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University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

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Vienna, Austria
matthias.aschenbrenner@univie.ac.at

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Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

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Department of Mathematics
Kyoto University
Kyoto 606-8502, Japan
atsushi.ichino@gmail.com

Kefeng Liu
School of Sciences
Chongqing University of Technology
Chongqing 400054, China
liu@math.ucla.edu

Sucharit Sarkar
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
sucharit@math.ucla.edu

Dimitri Shlyakhtenko
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
shlyakht@ipam.ucla.edu

Ruixiang Zhang
Department of Mathematics
University of California
Berkeley, CA 94720-3840
ruixiang@berkeley.edu

PRODUCTION

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LOCAL SMOOTHING ESTIMATES FOR BILINEAR FOURIER INTEGRAL OPERATORS

DUVÁN CARDONA

We formulate a local smoothing conjecture for bilinear Fourier integral operators in every dimension $d \geq 2$, derived from the celebrated linear case due to Sogge, which we refer to as the *bilinear smoothing conjecture*. We show that the linear local smoothing conjecture implies this bilinear version. As a consequence of our approach and due to the recent progress on the subject, we establish local smoothing estimates for bilinear Fourier integral operators in dimension $d = 2$, that is, on $\mathbb{R}^2_x \times \mathbb{R}_t$. Also, partial progress is presented for the high-dimensional case $d \geq 3$. In particular, our method allows us to deduce that the bilinear local smoothing conjecture holds for all odd dimensions d.

1. Introduction

We propose a conjecture on local smoothing estimates for bilinear Fourier integral operators (FIOs) derived from the linear case observed by Sogge [1991]. Recent progress in the field (see [Beltran et al. 2020; Gao et al. 2023]) allows our methods to fully prove the conjecture for $d = 2$ and to advance the problem partially for $d \geq 3$. In particular, we deduce that the bilinear local smoothing conjecture is true when d is odd. We first recall the linear local smoothing conjecture [Sogge 1991] and then introduce its bilinear form. Our main result is Theorem 2.6; our approach investigates the smoothing properties of high and of low frequencies of a bilinear Fourier integral operator. For the analysis of the high frequency portion, among

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other things, we make use of a boundedness theorem for maximal functions proved by Bourgain [1985].

According to Hörmander [1971], an FIO T of order $m \in \mathbb{R}$, with real phase function ϕ, is a continuous linear operator $T : C_0^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d \times \mathbb{R})$ of the form

$$T_a^\phi f(x,t) = \int_{\mathbb{R}^d} e^{i\phi(x,t,\xi)} a(x,t,\xi)\hat{f}(\xi)\, d\xi, \tag{1-1}$$

where $\hat{f}$ is the Fourier transform of $f \in C_0^\infty(\mathbb{R}^d)$. The symbol $a := a(x,t,\xi)$ is of Kohn–Nirenberg order $m \in \mathbb{R}$, that is, a satisfies the symbol inequalities

$$|\partial_x^\beta \partial_\xi^\alpha a(x,t,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|} \quad \text{for all } \alpha,\beta \in \mathbb{N}_0. \tag{1-2}$$

The symbol a has compact support in (x,t). We assume that in an open neighbourhood V of $\pi(\mathrm{supp}(a)) := \{(x,t) : (x,t,\xi) \in \mathrm{supp}(a)\}$ (with $\overline{V}$ being a compact set) the real-valued phase function ϕ is homogeneous of degree 1 in $\xi \neq 0$, smooth in $(x,t,\xi) \in V \times (\mathbb{R}^d \setminus \{0\})$, and satisfies the nondegeneracy condition

$$\det\big((\partial^2_{x,\xi}\phi(x,t,\eta))\big)_{1\le i,j\le d} \neq 0 \quad \text{for all } (x,t,\xi) \in \overline{V} \times (\mathbb{R}^d \setminus \{0\}). \tag{1-3}$$

The corresponding L^p-theory for these operators can be summarised as follows; see the works of Tao [2004] and Seeger, Sogge and Stein [Seeger et al. 1991].

Theorem 1.1. *Let $t \in \mathbb{R}$ be fixed in the support of the symbol $a := a(x,t,\xi)$. Let $T = T^{\phi(\cdot,t,\cdot)}_{a(\cdot,t,\cdot)}$ be a Fourier integral operator of order $m \in \mathbb{R}$. Then*:

(1) *If $m = -(d-1)/2$ and T is of weak $(1,1)$ type, then there is $C_t > 0$ such that*

$$\|Tf\|_{L^{1,\infty}(\mathbb{R}^d_x)} \le C_t \|f\|_{L^1(\mathbb{R}^d_x)} \quad \textit{for all } f \in L^1(\mathbb{R}^d). \tag{1-4}$$

(2) *Let $1 < p < \infty$. Let T be an FIO of order $m = -(d-1)\big|\frac{1}{2} - \frac{1}{p}\big|$. Then T is bounded from $L^p(\mathbb{R}^d_x)$ to $L^p(\mathbb{R}^d_x)$. If $p = 1$, then T is bounded from the Hardy space $H^1(\mathbb{R}^d_x)$ to $L^1(\mathbb{R}^d_x)$. As in* (1-4), *these operator norms depend on t.*

The case $p = 2$ in Theorem 1.1 was proved in [Duistermaat and Hörmander 1972]; see also [Èskin 1970]. Other references related to the global boundedness of Fourier integral operators are presented in Section 3. For now, we emphasise that, in terms of the L^p-regularity, Theorem 1.1 shows that Fourier integral operators lose $s = (d-1)\big|\frac{1}{2} - \frac{1}{p}\big|$ derivatives. In other words, on the scale of Sobolev spaces L^p_s, a Fourier integral operator $T = T^{\phi(\cdot,t,\cdot)}_{a(\cdot,t,\cdot)}$ of order $m \in \mathbb{R}$ satisfies the time-dependent estimate

$$\|Tf\|_{L^p_s(\mathbb{R}^d_x)} \lesssim_t \|f\|_{L^p_{s+m+(d-1)|1/2-1/p|}(\mathbb{R}^d_x)}, \quad 1 < p < \infty. \tag{1-5}$$

It was first observed by Sogge [1991], in the context of the wave equation, that averaging in time on the left-hand side of (1-5) allows one to improve the loss of

regularity. This observation led to the formulation of the (linear) *local smoothing conjecture* for the wave operator. If true, this conjecture would represent a remarkable breakthrough, as it formally implies several other major open problems in harmonic analysis, including the Bochner–Riesz, Fourier restriction, and Kakeya conjectures; see [Tao 1999].

In the setting of Fourier integral operators, the main assumption on the phase to formulate the linear local smoothing conjecture is the following *cinematic curvature condition*.

Definition 1.2 (cinematic curvature condition). We say that

$$\phi := \phi(x,t,\xi) \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}_t \times (\mathbb{R}^d \setminus \{0\}))$$

satisfies the *cinematic curvature condition* on a compact set $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}_t$ if the following conditions are satisfied:

(i) The phase ϕ is nondegenerate, that is,

$$\operatorname{rank}(\partial^2_{x,\eta}\phi(x,t,\eta)) = d, \quad (x,t) \in \mathcal{K}, \quad \eta \in \mathbb{R}^d \setminus \{0\}. \tag{1-6}$$

(ii) Consider the Gauss mapping $G : \mathcal{K} \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{S}^d$, where

$$G(x,t,\eta) = \frac{G_0(x,t,\eta)}{|G_0(x,t,\eta)|} \quad \text{and} \quad G_0(x,t,\eta) := \bigwedge_{1\le j\le d} \partial_{\eta_j}\partial_{x,t}\phi(x,t,\eta).$$

The rank condition

$$\operatorname{rank}\big(\partial^2_{\eta\eta}\langle \partial_{x,t}\phi(x,t,\eta), G(z,\eta_0)\rangle|_{\eta=\eta_0}\big) = d-1 \tag{1-7}$$

holds for every $(x,t,\eta_0) \in \mathcal{K} \times \mathbb{R}^d$, $\eta_0 \neq 0$.

Adapted to Fourier integral operators, the linear *smoothing conjecture* can be stated as follows. In order to formulate it we use, for its critical index, the notation

$$\bar{p}_d := \begin{cases} \frac{2(d+1)}{d-1} & \text{if } d \text{ is odd,} \\ \frac{2(d+2)}{d} & \text{if } d \text{ is even.} \end{cases} \tag{1-8}$$

Conjecture 1.3 (linear smoothing conjecture for FIOs). Let $d \ge 2$. Let T^ϕ_a be a Fourier integral operator of order $m \in \mathbb{R}$, and with nondegenerate phase function satisfying the *cinematic curvature condition*; see Definition 1.2. For any $p \ge \bar{p}_d$, and $\sigma < \frac{1}{p}$,

$$\|T^\phi_a f\|_{L^p(\mathbb{R}^d \times \mathcal{I})} \lesssim \|f\|_{L^p_s(\mathbb{R}^d)}, \quad s = m + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma. \tag{1-9}$$

Remark 1.4. Examples constructed by Bourgain [1991; 1995] can be used to identify Fourier integral operators that satisfy the cinematic curvature condition but for which local smoothing fails for all $\sigma < \frac{1}{p}$, whenever $p < \bar{p}_d$. Compared

d	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\bar{p}_d$	4	4	3	3	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{12}{5}$	$\frac{12}{5}$	$\frac{7}{3}$	$\frac{7}{3}$	$\frac{16}{7}$	$\frac{16}{7}$	$\frac{9}{4}$	$\frac{9}{4}$	$\frac{20}{9}$	$\frac{20}{9}$	$\frac{11}{5}$

Table 1. Critical index $\bar{p}_d$ for dimensions $d = 2$ to 20.

with Theorem 1.1, Conjecture 1.3 claims a gain of regularity of order $\sigma < \frac{1}{p}$. We illustrate some values of the critical index $\bar{p}_d$ in Table 1.

Let us introduce the required notation to formulate the bilinear counterpart to Conjecture 1.3 by adopting the notion of bilinear Fourier integral operators as studied, e.g., by Grafakos and Peloso [2010], and for the form of the operators formulated here, see the work of Rodríguez-López, Rule, and Staubach [Israelsson et al. 2021].

Let $a \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}^d_\xi \times \mathbb{R}^d_\eta)$ be a *bilinear symbol*. We say that a has order $m \in \mathbb{R}$ if it satisfies the estimates

$$|\partial^\beta_{x,t} \partial^{\alpha_1}_\xi \partial^{\alpha_2}_\eta a(x,t,\xi,\eta)| \lesssim_{\beta,\alpha_1,\alpha_2} (1+|\xi|+|\eta|)^{m-|\alpha_1|-|\alpha_2|}. \tag{1-10}$$

The Kohn–Nirenberg bilinear class $S^m_{1,0} := S^m_{1,0}(\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}^d_\xi \times \mathbb{R}^d_\eta)$ is the set formed by those bilinear symbols of order m.

Let us consider the *bilinear Fourier integral operator*

$$T^{\phi_1,\phi_2}_a(f,g)(x,t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i(\phi_1(x,t,\xi)+\phi_2(x,t,\eta))} a(x,t,\xi,\eta) \hat{f}(\xi)\hat{g}(\eta)\, d\xi\, d\eta,$$

where the symbol a is compactly supported in $(x,t) \in \mathbb{R}^d \times \mathbb{R}$, and each phase function $\phi_j(x,t,\xi) \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}_t \times (\mathbb{R}^d \setminus \{0\}))$ is positively homogeneous of order 1 in $\xi \in \mathbb{R}^d \setminus \{0\}$, and they satisfy the cinematic curvature condition; see Definition 1.2.

Here, we will denote by $\mathcal{I} = [a,b]$ an arbitrary nonempty finite interval and then $-\infty < a < b < \infty$. To slightly reduce our analysis, since the symbol $a := a(x,t,\xi,\eta)$ is assumed to be compactly supported in (x,t), and without loss of generality we will choose a compact set $K \subset \mathbb{R}^d$ such that

$$(x,t) - \operatorname{supp}(a) := \{(\tilde{x},\tilde{t}) \in \mathbb{R}^d \times \mathbb{R} : (\tilde{x},\tilde{t}) \in \operatorname{supp}(a)\} \subset K \times \mathcal{I}, \tag{1-11}$$

increasing the length of $\mathcal{I}$ if needed. It becomes clear that

$$\|T^{\phi_1,\phi_2}_a(f,g)\|_{L^p(\mathbb{R}^d_x \times \mathbb{R}_t)} = \|T^{\phi_1,\phi_2}_a(f,g)\|_{L^p(K \times \mathcal{I})}. \tag{1-12}$$

This fact will not be used in our formulation of the bilinear smoothing conjecture but will be crucial in our further analysis. Based on the linear smoothing conjecture for Fourier integral operators, it is natural to suppose that the following result holds in any dimension $d \geq 2$.

Conjecture 1.5 (bilinear smoothing conjecture for FIOs). Let $d \geq 2$. For any $p_1, p_2 \geq \bar{p}_d$, let $\sigma_1 < \frac{1}{p_1}$, $\sigma_2 < \frac{1}{p_2}$. Let $T_a^{\phi_1,\phi_2}$ be a bilinear Fourier integral operator, with nondegenerate phase functions satisfying the cinematic curvature condition. Let $a \in S^m_{1,0}$ be its symbol, having order $m < 0$, and being compactly supported in $(x,t) \in \mathbb{R}^d \times \mathbb{R}$. Then

$$\|T_a^{\phi_1,\phi_2}(f,g)\|_{L^p(\mathbb{R}^d_x \times \mathbb{R}_t)} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d)}, \tag{1-13}$$

where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad s_i = m_i + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma_i, \quad m = m_1 + m_2, \quad m_1 \leq 0, m_2 < 0.$$

Remark 1.6. In the above display one has that $p \geq \frac{1}{2}\bar{p}_d$. Note that there is no loss of generality in assuming that $m < 0$ in Conjecture 1.3; see Remark 2.5. Moreover, if $a(x,t,\xi,\eta)$ has compact support in ξ, there is no restriction in assuming that $m_2 < 0$ has arbitrarily large absolute value. Conversely, if $a(x,t,\xi,\eta)$ has compact support in η, there is no restriction in assuming that $m_1 < 0$ has arbitrarily large absolute value.

Analysing the relationship between the two conjectures above is our main goal. Indeed, we prove the following result.

Theorem 1.7. *The linear smoothing Conjecture 1.3 implies the bilinear smoothing Conjecture 1.5.*

The proof of Theorem 1.7 will be deduced from a more qualitative version of it, see Theorem 2.6, which in particular will prove that any progress in the linear smoothing conjecture for FIOs implies a counterpart for bilinear FIOs in any dimension $d \geq 2$.

The local smoothing conjecture has been verified in dimension $d = 2$ by Guth, Wang, and Zhang [Guth et al. 2020] for solutions of the wave equation. Moreover, since the linear smoothing conjecture for FIOs has been proved in dimension $d = 2$ by Gao, Liu, Miao, and Xi [Gao et al. 2023], and for odd d by Beltran, Hickman, and Sogge [Beltran et al. 2020], their results, combined with Theorem 1.7, imply:

Corollary 1.8. *For $d = 2$ the bilinear local smoothing conjecture holds. In other words, for any $p_1, p_2 \geq 4$, let $\sigma_1 < \frac{1}{p_1}$, $\sigma_2 < \frac{1}{p_2}$. Let $a \in S^m_{1,0}$ be a symbol of order $m < 0$, being compactly supported in $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$. Then*

$$\|T_a^{\phi_1,\phi_2}(f,g)\|_{L^p(\mathbb{R}^2_x \times \mathbb{R}_t)} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^2)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^2)}, \tag{1-14}$$

where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad s_i = m_i + \frac{1}{2} - \frac{1}{p_i} - \sigma_i, \quad m = m_1 + m_2, \quad m_1 \leq 0, m_2 < 0. \tag{1-15}$$

On the other hand, if d is odd, the bilinear local smoothing conjecture holds. This means that for any

$$p_1, p_2 \geq \bar{p}_d := \frac{2(d+1)}{d-1}, \tag{1-16}$$

and for every $\sigma_1 < \frac{1}{p_1}$, $\sigma_2 < \frac{1}{p_2}$, an operator $T_a^{\phi_1,\phi_2}$ having order $m < 0$, with nondegenerate phase functions satisfying the cinematic curvature condition, satisfies

$$\|T_a^{\phi_1,\phi_2}(f,g)\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d)}\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d)}, \tag{1-17}$$

where

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}, \quad s_i = m_i+(d-1)\left(\frac{1}{2}-\frac{1}{p_i}\right)-\sigma_i, \quad m=m_1+m_2, \quad m_1\leq 0, m_2<0.$$

There is no sense of thinking about a calculus of bilinear FIOs, however, each bilinear FIO has encoded its own *calculus* of linear FIOs when investigating its inner structure [Rodríguez-López et al. 2014]. We will adopt in our further analysis the structural properties introduced by Rodríguez-López, Rule, and Staubach [Rodríguez-López et al. 2014] for bilinear Fourier integral operators. These structural properties serve as a substitute for the calculus of Fourier integral operators due to Hörmander [1971] in this setting. A natural question is to what extent a bilinear FIO is approximately the composition (or the product) of two linear FIOs. Indeed, unless the bilinear symbol of an operator has compact support in one of the two frequency variables (ξ,η), a bilinear FIO cannot be written as the *composition* of two linear FIOs. Their method consists of making a careful continuous decomposition of the high frequencies of a bilinear FIO $T_a^{\phi_1,\phi_2}$. Roughly, it expresses the high frequency portion of the operator as a sum of two operators of the form (see (46) of [Rodríguez-López et al. 2014, page 18])

$$\begin{aligned}\widetilde{T}_{\tilde{a}}^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g)(x,s)&\\ = \int_0^1\iint T^{\phi_1}_{\nu_1^{t,u}} f(x,s)\, T^{\phi_2}_{\mu_1^{t,v}} g(x,s)\frac{m(t,x,s,U)}{(1+|U|^2)^N}\,du\,dv\,\frac{dt}{t},&\end{aligned} \tag{1-18}$$

where $|U|^2 = |u|^2 + |v|^2$. The operators $T^{\phi_1}_{\nu_1^{t,u}}$ and $T^{\phi_2}_{\mu_1^{t,v}}$ are approximations of linear Fourier integral operators. In particular, Rodríguez-López et al. [2014] have used this representation to establish a bilinear extension of the celebrated Seeger–Sogge–Stein theorem [Seeger et al. 1991].

In view of the linear smoothing property for FIOs proved in dimension $d = 2$ in [Gao et al. 2023], it is of interest for us to ask whether or not the methods in [Rodríguez-López et al. 2014] can be applied to smoothing estimates in the bilinear context considering the heavy machinery used in their analysis from various parts of harmonic analysis; see, e.g., the discussion in [Rodríguez-López et al. 2014, page 3].

One of the main differences of bilinear smoothing estimates compared with the analysis for time-dependent boundedness theorems as in [Rodríguez-López et al. 2014; Seeger et al. 1991] came from the fact that we cannot make use of endpoint estimates at infinity involving the BMO space, or at $p = 1$, and then involving the Hardy space H^1. In our case for the range of values $p_1, p_2 \geq \bar{p}_d$, we have to deal entirely with the $L^{p_1}_{s_1} \times L^{p_2}_{s_2} \to L^p$-estimate. Making use of square-maximal functions as in [Rodríguez-López et al. 2014] we deal with the complexity of working with the direct $L^{p_1}_{s_1} \times L^{p_2}_{s_2} \to L^p$-boundedness theorem by applying, in a certain portion of the continuous decomposition of the operator, a well-known theorem for estimations of maximal functions, due to Bourgain [1985; 1986]; see Theorem 4.2. In comparison with the previous methods in the literature this is the main novelty of this paper.

This paper is organised as follows. In Section 2 we present our main result (see Theorem 2.6) about smoothing estimates for bilinear FIOs. Section 3 is dedicated to providing preliminaries about the inner structure of these operators. Section 4 is dedicated to the proof of Theorem 2.6.

2. Smoothing estimates for bilinear FIOs

We formulate our main result in Theorem 2.6 presented below. First, we introduce two necessary definitions. Throughout, we denote by $\mathscr{S}(\mathbb{R}^d)$ the Schwartz class on $\mathbb{R}^d$.

Definition 2.1. Let $d \geq 2$, and let $2 \leq p^* \leq \infty$. Let T^{ϕ}_a be a Fourier integral operator of order $m \in \mathbb{R}$, and with nondegenerate phase function satisfying the cinematic curvature condition; see Definition 1.2. If for a fixed Lebesgue index p, with $p \geq p^*$, there exists $s = s(p, m, d)$ such that

$$\|T^{\phi}_a f\|_{L^p(\mathbb{R}^d \times \mathcal{I})} \lesssim \|f\|_{L^p_s(\mathbb{R}^d)} \quad \text{for all } \ f \in C^\infty_0(\mathbb{R}^d), \tag{2-1}$$

we say that the operator T satisfies the *linear local smoothing property* (LLSP) with parameters (p^*, p, s).

Remark 2.2. The local smoothing conjecture claims that any Fourier integral operator of order $m \in \mathbb{R}$ with phase satisfying the cinematic curvature condition should satisfy the linear local smoothing property (LLSP) with parameters $(\bar{p}_d, p, s)$, where

$$s = m + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma,$$

and $\sigma < \frac{1}{p}$.

Definition 2.3. Let $d \geq 2$, and let $2 \leq p^* \leq \infty$. Let T^{ϕ_1,ϕ_2}_a be a bilinear Fourier integral operator of order $m \in \mathbb{R}$, and with nondegenerate phase functions ϕ_1, ϕ_2,

satisfying the cinematic curvature condition. If for a fixed pair (p_1, p_2), with $p_1, p_2 \geq p^*$, there exist $s_1 = s(p_1, m, d)$ and $s_2 = s(p_2, m, d)$ such that

$$\|T_a^{\phi_1,\phi_2} f\|_{L^p(\mathbb{R}^d \times \mathcal{I})} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d)} \quad \text{for all } f, g \in C_0^\infty(\mathbb{R}^d), \tag{2-2}$$

with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we say that $T_a^{\phi_1,\phi_2}$ satisfies the *bilinear local smoothing property* (BLSP) with parameters $(p^*, p, p_1, p_2, s_1, s_2)$.

Remark 2.4. The bilinear local smoothing conjecture claims that any Fourier integral operator of order $m = m_1 + m_2 < 0$, $m_2 < 0$, with phases satisfying the cinematic curvature condition should satisfy the bilinear local smoothing property with parameters $(\bar{p}_d, p, p_1, p_2, s_1, s_2)$, where

$$s_i = m_i + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma_i,$$

where σ_i is any number satisfying $\sigma_i < \frac{1}{p_i}$.

Remark 2.5. There is no restriction in assuming that $m < 0$ in Conjecture 1.3. Indeed, if $m_0 \geq 0$ is the order of a Fourier integral operator T_a^ϕ, one can take s_0 such that $m = m_0 - s_0 < 0$, and to replace f by $f_{s_0} = (1-\Delta)^{s_0/2}$ to get for the FIO $\widetilde{T} = T(1-\Delta)^{-s_0/2}$ the smoothing property

$$\|\widetilde{T} f_{s_0}\|_{L^p(\mathbb{R}^d \times \mathcal{I})} \lesssim \|(1-\Delta)^{s_0/2} f\|_{L^p_{\tilde{s}}(\mathbb{R}^d)}, \quad \tilde{s} = m + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma.$$

The previous estimate is equivalent to

$$\|Tf\|_{L^p(\mathbb{R}^d \times \mathcal{I})} \lesssim \|f\|_{L^p_{\tilde{s}+s_0}(\mathbb{R}^d)}, \quad \tilde{s} + s_0 = m_0 + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma.$$

This fact and our evidence justify our choice of assuming that the order m is negative in Conjecture 1.5. Moreover, according to our methodology, we decompose the operator $T_a^{\phi_1,\phi_2}$ as

$$\begin{aligned} T_a^{\phi_1,\phi_2}(f,g) = {} & T_a^{\phi_1,\phi_2}(\mu(D)f, \mu(D)g) \\ & + T_a^{\phi_1,\phi_2}(\mu(D)f, (1-\mu)(D)g) + T_a^{\phi_1,\phi_2}\big((1-\mu(D))f, g\big), \end{aligned}$$

where μ is supported outside of a ball centred at the origin. The second and the third terms are the low-frequency portions of the operator. Curiously, for the analysis of the low-frequencies we need the hypothesis $m_1 \leq 0$, while for the analysis of the high frequencies we need to assume that $m_2 < 0$.

Theorem 2.6. *Let $d \geq 2$, and let $2 \leq p^* \leq \infty$. Consider the following hypothesis*:

(H1) *Assume that a pair of real numbers (m_1, m_2) and a pair of Lebesgue indices (p_1, p_2) are such that $p_i \geq p^*$ and satisfy that any Fourier integral operator of order m_i satisfying the cinematic curvature condition also satisfies the LLSP with parameters (p^*, p_i, s_i) for some $s_i = s_i(p_i, m_i, d)$.*

Then, if $m := m_1 + m_2$, $m_1 \le 0$, *and* $m_2 < 0$, *every bilinear Fourier integral operator* $T_a^{\phi_1,\phi_2}$ *of order* m *with nondegenerate phase functions satisfying the cinematic curvature condition satisfies* (*BLSP*) *with parameters* $(p^*, p, p_1, p_2, s_1, s_2)$. *In other words*,

$$\|T_a^{\phi_1,\phi_2} f\|_{L^p(\mathbb{R}^d \times \mathcal{I})} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d)}, \tag{2-3}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 < p < \infty$.

Proof of Theorem 1.7. The theorem follows from Theorem 2.6 taking also into account Remarks 2.2 and 2.4. □

Proof of Corollary 1.8. The proof follows from Theorem 2.6 and the solution of the linear smoothing conjecture in dimension $d = 2$ due to [Gao et al. 2023]; also for d odd, this conjecture has been verified in [Beltran et al. 2020]. For the numerical details used here for this conjecture with d odd, see [Gao et al. 2023, page 1926]. This fact combined with Theorem 2.6 proves the statement in this case. □

3. Preliminaries

In this section, we present the structural properties of bilinear FIOs as introduced in [Rodríguez-López et al. 2014]. As mentioned before, these structural properties replace the calculus of Fourier integral operators due to Hörmander [1971] in this setting in the sense that we want to understand bilinear operators as twisted versions of compositions or products of linear FIOs. In other words, each bilinear FIO induces its own *calculus* of linear FIOs when investigating its inner structure [Rodríguez-López et al. 2014].

Standard notation such as $\psi(D)$ and $\psi(tD)$ will be used later for the corresponding multipliers with symbols $\psi := \psi(\eta)$ and $\psi(t\,\cdot) := \psi(t\eta)$, respectively.

3.1. ***Structural properties of bilinear FIOs.*** First, we will review the structure of the bilinear operator $T_a^{\phi_1,\phi_2}$ given by

$$T_a^{\phi_1,\phi_2}(f,g)(x,t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i(\phi_1(x,t,\xi)+\phi_2(x,t,\eta))} a(x,t,\xi,\eta)\, \hat{f}(\xi)\hat{g}(\eta)\, d\xi\, d\eta. \tag{3-1}$$

The first step will make a reduction of our analysis in low frequencies and in high frequencies. For this we follow the approach and notation in [Rodríguez-López et al. 2014, Sections 3–4]. Some remarks are in order.

Remark 3.1 (see [Rodríguez-López et al. 2014, page 12]). We will use the structure of the operator $T_a^{\phi_1,\phi_2}$ to decompose it in a suitable continuous form. The phase

functions ϕ_j are smooth, positively homogeneous of order one in the frequency variable and they are nondegenerate. By the mean value theorem

$$|\nabla_x \phi_j(x,t,\xi)| \lesssim |\xi| \quad \text{for all } (x,t,\xi) \in \mathcal{K}\times(\mathbb{R}^d\setminus\{0\}) = K\times\mathcal{I}\times(\mathbb{R}^d\setminus\{0\}). \tag{3-2}$$

In addition, the phases are nondegenerate and then one has the lower bound

$$|\xi| \lesssim |\nabla_x \phi_j(x,t,\xi)| \quad \text{for all } (x,t,\xi) \in \mathcal{K}\times(\mathbb{R}^d\setminus\{0\}) = K\times\mathcal{I}\times(\mathbb{R}^d\setminus\{0\}). \tag{3-3}$$

Therefore, there exists $\lambda > 0$ such that on $K\times\mathcal{I}\times(\mathbb{R}^d\setminus\{0\})$ one has the estimate

$$\lambda|\xi| \le |\nabla_x \phi_j(x,t,\xi)| \le \frac{|\xi|}{\lambda}. \tag{3-4}$$

We consider a smooth function μ such that $\mu \equiv 0$ on $\{\xi\in\mathbb{R}^d : |\xi| \le \frac{1}{4\lambda}\}$ and $\mu \equiv 1$ on $\{\xi\in\mathbb{R}^d : |\xi| \ge \frac{1}{3\lambda}\}$.

Remark 3.2. We decompose the operator $T_a^{\phi_1,\phi_2}$ as

$$T_a^{\phi_1,\phi_2}(f,g) = T_a^{\phi_1,\phi_2}(\mu(D)f, \mu(D)g) \tag{3-5}$$

$$+ T_a^{\phi_1,\phi_2}(\mu(D)f, (1-\mu)(D)g) + T_a^{\phi_1,\phi_2}\big((1-\mu(D))f, g\big). \tag{3-6}$$

The second and the third term are the low-frequency portions of the operator. That these parts satisfy the $L^{p_1}_{s_1}\times L^{p_2}_{s_2}\to L^p$ estimate requires an analysis based on understanding the low frequency portion of the operator as an almost composition of linear FIOs; see Section 4.1.

Remark 3.3 (see [Rodríguez-López et al. 2014, Section 4]). For the analysis of the high-frequency portion $T_a^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g)$ one can introduce two smooth cut-off functions $\nu, \chi \in C_0^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ such that $\chi(\xi,\eta) = 1$ for $|\xi|+|\eta|\le 1$, $\chi(\xi,\eta) = 0$ for $|\xi|+|\eta|\ge 2$, $\nu(\xi,\eta) = 0$ for $\lambda^2|\xi| \le 16|\eta|$, and $\nu(\xi,\eta) = 1$ for $64|\eta| \le \lambda^2|\xi|$. Let us define

$$a_1 := (1-\chi)\nu a, \quad a_2 := (1-\chi)(1-\nu)a. \tag{3-7}$$

We have that a_1 and a_2 are bilinear symbols of order m, and

$$T_a^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g) = T_{a_1}^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g) + T_{a_2}^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g).$$

We have the representation (see equation (43) in [Rodríguez-López et al. 2014, page 16])

$$T_{a_1}^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g)(x,s) = \int_0^1\iint T^{\phi_1}_{\nu_1^{t,u}} f(x,s) T^{\phi_2}_{\mu_1^{t,v}} g(x,s) \frac{m(t,x,s,u,v)}{(1+|u|^2+|v|^2)^N}\, du\, dv\, \frac{dt}{t}. \tag{3-8}$$

The operator $T^{\phi_1}_{\nu_1^{t,u}}$ has the factorisation modulo an error term (see equation (54) in [Rodríguez-López et al. 2014, page 20])

$$T^{\phi_1}_{\nu_1^{t,u}} = t^{-m_2} Q_t^u T^{\phi_1}_{\nu_{m_1}} + t^{-m_2} E_t^1, \tag{3-9}$$

where:

• Q_t^u is a Fourier multiplier with symbol (see equation (54) in [Rodríguez-López et al. 2014, page 21])

$$\xi \mapsto |t\xi|^{m_2} \widehat{\psi}(t\xi) e^{itu\cdot\xi} \in \mathscr{S}(\mathbb{R}^n). \tag{3-10}$$

The function $\psi \in \mathscr{S}(\mathbb{R}^n)$ is chosen to be smooth, whose Fourier transform is supported on the annulus $\left\{\xi \in \mathbb{R}^d : \frac{1}{2} \le |\xi| \le 2\right\}$ and such that

$$\int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} = 1 \quad \text{for all } t > 0 \text{ and for all } \xi \neq 0.$$

• The operator $T^{\phi_1}_{\nu_{m_1}}$ is a Fourier integral operator of order m_1 with phase ϕ_1 and with symbol (see equation (55) in [Rodríguez-López et al. 2014, page 21])

$$(x,s,\xi) \mapsto \nu_{m_1}(x,\xi) = \mu(\xi)\chi(x)|\nabla_x \phi_1(x,s,\xi)|^{m_1} \in S^{m_1}_{1,0}(\mathbb{R}^d_x \times \mathbb{R}_s \times \mathbb{R}^d_\xi). \tag{3-11}$$

The operator E_t^1 has the same phase ϕ_1 of the operator $T^{\phi_1}_{\nu_{m_1}}$ and a symbol in the class $S^{m_1-(1/2-\varepsilon)}_{1,0}(\mathbb{R}^d_x \times \mathbb{R}_s \times \mathbb{R}^d_\xi)$, with seminorms bounded by the product of t^ε, $\varepsilon \in \left(0, \frac{1}{2}\right)$, with a polynomial expression in $|u|$.

On the other hand, $T^{\phi_2}_{\mu_1^{t,v}}$, modulo an error term, admits the factorisation (see equation (56) in [Rodríguez-López et al. 2014, page 21])

$$t^{-m_2} T^{\phi_2}_{\mu_1^{t,v}} = P_t^v T^{\phi_2}_{\mu_1^t} + E_t^2,$$

where:

(i) The symbol of P_t^v is given by

$$\eta \mapsto \widehat{\theta}(t\eta) e^{itv\cdot\eta} \in S^0_{1,0}(\mathbb{R}^d_\eta).$$

(ii) $T^{\phi_2}_{\mu_1^t}$ is an FIO of order $m_2 < 0$, with phase function ϕ_2, and with symbol

$$(x,s,\eta) \mapsto \mu_1^t(x,s,\eta) = \mu(\eta)\chi(x)\widehat{\theta}_1(t\nabla_x \phi_2(x,s,\eta)) t^{-m_2} \in S^{m_2}_{1,0}(\mathbb{R}^d_x \times \mathbb{R}_s \times \mathbb{R}^d_\eta),$$

uniformly in $t \in (0,1]$. Here, E_t^2 is a Fourier integral operator with phase ϕ_2 and a symbol in the class $S^{m_2-(1/2-\varepsilon)}_{1,0}(\mathbb{R}^d_x \times \mathbb{R}_s \times \mathbb{R}^d_\eta)$, with seminorms that behaves polynomially in $|v|$.

3.2. ***Decomposition à la Coifman–Meyer.*** Some remarks are in order.

Remark 3.4. The core of the proof of the structural properties for bilinear FIOs lies in the representation in (3-8) for $T_{a_1}^{\phi_1,\phi_2}$,[1] that is (see [Rodríguez-López et al. 2014, page 21]),

$$T_{a_1}^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g)(x,s) = \iiint_0^1 T_{\nu_1^{t,u}}^{\phi_1} f(x,s) T_{\mu_1^{t,v}}^{\phi_2} g(x,s) \frac{m(t,x,s,u,v)}{(1+|u|^2+|v|^2)^N}\, du\, dv\, \frac{dt}{t}. \tag{3-12}$$

About the $\frac{dt}{t}$-integral in (3-12), by writing $m(t,x,s) := m(t,x,s,u,v)$, one has

$$\int_0^1 T_{\nu_1^{t,u}}^{\phi_1} f(x,s) T_{\mu_1^{t,v}}^{\phi_2} g(x,s) m(t,x,s)\, \frac{dt}{t} = I_1 + I_2 + I_3,$$

with

$$I_1 = \int_0^1 Q_t^u T_{\nu_{m_1}}^{\phi_1} f(x,s) P_t^v T_{\mu_1^t}^{\phi_2} g(x,s) m(t,x,s)\, \frac{dt}{t},$$

$$I_2 = \int_0^1 Q_t^u T_{\nu_{m_1}}^{\phi_1} f(x,s) E_t^2 g(x,s) m(t,x,s)\, \frac{dt}{t},$$

$$I_3 = \int_0^1 E_t^1 f(x,s) t^{-m_2} T_{\mu_1^{t,v}}^{\phi_2} g(x,s) m(t,x,s)\, \frac{dt}{t}.$$

Remark 3.5. For $m < 0$, and with $m_2 < 0$, we have the representation (see [Rodríguez-López et al. 2014, Case II on page 24, and page 26])

$$I_1 = \sum_{j=1}^{3} \int_\kappa^\infty \int_0^1 \chi_{(0,1)}(rt)(Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(P_t^v S_{r,t}^j g)(x,s) m(t,x,s)\, \frac{dt\, dr}{tr},$$

where $S_{r,t}^1 = r^{m_2} P_{1,t} Q_{tr} T_\gamma^{\phi_2}$, with $T_\gamma^{\phi_2}$ being a Fourier integral operator of order m_2 with symbol

$$\gamma(x,s,\eta) = \chi(x)\mu(\eta)|\nabla_x \phi_2(x,s,\eta)|^{m_2} \in S_{1,0}^{m_2}(\mathbb{R}_x^d \times \mathbb{R}_s \times \mathbb{R}_\eta^d), \tag{3-13}$$

and $P_{1,t}$ is the Fourier multiplier with symbol $\widehat{\theta}_1(t\eta)$. The operators $S_{r,t}^2$, $S_{r,t}^3$, have the form

$$S_{r,t}^2 g(x,s) = s^{m_2} P_{1,t}\bigg(\int e^{i\phi_2(\cdot,s,\eta)} R_{tr}^2(\cdot,s,\eta)\hat{g}(\eta)\, d\eta \bigg)(x),$$

$$S_{r,t}^3 g(x,s) = s^{m_2} \bigg(\int e^{i\phi_2(x,s,\eta)} R_t^1(x,s,\eta)\hat{g}(\eta)\, d\eta \bigg),$$

[1] A similar representation holds for $T_{a_2}^{\phi_1,\phi_2}$. So, we will just analyse one of these two terms in the proof of our main theorem.

where $R_t^1(x,s,\eta), R_t^1(x,s,\eta) \in S^{m_2-(1/2-\varepsilon)}_{1,0}(\mathbb{R}^d_x \times \mathbb{R}_s \times \mathbb{R}^d_\eta)$. Also, these symbols have seminorms that are bounded by t^ε and by $(tr)^\varepsilon$, with $\varepsilon \in \left(0, \frac{1}{2}\right)$, respectively.

Remark 3.6. The operators P_t^v and Q_t^u are uniformly $L^p(\mathbb{R}^d_x)$-bounded with operators norms growing polynomially in $|u|$ and $|v|$, respectively; see, e.g., [Stein 1993, page 161].

Having understood the *calculus* of Fourier integral operators inside a bilinear FIO we will start the analysis of low frequencies in the next section.

4. Proof of Theorem 2.6

In this section we prove our main Theorem 2.6. We split our analysis in two parts. First, we investigate the smoothing properties of the low frequencies of a bilinear FIO. This portion of the operator is approximately the composition of two linear Fourier integral operators. As for the portion of the operator formed by the high frequencies, the idea is to express it in terms of *paraproducts* à la [Coifman and Meyer 1978].

4.1. ***Analysis of the low frequencies.*** For the analysis of low frequencies it suffices to prove that a Fourier integral operator

$$T_a^{\phi_1,\phi_2}(f,g)(x,t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i(\phi_1(x,t,\xi)+\phi_2(x,t,\eta))} a(x,t,\xi,\eta)\, \hat{f}(\xi)\hat{g}(\eta)\, d\xi\, d\eta,$$

with a symbol $a := a(x,t,\xi,\eta)$ compactly supported in ξ or in η, satisfies the estimate

$$\|T_{a_1}^{\phi_1,\phi_2}(f,g)\|_{L^p(K\times\mathcal{I})} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}. \tag{4-1}$$

Here

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad s_i = m_i + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma_i, \quad m = m_1 + m_2, \quad m_1 \le 0,\ m_2 < 0.$$

Without loss of generality, let us assume that $f, g \in C_0^\infty(\mathbb{R}^n)$, and that the symbol a is compactly supported in (x,t,ξ). The following remark will be useful in our further analysis.

Remark 4.1. Assume that the parameters s_i are of the form

$$s_i = m_i + \tilde{s}_i(p_i, d). \tag{4-2}$$

Let N be an arbitrary integer. Observe that if the symbol $a(s,t,\xi,\eta)$ has compact support in ξ, then there is no restriction in Theorem 2.6 in assuming that $m_2 < 0$ has arbitrarily large absolute value; in particular, we may assume that $|m_2| > N$.

Similarly, if $a(s,t,\xi,\eta)$ has compact support in η, there is no restriction in assuming that $m_1 < 0$ has arbitrarily large absolute value.

To justify this claim, namely that it is enough to consider the conclusion of Theorem 2.6 when $|m_2|$ is large, we consider the first case, where $a(s,t,\xi,\eta)$ has compact support in ξ.

The idea is to consider $m_2 \leq 0$ with small absolute value, and reduce the proof to the case where $m_2 \leq 0$ has large absolute value.

Let $m = m_1 + m_2$, $m_2 < 0$ be the order of a bilinear FIO $T_a^{\phi_1,\phi_2}$. One can take s_0 such that $m_2' = m_2 - s_0 < -N$, and can replace g by $g_{s_0} = (1-\Delta)^{s_0/2} g$ to get for the bilinear FIO

$$\widetilde{T}_a^{\phi_1,\phi_2}(\cdot,\cdot) = T_a^{\phi_1,\phi_2}(\cdot, (1-\Delta_x)^{-s_0/2}\cdot) \tag{4-3}$$

of order $m - s = (m_1 - s_0) + m_2 = m_1 + m_2 = m_1 + (m_2 - s_0) = m_1 + m_2'$ that

$$\|T_{a_1}^{\phi_1,\phi_2}(f,g)\|_{L^p(K\times\mathcal{I})} = \|\widetilde{T}_{a_1}^{\phi_1,\phi_2}(f,g_{s_0})\|_{L^p(K\times\mathcal{I})} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g_{s_0}\|_{L^{p_2}_{s_2'}(\mathbb{R}^d_x)},$$

where

$$s_1 = m_1 + \tilde{s}_1(p_1,d), \quad s_2' = m_2 - s_0 + \tilde{s}_2(p_1,d).$$

Observe that

$$s_2 := s_2' + s_0 = m_2 + \tilde{s}_2(p_1,d) \tag{4-4}$$

does not depend on s_0. Note also that in the case of the linear smoothing conjecture one has

$$\tilde{s}_1(p_i,d) = (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma_i, \quad \sigma_i < \frac{1}{p_i}.$$

It is clear that

$$\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g_{s_0}\|_{L^{p_2}_{s_2'}(\mathbb{R}^d_x)} = \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2'+s_0}(\mathbb{R}^d_x)} = \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.$$

Observe that we have used that the order of the FIO $T_a^{\phi_1,\phi_2}(\cdot,(1-\Delta_x)^{-s_0/2}\cdot)$ is $m - s = m_1 - s + m_2$ since its symbol is given by

$$a(x,t,\xi,\eta)\langle\eta\rangle^{-s_0} \sim a(x,t,\xi,\eta)\langle\xi,\eta\rangle^{-s_0},$$

and is compactly supported in ξ. The previous analysis shows that, under the condition of the symbol to be compactly supported in one of the two frequency variables, if the bilinear smoothing estimate holds for m_2 with $|m_2|$ large enough, then the same property holds with any $m_2 < 0$.

We return our attention to the analysis of the low frequencies. For the proof of (4-1) let us combine the proof of the L^2-boundedness of pseudodifferential operators of order zero [Stein 1993, page 234] and the proof of Lemma 3.1 in [Rodríguez-López et al. 2014, page 12]. Let ψ be a smooth cut-off function that

is equal to 1 on the support of a in ξ. The support of the symbol a in ξ will be denoted by $\mathfrak{S}$. Then

$$T_{a_1}^{\phi_1,\phi_2}(f,g)(x,t)=\int e^{i\phi_1(x,t,\xi)}\mathfrak{a}_g(x,t,\xi)\hat{f}(\xi)\,d\xi=T_{\mathfrak{a}_g}^{\phi_1}(f)(x,t), \tag{4-5}$$
$$(x,t,\xi)\in K\times\mathcal{I}\times\mathfrak{S},$$

where

$$\mathfrak{a}_g(x,t,\xi)=\int e^{i\phi_2(x,t,\eta)}\psi(\xi)a(x,t,\xi,\eta)\hat{g}(\eta)\,d\eta=T^{\phi_2}_{\psi(\xi)a(\cdot,\cdot,\xi,\cdot)}g(x,t). \tag{4-6}$$

Observe that the support of $\mathfrak{a}_g(x,t,\xi)$ in (x,t) coincides with the support of a in (x,t) and then it is contained in $K\times\mathcal{I}$. Using the Fourier inversion formula,

$$\begin{aligned}T_{\mathfrak{a}_g}^{\phi_1}(f)(x,t)&=\int e^{i\phi_1(x,t,\xi)}\mathfrak{a}_g(x,t,\xi)\zeta(\xi)\hat{f}(\xi)\,d\xi\\&=C_d\int e^{i\phi_1(x,t,\xi)}\int\hat{\mathfrak{a}}_g(x,t,\lambda)e^{i\xi\cdot\lambda}\,d\lambda\zeta(\xi)\hat{f}(\xi)\,d\xi\\&=C_d\int\hat{\mathfrak{a}}_g(x,t,\lambda)\int e^{i\phi_1(x,t,\xi)}\zeta(\xi)\widehat{f(\cdot-\lambda)}(\xi)\,d\xi\,d\lambda\\&=C_d\int\hat{\mathfrak{a}}_g(x,t,\lambda)T_\zeta^{\phi_1}(f(\cdot-\lambda))(x,t)\,d\lambda.\end{aligned}$$

Note that $\hat{\mathfrak{a}}_g(x,t,\lambda)$ has the same compact support in x as $\mathfrak{a}_g(x,t,\lambda)$. Let us consider a function $\chi\in C_0^\infty$ such that $\chi=1$ on $\operatorname{supp}_x(\hat{\mathfrak{a}}_g)$. Then,

$$\begin{aligned}T_{\mathfrak{a}_g}^{\phi_1}(f)(x,t)&=C_d\int\hat{\mathfrak{a}}_g(x,t,\lambda)\chi(x)T_\zeta^{\phi_1}(f(\cdot-\lambda))(x,t)\,d\lambda\\&=C_d\int\hat{\mathfrak{a}}_g(x,t,\lambda)T_{\chi\zeta}^{\phi_1}(f(\cdot-\lambda))(x,t)\,d\lambda.\end{aligned}$$

Hölder's inequality implies that

$$\begin{aligned}&\|T_{\mathfrak{a}_g}^{\phi_1}(f)(x,t)\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)}\\&\qquad\lesssim\int\|\hat{\mathfrak{a}}_g(x,t,\lambda)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\|T_{\chi\zeta}^{\phi_1}(f(\cdot-\lambda))(x,t)\|_{L^{p_1}(\mathbb{R}^d_x\times\mathbb{R}_t)}\,d\lambda\\&\qquad\lesssim\int\|\hat{\mathfrak{a}}_g(x,t,\lambda)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\|f(\cdot-\lambda)\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\,d\lambda.\end{aligned}$$

Observe that, as remarked by the referee, since $T_{\chi\zeta}^{\phi_1}$ has a symbol of order $-\infty$,

$$\|T_{\chi\zeta}^{\phi_1}(f(\cdot-\lambda))(x,t)\|_{L^{p_1}(\mathbb{R}^d_x\times\mathbb{R}_t)}\lesssim\|f(\cdot-\lambda)\|_{L^{p_1}_{s_1}}=\|((1-\Delta_x)^{s_1/2}f)(\cdot-\lambda)\|_{L^{p_1}}.$$

Indeed, the operator $(1-\Delta_x)^{s_1/2}$ and the translation operator $\tau_\lambda: f\mapsto f(\cdot-\lambda)$ commute. Since the Lebesgue norms are also invariant by the action of τ_λ,

$$\|T_{\chi\zeta}^{\phi_1}(f(\cdot-\lambda))(x,t)\|_{L^{p_1}(\mathbb{R}^d_x\times\mathbb{R}_t)}\lesssim\|f\|_{L^{p_1}_{s_1}}.$$

In order to estimate the norm $\|\hat{\mathfrak{a}}_g(x,t,\lambda)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}$, observe that

$$\hat{\mathfrak{a}}_g(x,t,\lambda)=\int e^{i\lambda\xi}\mathfrak{a}_g(x,t,\lambda)\,dx=\langle\lambda\rangle^{-2N}\int e^{i\lambda\xi}(I-\Delta_\xi)^N\mathfrak{a}_g(x,t,\xi)\,d\xi.$$

Then

$$\|\hat{\mathfrak{a}}_g(x,t,\lambda)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\lesssim\langle\lambda\rangle^{-2N}\int_{\mathfrak{S}}\|(I-\Delta_\xi)^N\mathfrak{a}_g(x,t,\xi)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\,d\xi,$$

where we have denoted the support of the symbol a in ξ by $\mathfrak{S}$. We can compute

$$(I-\Delta_\xi)^N\mathfrak{a}_g(x,t,\xi)=(I-\Delta_\xi)^N(\psi(\xi)T^{\phi_2}_{a(\cdot,\cdot,\xi,\cdot)}g(x,t))=T^{\phi_2}_{\varkappa(\xi)}g(x,t).$$

Note that the symbol of $T^{\phi_2}_{\varkappa(\xi)}$ is given by

$$\varkappa(\xi):=(1-\Delta_\xi)^N[\psi(\xi)a(\cdot,\cdot,\xi,\cdot)],$$

and is a linear combination of symbols of order $\leq m=m_1+m_2\leq m_2$, where we have used that $m_1\leq 0$. Thus, $\varkappa$ is a symbol of order m_2 and we have the smoothing estimate

$$\|T^{\phi_2}_{\varkappa(\xi)}g(x,t)\|_{L^{p_2}}\lesssim\|g\|_{L^{p_2}_{s_2}}.$$

By taking $N>d/2$,

$$\|T^{\phi_1}_{\mathfrak{a}_g}(f)(x,t)\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)}\lesssim\int\|\hat{\mathfrak{a}}_g(x,t,\lambda)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\,d\lambda.$$

Summarising, we have proved that

$$\begin{aligned}\|T^{\phi_1,\phi_2}_{a_1}(f,g)\|_{L^p(K\times\mathcal{I})}&\\
&\lesssim\int\langle\lambda\rangle^{-2N}\int_{\mathfrak{S}}\|(I-\Delta_\xi)^N\mathfrak{a}_g(x,t,\xi)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\,d\xi\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\,d\lambda\\
&=\int\langle\lambda\rangle^{-2N}\int_{\mathfrak{S}}\|T^{\phi_2}_{\varkappa(\xi)}g\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\,d\lambda\\
&\lesssim\int\langle\lambda\rangle^{-2N}\,d\lambda\sup_{\xi\in\mathfrak{S}}\|T_{\varkappa(\xi)}\|_{\mathscr{B}(L^{p_2}_{s_2},L^{p_2})}|\mathfrak{S}|\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\\
&\lesssim_{K\times\mathcal{I}\times\mathfrak{S}}\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.\end{aligned}$$

Thus, the previous estimate proves (4-1).

4.2. ***An estimate due to Bourgain.*** In our further analysis for high-frequencies we will make a continuous decomposition of the operator $T^{\phi_1,\phi_2}_{a_1}$ and we will estimate its norm using certain maximal functions. Of crucial interest for us will be the next estimate due to Bourgain.

Theorem 4.2 [Bourgain 1986]. *Let $K \in L^1(\mathbb{R}^d)$ be such that its Fourier transform $\widehat{K}$ is differentiable. Define*

$$\alpha_j = \sup_{|\xi|\sim 2^j} |\widehat{K}(\xi)|, \quad \beta_j = \sup_{|\xi|\sim 2^j} |\langle \nabla \widehat{K}(\xi), \xi\rangle|, \tag{4-7}$$

where $\nabla K = (\partial_{x_i} K)_{1\le i\le d}$ denotes the gradient of K, and $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$ denotes the inner product on $\mathbb{R}^d$. Then, we have the estimate, on the maximal operator associated to K,

$$\Big\| \sup_{t>0} |f * K_t| \Big\|_{L^2(\mathbb{R}^d)} \le C\,\Gamma(K)\|f\|_{L^2(\mathbb{R}^d)} \tag{4-8}$$

for every $f \in \mathscr{S}(\mathbb{R}^d)$, where

$$\Gamma(K) = \sum_{j\in\mathbb{Z}} \alpha_j^{1/2}(\alpha_j + \beta_j)^{1/2}. \tag{4-9}$$

There are several reasons Theorem 4.2 is interesting by itself. First, the constant $\Gamma(K)$ in (4-9) is sharp in some sense; see [Bourgain 1986, page 1475]. On the other hand, it has been useful in the analysis of the L^2-theory of maximal functions, for convex bodies and for the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$; see, e.g., the introduction of [Cardona 2024].

Remark 4.3 [Stein 1993, pages 27–29]. Recall that if $\psi \in \mathscr{S}(\mathbb{R}^d)$ is such that its Fourier transform $\widehat{\psi}$ is supported in an annulus isolating the origin (and then $\widehat{\psi}(0) = \int \psi(x)\,dx = 0$), one has the L^p-boundedness property

$$\left\| \left(\int_{\mathbb{R}^d_x} |u * \psi_t(x)|^2\,dx \right)^{1/2} \right\|_{L^p(\mathbb{R}^d_x)} \lesssim \|u\|_{L^p(\mathbb{R}^d_x)}, \quad \psi_t := t^{-d}\psi(t^{-1}\cdot). \tag{4-10}$$

This fact will be used later.

4.3. ***Analysis of the high frequencies.*** The analysis of the high frequencies consists in proving that

$$\|T_a^{\phi_1,\phi_2}(\mu(D)f, \mu(D)g)\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d)}\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d)}, \tag{4-11}$$

where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad s_i = m_i + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma_i, \quad m = m_1 + m_2, \quad m_i < 0.$$

This is the remaining main step in the proof of Theorem 2.6.

Proof of Theorem 2.6. Since the analysis of the low frequencies has been carried out in Section 4.1, according to the notation above, we need to prove the estimate

$$\|I_1\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)} + \|I_2\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)} + \|I_3\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.$$

We recall that the first term I_1 can be decomposed as (see Remark 3.5)

$$I_1 = \sum_{j=1}^{3} \int_{\kappa}^{\infty} \int_0^1 \chi_{(0,1)}(rt)(Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(P_t^v S_{r,t}^j g)(x,s) m(t,x,s) \frac{dt\, dr}{tr}.$$

The operators P_t^v and Q_t^u are uniformly $L^p(\mathbb{R}_x^d)$-bounded with operators norms growing polynomially in $|u|$ and $|v|$, respectively (see Remark 3.6). Consequently, the following $L^{p_2}(\mathbb{R}_x^d)$-estimates are satisfied:

- $\|P_t^v S_{r,t}^3 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d)} \lesssim_v \|S_{r,t}^3 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d)}$,
- $\|P_t^v S_{r,t}^2 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d)} \lesssim_v \|S_{r,t}^2 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d)}$,
- $\|Q_t^u T_{\nu_{m_1}}^{\phi_1} f(x,s)\|_{L^{p_1}(\mathbb{R}_x^d)} \lesssim_u \|T_{\nu_{m_1}}^{\phi_1} f(x,s)\|_{L^{p_1}(\mathbb{R}_x^d)}$.

As a consequence, the following $L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)$-estimates hold:

- $\|P_t^v S_{r,t}^3 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim_v \|S_{r,t}^3 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)}$,
- $\|P_t^v S_{r,t}^2 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim_v \|S_{r,t}^2 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)}$,
- $\|Q_t^u T_{\nu_{m_1}}^{\phi_1} f(x,s)\|_{L^{p_1}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim_u \|T_{\nu_{m_1}}^{\phi_1} f(x,s)\|_{L^{p_1}(\mathbb{R}_x^d \times \mathbb{R}_s)}$.

Since the operators $S_{r,t}^1$, $S_{r,t}^2$ and $S_{r,t}^3$ are Fourier integral operators with phase function $\phi_2(x,s,\eta)$ and of order $\leq m_2$, and the order of the Fourier integral operator $T_{\nu_{m_1}}^{\phi_1}$ equals m_1, in view of the hypothesis (H1) of Theorem 2.6, we have that

- $\|S_{r,t}^3 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim r^{m_2} t^{\varepsilon} \|g\|_{L_{s_2}^{p_2}(\mathbb{R}_x^d)}$,
- $\|S_{r,t}^2 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim r^{m_2+\varepsilon} t^{\varepsilon} \|g\|_{L_{s_2}^{p_2}(\mathbb{R}_x^d)}$,
- $\|T_{\nu_{m_1}}^{\phi_1} f(x,s)\|_{L^{p_1}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim \|f\|_{L_{s_1}^{p_1}(\mathbb{R}_x^d)}$.

Consequently, we have proved that, for $\varepsilon \in \left(0, \frac{1}{2}\right)$,

- $\|P_t^v S_{r,t}^3 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim r^{m_2} t^{\varepsilon} \|g\|_{L_{s_2}^{p_2}(\mathbb{R}_x^d)}$,
- $\|P_t^v S_{r,t}^2 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim r^{m_2+\varepsilon} t^{\varepsilon} \|g\|_{L_{s_2}^{p_2}(\mathbb{R}_x^d)}$,
- $\|Q_t^u T_{\nu_{m_1}}^{\phi_1} f(x,s)\|_{L^{p_1}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim \|f\|_{L_{s_1}^{p_1}(\mathbb{R}_x^d)}$.

Below,

$$\|m\|_{L^\infty} := \|m\|_{L_U^\infty}$$

denotes the L^∞-norm of m as a function of (t,x,s). So m is bounded polynomially

as a function of $U = 1 + |u| + |v|$. Observe that

$$\begin{aligned}
&\Big\| \int_\kappa^\infty \int_0^1 \chi_{(0,1)}(rt)(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^v S^3_{r,t} g)(x,s) m(t,x,s)\, \frac{dt\,dr}{tr} \Big\|_{L^p(K\times\mathcal{I})} \\
&\qquad \le \int_\kappa^\infty \int_0^{1/r} \|(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^v S^3_{r,t} g)(x,s) m(t,x,s)\|_{L^p(K\times\mathcal{I})}\, \frac{dt\,dr}{tr} \\
&\qquad \lesssim \int_\kappa^\infty \int_0^{1/r} \|(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)\|_{L^{p_1}(K\times\mathcal{I})} \\
&\qquad\qquad \times \|(P_t^v S^3_{r,t} g)(x,s)\|_{L^{p_2}(K\times\mathcal{I})} \|m\|_{L^\infty}\, \frac{dt\,dr}{tr} \\
&\qquad \lesssim \int_\kappa^\infty \int_0^{1/r} r^{m_2} t^\varepsilon\, \frac{dt\,dr}{tr} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.
\end{aligned}$$

We have used that

$$\int_\kappa^\infty \int_0^{1/r} r^{m_2} t^\varepsilon\, \frac{dt\,dr}{tr} < \infty, \tag{4-12}$$

since $m_2 < 0$. Also, we have the estimations

$$\begin{aligned}
&\Big\| \int_\kappa^\infty \int_0^1 \chi_{(0,1)}(rt)(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^v S^2_{r,t} g)(x,s) m(t,x,s)\, \frac{dt\,dr}{tr} \Big\|_{L^p(K\times\mathcal{I})} \\
&\qquad \le \int_\kappa^\infty \int_0^{1/r} \|(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^v S^2_{r,t} g)(x,s) m(t,x,s)\|_{L^p(K\times\mathcal{I})}\, \frac{dt\,dr}{tr} \\
&\qquad \lesssim \int_\kappa^\infty \int_0^{1/r} \|(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)\|_{L^{p_1}(K\times\mathcal{I})} \\
&\qquad\qquad \times \|(P_t^v S^2_{r,t} g)(x,s)\|_{L^{p_2}(K\times\mathcal{I})} \|m\|_{L^\infty}\, \frac{dt\,dr}{tr} \\
&\qquad \lesssim \int_\kappa^\infty \int_0^{1/r} r^{m_2+\varepsilon} t^\varepsilon\, \frac{dt\,dr}{tr} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.
\end{aligned}$$

Now, since $S^1_{r,t} = r^{m_2} P_{1,t} Q_{tr} T^{\phi_2}_\gamma$, in order to finish the estimate of I_1 we need to deal with the norm

$$\begin{aligned}
&\Big\| \int_\kappa^\infty r^{m_2} \int_0^1 \chi_{(0,1)}(rt)(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s) \\
&\qquad\qquad \times (P_t^v [P_{1,t} Q_{tr} T^{\phi_2}_\gamma] g)(x,s) m(t,x,s)\, \frac{dt\,dr}{tr} \Big\|_{L^p(K\times\mathcal{I})}.
\end{aligned}$$

To estimate the previous norm let us take an arbitrary $h \in L^{p'}(K\times\mathcal{I})$ assumed to be supported in $K\times\mathcal{I}$, when considering its extension by zero to $\mathbb{R}^d\times\mathbb{R}$. It suffices

to prove that

$$|\mathscr{I}| = \left| \int_{K\times\mathcal{I}} \int_0^1 h(x,s)(Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(\widetilde{P}_t^v [R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s) \frac{dt}{t}\, dx\, ds \right|$$
$$\lesssim \|h\|_{L^{p'}(K\times\mathcal{I})} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)},$$

where

$$R_t(G)(x) = \int_\kappa^{1/t} r^{m_2} Q_{tr}(G)(x) \frac{dr}{r} \quad \text{and} \quad \widetilde{P}_t^v := P_t^v P_{1,t}.$$

Let us write

$$\mathscr{I} = \int_{K\times\mathcal{I}} \int_0^1 h(x,s)(Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(\widetilde{P}_t^v [R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s) \frac{dt}{t}\, dx\, ds$$
$$= I + II,$$

where I and II will be defined later but the idea will be to transfer a little bit of the information of the multipliers Q_t^u and $\widetilde{P}_t^u$ to the product hm. First, we assume without loss of generality that the x-support of h is contained in the x-support of $m(t,x,s)$. We write

$$\widetilde{P}_t^v = P_t^{v,2} + Q_t^{v,2},$$

where the symbol of $P_t^{v,2}$ is given by $\widehat{\theta}_2(t\xi)e^{it\xi\cdot v}$. The support of $\widehat{\theta}_2$ is chosen small enough to assure that, by choosing a suitable radial function $\widehat{\psi}_2$ supported in an annulus, the identity

$$(Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s)$$
$$= \widetilde{Q}_t (Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s)$$

holds. We have denoted by $\widetilde{Q}_t$ the multiplier with symbol $\widehat{\psi}(t\xi)$. In a similar way, we can choose a radial function $\widehat{\theta}_3$ supported in a ball such that

$$(Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s)$$
$$= \widetilde{P}_t (Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s),$$

with $\widetilde{P}_t$ denoting the multiplier with symbol $\widehat{\theta}_3(t\xi)$. We have that

$$|\mathscr{I}| = \left| \int_{K\times\mathcal{I}} \int_0^1 h(x,s)(Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(\widetilde{P}_t^v [R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s) \frac{dt}{t}\, dx\, ds \right|$$
$$\leq \left| \int_{K\times\mathcal{I}} \int_0^1 h(x,s)(Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s) \frac{dt}{t}\, dx\, ds \right|$$
$$+ \left| \int_{K\times\mathcal{I}} \int_0^1 h(x,s)(Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \right.$$
$$\left. \times m(t,x,s) \frac{dt}{t}\, dx\, ds \right|$$

$$
\begin{aligned}
&= \left| \int_{K\times\mathcal{I}} \int_0^1 \widetilde{Q}_t (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \right. \\
&\qquad \left. \times h(x,s) m(t,x,s) \frac{dt}{t}\, dx\, ds \right| \\
&\quad + \left| \int_{K\times\mathcal{I}} \int_0^1 \widetilde{P}_t (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \right. \\
&\qquad \left. \times h(x,s) m(t,x,s) \frac{dt}{t}\, dx\, ds \right| \\
&= \left| \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \right. \\
&\qquad \left. \times \widetilde{Q}_t (h(\cdot,s) m(t,\cdot,s))(x) \frac{dt}{t}\, dx\, ds \right| \\
&\quad + \left| \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \right. \\
&\qquad \left. \times \widetilde{P}_t (h(\cdot,s) m(t,\cdot,s))(x) \frac{dt}{t}\, dx\, ds \right|.
\end{aligned}
$$

Let us define

$$
\begin{aligned}
I &= \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \\
&\qquad \times \widetilde{Q}_t (h(\cdot,s) m(t,\cdot,s))(x) \frac{dt}{t}\, dx\, ds, \\
II &= \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \\
&\qquad \times \widetilde{P}_t (h(\cdot,s) m(t,\cdot,s))(x) \frac{dt}{t}\, dx\, ds.
\end{aligned}
$$

Now, consider the multiplication operator

$$
M_m G \equiv M_m^{t,s} G := m(t,x,s) G. \tag{4-13}
$$

The idea will be to estimate I and II by carrying out an analysis over the commutators $[\widetilde{P}_t, M_m]$ and $[\widetilde{Q}_t, M_m]$.

Starting with I,

$$
\begin{aligned}
I &= \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s) \\
&\qquad \times (\widetilde{Q}_t h(\cdot,s))(x) \frac{dt}{t}\, dx\, ds \\
&\quad + \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \\
&\qquad \times ([\widetilde{Q}_t, M_m] h(\cdot,s))(x) \frac{dt}{t}\, dx\, ds \\
&= I_4 + I_5,
\end{aligned} \tag{4-14}
$$

with

$$I_5 = \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times [\widetilde{Q}_t, M_m](h(\cdot,s))(x)\,\frac{dt}{t}\,dx\,ds.$$

Similarly,

$$\begin{aligned} II &= \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s)m(t,x,s) \times (\widetilde{P}_t h(\cdot,s))(x)\,\frac{dt}{t}\,dx\,ds \\ &\quad + \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times [\widetilde{P}_t, M_m](h(\cdot,s))(x)\,\frac{dt}{t}\,dx\,ds \\ &= I_6 + I_7, \end{aligned}$$

with

$$I_7 = \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{v_{m_1}} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times [\widetilde{P}_t, M_m]h(\cdot,s)(x)\,\frac{dt}{t}\,dx\,ds.$$

The terms involving the commutators, namely I_5 and I_7, can be estimated due to the $L^{p'}$-boundedness of $[\widetilde{P}_t, M_m]$ and of $[\widetilde{Q}_t, M_m]$. Indeed, they satisfy the boundedness properties

$$\begin{aligned} \|[\widetilde{P}_t, M_m]h(\cdot,s)\|_{L^{p'}(\mathbb{R}^d_x)} &\lesssim t\|h(\cdot,s)\|_{L^{p'}(\mathbb{R}^2_x)}, \\ \|[\widetilde{Q}_t, M_m]h(\cdot,s)\|_{L^{p'}(\mathbb{R}^d_x)} &\lesssim t\|h(\cdot,s)\|_{L^{p'}(\mathbb{R}^d_x)}. \end{aligned}$$

In view of the hypothesis (H1) in Theorem 2.6,

$$\|P_t^{v,2}[R_t T_\gamma^{\phi_2}]g\|_{L^{p_2}(K\times\mathcal{I})} \lesssim C(v)\|T_\gamma^{\phi_2} g\|_{L^{p_2}(\mathbb{R}^d\times\mathbb{R})} \lesssim C(v)\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}, \tag{4-15}$$

and also

$$\|Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g\|_{L^{p_2}(K\times\mathcal{I})} \lesssim C(v)\|T_\gamma^{\phi_2} g\|_{L^{p_2}(\mathbb{R}^d\times\mathbb{R})} \lesssim C(v)\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}. \tag{4-16}$$

Thus, the analysis above implies that

$$|I_5| + |I_7| \lesssim C(u,v,\varepsilon)\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}\|h\|_{L^{p'}(\mathbb{R}^d_x\times\mathbb{R}_s)}.$$

In order to finish our analysis of $\mathscr{I}$ let us estimate the terms I_4 and I_6. By the

Cauchy–Schwarz inequality,

$$
\begin{aligned}
|I_4| &= \left| \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \right. \\
&\qquad \left. \times m(t,x,s)(\widetilde{Q}_t h(\cdot,s))(x)\,\frac{dt}{t}\,dx\,ds \right| \\
&= \left| \int_{\mathcal{I}}\int_K \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \right. \\
&\qquad \left. \times m(t,x,s)(\widetilde{Q}_t h(\cdot,s))(x)\,\frac{dt}{t}\,dx\,ds \right| \\
&\le \left(\int_{\mathcal{I}}\int_K \int_0^1 |(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)|^2\,\frac{dt}{t}\,dx\,ds \right)^{1/2} \\
&\quad \times \left(\int_{\mathcal{I}}\int_K \int_0^1 |(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s)(\widetilde{Q}_t h(\cdot,s))(x)|^2\,\frac{dt}{t}\,dx\,ds \right)^{1/2}.
\end{aligned}
$$

For the first term, Minkowski's integral inequality and Remark 4.3 imply that

$$
\begin{aligned}
&\left(\int_{\mathcal{I}}\int_K \int_0^1 |(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)|^2\,\frac{dt}{t}\,dx\,ds \right)^{1/2} \\
&\qquad \le \int_{\mathcal{I}}\int_K \left(\int_0^1 |(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)|^2\,\frac{dt}{t} \right)^{1/2} dx\,ds \\
&\qquad \le \int_{\mathcal{I}} \left(\int_K \left(\int_0^1 |(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)|^2\,\frac{dt}{t} \right)^{p_1/2} dx \right)^{1/p_1} \left(\int_K dx \right)^{1/p_1'} ds \\
&\qquad \lesssim \int_{\mathcal{I}} \left\| \left(\int_0^1 |(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)|^2\,\frac{dt}{t} \right)^{1/2} \right\|_{L^{p_1}(K)} ds \\
&\qquad \lesssim \int_{\mathcal{I}} \|T^{\phi_1}_{\nu_{m_1}} f\|_{L^{p_1}}\,ds \\
&\qquad \lesssim \left(\int_{\mathcal{I}} \|T^{\phi_1}_{\nu_{m_1}} f\|_{L^{p_1}}^{p_1}\,ds \right)^{1/p_1} \left(\int_{\mathcal{I}} ds \right)^{1/p_1'} ds.
\end{aligned}
$$

For the second factor, observe that

$$
\begin{aligned}
&\left(\int_{\mathcal{I}}\int_K \int_0^1 |(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s)(\widetilde{Q}_t h(\cdot,s))(x)|^2\,\frac{dt}{t}\,dx\,ds \right)^{1/2} \\
&\quad \le \|m\|_{L^\infty} \left(\int_{\mathcal{I}}\int_K \int_0^1 |(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s)(\widetilde{Q}_t h(\cdot,s))(x)|^2\,\frac{dt}{t}\,dx\,ds \right)^{1/2}.
\end{aligned}
$$

Let

$$
A_t = (P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s), \quad B_t = (\widetilde{Q}_t h(\cdot,s))(x).
$$

We can estimate the integral involving these two terms using repeatedly Hölder's inequality after using Minkowski's inequality. Indeed,

$$\begin{aligned}
&\Big(\int_{\mathcal{I}}\int_K\int_0^1 |A_tB_t|^2\,\frac{dt}{t}\,dx\,ds\Big)^{1/2}\\
&\quad\le \int_{\mathcal{I}}\int_K\Big(\int_0^1 |A_tB_t|^2\,\frac{dt}{t}\Big)^{1/2}\,dx\,ds\\
&\quad\le \int_{\mathcal{I}}\int_K \sup_{0<t<1}|A_t|\Big(\int_0^1 |B_t|^2\,\frac{dt}{t}\Big)^{1/2}\,dx\,ds\\
&\quad\le \Big(\int_{\mathcal{I}}\int_K \sup_{0<t<1}|A_t|^p\,dx\,ds\Big)^{1/p}\Big(\int_{\mathcal{I}}\int_K\Big(\int_0^1 |B_t|^2\,\frac{dt}{t}\Big)^{p'/2}\,dx\,ds\Big)^{1/p'}\\
&\quad\le \Big(\Big(\int_{\mathcal{I}}\int_K \sup_{0<t<1}|A_t|^{pp_2/p}\,dx\,ds\Big)^{p/p_2}\Big(\int_{K\times\mathcal{I}}dx\,ds\Big)^{1/(p_2/p)'}\Big)^{1/p}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\Big\|\Big(\int_0^1 |B_t|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^{p'}}\\
&\quad\lesssim \big\|\sup_{0<t<1}|A_t|\big\|_{L^{p_2}}\Big\|\Big(\int_0^1 |B_t|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^{p'}}.
\end{aligned}$$

Note that the use of Hölder's inequality before is justified by the fact that $\frac{p_2}{p}\ge 1$, since $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}\ge\frac{1}{p_2}$. In view of the hypothesis (H1) in Theorem 2.6 and of Remark 4.3,

$$\begin{aligned}
\big\|\sup_{0<t<1}|A_t|\big\|_{L^{p_2}}\Big\|\Big(\int_0^1 |B_t|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^{p'}} &\lesssim C(v)\|T^{\phi_2}_\gamma g\|_{L^{p_2}(\mathbb{R}^d\times\mathbb{R})}\|h\|_{L^{p'}(\mathbb{R}^d\times\mathbb{R})}\\
&\lesssim C(v)\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}\|h\|_{L^{p'}(\mathbb{R}^d_x\times\mathbb{R}_s)}.
\end{aligned}$$

Observe that we have used that the estimate

$$\big\|\sup_{0<t<1}|A_t|\big\|_{L^{p_2}}\lesssim C(v)\|T^{\phi_2}_\gamma g\|_{L^{p_2}(\mathbb{R}^d\times\mathbb{R})} \tag{4-17}$$

holds. Indeed, in order to prove this fact observe that

$$A_t=(P_t^{v,2}[R_tT^{\phi_2}_\gamma]g)(x,s)=R_t(P_t^{v,2}[T^{\phi_2}_\gamma]g)(x,s).$$

Now, the action of the multiplier $R_tP_t^{v,2}$ on $G=G_s=T^{\phi_2}_\gamma g(\cdot,s)$ is given by

$$\begin{aligned}
R_t(P_t^{v,2}G)(x)&=\int_\kappa^{1/t} s^{m_2}Q_{ts}P_t^{v,2}G(x)\,\frac{ds}{s}\\
&=\int_\kappa^{1/t} s^{m_2}\int e^{ix\cdot\xi}\widehat{\psi}(ts\xi)\widehat{\theta}_2(t\xi)e^{it\xi\cdot v}\widehat{G}(\xi)\,d\xi\,\frac{ds}{s}.
\end{aligned}$$

Since the supports of $\widehat{\psi}$ and of $\widehat{\theta}_2$ are contained in small annuli, on the support of these functions,

$$|\xi| \asymp \frac{1}{ts} \asymp \frac{1}{t},$$

which implies that $s \asymp 1$, and the integral in s is supported on an interval around $s = 1$ and on an annulus isolating $s = 0$. So

$$\begin{aligned}
|R_t(P_t^{v,2}G)(x)| &\lesssim \int_{s\asymp 1} s^{m_2} \left| \int e^{ix\cdot\xi} \widehat{\psi}(ts\xi)\widehat{\theta}_2(t\xi) e^{it\xi\cdot v} \widehat{G}(\xi)\, d\xi \right| \frac{ds}{s} \\
&\lesssim \int_{s\asymp 1} \left| \int e^{ix\cdot\xi} \widehat{\psi}(ts\xi)\widehat{\theta}_2(t\xi) e^{it\xi\cdot v} \widehat{G}(\xi)\, d\xi \right| ds \\
&= \int_{s\asymp 1} |\widehat{\psi}(tsD) P_t^{v,2} G(x)|\, ds.
\end{aligned}$$

Let θ_2^v be the kernel with Fourier transform $\widehat{\theta}_2^v(\cdot)$, i.e., it is the symbol of $P_t^{v,2}$, that is,

$$\widehat{\theta}_2^v(t\xi) := \widehat{\theta}_2(t\xi) e^{it\xi\cdot v}.$$

Then

$$\begin{aligned}
&\int_{s\asymp 1} |\widehat{\psi}(tsD) P_t^{v,2} G(x)|\, ds \\
&\qquad = \int_{s\asymp 1} |G * (s^{-n} t^{-n} (\psi(s^{-1}\cdot) * (\theta_2^v)(t^{-1}\cdot)))(x)|\, ds \\
&\qquad = \int_{s\asymp 1} \left| \int G(x-y) s^{-n} t^{-n} (\psi(s^{-1}\cdot) * (\theta_2^v))(t^{-1} y)) dy \right| ds \\
&\qquad \asymp \int_{s\asymp 1} \left| \int G(x-y) t^{-n} (\psi(s^{-1}\cdot) * (\theta_2^v))(t^{-1} y))\, dy \right| ds \\
&\qquad \le \|G\|_{L^\infty} \int_{s\asymp 1} t^{-n} \left| \int (\psi(s^{-1}\cdot) * (\theta_2^v))(t^{-1} y))\, dy \right| ds \\
&\qquad \le \|G\|_{L^\infty} \int_{s\asymp 1} \left| \int (\psi(s^{-1}\cdot) * (\theta_2^v)(y))\, dy \right| ds.
\end{aligned}$$

As a consequence,

$$\Big\| \sup_{0<t<1} |R_t(P_t^{v,2}G)| \Big\|_{L^\infty} \lesssim \|G\|_{L^\infty}. \tag{4-18}$$

Now, let us prove the estimate

$$\Big\| \sup_{0<t<1} |R_t(P_t^{v,2}G)| \Big\|_{L^2} \lesssim \|G\|_{L^2}. \tag{4-19}$$

For this we will use Bourgain's lemma (see Theorem 4.2). We proceed with the proof of (4-19) in the following remark.

