* λ is a Poisson algebra $(A, [\ ,\ ]_A, \cdot_A)$ equipped with a Rota–Baxter operator P of weight λ.

Example 4.1. Let $(A, [\ ,\ ]_A, \cdot_A)$ be a three-dimensional Poisson algebra with a basis $\{e_1, e_2, e_3\}$ whose nonzero products are given by

$$[e_1, e_2]_A = e_3, \quad e_1 \cdot_A e_2 = e_3. \tag{4-1}$$

Let $P : A \to A$ be the linear map given by

$$P(e_1) = e_1, \quad P(e_2) = 2e_2, \quad P(e_3) = \tfrac{1}{2}e_3.$$

Then $(A, [\ ,\ ]_A, \cdot_A, P)$ is a Rota–Baxter Poisson algebra of weight 1.

Lemma 4.2. *Let* $(A, [\ ,\ ]_A, \cdot_A, P)$ *be a Rota–Baxter Poisson algebra of weight* λ. *Define a bracket* $[\ ,\ ]_P : A \otimes A \to A$ *by*

$$[a, b]_P = [P(a), b]_A + [a, P(b)]_A + \lambda[a, b]_A, \tag{4-2}$$

and a product $\cdot_P : A \otimes A \to A$ *by*

$$a \cdot_P b = P(a) \cdot_A b + a \cdot_A P(b) + \lambda a \cdot_A b. \tag{4-3}$$

Then $(A, [\ ,\ ]_P, \cdot_P)$ *is a Poisson algebra, called the* ***descendent Poisson algebra*** *of* $(A, [\ ,\ ]_A, \cdot_A, P)$, *and* P *is a Poisson algebra homomorphism from* $(A, [\ ,\ ]_P, \cdot_P)$ *to* $(A, [\ ,\ ]_A, \cdot_A)$.

Proof. The proof is straightforward. □

Equipping quadratic Poisson algebras with Rota–Baxter operators satisfying compatibility conditions, we now introduce the notion of quadratic Rota–Baxter Poisson algebras. We then show that a quadratic Rota–Baxter Poisson algebra of zero weight induces a triangular Poisson bialgebra, and that there is a one-to-one correspondence between factorizable Poisson bialgebras and quadratic Rota–Baxter Poisson algebras of nonzero weight.

Definition 4.3. The triple $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, P)$ is called a *quadratic Rota–Baxter Poisson algebra of weight* λ if $(A, [\ ,\ ]_A, \cdot_A, \mathfrak{B})$ is a quadratic Poisson algebra and $(A, [\ ,\ ]_A, \cdot_A, P)$ is a Rota–Baxter Poisson algebra of weight λ satisfying the compatibility condition

$$\mathfrak{B}(a,\, P(b)) + \mathfrak{B}(P(a), b) + \lambda\mathfrak{B}(a, b) = 0, \quad \forall a, b \in A. \tag{4-4}$$

Proposition 4.4. *Let $(A, [\ ,\ ]_A, \cdot_A, \mathfrak{B})$ be a quadratic Poisson algebra and let $P : A \to A$ be a linear map. Then $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, P)$ is a quadratic Rota–Baxter Poisson algebra of weight λ if and only if $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, \tilde{P})$ is a quadratic Rota–Baxter Poisson algebra of weight λ, where $\tilde{P} := -\lambda\,\mathrm{id} - P$.*

Proof. It is easy to show that P is a Rota–Baxter operator of weight λ if and only if $\tilde{P}$ is a Rota–Baxter operator of weight λ. Moreover, we have, for all $a, b \in A$,

$$\mathfrak{B}(\tilde{P}(a), b) + \mathfrak{B}(a,\, \tilde{P}(b)) + \lambda\mathfrak{B}(a, b) = -\mathfrak{B}(P(a), b) - \mathfrak{B}(a,\, P(b)) - \lambda\mathfrak{B}(a, b).$$

Thus, P satisfies (4-4) if and only if $\tilde{P}$ satisfies (4-4). □

Proposition 4.5. *Let $(A, [\ ,\ ]_A, \cdot_A, P)$ be a Rota–Baxter Poisson algebra of weight λ. Then $\big((A \ltimes_{-\mathrm{ad}^*, L^*} A^*, [\ ,\ ], \cdot\,), \mathfrak{B}_d, P + \tilde{P}^*\big)$ is a quadratic Rota–Baxter Poisson algebra of weight λ, where $\mathfrak{B}_d$ is defined by (2-16) and $\tilde{P} := -\lambda\,\mathrm{id} - P$.*

Proof. The proof is straightforward. □

Example 4.6. Building on Example 4.1, let $(A, [\ ,\ ]_A, \cdot_A, P)$ be a Rota–Baxter Poisson algebra of weight 1. Let $\{e_1^*, e_2^*, e_3^*\}$ be the dual basis of $\{e_1, e_2, e_3\}$. Then $(A \ltimes_{-\mathrm{ad}^*, L^*} A^*, [\ ,\ ], \cdot\,)$ is a Poisson algebra whose nonzero products are given by

$$\begin{gathered} [e_1, e_2] = e_3, \quad [e_3^*, e_1] = e_2^*, \quad [e_3^*, e_2] = -e_1^*, \\ e_1 \cdot e_2 = e_3, \qquad e_3^* \cdot e_1 = e_2^*, \qquad e_3^* \cdot e_2 = e_1^*. \end{gathered} \tag{4-5}$$

By Proposition 4.5, we obtain a quadratic Rota–Baxter Poisson algebra

$$\big((A \ltimes_{-\mathrm{ad}^*, L^*} A^*, [\ ,\ ], \cdot\,), \mathfrak{B}_d, P + \tilde{P}^*\big)$$

of weight 1, where $\mathfrak{B}_d$ is defined by

$$\mathfrak{B}_d(e_i, e_j) = \mathfrak{B}_d(e_i^*, e_j^*) = 0, \quad \mathfrak{B}_d(e_i^*, e_j) = \mathfrak{B}_d(e_i, e_j^*) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for $i, j = 1, 2, 3$, and $\tilde{P}^*$ is given by

$$\tilde{P}^*(e_1^*) = -2e_1^*, \quad \tilde{P}^*(e_2^*) = -3e_2^*, \quad \tilde{P}^*(e_3^*) = -\tfrac{3}{2}e_3.$$

Lemma 4.7. *Let A be a vector space and $\mathfrak{B}$ be a nondegenerate symmetric bilinear form. Let $I_{\mathfrak{B}} : A^* \to A$ be the induced linear isomorphism by $\mathfrak{B}$ and $r_{\mathfrak{B}} \in A \otimes A$ be the 2-tensor form of $I_{\mathfrak{B}}$ given by (2-18). Suppose that $r \in A \otimes A$. Then $P_r := r_+ \circ I_{\mathfrak{B}}^{-1}$ satisfies (4-4) if and only if $r + \tau(r) = -\lambda r_{\mathfrak{B}}$ where $\lambda \in \mathbb{K}$.*

Proof. For all $a, b \in A$, there exist x^*, $y^* \in A^*$ such that $I_{\mathfrak{B}}(x^*) = a$, $I_{\mathfrak{B}}(y^*) = b$. Then

$$\begin{aligned}\mathfrak{B}\,(P_r(a), b) = \mathfrak{B}\,(b, P_r(a)) = \langle I_{\mathfrak{B}}^{-1}(b), (r_+ \circ I_{\mathfrak{B}}^{-1})(a)\rangle &= \langle r, x^* \otimes y^*\rangle\,,\\ \mathfrak{B}(a, P_r(b)) = \langle I_{\mathfrak{B}}^{-1}(a), (r_+ \circ I_{\mathfrak{B}}^{-1})(b)\rangle &= \langle r, y^* \otimes x^*\rangle = \langle \tau(r), x^* \otimes y^*\rangle\,,\\ \lambda\mathfrak{B}(a, b) = \lambda\mathfrak{B}(b, a) = \lambda\langle I_{\mathfrak{B}}^{-1}(b), (I_{\mathfrak{B}} \circ I_{\mathfrak{B}}^{-1})(a)\rangle &= \lambda\langle r_{\mathfrak{B}}, x^* \otimes y^*\rangle\,.\end{aligned}$$

Hence, $r + \tau(r) = -\lambda r_{\mathfrak{B}}$ if and only if P_r satisfies (4-4). □

As a direct consequence, a quadratic Rota–Baxter Poisson algebra of zero weight gives rise to a triangular Poisson bialgebra in the following sense.

