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HAO ZHUANG

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SYMPLECTIC SEMI-CHARACTERISTICS

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We define and study the symplectic semi-characteristic of a closed $4n$-dimensional symplectic manifold, based on the even-degree part of the primitive cohomology. Using a vector field with nondegenerate zero points, we prove a counting formula for the symplectic semi-characteristic. As corollaries of the counting formula, we obtain a vanishing property and the fact that the definition of the symplectic semi-characteristic is independent of the choice of symplectic forms.

1. Introduction

In three consecutive papers [18; 19; 20], Tsai, Tseng, and Yau introduced and studied the p-filtered cohomology groups $F^pH^k(M,\omega)$ $(0\le k\le 1+2p+\dim M)$ of a symplectic manifold (M,ω). The $p=0$ case in [18, (3.14), (3.22)] and [19, (1.5), (1.6)] is called the *primitive cohomology* of (M,ω), and the $p\ge 1$ case in [20, (1.2), Theorem 3.1] is generalized from it. Different from the classical de Rham cohomology, this p-filtered cohomology includes information about the symplectic form ω. Thus, an important application of it is in distinguishing among different symplectic structures; examples can be found in [19, Section 4] (computing the primitive cohomology groups) and [20, Section 6] (computing the product structures). Tanaka and Tseng [16, Theorem 1.1] proved that the mapping cone complex determined by the map $\wedge\omega^{p+1}$ between de Rham complexes computes the p-filtered cohomology.

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In this paper, we focus on the primitive cohomology. Our work starts from an interesting fact: When the symplectic manifold (M, ω) is closed, the Euler characteristic

$$\dim F^0 H^{\mathrm{even}}(M, \omega) - \dim F^0 H^{\mathrm{odd}}(M, \omega) \tag{1-1}$$

of the primitive cohomology is equal to zero. This was originally proved in [18, Proposition 3.26] and [19, Proposition 3.7] by means of duality between cohomology groups. Recently, using Tanaka and Tseng's mapping cone complex, Clausen, Tang, and Tseng proved the symplectic Morse inequality [7, Theorem 1.4], also showing that the value of (1-1) is zero. Intuitively speaking, we may say that the even-degree part of the primitive cohomology contains the same amount of information as the odd-degree part.

We then ask whether $F^0 H^{\mathrm{even}}(M, \omega)$ is an obstruction to some geometric object. Here, to some extent, we are motivated by the Kervaire semi-characteristic in a similar scenario. Recall that for any odd-dimensional closed oriented manifold N, the Euler characteristic of the de Rham cohomology of N is zero. Let b_k be the dimension of the k-th de Rham cohomology group of N. The $\mathbb{Z}_2$-valued Kervaire semi-characteristic of N [11, Introduction; 12, Section 1; 1, Section 4] is defined to be

$$\sum_{k \text{ even}} b_k \mod 2. \tag{1-2}$$

When $\dim N = 4n+1$, the $\mathbb{Z}_2$-valued Kervaire semi-characteristic satisfies Atiyah's vanishing theorem [1, Theorem 4.1] and Zhang's counting formula [23, Theorem 1.3]. Both the vanishing theorem and the counting formula involve two vector fields, showing that the $\mathbb{Z}_2$-valued Kervaire semi-characteristic is an obstruction to a certain pair of vector fields. Now, back to our symplectic situation, we state our main question:

Question 1.1. For a closed symplectic manifold (M, ω), what geometric object(s) on M does the even-degree part of the primitive cohomology of (M, ω) obstruct? Is it a pair of vector fields or something else?

Here we answer this question for $4n$-dimensional closed symplectic manifolds.

Assumption 1.2. Throughout this paper, when not stated otherwise, (M, ω) is a $4n$-dimensional closed symplectic manifold M equipped with a symplectic form ω.

We recall Tanaka and Tseng's mapping cone complex $(C^*(M, \omega), \partial_C)$, which computes the primitive cohomology of (M, ω). Let $\Omega^k(M)$ be the space of smooth k-forms on M. The space of k-cochains [16, Section 3.1; 7, Definition 1.1] is

$$C^k(M, \omega) := \Omega^k(M) \oplus \Omega^{k-1}(M) \quad (k = 0, 1, \dots, 4n+1).$$

Let d be de Rham exterior differentiation. The boundary map is

$$\partial_C : C^k(M,\omega) \to C^{k+1}(M,\omega) \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \mapsto \begin{bmatrix} d & \omega \\ 0 & -d\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix} d\alpha + \omega\wedge\beta \\ -d\beta\end{bmatrix}.$$

Here, we write the pair $(\alpha, \beta) \in \Omega^k(M) \oplus \Omega^{k-1}(M)$ as a column for the convenience of using matrices and operators later.

The k-th cohomology group of $(C^*(M,\omega), \partial_C)$ is exactly $F^0H^k(M,\omega)$.

Definition 1.3. Let b_k^ω be the dimension of $F^0H^k(M,\omega)$. The *symplectic semi-characteristic* of (M,ω) is the $\mathbb{Z}_2$-valued number

$$\kappa(M,\omega) := \sum_{k \text{ even}} b_k^\omega \bmod 2. \tag{1-3}$$

To state our main result, we review the definition of nondegenerate vector fields. Let V be a smooth vector field on M. Following [5, Section 1.6], at each zero point p of V, we define a homomorphism

$$\Phi_p : T_pM \to T_pM, \quad v \mapsto [V, \tilde{v}](p),$$

where $\tilde{v}$ is a vector field extending the tangent vector v and $[\,\cdot\,,\cdot\,]$ is the Lie bracket between vector fields. This Φ_p is independent of the extension $\tilde{v}$ since $V(p) = 0$.

Definition 1.4. A smooth vector field V on M is called *nondegenerate* if either V vanishes nowhere or Φ_p is invertible for each zero point p of V.

Such a nondegenerate vector field always exists because by [13, Theorem 6.6], there is always a Morse function on M. Now, our main result is as follows:

Theorem 1.5 (compare [23, Theorem 1.3]). *Let V be a smooth nondegenerate vector field on M. The counting formula for the symplectic semi-characteristic under Assumption 1.2 is*

$$\kappa(M,\omega) = \textit{the number of zero points of } V \bmod 2. \tag{1-4}$$

Remark 1.6. By the Poincaré–Hopf index formula [24, Theorem 4.5], formula (1-4) also equals (mod 2) the Euler characteristic of the de Rham cohomology of M.

Remark 1.7. A special situation is when the de Rham cohomology class of ω is integral. Then, by [16, Theorem 7.1], Theorem 1.5 computes the classical Kervaire semi-characteristic (1-2) of the circle bundle over M induced by the line bundle associated with ω.

The main idea of the proof is that we find a skew-adjoint operator, as in Zhang's construction [23, (1.1)]. Then, we show that $\kappa(M,\omega)$ is equal to the Atiyah–Singer mod 2 index Definition 2.5 of this operator. Next, as in [23, (2.1)], we apply a Witten deformation [22, Section 2] and Bismut and Lebeau's asymptotic analysis

[6, Chapters VIII–X; 24, Chapters 4-7] to the operator, compute its mod 2 index, and then obtain Theorem 1.5.

A corollary of Theorem 1.5 is an Atiyah-type vanishing property:

Corollary 1.8 (compare [1, Theorem 4.1]). *The semi-characteristic $\kappa(M, \omega)$ vanishes when there is a nonvanishing smooth vector field on M.*

Another way to prove Corollary 1.8 without using Theorem 1.5 is described in Remark 4.2.

The converse of Corollary 1.8 is not true; see Example 5.2. This contrasts with the Euler characteristic of the de Rham cohomology of M.

Finally, although we have used the symplectic form ω to define $\kappa(M, \omega)$, a nondegenerate vector field always exists and is independent of ω. Thus:

Corollary 1.9. *The definition of $\kappa(M, \omega)$ is independent of the chosen symplectic form.*

Remark 1.10. The definition of the symplectic semi-characteristic can be assigned to any closed symplectic manifold without assuming that $\dim M = 4n$. However, by Example 5.4, the counting formula does not work when the dimension is $4n+2$. Thus, the $(4n+2)$-dimensional part of Question 1.1 is still open.

Outline. In Section 2, we review Clifford actions and find a skew-adjoint operator so that $\kappa(M, \omega)$ equals the Atiyah–Singer mod 2 index of this operator. In Section 3, we carry out necessary analytic details about this operator. In Section 4, we prove Theorem 1.5 based on those analytic details. In Section 5, we give some examples, and Section 6 describes further developments.

2. Clifford actions and operators

In this section, we clarify technical details about the Clifford actions of tangent vectors and introduce the skew-adjoint operator that we will work with.