Remark 4.4. Note that

$$\begin{aligned} R_t(P_t^{v,2}G)(x) &= \int_{s\asymp 1} s^{m_2} \int e^{ix\cdot\xi}\widehat{\psi}(ts\xi)\widehat{\theta}_2(t\xi)e^{it\xi\cdot v}\widehat{G}(\xi)d\xi\,\frac{ds}{s} \\ &= \int_{s\asymp 1} s^{m_2}\widehat{\psi}(tsD)P_t^{v,2}G(x)\,\frac{ds}{s} \\ &= \int_{s\asymp 1} s^{m_2} G * \big(s^{-n}t^{-n}(\psi(s^{-1}\cdot) * (\theta_2^v)(t^{-1}\cdot))\big)(x)\,\frac{ds}{s} \\ &= \left(G * \int_{s\asymp 1} s^{m_2}s^{-n}t^{-n}(\psi(s^{-1}\cdot) * (\theta_2^v)(t^{-1}\cdot))\,\frac{ds}{s}\right)(x). \end{aligned}$$

We have that the kernel K_t of $R_t P_t^{v,2}$ is given by

$$K_t(x) = \int_{s\asymp 1} s^{m_2}\big(s^{-n}t^{-n}(\psi(s^{-1}\cdot) * (\theta_2^v)(t^{-1}x))\big)\,\frac{ds}{s}. \tag{4-20}$$

Let us write

$$K(x) = \int_{s\asymp 1} s^{m_2}s^{-n}(\psi(s^{-1}\cdot) * (\theta_2^v))(x)\,\frac{ds}{s}.$$

Its Fourier transform is

$$\widehat{K}(\xi) = \int_{s\asymp 1} s^{m_2-n}\mathscr{F}(\psi(s^{-1}\cdot) * (\theta_2^v))(\xi))\,\frac{ds}{s}, \tag{4-21}$$

and

$$\begin{aligned} \alpha_j := \sup_{|\xi|\sim 2^j} |\widehat{K}(\xi)| &\le \int_{s\asymp 1} \|\mathscr{F}(\psi(s^{-1}\cdot) * (\theta_2^v))\|_{L^\infty} s^{m_2-n}\,\frac{ds}{s} \\ &\lesssim \int_{s\asymp 1} \|\psi(s^{-1}\cdot) * (\theta_2^v)\|_{L^1}\,\frac{ds}{s}. \end{aligned}$$

This shows that $\sup_j \sup_{|\xi|\sim 2^j} |\widehat{K}(\xi)|$ is bounded. However, due to the properties of the support of ψ and θ_2^v we can improve this bound as

$$\alpha_j \lesssim \sup_{|\xi|\sim 2^j} \int_{s\asymp 1} |\widehat{\psi}(s\xi)\widehat{\theta}_2^v(\xi)|\,\frac{ds}{s} \lesssim \sup_{|\xi|\sim 2^j} \min\{|\xi|, |\xi|^{-1}\} = 2^{-|j|}.$$

On the other hand

$$\nabla\widehat{K}(\xi) = \left(\int_{s\asymp 1} s^{m_2-n}\mathscr{F}((\psi(s^{-1}\cdot) * (\theta_2^v))ix_j)(\xi))\,\frac{ds}{s}\right)_{1\le j\le d}.$$

Therefore, we have the estimate

$$\sup_{|\xi|\sim 2^j} |\langle\nabla\widehat{K}(\xi), \xi\rangle| \le \sum_{r=1}^{d} \int_{s\asymp 1} s^{m_2-n}|\mathscr{F}((\psi(s^{-1}\cdot) * (\theta_2^v))ix_r)(\xi))|\,|\xi_r|\,\frac{ds}{s}.$$

Observe that, using integration by parts,

$$\begin{aligned}\mathscr{F}\big((\psi(s^{-1}\cdot)*(\theta_2^v)ix_r)(\xi)\big)\xi_r &= \int e^{ix\xi}(\psi(s^{-1}\cdot)*(\theta_2^v))(x)ix_r\xi_r\,dx\\ &= \int \partial_{x_r}(e^{ix\xi})(\psi(s^{-1}\cdot)*(\theta_2^v))(x)x_r\,dx\\ &= -\int e^{ix\xi}\partial_{x_r}[(\psi(s^{-1}\cdot)*(\theta_2^v)(x)]x_r\,dx.\end{aligned}$$

We have proved that

$$\begin{aligned}\sup_{|\xi|\sim 2^j}|\langle\nabla\widehat{K}(\xi),\xi\rangle| &\le \sum_{r=1}^d\int_{s\asymp 1} s^{m_2-n}\big\|\mathscr{F}\big(\partial_{x_r}[(\psi(s^{-1}\cdot)*(\theta_2^v))(x)]x_r\big)\big\|_{L^\infty}\frac{ds}{s}\\ &\le \sum_{r=1}^d\int_{s\asymp 1}\|\partial_{x_r}[(\psi(s^{-1}\cdot)*(\theta_2^v)(x)]x_r\|_{L^1}\frac{ds}{s}<\infty.\end{aligned}$$

We have included the previous computation to emphasise the cancellation of the quantity $|\xi_r|\lesssim 2^j$ after using integration by parts which we found interesting. Nevertheless, one can improve this bound. Indeed, due to the properties of the support of ψ and θ_2^v,

$$Z_r := (\psi(s^{-1}\cdot)*(\theta_2^v))ix_r\in\mathscr{S}(\mathbb{R}^d),$$

and

$$\begin{aligned}\beta_j &\lesssim \sup_{|\xi|\sim 2^j}\sum_{r=1}^d\int_{s\asymp 1}|\mathscr{F}(Z_r)(\xi)|\,|\xi_r|\,ds\\ &\asymp \sup_{|\xi|\sim 2^j}\sum_{r=1}^d\int_{s\asymp 1}|\mathscr{F}(\partial_{x_r}Z_r)(\xi)|\,ds\lesssim \sup_{|\xi|\sim 2^j}\min\{|\xi|,|\xi|^{-1}\}=2^{-|j|}.\end{aligned}$$

As a consequence,

$$\Gamma(K) := \sum_{j\in\mathbb{Z}}\alpha_j^{1/2}(\alpha_j+\beta_j)^{1/2}<\infty. \tag{4-22}$$

In view of Theorem 4.2 we have proved that

$$G\mapsto S^*G := \sup_{0<t<1}|R_t(P_t^{v,2}G)| : L^2\to L^2$$

is bounded.

So, interpolating the L^p-estimates (4-18) and (4-19) for the sublinear operator S^*, defined via

$$S^*G := \sup_{0<t<1}|R_t(P_t^{v,2}G)|,$$

in view of the Marcinkiewicz interpolation theorem we have proved that for any $2 \leq \kappa \leq \infty$,

$$\Big\| \sup_{0<t<1} |R_t(P_t^{v,2} G)| \Big\|_{L^\kappa} \lesssim C_\kappa(v) \|G\|_{L^\kappa}, \tag{4-23}$$

with a polynomial constant $C_\varkappa(v)$ in v. With $\kappa = p_2$ we have proved (4-17) once we take the L^{p_2}-norm in the s-variable on both sides of (4-23). As a consequence,

$$|I_4| \lesssim \|h\|_{L^{p'}(K\times\mathcal{I})} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)},$$

since h is assumed to be supported in $K \times I$. Since the same analysis done before for I_4 also works for I_6,

$$|I_4| + |I_6| \lesssim \|h\|_{L^{p'}(K\times\mathcal{I})} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.$$

We have completed the proof of the estimate

$$|\mathscr{I}| \lesssim \|h\|_{L^{p'}(K\times\mathcal{I})} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.$$

It follows that

$$\|I_1\|_{L^p(K\times\mathcal{I})} \lesssim_{u,v} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.$$

To estimate I_2, note that the hypothesis (H1) in Theorem 2.6 implies that

$$\|Q_t^u T^{\phi_1}_{\nu_{m_1}} f(x,s)\|_{L^{p_1}(K\times\mathcal{I})} \lesssim_u \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}.$$

Then Hölder's inequality gives

$$\begin{aligned}
\|I_2\|_{L^p(K\times\mathcal{I})} &\leq \int_0^1 \|Q_t^u T^{\phi_1}_{\nu_{m_1}} f(x,s) E_t^2 g(x,s) m(t,x,s)\|_{L^p(K\times\mathcal{I})} \frac{dt}{t} \\
&\leq \int_0^1 \|Q_t^u T^{\phi_1}_{\nu_{m_1}} f(x,s)\|_{L^{p_1}(K\times\mathcal{I})} \|E_t^2 g(x,s) m(t,x,s)\|_{L^{p_2}(K\times\mathcal{I})} \frac{dt}{t} \\
&\leq \int_0^1 \|Q_t^u T^{\phi_1}_{\nu_{m_1}} f(x,s)\|_{L^{p_1}(K\times\mathcal{I})} \|E_t^2 g(x,s)\|_{L^{p_2}(K\times\mathcal{I})} \|m\|_{L^\infty} \frac{dt}{t} \\
&\leq \int_0^1 \|E_t^2 g(x,s)\|_{L^{p_2}(K\times\mathcal{I})} \|m\|_{L^\infty} \frac{dt}{t} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \\
&\lesssim_u \int_0^1 t^{\varepsilon-1} \|m\|_{L^\infty} dt \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.
\end{aligned}$$

The same analysis above applies for I_3. Remember that $\|m\|_{L^\infty} := \|m\|_{L^\infty_U}$ denotes the L^∞-norm of m as a function of (t,x,s) and then it is bounded polynomially as a function of $U = 1 + |u| + |v|$. As a consequence, taking N large enough, we

have from (3-8) that

$$\begin{aligned}\|T^{\phi_1,\phi_2}_{a_1}(\mu(D)f,\mu(D)g)(x,s)\|_{L^p(K\times\mathcal{I})}&\\ &\leq \iint \frac{\|m\|_{L^\infty_U}}{(1+|u|^2+|v|^2)^N}\,du\,dv\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}\\ &\lesssim_N \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.\end{aligned}$$

The proof of Theorem 2.6 is complete. □

Appendix: Bibliographic discussion

The study of the continuity properties for FIOs has attracted considerable attention due to their applications to hyperbolic PDEs. Initial enquiries into global L^p boundedness results can be traced to the pioneering works [Asada and Fujiwara 1978; Peral 1980; Miyachi 1980]. Subsequent research has significantly expanded these results to a wide classes of symbols [Ruzhansky and Sugimoto 2006; 2011; Coriasco and Ruzhansky 2014; Coriasco et al. 2016]. A notable line of enquiry involves establishing global criteria for FIOs within Hörmander symbol classes $S^m_{\rho,\delta}$, specifically by relaxing the traditional $\delta < \rho$ condition in the setting of pseudo-differential operators to allow broader parameter ranges $0 \leq \delta < 1$ and $0 < \rho \leq 1$ [Dos Santos Ferreira and Staubach 2014; Kenig and Staubach 2007; Israelsson et al. 2021; 2023; Rodríguez-López et al. 2014; 2015]. This development was partly inspired by analogous results in the theory of pseudodifferential operators, which demonstrated boundedness without the $\delta < \rho$ restriction [Delgado 2006; Álvarez and Hounie 1990]. For general pseudodifferential operators on graded Lie groups we refer to the work of D. Cardona with M. Ruzhansky and J. Delgado [Cardona et al. 2021]. The question of global parametrisation for Lagrangian distributions has also been explored through alternative geometric and analytical approaches [Laptev et al. 1994; Ruzhansky 2001]. Additionally, global definitions of FIOs have been formulated in specialised settings, such as on compact Lie groups using the representation theory of these groups [Cardona and Ruzhansky 2026].

The systematic theory of FIOs with complex-phase functions originated from [Melin and Sjöstrand 1974], motivated by constructing parametrices for partial differential operators with complex principal symbols [Melin and Sjöstrand 1976]. This framework has found significant application in several complex variables, where the singular structure of the Bergman kernel can be described via asymptotic expansions involving complex-phase FIOs [Boutet de Monvel and Sjöstrand 1976]. The comprehensive analysis of the Bergman kernel itself represents another major area of study [Fefferman 1974; 1976].

A distinct and highly active research direction concerns smoothing estimates for FIOs, particularly following the resolution of the decoupling conjecture by Bourgain

and Demeter [2015]. The conjecture regarding the inherent smoothing properties of FIOs was formally posed by Sogge [1991]. While a complete survey of this extensive field is beyond our scope, the literature includes influential early results [Bourgain 1991] and numerous recent advances that explore various aspects of this smoothing phenomenon, employing diverse methods from harmonic analysis; see [Minicozzi and Sogge 1997; Guth et al. 2020; Beltran et al. 2020; Gao et al. 2023].

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DUVÁN CARDONA
DEPARTMENT OF MATHEMATICS
KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAN
KINGDOM OF SAUDI ARABIA
duvan.sanchez@kfupm.edu.sa

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SECOND-ORDER ESTIMATES FOR χ-SEMICONVEX SOLUTIONS OF HESSIAN EQUATIONS ON HERMITIAN MANIFOLDS

XIAOJUAN CHEN, QIANG TU AND NI XIANG

We establish the modified concavity inequality for complex Hessian equations under the semiconvexity assumption inspired by Lu (2023) and Zhang (2025) for the real case. Then second-order estimates for admissible solutions of complex Hessian equations on compact Hermitian manifolds with both sides of equations depending on gradient terms are obtained by taking advantage of this crucial inequality.

1. Introduction

Let (M, ω) be a compact complex manifold of complex dimension $n \geq 2$ without boundary, and ω be the Hermitian metric on M. For any $u \in C^\infty(M)$, we consider a smooth real (1, 1)-form $\chi'(z, u)$ satisfying the positivity condition $\chi' \geq \varepsilon\omega$ for some $\varepsilon > 0$. Given any smooth (1, 0)-form a on M, we obtain a new real (1, 1)-form

$$\chi(z, u) = \chi'(z, u) + \sqrt{-1}a \wedge \bar{\partial}u - \sqrt{-1}\bar{a} \wedge \partial u + \sqrt{-1}\partial\bar{\partial}u. \tag{1-1}$$

We study second-order estimates for the complex Hessian equation

$$\chi^k \wedge \omega^{n-k} = \psi(z, Du, u)\omega^n \tag{1-2}$$

for $1 \leq k \leq n$ since the a priori estimates are crucial in the use of the continuity method, which is a natural approach to solve the equation (1-2). Here $\psi(z, v, u) \in C^\infty((T^{1,0}(M))^* \times \mathbb{R})$ is a given positive function, and D is the Chern connection with respect to the metric ω.

The complex Hessian equation (1-2) without gradient term on both sides (i.e., $\chi = \chi' + \sqrt{-1}\partial\bar{\partial}u$, $\psi = \psi(z, u)$) has been extensively studied due to its geometric applications such as J-flow (see [Song and Weinkove 2008]) and quaternionic geometry (see [Alesker and Verbitsky 2010]). Li [2004], and later Błocki [2005], considered the Dirichlet problem for complex Hessian equations on domain in $\mathbb{C}^n$. On compact Kähler manifolds, Hou [2009] proved the existence of a smooth admissible solution of (1-2) when $\chi' = \omega$, $a = 0$ and $\psi = \psi(z)$ by assuming the

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nonnegativity of the holomorphic bisectional curvature of ω. The curvature assumption was subsequently removed by Hou, Ma and Wu [Hou et al. 2010] and Dinew and Kołodziej [2014]. On compact Hermitian manifolds, the authors in [Cherrier 1987; Hanani 1996; Guan and Li 2009; Zhang 2010; Tosatti and Weinkove 2010; Zhang and Zhang 2011] studied complex Monge–Ampère equations. For complex Hessian equations or more general fully nonlinear elliptic equations, the related results can be found in [Sun 2017; Zhang 2017; Kołodziej and Nguyen 2016; Székelyhidi 2018].

Furthermore, complex Hessian equations with gradient terms on the left-hand side arise from the Gauduchon conjecture, which was studied by Székelyhidi, Tosatti and Weinkove [Székelyhidi et al. 2017], see also [Guan and Nie 2023] for related work. Complex Monge–Ampère equations with gradient terms inside the determinant can be found in the study of the Calabi–Yau equation on certain symplectic non-Kähler 4-manifolds, see Fino, Li, Salamon and Vezzoni's work [Fino et al. 2013]. See also [Tosatti and Weinkove 2021], where they solved complex Monge–Ampère equations with gradient terms left open by Yuan [2018]. More results for complex Hessian equations or general fully nonlinear elliptic equations with linear gradient terms on the left-hand side can be found in the works of Zhang [2010], Feng, Ge and Zheng [Feng et al. 2019] and Yuan [2018; 2020].

In the above works, the right-hand side function ψ is independent of Du. In fact, (1-2) with $\psi = \psi(z, Du, u)$ has been relatively less studied. An important case corresponding to $k = n = 2$ (i.e., complex Monge–Ampère equations in two dimensions) was solved in [Fu and Yau 2007; 2008] for a Strominger system on a toric fibration over a K3 surface. For general dimension n, the corresponding results on compact Kähler manifolds or Hermitian manifolds can be found in the works of Phong, Picard and Zhang [Phong et al. 2016; 2017; 2019; 2021] and Chu, Huang and Zhu [Chu et al. 2019a; 2019b]. Further, complex Hessian equations with gradient terms on both sides were studied in [Dong and Li 2021] on Hermitian manifolds, and second-order estimates were obtained for $\chi \in \Gamma_n(M)$. Subsequently, Dong [2021a; 2021b] proved second-order estimates under a weaker convexity assumption $\chi \in \Gamma_{k+1}(M)$ for (1-2) and a class of general Hessian equations on Hermitian manifolds, respectively.

From an analysis point of view, a natural problem is whether we can weaken the assumption $\chi \in \Gamma_{k+1}(M)$ in [Dong 2021b] to a more natural semiconvex assumption inspired by Lu [2023] and Zhang [2025]. Now we define χ-semiconvex function on Hermitian manifolds, which generalizes the definition of semiconvex function for the real case.

Definition 1.1. Let $\chi \in \mathcal{A}^{1,1}(M)$ be a smooth real $(1, 1)$-form defined by (1-1). If there exists a constant $A > 0$ such that for the Hermitian metric ω on M,

$$\chi + A\omega > 0,$$

then χ is semiconvex. We say the function u is χ-semiconvex if χ is semiconvex.

Definition 1.2. A function $u \in C^2(M)$ is called an admissible solution of (1-2) if $\chi \in \Gamma_k(M)$. The Gårding's cone on M is defined by

$$\Gamma_k(M) = \{\chi \in \mathcal{A}^{1,1}(M) : \sigma_i(\chi) > 0 \text{ for all } 1 \le i \le k\},$$

where $\mathcal{A}^{1,1}(M)$ denotes the space of real smooth $(1,1)$-forms on a Hermitian manifold (M, ω).

The main theorem is as follows.

Theorem 1.3. *Let (M, ω) be a compact Hermitian manifold of complex dimension n. Suppose $\chi'(z, u) \ge \varepsilon\omega$ and $u \in C^\infty(M)$ is an admissible, χ-semiconvex solution of* (1-2)*. Then we have uniform second-order estimates*

$$|D\bar{D}u| \le C,$$

where D is the Chern connection with respect to the metric ω, $|D\bar{D}u|$ is the second-order derivative of u with respect to the Chern connection of ω and C is a uniform constant depending on (M, ω), n, k, ε, χ', ψ, a, A, $\sup_M |u|$ and $\sup_M |Du|$.

It is easy to see that $\sigma_{k+1}(\chi) > -A$ for some positive constant A implies the χ-semiconvex condition according to Lemma 2.2 in [Zhang 2025].

Corollary 1.4. *Let (M, ω) be a compact Hermitian manifold of complex dimension n. Suppose $\chi'(z, u) \ge \varepsilon\omega$ and $u \in C^\infty(M)$ is an admissible solution of* (1-2) *with $\sigma_{k+1}(\chi) > -A$ for some positive constant A. Then we have uniform second-order estimates*

$$|D\bar{D}u| \le C,$$

where C is a uniform constant depending on (M, ω), n, k, ε, χ', ψ, a, A, $\sup_M |u|$ and $\sup_M |Du|$.

Due to the presence of Du in both sides of (1-2), the principal challenge in establishing second-order estimates is managing terms such as $|DDu|^2$, $|D\bar{D}u|^2$ and some bad third-order terms when differentiating the equation twice. However, we cannot directly control bad terms as in the real case owing to complex conjugacy. Furthermore, in comparison to calculations on Kähler manifolds, a greater number of undesirable third-order terms in the form of $T * D^3u$ are produced, here T is the torsion of ω on Hermitian manifolds. To surmount these obstacles, it is essential to establish a concavity inequality. Lu [2023] established curvature estimates for semiconvex solutions of real Hessian equations based on the concavity inequality

$$-\sum_{p\neq q} \sigma_k^{pp,qq}\xi_p\xi_q + \frac{\left(\sum_i \sigma_k^{ii}\xi_i\right)^2}{\sigma_k} + \delta_0 \sum_{i>l} \frac{\sigma_k^{ii}\xi_i^2}{\lambda_1} \ge (1-\epsilon)\frac{\sigma_k\xi_1^2}{\lambda_1^2}, \tag{1-3}$$

where $\xi = (\xi_1, \ldots, \xi_n)$ is an arbitrary vector in $\mathbb{R}^n$. Recently, Zhang [2025] improved the concavity inequality (1-3) and derived the following lemma.

Lemma 1.5 [Zhang 2025]. *Assume $\lambda=(\lambda_1,\dots,\lambda_n)\in\Gamma_k\subset\mathbb{R}^n$, $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$ and $\lambda_n>-A$ for a constant $A>0$. Then there exists some constant $C>0$ depending on n, k, σ_k, A and small constant $\delta_0>0$ depending on k such that if $\lambda_1\geq C$, then*

$$-\sum_{p\neq q}\sigma_k^{pp,qq}\xi_p\xi_q+K\frac{\left(\sum_i\sigma_k^{ii}\xi_i\right)^2}{\sigma_k}+2\sum_{i>1}\frac{\sigma_k^{ii}\xi_i^2}{(\lambda_1+A+1)}\geq(1+\delta_0)\frac{\sigma_k^{11}\xi_1^2}{\lambda_1} \tag{1-4}$$

for some sufficiently large $K>0$ (depending on δ_0), where $\xi=(\xi_1,\dots,\xi_n)$ is an arbitrary vector in $\mathbb{R}^n$.

By comparing (1-3) and (1-4), we see that each of these two inequalities has its own advantages. Specifically, we find that the term $\delta_0\sum_{i>l}\sigma_k^{ii}\xi_i^2/\lambda_1$ in (1-3) can be easier to control than the term $2\sum_{i>1}\sigma_k^{ii}\xi_i^2/(\lambda_1+A+1)$ in (1-4) since δ_0 can be chosen arbitrarily small. However, $\sigma_k\xi_1^2/\lambda_1^2$ in (1-3) will lead to the bad term $\sigma_k(\lambda|1)$, which is unmanageable term without the assumption $\chi\in\Gamma_{k+1}(M)$. It seems that the term $(1+\delta_0)\sigma_k^{11}\xi_1^2/\lambda_1\sigma_k$ in (1-4) is a higher choice. For complex Hessian equations, some good third-order terms will be lost due to complex conjugacy. Therefore, we will fully absorb the processing techniques of the above two inequalities, and optimize the coefficients in (1-4) from 2 to $1-\varepsilon_0$ for some small $\varepsilon_0>0$ while retaining the good terms. Then we establish the modified concavity inequality.

Lemma 1.6. *Let (M,ω) be a compact Hermitian manifold of complex dimension n, $W=(\omega_{p\bar{q}})$ be a Hermitian tensor and its eigenvalues $\lambda(W)=(\lambda_1,\dots,\lambda_n)\in\Gamma_k(M)$ satisfying $\lambda_1\geq\cdots\geq\lambda_n>-A$ for a constant $A>0$. For any $\varepsilon_0\in(0,1)$, if $\lambda_1\geq C(A)$ sufficiently large, then for $j=1,\dots,n$ and some positive constant K depending only on k,*

$$\begin{aligned}-\sum_{p\neq q}\frac{\sigma_k^{p\bar{p},q\bar{q}}\omega_{p\bar{p}j}\omega_{q\bar{q}\bar{j}}}{\sigma_k}+K\frac{|D_j\sigma_k|^2}{\sigma_k^2}+(1-\varepsilon_0)\sum_{i>1}\frac{\sigma_k^{i\bar{i}}|\omega_{i\bar{i}j}|^2}{\lambda_1\sigma_k}&\\ \geq(1-\varepsilon_0)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{\lambda_1\sigma_k}.&\end{aligned} \tag{1-5}$$

Remark 1.7. The selection of auxiliary functions is crucial in dealing with third-order terms. Dong [2021b] employed all eigenvalues of the solution to obtain additional good third-order terms to control bad terms. In our test function, we can choose the largest eigenvalue as the primary term with the help of the concavity inequality (1-5) and the techniques in [Chu 2021; Lu 2023].

The organization of the paper is as follows. In Section 2 we start with some properties for k-th elementary symmetric function and some basic calculations for complex Hessian equations on Hermitian manifolds. In Section 3 we prove the crucial concavity inequality. In Section 4 we give the proof of Theorem 1.3.

2. Preliminaries

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, the k-th elementary symmetric function is defined by

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

We also set $\sigma_0 = 1$ and $\sigma_k = 0$ for $k > n$ or $k < 0$. The Gårding's cone is defined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0 \text{ for all } 1 \le i \le k\}.$$

Define $\sigma_k(\lambda|i)$ to be the k-th elementary function with $\lambda_i = 0$ and $\sigma_k(\lambda|ij)$ to be the k-th elementary function with $\lambda_i = \lambda_j = 0$. We list some properties of σ_k which will be used later.

Proposition 1. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $1 \le k \le n$. Then we have*

(1) $\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n$;

(2) $\sigma_{k-1}(\lambda|i) > 0$ *for* $\lambda \in \Gamma_k$ *and* $1 \le i \le n$;

(3) $\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i)$ *for* $1 \le i \le n$;

(4) $\sum_{i=1}^n \partial\big[\frac{\sigma_k}{\sigma_l}\big]^{1/(k-l)}/\partial\lambda_i \ge \big[\frac{C_n^k}{C_n^l}\big]^{1/(k-l)}$ *for* $\lambda \in \Gamma_k$ *and* $0 \le l < k$;

(5) $\big[\frac{\sigma_k}{\sigma_l}\big]^{1/(k-l)}$ *are concave in* Γ_k *for* $0 \le l < k$;

(6) *If* $\lambda \in \Gamma_k$, $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, *then* $\sigma_{k-1}(\lambda|1) \le \sigma_{k-1}(\lambda|2) \le \cdots \le \sigma_{k-1}(\lambda|n)$;

(7) $\sum_{i=1}^n \sigma_{k-1}(\lambda|i) = (n-k+1)\sigma_{k-1}(\lambda)$;

(8) *For* $\lambda \in \Gamma_k$ *and* $\lambda_1 \ge \cdots \ge \lambda_n$, $\lambda_1 \sigma_{k-1}(\lambda|1) \ge \frac{k}{n}\sigma_k(\lambda)$.

Proof. All the properties are well known. For example, see Chapter XV in [Lieberman 1996] or [Huisken and Sinestrari 1999] for (1), (2), (3), (6) and (7); see Lemma 2.2.19 in [Gerhardt 2006] for (4); see [Caffarelli et al. 1985; Lieberman 1996] for (5); see Lemma 2 in [Dong 2021b] for (8). □

The generalized Newton–Maclaurin inequality is as follows.

Proposition 2. *For $\lambda \in \Gamma_m$ and $m > l \ge 0$, $r > s \ge 0$, $m \ge r$, $l \ge s$, we have*

$$\left[\frac{\sigma_m(\lambda)/C_n^m}{\sigma_l(\lambda)/C_n^l}\right]^{1/(m-l)} \le \left[\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s}\right]^{1/(r-s)}.$$

Proof. See [Spruck 2005]. □

To be more clear, we shall rewrite (1-2) in a local coordinate chart. Let $\lambda\{a_{i\bar{j}}\}$ be the eigenvalues of a Hermitian matrix $\{a_{i\bar{j}}\}$. Define $\sigma_k(a_{i\bar{j}}) = \sigma_k(\lambda\{a_{i\bar{j}}\})$. This definition can be naturally extended to complex manifolds. Let $\mathcal{A}^{1,1}(M)$ be the space

of real smooth $(1,1)$-forms on Hermitian manifold (M,ω). For any $\chi \in \mathcal{A}^{1,1}(M)$, we write, in a local coordinate chart $(z^1,\dots,z^n)$,

$$\chi = \sqrt{-1}\chi_{i\bar{j}}dz^i \wedge d\bar{z}^j,$$

and define

$$\sigma_k(\chi) = C_n^k \frac{\chi^k \wedge \omega^{n-k}}{\omega^n}.$$

In particular, in a local normal coordinate system $\omega_{i\bar{j}} = \delta_{ij}$ with the Hermitian metric $\omega = \{\omega_{i\bar{j}}\}$, equation (1-2) can be rewritten by

$$\sigma_k(\chi) = \sigma_k(\chi'_{i\bar{j}} + u_{i\bar{j}} + a_i u_{\bar{j}} + a_{\bar{j}} u_i) = \psi(z, Du, u), \tag{2-1}$$

where a_i is the component of the smooth $(1,0)$-form a on M.

Lemma 2.1. *If $F(A) = f(\lambda_1,\dots,\lambda_n)$ is a symmetric function of the eigenvalues of a Hermitian matrix $A = \{b_{i\bar{j}}\}$, then at a diagonal matrix A with distinct eigenvalues, we have*

$$F^{i\bar{j}} = \delta_{ij} f_i, \tag{2-2}$$

$$F^{i\bar{j},r\bar{s}}\xi_{i\bar{j}k}\xi_{r\bar{s}\bar{k}} = \sum_{i,j} f_{ij}\xi_{i\bar{i}k}\xi_{j\bar{j}\bar{k}} + \sum_{p\neq q}\frac{f_p - f_q}{\lambda_p - \lambda_q}|\xi_{p\bar{q}k}|^2, \tag{2-3}$$

where $F^{i\bar{j}} = \frac{\partial F}{\partial b_{i\bar{j}}}$, $F^{i\bar{j},r\bar{s}} = \frac{\partial^2 F}{\partial b_{i\bar{j}}\partial b_{r\bar{s}}}$ and $\xi_{i\bar{j}k}$ is an arbitrary tensor.

Proof. See [Ball 1984; Dong 2021b]. □

In local complex coordinates $(z_1,\dots,z_n)$, the subscripts of a function u always denote the covariant derivatives of u with respect to the Chern connection of ω in the directions of the local frame $(\partial/\partial z^1,\dots,\partial/\partial z^n)$. Namely,

$$u_i = D_i u = D_{\partial/\partial z^i} u, \quad u_{i\bar{j}} = D_{\partial/\partial\bar{z}^j} D_{\partial/\partial z^i} u, \quad u_{i\bar{j}\ell} = D_{\partial/\partial z^\ell} D_{\partial/\partial\bar{z}^j} D_{\partial/\partial z^i} u.$$

Next, we list some well-known results on Hermitian manifolds.

Lemma 2.2. *We have*

$$\begin{aligned} u_{pj} &= u_{jp} + T^k_{pj}u_k, & u_{i\bar{j}\ell} &= u_{i\ell\bar{j}} - u_p R_{\ell\bar{j}i}{}^p, \\ u_{p\bar{j}\bar{m}} &= u_{p\bar{m}\bar{j}} - \overline{T^q_{mj}}u_{p\bar{q}}, & u_{i\bar{q}\ell} &= u_{\ell\bar{q}i} - T^p_{\ell i}u_{p\bar{q}}, \end{aligned} \tag{2-4}$$

and

$$u_{i\bar{j}\ell\bar{m}} = u_{\ell\bar{m}i\bar{j}} + u_{p\bar{j}}R_{\ell\bar{m}i}{}^p - u_{p\bar{m}}R_{i\bar{j}\ell}{}^p - T^p_{\ell i}u_{p\bar{m}\bar{j}} - \overline{T^q_{mj}}u_{\ell\bar{q}i} - T^p_{i\ell}\overline{T^q_{mj}}u_{p\bar{q}}, \tag{2-5}$$

where T^k_{pj} and $R_{\ell\bar{j}i}{}^p$ are the components of the torsion tensor and curvature tensor, respectively.

Proof. See [Tosatti and Weinkove 2019]. □

Define

$$\sigma_k^{p\bar{q}} = \frac{\partial \sigma_k}{\partial \chi_{p\bar{q}}}, \quad \sigma_k^{p\bar{q},r\bar{s}} = \frac{\partial^2 \sigma_k}{\partial \chi_{p\bar{q}} \partial \chi_{r\bar{s}}}, \quad \mathcal{F} = \sum_p \sigma_k^{p\bar{p}}. \tag{2-6}$$

We also use the notation, as in [Phong et al. 2019],

$$|DDu|^2_{\sigma\omega} = \sigma_k^{p\bar{q}} \omega^{m\bar{\ell}} u_{mp} u_{\bar{\ell}\bar{q}}, \quad |D\overline{D}u|^2_{\sigma\omega} = \sigma_k^{p\bar{q}} \omega^{m\bar{\ell}} u_{p\bar{\ell}} u_{m\bar{q}}, \tag{2-7}$$

and

$$|\eta|^2_\sigma = \sigma_k^{p\bar{q}} \eta_p \eta_{\bar{q}} \tag{2-8}$$

for any 1-form η, where $\{\omega^{m\bar{l}}\}$ is the inverse of the metric ω.

The following basic calculations are carried out at a point z on the manifold M, and we use coordinates such that at this point $\omega = \sqrt{-1}\sum_{k,\ell} \delta_{k\ell} dz^k \wedge d\bar{z}^\ell$ and $\{\chi_{i\bar{j}}\}$ is diagonal. Note that $\{\sigma_k^{i\bar{j}}\}$ is diagonal when $\{\chi_{i\bar{j}}\}$ is diagonal. Combining with (2-4) and (2-5), we derive the following two inequalities which will be used later.

Lemma 2.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of $\{\chi_{i\bar{j}}\}$ with the ordering $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$\begin{aligned} &\sigma_k^{p\bar{q}} D_{\bar{q}} D_p \chi_{i\bar{j}} \\ &\quad \geq -\sigma_k^{p\bar{q},r\bar{s}} D_{\bar{j}} \chi_{r\bar{s}} D_i \chi_{p\bar{q}} + \sum_\ell \psi_{v_\ell} D_\ell \chi_{i\bar{j}} + \sum_\ell \psi_{\bar{v}_\ell} D_{\bar{\ell}} \chi_{i\bar{j}} \\ &\qquad - \sigma_k^{p\bar{q}} (T^a_{pi} u_{a\bar{q}\bar{j}} + \overline{T^a_{qj}} u_{p\bar{a}i}) - C(1 + |DDu|^2 + |D\overline{D}u|^2 + \lambda_1 \mathcal{F} + \lambda_1) \\ &\qquad\qquad + \sigma_k^{p\bar{q}} (a_i u_{\bar{j}p\bar{q}} + a_{\bar{j}} u_{ip\bar{q}} - a_p u_{\bar{q}i\bar{j}} - a_{\bar{q}} u_{pi\bar{j}}) \\ &\qquad\qquad + \sigma_k^{p\bar{q}} (a_{ip} u_{\bar{j}\bar{q}} + a_{\bar{j}\bar{q}} u_{ip} - a_{pi} u_{\bar{q}\bar{j}} - a_{\bar{q}\bar{j}} u_{pi}), \end{aligned} \tag{2-9}$$

and

$$\begin{aligned} &\sigma_k^{p\bar{q}} |Du|^2_{p\bar{q}} \\ &\quad \geq 2\,\mathrm{Re}\left\{ \sum_{p,m} (u_{\bar{p}} u_{mp} + u_p u_{m\bar{p}}) \psi_{v_m} \right\} + \tfrac{1}{2} |DDu|^2_{\sigma\omega} + \tfrac{1}{2} |D\overline{D}u|^2_{\sigma\omega} - C - C\mathcal{F}, \end{aligned} \tag{2-10}$$

where a_{ip} is the covariant derivative of a_i.

Proof. See the proof of (19) and (23) in Section 2 of [Dong 2021b]. □

3. A concavity inequality

In this section, we provide a proof of the concavity inequality in the complex case. Since it is similar to the real case in [Zhang 2025], we mainly highlight the differences.

Proof of Lemma 1.6. By (3.27) in [Zhang 2025], we have for $K \geq (k+1)^2$,

$$-\sum_{p\neq q}\frac{\sigma_k^{p\bar{p},q\bar{q}}\omega_{p\bar{p}j}\omega_{q\bar{q}\bar{j}}}{\sigma_k}+K\frac{|D_j\sigma_k|^2}{\sigma_k^2} \geq \frac{(k+2)^2}{(k+1)(k+3)}\frac{|\omega_{1\bar{1}j}|^2}{\lambda_1^2}-\frac{C}{\lambda_1^{1/(k-1)}}\sum_{i>1}\frac{\sigma_k^{i\bar{i}}|\omega_{i\bar{i}j}|^2}{\lambda_1\sigma_k}, \tag{3-1}$$

where C is a positive constant depending on n, k, σ_k, A. Recall that

$$\sigma_k=\lambda_1\sigma_k^{1\bar{1}}+\sigma_k(\lambda|1). \tag{3-2}$$

We note that if

$$\sigma_k(\lambda|1)\geqslant -\frac{\sigma_k}{2(k+2)^2-1}, \tag{3-3}$$

then (3-1) implies (1-5) by choosing $\lambda_1\geq\left(\frac{C}{1-\varepsilon_0}\right)^{k-1}$.

Define $c_0=\frac{1}{2(k+2)^2-1}$ and next we only need to consider the case

$$\sigma_k(\lambda|1)<-c_0\sigma_k. \tag{3-4}$$

Under this assumption, referring to the deduction of (3.73) in [Zhang 2025],

$$\begin{aligned}
&-\sum_{p\neq q}\frac{\sigma_k^{p\bar{p},q\bar{q}}\omega_{p\bar{p}j}\omega_{q\bar{q}\bar{j}}}{\sigma_k}+K\frac{|D_j\sigma_k|^2}{\sigma_k^2}+(1-\varepsilon_0)\sum_{i>1}\frac{\sigma_k^{i\bar{i}}|\omega_{i\bar{i}j}|^2}{\lambda_1\sigma_k}\\
&\geq (k+1-l)\left(1-\varepsilon_0-\frac{C_1}{M}\right)\left(1-\frac{C_1}{M}\right)(1+a)^2\frac{\lambda_1\sigma_k^{1\bar{1}}-\sigma_k}{\lambda_1^2\sigma_k}|\omega_{1\bar{1}j}|^2\\
&\qquad+\frac{(k+2)^2}{(k+1)(k+3)}\frac{|\omega_{1\bar{1}j}|^2}{\lambda_1^2}\\
&\quad+\frac{2a^2|\omega_{1\bar{1}j}|^2}{\lambda_1}\left(1-\frac{C_2}{M}\right)\frac{\lambda_1\sigma_k^{1\bar{1}}-\sigma_k}{\lambda_1\sigma_k}-C_3(a^2+(1-\varepsilon_0)(1+a)^2)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{M\lambda_1}\\
&\geq\frac{2(a^2+(1-\varepsilon_0)(1+a)^2)|\omega_{1\bar{1}j}|^2}{\lambda_1}\left(1-\frac{C_2}{(1-\varepsilon_0)M}\right)\frac{\lambda_1\sigma_k^{1\bar{1}}-\sigma_k}{\lambda_1\sigma_k}\\
&\qquad+\frac{(k+2)^2}{(k+1)(k+3)}\frac{|\omega_{1\bar{1}j}|^2}{\lambda_1^2}-C_3(a^2+(1-\varepsilon_0)(1+a)^2)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{M\lambda_1}\\
&\geq(2-\varepsilon_0)\frac{(a^2+(1-\varepsilon_0)(1+a)^2)|\omega_{1\bar{1}j}|^2}{\lambda_1}\frac{\lambda_1\sigma_k^{1\bar{1}}-\sigma_k}{\lambda_1\sigma_k}+\frac{(k+2)^2}{(k+1)(k+3)}\frac{|\omega_{1\bar{1}j}|^2}{\lambda_1^2}\\
&\qquad+\frac{(a^2+(1-\varepsilon_0)(1+a)^2)|\omega_{1\bar{1}j}|^2}{\lambda_1}\left[\frac{\varepsilon_0}{2}\frac{\lambda_1\sigma_k^{1\bar{1}}-\sigma_k}{\lambda_1\sigma_k}-\frac{C_3'\sigma_k^{1\bar{1}}}{\sigma_k M}\right],
\end{aligned}$$

by assuming $\lambda_1 > M^k$ and M large enough such that $1 - \frac{C_2}{(1-\varepsilon_0)M} \geq 1 - \frac{\varepsilon_0}{4}$. Here C_1, C_2, C_3, C_3' are positive constants depending on n, k, σ_k, A.

Under the assumption

$$\sigma_k(\lambda|1) < -c_0\sigma_k,$$

(3-2) implies that

$$\frac{\sigma_k^{1\bar{1}}}{\sigma_k} = \frac{1}{\lambda_1} - \frac{\sigma_k(\lambda|1)}{\lambda_1\sigma_k} > \frac{c_0+1}{\lambda_1}.$$

Then by choosing $M \geq \frac{2C_3'(c_0+1)}{c_0\varepsilon_0}$, we get

$$\begin{aligned}\frac{\varepsilon_0}{2}\frac{\lambda_1\sigma_k^{1\bar{1}} - \sigma_k}{\lambda_1\sigma_k} - \frac{C_3'\sigma_k^{1\bar{1}}}{\sigma_k M} &= \frac{\sigma_k^{1\bar{1}}}{\sigma_k}\left(\frac{\varepsilon_0}{2} - \frac{C_3'}{M}\right) - \frac{\varepsilon_0}{2\lambda_1} \\ &> \frac{1}{\lambda_1}\left[\left(\frac{\varepsilon_0}{2} - \frac{C_3'}{M}\right)(c_0+1) - \frac{\varepsilon_0}{2}\right] \geq 0.\end{aligned}$$

Thus

$$\begin{aligned}(3\text{-}5)\quad &-\sum_{p\neq q}\frac{\sigma_k^{p\bar{p},q\bar{q}}\omega_{p\bar{p}j}\omega_{q\bar{q}\bar{j}}}{\sigma_k} + K\frac{|D_j\sigma_k|^2}{\sigma_k^2} + (1-\varepsilon_0)\sum_{i>1}\frac{\sigma_k^{i\bar{i}}|\omega_{i\bar{i}j}|^2}{\lambda_1\sigma_k} \\ &\geq (2-\varepsilon_0)\frac{(a^2+(1-\varepsilon_0)(1+a)^2)|\omega_{1\bar{1}j}|^2}{\lambda_1}\frac{\lambda_1\sigma_k^{1\bar{1}} - \sigma_k}{\lambda_1\sigma_k} + \frac{(k+2)^2}{(k+1)(k+3)}\frac{|\omega_{1\bar{1}j}|^2}{\lambda_1^2} \\ &= (2-\varepsilon_0)(a^2+(1-\varepsilon_0)(1+a)^2)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{\lambda_1\sigma_k} \\ &\qquad + \left(\frac{(k+2)^2}{(k+1)(k+3)} - (2-\varepsilon_0)[a^2+(1-\varepsilon_0)(1+a)^2]\right)\frac{|\omega_{1\bar{1}j}|^2}{\lambda_1^2}.\end{aligned}$$

If

$$\frac{(k+2)^2}{(k+1)(k+3)} \geq (2-\varepsilon_0)[a^2+(1-\varepsilon_0)(1+a)^2],$$

(3-5) implies that

$$\begin{aligned}&-\sum_{p\neq q}\frac{\sigma_k^{p\bar{p},q\bar{q}}\omega_{p\bar{p}j}\omega_{q\bar{q}\bar{j}}}{\sigma_k} + K\frac{|D_j\sigma_k|^2}{\sigma_k^2} + (1-\varepsilon_0)\sum_{i>1}\frac{\sigma_k^{i\bar{i}}|\omega_{i\bar{i}j}|^2}{\lambda_1\sigma_k} \\ &\qquad\qquad \geq (2-\varepsilon_0)(a^2+(1-\varepsilon_0)(1+a)^2)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{\lambda_1\sigma_k} \\ &\qquad\qquad \geq (1-\varepsilon_0)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{\lambda_1\sigma_k}.\end{aligned}$$

In the last inequality, we use the fact that

$$a^2+(1-\varepsilon_0)(1+a)^2=(2-\varepsilon_0)\left[\left(a+\frac{1-\varepsilon_0}{1+\varepsilon_0}\right)^2-\frac{(1-\varepsilon_0)^2}{(2-\varepsilon_0)^2}+\frac{1-\varepsilon_0}{2-\varepsilon_0}\right]$$
$$\geq 1-\varepsilon_0-\frac{(1-\varepsilon_0)^2}{2-\varepsilon_0}=\frac{1-\varepsilon_0}{2-\varepsilon_0}.$$

If

$$\frac{(k+2)^2}{(k+1)(k+3)}<(2-\varepsilon_0)[a^2+(1-\varepsilon_0)(1+a)^2],$$

(3-5) implies that

$$-\sum_{p\neq q}\frac{\sigma_k^{p\bar{p},q\bar{q}}\omega_{p\bar{p}j}\omega_{q\bar{q}\bar{j}}}{\sigma_k}+K\frac{|D_j\sigma_k|^2}{\sigma_k^2}+(1-\varepsilon_0)\sum_{i>1}\frac{\sigma_k^{i\bar{i}}|\omega_{i\bar{i}j}|^2}{\lambda_1\sigma_k}$$
$$>\left(1+\frac{1}{(k+1)(k+3)}\right)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{\lambda_1\sigma_k}>(1-\varepsilon_0)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{\lambda_1\sigma_k}.$$

Hence we obtain (1-5) and complete the proof. □

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. Consider the auxiliary function

$$Q=\log\lambda_1+\varphi(|Du|^2)+\phi(u),$$

where λ_1 is the largest eigenvalue of the matrix $\{\chi_{i\bar{j}}\}$ with respect to ω.

Define

$$\varphi(s)=e^{Ns},\quad 0\leq s\leq S-1,$$

and

$$\phi(t)=e^{\Lambda(-t+T)},\quad -T+1\leq t\leq T-1.$$

Here we set

$$S=\sup_M|Du|^2+1,\quad T=\sup_M|u|+1,$$

and Λ, $N>1$ are large constants to be determined later. Clearly, φ, ϕ satisfy

$$\varphi'=N\varphi>0,\quad \varphi''=\frac{(\varphi')^2}{\varphi},\qquad \phi'=-\Lambda\phi<0,\quad \phi''=\frac{(\phi')^2}{\phi},$$

and

$$\varphi''-2\phi''\frac{(\varphi')^2}{(\phi')^2}=\varphi''-2\frac{(\varphi')^2}{\phi}=N^2e^{Ns}-\frac{2N^2e^{2Ns}}{e^{\Lambda(-t+T)}}>0, \tag{4-1}$$

when $\Lambda\gg N>1$. Suppose Q attains its maximum at $z_0\in M$. Denote by $(\lambda_1,\lambda_2,\ldots,\lambda_n)$ the eigenvalues of the matrix $\{\chi_{i\bar{j}}\}$ with respect to ω. Without

loss of generality, assume λ_1 has multiplicity m, we choose the coordinate system centered at z_0 such that $\omega = \sqrt{-1}\delta_{kl}\, dz^k \wedge d\bar{z}^l$ and $\{\chi_{i\bar{j}}\}$ is diagonal with

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m \geq \lambda_{m+1} \geq \cdots \geq \lambda_n.$$

We now apply a perturbation argument (see [Székelyhidi 2018]). Let B be a diagonal matrix such that $B_{1\bar{1}} = 0$ and $0 < B_{2\bar{2}} < \cdots < B_{n\bar{n}}$ are small, satisfying $B_{n\bar{n}} < 2B_{2\bar{2}}$. Define $\widetilde{\chi}_{i\bar{j}} = \chi_{i\bar{j}} - B_{i\bar{j}}$ and denote its eigenvalues by $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n)$. It follows that $\lambda_1 \geq \tilde{\lambda}_1$ near z_0 and, at z_0,

$$\tilde{\lambda}_1 = \lambda_1, \qquad \tilde{\lambda}_i = \lambda_i - B_{i\bar{i}} \quad \text{for } i > 1.$$

Thus $\tilde{\lambda}_1 > \tilde{\lambda}_2$ at z_0, which implies $\tilde{\lambda}_1$ is smooth at z_0. We consider the new function

$$\widetilde{Q} = \log \tilde{\lambda}_1 + \varphi(|Du|^2) + \phi(u).$$

It still achieves a local maximum at z_0. Then at z_0, we have

$$0 = \widetilde{Q}_i = \frac{\tilde{\lambda}_{1,i}}{\lambda_1} + \varphi' D_i(|Du|^2) + \phi' u_i, \tag{4-2}$$

and

$$\begin{aligned} 0 \geq \sigma_k^{i\bar{i}} \widetilde{Q}_{i\bar{i}} = {} & \sigma_k^{i\bar{i}} (\log \tilde{\lambda}_1)_{i\bar{i}} + \varphi'' \sigma_k^{i\bar{i}} |D_i(|Du|^2)|^2 \\ & + \varphi' \sigma_k^{i\bar{i}} |Du|^2_{i\bar{i}} + \phi'' \sigma_k^{i\bar{i}} |u_i|^2 + \phi' \sigma_k^{i\bar{i}} u_{i\bar{i}}. \end{aligned} \tag{4-3}$$

The following calculations are all at z_0. For convenience, we will use a unified notation C to denote a constant depending on (M, ω), n, k, ε, χ', ψ, a, A, $\sup_M |u|$ and $\sup_M |Du|$.

First, we deal with the term $\sigma_k^{i\bar{i}} (\log \tilde{\lambda}_1)_{i\bar{i}}$ in (4-3). Thus

$$\tilde{\lambda}_{1,i} = \frac{\partial \tilde{\lambda}_1}{\partial \widetilde{\chi}_{p\bar{q}}} D_i \widetilde{\chi}_{p\bar{q}} = \delta_{1p}\delta_{1q} D_i \widetilde{\chi}_{p\bar{q}} = D_i \widetilde{\chi}_{1\bar{1}} = D_i \chi_{1\bar{1}} - D_i B_{1\bar{1}},$$

and

$$\begin{aligned} \tilde{\lambda}_{1,i\bar{i}} &= \frac{\partial \tilde{\lambda}_1}{\partial \widetilde{\chi}_{p\bar{q}}} D_{\bar{i}} D_i \widetilde{\chi}_{p\bar{q}} + \frac{\partial^2 \tilde{\lambda}_1}{\partial \widetilde{\chi}_{p\bar{q}} \partial \widetilde{\chi}_{r\bar{s}}} D_i \widetilde{\chi}_{p\bar{q}} D_{\bar{i}} \widetilde{\chi}_{r\bar{s}} \\ &= \delta_{1p}\delta_{1q} D_{\bar{i}} D_i \widetilde{\chi}_{p\bar{q}} + \left[(1-\delta_{1p}) \frac{\delta_{1q}\delta_{1r}\delta_{ps}}{\tilde{\lambda}_1 - \tilde{\lambda}_p} + (1-\delta_{1r}) \frac{\delta_{1s}\delta_{1p}\delta_{qr}}{\tilde{\lambda}_1 - \tilde{\lambda}_r} \right] D_i \widetilde{\chi}_{p\bar{q}} D_{\bar{i}} \widetilde{\chi}_{r\bar{s}} \\ &= D_{\bar{i}} D_i \chi_{1\bar{1}} + D_{\bar{i}} D_i B_{1\bar{1}} + \sum_{p>1} \frac{|D_i \chi_{p\bar{1}}|^2 + |D_i \chi_{1\bar{p}}|^2}{\lambda_1 - \tilde{\lambda}_p} \\ &\quad + \frac{\partial^2 \tilde{\lambda}_1}{\partial \widetilde{\chi}_{p\bar{q}} \partial \widetilde{\chi}_{r\bar{s}}} D_i B_{p\bar{q}} D_{\bar{i}} B_{r\bar{s}} - 2\,\mathrm{Re} \sum_{p>1} \frac{D_i \chi_{p\bar{1}} D_{\bar{i}} B_{1\bar{p}} + D_i \chi_{1\bar{p}} D_{\bar{i}} B_{p\bar{1}}}{\lambda_1 - \tilde{\lambda}_p}. \end{aligned}$$

We can absorb the terms $D_i\chi_{p\bar{1}}D_{\bar{i}}B_{1\bar{p}}$ by

$$|D_i\chi_{p\bar{1}}D_{\bar{i}}B_{1\bar{p}}| \le \frac{\beta}{4}|D_i\chi_{p\bar{1}}|^2 + \frac{C}{\beta}|B_{n\bar{n}}|^2,$$

using that the derivative of B is controlled by $B_{n\bar{n}}$. Here β is a positive constant to be determined later.

Moreover, for $p > 1$ there is

$$\frac{1}{\lambda_1 - \tilde{\lambda}_p} \le \frac{1}{B_{2\bar{2}}} < \frac{2}{B_{n\bar{n}}},$$

and so

$$\frac{|B_{n\bar{n}}|^2}{\lambda_1 - \tilde{\lambda}_p} + \frac{\partial^2\tilde{\lambda}_1}{\partial\widetilde{\chi}_{p\bar{q}}\partial\widetilde{\chi}_{r\bar{s}}}D_iB_{p\bar{q}}D_{\bar{i}}B_{r\bar{s}} = O(B_{n\bar{n}}).$$

Then by choosing B small enough,

$$\tilde{\lambda}_{1,i\bar{i}} \ge D_{\bar{i}}D_i\chi_{1\bar{1}} + \left(1 - \frac{\beta}{2}\right)\sum_{p>1}\frac{|D_i\chi_{p\bar{1}}|^2 + |D_i\chi_{1\bar{p}}|^2}{\lambda_1 - \tilde{\lambda}_p} - \frac{C}{\beta}.$$

Using Cauchy–Schwarz inequality, we have

$$|\sigma_k^{p\bar{q}}(a_{ip}u_{\bar{j}\bar{q}} + a_{\bar{j}\bar{q}}u_{ip} - a_{pi}u_{\bar{q}\bar{j}} - a_{\bar{q}\bar{j}}u_{pi})| \le \tfrac{1}{4}|D\bar{D}u|^2_{\sigma\omega} + \tfrac{1}{4}|DDu|^2_{\sigma\omega} + C\mathcal{F}. \tag{4-4}$$

Without loss of generality, we assume $\lambda_1 \ge 1$. Then combining with (2-9) and (4-4),

$$\begin{aligned}
\sigma_k^{i\bar{i}}(\log\tilde{\lambda}_1)_{i\bar{i}} &= \frac{\sigma_k^{i\bar{i}}\tilde{\lambda}_{1,i\bar{i}}}{\lambda_1} - \frac{\sigma_k^{i\bar{i}}|\tilde{\lambda}_{1,i}|^2}{\lambda_1^2} \\
&\ge \frac{\sigma_k^{i\bar{i}}D_{\bar{i}}D_i\chi_{1\bar{1}}}{\lambda_1} + \left(1 - \frac{\beta}{2}\right)\sum_{p>1}\sigma_k^{i\bar{i}}\frac{|D_i\chi_{p\bar{1}}|^2 + |D_i\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1 - \tilde{\lambda}_p)} \\
&\qquad - \left(1 + \frac{\beta}{2}\right)\frac{\sigma_k^{i\bar{i}}|D_i\chi_{1\bar{1}}|^2}{\lambda_1^2} - \frac{C}{\beta}\mathcal{F} \\
&\ge -\frac{\sigma_k^{i\bar{j},r\bar{s}}D_1\chi_{i\bar{j}}D_{\bar{1}}\chi_{r\bar{s}}}{\lambda_1} + \frac{1}{\lambda_1}\left(\sum_\ell \psi_{v_\ell}D_\ell\chi_{1\bar{1}} + \sum_\ell \psi_{\bar{v}_\ell}D_{\bar{\ell}}\chi_{1\bar{1}}\right) \\
&\qquad - \frac{1}{\lambda_1}\sigma_k^{i\bar{i}}(T^a_{i1}u_{a\bar{i}\bar{1}} + \overline{T^a_{i1}}u_{i\bar{a}1}) - \frac{1}{\lambda_1}\left(\tfrac{1}{4}|DDu|^2_{\sigma\omega} + \tfrac{1}{4}|D\bar{D}u|^2_{\sigma\omega}\right) \\
&\qquad + \frac{1}{\lambda_1}\sigma_k^{i\bar{i}}(a_1u_{\bar{1}i\bar{i}} + a_{\bar{1}}u_{1i\bar{i}} - a_iu_{\bar{i}1\bar{1}} - a_{\bar{i}}u_{i1\bar{1}}) \\
&\qquad + \left(1 - \frac{\beta}{2}\right)\sum_{p>1}\sigma_k^{i\bar{i}}\frac{|D_i\chi_{p\bar{1}}|^2 + |D_i\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1 - \tilde{\lambda}_p)} \\
&\qquad - \left(1 + \frac{\beta}{2}\right)\frac{\sigma_k^{i\bar{i}}|D_i\chi_{1\bar{1}}|^2}{\lambda_1^2} - \frac{C}{\lambda_1}(1 + |DDu|^2 + |D\bar{D}u|^2) - C - \frac{C}{\beta}\mathcal{F}.
\end{aligned}$$

Using (2-10) and (4-2), we get

$$
\begin{aligned}
(4\text{-}5)\quad & \frac{1}{\lambda_1}\bigg(\sum_\ell \psi_{v_\ell} D_\ell \chi_{1\bar 1} + \sum_\ell \psi_{\bar v_\ell} D_{\bar\ell}\chi_{1\bar 1}\bigg) + \varphi' \sigma_k^{i\bar i} |Du|^2_{i\bar i} \\
&\geq \frac{1}{\lambda_1}\bigg(\sum_\ell \psi_{v_\ell} D_\ell \chi_{1\bar 1} + \sum_\ell \psi_{\bar v_\ell} D_{\bar\ell}\chi_{1\bar 1}\bigg) + \frac{\varphi'}{2}|DDu|^2_{\sigma\omega} + \frac{\varphi'}{2}|D\overline{D}u|^2_{\sigma\omega} \\
&\qquad + 2\varphi' \mathrm{Re}\bigg\{\sum_{p,m} \psi_{v_m}(u_{\bar p} u_{mp} + u_p u_{m\bar p})\bigg\} - C\varphi' - C\varphi'\mathcal{F} \\
&\geq -\phi' \sum_m (\psi_{v_m} u_m + \psi_{\bar v_m} u_{\bar m}) + \frac{\varphi'}{2}|DDu|^2_{\sigma\omega} + \frac{\varphi'}{2}|D\overline{D}u|^2_{\sigma\omega} - C\varphi' - C\varphi'\mathcal{F} \\
&\geq \frac{\varphi'}{2}|DDu|^2_{\sigma\omega} + \frac{\varphi'}{2}|D\overline{D}u|^2_{\sigma\omega} + C\phi' - C\varphi' - C\varphi'\mathcal{F}.
\end{aligned}
$$

Since $\lambda_i \geq -A$ for any $i = 1, \ldots, n$, and if we further assume that $\lambda_1 \geq \max\{1, A\}$ large enough,

$$
(4\text{-}6)\qquad \frac{1}{\lambda_1 - \tilde\lambda_i} = \frac{1}{\lambda_1 - \lambda_i + 1} \geq \frac{1}{3\lambda_1}.
$$

Then by the Cauchy–Schwarz inequality, for any $\beta \in (0, 1)$,

$$
\begin{aligned}
(4\text{-}7)\quad & \frac{1}{\lambda_1}\sigma_k^{i\bar i}|T^a_{i1} u_{a\bar i\bar 1} + \overline{T^a_{i1}} u_{i\bar a 1}| \\
&= \frac{2}{\lambda_1}\sigma_k^{i\bar i}|\mathrm{Re}\{\overline{T^p_{i1}} u_{i\bar p 1}\}| \leq \frac{2}{\lambda_1}\sigma_k^{i\bar i}|\overline{T^p_{i1}}(u_{1\bar p i} + T^q_{i1} u_{q\bar p})| \\
&\leq \frac{2}{\lambda_1}\sigma_k^{i\bar i}|\overline{T^p_{i1}} D_i \chi_{1\bar p}| + \frac{C}{\lambda_1}(|DDu|^2_{\sigma\omega} + |D\overline{D}u|^2_{\sigma\omega}) + C\mathcal{F} \\
&\leq \frac{\beta}{12}\sum_p \frac{\sigma_k^{i\bar i}|D_i \chi_{1\bar p}|^2}{\lambda_1^2} + \frac{C}{\beta}\mathcal{F} + \frac{C}{\lambda_1}(|DDu|^2_{\sigma\omega} + |D\overline{D}u|^2_{\sigma\omega}) \\
&\leq \frac{\beta}{4}\frac{\sigma_k^{i\bar i}|D_i \chi_{1\bar 1}|^2}{\lambda_1^2} + \frac{\beta}{4}\sum_{p>1}\frac{\sigma_k^{i\bar i}|D_i \chi_{1\bar p}|^2}{\lambda_1(\lambda_1 - \tilde\lambda_p)} + \frac{C}{\beta}\mathcal{F} + \frac{C}{\lambda_1}(|DDu|^2_{\sigma\omega} + |D\overline{D}u|^2_{\sigma\omega}).
\end{aligned}
$$

We also have

$$
\begin{aligned}
(4\text{-}8)\quad & \frac{1}{\lambda_1}\sigma_k^{i\bar i}|a_1 u_{\bar 1 i\bar i} + a_{\bar 1} u_{1i\bar i} - a_i u_{\bar i 1\bar 1} - a_{\bar i} u_{i1\bar 1}| \\
&\leq \frac{\beta}{12}\bigg(\frac{\sigma_k^{i\bar i}|D_i \chi_{1\bar i}|^2}{\lambda_1^2} + \frac{\sigma_k^{i\bar i}|D_i \chi_{1\bar 1}|^2}{\lambda_1^2}\bigg) + \frac{C}{\lambda_1}(|DDu|^2_{\sigma\omega} + |D\overline{D}u|^2_{\sigma\omega}) + \frac{C}{\beta}\mathcal{F} \\
&\leq \frac{\beta}{12}\sum_p \frac{\sigma_k^{i\bar i}|D_i \chi_{1\bar p}|^2}{\lambda_1^2} + \frac{C}{\beta}\mathcal{F} + \frac{C}{\lambda_1}(|DDu|^2_{\sigma\omega} + |D\overline{D}u|^2_{\sigma\omega}) \\
&\leq \frac{\beta}{4}\frac{\sigma_k^{i\bar i}|D_i \chi_{1\bar 1}|^2}{\lambda_1^2} + \frac{\beta}{4}\sum_{p>1}\frac{\sigma_k^{i\bar i}|D_i \chi_{1\bar p}|^2}{\lambda_1(\lambda_1 - \tilde\lambda_p)} + \frac{C}{\beta}\mathcal{F} + \frac{C}{\lambda_1}(|DDu|^2_{\sigma\omega} + |D\overline{D}u|^2_{\sigma\omega}).
\end{aligned}
$$

Using the critical equation (4-2), we obtain

$$\phi''\sigma_k^{i\bar{i}}|u_i|^2 \geq \frac{\phi''}{2(\phi')^2}\frac{\sigma_k^{i\bar{i}}|D_i\chi_{1\bar{1}}|^2}{\lambda_1^2} - \phi''\left(\frac{\varphi'}{\phi'}\right)^2\sigma_k^{i\bar{i}}|D_i(|Du|^2)|^2, \tag{4-9}$$

and

$$\begin{aligned}
&|\phi'\sigma_k^{i\bar{i}}(a_iu_{\bar{i}}+a_{\bar{i}}u_i)| \\
&\quad = 2|\phi'\sigma_k^{i\bar{i}}\,\mathrm{Re}\{a_iu_{\bar{i}}\}| = 2\left|\sigma_k^{i\bar{i}}\,\mathrm{Re}\left\{a_i\left(\frac{D_{\bar{i}}\chi_{1\bar{1}}}{\lambda_1}+\varphi' D_{\bar{i}}(|Du|^2)\right)\right\}\right| \\
&\quad \leq \frac{\phi''}{8(\phi')^2}\sigma_k^{i\bar{i}}\left|\frac{D_{\bar{i}}\chi_{1\bar{1}}}{\lambda_1}+\varphi' D_{\bar{i}}(|Du|^2)\right|^2 + \frac{C(\phi')^2}{\phi''}\mathcal{F} \\
&\quad \leq \frac{\phi''}{4(\phi')^2}\frac{\sigma_k^{i\bar{i}}|D_{\bar{i}}\chi_{1\bar{1}}|^2}{\lambda_1^2} + (\varphi')^2\frac{\phi''}{4(\phi')^2}\sigma_k^{i\bar{i}}|D_i(|Du|^2)|^2 + \frac{C(\phi')^2}{\phi''}\mathcal{F}.
\end{aligned} \tag{4-10}$$

Since $\chi'(z,u)\geq\varepsilon\omega$,

$$\begin{aligned}
-\sigma_k^{p\bar{q}}u_{p\bar{q}} &= \sigma_k^{p\bar{q}}(\chi'_{p\bar{q}}+a_pu_{\bar{q}}+a_{\bar{q}}u_p-\chi_{p\bar{q}}) \\
&\geq \varepsilon\mathcal{F}+\sigma_k^{p\bar{q}}(a_pu_{\bar{q}}+a_{\bar{q}}u_p)-k\psi.
\end{aligned} \tag{4-11}$$

Combining (4-3)–(4-11), we get

$$\begin{aligned}
0 \geq\ & -\frac{1}{\lambda_1}\sigma_k^{i\bar{j},r\bar{s}}D_1\chi_{i\bar{j}}D_{\bar{1}}\chi_{r\bar{s}} + \left(\frac{\varphi'}{2}-\frac{C}{\lambda_1}\right)(|DDu|^2_{\sigma\omega}+|D\bar{D}u|^2_{\sigma\omega}) \\
&+(1-\beta)\sum_{p>1}\sigma_k^{i\bar{i}}\frac{|D_i\chi_{p\bar{1}}|^2+|D_i\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)} - \left(1+\beta-\frac{\phi''}{4(\phi')^2}\right)\frac{\sigma_k^{i\bar{i}}|D_i\chi_{1\bar{1}}|^2}{\lambda_1^2} \\
&+\left(\varphi''-2\phi''\frac{(\varphi')^2}{(\phi')^2}\right)\sigma_k^{i\bar{i}}|D_i(|Du|^2)|^2 - \frac{C}{\lambda_1}(|DDu|^2+|D\bar{D}u|^2) \\
&+\left(-\varepsilon\phi'-C\varphi'-\frac{C(\phi')^2}{\phi''}-\frac{C}{\beta}\right)\mathcal{F}+C\phi'-C\varphi'-C.
\end{aligned} \tag{4-12}$$

According to (2-3),

$$\begin{aligned}
&-\frac{1}{\lambda_1}\sigma_k^{i\bar{j},r\bar{s}}D_1\chi_{i\bar{j}}D_{\bar{1}}\chi_{r\bar{s}} + (1-\beta)\sum_{p>1}\sigma_k^{i\bar{i}}\frac{|D_i\chi_{p\bar{1}}|^2+|D_i\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)} \\
&\quad \geq -\frac{1}{\lambda_1}\sum_{p\neq q}\sigma_k^{p\bar{p},q\bar{q}}D_1\chi_{p\bar{p}}D_{\bar{1}}\chi_{q\bar{q}} + \frac{1}{\lambda_1}\sum_{p\neq q}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{q\bar{q}}}{\lambda_q-\lambda_p}|D_1\chi_{p\bar{q}}|^2 \\
&\qquad + (1-\beta)\sum_{p>1}\sigma_k^{1\bar{1}}\frac{|D_1\chi_{p\bar{1}}|^2+|D_1\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)} \\
&\qquad + (1-\beta)\sum_{p>1}\sigma_k^{p\bar{p}}\frac{|D_p\chi_{p\bar{1}}|^2+|D_p\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}
\end{aligned} \tag{4-13}$$

$$\geq -\frac{1}{\lambda_1}\sum_{p\neq q}\sigma_k^{p\bar{p},q\bar{q}}D_1\chi_{p\bar{p}}D_{\bar{1}}\chi_{q\bar{q}}+\sum_{p>m}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{1\bar{1}}}{\lambda_1(\lambda_1-\lambda_p)}|D_1\chi_{p\bar{1}}|^2$$
$$+(1-\beta)\sum_{p>1}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}+(1-\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}.$$

On the one hand, by Cauchy–Schwarz inequality, for any $\beta\in(0,1)$,

$$\sum_{p>m}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{1\bar{1}}}{\lambda_1(\lambda_1-\lambda_p)}|D_1\chi_{p\bar{1}}|^2+(1-\beta)\sum_{p>1}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}-(1-3\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{1}}|^2}{\lambda_1^2}$$
$$\geq(1-\beta)\sum_{p>m}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{1\bar{1}}}{\lambda_1(\lambda_1-\lambda_p)}|D_1\chi_{p\bar{1}}|^2+(1-\beta)\sum_{p>1}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}$$
$$-(1-3\beta)(1+\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1^2}-C(1-3\beta)\left(1+\frac{1}{\beta}\right)\mathcal{F}$$
$$\geq(1-\beta)\Bigg[\sum_{p>m}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{1\bar{1}}}{\lambda_1(\lambda_1-\lambda_p)}|D_1\chi_{p\bar{1}}|^2$$
$$+\sum_{p>1}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}-(1-\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1^2}\Bigg]-\frac{C}{\beta}\mathcal{F}$$
$$\geq-\frac{C}{\beta}\mathcal{F}.$$

In the last inequality, we use the fact

$$\sum_{p>m}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{1\bar{1}}}{\lambda_1(\lambda_1-\lambda_p)}|D_1\chi_{p\bar{1}}|^2+\sum_{p>1}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}-(1-\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1^2}$$
$$=\sum_{p>m}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{1\bar{1}}}{\lambda_1(\lambda_1-\lambda_p)}|D_1\chi_{p\bar{1}}|^2+\sum_{1<i\leq m}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{i\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_i)}+\sum_{p>m}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}$$
$$-(1-\beta)\sum_{1<i\leq m}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{i\bar{1}}|^2}{\lambda_1^2}-(1-\beta)\sum_{p>m}\frac{\sigma_k^{p\bar{p}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1^2}$$
$$=\sum_{1<i\leq m}\frac{\beta\lambda_1+(1-\beta)\tilde{\lambda}_i}{\lambda_1^2(\lambda_1-\tilde{\lambda}_i)}\sigma_k^{1\bar{1}}|D_1\chi_{i\bar{1}}|^2+\sum_{p>m}\frac{\beta\lambda_1+(1-\beta)\lambda_p}{\lambda_1^2(\lambda_1-\lambda_p)}\sigma_k^{p\bar{p}}|D_1\chi_{p\bar{1}}|^2$$
$$+\sum_{p>m}\frac{\tilde{\lambda}_p-\lambda_p}{\lambda_1(\lambda_1-\lambda_p)(\lambda_1-\tilde{\lambda}_p)}\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2$$

$$\geq \sum_{1<i\leq m} \frac{\beta\lambda_1+(1-\beta)\tilde{\lambda}_i}{\lambda_1^2(\lambda_1-\tilde{\lambda}_i)}\sigma_k^{1\bar{1}}|D_1\chi_{i\bar{1}}|^2+\sum_{p>m}\frac{\beta\lambda_1+(1-\beta)\tilde{\lambda}_p}{\lambda_1^2(\lambda_1-\tilde{\lambda}_p)}\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2$$
$$\geq 0,$$

by choosing $\lambda_1 \geq \frac{1-\beta}{\beta}(A+1)$ large enough such that $\beta\lambda_1+(1-\beta)\tilde{\lambda}_i \geq 0$ for any i.