Proposition 4.8. *Let* $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, P)$ *be a quadratic Rota–Baxter Poisson algebra of weight* 0 *and* $I_{\mathfrak{B}} : A^* \to A$ *be the induced linear isomorphism by* $\mathfrak{B}$. *Then* $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ *is a triangular Poisson bialgebra where* $r \in A \otimes A$ *is the* 2*-tensor form of* $P \circ I_{\mathfrak{B}}$ *given by*

$$\langle r, x^* \otimes y^*\rangle := \langle (P \circ I_{\mathfrak{B}})(x^*), y^*\rangle\,, \quad \forall x^*, y^* \in A^*. \tag{4-6}$$

Proof. This follows from Proposition 2.20 and Lemma 4.7 with $r + \tau(r) = 0$. □

The following theorem shows that a factorizable Poisson bialgebra naturally gives rise to a quadratic Rota–Baxter Poisson algebra of nonzero weight.

Theorem 4.9. *Let* $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ *be a factorizable Poisson bialgebra with* $I_r = r_+ - r_-$. *Then* $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, P)$ *is a quadratic Rota–Baxter Poisson algebra of weight* λ, *where* $P : A \to A$ *is given by*

$$P = -\lambda r_+ \circ I_r^{-1}, \quad \lambda \neq 0,$$

and the bilinear form $\mathfrak{B} \in \otimes^2 A^*$ *is defined by*

$$\mathfrak{B}(a, b) = -\lambda\langle I_r^{-1}(a), b\rangle\,, \quad \forall a, b \in A.$$

Proof. Clearly, $\mathfrak{B}$ is a nondegenerate symmetric bilinear form and $I_{\mathfrak{B}}^{-1} = -\lambda I_r^{-1}$. Immediately, we have $I_r = -\lambda I_{\mathfrak{B}}$ and thus $r + \tau(r) = -\lambda r_{\mathfrak{B}}$. Noting that the symmetric part of r is (ad, L)-invariant, we have $r_{\mathfrak{B}}$ is (ad, L)-invariant and thus by Proposition 2.17, $(A, [\ ,\ ]_A, \cdot_A, \mathfrak{B})$ is a quadratic Poisson algebra. Moreover, it is clear that P is a Rota–Baxter operator of weight λ on the Poisson algebra $(A, [\ ,\ ]_A, \cdot_A)$ by Proposition 2.20 and P satisfies (4-4) by Lemma 4.7. Thus, $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, P)$ is a quadratic Rota–Baxter Poisson algebra of weight λ. □

Corollary 4.10. *Let* $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ *be a factorizable Poisson bialgebra with* $I_r = r_+ - r_-$, *and* $P = -\lambda r_+ \circ I_r^{-1}$ *be the induced Rota–Baxter operator of weight* $\lambda \neq 0$. *Let* $(A, [\ ,\ ]_P, \cdot_P)$ *be the descendent Poisson algebra of the Rota–Baxter algebra* $(A, [\ ,\ ]_A, \cdot_A, P)$. *Then* $((A, [\ ,\ ]_P, \cdot_P), (A^*, [\ ,\ ]_{I_r}, \cdot_{I_r}))$ *is a Poisson* *bialgebra, where, for all* $x^*, y^* \in A^*$,

$$[x^*, y^*]_{I_r} := -\lambda I_r^{-1}\big([\lambda^{-1} I_r(x^*), \lambda^{-1} I_r(y^*)]_A\big),$$
$$x^* \cdot_{I_r} y^* := -\lambda I_r^{-1}\big((\lambda^{-1} I_r(x^*)) \cdot_A (\lambda^{-1} I_r(y^*))\big).$$

Moreover, $-\lambda^{-1} I_r : A^* \to A$ *is a Poisson bialgebra isomorphism*

$$\big((A^*, [\ ,\ ]_r, \cdot_r), (A, [\ ,\ ]_A, \cdot_A)\big) \to \big((A, [\ ,\ ]_P, \cdot_P), (A^*, [\ ,\ ]_{I_r}, \cdot_{I_r})\big),$$

where $[\ ,\ ]_r,\ \cdot_r : A^* \otimes A^* \to A^*$ *are defined in* (2-3) *and* (2-4).

Proof. For all $x^*, y^* \in A^*$, setting $a = I_r(x^*)$ and $b = I_r(y^*)$, we have

$$\begin{aligned}
&-\lambda^{-1} I_r([x^*, y^*]_r) \\
&\quad \overset{(2\text{-}3)}{=} -\lambda^{-1} I_r\big(-\mathrm{ad}^*(r_+(x^*))y^* + \mathrm{ad}^*(r_-(y^*))x^*\big) \\
&\quad \overset{(2\text{-}12)}{=} -\lambda^{-1}\big([r_+(x^*), I_r(y^*)]_A - [r_-(y^*), I_r(x^*)]_A\big) \\
&\quad = \lambda^{-2}\big([P I_r(x^*), I_r(y^*)]_A + [I_r(x^*), P I_r(y^*)]_A + \lambda[I_r(x^*), I_r(y^*)]_A\big) \\
&\quad \overset{(4\text{-}2)}{=} [-\lambda^{-1} I_r(x^*), -\lambda^{-1} I_r(y^*)]_P.
\end{aligned}$$

Similarly, we have

$$-\lambda^{-1} I_r(x^* \cdot_r y^*) = (-\lambda^{-1} I_r(x^*)) \cdot_P (-\lambda^{-1} I_r(y^*)).$$

Hence, $-\lambda^{-1} I_r : (A^*, [\ ,\ ]_r, \cdot_r) \to (A, [\ ,\ ]_P, \cdot_P)$ is a Poisson algebra homomorphism.

Noting that $\lambda^{-1} I_r^* = \lambda^{-1} I_r$, we have

$$-\lambda^{-1} I_r^*([x^*, y^*]_{I_r}) = \big[-\lambda^{-1} I_r(x^*), -\lambda^{-1} I_r(y^*)\big]_A = \big[-\lambda^{-1} I_r^*(x^*), -\lambda^{-1} I_r^*(y^*)\big]_A,$$
$$-\lambda^{-1} I_r^*(x^* \cdot_{I_r} y^*) = (-\lambda^{-1} I_r(x^*)) \cdot_A (-\lambda^{-1} I_r(y^*)) = (-\lambda^{-1} I_r^*(x^*)) \cdot_A (-\lambda^{-1} I_r^*(y^*)),$$

which means that $-\lambda^{-1} I_r^* : (A^*, [\ ,\ ]_{I_r}, \cdot_{I_r}) \to (A, [\ ,\ ]_A, \cdot_A)$ is also a Poisson algebra homomorphism. Since $\big((A^*, [\ ,\ ]_r, \cdot_r), (A, [\ ,\ ]_A, \cdot_A)\big)$ is a Poisson bialgebra, so is $((A, [\ ,\ ]_P, \cdot_P), (A^*, [\ ,\ ]_{I_r}, \cdot_{I_r}))$, by Proposition 2.6. Clearly, $-\lambda^{-1} I_r$ is an isomorphism of Poisson bialgebras. □

As a counterpart to Theorem 4.9, the following theorem shows that a quadratic Rota–Baxter Poisson algebra of nonzero weight induces a factorizable Poisson bialgebra, thereby refining the one-to-one correspondence between factorizable Poisson bialgebras and quadratic Rota–Baxter Poisson algebras of nonzero weight.