After choosing an almost complex structure J on M, we let g be the Riemannian metric on M:

$$g(\cdot\,,\cdot) = \omega(\cdot\,, J\cdot).$$

We equip M with the orientation $\omega \wedge \cdots \wedge \omega$ and let $\star$ be the Hodge star operator. Letting $\mathrm{dvol} = \star 1$ be the volume form of M, we define the L^2-norm (and inner product) by

$$\|\alpha\| = \left(\int_M g(\alpha, \alpha)\, \mathrm{dvol}\right)^{1/2} \tag{2-1}$$

on $\Omega^k(M)$. We require that $\Omega^k(M) \perp \Omega^\ell(M)$ when $k \neq \ell$. For pairs of forms on

$C^k(M,\omega) = \Omega^k(M) \oplus \Omega^{k-1}(M)$, following [7, (2.2)], we define

$$g\left(\begin{bmatrix}\alpha\\ \beta\end{bmatrix}, \begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}\right) = g(\alpha,\alpha') + g(\beta,\beta').$$

As in (2-1), we have the L^2-norm (and inner product)

$$\left\| \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \right\| = \left(\int_M g(\alpha,\alpha)\,\mathrm{dvol} + \int_M g(\beta,\beta)\,\mathrm{dvol}\right)^{1/2} \tag{2-2}$$

on $C^k(M,\omega)$. We require $C^k(M,\omega) \perp C^\ell(M,\omega)$ when $k \neq \ell$.

Let d^* be the formal adjoint of d with respect to the inner product induced by (2-1), and

$$\omega^* \lrcorner : \Omega^k(M) \to \Omega^{k-2}(M)$$

be the adjoint of

$$\omega\wedge : \Omega^k(M) \to \Omega^{k+2}(M), \quad \alpha \mapsto \omega \wedge \alpha,$$

with respect to the same inner product. For convenience, we will omit the "$\lrcorner$" after ω^* and the "$\wedge$" after ω when there is no ambiguity. Recall the mapping cone complex

$$\partial_C : \Omega^k(M) \oplus \Omega^{k-1}(M) \to \Omega^{k+1}(M) \oplus \Omega^k(M)(\alpha,\beta) \mapsto \begin{bmatrix} d & \omega \\ 0 & -d\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}. \tag{2-3}$$

The formal adjoint of ∂_C is

$$\partial_C^* = \begin{bmatrix} d^* & 0 \\ \omega^* & -d^*\end{bmatrix}$$

with respect to the inner product induced by (2-2).

Proposition 2.1. *The kernel of the Dirac type operator $\partial_C + \partial_C^*$ and that of the Laplacian $(\partial_C + \partial_C^*)^2$ are isomorphic to the primitive cohomology of (M,ω). In particular,*

$$\ker\left((\partial_C + \partial_C^*)^2 : C^k(M,\omega) \to C^k(M,\omega)\right)$$

is isomorphic to the k-th primitive cohomology group.

Proof. We can check that (2-3) defines an elliptic complex [14, Definition 10.4.28]. By the properties of such a complex [14, Theorem 10.4.30], the complex defined by (2-3) satisfies the Hodge decomposition theorem. Therefore, the kernel of $(\partial_C + \partial_C^*)^2|_{C^k(M,\omega)}$ is isomorphic to the k-th primitive cohomology group. □

For any (globally or locally defined) vector field Y on M, we have two Clifford actions:

$$\hat{c}(Y) = Y^* \wedge + Y \lrcorner \quad \text{and} \quad c(Y) = Y^* \wedge - Y \lrcorner.$$

Given any oriented local orthonormal frame $e_1, \ldots, e_{4n}$ of TM, the Clifford action of the volume form dvol is expressed as

$$\hat{c}(\mathrm{dvol}) = \hat{c}(e_1) \cdots \hat{c}(e_{4n}).$$

This is independent of the choice of oriented local orthonormal frames. Following [1, Section 3], we lay out some interactions between the Hodge star, Clifford actions, and differential forms. Recall that the dimension of M is $4n$.

Lemma 2.2. *For all $\alpha \in \Omega^k(M)$, we have $\hat{c}(\mathrm{dvol})\alpha = (-1)^{k(k+1)/2} \star \alpha$.*

Proof. Let $e_1, \ldots, e_{4n}$ be an oriented local orthonormal frame. Suppose $\alpha = e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}$ and choose indices $j_1, \ldots, j_{4n-k}$ so that

$$e^*_{i_1} \wedge \cdots \wedge e^*_{i_k} \wedge e^*_{j_1} \wedge \cdots \wedge e^*_{j_{4n-k}} = e^*_1 \wedge \cdots \wedge e^*_{4n}.$$

Then we have

$$\begin{aligned}
\hat{c}(\mathrm{dvol})\alpha &= \hat{c}(e_{i_1})\hat{c}(e_{i_2}) \cdots \hat{c}(e_{i_k})\hat{c}(e_{j_1}) \cdots \hat{c}(e_{j_{4n-k}})\alpha \\
&= (-1)^{k(4n-k)}\hat{c}(e_{j_1}) \cdots \hat{c}(e_{j_{4n-1}})\hat{c}(e_{i_1})\hat{c}(e_{i_2}) \cdots \hat{c}(e_{i_k})\alpha \\
&= (-1)^{k(4n-k)+\frac{1}{2}(0+k-1)k} \star (e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}) \\
&= (-1)^{k(k+1)/2} \star \alpha.
\end{aligned}$$

The case of general α is straightforward. □

Lemma 2.3. *For all $\alpha \in \Omega^k(M)$, we have $\hat{c}(\mathrm{dvol})(\omega^* \lrcorner \alpha) = -\omega \wedge \hat{c}(\mathrm{dvol})\alpha$.*

Proof. Using the equality $\omega^* \lrcorner \alpha = (-1)^k \star \omega \star \alpha$ from [7, Section 2.1], we find

$$\begin{aligned}
\hat{c}(\mathrm{dvol})(\omega^* \lrcorner \alpha) &= \hat{c}(\mathrm{dvol})((-1)^k \star \omega \star \alpha) \\
&= (-1)^{\frac{1}{2}(k-2+1)(k-2)} \star ((-1)^k \star \omega \star \alpha) \quad \text{(by Lemma 2.2)} \\
&= (-1)^{\frac{1}{2}(k-2+1)(k-2)} \star \big((-1)^k \star \big(\omega \wedge (-1)^{\frac{1}{2}k(k+1)}\hat{c}(\mathrm{dvol})\alpha\big)\big) \\
&= (-1)^{k^2+1} \star\star\big(\omega \wedge (\hat{c}(\mathrm{dvol})\alpha)\big).
\end{aligned}$$

When k is odd, $\omega \wedge \hat{c}(\mathrm{dvol})\alpha$ is an odd-degree form, making $\star\star = -1$ and then $(-1)^{k^2+1}\star\star = -1$. Similarly, when k is even, we have $\star\star = 1$ and then $(-1)^{k^2+1}\star\star = -1$. Thus, we obtain $\hat{c}(\mathrm{dvol})(\omega^* \lrcorner \alpha) = -\omega \wedge (\hat{c}(\mathrm{dvol})\alpha)$. □

We set $\Omega^{\mathrm{even}}(M) := \bigoplus_{k=0}^{2n} \Omega^{2k}(M)$ and $\Omega^{\mathrm{odd}}(M) := \bigoplus_{k=0}^{2n} \Omega^{2k-1}(M)$. We also set

$$\underline{\Omega}(M) := \Omega^{\mathrm{even}}(M) \oplus \Omega^{\mathrm{odd}}(M) \quad \text{and} \quad \overline{\Omega}(M) := \Omega^{\mathrm{odd}}(M) \oplus \Omega^{\mathrm{even}}(M). \tag{2-4}$$

Using Lemmas 2.2 and 2.3, we obtain a skew-adjoint operator as follows.

Proposition 2.4. *Abbreviate $\hat{c}(\mathrm{dvol})$ as $\hat{c}_v$. The operator*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_v & \\ & \hat{c}_v \end{bmatrix} \begin{bmatrix} d+d^* & \omega \\ \omega^* & -d-d^* \end{bmatrix} \tag{2-5}$$

on $\underline{\Omega}(M)$ is skew-adjoint.