On the other hand, by Lemma 1.6 and choosing $\varepsilon_0 = 3\beta$ and $\lambda_1 \geq \frac{1-2\beta}{\beta}(A+1)$ large enough, we derive

$$\begin{aligned}
&-\frac{1}{\lambda_1}\sum_{p\neq q}\sigma_k^{p\bar{p},q\bar{q}}D_1\chi_{p\bar{p}}D_{\bar{1}}\chi_{q\bar{q}}+(1-\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}-(1-3\beta)\frac{\sigma_k^{1\bar{1}}|D_1\chi_{1\bar{1}}|^2}{\lambda_1^2}\\
&\quad\geq -K\frac{|D_1\psi|^2}{\lambda_1\sigma_k}+(1-\varepsilon_0)\frac{\sigma_k^{1\bar{1}}|D_1\chi_{1\bar{1}}|^2}{\lambda_1^2}-(1-\varepsilon_0)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_1\chi_{p\bar{p}}|^2}{\lambda_1^2}\\
&\qquad\qquad+(1-\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}-(1-3\beta)\frac{\sigma_k^{1\bar{1}}|D_1\chi_{1\bar{1}}|^2}{\lambda_1^2}\\
&\quad\geq -\frac{CK}{\lambda_1}(|DDu|^2+|D\bar{D}u|^2)+(1-\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}\\
&\qquad\qquad-(1-3\beta)(1+\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{p}}|^2}{\lambda_1^2}-\frac{C}{\beta}\mathcal{F}\\
&\quad\geq \sum_{p>1}\frac{\beta\lambda_1+(1-2\beta)\tilde{\lambda}_p}{\lambda_1^2(\lambda_1-\tilde{\lambda}_p)}\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{p}}|^2-\frac{CK}{\lambda_1}(|DDu|^2+|D\bar{D}u|^2)-\frac{C}{\beta}\mathcal{F}\\
&\quad\geq -\frac{CK}{\lambda_1}(|DDu|^2+|D\bar{D}u|^2)-\frac{C}{\beta}\mathcal{F}.
\end{aligned}$$

Since $\sigma_k^{i\bar{i}} \geq \sigma_k^{1\bar{1}} \geq \frac{k\sigma_k}{n\lambda_1} \geq \frac{C}{\lambda_1}$ for any i,

$$|DDu|^2_{\sigma\omega}+|D\bar{D}u|^2_{\sigma\omega} \geq \frac{C}{\lambda_1}(|DDu|^2+|D\bar{D}u|^2) \geq \frac{C}{\lambda_1}|DDu|^2+C\lambda_1. \tag{4-14}$$

Combining (4-1) and (4-12)–(4-14), we obtain

$$\begin{aligned}
0 \geq{}& \left(\frac{\phi''}{4(\phi')^2}-4\beta\right)\frac{\sigma_k^{i\bar{i}}|D_i\chi_{1\bar{1}}|^2}{\lambda_1^2}+\left(\frac{\varphi'}{2}-\frac{C}{\lambda_1}\right)(|DDu|^2_{\sigma\omega}+|D\bar{D}u|^2_{\sigma\omega})\\
&-\frac{C(K+1)}{\lambda_1}(|DDu|^2+|D\bar{D}u|^2)+\left(-\varepsilon\phi'-C\varphi'-\frac{C(\phi')^2}{\phi''}-\frac{C}{\beta}\right)\mathcal{F}\\
&+C\phi'-C\varphi'-C\\
\geq{}& \left(\frac{C\varphi'}{\lambda_1}-\frac{C(K+1)}{\lambda_1}\right)|DDu|^2+(C\varphi'-C(K+1))\lambda_1\\
&+(-\varepsilon\phi'-C\varphi'-C\phi)\mathcal{F}+C\phi'-C\varphi'-C,
\end{aligned} \tag{4-15}$$

by choosing $\varepsilon_0 = 3\beta = \frac{3}{16e^{\Lambda(-u(z_0)+T)}} = \frac{3\phi''}{16(\phi')^2}(z_0)$ and $\lambda_1 \geq \frac{4C}{N}$ large enough such that $\frac{\varphi'}{2} - \frac{C}{\lambda_1} \geq \frac{\varphi'}{4}$. Then choosing N large enough such that $C\varphi' - C(K+1) \geq 1$ and $\Lambda \gg N$ such that $-\varepsilon\phi' - C\varphi' - C\phi = \varepsilon\Lambda\phi - CN\varphi - C\phi > 0$. Thus (4-15) implies that

$$\lambda_1 \leq -C\phi' + C\varphi' + C.$$

The proof of Theorem 1.3 is completed.

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XIAOJUAN CHEN
INSTITUTE OF MATHEMATICS
HUNAN UNIVERSITY
CHANGSHA
CHINA
cxj@hnu.edu.cn

QIANG TU
HUBEI KEY LABORATORY OF APPLIED MATHEMATICS
HUBEI UNIVERSITY
WUHAN
CHINA
qiangtu@hubu.edu.cn

NI XIANG
HUBEI KEY LABORATORY OF APPLIED MATHEMATICS
HUBEI UNIVERSITY
WUHAN
CHINA
nixiang@hubu.edu.cn

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HIGHER-RANK INSTANTONS SHEAVES ON FANO THREEFOLDS

GAIA COMASCHI AND DANIELE FAENZI

We define higher-rank instanton sheaves on smooth Fano threefolds X of Picard rank one and show that their topological type depends on two integers, namely the rank n (or half of it if the Fano index of X is odd) and the charge k. With the possible exception of Fano threefolds of index one and genus at most 3, or genus 4 contained in a singular quadric, we determine the minimal charge k_0 of slope-stable n-instanton bundles as an integer depending only on H_X^3 and n, and we prove the existence of slope-stable n-instanton bundles of charge $k \geq k_0$.

Next, we study several properties of a general element of an irreducible component, which we call the main component, of the moduli space of instanton sheaves. This component is defined inductively on the rank and the charge, a natural procedure based on elementary transformations along rational curves, starting with minimal instantons. The questions we address include the generic splitting over certain rational curves contained in X, as well as stable restriction to a K3 section S of X. We obtain applications to Lagrangian subvarieties of moduli spaces of sheaves on S.

Finally, we study the acyclic extension associated with an instanton sheaf on Fano threefolds with curvilinear Kuznetsov component and give a monadic description in the case $H^3(X, \mathbb{Z}) = 0$.

1.	Introduction	294
2.	Generalities	298
3.	Instantons on Fano threefolds of even index	308
4.	Instantons on Fano threefolds of odd index	316
5.	Stability of restriction to $K3$ surfaces	336
6.	Fano threefolds with curvilinear Kuznetsov component	341
7.	Monadic description of instantons	350
Acknowledgements		355
References		356

MSC2020: primary 14F06, 14F08, 14J60; secondary 14D21.

Keywords: instanton bundle, Fano threefold, moduli space of instantons, restriction of stable sheaves, curvilinear Kuznetsov component, monads.

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1. Introduction

The notion of instanton bundle in higher-dimensional algebraic geometry appeared in the context of Yang–Mills gauge theory. Indeed, via Penrose's twistor transform, anti-self-dual solutions (gauge instantons) of the Yang–Mills equation on the 4-dimensional sphere S^4 correspond, up to a reality condition, to certain complex algebraic vector bundles on the complex projective space $\mathbb{P}^3$, which we will call instanton bundles. This was originally carried out in [Atiyah et al. 1978], where special emphasis was put on the case of instantons of rank 2 and on the resulting interpretation of the moduli space of instantons in terms of linear algebra data. The ADHM data has been thoroughly investigated to study instantons as a higher-rank analogue of the Hilbert scheme on smooth projective surfaces, where deep results have been obtained on the topology of the moduli space of instantons in connection with Donaldson invariants; see, for instance, [Göttsche et al. 2008; Nakajima and Yoshioka 2005].

In higher dimension, the situation is less clear. Several notions of instanton sheaves have been explored in various settings. Some definitions have been given in higher rank (see [Jardim 2006]) and over some other varieties besides the projective space; see, for example, [Faenzi 2014; Brambilla and Faenzi 2011; Kuznetsov 2012; Antonelli and Casnati 2023; Malaspina et al. 2020]. One may extend the definition to include sheaves which are not locally free or even move to the derived category and look at complexes of sheaves; see [Comaschi and Jardim 2024; Comaschi et al. 2024]. The definition of instanton, in all these generalisations, essentially consists of three main ingredients: the choice of topological invariants, constraints on the behaviour of the intermediate cohomology and a notion of stability.

However, in many cases the basic question of nonemptiness of instanton moduli spaces remains unclear. Notably, is it not known whether there exist stable instanton bundles for given choices of topological invariants, notably for Fano threefolds.

Once nonemptiness of the moduli space is proved for a given topological datum, it is natural to ask what the main features of a sufficiently general element in one component of the moduli space are. Among these questions, one could ask about the splitting type of the bundle on rational curves on X. For instance, does the bundle have *generic splitting*? Trivial splitting over a generic line is conjectured [Kuznetsov 2012, Conjecture 3.16] for an instanton bundle of rank 2 when $i_X = 2$. Further natural questions are smoothness and irreducibility of the moduli space of instanton bundles. A positive answer for $\mathbb{P}^3$ is provided in [Jardim and Verbitsky 2014] for instantons with trivial generic splitting. Irreducibility of the moduli space of instanton bundles of rank 2 on $\mathbb{P}^3$ is proved in [Tikhomirov 2012; 2013]. These questions have been explored in some more specific situations, for example symplectic bundles on $\mathbb{P}^3$, see [Bruzzo et al. 2012], or in special cases; e.g., see [Andrade et al. 2024] for instantons of rank 3 and charge 2 on $\mathbb{P}^3$.

We focus on instanton sheaves on Fano threefolds of Picard rank one and provide partial answers to some of the questions above, mainly about nonemptiness of the moduli space of slope-stable instanton bundles of given rank and charge. We actually construct an irreducible component of this moduli space, whose general element is unobstructed and well behaved with respect to restrictions to rational curves and to anticanonical sections.

Let us start with our definition of instanton sheaf — we refer to Remark 2.3 for a discussion of related notions.

Definition 1.1. Let X be a Fano threefold of Picard rank one and index i_X equal to $2q_X + r_X$, with $r_X \in \{0, 1\}$. An *instanton sheaf* is a torsion-free sheaf E on X such that the following conditions hold.

(i) $\mathrm{ch}(E) = \mathrm{ch}\big(\mathrm{R}\mathcal{H}om(E, \mathcal{O}_X(-r_X))\big)$.

(ii) $H^1(E(-q_X - t)) = H^2(E(-q_X + t)) = 0$ for $t \in \{0, 1\}$.

(iii) E is slope-semistable.

The first observation is that the topological constraints on the Chern characters imply that $\mathrm{ch}(E)$ depends on two integers, namely, the (half)rank and an invariant k depending on the second Chern class, which we refer to as the *charge* of the instanton. As it turns out, an instanton E can be of any rank when i_X is even, while E must be of even rank when i_X is odd. With this in mind, we write $\mathrm{rk}(E) = n$ when i_X is even and $\mathrm{rk}(E) = 2n$ when i_X is odd. Then one has

$$\mathrm{ch}(E) = n\,\mathrm{ch}(F_0) - k\,\mathrm{ch}(\mathcal{O}_l(q_X - 1)), \tag{1}$$

where $l \subset X$ is a line and F_0 is a slope-stable instanton bundle of the minimal possible charge and rank. Let us mention that $F_0 = \mathcal{O}_X$ when i_X is even, while when i_X is odd, F_0 is a rank-2 bundle playing an important role in the geometry of X. Namely, when $i_X = 3$, so that X is a quadric threefold, F_0 is the spinor bundle on X, while, for $i_X = 1$, F_0 is a Mukai bundle; see [Bayer et al. 2026] for the most recent developments on this.

An instanton E satisfying (1) will be referred to as an (n, k)-*instanton*. We will frequently write $\gamma(n, k)$ for the Chern character of an (n, k)-instanton. We will write $\mathrm{MI}_X(n, k)$ for the moduli space of Gieseker-semistable (n, k)-instantons, seen as an open piece of Maruyama's moduli space $\mathrm{M}_X(\gamma(n, k))$ of semistable sheaves on X with Chern character $\gamma(n, k)$. Here, an S-equivalence class is an instanton if some representative of it is an instanton sheaf. We use a similar notation when working over an anticanonical section S of X.

Studying nonemptiness of $\mathrm{MI}_X(n, k)$ involves two main steps: The first is to identify which are the vectors $v \in \bigoplus_{i=0}^{3} H^{i,i}(X, \mathbb{Z}) \simeq \mathbb{Z}^4$ for which instantons E on X with $\mathrm{ch}(E) = v$ may exist. The second step involves the actual construction

of the sheaves. Constructing strictly slope-semistable nonlocally free instantons presents no serious difficulty. Indeed, to this aim it is sufficient to consider kernels of surjections $E^n \twoheadrightarrow T$ for T a purely one-dimensional sheaf with $H^i(T(-q_X)) = 0$, $i = 0, 1$, and E a slope-stable rank-2 instanton bundle. Much greater difficulties arise in showing the existence of *locally free slope-stable* instantons. Our main goal is to show that these difficulties can be overcome. Indeed, we show the following result.

Main Theorem 1. *Let X be a smooth Fano threefold with $\mathrm{Pic}(X) \simeq \langle H_X \rangle$ and $-K_X = i_X H_X$. When the index i_X of X is one, put g for the genus of X. Let k_0^n be the integer defined as*

($\heartsuit$) $k_0^n = \lceil \frac{n}{2} \rceil$ *if* $i_X = 4$;

($\clubsuit$) $k_0^n = \lceil \frac{n}{3} \rceil$ *if* $i_X = 3$;

($\diamondsuit$) $k_0^n = n$ *if* $i_X = 2$;

($\spadesuit$) $k_0^1 = 0$, $k_0^n = 1$ *for g odd and* $k_0^n = n$ *for g even if* $n \geq 2$, *in the case* $i_X = 1$.

Assume that, if $i_X = 1$, then $g \geq 4$ and, when $g = 4$, X is not contained in a singular quadric.

Then, for all $n \geq 2 - r_X$ and $k \geq k_0^n$, X supports unobstructed slope-stable (n, k)-instanton bundles whose general splitting on lines is 0^n when i_X is even and $(0^n, -1^n)$ when i_X is odd.

Also, for all $n \geq 2$, there are no slope-stable (n, k)-instanton sheaves with $k < k_0^n$.

Here, the splitting of a rank-n bundle E over a rational curve C is denoted by the string of integers $(d_1, \ldots, d_n)$, to indicate that $E|_C \simeq \bigoplus \mathcal{O}_{\mathbb{P}^1}(d_i)$. It should be noted that, over manifolds of dimension 3 and higher, having a necessary and sufficient condition on the topological invariants that guarantees nonemptiness of the moduli space of stable vector bundles with such invariants is in general quite rare. Our result shows that moduli of instanton bundles are nicely behaved in this respect. Still, as mentioned, we don't know much about many basic properties of such moduli space — singularities, connectedness, number of irreducible components, dimension of the components: all these points remain unclear. However, we do construct a generically smooth component of the moduli space, whose general element behaves nicely in terms of stability of restriction, and, to some extent, splitting over rational curves and vanishing of cohomology.

The result is based on an inductive argument. Assume by induction that a slope-stable (n, k)-instanton bundle E_n^k on a Fano threefold X is available. Whenever $\mathrm{Ext}^1(E_n^k, F_0) \neq 0$, we can produce a strictly slope-semistable $(n+1, k)$-instanton E_{n+1}^k. This is actually always the case for values of k greater than k_0^n. We will then prove that such an E_{n+1}^k deforms to a slope-stable instanton.

When $i_X > 1$ the results we obtain are somehow optimal, as we are able to determine, for each rank, the minimal value k_0^n of the charge of an instanton bundle,

and we construct slope-stable instanton bundles of charge $\geq k_0^n$. Also, the bundles we construct this way admit global sections after a twist by $\mathcal{O}_X(1)$. The base case of the induction is worth commenting. Indeed, it is given by rank-2 't Hooft bundles, namely rank-2 instanton bundles E such that $h^0(E(1)) > 0$. Rank-2 't Hooft bundles of charge $\geq k_0$ on Fano threefolds of index ≥ 2 can be obtained via the Serre correspondence. This is well known for $i_X = 3, 4$, where the bundles are obtained from sets of disjoint lines. For Del Pezzo threefolds we will construct families of elliptic curves that give rise to rank-2 't Hooft instantons.

The case $i_X = 1$, treated in Section 4.2, turns out to be a bit more difficult, even in rank 2. We will extend some known results on the existence of rank-2 instanton bundles of arbitrary charge to all Fano threefolds of index one, provided that H_X is very ample. For rank 2, the only point to address is the existence for arbitrary charge, which was known under the additional assumption that X is ordinary, that is to say, X carries an unobstructed line. We will then focus on the higher-rank range. For $n \geq 2$, letting $k_0^n = 1$ for g odd and $k_0^n = n$ for g even, we will prove the existence of slope-stable (n, k)-instanton bundles for all $k \geq k_0^n$. We are able to prove that k_0^n is indeed the minimal value of k for slope-stable instantons when $g \geq 4$, as the cases with $g \leq 3$ present some extra difficulty that we do not address here. Again we proceed by induction, constructing slope-semistable instantons at first and then showing that these always deform to slope-stable ones. One difference compared to the previous cases essentially lies in the moduli space where such deformation takes place.

Next, we look at the restrictions $E|_S$, where E is an instanton on X and S is a K3 surface obtained as general anticanonical divisor. We show that for the slope-stable instanton bundles E constructed via the techniques presented in the previous sections, $E|_S$ is slope-stable as well. We show the following result.

Main Theorem 2. *Let X be a smooth Fano threefold of Picard number one. Assume that if $i_X = 1$, $g \geq 4$ and X is not contained in a singular quadric for $g = 4$. Let S be a general element in the linear system $|\mathcal{O}_X(-K_X)|$. Then for all n, k with $n \geq 2 - r_X$, $k \geq k_0^n$, the following holds.*

(i) *There exists an open nonempty subscheme $\mathcal{R}_S(n, k) \subset \mathrm{MI}_X(n, k)$ parametrising (n, k)-instanton bundles E on X satisfying the statement of Main Theorem 1 and such that the restriction $E|_S$ is slope-stable.*

(ii) *The image of $\mathcal{R}_S(n, k)$ in the moduli space $\mathrm{M}_S(n, k)$ under the restriction map $E \mapsto E|_S$ is a Lagrangian subvariety for the standard symplectic structure on $\mathrm{M}_S(n, k)$.*

We conclude the article by describing some properties of instantons on certain Fano threefolds X whose derived category has a distinctive structure. In Section 6, we consider the case of Fano threefolds whose derived category contains a component equivalent to the derived category of a curve Γ. We say that X has a curvilinear

Kuznetsov component. These are the Del Pezzo threefold Y_4 and the prime Fano threefolds X_g, $g \in \{7, 9, 10\}$. We first show that slope-stable (n, k)-instantons E on X correspond to certain vector bundles on Γ. Then we prove that for Y_4 and X_{10} these bundles are simple. On X_7, X_9 the same holds for E belonging to a subscheme of $\mathrm{MI}_X(n, k)$ which is dense in the component we constructed. Finally, we describe the canonical resolutions of the instanton itself in terms of the Fourier–Mukai transform of these bundles.

The paper is organised as follows. In the next section, we provide some standard material on Fano threefolds and instantons. In Section 3 we look at constructions of instantons on Fano threefolds of even Fano index, first on $\mathbb{P}^3$ and then on Del Pezzo threefolds, and prove Main Theorem 1 in this case. In Section 4 we look at Fano threefolds of odd Fano index, construct instantons on them, and finish the proof of Main Theorem 1. We first deal with a quadric threefold and then with prime Fano threefolds. Next we study stability of restriction to K3 sections in Section 5 and prove Main Theorem 2. Finally in Sections 6 and 7, we shift our focus to Fano threefolds X whose derived category has a nice description, either because $D^b(X)$ has a curvilinear Kuznetsov component (such category is equivalent to $D^b(\Gamma)$, with Γ a smooth projective curve) or because $D^b(X)$ has a full exceptional collection. We prove that, on such varieties, instantons, just as in the rank-2 case, admit a description given via acyclic extensions in terms of sheaves on the curve Γ, or in terms of monads.

2. Generalities

2.1. ***Preliminaries.*** Let X be a smooth Fano threefold of Picard rank 1. We denote by H_X the ample generator of $\mathrm{Pic}(X)$. We recall that the index of X is the integer i_X such that the canonical line bundle K_X is isomorphic to $\mathcal{O}_X(-i_X)$. It is well known that $i_X \in \{1, 2, 3, 4\}$, essentially by Kobayashi and Ochiai's theorem [1973]; see also [Parshin and Shafarevich 1998]. Let us now denote by q_X and r_X the integers such that $i_X = 2q_X + r_X$, with $r_X \in \{0, 1\}$. Notice that $q_X \in \{0, 1, 2\}$. We will often write the Chern classes of a coherent sheaf E on X as integers, omitting to mention the generator of the corresponding cohomology group, which we fix to be the class H_X of a hyperplane section H_X, the class l_X of a line, or the class p_X of a point, for $H^{1,1}(X)$, $H^{2,2}(X)$, or $H^{3,3}(X)$, respectively.

Given a coherent sheaf E on X, we recall that E is said to be Gieseker-*stable* (resp. -*semistable*) if it is pure and, for any proper subsheaf $F \subset E$,

$$p_F(t) < p_E(t) \ \ (\text{resp. } \leq p_E(t)).$$

Here, $p_E(t)$ denotes the *reduced Hilbert polynomial* of the sheaf E, namely its Hilbert polynomial divided by its rank. See [Huybrechts and Lehn 2010, §I.1.2] for more details on this notion. If E is torsion-free, we say that E is *slope-stable*

(resp. *slope-semistable*) if for every proper saturated subsheaf $F \subset E$,

$$\mu(F) := \frac{c_1(F)}{\mathrm{rk}(F)} < \frac{c_1(E)}{\mathrm{rk}(E)} := \mu(E) \quad \left(\text{resp. } \leq \frac{c_1(E)}{\mathrm{rk}(E)} := \mu(E)\right).$$

According to the convention mentioned above, the first Chern class of a coherent sheaf E is an integer, so that the *slope* $\mu(E)$ is a rational number. Sometimes we will only say "stable" or "semistable" to mean "Gieseker-stable" or "Gieseker-semistable".

We said that our notion of instanton sheaf is based on three conditions. The first one is about the Chern character, the second one is a cohomological vanishing, and the third one is a stability condition. From Definition 1.1(ii)–(iii) we deduce the following result.

Lemma 2.1. *Let E be an instanton sheaf. Then*

(i) $h^i(E(-q_X)) = 0$ *for all integers* i;

(ii) $h^1(E(-q_X - t)) = h^2(E(-q_X + t)) = 0$ *for all integers* $t \geq 0$.

Proof. The slope-semistability assumption ensures that $\hom(\mathcal{O}_X(q_X), E) = 0$ and $h^3(E(-q_X)) = \hom(E, \mathcal{O}_X(-q_X - r_X)) = 0$; hence we have

$$h^i(E(-q_X)) = 0 \quad \text{for all integers } i. \tag{2}$$

Consider now a general hyperplane section $S := X \cap H$ for $H \in |\mathcal{O}_X(1)|$ and the short exact sequence

$$0 \to E(-1) \to E \to E|_S \to 0.$$

The previous vanishing implies $h^0(E(-q_X - t)) = h^3(E(-q_X + t)) = 0$ for all $t \geq 0$; hence from Definition 1.1(ii) we deduce, inductively, that

$$h^1(E(-q_X - t)) = h^2(E(-q_X + t)) = 0 \quad \text{for all integers } t \geq 0. \qquad \square$$

Remark 2.2. Definition 1.1(i)–(ii) can be rephrased in categorical terms. Indeed we consider the bounded derived category of coherent sheaves $D^b(X)$. If E is an instanton sheaf on X, then (2) says that

$$E \in \mathcal{O}_X(q_X)^\perp := \{F \in D^b(X) \mid \mathrm{Ext}^\bullet(\mathcal{O}_X(q_X), F) = 0\}.$$

Moreover, the functor $\Phi : F \mapsto \mathrm{R}\mathcal{H}om(F, \mathcal{O}_X(-r_X))$ induces an antiequivalence of $\mathcal{O}_X(q_X)^\perp$. Indeed, if F is an object of $\mathcal{O}_X(q_X)^\perp$, then, for all integers i,

$$\mathrm{Ext}^i(\mathcal{O}_X(q_X), \Phi(F)) \simeq \mathrm{Ext}^i(F, \mathcal{O}_X(-q_X - r_X)) \simeq \mathrm{Ext}^{3-i}(\mathcal{O}_X(q_X), F)^* = 0.$$

Hence, Definition 1.1 tells us that an instanton sheaf E belongs to $\mathcal{O}_X(q_X)^\perp$ and that $\Phi(E)$ is an object in $\mathcal{O}_X(q_X)^\perp$ such that $\mathrm{ch}(E) = \mathrm{ch}(\Phi(E))$.

Remark 2.3. Let us discuss briefly the conditions of Definition 1.1, also with respect to similar notions in the literature.

(i) Let E be a torsion-free sheaf satisfying Definition 1.1(i)–(iii). For all the applications envisaged here, including Theorems 1 and 2, as well as Theorems 6.4 and 7.1, Definition 1.1(ii) can be replaced by the more economical condition

$$h^1(E(-q_X)) = 0 \quad \text{if } i_X \in \{3, 4\},$$

or, in the case $i_X \in \{1, 2\}$, by

$$h^1(E(-q_X)) = h^1(E(-1-q_X)) = h^2(E(1-q_X)) = 0.$$

Indeed, $\chi(E(-q_X)) = 0$ follows from Definition 1.1(i) and slope-semistability easily implies $h^0(E(-q_X)) = h^3(E(-q_X)) = 0$, so $h^2(E(-q_X)) = 0$ follows. Also, by slope-semistability, $h^0(E(1-q_X)) = 0$ if $i_X \geq 3$, which gives back Definition 1.1(ii), by the basic argument of Lemma 2.1.

(ii) Let E be a torsion-free sheaf satisfying Definition 1.1(i). It follows that Definition 1.1(ii)–(iii) are satisfied if $h^1(E(-q_X)) = 0$, and $E|_S$ is slope-stable, for some hyperplane section S of X. Semistability on S is intended with respect to the restriction of H_X to S. If E is slope-semistable of rank two and S is general enough, by [Maruyama 1980], $E|_S$ is slope-semistable; see Remark 2.4.

(iii) A different definition of instanton sheaf was given in [Antonelli and Casnati 2023], encompassing previous work, e.g., [Faenzi 2014; Jardim and Miró-Roig 2008; Kuznetsov 2012]. A comparison between that notion and our definition (at least for locally free instanton sheaves of rank 2) is given in [Antonelli and Casnati 2023, §1]. The two notions are similar, but not identical in general.

In summary, if E is an instanton sheaf as in Definition 1.1, $\mathcal{E} = E(2-q_X)$ is an instanton in the sense of [Antonelli and Casnati 2023] if $i_X \in \{3, 4\}$. For $i_X \in \{1, 2\}$, this happens under the further condition $h^0(E(1-q_X)) = h^3(E(-1-q_X)) = 0$. Note that this is not satisfied for minimal instantons if $i_X \in \{1, 2\}$. For $i_X = 1$, we see that actually instanton bundles E as defined here will not satisfy $h^0(E(1)) = 0$ if the charge is relatively small with respect to the rank.

Conversely, if $\mathcal{E}$ is an *ordinary instanton* on X, in the sense of [Antonelli and Casnati 2023], which is slope-semistable, then $E = \mathcal{E}(q_X - 2)$ is an instanton sheaf according to Definition 1.1. The two notions of charge, however, are typically off by a constant that depends on the topological invariants of X.

(iv) Later on we will also use torsion instantons; see Definition 2.9. These have been studied on Fano threefolds notably in [Comaschi and Jardim 2024].

2.2. ***Rank-2 instantons.*** Before dealing with the higher-rank range, which is our main focus, we recall some basic facts about the rank-2 instantons. This case has been largely investigated, notably on $\mathbb{P}^3$, starting with [Atiyah et al. 1978], where special bundles, called 't Hooft instantons, are also introduced. In this subsection we introduce a similar construction for Fano threefolds of Picard number one.

Remark 2.4. In the rank-2 case, the cohomological conditions of Definition 1.1 can be reduced just to $H^i(E(-q_X)) = 0$, $i = 1, 2$. This is because, since $\mathrm{rk}(E) = 2$, given a general hyperplane section $S \subset X$, the sheaf $E|_S$ is still slope-semistable, according to [Maruyama 1980]. Therefore $h^0(E(-n)) = h^3(E(-i_X + n)) = 0$ for all $n \geq 0$, which allows one to show, inductively

$$H^1(E(-q_X - n)) = H^2(E(-q_X + n)) = 0 \quad \text{for all } n \geq 0.$$

If moreover E is locally free it actually suffices to require $H^1(E(-q_X)) = 0$ since by Serre duality, and as $E^* \simeq E(r_X)$, we have $H^2(E(-q_X)) \simeq H^1(E(-q_X))^*$.

We recall an effective and widely applied technique to construct rank-2 bundles on a threefold X, namely *Serre's correspondence*. It establishes a correspondence between pairs (E, s) for E a rank-2 vector bundle and $s \in H^0(E)$ a global section with torsion-free cokernel and pairs (e, C) for C a locally complete intersection curve (l.c.i. for short) and $e \in \mathrm{Ext}^1(\mathcal{I}_C(c_1(E)), \mathcal{O}_X)$ a nonvanishing extension class corresponding to

$$0 \to \mathcal{O}_X \xrightarrow{s} E \to \mathcal{I}_C(c_1(E)) \to 0.$$

Under the cohomological vanishings used below, Serre duality and adjunction give $\mathrm{Ext}^1\big(\mathcal{I}_C(c_1(E)), \mathcal{O}_X\big) \simeq H^0(\mathcal{O}_C)$. Note that this can be generalised to torsion-free sheaves; see [Comaschi and Jardim 2024].

In what follows we will need to construct rank-2 slope-stable and *unobstructed* instantons E, namely $\mathrm{Ext}^2(E, E) = 0$. This can be done via the Serre correspondence starting from l.c.i. curves that are unobstructed as well, which in this context means that $h^1(\mathcal{N}_{C/X}) = 0$. We recall that the unobstructedness of a curve $C \subset X$ is actually equivalent to the unobstructedness of its sheaf of ideals $\mathcal{I}_C$. This is due to the following result.

Lemma 2.5. *Let C be a locally complete intersection curve on a smooth projective threefold X such that $H^p(\mathcal{O}_X) = 0$ for $p > 0$. Then $H^i(\mathcal{N}_C) \simeq \mathrm{Ext}^{i+1}(\mathcal{I}_C, \mathcal{I}_C)$ for $i = 0, 1$.*

Proof. Consider the exact sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0.$$

Applying $\mathcal{H}om(\mathcal{I}_C, \cdot)$ to it and recalling the sheaf $\mathcal{I}_C$ has homological dimension 1, we get the exact sequence of $\mathcal{O}_C$ modules

$$0 \to \mathcal{N}_C \to \mathcal{E}xt^1(\mathcal{I}_C, \mathcal{I}_C) \to \mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X) \xrightarrow{\gamma} \mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_C) \to 0. \tag{3}$$

We also get that $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_X) \simeq \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_X) \simeq \omega_C(-K_X)$ is a line bundle on C and the same holds for $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{O}_C) \simeq \mathcal{E}xt^2(\mathcal{O}_C, \mathcal{O}_C)$. Therefore the surjection γ in (3) must actually be an isomorphism. As a consequence $\mathcal{N}_C \simeq \mathcal{E}xt^1(\mathcal{I}_C, \mathcal{I}_C)$.

Using $\mathcal{H}om(\mathcal{I}_C, \mathcal{I}_C) \simeq \mathcal{O}_X$ and $\mathcal{E}xt^1(\mathcal{I}_C, \mathcal{I}_C) \simeq \mathcal{N}_C$, the local-to-global spectral sequence

$$H^p(\mathcal{E}xt^q(\mathcal{I}_C, \mathcal{I}_C)) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{I}_C, \mathcal{I}_C)$$

gives the exact sequence

$$H^1(\mathcal{O}_X) \to \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C) \to H^0(\mathcal{N}_C) \to H^2(\mathcal{O}_X) \\ \to \mathrm{Ext}^2(\mathcal{I}_C, \mathcal{I}_C) \to H^1(\mathcal{N}_C) \to H^3(\mathcal{O}_X).$$

Since $H^p(\mathcal{O}_X) = 0$ for $p > 0$, we obtain

$$\mathrm{Ext}^1(\mathcal{I}_C, \mathcal{I}_C) \simeq H^0(\mathcal{N}_C), \quad \mathrm{Ext}^2(\mathcal{I}_C, \mathcal{I}_C) \simeq H^1(\mathcal{N}_C). \qquad \square$$

We use the Serre correspondence to construct rank-2 *'t Hooft bundles*. These are rank-2 instanton bundles with $h^0(E(1)) > 1$, which arise from locally complete intersection curves C of arithmetic genus $p_a(C) = 3-2(q_X+r_X)$, i.e., $p_a(C) = -1$ for $i_X = 3, 4$ and $p_a(C) = 1$ for $i_X = 1, 2$. This is a generalisation of the classical 't Hooft instantons of [Atiyah et al. 1978], which arise when C is the union of disjoint lines in $\mathbb{P}^3$. Incidentally, we recall that classical 't Hooft bundles have a distinguished description in terms of the corresponding solutions of the Yang Mills equations; see [Atiyah 1978] — we don't know if anything similar could be said for the bundles we construct here.

Lemma 2.6. *Let X be a Fano threefold of Picard rank one of index $i_X \in \{3, 4\}$ (resp. $i_X \in \{1, 2\}$), $C \subset X$ an l.c.i. curve of arithmetic genus $p_a(C) = 1 - \deg(C)$ (resp. a connected l.c.i. curve of $p_a(C) = 1$), and E a rank-2 vector bundle arising from a nonvanishing extension*

$$e \in \mathrm{Ext}^1(\mathcal{I}_C(1 - r_X), \mathcal{O}_X(-1)) \simeq H^0(\mathcal{O}_C).$$

Then E is a rank-2 't Hooft bundle and the following hold:

(i) *If C is unobstructed then E is unobstructed.*

(ii) *If i_X is odd, E is slope-stable.*

(iii) *If i_X is even, E is Gieseker-semistable if and only if it is slope-stable if and only if C is nondegenerate.*

(iv) *The general line $l \subset X$ is not jumping for E, that is to say, $h^1(E|_l(-1)) = 0$.*

Proof. Consider the short exact sequence, defined by the extension class e,

$$0 \to \mathcal{O}_X(-1) \to E \to \mathcal{I}_C(1 - r_X) \to 0. \tag{4}$$

We have $H^1(E(-q_X)) = H^1(\mathcal{I}_C(1 - r_X - q_X))$. Indeed, if $i_X \in \{3, 4\}$, then $r_X + q_X = 2$; hence this term vanishes. If $i_X \in \{1, 2\}$, then $r_X + q_X = 1$ and we get

$H^1(\mathcal{I}_C) = 0$ since we are assuming that C is connected. We can thus conclude that E is an instanton bundle.

If i_X is odd, since $c_1(E) = -1$, the slope-semistability of E is equivalent to its slope-stability which is equivalent to $H^0(E) = 0$. This always holds since $H^0(\mathcal{I}_C) = 0$. If i_X is even, E is slope-semistable (resp. slope-stable) if and only if $H^0(E(-1)) = 0 = H^0(\mathcal{I}_C)$ (resp. $H^0(E) = 0 = H^0(\mathcal{I}_C(1))$). Since a direct computation shows that $p_{\mathcal{O}_X} > p_E$ we conclude that E is Gieseker-semistable if and only if it is slope-stable which holds if and only if C is not contained in any hyperplane.

Let us now prove (i). From Lemma 2.5, the unobstructedness of C is equivalent to $\mathrm{Ext}^2(\mathcal{I}_C, \mathcal{I}_C) = 0$. Then, we apply the functors $\mathrm{Hom}(\cdot\,, E)$ and $\mathrm{Hom}(\mathcal{I}_C, \cdot\,)$ to the short exact sequence (4). Using Lemma 2.1, this gives the exact sequences

$$\mathrm{Ext}^1(\mathcal{O}_X(-1), E) \to \mathrm{Ext}^2(\mathcal{I}_C(1-r_X), E) \to \mathrm{Ext}^2(E, E) \to 0,$$
$$\mathrm{Ext}^2(\mathcal{I}_C(1-r_X), \mathcal{O}_X(-1)) \to \mathrm{Ext}^2(\mathcal{I}_C(1-r_X), E) \to \mathrm{Ext}^2(\mathcal{I}_C, \mathcal{I}_C) \to 0.$$

Note that $\mathrm{Ext}^2(\mathcal{I}_C(1-r_X), \mathcal{O}_X(-1))^* \simeq H^1(\mathcal{I}_C(2-i_X-r_X))$. For $i_X \in \{3, 4\}$ we find $H^1(\mathcal{I}_C(-2)) = 0$ whilst for $i_X \in \{1, 2\}$ we have $H^1(\mathcal{I}_C) = 0$, as C is connected. Therefore if we assume C unobstructed, $\mathrm{ext}^2(\mathcal{I}_C, \mathcal{I}_C) = 0$ by Lemma 2.5, so that $\mathrm{ext}^2(\mathcal{I}_C(1-r_X), E) = 0$ and $\mathrm{ext}^2(E, E) = 0$ as well.

It remains to show that a general line is not jumping for E. When i_X is odd, take a line $l \subset X$ disjoint from the curve C, which is an open condition on the family of lines on X. Restricting (4) to l we get that $E|_l$ is an extension of $\mathcal{O}_l$ by $\mathcal{O}_l(-1)$ so that $E|_l \simeq \mathcal{O}_l \oplus \mathcal{O}_l(-1)$. When i_X is even, we choose a line l meeting C transversely at one point $p \in C$. This time, restricting (4) to l, we get

$$0 \to \mathcal{O}_l(-1) \to E|_l \to \mathcal{O}_l \oplus \mathcal{O}_p \to 0.$$

Since $E|_l$ is locally free, the kernel of the induced surjection $E|_l \twoheadrightarrow \mathcal{O}_l \oplus \mathcal{O}_p \twoheadrightarrow \mathcal{O}_l$ must be $\mathcal{O}_l$ yielding $E|_l \simeq \mathcal{O}_l^2$. By semicontinuity the same will hold for the general line on X. □

The Serre correspondence is particularly effective to construct instanton bundles of low charges, in particular those having minimal charge (meaning the minimal one allowed by the constraints in Definition 1.1), or having charge slightly greater than the minimal one. In the upcoming sections we will describe in detail the behaviour of the slope-stable rank-2 instantons having the lowest possible charge. We will see that these bundles are actually 't Hooft instantons and for each value of i_X we will also describe the corresponding curves on X.

2.3. ***Instanton bundles of higher rank.*** We choose to focus now our attention on instanton sheaves of higher rank.

2.3.1. *The Chern character of instantons.* To begin with, let us look at their Chern character. From now on we will denote by F_0 a minimal slope-stable instanton on X, in the sense that it has the minimal possible values of rank and charge admitted by Definition 1.1. As already pointed out in the previous section, $F_0 \simeq \mathcal{O}_X$ for i_X even whilst F_0 is a locally free rank-2 instanton for i_X odd. Note that the main features of these bundles in the odd index cases will be summarised in Sections 4.1 and 4.2.

Lemma 2.7. *Let E be an instanton. Then there is a pair of integers (n, k) such that*

$$\operatorname{ch}(E) = n \operatorname{ch}(F_0) - k \operatorname{ch}(\mathcal{O}_l(q_X - 1)) \tag{5}$$

for F_0 a minimal instanton.

Proof. Let us start with the case i_X even. For an instanton E, the condition $\operatorname{ch}(E) = \operatorname{ch}(\mathrm{R}\mathcal{H}om(E, \mathcal{O}_X))$ implies that the odd Chern classes of E vanish. Hence $\operatorname{ch}(E)$ has the form $(n, 0, -k, 0) = n \operatorname{ch}(F_0) - k \operatorname{ch}(\mathcal{O}_l(q_X - 1))$ for $n = \operatorname{rk}(E)$ and $k = c_2(E)$.

Assume now that i_X is odd and let $\operatorname{ch}_j(E)$ be the j-th Chern character of a nonzero instanton sheaf E for $j \in \{0, \dots, 3\}$. The condition $\operatorname{ch}(E) = \operatorname{ch}\big(\mathrm{R}\mathcal{H}om(E, \mathcal{O}_X(-1))\big)$ imposes

$$c_1(E) = -\tfrac{1}{2} \operatorname{rk}(E) H_X, \quad \operatorname{ch}_3(E) = -\tfrac{1}{12} c_1(E) H_X^2 - \tfrac{1}{2} \operatorname{ch}_2(E). \tag{6}$$

Note that the first equation implies that the rank of E must be even; hence $\operatorname{rk}(E) = 2n$ for some positive integer n. Let now E^m be a rank-2 instanton of second Chern class $c_2(E^m) = m$. From the conditions above $c_1(E^m) = -H_X$ and $c_3(E^m) = 0$. Then, going back to our arbitrary instanton sheaf E of rank $2n$, we check directly that, under the condition (6), we have

$$\operatorname{ch}(E) - n \operatorname{ch}(E^m) = -k_m l_X + \tfrac{1}{2} k_m p_X = -k_m \operatorname{ch}(\mathcal{O}_l(q_X - 1)),$$

where $k_m = n \operatorname{ch}_2(E^m) - \operatorname{ch}_2(E)$ is an integer that depends on m. Setting $E^m = F_0$, we obtain (5). □

Remark 2.8. The invariance of the Chern character of an instanton E with respect to the functor $E \mapsto \mathrm{R}\mathcal{H}om(E, \mathcal{O}_X(-r_X))$ also tells us that

$$\chi(E, F_0) = \chi(F_0, E).$$

From now on, an instanton E on X whose Chern character can be expressed as in (5) will be referred to as an (n, k)-*instanton*. We denote the Chern character $\operatorname{ch}(E) \in \bigoplus_{i \geq 0} H^{i,i}(X, \mathbb{Z})$ of an (n, k)-instanton by $\gamma(n, k)$. The number k will be referred to as the *charge* of the instanton. This will coincide with the second Chern class of E when i_X is even whilst when i_X is odd we have

$$k = \tfrac{1}{2} n d_X - \tfrac{1}{2} n^2 d_X - n c_2(F_0) + c_2(E), \quad \text{where } d_X = H_X^3.$$

2.3.2. *Existence of slope-semistable instanton sheaves of higher rank.* The main existence issue we will deal with is the following: for which values of n and k does X carry an (n,k)-instanton? If one just wants to look at slope-semistable, not necessarily locally free (n,k)-instantons, it is actually quite easy to answer this question. Sheaves F of this kind can indeed be obtained by taking kernels of surjections $E \twoheadrightarrow T$ for E and T, respectively, a slope-semistable and a *torsion instanton*, in which case we will say that F is the *elementary transformation* of E along T. We refer to [Comaschi and Jardim 2024] for a more detailed discussion of the features of the instanton sheaves obtained using this technique.

Definition 2.9. Let X be a Fano threefold of Picard rank one. A *torsion* instanton is a coherent sheaf T of pure dimension one and such that $h^i(T(-q_X)) = 0$ for $i = 0, 1$. The integer $d = \deg(\mathrm{Supp}(T))$ is called the *degree* of T.

Lemma 2.10. *Let X be a Fano threefold. Then for all $n \geq 2$ and $k \geq 0$, X carries a strictly slope-semistable (n,k)-instanton.*

Proof. Instantons E satisfying Lemma 2.10 can be obtained via elementary transformation $F_0^n \twoheadrightarrow T$ for T a torsion instanton of degree k. A torsion instanton T of such a kind always exists: it will be enough to take $T \simeq \bigoplus_{i=1}^{k} \mathcal{O}_{l_i}((q_X - 1))$ for disjoint lines $l_1, \ldots, l_k$ on X, with the additional assumption that none of these lines is jumping for F_0, i.e., $h^1(F_0|_{l_i}(-1)) = 0$, for i_X odd. That a general line on X indeed satisfies this condition is ensured by the fact that F_0 is a 't Hooft bundle; see Lemma 2.6. A more detailed description of the corresponding curves will be presented later on.

A sheaf E obtained in this way is slope-semistable since this holds for its double dual, as indeed we have $E^{**} \simeq F_0^n$. Since the kernel of the composition $F_0 \hookrightarrow F_0^n \to T$ injects into E, E is strictly slope-semistable. It is immediate to check that $h^1(E(-q_X - t)) = H^2(E(-q_X + t)) = 0$ for $t \in \{0, 1\}$, so we conclude that E is an (n,k)-instanton. □

2.3.3. *Minimal charge of slope-stable instanton bundle of higher rank.* Things become more intricate after strengthening our stability assumption. When we pass to the higher-rank cases we can immediately observe that checking the stability of sheaves becomes typically much more tricky. Here, we look for *slope-stable instanton bundles*; slope-stability is among the strongest conditions that one can impose. To accomplish this task we go through two main steps. First, for each n, we need to determine the minimal value of k for which (n,k)-instantons might exist. If we assume $i_X \geq 2$, whenever n is fixed, it is easy to establish a first lower bound for the value of k that must hold if we hope for the existence of (n,k)-instantons that are at least Gieseker-semistable.

Lemma 2.11. *Let E_n^k be an (n,k)-instanton on a smooth Fano threefold of Picard number one and index $i_X \geq 2$. Assume that E_n^k is Gieseker-semistable if i_X has*

even index and that E_n^k is slope-semistable if $i_X = 3$. Then $k \geq \lceil \frac{n}{q_X} \rceil$ and $k \geq 0$, respectively.

Proof. The inequality is obtained by imposing $\chi(E_n^k) \leq 0$. Indeed, if $\chi(E_n^k) > 0$, then E_n^k admits a nonzero global section, because Lemma 2.1 gives $h^2(E_n^k) = 0$. But this is impossible due to the semistability assumption on E_n^k. □

Trying to establish a lower bound for k when $i_X = 1$ in a similar way is not effective. Indeed, when $i_X = 1$, any instanton sheaf E satisfies $\chi(E) = 0$.

We will see that in the even index cases, this lower bound is actually sharp; whilst for the odd indices further constraints will be required.

The second main step consists in the actual construction of instantons for all values of k greater than the minimal one.

Remark 2.12. The technique of the elementary transformation that we applied in Lemma 2.10 is also used in [Faenzi 2014] to prove the existence of rank-2 instanton bundles of high charge on Del Pezzo threefolds. Performing the elementary transformation of a slope-stable instanton bundle E along a line $l \subset X$ which is not a jumping line of E, we get

$$0 \to F \to E \to \mathcal{O}_l \to 0.$$

Here, F is a slope-stable nonlocally free instanton sheaf of charge $c_2(F) = c_2(E) + 1$, with $F^{**} \simeq E$.

It turns out the sheaf F deforms flatly to a locally free instanton. This is proved by showing the dimension of the nonlocally free deformations of F is strictly less than the dimension of the Gieseker-moduli space at $[F]$. We will recall the key ingredients of this argument in Section 4.2, where we will adapt it to prove the existence of rank-2 instanton bundles on Fano threefolds of index 1; see Proposition 4.9.

It is natural to ask if a similar approach can be exploited in the higher rank cases, but unfortunately the answer is negative. The reasons for the effectiveness of this technique in the rank-2 case are essentially two. First, the fact that for any nonlocally free rank-2 instanton E, the sheaves E^{**} and E^{**}/E are, respectively, an instanton bundle and a torsion instanton. This affords an estimate of the dimension of the nonlocally free deformations of E. Second, for rank-2 instantons, local freeness is equivalent to reflexivity. But both these properties no longer hold for rank greater than 2; see [Comaschi and Jardim 2024] for details.

2.3.4. *Slope-destabilising subsheaves and unobstructedness.* Starting from a slope-stable (n, k)-instanton E_n^k on a Fano threefold X, we will construct a strictly slope-semistable $(n+1, k)$-instanton E_{n+1}^k arising from an extension $\mathrm{Ext}^1(E_n^k, \mathcal{O}_X)$ for i_X even and from an extension $\mathrm{Ext}^1(E_n^k, F_0)$ for i_X odd. The fact that E_{n+1}^k is obtained in this way enables us to classify all its slope-destabilising subsheaves. This is due to the following result.

Lemma 2.13. *Let X be an m-dimensional projective variety of Picard rank one and let E be a sheaf arising from a nonzero extension $e \in \mathrm{Ext}^1(F, G)$, with F and G slope-stable vector bundles with $\mu(F) = \mu(G) = \mu(E)$ and $G \not\simeq F$. Then E is a strictly slope-semistable vector bundle and F is, up to isomorphism, the unique torsion-free quotient of E having slope $\mu(E)$.*

Proof. By construction E fits in the short exact sequence

$$0 \to G \to E \to F \to 0.$$

So E cannot be slope-stable, since it contains G. If E were unstable, it would admit a slope-semistable subsheaf L with $\mathrm{rk}(L) < \mathrm{rk}(E)$, $\mu(L) > \mu(E)$. But the slope-stability of F implies that $\mathrm{Hom}(L, F) = 0$ so that L would inject into G, which is clearly impossible as G is slope-stable as well. Therefore E is strictly slope-semistable.

To prove the lemma we need to show that every subsheaf $L \hookrightarrow E$ of slope $\mu(E)$ such that E/L is torsion-free is equal to G. For such a subsheaf, which must be slope-semistable, consider the composition $L \to F$, denote by Q its image and by K the kernel of $L \twoheadrightarrow Q$.

Let us suppose that $Q \neq 0$. Since F is slope-stable, we must have the equalities $\mu(Q) = \mu(L) = \mu(K) = \mu(G) = \mu(L) = \mu(E)$. Now, if $K \neq 0$, we must have $G \simeq K$, for otherwise G/K would be a torsion subsheaf of E/L. So $E/L \simeq F/Q$ which is torsion unless $F \simeq Q$, so $E \simeq L$ as well which is not possible since L is a proper subsheaf of E.

Then we would have $K = 0$, so $L \simeq Q$ and E/L would be an extension of F/L by G. Note that $F/L \neq 0$, since otherwise $E \twoheadrightarrow G$ which would imply that the sequence splits. Note that, since F is slope-stable and $\mu(F) = \mu(L)$, we must have $\mathrm{rk}(F) = \mathrm{rk}(L)$ and $c_1(F) = c_1(L)$, so $\mathrm{codim}(\mathrm{Supp}(F/L)) \geq 2$. Hence $\mathrm{Ext}^1(F/L, G) \simeq \mathrm{Ext}^{m-1}(G, F/L \otimes \omega_X)^* \simeq H^{m-1}(G^* \otimes F/L \otimes \omega_X)^* = 0$ which again contradicts the torsion-freeness of E/L.

Then we must have $Q = 0$, which gives $K \simeq L$; hence $L \hookrightarrow G$. Again, since G is slope-stable and $\mu(L) = \mu(G)$, we get $\mathrm{rk}(L) = \mathrm{rk}(G)$, whence G/L is torsion. But G/L injects into E/L, which is torsion-free, so $G = L$. □

The way we use the previous lemma is to characterise the strictly slope-semistable deformations of E^k_{n+1}, in the sense that a general deformation will also admit a slope-stable (n, k)-instanton as its unique torsion-free quotient having the same slope. This allows one to determine the dimension of the family of strictly slope-semistable deformations of E^k_{n+1}. Based on this, we will prove that E^k_{n+1} deforms to a slope-stable instanton.

In order to control the deformations of E^k_{n+1}, we must check its unobstructedness. This will be proved inductively: starting from an unobstructed (n, k)-instanton E^k_n on

a Fano threefold of even index (resp. E_n^k such that $\mathrm{Ext}^2(F_0, E_n^k) = \mathrm{Ext}^2(E_n^k, F_0) = 0$ on a Fano threefold X of odd index) we get the unobstructedness of the extensions of E_n^k by $\mathcal{O}_X$ (resp. F_0).

Lemma 2.14. *Let E and F be coherent sheaves on a scheme X. Suppose that E and F are unobstructed and that $\mathrm{Ext}^2(E, F) = 0 = \mathrm{Ext}^2(F, E)$. Let $e \in \mathrm{Ext}^1(F, E)$, $e \neq 0$, be an extension class corresponding to the nonsplit exact sequence*

$$0 \to F \to E' \to E \to 0.$$

Then E' is unobstructed.

Proof. Since $\mathrm{Ext}^2(F, F) = 0 = \mathrm{Ext}^2(E, F)$, we have that $\mathrm{Ext}^2(E', F)$ vanishes as well. Therefore $\mathrm{Ext}^2(E', E')$ injects into $\mathrm{Ext}^2(E', E)$. But the vanishing of $\mathrm{Ext}^2(F, E)$ implies that $\mathrm{Ext}^2(E, E)$ surjects onto $\mathrm{Ext}^2(E', E)$ so that the latter must be zero as we are assuming E unobstructed. Thus, $\mathrm{Ext}^2(E', E') = 0$. □

If $i_X \neq 1$, we will be able to prove the existence of slope-stable instanton bundles that also admit global sections after a twist by $\mathcal{O}_X(1)$. We remark that for most values of k, (n, k)-instantons E such that $h^0(E(1)) \neq 0$ form proper closed subsets of their moduli space.

3. Instantons on Fano threefolds of even index

We will begin by treating the case when i_X is even. Recall that in this context $F_0 \simeq \mathcal{O}_X$ and the Chern character $\gamma(n, k)$ of an (n, k)-instanton E_n^k has the form

$$\gamma(n, k) = n \,\mathrm{ch}(\mathcal{O}_X) - k \,\mathrm{ch}(\mathcal{O}_l(q_X - 1)) = (n, 0, -k, 0). \tag{7}$$

The integer n is just the rank of E while k is the second Chern class of E. We saw that the instanton conditions impose $k \geq \lceil \frac{n}{q_X} \rceil$; see Lemma 2.11 — recall that $q_X = 2$ if $X = \mathbb{P}^3$ and $q_X = 1$ if X is a Del Pezzo threefold.

We will show that this lower bound is sharp, constructing inductively slope-stable (n, k)-instanton bundles for all $k \geq \lceil \frac{n}{q_X} \rceil$. These will be obtained by deforming strictly slope-semistable (n, k)-instantons arising as extensions of a slope-stable $(n-1, k)$-instanton by $\mathcal{O}_X$.

3.1. ***Instantons of high rank on $\mathbb{P}^3$.*** We first direct our attention towards the case $X = \mathbb{P}^3$, that is to say, $i_X = 4$. The first thing that we need to do is to recall the well-known construction of slope-stable unobstructed rank-2 instanton bundles of charge $k \geq 1$ that will provide us the base of our inductive argument. We just pick the construction from Lemma 2.6, through unobstructed curves of degree $k+1$ and arithmetic genus $-k$, for instance, the union of $k+1$ disjoint lines, giving "classical" 't Hooft bundles.

Lemma 3.1. *For $k > 0$, let C_k be the union of $k+1$ disjoint lines and let E_2^k be a 't Hooft bundle arising from a nonzero extension class in* $\mathrm{Ext}^1(\mathcal{I}_{C_k}(1), \mathcal{O}_{\mathbb{P}^3}(-1))$. *Then E_2^k is slope-stable and unobstructed and has trivial splitting on a sufficiently general line.*

Proof. This is Lemma 2.6. The only thing left to check is the unobstructedness of C_k; but this is immediate since, if C_k is supported on $l_0, \dots, l_k$, then $\mathcal{N}_{C_k} \simeq \bigoplus_{i=0}^k \mathcal{O}_{l_i}^2(1)$, and hence $H^1(\mathcal{N}_{C_k/\mathbb{P}^3}) = 0$. □

The minimal charge for a Gieseker-semistable instanton is 1. Moreover, for a rank-2 instanton E of charge 1, $\chi(E(1)) > 0$. We conclude that a minimal rank-2 instanton indeed is a 't Hooft instanton. For higher rank, we have the following result.

Theorem 3.2. *Let n, k be integers with $n \geq 2$ and $k \geq \lceil \frac{n}{2} \rceil$. Then there exists a slope-stable unobstructed (n,k)-instanton bundle E on $\mathbb{P}^3$ satisfying the following*:

(i) *The cokernel sheaf F of a general morphism $\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s} E$ is unobstructed, slope-stable, torsion-free, of homological dimension at most* 1, *and satisfies*

$$\begin{aligned} \mathrm{ext}^i(F, \mathcal{O}_{\mathbb{P}^3}(-j)) = 0 \quad & \text{for } i \neq 1,\ j = 0, 1; \\ h^i(F) = 0 \quad & \text{for } i \neq 1; \\ h^i(F(1)) = 0 \quad & \text{for } i \geq 2. \end{aligned}$$

(ii) *For all integers q with $1 \leq q \leq n-1$ we have $H^0\big(\bigwedge^q E\big) = 0$.*

(iii) *For a general line $l \subset \mathbb{P}^3$, E has trivial splitting over l, namely $E|_l \simeq \mathcal{O}_l^n$.*

Before passing to the actual proof of Theorem 3.2 let us summarise its main steps. To begin with we review the base of the induction (Step 1), given by rank-2 't Hooft instanton bundles satisfying Theorem 3.2.

We then pass to the inductive step (Step 2): from an (n,k)-instanton bundle E_n^k satisfying Theorem 3.2, we construct a strictly slope-semistable, but Gieseker-unstable $(n+1,k)$-instanton E_{n+1}^k from a nonsplit extension class in $\mathrm{Ext}^1(E_n^k, \mathcal{O}_{\mathbb{P}^3})$.

Next, we lift a general $s \in H^0(E_n^k(1))$ to $H^0(E_{n+1}^k(1))$ and we prove that the cokernel F_{n+1}^k of the corresponding morphism $\mathcal{O}_{\mathbb{P}^3}(-1) \to E_{n+1}^k$ is unobstructed and slope-stable (see Steps 3 and 4). By construction F_{n+1}^k arises from a nonzero element in $\mathrm{Ext}^1(F_n^k, \mathcal{O}_{\mathbb{P}^3})$ for $F_n^k := \mathrm{coker}(s)$.

In Step 5 we show F_{n+1}^k deforms, in the Gieseker-moduli space $\mathrm{M}_{\mathbb{P}^3}(\mathrm{ch}(F_{n+1}^k))$, to a sheaf admitting no quotient of type F_n^k.

Finally, via universal extensions, we will construct a projective bundle over $\mathrm{M}_{\mathbb{P}^3}(\mathrm{ch}(F_{n+1}^k))$ that will play the role of the moduli space of extensions of sheaves in $\mathrm{M}_{\mathbb{P}^3}(\mathrm{ch}(F_{n+1}^k))$ by $\mathcal{O}_{\mathbb{P}^3}(-1)$ (see Step 6). Thanks to the conclusions drawn in Step 5, we show that, in this moduli space, $[E_{n+1}^k]$ deforms to a slope-stable $(n+1,k)$-instanton bundle.

Proof. We prove the theorem by induction on the rank n.

Step 1 (check the base of the induction). For $n = 2$, the theorem holds due to the existence of the rank-2 't Hooft bundles. By Lemma 3.1, 't Hooft E_2^k of rank-2 is unobstructed, slope-stable (which is equivalent to satisfying (ii)) and satisfies (iii). A nonzero global section s of $E_2^k(1)$ vanishes on a locally complete intersection curve C_k of degree $k+1$ and arithmetic genus $-k$. These are supported on $\leq k+1$ disjoint lines and will consist of the union of $k+1$ disjoint lines, generically. We checked that the curves C_k of this kind are unobstructed and $\mathcal{I}_{C_k}(1)$ satisfies (i).

Step 2 (construction of an instanton of higher rank). Let us now pass to the proof of the induction step. Consider integers $n \geq 2$ and $k \geq \lceil \frac{n+1}{2} \rceil$. By induction there exists an instanton bundle E_n^k of rank n, charge k and satisfying the theorem. We have $\mathrm{ext}^3(E_n^k, \mathcal{O}_{\mathbb{P}^3}) = 0$ by stability and $\mathrm{ext}^2(E_n^k, \mathcal{O}_{\mathbb{P}^3}) = 0$ by (ii). Since $\chi(E_n^k, \mathcal{O}_{\mathbb{P}^3}) = n - 2k < 0$ we deduce that there exists the nonsplit extension

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to E_{n+1}^k \to E_n^k \to 0, \tag{8}$$

with E_{n+1}^k a strictly slope-semistable (and Gieseker-unstable) instanton bundle.

We claim that E_{n+1}^k deforms to an instanton bundle satisfying the statement of the theorem. To begin with, we observe that, due to Lemma 2.14, E_{n+1}^k is unobstructed. Note also that, since E_n^k satisfies (iii), the same holds for E_{n+1}^k; to see this, it suffices to restrict (8) to a general line $l \subset \mathbb{P}^3$.

We take a general global section $s' \in H^0(E_n^k(1))$ and we lift it to a global section $s \in H^0(E_{n+1}^k(1))$. Since s does not lie in the image of $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \hookrightarrow H^0(E_{n+1}^k(1))$, $\mathrm{coker}(s)$ is a torsion-free sheaf F_{n+1}^k.

Step 3 (check the unobstructedness of F_{n+1}^k). By construction, the sheaf F_{n+1}^k fits in the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \mathcal{O}_{\mathbb{P}^3}(-1) & = & \mathcal{O}_{\mathbb{P}^3}(-1) & & \\
 & & & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & E_{n+1}^k & \longrightarrow & E_n^k & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & F_{n+1}^k & \longrightarrow & F_n^k & \longrightarrow & 0
\end{array} \tag{9}$$

Since s' is general, we may assume that F_n^k satisfies (i). From the diagram we compute that F_{n+1}^k has homological dimension ≤ 1 and that F_{n+1}^k satisfies all cohomological conditions of (i) except for $h^0(F_{n+1}^k)$, as actually $h^0(F_{n+1}^k) = 1$. Notice that since F_n^k satisfies (i), F_{n+1}^k is unobstructed by Lemma 2.14.

Step 4 (show that F_{n+1}^k is slope-stable). We use the last row of (9). Suppose, by contradiction, that F_{n+1}^k is not slope-stable, and let $L \subset F_{n+1}^k$ be a saturated

slope-semistable destabilising subsheaf. Since $c_1(F^k_{n+1}) = 1$, we have $c_1(L) \geq 1$. Let I be the image of L in F^k_n. Since the kernel of $L \to I$ is contained in $\mathcal{O}_{\mathbb{P}^3}$, we have $c_1(I) \geq 1$. If $0 < \mathrm{rk}(I) < n-1$, then

$$\mu(I) \geq \frac{1}{\mathrm{rk}(I)} > \frac{1}{n-1} = \mu(F^k_n),$$

contradicting the slope-stability of F^k_n.

Assume therefore that $\mathrm{rk}(I) = n-1$. Then $\mathrm{rk}(L) = n-1$, the map $L \to I$ is injective and $c_1(L) = c_1(I) = 1$. Since L is saturated, $M := F^k_{n+1}/L$ is a torsion-free rank-one sheaf with $c_1(M) = 0$. The inclusion $\mathcal{O}_{\mathbb{P}^3} \hookrightarrow F^k_{n+1}$ induces an injection $\mathcal{O}_{\mathbb{P}^3} \hookrightarrow M$. As $M^{**} \simeq \mathcal{O}_{\mathbb{P}^3}$, this forces $M \simeq \mathcal{O}_{\mathbb{P}^3}$. Thus the inclusion $\mathcal{O}_{\mathbb{P}^3} \hookrightarrow F^k_{n+1}$ admits a retraction, so the defining extension of F^k_{n+1} splits, a contradiction.

Consequently $I = 0$, so L is contained in $\mathcal{O}_{\mathbb{P}^3}$, contradicting $c_1(L) \geq 1$.

Step 5 (deform F^k_{n+1} to a slope-stable sheaf having no quotient of type F^k_n). Next, we claim that F^k_{n+1} deforms to a slope-stable sheaf having no global sections. A general deformation F of F^k_{n+1} is an unobstructed slope-stable sheaf of homological dimension ≤ 1, satisfying $\mathrm{ext}^i(F, \mathcal{O}_{\mathbb{P}^3}(-j)) = 0$ for $i \neq 1$, $h^i(F(j)) = 0$ for $i \geq 2$ and $j = 0, 1$. Let us check that $h^0(F) = 0$. Assuming the contrary, we get the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3} \to F \to F' \to 0. \tag{10}$$

Here, F' has $c_1(F') = c_1(F) = 1$ and admits no quotients of degree ≤ 0, due to the slope-stability of F, hence it is slope-stable. From (10) we deduce that, for all $j \in \{0, 1\}$, one has $\mathrm{ext}^i(F', \mathcal{O}_{\mathbb{P}^3}(-j)) = 0$ for $i \neq 1$ and $h^i(F'(j)) = 0$ for $i \geq 2$. We can readily prove that, as F is unobstructed, the same holds for F'. Thus the sheaf F' represents a smooth point in $\mathrm{M}_{\mathbb{P}^3}(v)$, the moduli space of Gieseker-semistable torsion-free sheaves having Chern character $v = \mathrm{ch}(F^k_n)$.