Theorem 4.11. *Let* $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, P)$ *be a quadratic Rota–Baxter Poisson algebra of weight* $\lambda \neq 0$, *and* $I_{\mathfrak{B}} : A^* \to A$ *be the induced linear isomorphism by* $\mathfrak{B}$. *Let* $r \in A \otimes A$ *be the* 2*-tensor form of* $P \circ I_{\mathfrak{B}}$ *given by* (4-6). *Then* r *is a solution of the Poisson Yang–Baxter equation in* $(A, [\ ,\ ]_A, \cdot_A)$ *and gives rise to a factorizable Poisson bialgebra* $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$.

Proof. Let $r_{\mathfrak{B}} \in A \otimes A$ be the 2-tensor form of $I_{\mathfrak{B}}$ in (2-18). By Proposition 2.17 and Lemma 4.7, $r_{\mathfrak{B}}$ is (ad, L)-invariant and $r + \tau(r) = -\lambda r_{\mathfrak{B}}$, which shows that the symmetric part of r is also (ad, L)-invariant and $I_r = -\lambda I_{\mathfrak{B}}$ is a linear isomorphism. By Proposition 2.20, r satisfies the Poisson Yang–Baxter equation, since P is a Rota–Baxter operator of weight λ on $(A, [\ ,\ ]_A, \cdot_A)$. Thus, $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ is a factorizable Poisson bialgebra. □

Corollary 4.12. *Let $(A, [\ ,\ ]_A, \cdot_A)$ be a Poisson algebra, $\{e_1, e_2, \dots, e_n\}$ be a basis of A and $\{e_1^*, e_2^*, \dots, e_n^*\}$ be the dual basis of A^*. Then $r = \sum_i e_i^* \otimes e_i$ is a solution of the Poisson Yang–Baxter equation in $(A \ltimes_{-\mathrm{ad}^*, L^*} A^*, [\ ,\ ], \cdot)$ and gives rise to a factorizable Poisson bialgebra $(A \ltimes_{-\mathrm{ad}^*, L^*} A^*, [\ ,\ ], \cdot, \delta_r, \Delta_r)$.*

Proof. Clearly, $\mathrm{id} : A \to A$ is a Rota–Baxter operator of weight -1 on $(A, [\ ,\ ]_A, \cdot_A)$. By Proposition 4.5, $\big((A \ltimes_{-\mathrm{ad}^*, L^*} A^*, [\ ,\ ], \cdot), \mathfrak{B}_d, \mathrm{id} + 0^*\big)$ is then a quadratic Rota–Baxter Poisson algebra of weight -1, where $\mathfrak{B}_d$ is defined by (2-16). Now the conclusion follows from Theorem 4.11. □

Example 4.13. Let $(A \ltimes_{-\mathrm{ad}^*, L^*} A^*, [\ ,\ ], \cdot)$ be the Poisson algebra in Example 4.6. By Corollary 4.12, $r = \sum_{i=1}^{3} e_i^* \otimes e_i$ is a solution of the Poisson Yang–Baxter equation in $(A \ltimes_{-\mathrm{ad}^*, L^*} A^*, [\ ,\ ], \cdot)$ and then $(A \ltimes_{-\mathrm{ad}^*, L^*} A^*, [\ ,\ ], \cdot, \delta_r, \Delta_r)$ is a factorizable Poisson bialgebra where δ_r and Δ_r are explicitly defined by

$$\begin{aligned} &\delta_r(e_1) = \delta_r(e_2) = \delta_r(e_3) = 0, \qquad \delta_r(e_1^*) = \delta_r(e_2^*) = 0, \\ &\delta_r(e_3^*) = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*, \end{aligned} \tag{4-7}$$

$$\begin{aligned} &\Delta_r(e_1) = \Delta_r(e_2) = \Delta_r(e_3) = 0, \quad \Delta_r(e_1^*) = \Delta_r(e_2^*) = 0, \\ &\Delta_r(e_3^*) = e_1^* \otimes e_2^* + e_2^* \otimes e_1^*. \end{aligned} \tag{4-8}$$

Proposition 4.14. *Let $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ be a factorizable Poisson bialgebra, which corresponds to a quadratic Rota–Baxter Poisson algebra $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, P)$ of weight $\lambda \neq 0$ via Theorems 4.9 and 4.11. Then the factorizable Poisson bialgebra $(A, [\ ,\ ]_A, \cdot_A, \delta_{\tau(r)}, \Delta_{\tau(r)})$ corresponds to the quadratic Rota–Baxter Poisson algebra $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, \tilde{P})$ of weight $\lambda \neq 0$ where $\tilde{P} := -\lambda \mathrm{id} - P$. In conclusion, we have the following commutative diagram*:

$$\begin{array}{ccc} (A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r) & \xleftrightarrow{\text{Proposition 3.2}} & (A, [\ ,\ ]_A, \cdot_A, \delta_{\tau(r)}, \Delta_{\tau(r)}) \\ \text{Theorem 4.11} \uparrow \;\; \downarrow \text{Theorem 4.9} & & \text{Theorem 4.11} \uparrow \;\; \downarrow \text{Theorem 4.9} \\ ((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, P) & \xleftrightarrow{\text{Proposition 4.4}} & ((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, \tilde{P}) \end{array}$$

Proof. By Theorem 4.9, the factorizable Poisson bialgebra $(A, [\ ,\ ]_A, \cdot_A, \delta_{\tau(r)}, \Delta_{\tau(r)})$ induces a quadratic Rota–Baxter Poisson algebra $(A, [\ ,\ ]_A, \cdot_A, \mathfrak{B}', P')$ of weight

$\lambda \neq 0$, where

$$\mathfrak{B}'(a, b) = -\lambda \langle I_{\tau(r)}^{-1}(a), b\rangle = -\lambda \langle I_r^{-1}(a), b\rangle = \mathfrak{B}(a, b), \quad \forall a, b \in A,$$

and

$$P' = -\lambda(\tau(r))_+ \circ I_{\tau(r)}^{-1} = \lambda r_- \circ I_r^{-1} = \lambda r_+ \circ I_r^{-1} - \lambda I_r \circ I_r^{-1} = -P - \lambda \mathrm{id}_A = \tilde{P}.$$

Thus, $(A, [\ ,\ ]_A, \cdot_A, \delta_{\tau(r)}, \Delta_{\tau(r)})$ induces a quadratic Rota–Baxter Poisson algebra $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, \tilde{P})$ of weight λ by Theorem 4.9. By a similar argument, we show the converse: the quadratic Rota–Baxter Poisson algebra $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, \tilde{P})$ of weight λ induces the factorizable Poisson bialgebra $(A, [\ ,\ ]_A, \cdot_A, \delta_{\tau(r)}, \Delta_{\tau(r)})$ via Theorem 4.11. □

5. Quasitriangular Poisson bialgebras via quasitriangular differential ASI bialgebras

In this section, we generalize the construction of Poisson algebras from commutative algebras with a pair of commuting derivations to the context of quasitriangular bialgebras. We establish the quasitriangular and factorizable theories for differential ASI bialgebras, and then construct quasitriangular and factorizable Poisson bialgebras from quasitriangular and factorizable (commutative and cocommutative) differential ASI bialgebras respectively.

Recall that an *associative coalgebra* (A, Δ) is a vector space A with a linear map $\Delta : A \to A \otimes A$ satisfying the coassociative law

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta.$$

An associative coalgebra (A, Δ) is called *cocommutative* if $\Delta = \tau\Delta$.