Proof. Using Lemma 2.2 and [21, Definition 6.1(2)], for all $\alpha \in \Omega^k(M)$, we have

$$\begin{aligned} \hat{c}_v(d+d^*)\alpha &= (-1)^{\frac{1}{2}(k+1)(k+2)} \star d\alpha + (-1)^{\frac{1}{2}(k-1)k} \star (-1) \star d \star \alpha \\ &= (-1)^{\frac{1}{2}(k+1)(k+2)} \star d\alpha + (-1)^{\frac{1}{2}(k-1)k+1} (-1)^{(4n-k+1)(4n-4n+k-1)} d \star \alpha \\ &= (-1)^{\frac{1}{2}(k+1)(k+2)} \star d\alpha + (-1)^{\frac{1}{2}k(k+1)} d \star \alpha. \end{aligned}$$

Similarly,

$$\begin{aligned} (d+d^*)\hat{c}_v\alpha &= d(-1)^{\frac{1}{2}k(k+1)} \star \alpha + d^*(-1)^{\frac{1}{2}k(k+1)} \star \alpha \\ &= (-1)^{\frac{1}{2}k(k+1)} d \star \alpha + (-1) \star d \star (-1)^{\frac{1}{2}k(k+1)} \star \alpha \\ &= (-1)^{\frac{1}{2}k(k+1)} d \star \alpha + (-1)^{\frac{1}{2}(k+1)(k+2)} \star d\alpha. \end{aligned}$$

Thus, we have

$$(d+d^*)\hat{c}_v = \hat{c}_v(d+d^*).$$

Now, since $\hat{c}_v^* = \hat{c}(e_{4n}) \cdots \hat{c}(e_1) = \hat{c}(e_1) \cdots \hat{c}(e_{4n}) = \hat{c}_v$, we find

$$\begin{aligned} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_v & \\ & \hat{c}_v \end{bmatrix} \begin{bmatrix} d+d^* & \omega \\ \omega^* & -d-d^* \end{bmatrix} \right)^* &= \begin{bmatrix} \hat{c}_v\omega^* & -\hat{c}_v(d+d^*) \\ \hat{c}_v(d+d^*) & \hat{c}_v\omega \end{bmatrix}^* \\ &= \begin{bmatrix} \omega\hat{c}_v & (d+d^*)\hat{c}_v \\ -(d+d^*)\hat{c}_v & \omega^*\hat{c}_v \end{bmatrix} \\ &= -\begin{bmatrix} \hat{c}_v\omega^* & -\hat{c}_v(d+d^*) \\ \hat{c}_v(d+d^*) & \hat{c}_v\omega \end{bmatrix} \\ &= -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_v & \\ & \hat{c}_v \end{bmatrix} \begin{bmatrix} d+d^* & \omega \\ \omega^* & -d-d^* \end{bmatrix}, \end{aligned}$$

where the second-to-last equality is justified by Lemma 2.3. Thus, the operator (2-5) is skew-adjoint. □

We recall the definition of the Atiyah–Singer mod 2 index. In [2, Theorem A], this invariant was defined for real Fredholm skew-adjoint operators. However, by functional calculus [10, Definition 1.13], one derives a version for real elliptic skew-adjoint operators:

Definition 2.5 [24, (7.5)]. Given a real elliptic skew-adjoint operator D, its Atiyah–Singer mod 2 index is the $\mathbb{Z}_2$-valued number

$$\mathrm{ind}_2 D := \dim \ker D \mod 2.$$

According to the definition of $\kappa(M,\omega)$ and the identification between kernels and cohomology groups, we have:

Corollary 2.6. *Keep the notation* $\hat{c}_v := \hat{c}(\mathrm{dvol})$. *The Atiyah–Singer mod* 2 *index of*

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{c}_v & \\ & \hat{c}_v \end{bmatrix} \begin{bmatrix} d+d^* & \omega \\ \omega^* & -d-d^* \end{bmatrix} \tag{2-6}$$

on $\underline{\Omega}(M)$ *is equal to* $\kappa(M,\omega)$.

Proof. This is verified using Definition 1.3 and Propositions 2.1 and 2.4. With $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, the operator (2-6) preserves the parity of the grading, mapping $\underline{\Omega}(M)$ into $\underline{\Omega}(M)$ and $\overline{\Omega}(M)$ into $\overline{\Omega}(M)$. When we restrict (2-6) to $\underline{\Omega}(M)$, its kernel counts half of the b_k^{ω}. □

Similar to the Fredholm index [8, Theorem 3.11], and as stated in [2, Proposition 5.1] and [3, Section 2], the mod 2 index of a real skew-adjoint elliptic operator on a compact manifold is homotopy invariant:

Proposition 2.7. *Given a real skew-adjoint elliptic operator* D *on* M, *the index* $\mathrm{ind}_2 D$ *is invariant under a continuous deformation of* D.

Remark 2.8. The invariance of ind_2 in [2, Proposition 5.1] is for bounded real skew-adjoint Fredholm operators. It is extended to the elliptic operators on M as follows [1, Section 4]: for D as in Proposition 2.7, the operator $1+(-D^2)$ is self-adjoint and positive. We have the compact operator

$$(1+(-D^2))^{-1/2}$$

defined by functional calculus [10, Definition 1.13]. Then,

$$D \circ (1+(-D^2))^{-1/2}$$

is a bounded real skew-adjoint Fredholm operator.

The next proposition gives us the skew-adjoint operator similar to [23, (1.1)].

Proposition 2.9. *The Atiyah–Singer mod* 2 *index of the skew-adjoint operator*

$$\begin{bmatrix} \frac{1}{2}(\omega^*-\omega) & -d-d^* \\ d+d^* & \frac{1}{2}(\omega-\omega^*) \end{bmatrix} \tag{2-7}$$

on $\underline{\Omega}(M)$ *is equal to* $\kappa(M,\omega)$.

Proof. We keep the notation $\hat{c}_v := \hat{c}(\mathrm{dvol})$. By Lemma 2.3, the operator

$$\frac{1}{2}\begin{bmatrix} \hat{c}_v(\omega^*+\omega) & \\ & \hat{c}_v(\omega+\omega^*) \end{bmatrix}$$

is skew-adjoint on $\underline{\Omega}(M)$. Then, by Corollary 2.6, we find

$$\begin{aligned}\kappa(M,\omega) &= \operatorname{ind}_2\left(\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\hat{c}_v&\\&\hat{c}_v\end{bmatrix}\begin{bmatrix}d+d^*&\omega\\ \omega^*&-d-d^*\end{bmatrix}\right)\\
&= \operatorname{ind}_2\begin{bmatrix}\hat{c}_v\omega^*&-\hat{c}_v(d+d^*)\\ \hat{c}_v(d+d^*)&\hat{c}_v\omega\end{bmatrix}\\
&= \operatorname{ind}_2\left(\begin{bmatrix}\hat{c}_v\frac{1}{2}(\omega^*-\omega)&-\hat{c}_v(d+d^*)\\ \hat{c}_v(d+d^*)&\hat{c}_v\frac{1}{2}(\omega-\omega^*)\end{bmatrix}+\frac{1}{2}\begin{bmatrix}\hat{c}_v(\omega^*+\omega)&\\&\hat{c}_v(\omega+\omega^*)\end{bmatrix}\right)\\
&= \operatorname{ind}_2\begin{bmatrix}\hat{c}_v\frac{1}{2}(\omega^*-\omega)&-\hat{c}_v(d+d^*)\\ \hat{c}_v(d+d^*)&\hat{c}_v\frac{1}{2}(\omega-\omega^*)\end{bmatrix} = \operatorname{ind}_2\begin{bmatrix}\frac{1}{2}(\omega^*-\omega)&-d-d^*\\ d+d^*&\frac{1}{2}(\omega-\omega^*)\end{bmatrix}.\end{aligned}$$

The second-to-last equality is justified by Proposition 2.7. □

3. Symplectic Witten deformation

In this section, we study the symplectic Witten deformation of the skew-adjoint operator (2-7) on $C^{\text{even}}(M,\omega) := \underline{\Omega}(M)$.

We let V be a nondegenerate smooth vector field on M. This means around each zero point p of V, given any small local chart with coordinates $x_1, y_1, \ldots, x_{2n}, y_{2n}$ satisfying $x_1(p) = \cdots = y_{2n}(p) = 0$, there is an $\mathbb{R}^{4n}$-valued smooth function B on the chart with order

$$O(x_1^2 + y_1^2 + \cdots + x_{2n}^2 + y_{2n}^2)$$

and a matrix $A \in \mathrm{GL}_{4n}(\mathbb{R})$ such that

$$V(x_1, y_1, \ldots, x_{2n}, y_{2n}) = \left[\partial_{x_1}, \partial_{y_1}, \ldots, \partial_{x_{2n}}, \partial_{y_{2n}}\right]\left(A\begin{bmatrix}x_1\\y_1\\ \vdots\\x_{2n}\\y_{2n}\end{bmatrix}+B\right). \tag{3-1}$$

Here, ∂_{x_i} and ∂_{y_i} are the local coordinate vector fields. For convenience, we let

$$\boldsymbol{x} = \begin{bmatrix}x_1\\y_1\\ \vdots\\x_{2n}\\y_{2n}\end{bmatrix}, \qquad \boldsymbol{x}^{\mathsf{T}} = [x_1, y_1, \ldots, x_{2n}, y_{2n}],$$

$$|\boldsymbol{x}| = \sqrt{x_1^2 + y_1^2 + \cdots + x_{2n}^2 + y_{2n}^2}.$$

Alternatively, if we write $\boldsymbol{x} = [\varphi_1, \ldots, \varphi_{4n}]^{\mathsf{T}}$, we have $|\boldsymbol{x}| = \sqrt{\varphi_1^2 + \cdots + \varphi_{4n}^2}$.