The assumption that a general deformation F has nonvanishing global sections would then lead to $\mathrm{ext}^1(F^k_{n+1}, F^k_{n+1}) \leq \mathrm{ext}^1(F^k_n, F^k_n) + \mathrm{ext}^1(F^k_n, \mathcal{O}_{\mathbb{P}^3}) - 1$. However, this is a contradiction, for a direct computation shows, by our choice of k

$$\begin{aligned}\mathrm{ext}^1(F^k_{n+1}, F^k_{n+1}) - \mathrm{ext}^1(F^k_n, F^k_n) - \mathrm{ext}^1(F^k_n, \mathcal{O}_{\mathbb{P}^3}) + 1 \\ = -\chi(F^k_{n+1}, F^k_{n+1}) + \chi(F^k_n, F^k_n) + \chi(F^k_n, \mathcal{O}_{\mathbb{P}^3}) + 1 \\ = -\chi(\mathcal{O}_{\mathbb{P}^3}, F^k_n) = -\chi(E^k_n) = 2k - n > 0.\end{aligned}$$

Step 6 (use the universal extension to construct a slope-stable deformation of E^k_{n+1}). Summarising the previous steps, we proved that F^k_{n+1} deforms to a slope-stable sheaf having no global sections, so we may consider a family of sheaves $\mathbf{F}$ over $U \times \mathbb{P}^3$, where U is an appropriate open subset U of $\mathrm{M}_{\mathbb{P}^3}(\mathrm{ch}(F^k_{n+1}))$, specialising to F^k_{n+1} and all sheaves $\mathbf{F}_t$ corresponding to points $t \in U$ satisfy (i). Let us use it

to construct a family $\mathbf{E}$ specialising E_{n+1}^k and whose general element satisfies the theorem. Consider the relative Ext sheaf

$$\mathcal{E} := \mathcal{E}xt^1_{p_1}(\mathbf{F}, \mathcal{O}_{U\times\mathbb{P}^3}(-1)).$$

Here, p_1 is the projection onto the first factor, $\mathcal{E}xt^1_{p_1}(\cdot,\cdot) = R^1(p_{1*}\mathcal{H}om(\cdot,\cdot))$ and for a coherent sheaf $\mathcal{F}$ on $U\times\mathbb{P}^3$ we define $\mathcal{F}(n) := \mathcal{F}\otimes p_2{}^*\mathcal{O}_{\mathbb{P}^3}(n)$. Since $\mathrm{Ext}^2(\mathbf{F}_t, \mathcal{O}_{\mathbb{P}^3}(-1)) = 0$ for all t in U, we get $\mathcal{E}xt^2_{p_1}(\mathbf{F}, \mathcal{O}_{U\times\mathbb{P}^3}(-1)) = 0$ and $\mathcal{E}xt^1_{p_1}(\mathbf{F}, \mathcal{O}_{U\times\mathbb{P}^3}(-1))$ commutes with base change (see [Lange 1983, Theorem 1.4; Lønsted and Kleiman 1979, Theorem A.5]).

Moreover, since for all $t\in U$,

$$\mathrm{ext}^1(\mathbf{F}_t, \mathcal{O}_{\mathbb{P}^3}(-1)) = -\chi(\mathbf{F}_t, \mathcal{O}_{\mathbb{P}^3}(-1)),$$

$\mathcal{E}xt^1_{p_1}(\mathbf{F}, \mathcal{O}_{U\times\mathbb{P}^3}(-1))$ is locally free on U. Therefore $p_{1*}(\mathcal{H}om(\mathbf{F}, \mathcal{O}_{U\times\mathbb{P}^3}(-1))$ commutes with base change as well (due to [Lange 1983, Theorem 1.4]) and is, a fortiori, the zero sheaf.

Applying now [Lange 1983, Corollary 4.5], we get the existence of a universal extension, on $\mathbb{P}(\mathcal{E})\times\mathbb{P}^3$,

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})\times\mathbb{P}^3} \otimes p_1{}^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathbf{E} \to \widehat{\mathbf{F}} \to 0, \tag{11}$$

where the sheaf $\widehat{\mathbf{F}}$ on $\mathbb{P}(\mathcal{E})\times\mathbb{P}^3$ is the pullback of $\mathbf{F}$.

We obtain a family $\mathbf{E}$ of instanton bundles over $\mathbb{P}(\mathcal{E})$. In view of the unobstructedness of E_{n+1}^k and semicontinuity of Ext, up to shrinking U, we can assume that this family consists of unobstructed instantons. By construction, a general element of the family satisfies (i).

Finally let us show that, for x general in $\mathbb{P}(\mathcal{E})$ we have $H^0\big(\bigwedge^q \mathbf{E}_x\big) = 0$ for $q = 1,\dots,n-1$. Taking exterior powers of the exact sequence (8), we obtain the exact sequences

$$0 \to \bigwedge^{q-1} E_n^k \to \bigwedge^q E_{n+1}^k \to \bigwedge^q E_n^k \to 0$$

for $2\le q\le n-1$. Due to the inductive hypothesis, we deduce the vanishing of $H^0\big(\bigwedge^q E_{n+1}^k\big)$ for $2\le q\le n-1$. Still from (8), we also check that

$$H^0\bigg(\bigwedge^n E_{n+1}^k\bigg) \simeq \mathrm{Hom}(E_{n+1}^k, \mathcal{O}_{\mathbb{P}^3}) = 0.$$

Indeed, applying the functor $\mathrm{Hom}(\cdot, \mathcal{O}_{\mathbb{P}^3})$, we get that, by construction, the induced morphism $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}) \to \mathrm{Ext}^1(E_n^k, \mathcal{O}_{\mathbb{P}^3})$ is injective, for this is nothing but the map sending the identity to the extension class defining (8).

Consider a general deformation $\mathbf{E}_x$ of E_{n+1}^k. By semicontinuity $H^0\big(\bigwedge^q \mathbf{E}_x\big) = 0$ for $2\le q\le n-1$. In addition to this, by construction, $\mathbf{E}_x$ fits in

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathbf{E}_x \to F \to 0,$$

with $F = \mathbf{F}_{\pi(x)}$, where $\pi : \mathbb{P}(\mathcal{E}) \to U$ is the projection. Since we are assuming x general, the sheaf F will have no nonzero global section. Therefore the same holds for $\mathbf{E}_x$. So, finally, $H^0\big(\bigwedge^q \mathbf{E}_x\big) = 0$ for $1 \le q \le n-1$. This implies slope-stability of $\mathbf{E}_x$ by Hoppe's criterion; see, for instance, [Jardim et al. 2017, Theorem 3]. □

An immediate consequence of the theorem is that the moduli space $\mathrm{MI}_{\mathbb{P}^3}(n,k)$ of (n,k)-instanton bundles is nonempty and that it has an irreducible component whose general point corresponds to a slope-stable and unobstructed bundle.

Corollary 3.3. *For all $n \ge 2$, $k \ge \lceil \frac{n}{2} \rceil$, there is a generically smooth $(1-n^2+4nk)$-dimensional component of the moduli space $\mathrm{MI}_{\mathbb{P}^3}(n,k)$ of (n,k)-instantons whose general point is a slope-stable locally free instanton having general splitting 0^n.*

Looking at the sheaves arising as cokernels of the (twisted) global sections of the instanton bundles constructed in the previous theorem, we get the following result.

Corollary 3.4. *For all $n \ge 2$, $k \ge \lceil \frac{n}{2} \rceil$, the moduli space*

$$\mathrm{M}_{\mathbb{P}^3}\big(\gamma(n,k) - \mathrm{ch}(\mathcal{O}_{\mathbb{P}^3}(-1))\big)$$

of Gieseker-semistable torsion-free sheaves with Chern character

$$\gamma(n,k) - \mathrm{ch}(\mathcal{O}_{\mathbb{P}^3}(-1))$$

has a generically smooth component of dimension $n(4-n)+4k(n-1)$ whose general point is a slope-stable sheaf.

3.2. ***Instanton bundles of high rank on Del Pezzo threefolds.*** Now, we focus on Fano threefolds of index 2, called Del Pezzo threefolds, and Picard number 1. We recall that there exist five families of deformation equivalent threefolds Y_d of this kind, distinguished by the degree d of the ample generator of the Picard group, with $d \in \{1,2,3,4,5\}$. For the rest of the section, Y will denote any Fano threefold of index 2 and d will denote its degree.

As we pointed out, on a Fano threefold Y_d of index 2, the Chern character $\gamma(n,k)$ of an (n,k)-instanton E_n^k has the same form as on $\mathbb{P}^3$, $\gamma(n,k) = (n,0,-k,0)$. We will prove that, for all $n \ge 2$, Y carries slope-stable (n,k)-instanton bundles for all $k \ge n$. This was the lower bound for k determined in Lemma 2.11. We adopt an inductive method almost equivalent to the one used on the projective space. Also this time the base of the induction will consist of the rank-2 bundles analogous to 't Hooft bundles, as shown in Lemma 2.6. They are obtained via Serre's correspondence from elliptic curves. More precisely a rank-2 't Hooft bundle of charge k corresponds to an l.c.i. unobstructed and nondegenerate elliptic curve of degree $d+k$. Given an integer d, we let $\mathrm{Hilb}_{dt}(Y)$ be the Hilbert scheme of closed subschemes of Y having Hilbert polynomial dt, so that curves (i.e., purely one-dimensional closed subschemes of Y) of arithmetic genus 1 and degree d give points of this Hilbert scheme. We will prove the following result:

Proposition 3.5. *For all $k \geq 2$, there is a generically smooth $2(d+k)$-dimensional component $\mathcal{H}_{d+k}$ of $\mathrm{Hilb}_{(d+k)t}(Y)$ whose generic point is a smooth elliptic curve which is nondegenerate and unobstructed.*

We will prove the proposition by induction on k. From a smooth, unobstructed elliptic curve $C \subset Y$ of degree d_C, we construct a smooth, unobstructed elliptic curve of degree $d_C + 1$, by *smoothing* a nodal curve C' consisting of the union of C and of an *ordinary line*, i.e., such that $\mathcal{N}_l \simeq \mathcal{O}_l^2$, meeting C quasitransversely (meaning with distinct tangent vectors) at a point p.

Lemma 3.6. *Let $C \subset Y$ be a smooth, unobstructed elliptic curve on Y and let $l \subset Y$ be an ordinary line meeting C quasitransversely at a point. Then $C \cup l$ deforms to a smooth and unobstructed curve.*

Proof. We apply [Hartshorne and Hirschowitz 1985, Theorem 4.1] and we refer to [loc. cit.] for further details. Denote by $C' := C \cup l$ and by p the intersection point $p := C \cap l$. The normal sheaf $\mathcal{N}_{C'}$ of C' fits in the short exact sequence

$$0 \to \mathcal{N}_{C'} \to \mathcal{N}_{C'}|_C \oplus \mathcal{N}_{C'}|_l \to \mathcal{N}_{C'} \otimes \mathcal{O}_p \to 0. \tag{12}$$

Let us consider the restriction $\mathcal{N}_{C'}|_C$. This is a rank-2 vector bundle on C fitting in

$$0 \to \mathcal{N}_C \to \mathcal{N}_{C'}|_C \to \mathcal{O}_p \to 0; \tag{13}$$

since by assumption C is unobstructed, $H^1(\mathcal{N}_{C'}|_C) = 0$ and $H^0(\mathcal{N}_{C'}|_C) \to H^0(\mathcal{O}_p)$ is surjective. According to [Hartshorne and Hirschowitz 1985], the restriction $\mathcal{N}'_C|_l$ fits in the exact sequence of $\mathcal{O}_l$ modules

$$0 \to \mathcal{N}_{C'}|_l(-p) \to \mathcal{N}_{C'}|_l \to \mathcal{N}_{C'} \otimes \mathcal{O}_p \to 0; \tag{14}$$

since also $\mathcal{N}_{C'}|_l$ arises from an extension class in $\mathrm{Ext}^1(\mathcal{O}_p, \mathcal{N}_l)$ and since l is ordinary, $h^1(\mathcal{N}_{C'}|_l(-p)) = 0$ which implies that $h^1(\mathcal{N}_{C'}|_l) = 0$ as well and the map $H^0(\mathcal{N}_{C'}|_l) \to H^0(\mathcal{N}_{C'} \otimes \mathcal{O}_p)$ is surjective. These results, together with (12), lead us to conclude that $H^1(\mathcal{N}_{C'}) = 0$ and $H^0(\mathcal{N}_{C'}) \to H^0(\mathcal{O}_p)$ is surjective. Then, applying [Hartshorne and Hirschowitz 1985, Proposition 1.1], we conclude that C' is a smoothable and unobstructed elliptic curve. □

Proof of Proposition 3.5. We argue by induction on k. For $k = 2$ this can be found, e.g., in [Faenzi 2014, Proof of Theorem D]. We sketch here the key steps leading to the result. To begin with we consider a general hyperplane section S of Y, so S is a smooth Del Pezzo surface of degree d. It is easy to construct a smooth elliptic curve C of degree $d+2$ on S that is unobstructed on Y. This is done by realising S as the blow-up of $\mathbb{P}^2$ along $9-d$ points $p_1, \ldots, p_{9-d}$, and pulling back a general plane cubic passing through $7-d$ among $p_1, \ldots, p_{9-d}$. For such a curve we have $h^0(\mathcal{N}_C) = 2d+4$ and $h^1(\mathcal{N}_C) = 0$. However, a dimension count shows that the family of elliptic curves of degree $d+2$ which are contained in a

hyperplane section has dimension $2d+3$. Accordingly, C belongs to a generically smooth $(2d+4)$-dimensional component $\mathcal{H}_{d+2}$ whose general element is a smooth, unobstructed and nondegenerate curve.

Let us pass to the inductive step. For $k>2$, consider a smooth and unobstructed curve C of degree $d_C := d+k$, corresponding to a general point in $\mathcal{H}_{d_C}$. Recall that for a general point $p \in Y$ we have a finite number of lines passing through p. Since C is general, we can find an ordinary line l meeting C transversely at a point $p \in C$. The union $C' = C \cup l$ is an elliptic curve of degree d_C+1. According to Lemma 3.6, this curve is unobstructed. From the short exact sequences (12), (13) and (14), $h^0(\mathcal{N}_{C'}) = h^0(\mathcal{N}_{C'|C}) + h^0(\mathcal{N}_{C'|l}) - 2 = h^0(\mathcal{N}_C) + h^0(\mathcal{N}_l) = 2(d_C+1) = 2(d+k+1)$. Then C' defines a smooth point of a $2(d+k+1)$-dimensional component $\mathcal{H}_{d+k+1}$ of $\mathrm{Hilb}_{(d+k+1)t}$. Since, still according to Lemma 3.6, C' deforms to a smooth curve $\widetilde{C}$, a general point of $\mathcal{H}_{d+k+1}$ corresponds to a smooth elliptic curve on Y. Finally, since C' is unobstructed and nondegenerate, assuming $\widetilde{C}$ general enough, by semicontinuity, the same holds for $\widetilde{C}$. □

Lemma 3.7. *For all $k \geq 2$ there exists an unobstructed slope-stable rank-2 instanton bundle E_2^k fitting in the short exact sequence*

$$0 \to \mathcal{O}_Y(-1) \to E \to \mathcal{I}_C(1) \to 0, \tag{15}$$

with C a smooth, nondegenerate and unobstructed curve in $\mathcal{H}_{d+k}$. Moreover such a bundle E_2^k has splitting $(0,0)$ on the general line.

Proof. Once again this is a consequence of Lemma 2.6. □

Again, actually all minimal rank-2 instantons E (i.e., rank-2 instantons having charge 2) are 't Hooft instantons, since $\chi(E(1)) > 0$. We can finally adapt the argument of Theorem 3.2 to check the existence of slope-stable instanton bundles of rank r for all $r \geq 2$. We omit the detailed proof since there is no major modification with respect to Theorem 3.2.

Theorem 3.8. *For all $n \geq 2$ and $k \geq n$ there exists a slope-stable unobstructed (n,k)-instanton bundle E on Y satisfying the following*:

(i) *The cokernel sheaf F of a general map $\mathcal{O}_Y(-1) \xrightarrow{s} E$ is an unobstructed slope-stable torsion-free sheaf of homological dimension at most 1 such that*

$$\mathrm{ext}^i(F, \mathcal{O}_Y(-j)) = 0 \quad \textit{for all } i \neq 1 \textit{ and } j \in \{0,1\},$$
$$h^i(F) = 0 \quad \textit{for all } i \neq 1.$$

(ii) *For any integer q with $1 \leq q \leq n-1$, we have $h^0\big(\bigwedge^q E\big) = 0$.*

(iii) *For a general line $l \subset Y$, we have $E|_l \simeq \mathcal{O}_l^n$.*

Proof. The steps of the proof are equivalent to the ones we used for Theorem 3.2. □

Corollary 3.9. *For all $n \geq 2$, $k \geq n$, there exists a generically smooth $(2kn-n^2+1)$-dimensional component of the moduli space $\mathrm{MI}_Y(n,k)$ of (n,k)-instanton bundles whose general point is a slope-stable locally free instanton with general splitting 0^n.*

Corollary 3.10. *For all $n \geq 2$, $k \geq n$, the Gieseker moduli space*

$$\mathrm{M}_Y\big(\gamma(n,k) - \mathrm{ch}(\mathcal{O}_Y(-1))\big)$$

of Gieseker-semistable torsion-free sheaves with Chern character

$$\gamma(n,k) - \mathrm{ch}(\mathcal{O}_Y(-1))$$

admits a generically smooth component of dimension $n(d+2-n)+2k(n-1)$ whose general point is a slope-stable sheaf.

4. Instantons on Fano threefolds of odd index

Now we deal with Fano threefolds of odd index. In these cases the invariance of the Chern character with respect to $\mathrm{R}\mathcal{H}om(\,\cdot\,, \mathcal{O}_X(-r_X))$ imposes several constraints. Notably, the rank r of an instanton sheaf must be even, so we write $r = 2n$. To prove the existence of slope-stable instanton bundles, we will need to modify nontrivially the methods applied in the even-index cases. The central idea remains to construct strictly slope-semistable sheaves that are not Gieseker-stable, more specifically having F_0 as their maximal destabilising subsheaf, and then to prove that these deform to slope-stable ones. For i_X odd the strictly slope-semistable sheaves that we construct will arise from extensions of slope-stable instantons by a minimal instanton F_0, which this time will no longer be decomposable. Note that this is consistent with the requirement $r = 2n$.

4.1. ***Instantons of high rank on 3-dimensional quadrics.*** We start by treating the case of the smooth quadric hypersurface $Q \subset \mathbb{P}^4$, the only smooth Fano threefold of index 3. On the hypersurface Q, the minimal instanton is the *spinor bundle* $\mathcal{S}$. This is the only rank-2 slope-stable instanton bundle of charge 1. The Chern character $\gamma(n,k) \in \bigoplus_{i=0}^{3} H^{i,i}(Q,\mathbb{Z}) \simeq \mathbb{Z}^4$ of an (n,k)-instanton E_n^k on Q has the form

$$\gamma(n,k) = n\,\mathrm{ch}(\mathcal{S}) - k\,\mathrm{ch}(\mathcal{O}_l) = \big(2n, -n, -k, \tfrac{1}{2}k - \tfrac{1}{6}n\big).$$

We recall some well-known features of the spinor bundle that will be needed later — for further details we refer, e.g., to [Arrondo and Sols 1989; Ottaviani 1988]. The bundle $\mathcal{S}$ is exceptional, namely it is simple and $\mathrm{Ext}^p(\mathcal{S},\mathcal{S}) = 0$ for $p = 1, 2, 3$. Via the Serre correspondence, $\mathcal{S}$ can be obtained from any line $l \subset Q$ and a nonzero extension $e \in \mathrm{Ext}^1(\mathcal{I}_l, \mathcal{O}_Q(-1)) \simeq H^1(\mathcal{O}_l(-2)) \simeq \mathbb{C}$:

$$0 \to \mathcal{O}_Q(-1) \to \mathcal{S} \to \mathcal{I}_l \to 0. \tag{16}$$

The spinor bundle $\mathcal{S}$ is uniform, namely, given any line $l \subset Q$, we have the isomorphism $\mathcal{N}_{l/X} \simeq \mathcal{S}(1)|_l \simeq \mathcal{O}_l \oplus \mathcal{O}_l(1)$; hence $\mathcal{S}$ has splitting type $(-1, 0)$ on l. The Hilbert scheme of lines $\mathrm{Hilb}_{t+1}(Q)$ is identified with $\mathbb{P}(H^0(\mathcal{S}(1))) \simeq \mathbb{P}^3$. The dual $\mathcal{S}^* \simeq \mathcal{S}(1)$ is an Ulrich bundle, that is to say, it satisfies $H^i(\mathcal{S}(-i)) = 0$ for $i < 3$ and $H^i(\mathcal{S}^*(-i)) = 0$ for $i > 0$. The bundle $\mathcal{S}^*$ is globally generated and the surjective evaluation map $\mathcal{O}_Q^4 \simeq \mathcal{O}_Q \otimes H^0(\mathcal{S}(1)) \to \mathcal{S}(1)$ induces an embedding $Q \hookrightarrow \mathbb{G}(2, 4)$ whose image is a hyperplane section of $\mathbb{G}(2, 4)$. In this way, $\mathcal{S}$ is identified with the pullback of the tautological rank-2 vector bundle $\mathcal{U}$. Pulling back the tautological exact sequence on $\mathbb{G}(2, 4)$ one sees that $\mathcal{S}$ fits in

$$0 \to \mathcal{S} \to \mathcal{O}_Q^4 \to \mathcal{S}^* \to 0. \tag{17}$$

Next, we determine the minimal charge of Gieseker-stable (n, k)-instantons.

Lemma 4.1. *Let E_n^k be a Gieseker-stable (n, k)-instanton on Q, not isomorphic to $\mathcal{S}$. Then*

$$k \geq \left\lceil \tfrac{1}{3} n \right\rceil.$$

Proof. Applying Riemann–Roch we compute $\chi(\mathcal{S}, E_n^k) = \chi(E_n^k, \mathcal{S}) = n - 3k$ and $\chi(E_n^k) = -k$. By definition $H^2(E_n^k) = 0$. Also, stability of the involved sheaves implies $\mathrm{Ext}^3(\mathcal{S}^*, E_n^k) = 0$.

Then, from (17) we compute $\mathrm{Ext}^2(\mathcal{S}, E_n^k) = 0$. This means that whenever $\chi(\mathcal{S}, E_n^k) > 0$ there exists a morphism $\mathcal{S} \to E_n^k$. Looking at this morphism one sees that, if $k \geq 0$, a necessary condition to have slope-semistability (see Lemma 2.11) is $p_{\mathcal{S}} \geq p_{E_n^k}$ with equality holding if and only if $k = 0$.

This proves, unless $E_n^k \simeq \mathcal{S}$, Gieseker-stability requires $3k \geq n$; hence $k \geq \left\lceil \frac{n}{3} \right\rceil$. □

The short exact sequence (16) tells us that $\mathcal{S}$ might be thought of as a minimal 't Hooft bundle on Q. From Lemma 3.1, applying the construction to curves supported on disjoint lines, we get the following result. This is also studied in [Faenzi 2014, §2.3].

Lemma 4.2. *For all $k \geq 0$, the quadric threefold Q carries rank-2 't Hooft slope-stable and unobstructed instanton bundles E of charge k. Also, a general bundle of this kind satisfies $H^1(E|_{R_d}) = 0$ for a general rational curve R_d of degree $d = 1, 2$.*

Proof. Consider the union C of $k+1$ disjoint lines $l_0, \ldots, l_k$ lying on Q. Then C is an l.c.i. curve of degree $k+1$ and arithmetic genus $-k$. Note that unobstructedness holds obviously on each line on Q and this implies unobstructedness of the curve C as well. From Lemma 3.1, a vector bundle E arising from a nowhere vanishing extension class $[e] \in \mathrm{Ext}^1(\mathcal{I}_C, \mathcal{O}_Q(-1)) \simeq \mathbb{C}^{k+1}$ is a slope-stable 't Hooft bundle which is unobstructed. Finally let us check that a general line and a general conic are not jumping for E.

For a line l such that $l \cap l_i = \varnothing$ for all $i \in \{0, \dots, k\}$, we have $\mathcal{I}_C \otimes \mathcal{O}_l \simeq \mathcal{O}_l$; hence

$$E|_l \simeq \mathcal{O}_l \oplus \mathcal{O}_l(-1).$$

A general line $l \subset Q$ is disjoint from each line l_i. Indeed, in the projective space $\mathbb{P}^3$ of lines in Q, those meeting a fixed line l_i are parametrised by a plane $\mathbb{P}^2_{l_i}$; hence the aforementioned lines correspond to points in the dense open subscheme $\mathbb{P}^3 \setminus \bigcup_{i=0}^k \mathbb{P}^2_{l_i}$.

Assuming now C general, we suppose that the points $[l_i] \in \mathbb{P}^3$, for $i = 0, \dots, k$, are in general position — in particular no triples of points are aligned. Choose one of these points, say $[l_0]$, and a general point $[l'] \in \mathbb{P}^3$, distinct from $[l_i]$, $i = 1, \dots, k$. A smooth conic R_2 in $\langle l_0, l' \rangle \cap Q$ meets C in just one point p so that

$$\mathcal{I}_C \otimes \mathcal{O}_{R_2} \simeq \mathcal{O}_{R_2}(-p) \oplus \mathcal{O}_p.$$

Accordingly $E|_{R_2} \simeq \mathcal{O}_{R_2}(-p)^{\oplus 2}$. Since the vanishing of $H^1(E|_{R_2})$ is an open condition on the Hilbert scheme of conics on Q, we conclude that a general conic on Q is not jumping for E. □

The existence of rank-2 't Hooft bundles suggests using them as the base of an inductive argument inspired by Theorem 3.2, affording slope-stable instanton bundles of any rank $r \geq 2$ that still admit a global section after a twist by $\mathcal{O}_Q(1)$. The inductive step will be based on the fact that, whenever E_n^k has charge k greater than the minimum k_0^n, we have $\operatorname{Ext}^1(E_n^k, \mathcal{S}) > 0$. This allows us to construct strictly slope-semistable bundles, which then deform to slope-stable instantons.

Theorem 4.3. *For all $n \geq 1$ and $k \geq \lceil \frac{n}{3} \rceil$ there exists a slope-stable unobstructed 't Hooft (n, k)-instanton bundle E on Q satisfying the following*:

(i) *We have* $\operatorname{ext}^i(E, \mathcal{S}) = 0$ *for* $i \neq 1$.

(ii) *For $s \in H^0(E(1))$ general, the cokernel sheaf of $\mathcal{O}_Q(-1) \xrightarrow{s} E$ is an unobstructed slope-stable torsion-free sheaf F of homological dimension at most* 1 *such that*

$$\operatorname{ext}^2(F, \mathcal{S}) = 0, \qquad \operatorname{Hom}(\mathcal{S}, F) = 0, \qquad \operatorname{Ext}^i(F, \mathcal{O}_Q(-1)) = 0 \quad \text{for } i \neq 1.$$

(iii) *For a general rational curve $R_d \subset Q$ of degree $d = 1, 2$, we have $H^1(E|_{R_d}) = 0$.*

Proof. The proof is similar in spirit to those of Theorems 3.2 and 3.8. However, there are several substantial differences, so we carry out the argument in detail.

We construct slope-semistable instanton bundles E_{n+1}^k via nonsplit extensions in $\operatorname{Ext}^1(E_n^k, \mathcal{S})$ for E_n^k an (n, k)-instanton satisfying the theorem. We will use a different method to show that the cokernel F_{n+1}^k of a global section of E_{n+1}^k, obtained by lifting a general $s \in H^0(E_n^k(1))$, flatly deforms to a sheaf that does not admit $\mathcal{S}$ as a subsheaf. By construction F_{n+1}^k will be an extension of F_n^k by $\mathcal{S}$,

where F_n^k is the cokernel of s. This time we will not be able to rely on Hoppe's criterion, so we will need to slightly adjust our approach.

Let us briefly outline the main idea behind this step of the proof, as this type of argument will be widely used throughout the rest of the paper. The locus of points in $\mathrm{M}_Q(\mathrm{ch}(F_{n+1}^k))$ corresponding to sheaves admitting a quotient having Chern character $\mathrm{ch}(F_n^k)$ is a closed subscheme $Z \subset \mathrm{M}_Q(\mathrm{ch}(F_{n+1}^k))$. To show the existence of the aforementioned deformation of F_{n+1}^k, we will prove that there is a unique component of $\mathrm{M}_Q(\mathrm{ch}(F_{n+1}^k))$ passing through $[F_{n+1}^k]$. We show that this component is smooth at $[F_{n+1}^k]$ and of dimension $\mathrm{ext}^1(F_{n+1}^k, F_{n+1}^k)$. We then prove that it cannot be entirely contained in Z. For the latter, it is enough to show that, at least locally around $[F_{n+1}^k]$, one has $\dim(Z) < \mathrm{ext}^1(F_{n+1}^k, F_{n+1}^k)$.

Step 1 (check the base of the induction). For $n = 1$ we proved that the 't Hooft bundles constructed in Lemma 4.2 are unobstructed, slope-stable and have the expected splitting on a general line. One checks that, given the union C_m of m disjoint lines, the ideal sheaf $\mathcal{I}_{C_m}$ satisfies (ii), so that for any 't Hooft bundle E constructed from C_m we have $\mathrm{Ext}^2(E, \mathcal{S}) = 0$ and, moreover, a general section $s \in H^0(E(1))$ satisfies (ii) as well. All these conditions are of course open.

For the inductive step, suppose that the theorem holds for a given integer n. Let us take an integer k such that $k \geq \lceil \frac{n+1}{3} \rceil$ and prove the result for $(n+1, k)$-instantons of all such k.

Step 2 (construction of an instanton of higher rank). By the inductive assumption, there exists an (n, k) 't Hooft bundle E_n^k satisfying the theorem and moreover such a bundle satisfies

$$\chi(E_n^k, \mathcal{S}) = n - 3k \leq -1.$$

Therefore, there exists a nonzero element $[e] \in \mathrm{Ext}^1(E_n^k, \mathcal{S})$ which defines the nonsplit short exact sequence

$$0 \to \mathcal{S} \to E_{n+1}^k \to E_n^k \to 0. \tag{18}$$

Here, E_{n+1}^k is a strictly slope-semistable, Gieseker-unstable $(n+1, k)$-instanton. Applying Lemmas 2.13 and 2.14, we deduce that E_{n+1}^k is unobstructed and that E_n^k is its only torsion-free quotient having slope $-\frac{1}{2}$. From the short exact sequence (18) it is also immediate to check that, since E_n^k satisfies (iii), the same holds for E_{n+1}^k.

Our final goal is to deform E_{n+1}^k to an unobstructed slope-stable 't Hooft instanton. We start by looking at the cokernel sheaf F_{n+1}^k of a global section of E_{n+1}^k.

Step 3 (check unobstructedness of F_{n+1}^k). As for the even-index case, we start by lifting a global section $s \in H^0(E_n^k(1))$ satisfying (ii) to $s' \in H^0(E_{n+1}^k(1))$. Note

that taking such a lift is possible due to the vanishing of $H^1(\mathcal{S}(1))$. This leads to the commutative diagram

$$\begin{array}{ccccccccc} & & & & \mathcal{O}_Q(-1) & = & \mathcal{O}_Q(-1) & & \\ & & & & \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} & & \\ 0 & \longrightarrow & \mathcal{S} & \longrightarrow & E^k_{n+1} & \longrightarrow & E^k_n & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{S} & \longrightarrow & F^k_{n+1} & \longrightarrow & F^k_n & \longrightarrow & 0 \end{array} \tag{19}$$

Note that the vanishing $\operatorname{Ext}^2(F^k_n, F^k_n) = 0 = \operatorname{Ext}^2(F^k_n, \mathcal{S}) = 0$ is ensured by the inductive hypothesis. Also, we have $\operatorname{Ext}^2(\mathcal{S}, F^k_n) = 0$, since $\operatorname{Ext}^2(\mathcal{S}, E^k_n) = 0$; see the proof of Lemma 4.1. Then, the sheaf F^k_{n+1} is unobstructed due to Lemma 2.14.

Step 4 (show that F^k_{n+1} is slope-stable). From the last row of diagram (19) we readily check that F^k_{n+1} does not admit subsheaves of degree ≤ 0. Therefore the slope of any subsheaf L of F^k_{n+1} having even rank $2m$ for $m \le n$ or odd rank $2m+1$ for $m \le n-1$, and degree $-d$, $d > 0$, can be written, respectively, in the form

$$\mu(L) = -\frac{1}{2} + \frac{m-d}{2m} \quad \text{or} \quad \mu(L) = -\frac{1}{2} + \frac{2(m-d)+1}{2(2m+1)}.$$

Let us suppose now that L is a slope-semistable subsheaf of F^k_{n+1} of rank $\le 2n$. If the composition $L \to F^k_n$ is 0, L injects into $\mathcal{S}$. Hence $\mu(L) \le -\frac{1}{2} < \mu(F^k_{n+1})$, so this case is excluded. Then the induced map $L \to F^k_n$ is nonzero. Let us denote by $G \subset F^k_n$ the image of this map. If $\operatorname{rk}(G) < 2n-1$, by the stability assumption on F^k_n, we must have

$$\mu(L) < \mu(F^k_n) = -\frac{1}{2} + \frac{1}{2(2n-1)}.$$

Now, if L has even rank $2m$ and degree $-d$, the last inequality leads to

$$m - d < \frac{m}{2n-1} \le 1,$$

so that $m - d \le 0$ and $\mu(L) \le -\frac{1}{2} < \mu(F^k_{n+1})$, so L would not destabilise F^k_{n+1}. Then L has odd rank $2m+1$, so the inequality yields

$$2(m-d)+1 < \frac{2m+1}{2n-1} \le 1,$$

so that $(2(m-d)+1)/(2(2m+1)) \le 0$ and consequently $\mu(L) \le -\frac{1}{2} < \mu(F^k_{n+1})$, which is excluded.

To finish the proof of the slope-stability, it remains to treat the case $\operatorname{rk}(G) = 2n-1$ and $\mu(G) = \mu(F^k_n)$, which implies $\dim(\operatorname{Supp}(F^k_n/G)) \le 1$. Denote by K the kernel

of the surjection $L \to G$. We have $K \neq 0$, since otherwise $F^k_{n+1}/L \simeq \mathcal{S} \oplus F^k_n/G$, as $\operatorname{ext}^1(F^k_n/G, \mathcal{S}) = h^2(\mathcal{S} \otimes F^k_n/G(-2)) = 0$. But in this case we would get the existence of a surjection $F^k_{n+1} \to \mathcal{S}$ which is impossible since the last row of (19) does not split. Therefore $\operatorname{rk}(K) = 1$,

$$\mu(F^k_{n+1}/L) = \mu(\mathcal{S}/K) \geq 0 > \mu(F^k_{n+1})$$

so that L does not destabilise F^k_{n+1}.

In the next step, we use our main intuition on the construction of slope-stable deformations, essentially based on a parameter count.

Step 5 (deform F^k_{n+1} to a slope-stable sheaf having no quotient of type F^k_n). Let us denote by v and v', the Chern characters of F^k_{n+1} and F^k_n, respectively. We show that F^k_{n+1} deforms, in the Gieseker moduli space $\mathrm{M}_Q(v)$ of semistable sheaves with Chern character v, to a slope-stable sheaf that admits no torsion-free quotients with Chern character v'.

The locus $Z_{v'} \subset \mathrm{M}_Q(v)$ parametrising sheaves admitting a torsion-free quotient of Chern character v' is closed in $\mathrm{M}_Q(v)$ (see [Huybrechts and Lehn 2010, §2.3]). Note that $[F^k_{n+1}]$ is a smooth point of this moduli space, in particular it belongs to a unique irreducible component Y of $\mathrm{M}_Q(v)$. Hence, to prove our claim, it is enough to show that $\dim(Z_{v'} \cap Y) < \operatorname{ext}^1(F^k_{n+1}, F^k_{n+1})$, as this will ensure that $Y \not\subset Z_{v'}$.

We observe that actually, for $[F] \in Z_{v'}$, the kernel L of a surjection $F \twoheadrightarrow T$ with $\operatorname{ch}(T) = v'$ must be exactly $\mathcal{S}$. Indeed, one first observes that, if L was not slope-stable, it would admit a rank-1 torsion-free subsheaf of degree ≥ 0, which is clearly impossible because of the semistability of F. Then L must be slope-stable and the claim follows, since $\mathcal{S}$ is the unique slope-stable torsion-free sheaf on Q with Chern character $\operatorname{ch}(\mathcal{S})$, as we already recalled — see, e.g., [Arrondo and Sols 1989].

Suppose, by contradiction, that $Y \subset Z_{v'}$ and let $\mathbb{F}_t$ be a deformation of F^k_{n+1} in Y. For t general, $\mathbb{F}_t$ would then be a slope-stable sheaf satisfying (ii) fitting in

$$0 \to \mathcal{S} \to \mathbb{F}_t \to F' \to 0 \tag{20}$$

for F' an unobstructed slope-stable sheaf in $\mathrm{M}_Q(v')$. But the family of sheaves fitting in a short exact sequence of this form has dimension equal to at most $\operatorname{ext}^1(F^k_n, F^k_n) + \operatorname{ext}^1(F^k_n, \mathcal{S}) - 1$. So the assumption $Y \subset Z_{v'}$ implies

$$\operatorname{ext}^1(F^k_{n+1}, F^k_{n+1}) \leq \operatorname{ext}^1(F^k_n, F^k_n) + \operatorname{ext}^1(F^k_n, \mathcal{S}) - 1.$$

However, a direct computation shows that

$$\begin{aligned}\operatorname{ext}^1(F^k_{n+1}, F^k_{n+1}) - \operatorname{ext}^1(F^k_n, F^k_n) - \operatorname{ext}^1(F^k_n, \mathcal{S}) + 1 \\ = -\chi(F^k_{n+1}, F^k_{n+1}) + \chi(F^k_n, F^k_n) + \chi(F^k_n, \mathcal{S}) + 1 = 3k - n > 0,\end{aligned}$$

by the assumptions on n and k. So this is a contradiction and our step is thus proved.

Step 6 (use the universal extension to construct a slope-stable deformation of E_{n+1}^k). Applying the same arguments as in the proof of Theorem 3.2 we can now construct a family of instantons specialising to E_{n+1}^k using universal extensions.

Substep 6.1 (define the universal extension). We consider a family defined by a coherent sheaf $\mathbf{F}$ on $Q \times U$, where $U \subset \mathrm{M}_Q(v)$ is an open neighbourhood of $[F_{n+1}^k]$, whose general element satisfies (ii) and does not lie in $Z_{v'}$. Since $\operatorname{ext}^i(\mathbf{F}_t, \mathcal{O}_Q(-1)) = 0$ for all $i \neq 1$, we deduce that $\mathcal{E} := \mathcal{E}xt^1_{p_1}(\mathbf{F}, \mathcal{O}_{U\times Q}(-1))$ commutes with base change and is locally free. Also, there is a universal extension, on $\mathbb{P}(\mathcal{E}) \times Q$,

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})\times Q} \otimes p_1^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathbf{E} \to \widehat{\mathbf{F}} \to 0, \tag{21}$$

where $\widehat{\mathbf{F}}$ is the pullback of $\mathbf{F}$ to $\mathbb{P}(\mathcal{E}) \times Q$ — we refer to the proof of Theorem 3.2 and [Lange 1983; Lønsted and Kleiman 1979; Jardim et al. 2017] for further details.

To conclude this step our goal is to show that, for $x \in \mathbb{P}(\mathcal{E})$ general enough, $\mathbf{E}_x$ is slope-stable.

Substep 6.2 (make preliminary observations about slope-semistability and duality). We first note that on $\mathbb{P}(\mathcal{E})$ there exists an open dense subset V of points x such that $\mathbf{E}_x$ is a slope-semistable vector bundle — recall that slope-semistability and local freeness are open conditions. Since E_{n+1}^k is a slope-semistable vector bundle, the point corresponding to it belongs to V. Up to replacing $\mathbb{P}(\mathcal{E})$ by V, we can therefore suppose that $\mathbf{E}$ is a family of slope-semistable vector bundles.

We also note that the slope-stability of $\mathbf{E}_x$ is equivalent to the slope-stability of $\mathbf{E}_x^*$. Indeed, consider the dual family $\mathbf{E}^* := \mathcal{H}om(\mathbf{E}, \mathcal{O}_{\mathbb{P}(\mathcal{E})\times X})$. Since for all $x \in \mathbb{P}(\mathcal{E})$, $\mathbf{E}_x$ is locally free we have that $(\mathbf{E}^*)_x \simeq \mathbf{E}_x^*$.

Substep 6.3 (consider the Hilbert polynomials of potentially destabilising quotients). For any $x \in \mathbb{P}(\mathcal{E})$, the sheaf $(\mathbf{E}^*)_x$ is strictly slope-semistable if and only if it admits a torsion-free quotient having Hilbert polynomial R such that $\mu(R) = \frac{1}{2}$. Then we consider the set of polynomials

$$A := \big\{ R \in \mathbb{Q}[t] \mid \mu(R) = \tfrac{1}{2} \text{and there exists } x \in \mathbb{P}(\mathcal{E}) \text{ such that } \mathbf{E}_x^* \text{ admits a torsion-free quotient with Hilbert polynomial } R \big\}.$$

The set A is finite; see [Huybrechts and Lehn 2010, Lemma 1.7.9, Proposition 2.3.1].

Incidentally, we warn the reader that in [loc. cit.] the slope is expressed in terms of the coefficients of the Hilbert polynomial $R = \sum_{i=0}^3 \frac{\alpha_i(R)}{i!} t^i$. On Q, if $R = P_F$,

$$\mu(F) = \frac{\alpha_2(R)}{\alpha_3(R)} - \frac{3}{2}.$$

Hence fixing the slope in our convention is equivalent to fixing it in the convention of [loc. cit.], and the statement still holds.

Thus the locus W^{ss} of points $x \in \mathbb{P}(\mathcal{E})$ such that $\mathbf{E}_x$ is strictly slope-semistable is closed. To see this, note that the locus W^{ss} is the union of closed subschemes W_R, $R \in A$, where W_R is the image of the natural morphism $\mathrm{Quot}(\mathbf{E}^*, R) \to \mathbb{P}(\mathcal{E})$, the closedness of W_R being ensured by the projectivity of this morphism.

Substep 6.4 (restrict the set of Hilbert polynomials of destabilising quotients to a singleton). We claim that, up to shrinking U, we can suppose that actually A consists of the single element $\{P_{\mathcal{S}^*}\}$. Let $x_0 \in \mathbb{P}(\mathcal{E})$ be the point corresponding to $[E^k_{n+1}]$ and set $A_{x_0} := \{R \in A \mid x_0 \in W_R\}$. The first assertion is that, without loss of generality, we may consider only the polynomials R of A such that $x_0 \in W_R$. Indeed, if x_0 does not lie in W_R, we may replace $\mathbb{P}(\mathcal{E})$ with $\bigcap_{R \notin A_{x_0}} \mathbb{P}(\mathcal{E}) \setminus W_R$. Indeed, the latter is an open dense neighbourhood of the point x_0.

The second assertion is that by Lemma 2.13 the polynomials in A_{x_0} are necessarily of the form $P_{\mathcal{S}^*} + l$, where $l = at + b$ a polynomial of degree at most one such that $l \gg 0$, i.e., either $a > 0$, or $a = 0$ and $b > 0$. Indeed, we note that these are the only possible Hilbert polynomials of quotients of E^*_{n+1}, as their torsion-free part must be $\mathcal{S}^*$ and their torsion part can have at most dimension one, due to semistability. Then, if R is not of this form we would have that $x_0 \notin W_R$, against the first assertion.

The third assertion is that a rank-2 torsion-free sheaf with Hilbert polynomial $P_{\mathcal{S}^*} + l$ with l as described above is slope-unstable. Indeed, there is no Gieseker-semistable (hence slope-semistable) rank-2 torsion-free sheaf G on Q with $c_1 = 1$ and $P_G \gg P_{\mathcal{S}^*}$. To see this, notice that if $a > 0$, we would have $c_2(G) < c_2(\mathcal{S}^*)$ which would contradict the Bogomolov inequality. So we should have $a = 0$, which implies $G^{**} \simeq \mathcal{S}^*$ so that $b = -\mathrm{length}(\mathcal{S}^*/G) \leq 0$.

Therefore, since we are assuming that each sheaf in $\mathbb{P}(\mathcal{E})$ is slope-semistable, there is no $x \in \mathbb{P}(\mathcal{E})$ such that $\mathbf{E}^*_x$ admits a torsion-free quotient of Hilbert polynomial $P_{\mathcal{S}^*} + l \gg P_{\mathcal{S}^*}$. This ends the proof of our claim $A = \{P_{\mathcal{S}^*}\}$.

Since we know that $W_{P_{\mathcal{S}^*}}$ is a proper closed subscheme of $\mathbb{P}(\mathcal{E})$ (as $Z_{v'} \subsetneq U$), we can conclude that a general element of $\mathbf{E}$ is slope-stable.

Finally, by semicontinuity, as E^k_{n+1} has splitting $(-1^{n+1}, 0^{n+1})$ on the general line, the same will hold for $x \in \mathbb{P}(\mathcal{E})$ general. The statement about conics has the same proof. □

Consider the moduli space $\mathrm{MI}_Q(n, k)$ of (n, k)-instanton bundles. Theorem 4.3 shows that $\mathrm{MI}_Q(n, k)$ is nonempty and that it contains points corresponding to unobstructed and slope-stable bundles. We deduce the next result.

Corollary 4.4. *There is a generically smooth* $(1-n^2+6nk)$*-dimensional component of the moduli space* $\mathrm{MI}_Q(n, k)$ *of* (n, k)*-instanton bundles whose general point is a slope-stable locally free instanton having general splitting* $(0^n, -1^n)$.

As we also constructed $E \in \mathrm{MI}_Q(n,k)$ admitting a global section $\mathcal{O}_Q(-1) \to E$ whose cokernel is unobstructed and slope-stable, we deduce the following result.

Corollary 4.5. *For all $n \geq 2$, $k \geq \lceil \frac{n}{3} \rceil$, there exists a generically smooth component of the Gieseker moduli space $\mathrm{M}_Q\big(\gamma(n,k) - \mathrm{ch}(\mathcal{O}_Q(-1))\big)$, with dimension equal to $n(4-n)+k(6n-1)$, whose general point is a slope-stable sheaf.*

4.2. ***Instantons on Fano threefolds of index 1.*** We finally treat the case of the Fano threefolds of index one (also referred to as prime Fano threefolds). We restrict the framework to *nonhyperelliptic* prime Fano threefolds, which means that the ample generator H_X of the Picard group is actually very ample. We recall that prime Fano threefolds are classified, up to deformation equivalence, by their *genus* g: this is the integer defined as the genus of a codimension 2 smooth linear section of X, or equivalently by the equality $H_X^3 = 2g-2$. The genus g of a nonhyperelliptic smooth prime Fano threefold X takes values in $\{3,\dots,12\} \setminus \{11\}$. Also, when $g=3$, X must be a quartic hypersurface in $\mathbb{P}^4$.

All through the current section, X, or X_g, denotes a prime Fano threefold with very ample generator of $\mathrm{Pic}(X)$ and g denotes its genus. As we saw in Section 2, given an instanton E on X, there is a unique pair of integers (n,k) such that

$$\gamma(n,k) := \mathrm{ch}(E) = n\,\mathrm{ch}(F_0) - k\,\mathrm{ch}(\mathcal{O}_l(-1)),$$

where F_0 is a minimal instanton bundle of rank 2. In particular $\mathrm{rk}(E) = 2n$. When g is even, the minimal instanton F_0 is a Mukai bundle according to the terminology used in [Bayer et al. 2026]. We recall a few basic properties of F_0. Put

$$m_g = \lceil \tfrac{1}{2} g \rceil + 1.$$

In the next lemma we recollect results from [Ciliberto et al. 2023; 2024; Bayer et al. 2026] for the case where g is even. To state it, we recall that a vector bundle E on a threefold X is said to be ACM if it has no intermediate cohomology, namely $H^i(E(m)) = 0$ for all integers m and for $i = 1, 2$.

Lemma 4.6. *Let X be a smooth prime Fano threefold of genus g and such that $-K_X$ is very ample. Then the following hold*:

(i) *There are no slope-semistable rank-2 sheaves on X with $c_1 = -1$, $c_2 < m_g$ and $c_3 = 0$.*

(ii) $\mathrm{M}_X(\gamma(1,0))$ *is nonempty of dimension δ, where δ is the remainder of the division of g by 2. Every point in $\mathrm{M}_X(\gamma(1,0))$ corresponds to a slope-stable locally free instanton bundle.*

(iii) $\mathrm{M}_X(\gamma(1,0))$ *admits a reduced component whose general point is an unobstructed bundle except for the case where $g=4$ and X is contained in a singular quadric.*

(iv) *If $g \geq 6$, for any pair of points $[F_0]$, $[F_0']$ in $\mathrm{M}_X(\gamma(1,0))$, $\operatorname{ext}^2(F_0, F_0') = 0$. In particular $\mathrm{M}_X(\gamma(1,0))$ is smooth.*

(v) *If $g \geq 6$, for all $[F_0] \in \mathrm{M}_X(\gamma(1,0))$, $F_0(1)$ is globally generated and ACM.*

It may be useful to display the Chern character of an (n,k)-instanton on X. It has the form

$$\begin{aligned}(22)\quad \gamma(n,k) &= \operatorname{ch}(E_n^k)\\ &= 2n - nH_X + (n(g-2-m_g)-k)l_X + \left(\tfrac{1}{2}(k+m_g) + \tfrac{1}{3}(1-g)\right)p_X.\end{aligned}$$

4.2.1. *Rank-2 instanton bundles.* The first step towards constructing instantons of high rank is the rank-2 case, since these are the base of our inductive argument. Recall that for rank-2 vector bundles of degree -1, slope-stability and Gieseker-stability coincide. Also, recall the terminology of an *ordinary threefold*. A line l contained in a smooth Fano threefold of Picard rank one and index one has normal bundle $\mathcal{O}_l \oplus \mathcal{O}_l(-1)$, so l is *ordinary*, or $\mathcal{O}_l(1) \oplus \mathcal{O}_l(-2)$, in which case l is obstructed. In the latter case, l is a singular point of the Hilbert scheme of lines in X, which is of pure dimension 1. We say that X is *ordinary* if X contains an ordinary line. By [Gruson et al. 2006; Prokhorov 1990], the threefold X is ordinary if $g \geq 9$ and X is not the Mukai–Umemura threefold.

The existence of rank-2 instanton bundles is known in the following cases:

Theorem 4.7. *Let X be a Fano threefold with $\operatorname{Pic}(X) = \langle -K_X \rangle$ and let $-K_X$ be very ample.*

- *There exists a slope-stable unobstructed instanton bundle on X with $c_2 = m$ for $m \in \{m_g, \dots, g+3\}$ except for $g = 4$, $c_2 = m_g$ and X contained in a singular quadric.*
- *There exists a slope-stable ACM instanton bundle with $c_2 = m$ on X for $m \in \{m_g, \dots, g+3\}$.*
- *If X is ordinary, there exists a slope-stable unobstructed rank-2 instanton bundle of $c_2 = m$ for all $m \geq m_g$.*

The existence result when X carries ordinary lines, presented in [Faenzi 2014] and going back to [Brambilla and Faenzi 2011], is obtained inductively, performing elementary transformations of rank-2 unobstructed slope-stable instanton bundles along $\mathcal{O}_l(-1)$ for $l \subset X$ an ordinary line; see Remark 2.12.

The existence for charges ranging in $\{m_g, \dots, g+3\}$ can be found in [Ciliberto et al. 2023; 2024] and is based on the proof of the existence of smooth, unobstructed elliptic curves on X of degree $d \in \{m_g, \dots, g+3\}$. The instanton bundles are obtained from these elliptic curves via the Serre construction.

Combining the results of [Ciliberto et al. 2023; 2024] and adopting the method used in [Faenzi 2014], we can prove the existence of rank-2 instanton bundles of

charge greater than the minimal one on the Fano threefolds of index one and genus at least three. This can be done using the technique of the elementary transformation used in [Faenzi 2014], with the only difference being that we will be performing these along torsion instantons (see Definition 2.9) supported on unobstructed conics, instead of lines. Note that X might indeed fail to carry unobstructed lines (so X is *exotic*) but it will contain unobstructed smooth conics. The torsion instantons supported on such a curve C are of the form $\mathcal{O}_C(-p)$ for a point $p \in C$. Since we are assuming H_X very ample, the torsion instantons supported on lines and smooth conics are elements of the Gieseker moduli spaces $\mathrm{M}_X\big(0, 0, d, -\frac{d}{2}\big)$ of purely one-dimensional sheaves with Hilbert polynomial dt, with $d \leq 2$.

Lemma 4.8. *Let $d \leq 2$ be an integer and Q be a coherent sheaf of pure dimension one having Hilbert polynomial dt and such that $h^i(Q) = 0$ for $i = 0, 1$. Then one of the following holds*:

(i) $Q \simeq \mathcal{O}_l(-1)$ *for a line $l \subset X$, or*

(ii) $Q \simeq \mathcal{O}_C(-p)$ *for a smooth conic $C \subset X$ and a point $p \in C$, or*

(iii) *there are two possibly coincident lines l, l' on X such that Q fits in*

$$0 \to \mathcal{O}_l(-1) \to Q \to \mathcal{O}_{l'}(-1) \to 0.$$

Proof. We start by recalling that, for a curve, being Cohen–Macaulay is equivalent to having neither embedded nor isolated points. A one-dimensional sheaf R is supported on a Cohen–Macaulay curve if $h^0(R(t)) = 0$ for some integer t. Looking at Q, since $\chi(Q(1)) = d > 0$, we have $h^0(Q(1)) > 0$, therefore there exists a nonzero morphism $\mathcal{O}_X(-1) \to Q$ whose image must be of the form $\mathcal{O}_D(-1)$ for a Cohen–Macaulay curve $D \subset \mathrm{Supp}(Q)$.

In particular $\deg(D) \leq 2$. If $\deg(D) = 1$, D must be a line l; hence Q fits in

$$0 \to \mathcal{O}_l(-1) \to Q \to Q' \to 0.$$

Here, Q' has Hilbert polynomial $(d-1)t$. If $d = 1$, then $Q' = 0$ and $Q \simeq \mathcal{O}_l(-1)$. If $d = 2$, since $H^i(Q') = 0$ for $i = 0, 1$, we get that Q' is a sheaf of pure dimension one and $P_{Q'}(t) = t + 1$. Applying the same argument as above, we conclude that $Q' \simeq \mathcal{O}_l'(-1)$ for a line l'.

In the case $\deg(D) = 2$, so that, necessarily $d = 2$, Q fits in

$$0 \to \mathcal{O}_D(-1) \to Q \to Z \to 0$$

for Z a zero-dimensional scheme of length $1 + p_a(D)$. This implies $p_a(D) \geq -1$. If D is integral it is a smooth conic, therefore Z consists of a point $p \in D$, as otherwise $Q \simeq \mathcal{O}_p \oplus \mathcal{O}_D(-1)$ contradicting the fact that Q has pure dimension one. Then Q is a degree-1 line bundle on D. Finally, we show that if D is not integral, Q is always of the form (iii).

If $p_a(D) = -1$, $Q \simeq \mathcal{O}_D(-1)$ and this follows from the fact that, for a Cohen–Macaulay degree-2 curve D of arithmetic genus -1, $\mathcal{O}_D$ arises as a nonsplit extension $\mathrm{Ext}^1(\mathcal{O}_{l'}, \mathcal{O}_l)$ for two possibly coincident lines l, l'.

It remains to analyse the case when D is a singular plane conic and $Z \simeq \mathcal{O}_p$ for $p \in D$. For a curve D of this kind, $\mathcal{O}_D(-1)$ fits in

$$0 \to \mathcal{O}_l(-2) \to \mathcal{O}_D(-1) \to \mathcal{O}_{l'}(-1) \to 0$$

for two possibly coincident lines l, l'. Then we have the commutative diagram

$$\begin{array}{ccccccccc} & & & & \mathcal{O}_l(-2) & = & \mathcal{O}_l(-2) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_D(-1) & \longrightarrow & Q & \longrightarrow & \mathcal{O}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_{l'}(-1) & \longrightarrow & Q' & \longrightarrow & \mathcal{O}_p & \longrightarrow & 0 \end{array} \tag{23}$$

If $Q' \simeq \mathcal{O}_{l'}(-1) \oplus \mathcal{O}_p$, the sheaf Q surjects onto $\mathcal{O}_{l'}(-1)$ and the kernel of this surjection must necessarily be $\mathcal{O}_l(-1)$. If $Q' \simeq \mathcal{O}_{l'}$, we get $\mathrm{Hom}(\mathcal{O}_{l'}(-1), Q) \neq 0$, indeed $\hom(\mathcal{O}_{l'}(-1), \mathcal{O}_l(-2)) > \mathrm{ext}^1(\mathcal{O}_{l'}(-1), \mathcal{O}_l(-2))$. This last inequality is obvious if $l \neq l'$. For $l = l'$ it holds since, for a line l supporting a nonreduced conic, we must have $\mathcal{I}_l|_l \simeq \mathcal{O}_l(-1) \oplus \mathcal{O}_l(2)$ and $\mathrm{Hom}(\mathcal{I}_l, \mathcal{O}_l(-1)) \simeq \mathrm{Ext}^1(\mathcal{O}_l, \mathcal{O}_l(-1)) \simeq \mathbb{C}$. We refer also to [Brambilla and Faenzi 2011, Lemma 4.2] for this argument. Note that this is consistent with the fact that there exists a unique structure of a conic supported on l. A morphism $\mathcal{O}_{l'}(-1) \to Q$ must be injective and again we find that its cokernel must be $\mathcal{O}_l(-1)$. □

Using this result we can prove the existence of rank-2 instanton bundles having charge strictly greater than the minimal one.

Proposition 4.9. *Let X be a Fano threefold of index one and such that $-K_X$ is very ample. Moreover, assume that X is not contained in a singular quadric for $g = 4$. Then, for all $k \geq 0$, there exists a $(1, k)$-instanton bundle E satisfying the following*:

(i) *E is slope-stable and unobstructed.*

(ii) *For a sufficiently general smooth rational curve $R_d \subset X$, of degree $d \in \{1, 2\}$, we have $H^1(E|_{R_d}) = 0$.*

If $g = 4$ and X is contained in a singular quadric, the same statement holds for all $k \geq 1$.

Proof. We prove the result by induction on k. We know from Theorem 4.7 the existence of a $(1, k)$-instanton satisfying (i) for $k \in \{0, 1\}$ except for the case where $g = 4$ and X is contained in a singular quadric, in which case the result holds for

$k \in \{1, 2\}$. A $(1, k)$-instanton as just described is always a 't Hooft instanton, as $\chi(E(1)) > 0$ and $h^2(E(1)) = 0$. Thus, it fits in the short exact sequence

$$0 \to \mathcal{O}_X(-1) \to E \to \mathcal{I}_C \to 0$$

for C an l.c.i. elliptic curve on X of degree $m_g + k$.

From this short exact sequence one checks that a general line and a general conic are not jumping for E. Indeed, for any line $l \subset X$ disjoint from C, it is immediate to compute $E|_l \simeq \mathcal{O}_l(-1) \oplus \mathcal{O}_l$. To verify that the same holds for a general conic R_2, we recall the following facts.

First, for a general point $p \in X$, the family of l.c.i. elliptic curves passing through p is parametrised by a smooth codimension 2 subscheme $\mathcal{H}_p$ of the Hilbert scheme $\mathcal{H}_{m_g+k,1}$ of l.c.i. elliptic curves of degree $m_g + k$.

Second, for such a point p,

$$h^0(\mathcal{N}_{C/X}(-p)) = h^0(\mathcal{N}_{C/X}) - 2, \quad h^1(\mathcal{N}_{C/X}(-p)) = 0 \quad \text{for all curves } C \text{ in } \mathcal{H}_p, \tag{24}$$

so that for all $(1, k)$-instantons E,

$$h^0(E(1) \otimes \mathcal{I}_p) = h^0(E(1)) - 2, \qquad h^i(E(1) \otimes \mathcal{I}_p) = 0 \quad \text{for } i \neq 0.$$

Note that this argument fails a priori for $(1, 1)$-instantons on a prime Fano threefold X of genus $g = 3$ and for $(1, 2)$-instantons on X of genus 4 — we need to look at this situation to treat the case when X is contained in a singular quadric. Indeed, for a general bundle E of this kind we have $h^0(E(1)) = 2$. Nevertheless we do have that for $p \in X$ general, $\mathcal{H}_p$ is nonempty. Indeed, we argue as follows. Note that for a bundle E of such a kind and a point $p \in X$ we get the short exact sequence

$$0 \to \mathcal{I}_p \to E(1) \otimes \mathcal{I}_p \to \mathcal{I}_C(1) \otimes \mathcal{I}_p \to 0.$$

We deduce that for a general curve $[C] \in \mathcal{H}_{3+k,1}$ (so that, in particular the curve might be assumed to be normal, according to [Ciliberto et al. 2023; 2024]), if E is a $(1, k)$-instanton bundle obtained from C by the Serre correspondence, given a point p, one has $h^0(E(1) \otimes \mathcal{I}_p) \neq 0$ only if p belongs to the linear span $\langle C \rangle$ of C. More precisely, one can identify the pencil $\mathbb{P}\big(H^0(E(1))\big)$ with a complete linear system of elliptic curves on a hyperplane section S_E of X admitting an elliptic fibration, and for S_E general, $H^0(E(1) \otimes \mathcal{I}_p)$ corresponds to the unique elliptic curve in this linear system passing through p. Thus a general element in $\mathcal{H}_p$ corresponds to a hyperplane section through p admitting an elliptic fibration. We can then conclude that for p general $\mathcal{H}_p$ is a smooth codimension 2 subscheme of $\mathcal{H}_{3+k,1}$ and that its general element satisfies (24).

Third, still assuming p general, we also have that a general point $q \in X$ distinct from p imposes two linear conditions on each vector space $H^0(E(1))$ that are

independent of $H^0(E(1)\otimes\mathcal{I}_p)$. Therefore the degree m_g+k elliptic curves passing through both p and q are parametrised by a codimension-4 subscheme of $\mathcal{H}_{m_g+k,1}$.

As a consequence of these facts, for a general conic R_2 (which, without loss of generality, we can assume to be unobstructed), and for a general point $p\in R_2$, a general element $[C]\in\mathcal{H}_p$ satisfies $C\cap R_2=\{p\}$ so that $\mathcal{I}_C\otimes\mathcal{O}_{R_2}\simeq\mathcal{O}_{R_2}(-p)\oplus\mathcal{O}_p$. For an instanton E, obtained through the Serre correspondence from such a curve C, $E|_{R_2}$ is an extension of two degree -1 line bundles on R_2, so that $E|_{R_2}\simeq\mathcal{O}_{R_2}(-p)^2$ or, equivalently, $h^1(E|_{R_2})=0$. To show that the proposition holds for higher values of k it will be sufficient to prove the following claim:

Claim. *Let E, R_2, ϕ be, respectively, a $(1,k)$-instanton bundle satisfying* (i), (ii), *an unobstructed smooth conic $R_2\subset X$ that is not jumping for E and an epimorphism $\phi: E\twoheadrightarrow\mathcal{O}_{R_2}(-p)$, $p\in R_2$. Then $F:=\ker(\phi)$ deforms to a slope-stable and unobstructed $(1,k+2)$-instanton bundle which satisfies* (i), (ii).

Let us prove the claim (we follow essentially [Faenzi 2014]). By construction,

$$0\to F\to E\to\mathcal{O}_{R_2}(-p)\to 0. \tag{25}$$

One checks that F is slope-stable, since $E=F^{**}$ and E is slope-stable by assumption. Also, $H^i(F)=0$ for $i=1,2$. Note that from (25), F satisfies (ii). Indeed, there is a nonempty open subset of the Hilbert scheme of lines (or conics) in X such that a curve R' of such subset is disjoint from R_2. For the curve R', we have $E|_{R'}\simeq F|_{R'}$.

Let us check unobstructedness of F. Applying $\mathrm{Hom}(\cdot,F)$ and $\mathrm{Hom}(E,\cdot)$ to (25), by the assumptions on E, we have

$$\mathrm{Ext}^2(E,E)=0,\quad \mathrm{Ext}^2(E,\mathcal{O}_{R_2}(-p))\simeq H^2(E^*|_{R_2}(-p))=0,$$

so that $\mathrm{Ext}^2(E,F)=0$. Restricting (25) to R_2 we check the vanishing of

$$\mathrm{Ext}^3(\mathcal{O}_{R_2}(-p),F)\simeq\mathrm{Hom}(F,\mathcal{O}_{R_2}(-3p))^*=0.$$

This shows $\mathrm{Ext}^2(F,F)=0$.

Consider now the vector $\gamma(1,k+2)$ in $\bigoplus_{0\le i\le 3}H^{i,i}(X)$; see (22). We have that F corresponds to a smooth point in the Gieseker–Maruyama moduli space $\mathrm{M}_X(\gamma(1,k+2))$. Since F is unobstructed, it can indeed be deformed as a smooth point of this moduli space. So there exists a family $\mathbf{F}$, i.e., a sheaf on $X\times T$, flat over a smooth curve T with a marked point $0\in T$, whose central fibre $\mathbf{F}_0$ is isomorphic to F and whose general element $\mathbf{F}_t$ satisfies the cohomological vanishing of Definition 1.1 and (i), (ii). Indeed, all these conditions are open.

We show that for t sufficiently general, $\mathbf{F}_t$ is locally free; hence it is an instanton bundle. If $\mathbf{F}_t$ is not locally free, by Theorem 24 of [Comaschi and Jardim 2024]

it would fit in the short exact sequence

$$0 \to \mathbf{F}_t \to \mathbf{F}_t^{**} \to Q \to 0, \tag{26}$$

where the double dual $\mathbf{F}_t^{**}$ is a slope-stable $(1, k+2-d)$-instanton bundle and Q is a torsion instanton of degree d.

By semicontinuity of cohomology, we find that, for $n \ll 0$,

$$h^2(\mathbf{F}_t(n)) = h^1(Q(n)) = -\chi(Q(n)) = -dn \leq -2n;$$

hence $d \leq 2$. From Lemma 4.8, we know that Q can only belong to three families of 1-dimensional sheaves. We show that all the corresponding families of nonlocally free instantons are divisorial in $\mathrm{MI}_X(1, k+2)$. This suffices to conclude that for $t \in T$ general, $\mathbf{F}_t$ is an instanton bundle.

If $d = 1$, we have $Q \simeq \mathcal{O}_l(-1)$ for a line $l \subset X$. Sheaves of this kind belong to a family of dimension one, therefore the general nonlocally free deformations of F that are singular along a line are parametrised by a family of dimension at most $\mathrm{ext}^1(\mathbf{F}_t^{**}, \mathbf{F}_t^{**}) + 1$ for $\mathbf{F}_t^{**}$ a slope-stable unobstructed $(1, k+1)$-instanton bundle. But by Riemann–Roch we compute that $\mathrm{ext}^1(\mathbf{F}_t, \mathbf{F}_t) = \mathrm{ext}^1(\mathbf{F}_t^{**}, \mathbf{F}_t^{**}) + 2$.

If $d = 2$, $c_2(\mathbf{F}_t^{**}) = m$ and Q is either a degree -1 line bundle on a smooth conic or it arises from an extension class in $\mathrm{Ext}^1(\mathcal{O}_{l'}(-1), \mathcal{O}_l(-1))$. As $c_2(\mathbf{F}_t) = c_2(\mathbf{F}_t^{**}) + 2$, by Riemann–Roch we compute that $\mathrm{ext}^1(\mathbf{F}_t, \mathbf{F}_t) = \mathrm{ext}^1(\mathbf{F}_t^{**}, \mathbf{F}_t^{**}) + 4$. But the families of instantons that are singular along a degree-2 curve have dimension $\mathrm{ext}^1(\mathbf{F}_t^{**}, \mathbf{F}_t^{**}) + 3$. Indeed, Q varies in a 2-dimensional family, and we have a one-dimensional family of epimorphisms $\mathbf{F}_t^{**} \twoheadrightarrow Q$. We can thus conclude that a general deformation of F is an instanton bundle that, by semicontinuity, satisfies (i) and (ii). □

Remark 4.10. The hypothesis that $-K_X$ is very ample is necessary for the argument used in Proposition 4.9. Indeed, this assumption ensures that a curve of degree 2 in X is rational, which implies that the dimension of the family of torsion instantons on X supported on degree-2 curves is 2. This will no longer be true if $-K_X$ is not very ample. For example, if X is a degree-2 covering of a Fano threefold Y, we have irrational curves of degree 2 obtained by pulling back lines on Y. The torsion instantons supported on a curve of this kind are not all isomorphic; in fact, they vary in a family of dimension one.

4.2.2. *Instanton bundles of higher rank.* Let us now pass to the higher-rank range. The approach will be similar to what we did earlier, namely, we produce strictly slope-semistable instanton bundles that can then be deformed into slope-stable instantons.

However, this time we meet some additional difficulties. In the first place, with our approach, we can no longer hope to prove the existence of instanton bundles E

such that $h^0(E(1)) \neq 0$: in the rank-2 cases this indeed only holds for values of c_2 varying in $\{m_g, \dots, g+3\}$. The second difficulty, actually the main one, is the identification of the minimal charge: we are indeed able to determine its exact value only when $g \geq 4$. This computation is a consequence of the following lemma.

Lemma 4.11. *Let E be an (n,k)-instanton on X.*

(a) *If $g \geq 6$ then*

(i) $\operatorname{ext}^2(F_0, E) = 0$;

(ii) $\operatorname{ext}^2(E, F_0) = 0$.

Items (i) *and* (ii) *hold for $g \in \{4,5\}$ as well if E is slope-stable and Gieseker-semistable, respectively, and not a minimal instanton.*

(b) *If $g \geq 6$, then $k \geq 0$. The same holds for $g \in \{4,5\}$ and E Gieseker-semistable.*

(c) *If $g \geq 6$ and E is Gieseker-stable and not a minimal instanton, then $k \geq 1$ for g odd and $k \geq n$ for g even. The same holds for $g \in \{4,5\}$ if E is slope-stable.*

Proof of (a). Let us consider a minimal rank-2 instanton bundle F_0. Let δ be the remainder of the division of g by 2. It is well known (see, for instance, [Brambilla and Faenzi 2011]) that the evaluation map $H^0(F_0^*) \otimes \mathcal{O}_X \to F_0^*$ is surjective, in other words, $F_0^* = F_0(1)$ is globally generated. Also, the evaluation map has a locally free kernel K_0, so that, in view of the equality $m_g + 1 - \delta = h^0(F_0(1)) = \chi(F_0(1))$, we have the short exact sequence of vector bundles

$$0 \to K_0 \to \mathcal{O}_X^{m_g+1-\delta} \to F_0^* \to 0. \tag{27}$$

The vector bundle K_0 is slope-stable. We reproduce here a well-known argument to show it, for the reader's convenience. From (27) we can check that the bundle K_0 satisfies $H^0\big(\bigwedge^i K_0\big) = 0$ for $i = 1, \dots, \operatorname{rk}(K_0)$. Indeed, from its very definition, first of all we have $h^0(K_0) = 0$. Then, all the other vanishing results follow recursively from the inclusion

$$\bigwedge^i K_0 \hookrightarrow \mathcal{O}_X^{m_g+1-\delta} \otimes \bigwedge^{i-1} K_0,$$

valid for all integers $i > 0$. Such an inclusion, in turn, comes from tensoring (27) with $\bigwedge^{i-1} K_0$ and using the inclusion $\bigwedge^i K_0 \hookrightarrow K_0 \otimes \bigwedge^{i-1} K_0$. Therefore K_0 is slope-stable by Hoppe's criterion; see [Jardim et al. 2017, Theorem 3]. Note that the assumption $g \geq 6$ guarantees $m_g + 1 - \delta \geq 5$ so $\operatorname{rk}(K_0) \geq 3$; hence

$$\mu(K_0^*) = \frac{1}{\operatorname{rk}(K_0)} \leq \frac{1}{3}. \tag{28}$$

Let us now take an instanton sheaf E and apply $\operatorname{Hom}(\cdot, E)$ to the dual of (27). Doing so we get an isomorphism $\operatorname{Ext}^2(F_0, E) \simeq \operatorname{Ext}^3(K_0^*, E) \simeq \operatorname{Hom}(E(1), K_0^*)^*$. Now, since $E(1)$ is slope-semistable of slope $\frac{1}{2}$, by (28) we get $\operatorname{Hom}(E(1), K_0^*) = 0$

and (a) is proved. Applying $\mathrm{Hom}(E, \cdot)$ to (27) tensored by $\mathcal{O}_X(-1)$ we get that, as $\mathrm{Ext}^2(E, \mathcal{O}_X(-1)) \simeq H^1(E)^* = 0$, $\mathrm{Ext}^2(E, F_0)$ injects into the vector space $\mathrm{Ext}^3(E, K_0(-1)) \simeq \mathrm{Hom}(K_0, E)^*$. As $c_1(K_0) = -1$ and $\mathrm{rk}(K_0) \geq 3$ for $g \geq 6$, the slope-semistability of E ensures that $\hom(K_0, E) = 0$.