Definition 5.1 [4]. An *antisymmetric infinitesimal bialgebra* or simply an *ASI bialgebra* is a triple $(A, \cdot_A, \Delta)$ consisting of a vector space A and linear maps $\cdot_A : A \otimes A \to A$ and $\Delta : A \to A \otimes A$ such that

(1) $(A, \cdot_A)$ is an associative algebra,

(2) (A, Δ) is an associative coalgebra, and

(3) the following equations hold for all $a, b \in A$:

$$\Delta(a \cdot_A b) = (R(b) \otimes \mathrm{id})\Delta(a) + (\mathrm{id} \otimes L(a))\Delta(b),$$
$$(L(a) \otimes \mathrm{id} - \mathrm{id} \otimes R(a))\Delta(b) = \tau\left((\mathrm{id} \otimes R(b) - L(b) \otimes \mathrm{id})\Delta(a)\right).$$

Definition 5.2. Let $(A, \cdot_A)$ be an associative algebra. A linear map $\partial : A \to A$ is called a *derivation* if the Leibniz rule is satisfied:

$$\partial(a \cdot_A b) = \partial(a) \cdot_A b + a \cdot_A \partial(b), \quad \forall a, b \in A.$$

A *differential algebra* is a triple $(A, \cdot_A, \Phi)$, where $(A, \cdot_A)$ is an associative algebra and $\Phi = \{\partial_i : A \to A\}_{i=1}^m$ is a tuple of commuting derivations. A differential algebra $(A, \cdot_A, \Phi)$ is called *commutative* if $(A, \cdot_A)$ is commutative.

Definition 5.3 [10]. Let (A, Δ) be an associative coalgebra. A linear map $\vartheta : A \to A$ is called a *coderivation* on (A, Δ) if

$$\Delta\vartheta = (\vartheta \otimes \mathrm{id} + \mathrm{id} \otimes \vartheta)\Delta. \tag{5-1}$$

A *differential coalgebra* is a triple (A, Δ, Ψ), consisting of an associative coalgebra (A, Δ) and a tuple of commuting coderivations $\Psi = \{\vartheta_k : A \to A\}_{k=1}^m$. A differential coalgebra (A, Δ, Ψ) is called *cocommutative* if (A, Δ) is cocommutative.

Definition 5.4 [20]. A *differential antisymmetric infinitesimal bialgebra* or simply a *differential ASI bialgebra* is a quintuple $(A, \cdot_A, \Delta, \Phi, \Psi)$ with these properties:

(1) $(A, \cdot_A, \Delta)$ is an ASI bialgebra.

(2) $(A, \cdot_A, \Phi = \{\partial_k\}_{k=1}^m)$ is a differential algebra.

(3) $(A, \Delta, \Psi = \{\vartheta_k\}_{k=1}^m)$ is a differential coalgebra.

(4) $(A, \cdot_A, \Phi)$ is Ψ*-admissible*, that is, for all $a, b \in A$ and $k = 1, \ldots, m$ we have

$$\vartheta_k(a) \cdot_A b = a \cdot_A \partial_k(b) + \vartheta_k(a \cdot_A b), \tag{5-2}$$

$$a \cdot_A \vartheta_k(b) = \partial_k(a) \cdot_A b + \vartheta_k(a \cdot_A b). \tag{5-3}$$

(5) (A, Δ^*, Ψ^*) is Φ^**-admissible*, that is, for all $k = 1, \ldots, m$ we have

$$(\partial_k \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \vartheta_k)\Delta + \Delta\partial_k, \tag{5-4}$$

$$(\mathrm{id} \otimes \partial_k)\Delta = (\vartheta_k \otimes \mathrm{id})\Delta + \Delta\partial_k. \tag{5-5}$$

A differential ASI bialgebra $(A, \cdot_A, \Delta, \Phi, \Psi)$ is called *commutative and cocommutative* if $(A, \cdot_A, \Phi)$ is commutative and (A, Δ, Ψ) is cocommutative.

Definition 5.5. Let $(A, \cdot_A, \Phi = \{\partial_k\}_{k=1}^m)$ be a differential algebra. Suppose that $r \in A \otimes A$ and that $\Psi = \{\vartheta_k : A \to A\}_{k=1}^m$ is a set of commuting linear maps. The Ψ*-admissible associative Yang–Baxter equation* or Ψ*-admissible AYBE* in $(A, \cdot_A, \Phi)$ is the set of equations

$$\boldsymbol{A}(r) = 0,$$

$$(\partial_k \otimes \mathrm{id} - \mathrm{id} \otimes \vartheta_k)(r) = 0, \quad \forall k = 1, \ldots, m, \tag{5-6}$$

$$(\vartheta_k \otimes \mathrm{id} - \mathrm{id} \otimes \partial_k)(r) = 0, \quad \forall k = 1, \ldots, m. \tag{5-7}$$

Lemma 5.6. *Let $(A, \cdot_A, \Phi = \{\partial_k\}_{k=1}^m)$ be a differential algebra. Suppose that $r \in A \otimes A$ and $\Psi = \{\vartheta_k : A \to A\}_{k=1}^m$ is a set of commuting linear maps. Then equations* (5-6)–(5-7) *hold if and only if*

$$\partial_i r_+ = r_+\vartheta_i^*, \quad \partial_i r_- = r_-\vartheta_i^*, \quad \forall i = 1, 2, \ldots, m.$$

Proof. For all $x^*, y^* \in A^*$, we have

$$\langle \partial_i r_+(x^*), y^* \rangle = \langle r, x^* \otimes \partial_i^*(y^*) \rangle = \langle (\mathrm{id} \otimes \partial_i)(r), x^* \otimes y^* \rangle,$$
$$\langle r_+(\partial_i^*(x^*)), y^* \rangle = \langle r, \partial_i^*(x^*) \otimes y^* \rangle = \langle (\partial_i \otimes \mathrm{id})(r), x^* \otimes y^* \rangle,$$
$$\langle \partial_i r_-(x^*), y^* \rangle = -\langle r, \partial_i^*(y^*) \otimes x^* \rangle = -\langle (\partial_i \otimes \mathrm{id})(r), y^* \otimes x^* \rangle,$$
$$\langle r_-(\partial_i^*(x^*)), y^* \rangle = -\langle r, y^* \otimes \partial_i^*(x^*) \rangle = -\langle (\mathrm{id} \otimes \partial_i)(r), y^* \otimes x^* \rangle,$$

which completes the proof. □

Lemma 5.7. *Let $(A, \cdot_A, \Phi)$ be a Ψ-admissible differential algebra and $r \in A \otimes A$. If $r \in A \otimes A$ is a solution of the Ψ-admissible AYBE in $(A, \cdot_A, \Phi)$ and the symmetric part of r satisfies* (2-10), *then $(A, \cdot_A, \Delta, \Phi, \Psi)$ is a differential ASI bialgebra, where $\Delta : A \to A \otimes A$ is defined by*

$$\Delta(a) = (\mathrm{id} \otimes L(a) - R(a) \otimes \mathrm{id})(r), \quad \forall a \in A. \tag{5-8}$$

Proof. This follows from [20, Corollary 4.4]. □

Definition 5.8. Let $(A, \cdot_A, \Phi)$ be a differential algebra and $r \in A \otimes A$. Then r is called *L-invariant* if the following equation holds:

$$(L(a) \otimes \mathrm{id} - \mathrm{id} \otimes R(a))(r) = 0, \quad \forall a \in A. \tag{5-9}$$

Recall that a differential ASI bialgebra $(A, \cdot_A, \Delta, \Phi, \Psi)$ is called *coboundary* if Δ is defined by (5-8) for some $r \in A \otimes A$. A coboundary differential ASI bialgebra is denoted by $(A, \cdot_A, \Delta_r, \Phi, \Psi)$.

Definition 5.9. Let $(A, \cdot_A, \Phi)$ be a Ψ-admissible differential algebra. If r is a solution of the Ψ-admissible AYBE in $(A, \cdot_A, \Phi)$ and the symmetric part of $r \in A \otimes A$ is L-invariant, then the coboundary differential ASI bialgebra $(A, \cdot_A, \Delta_r, \Phi, \Psi)$ induced by r is called a *quasitriangular differential ASI bialgebra*. In particular, if r is antisymmetric, then $(A, \cdot_A, \Delta_r, \Phi, \Psi)$ is called a *triangular differential ASI bialgebra*. A quasitriangular differential ASI bialgebra $(A, \cdot_A, \Delta_r, \Phi, \Psi)$ is called *factorizable* if the symmetric part of r is nondegenerate.