Lemma 3.1. *There is a smooth vector field X on M such that the zero set of X is the same as the zero set of V, and*

$$X = \left[\partial_{x_1}, \partial_{y_1}, \ldots, \partial_{x_{2n}}, \partial_{y_{2n}}\right] A\boldsymbol{x}$$

near each zero point p.

Proof. We will use a cutoff function to modify the vector field V near its zeros so that it agrees with a standard local model. We find a constant $C > 0$ such that

$$|B| \leq C\,|\boldsymbol{x}|^2 \tag{3-2}$$

on the local chart centered at p. Viewing A as an operator on the linear space $\mathbb{R}^{4n}$, we let $\|A\|$ be its operator norm. We choose a bump function σ such that

$$\operatorname{supp}(\sigma) \subseteq \left\{(x_1, \dots, y_{2n}) : (x_1^2 + \cdots + y_{2n}^2)^{1/2} < \|A^{-1}\|^{-1} \cdot C^{-1}\right\} \tag{3-3}$$

and $\sigma = 1$ near p. Now, we show that

$$\begin{aligned} X &= \sigma V + (1-\sigma)\left[\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_{2n}}, \partial_{y_{2n}}\right] A\boldsymbol{x} \\ &= \left[\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_{2n}}, \partial_{y_{2n}}\right](A\boldsymbol{x} + (1-\sigma)B) \end{aligned}$$

is the vector field we need. Indeed, by (3-2) and (3-3),

$$|\boldsymbol{x} + (1-\sigma)A^{-1}B| \geq |\boldsymbol{x}| - \|A^{-1}\| \cdot C \cdot |\boldsymbol{x}|^2 \geq 0.$$

The last "$\geq$" becomes "$=$" if and only if $\boldsymbol{x}$ is zero. Thus,

$$A\boldsymbol{x} + (1-\sigma)B = A \cdot (\boldsymbol{x} + (1-\sigma)A^{-1}B) = 0$$

if and only if we are at a zero point p of V. Therefore, the zero set of X coincides that of V. □

Inspired by [22, Section 2; 6, Chapters VIII–X; 24, Section 7.3; 23, Section 2], for a parameter $T > 0$, we use the vector field X to set up the Witten deformation

$$\mathbb{D}_T := \begin{bmatrix} \frac{1}{2}(\omega^* - \omega) & -d - d^* - T\hat{c}(X) \\ d + d^* + T\hat{c}(X) & \frac{1}{2}(\omega - \omega^*) \end{bmatrix} \tag{3-4}$$

of the operator (2-7) on $\underline{\Omega}(M)$. Let $\varepsilon > 0$ be a sufficiently small number. Around each zero point p of X, we choose a chart

$$U = \left\{(x_1, \dots, y_{2n}) : x_1^2 + y_1^2 + \cdots + x_{2n}^2 + y_{2n}^2 < (4\varepsilon)^2\right\} \tag{3-5}$$

centered at p and satisfying

(1) $\omega|_U = dx_1 \wedge dy_1 + \cdots + dx_{2n} \wedge dy_{2n}$,

(2) $g(\cdot,\cdot)|_U = dx_1^2 + dy_1^2 + \cdots + dx_{2n}^2 + dy_{2n}^2$, where g is the metric, and

(3) $X|_U = \left[\partial_{x_1}, \partial_{y_1}, \dots, \partial_{x_{2n}}, \partial_{y_{2n}}\right]A\boldsymbol{x}$.

We can obtain (1)–(3) as follows: Using [15, Theorem 8.1], we first choose a Darboux chart U centered at the zero point p with coordinates $x_1, y_1, \dots, x_{2n}, y_{2n}$ such that

$$\omega|_U = dx_1 \wedge dy_1 + \cdots + dx_{2n} \wedge dy_{2n}.$$

We then construct a metric g' on M such that

$$g'(\cdot\,,\cdot)|_U = dx_1^2 + dy_1^2 + \cdots + dx_{2n}^2 + dy_{2n}^2.$$

Next, following the proof of [15, Proposition 12.3], we use the polar decomposition together with g' to construct the almost complex structure J. Then, we let $g(\cdot\,,\cdot) = \omega(\cdot\,, J\cdot)$. The two metrics $g(\cdot\,,\cdot)$ and $g'(\cdot\,,\cdot)$ are different, but checking the polar decomposition, we have

$$g(\cdot\,,\cdot)|_U = g'(\cdot\,,\cdot)|_U = dx_1^2 + dy_1^2 + \cdots + dx_{2n}^2 + dy_{2n}^2.$$

Finally, the vector field X is guaranteed by Lemma 3.1.

Let

$$\boldsymbol{e}_i = [\underbrace{0,\ldots,0}_{2i-2},1,0,0,\ldots,0]^{\mathsf{T}} \quad\text{and}\quad \boldsymbol{f}_i = [\underbrace{0,\ldots,0,0}_{2i-1},1,0,\ldots,0]^{\mathsf{T}}$$

Inside U, we find that

$$\begin{aligned}(d+d^*+T\hat{c}(X))^2 = -\sum_{i=1}^{2n}\partial_{x_i}^2 - \sum_{i=1}^{2n}\partial_{y_i}^2 + T\sum_{i=1}^{2n} c(\partial_{x_i})\hat{c}\left([\partial_{x_1},\partial_{y_1},\ldots,\partial_{x_{2n}},\partial_{y_{2n}}]\,A\boldsymbol{e}_i\right)\\ + T\sum_{i=1}^{2n} c(\partial_{y_i})\hat{c}\left([\partial_{x_1},\partial_{y_1},\ldots,\partial_{x_{2n}},\partial_{y_{2n}}]\,A\boldsymbol{f}_i\right) + T^2\boldsymbol{x}^{\mathsf{T}}A^*A\boldsymbol{x}.\end{aligned}$$

Now, on $\mathbb{R}^{4n}$ with coordinates denoted by $x_1, y_1, \ldots, x_{2n}, y_{2n}$, we let

$$X_0 = [\partial_{x_1},\partial_{y_1},\ldots,\partial_{x_{2n}},\partial_{y_{2n}}]\,A\boldsymbol{x}.$$

Meanwhile, using the standard Euclidean metric

$$g_0 := dx_1^2 + dy_1^2 + \cdots + dx_{2n}^2 + dy_{2n}^2$$

on $\mathbb{R}^{4n}$, we have the L^2-norm (and inner product)

$$\|\alpha\| = \left(\int_{\mathbb{R}^{4n}} g_0(\alpha,\alpha)\, dx_1\wedge dy_1\wedge\cdots\wedge dx_{2n}\wedge dy_{2n}\right)^{1/2} \tag{3-6}$$

on the space $\Omega^k(\mathbb{R}^{4n})$ of smooth k-forms on $\mathbb{R}^{4n}$. For the standard symplectic form

$$\omega_0 = dx_1\wedge dy_1 + \cdots + dx_{2n}\wedge dy_{2n}$$

on $\mathbb{R}^{4n}$, we let

$$\omega_0^* \lrcorner = \partial_{y_1}\lrcorner\,\partial_{x_1}\lrcorner + \cdots + \partial_{y_{2n}}\lrcorner\,\partial_{x_{2n}}\lrcorner$$

be the adjoint of $\omega_0\wedge$.

Let L be the operator with the expression

$$-\sum_{i=1}^{2n}\partial_{x_i}^2-\sum_{i=1}^{2n}\partial_{y_i}^2+T\sum_{i=1}^{2n}c(\partial_{x_i})\hat{c}\left(\left[\partial_{x_1},\partial_{y_1},\dots,\partial_{x_{2n}},\partial_{y_{2n}}\right]A\boldsymbol{e}_i\right) \\ +T\sum_{i=1}^{2n}c(\partial_{y_i})\hat{c}\left(\left[\partial_{x_1},\partial_{y_1},\dots,\partial_{x_{2n}},\partial_{y_{2n}}\right]A\boldsymbol{f}_i\right)+T^2\boldsymbol{x}^{\mathsf{T}}A^*A\boldsymbol{x}$$

but defined on the space $\bigoplus_{k=0}^{4n}\Omega^k(\mathbb{R}^{4n})$ of smooth forms on $\mathbb{R}^{4n}$. As in [24, (4.23)], we let

$$L'=-\sum_{i=1}^{2n}\partial_{x_i}^2-\sum_{i=1}^{2n}\partial_{y_i}^2-T\cdot\operatorname{trace}(\sqrt{A^*A})+T^2\boldsymbol{x}^{\mathsf{T}}A^*A\boldsymbol{x}$$

and

$$L''=\operatorname{trace}(\sqrt{A^*A})+\sum_{i=1}^{2n}c(\partial_{x_i})\hat{c}\left(\left[\partial_{x_1},\partial_{y_1},\dots,\partial_{x_{2n}},\partial_{y_{2n}}\right]A\boldsymbol{e}_i\right) \\ +\sum_{i=1}^{2n}c(\partial_{y_i})\hat{c}\left(\left[\partial_{x_1},\partial_{y_1},\dots,\partial_{x_{2n}},\partial_{y_{2n}}\right]A\boldsymbol{f}_i\right).$$

Then, $L=L'+T\cdot L''$. Indeed, L' is the (rescaled) harmonic oscillator [17, Chapter 8, Section 6] on the space of square-integrable functions on $\mathbb{R}^{4n}$, and L'' is a nonnegative operator on the (real) vector space spanned by

$$\left\{dx_{i_1}\wedge\cdots\wedge dx_{i_r}\wedge dy_{j_1}\wedge\cdots\wedge dy_{j_s}: \\ 0\le i_1<\cdots<i_r\le 2n,\ 0\le j_1<\cdots<j_s\le 2n,\ 0\le r,s\le 2n\right\}. \tag{3-7}$$

The following result was proved using the properties of harmonic oscillators [loc. cit.] and equations (4.23)–(4.25) and Lemma 4.8 of [24].