Finally, let us prove item (a) for $g \in \{4, 5\}$, in which case $m_g + 1 - \delta = 4$ and $K_0 \simeq K_0^*(-1)$ is also a minimal instanton. If E is slope-stable and not a minimal instanton, $\mathrm{Hom}(E(1), K_0^*) = 0$ which again allows us to conclude that $\mathrm{Ext}^2(F_0, E) = 0$. If $\mathrm{Ext}^2(E, F_0) \neq 0$, we would have $\mathrm{Hom}(K_0, E) \neq 0$. Now, such a morphism must necessarily be injective which cannot happen if $k \geq 1$ and E is Gieseker-semistable. □

Remark 4.12. We cannot hope to get such a strong statement for the case $g = 3$, namely for $X \subset \mathbb{P}^4$ a smooth quartic hypersurface. For now, note that in this case the cokernel of the coevaluation map $F_0 \hookrightarrow \mathcal{O}_X^3$ is $\mathcal{I}_l(1)$, where $l \subset X$ is a line. Indeed, already in rank 2, taking a rank-2 instanton E corresponding to an elliptic curve C' containing l gives $\hom(E, \mathcal{I}_l) = \mathrm{ext}^3(\mathcal{I}_l(1), E) = \mathrm{ext}^2(F_0, E) \neq 0$.

End of the proof of Lemma 4.11. To prove points (b) and (c) we start by computing $\chi(E, F_0) = \chi(F_0, E)$. Setting again δ to be the remainder of the division of g by 2, this is done by applying Riemann–Roch, which leads to

$$\chi(E, F_0) = \chi(F_0, E) = (1 - \delta)n - k.$$

Let us prove item (b). We argue by induction on n. For $n = 1$ the statement holds due to Lemma 4.6. Let us suppose now that $n > 1$. If $g \geq 6$, $\mathrm{ext}^2(F_0, E) = 0$, so that $k < 0$ implies $\hom(F_0, E) > 0$. A nonzero morphism $F_0 \to E$ must necessarily be injective so that its cokernel should be a $(n-1, k)$-instanton, which is not possible due to the inductive hypothesis. For the cases $g \in \{4, 5\}$ the same argument applies if $\mathrm{ext}^2(F_0, E) = 0$, which in particular is true for E slope-stable. Then, we assume $\mathrm{ext}^2(F_0, E) \neq 0$ so that, by (27) $\mathrm{ext}^3(K_0^*, E) = \hom(E, K_0) \neq 0$. But assuming E Gieseker-semistable, a morphism $E \to K_0$ can exist only for $k \geq 0$.

Finally, let us prove (c). Let E be an (n, k)-instanton satisfying the statement. In particular, by item (a), $\mathrm{ext}^2(F_0, E) = 0$. The stability assumption on E and the fact that E is not a minimal instanton ensure that $\mathrm{Hom}(F_0, E) = 0$ so that $\chi(F_0, E) < 0$. This proves that, when g is even, we must have $k \geq n$. So (c) is proved in this case. When g is odd, this gives $k \geq 0$.

To prove $k \geq 1$ we need a little extra work. We consider $\gamma = \gamma(1, 0) = \mathrm{ch}(F_0)$ and we look at the moduli space $M := \mathrm{M}_X(\gamma)$. Assume g is odd, hence $\chi(E, F_0) = -k$. It is proved in [Brambilla and Faenzi 2011] that, when g is odd, M is a connected projective curve, possibly singular and/or reducible, and that M is a fine moduli space. Therefore, there exists a universal sheaf $\mathcal{E}$ over $X \times M$ such that any closed point y of M corresponds to a sheaf $\mathcal{E}_y$ which is a minimal instanton F_0. Denoting

by p and q the projections from $X \times M$ onto X and M, by Lemma 4.11, we have

$$R^i q_*(p^*(E) \otimes \mathcal{E}^*) = 0 \quad \text{for } i \neq 0, 1. \tag{29}$$

Also, $q_*(p^*(E) \otimes \mathcal{E}^*) = 0$, as $H^0(E \otimes \mathcal{E}_y^*) = 0$ if y is a sufficiently general point of each component of M. On the other hand, in view of (29), we may use base change of cohomology, so that the sheaf $R^1 q_*(p^*(E) \otimes \mathcal{E}^*)$ is supported at a point $y \in M$ if and only if

$$\operatorname{ext}^1_X(\mathcal{E}_y, E) = \hom_X(\mathcal{E}_y, E) \neq 0,$$

where the equality of the first two quantities follows from Riemann–Roch and Lemma 4.11. Note that, if $E \simeq \mathcal{E}_z$ for some point $z \in M$, then

$$\operatorname{ext}^1_X(\mathcal{E}_y, \mathcal{E}_z) = \hom_X(\mathcal{E}_y, \mathcal{E}_z) = 1 \iff z = y,$$

otherwise, if $z \neq y$, we have $\operatorname{ext}^1_X(\mathcal{E}_y, \mathcal{E}_z) = \hom_X(\mathcal{E}_y, \mathcal{E}_z) = 0$. This implies that $R^1 q_*(p^*(\mathcal{E}_z) \otimes \mathcal{E}^*) \simeq \mathcal{O}_z$. Since $\operatorname{ch}(E) = n \operatorname{ch}(F_0) = n \operatorname{ch}(\mathcal{E}_z)$, we deduce, via Grothendieck–Riemann–Roch, that

$$\operatorname{ch}(R^1 q_*(p^*(E) \otimes \mathcal{E}^*)) = n \operatorname{ch}(\mathcal{O}_z).$$

Therefore, the support of $R^1 q_*(p^*(E) \otimes \mathcal{E}^*)$ is not empty; hence there is $y \in M$ such that $\hom_X(\mathcal{E}_y, E) \neq 0$. This is impossible if E is slope-stable and not isomorphic to $\mathcal{E}_y$. This finishes the proof of (c). □

For all $n \geq 2$, we finally prove the existence of slope-stable (n, k)-instanton bundles for any values of $k \geq n$ for g even (hence for any values of k greater than or equal to the minimal one), provided that for $g = 4$, X is not contained in a singular quadric, and for any $k \geq 1$ for g odd and greater than 3. Once again, we construct, inductively, strictly slope-semistable bundles that can be deformed into slope-stable ones. The primary difference with respect to the cases where the index is greater than 1 lies in the moduli space where this deformation occurs. Indeed, since for $i_X = 1$, rank-2 't Hooft bundles do not necessarily exist for arbitrary values of the charge, our inductive method cannot produce sheaves that admit global sections only after a twist by $\mathcal{O}_X(1)$. For this reason the deformation will take place in a moduli space of simple sheaves rather than in a moduli space of extensions.

Theorem 4.13. *Let X be a Fano threefold with* $\operatorname{Pic}(X) = \langle -K_X \rangle$ *and genus $g \geq 4$. Assume that X is not contained in a singular quadric whenever $g = 4$. Then, for all $n \geq 1$ and all $k \geq n$ (resp. $k \geq 1$) if g even (resp. odd), there exists an (n, k)-instanton bundle E_n^k such that the following hold*:

(i) *E_n^k is slope-stable and unobstructed.*

(ii) *For a general rational curve $R_d \subset X$ of degree $d \in \{1, 2\}$, $H^1(E_n^k|_{R_d}) = 0$.*

(iii) $\operatorname{Ext}^2(F_0, E_n^k) = \operatorname{Ext}^2(E_n^k, F_0) = 0$ *for all minimal instantons F_0.*

The main idea behind the proof is the same as the one used for indices $\{2, 3, 4\}$, i.e., to construct inductively some $(n+1, k)$-instantons, which would be unobstructed and strictly slope-semistable, via nonsplit extensions in $\mathrm{Ext}^1(E_n^k, F_0)$, where E_n^k is an (n, k)-instanton coming from the previous step of the induction, and then to show that they deform into slope-stable instantons.

The key difference lies in the moduli space where this deformation takes place and the main reason for this difference is that on X we do not have existence of rank-2 't Hooft bundles for every value of the charge. This time, we will deform E_{n+1}^k in a moduli space of simple sheaves $\mathrm{Spl}_X(\gamma(n+1, k))$. To achieve this, we will once again use a dimension counting argument to show that the only component of $\mathrm{Spl}_X(\gamma(n+1, k))$ through $[E_{n+1}^k]$ cannot be entirely contained in the closed subscheme $Z_{\gamma(n,k)} \subset \mathrm{Spl}_X(\gamma(n+1, k))$, which parametrises sheaves admitting quotients with Chern character $\gamma(n, k)$.

Proof. The argument is by induction, namely, for given n, we take care of the minimal value of k and work by induction on k; this allows us to settle the induction step and move to higher n.

Step 1 (check the base of the induction). For $n = 1$, (i) and (ii) hold due to Proposition 4.9 whilst (iii) is true by Lemma 4.11.

Step 2 (construction of an instanton of higher rank). Suppose now that the theorem holds for $n \geq 1$ and let k be an integer $\geq n+1$ (resp. ≥ 1) for g even (resp. odd). By the inductive hypothesis there exists a slope-stable (n, k)-instanton E_n^k satisfying Theorem 4.13. Since $\chi(E_n^k, F_0) < 0$ and $\mathrm{Ext}^3(E_n^k, F_0) = 0$ by semistability, we have $\mathrm{ext}^1(E_n^k, F_0) \neq 0$; hence there exists a nonsplit extension

$$0 \to F_0 \to E_{n+1}^k \to E_n^k \to 0. \tag{30}$$

The sheaf E_{n+1}^k is a $(n+1, k)$-instanton bundle and from Lemma 2.13 we know that E_{n+1}^k is a strictly slope-semistable sheaf and that moreover E_n^k is its only torsion-free quotient having slope $-\frac{1}{2}$. The sheaf E_{n+1}^k is unobstructed, as it follows from Lemma 2.14 and it satisfies (ii) since this holds for both F_0 and E_n^k.

Note that the sheaf E_{n+1}^k is not Gieseker-stable, as it is destabilised by F_0. Still, we may consider it as a point in the moduli space $\mathrm{Spl}_X(\gamma(n+1, k))$ of simple sheaves having Chern character $\gamma(n+1, k)$. For an account on this moduli space we refer to [Altman and Kleiman 1980; Kosarew and Okonek 1989].

The fact that E_{n+1}^k is simple can easily be proved by the short exact sequence (30). Indeed, the slope-stability of F_0 and E_n^k implies $\mathrm{Hom}(E_n^k, F_0) = 0 = \mathrm{Hom}(F_0, E_n^k)$. Accordingly, we get $\mathrm{Hom}(E_{n+1}^k, F_0) = 0$, as otherwise the short exact sequence would split, and

$$\mathrm{Hom}(E_{n+1}^k, E_{n+1}^k) \simeq \mathrm{Hom}(E_{n+1}^k, E_n^k) \simeq \mathrm{Hom}(E_n^k, E_n^k) \simeq \mathbb{C}.$$

Step 3 (deform E^k_{n+1} to a slope-stable instanton in the moduli space of simple sheaves). The unobstructedness of E^k_{n+1} ensures that the corresponding point $[E^k_{n+1}]$ belongs to a unique component Y of $\mathrm{Spl}_X(\gamma(n+1,k))$ which has dimension $\mathrm{ext}^1(E^k_{n+1}, E^k_{n+1})$. Applying the same argument in the proof of Theorem 4.3, the strictly slope-semistable simple sheaves are parametrised by a closed subscheme $Z \subset \mathrm{Spl}_X(\gamma(n+1,k))$. Thus, in order to prove that $[E^k_{n+1}]$ deforms to a slope-stable sheaf, it will be enough to prove that $Y \cap Z$ is a strict closed subscheme of Y.

Next, notice the following fact. A torsion-free sheaf with Chern character $\gamma(n+1,k)$ is strictly slope-semistable if and only if it admits a torsion-free quotient with slope $-\frac{1}{2}$. We deduce that Z can be written as a finite union $Z = \bigcup_{v\in A} Z_v$ where each Z_v is the closed subscheme parametrising sheaves admitting a torsion-free quotient with Chern character $v = (v_0, \ldots, v_3)$ with $\frac{v_1}{v_0} = -\frac{1}{2}$. Here, the finiteness of A follows from [Huybrechts and Lehn 2010, Lemma 1.7.9].

Without loss of generality, we can assume that the set A is actually the singleton $\{\gamma(n,k)\}$. Since on X there are no slope-semistable rank-2 torsion-free sheaves with $c_1 = -1$ and Hilbert polynomial $P > P_{F_0}$, the same arguments presented in the proof of Theorem 4.3, Steps 6.2–6.4, allow us to conclude what follows: a general strictly slope-semistable deformation E' of E^k_{n+1} is a locally free sheaf such that the only torsion-free quotients of slope $\frac{1}{2}$ of the dual E'^* have Hilbert polynomial $P_{F_0^*}$. But as every slope-semistable rank-2 torsion-free sheaf with Hilbert polynomial $P_{F_0^*}$ is the dual of a minimal instanton, this is equivalent to saying that all torsion-free quotients of strictly slope-semistable deformations of E^k_{n+1} are (n,k)-instantons, so that $A = \{\gamma(n,k)\}$.

Thus, $Y \subset Z$ if and only if $Y \subset Z_{\gamma(n,k)}$. Suppose this is the case. Since all slope-semistable torsion-free sheaves with Chern character $\gamma(1,0)$ are actually slope-stable instanton bundles (in view of [Brambilla and Faenzi 2011, Proposition 3.4]), we deduce that all closed points of $Z_{\gamma(n,k)}$ correspond to sheaves arising from extension classes in $\mathrm{Ext}^1(F, F_0)$ for F an (n,k)-instanton. Let us consider now an open neighbourhood $U \subset Y$ of $[E^k_{n+1}]$. Up to shrinking U, we can suppose that the latter is disjoint from Z_v for every $v \in A$, $v \neq \gamma(n,k)$. This implies that given a sheaf E corresponding to a point in U, all torsion-free quotients of E having Chern character $\gamma(n,k)$ must be slope-stable. Thus, all points $[E]$ in U admit a unique (up to isomorphism) torsion-free slope-stable quotient F with Chern character $\gamma(n,k)$. This will in particular ensure that F satisfies $\mathrm{ext}^2(F_0, F) = \mathrm{ext}^2(F, F_0) = 0$ (this vanishing is obtained by applying Lemma 4.11). Summing up,

$$\dim(Z_{\gamma(n,k)}) = \dim\big(\mathrm{M}_X(\gamma(1,0))\big) + \mathrm{ext}^1(F,F) + \mathrm{ext}^1(F,F_0) - 1, \tag{31}$$

which from a direct computation would lead to

$$\mathrm{ext}^1(E^k_{n+1}, E^k_{n+1}) - \dim(Z_{\gamma(n,k)}) = k - n\chi(F_0,F_0) = -\chi(E^k_n, F_0),$$

which is strictly greater than zero by the assumptions on n and k. This contradicts our assumption $U \subset Z_{\gamma(n,k)}$. A general deformation of $[E^k_{n+1}]$ in $\mathrm{Spl}_X(\gamma(n+1,k))$ is therefore a slope-stable instanton bundle. Since moreover E^k_{n+1} is unobstructed and satisfies (ii), as it can be checked immediately from the defining extension, and (iii), the same will hold, by semicontinuity, for a general deformation. □

This ends the proof of the main theorem from the introduction. Also this time, as a direct consequence of Theorem 4.13, we get the following.

Corollary 4.14. *If g is even (resp. odd) for all $n \geq 2$, $k \geq n$ (resp. $k \geq 1$) there exists a generically smooth component of the moduli space $\mathrm{MI}_X(n,k)$ of dimension $1-n^2+2nk$ (resp. $(1+2nk)$) of (n,k)-instanton bundles whose general point is a slope-stable locally free instanton having general splitting $(0^n,-1^n)$.*

The existence of ACM instantons on prime Fano threefolds is proved in [Ciliberto et al. 2024] and [Brambilla and Faenzi 2011] (in the ordinary case), as we recollected in Theorem 4.7. We deduce the following:

Corollary 4.15. *Let X be a prime Fano threefold of genus $g \geq 4$ and assume that X is not contained in a singular quadric for $g = 4$.*

- *If g is odd, there exists a slope-stable ACM (n,k)-instanton bundle for all $n \geq 2$ and all $k \in \{1,\dots,g+3-m_g\}$.*
- *If g is even, there exists a slope-stable ACM (n,k)-instanton bundle for all $n \in \{2,\dots,g+3-m_g\}$ and all $k \in \{n,\dots,g+3-m_g\}$.*

Proof. This is obtained by applying the construction of the proof of Theorem 4.13, just noticing that an extension of ACM bundles is ACM and that being ACM is an open condition. □

5. Stability of restriction to $K3$ surfaces

Let X be a Fano threefold of index i_X and Picard number one and let S be an element in the linear system $|\mathcal{O}_X(-K_X)|$. Then, if S is general enough, it is a smooth $K3$ surface of Picard rank one. Let now E be an (n,k)-instanton bundle. We investigate the stability of the restriction $E|_S$. We set k_0^n to be the integer defined in Main Theorem 1. We move on to show Main Theorem 2.

Theorem 5.1. *Let X be a Fano threefold of index i_X. If $i_X = 1$, assume that $g \geq 4$ and that X is not contained in a singular quadric when $g = 4$. Let S be a general element in the linear system $|\mathcal{O}_X(-K_X)|$. Then for all integers n, k with $n \geq 2 - r_X$, $k \geq k_0^n$, there exists an open nonempty subscheme $\mathcal{R}_S(n,k) \subset \mathrm{MI}_X(n,k)$ parametrising slope-stable (n,k)-instanton E on X such that $E|_S$ is slope-stable.*

Proof. We prove the result by induction on n in several steps.

Step 1 (the base of the induction). Let $S \subset X$ be a smooth anticanonical divisor having Picard rank one — these conditions are satisfied for S general in $|\mathcal{O}_X(-K_X)|$. Denote by H_S the restriction of $H_X \in |\mathcal{O}_X(1)|$ to S, so that $\mathrm{Pic}(S) \simeq \mathbb{Z}H_S$. For any $n = 2 - r_X$ and $k \geq k_0^n$, we consider a slope-stable (n, k)-instanton E.

Without loss of generality, we may assume that E satisfies Theorems 3.2, 4.3, 3.8, 4.13 for $i_X = 4, 3, 2, 1$, respectively. Consider the short exact sequence

$$0 \to E(-i_X) \to E \to E|_S \to 0. \tag{32}$$

As $\mathrm{Pic}(S) \simeq \mathbb{Z}$ and $E|_S$ is a rank-2 vector bundle with $c_1(E|_S) = -r_X H_S$, we see that $E|_S$ is slope-stable. Indeed from (32), since $H^0(E)$ and $H^1(E(-i_X))$ both vanish, $H^0(E|_S) = 0$, which is enough to prove slope-stability since $\mathrm{Pic}(S) \simeq \mathbb{Z}H_S$.

So Step 1 is proved and we can pass to the induction step.

Step 2 (constructing and restricting an instanton of rank $n+1$). For $n \geq 2 - r_X$, $k \geq k_0^{n+1}$ we take a slope-stable (n, k)-instanton bundle E_n^k satisfying the statement of Theorem 5.1. Let F_0 be a minimal instanton on X. Incidentally, recall that this implies $F_0 \simeq \mathcal{O}_X$ when i_X is even, while, when i_X is odd, F_0 is a minimal rank-2 instanton bundle.

Note that $\mathrm{ext}^1(E_n^k, F_0(-i_X)) = \mathrm{ext}^2(F_0, E_n^k) = 0$. This is clear for even index since in that case $F_0 \simeq \mathcal{O}_X$. For odd index this is shown in Lemmas 4.1 and 4.11.

Then, by the short exact sequence

$$0 \to F_0(-i_X) \to F_0 \to F_0|_S \to 0,$$

we see that the linear map

$$\mathrm{Ext}^1(E_n^k, F_0) \to \mathrm{Ext}^1(E_n^k, F_0|_S) \simeq H^1(E_n^{k*} \otimes_{\mathcal{O}_S} F_0|_S) \simeq \mathrm{Ext}^1(E_n^k|_S, F_0|_S)$$

is injective.

So, given a nonzero element e in $\mathrm{Ext}^1(E_n^k, F_0)$ and denoting by E the $(n+1, k)$-instanton defined by e, we have that $E|_S$ fits in the nonsplit short exact sequence, defined by $e|_S$,

$$0 \to F_0|_S \to E|_S \to E_n^k|_S \to 0.$$

Since by assumption $E_n^k|_S$ is slope-stable and $\mathrm{Pic}(S) \simeq \mathbb{Z}H_S$, Lemma 2.13 implies that $E|_S$ is strictly slope-semistable and $E_n^k|_S$ is its only torsion-free quotient having slope $-\frac{1}{2}r_X$.

We have thus constructed an $(n+1, k)$-instanton and checked its restriction to S. The next goal is to show that, for a general deformation E' of E, $E'|_S$ can be "slope-destabilised" at most by $F_0|_S$. The proof will essentially consist of a dimension argument, which will involve explaining this last sentence rigorously.

Step 3 (the case $i_X \neq 1$). Let us suppose, at first, that $i_X > 1$. From the proofs of Theorems 3.2, 4.3, 3.8, we know that E is given by a point in a projective bundle $\mathbb{P}(\mathcal{E}) \to U$ over an open subscheme U of $\mathrm{M}_X(v)$, the moduli space of torsion-free Gieseker-semistable sheaves having Chern character

$$v := \gamma(n, k) - \mathrm{ch}(\mathcal{O}_X(-1)) + \mathrm{ch}(F_0).$$

Moreover there exists an open neighbourhood $\mathcal{V}$ of $[E]$ in $\mathbb{P}(\mathcal{E})$ admitting a family $\mathbb{E} \in \mathrm{Coh}(\mathcal{V} \times X)$ such that, for all $x \in \mathcal{V}$, distinct from $[E]$, the bundle $\mathbb{E}_x$ satisfies Theorems 3.2, 4.3, 3.8 for $i_X = 4, 3, 2$ respectively. Let us now denote by $\mathbb{E}|_S$ the pullback of $\mathbb{E}$ to $\mathcal{V} \times S$. The locus of points $x \in \mathcal{V}$ such that $(\mathbb{E}|_S)_x$ is not slope-stable is closed in $\mathcal{V}$. Indeed, let A be the set of degree-2 polynomials $P \in \mathbb{Q}[t]$ with $\mu(P) = -\frac{1}{2} r_X$ and such that there exists $x \in \mathcal{V}$ for which $(\mathbb{E}|_S)_x$ admits a pure 2-dimensional quotient F with $P_F = P$. The set A is finite. For every $P \in A$, the points $x \in \mathcal{V}$ such that $(\mathbb{E})|_{S_x}$ admits a pure 2-dimensional quotient F with $P_F = P$ define a closed subscheme $Z_P \subset \mathcal{V}$; we refer again to [Huybrechts and Lehn 2010, §2.3].

Notice that for $P \neq P_{E_n^k|_S}$, the closed subset Z_P is strictly contained in $\mathcal{V}$, as $[E]$ itself does not belong to Z_P. Moreover the only Gieseker-stable sheaf on S with $\mathrm{ch}(F_0|_S)$ is $F_0|_S$ itself, see, for instance, [Huybrechts and Lehn 2010, Theorem 6.1.6], as $F_0|_S$ is Gieseker-stable and $\chi(F_0|_S, F_0|_S) = 2$. We deduce that for a general $x \in \mathcal{V}$, letting $E' := \mathbb{E}_x$, $(E'|_S)$ is not slope-stable only if $\mathrm{Hom}(F_0|_S, E'|_S) \neq 0$.

Let us check that this is a contradiction. From (32), applying $\mathrm{Hom}(F_0, \cdot)$,

$$0 \to \mathrm{Hom}(F_0, E') \to \mathrm{Hom}(F_0, E'|_S) \to \mathrm{Ext}^1(F_0, E'(-i_X)).$$

Then, it suffices to show that both the left- and right-side terms of this sequence vanish. However, this is easy to check: First, $\mathrm{Hom}(F_0, E') = 0$ by slope-stability. Second, $\mathrm{Ext}^1(F_0, E'(-i_X)) \simeq \mathrm{Ext}^2(E', F_0)^*$, and this vanishes by the instanton cohomological vanishing when i_X is even, while for $i_X = 3$ this vanishing follows by the assumption that E' satisfies Theorem 4.3. This concludes Step 3 and proves the theorem for $i_X \neq 1$.

Step 4 (the case $i_X = 1$). For the case $i_X = 1$ the minimal instanton $F_0|_S$ may fail to be isolated in its moduli space — actually this occurs when the genus is odd. So the previous argument needs some further adjustment before we can get to the same conclusion.

Let $\mathcal{T} \to \mathcal{V} \subset \mathrm{Spl}(\gamma(n+1, k))$ be an étale neighbourhood of $[E]$ admitting a family of sheaves $\mathcal{E}$ over $\mathcal{T} \times X$ specialising to E. Consider then the pullback of $\mathcal{E}$ to $\mathcal{T} \times S$, which will be simply denoted by $\mathcal{E}|_S$. We claim that, up to shrinking $\mathcal{V}$, we can suppose that for all $x \in \mathcal{T}$, the sheaf $\mathcal{E}_x|_S$ is slope-semistable and that moreover it can at most be slope-destabilised by the restriction to S of a minimal instanton on X.

Recall that, as we learned from the proof of Theorem 4.13, up to shrinking $\mathcal{V}$, all slope-semistable deformations of E belong to the closed subscheme $Z_{\gamma(n,k)}$ image of $\mathrm{Quot}(\mathcal{E}, \gamma(n,k))$. Since by semicontinuity we can assume that for every point $x \in \mathcal{T}$, $\mathcal{E}_x|_S$ is simple, we get a well-defined restriction map

$$\rho : \mathcal{T} \to \mathrm{Spl}_S(\gamma_S(n+1,k)), \quad E \mapsto E|_S.$$

Up to replacing $\mathrm{Spl}_S(n+1,k)$ by an appropriate neighbourhood $\mathcal{U}$ of $[E|_S]$ we suppose the existence of a Poincaré bundle $\mathbf{E}$ on $\mathcal{U}$. This allows us to define a closed subscheme $W_{\gamma_S(n,k)}$, image of $\mathrm{Quot}(\mathbf{E}, \gamma_S(n,k)) \xrightarrow{\pi} \mathcal{U}$, that parametrises bundles on S having quotients with Chern character $\gamma_S(n,k)$. The same arguments used to characterise the general slope-semistable deformations of E lead us to conclude that a general deformation of $E|_S$ is not slope-stable if and only if it admits a quotient having Chern character $\gamma_S(n,k)$. Note that this relies on the fact that, on S as well as on X, there are no semistable torsion-free rank-2 sheaves having Hilbert polynomial strictly greater than $P_{F_0|_S}$.

The deformations of E whose restriction to S admits a quotient having Chern character $\gamma_S(n,k) := \mathrm{ch}(E_n^k|_S)$ are parametrised by the closed subscheme $Z_{\gamma_S(n,k)}$ image of $\mathrm{Quot}(\mathcal{E}|_S, \gamma_S(n,k)) \xrightarrow{\pi} \mathcal{T}$. This closed subscheme is the preimage $\rho^{-1}(W_{\gamma_S(n,k)})$. The characterisation of $W_{\gamma(n,k)}$ presented above allows us to conclude that $Z_{\gamma_S(n,k)}$ parametrises sheaves whose restriction to S is strictly μ-semistable.

By construction, we have $Z_{\gamma(n,k)} \subset Z_{\gamma_S(n,k)}$. We prove that these schemes actually coincide at the point $[E]$. From $Z_{\gamma_S(n,k)} = \rho^{-1}(W_{\gamma_S(n,k)})$, we get

$$\dim T_{[E]} Z_{\gamma_S(n,k)} \leq \dim(d_{[E]}\rho^{-1} T_{[E|_S]} W_{\gamma_S(n,k)}) = \dim(T_{[E|_S]} W_{\gamma_S(n,k)} \cap \mathrm{Ext}^1(E,E)).$$

To understand the right side of this equation we recall that, due to the vanishing of $\mathrm{Ext}^2(E,E)$, the space $\mathrm{Ext}^1(E,E)$ injects via $d_{[E]}\rho$ into $\mathrm{Ext}^1(E|_S, E|_S)$, because $d_{[E]}\rho$ identifies with the linear map $\mathrm{Ext}^1(E,E) \to \mathrm{Ext}^1(E, E_S)$ induced by $E \twoheadrightarrow E_S$. This last point was already observed in [Tyurin 2004].

Note also that the cokernel of this injection is $\mathrm{Ext}^2(E, E(-1)) \simeq \mathrm{Ext}^1(E,E)^*$ which tells us that $\mathrm{Ext}^1(E,E)$ is a maximal isotropic subspace of $\mathrm{Ext}^1(E|_S, E|_S)$ with respect to the skew-symmetric form ω defined by the Yoneda pairing. We explicitly compute the dimension of $T_{[E|_S]} W_{\gamma_S(n,k)} \cap \mathrm{Ext}^1(E,E)$. Denote by q_S the epimorphism $E|_S \twoheadrightarrow E_n^k|_S$. We recall that the tangent space to the Quot scheme $\mathrm{Quot}(\mathbf{E}, \gamma_S(n,k))$ at the point $q_S \in \pi^{-1}(E|_S)$ fits in the exact sequence (see, e.g., [Huybrechts and Lehn 2010], Proposition 2.2.7)

$$0 \to \mathrm{Hom}(F_0|_S, E_n^k|_S) \to T_{q_S}\,\mathrm{Quot}(\mathbf{E}, \gamma_S(n,k)) \xrightarrow{d_{q_S}\pi} T_{[E|_S]}\mathcal{U} \to \mathrm{Ext}^1(F_0|_S, E_n^k|_S).$$

As $T_{[E|_S]}\mathcal{U} \simeq \mathrm{Ext}^1(E|_S, E|_S)$ and $\mathrm{Hom}(F_0|_S, E_n^k|_S) = 0$ by the inductive hypothesis, we deduce that $T_{[E|_S]} W_{\gamma_S(n,k)}$ identifies with the kernel of the linear map

$$\sigma_S : \mathrm{Ext}^1(E|_S, E|_S) \simeq T_{E|_S}\mathcal{U} \to \mathrm{Ext}^1(F_0|_S, E_k^n|_S).$$

Observe that σ_S is nothing but the composition of the canonical linear maps $\mathrm{Ext}^1(E|_S, E|_S) \to \mathrm{Ext}^1(E|_S, E_n^k|_S)$ and $\mathrm{Ext}^1(E|_S, E_n^k|_S) \to \mathrm{Ext}^1(F_0|_S, E_n^k|_S)$; see, e.g., [Huybrechts and Lehn 2010, Section 2.2].

Therefore, the space $T_{[E|_S]} W_{\gamma_S(n,k)} \cap \mathrm{Ext}^1(E, E)$ is the kernel of the restriction of σ_S to $\mathrm{Ext}^1(E, E)$. But the latter factors through $\mathrm{Ext}^1(F_0, E)$, giving rise to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^1(E, E) & \longrightarrow & \mathrm{Ext}^1(E|_S, E|_S) & \xrightarrow{\sigma} & \mathrm{Ext}^1(E, E)^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma_S} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}^1(F_0, E_n^k) & \longrightarrow & \mathrm{Ext}^1(F_0|_S, E_n^k|_S) & \longrightarrow & \mathrm{Ext}^1(E_n^k, F_0)^* & \longrightarrow & 0 \end{array} \tag{33}$$

where the exactness of the rows is due to the vanishing of $\mathrm{ext}^2(E, E)$, $\mathrm{ext}^2(F_0, E_n^k)$ and $\mathrm{ext}^2(E_n^k, F_0)$.

Let us now look at the first column. The morphism σ is nothing but the composition of the natural linear maps $\mathrm{Ext}^1(E, E) \to \mathrm{Ext}^1(E, E_n^k)$ and $\mathrm{Ext}^1(F_0, E_n^k)$ induced by $E \xrightarrow{q} E_n^k \to 0$. Since $\mathrm{ext}^2(E, F_0) = \mathrm{ext}^2(E_n^k, E_n^k) = 0$, σ is surjective. As moreover

$$\mathrm{Hom}(F_0, E_n^k) = 0 \quad \text{and} \quad T_{[E]}\mathcal{T} \simeq \mathrm{Ext}^1(E, E)$$

by [Huybrechts and Lehn 2010, Proposition 2.2.7], we get the short exact sequence

$$0 \to T_{[E]} Z_{\gamma(n,k)} \to \mathrm{Ext}^1(E, E) \to \mathrm{Ext}^1(F_0, E_n^k) \to 0.$$

Now, from the diagram, we identify the linear space $T_{[E|_S]} W_{\gamma_S(n,k)} \cap \mathrm{Ext}^1(E, E)$ with $\ker(\sigma) \simeq T_{[E]} Z_{\gamma(n,k)}$. This vector space has dimension $\mathrm{ext}^1(E, E) - \mathrm{ext}^1(F_0, E_n^k)$.

But we know from (31) in the proof of Theorem 4.13 that the local dimension of $Z_{\gamma(n,k)}$ at $[E]$ is at least $\mathrm{ext}^1(E, E) - \mathrm{ext}^1(F_0, E_n^k) = \mathrm{ext}^1(E, E) - \chi(F_0, E_n^k)$. We conclude that $Z_{\gamma(n,k)}$ and $Z_{\gamma_S(n,k)}$ coincide at $[E]$, and so the same will hold in a neighbourhood of this point. A general deformation of E is thus slope-stable and the same will hold for its restriction to S. □

This proves Theorem 5.1. Finally, this argument applies to every $E \in \mathcal{R}_S(n, k)$. Indeed, from

$$0 \to E(K_X) \to E \to E|_S \to 0$$

and Serre duality, using $\mathrm{Ext}^2(E, E) = 0$, we obtain an exact sequence

$$0 \to \mathrm{Ext}^1(E, E) \to \mathrm{Ext}^1(E|_S, E|_S) \to \mathrm{Ext}^1(E, E)^* \to 0.$$

The image of $\mathrm{Ext}^1(E, E)$ is isotropic for the symplectic form on $\mathrm{Ext}^1(E|_S, E|_S)$ defined by the Yoneda pairing, since $\mathrm{Ext}^2(E, E) = 0$; hence it is maximal isotropic. Thus the image of the restriction map is Lagrangian. This proves Main Theorem 2.

6. Fano threefolds with curvilinear Kuznetsov component

We briefly treat Fano threefolds X whose Kuznetsov component $\mathrm{Ku}(X) \subset D^b(X)$ is equivalent to the derived category $D^b(\Gamma)$ of a curve Γ. In this case, we say that X has a curvilinear Kuznetsov component. These are the Del Pezzo threefold Y_4 of degree 4 and the prime Fano threefolds X_g of genus $g \in \{7, 9, 10\}$.

6.1. ***Reminders about Fano threefolds with curvilinear Kuznetsov component.*** We recall that the Kuznetsov component $\mathrm{Ku}(X)$ can be thought of as the "nontrivial" component of $\mathcal{O}_X(q_X)^\perp$, or equivalently of ${}^\perp\mathcal{O}_X(-q_X)$. If X is a threefold of one of the types mentioned at the beginning of this section, $\mathcal{O}_X(q_X)^\perp = \langle \mathrm{Ku}(X), \mathcal{F}\rangle$ where $\mathcal{F}$ is an exceptional bundle and $\mathrm{Ku}(X)$ is an indecomposable triangulated category. This situation is quite different from the case when $H^3(X) = 0$, where $\mathrm{Ku}(X)$ is strongly generated by a full exceptional collection. We will treat this case later on.

It turns out that the curve Γ is identified with a certain fine moduli space of vector bundles on X. This moduli space is fine and admits a universal bundle $\mathcal{E}$ on $X \times \Gamma$. So, we will implicitly identify points of Γ with vector bundles from that moduli space without further mention. The equivalence $D^b(\Gamma) \to \mathrm{Ku}(X)$ is induced by the Fourier–Mukai functor

$$\Phi(-) := Rp_*(q^*(-) \otimes \mathcal{E}),$$

having $\mathcal{E}$ as its kernel. The functor Φ is fully faithful and admits a left and a right adjoint Φ^*, $\Phi^!$ defined as

$$\begin{aligned} &\Phi^! : D^b(X) \to D^b(\Gamma), \quad \Phi^!(-) := Rq_*(p^*(-) \otimes \mathcal{E}^* \otimes \omega_\Gamma)[1]; \\ &\Phi^* : D^b(X) \to D^b(\Gamma), \quad \Phi^*(-) := Rq_*(p^*(-) \otimes \mathcal{E}^* \otimes \omega_X)[3]. \end{aligned} \tag{34}$$

The equivalence $\Phi : D^b(\Gamma) \to \mathrm{Ku}(X)$ lets us decompose $D^b(X)$ as

$$D^b(X) = \langle \mathcal{O}_X(-q_X), \mathcal{F}^*, \Phi(D^b(\Gamma))\rangle. \tag{35}$$

Recall that we are dealing with Fano threefolds which are either isomorphic to Y_4 or to X_g for $g \in \{7, 9, 10\}$. For each of these threefolds X, we will recall the main features of the curve Γ, of the sheaf $\mathcal{E}$ and of the derived category $D^b(X)$. We will then study the objects $\Phi^!(E(r_X))$ for E an (n, k)-instanton on X. We will essentially mimic what is done in the rank-2 cases, treated, e.g., in [Faenzi 2014; Brambilla and Faenzi 2013; 2014; Kuznetsov 2006; Kuznetsov et al. 2018; Kapustka and Ranestad 2013; Faenzi and Verra 2025]. We will adopt these as the main references throughout the current section.

6.1.1. *The Del Pezzo threefold of degree* 4. A Del Pezzo threefold Y_4 is the complete intersection of 2 quadrics in $\mathbb{P}^5$. In this case, the curve Γ is a smooth genus 2 curve,

a double cover of the line $\mathbb{P}^1$ corresponding to the pencil of quadrics defining X, ramified along 6 points given by the singular quadrics in the pencil. As it turns out, the curve Γ is isomorphic to the moduli space $\mathrm{M}_{Y_4}(v)$ of Gieseker-stable rank-2 sheaves having Chern character $v \in \oplus H^{i,i}(Y_4)$, $v = 2 + H_{Y_4} - \frac{1}{3} p_{Y_4}$. The universal bundle $\mathcal{E}$ of this moduli space has the topological invariants

$$c_1(\mathcal{E}) = H_{Y_4} + N, \quad c_2(\mathcal{E}) = 2l_{Y_4} + H_{Y_4} M + \eta,$$

where M and N are divisors on Γ of respective degrees m and $2m-1$. Here m is an arbitrary integer and η a class in $H^3(Y_4) \otimes H^1(\Gamma)$ such that $\eta^2 = 4$. In order to simplify the upcoming computations, we set $m = 2$. In addition to this, for all $y \in \Gamma$, $\mathcal{E}_y$ is a slope-stable ACM and globally generated bundle. The evaluation morphism gives rise to the short exact sequence

$$0 \to \mathcal{E}_{\bar{y}}(-1) \to H^0(\mathcal{E}_y) \otimes \mathcal{O}_{Y_4} \to \mathcal{E}_y \to 0, \tag{36}$$

where $\bar{y}$ is the conjugate point to y in the double cover $\Gamma \to \mathbb{P}^1$. The derived category $D^b(Y_4)$ of Y_4 admits the semiorthogonal decomposition

$$D^b(Y_4) = \langle \mathcal{O}_{Y_4}(-1), \mathcal{O}_{Y_4}, \Phi(D^b(\Gamma)) \rangle.$$

6.1.2. *Prime Fano threefolds of genus* 10. Let us consider a 7-dimensional complex vector space $V \simeq \mathbb{C}^7$ and a general 3-form $\omega \in \bigwedge^3 V^*$. The stabiliser of ω in $GL(V)$ is the 14-dimensional group $G_2 \subset \mathrm{SO}(V)$. Let us consider the Grassmannian $\mathrm{Gr}(2, V)$ and the tautological rank-2 bundle $\mathcal{U}$. The form ω defines a G_2-invariant global section s_ω of the rank-5 vector bundle $\mathcal{U}^\perp(1)$. The zero locus of s_ω is a smooth G_2-homogeneous 5-dimensional Fano manifold, namely the G_2 Grassmannian $G_2\,\mathrm{Gr}(2, V)$, which is the adjoint variety of type G_2. The vector space $\bigwedge^2 V$ decomposes as the direct sum of two irreducible G_2 representations $\bigwedge^2 V \simeq V \oplus W$ with W isomorphic to the Lie algebra $\mathfrak{g}_2$. The G_2 Grassmannian $G_2\,\mathrm{Gr}(2, V)$ is nondegenerate in $\mathbb{P}(W)$ and its projective dual $G_2\,\mathrm{Gr}(2, V)^\vee$ is a sextic hypersurface in $\mathbb{P}(W^*)$.

Consider now a general line $\mathbb{P}(A) \simeq \mathbb{P}^1 \subset \mathbb{P}(W^*)$. Note that the linear section $\mathbb{P}(A^\perp) \cap G_2\,\mathrm{Gr}(2, V)$ is a smooth prime Fano threefold $X_{10} \subset \mathbb{P}^{11}$ of genus 10. In the pencil $\mathbb{P}(A)$ there exist six singular hyperplane sections given by the points in $\mathbb{P}(A) \cap G_2\,\mathrm{Gr}(2, V)^\vee$. We consider the curve Γ of genus 2 obtained as a double cover $\Gamma \to \mathbb{P}(A)$, ramified along these six points.

The curve Γ identifies with the fine moduli space $\mathrm{M}_{X_{10}}(v)$ of slope-stable sheaves with Chern character $v = 3 + 2H_{X_{10}} + 9l_{X_{10}} + \frac{1}{2} p_{X_{10}}$. The universal sheaf $\mathcal{E}$ on $X \times \Gamma$ has rank 3 and Chern classes

$$c_1(\mathcal{E}) = 2H_{X_{10}} + K_C, \quad c_2(\mathcal{E}) = 27l_{X_{10}} + 3H_{X_{10}} p_C + \eta, \quad c_3(\mathcal{E}) = 7p_{X_{10}} + 22l_{X_{10}} p_C,$$

where η is a class in $H^3(X_{10}) \otimes H^1(\Gamma)$ such that $\eta^2 = 4$. For all $y \in \Gamma$, $\mathcal{E}_y$ is locally free, slope-stable, globally generated and fits in the exact sequence

$$0 \to \mathcal{E}_y(-1) \to \mathcal{O}_X^6 \to \mathcal{U}^{*3} \to \mathcal{E}_y \to 0 \tag{37}$$

for $\mathcal{U}$ the restriction to X_{10} of the rank-2 tautological bundle on $\mathrm{Gr}(2, V)$; see [Kapustka and Ranestad 2013; Faenzi and Verra 2025]. We recall that $c_1(\mathcal{U}) = -1$. The derived category $D^b(X_{10})$ decomposes as

$$D^b(X_{10}) = \langle \mathcal{O}_{X_{10}}, \mathcal{U}^*, \Phi(D^b(\Gamma)) \rangle.$$

6.1.3. *Prime Fano threefolds of genus* 9. Let us consider a 6-dimensional complex vector space V endowed with a nondegenerate skew-symmetric form ω. The maximal ω-isotropic linear subspaces of V are parametrised by the Lagrangian Grassmannian $\mathrm{LG}(3, 6) \subset \mathbb{P}^{13}$, a smooth 6-dimensional projective variety. For a general $\mathbb{P}(A) \subset \mathbb{P}^{13*}$, $\mathbb{P}(A) \simeq \mathbb{P}^2$, the linear section $\mathrm{LG}(3, 6) \cap \mathbb{P}(A^\perp)$ is a smooth prime Fano threefold X_9 of genus 9, nondegenerate in $\mathbb{P}^{10}$.

The projective dual $\mathrm{LG}(3, 6)^\vee$ to $\mathrm{LG}(3, 6)$ is a quartic hypersurface in $\mathbb{P}^{13*}$ whose linear section $\mathrm{LG}(3, 6)^\vee \cap \mathbb{P}(A)$ is a smooth (by generality of $\mathbb{P}(A)$) plane curve Γ of degree 4, defined as the homological projective dual to X_9.

The curve Γ identifies with the fine moduli space $\mathrm{M}_{X_9}(v) \simeq \mathrm{M}_{X_9}(\gamma(1, 0))$ of slope-stable sheaves with Chern character $v = 2 + H_{X_9} + 2l_{X_9} - \frac{1}{3}p_{X_9}$. The isomorphism $\mathrm{M}_{X_9}(v) \simeq \mathrm{M}_{X_9}(\gamma(1, 0))$ is given by $\mathcal{E}_y \mapsto \mathcal{E}_y^* \simeq \mathcal{E}_y(-1)$. We have the invariants

$$c_1(\mathcal{E}) = H_{X_9} + N, \quad c_2(\mathcal{E}) = 6L_{X_9} + H_{X_9}M + \eta.$$

Here, M and N are divisors on Γ of degrees m and $2m-1$ and $\eta \in H^3(X_9) \otimes H^1(\Gamma)$ satisfies $\eta^2 = 6$. The integer m can be chosen arbitrarily; however, we fix $m = 3$, which will make some computations easier. For each point $y \in \Gamma$, $\mathcal{E}_y$ is a slope-stable ACM bundle on X_9 fitting in

$$0 \to \mathcal{O}_{X_9} \to \mathcal{U}^* \to \mathcal{E}_y \to \mathcal{O}_C \to 0, \tag{38}$$

where $C \subset X_9$ is a conic and $\mathcal{U}$ is the restriction to X_9 of the rank-3 tautological bundle on $\mathrm{LG}(3, 6)$. We recall that $\mathcal{U}$ is a rank-3 vector bundle of first Chern class $c_1(\mathcal{U}) = -1$. We decompose $D^b(X_9)$ as

$$D^b(X_9) = \langle \mathcal{O}_{X_9}, \mathcal{U}^*, \Phi(D^b(\Gamma)) \rangle.$$

6.1.4. *Prime Fano threefolds of genus* 7. Consider the orthogonal Grassmannian $\mathrm{OG}(5, 10)$ parametrising 4-dimensional linear subspaces contained in a smooth 8-dimensional quadric hypersurface Q in $\mathbb{P}^9$. The variety $\mathrm{OG}(5, 10)$ is smooth, of dimension 10 and has 2 connected components $\Sigma_\pm$. Each component spans a 15-dimensional linear space $\langle \Sigma_\pm \rangle \simeq \mathbb{P}(V_\pm) \simeq \mathbb{P}^{15}$. We denote by $\mathcal{U}_\pm$ the restriction

to $\Sigma_\pm$ of the tautological rank-5 bundle on OG(5, 10). For a sufficiently general linear subspace $\mathbb{P}(A) \simeq \mathbb{P}^8$ of $\mathbb{P}(V_+)$, the resulting linear section $\mathbb{P}(A) \cap \Sigma_+$ is a smooth prime Fano threefold X_7 of genus 7.

Let us consider the orthogonal $\mathbb{P}(A^\perp) \simeq \mathbb{P}^6 \subset \mathbb{P}(V_-)$. By the generality assumption on A, $\mathbb{P}(A^\perp) \cap \Sigma_- \subset \mathbb{P}(V_-)$ is a smooth curve Γ. This curve has genus 7 and is referred to as the homological projective dual to X_7. The curve Γ is the moduli space of Gieseker-stable bundles with Chern character $v \in \bigoplus_i H^{i,i}(X_7)$, $v = 2 + H_{X_7} + l_{X_7} - \frac{1}{2} p_{X_7}$. So $\Gamma \simeq \mathrm{M}_{X_7}(v) \simeq \mathrm{M}_{X_7}(\gamma(1,0))$. As in the case $g = 9$, all bundles in this moduli space are duals of minimal rank-2 instantons. The universal bundle $\mathcal{E}$ on $X \times \Gamma$ has Chern classes

$$c_1(\mathcal{E}) = H_{X_7} + H_\Gamma, \quad c_2(\mathcal{E}) = \tfrac{7}{12} H_{X_7} H_\Gamma + 5 l_{X_7} + \eta$$

for $\eta \in H^3(X_7) \otimes H^1(\Gamma)$ a class such that $\eta^2 = 14$. Moreover it fits in

$$0 \to \mathcal{E}^* \to \mathcal{U}_- \to \mathcal{U}_+^* \to \mathcal{E} \to 0 \tag{39}$$

for $\mathcal{U}_\pm$ the pullback of $\mathcal{U}_\pm$ to $X \times \Gamma \subset \Sigma_+ \times \Sigma_-$. The restriction of $\mathcal{U}_+$ to X will still be denoted by $\mathcal{U}_+$. This is a rank-5 vector bundle on X having $c_1(\mathcal{U}_+) = -2$. As already recalled in Section 4.2, for all $y \in \Gamma$, $\mathcal{E}_y$ is a locally free slope-stable and globally generated bundle. The derived category $D^b(X_7)$ decomposes as

$$D^b(X_7) = \langle \mathcal{O}_{X_7}, \mathcal{U}_+^*, \Phi(D^b(\Gamma)) \rangle.$$

6.2. ***The canonical resolution of instantons.*** For a slope-stable (n, k)-instanton E on X, we look at the object $\Phi^!(E(r_X))$ in $D^b(\Gamma)$. Let us denote by $\mathcal{F}$ the exceptional object on X such that ${}^\perp\mathcal{O}_X(-q_X) = \langle \mathcal{F}^*, \Phi(D^b(\Gamma)) \rangle$. We recall that $\mathcal{F} \simeq \mathcal{O}_X$ for $X = Y_4$, $\mathcal{F} \simeq \mathcal{U}$ for X_g, $g \in \{9, 10\}$ and $\mathcal{F} \simeq \mathcal{U}_+$ for X_7. In the following lemma, X is of type Y_4 or X_g with $g \in \{7, 9, 10\}$.

Lemma 6.1. *Let E be a slope-stable (n, k)-instanton on X with $k \geq 1$.*

(i) *We have* $\mathrm{ext}^i(\mathcal{E}_y, E(r_X)) = 0$ *for all* $i \neq 1$ *and all* $y \in \Gamma$.

(ii) *If $X = Y_4$ or $X = X_{10}$, then* $\hom(\mathcal{F}^*, E(r_X)) = 0$.

Proof. The proof is essentially equivalent to the proofs of [Brambilla and Faenzi 2013, Lemmas 4.3–4.4; 2014, Lemmas 5.3–5.4]. For (i), let $y \in \Gamma$ be an arbitrary point. The vanishing of $\mathrm{ext}^i(\mathcal{E}_y, E(r_X))$ for $i = 0, 3$ is an immediate consequence of stability. The vanishing of $\mathrm{ext}^2(\mathcal{E}_y, E(1))$ for X_7 and X_9 were shown in Lemma 4.11. Let us prove that $\mathrm{ext}^2(\mathcal{E}_y, E(r_X)) = h^2(E(r_X) \otimes \mathcal{E}_y^*) = 0$ also on Y_4 and X_{10}. For Y_4 it suffices to tensor (36) by $E(-1)$. As $\mathcal{E}_y^* \simeq \mathcal{E}_y(-1)$ we then use that $H^2(E(-1)) = 0$ and $H^3(E \otimes \mathcal{E}_y(-2)) = 0$, again by stability. For X_{10} let us consider the image $\mathcal{G}$ of the morphism $\mathcal{O}_{X_{10}}^6 \to \mathcal{U}^{*3}$ appearing in the complex (37). As $H^2(E) = 0$, $\mathrm{Ext}^2(\mathcal{E}_y(-1), E)$ injects into $\mathrm{Ext}^3(\mathcal{G}, E) \simeq \mathrm{Hom}(E, \mathcal{G}(-1))^*$. But $\mathrm{Hom}(E, \mathcal{G}(-1))$

injects into $\mathrm{Hom}(E, \mathcal{U}^3) \simeq \mathrm{Hom}(E, \mathcal{U}^*(-1)^3)$ which vanishes by slope-stability, since we are assuming $k \geq 1$.

Also (ii) is a direct consequence of the slope-stability assumption. Indeed, for $X = Y_4$ or $X = X_{10}$, $\mathcal{F}$ is a minimal instanton, in particular $\mathcal{F}^*$ and $E(r_X)$ are slope-stable of the same slope, but $E(r_X)$ is not isomorphic to $\mathcal{F}^*$ since $k \geq 1$. □

For the cases $X = X_g$, $g \in \{7, 9\}$, we will show that the vanishing $\mathrm{Hom}(\mathcal{F}^*, E(1))$ holds at least on an open nonempty subscheme $\mathcal{V}(n, k) \subset \mathrm{M}_X(\gamma(n, k))$. To prove this we need a preliminary lemma.

Lemma 6.2. *Let E be a slope-stable (n, k)-instanton bundle on X_g, $g \in \{7, 9\}$, for $n \geq 2$. Then a nonzero morphism $\mathcal{F}^* \to E(1)$ is injective and its cokernel is torsion-free. The same holds for any nonzero morphism $\mathcal{F}^*|_S \to E(1)|_S$ for $S \in |\mathcal{O}_X(1)|$ general and E a general slope-stable instanton bundle belonging to $\mathcal{R}_S(n, k) \subset \mathrm{MI}_X(n, k)$.*

Proof. We first prove that any nonzero morphism $\mathcal{F}^* \to E(1)$ is injective. Let us look at $X = X_9$. We have $\mathcal{F}^* \simeq \mathcal{U}^*$ is slope-stable of rank 3 and $c_1 = 1$. Since we are assuming that E is slope-stable and that $\mathrm{rk}(E) \geq 4$, we see that for a nonzero morphism $f : \mathcal{U}^* \to E(1)$, $\mathrm{im}(f)$ must have rank 3. Since $\mathcal{U}^*$ is torsion-free we conclude that $\mathrm{im}(f) \simeq \mathcal{U}^*$.

Let us pass to the case $g = 7$. Recall that $\mathcal{F}^* \simeq \mathcal{U}_+^*$ is slope-stable with slope $\frac{2}{5}$. Consider a nonzero map $f : \mathcal{F}^* \to E(1)$. Suppose, by contradiction, that f is not injective. By stability of $\mathcal{F}$ and E, we see that $\mathrm{im}(f)$ must have slope $\frac{1}{2}$, so, since $n \geq 2$ and E is slope-stable, we must have $\mathrm{rk}(\mathrm{im}(f)) = 4$. If $n \geq 3$, this contradicts stability of E, so we look at the case $n = 2$. Note that $\ker(f)$ is reflexive of rank one and slope 0, so $\ker(f) \simeq \mathcal{O}_X$, so that $\mathrm{ch}(\mathrm{im}(f)) = \mathrm{ch}(\mathcal{F}^*) - 1$. Therefore, $\mathrm{coker}(f)$ is a torsion sheaf with $c_1(\mathrm{coker}(f)) = 0$ and $c_2(\mathrm{coker}(f)) = (k-2)l$, which is impossible if $k \geq 3$. For $k = 2$, we get that $\mathrm{coker}(f)$ is a torsion sheaf with $c_1(\mathrm{coker}(f)) = c_2(\mathrm{coker}(f)) = 0$ and $c_3(\mathrm{coker}(f)) = -2p$, where p is a point of X, which is again impossible. This ends the proof of the injectivity of nonzero morphisms $\mathcal{F}^* \to E(1)$.

Given a morphism $f : \mathcal{F}^* \hookrightarrow E(1)$, we denote by Q its cokernel. Let us check that Q is torsion-free. If not, then f factors through an injective morphism $f' : G \to E(1)$ whose cokernel is the torsion-free part of Q. This means that G is an extension of $T(Q)$, the torsion part of Q by $\mathcal{F}^*$. Due to the stability of $E(1)$, $\dim(T(Q)) \leq 1$. But then we have $\mathrm{ext}^1(T(Q), \mathcal{F}^*) = \mathrm{ext}^2(\mathcal{F}^*, T(Q)(-1)) = h^2(\mathcal{F} \otimes T(Q)(-1)) = 0$. Therefore we would have that G has torsion which is impossible since it injects into $E(1)$.

Let us consider now a smooth hyperplane section $S \in |\mathcal{O}_X(1)|$ such that $\mathrm{Pic}(S) \simeq \mathbb{Z}$ and assume that $E \in \mathcal{R}_s(n, k)$, so that $E|_S$ is slope-stable. Then the same arguments presented for X allow us to deal with the cases $S \subset X_9$ and $S \subset X_7$, except when

$(n, k) = (2, 2)$. In this case, a noninjective morphism $\mathcal{U}_+^*|_S \to E(1)|_S$ would lead to the short exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{U}_+^*|_S \to E(1)|_S \to 0.$$

But by the generality assumption on E, we can assume that E is ACM; see Corollary 4.15. This implies that the graded module $H^1_*(E|_S) := \bigoplus H^1(E_S(m))$ is zero, which would lead to $\mathcal{U}_+^*|_S \simeq \mathcal{O}_S \oplus E|_S(1)$ which is impossible. The proof of the torsion-freeness of the cokernel of a morphism $\mathcal{F}^*|_S \to E|_S(1)$ is carried out analogously to the 3-dimensional case. □

Proposition 6.3. *Let $X = X_g$, $g \in \{7, 9\}$. Then, for all $n \geq 1$, $k \geq 2$ there exists an open nonempty subscheme $\mathcal{V}(n, k) \subset \mathrm{M}_X(\gamma(n, k))$ parametrising slope-stable (n, k)-instantons E on X such that* $\mathrm{Hom}(\mathcal{F}^*, E(1)) = 0$ *and for $S \in |\mathcal{O}_X(1)|$ general,* $\mathrm{Hom}(\mathcal{F}^*|_S, E(1)|_S) = 0$.

Proof. We work by induction on n. For the rest of the proof S will be a smooth hyperplane section of X with Picard rank one. For $n = 1$, let E be a $(1, k)$ slope-stable instanton bundle with $k \geq 2$. Let us suppose at first that $X = X_9$. Then $\hom(\mathcal{U}^*, E(1)) = 0$ by [Brambilla and Faenzi 2013, Lemma 4.4]. An argument similar to the one presented in [loc. cit.] also allows us to prove that, assuming E is general, we can additionally suppose that $\hom(\mathcal{U}^*|_S, E(1)|_S) = 0$. Indeed, by the assumptions on S, $E|_S$ is slope-stable due to the vanishing of $H^0(E|_S)$.

The kernel of a nonzero morphism $f : \mathcal{U}^*|_S \to E(1)|_S$ must be a rank-1 reflexive sheaf, so that $\ker(f) \simeq \mathcal{O}_S$, as $\mathrm{Pic}(S) \simeq \mathbb{Z}H_S$. Therefore $\mathrm{ch}(\mathrm{im}(f)) = (2, 1, 0)$ and $\mathrm{ch}(\mathrm{coker}(f)) = (0, 0, 2-k)$. Now, $\mathrm{coker}(f)$ is a torsion sheaf whose length equals $\mathrm{ch}_2(\mathrm{coker}(f))$ so that $k \leq 2$. The case $k = 2$ can be excluded as well. Indeed, let E be a $(1, 2)$-instanton bundle on X. Since we are assuming E general, we might suppose that E is ACM. Assume now, by contradiction, that $E(1)|_S$ fits in

$$0 \to \mathcal{O}_S \to \mathcal{U}^*|_S \to E(1)|_S \to 0.$$

Since $h^2(\mathcal{U}^*|_S) = h^0(\mathcal{U}|_S) = 0$ and $h^2(\mathcal{O}_S) = h^0(\mathcal{O}_S) = 1$, we would get

$$h^1(E(1)|_S) \neq 0.$$

Indeed, otherwise, since the sequence cannot split, we would have

$$\mathrm{ext}^1(E(1)|_S, \mathcal{O}_S) = h^1(E|_S) \neq 0,$$

which is impossible since if E is ACM, $H^1_*(E|_S) := \bigoplus H^1(E|_S(m)) = 0$.

For $X = X_7$ the vanishing of $\mathrm{Hom}(\mathcal{U}_+^*, E(1))$ for a $(1, k)$-instanton, for $k \geq 2$ is shown in [Brambilla and Faenzi 2013, Lemma 5.3]. For an arbitrary point $y \in \Gamma$, we have the short exact sequence, of bundles on S,

$$0 \to \mathcal{G}_y|_S \to \mathcal{U}_+^*|_S \to \mathcal{E}_y|_S \to 0. \tag{40}$$

The vector bundle $\mathcal{U}_+^*|_S$ is slope-stable, see, e.g., [Mukai 1988, Theorem 3.3], and the same holds for $E|_S$, which is still ensured by $H^0(E|_S)=0$ and $\mathcal{G}_y|_S$. For the stability of the latter it is enough to notice that $H^0(\mathcal{G}_y|_S(-1)) \hookrightarrow H^0(\mathcal{U}_+^*|_S(-1))=0$, the vanishing being ensured by the slope-stability of $\mathcal{U}_+^*|_S$. This gives that $\mathcal{G}_y$ cannot be destabilised by rank-1 subsheaves, $H^0(\mathcal{G}_y^*|_S)=0$ since $H^0(\mathcal{U}_+|_S)=0$ (again by stability) and $H^1(\mathcal{E}_y^*|_S)=H^1(\mathcal{E}_y(-1))=0$ since $\mathcal{E}_y$ is ACM. This, as recalled, implies $H^1_*(\mathcal{E}_y|_S)=0$. This means that $\mathcal{G}_y|_S$ cannot be destabilised by rank-2 subsheaves either. Hence $\mathcal{G}_y|_S$ is a slope-stable rank-3 bundle of degree 1 and the same arguments used for $\mathcal{U}^*$ on X_9 allow us to conclude that $\mathrm{Hom}(\mathcal{G}_y|_S, E(1)|_S)=0$. Indeed, again we would have that the kernel of a nonzero map $\mathcal{G}_y|_S \to E(1)|_S$ must be $\mathcal{O}_S$ and hence a cokernel with Chern character $(0,0,2-k)$, which cannot occur. In conclusion, $\mathrm{Hom}(\mathcal{U}_+^*|_S, E(1)|_S)$ vanishes as well.

Let us now pass to the inductive step. We consider $n>1$, $k\geq 2$ and we take a $(n+1,k)$-instanton E obtained as a nontrivial extension $e\in \mathrm{Ext}^1(E_n^k, F_0)$ for a slope-stable (n,k)-instanton bundle E_n^k belonging to $\mathcal{R}_S(n,k)$ and satisfying the proposition. By the assumptions on E_n^k the restriction $e|_S$ does not split and $E_n^k|_S$ is slope-stable. Accordingly, by Lemma 2.13, $E|_S$ is strictly slope-semistable and $F_0|_S$ is its only slope-semistable subbundle of slope $-\frac{1}{2}$. All subsheaves of $E|_S$ having slope $-\frac{1}{2}$ will indeed be elementary transformation of $F_0|_S$ along a zero-dimensional scheme.

Let us consider now an étale neighbourhood $\mathcal{T}$ of $[E]$ in $\mathrm{Spl}_X(\gamma(n+1,k))$ admitting a universal bundle $\mathcal{E}\in \mathrm{Coh}(\mathcal{T}\times X)$. Denote by $\mathcal{E}|_S$ the pullback of $\mathcal{E}$ to $\mathcal{T}\times S$. Let $P:=P_E-P_{\mathcal{U}^*(-1)}$, $P_S:=P_{E|_S}-P_{\mathcal{U}^*(-1)|_S}$ and denote by $Z_{P_S}\subset\mathcal{T}$ (resp. $Z_P\subset\mathcal{T}$) the closed subscheme parametrising points $x\in\mathcal{T}$ such that $(\mathcal{E}|_S)_x$ admits a pure 2-dimensional quotient with Hilbert polynomial P_S (resp. $\mathcal{E}_x$ admits a torsion-free quotient with Hilbert polynomial P). We claim that $[E]\notin Z_{P_S}$. Suppose, by contradiction, that $E|_S$ fits in the short exact sequence of $\mathcal{O}_S$-modules

$$0\to K\to E|_S\to Q\to 0,$$

with Q a torsion-free $\mathcal{O}_S$ module with Hilbert polynomial P_S. Notice that the torsion-freeness of Q ensures that K is locally free; see, e.g., [Hartshorne 1980, Proposition 1.1].

We claim that K is slope-stable. Notice that, due to the slope-semistability of $E|_S$, the only possible destabilising subsheaves L of K must have slope $-\frac{1}{2}$, as for the odd rank cases we must have $\mu(L)<-\frac{m}{2m+1}$ with $m=0$ for X_9 and $m=0,1$ for X_7. But then $L\hookrightarrow E|_S$ factors through $F_0|_S$ due to the slope-stability of both $F_0|_S$ and $E_n^k|_S$ and Lemma 2.13. So $\mathrm{rk}(L)=2$ and $T:=F_0|_S/L$ is zero-dimensional. Notice that the latter is nothing but the torsion part of $E|_S/L$ — here the dimension count is due to the slope-semistability of $E|_S$. But this would mean that $L^{**}\simeq F_0|_S$ and since K is a vector bundle, this would imply that also the inclusion $L\hookrightarrow K$ factors through $F_0|_S$.

Thus, to prove that K is slope-stable, it suffices to check that there is no inclusion $F_0|_S \hookrightarrow K$. Note that such an inclusion would lead to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_0|_S & \longrightarrow & K & \longrightarrow & I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_0|_S & \longrightarrow & E|_S & \longrightarrow & E_n^k|_S & \longrightarrow & 0 \end{array} \tag{41}$$

This would provide the short exact sequence

$$0 \to I \to E_n^k|_S \to Q \to 0.$$

Note that, since Q is torsion-free, I must be locally free.

Let us show that, when $X \simeq X_7$ or $X \simeq X_9$, this cannot happen.

• We first treat the case X_9. We get that I must be a line bundle. But this is not possible since $\mathrm{ch}(I) = (1, -1, 6)$.

• Now let us focus on the case X_7. We have $\mathrm{ch}(I) = (3, -2, 5)$, so that $n \geq 2$, $\mathrm{rk}(E_n^k|_S) \geq 4$. We claim that we can assume that $E_n^k|_S$ admits no injective morphisms from vector bundles with Chern character $\mathrm{ch}(I)$ (which is the same as requiring that $E_n^k|_S$ has no torsion-free quotient with Chern character $\mathrm{ch}(E_n^k|_S/I)$). This will be enough to prove that also on $S \subset X_7$ we have slope-stability of K.

We will prove the claim by induction on n. For $n = 1$ this is obvious. Supposing now that the claim holds for a general slope-stable instanton bundle E_n^k, $k \geq 2$, such that $E_n^k|_S$ is slope-stable, we show that it holds for any E arising from $\mathrm{Ext}^1(E_n^k, F_0)$. Suppose, by contradiction, that $E|_S$ admits a subbundle I with $\mathrm{ch}(I) = (3, -2, 5)$. Then using the same arguments previously presented, I must be slope-stable.

Indeed, if I is not slope-stable, it can only be destabilised by a rank-2 sheaf F with $c_1(-1)$. But then we have an injection $F \hookrightarrow F_0|_S$ and an isomorphism $F^{**} \simeq F_0|_S$, which gives an injection $F_0|_S \hookrightarrow I$. We obtain a rank-1 subsheaf L of $E_n^k|_S$ with Chern character $\mathrm{ch}(L) = (1, -1, 4)$. But since I is locally free, $E|_S/I \simeq E_{n+1}^k|_S/L$ is torsion-free so that L must be a line bundle, which is impossible due to its Chern character. But if I is a slope-stable bundle, admitting no injective morphism into $E_n^k|_S$ (which is true by the inductive hypothesis) is equivalent to $\mathrm{Hom}(I, E_{n+1}^k|_S) = 0$, as it can be shown with arguments equivalent to the ones presented in Lemma 6.2. But then $I \hookrightarrow E|_S$ should factor through $F_0|_S$, which is impossible. A general deformation E' of E will satisfy that $E'|_S$ is slope-stable and admits no injective morphisms from vector bundles with Chern character $\mathrm{ch}(I)$.

Summing up, K is a slope-stable vector bundle. But as $\chi(K, K) = 2$, we must have $K \simeq \mathcal{F}^*|_S$; see, e.g., [Mukai 1987]. By the inductive hypothesis $\mathrm{Hom}(\mathcal{F}^*|_S, E_n^k|_S) = 0$ so that $\mathcal{F}^*|_S \simeq K \hookrightarrow E|_S$ would factor through $F_0|_S$. But

this cannot happen. Therefore we can conclude that $[E] \notin Z_{P_S}$, hence $[E] \notin Z_P$. So, for a general point $x \in \mathcal{T}$ we have $x \notin Z_{P_S}$ and $x \notin Z_P$. Assuming that moreover $\mathcal{E}_x$ is slope-stable and that the same holds for its restriction to S, we conclude that, from Lemma 6.2, $\mathrm{Hom}(\mathcal{F}^*, E') = \mathrm{Hom}(\mathcal{F}^*|_S, E'|_S) = 0$. □

Applying the above results we can construct canonical resolutions for instantons.

Theorem 6.4. *Let E be a slope-stable (n,k)-instanton on X. Set $F := \Phi^!(E(r_X))$.*

(i) *F is a vector bundle on Γ of rank r and degree d, where (r,d) takes the values*

(42)

X	r	d
Y_4	k	0
X_{10}	k	k
X_9	k	$n+k$
X_7	k	$5k+n$

(ii) *The sheaf $E(r_X)$ fits in the short exact sequence*

$$0 \to \mathrm{ext}^{1+r_X}(E(r_X), \mathcal{F})^* \otimes \mathcal{F}^* \to \Phi(F) \to E(r_X) \to 0.$$

(iii) *If $X = Y_4$ or $X = X_{10}$ and $k \geq 1$, F is simple. The same holds on X_g, $g \in \{7,9\}$, with the additional assumption that $E \in \mathcal{V}(n,k)$, $k \geq 2$.*

Proof. Item (i) is an immediate consequence of Lemma 6.1. Since we have $h^i(E \otimes \mathcal{E}_y^*) = 0$ for all $i \neq 1$ and all $y \in \Gamma$, $F := \Phi^!(E(r_X))$ is a sheaf. Moreover, as $h^1(E(r_X) \otimes \mathcal{E}_y^*) = -\chi(\mathcal{E}_y, E(r_X))$, we conclude that F is locally free of rank $-\chi(\mathcal{E}_y, E(r_X))$. The computations of the ranks and degrees appearing in the table (42) are a direct application of Riemann–Roch and Grothendieck–Riemann–Roch theorem.