Example 5.10. By Example 4.6, there is a six-dimensional commutative algebra $(A \oplus A^*, \cdot)$ on $A \oplus A^*$ with the nonzero products of $\cdot$ given by (4-5). Let $\partial_1, \partial_2 : A \oplus A^* \to A \oplus A^*$ be linear maps given by

$$\partial_1(e_1) = e_1, \quad \partial_1(e_2) = 0, \quad \partial_1(e_3) = e_3, \quad \partial_1(e_1^*) = 0, \quad \partial_1(e_2^*) = e_2^*, \quad \partial_1(e_3^*) = 0,$$
$$\partial_2(e_1) = 0, \quad \partial_2(e_2) = e_2, \quad \partial_2(e_3) = e_3, \quad \partial_2(e_1^*) = e_1^*, \quad \partial_2(e_2^*) = 0, \quad \partial_2(e_3^*) = 0.$$

Let $\Phi = \{\partial_1, \partial_2\}$ and $\Psi = \{-\partial_1, -\partial_2\}$. Then $(A \oplus A^*, \cdot, \Phi)$ is a Ψ-admissible differential algebra. Moreover, $r = \sum_{i=1}^{3} e_i^* \otimes e_i$ is a solution of the Ψ-admissible AYBE in $(A \oplus A^*, \cdot)$ and $(A \oplus A^*, \cdot, \Delta_r, \Phi, \Psi)$ is a commutative and cocommutative factorizable differential ASI bialgebra with Δ_r given by (4-8).

The next proposition justifies the term "factorizable differential ASI bialgebra."

Definition 5.11. Let $(A, \cdot_A, \Phi_A = \{\partial_{A,i}\}_{i=1}^m)$ and $(B, \cdot_B, \Phi_B = \{\partial_{B,i}\}_{i=1}^m)$ be two differential algebras. A linear map $\varphi : (A, \cdot_A, \Phi_A) \to (B, \cdot_B, \Phi_B)$ is called a *homomorphism* of differential algebras if $\varphi : (A, \cdot_A, \Phi_A) \to (B, \cdot_B, \Phi_B)$ is a homomorphism of associative algebras satisfying

$$\varphi \circ \partial_{A,i} = \partial_{B,i} \circ \varphi, \quad \forall i = 1, 2, \ldots, m.$$

If in addition φ is a linear isomorphism, φ is called an *isomorphism* of differential algebras.

Proposition 5.12. *Let* $(A, \cdot_A, \Delta_r, \Phi, \Psi)$ *be a factorizable differential ASI bialgebra. Then* $\operatorname{Im}(r_+ \oplus r_-)$ *is a differential subalgebra of the differential algebra* $A \oplus A$*, which is isomorphic to the differential algebra* $(A^*, \cdot_r, \Psi^*)$*, where* $\cdot_r : A^* \otimes A^* \to A^*$ *is defined by*

$$x^* \cdot_r y^* = R^*(r_+(x^*))y^* + L^*(r_-(y^*))x^*, \quad \forall x^*, y^* \in A^*. \tag{5-10}$$

Every $a \in A$ *has a unique decomposition* $a = a_+ - a_-$ *with* $(a_+, a_-) \in \operatorname{Im}(r_+ \oplus r_-)$.

Proof. By [27, Proposition 3.2], both $r_+, r_- : (A^*, \cdot_r) \to (A, \cdot_A)$ are associative algebra homomorphisms, $\operatorname{Im}(r_+ \oplus r_-)$ is an associative subalgebra of the direct sum associative algebra $A \oplus A$, which is isomorphic to the associative algebra $(A^*, \cdot_r)$, and every $a \in A$ has a unique decomposition $a = a_+ - a_-$ with $(a_+, a_-) \in \operatorname{Im}(r_+ \oplus r_-)$. By Lemma 5.6, we know that both $r_+, r_- : (A^*, \cdot_r, \Psi^*) \to (A, \cdot_A, \Phi)$ are differential algebra homomorphisms. Therefore, $\operatorname{Im}(r_+ \oplus r_-)$ is a differential subalgebra of the direct sum differential algebra $A \oplus A$, which is isomorphic to the differential algebra $(A^*, \cdot_r, \Psi^*)$. □

Let $(A, \cdot_A, \Delta, \Phi, \Psi)$ be an arbitrary differential ASI bialgebra and $(\mathfrak{A}, \cdot_{\mathfrak{A}})$ be the associative algebra structure on $A \oplus A^*$ obtained from the matched pair of associative algebras $(A, A^*, R^*_{\cdot_A}, L^*_{\cdot_A}, R^*_{\cdot_{A^*}}, L^*_{\cdot_{A^*}})$ [20]. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of A and $\{e_1^*, e_2^*, \ldots, e_n^*\}$ be the dual basis of A^*. Define

$$r = \sum_i e_i \otimes e_i^* \in A \otimes A^* \subset \mathfrak{A} \otimes \mathfrak{A}$$

and

$$\Delta_r(u) = (\mathrm{id} \otimes L_{\cdot_{\mathfrak{A}}}(u) - R_{\cdot_{\mathfrak{A}}}(u) \otimes \mathrm{id})(r), \quad \forall u \in \mathfrak{A}.$$

Then $(\mathfrak{A}, \cdot_{\mathfrak{A}}, \Delta_r, \Phi + \Psi^*, \Psi + \Phi^*)$ is a coboundary differential ASI bialgebra (see [20, Theorem 4.5] for details).

Theorem 5.13. *The differential ASI bialgebra* $(\mathfrak{A}, \cdot_{\mathfrak{A}}, \Delta_r, \Phi + \Psi^*, \Psi + \Phi^*)$ *is a factorizable quasitriangular differential ASI bialgebra.*

Proof. The proof of Theorem 4.5 in [20] implies that r is a solution of the $(\Psi+\Phi^*)$-admissible AYBE in $(\mathfrak{A}, \cdot_{\mathfrak{A}}, \Phi+\Psi^*)$. By suitably modifying the same proof, one can show that the symmetric part of r is L-invariant. By the proof of Theorem 3.4, I_r is an isomorphism of vector spaces. Hence, the differential ASI bialgebra $(\mathfrak{A}, \cdot_{\mathfrak{A}}, \Delta_r, \Phi+\Psi^*, \Psi+\Phi^*)$ is factorizable. □

Definition 5.14. A bilinear form $\mathfrak{B}(\ ,\ )$ on an associative algebra $(A, \cdot_A)$ is called *invariant* if

$$\mathfrak{B}(a \cdot_A b, c) = \mathfrak{B}(a, b \cdot_A c), \quad \forall a, b, c \in A.$$

A *Frobenius algebra* $(A, \cdot_A, \mathfrak{B})$ is an associative algebra $(A, \cdot_A)$ with a nondegenerate invariant bilinear form $\mathfrak{B}(\ ,\ )$. A Frobenius algebra $(A, \cdot_A, \mathfrak{B})$ is called *symmetric* if $\mathfrak{B}(\ ,\ )$ is symmetric.

Definition 5.15. A *differential Frobenius algebra* is a quadruple $(A, \cdot_A, \Phi, \mathfrak{B})$, where $(A, \cdot_A, \Phi = \{\partial_k\}_{k=1}^m)$ is a differential algebra and $(A, \cdot_A, \mathfrak{B})$ is a Frobenius algebra. It is called *symmetric* if $\mathfrak{B}$ is symmetric. For all $k = 1, \dots, m$, let $\hat{\partial}_k$ be the adjoint linear operator of ∂_k under the nondegenerate bilinear form $\mathfrak{B}$:

$$\mathfrak{B}(\partial_k(a), b) = \mathfrak{B}(a, \hat{\partial}_k(b)), \quad \forall a, b \in A.$$

We call $\hat{\Phi} := \{\hat{\partial}_k\}_{k=1}^m$ the *adjoint of* $\Phi = \{\partial_k\}_{k=1}^m$ *with respect to* $\mathfrak{B}$.