Proposition 3.2 [24, Proposition 4.9]. *For any $T>0$, the kernel of L is one-dimensional and generated by*

$$\rho=\exp\left(-\tfrac{1}{2}T\boldsymbol{x}^{\mathsf{T}}\sqrt{A^*A}\boldsymbol{x}\right)\cdot\delta, \tag{3-8}$$

where δ is a certain linear combination (with real coefficients independent of T) of elements of (3-7). *The grading of δ is even if* $\det A>0$ *and odd if* $\det A<0$. *Each nonzero eigenvalue of L has the expression $\alpha\cdot T$, where α is a positive constant independent of T.*

Noticing that $\omega_0^*\lrcorner-\omega_0\wedge$ is skew-symmetric, we have:

Proposition 3.3. *There exists a unique smooth form η on $\mathbb{R}^{4n}$ such that*

$$\tfrac{1}{2}(\omega_0^*-\omega_0)\rho=\left(d+d^*+T\hat{c}(X_0)\right)\eta$$

and $\eta\perp\rho$. Here, $d+d^*$ is defined on $\bigoplus_{k=0}^{4n}\Omega^k(\mathbb{R}^{4n})$according to the L^2-norm of forms.

Proof. Recall the definition (3-8) of ρ. We notice that

$$g_0\left((\omega_0^* - \omega_0)\rho, \rho\right) = 0.$$

Therefore, $\frac{1}{2}(\omega_0^* - \omega_0)\rho$ is orthogonal to the kernel of L on $\bigoplus_{k=0}^{4n}\Omega^k(\mathbb{R}^{4n})$. Since $d + d^* + T\hat{c}(X_0)$ preserves the eigenspaces of L, we find

$$\eta = L^{-1} \circ \left(d + d^* + T\hat{c}(X_0)\right)\left(\tfrac{1}{2}(\omega_0^* - \omega_0)\rho\right). \tag{3-9}$$

Here, L^{-1} is the inverse of L restricted to the orthogonal complement of $\ker(L)$. The kernel of L is given by Proposition 3.2. See [6, (10.17)] for details about the inverse map L^{-1}. □

The next proposition will be used in the estimates of the spectrum of $-\mathbb{D}_T^2$.

Proposition 3.4. *There is a constant $C_1 \geq 0$ independent of T such that*

$$\|\eta\| = C_1 T^{-1/2} \cdot \|\rho\|. \tag{3-10}$$

Here, the L^2-norm is that on the space of forms on $\mathbb{R}^{4n}$.

Proof. If $\eta = 0$, we choose $C_1 = 0$. If $\eta \neq 0$, by Proposition 3.3, $\frac{1}{2}(\omega_0^* - \omega_0)\rho$ is nonzero, and we look at (3-9). We write $\frac{1}{2}(\omega_0^* - \omega_0)\rho$ as a finite sum of eigenvectors of L:

$$\tfrac{1}{2}(\omega_0^* - \omega_0)\rho = \sum_i K_i \cdot \exp\left(-\tfrac{1}{2}T\boldsymbol{x}^{\mathsf{T}}\sqrt{A^*A}\boldsymbol{x}\right) \cdot \delta_i,$$

where each K_i is a constant independent of T and each δ_i is an eigenvector of L'' in the span of (3-7), associated with an eigenvalue $\lambda_i > 0$. These δ_i and λ_i satisfy

$$g_0(\delta_i, \delta_j) = 0 \quad \text{and} \quad \lambda_i \neq \lambda_j \quad \text{when } i \neq j.$$

Then, we apply $L^{-1} \circ (d + d^* + T\hat{c}(X_0))$ to $\frac{1}{2}(\omega_0^* - \omega_0)\rho$. Since $d + d^* + T\hat{c}(X_0)$ preserves the eigenspaces of L, we obtain

$$\begin{aligned} &L^{-1} \circ (d + d^* + T\hat{c}(X_0))\left(\tfrac{1}{2}(\omega_0^* - \omega_0)\rho\right) \\ &\qquad = \sum_i \frac{1}{\lambda_i T} \cdot (d + d^* + T\hat{c}(X_0))\left(K_i \cdot \exp\left(-\tfrac{1}{2}T\boldsymbol{x}^{\mathsf{T}}\sqrt{A^*A}\boldsymbol{x}\right) \cdot \delta_i\right). \end{aligned}$$

One step further, considering the effect of $d + d^* + T\hat{c}(X_0)$, we find

$$\begin{aligned} \eta &= L^{-1} \circ (d + d^* + T\hat{c}(X_0))\left(\tfrac{1}{2}(\omega_0^* - \omega_0)\rho\right) \\ &= \sum_i \frac{1}{\lambda_i T} \cdot T \cdot \exp\left(-\tfrac{1}{2}T\boldsymbol{x}^{\mathsf{T}}\sqrt{A^*A}\boldsymbol{x}\right) \cdot \left(\sum_{j=1}^{2n} K_{ij} \cdot x_j \cdot \delta_{ij} + \sum_{j=1}^{2n} \tilde{K}_{ij} \cdot y_j \cdot \tilde{\delta}_{ij}\right) \\ &= \sum_i \frac{1}{\lambda_i} \cdot \exp\left(-\tfrac{1}{2}T\boldsymbol{x}^{\mathsf{T}}\sqrt{A^*A}\boldsymbol{x}\right) \cdot \left(\sum_{j=1}^{2n} K_{ij} \cdot x_j \cdot \delta_{ij} + \sum_{j=1}^{2n} \tilde{K}_{ij} \cdot y_j \cdot \tilde{\delta}_{ij}\right), \end{aligned}$$

where the K_{ij} and $\tilde{K}_{ij}$ are constants independent of T and the δ_{ij} and $\tilde{\delta}_{ij}$ are certain linear combinations (with real coefficients independent of T) of elements of (3-7). Thus, (3-10) is essentially the relation between

$$\left(\int_{\mathbb{R}^{4n}} x_i^2 \exp\left(-T|\boldsymbol{x}|^2\right) dx_1\, dy_1 \cdots dx_{2n}\, dy_{2n}\right)^{1/2} = \frac{\pi^n}{T^n} \cdot \frac{1}{\sqrt{2T}}$$

and

$$\left(\int_{\mathbb{R}^{4n}} \exp\left(-T|\boldsymbol{x}|^2\right) dx_1\, dy_1 \cdots dx_{2n}\, dy_{2n}\right)^{1/2} = \frac{\pi^n}{T^n}.$$

Their ratio gives us the factor $T^{-1/2}$. □

Remark 3.5. In the standard Witten deformation, the form ρ functions as a model for eigenforms associated with small eigenvalues of the deformed Laplacian. In this paper, the pair (ρ, η) plays a similar role in the mapping cone Witten deformation.