Let us now prove (ii). We first deal with the case where $X \simeq Y_4$ so that $i_X = 2$ and $r_X = 0$. For an (n,k)-instanton on Y_4, we consider the right mutation $\widetilde{E} := \mathbb{R}_{\mathcal{O}_X}(E)$, defined as the kernel of the evaluation map $E \to \bigoplus_i \mathrm{Ext}^i(E, \mathcal{O}_X) \otimes \mathcal{O}_X[i]$. By construction $\widetilde{E} \in {}^{\perp}\langle \mathcal{O}_X(-1), \mathcal{O}_X \rangle \simeq \Phi(D^b(\Gamma))$ and as moreover $\mathrm{Ext}^i(E, \mathcal{O}_X)$ is concentrated in degree 1, we get that $\widetilde{E}$ is a sheaf, actually it is a (k,k)-instanton, fitting in

$$0 \to \mathcal{O}_X^{k-n} \to \widetilde{E} \to E \to 0. \tag{43}$$

Let us now see how to obtain a similar complex when $i_X = 1$, so that $q_X = 0$ and $r_X = 1$. Given an (n,k)-instanton E on X, we consider the right mutation $\widetilde{E}(1) := \mathbb{R}_{\langle \mathcal{O}_X, \mathcal{F}^* \rangle}(E(1))$ of $E(1)$ by the exceptional pair $\langle \mathcal{O}_X, \mathcal{F}^* \rangle$. By definition this is the kernel of the evaluation morphism

$$E(1) \to \bigoplus_i \mathrm{Ext}^i(E(1), \mathcal{O}_X)^* \otimes \mathcal{O}_X[i] \oplus \mathrm{Ext}^{i-1}(E(1), \mathcal{F})^* \otimes \mathcal{F}^*[i-1].$$

We note that $\mathrm{Ext}^i(E,\mathcal{F})=0$ except for $i=2$. Indeed, $\mathrm{ext}^i(E(1),\mathcal{F})=0$ for $i=0,3$ by stability of $\mathcal{F}$ and E and since $k\geq 1$. The stability assumption also leads to $\mathrm{ext}^1(E(1),\mathcal{F})=0$. To verify this recall that, for $g=7,9,10$, we have the tautological exact sequence

$$0\to\mathcal{F}\to\mathcal{O}_X\otimes V\to\mathcal{Q}\to 0,$$

where $\mathcal{Q}$ is the slope-stable exceptional bundle obtained by restricting to X the quotient bundle on the Grassmannian containing X in view of the embedding provided by $\mathcal{F}^*$. Also, $\mathcal{Q}\simeq\mathcal{F}^*$ if $g=7,9$. Then, we notice the equalities $\mathrm{ext}^1(E(1),\mathcal{O}_X)=h^2(E)=0$ and $\hom(E(1),\mathcal{Q})=0$ by stability. So applying $\mathrm{Hom}(E(1),\cdot)$ we get $\mathrm{ext}^1(E(1),\mathcal{F})=0$. Therefore we get that $\widetilde{E}(1)$ is a coherent sheaf which, by construction, belongs to $\Phi(D^b(\Gamma))={}^{\perp}\langle\mathcal{O}_X,\mathcal{F}^*\rangle$ and fits in

$$0\to\mathcal{F}^*\otimes\mathrm{Ext}^2(E(1),\mathcal{F})^*\to\widetilde{E}(1)\to E(1)\to 0. \tag{44}$$

So the sheaf $\widetilde{E}(r_X)$ lies in $\Phi(D^b(\Gamma))$. Accordingly, $\widetilde{E}(r_X)\simeq\Phi(G)$ for some $G\in D^b(\Gamma)$. Thus, applying $\Phi^!$ to the sequences (43), (44) and using that $\Phi^!(\mathcal{F}^*)=0$,

$$G\simeq\Phi^!(\Phi(G))\simeq\Phi^!(\widetilde{E}(r_X))=F.$$

This shows that the complexes (43) and (44) are of the form described in (ii). For the proof of (iii), notice that whenever E is an (n,k)-instanton with $k\geq 1$ on Y_4 or X_{10}, or an (n,k)-instanton on X_g, $g\in\{7,9\}$, belonging to $\mathcal{V}(n,k)$, $k\geq 2$, we have $\mathrm{Hom}(\mathcal{F}^*,E(r_X))=0$.

By adjunction

$$\mathrm{Hom}(F,F)\simeq\mathrm{Hom}(F,\Phi^!(E(r_X)))\simeq\mathrm{Hom}(\Phi(F),E(r_X))\simeq\mathrm{Hom}(\widetilde{E},E).$$

Applying $\mathrm{Hom}(\cdot,E(r_X))$ to (43) and (44) we see that this last vector space is one-dimensional since $E(r_X)$ is simple and $\mathrm{Hom}(\mathcal{F}^*,E(r_X))=0$. □

7. Monadic description of instantons

In this final section we look at Fano threefolds X of Picard number one and trivial intermediate Jacobian, i.e., $H^3(X,\mathbb{Z})=0$. For each value of the index $i_X\in\{1,\dots,4\}$ there exists a unique family of Fano threefolds satisfying the condition $H^3(X,\mathbb{Z})=0$. These are:

(♡) For $i_X=4$, $X\simeq\mathbb{P}^3$.

(♣) For $i_X=3$, $X\simeq Q\subset\mathbb{P}^4$.

(◇) For $i_X=2$, $X=Y_5$ is the degree-5 Del Pezzo threefold. This is the projective variety obtained as a general codimension-3 linear section of $\mathbb{G}(2,U_5)\subset\mathbb{P}^9$ where $U_5\simeq\mathbb{C}^5$.

(♠) For $i_X = 1$, $X = X_{12}$ is the prime Fano threefold of genus 12. The threefold X identifies with a closed subvariety of the Hilbert scheme $\mathrm{Hilb}_{3t+1}(\mathbb{P}(U_4))$ of twisted cubic curves in $\mathbb{P}(U_4) \simeq \mathbb{P}^3$, where U_4 is a 4-dimensional vector space. The subvariety X corresponds to twisted cubics whose defining ideal is annihilated by a fixed 2-dimensional net of quadrics $\mathbb{P}^2 \simeq \mathbb{P}(U_3) \subset \mathbb{P}(S^2U_4)$.

The triviality of the intermediate Jacobian of X turns out to be equivalent to the fact that the derived category $D^b(X)$ of X admits a full exceptional collection. We use, in addition to Beilinson's theorem for $\mathbb{P}^3$ and Kapranov's theorem for the quadric, the full exceptional collections described in [Orlov 1991; Kuznetsov 1996; Faenzi 2005; Faenzi 2007]. In summary,

$$D^b(X) = \langle \mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1, \mathcal{O}_X(q_X) \rangle$$

for vector bundles $\mathcal{F}_i$, $i \in \{-1, 0, 1\}$, such that

$$\mathcal{F}_i^* \simeq \mathcal{F}_{-i}(r_X), \quad \langle \mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1 \rangle = \mathcal{O}_X(q_X)^{\perp}. \tag{45}$$

This structure of the triangulated category $\mathcal{O}_X(q_X)^{\perp}$ ensures that any object $F \in \mathcal{O}_X(q_X)^{\perp}$ is quasi-isomorphic to the homology of a complex whose terms are all of the form $\mathcal{F}_i^{\oplus n_{i,j}}$, for some integers $n_{i,j}$, with $i \in \{-1, 0, 1\}$ and $j \in \mathbb{Z}$.

To compute these integers, we use the dual exceptional collection

$$\langle \mathcal{G}_{-1}, \mathcal{G}_0, \mathcal{G}_1 \rangle = {}^{\perp}\mathcal{O}_X(-q_X)$$

with respect to $\langle \mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1 \rangle = \mathcal{O}_X(q_X)^{\perp}$. The objects $\mathcal{G}_i$, with $i \in \{-1, 0, 1\}$, are exceptional vector bundles satisfying, for $i \in \{-1, 0, 1\}$,

$$\mathcal{G}_{-i}^*(-q_X) \simeq \mathrm{L}_{\mathcal{O}_X}(\mathcal{G}_i(q_X))[-1], \tag{46}$$

where L is the left mutation functor. The mutation functor is characterised by the property that, for any pair of objects $\mathcal{F}$ and $\mathcal{G}$, there is a distinguished triangle

$$\mathrm{L}_{\mathcal{F}}(\mathcal{G})[-1] \to \mathrm{Ext}^{\bullet}_X(\mathcal{F}, \mathcal{G}) \otimes \mathcal{F} \to \mathcal{G} \to \mathrm{L}_{\mathcal{F}}(\mathcal{G}),$$

where the map $\mathrm{Ext}^{\bullet}_X(\mathcal{F}, \mathcal{G}) \otimes \mathcal{F} \to \mathcal{G}$ is the natural evaluation.

Given a coherent sheaf F on X, we write down the cohomology table of F with respect to the collection mentioned above:

(47)

$\mathcal{F}_{-1}$	$\mathcal{F}_0$	$\mathcal{F}_1$
$h^3(F \otimes \mathcal{G}_{-1})$	$h^3(F \otimes \mathcal{G}_0)$	$h^3(F \otimes \mathcal{G}_1)$
$h^2(F \otimes \mathcal{G}_{-1})$	$h^2(F \otimes \mathcal{G}_0)$	$h^2(F \otimes \mathcal{G}_1)$
$h^1(F \otimes \mathcal{G}_{-1})$	$h^1(F \otimes \mathcal{G}_0)$	$h^1(F \otimes \mathcal{G}_1)$
$h^0(F \otimes \mathcal{G}_{-1})$	$h^0(F \otimes \mathcal{G}_0)$	$h^0(F \otimes \mathcal{G}_1)$

The sheaf F is isomorphic to the homology of a complex $C_F^\bullet \in D^b(X)$ whose terms are

$$C_F^l = \bigoplus_{i-j=l+1} H^i(F \otimes \mathcal{G}_j) \otimes \mathcal{F}_j. \tag{48}$$

We give below an explicit description of the collections $\mathcal{F}_i, \mathcal{G}_i$ for $i \in \{-1, 0, 1\}$, on each of the Fano threefolds X with $H^3(X, \mathbb{Z}) = 0$.

The projective 3-space. For $X = \mathbb{P}^3$, we just use the classical exceptional collection:

$$\mathcal{F}_{-1} = \mathcal{O}_{\mathbb{P}^3}(-1), \quad \mathcal{F}_0 = \mathcal{O}_{\mathbb{P}^3}, \quad \mathcal{F}_1 = \mathcal{O}_{\mathbb{P}^3}(1);$$
$$\mathcal{G}_{-1} = \mathcal{T}_{\mathbb{P}^3}(-3), \quad \mathcal{G}_0 = \Omega_{\mathbb{P}^3}, \quad \mathcal{G}_1 = \mathcal{O}_{\mathbb{P}^3}(-1).$$

Of course, this agrees with Beilinson's exceptional collection, appropriately twisted for $\Omega^2_{\mathbb{P}^3}(1) \simeq \mathcal{T}_{\mathbb{P}^3}(-3)$. All these vector bundles are slope-stable, with $\mu(\mathcal{G}_{-1}) = -\frac{5}{3}$, $\mu(\mathcal{G}_0) = -\frac{4}{3}$.

The quadric threefold. For a smooth quadric threefold Q_3 inside $\mathbb{P}^4$, we use a version of Kapranov's exceptional collection:

$$\mathcal{F}_{-1} = \mathcal{O}_Q(-1), \quad \mathcal{F}_0 = \mathcal{S}, \quad \mathcal{F}_1 = \mathcal{O}_Q;$$
$$\mathcal{G}_{-1} = \mathcal{T}_{\mathbb{P}^4}|_Q(-2), \quad \mathcal{G}_0 = \mathcal{S}, \quad \mathcal{G}_1 = \mathcal{O}_Q.$$

Here, as before, $\mathcal{S}$ is the spinor bundle, which is slope-stable of rank 2 and slope $-\frac{1}{2}$. The vector bundle $\mathcal{T}_{\mathbb{P}^4}|_Q(-2)$ is slope-stable of slope $-\frac{3}{4}$.

The quintic Del Pezzo threefold. The smooth quintic Del Pezzo threefold $Y = Y_5$ sits as a linear section in the Grassmannian $\mathbb{G}(2, U_5)$, where U_5 is a 5-dimensional vector space. The bundles $\mathcal{U}_Y$ and $\mathcal{Q}_Y$ are obtained by restricting to Y the rank-2 tautological subbundle and the rank-3 tautological quotient bundle on $\mathbb{G}(2, U_5)$. The linear section Y is given by the choice of a 3-dimensional vector space $U_3 \subset \bigwedge^2 U_5^*$. We have a natural identification $U_5 = H^0(\mathcal{U}_Y^*)^*$.

The next ingredient is a vector bundle $\mathcal{R}_Y$ of rank 9, introduced in [Faenzi 2014, §3.2.2]. It is defined as the kernel of the natural evaluation morphism $U_5^* \otimes \mathcal{U}_Y \to \mathcal{O}_Y$, so it fits into

$$0 \to \mathcal{R}_Y \to U_5^* \otimes \mathcal{U}_Y \to \mathcal{O}_Y \to 0. \tag{49}$$

We have $h^0(\mathcal{R}_Y(1)) = 18$ and the vector bundle $\mathcal{R}_Y(1)$ is globally generated. The isomorphism (46) is given by the evaluation map of global sections, which provides the natural exact sequence

$$0 \to \mathcal{R}_Y^*(-1) \to U_{18} \otimes \mathcal{O}_Y \to \mathcal{R}_Y(1) \to 0.$$

Here, $U_{18} = H^0(\mathcal{R}_Y(1))$ is an 18-dimensional vector space. To see this sequence explicitly, we note that U_3 is naturally contained in $U_5^* \otimes U_5^*$ and, using [Faenzi 2014, exact sequence (10)], we identify the quotient $U_{18} = U_5^* \otimes U_5^* / U_3$ with $H^0(\mathcal{R}_Y(1))$ and the kernel of the evaluation of global sections of $\mathcal{R}_Y(1)$ with $\mathcal{R}_Y^*(-1)$. In summary, for $Y = Y_5$, we have

$$\begin{aligned} &\mathcal{F}_{-1} = \mathcal{U}_Y, && \mathcal{F}_0 = \mathcal{O}_Y, \quad \mathcal{F}_1 = \mathcal{U}_Y^\vee, \\ &\mathcal{G}_{-1} = \mathcal{Q}_Y(-1), && \mathcal{G}_0 = \mathcal{R}_Y, \quad \mathcal{G}_1 = \mathcal{U}_Y. \end{aligned}$$

We know $\mathcal{U}_Y$ and $\mathcal{Q}_Y(-1)$ are slope-stable [Faenzi 2005], with $\mu(\mathcal{Q}_Y(-1)) = -\frac{2}{3}$, $\mu(\mathcal{U}_Y) = -\frac{1}{2}$.

Claim. *The bundle $\mathcal{R}_Y$ is slope-stable, with slope $-\frac{5}{9}$.*

Proof. The statement about $\mathcal{U}_Y$ is well known. The slope of the bundle $\mathcal{R}_Y$ is computed immediately from (49). To check stability, suppose by contradiction that a subsheaf K destabilises $\mathcal{R}_Y$. We may assume that K is slope-stable, and, up to passing to double dual, that K is reflexive. In view of (49), since $\mathcal{U}_Y$ is stable of slope $-\frac{1}{2}$ and K injects into the polystable bundle $\mathcal{U}_Y^{\oplus 5}$, we see that K must have slope $-\frac{1}{2}$ and that the reflexive hull of K is isomorphic to $\mathcal{U}_Y$. Then K must split off a copy of $\mathcal{U}_Y$ from $\mathcal{R}_Y$, which cannot happen, since $\mathcal{R}_Y$ is exceptional, hence indecomposable. □

***Prime Fano threefolds of genus* 12.** Consider $X = X_{12}$, a smooth prime Fano threefold of genus 12. We mentioned that X is a closed subvariety of the Hilbert scheme of cubics in $\mathbb{P}(U_4)$. More precisely, X parametrises twisted cubics whose ideals are generated by quadrics lying in the kernel of $S^2 U_4^* \to U_3^*$. Writing U_7 for the 7-dimensional kernel of this map, we get that X embeds in $\mathbb{G}(3, U_7)$. The threefold X is cut out as a subvariety of $\mathbb{G}(3, U_7)$ as the vanishing locus of a net of skew-symmetric 2-forms, according to Mukai's famous description. Then we get vector bundles $\mathcal{U}_X$ and $\mathcal{Q}_X$ by restricting to X the tautological rank-3 subbundle, and the tautological rank-4 quotient bundle on $\mathbb{G}(3, U_7)$.

Also, X can be seen as a subscheme of the moduli space of twisted cubics in $\mathbb{P}^3 = \mathbb{P}(U_4)$, where U_4 is a 4-dimensional vector space. Then the minimal instanton F_0, which is denoted here by $\mathcal{T}$, is the rank-2 vector bundle whose fibre over a twisted cubic curve $[C] \in X$ is the two-dimensional linear space generated by the linear syzygies of the quadrics defining I_C.

Finally, we have a rank-10 vector bundle $\mathcal{R}_X$, which is the dual of the right mutation of $\mathcal{T}$ with respect to $\mathcal{U}_X$, so that the dual of the fibre of $\mathcal{R}_X$ at $[C]$ represents the 10-dimensional space of cubics of $\mathbb{P}^3$ that vanish on C. One has the exact sequence

$$0 \to \mathcal{R}_X \to U_4^* \otimes \mathcal{U}_X^* \to \mathcal{T}^* \to 0. \tag{50}$$

The bundle $\mathcal{R}_X$ is introduced and studied in [Faenzi 2007], where it is denoted by M in [loc. cit.] The mutation induced by global sections of $\mathcal{R}_X = \mathcal{G}_0$, see (46), here takes the form

$$0 \to \mathcal{R}_X^* \to S^3 U_4^* \otimes \mathcal{O}_X \to \mathcal{R}_X \to 0, \tag{51}$$

where the map $\mathcal{O}_X^{20} \to \mathcal{R}_X$ is the canonical evaluation of global sections. All this is proved in [Faenzi 2007, Lemma 6.7]. In summary, we have, for $X = X_{12}$,

$$\begin{aligned} &\mathcal{F}_{-1} = \mathcal{U}_X^*(-1), \quad \mathcal{F}_0 = \mathcal{T}, \qquad \mathcal{F}_1 = \mathcal{U}_X; \\ &\mathcal{G}_{-1} = \mathcal{Q}_X, \qquad\quad \mathcal{G}_0 = \mathcal{R}_X, \quad \mathcal{G}_1 = \mathcal{U}_X^*. \end{aligned}$$

The bundles $\mathcal{U}_X$ and $\mathcal{Q}_X$ are slope-stable (see [Faenzi 2007, Lemma 6.2]), with $\mu(\mathcal{Q}_X) = \frac{1}{4}$, $\mu(\mathcal{U}_X^*) = \frac{1}{3}$.

Claim. *The bundle $\mathcal{R}_X$ is slope-stable, with slope $\frac{3}{10}$.*

Proof. One computes the slopes of $\mathcal{R}_X$ from (51). A destabilising reflexive, stable subsheaf K of $\mathcal{R}_X^*$ sits in the polystable bundle $\mathcal{U}_X^{\oplus 4}$ by (50). Since there is no rational number $\frac{p}{q}$ with $q \leq 9$ with $-\frac{1}{3} \leq \frac{p}{q} \leq -\frac{3}{10}$, we get that K splits off a summand $\mathcal{U}_X$ from $\mathcal{R}_X$, which is impossible, since $\mathcal{R}_X$ is exceptional. □

Using the complex (48), one obtains a monadic representation of an instanton bundle. We use arguments similar to those in [Faenzi 2014] to determine the cohomology table (54) for nonlocally free and higher-rank instantons. This leads to the next result.

Theorem 7.1. *Let E be a Gieseker-semistable (n, k)-instanton on a Fano threefold X such that $H^3(X) = 0$. Suppose that $k \geq 2$ for $X = Y_5$ and $k \geq 1$ for $X = X_{12}$. Then E is the homology of a monad of the form*

$$\mathcal{F}_{-1}^{\oplus k} \to W \otimes \mathcal{F}_0 \to \mathcal{F}_1^{\oplus k}, \tag{52}$$

where W is a complex vector space of dimension $w := \dim(W)$, with

(53)

i_X	w
4	$2k+n$
3	$k+n$
2	$4k+n$
1	$3k+n$

Proof. To start with, we recall that, as observed at the beginning of the section, a Gieseker-semistable (n, k)-instanton E lies in $\mathcal{O}_X(q_X)^\perp = \langle \mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1 \rangle$.

We now adapt to the higher-rank case the arguments presented in [Faenzi 2014]. Doing so, we show the cohomology table of a Gieseker-semistable (n, k)-instanton

satisfying the hypotheses of the statement, with respect to the collection $\mathcal{G}_{-1}, \mathcal{G}_0, \mathcal{G}_1$, has the form

(54)

$\mathcal{F}_{-1}$	$\mathcal{F}_0$	$\mathcal{F}_1$
0	0	0
0	0	0
k	w	k
0	0	0

The complex (48) reduces therefore to a monad of the form appearing in (52).

To exhibit the table (54), it is enough to prove the vanishing results appearing in the table. Indeed, the dimension of the remaining cohomology group is computed using Riemann–Roch, relying on the Chern character $\gamma(n, k)$, which is computed by (1). To get the vanishing results appearing in the table, we only need that E is slope-semistable and satisfies $H^1(E(-q_X)) = 0$.

We first check $h^0(E \otimes \mathcal{G}_1) = 0$, which implies the vanishing results summarised in the bottom row of (54). By stability of the involved sheaves, this equation is true if $\mu(\mathcal{G}_1) < -\mu(E) = \frac{1}{2} r_X$. However, we have $\mu(\mathcal{G}_1) < 0$ if $r_X = 0$, while $\mu(\mathcal{G}_1) = 0 < \frac{1}{2} = -\mu(E)$ for $i_X = 3$ and $\mu(\mathcal{G}_1) = \frac{1}{3} < \frac{1}{2} = -\mu(E)$ for $i_X = 1$.

Next, we check the top row of (54). Note that it is enough to show $h^3(E \otimes \mathcal{G}_{-1}) = 0$. By slope-semistability of E and $\mathcal{G}_{-1}$, coupled with Serre duality, this follows once we show $-\mu(E) < \mu(\mathcal{G}_{-1}) + i_X$, which in turn is checked immediately.

Finally, we have to justify $h^2(E \otimes \mathcal{G}_i) = 0$ for $i \in \{-1, 0, 1\}$. We use the mutation formula (46) to write down the exact sequence

$$0 \to \mathcal{G}_{-i}^*(-2q_X) \to H^0(\mathcal{G}_i(q_X)) \otimes \mathcal{O}_X(-q_X) \to \mathcal{G}_i \to 0 \quad \text{for } i \in \{-1, 0, 1\}.$$

We tensor the above sequence with E and use the vanishing $h^p(E(-q_X)) = 0$ for $p = 2, 3$ to get $h^2(E \otimes \mathcal{G}_i) = h^3(E \otimes \mathcal{G}_{-i}^*(-2q_X))$ for $i \in \{-1, 0, 1\}$. Again, this vanishes in view of Serre duality, because of stability of E and $\mathcal{G}_{-i}$, together with the calculation

$$-\mu(E) + \mu(\mathcal{G}_{-i}) + 2q_X - i_X = \mu(\mathcal{G}_{-i}) - \tfrac{1}{2} r_X < 0 \quad \text{for } i \in \{-1, 0, 1\}. \qquad \square$$

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GAIA COMASCHI
UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR
CNRS, LMAP UMR 5142
PAU
FRANCE
gaia.comaschi@gmail.com

DANIELE FAENZI
INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UMR CNRS 5584
UNIVERSITÉ BOURGOGNE EUROPE
DIJON
FRANCE
daniele.faenzi@u-bourgogne.fr

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ARTIN SHAPES

NIKITA KARPENKO AND GUANGZHAO ZHU

We introduce, and study examples of, a notion of the Artin shape for a motive related to a projective homogeneous variety. We apply it to the problem of finding the complete motivic decomposition of the variety. Our examples cover unitary involution varieties as well as some varieties given by a quadratic Weil transfer. Some of the decompositions obtained dispel prior expectations on how motivic decompositions of projective homogeneous varieties can look like.

1. The scope

We consider Grothendieck's Chow motives (defined as in [Elman et al. 2008, §64]) over a field with coefficients in $\mathbb{F} := \mathbb{Z}/p\mathbb{Z}$, where p is a prime number. Following a tradition, we also write $\mathbb{F}$ for the motive of the spectrum of the base field and call a *Tate motive* its (Tate) shift $\mathbb{F}\{i\}$ with any $i \in \mathbb{Z}$. Another important type of objects are direct summands in the spectra of étale algebras, called *Artin motives*.

Let M be a direct summand in the motive of a projective homogeneous variety under a reductive group G over a field F and let B be the variety of Borel subgroups in G. By [Chernousov et al. 2005, Theorem 7.5], over the function field $F(B)$, M becomes a direct sum of (Tate) shifts of indecomposable Artin motives which we call the *Artin shape* (or simply the *shape*) of M. As shown in [Chernousov and Merkurjev 2006] (see also [Karpenko 2013, Corollary 2.6]), the decomposition we use to define the Artin shape is unique. The motive M itself decomposes into a finite direct sum of indecomposables and such a decomposition, called *complete*, is also unique.

The shape of the entire motive $M(X)$ of a projective G-homogeneous variety X, which we also call the *motivic shape* of X, contains the Artin motive given by the Tate motive $\mathbb{F} := M(\mathrm{Spec}(F))$ with no shift (i.e., with the shift 0), whereas all remaining Artin motives in the shape have positive shifts. It follows that the

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complete motivic decomposition of X contains precisely one summand $U(X)$, called the *upper motive* of X, whose shape satisfies the same property.

By [Karpenko 2013, Corollary 2.15], the upper motives of projective homogeneous varieties satisfy a very efficient isomorphism criterion: $U(X) \simeq U(X')$ if and only if each of the varieties $X_{F(X')}$ and $X'_{F(X)}$ possesses a closed point of prime-to-p degree.

A description of summands that can appear in the complete decomposition of $M(X)$, already previously available in a different situation (see [Karpenko 2010, Theorem 1.1]), has been provided in [Clercq et al. 2025] for *p'-inner p-consistent* G, where G is *p'-inner* if it acquires inner type over a finite Galois field extension E/F of prime-to-p degree, and G is *p-consistent* if its *higher Tits p-indexes*, defined in [De Clercq and Garibaldi 2017], are stable under the action of the Galois group of E/F on the Dynkin diagram of G:

Theorem 1.1 [Clercq et al. 2025, Theorem 6.3 and Remark 5.14]. *Every summand in the complete motivic decomposition of a projective homogeneous variety X under a p'-inner p-consistent group G is a shift of the tensor product $U(Y) \otimes A$, where Y is a projective G-homogeneous variety with $X(F(Y)) \neq \varnothing$ and A is an Artin motive isomorphic to an indecomposable summand in the motive of the spectrum of the F-algebra L given by an intermediate field L in E/F.*

We work out several new applications for that result.

Assume that p is odd and fix a quadratic Galois field extension L/F. The motive of the spectrum of the F-algebra L splits off the Tate motive $\mathbb{F}$; we write A for the complementary summand. This A is an indecomposable Artin motive nonisomorphic to $\mathbb{F}$ over F but becoming isomorphic to $\mathbb{F}$ over L.

Let D be a degree-p central division L-algebra, write X for its Severi–Brauer L-variety, and consider the F-variety $R(X)$ given by the Weil transfer of X. The variety $R(X)$ is projective homogeneous under the reductive group $G = R(\mathrm{Aut}(D))$ given by the Weil transfer of the automorphism group of D. This G acquires inner type over L. Since p is odd, the degree of the field extension L/F is prime to p. Therefore G is p'-inner. However, G is not necessarily p-consistent: it is p-consistent if and only if the algebra D is *balanced* in the sense of [Karpenko and Reichstein 2015, §4] meaning that the algebra $\breve{D}$ conjugate to D is Brauer-equivalent to a tensor power of D. Note that if this is the r-th tensor power, then $r^2 \equiv 1 \bmod p$ implying that $r \equiv \pm 1 \bmod p$. It follows that $\breve{D}$ is isomorphic either to D or to its opposite algebra. (And the two conditions are mutually exclusive.) Considering the norm algebra of D, defined in [Knus et al. 1998, §3B], one translates the first condition as $D \simeq D'_L$ for a degree-p central division F-algebra D'. The second condition means that the norm algebra is Brauer-trivial and translates by [Knus et al. 1998, Theorem 3.1(2)] as "D admits a unitary F-involution".

We are going to investigate the shape of the *upper motive* $U(R(X))$ as well as the complete motivic decompositions of the variety $R(X)$ and of another related variety in these two cases. Our main results are Theorems 2.3, 4.5, and 4.3.

2. The first case

Assume that we are in the first case, i.e., $D \simeq D'_L$, and let X' be the Severi–Brauer F-variety of D'. Since $R(X)_L \simeq X \times X$ and the degree of L/F is coprime with p, we have $U(R(X)) \simeq U(X')$ by the isomorphism criterion mentioned in Section 1.

By [Karpenko 2013, Corollary 2.22] (see also [Karpenko 1995, Theorem 2.2.1]), the motive $M(X')$ is indecomposable. Therefore $U(X') = M(X')$. The variety X' is projective homogeneous under the group $G' := \mathrm{Aut}(D')$. Over the function field of the corresponding variety of Borel subgroups, X' becomes isomorphic to a $(p-1)$-dimensional projective space $\mathbb{P}$ whose motivic shape is

$$\mathbb{F} \oplus \mathbb{F}\{1\} \oplus \cdots \oplus \mathbb{F}\{p-1\}.$$

So, this is also the shape of the upper motive $U(R(X))$ we started with. It can be represented "graphically" as

$$\begin{array}{cccc} \{0\} & \{1\} & \cdots & \{p-1\} \\ \mathbb{F} & \mathbb{F} & \cdots & \mathbb{F} \end{array} \quad \text{or simply} \quad \mathbb{F}\ \mathbb{F}\ \cdots\ \mathbb{F}.$$

The graphical shape of $M(R(X))$ is

$$\begin{array}{ccccccccccccc} \mathbb{F} & & \mathbb{F} & & \cdots & & \mathbb{F} & & & & & & \\ & A & & A & & \cdots & & A & & & & & \\ & & \mathbb{F} & & \mathbb{F} & & \cdots & & \mathbb{F} & & & & \\ & & & A & & A & & \cdots & & A & & & \\ & & & & \ddots & & & & & & \ddots & & \\ & & & & & & \mathbb{F} & & \mathbb{F} & & \cdots & & \mathbb{F} \end{array} \tag{2.1}$$

(with p variously shifted lines, each of length p), where A is the Artin motive defined in Section 1. Indeed, this shape is given by the complete motivic decomposition of the Weil transfer $R(\mathbb{P})$. The latter, as explained in [Karpenko 2000, Proposition 5.6], is determined by the action of the Galois group of L/F on the Chow group of the product $R(\mathbb{P})_L = \mathbb{P} \times \mathbb{P}$, performing via exchange of the factors. The Chow group of each of the two (isomorphic) motives $\mathbb{F}_L$ and A_L is 1-dimensional, the first one has the trivial Galois action whereas the second has the nontrivial one. The standard basis of the codimension i Chow group of the product $\mathbb{P} \times \mathbb{P}$ is permuted by the action; each invariant basis element corresponds to $\mathbb{F}\{i\}$, whereas each pair of permuted basis elements corresponds to $\mathbb{F}\{i\} \oplus A\{i\}$.

By Theorem 1.1, the complete decomposition of $M(R(X))$ consists of shifts of $U(R(X))$ and of $U(R(X)) \otimes A$. It follows that the complete decomposition

of $M(R(X))$ (over F) is

$$\begin{aligned}(2.2)\quad M(R(X)) &\simeq U(R(X))\oplus\big(U(R(X))\otimes A\big)\{1\}\\ &\quad\oplus U(R(X))\{2\}\oplus\big(U(R(X))\otimes A\big)\{3\}\oplus\cdots\oplus U(R(X))\{p-1\}\\ &=\Big(\bigoplus_{i=0}^{(p-1)/2} U(R(X))\{2i\}\Big)\oplus\Big(\bigoplus_{i=1}^{(p-1)/2}\big(U(R(X))\otimes A\big)\{2i-1\}\Big).\end{aligned}$$

We put together the main result of this section dispelling some prior expectations on how motivic decompositions of projective homogeneous varieties can look like. Indeed, for the very first time a projective homogeneous variety is found whose motive is not a direct sum of shifts of *corestrictions* (in the sense of [Clercq et al. 2025, §2]) of upper motives.

Theorem 2.3. *For an odd prime p, let X' be the Severi–Brauer variety of a degree-p central division algebra over a field F. For a quadratic Galois field extension L/F, formula* (2.2) *provides the complete motivic decomposition for the Weil transfer $R(X)$ of the L-variety $X := X'_L$. The decomposition holds in the category of Chow motives with coefficients in $\mathbb{Z}/p\mathbb{Z}$.*

3. The second case

Now let us assume that D admits a unitary F-involution. First of all, let us note that the shape of $M(R(X))$ is still given by (2.1): the proof remains unchanged. What we want to figure out is the shape of $U(R(X))$ and the complete decomposition of $M(R(X))$. Concerning the latter, by Theorem 1.1 once again we still know a priori that $M(R(X))$ is a direct sum of shifts of $U(R(X))$ and of $U(R(X))\otimes A$: indeed, $Y(L(X))\neq\varnothing$ for any projective G-homogeneous variety Y (note that the group $G=R(\mathrm{Aut}(D))$ becomes $\mathrm{Aut}(D)\times\mathrm{Aut}(\breve{D})$ over L whereas $\breve{D}$ is now the opposite algebra of D), so that the upper motive of Y is isomorphic to $U(X)$ as long as the condition $X(F(Y))\neq\varnothing$ of Theorem 1.1 is satisfied.

By [Clercq et al. 2025, Lemma 5.10], the motive $U(R(X))$ is still indecomposable over L so that $U(R(X))_L\simeq U(R(X)_L)$. Since $U(R(X)_L)\simeq U(X)=M(X)$ has the shape $\mathbb{F}\ \mathbb{F}\cdots\mathbb{F}$, we conclude that the shape of $U(R(X))$ looks like

$$\mathbb{F}\ *\ *\cdots*,$$

where each $*$ is either $\mathbb{F}$ or A. We will be able to precisely determine the shape in the next section based on the following preliminary result:

Lemma 3.1. *The shape of the upper motive $U(R(X))$ is either $\mathbb{F}\ \mathbb{F}\cdots\mathbb{F}$ (as in Section 2) or $\mathbb{F}\ A\ \mathbb{F}\ A\cdots\mathbb{F}\ A\ \mathbb{F}$ (i.e., $(p+1)/2$ exemplars of $\mathbb{F}$ alternated with $(p-1)/2$ exemplars of A). The complete decomposition of $M(R(X))$ in the first*

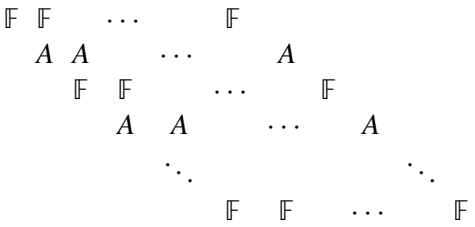
case is (2.2); *in the second case it is given by*

$$M(R(X)) \simeq U(R(X)) \oplus U(R(X))\{1\} \oplus \cdots \oplus U(R(X))\{p-1\} = \bigoplus_{i=0}^{p-1} U(R(X))\{i\}. \tag{3.2}$$

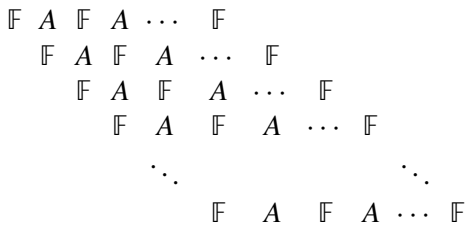
Proof. Note that, by (2.1), the shape of $U := U(R(X))$ takes the first (or second) indicated value if and only if the complete decomposition of $M := M(R(X))$ does.

Let a be the number of A in the graphical shape of U and let b be the number of shifts of $U \otimes A$ in the complete decomposition of M. Then $ab + (p-a)(p-b)$ is the number of $\mathbb{F}$ in the graphical shape of M. Since this number is divisible by p whereas both a and b are smaller than p, we conclude that $a = 0$ or $b = 0$. □

At this point, we do not know yet if decomposition (3.2) actually occurs, but, as already mentioned, we will prove that it does in the next section. Let us illustrate this result by noticing that decompositions (2.2) and (3.2) correspond to the two different graphical representations



and



of the motivic shape of $R(X)$.

Let us also note that the ratio

$$\frac{\text{number of } \mathbb{F}}{\text{number of } A} \tag{3.3}$$

in the second possible graphical shape of $U(R(X))$ equals $(p+1)/(p-1)$ and is the same as in the graphical shape of $M(R(X))$. Note that the similar ratio for $U(R(X)) \otimes A$ equals $(p-1)/(p+1)$ which is lower. This gives another proof of (3.2) in the second case of Lemma 3.1: we already know that $M(R(X))$ is a direct sum of shifts of $U(R(X))$ and of $U(R(X)) \otimes A$, but due to the ratio considerations, the tensor product with A cannot appear.

4. The unitary involution variety

In the situation of Section 3, let us fix a unitary F-involution τ on the L-algebra D. The reductive F-group $G := \mathrm{Aut}_L(D,\tau)$ (the projective unitary group of (D,τ)) is p'-inner p-consistent; over L it acquires inner type becoming isomorphic to the group $\mathrm{Aut}(D)$. Let us determine the complete motivic decomposition of the corresponding *involution variety* Y — the variety of rank-1 (over L) right isotropic ideals in D. We start with some information on its motivic shape:

Proposition 4.1. *Given any $i = 1, 2, \dots, (p-1)/2$, the motivic shape of the variety Y_i of rank-i (over L) right isotropic ideals in the algebra D contains $(b_i + a_i)/2$ shifts of the Tate motive $\mathbb{F}$ and of $(b_i - a_i)/2$ shifts of the Artin motive A, where*

$$a_i = 2^i \cdot \binom{(p-1)/2}{i} \quad \text{and} \quad b_i = \binom{p}{i} \cdot \binom{p-i}{i}.$$

Proof. Here is the meaning of the integers a_i and b_i appearing in the statement of Proposition 4.1. The integer b_i is the *rank* of $M(Y_i)$ defined as the number of Tate motives in the complete decomposition of $M(Y_i)$ over an algebraic closure. (Since the shape of $M(Y_i)$ consists of shifts of motives $\mathbb{F}$ and A which become Tate motives over an algebraic closure, b_i is just the number of motives in the shape.) To get the formula for b_i, note that over a separable closure of the base field, Y_i becomes the variety of flags in a p-dimensional vector space, where the flags are given by an i-plane contained in a $(p-i)$-plane.

To explain the meaning of a_i, we first note that over $F(B)$ the motive $M(Y_i)$ is also a direct sum of shifts of the Tate motive $\mathbb{F}$ and of the motive of spectrum of the $F(B)$-algebra $L(B)$; the integer a_i is the number of shifts of $\mathbb{F}$. It follows from [Karpenko 2012b, Lemma 7.1] for unitary grassmannians and the similar formula [Karpenko 2012a, Formula 2.6] for orthogonal grassmannians that a_i coincides with the rank of the motive of an orthogonal grassmannian of totally isotropic i-planes in a p-dimensional nondegenerate quadratic form, indicated, e.g., in [Buch et al. 2009, Proof of Theorem 2.2] (see also [Buch et al. 2009, Proof of Theorem 1.2]). □

Corollary 4.2. *The number of $\mathbb{F}$ in the motivic shape of $Y = Y_1$ is $(p+1)(p-1)/2$ whereas the number of A is $(p-1)^2/2$. In particular, ratio* (3.3) *for the motivic shape of $Y = Y_1$ is equal to $(p+1)/(p-1)$.* □

The involution variety Y is a subvariety of X. By the isomorphism criterion mentioned in Section 1, $U(Y)$ is isomorphic to the upper motive $U(R(X))$ because the algebra D splits over the function field of the L-variety $R(X)_L$. Therefore, the two possible shapes of $U(Y)$ are listed in Lemma 3.1.

Theorem 4.3. *Let p be an odd prime, L/F a quadratic Galois field extension, D a degree-p central division L-algebra endowed with a unitary F-involution τ, and Y*

the corresponding involution variety (*over* F). *Then the shape of* $U(Y)$ *is as in the second case of Lemma 3.1 whereas the complete motivic decomposition of* Y *is*

$$M(Y) \simeq \bigoplus_{i=0}^{p-2} U(Y)\{i\}. \tag{4.4}$$

In contrast to (3.2), i in (4.4) ranges up to $p-2$ only.

Proof of Theorem 4.3. By Theorem 1.1, every summand in the complete motivic decomposition of Y is a shift of $U(Y)$ or of $U(Y)\otimes A$: since the algebra D splits over the function field of the L-variety Y_L, every projective G-homogeneous variety acquires a closed point of prime-to-p degree over $F(Y)$. Therefore, if $U(Y)$ has the first shape of Lemma 3.1, the number of $\mathbb{F}$ in the shape of $M(Y)$ is divisible by p. On the other hand, it is not by Corollary 4.2. This contradiction proves the statement of Theorem 4.3 on the shape of $U(Y)$.

Now, by the same argument as in Section 3, the complete decomposition of $M(Y)$ consists of shifts of $U(Y)$ only. Over L, the variety Y becomes a rank-$(p-2)$ projective bundle over the Severi–Brauer variety X, so the motive of Y decomposes over L as in (4.4). It follows that the same decomposition holds already over F. □

For illustrative purposes, we draw the shape of the motive of Y that reflects its complete decomposition as

$$\begin{array}{ccccccccccccc}
\mathbb{F} & A & \mathbb{F} & A & \cdots & \mathbb{F} & & & & & & & \\
 & \mathbb{F} & A & \mathbb{F} & A & \cdots & \mathbb{F} & & & & & & \\
 & & \mathbb{F} & A & \mathbb{F} & A & \cdots & \mathbb{F} & & & & & \\
 & & & \mathbb{F} & A & \mathbb{F} & A & \cdots & \mathbb{F} & & & & \\
 & & & & \ddots & & & & & \ddots & & & \\
 & & & & & & \mathbb{F} & A & \mathbb{F} & A & \cdots & \mathbb{F}
\end{array}$$

where there are $p-1$ variously shifted lines, each of length p.

With Theorem 4.3, we also proved our last main result:

Theorem 4.5. *In the situation of Theorem 4.3, let* X *be the Severi–Brauer variety of* D *and let* $R(X)$ *be its Weil transfer to* F. *Then its upper motive* $U(R(X))$ *has the second shape of Lemma 3.1 whereas its total motive* $M(R(X))$ *has the complete decomposition* (3.2). □

Decompositions of Theorem 4.5 and 4.3 put together yield

$$M(R(X)) \simeq M(Y) \oplus U(Y)\{p-1\}.$$

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NIKITA KARPENKO
MATHEMATICAL & STATISTICAL SCIENCES
UNIVERSITY OF ALBERTA
EDMONTON
CANADA
karpenko@ualberta.ca

GUANGZHAO ZHU
MATHEMATICAL & STATISTICAL SCIENCES
UNIVERSITY OF ALBERTA
EDMONTON
CANADA
guangzha@ualberta.ca

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L-PACKETS AND THE GENERIC ARTHUR PACKET CONJECTURES FOR EVEN UNITARY SIMILITUDE GROUPS

YEANSU KIM, MUTHU KRISHNAMURTHY AND FREYDOON SHAHIDI

We establish the generic local Langlands correspondence by showing the equality of the Langlands–Shahidi L-functions and Artin L-functions in the case of even unitary similitude groups. As an application, we prove both weak and strong versions of the generic Arthur packet conjectures in the cases of even unitary similitude groups and even unitary groups. We further describe (not necessarily generic) L-packets for even unitary similitude groups and establish their expected properties, including Shahidi's conjecture, the finiteness of L-packets, and other related results.

1. Introduction

Let $\boldsymbol{G}$ be a connected reductive group defined over a nonarchimedean local field F of characteristic zero. The local Langlands conjecture predicts that each irreducible admissible complex representation of the group $\boldsymbol{G}(F)$ is associated with a Langlands parameter, such that each parameter corresponds to a finite set of equivalence classes of irreducible admissible representations, called an L-packet. A central problem in the Langlands program is to study the structure and properties of these L-packets.

The primary objective of this paper is to establish a strong version of the generic Arthur packet conjecture for certain unitary groups. Namely, let $\boldsymbol{G}_n := \mathrm{GU}(n, n)$ (resp. $\boldsymbol{H}_n := U(n, n)$) denote the quasisplit even unitary similitude group (resp. even unitary group) defined with respect to a quadratic extension E/F of local fields as in Section 2.1. Following the notation used throughout this paper, we let $G := \boldsymbol{G}_n(F)$ and $H := \boldsymbol{H}_n(F)$ denote their respective groups of F-points. For an Arthur parameter ψ of G or H, let ϕ_ψ be the corresponding Langlands parameter (see Section 2.4 for a review of these definitions). Let Π_{ϕ_ψ} denote the L-packet associated with ϕ_ψ, as defined in the generic case in Definition 3.4 and in the

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general case in Definition 4.2. We refer the reader to Section 2.4 and the references therein for details regarding the relationship between an Arthur packet Π_ψ and the corresponding L-packet Π_{ϕ_ψ}.

Theorem 1.1 (Theorems 3.10 and 3.11). *If the L-packet Π_{ϕ_ψ} for G (resp. H) has a generic member, then Π_{ϕ_ψ} is a tempered L-packet for G (resp. H).*

Theorem 1.1 provides a strong version of the generic Arthur packet conjecture and can be viewed as a local version of the generalized Ramanujan conjecture. To further explain our result, we fix an additive character ψ of F and set $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$. We apologize for this conflict in notation, but the meaning of "ψ" should be clear from the context.

The proof of Theorem 1.1 proceeds in three steps. First, we relate the Langlands parameters to the local stable base change lift of generic representations. Let ${}^L G = \widehat{G} \rtimes W_F$ be the L-group of G. The base change lift from G to $\mathrm{GL}_{2n}(E) \times E^\times$ is associated with a homomorphism of L-groups

$$\mathrm{BC} : {}^L G \to {}^L(\mathrm{Res}_{E/F}(\mathrm{GL}_{2n} \times \mathrm{GL}_1)),$$

which, on the level of Langlands parameters, corresponds to the restriction $\phi_\pi|_{W_E'}$. Similarly, for the unitary group H, the corresponding map of L-groups is denoted by $\mathrm{BC}^U : {}^L H \to {}^L(\mathrm{Res}_{E/F}\mathrm{GL}_{2n})$. A stable base change lift $\pi \mapsto \mathrm{BC}^U(\pi)$, mapping representations of H to those of $\mathrm{GL}_{2n}(E)$, was established in [Kim and Krishnamurthy 2005] for the generic case and subsequently in full generality in [Mok 2015]. Furthermore, the L-parameter ϕ_π constructed in [Mok 2015] for tempered representations satisfies $\phi_{\mathrm{BC}^U(\pi)} = \phi_\pi|_{W_E'}$.

These results imply the existence of a lift $\pi \mapsto \mathrm{BC}(\pi)$ (see Section 4.2) mapping irreducible admissible representations π of G to those of $\mathrm{GL}_{2n}(E) \times E^\times$ in a manner compatible with the restriction of ϕ_π to W_E'. This allows us to translate the study of L-functions for G defined via the Langlands–Shahidi method into the theory of Rankin–Selberg L-functions for general linear groups over E.

Theorem 1.2 (Theorem 3.1). *Let π be an irreducible admissible generic representation of G. Then, there exists a Langlands parameter ϕ_π such that for any irreducible admissible generic representation ρ of $\mathrm{GL}_k(E)$, $k \geq 1$, we have*

$$L(s, \rho \times \pi) = L(s, \phi_\rho \otimes \phi_\pi|_{W_E'}) \quad \textit{and} \quad \gamma(s, \rho \times \pi, \psi) = \gamma(s, \phi_\rho \otimes \phi_\pi|_{W_E'}, \psi_E),$$

where ϕ_ρ is the Langlands parameter associated with ρ via the local Langlands correspondence for general linear groups [Harris and Taylor 2001; Henniart 2000].

The second step is to prove a weak version of Theorem 1.1, namely, Theorem 1.3, which follows from an application of [Shahidi 2011, Theorem 5.1] together with the first step. Shahidi [2011] proved the weak version of the generic Arthur packet

conjecture for any quasisplit group $\boldsymbol{G}$, assuming the equality of the Langlands–Shahidi and Artin L-functions. As a result, Theorem 1.2 implies the following for the group G:

Theorem 1.3 (Theorem 3.7). *If the L-packet $\Pi_{\phi_\psi}(G)$ has a generic member, then ϕ_ψ is a tempered L-parameter.*

The third step of the proof is to strengthen the result using the properties of L-packets. First, we prove that temperedness is preserved under the local functorial lift (Lemma 3.9). We emphasize that the "if" part of this lemma — that if the local functorial lift of an irreducible admissible representation π of G is tempered, then π is tempered — is essential to the proof. Second, we show that if a generic member of an L-packet is tempered, then all members of that packet are tempered (Lemma 3.6).

In the descriptions of Theorems 1.2 and 1.3, we must specify the definition of the L-packets. The second purpose of this paper is to describe the L-packets for G and their corresponding local L-functions so that Theorem 1.1 becomes unconditional. For unitary groups, Mok [2015, Theorem 2.5.1(b)] explicitly describes the Arthur packets, including the definition of tempered L-packets. We define L-packets in the general case for H following the Langlands classification (see [Arthur 1984] or Section 4). We then utilize this to construct L-packets for G. More precisely, we construct the stable base change lifts $\mathrm{BC}(\pi)$ in Section 4.2 using the restriction method, allowing us to define L-packets for G as follows.

Definition 1.4. Let π_0 be an irreducible admissible representation of G. We define the local L-packet containing π_0 as the set of irreducible admissible representations π of G such that

(1) $\omega_\pi = \omega_{\pi_0}$, and

(2) $\mathrm{BC}(\pi) \cong \mathrm{BC}(\pi_0)$.

Remark 1.5. This construction allows us to define Rankin product L-functions for $G \times \mathrm{GL}_k(E)$ in the nongeneric case as well (Definition 4.4).

A natural question is whether our definition of L-packets for G satisfies expected properties, such as Shahidi's conjecture [1990] regarding the existence of generic members and compatibility with the Langlands classification. We conclude the paper with the following result.

Proposition 1.6 (Proposition 4.5). *The L-packets for G satisfy these properties*:

(1) *Each L-packet for G consists of finitely many members.*

(2) *Every tempered L-packet contains a generic representation.*

(3) *The L-functions are compatible with the Langlands classification.*

(4) *The L-functions defined in Definition 4.4 agree with the L-functions obtained via the Langlands–Shahidi method in the generic case.*

2. Preliminaries

2.1. ***Unitary similitude groups.*** For any nonarchimedean local field $\mathfrak{k}$ of characteristic zero, we write $\mathfrak{o}_\mathfrak{k}$ to denote its ring of integers and $\mathfrak{p}_\mathfrak{k}$ to denote the unique maximal ideal in $\mathfrak{o}_\mathfrak{k}$. Let $\varpi_\mathfrak{k}$ be a uniformizer in $\mathfrak{k}$, that is, $\mathfrak{p}_\mathfrak{k} = \varpi_\mathfrak{k}\mathfrak{o}_\mathfrak{k}$ and let $q_\mathfrak{k} := |\mathfrak{o}_\mathfrak{k}\backslash\mathfrak{p}_\mathfrak{k}|$ be the residual characteristic of $\mathfrak{k}$. We fix the absolute value $\|\cdot\|_\mathfrak{k}$ that satisfies $\|\varpi\|_\mathfrak{k} = q_\mathfrak{k}^{-1}$. Now fix F, a nonarchimedean local field of characteristic zero. Let $\bar{F}$ be a (separable) algebraic closure of F and let Γ denote the Galois group of $\bar{F}/F$. Let $E \subseteq \bar{F}$ be a quadratic extension of F and let θ denote the corresponding nontrivial Galois automorphism. For $x \in E$, we write $\bar{x}$ to denote $\theta(x)$. We may write $E = F(\beta)$ for some $\beta \in E^\times$ satisfying $\bar{\beta} = -\beta$. Let $\psi : F \to \mathbb{C}$ be a nontrivial additive character of F of level 0, i.e., $\ker \psi \subset \mathfrak{o}_F$ but $\mathfrak{p}_F^{-1} \not\subset \ker \psi$. Then $\psi_E : E \to \mathbb{C}$ given by $\psi_E(x) := \psi(\mathrm{Tr}_{E/F}(x))$ is a nontrivial additive character of E which is trivial on $\mathfrak{o}_E$. For any algebraic F-group $\boldsymbol{G}$, we write G to denote $\boldsymbol{G}(F)$, the corresponding group of F-points. In addition, given any field K and an embedding $\tau : F \to K$, we write $\boldsymbol{G} \times_{F,\tau} K$ to denote the base change of $\boldsymbol{G}$ from F to K through τ. If $K \supset F$ and $\tau = \iota$ is the inclusion map, then we also write $\boldsymbol{G}/_K$ to denote the group $\boldsymbol{G} \times_{F,\iota} K$ obtained by extension of scalars. We do not typeset matrix groups in "boldface." For example, we write GL_n, $n \geq 1$, to denote the general linear group, B_n for the Borel subgroup of upper triangular matrices in GL_n, and U_n for the unipotent radical of B_n, and write their F-points explicitly as $\mathrm{GL}_n(F)$, $B_n(F)$, and $U_n(F)$, respectively. For $n \geq 1$, let J_n be the $n \times n$ matrix $J_n := (a_{i,j})_{i,j}$ with $a_{i,j} = 1$ whenever $i + j = n + 1$, and 0 otherwise. Put

$$J'_{2n} = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}.$$

Let $\boldsymbol{G}_n := \mathrm{GU}(n, n)$ be the quasisplit unitary similitude group in $2n$ variables defined with respect to E/F and J'_{2n} [Rogawski 1992]. Here, $\beta J'_{2n}$ must be viewed as the underlying Hermitian form. The group is the F-form of $\mathrm{GL}_{2n} \times \mathrm{GL}_1$ determined by the 1-cocycle $(\phi_\sigma)_{\sigma\in\Gamma}$, where

$$\phi_\sigma(g, \eta) := \begin{cases} (g, \eta) & \text{if } \sigma|_E = 1, \\ (\eta(J'_{2n})^{-1}({}^t g^{-1})J'_{2n}, \eta) & \text{if } \sigma|_E = \theta. \end{cases}$$

The Galois action on $\boldsymbol{G}_n(\bar{F}) = \mathrm{GL}_{2n}(\bar{F}) \times \mathrm{GL}_1(\bar{F})$ is the twisted Galois action $(g, \eta) \mapsto \phi_\sigma(\sigma(g, \eta))$. For any F-algebra $\mathcal{A}$,

$$\boldsymbol{G}_n(\mathcal{A}) = \{g \in \mathrm{GL}_{2n}(E \otimes_F \mathcal{A}) : {}^t\bar{g} J'_{2n} g = \eta(g) J'_{2n} \text{ for some } \eta(g) \in \mathcal{A}^\times\}, \tag{2-1}$$

where $g \mapsto \bar{g}$ denotes the action on $E \otimes_F \mathcal{A}$ obtained from the action of θ on E by linear extension. This defines a homomorphism $\eta : \boldsymbol{G}_n \to \mathbb{G}_m$ of F-groups called the *similitude* character. The kernel of this homomorphism is the usual (quasisplit)

unitary group $\boldsymbol{H}_n := U(n,n)$. We fix the F-Borel subgroup $\boldsymbol{B}$ of $\boldsymbol{G}_n$ consisting of upper triangular matrices and write $\boldsymbol{B} = \boldsymbol{T}\boldsymbol{U}$, where $\boldsymbol{T}$ is a maximal F-torus of $\boldsymbol{G}_n$ and $\boldsymbol{U}$ is the unipotent radical of $\boldsymbol{B}$ consisting of upper triangular unipotent matrices. Their F-points are given by

$$T = \{t = \mathrm{diag}(x_1, \ldots, x_n, y_1, \ldots, y_n) \mid x_i, y_i \in E^\times,\ x_{n-i+1}\bar{y}_i = \eta \in F^\times,\ 1 \le i \le n\},$$

(here $\eta = \eta(t)$) and

$$U = \left\{ n(u,v) = \begin{pmatrix} u & v \\ & \bar{u}^\star \end{pmatrix} \,\middle|\, u \in U_n(E),\ v \in \mathrm{M}_{n\times n}(E),\ J_n u^{-1} v = {}^t\bar{v}\,{}^t\bar{u}^{-1} J_n \right\}, \tag{2-2}$$

where, for any $n \times n$ matrix, we use $g^\star$ to denote $g^\star = J_n^{-1}({}^t g^{-1}) J_n$. Let $\boldsymbol{A}_0$ be the maximal F-split subtorus of $\boldsymbol{T}$. Then its F-points are given by

$$A_0 = \{\mathrm{diag}(x_1, \ldots, x_n, y_1, \ldots, y_n) \mid x_i, y_i \in F^\times \text{ satisfying } x_{n-i+1} y_i = \eta,\ 1 \le i \le n\}.$$

The groups $\boldsymbol{H}_n \subset \boldsymbol{G}_n$ share the same derived group and there is an exact sequence

$$1 \to \boldsymbol{H}_n \to \boldsymbol{G}_n \xrightarrow{\eta} \mathrm{GL}_1 \to 1.$$

Let $Z = Z_n$ denote the center of the group $G := \boldsymbol{G}_n(F)$. It is given by

$$\mathrm{diag}(z, \ldots, z) \in \mathrm{GL}_{2n}(E), \quad \eta = z\bar{z},$$

and its intersection Z_H with $H := \boldsymbol{H}_n(F)$ consists of all $z \in E^\times$ satisfying $z\bar{z} = 1$. It follows that η induces the isomorphism

$$G/ZH \cong F^\times / N_{E/F}(F^\times). \tag{2-3}$$

Let $\widetilde{\Sigma}$ denote the set of (absolute) roots of $\boldsymbol{G}_n$ with respect to $\boldsymbol{T}$. The choice of $\boldsymbol{B}$ then determines a set of positive roots $\widetilde{\Sigma}^+$ and let $\tilde{\Delta}$ denote the corresponding set of simple roots. Let $\widetilde{\Sigma}|_{\boldsymbol{A}_0} = \Sigma$, $\widetilde{\Sigma}^+|_{\boldsymbol{A}_0} = \Sigma^+$ and $\tilde{\Delta}|_{\boldsymbol{A}_0} = \Delta$. Then Σ is the set of roots of $\boldsymbol{G}_n$ relative to $\boldsymbol{A}_0$ and Δ is the corresponding set of simple roots. The elements of Σ are called *relative roots* of $\boldsymbol{G}_n$ (with respect to $\boldsymbol{A}_0$). (See [Borel 1991, §21].) The group Γ acts on $\widetilde{\Sigma}$ and $\tilde{\Delta}$ with $\mathrm{Gal}(\bar{F}/E)$ acting trivially. Thus we get an action of $\mathrm{Gal}(E/F)$ on $\widetilde{\Sigma}$ and $\tilde{\Delta}$ partitioning them into finitely many $\mathrm{Gal}(E/F)$-orbits; these orbits are in one-to-one correspondence with elements in Σ and Δ, respectively. The root system determined by $\widetilde{\Sigma}$ is of type ${}^2A_{2n-1}$ and the root system determined by Σ is of type C_n.

For a field extension $F' \supset F$, let us describe $\boldsymbol{G}_n(F')$ explicitly. Since E is separable, $E \otimes_F F'$ is either

(i) a field containing an isomorphic copy of E, or

(ii) a (direct) product of two fields each containing an isomorphic copy of E.

In case (i), the group $\boldsymbol{G}_n(F')$ is given as in (2-1). In case (ii), $E \otimes_F F' = F' \times F'$ with associated embeddings ι_1, ι_2 from $E \to F'$, where the i-th copy of F' in this product corresponds to ι_i, $i = 1, 2$. The nontrivial Galois action is then given by swapping two factors, i.e., $(x, y) \mapsto (y, x)$. Consequently

$$\begin{aligned}\boldsymbol{G}_n(F') &= \{(g, g') \in \mathrm{GL}_{2n}(F' \oplus F') : {}^t g' J'_{2n} g = \eta J'_{2n},\ \eta \in F'^{\times}\} \\ &\cong \{(g, \eta) \in \mathrm{GL}_{2n}(F') \times \mathrm{GL}_1(F')\}.\end{aligned}$$

Choosing a distinguished embedding ι_1, we see that $\boldsymbol{G}_n \times_{F,\iota_1} F' \cong \mathrm{GL}_{2n} \times \mathrm{GL}_1$ via projection onto the first factor. In particular, $\boldsymbol{G}_n/_E \cong \mathrm{GL}_{2n} \times \mathrm{GL}_1$ determined by the identity map ι_1. The maximal torus $\boldsymbol{T}/_E \subset \boldsymbol{G}_n/_E$ corresponds to the diagonal torus $(\mathrm{GL}_1)^{2n} \times \mathrm{GL}_1$ under this isomorphism. Thus

$$t \mapsto \mathrm{diag}(x_1, x_2, \ldots, x_{2n}) \times \eta \in \mathrm{GL}_{2n} \times \mathrm{GL}_1 \tag{2-4}$$

represents a generic element of the maximal torus $\boldsymbol{T}/_E$. For $1 \le i \le 2n$, let e_i be the character of $\boldsymbol{T}$ that sends $t \mapsto x_i$. Let e_0 be the character that sends $t \mapsto \eta$. The group $X(\boldsymbol{T})_E$ of E-rational characters of $\boldsymbol{T}$ is $\bigoplus_{i=1}^{2n} \mathbb{Z}e_i \oplus \mathbb{Z}e_0$ and the nontrivial Galois action is given by $e_0 \mapsto e_0$, $e_i \mapsto e_0 - e_{2n+1-i}$, $1 \le i \le 2n$. Put $\tilde{\alpha}_i := e_i - e_{i+1}$, $1 \le i \le 2n-1$, then $\tilde{\Delta} = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_{2n-1}\}$ whose restriction to A_0 gives Δ, the corresponding set of simple relative roots.

The set of all standard parabolic F-subgroups is in bijection with the set of all subsets Δ [Borel 1991, Proposition 21.12]. We write $\Delta^{\boldsymbol{M}'} \subset \Delta \leftrightarrow \boldsymbol{P}' = \boldsymbol{M}'\boldsymbol{N}'$ to denote this bijection, where $\boldsymbol{M}' \supset \boldsymbol{T}$ is the corresponding standard F-Levi subgroup and $\boldsymbol{N}' \subset \boldsymbol{U}$ is the unipotent radical of $\boldsymbol{P}'$. Here, $\Delta^{\boldsymbol{M}'}$ is the analogue of Δ obtained from replacing $\boldsymbol{G}_n$ by $\boldsymbol{M}'$ in the definitions. The standard Levi F-subgroups $\boldsymbol{M}'$ are of the following shape: We use $R_{E/F}$ to denote the functor of restriction of scalars, and set $G_0 := \mathrm{GU}(0, 0) = \mathrm{GL}_1$. Then for $\Delta^{\boldsymbol{M}'} \subset \Delta$, there are integers $n_1, \ldots, n_r > 0$ and $m \ge 0$ so that $n_1 + n_2 + \cdots + n_r + m = n$, and

$$\boldsymbol{M}' \cong R_{E/F}\,\mathrm{GL}_{n_1} \times R_{E/F}\,\mathrm{GL}_{n_2} \times \cdots \times R_{E/F}\,\mathrm{GL}_{n_r} \times \boldsymbol{G}_m.$$

By [loc. cit.] any parabolic F-subgroup of $\boldsymbol{G}_n$ is conjugate to one and only one standard $\boldsymbol{P}'$ by an element of G_n.

2.2. ***The Langlands–Shahidi method and relevant L-functions.*** We consider the case ${}^2A_{2n-1} - 4$ in [Shahidi 1988] in the context of unitary similitude groups and describe the resulting L-functions. (See [Kim and Krishnamurthy 2005] where this case was studied for unitary groups.) Put $\tilde{\theta} = \tilde{\Delta} - \{\tilde{\alpha}_i, \tilde{\alpha}_{n+i}\}$, $1 \le i \le n-1$, and let $\theta \subset \Delta$ be the corresponding subset of simple relative roots. Then $\Delta \backslash \theta = \alpha_i$, where α_i is the restriction of $\tilde{\alpha}_i$ to $\boldsymbol{A}_0$. Let $\boldsymbol{M} = \boldsymbol{M}_\theta$ be the standard maximal Levi F-subgroup of $\boldsymbol{G}_n$ associated to θ under the above mentioned correspondence. Then $\boldsymbol{M} \cong R_{E/F}\,\mathrm{GL}_k \times \boldsymbol{G}_m$, $k + m = n$, with $k \le n-1$. Let $\boldsymbol{P} \supset \boldsymbol{B}$ be the maximal

F-parabolic (standard) subgroup of $\boldsymbol{G}$ with Levi-component $\boldsymbol{M}$ and let $\boldsymbol{N}$ be the corresponding unipotent radical. Then as a subgroup of $\boldsymbol{G}_n(F)$,

$$M = \left\{ m(g,h) = \left[\begin{array}{c|c|c} \eta_h g & & \\ \hline & h & \\ \hline & & \bar{g}^\star \end{array}\right] \;\middle|\; g \in \mathrm{GL}_k(E),\ h \in \boldsymbol{G}_m(F) \right\};$$

and

$$N = \left\{ \left[\begin{array}{c|c|c} I_k & X & Y \\ \hline & I_{2m} & Z \\ \hline & & I_k \end{array}\right] \;\middle|\; {}^t\overline{X}(-1)^k J_k + J_{2m} Z = 0,\ J_k Y + {}^t\overline{Y}(-1)^k J_k = -{}^t\overline{Z} J_{2m} Z \right\}.$$

There is a unique Weyl group element $\widetilde{w_0}$ (for $\boldsymbol{A}_0$ in $\boldsymbol{G}_n$) satisfying $\widetilde{w_0}(\theta) \subset \Delta$ while $\widetilde{w_0}(\alpha_n) < 0$ which may be represented by the element

$$w_0 := \left[\begin{array}{c|c|c} & & I_k \\ \hline & I_{2m} & \\ \hline (-1)^k I_k & & \end{array}\right].$$

Observe that $\boldsymbol{P}$ is self-associate, meaning $w_0 \boldsymbol{M} w_0^{-1} = \boldsymbol{M}$.

Next, for a field extension $F' \supset F$, let us write $\mathrm{GL}_k(E \otimes_F F') \times \boldsymbol{G}_m(F')$ as a subgroup of $\boldsymbol{G}_n(F')$. As in Section 2.1, there are two possibilities, (i) and (ii), for $E \otimes_F F'$. In case (i), the group $\boldsymbol{G}_n(F')$ is given as in (2-1) and

$$\mathrm{GL}_k(E \otimes_F F') \times \boldsymbol{G}_m(F') \ni (g,h) \mapsto m(g,h) \in \boldsymbol{G}_n(F').$$

In case (ii), $E \otimes_F F' = F' \times F'$ with associated embeddings ι_1, ι_2 from $E \to F'$ and $\boldsymbol{G}_n/_{F'} \cong \mathrm{GL}_{2n} \times \mathrm{GL}_1$ via ι_1. Then $\boldsymbol{M}(F')$ as a subgroup of $\mathrm{GL}_{2n}(F') \times \mathrm{GL}_1(F')$ is given by

$$(g_1, g_2, h, \eta) \mapsto \left(\left[\begin{array}{c|c|c} \eta g_1 & & \\ \hline & h & \\ \hline & & g_2^\star \end{array}\right], \eta \right) \in \mathrm{GL}_{2n}(F') \times \mathrm{GL}_1(F'), \tag{2-5}$$

where $g_1, g_2 \in \mathrm{GL}_k(F')$, $h \in \mathrm{GL}_{2m}(F')$, and $\eta \in F'^\times$.

2.2.1. *L-groups and the adjoint action.* For any connected reductive group $\boldsymbol{G}$ defined over F, let ${}^L G = \widehat{G} \rtimes W_F$ be its L-group [Borel 1979] with $\widehat{G}$ being its connected component. Here, W_F is the Weil group of F which is a dense subgroup of Γ_F. Its action on $\widehat{G}$ is inherited from the action of Γ_F on $\widehat{G}$. Let Φ_n denote the $n \times n$ matrix $(\Phi_n)_{ij} = (-1)^{i+1}\delta_{i,n-j+1}$. Since $\boldsymbol{G}_n/_{\bar{F}} = \mathrm{GL}_{2n} \times \mathrm{GL}_1$, we have $\widehat{G}_n = \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$. We identify the maximal torus $\widetilde{T} \subset \widehat{G_n}$ with $(\mathbb{C}^\times)^{2n} \times \mathbb{C}^\times$ as in (2-4). Let $\widetilde{B} \supset \widetilde{T}$ be the corresponding Borel subgroup of upper triangular matrices. We use the standard F-splitting $\{\widetilde{B}, \widetilde{T}, \{E_{i,i+1}\}\}$ with root vectors $\{E_{i,i+1} \mid 1 \le i \le 2n-1\}$ and define the L-group of G_n as ${}^L G_n =$

$\mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \rtimes \Gamma_F$. The Galois action factors through $\mathrm{Gal}(E/F)$ and the nontrivial Galois automorphism θ sends $(g, \lambda) \mapsto (\Phi_{2n}^{-1}\, {}^t g^{-1} \Phi_{2n}, \lambda \det(g))$. The embedding $\widetilde{M} = \mathrm{GL}_k(\mathbb{C}) \times \mathrm{GL}_k(\mathbb{C}) \times \mathrm{GL}_{2m}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \hookrightarrow \widehat{G_n}$ is given by (2-5). For $(g_1, g_2, h, \eta) \in \widetilde{M}$, the nontrivial Galois action interchanges $g_1 \leftrightarrow g_2$, sends $h \mapsto h^\star$ and $\eta \mapsto \eta \det(h)$.

Suppose ${}^L N$ is the L-group of $\boldsymbol{N}$ as defined in [Borel 1979], and let ${}^L\mathfrak{n}$ denote the complex Lie algebra of ${}^L N$. Then

$$ {}^L\mathfrak{n} = \left\{ \left(\begin{array}{c|c|c} \mathbb{0} & X & Y \\ \hline \mathbb{0} & \mathbb{0} & Z \\ \hline \mathbb{0} & \mathbb{0} & \mathbb{0} \end{array} \right) \;\middle|\; X \in \mathrm{M}_{k\times 2m}(\mathbb{C}),\ Y \in \mathrm{M}_{k\times k}(\mathbb{C}),\ Z \in M_{2m\times k}(\mathbb{C}) \right\}. $$

Let r denote the adjoint action of $\widetilde{M} \rtimes W_F$ on ${}^L\mathfrak{n}$ — an element $(g_1, g_2, h, \eta) \in \widetilde{M}$ acts by sending

$$ X \mapsto \eta g_1 X h^{-1}, \quad Y \mapsto \eta g_1 Y g_2^{-1}, \quad Z \mapsto h Z (g_2^\star)^{-1}; $$

and the Galois action is given according to its action on the roots. For $s \geq 1$, let ρ_s denote the standard representation of $\mathrm{GL}_s(\mathbb{C})$ and 1_s the trivial representation of $\mathrm{GL}_s(\mathbb{C})$. Then $r = r_1 \oplus r_2$, where

$$ \begin{aligned} r_1|_{\widetilde{M}} &= \rho_k \otimes 1_k \otimes \tilde{\rho}_{2m} \otimes \rho_1 \oplus 1_k \otimes \rho_k \otimes \rho_{2m} \otimes 1_1, \\ r_2|_{\widetilde{M}} &= \rho_k \otimes \tilde{\rho}_k \otimes 1_{2m} \otimes \rho_1. \end{aligned} $$

Let As denote the n^2-dimensional representation of the L-group ${}^L(R_{E/F}\,\mathrm{GL}_n) = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \rtimes \Gamma_F$ given by

$$ \mathrm{As}(x, y, \gamma) = \begin{cases} x \otimes y & \text{if } \gamma \text{ restricted to } E \text{ is trivial,} \\ y \otimes x & \text{if } \gamma \text{ restricted to } E \text{ is not trivial.} \end{cases} $$

Then as a representation of ${}^L(R_{E/F}\,\mathrm{GL}_n) \times \mathrm{GL}_{2m}(\mathbb{C}) \times \mathbb{C}^\times$,

$$ r_2 = \mathrm{As} \otimes 1_{2m} \otimes \rho_1. $$

Hence r_2 is referred to as the *twisted Asai representation*.