Note that $\hat{\Phi}$ is admissible to $(A, \cdot_A, \Phi)$, by [20, Proposition 3.3].

Definition 5.16. The triple $((A, \cdot_A, \Phi = \{\partial_k\}_{k=1}^m), \mathfrak{B}, P)$ is called a *symmetric Rota–Baxter differential Frobenius algebra of weight* λ if $(A, \cdot_A, \Phi, \mathfrak{B})$ is a symmetric differential Frobenius algebra and $(A, \cdot_A, P)$ is a Rota–Baxter algebra of weight λ satisfying the compatibility condition given by (4-4) such that

$$\partial_i P = P \partial_i, \quad \forall i = 1, 2, \dots, m.$$

Lemma 5.17. *Let* $\big((A, \cdot_A, \Phi = \{\partial_k\}_{k=1}^m), \mathfrak{B}, P\big)$ *be a symmetric Rota–Baxter differential Frobenius algebra of weight* λ, *and* $I_{\mathfrak{B}} : A^* \to A$ *be the induced linear isomorphism by* $\mathfrak{B}$. *Let* $r \in A \otimes A$ *be the* 2*-tensor form of* $P \circ I_{\mathfrak{B}}$ *given by* (4-6) *and* $\hat{\Phi} = \{\hat{\partial}_k\}_{k=1}^m$ *be the adjoint of* Φ *with respect to* $\mathfrak{B}$. *Then*

$$\partial_i I_{\mathfrak{B}} = I_{\mathfrak{B}} \hat{\partial}_i^*, \quad \partial_i r_+ = r_+ \hat{\partial}_i^*, \quad \forall i = 1, 2, \dots, m.$$

Proof. For all $a \in A$ and $x^* \in A^*$, we have

$$\mathfrak{B}(\partial_i I_{\mathfrak{B}}(x^*), a) = \mathfrak{B}(I_{\mathfrak{B}}(x^*), \hat{\partial}_i(a)) = \langle x^*, \hat{\partial}_i(a)\rangle = \langle \hat{\partial}_i^*(x^*), a\rangle = \mathfrak{B}(I_{\mathfrak{B}} \hat{\partial}_i^*(x^*), a),$$

that is, $\partial_i I_{\mathfrak{B}} = I_{\mathfrak{B}} \hat{\partial}_i^*$. Therefore,

$$\partial_i r_+ = \partial_i P I_{\mathfrak{B}} = P \partial_i I_{\mathfrak{B}} = P I_{\mathfrak{B}} \hat{\partial}_i^* = r_+ \hat{\partial}_i^*,$$

completing the proof. □

Proposition 5.18. *Let $((A, \cdot_A, \Phi), \mathfrak{B}, P)$ be a symmetric Rota–Baxter differential Frobenius algebra of weight* 0 *and $I_{\mathfrak{B}} : A^* \to A$ be the induced linear isomorphism by $\mathfrak{B}$. Then $(A, \cdot_A, \Delta_r, \Phi, \hat{\Phi})$ is a triangular differential ASI bialgebra where $r \in A \otimes A$ is the* 2*-tensor form of $P \circ I_{\mathfrak{B}}$ given by* (4-6) *and $\hat{\Phi}$ is the adjoint of Φ with respect to $\mathfrak{B}$.*

Proof. It follows from Lemma 4.7 that $r + \tau(r) = 0$. Similarly to Proposition 2.20, we show that $\boldsymbol{A}(r) = 0$. On the other hand, setting $\Phi = \{\partial_k\}_{k=1}^m$, by Lemma 5.17, we have $\partial_i r_+ = r_+ \hat{\partial}_i^*$. Therefore, by Lemma 5.6, we show that r is a solution of the $\hat{\Phi}$-admissible AYBE in $(A, \cdot_A, \Phi)$. Noting that $\hat{\Phi}$ is admissible to $(A, \cdot_A, \Phi)$, we complete the proof. □

Theorem 5.19. *Let $(A, \cdot_A, \Delta_r, \Phi = \{\partial_k\}_{k=1}^m, \Psi = \{\eth_k\}_{k=1}^m)$ be a factorizable differential ASI bialgebra with $I_r = r_+ - r_-$. Then $((A, \cdot_A, \Phi), \mathfrak{B}, P)$ is a symmetric Rota–Baxter differential Frobenius algebra of weight λ such that the adjoint of Φ with respect to $\mathfrak{B}$ is Ψ, where*

$$P = -\lambda r_+ \circ I_r^{-1}, \quad \lambda \neq 0$$

and the bilinear form $\mathfrak{B} \in \otimes^2 A^*$ is defined by

$$\mathfrak{B}(a, b) = -\lambda \langle I_r^{-1}(a), b \rangle, \quad \forall a, b \in A.$$

Proof. It follows from [27, Corollary 4.8] that $(A, \cdot_A, \mathfrak{B})$ is a symmetric Frobenius algebra and $(A, \cdot_A, P)$ is a Rota–Baxter algebra of weight λ such that (4-4) holds. For all $a, b \in A$, we have

$$\mathfrak{B}(\partial_i(a), b) = -\lambda \langle I_r^{-1}(\partial_i(a)), b \rangle = -\lambda \langle I_r^{-1} \partial_i I_r I_r^{-1}(a), b \rangle,$$
$$\mathfrak{B}(a, \eth_i(b)) = -\lambda \langle I_r^{-1}(a), \eth_i(b) \rangle = -\lambda \langle I_r^{-1} I_r \eth_i^* I_r^{-1}(a), b \rangle,$$

and by Lemma 5.6, we have

$$\partial_i I_r = \partial_i(r_+ - r_-) = (r_+ - r_-)\eth_i^* = I_r \eth_i^*.$$

Therefore, $\mathfrak{B}(\partial_i(a), b) = \mathfrak{B}(a, \eth_i(b))$. Furthermore, we have

$$\partial_i P = -\lambda \partial_i r_+ I_r^{-1} = -\lambda r_+ \eth_i^* I_r^{-1} = -\lambda r_+ I_r^{-1} \partial_i = P \partial_i.$$

The proof is complete. □

Theorem 5.20. *Let $((A, \cdot_A, \Phi = \{\partial_k\}_{k=1}^m), \mathfrak{B}, P)$ be a symmetric Rota–Baxter differential Frobenius algebra of weight $\lambda \neq 0$, and $I_{\mathfrak{B}} : A^* \to A$ be the induced linear isomorphism by $\mathfrak{B}$. Let $r \in A \otimes A$ be the* 2*-tensor form of $P \circ I_{\mathfrak{B}}$ given by* (4-6) *and $\hat{\Phi}$ be the adjoint of Φ with respect to $\mathfrak{B}$. Then r is a solution of the $\hat{\Phi}$-admissible AYBE in $(A, \cdot_A, \Phi)$ and gives rise to a factorizable differential ASI bialgebra $(A, \cdot_A, \Delta_r, \Phi, \hat{\Phi})$.*

Proof. By Lemma 4.7, $r+\tau(r)$ is equal to $-\lambda r_{\mathfrak{B}}$ and $I_r=-\lambda I_{\mathfrak{B}}$. Then by Lemma 5.17, we have

$$\partial_i r_+ = r_+\hat{\partial}_i^* \quad \text{and} \quad \partial_i r_- = \partial_i(r_+ - I_r) = r_+\hat{\partial}_i^* - I_r\hat{\partial}_i^* = r_-\hat{\partial}_i^*.$$

Similarly to Theorem 4.11, we can show that $\boldsymbol{A}(r)=0$ and the symmetric part of r is L-invariant. Hence, by Lemma 5.6, r is a solution of the $\hat{\Phi}$-admissible AYBE in $(A,\cdot_A,\Phi)$, which completes the proof. □

We now turn to the constructions of quasitriangular and factorizable Poisson bialgebras from quasitriangular and factorizable (commutative and cocommutative) differential ASI bialgebras.