Now, using (3-6), we define the L^2-norm (and inner product)

$$\left\| \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\| = \left(\|\alpha\|^2 + \|\beta\|^2\right)^{1/2} \tag{3-11}$$

on $\underline{\Omega}(\mathbb{R}^{4n})$. (See (2-4) and recall the matrix A in (3-1) associated with the zero point p.) When $\det A > 0$, we study the orthogonal complement of $\operatorname{span}_{\mathbb{R}}\left(\left[\begin{smallmatrix} \rho \\ \eta \end{smallmatrix}\right]\right)$ in $\underline{\Omega}(\mathbb{R}^{4n})$ under the inner product induced by (3-11). Let $\left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] \in \underline{\Omega}(\mathbb{R}^{4n})$ be an L^2-element orthogonal to $\left[\begin{smallmatrix} \rho \\ \eta \end{smallmatrix}\right]$. We write

$$\alpha = r\rho + \alpha' \quad \text{and} \quad \beta = s\eta + \beta',$$

with $\alpha' \perp \rho$ and $\beta' \perp \eta$. Then, we have

$$r\|\rho\|^2 + s\|\eta\|^2 = 0. \tag{3-12}$$

Let $\|\cdot\|_1$ be the first Sobolev norm (see [14, Definition 10.2.7]) induced by (3-6). If $\|\alpha\|_1 < \infty$ and $\|\beta\|_1 < \infty$, we find that $\|\alpha'\|_1 < \infty$, $\|\beta'\|_1 < \infty$, and then

$$\begin{aligned} &\left\| \begin{bmatrix} \frac{1}{2}(\omega_0^* - \omega_0) & -d - d^* - T\hat{c}(X_0) \\ d + d^* + T\hat{c}(X_0) & \frac{1}{2}(\omega_0 - \omega_0^*) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right\| \\ &\geq \left\| \begin{bmatrix} & -d - d^* - T\hat{c}(X_0) \\ d + d^* + T\hat{c}(X_0) & \end{bmatrix} \begin{bmatrix} r\rho + \alpha' \\ s\eta + \beta' \end{bmatrix} \right\| \\ &\qquad\qquad - \left\| \frac{1}{2} \begin{bmatrix} \omega_0^* - \omega_0 & \\ & \omega_0 - \omega_0^* \end{bmatrix} \begin{bmatrix} r\rho + \alpha' \\ s\eta + \beta' \end{bmatrix} \right\| \\ &\geq \left\| \begin{bmatrix} (-d - d^* - T\hat{c}(X_0))(s\eta + \beta') \\ (d + d^* + T\hat{c}(X_0))(r\rho + \alpha') \end{bmatrix} \right\| - C_2 \left\| \begin{bmatrix} r\rho + \alpha' \\ s\eta + \beta' \end{bmatrix} \right\| \\ &= \left(\|(d + d^* + T\hat{c}(X_0))(s\eta + \beta')\|^2 + \|(d + d^* + T\hat{c}(X_0))\alpha'\|^2\right)^{1/2} - C_2 \left\| \begin{bmatrix} r\rho + \alpha' \\ s\eta + \beta' \end{bmatrix} \right\|, \end{aligned} \tag{3-13}$$

since $(d+d^*+T\hat{c}(X_0))\rho=0$. Using $\|\alpha'\|_1<\infty$, $\|\beta'\|_1<\infty$, and Proposition 3.2, we see that the right-hand side of (3-13) is

$$\begin{aligned}
&\geq C_3\sqrt{T}\|s\eta+\beta'\|+C_3\sqrt{T}\|\alpha'\|-C_2\left\|\begin{bmatrix} r\rho+\alpha' \\ s\eta+\beta' \end{bmatrix}\right\| \\
&= C_3\sqrt{T}\sqrt{\|s\eta\|^2+\|\beta'\|^2}+C_3\sqrt{T}\|\alpha'\|-C_2\left\|\begin{bmatrix} r\rho+\alpha' \\ s\eta+\beta' \end{bmatrix}\right\| \\
&\geq \mathfrak{X},
\end{aligned}$$

where

$$\mathfrak{X}:=C_4\sqrt{T}\|s\eta\|+C_4\sqrt{T}\|\beta'\|+C_3\sqrt{T}\|\alpha'\|-C_2\left\|\begin{bmatrix} r\rho+\alpha' \\ s\eta+\beta' \end{bmatrix}\right\|. \tag{3-14}$$

We estimate $\mathfrak{X}$ in two complementary cases. If $\eta=0$, then by (3-12), we find $r=0$ and then

$$\mathfrak{X}=C_4\sqrt{T}\|\beta'\|+C_3\sqrt{T}\|\alpha'\|-C_2\left\|\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}\right\| \geq C_5\sqrt{T}\left\|\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}\right\| = C_5\sqrt{T}\left\|\begin{bmatrix} r\rho+\alpha' \\ s\eta+\beta' \end{bmatrix}\right\|.$$

If $\eta\neq 0$, then

$$\begin{aligned}
\mathfrak{X}&=\frac{1}{2}C_4\sqrt{T}\|s\eta\|+\frac{1}{2}C_4\sqrt{T}\frac{|r|\cdot\|\rho\|^2}{\|\eta\|^2}\|\eta\|+C_4\sqrt{T}\|\beta'\|+C_3\sqrt{T}\|\alpha'\|-C_2\left\|\begin{bmatrix} r\rho+\alpha' \\ s\eta+\beta' \end{bmatrix}\right\| \\
&=\frac{1}{2}C_4\sqrt{T}\|s\eta\|+\frac{1}{2}C_4\sqrt{T}\|r\rho\|C_1^{-1}\sqrt{T}+C_4\sqrt{T}\|\beta'\|+C_3\Big[\sqrt{T}\|\alpha'\|-C_2\left\|\begin{bmatrix} r\rho+\alpha' \\ s\eta+\beta' \end{bmatrix}\right\| \\
&\geq C_5\sqrt{T}\left\|\begin{bmatrix} r\rho+\alpha' \\ s\eta+\beta' \end{bmatrix}\right\|.
\end{aligned}$$

This takes care of the case $\det A>0$. When $\det A<0$, we replace $\left[\begin{smallmatrix}\rho\\ \eta\end{smallmatrix}\right]$ by $\left[\begin{smallmatrix}\eta\\ \rho\end{smallmatrix}\right]$ and repeat the argument. Then, we summarize:

Proposition 3.6. *There exists a constant $C_5>0$ such that when* $\det A>0$ (*resp. when* $\det A<0$), *for all sufficiently large T, we have*

$$\left\|\begin{bmatrix} \frac{1}{2}(\omega_0^*-\omega_0) & -d-d^*-T\hat{c}(X_0) \\ d+d^*+T\hat{c}(X_0) & \frac{1}{2}(\omega_0-\omega_0^*) \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right\| \geq C_5\sqrt{T}\left\|\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right\|$$

whenever $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\in\underline{\Omega}(\mathbb{R}^{4n})$ is orthogonal to $\left[\begin{smallmatrix}\rho\\ \eta\end{smallmatrix}\right]$ $\big($resp. $\left[\begin{smallmatrix}\eta\\ \rho\end{smallmatrix}\right]\big)$ and satisfies $\|\alpha\|_1<\infty$ and $\|\beta\|_1<\infty$.

Following [6] and [24], based on Propositions 3.2–3.6, we apply the asymptotic analysis to carry out the estimates about $\mathbb{D}_T$. Recall from (3-5) the chart U around each zero point p of X. For each zero point p, we pick a bump function $\gamma: M\to\mathbb{R}$ such that

$$\operatorname{supp}(\gamma)\subseteq U(2\varepsilon):=\big\{(x_1,\dots,y_{2n}): x_1^2+\cdots+y_{2n}^2<(2\varepsilon)^2\big\},$$

and $\gamma = 1$ on
$$U(\varepsilon) := \big\{(x_1, \dots, y_{2n}) : x_1^2 + \cdots + y_{2n}^2 < \varepsilon^2\big\}.$$
For each zero point p, we let
$$\rho_p = \gamma \cdot \exp\big(-\tfrac{1}{2} T \boldsymbol{x}^{\mathsf{T}} \sqrt{A^* A}\boldsymbol{x}\big) \cdot \delta$$
and $\eta_p = \gamma \cdot \eta$. As in [6, Definition 9.4] and [24, (4.36)], we let
$$\begin{aligned} E_{T,0} &:= \operatorname{span}_{\mathbb{R}}\big\{\big[\begin{smallmatrix}\rho_p\\ \eta_p\end{smallmatrix}\big] : p \text{ is a zero point of } X\big\},\\ E_{T,1} &:= \operatorname{span}_{\mathbb{R}}\big\{\big[\begin{smallmatrix}\eta_p\\ \rho_p\end{smallmatrix}\big] : p \text{ is a zero point of } X\big\},\\ E_T &:= E_{T,0} \oplus E_{T,1}. \end{aligned}$$

Let $E_T^\perp$ be the orthogonal complement of E_T in $\underline{\Omega}(M)$ and p_T (resp. $p_T^\perp$) be the orthogonal projection from $\underline{\Omega}(M)$ to E_T (resp. $E_T^\perp$).