2.2.2. *Local factors.* We continue with the pair $(\boldsymbol{G} = \boldsymbol{G}_n, \boldsymbol{P})$, $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{N}$, as above. The character ψ defines a nondegenerate character ψ^G of U via

$$ n(u, v) \mapsto \psi((u_{1,2} + \bar{u}_{1,2}) + \cdots + (u_{n-1,n} + \bar{u}_{n-1,n}) + v_{n,1}), $$

where the matrix $n(u, v)$ is as in (2-2). It is compatible with the Weyl group representative w_0 in the sense that $\psi^G(w_0^{-1} u w_0) = \psi^G(u)$, $u \in U \cap M$. By a *generic* representation of M (see [Shahidi 1988, Section 3]), we mean generic with respect to the restriction of ψ^G to $U \cap M$. Let σ be an irreducible admissible generic representation of $\mathrm{GL}_k(E)$ and let τ be an irreducible admissible generic representation of G_m. Put $\pi = \sigma \times \tau$ to denote the corresponding generic representation of M.

The Langlands–Shahidi method [Shahidi 1990] applied to the triple (G, M, π) then gives rise to the local factors $L(s, \sigma \times \tau, r_i)$ and $\gamma(s, \sigma \times \tau, r_i, \psi)$, $i = 1, 2$. We write $L(s, \sigma \times \tau)$ and $\gamma(s, \sigma \times \tau, \psi)$ to denote the factors corresponding to the representation r_1. Similarly, we have the local factors $L^U(s, \sigma \times \tau')$ and $\gamma^U(s, \sigma \times \tau', \psi)$ associated with the triplet $(H, M \cap H, \sigma \times \tau')$ for any irreducible admissible generic representation τ' of H_m. (Here, $H = H_n$.)

2.2.3. *The unramified calculation.* Assume in this section that τ, σ, and E/F are all unramified. Write $\tau = \tau(\nu_1, \nu_2, \ldots, \nu_m, \nu_0)$ as the spherical constituent of a unramified principal series representation, where $\nu_1, \nu_2, \ldots, \nu_m$ are unramified characters of $E^\times$, and ν_0 is an unramified character of $F^\times$. Its central character ω_τ is given by $\omega_\tau = \nu_1\nu_2 \cdots \nu_m(\nu_0 \circ N_{E/F})$. Similarly, write $\sigma = \sigma(\mu_1, \mu_2, \ldots, \mu_k)$, where $\mu_1, \mu_2, \ldots, \mu_k$, are unramified characters of $E^\times$. Let χ be the corresponding unramified character of the maximal torus $T \subset M$ so that π is the spherical constituent of the unramified principal series representation of M induced from χ. Let $\tilde{\rho}$ be half the sum of the absolute roots generating $\boldsymbol{N}$, and let $\tilde{\alpha}'$ be $\frac{\tilde{\rho}}{(\tilde{\alpha}_i^\vee, \tilde{\rho})}$, where $\tilde{\alpha}_i$ is one of the two simple roots outside $\tilde{\theta}$. Let α' denote the restriction of $\tilde{\alpha}'$ to the maximal split torus $\boldsymbol{A}_0$. Then one has the general formula

$$L(s, \pi, r_i) = \prod_{\substack{\gamma \in \Sigma^+ - \Sigma_\theta^+ \\ \langle \alpha', \gamma \rangle = i}} \left(1 - \chi(\gamma^\vee(\varpi_F)) q_F^{-n_\gamma s}\right)^{-1}, \quad i = 1, 2,$$

where $\langle \alpha', \gamma \rangle = \frac{2(\tilde{\gamma}, \tilde{\alpha}')}{(\tilde{\gamma}, \tilde{\gamma})}$ with $\tilde{\gamma}$ any absolute root restricting to γ and n_γ is the number of absolute roots restricting to γ. From this we obtain

$$L(s, \pi, r_1) = \prod_{i=1}^{k} \prod_{j=1}^{m} L_E(s, \bar{\omega}_\tau \mu_i \nu_j^{-1}) L_E(s, \bar{\omega}_\tau \mu_i \nu_j),$$

$$L(s, \pi, r_2) = \prod_{i=1}^{k} L_F(s, \nu_0 \mu_i|_{F^\times}) \prod_{1 \le i < j \le k} L_E(s, (\nu_0 \circ N_{E/F}) \mu_i \mu_j).$$

Here, for $\mathfrak{k} = E$ or $\mathfrak{k} = F$ and an unramified character η of $\mathfrak{k}^\times$, $L_\mathfrak{k}(s, \eta) = (1 - \eta(\varpi_\mathfrak{k}) q_\mathfrak{k}^{-s})^{-1}$, and the character $\bar{\omega}_\tau$ is the Galois conjugate of ω_τ given by $\bar{\omega}_\tau(x) = \omega_\tau(\bar{x})$, $x \in E^\times$.

2.3. ***Base change lifts from even unitary similitude groups in the generic case.*** We recall the notion of base change lifts (or transfers) from similitude unitary groups to general linear groups. Let K/k be a quadratic extension of number fields (or local fields). Consider the groups $\boldsymbol{G} = \boldsymbol{G}_n$ and $\boldsymbol{H} = \boldsymbol{H}_n$ defined with respect to K/k as in Section 2.1. Put $\widetilde{\boldsymbol{G}} = \mathrm{Res}_{K/k}(\mathrm{GL}_{2n} \times \mathrm{GL}_1)$ and $\widetilde{\boldsymbol{H}} = \mathrm{Res}_{K/k}(\mathrm{GL}_{2n})$. The identification $\boldsymbol{G}_n \simeq_K \mathrm{GL}_{2n} \times \mathrm{GL}_1$ identifies $\mathrm{Res}_{K/k}\boldsymbol{G}_n$ with $\widetilde{\boldsymbol{G}}$ and similarly for the group $\widetilde{\boldsymbol{H}}$. We introduced the dual group ${}^L G$ in Section 2.2.1, the corresponding group for the unitary group $\boldsymbol{H} = \boldsymbol{H}_n$ is given by ${}^L H = \mathrm{GL}_{2n}(\mathbb{C}) \rtimes \Gamma_k$ with the same

Galois action on $\mathrm{GL}_{2n}(\mathbb{C})$. Suppose $H \subset G$ denote the subgroups consisting of the $\mathbb{A}_k$-points (or k-points) of the algebraic groups $\boldsymbol{H} \subset \boldsymbol{G}$, respectively. The natural projection map from

$$^L G \to {}^L H, \quad (g, \lambda) \rtimes \gamma \mapsto g \rtimes \gamma,$$

corresponds to taking irreducible constituents of $\pi|_H$ for an irreducible admissible representation π of G. We note $^L\widetilde{G} = \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) \rtimes \Gamma_k$ with the Galois action factoring through the Galois group $\mathrm{Gal}(K/k)$ and the nontrivial Galois automorphism sends $(g_1, \lambda_1) \times (g_2, \lambda_2) \mapsto (g_2, \lambda_2) \times (g_1, \lambda_1)$. Similarly, $^L\widetilde{H} = \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C}) \rtimes \Gamma_k$ with the nontrivial action given by $(g_1, g_2) \mapsto (g_2, g_1)$.

The embedding $\widehat{G} \hookrightarrow \widehat{\widetilde{G}}$ given by $(g, \lambda) \mapsto ((g, \lambda), \theta(g, \lambda))$ extends to a morphism of L-groups

$$\mathrm{BC} : {}^L G \to {}^L\widetilde{G}$$

called the stable base change map (see [Kim and Krishnamurthy 2008, Section 2]). There is a similarly defined map $\mathrm{BC}^U : {}^L H \to {}^L\widetilde{H}$ (see Section 3 in [Kim and Krishnamurthy 2005] for more details).

2.3.1. *Local lift in the generic case.* We deduce the existence of local lifts for generic representations of similitude unitary groups from the corresponding results in [Kim and Krishnamurthy 2005]. Let E/F be a quadratic extension of p-adic fields, as defined in Section 2.1. Define $G := \boldsymbol{G}(F)$ and $H := \boldsymbol{H}(F)$. Recall $ZH \subset G$ is a subgroup of index 2 (see (2-3)). By [37, Section 6], the structural properties of the restriction $\pi|_H$ are governed by the relation between the groups G and H. Note that both groups share the same derived group $\mathrm{SU}(n, n)$, which satisfies the key prerequisite of [Xu 2016]. According to [Xu 2016, Corollary 6.7], the multiplicity m of an irreducible constituent $\pi_1 \subset \pi|_H$ satisfies $m^2 = |X(\pi)|/[G : G(\pi_1)]$, where $G(\pi_1)$ is the stabilizer of π_1 in G, and $X(\pi)$ is the group of characters ω of G/ZH such that $\pi \otimes \omega \cong \pi$. Since $[G : ZH] = 2$, the size of the character group $|X(\pi)|$ is bounded by 2. Because m is a positive integer, m^2 cannot equal 2, forcing $m = 1$. This proves that the restriction is always multiplicity-free.

Furthermore, this yields $|X(\pi)| = [G : G(\pi_1)]$, which naturally dictates the splitting behavior based on whether $\pi \simeq \pi \otimes \delta_{E/F}$ ($|X(\pi)| = 2$) or not ($|X(\pi)| = 1$). To be precise, we establish the following key points:

(1) For any $\pi \in \Pi(G)$, the restriction $\pi|_H$ is multiplicity-free.

(2) The restriction $\pi|_H$ is irreducible unless $\pi \simeq \pi \otimes \delta_{E/F}$, in which case it decomposes as

$$\pi|_H = \pi_1 \oplus \pi_1 \circ \mathrm{Ad}(g),$$

where $g = \mathrm{diag}(aI_n, I_n) \in G$ with $a \notin N_{E/F}(E^\times)$.

When π is ψ-generic, we fix π_1 as the unique ψ-generic constituent of $\pi|_H$.

Definition 2.1. An irreducible admissible representation $\Pi \otimes \chi$ of the group $\widetilde{\boldsymbol{G}}(F) \cong \mathrm{GL}_{2n}(E) \times \mathrm{GL}_1(E)$ is called the *local base change lift* of π if

- $\chi = \bar{\omega}_\pi$,
- the central character ω_Π of Π satisfies

$$\omega_\Pi(z) = \omega_\pi(z/\bar{z}),$$

- for all irreducible generic representations σ of $\mathrm{GL}_m(E)$,

$$\gamma^U(s, (\sigma \otimes \bar{\omega}_\pi) \times \pi_1, \psi) = \lambda(E/F, \psi)^{2mn} \gamma(s, (\sigma \otimes \chi) \times \Pi, \psi_E), \tag{2-6}$$

$$L^U(s, (\sigma \otimes \bar{\omega}_\pi) \times \pi_1) = L(s, (\sigma \otimes \chi) \times \Pi). \tag{2-7}$$

Here, $\lambda(E/F, \psi)$ denotes Langlands' λ-function associated with the pair $(E/F, \psi)$.

Remark 2.2. Suppose E/F, π and σ are all unramified. Let $\pi = \pi(\nu_1, \nu_2, \ldots, \nu_m, \nu_0)$ be the spherical constituent of a unramified principal series representation and let $\sigma = \sigma(\mu_1, \mu_2, \ldots, \mu_k)$, where $\mu_1, \mu_2, \ldots, \mu_k$ are unramified characters of $E^\times$, as in Section 2.2.3. Let Π be the unique unramified quotient $\Pi(\nu_1, \ldots, \nu_n, \nu_n^{-\theta}, \ldots, \nu_n^{-\theta})$. It follows from the unramified calculation in Section 2.2.3 that $\Pi \otimes \bar{\omega}_\pi$ is the local base change lift of π.

For ramified representations, we deduce the existence of local lifts from the corresponding results for unitary groups in [Kim and Krishnamurthy 2005].

Theorem 2.3. *Assume E/F is a quadratic extension of p-adic fields. Let π be a supercuspidal generic representation of G and let ω_π be its central character. Then it has a local lift which is unique and of the form $(\rho_1 \boxplus \cdots \boxplus \rho_k) \otimes \bar{\omega}_\pi$, where ρ_i's are supercuspidal representations of $\mathrm{GL}_{n_i}(E)$ for all $i = 1, \ldots, k$ satisfying $\rho_i \cong \rho_i^\epsilon$, $\rho_i \not\cong \rho_j$ for $i \neq j$, and $L(s, \rho_i, r_A \otimes \delta_{E/F})$ has a pole at $s = 0$. Here, $\rho_i^\epsilon = \tilde{\bar{\rho}}_i$ denotes the conjugate dual of ρ_i.*

Proof. Let π' be an irreducible admissible generic constituent of $\pi|H$ and let Π be its local lift as in [Kim and Krishnamurthy 2005, Theorem 8.4]; it is of the form $\rho_1 \boxplus \cdots \boxplus \rho_k$ with ρ_i's as stated above. By construction, Π is a local component of a global automorphic representation whose unramified components are local base change lifts as in Remark 2.2 (see [loc. cit.]) The multiplicity one theorem for idele class characters then implies that ω_Π is given by $\omega_\Pi(z) = \omega_\pi(z/\bar{z})$. We conclude $\Pi \otimes \bar{\omega}_\pi$ is a local lift of π. If $\Pi' \otimes \bar{\omega}_\pi$ were another candidate lift (generic) of π, then it follows that

$$\gamma(s, (\sigma \otimes \bar{\omega}_\pi) \times \Pi, \psi_E) = \gamma(s, (\sigma \otimes \bar{\omega}_\pi) \times \Pi', \psi_E)$$

for all irreducible generic representations σ of $\mathrm{GL}_m(E)$. We now appeal to the local converse theorem for general linear groups [Henniart 1993] to conclude $\Pi \simeq \Pi'$. $\square$

2.3.2. *Global lift in the generic case.*

Definition 2.4. Let $\pi = \bigotimes_v \pi_v$ be a globally generic irreducible cuspidal representation of $\boldsymbol{G}(\mathbb{A}_k)$ with respect to an additive character $\psi = \bigotimes_v \psi_v$ of $\mathbb{A}_k/k$. We say an automorphic representation $\Pi \otimes \chi = \bigotimes_v (\Pi_v \otimes \chi_v)$ of $\widetilde{\boldsymbol{G}}(\mathbb{A}_k) \cong \mathrm{GL}_{2n}(\mathbb{A}_K) \times \mathrm{GL}_1(\mathbb{A}_K)$ is the strong base change lift of π if for every v, $\Pi_v \otimes \chi_v$ is a local base change lift of π_v, in the above sense. It is said to be a weak base change lift if $\Pi_v \otimes \chi_v$ is a local lift of π_v for almost all places v.

Theorem 2.5. *Let π be a globally ψ-generic cuspidal representation of $\boldsymbol{G}(\mathbb{A}_k)$ with central character ω_π. There exists a strong base change lift of π and is necessarily of the form*

$$\mathrm{BC}(\pi) = (\sigma_1 \boxplus \cdots \boxplus \sigma_k) \otimes \bar{\omega}_\pi, \tag{2-8}$$

where σ_i's are unitary cuspidal representation of $\mathrm{GL}_{n_i}(\mathbb{A}_K)$ such that $\sigma_i \not\cong \sigma_j$ for $i \neq j$ and $L(s, \sigma_i, \mathrm{As}_{K/k} \otimes \delta_{K/k})$ has a simple pole at $s = 1$, $1 \leq i \leq k$.

Proof. Let π' be the cuspidal constituent of $\pi|_{\mathbb{H}(\mathbb{A}_k)}$ that is globally generic with respect to ψ. Let $\Pi' = \mathrm{BC}^U(\pi')$ be the strong base change lift of π' as in [Kim and Krishnamurthy 2005, Theorem 8.10]; it is of the form $\sigma_1 \boxplus \cdots \boxplus \sigma_k$ with $\sigma_i' s$ satisfying the stated condition. Put $\Pi = \Pi' \otimes \bar{\omega}_\pi$. Then by construction Π_v is a local base change lift of π_v for every v. On the other hand, as shown in [Kim and Krishnamurthy 2008, Lemma 5.4], any other strong base change lift Π_1 of π is necessarily of the form $\Pi_1' \otimes \bar{\omega}_\pi$ with $\Pi_{1,v}' \simeq \mathrm{BC}^U(\pi_v')$ for almost all v. This means Π_1' is a weak base change lift of π' (in the context of unitary groups). We can now appeal to [Kim and Krishnamurthy 2005, Theorem 6.8] to conclude that Π_1' is a full induced representation from unitary cuspidal representations of general linear groups. Hence by multiplicity one theorem, $\Pi_v' \simeq \Pi_{1,v}'$ for all v. □

Remark 2.6. If v is a place where the associated data are all unramified, the calculation in Section 2.2.3 implies that

$$\gamma^U(s, (\sigma_v \otimes \bar{\omega}_{\pi_v}) \times \pi_v', \psi_v) = \gamma(s, \sigma_v \times \pi_v, \psi_v),$$
$$L^U(s, (\sigma_v \otimes \bar{\omega}_{\pi_v}) \times \pi_v') = L(s, \sigma_v \times \pi_v).$$

This also holds at a split place v (see [Kim and Krishnamurthy 2005, Remark 6.1]). In this case, the representation π_v is identified with $\tau_v \otimes \xi_v$ as a representation of $\mathrm{GL}_n(F_v) \times F_v^\times$. Under this identification, $\mathrm{BC}(\pi_v) \simeq (\tau_v \otimes \tilde{\tau}_v) \otimes \bar{\omega}_{\pi_v}$. If w_1, w_2, are the places lying above v, it follows from [Kim and Krishnamurthy 2008, Section 2] that the local component of ω_π at w_1 is $\omega_{\tau_v}\xi_v$ and at w_2 is ξ_v. For the remaining finite places v, where π_v is ramified, one can establish the above equalities when π_v is supercuspidal via the local-to-global principal as in [Shahidi 1990, Section 5]. The case of a general π_v will then follow from the multiplicative property of these local factors.

2.4. ***Generic Arthur packet conjectures.*** We begin with a review of the concept of L-packets and A-packets in representation theory of p-adic groups. For F as in Section 2, let W_F' be the group $W_F' = W_F \times \mathrm{SL}_2(\mathbb{C})$, where as usual W_F is the Weil group of F. Let $I_F \subset W_F$ be the inertia subgroup of F and let us fix a geometric Frobenius element $\mathrm{Fr} \in W_F$ which satisfies $\mathrm{Fr}(x^q) = x$ for x in the residue field of the maximal unramified extension of F. We write $\|\cdot\| : W_F \to \mathbb{C}^\times$ to denote the unramified character of W_F that takes the value q^{-1} on Fr. For a connected reductive group $\boldsymbol{G}$ defined over F, let $\Pi(G)$ denote the set of equivalence classes of irreducible representations of G, where $G := \boldsymbol{G}(F)$. A Langlands parameter is a homomorphism ϕ from $W_F' \to {}^L G$ satisfying

(1) $\phi : \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ is algebraic,

(2) ϕ is trivial on an open subgroup of I_F and $\phi(\mathrm{Fr})$ is semisimple,

(3) ϕ is compatible with the projection onto the Weil group W_F.

Two such parameters ϕ, ϕ' are said to be equivalent if there is $g \in \widehat{G}$ such that $\phi' = \mathrm{Ad}(g) \circ \phi$. Let $\Phi(G)$ be the set of equivalence classes of Langlands parameters. The local Langlands conjecture asserts that to each ϕ in $\Phi(G)$ corresponds a finite set $\Pi_\phi(G)$ of equivalence classes of irreducible admissible representations of the group G called L-packets which partition the set $\Pi(G)$. A Langlands parameter ϕ is said to be tempered if the image $\phi(W_F)$ of W_F is bounded in $\widehat{G}$. We write $\Phi_{\mathrm{temp}}(G) \subset \Phi(G)$ to denote the subset of equivalence classes of all tempered parameters. Let $\Pi_{\mathrm{temp}}(G)$ denote the subset of $\Pi(G)$ that consists of tempered representations, in the sense that their characters are tempered distributions on G.

Let us now recall the notion of A-packets as introduced by Arthur. For a detailed account, see [1989; 2013]. An Arthur parameter ψ is a continuous homomorphism

$$\psi : W_F' \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

such that the restriction of ψ to W_F' belongs to $\Phi_{\mathrm{temp}}(G)$ and the restriction of ψ to the supplemental $\mathrm{SL}_2(\mathbb{C})$ factor is algebraic. Let $\Psi(G)$ denote the set of all Arthur parameters subject to the same equivalence as above. Arthur also introduced a larger set denoted as $\Psi^+(G)$ whose elements are equivalence classes of parameters ψ but without the requirement that ψ be bounded upon restriction to W_F'. For each Arthur parameter ψ, there is an associated Langlands parameter ϕ_ψ given by the formula

$$\phi_\psi(w) = \psi\left(w, \begin{pmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{pmatrix}\right), \quad w \in W_F',$$

where $\|\cdot\|$ on W_F' is the pullback of the above defined unramified character of W_F.

Arthur [2013] conjectured that for each ψ in $\Psi(G)$, there exists an associated finite set $\Pi_\psi(G)$ of irreducible admissible representations of G, referred to as the A-packet of ψ, satisfying several defining properties. Primarily, A-packets are

constrained to comprise solely unitary representations and must encapsulate the associated L-packet, meaning $\Pi_{\phi_\psi}(G) \subset \Pi_\psi(G)$. A pivotal aspect of the A-packet of ψ involves the presence of a pairing between a naturally affiliated finite group A_ψ and the set $\Pi_\psi(G)$, thereby facilitating a parameterization of $\Pi_\psi(G)$ in terms of irreducible characters of the group A_ψ. Contrary to L-packets, A-packets are neither disjoint nor do they partition the set of equivalence classes of irreducible unitary representations of G.

When $\boldsymbol{G}$ is a quasisplit classical group — such as a symplectic, special orthogonal or unitary group — Arthur's conjecture and consequently the local Langlands conjecture have been proven. Arthur [2013] established these results for symplectic and orthogonal groups, while Mok [2015] extended them to quasisplit unitary groups.

We now introduce the generic Arthur packet conjecture.

Conjecture 2.7. Let ψ and ϕ_ψ be as above and let $\Pi_{\phi_\psi}(G)$ be the L-packet associated to ϕ_ψ. Assume $\Pi_{\phi_\psi}(G)$ has a generic member with respect to some Whittaker datum. Then ϕ_ψ is tempered.

Remark 2.8. Assuming the Langlands–Shahidi and Artin L-functions are equal, Shahidi [2011, Theorem 5.1] proved Conjecture 2.7 for any (quasisplit) $\boldsymbol{G}$. For additional progress on Conjecture 2.7 in the context of classical groups and GSpin groups, see [Ban 2006; Jantzen and Liu 2014; Liu 2011; Heiermann and Kim 2017; Kim 2021].

Definition 2.9. An L-packet $\Pi_\phi(G)$ is said to be tempered if all its members are tempered representations.

Here is another version of the above conjecture:

Conjecture 2.10. Let ψ, ϕ_ψ, and $\Pi_{\phi_\psi}(G)$ be as in Conjecture 2.7. If $\Pi_{\phi_\psi}(G)$ has a generic member with respect to any Whittaker datum, then it is a tempered L-packet.

This paper aims to unconditionally prove these conjectures for unitary and similitude unitary groups of even rank.

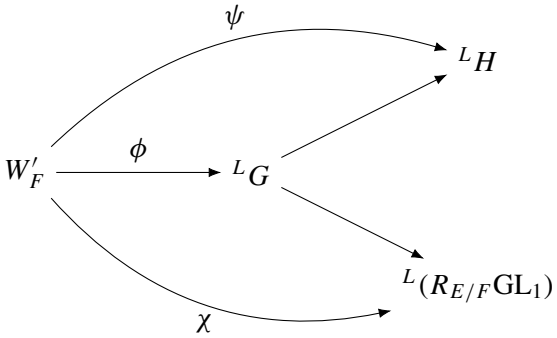
3. The local conjectures for unitary similitude groups

For a quadratic extension E/F of nonarchimedean local fields, we set $G = \boldsymbol{G}_n(F)$ and $H = \boldsymbol{H}_n(F)$ as before. There is a natural relationship between L-parameters of G and those of H, as explained in [Langlands and Ramakrishnan 1992].

Namely, recall the projection maps from the L-group ${}^L G$ into ${}^L H$ and into ${}^L(R_{E/F}\mathrm{GL}_1)$. On the connected component of ${}^L G$ these maps are given by

$$(A, y) \mapsto A \quad \text{and} \quad (A, y) \mapsto ((\det A)y, y),$$

respectively. Any parameter $\phi : W'_F \to {}^L G$ gives rise to a pair (ψ, χ) via these projections:



Note that χ is equivalent to a character of $E^\times$ via class field theory; by abuse of notation, we also write it as χ. In the other direction, given parameters (ψ, χ), we define the corresponding $\phi : W'_F \to {}^L G$ as follows: On W'_E it is given by

$$x \mapsto (\psi(x), \chi(\bar{z})), \quad x \in W'_E,$$

where $x \mapsto z$ is given by the class field theory map. For a fixed representative $w_c \in W_F/W_E$, suppose that $\psi(w_c) = a \rtimes \gamma$ and $\chi(w_c) = (x, y) \rtimes \gamma$. Then $\phi(w_c) = (a, z) \rtimes \gamma$ where z is such that $\det(a)z^2 = xy$. In what follows, we will consider $\phi|_{W'_E}$ as a homomorphism from $W'_E \to \mathrm{GL}_{2n}(\mathbb{C})$. First, we identify $\phi|_{W'_E}$ as a homomorphism from $W'_E \to \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$ (see [Borel 1979, Lemma 4.5]). Then apply the representation $(g, \lambda) \mapsto \lambda g$. A characteristic feature of the stable base change map BC is that it corresponds to restriction to W'_E on the dual side.

3.1. *Construction of generic L-packets.*

Theorem 3.1. *Let π be an irreducible admissible generic representation of G. Then, there exists a Langlands parameter $\phi_\pi : W'_F \to {}^L G$ such that for any irreducible admissible generic representation ρ of $\mathrm{GL}_k(E)$, $k \geq 1$, we have*

$$L(s, \rho \times \pi) = L(s, \phi_\rho \otimes (\phi_\pi|_{W'_E})) \quad \text{and} \quad \gamma(s, \rho \times \pi, \psi) = \gamma(s, \phi_\rho \otimes (\phi_\pi|_{W'_E}), \psi_E),$$

where ϕ_ρ is the Langlands parameter associated to ρ through the local Langlands correspondence for general linear groups [Harris and Taylor 2001; Henniart 2000].

Proof. Let us first assume π is supercuspidal. In that case, if π' is the irreducible admissible generic constituent of $\pi|H$, it is also supercuspidal. Let Π' be the local lift of π' to $\mathrm{GL}_{2n}(E)$ as in the proof of Theorem 2.3. Then, it is of the form $\Pi' = \rho_1 \boxplus \cdots \boxplus \rho_k$, where the ρ_i's are mutually nonisomorphic conjugate self-dual supercuspidal representations of $\mathrm{GL}_{n_i}(E)$ and satisfying $L(s, \rho_i, r_A \otimes \delta_{E/F})$ has a pole at $s = 0$, $1 \leq i \leq k$. The Langlands parameter $\phi_{\Pi'} : W'_E \to \mathrm{GL}_{2n}(\mathbb{C})$ associated to Π' is given by $\phi_{\rho_1} \oplus \cdots \oplus \phi_{\rho_k}$, where ϕ_{ρ_i} is the Langlands parameter associated

to ρ_i through the local Langlands correspondence for $\mathrm{GL}_{n_i}(E)$. In the terminology of [Mok 2015, §2.2], the parameter $\phi_{\Pi'}$ is conjugate self-dual with parity -1. Let $\phi_{\pi'} : W'_F \to {}^L H$ be the parameter associated to $\phi_{\Pi'}$ via [Mok 2015, Lemma 2.2.1]. If one takes the character χ_κ in [loc. cit.] to be trivial, then the associated map $\xi_{\chi_\kappa,*}$ from $\Phi(H)$ to $\Phi(\mathrm{GL}_{2n}(E))$ as defined by Mok coincides with our base change map BC^U. In particular, the restriction of $\phi_{\pi'}$ to W'_E recovers $\phi_{\Pi'}$, i.e.,

$$\phi_{\pi'}|_{W'_E} = \phi_{\Pi'}.$$

Now, let $\phi_\pi : W'_F \to {}^L G$ be the parameter associated with the pair $(\phi_{\pi'}, \omega_\pi)$ in the sense described above. It follows from Theorem 2.3 and Remark 2.6 that

$$L(s, \rho \times \pi) = L^U(s, (\rho \otimes \bar{\omega}_\pi) \times \pi') = L(s, \phi_\rho \otimes (\phi_\pi|_{W'_E})),$$
$$\gamma(s, \rho \times \pi, \psi_F) = \gamma^U(s, (\rho \otimes \bar{\omega}_\pi) \times \pi', \psi_F) = \gamma(s, \phi_\rho \otimes (\phi_\pi|_{W'_E}), \psi_E).$$

For nonsupercuspidal admissible representation, we follow the construction of [Heiermann 2006a]. Namely, first consider the discrete series case $\pi = \pi_{\mathrm{ds}}$. There exist a parabolic subgroup $P = MN$ of G, an irreducible supercuspidal representation π_{sc} of M, and a point λ in the real vector space $a_M^* = X(M)_F \otimes \mathbb{R}$ such that π_{ds} is a subquotient of $\mathrm{Ind}_P^G(\pi_{\mathrm{sc}} \otimes \chi_\lambda)$. (Here, $X(M)_F$ is the group of rational characters of M and χ_λ is the usual unramified quasicharacter associated to the point λ.) If $\phi_{\pi_{\mathrm{sc}}}$ is the Langlands parameter attached to π_{sc} as above, then the Langlands parameter $\phi_{\pi_{\mathrm{ds}}}$ is given by

$$\phi_{\pi_{\mathrm{ds}}}(\gamma, h) = \phi_{\pi_{sc}}(\gamma, 1)\phi_{\pi_{sc},\lambda}(h) \quad \text{for } \gamma \in W_F \text{ and } h \in \mathrm{SL}_2(\mathbb{C})$$

as explained in [Heiermann 2006a, Section 6.2]. The notation $\phi_{\pi_{sc},\lambda}(h)$ is explained in [loc. cit.].

To pass from discrete series to an irreducible admissible representation of G, we use Langlands classification. In particular, there is a parabolic subgroup $P_1 = M_1 N_1$ of G, a tempered representation π_t of M_1 and a point λ_{M_1} in $a_{M_1}^*$ in the positive Weyl chamber of P_1 such that π is the Langlands quotient of the induced representation $\mathrm{Ind}_{P_1}^G(\pi_t \otimes \chi_{\lambda_{M_1}})$. Furthermore, there is a parabolic subgroup $P_2 = M_2 N_2$ of $\boldsymbol{M}_1$ and a discrete series π_{ds} such that $\pi_t \hookrightarrow \mathrm{Ind}_{P_2 \cap M_1}^{M_1}(\pi_{\mathrm{ds}})$. Note that we constructed the Langlands parameter $\phi_{\pi_{\mathrm{ds}}}$ that corresponds to π_{ds}. Then in [Heiermann 2006a, Section 6.2] the Langlands parameter $\psi_\pi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$ is constructed as

$$(\gamma, h) \mapsto s_{M_1}^{v_F(\gamma)} \phi_{\pi_{\mathrm{ds}}}(\gamma, h),$$

where s_{M_1} is a semisimple element in $C({}^L M_1)^\circ$ (the connected component of the center of the group ${}^L M_1$) that corresponds to λ_{M_1} through the local Langlands correspondence for tori. Note that we have the equalities of local factors [Heiermann 2006b, Theorem 4.4]. □

Suppose π is an irreducible admissible generic representation of G and $\phi = \phi_\pi$ as above. Recall we have the conjectural set $\Pi_\phi(G)$. The following result characterizes all generic members in this set in terms of L-functions from the Langlands–Shahidi method.

Theorem 3.2. *Let π_1 (resp. π_2) be an irreducible admissible generic representation of G and let ϕ_{π_1} (resp. ϕ_{π_2}) be its Langlands parameter as constructed in Theorem 3.1. Then, $\phi_{\pi_1} \cong \phi_{\pi_2}$ if and only if for each irreducible admissible generic representation ρ of $\mathrm{GL}_r(F)$ we have*

$$L(s, \rho \times \pi_1) = L(s, \rho \times \pi_2) \quad \text{and} \quad \gamma(s, \rho \times \pi_1, \psi) = \gamma(s, \rho \times \pi_2, \psi).$$

Proof. Assume $\phi_{\pi_1} \cong \phi_{\pi_2}$. Theorem 3.1 directly tells us the equalities of local factors. We now assume the equality of local factors. We have $L(s, \rho \times \Pi_1) = L(s, \rho \times \Pi_2)$ for any irreducible admissible generic representation ρ of $\mathrm{GL}_r(F)$, where Π_1 (resp. Π_2) is local functorial lift of π_1 (resp. π_2). Then, the local converse theorem for general linear groups implies that $\Pi_1 \cong \Pi_2$ [Henniart 1993]. Note that construction of the Langlands parameter in the proof of Theorem 3.1 only depends on its local functorial lift. Therefore, $\phi_{\pi_1} \cong \phi_{\pi_2}$. □

This points to a way of defining the generic L-packet $\Pi_\phi(G)$ under the following assumption.

Assumption 3.3. Assume there exists a theory of local factors that extends the Langlands–Shahidi construction to nongeneric representations π. Specifically, for any irreducible admissible representation ρ of the group $\mathrm{GL}_r(E)$, there exists local factors $L'(s, \rho \times \pi)$ and $\gamma'(s, \rho \times \pi, \psi)$ that are compatible with the Langlands classification.

Definition 3.4 (generic L-packets). For any fixed irreducible admissible generic representation π_0 of G with L-parameter $\phi = \phi_{\pi_0}$, let $\Pi_\phi(G)$ be the set of equivalence classes of irreducible admissible representations π such that for each irreducible admissible representation ρ of $\mathrm{GL}_r(E)$ we have

$$L'(s, \rho \times \pi) = L(s, \rho \times \pi_0) \quad \text{and} \quad \gamma'(s, \rho \times \pi, \psi) = \gamma(s, \rho \times \pi_0, \psi).$$

Remark 3.5. One main purpose is to remove Assumption 3.3 so that applications of our main theorems, i.e., Theorems 3.10 and 3.11, become completely unconditional. We remove it in Section 4.

We prove the following property for generic L-packets in the case of even unitary similitude groups:

Lemma 3.6. *If an L-packet $\Pi_\phi(G)$ contains a tempered representation that is also generic, then all members in $\Pi_\phi(G)$ are tempered as well (i.e., $\Pi_\phi(G)$ is a tempered L-packet).*

Proof. Let π_0 be an irreducible tempered generic representation in $\Pi_\phi(G)$ and suppose that there exists a nontempered representation $\pi \in \Pi_\phi(G)$. Since π and π_0 are in the same L-packet, we have $L'(s, \rho \times \pi) = L(s, \rho \times \pi_0)$ for any irreducible admissible representation ρ of $\mathrm{GL}_s(F)$, $s \geq 1$.

The Langlands classification implies that π can be considered to be the unique quotient of the induced representation

$$\nu^{r_1}\tau_1 \times \cdots \times \nu^{r_m}\tau_m \rtimes \pi_t,$$

where τ_i is a tempered representation of $\mathrm{GL}_{n_i}(F)$, $n_i \geq 1$ for each $i = 1, \ldots, m$ with $r_1 > r_2 > \cdots > r_m > 0$, and π_t is a tempered representation of G_{n_t} (n_t and n_0 are in the same parity). Since π is nontempered, $m \geq 1$.

Since L-functions follow the Langlands classification (Assumption 3.3), we have

$$\begin{aligned} L(s, \rho \times \pi_0) &= L'(s, \rho \times \pi) \\ &= L'(s, \rho \times \pi_t) \prod_{i=1}^{m} L(s + r_i, \rho \times (\bar{\omega}_{\pi_0} \otimes \tau_i)) L(s - r_i, \rho \times (\bar{\omega}_{\pi_0} \otimes \widetilde{\bar{\tau}_i})) \end{aligned}$$

for any admissible representation ρ of $\mathrm{GL}_s(F)$, $s \geq 1$. If we let ρ be a contragredient representation of $\bar{\omega}_{\pi_0} \otimes \widetilde{\bar{\tau}_i}$, then $L(s - r_j, \rho \times (\bar{\omega}_{\pi_0} \otimes \widetilde{\bar{\tau}_i}))$ has a pole at $s = r_j > 0$. This also implies that $L(s, \rho \times \pi_0) = L'(s, \rho \times \pi)$ has a pole at $s = r_j > 0$ when ρ is a contragredient representation of $\bar{\omega}_{\pi_0} \otimes \widetilde{\bar{\tau}_i}$. This contradicts that $L(s, \rho \times \pi_0)$ is holomorphic for $\mathrm{Re}(s) > 0$ since both ρ and π_0 are tempered and generic [Heiermann and Opdam 2013]; therefore, $m = 0$. Thus all members in $\Pi_\phi(G)$ are tempered. □

3.1.1. *A weak version of the generic Arthur packet conjectures for even unitary similitude groups.* Thanks to [Shahidi 2011, Theorem 5.1], Theorem 3.1 implies Conjecture 2.7 in the case of even unitary similitude groups.

Theorem 3.7. *Let ψ be an Arthur parameter for G and ϕ_ψ be the corresponding Langlands parameter. Assume there exists an L-packet $\Pi_{\phi_\psi}(G)$ corresponding to the Langlands parameter ϕ_ψ. If $\Pi_{\phi_\psi}(G)$ has a generic member, then ϕ_ψ is a tempered Langlands parameter.*

We now prove the strong version of the generic Arthur packet conjecture (namely, Conjecture 2.10) for even unitary similitude groups. We first study L-packets through the local Langlands correspondence. Then we prove several properties of L-packets and Langlands parameters in the following subsections.

3.1.2. *A strong version of the generic Arthur packet conjecture for even unitary similitude groups.* We first prove some embedding properties of discrete series of G.

Proposition 3.8. *Let π_{ds} be an irreducible discrete series generic representation of G. Then π_{ds} can be embedded into the induced representation*

$$\pi_{\mathrm{ds}} \hookrightarrow \delta_1 \times \cdots \times \delta_r \times \delta_1' \times \cdots \times \delta_k' \rtimes \pi_{\mathrm{cusp}}, \tag{3-1}$$

where $\delta_i = \delta([\nu^{a_i}\rho_i, \nu^{b_i}\rho_i])$, $a_i \le 0$, $a_i + b_i > 0$, ρ_i *is an irreducible unitary supercuspidal representation of* $\mathrm{GL}_{n_i}(E)$, $n_i \ge 1$ *for* $i = 1, \dots, r$, $\delta'_j = \delta([\nu^{a_{\rho'_j}}\rho'_j, \nu^{b^{(j)}}\rho'_j])$ *are as in* [Kim and Matić 2019, (4.5)] (*that is*, δ'_j *are from the classification of strongly positive representations*), *and* π_{cusp} *is an irreducible supercuspidal generic representation of* G_{n_c}, $n_c \ge 1$. *Furthermore, each* $a_{\rho'_j}$ *is in* $\left\{\frac{1}{2}, 1\right\}$ *for* $j = 1, \dots, k$.

Proof. The proof is similar to [Kim 2015, Section 6] and we briefly explain the main ideas of the proof instead of repeating similar arguments. In the case when π_{ds} is strongly positive, the theorem follows with $r = 0$. Suppose that π_{ds} is not strongly positive. In the same way as in the proof of [Kim 2015, Theorem 5.3], we conclude that there exist a sequence of segments $\Delta_1, \dots, \Delta_k$ satisfying $e(\Delta_1) \le \cdots \le e(\Delta_k)$ and an irreducible supercuspidal representation $\pi_{\mathrm{cusp}} \in R(G_{n_c})$, $n_c \ge 1$, such that we have $\pi_{\mathrm{ds}} \hookrightarrow \delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \pi_{\mathrm{cusp}}$. Write $\Delta_i := [\nu^{a_i}\rho_i, \nu^{b_i}\rho_i]$, where $\rho_i \in R(GL)$ is an irreducible unitary supercuspidal representation for $i = 1, \dots, k$. Let $a := \min\{a_i \mid 1 \le i \le k\}$ and let j be such that $a_j = a$. Then minimality of a_j implies that there exists an irreducible representation $\pi_1 \in R(G_{n_1})$, $n_1 \ge 1$, such that $\pi_{\mathrm{ds}} \hookrightarrow \delta(\Delta_j) \rtimes \pi_1$. We show that π_1 is a discrete series representation. Suppose that π_1 is not a discrete series representation. Then there exists an embedding of the form $\pi_1 \hookrightarrow \delta([\nu^{a'}\rho, \nu^{b'}\rho]) \rtimes \pi'$, where $a' + b' \le 0$. Then, minimality of a again implies that we have $\pi_{\mathrm{ds}} \hookrightarrow \delta([\nu^{a'}\rho, \nu^{b'}\rho]) \times \delta([\nu^{a_j}\rho_j, \nu^{b_j}\rho_j]) \rtimes \pi'$ which is a contradiction since π is a discrete series representation. We conclude that π_1 is also a discrete series representation. If π_1 is strongly positive, the theorem follows with $r = 1$. If not, in the same way as above, π_1 can be embedded into the representation $\delta([\nu^{a_{j'}}\rho_{j'}, \nu^{b_{j'}}\rho_{j'}]) \rtimes \pi_2$, where $a_{j'} \le 0$, $a_{j'} + b_{j'} > 0$ and π_2 is a discrete series representation. We repeat this argument until we get strongly positive. □

Lemma 3.9. *Let* π *be an irreducible admissible generic representation of* G. *Then*, π *is a tempered representation if and only if its local functorial lift* Π *is a tempered representation of* $\mathrm{GL}_{2n}(E)$.

Proof. Suppose that π is a nontempered admissible generic representation of G and its local functorial lift Π is a tempered representation. Then, due to [Heiermann and Muić 2007], there exists $r_1 > r_2 > \cdots > r_m > 0$ such that π can be written as the induced representation

$$\nu^{r_1}\tau_1 \times \cdots \times \nu^{r_m}\tau_m \rtimes \pi_t,$$

where τ_i is a tempered representation of $\mathrm{GL}_{n_i}(E)$, $n_i \ge 1$ for each $i = 1, \dots, m$, and π_t is a tempered generic representation of G_{n_t} for some $n_t < n$. Since π is nontempered, $m \ge 1$.

Now we consider the L-function $L(s, (\omega_\pi \otimes \widetilde{\tau_j}) \times \pi)$. Since the L-functions from the Langlands–Shahidi method are defined by the Langlands classification,

for any admissible representation ρ we have

$$L(s, \rho \times \pi) = L(s, \rho \times \pi_t) \prod_{i=1}^{m} L(s + r_i, \rho \times (\bar{\omega}_\pi \otimes \tau_i)) L(s - r_i, \rho \times (\bar{\omega}_\pi \otimes \widetilde{\tau_i})).$$

As in the proof of Lemma 3.6, we conclude that $L(s, \rho \times \Pi) = L(s, \rho \times \pi)$ has a pole at $s = r_j > 0$ when ρ is a contragredient representation of $\bar{\omega}_\pi \otimes \widetilde{\tau_j}$. This contradicts that $L(s, \rho \times \Pi)$ is holomorphic for $\mathrm{Re}(s) > 0$ since both ρ and Π are tempered and generic [Heiermann and Opdam 2013]; therefore, π is also a tempered representation.

For the sufficient condition we exactly follow the arguments given in [Kim and Krishnamurthy 2005, Proposition 8.5] and we skip the proof and briefly mention which tools that we need to apply those arguments. We use Theorem 2.3 and Proposition 3.8 to show the local functorial lift of discrete series of G is tempered. Then it is straightforward to show that the functorial lift of tempered representations is also tempered. □

We are ready to prove a strong version of the generic Arthur packet conjecture for even unitary similitude groups under Assumption 3.3. Let ψ, ϕ_ψ be as in Conjecture 2.7. Let $\Pi_{\phi_\psi}(G)$ denote the (generic) L-packet corresponding to ϕ_ψ (see Definition 3.4).

Theorem 3.10. *The generic L-packet $\Pi_{\phi_\psi}(G)$ is tempered.*

Proof. Let π be a generic member in $\Pi_{\phi_\psi}(G)$. Theorem 3.7 implies that the corresponding L-parameter $\phi_\psi : W_F' \to {}^L G = \widehat{G} \rtimes W_F$ is a tempered L-parameter; therefore, its image $\phi_\psi(W_F)$ in $\widehat{G}$ is bounded. It follows that $\mathrm{BC} \circ \phi_\psi = \phi_\psi|_{W_E'}$ also has a bounded image and thus a tempered L-parameter for $\mathrm{GL}_{2n}(E) \times \mathrm{GL}_1(E)$. By the local Langlands correspondence for general linear groups [Harris and Taylor 2001; Henniart 2000], there exists an irreducible tempered representation Π of $\mathrm{GL}_{2n}(E) \times \mathrm{GL}_1(E)$ corresponding to $\mathrm{BC} \circ \phi_\psi$ satisfying

$$\begin{aligned} \gamma(s, \rho \times \Pi, \psi_E) &= \gamma(s, \phi_\rho \otimes (\phi_\psi|_{W_E'}), \psi_E), \\ L(s, \rho \times \Pi) &= L(s, \phi_\rho \otimes (\phi_\psi|_{W_E'})) \end{aligned} \tag{3-2}$$

for any irreducible admissible representation ρ of $\mathrm{GL}_s(E)$, $s \geq 1$, where ϕ_ρ is the L-parameter corresponding to ρ. The local factors on the left side are Rankin–Selberg local factors [Jacquet et al. 1983], while those on the right side are Artin factors. Combined with Theorem 3.1, the equalities in (3-2) imply

$$\gamma(s, \rho \times \pi, \psi) = \gamma(s, \rho \times \Pi, \psi_E), \quad L(s, \rho \times \pi) = L(s, \rho \times \Pi).$$

Consequently, Π is a local functorial lift of π.

Since Π is tempered, Lemma 3.9 implies that π is a tempered representation as well. It remains to prove that all other members in $\Pi(\phi_\psi)$ are tempered as well. Since $\Pi(\phi_\psi)$ contains a tempered generic representation π, Lemma 3.6 implies that $\Pi(\phi_\psi)$ is a tempered L-packet. □

We now briefly explain how we can adapt the above idea to prove the generic Arthur packet conjecture in the case of even unitary groups.

Theorem 3.11. *Let $\psi \in \Psi(H)$ be an Arthur parameter for the even unitary group and let ϕ_ψ be its corresponding Langlands parameter. Suppose the L-packet $\Pi_{\phi_\psi}(H)$ has a generic member. Then $\Pi_{\phi_\psi}(H)$ is a tempered L-packet.*

Proof. The first step, i.e., equality of L-functions from the Langlands–Shahidi method and Artin L-functions is proved in Proposition 8.7 of [Kim and Krishnamurthy 2005]. Therefore, a weak version of the generic Arthur packet conjecture for H is true due to [Shahidi 2011, Theorem 5.1]. It now remains to prove a strong version of the conjecture. Let us point out three important steps of the proof. First, the construction of Langlands parameter for H is already described in the proof of Theorem 3.1. Second, we define L-packets exactly as in Definition 3.4 with the same Assumption 3.3. Third, Lemmas 3.6 and 3.9 can be generalized to even unitary groups since it depends on the properties of L-functions from the Langlands–Shahidi methods and classification of discrete series, which are available for H. Therefore, we can apply the argument in the proof of Theorem 3.10 to the case of even unitary groups. □

4. Removing Assumption 3.3

We continue with the notation as in Section 3. The purpose of this section is to make Theorems 3.10 and 3.11 unconditional. Namely, we define generic L-packets so that we can remove Assumption 3.3 for the groups G and H, respectively.

4.1. ***L-packets for H.*** Recall the L-packets for the group H defined in [Mok 2015, Theorem 2.5.1(b)]. Briefly, for an Arthur parameter $\psi \in \Psi(H)$, the Arthur packet $\Pi_\psi(H)$ is defined in [Mok 2015, Theorem 2.5.1(a)]. This includes definitions of L-packets that correspond to tempered Langlands parameter since ψ becomes a tempered Langlands parameter when it is trivial on the second $\mathrm{SL}_2(\mathbb{C})$-factor. Furthermore, L-packets, in general, are defined as follows: Recall in [Arthur 1984] that a member ϕ of $\Phi(H)$ determines a commuting pair $\phi_t \in \Phi_{\mathrm{temp}}(H)$ and a $\phi_+ \in \Phi(H)$ that satisfies

$$\phi(w) = \phi_t(w)\phi_+(w) \quad (w \in W_F'). \tag{4-1}$$

The centralizer of the image of ϕ_+ in ${}^L H$ is a Levi subgroup ${}^L M_H$. Note that ϕ_+ will be a map into ${}^L A_H$, where $\boldsymbol{A}_{H_n}$ is the split component of $\boldsymbol{M}_{H_n}$ Also, the image of ϕ_t

is in ${}^L M_H$ since ϕ_t and ϕ_+ commute. Let $\Pi_{\phi_t} = \Pi_{\phi_t}(M_H)$ be the tempered L-packet corresponding to ϕ_t through [Mok 2015, Theorem 2.5.1(b)]. Let $\nu : A_H \to \mathbb{C}^*$ be a character that corresponds to ϕ_+. For $\pi_t \in \Pi_{\phi_t}$, let $I(\nu, \pi_t) = \mathrm{Ind}_P^G(\pi_t \otimes \nu)$. We assume that this is a standard module after the conjugation of Weyl group elements. Then, L-packets for H in general are defined as follows:

Definition 4.1. Let ϕ, ϕ_t, ϕ_+, Π_{ϕ_t}, and ν be as above. The L-packet $\Pi_\phi(H)$ is defined as

$$\Pi_\phi(H) = \{J(\nu, \pi_t) \mid \pi_t \in \Pi_{\phi_t}\},$$

where $J(\nu, \pi_t)$ is unique Langlands quotient of a standard module $I(\nu, \pi_t)$.

4.2. ***Base change lifts and L-packets for G.*** Recall that we defined local lifts of generic representations of G in Section 2.3.1 using the Langlands–Shahidi method. In fact, we may define local lifts more generally using the endoscopic classification established in [Mok 2015]. Namely, suppose that π is an irreducible admissible tempered representation of G. As noted in Section 2.3.1, restriction $\pi|_H$ is irreducible or decomposes as

$$\pi|_H = \pi_1 \oplus \pi_1', \quad \text{where } \pi_1' = \pi_1 \circ \mathrm{Ad}(g),$$

with g is as in Section 2.3.1. If the restriction to H is irreducible, let ϕ be the Langlands parameter attached to $\pi|_H$. Otherwise, say ϕ is the Langlands parameter attached to π_1. Then $\phi' := \widehat{\mathrm{Ad}(g)} \circ \phi$ is the Langlands parameter attached to π_1'. Since $\mathrm{Ad}(g)$ actually defines an inner E-automorphism of the unitary group $H \times_F E \cong \mathrm{GL}_{2n}/E$, the corresponding dual morphism $\widehat{\mathrm{Ad}(g)} : \widehat{H} \to \widehat{H}$ is also an inner automorphism, given by conjugation by an element in $\widehat{H} = \mathrm{GL}_{2n}(\mathbb{C})$ following [Borel 1979]. In particular, when restricted to the Deligne–Weil group W_E' of E, the parameters $\phi|_{W_E'}$ and $\phi'|_{W_E'}$ are conjugate by an element in $\mathrm{GL}_{2n}(\mathbb{C})$. Consequently, as representations of $\mathrm{GL}_{2n}(E)$,

$$\mathrm{BC}^U(\pi_1) \simeq \mathrm{BC}^U(\pi_1').$$

Now, if the stable base change lift $\mathrm{BC}(\pi)$ from G exists, it must be of the form $\Pi \otimes \chi$, where χ is a character of $E^\times$. Since the projection ${}^L G \to {}^L H$ commutes with base change, we can argue as in [Kim and Krishnamurthy 2008, Lemma 5.4] (see [Rogawski 1992, Lemma 4.1.1]) to conclude $\Pi \simeq \mathrm{BC}^U(\pi_1)$ and $\chi = \bar{\omega}_\pi$. The above argument confirms that the choice of the irreducible constituent π_1 is irrelevant for the definition of the local lift. We can pass from a tempered representation to any irreducible admissible representation via the Langlands classification.

Definition 4.2. Let π be an irreducible admissible representation of G and ω_π be its central character. Let π' be an irreducible constituent of $\pi|_H$. Then the stable base change lift $\mathrm{BC}(\pi)$ of π is defined as $\mathrm{BC}^U(\pi') \otimes \bar{\omega}_\pi$.

This definition recovers Definition 2.1 in the generic case. We use these lifts to formalize the notion of L-packets for the group G.

Definition 4.3. Let π_0 be an irreducible admissible representation of G. We define the local L-packet containing π_0 as the set of all irreducible admissible representations π of G such that

(1) $\omega_\pi = \omega_{\pi_0}$,

(2) $\mathrm{BC}(\pi) \cong \mathrm{BC}(\pi_0)$.

Furthermore, we define local factors using the base change lift as follows.

Definition 4.4. Let π be an irreducible admissible representation of G. For any irreducible admissible representation σ of $\mathrm{GL}_m(E)$ we define the local factors as

$$\gamma(s, \sigma \times \pi, \psi) = \lambda(E/F, \psi)^{2mn} \gamma(s, \sigma \times \mathrm{BC}(\pi), \psi_E),$$
$$L(s, \sigma \times \pi) = L(s, \sigma \times \mathrm{BC}(\pi)).$$

Here, the local factors on the right-hand side are Rankin–Selberg local factors or, equivalently, those from the Langlands–Shahidi method [Jacquet et al. 1983; Shahidi 1984].

We now prove several important properties of L-packets, including Shahidi's conjecture.

Proposition 4.5. *The L-packets for G satisfy the following properties*:

(1) *An L-packet for G consists of a finite number of members.*

(2) *A tempered L-packet contains a generic representation.*

(3) *L-functions follow the Langlands classification.*

(4) *L-functions defined in Definition 4.4 agree with those from the Langlands–Shahidi method in the generic case.*

Proof. We first prove (1), i.e., finiteness of L-packets for G. Let π be an irreducible admissible unitarizable representation of G and Π be an L-packet that contains π. Let π_1 be another member in Π and let π' (resp. π_1') be an irreducible constituent in $\pi|_H$ (resp. $\pi_1|_H$) as in Definition 4.2. Then, it follows from the definition of Π that $\mathrm{BC}^U(\pi') \otimes \bar{\omega}_\pi \cong \mathrm{BC}(\pi) \cong \mathrm{BC}(\pi_1) \cong \mathrm{BC}^U(\pi_1') \otimes \bar{\omega}_{\pi_1} \cong \mathrm{BC}^U(\pi_1') \otimes \bar{\omega}_\pi$. Therefore, for any irreducible admissible representation σ of $\mathrm{GL}_k(E)$ we have

$$\gamma(s, \sigma \times \mathrm{BC}^U(\pi_1') \otimes \bar{\omega}_\pi, \psi_E) = \gamma(s, \sigma \times \mathrm{BC}^U(\pi') \otimes \bar{\omega}_\pi, \psi_E),$$
$$L(s, \sigma \times \mathrm{BC}^U(\pi_1') \otimes \bar{\omega}_\pi) = L(s, \sigma \times \mathrm{BC}^U(\pi') \otimes \bar{\omega}_\pi).$$

This implies that π_1' and π' are in the same L-packet for H. Since L-packets for H are finite and the restriction $\pi|_H$ has only finitely many irreducible constituents (at most two), it follows that the L-packet Π is also finite.

We now prove (2), i.e., Shahidi's conjecture for G. Suppose Π is a tempered L-packet for G. For $\pi_t \in \Pi$, take an irreducible component $\pi_t' \subset \pi_t|_H$ in the restriction. Let Π' be the corresponding tempered L-packet for H. Since Shahidi's conjecture for the group H is known to be true, we conclude Π' contains a generic member π_g'. Both π_g' and π_t' have the same central character given by the restriction of ω_{π_t} to the center $Z \cap H$ of H. Thus the pair (π_g', ω_{π_t}) defines a representation of the subgroup $ZH \subset G$. Since this is a subgroup of finite index, it follows from a standard argument using Frobenius reciprocity that there is an irreducible generic representation π_g of G with central character ω_{π_t} containing π_g' upon restriction to H. Since $\mathrm{BC}^U(\pi_g') \cong \mathrm{BC}^U(\pi_t')$, it follows that

$$\mathrm{BC}(\pi_g) \cong \mathrm{BC}(\pi_g') \otimes \bar{\omega}_{\pi_g} \cong \mathrm{BC}(\pi_t') \otimes \bar{\omega}_{\pi_t} \cong \mathrm{BC}(\pi_t).$$

We conclude π_g is a member of the L-packet Π.

We now prove (3). Let π be a nontempered representation of G and consider it the Langlands quotient of the induced representation

$$\nu^{r_1}\tau_1 \times \cdots \times \nu^{r_m}\tau_m \rtimes \pi_t,$$

where τ_i is an irreducible tempered representation of $\mathrm{GL}(E)$ for each $i = 1, \ldots, m$, and π_t is an irreducible tempered representation of G_{n_t} for some $n_t < n$.

The direct calculation implies that there is a nontempered representation π' of H that appears in $\pi|H$ such that it is the Langlands quotient of $\nu^{r_1}\tau_1 \times \cdots \times \nu^{r_m}\tau_m \rtimes \pi_t'$ for some irreducible tempered representation π_t' of H_{n_t} that appears in $\pi_t|H_{n_t}$ since $(\nu^{r_1}\tau_1 \times \cdots \times \nu^{r_m}\tau_m \rtimes \pi_t)|_H$ is $\nu^{r_1}\tau_1 \times \cdots \times \nu^{r_m}\tau_m \rtimes (\pi_t|_{H_{n_t}})$. Then, due to Definitions 4.2, 4.4 and the property that L-functions for H follow the Langlands classification, we have

$$\begin{aligned} L(s, \sigma \times \pi) &= L(s, \sigma \times \mathrm{BC}(\pi)) = L(s, \sigma \times \mathrm{BC}^U(\pi') \otimes \bar{\omega}_\pi) \\ &= L(s, \pi_t' \otimes \bar{\omega}_\pi \times \sigma) \prod_{i=1}^{m} L(s + r_i, \tau_i \otimes \bar{\omega}_\pi \times \sigma) L(s - r_i, \tilde{\bar{\tau}}_i \otimes \bar{\omega}_\pi \times \sigma). \end{aligned}$$

It follows that L-functions for G as in Definition 4.4 also follow the Langlands classification.

For (4), it follows from the proof of Theorem 3.1 and our construction of the stable base change lift $\mathrm{BC}(\pi)$. See also Remark 4.6 below. □

Remark 4.6. Definition 4.4 provides a new definition of L-functions for G in full generality, including nongeneric representations. In the generic case, this construction is compatible with Theorem 3.1. Furthermore, in the tempered case, Proposition 4.5 implies that every tempered L-packet contains a generic member; thus, we may extend the Langlands–Shahidi definition from the generic case to the tempered case. Finally, invoking the Langlands classification, we can

extend L-functions from the Langlands–Shahidi method in full generality. This is compatible with Definition 4.4 by virtue of Proposition 4.5. We note that the above definition admits a natural interpretation in terms of Langlands parameters: namely, if φ_π denotes the parameter of π, then the base change lift $\mathrm{BC}(\pi)$ corresponds to the restriction $\varphi_\pi|_{W'_E}$. In particular, the local factors defined above agree with the Artin L-functions attached to $\varphi_\pi|_{W'_E}$ via the local Langlands correspondence for general linear groups.

We conclude that the L-functions defined in this section follow the Langlands classification.

Corollary 4.7. *Assumption 3.3 holds.*

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YEANSU KIM
DEPARTMENT OF MATHEMATICS EDUCATION
CHONNAM NATIONAL UNIVERSITY
GWANGJU
SOUTH KOREA
ykim@jnu.ac.kr

MUTHU KRISHNAMURTHY
HARTREE PARTNERS LTD
HOUSTON, TX
UNITED STATES
mkrish1939@gmail.com

FREYDOON SHAHIDI
DEPARTMENT OF MATHEMATICS
PURDUE UNIVERSITY
WEST LAFAYETTE, IN
UNITED STATES
freydoon.shahidi.1@purdue.edu

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BOUNDEDNESS OF EISENSTEIN INTEGRALS ON SPLIT RANK-ONE SEMISIMPLE SYMMETRIC SPACES

SANJOY PUSTI AND ISWARYA SITIRAJU

We characterize the bounded left K-invariant normalized Eisenstein integrals on split rank-one semisimple symmetric spaces. As a consequence we prove the Hausdorff–Young inequality on these spaces. We prove a similar result for K-finite Eisenstein integrals on pseudo-Riemannian real hyperbolic spaces.

1. Introduction

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. Let ϕ_λ be the elementary spherical functions on the Riemannian symmetric space G/K, with spectral parameter λ in the complexified dual space denoted by $\mathfrak{a}_{\mathbb{C}}^*$. A celebrated theorem of Helgason and Johnson [1969] states that there exists a tube domain S_1 in the complexified dual space $\mathfrak{a}_{\mathbb{C}}^*$ such that ϕ_λ is bounded on G/K if and only if λ belongs to S_1. Moreover, $|\phi_\lambda(x)| \leq 1$ for $\lambda \in S_1$ and for all $x \in G$. In other words, the theorem characterizes the bounded elementary spherical functions on G/K. As a consequence, it follows that for a left K-invariant integrable function f on G/K, the spherical Fourier transform $\mathcal{F}f(\lambda)$ is holomorphic in the interior S_1° and continuous on S_1. Also, for $f \in L^1(K\backslash G/K)$,

$$|\mathcal{F}f(\lambda)| \leq \|f\|_1, \quad \lambda \in S_1.$$

This result was generalized to L^r functions in [Eguchi and Kumahara 1982] for $1 < r < 2$. A generalized Hausdorff–Young inequality and a generalized Riemann Lebesgue lemma has been shown to hold in S_r° in [Eguchi and Kumahara 1982] and a variant of the Hausdorff–Young inequality for spherical functions has been shown in [Cowling et al. 1993]. Similarly, a Hausdorff–Young inequality was proved for K-finite L^p functions on G/K in the domain S_r in [Eguchi et al. 1987]. An analogue of this Helgason–Johnson theorem is proved for the Jacobi functions $\varphi_{-i\lambda}^{(\alpha,\beta)}$ for $\alpha \geq \beta \geq -\frac{1}{2}$ by Flensted-Jensen and Koornwinder [1973].

Now, consider the triple $(\mathfrak{a}, \Sigma, m)$, where $\mathfrak{a}$ is a finite-dimensional real Euclidean vector space, a root system Σ and a multiplicity function m on Σ. Let ϕ_λ be the

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Heckmann–Opdam hypergeometric functions associated with the triple $(\mathfrak{a}, \Sigma, m)$. The Helgason–Johnson theorem has been extended to this case [Narayanan et al. 2014]. In this case, we have a tube S_1 in the complexified dual vector space $\mathfrak{a}_{\mathbb{C}}^*$ such that the function ϕ_λ is bounded on $\mathfrak{a}$ if and only if $\lambda \in S_1$. Moreover, the Fourier transform satisfies the generalized Riemann–Lebesgue lemma on S_1 and also a Hausdorff–Young inequality holds on a tube domain S_r depending on r with $1 \le r \le 2$. If the triple $(\mathfrak{a}, \Sigma, m)$ is associated with the Riemannian symmetric space G/K, then the function ϕ_λ coincides with the elementary spherical function on G/K, and thus the domain in which they are bounded is the same. This result was also proved for certain types of Heckman–Opdam hypergeometric functions for the root system BC [Narayanan and Pasquale 2022].

Let G be a real connected semisimple linear Lie group and H be a connected subgroup such that (G, H) is a symmetric pair. Let K be the maximal compact subgroup of G. The Eisenstein integrals are defined as the matrix coefficients of K-finite vectors with H-fixed vectors of parabolically induced representations. These functions were first introduced in [van den Ban 1992] and also studied in [van den Ban and Schlichtkrull 1997a; 1997b]. The left K-invariant Eisenstein integrals are the generalizations of the elementary spherical functions on the Riemannian symmetric spaces, that is, when $H = K$ (see [Harish-Chandra 1958]) and the spherical functions for the K_ϵ symmetric spaces introduced in [Oshima and Sekiguchi 1980].

Let $(\mathfrak{a}_q, \Sigma, m)$ be the root system associated with (G, H, K) and W be the Weyl group corresponding to the root system $(\mathfrak{a}_q, \Sigma)$. Let $\mathcal{W}$ be a specific subgroup of the Weyl group W. For a vector η in $\mathbb{C}^{\mathcal{W}}$ and the spectral parameter $\lambda \in \mathfrak{a}_{q,\mathbb{C}}^*$, the Eisenstein integrals are denoted by $E(\lambda, \eta)(\cdot)$. For $x \in G/H$, the function $\lambda \mapsto E(\lambda, \eta)(x)$ can be meromorphic in the complexified dual space $\mathfrak{a}_{q,\mathbb{C}}^*$. Hence, E is normalized using their asymptotic behavior such that there are no singularities on $i\mathfrak{a}_q^*$ and is denoted as E°. For the Riemannian case, the Eisenstein integral is the elementary spherical function ϕ_λ and the normalized Eisenstein integral is $\phi_\lambda / c(\lambda)$, where $c(\lambda)$ is the Harish-Chandra c-functions. In this article, we will study the normalized Eisenstein integrals on the split rank-one semisimple symmetric spaces.

Let $x_0 = eH$ be the base point of G/H. For a fixed $w \in \mathcal{W}$ and $\eta \in \mathbb{C}^{\mathcal{W}}$, the normalized Eisenstein integral

$$E_w^\circ(\lambda, \eta)(t) := E_w^\circ(\lambda, \eta)(a_t w \cdot x_0)$$

is given by

$$\text{(1-1)} \quad E_w^\circ(\lambda, \eta)(t) = \eta_w 2^{\lambda-\rho} \frac{\Gamma\left(\frac{\rho+\lambda}{2}\right)\Gamma\left(\frac{-\rho+\lambda+m_1^+ + m_2^+ + 1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{m_1^+ + m_2^+ + 1}{2}\right)} \times {}_2F_1\left(\frac{\rho+\lambda}{2}, \frac{\rho-\lambda}{2}; \frac{m_1^+ + m_2^+ + 1}{2}; -\sinh^2 t\right).$$

These normalized Eisenstein integrals $E^\circ_w(\lambda, \eta)$ are not entire functions. In fact, for each $R > 0$, the function $\lambda \mapsto E^\circ_w(\lambda, \eta)(t)$ has finitely many simple poles in $-a^*(R)$, where

$$a^*(R) = \{\lambda \in \mathbb{C} \mid \Re\lambda \leq R\}.$$

Moreover, there is a polynomial p_R (which is a finite product of linear polynomials) such that the function $\lambda \mapsto p_R(\lambda)E^\circ_w(\lambda, \eta)(t)$ is holomorphic on $-a^*(R)$. Then we characterize all those $\lambda \in -a^*(R)$ for which the function $t \mapsto p_R(\lambda)E^\circ_w(\lambda, \eta)(t)$ is bounded on $(0, \infty)$ (see Theorem 4.1 in Section 4).

Using [Flensted-Jensen and Koornwinder 1973, Lemma 2.1] it follows that for $m_1^+ \geq m_1^- + 2m_2^-$, the function

$$t \mapsto {}_2F_1\left(\frac{\lambda+\rho}{2}, \frac{-\lambda+\rho}{2}; \frac{m_1^+ + m_2^+ + 1}{2}; -\sinh^2 t\right)$$

is bounded on $(0, \infty)$ if and only if $|\Re\lambda| \leq \rho$. Thus, this result and (1-1) give characterizations of all λ for which $t \mapsto p_R(\lambda)E^\circ_w(\lambda, \eta)(t)$ is bounded on $(0, \infty)$, only for the case when $m_1^+ \geq m_1^- + 2m_2^-$. But for the case $m_1^+ < m_1^- + 2m_2^-$, we will not be able to use the result of Flensted-Jensen and Koornwinder [1973]. In this case, we mainly adapt the methods of [Narayanan et al. 2014]. First we get a series expansion of the normalized Eisenstein integrals at any λ and using this we characterize all the functions $t \mapsto p_R(\lambda)E^\circ_w(\lambda, \eta)(t)$ that are bounded.

Then as an application of the theorem we prove the Hausdorff–Young inequality (see Theorem 5.4) for this case. Also, we obtain similar (partial) results for the left K-finite normalized Eisenstein integral (see Theorems 6.5 and 6.7) for the (pseudo-Riemannian) real hyperbolic spaces $\mathrm{SO}_e(p, q)/\mathrm{SO}_e(p-1, q)$. We note that $\mathcal{W} = \{1\}$ when $q > 1$ and $\mathcal{W} = \{\pm 1\}$ when $q = 1$. Therefore, there is only one orbit of σ-minimal parabolic subgroup on the (pseudo-Riemannian) real hyperbolic spaces for $q > 1$ and two disjoint orbits of σ-minimal parabolic subgroup for $q = 1$. This makes the structure and analysis different for these two cases. We will provide the required preliminaries of these cases separately in Section 6 and then unify them to prove the results.

We organize the paper as follows: In Section 2, we give necessary preliminaries for the general semisimple symmetric spaces G/H and then give preliminaries for the split rank-one semisimple symmetric spaces. In Section 3 we deduce an explicit formula of the Harish-Chandra c-function. In Section 4, we prove the analogue of the Helgason–Johnson theorem on split rank-one semisimple symmetric spaces. In Section 5, we prove the Hausdorff–Young inequality and the Riemann–Lebesgue lemma. Finally, in Section 6, we prove similar results for fixed left K-type normalized Eisenstein integrals on pseudo-Riemannian real hyperbolic spaces $\mathrm{SO}_e(p, q)/\mathrm{SO}_e(p-1, q)$.

2. Preliminaries

We shall use the standard notation $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ for natural numbers, nonnegative integers, integers, real numbers and complex numbers, respectively. Given two nonnegative functions f and g on a domain D, we write $f \asymp g$ if there exists positive constants C_1 and C_2 such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ for all $x \in D$.

In this section, we describe the necessary preliminaries regarding harmonic analysis on semisimple symmetric spaces. These are standard and can be found, for example, in [Andersen 2001; van den Ban and Schlichtkrull 1997a; 1997b; Heckman and Schlichtkrull 1995; Rossmann 1979]. To make the article self-contained, we shall gather only those results which will be used.