Proposition 5.21 [7]. *Let $(A,\cdot_A,\Phi=\{\partial_1,\partial_2\})$ be a commutative differential algebra. Then $(A,[\ ,\ ]_A,\cdot_A)$ is a Poisson algebra, called the* ***induced Poisson algebra*** *of $(A,\cdot_A,\Phi)$, where $[\ ,\ ]: A\otimes A\to A$ is defined by*

$$[a,b]_A := \partial_1(a)\cdot_A\partial_2(b) - \partial_2(a)\cdot_A\partial_1(b),\quad \forall a,b\in A. \tag{5-11}$$

Lemma 5.22 [20]. *Let $(A,\cdot_A,\Phi=\{\partial_1,\partial_2\})$ be a $\Psi=\{\vartheta_1,\vartheta_2\}$-admissible commutative differential algebra and $(A,[\ ,\ ]_A,\cdot_A)$ be the induced Poisson algebra of $(A,\cdot_A,\Phi)$. Suppose that*

$$\vartheta_2(\partial_1(a))\cdot b = \vartheta_1(\partial_2(a))\cdot b,\quad \forall a,b\in A. \tag{5-12}$$

Then every solution of the Ψ-admissible AYBE in $(A,\cdot_A,\Phi)$ is a solution of the Poisson Yang–Baxter equation in the Poisson algebra $(A,[\ ,\ ]_A,\cdot_A)$.

Theorem 5.23. *If $(A,\cdot_A,\Delta_r,\Phi,\Psi)$ is a quasitriangular (resp. triangular, factorizable) commutative and cocommutative differential ASI bialgebra, and equation* (5-12) *holds, then $(A,[\ ,\ ]_A,\cdot_A,\delta_r,\Delta_r)$ is a quasitriangular (resp. triangular, factorizable) Poisson bialgebra through r, where $(A,[\ ,\ ]_A,\cdot_A)$ is the induced Poisson algebra of $(A,\cdot_A,\Phi)$ and $\delta_r: A\to A\otimes A$ is defined by*

$$\delta_r = (\vartheta_1\otimes\vartheta_2 - \vartheta_2\otimes\vartheta_1)\Delta_r. \tag{5-13}$$

Proof. For all $a\in A$, we have

$$\begin{aligned}&(\mathrm{ad}(a)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(a))(r+\tau(r))\\ &=\big((L(\partial_1(a))\partial_2 - L(\partial_2(a))\partial_1)\otimes\mathrm{id}+\mathrm{id}\otimes(L(\partial_1(a))\partial_2 - L(\partial_2(a))\partial_1)\big)(r+\tau(r)).\end{aligned}$$

By (5-2), this equals

$$\begin{aligned}&\big(L(\partial_1(a))\otimes\vartheta_2 - L(\partial_2(a))\otimes\vartheta_1 - \mathrm{id}\otimes\vartheta_2 L(\partial_1(a)) + \mathrm{id}\otimes\vartheta_1 L(\partial_2(a))\big)(r+\tau(r))\\ &\qquad + \big(\mathrm{id}\otimes L(\vartheta_2\partial_1(a)) - \mathrm{id}\otimes L(\vartheta_1\partial_2(a))\big)(r+\tau(r)).\end{aligned}$$

Using (5-12) this becomes

$$\big((\mathrm{id}\otimes\vartheta_2)(L(\partial_1(a))\otimes\mathrm{id}-\mathrm{id}\otimes L(\partial_1(a))) \\ +(\mathrm{id}\otimes\vartheta_1)(L(\partial_2(a))\otimes\mathrm{id}-\mathrm{id}\otimes L(\partial_2(a)))\big)(r+\tau(r))=0.$$

Hence, the symmetric part of r is (ad, L)-invariant. By Lemma 5.22, r is a solution of the Poisson Yang–Baxter equation in $(A,[\ ,\ ]_A,\cdot_A)$. Furthermore,

$$\begin{aligned}\delta_r(a) &= (\vartheta_1\otimes\vartheta_2-\vartheta_2\otimes\vartheta_1)\Delta_r(a)=(\vartheta_1\otimes\vartheta_2-\vartheta_2\otimes\vartheta_1)(\mathrm{id}\otimes L(a)-L(a)\otimes\mathrm{id})(r)\\ &\overset{(5\text{-}6)}{=}(\mathrm{id}\otimes\vartheta_2L(a)\partial_1-\mathrm{id}\otimes\vartheta_1L(a)\partial_2)(r)-(\vartheta_1L(a)\partial_2\otimes\mathrm{id}-\vartheta_2L(a)\partial_1\otimes\mathrm{id})(r)\\ &\overset{(5\text{-}2)}{=}(\mathrm{ad}(a)\otimes\mathrm{id}+\mathrm{id}\otimes\mathrm{ad}(a))(r).\end{aligned}$$

It is now obvious that the theorem holds. □

Example 5.24. Continuing with Example 5.10, there is a commutative and cocommutative factorizable differential ASI bialgebra $(A\oplus A^*,\cdot,\Delta_r,\Phi,\Psi)$. Note that (5-12) holds automatically. Thus by Theorem 5.23, it induces a factorizable Poisson bialgebra $(A\oplus A^*,[\ ,\ ],\cdot,\delta_r,\Delta_r)$ where the nonzero product of $[\ ,\ ]$ is defined by $[e_1,e_2]=e_3$ and δ_r is explicitly given by (4-8).

Let $\big(A,\cdot_A,\Delta_r,\Phi=\{\partial_1,\partial_2\},\Psi=\{\vartheta_1,\vartheta_2\}\big)$ be a commutative and cocommutative factorizable differential ASI bialgebra. Using Proposition 5.12, let

$$\big(\mathrm{Im}(r_+\oplus r_-),\cdot,\Phi+\Phi=\{\partial_1+\partial_1,\partial_2+\partial_2\}\big)$$

be the differential subalgebra of the differential algebra $A\oplus A$, which is isomorphic to the differential algebra $(A^*,\cdot_r,\Psi^*)$, where $\cdot_r:A^*\otimes A^*\to A^*$ is defined by (2-4). On the other hand, suppose that (5-12) holds. Let $(A,[\ ,\ ]_A,\cdot_A,\delta_r,\Delta_r)$ be the induced factorizable Poisson bialgebra in Theorem 5.23. Then by Proposition 3.3, let $(\mathrm{Im}(r_+\oplus r_-),[\ ,\ ],\cdot)$ be the Poisson subalgebra of the direct sum Poisson algebra $A\oplus A$, which is isomorphic to the Poisson algebra $(A^*,[\ ,\ ]_r,\cdot_r)$, where $[\ ,\ ]_r,\cdot_r:A^*\otimes A^*\to A^*$ are respectively defined by (2-3) and (2-4).