Recall the operator
$$\mathbb{D}_T := \begin{bmatrix} \frac{1}{2}(\omega^* - \omega) & -d - d^* - T\hat{c}(X) \\ d + d^* + T\hat{c}(X) & \frac{1}{2}(\omega - \omega^*) \end{bmatrix}$$
on $\underline{\Omega}(M)$. There is a constant $C_6 > 0$ such that
$$\begin{aligned} \left\| \mathbb{D}_T \begin{bmatrix} \rho_p \\ \eta_p \end{bmatrix} \right\| &= \left\| \begin{bmatrix} \frac{1}{2}(\omega^* - \omega) & -d - d^* - T\hat{c}(X) \\ d + d^* + T\hat{c}(X) & \frac{1}{2}(\omega - \omega^*) \end{bmatrix} \begin{bmatrix} \gamma\rho \\ \gamma\eta \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} -c(d\gamma)\eta \\ c(d\gamma)\rho + \frac{1}{2}(\omega - \omega^*)\gamma\eta \end{bmatrix} \right\| \\ &\le \left\| \begin{bmatrix} c(d\gamma)\eta \\ c(d\gamma)\rho \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 \\ \frac{1}{2}(\omega - \omega^*)\gamma\eta \end{bmatrix} \right\| \le C_6 \left\| \begin{bmatrix} \rho_p \\ \eta_p \end{bmatrix} \right\| \end{aligned}$$
when T is sufficiently large. Summarizing this estimate, we get:

Proposition 3.7. *There is a constant $C_6 > 0$ such that, when T is sufficiently large,*
$$\big\|\mathbb{D}_T \big[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\big]\big\| \le C_6 \big\|\big[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\big]\big\|$$
for all $\big[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\big] \in E_T$.

Now, if $\big[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\big] \in E_T^\perp$, we have the following estimate similar to those in Theorem 9.11 of [6] and Proposition 4.12 of [24]:

Proposition 3.8. *There exists a constant $C_7 > 0$ such that when T is sufficiently large,*
$$\big\|\mathbb{D}_T \big[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\big]\big\| \ge C_7 \sqrt{T} \big\|\big[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\big]\big\|$$
for all $\big[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\big] \in E_T^\perp$.

Proof. We perform three steps:

Step 1: If $\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]$ is supported outside all the $U(2\varepsilon)$'s, the minimum of $g(X, X)$ is greater than 0. Then, as in [24, Proposition 4.7], we find

$$\begin{aligned}
&\left\|\mathbb{D}_T\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\| \\
&\geq \left\|\begin{bmatrix} & -d-d^*-T\hat{c}(X)\\ d+d^*+T\hat{c}(X) & \end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\| - \left\|\begin{bmatrix}\frac{1}{2}(\omega^*-\omega) & \\ & \frac{1}{2}(\omega-\omega^*)\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\| \\
&\geq C_8T\left\|\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\| - C_9\left\|\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\|.
\end{aligned}$$

Step 2: If $\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]$ is supported inside the chart U centered at some zero point p, we view $\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]$ as an element in $\underline{\Omega}(\mathbb{R}^{4n})$. Let p'_T be the orthogonal projection from $\underline{\Omega}(\mathbb{R}^{4n})$ to the one-dimensional space generated by $\left[\begin{smallmatrix}\rho\\\eta\end{smallmatrix}\right]$. Letting $\langle\,\cdot\,,\cdot\,\rangle$ denote the inner product induced by (3-11), we have

$$\begin{aligned}
p'_T\begin{bmatrix}\alpha\\\beta\end{bmatrix} &= \frac{1}{\sqrt{\|\rho\|^2+\|\eta\|^2}}\left\langle\begin{bmatrix}\rho\\\eta\end{bmatrix},\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\rangle\cdot\left(\frac{1}{\sqrt{\|\rho\|^2+\|\eta\|^2}}\begin{bmatrix}\rho\\\eta\end{bmatrix}\right) \\
&= \frac{1}{\|\rho\|^2+\|\eta\|^2}\left\langle\begin{bmatrix}\rho\\\eta\end{bmatrix},\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\rangle\cdot\begin{bmatrix}\rho\\\eta\end{bmatrix} \\
&= \frac{1}{\|\rho\|^2+\|\eta\|^2}\int_M (1-\gamma)\cdot g\left(\begin{bmatrix}\rho\\\eta\end{bmatrix},\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right)\mathrm{dvol}\cdot\begin{bmatrix}\rho\\\eta\end{bmatrix},
\end{aligned}$$

since $\left\langle\left[\begin{smallmatrix}\gamma\rho\\\gamma\eta\end{smallmatrix}\right],\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\right\rangle = 0$. Then, we find, by Cauchy–Schwarz and comparing $\|\rho\|$ with $\exp(-C_{10}\varepsilon^2T)$,

$$\begin{aligned}
\left\|p'_T\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\| &= \frac{1}{\sqrt{\|\rho\|^2+\|\eta\|^2}}\left|\int_M (1-\gamma)\cdot g\left(\begin{bmatrix}\rho\\\eta\end{bmatrix},\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right)\mathrm{dvol}\right| \\
&\leq \frac{1}{\sqrt{\|\rho\|^2+\|\eta\|^2}}\cdot\exp(-C_{10}\varepsilon^2T)\int_{|x|\leq 4\varepsilon} g\left(\begin{bmatrix}\alpha\\\beta\end{bmatrix},\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right)^{1/2}\mathrm{dvol} \\
&\leq \frac{\sqrt{T}}{\sqrt{T+C_1^2}}\cdot\|\rho\|^{-1}\cdot\exp(-C_{10}\varepsilon^2T)\cdot\left\|\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\| \\
&\leq \exp(-C_{11}T)\cdot\left\|\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\|.
\end{aligned}$$

By Proposition 3.6, we find

$$\left\|\mathbb{D}_T\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\| \geq C_5\sqrt{T}\left\|\begin{bmatrix}\alpha\\\beta\end{bmatrix}-p'_T\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\| \geq C_5\sqrt{T}\cdot(1-\exp(-C_{11}T))\cdot\left\|\begin{bmatrix}\alpha\\\beta\end{bmatrix}\right\|.$$

Step 3: For a general $\left[\begin{smallmatrix}\alpha\\\beta\end{smallmatrix}\right]\in E_T^{\perp}$ supported on M, we combine what we have shown in Steps 1 and 2, following the standard procedure used in Step 3 of the proof of [24, Proposition 4.12]. □

Noticing that $\mathbb{D}_T$ is skew-adjoint, we have:

Proposition 3.9. *The operator* $-\mathbb{D}_T^2$ *is self-adjoint and nonnegative. When* T *is sufficiently large*, *the eigenvalues of* $-\mathbb{D}_T^2$ *lie in the union* $[0, C_6^2] \cup [C_7^2 T, +\infty)$.

Proof. This is a combination of Propositions 3.7 and 3.8, following the same pattern as in the proof of [25, Lemma 5.3]. Since there is no essential spectrum here, we only need a simplified procedure as in the proof of [26, Proposition 6.18]. □

4. The counting formula

We now prove the counting formula (1-4) stated in Theorem 1.5. Let $\tilde{E}_T$ be the sum of eigenspaces of $-\mathbb{D}_T^2$ on $\underline{\Omega}(M)$ associated with eigenvalues in $[0, C_6^2]$. Then,

$$\begin{aligned}\kappa(M,\omega) &= \operatorname{ind}_2\big(\mathbb{D}_T \text{ on } \underline{\Omega}(M)\big)\\ &= \dim\ker\big(-\mathbb{D}_T^2 \text{ on } \underline{\Omega}(M)\big) \bmod 2\\ &= \dim\ker\big(\mathbb{D}_T : \tilde{E}_T \to \tilde{E}_T\big) \bmod 2,\end{aligned}$$

since each eigenspace of $-\mathbb{D}_T^2$ is invariant under $\mathbb{D}_T$. By [9, Section 8.16], every $r \times r$ skew-symmetric matrix has Atiyah–Singer mod 2 index equal to the parity of r. Thus,

$$\kappa(M,\omega) = \dim \tilde{E}_T \bmod 2.$$

Now, to prove Theorem 1.5, we only need to show that $\dim E_T = \dim \tilde{E}_T$.

Proposition 4.1. *We have* $\dim E_T = \dim \tilde{E}_T$ *when* T *is sufficiently large.*

Proof. Recall the L^2-norm (2-2) on $\underline{\Omega}(M)$. We let

$$\tilde{P}_T : \underline{\Omega}(M) \to \tilde{E}_T$$

be the orthogonal projection to $\tilde{E}_T$. Then, for any $h \in E_T$, we obtain

$$\begin{aligned}\|h - \tilde{P}_T h\| &\le \frac{1}{C_7\sqrt{T}} \|\mathbb{D}_T(h - \tilde{P}_T h)\| \qquad \text{(by Proposition 3.9)}\\ &\le \frac{1}{C_7\sqrt{T}} \big(\|\mathbb{D}_T h\| + \|\mathbb{D}_T \tilde{P}_T h\|\big)\\ &\le \frac{1}{C_7\sqrt{T}} \cdot C_6 \cdot (\|h\| + \|h\|) \qquad \text{(by Proposition 3.9).}\end{aligned}$$

Thus, when T is large, $\tilde{P}_T$ maps E_T injectively into $\tilde{E}_T$, meaning that $\dim \tilde{E}_T \ge \dim E_T$.

We prove the opposite inequality by contradiction. As in [24, (5.32)], suppose that $\dim \tilde{E}_T > \dim E_T$. We pick some $\varphi \in \tilde{E}_T$ such that φ is orthogonal to the space $\tilde{P}_T E_T$. Let $\langle \cdot, \cdot \rangle$ denote the inner product induced by (2-2). For any zero point p of X and the associated $\left[\begin{smallmatrix}\rho_p\\ \eta_p\end{smallmatrix}\right]$ (or $\left[\begin{smallmatrix}\eta_p\\ \rho_p\end{smallmatrix}\right]$, depending on the sign of $\det A$), we have

$$\left\langle \varphi, \begin{bmatrix}\rho_p\\ \eta_p\end{bmatrix}\right\rangle = \left\langle \varphi, \begin{bmatrix}\rho_p\\ \eta_p\end{bmatrix}\right\rangle - \left\langle \varphi, \tilde{P}_T \begin{bmatrix}\rho_p\\ \eta_p\end{bmatrix}\right\rangle = \left\langle \varphi, \begin{bmatrix}\rho_p\\ \eta_p\end{bmatrix}\right\rangle - \left\langle \varphi, \begin{bmatrix}\rho_p\\ \eta_p\end{bmatrix}\right\rangle = 0,$$

the middle equality being a consequence of $\varphi \in \tilde{E}_T$. Thus, $\varphi \in E_T^{\perp}$. Using Proposition 3.8, we get

$$\|\mathbb{D}_T\varphi\| \geq C_7\sqrt{T}\|\varphi\|,$$

contradicting to the fact that $\varphi \in \tilde{E}_T$ (this space, we recall, is the sum of the eigenspaces of $-\mathbb{D}_T^2$ associated with eigenvalues in $[0, C_6^2]$.) Therefore, $\tilde{E}_T$ is isomorphic to E_T when T is sufficiently large. $\square$

Recall that X is an adjusted version of V. By Proposition 4.1, we finally have

$$\begin{aligned}\kappa(M,\omega) &= \dim \tilde{E}_T \mod 2\\ &= \dim E_T \mod 2\\ &= \text{the number of zero points of the adjusted vector field } X \mod 2\\ &= \text{the number of zero points of the original vector field } V \mod 2,\end{aligned}$$

and this completes the proof of Theorem 1.5.

Remark 4.2. We get Corollary 1.8 from Theorem 1.5. An alternative to using Theorem 1.5 involves applying Atiyah's perturbation technique from [1, Section 4] to prove Corollary 1.8 directly. Let V be a vector field with $g(V, V) = 1$ on M. We perturb the operator

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{c}(\mathrm{dvol}) & \\ & \hat{c}(\mathrm{dvol}) \end{bmatrix} \begin{bmatrix} d + d^* & \omega \\ \omega^* & -d - d^* \end{bmatrix}$$

on $\underline{\Omega}(M)$ into the operator

$$D' = D + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{c}(V) & \\ & -\hat{c}(V) \end{bmatrix} D \begin{bmatrix} \hat{c}(V) & \\ & -\hat{c}(V) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Once we verify that $\mathrm{ind}_2 D = \mathrm{ind}_2 D'$ and that $\ker D'$ admits a complex structure given by

$$\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{c}(V) & \\ & -\hat{c}(V) \end{bmatrix}\right)^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

we conclude that $\dim \ker D'$ is even, and therefore $\kappa(M, \omega) = 0$.

5. Examples

We illustrate with some examples, most of which have already been studied in other papers; we just adapt them to the computation of symplectic semi-characteristics.

Example 5.1. We equip $M = \mathbb{C}\mathrm{P}^2$ with the Fubini–Study form [15, Homework 12]. According to [7, Example 4.2],

$$b_0^\omega = 1, \quad b_2^\omega = 0, \quad b_4^\omega = 0,$$

meaning that $\kappa(M,\omega)=1$. These b_i^ω are computed using a Morse function with three critical points, together with the associated cone Morse cochain complex [7, Definition 1.2]. By the counting formula (1-4), we can also use the three critical points of this perfect Morse function to find $\kappa(M,\omega)=1$.

Example 5.2. Let $M=\mathbb{S}^2\times\mathbb{S}^2$ equipped with the standard symplectic structure. Recall that we have a height function h (see [4, Example 3.4]) on $\mathbb{S}^2$ with two critical points. Then,

$$f:\mathbb{S}^2\times\mathbb{S}^2\to\mathbb{R},\quad (p,q)\mapsto h(p)+h(q),$$

is a Morse function on $\mathbb{S}^2$ with four critical points. Thus, in this case, $\kappa(M,\omega)=0$.

As we know, the Euler characteristic of the de Rham cohomology of $\mathbb{S}^2\times\mathbb{S}^2$ is 4, meaning that $\mathbb{S}^2\times\mathbb{S}^2$ does not admit a nonvanishing vector field. However, as we see, its symplectic semi-characteristic is 0. Thus, in terms of judging the existence of nonvanishing vector fields, the symplectic semi-characteristic (1-3) of the primitive cohomology is a weak substitute for the Euler characteristic of the de Rham cohomology.

Example 5.3. As in [18, Section 3.4] and [16, (5.3)], we let $\sim$ be the identification

$$(x_1,x_2,x_3,x_4)\sim(x_1+a,x_2+b,x_3+c,x_4+d-bx_3)\quad(\text{when } a,b,c,d\in\mathbb{Z})$$

on $\mathbb{R}^4$. Then, the Kodaira–Thurston fourfold is equal to $\mathbb{R}^4/{\sim}$. Let $M=\mathbb{R}^4/{\sim}$ equipped with the symplectic form ω given in [18, (3.26)] and [16, (5.4)]. We know that $\kappa(M,\omega)=0$ from the tables of primitive cohomology groups in [18, Section 3.4] and [16, Section 5.4]. Since $\mathbb{R}^4/{\sim}$ has a globally defined tangent vector field ∂_{x_1}, we also obtain $\kappa(M,\omega)=0$ according to Corollary 1.8.

Example 5.4. We briefly mention the $(4n+2)$-dimensional case. Let $M=\mathbb{T}^2$ be equipped with the standard symplectic form. Since $\mathbb{T}^2$ is Kähler, we use the formula [7, (4.4)] to find $b_0^\omega=1$, $b_2^\omega=2$, and so $\kappa(M,\omega)=1$.

However, we know there is a height function [13, Part I, Section 1] with 4 nondegenerate critical points on $\mathbb{T}^2$. This means our Theorem 1.5 does not apply to the $(4n+2)$-dimensional case.

6. Discussion and perspectives

We now mention some possible extensions of this project. The deformation replaces $d+d^*$ by $d+d^*+T\hat{c}(V)$ and does not perturb ω. This preserves the symplectic information and relates $\kappa(M,\omega)$ to the primitive forms given by the Lefschetz decomposition. However, as $\kappa(M,\omega)$ is unchanged when replacing ω by another symplectic form, it is also natural to consider replacing ω by $\omega\wedge\cdots\wedge\omega$ or by other forms.

The form $\omega\wedge\cdots\wedge\omega$ gives the semi-characteristic of the 1-filtered cohomology [16; 20]. In a follow-up work [27], we use $\omega\wedge\omega$ on a $(4n+2)$-dimensional closed symplectic manifold. Then, the associated semi-characteristic vanishes, which is exactly the parity of the de Rham Euler characteristic. This provides more evidence that κ relies more on M than on the form.

Also, from an index-theoretic perspective and without involving the primitive cohomology, we may perturb ω into $s\omega$ and let s change from 0 to 1. The study could thus be done for any closed orientable manifold equipped with a closed homogeneous form. For this case, we will give the formula and the analysis in a future joint work with S. Xu, with assumptions on both $\dim M$ and the degree of the form. A similar perturbation using s^{-1} instead of s was carried out in a recent work [28] on the mapping cone Morse theory for any closed oriented manifold equipped with a closed homogeneous smooth form. This s^{-1} preserves the cohomology.

To conclude, we briefly discuss the K-theoretic background of this study. By [3, Theorem 2.3], the mod 2 index is equivalent to the map

$$KO^{-1}(TM) \to KO^{-1}(\text{point}) \cong \mathbb{Z}_2. \tag{6-1}$$

When $\dim M = 4n$, we have a skew-adjoint elliptic operator (3-4) whose skew-symbol class is in $KO^{-1}(TM)$ and then mapped to $\kappa(M, \omega)$. In the $(4n+2)$-dimensional case, if we continue the current pattern of construction, we cannot obtain a skew-adjoint elliptic operator on $\underline{\Omega}(M)$ whose skew-symbol class is mapped to $\kappa(M, \omega)$. Thus, in the $4n+2$ case, it could be worthwhile to study the geometric meaning of the image of the skew-symbol class of (2-7) under (6-1) [23, Theorem 3.2], and whether (6-1) can be generalized to give the $(4n+2)$-dimensional case a KO-valued index result of the semi-characteristic.

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HAO ZHUANG
BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH
PEKING UNIVERSITY
BEIJING
CHINA
hzhuang@pku.edu.cn

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