Corollary 5.25. *With the same conditions as above, the induced Poisson algebras of* $(\mathrm{Im}(r_+\oplus r_-),\cdot,\Phi+\Phi)$ *and* $(A^*,\cdot_r,\Psi^*)$ *are exactly* $(\mathrm{Im}(r_+\oplus r_-),[\ ,\ ],\cdot)$ *and* $(A^*,[\ ,\ ]_r,\cdot_r)$. *Thus we have the commutative diagram*

$$\begin{array}{ccccc}(A,\cdot_A,\Delta_r,\Phi,\Psi) & \xrightarrow{\text{Proposition 5.12}} & (\mathrm{Im}(r_+\oplus r_-),\cdot,\Phi+\Phi) & \xrightarrow{\simeq} & (A^*,\cdot_r,\Psi^*)\\ \big\downarrow{\scriptstyle\text{Theorem 5.23}} & & \big\downarrow{\scriptstyle\text{Proposition 5.21}} & & \big\downarrow{\scriptstyle\text{Proposition 5.21}}\\ (A,[\ ,\ ]_A,\cdot_A,\delta_r,\Delta_r) & \xrightarrow{\text{Proposition 3.3}} & (\mathrm{Im}(r_+\oplus r_-),[\ ,\ ],\cdot) & \xrightarrow{\simeq} & (A^*,[\ ,\ ]_r,\cdot_r)\end{array}$$

Proof. By Theorem 5.23, the induced Poisson algebra of $(A^*,\cdot_r,\Psi^*)$ is exactly $(A^*,[\ ,\ ]_r,\cdot_r)$.Noting that $(A,[\ ,\ ]_A,\cdot_A)$ is the induced Poisson algebra of $(A,\cdot_A,\Phi)$, it is straightforward that the induced Poisson algebra of $(\mathrm{Im}(r_+\oplus r_-),\cdot,\Phi+\Phi)$ is exactly $(\mathrm{Im}(r_+\oplus r_-),[\ ,\ ],\cdot)$. The proof is complete. □

Let $(A, \cdot_A, \Delta, \Phi = \{\partial_1, \partial_2\}, \Psi = \{\vartheta_1, \vartheta_2\})$ be a commutative and cocommutative differential ASI bialgebra. Let $(A, [\ ,\ ]_A, \cdot_A)$ be the induced Poisson algebra of $(A, \cdot_A, \Phi)$. Suppose that (5-12) and the following equation hold:

$$(\vartheta_2\partial_1 \otimes \mathrm{id})\Delta = (\vartheta_1\partial_2 \otimes \mathrm{id})\Delta. \tag{5-14}$$

Then $(A, [\ ,\ ]_A, \cdot_A, \delta, \Delta)$ is a Poisson bialgebra where $\delta : A \to A \otimes A$ is defined by $\delta = (\vartheta_1 \otimes \vartheta_2 - \vartheta_2 \otimes \vartheta_1)\Delta$ [20, Theorem 5.24]. Let $(\mathfrak{A}, \cdot_{\mathfrak{A}}, \Delta_r, \Phi + \Psi^*, \Psi + \Phi^*)$ be the factorizable differential ASI bialgebra obtained in Theorem 5.13 arising from $(A, \cdot_A, \Delta, \Phi, \Psi)$ and let $(\mathfrak{A}, [\ ,\ ]_{\mathfrak{A}}, \cdot_{\mathfrak{A}}, \delta_r, \Delta_r)$ be the factorizable Poisson bialgebra of Theorem 3.4 arising from $(A, [\ ,\ ]_A, \cdot_A, \delta, \Delta)$. Since the Poisson algebra structure $(\mathfrak{A}, [\ ,\ ]_{\mathfrak{A}}, \cdot_{\mathfrak{A}})$ is the induced Poisson algebra of $(\mathfrak{A}, \cdot_{\mathfrak{A}}, \Phi + \Psi^*)$ [20, Remark 5.25] and equation (5-12) also holds for the $(\Psi + \Phi^*)$-admissible commutative differential algebra $(\mathfrak{A}, \cdot_{\mathfrak{A}}, \Phi + \Psi^*)$, we have the following conclusion.

Corollary 5.26. *With the same conditions as above*, $(\mathfrak{A}, [\ ,\ ]_{\mathfrak{A}}, \cdot_{\mathfrak{A}}, \delta_r, \Delta_r)$ *is the induced factorizable Poisson bialgebra of the factorizable differential ASI bialgebra* $(\mathfrak{A}, \cdot_{\mathfrak{A}}, \Delta_r, \Phi + \Psi^*, \Psi + \Phi^*)$. *Thus*, *we have the following commutative diagram*:

$$\begin{array}{ccc}
(A, \cdot_A, \Delta, \Phi, \Psi) & \xrightarrow{\text{Theorem 5.13}} & (\mathfrak{A}, \cdot_{\mathfrak{A}}, \Delta_r, \Phi + \Psi^*, \Psi + \Phi^*) \\
\downarrow & & \downarrow \text{Theorem 5.23} \\
(A, [\ ,\ ]_A, \cdot_A, \delta, \Delta) & \xrightarrow{\text{Theorem 3.4}} & (\mathfrak{A}, [\ ,\ ]_{\mathfrak{A}}, \cdot_{\mathfrak{A}}, \delta_r, \Delta_r)
\end{array}$$

Proposition 5.27. *Let* $(A, \cdot_A, \Delta_r, \Phi, \Psi)$ *be a commutative and cocommutative factorizable differential ASI bialgebra*, *which corresponds to a symmetric Rota–Baxter differential Frobenius algebra* $((A, \cdot_A, \Phi), \mathfrak{B}, P)$ *of weight* $\lambda \neq 0$. *Suppose that* (5-12) *holds. Let* $(A, [\ ,\ ]_A, \cdot_A)$ *be the induced Poisson algebra of* $(A, \cdot_A, \Phi)$. *Then* $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, P)$ *is a quadratic Rota–Baxter Poisson algebra of weight* λ, *to which the induced factorizable Poisson bialgebra* $(A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r)$ *in Theorem 5.23 corresponds. Thus*, *we have the following commutative diagram*:

$$\begin{array}{ccc}
(A, \cdot_A, \Delta_r, \Phi, \Psi) & \xrightarrow{\text{Theorem 5.23}} & (A, [\ ,\ ]_A, \cdot_A, \delta_r, \Delta_r) \\
\text{Theorem 5.20}\ \uparrow\downarrow\ \text{Theorem 5.19} & & \text{Theorem 4.11}\ \uparrow\downarrow\ \text{Theorem 4.9} \\
((A, \cdot_A, \Phi), \mathfrak{B}, P) & \longrightarrow & ((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, P)
\end{array}$$

Proof. For all $a, b, c \in A$, we have

$$\begin{aligned}
&\mathfrak{B}([a, b]_A, c) \\
&\quad = \mathfrak{B}(\partial_1(a) \cdot_A \partial_2(b) - \partial_2(a) \cdot_A \partial_1(b), c) = \mathfrak{B}(a, \vartheta_1(\partial_2(b) \cdot_A c) - \vartheta_2(\partial_1(b) \cdot_A c)) \\
&\quad = \mathfrak{B}(a, \vartheta_1(\partial_2(b)) \cdot_A c - \partial_2(b) \cdot_A \partial_1(c) - \vartheta_2(\partial_1(b)) \cdot_A c + \partial_1(b) \cdot_A \partial_2(c)) \\
&\quad = \mathfrak{B}(a, [b, c]_A)
\end{aligned}$$

and

$$
\begin{aligned}
[P(a), P(b)]_A &= \partial_1(P(a)) \cdot_A \partial_2(P(b)) - \partial_2(P(a)) \cdot_A \partial_1(P(b)) \\
&= P(\partial_1(a)) \cdot_A P(\partial_2(b)) - P(\partial_2(a)) \cdot_A P(\partial_1(b)) \\
&= P\big(P(\partial_1(a)) \cdot_A \partial_2(b) + \partial_1(a) \cdot_A P(\partial_2(b)) + \lambda \partial_1(a) \cdot_A \partial_2(b)\big) \\
&\qquad - P\big(P(\partial_2(a)) \cdot_A \partial_1(b) + \partial_2(a) \cdot_A P(\partial_1(b)) + \lambda \partial_2(a) \cdot_A \partial_1(b)\big) \\
&= P\big(\partial_1(P(a)) \cdot_A \partial_2(b) + \partial_1(a) \cdot_A \partial_2(P(b)) + \lambda \partial_1(a) \cdot_A \partial_2(b)\big) \\
&\qquad - P\big(\partial_2(P(a)) \cdot_A \partial_1(b) + \partial_2(a) \cdot_A \partial_1(P(b)) + \lambda \partial_2(a) \cdot_A \partial_1(b)\big) \\
&= P\big([P(a), b]_A + [a, P(b)]_A + \lambda [a, b]_A\big).
\end{aligned}
$$

Therefore, $((A, [\ ,\ ]_A, \cdot_A), \mathfrak{B}, P)$ is a quadratic Rota–Baxter Poisson algebra of weight λ. Thus the proposition is true. □

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