Let G be a connected semisimple linear Lie group, H a connected subgroup of G and K be a maximal compact subgroup of G. Let θ be the Cartan involution and σ be a involution such that $H = G_e^\sigma$, where G^σ is the subgroup of fixed points for σ and G_e^σ is the identity component of G^σ. We assume that $\sigma\theta = \theta\sigma$. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$$

be the decomposition of $\mathfrak{g}$ induced by σ. Also let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the Cartan decomposition. Let

$$\mathfrak{g}_+ = \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q} \quad \text{and} \quad \mathfrak{g}_- = \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h}.$$

Then $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ and $G_+ = G_e^{\sigma\theta}$ has Lie algebra $\mathfrak{g}_+$ with maximal compact subgroup $K \cap H$. Let $\mathfrak{a}_q$ be the maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ and $A_q = \exp \mathfrak{a}_q$. Let $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+)$ be the corresponding set of restricted roots and let $W_{K\cap H}$ be the Weyl group associated with the root system $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+)$. Let $\mathfrak{a}_q^+$ be the corresponding positive Weyl chamber and $A_q^+ = \exp \mathfrak{a}_q^+$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}_q$. Then $\mathfrak{a} \cap \mathfrak{q} = \mathfrak{a}_q$. We denote the real dual of $\mathfrak{a}_q$ by $\mathfrak{a}_q^*$. For $\alpha \in \mathfrak{a}_q^*$, we define

$$\mathfrak{g}_\alpha = \{Y \in \mathfrak{g} \mid [H, Y] = \alpha(H)Y \text{ for all } H \in \mathfrak{a}_q\}.$$

Let $\Sigma = \Sigma(\mathfrak{a}_q, \mathfrak{g})$ be the set of $\alpha \in \mathfrak{a}_q^*$ such that $\mathfrak{g}_\alpha \neq \{0\}$. Then $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+) \subseteq \Sigma(\mathfrak{a}_q, \mathfrak{g})$. Let $\mathfrak{g}_\alpha^\pm = \mathfrak{g}_\alpha \cap \mathfrak{g}_\pm$. Let

$$W = W(\mathfrak{a}_q, \mathfrak{g}) = N_K(\mathfrak{a}_q)/Z_K(\mathfrak{a}_q)$$

be the Weyl group of $\mathfrak{a}_q$ in $\mathfrak{g}$ associated with the root system $\Sigma(\mathfrak{a}_q, \mathfrak{g})$. Let

$$\mathcal{W} = N_{K\cap H}(\mathfrak{a}_q) \backslash N_K(\mathfrak{a}_q)/Z_K(\mathfrak{a}_q) = W/W_{K\cap H}.$$

Let $m_\alpha = \dim \mathfrak{g}_\alpha$ be the multiplicity of α and let $m_\alpha^\pm = \dim \mathfrak{g}_\alpha^\pm$. Then

$$m_\alpha = m_\alpha^+ + m_\alpha^-$$

and m_α^+ is the multiplicity of α as a member of $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+)$. Let

$$J(a) = \prod_{\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})} (a^\alpha - a^{-\alpha})^{m_\alpha^+} (a^\alpha + a^{-\alpha})^{m_\alpha^-} \tag{2-1}$$

for $a \in A_q$. We define

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})} m_\alpha \alpha \in \mathfrak{a}_q^*.$$

Every element $g \in G$ has a decomposition as

$$g = kah, \quad k \in K,\ a \in A_q,\ h \in H, \tag{2-2}$$

and a is unique up to $W_{K \cap H}$. Consider the set

$$X_+ = \bigcup_{w \in \mathcal{W}} K A_q^+ w H.$$

When viewed as a subset of G/H, the set X_+ is open and dense. Then there exists a unique Haar measure on A_q such that, for $f \in L^1(X)$,

$$\int_{G/H} f(x)\, dx = \sum_{w \in \mathcal{W}} \int_K \int_{A_q^+} f(kawH) J(a)\, da\, dk, \tag{2-3}$$

where dk is the normalized Haar measure on K (see [Anker and Orsted 2005, Theorem 3.9, p. 24]). We will denote the complexification of $\mathfrak{a}_q$ as $\mathfrak{a}_{q,\mathbb{C}}$ and the dual of $\mathfrak{a}_{q,\mathbb{C}}$ as $\mathfrak{a}_{q,\mathbb{C}}^*$.

The σ-minimal parabolic subgroup P is a parabolic subgroup satisfying $\sigma\theta P = P$ and is minimal among all other parabolic subgroups satisfying this identity. We then have the following theorem [Rossmann 1979, Theorem 13]:

Theorem 2.1. *Let P be a σ-minimal parabolic subgroup of G and $\mathcal{W}$ be as defined above.*

(1) *For each $\omega \in \mathcal{W}$, the orbit $\mathcal{O}_\omega = P\omega H$ is open in G.*

(2) *The orbits $\mathcal{O}_\omega$ are mutually disjoint.*

(3) *The disjoint union $\bigcup_{\omega \in \mathcal{W}} \mathcal{O}_\omega$ is dense in G.*

Let $P = MAN$ be the Langlands decomposition. Let $\pi_{\xi,\lambda} = \mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$ be the parabolically induced representations on the space of continuous functions $C(K; \xi)$. The left K-invariant Eisenstein integrals, denoted by $E(\lambda)$, are linear combinations of matrix coefficients for the induced representations $\pi_{1,\lambda}$ with $\xi = 1$ and $\lambda \in \mathbb{C}$. Let $1_{-\lambda}$ be the K-fixed vector as a constant function 1 on K. For $\eta \in \mathbb{C}^{\mathcal{W}}$, $\lambda \in \mathbb{C}$ such that $\Re\lambda > \rho$ we define

$$j(\lambda, \eta)(x) = \begin{cases} a^{\lambda - \rho} \eta_w & \text{if } x = manwh \in \mathcal{O}_w, \\ 0 & \text{otherwise,} \end{cases}$$

where η_w is the w-th coordinate of η. Observe that $j(\lambda, \eta) \in C^{-\infty}(K; 1)^H$ is an H-fixed distribution of $\pi_{\lambda,1}$ for $\lambda \in \mathbb{C}$ such that $\Re\lambda > \rho$. It was proven in [van den Ban 1988, Theorem 5.10, p. 83] that:

Theorem 2.2. *The map $\lambda \mapsto j(\lambda, \eta)$ can be extended as a meromorphic function on $\mathbb{C}$ for all $\eta \in \mathbb{C}^{\mathcal{W}}$. Moreover, for almost all λ, $j(\lambda, \cdot) : \mathbb{C}^{\mathcal{W}} \to C^{-\infty}(K, 1)^H$ is a bijection.*

The proof of this theorem is given in [van den Ban 1988, Theorem 5.10, p. 83; Ólafsson 1987]. Thus, the Eisenstein integrals are defined as

$$E(\lambda, \eta)(x) = \langle 1_{-\lambda}, \pi_{\lambda,1}(x) j(\lambda, \eta)\rangle_{L^2(K)}.$$

The function $\lambda \mapsto E(\lambda, \eta)(x)$ may have singularities on $i\mathbb{R}$. Thus, we consider a normalization by their behavior at infinity which regularizes these functions on $i\mathbb{R}$. We denote these normalized Eisenstein integral as $E^\circ(\lambda, \cdot)(\cdot)$.

For $\lambda \in \mathfrak{a}_{q,\mathbb{C}}^*$, $\eta \in \mathbb{C}^{\mathcal{W}}$, Eisenstein integrals $E(\lambda, \eta)$ and normalized Eisenstein integrals $E^\circ(\lambda, \eta)$ are left K-invariant and right H-invariant meromorphic functions in λ. These functions have the following properties:

(1) These Eisenstein integrals and normalized Eisenstein integrals are K-invariant eigenfunctions for the invariant differential operators on G/H. More precisely,

$$DE(\lambda, \eta) = \gamma_q(D : \lambda)E(\lambda, \eta) \tag{2-4}$$

for all $D \in \mathbb{D}(G/H)$, $\lambda \in \mathfrak{a}_{q,\mathbb{C}}^*$. Here γ_q is the algebra homomorphism between $\mathbb{D}(G/H)$ and $\mathcal{S}(\mathfrak{a}_q)^W$.

(2) These are analytic functions on G/H.

(3) The functions $\lambda \mapsto E(\lambda, \eta)(x)$, $E^\circ(\lambda, \eta)(x)$ are meromorphic on $\mathfrak{a}_{q,\mathbb{C}}^*$ for fixed η and x.

(4) The Eisenstein integral is normalized in such way that $E^\circ(\cdot, \eta)$ has no singularities on $i\mathfrak{a}_q^*$.

(5) $\overline{E^\circ(-\bar{\lambda}, \eta)(x)} = E^\circ(-\lambda, \bar{\eta})(x)$.

Let $\mathfrak{a}_1$ be a maximal abelian subspace of $\mathfrak{q}$. The dimension of $\mathfrak{a}_1$ is called the *rank* of the semisimple symmetric spaces G/H. The following are the only rank-one connected non-Riemannian symmetric spaces up to covering (for case-by-case treatment of the Plancherel theorem, see [van Dijk 1986; Molchanov 1986]):

(1) the real hyperbolic spaces $\mathrm{SO}_e(p, q)/\mathrm{SO}_e(p-1, q)$, $p > 1$, $q > 0$;

(2) the complex hyperbolic spaces $\mathrm{SU}(p, q)/S(U(1) \times U(p-1, q))$, $p > 1$, $q > 0$;

(3) the quaternion hyperbolic spaces $\mathrm{Sp}(p, q)/(\mathrm{Sp}(1) \times \mathrm{Sp}(p-1, q))$, $p > 1$, $q > 0$;

(4) the octonion hyperbolic space $F_{4(-20)}/\mathrm{Spin}(1, 8)$;

(5) $\mathrm{SL}(n,\mathbb{R})/\mathrm{GL}_+(n-1,\mathbb{R})$, $n>1$;

(6) $\mathrm{Sp}(n,\mathbb{R})/(\mathrm{Sp}(1,\mathbb{R})\times\mathrm{Sp}(n-1,\mathbb{R}))$, $n>1$;

(7) $F_4(4)/\mathrm{Spin}(4,5)$.

The dimension of $\mathfrak{a}_q$ is called the *split rank* of the semisimple symmetric spaces G/H. For a complete list of split rank-one connected non-Riemannian symmetric spaces, we refer to [Oshima and Sekiguchi 1984, Table II, p. 462].

Remark 2.3. In the case of Riemannian symmetric spaces, we have $H=K$, $\mathfrak{p}=\mathfrak{q}$, and hence $\mathfrak{g}_+=\mathfrak{g}$, $\mathfrak{g}_-=0$. This implies $\mathfrak{a}=\mathfrak{a}_q$ and $\Sigma(\mathfrak{a}_q,\mathfrak{g})=\Sigma(\mathfrak{a}_q,\mathfrak{g}_+)$. So $W_{K\cap H}=W$, and hence $\mathcal{W}=\{1\}$. Also the Eisenstein integrals and normalized Eisenstein integrals becomes elementary spherical functions ϕ_λ and $c(\lambda)^{-1}\phi_\lambda$ respectively, where $c(\lambda)$ is the Harish-Chandra c-function.

For $R>0$, we define

$$\mathfrak{a}_q^*(R)=\{\lambda\in\mathfrak{a}_{q,\mathbb{C}}^* \mid \Re\langle\lambda,\alpha\rangle\le R \text{ for all } \alpha\in\Sigma^+(\mathfrak{a}_q,\mathfrak{g})\}.$$

Let S be the union of all the hyperplanes given by $\sigma_\mu=\{\lambda\in\mathfrak{a}_{q,\mathbb{C}}^* \mid \langle 2\lambda-\mu,\mu\rangle=0\}$ for $\mu\in\mathbb{N}(\Sigma^+(\mathfrak{a}_q,\mathfrak{g}))$. For $\lambda\notin S$, there exists Harish-Chandra series

$$\Phi_\lambda(a)=a^{\lambda-\rho}\sum_{\mu\in\mathbb{N}_0(\Sigma^+(\mathfrak{a}_q,\mathfrak{g}))}a^{-\mu}\Gamma_\mu(\lambda)$$

on A_q^+ with $\Gamma_\mu(\lambda)\in\mathbb{C}$, $\Gamma_0=1$ which solves (2-4). Then the set

$$X_R=\{\mu\in\mathbb{N}(\Sigma^+(\mathfrak{a}_q,\mathfrak{g})) \mid \sigma_\mu\cap\mathfrak{a}_q^*(R)\neq\varnothing\}$$

is finite. Let $q_R(\lambda)$ be the polynomial

$$q_R(\lambda)=\prod_{\mu\in X_R}\langle 2\lambda-\mu,\mu\rangle.$$

Then $q_R(\lambda)\Phi_\lambda(a)$ is holomorphic as a function of $\lambda\in\mathfrak{a}_q^*(R)$. Moreover, for each $\epsilon>0$, there exists $M>0$ such that

$$|q_R(\lambda)\Phi_\lambda(a)|\le M(1+|\lambda|)^c a^{|\Re\lambda|-\rho}$$

for all $a\in A_q$ with $\alpha(\log a)>\epsilon$ for all $\alpha\in\Sigma^+(\mathfrak{a}_q,\mathfrak{g})$ and all $\lambda\in\mathfrak{a}_q^*(R)$. Also [van den Ban and Schlichtkrull 1997a, Theorem 7.6] states that for each $s\in W$, there exists a unique *endomorphism valued* meromorphic function

$$\lambda\mapsto C^\circ(s:\lambda)\in\ \mathrm{End}(\mathbb{C}^{\mathcal{W}})$$

on $\mathfrak{a}_{q,\mathbb{C}}^*$ such that

$$E^\circ(\lambda:\eta)(aw)=\sum_{s\in\mathcal{W}}\Phi_{s\lambda}(a)(C^\circ(s:\lambda)\eta)_w \tag{2-5}$$

for $a \in A_q^+$, $w \in \mathcal{W}$, $\eta \in \mathbb{C}^{\mathcal{W}}$, as a meromorphic identity in $\lambda \in \mathfrak{a}_{q,\mathbb{C}}^*$, where $(\cdot)_w$ is the w-th component in $\mathbb{C}^{\mathcal{W}}$. Also,

$$C^\circ(s:\lambda)^* C^\circ(s:-\bar{\lambda}) = \mathrm{Id}$$

for all $s \in W$ and $\lambda \in \mathfrak{a}_{q,\mathbb{C}}^*$. Moreover, we have $C^\circ(1,\lambda) = \mathrm{Id}$ and (see [Anker and Orsted 2005, p. 121])

$$E^\circ(s\lambda,\eta) = E^\circ(\lambda, C^0(s\lambda,\eta)), \quad s \in W.$$

Let $\lambda \in \mathfrak{a}_{q,\mathbb{C}}^*$ with $\Re\lambda$ strictly dominant (that is, $\langle \Re\lambda, \alpha\rangle > 0$ for all $\alpha \in \Sigma^+(\mathfrak{a}_q, \mathfrak{g})$) and let $\omega \in \mathcal{W}$. Then, we have the asymptotics (see [Heckman and Schlichtkrull 1995, Part II, Proposition 7.7])

$$a^{\rho-\lambda} E^\circ(\lambda,\eta)(aw) \to \eta_\omega \tag{2-6}$$

as $a \to \infty$ in A_q^+ where η_ω is the ω-component of η. For $R > 0$ there exists finitely many roots $\{\alpha_i\}$ and real constants $\{c_i\}$ such that for all $\eta \in \mathbb{C}^{\mathcal{W}}$ the singular set of the function $\lambda \mapsto E^\circ(\lambda,\eta)$ in $-\mathfrak{a}_q^*(R)$ is contained in

$$\bigcup_{i=1}^{k} \{\lambda : \langle \lambda, \alpha_i\rangle = c_i\}.$$

Hence, the polynomial $p_R(\lambda) = \prod_{i=1}^k (\langle\lambda,\alpha_i\rangle - c_i)$ is such that $p_R(\lambda)E^\circ(\lambda,\eta)$ is holomorphic on a neighborhood of $-\mathfrak{a}_q^*(R)$ for all $\eta \in \mathbb{C}^{\mathcal{W}}$. The polynomial p_R cancels all the poles of $E^\circ(\lambda,\eta)$ in $-\mathfrak{a}_q^*(R)$. Moreover, the following estimate follows from [van den Ban 1992, Proposition 10.3 and Corollary 16.2] which says

$$|p_R(\lambda)E^\circ(\lambda:\eta)(a)| \le C(1+|\lambda|)^N e^{(s+|\Re\lambda|)|\log a|}\|\eta\| \tag{2-7}$$

for some constants C, N and s, for all $\lambda \in -\mathfrak{a}_q^*(R)$, $\eta \in \mathbb{C}^{\mathcal{W}}$ and $a \in A_q$.

For a K-invariant function f on G/H, the Fourier transform of f is defined as [Anker and Orsted 2005, Theorem 3.9]

$$\mathcal{F}f(\lambda)(\eta) = \int_X f(x)E^\circ(-\lambda:\eta)(x)\,dx$$

for $\eta \in \mathbb{C}^{\mathcal{W}}$. This $\mathcal{F}f(\lambda)$ is a linear form on $\mathbb{C}^{\mathcal{W}}$.

2.1. ***Split rank-one cases.*** We will prove our theorems in the split rank-one cases only. In this case the dimension of $\mathfrak{a}_q$ is one. The root system $\Sigma(\mathfrak{a}_q,\mathfrak{g})$ is $\{\pm\alpha\}$ or $\{\pm\alpha, \pm 2\alpha\}$. Let $H_0 \in \mathfrak{a}_q$ be the unique element such that $\alpha(H_0) = 1$. Let

$$m_1 = m_\alpha, \quad m_1^+ = m_\alpha^+, \quad m_1^- = m_\alpha^-; \quad m_2 = m_{2\alpha}, \quad m_2^+ = m_{2\alpha}^+, \quad m_2^- = m_{2\alpha}^-.$$

Then we have $m_1 = m_1^+ + m_1^-$ and $m_2 = m_2^+ + m_2^-$. Also the following are true (in split rank-one cases) [Sekiguchi 1990, p. 3]:

(1) $m_1^+ + m_1^- > 0$.

(2) If $m_2^- > 0$ then $m_1^+ = m_1^-$.

(3) $m_1^- = m_2^- = 0$ if and only if $(\mathfrak{g}, \mathfrak{h})$ is Riemannian.

(4) If $m_1^+ = m_2^- = 0$ then $(\mathfrak{g}, \mathfrak{h})$ is of $\mathfrak{k}_\epsilon$ type.

Also ρ is identified with $\rho(H_0)$ and

$$\rho = \tfrac{1}{2}(m_1 + 2m_2) = \tfrac{1}{2}(m_1^+ + m_1^- + 2m_2^+ + 2m_2^-).$$

Also $\mathcal{W} = \{1\}$ or $\mathcal{W} = \{\pm 1\}$. The Jacobian defined in (2-1) becomes

$$J(t) = (\sinh t)^{m_1^+} (\cosh t)^{m_1^-} (\sinh 2t)^{m_2^+} (\cosh 2t)^{m_2^-}. \tag{2-8}$$

Let $\Delta \in \mathbb{D}(G/H)$ be the Laplace–Beltrami operator on X. The radial part of Δ is given by [Sekiguchi 1990, pp. 8–9; Artemov and Molchanov 1990, Theorem 2.1]

$$\begin{aligned} L(\Delta) &= \frac{\partial^2}{\partial t^2} + (m_1^+ \coth t + m_1^- \tanh t + 2m_2^+ \coth 2t + 2m_2^- \tanh 2t)\frac{\partial}{\partial t} \\ &= \frac{1}{J(t)} \frac{\partial}{\partial t}\left(J(t) \frac{\partial}{\partial t} \right). \end{aligned}$$

Let $x_0 = eH$ be the base point of G/H. For a fixed $w \in \mathcal{W}$ and $\eta \in \mathbb{C}^{\mathcal{W}}$, the Eisenstein integral $E_w^\circ(\lambda, \eta)(t) := E_w^\circ(\lambda, \eta)(a_t w \cdot x_0)$ satisfies the differential equation

$$L(\Delta) E_w^\circ(\lambda, \eta)(t) = (\rho^2 - \lambda^2) E_w^\circ(\lambda, \eta)(t). \tag{2-9}$$

We now find an explicit formula for the Eisenstein integral $E_w^\circ(\lambda, \eta)(t)$ (see [Sekiguchi 1990, pp. 38–39]). If $m_2^- = 0$, by substituting $s = -\sinh^2 t$, the above differential equation can be transformed into a hypergeometric differential equation and $E_w^\circ(\lambda, \eta)$ satisfies

$$\left[-(1-s)s \frac{\partial^2}{\partial s^2} + \left(s(1+\rho) - \frac{1 + m_1^+ + m_2^+}{2} \right) \frac{\partial}{\partial s} \right] f = \frac{\lambda^2 - \rho^2}{4} f.$$

If $m_2^- > 0$, then from [Sekiguchi 1990, item (C.2), p. 32] we have $m_1^+ = m_1^-$. Thus by substituting $s = 2t$ and $r = -\sinh^2 s$ the differential equation (2-9) transforms into a hypergeometric differential equation and $E_w^\circ(\lambda, \eta)$ satisfies

$$\left[-(1-r)r \frac{\partial^2}{\partial r^2} + \left(r\left(1 + \frac{\rho}{2}\right) - \frac{1 + m_1^+ + m_2^+}{2} \right) \frac{\partial}{\partial r} \right] f = \frac{\lambda^2 - \rho^2}{16} f.$$

Since E_w is regular at $t = 0$, for all λ we have

$$E_w^\circ(\lambda, \eta)(t) = \text{const.}\, {}_2F_1\left(\frac{\rho + \lambda}{2}, \frac{\rho - \lambda}{2}; \frac{m_1^+ + m_2^+ + 1}{2}; -\sinh^2 t \right).$$

Using the property [Erdélyi et al. 1953, equation 17, p. 63]

$$
\begin{aligned}
{}_2F_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F(b,1-c+b;1-a+b;z^{-1}) \\
&\quad + \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F(a,1-c+a;1-b+a;z^{-1}),
\end{aligned} \tag{2-10}
$$

and

$$
\lim_{t\to\infty} e^{(\rho-\lambda)t}E_w^\circ(\lambda,\eta)(t) = \eta_w, \quad \Re\lambda > 0, \tag{2-11}
$$

we obtain, for $\Re\lambda > 0$,

$$
\begin{aligned}
E_w^\circ(\lambda,\eta)(t) &= \eta_w 2^{\lambda-\rho}\frac{\Gamma\left(\frac{\rho+\lambda}{2}\right)\Gamma\left(\frac{-\rho+\lambda+m_1^++m_2^++1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{m_1^++m_2^++1}{2}\right)} \\
&\quad \times {}_2F_1\left(\frac{\rho+\lambda}{2},\frac{\rho-\lambda}{2};\frac{m_1^++m_2^++1}{2};-\sinh^2 t\right).
\end{aligned} \tag{2-12}
$$

Let $a^*(R) = \{\lambda \in \mathbb{C} \mid \Re\lambda \le R\}$. We note that $\mathfrak{a}_q^*(R)$ becomes $a^*(R)$ in this case. We will continue to use this $(a^*(R))$ notation instead of $\mathfrak{a}_q^*(R)$.

From the expressions (2-12) of the normalized Eisenstein integrals we observe that, for a fixed $R > 0$, there exists a polynomial p_R such that $p_R(\lambda)E_w^\circ(\lambda,\eta)(t)$ is holomorphic in $-a^*(R)$. In fact, p_R is given by

$$
\begin{aligned}
p_R(\lambda) &= \prod_{\substack{k_1\in\mathbb{N}_0 \\ \rho+2k_1\le R}} (\lambda+\rho+2k_1) \\
&\quad \times \prod_{\substack{k_2\in\mathbb{N}_0 \\ -\rho+m_1^++m_2^++1+2k_1\le R}} (\lambda-\rho+m_1^++m_2^++1+2k_2).
\end{aligned} \tag{2-13}
$$

Proposition 2.4. (1) *There exists a unique absolutely convergent series for $\lambda \notin \frac{1}{2}\mathbb{N}$, for $t > 0$,*

$$
\Phi_\lambda(t) = e^{(\lambda-\rho)t}\sum_{m=0}^{\infty} e^{-mt}\Gamma_m(\lambda), \tag{2-14}
$$

with $\Gamma_m(\lambda) \in \mathbb{C}$ and $\Gamma_0 = 1$ which satisfies the differential equation

$$
L(\Delta)\Phi_\lambda(t) = (\lambda^2-\rho^2)\Phi_\lambda(t).
$$

(2) *Let $X_R = \{k \in \mathbb{N} : 0 < k \le 2R\}$. Then there exists a polynomial*

$$
q_R(\lambda) = \prod_{k=1}^{\lfloor 2R\rfloor}(2\lambda-k) \tag{2-15}
$$

such that $q_R\Phi_\lambda$ is holomorphic in $a^*(R)$.

We recall for a function $f \in C_c^\infty(K\backslash G/H)$, its Fourier transform is defined by

$$\begin{aligned}\mathcal{F}f(\lambda)(\eta) &= \int_X f(x)E^\circ(-\lambda,\eta)(x)\,dx \\ &= \sum_{w\in\mathcal{W}}\int_0^\infty f(a_t w\cdot e_1)E^0(-\lambda,\eta)(a_t w\cdot e_1)J(t)\,dt\end{aligned}$$

for all $\eta \in \mathbb{C}^{\mathcal{W}}$ and $J(t)$ is the Jacobian given in (2-8). Therefore, $p_R(-\lambda)\mathcal{F}f(\lambda)$ is holomorphic on $a^*(R)$ and

$$\mathcal{F}(-\lambda)(\eta) = \mathcal{F}(\lambda)(C^0(-1,\lambda)\eta).$$

The following proposition is about the Fourier transform of compactly supported K-invariant functions on X from [Heckman and Schlichtkrull 1995, Proposition 8.4].

Proposition 2.5. *Let $\phi \in C_c^\infty(K\backslash G/H)$ and $n \in \mathbb{N}_0$. Then there exists a constant $M > 0$ such that*

$$|p_R(-\lambda)\mathcal{F}\phi(\lambda)(\eta)| \le M(1+|\lambda|)^{-n}e^{r|\mathrm{Re}\lambda|}\|\eta\|$$

for all $\lambda \in a^*(R)$ and r depends on the size of the support of ϕ.

From [Heckman and Schlichtkrull 1995, Theorems 7.1 and 7.4] we have the following theorem:

Theorem 2.6. *For $f \in L^2(X)$,*

$$\left(\int_{i\mathfrak{a}_q^*} |\mathcal{F}(\lambda)(\eta)|^2\,d\lambda\right)^{1/2} \le \|f\|_{L^2(X)}\|\eta\|, \tag{2-16}$$

where $d\lambda$ is the Lebesgue measure on $i\mathfrak{a}_q^*$.

3. Harish-Chandra series and C^0-functions

Let G/H be a split rank-one semisimple symmetric spaces. In this section we will study Harish-Chandra series and will find explicit expression of C^0-functions.

We have the results below on Harish-Chandra series [Heckman and Schlichtkrull 1995, Proposition 7.5; van den Ban and Schlichtkrull 1997a, p. 66]: Let

$$\Psi_\lambda(t) = \sqrt{J(t)}\Phi_\lambda(t) = e^{\lambda t}\sum_{m=0}^\infty \widetilde{\Gamma}_m(\lambda)e^{-mt}. \tag{3-1}$$

The Ψ_λ satisfies

$$\Psi_\lambda''(t) + d(t)\Psi_\lambda(t) = (\lambda^2-\rho^2)\Psi_\lambda(t),$$

where

$$d(t) = J''/\sqrt{J} = \sum_{m=0}^\infty d_m e^{-mt},$$

and $d_0 = \rho^2$.

By substituting (3-1) into above we obtain the recursive formula

$$(\lambda^2-\rho^2)\widetilde{\Gamma}_m=(\lambda-m)^2\widetilde{\Gamma}_m-\sum_{i=0}^{m}\widetilde{\Gamma}_i d_{m-i}.$$

That is,

$$m(2\lambda-m)\widetilde{\Gamma}_m=\rho^2\widetilde{\Gamma}_m-\sum_{i=0}^{m}\widetilde{\Gamma}_i d_{m-i},$$

which implies that

$$m(2\lambda-m)\widetilde{\Gamma}_m=-\sum_{i=0}^{m-1}\widetilde{\Gamma}_i d_{m-i}.$$

Therefore, the poles of $\widetilde{\Gamma}_m$ lie in the set $\{\frac{1}{2},1,\dots,\frac{1}{2}m\}$. We also have

$$\Gamma_m(\lambda)=\sum_{i=0}^{m}b_i\widetilde{\Gamma}_{m-i}(\lambda),$$

where

$$(J(t))^{-1/2}=e^{-\rho t}\sum_{m=0}^{\infty}b_m e^{-mt}.$$

We thus obtain that the poles of Γ_m lie in $\{\frac{1}{2},1,\dots,\frac{1}{2}m\}$. Further we have the following [van den Ban and Schlichtkrull 1997a, Theorem 7.4]:

Theorem 3.1. *For $R>0$, there exists a polynomial q_R and constants $M,\chi>0$ depending on R such that*

$$|q_R(\lambda)\Gamma_m(\lambda)|\le M(1+m)^{\chi}(1+|\lambda|)^{\deg q_R}$$

for all $m\ge 0$ and $\lambda\in a^*(R)$.

The polynomial q_R, defined as in (2-15), is of the form

$$q_R(\lambda)=d\big(\lambda-\tfrac{1}{2}\big)(\lambda-1)\cdots\big(\lambda-\tfrac{1}{2}\lfloor 2R\rfloor\big),$$

where d is a fixed constant. Therefore if $\lambda_0\in\{\frac{1}{2},1,\dots,\frac{1}{2}m\}$, then

$$|(\lambda-\lambda_0)\Gamma_m(\lambda)|\le M(1+m)^{\chi} \tag{3-2}$$

for all λ in a small neighborhood of λ_0.

Next we give explicit expression of $C^0(-1,\lambda)$. Our method is based on the idea of [Sekiguchi 1990].

Lemma 3.2. (1) *If $\mathcal{W}=\{1\}$, then $C^0(-1,\lambda)=c(\lambda)$ is a complex number given by*

$$c(\lambda)=2^{2\lambda}\frac{\Gamma\big(\frac{\rho+\lambda}{2}\big)\Gamma(-\lambda)\Gamma\big(\frac{-\rho+\lambda+m_1^++m_2^++1}{2}\big)}{\Gamma\big(\frac{\rho-\lambda}{2}\big)\Gamma(\lambda)\Gamma\big(\frac{-\rho-\lambda+m_1^++m_2^++1}{2}\big)}. \tag{3-3}$$

(2) *If* $\mathcal{W} = \{\pm 1\}$, *then* $C^0(-1, \lambda)$ *is the* 2×2 *matrix given by* $C^0(-1, \lambda) = c(\lambda)I$ *where*

$$c(\lambda) = 2^{2\lambda} \frac{\Gamma\left(\frac{\rho+\lambda}{2}\right)\Gamma(-\lambda)\Gamma\left(\frac{\lambda-\rho+1+m_1^+ +m_2^+}{2}\right)}{\Gamma\left(\frac{\rho-\lambda}{2}\right)\Gamma(\lambda)\Gamma\left(\frac{-\rho-\lambda+1+m_1^+ +m_2^+}{2}\right)},$$

and I *is the* 2×2 *identity matrix.*

Proof. Let $\mathcal{W} = \{1\}$. Then in this case, $C^0(-1, \lambda) = c(\lambda)$ is a complex number. Without loss of generality we assume that $\eta = 1$. For $\Re\lambda < 0$, using (2-10), (2-12), (2-5) and

$$\lim_{t\to\infty} e^{(\rho+\lambda)t} E_1^\circ(\lambda, 1)(t) = \lim_{t\to\infty} e^{(\rho+\lambda)t}[\Phi_\lambda(t) + \Phi_{-\lambda}(t)c(\lambda)] = c(\lambda)$$

we obtain

$$c(\lambda) = 2^{2\lambda} \frac{\Gamma\left(\frac{\rho+\lambda}{2}\right)\Gamma(-\lambda)\Gamma\left(\frac{-\rho+\lambda+m_1^+ +m_2^+ +1}{2}\right)}{\Gamma\left(\frac{\rho-\lambda}{2}\right)\Gamma(\lambda)\Gamma\left(\frac{-\rho-\lambda+m_1^+ +m_2^+ +1}{2}\right)}.$$

Let $\mathcal{W} = \{\pm 1\}$. With respect to the basis e_1 and e_2, let $C^0(-1, \lambda) = [c_{ij}(\lambda)]$. Then for $t > 0$,

$$\begin{aligned}
E_1^\circ(\lambda, e_1)(t) &= \Phi_\lambda(t)(1) + \Phi_{-\lambda}(t)c_{11}(\lambda),\\
E_{-1}^\circ(\lambda, e_1)(t) &= \Phi_\lambda(t)(0) + \Phi_{-\lambda}(t)c_{21}(\lambda),\\
E_1^\circ(\lambda, e_2)(t) &= \Phi_\lambda(t)(0) + \Phi_{-\lambda}(t)c_{12}(\lambda),\\
E_{-1}^\circ(\lambda, e_2)(t) &= \Phi_\lambda(t)(1) + \Phi_{-\lambda}(t)c_{22}(\lambda).
\end{aligned}$$

Thus, for $\Re\lambda < 0$ we obtain from the expansion (2-5) that

$$\lim_{t\to\infty} e^{(\rho+\lambda)t} E_w^\circ(\lambda, \eta)(t) = [C^0(-1, \lambda)\eta]_w.$$

Identify that if $w = 1$, then $i = 1$ and if $w = -1$, then $i = 2$. Then the above says

$$\begin{aligned}
c_{ij} &= \lim_{t\to\infty} e^{(\rho+\lambda)t} E_w^\circ(\lambda, e_j)(t)\\
&= \delta_{ij} \frac{2^{\lambda-\rho}\Gamma\left(\frac{\lambda+\rho}{2}\right)\Gamma\left(\frac{\lambda-\rho+1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} {}_2F_1\left(\frac{\rho+\lambda}{2}, \frac{\rho-\lambda}{2}; \frac{1}{2}; -\sinh^2 t\right).
\end{aligned}$$

Immediately, we obtain that $c_{ij} = 0$ if $i \neq j$. Now, substituting (2-10) in the above we obtain the entries

$$\begin{aligned}
c_{11}(\lambda) = c_{22}(\lambda) &= 2^{2\lambda} \frac{\Gamma(\frac{\rho+\lambda}{2})\Gamma(-\lambda)\Gamma\left(\frac{\lambda-\rho+1+m_1^+ +m_2^+}{2}\right)}{\Gamma\left(\frac{\rho-\lambda}{2}\right)\Gamma(\lambda)\Gamma\left(\frac{-\rho-\lambda+1+m_1^+ +m_2^+}{2}\right)},\\
c_{12}(\lambda) = c_{21}(\lambda) &= 0.
\end{aligned} \tag{3-4}$$

Thus, $C^0(-1,\lambda) = c(\lambda)\,\mathrm{Id}$ where

$$c(\lambda) = 2^{2\lambda}\frac{\Gamma\left(\frac{\rho+\lambda}{2}\right)\Gamma(-\lambda)\Gamma\left(\frac{\lambda-\rho+1+m_1^+ +m_2^+}{2}\right)}{\Gamma\left(\frac{\rho-\lambda}{2}\right)\Gamma(\lambda)\Gamma\left(\frac{-\rho-\lambda+1+m_1^+ +m_2^+}{2}\right)}. \qquad \square$$

We observe that $C^0(-1,\lambda)$ is a diagonal matrix with same diagonal entries $c(\lambda)$. Therefore, we can write (2-5) as

$$E_w^\circ(\lambda,\eta)(t) = \Phi_\lambda(t)\eta_w + c(\lambda)\eta_w\Phi_{-\lambda}(t). \tag{3-5}$$

Since $E_w^0(-\lambda,\eta)(t) = E_w^0(\lambda, C^0(-1,\lambda)\eta)$, we obtain

$$E_w^0(\lambda,\eta)(t) = \eta_w 2^{\lambda-\rho}\frac{\Gamma\left(\frac{\lambda+\rho}{2}\right)\Gamma\left(\frac{\lambda-\rho+m_1^+ +m_2^+ +1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{m_1^+ +m_2^+ +1}{2}\right)} \times {}_2F_1\left(\frac{\lambda+\rho}{2}, \frac{-\lambda+\rho}{2}; \frac{q}{2}; -\sinh^2 t\right) \tag{3-6}$$

for $\Re\lambda < 0$. Thus, for $\Re\lambda < 0$ the singularities are contained in

$$\{-\rho-2k_1 : k_1 \geq 0\} \cup \{\rho - m_1^+ - m_2^+ - 1 - 2k_2 : k_2 \geq 0 \text{ and } \rho - m_1^+ - m_2^+ - 1 - 2k_2 < 0\}.$$

From the previous results we have the following:

Lemma 3.3. (1) *The singularities of Γ_m are at most simple poles and contained in $\left\{\frac{1}{2}, 1, \ldots, \frac{1}{2}m\right\}$.*

(2) *The singularities of Φ_λ (as a function of λ) are at most simple poles and contained in $\frac{1}{2}\mathbb{N}$.*

(3) *The poles of $c(\lambda)$ are simple and equal to*

$$\{-\rho - 2k_1 : k_1 \in \mathbb{N}_0\} \cup \{\rho - 1 - m_1^+ - m_2^+ - 2k_2 : k_2 \in \mathbb{N}_0\} \cup \{k_3 : k_3 \in \mathbb{N}\}.$$

(4) *The poles of $E_w^\circ(\cdot,\eta)(t)$ are simple and equal to*

$$\{-\rho - 2k_1 : k_1 \in \mathbb{N}_0\} \cup \{\rho - 1 - m_1^+ - m_2^+ - 2k_2 : k_2 \in \mathbb{N}_0 \text{ and } \rho - 1 - m_1^+ - m_2^+ - 2k_2 \neq 0\}.$$

(5) *The set of zeros of $c(\lambda)$ is equal to*

$$\{\rho + 2k_1 : k_1 \in \mathbb{N}_0\} \cup \{-k_2 : k_2 \in \mathbb{N}\} \cup \{-\rho + 1 + m_1^+ + m_2^+ + 2k_3 : k_3 \in \mathbb{N}_0\}.$$

(6) *The set of zeros of $p_R(\lambda)E_w^\circ(\cdot,\eta)(t)$ is equal to $\{-k : k \in \mathbb{N}_0\}$.*

Remark 3.4. Suppose there exists $k_2 \in \mathbb{N}_0$ such that

$$\rho - 1 - m_1^+ - m_2^+ - 2k_2 = 0.$$

Then, the numerator of

$$\frac{\Gamma\left(\frac{\lambda+\rho}{2}\right)\Gamma\left(\frac{\lambda-\rho+m_1^+ +m_2^+ +1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{m_1^+ +m_2^+ +1}{2}\right)}$$

has a pole at $\lambda = 0$. On the other hand, it follows that $\lambda = 0$ is also a pole of the denominator. Hence, the singularity at $\lambda = 0$ is removable. Therefore, $E_w^\circ(\lambda, \eta)(t)$ is holomorphic at $\lambda = 0$, and consequently holomorphic on the line $i\mathbb{R}$.

If $\rho - 1 - m_1^+ - m_2^+ - 2k_2 \neq 0$ for all $k_2 \in \mathbb{N}_0$, then from the expression of E_w° it is holomorphic on $i\mathbb{R}$.

The Fourier inversion formula for the general rank cases is discussed in [van den Ban and Schlichtkrull 1999, Theorem 7.1] and the explicit formula given below for the split rank-one symmetric spaces is provided in [Anker and Orsted 2005, equation 5-13, p. 128].

Theorem 3.5 (Fourier inversion). *For $f \in C_c^\infty(K\backslash G/H)$ the Fourier inversion formula is given by*

$$f(a_t w\cdot x_0) = \int_{i\mathbb{R}} E_w^\circ(\lambda, \mathcal{F}f(\lambda))(t)\, d\lambda + 4\pi i \sum_{\mu_k \in L} \operatorname{Res}_{\lambda=-\mu_k}[\mathcal{F}f(\lambda)\Phi_\lambda(t)]_w, \tag{3-7}$$

where $L = \{\mu_k = \rho - 1 - m_1^+ - m_2^+ - 2k : k \in \mathbb{N}_0$ and $\mu_k > 0\}$.

4. The Helgason–Johnson theorem

In this section we prove an analogue of the Helgason–Johnson theorem for split rank-one semisimple symmetric spaces. Let

$$S_1 = \{\lambda \in \mathbb{C} : |\Re\lambda| \leq \rho\}.$$

Let $\alpha, \beta, \lambda \in \mathbb{C}$ with $\alpha \neq -1, -2, \ldots$ and $\rho = \alpha + \beta + 1$. From [Koornwinder 1975, equation 2.3] the Jacobi functions with parameters α, β and λ are defined as

$$\varphi_{-i\lambda}^{(\alpha,\beta)}(t) = {}_2F_1\left(\frac{\rho+\lambda}{2}, \frac{\rho-\lambda}{2}; \alpha+1; -\sinh^2 t\right).$$

Then the normalized Eisenstein integral can also be written in the form of Jacobi functions for $\Re\lambda > 0$ with $\alpha = (m_1^+ + m_2^+ + 1)/2 - 1$, $\beta = (m_1^- + m_2^+ + 2m_2^- + 1)/2 - 1$ and $\rho = \alpha + \beta + 1$ as

$$E_w^\circ(\lambda, \eta)(t) = \eta_w 2^{\lambda-\rho} \frac{\Gamma\left(\frac{\lambda+\rho}{2}\right)\Gamma\left(\frac{\lambda-\rho+m_1^+ +m_2^+ +1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{m_1^+ +m_2^+ +1}{2}\right)} \varphi_{-i\lambda}^{(\alpha,\beta)}(t).$$

Thus, from [Koornwinder 1975, Lemma 2.3] and properties of the Gamma function, we obtain the estimate, for each fixed $R > 0$,

$$|p_R(\lambda)E_w^\circ(\lambda, \eta)(t)| \lesssim \|\eta\|(1 + |\lambda|)^d(1 + t)e^{(|\Re\lambda|-\rho)t} \tag{4-1}$$

for $d = \deg p_R$ and $\lambda \in -a^*(R)$. Therefore, for each fixed $\eta \in \mathbb{C}^{\mathcal{W}}$,

$$p_R(\lambda)E_w^\circ(\lambda, \eta)(\cdot)$$

is bounded if $\lambda \in -a^*(R)$ and $|\Re\lambda| < \rho$. It is also known that (see [Flensted-Jensen and Koornwinder 1973, Lemma 2.1]) $\varphi_{-i\lambda}^{(\alpha,\beta)}(t)$ is bounded in t if and only if $\lambda \in S_1$ for $\alpha \geq \beta \geq -\frac{1}{2}$. This gives characterization of all λ for which $t \mapsto p_R(\lambda)E_w^\circ(\lambda, \eta)(t)$ is bounded on $(0, \infty)$, only for the case when $m_1^+ \geq m_1^- + 2m_2^-$. In this section we prove a similar result for general m_1, m_2.

Moreover, we observe that from the asymptotic behavior (2-11), if $\Re\lambda > 0$ and λ is not a pole of E_w° we have

$$\lim_{t\to\infty} |e^{(\rho-\lambda)t}E_w^\circ(\lambda, \eta)(t) - \eta_w| = 0.$$

This implies that

$$|E_w^\circ(\lambda, \eta)(t)| \asymp |\eta_w|e^{(\Re\lambda-\rho)t}.$$

Thus, for the case when $m_1^- + 2m_2^- \leq m_1^+ + 2$, where there are no poles of $E_w^\circ(\cdot, \eta)(t)$ for $\Re\lambda > 0$, the characterization of boundedness (for $\Re\lambda > 0$) also follows naturally from this asymptotic behavior. In particular, E_w° is bounded if $0 < \Re\lambda \leq \rho$ and unbounded if $\Re\lambda > \rho$. However, we have the main difficulties in the other cases.

Theorem 4.1. *Fix $R > 0$, $w \in \mathcal{W}$ and $\eta \in \mathbb{C}^{\mathcal{W}}$.*

(1) *Let $\lambda_0 \in (S_1 \cap -a^*(R)) \cup -\mathbb{N}_0$. Then the function $t \mapsto p_R(\lambda_0)E_w^\circ(\lambda_0, \eta)(t)$ is bounded on $(0, \infty)$.*

(2) *For $\lambda_0 \in -a^*(R)$ and for nonzero η_w, if the function $t \mapsto p_R(\lambda_0)E_w^\circ(\lambda_0, \eta)(t)$ is bounded on $(0, \infty)$, then $\lambda_0 \in S_1 \cup -\mathbb{N}_0$.*

Remark 4.2. From this theorem above, we conclude that for a fixed $R > \rho$, a nonzero η_w and $\lambda_0 \in -a^*(R)$, the function $p_R(\lambda_0)E_w^\circ(\lambda_0, \eta)(\cdot)$ is bounded on $(0, \infty)$ if and only if $\lambda_0 \in S_1 \cup -\mathbb{N}_0$.

Proof of Theorem 4.1. Throughout the proof we will assume that $\eta_w \neq 0$.

Case 1. $\Re\lambda_0 \geq 0$: For a fixed $R > 0$, let p_R be the polynomial given in (2-13). Let

$$F_w(\lambda, \eta)(t) = p_R(\lambda)E_w^\circ(\lambda, \eta)(t).$$

This is holomorphic at λ_0. Then using (2-14) and (3-5) we have

$$\begin{aligned} F_w(\lambda_0, \eta)(t) &= \frac{d}{d\lambda}\Big|_{\lambda=\lambda_0} (\lambda - \lambda_0)F_w(\lambda, \eta)(t) \\ &= \frac{d}{d\lambda}\Big|_{\lambda=\lambda_0} \sum_{m=0}^{\infty} \eta_w \big[e^{(\lambda-\rho)t}(\lambda - \lambda_0)p_R(\lambda)\Gamma_m(\lambda) \\ &\qquad + e^{(-\lambda-\rho)t}(\lambda - \lambda_0)p_R(\lambda)c(\lambda)\Gamma_m(-\lambda)\big]e^{-mt}. \end{aligned}$$

If $\lambda_0 \in \frac{1}{2}\mathbb{N}_0$, then the series expansion of Φ_{λ_0} and (hence) the series expansion of $F_w(\lambda_0, \eta)(t)$ is not valid. Thus, to get the series expansion we multiply by $\lambda - \lambda_0$ and take derivative at λ_0. To make it uniform, we use the above expression for all λ_0. Let

$$a_m(\lambda) = (\lambda - \lambda_0)\Gamma_m(\lambda) \quad \text{and} \quad b(\lambda) = (\lambda - \lambda_0)p_R(\lambda)c(\lambda).$$

It follows from Lemma 3.3 that a_m and b are holomorphic at λ_0. Then $F_w(\lambda_0, \eta)(t)$ can be written as

$$F_w(\lambda_0, \eta)(t) = p_0(t) + \sum_{m=1}^{\infty} p_m(t),$$

where

$$p_0(t) = \eta_w[e^{(\lambda_0-\rho)t}\, p_R(\lambda_0) + e^{(-\lambda_0-\rho)t} b(\lambda_0)(-t) + e^{(-\lambda_0-\rho)t} b'(\lambda_0)]$$

and

$$\begin{aligned} p_m(t) = \eta_w\big[&e^{(\lambda_0-\rho)t} a_m(\lambda_0)\, p_R'(\lambda_0) + e^{(\lambda_0-\rho)t} a_m'(\lambda_0) p_R(\lambda_0) \\ &+ e^{(\lambda_0-\rho)t} a_m(\lambda_0) p_R(\lambda_0) t - e^{(-\lambda_0-\rho)t} b(\lambda_0)\Gamma_m(-\lambda_0)t \\ &+ e^{(-\lambda_0-\rho)t} b'(\lambda_0)\Gamma_m(-\lambda_0) - e^{(-\lambda_0-\rho)t} b(\lambda_0)\Gamma_m'(-\lambda_0)\big]e^{-mt}. \end{aligned}$$

First we will find the estimates of a_m, a_m', b, b', Γ_m and Γ_m'. From Theorem 3.1 we can choose an $R' > \rho$ and obtain

$$|\Gamma_m(-\lambda_0)| \le M \frac{(1 + |\lambda_0|)^{\deg q_{R'}}}{|q_{R'}(-\lambda_0)|}(1+m)^{\chi}.$$

We note that Γ_m has no pole on $\Re\lambda < 0$. Since $q_{R'}(\lambda)$ has no zeroes in $\Re\lambda < 0$ and $(1 + |\lambda|)^{\deg q_{R'}}/|q_R(\lambda)|$ is bounded uniformly in the region $\Re\lambda < 0$, we have

$$|\Gamma_m(-\lambda_0)| \le M(1+m)^{\chi}.$$

Therefore, for a fixed $t_0 > 0$, we obtain

$$|\Gamma_m(-\lambda_0)| \le M_{t_0} e^{mt_0}.$$

Now, we shall calculate the estimate of a_m. Since a_m is holomorphic at λ_0, integrating over a circle γ_ϵ centered at λ_0 with radius $\epsilon < \frac{1}{2}$ and using (3-2) we get

$$|a_m(\lambda_0)| \le M \oint_{\gamma_\epsilon} \frac{|\lambda - \lambda_0|\,|\Gamma_m(\lambda)|}{|\lambda - \lambda_0|}\,|d\gamma| \le \frac{M}{2}(1+m)^{\chi}.$$

Now, we have to estimate $a_m'(\lambda_0)$. Since $a_m(\lambda)$ is holomorphic at λ_0, we obtain from Cauchy's integral formula that

$$a_m'(\lambda_0) = \frac{1}{2\pi i} \oint_{\gamma_\epsilon} \frac{a_m(\lambda)}{(\lambda - \lambda_0)^2}\, d\gamma.$$

Since $\Gamma_m(\lambda)$ is holomorphic on the closed curve γ_ϵ,

$$|a_m'(\lambda_0)| \le \oint_{\gamma_\epsilon} \frac{|(\lambda-\lambda_0)\Gamma_m(\lambda)|}{|\lambda-\lambda_0|^2}\,|d\gamma| \le M(1+m)^\chi \oint_{\gamma_\epsilon} \frac{1}{|\lambda-\lambda_0|}\,|d\gamma| \le M(1+m)^\chi.$$

If λ_0 is not a pole of $c(\lambda)$ then $b(\lambda_0)=0$. Suppose that λ_0 is a pole of $c(\lambda)$. Then $\lambda_0=\mu_k=\rho-1-m_1^+-m_2^+-2k>0$ for some $k\in\mathbb{N}_0$. Using the property $\Gamma(z+1)=z\Gamma(z)$, we have

$$\begin{aligned}\Gamma\left(\frac{-\rho+\lambda+1+m_1^++m_2^+}{2}\right) &= \frac{\Gamma\left(\frac{-\rho+\lambda+1+m_1^++m_2^+}{2}+k+1\right)}{\prod_{i=0}^k\left(\frac{-\rho+\lambda+1+m_1^++m_2^+}{2}+i\right)}\\ &= 2^{k+1}\frac{\Gamma\left(\frac{\lambda-\mu_k}{2}+1\right)}{\prod_{i=0}^k(\lambda-\mu_i)}.\end{aligned}$$

Thus, using $2^{1-2z}\sqrt{\pi}\Gamma(2z)=\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$ and $\Gamma(z+\alpha)/\Gamma(z+\beta)\sim z^{\alpha-\beta}$ [Erdélyi et al. 1953, Section 1.18, equation (3)], we obtain that, for $d=\deg p_R$,

$$\begin{aligned}|b(\lambda)| &\le \left|(\lambda-\mu_k)p_R(\lambda)2^{2\lambda}\frac{\Gamma\left(\frac{\rho+\lambda}{2}\right)\Gamma(-\lambda)\Gamma\left(\frac{\lambda-\rho+1+m_1^++m_2^+}{2}\right)}{\Gamma\left(\frac{\rho-\lambda}{2}\right)\Gamma(\lambda)\Gamma\left(\frac{-\rho-\lambda+1+m_1^++m_2^+}{2}\right)}\right|\\ &\le \left|\frac{(\lambda-\mu_k)p_R(\lambda)\Gamma\left(\frac{\lambda-\mu_k}{2}+1\right)}{\prod_{i=0}^k(\lambda-\mu_i)}\frac{2^{2\lambda+k+1}\Gamma\left(\frac{\rho+\lambda}{2}\right)\Gamma(-\lambda)}{\Gamma\left(\frac{\rho-\lambda}{2}\right)\Gamma(\lambda)\Gamma\left(\frac{-\rho-\lambda+1+m_1^++m_2^+}{2}\right)}\right|\\ &\le \left|\frac{p_R(\lambda)\Gamma\left(\frac{\lambda-\mu_k}{2}+1\right)}{\prod_{i=0}^{k-1}(\lambda-\mu_i)}\frac{2^{2\lambda+k+1}\Gamma\left(\frac{\rho+\lambda}{2}\right)\Gamma(-\lambda)}{\Gamma\left(\frac{\rho-\lambda}{2}\right)\Gamma(\lambda)\Gamma\left(\frac{-\rho-\lambda+1+m_1^++m_2^+}{2}\right)}\right|\\ &\le M(1+|\lambda|)^d.\end{aligned}$$

Thus,

$$|b(\lambda_0)| \le M(1+|\lambda_0|)^d,$$

and by Cauchy's integral formula we have

$$|b'(\lambda_0)| \le M(1+|\lambda_0|)^d.$$

Now, we will estimate $\Gamma_m'(-\lambda_0)$. Since Γ_m is holomorphic at $-\lambda_0$, using Theorem 3.1, we get

$$|\Gamma_m'(-\lambda_0)| \le \oint_{\gamma} \frac{|\Gamma_m(-\lambda)|}{|\lambda+\lambda_0|^2}\,|d\gamma| \le \tfrac{1}{2}M(1+m)^\chi,$$

where γ is a circle centered at $-\lambda_0$ with radius $<\frac{1}{2}$.

Hence, for a fixed $t_0>0$,

$$|\Gamma_m(-\lambda_0)|, \quad \Gamma_m'(-\lambda_0), \quad |a_m(\lambda_0)|, \quad |a_m'(\lambda_0)|, \quad |b(\lambda_0)|, \quad |b'(\lambda_0)|$$

all are dominated by e^{mt_0}. Let $\deg p_R = d$. Then, we have

$$\begin{aligned}
&|F_w(\lambda,\eta)(t)e^{(-\lambda_0+\rho)t} - \eta_w p_R(\lambda_0)| \\
&\quad\le e^{-2\Re\lambda_0 t}|\eta_w b(\lambda_0)t| + e^{-2\Re\lambda_0 t}|\eta_w b'(\lambda_0)| \\
&\qquad + \sum M|\eta_w|\big[(1+m)^{\chi+d} + t(1+m)^{\chi+d} + e^{-2\Re\lambda_0 t}t(1+m)^{\chi+d} \\
&\qquad\qquad + e^{-2\Re\lambda_0 t}(1+m)^{\chi+d}\big]e^{-mt} \\
&\quad\le e^{-2\Re\lambda_0 t}|\eta_w b(\lambda_0)t| + e^{-2\Re\lambda_0 t}|\eta_w b'(\lambda_0)| + Mt\sum_{m=1}^{\infty}|\eta_w|e^{-m(t-t_0)} \\
&\quad\le e^{-2\Re\lambda_0 t}|\eta_w b(\lambda_0)t| + e^{-2\Re\lambda_0 t}|\eta_w b'(\lambda_0)| + M|\eta_w|te^{-(t-t_0)}\left(\sum_{m=0}^{\infty}e^{-m(t-t_0)}\right) \\
&\quad\le e^{-2\Re\lambda_0 t}|\eta_w b(\lambda_0)t| + e^{-2\Re\lambda_0 t}|\eta_w b'(\lambda_0)| + M|\eta_w|te^{-(t-t_0)}\frac{1}{1-e^{-(t-t_0)}}.
\end{aligned}$$

This implies that, for $\Re\lambda_0 > 0$,

$$|F_w(\lambda,\eta)(t)e^{(-\lambda_0+\rho)t} - \eta_w p_R(\lambda_0)| \to 0$$

as $t\to\infty$. Therefore, if λ_0 is not a zero of p_R (or, equivalently if it is not a pole of $E_w^\circ(\lambda,\eta)(t)$) then

$$|F_w(\lambda_0,\eta)(t)| \asymp |p_R(\lambda_0)\eta_w|e^{(\Re\lambda_0-\rho)t}$$

as $t\to\infty$. Hence, if $\Re\lambda_0 > 0$ and if λ_0 is not a pole of $E_w^\circ(\lambda,\eta)(t)$ then

$$|E_w^\circ(\lambda_0,\eta)(t)| \asymp |\eta_w|e^{(\Re\lambda_0-\rho)t} \tag{4-2}$$

as $t\to\infty$.

Subcase 1. $\Re\lambda_0 > \rho$: We observe that $E_w^\circ(\lambda,\eta)(t)$ has no pole for $\Re\lambda > \rho$. Therefore, it follows from (4-2) that for $\Re\lambda_0 > \rho$, the function $E_w^\circ(\lambda_0,\eta)(\,\cdot\,)$ is not bounded.

Subcase 2. $\Re\lambda_0 = \rho$: We also observe that $E^\circ(\lambda,t)$ has no pole for $\Re\lambda = \rho$. Therefore, it follows from (4-2) that for $\Re\lambda_0 = \rho$, the function $E^\circ(\lambda_0,\,\cdot\,)$ is bounded.

Subcase 3. $0 \le \Re\lambda_0 < \rho$: By using the estimates of $|\Gamma_m(-\lambda_0)|$, $|\Gamma_m'(-\lambda_0)|$, $|a_m(\lambda_0)|$, $|a_m'(\lambda_0)|$, $p|b(\lambda_0)|$ and $|b'(\lambda_0)|$ we obtain that

$$|p_m(t)| \le M|\eta_w|e^{(\Re\lambda_0-\rho)t}[(1+m)^\chi(1+t)(1+|\lambda_0|)^d]e^{-mt}.$$

By using $(1+m)^\chi \le (m+\chi)!/m!$, we obtain

$$\sum_{m=0}^{\infty}(1+m)^\chi e^{-mt} \le \sum_{m=0}^{\infty}\frac{(m+\chi)!}{m!}e^{-mt} = \chi!(1-e^{-t})^{-\chi-1}. \tag{4-3}$$

Using the above,

$$|F_w(\lambda_0, \eta)(t)| \le M|\eta_w|(1+|\lambda_0|)^d(1+t)e^{(\Re\lambda_0-\rho)t}.$$

Thus, in this case $F_w(\lambda_0, \eta)(t)$ is a bounded function in t. Note that, when $\Re\lambda_0 = 0$, E_w° has no singularities. Thus, for $\lambda_0 = ix$ with $x \neq 0$, we have (from (4-2))

$$|E_w^\circ(ix, \eta)(t)| \asymp |\eta_w|e^{-\rho t}.$$

Case 2. $\lambda_0 \in -\mathbb{N}_0$: From Lemma 3.3, $\lambda_0 = -k$ is a zero of $p_R E_w^\circ$. Thus, $p_R(\lambda_0)E_w^\circ(\lambda_0, \eta)(t) = 0$ for all $t > 0$.

Case 3. $\Re\lambda_0 < 0$: In this case we assume that λ_0 is not a negative integer and $-R < \Re\lambda_0 < 0$. Then using (2-14) and (3-5),

$$F_w(\lambda_0, \eta)(t) = \frac{d}{d\lambda}\Big|_{\lambda=\lambda_0} \eta_w \sum_{m=0}^{\infty} \big[e^{(\lambda-\rho)t}(\lambda-\lambda_0)p_R(\lambda)\Gamma_m(\lambda) + e^{(-\lambda-\rho)t}(\lambda-\lambda_0)p_R(\lambda)c(\lambda)\Gamma_m(-\lambda)\big]e^{-mt}. \tag{4-4}$$

Let

$$g_m(\lambda) = (\lambda-\lambda_0)p_R(\lambda)\Gamma_m(\lambda), \quad h(\lambda) = p_R(\lambda)c(\lambda), \quad d_m(\lambda) = (\lambda-\lambda_0)\Gamma_m(-\lambda).$$

We observe that all these are holomorphic at λ_0.

Again we write $F_w(\lambda_0, \eta)(t)$ as

$$F_w(\lambda_0, \eta)(t) = p_0(t) + \sum_{m=1}^{\infty} p_m(t),$$

with

$$p_0(t) = \eta_w e^{(\lambda_0-\rho)t}p_R(\lambda_0) + \eta_w e^{(-\lambda_0-\rho)t}h(\lambda_0)$$

and

$$p_m(t) = \eta_w\big[e^{(\lambda_0-\rho)t}g_m'(\lambda_0) + e^{(\lambda_0-\rho)t}tg_m(\lambda_0) + e^{(-\lambda_0-\rho)t}d_m(\lambda_0)h'(\lambda_0) + e^{(-\lambda_0-\rho)t}d_m'(\lambda_0)h(\lambda_0) + e^{(-\lambda_0-\rho)t}(-t)d_m(\lambda_0)h(\lambda_0)\big].$$

Similar to case $\Re\lambda_0 > 0$, the functions $g_m(\lambda)$, $h(\lambda)$ and $d_m(\lambda)$ can be dominated by e^{mt_0} for some $t_0 > 0$. Then calculating similarly as above we get that

$$|F_w(\lambda_0, \eta)(t)e^{(\lambda_0+\rho)t} - \eta_w h(\lambda_0)| \to 0$$

as $t \to \infty$. We have $p_R(\lambda)E_w^\circ(\lambda, \eta)(t) \equiv 0$ if and only if $\lambda_0 \in -\mathbb{N}_0$ as the polynomial p_R captures the poles of E° described in Lemma 3.3. It follows that when $\lambda_0 \notin -\mathbb{N}_0$ and $\Re\lambda_0 < 0$ we have $h(\lambda_0) \neq 0$.

Subcase 1. $-R \le \Re\lambda_0 < -\rho$ and $\lambda_0 \notin -\mathbb{N}_0$: In this case, we first observe (from the definition of h) that $h(\lambda_0) \neq 0$. We have

$$|F_w(\lambda_0, \eta)(t)| \asymp |\eta_w|e^{-(\Re\lambda_0+\rho)t}$$

as $t \to \infty$. This implies $F_w(\lambda_0, \eta)(\,\cdot\,)$ is unbounded.

Subcase 2. $\Re\lambda_0 = -\rho$: Suppose ρ is not a natural number. Then h is nonzero at $\Re\lambda_0 = -\rho$ and

$$|F_w(\lambda_0, \eta)(t)| \asymp |\eta_w|.$$

Thus, $F_w(\lambda_0, \eta)(\cdot)$ is bounded.

If ρ is a natural number and $\Re\lambda_0 = -\rho$, $\lambda_0 \neq -\rho$, then also $h(\lambda_0)$ is nonzero. Therefore, $F_w(\lambda_0, \eta)(\cdot)$ is bounded in this case as well.

Subcase 3. $-\rho < \Re\lambda_0 < 0$ and $-R < \Re\lambda_0$: It follows from (4-4) that

$$|F_w(\lambda_0, \eta)(t)| \le C|h(\lambda_0)|(1+t)e^{-(\Re\lambda_0+\rho)t}$$

as $t \to \infty$. If $-\rho \le \Re\lambda_0 < 0$, then $\Re\lambda_0 + \rho \ge 0$; hence $F_w(\lambda_0, \eta)(t)$ is bounded. □

From the proof, we get the following improved estimate of normalized Eisenstein integrals (see (2-7)):

Corollary 4.3. *For $R > 0$ and a fixed $\eta \in \mathbb{C}^{\mathcal{W}}$, there exists a constant $M > 0$ such that for all $\lambda_0 \in -a^*(R)$ and $t > 0$,*

$$|p_R(\lambda_0)E_w^\circ(\lambda_0, \eta)(t)| \le M\|\eta\|(1+|\lambda_0|)^{\deg p_R}(1+t)e^{(|\Re\lambda_0|-\rho)t}. \tag{4-5}$$

Moreover, for each fixed λ_0:

(1) *If λ_0 is a pole of E_w°, then*

$$|p_R(\lambda_0)E_w^\circ(\lambda_0, \eta)(t)| \le M\|\eta\|(1+|\lambda_0|)^{\deg p_R}(1+t)e^{(|\Re\lambda_0|-\rho)t}.$$

(2) *If λ_0 is a nonpositive integer, then $p_R(\lambda_0)E_w^\circ(\lambda_0, \eta)(t) = 0$.*

(3) *If λ_0 is not a pole of E° and also not a nonpositive integer, then*

$$|E_w^\circ(\lambda_0, \eta)(t)| \asymp M|\eta_w|e^{(|\Re\lambda_0|-\rho)t}.$$

The above estimate (4-5) has been obtained in [Andersen 2001] for $R = \frac{1}{4}$ using Jacobi functions for real hyperbolic spaces $\mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q)$ with $q > 1$ and in the general case we had only a rough estimate (2-7).

Corollary 4.4. *For $R > 0$ and a fixed $\eta \in \mathbb{C}^{\mathcal{W}}$, there exists a constant $M > 0$ such that for all $\lambda \in -a^*(R)$ and $t > 0$,*

$$\left|\left(\frac{\partial}{\partial\lambda}\right)^n p_R(\lambda)E_w^\circ(\lambda, \eta)(t)\right| \le M\|\eta\|(1+|\lambda|)^{\deg p_R}(1+t)^{n+1}e^{(|\Re\lambda|-\rho)t}.$$

The proof of this follows from (4-5) and Cauchy's integral formula by integrating over a circle with radius $\frac{1}{1+t}$.

5. Hausdorff–Young inequality and Riemann–Lebesgue lemma

Our version of the Helgason–Johnson theorem says that $p_R(\lambda)E_w^\circ(\lambda,\eta)(t)$ is bounded in t for a fixed $\lambda \in -a^*(R) \cap S_1$. As a consequence of this theorem and Corollary 4.3, we have that for $f \in L^1(K\backslash G/H)$ the Fourier transform $\mathcal{F}f$ extends meromorphically to S_1. Moreover, for a fixed $R > 0$, the function $p_R(-\lambda)\mathcal{F}f(\lambda)$ is holomorphic on $S_1^0 \cap a^*(R)$ and continuous on $S_1 \cap a^*(R)$. Now, we consider the region

$$S_r = \{z \in \mathbb{C} : |\Re z| \le (2/r - 1)\rho\}$$

for $1 \le r \le 2$. Observe that $J(t) \asymp e^{2\rho t}$. Let r' be such that $1/r' + 1/r = 1$. Define $C_c^\infty(i\mathfrak{a}_q^*)$ to be the space of all complex valued smooth functions with compact support on $i\mathfrak{a}_q^*$.

Proposition 5.1. *Let $1 \le r \le 2$. For $\lambda \in S_r^\circ \cap -a^*(R)$, $w \in \mathcal{W}$ and $\eta \in \mathbb{C}^{\mathcal{W}}$, the function $t \mapsto p_R(\lambda)E_w^\circ(\lambda,\eta)(t)$ is in $L^{r'}((0,\infty), J(t)\,dt)$, where $1/r + 1/r' = 1$.*

Proof. Using the estimate of Corollary 4.3 we have

$$\begin{aligned}\left(\int_0^\infty |p_R(\lambda)E_w^\circ(\lambda,\eta)(t)|^{r'} J(t)\,dt\right)^{1/r'} &\\ \le M\|\eta\|\left(\int_0^\infty ((1+|\lambda|)^d(1+t)e^{(|\Re\lambda|-\rho)t})^{r'} e^{2\rho t}\,dt\right)^{1/r'} &\\ \le M\|\eta\|(1+|\lambda|)^d\left(\int_0^\infty (1+t)^{r'} e^{r'(|\Re\lambda|-\rho)t+2\rho t}\,dt\right)^{1/r'}.&\end{aligned}$$

For the integral to converge we must have that

$$r'(|\Re\lambda| - \rho) + 2\rho < 0.$$

That is,

$$|\Re\lambda| < (1 - 2/r')\rho = (2/r - 1)\rho,$$

which implies that if $\lambda \in S_r^\circ$ then $p_R(\lambda)E_w^\circ(\lambda,\eta)(t)$ is in $L^{r'}$ for any $\eta \in \mathbb{C}^{\mathcal{W}}$ and $w \in \mathcal{W}$. □

Proposition 5.2. *For $1 \le r \le 2$ and $f \in L^r(K\backslash G/H)$, the Fourier transform exists on S_r and extends meromorphically to the strip S_r. Furthermore, for a fixed $R > 0$, the function $p_R(-\lambda)\mathcal{F}f(\lambda)\eta$ is holomorphic on $S_r^0 \cap a^*(R)$ and has continuous extension on $S_r \cap a^*(R)$ for any $\eta \in \mathbb{C}^{\mathcal{W}}$.*

Proof. By Proposition 5.1 and Hölder's inequality, for $\lambda \in S_r \cap a^*(R)$,

$$\begin{aligned}|p_R(-\lambda)\mathcal{F}f(\lambda)\eta| &\le \sum_{w\in\mathcal{W}} \int_0^\infty |f(a_t w \cdot x_0)\, p_R(-\lambda)E_w^\circ(-\lambda,\eta)(t)|J(t)\,dt \\ &\le M\|f\|_{L^r(K\backslash G/H)}\|\eta\|(1+|\lambda|)^d.\end{aligned}$$

Thus, $p_R(-\lambda)\mathcal{F}f(\lambda)\eta$ exists and is continuous on the strip $S_r \cap a^*(R)$. Moreover, using Corollary 4.4 we obtain

$$\begin{aligned}
&\left|\frac{\partial^m}{\partial\lambda^m} p_R(-\lambda)\mathcal{F}f(\lambda)\eta\right| \\
&\quad\le \sum_{w\in\mathcal{W}} \int_0^\infty \left| f(a_t w\cdot x_0)\frac{\partial^m}{\partial\lambda^m} p_R(-\lambda)E_w^\circ(-\lambda,\eta)(t)\right| J(t)\,dt \\
&\quad\le M\|f\|_{L^r(K\backslash G/H)}\|\eta\|(1+|\lambda|)^d\left(\int_0^\infty (1+t)^{(m+1)r'} e^{r'(|\Re\lambda|-\rho)t+2\rho t}\,dt\right)^{1/r'},
\end{aligned}$$

which converges if and only if $\lambda \in S_r^0 \cap a^*(R)$. Thus, the function $p_R(-\lambda)\mathcal{F}f(\lambda)\eta$ is holomorphic in $S_r^0 \cap a^*(R)$. □

Then we have the following lemma, whose proof follows from the proof of Proposition 5.2.

Lemma 5.3. *Let $\lambda_0 \in S_1 \cap a^*(R)$ be a real number, $w \in \mathcal{W}$, $\eta \in \mathbb{C}^{\mathcal{W}}$ and $d = \deg p_R$. Then, we have*

$$\sup_{\lambda\in\mathbb{R}} \left|\frac{p_R(-(\lambda_0+i\lambda))\mathcal{F}f(\lambda_0+i\lambda)\eta}{(1+\lambda_0+i\lambda)^d}\right| \le M\|f\|_1\|\eta\|, \quad f \in L^1(K\backslash G/H),$$

for some $M > 0$ and for any $\eta \in \mathbb{C}^{\mathcal{W}}$.

Now we prove an analogue of the Hausdorff–Young inequality.

Theorem 5.4. *Let $1 \le r \le 2$ and $\lambda_0 \in S_r \cap a^*(R)$ be a real number. Then there exists $M > 0$ such that*

$$\left(\int_{\mathbb{R}} \left|\frac{p_R(-(\lambda_0+i\lambda))\mathcal{F}f(\lambda_0+i\lambda)\eta}{(1+\lambda_0+i\lambda)^d}\right|^{r'} d\lambda\right)^{1/r'} \le M\|f\|_r\|\eta\|, \quad f \in L^r(K\backslash G/H),$$

for any $\eta \in \mathbb{C}^{\mathcal{W}}$, with $d = \deg p_R$.

Proof. We will equip the space $i\mathbb{R}$ with Lebesgue measure $d\lambda$. Let $1 \le r \le 2$ and fix an arbitrary $\lambda_0 \in S_r \cap a^*(R)$ such that $\lambda_0 \in \mathbb{R}$. Write $\mu = (2/r-1)^{-1}\lambda_0$. For every complex number z such that $0 \le \Re z \le 1$ and $f \in C_c^\infty(K\backslash G/H)$, $\lambda \in i\mathbb{R}$, we define a family of operators $T_z f(\lambda)$ by

$$T_z f(\lambda)(\eta) = \frac{p_R(-\lambda - z\mu)\mathcal{F}f(\lambda+z\mu)\eta}{(1+\lambda+z\mu)^d}, \quad \eta \in \mathbb{C}^{\mathcal{W}}.$$

Fix $\eta \in \mathbb{C}^{\mathcal{W}}$. Let $z = 1+iy$. Then using Lemma 5.3 we get that

$$\begin{aligned}
\|T_{1+iy}f(\,\cdot\,)(\eta)\|_{L^\infty(i\mathbb{R})} &\le \sup_{\lambda\in\mathbb{R}} \left|\frac{p_R(-i\lambda - iy\mu - \mu)\mathcal{F}f(i\lambda+iy\mu+\mu)\eta}{(1+i\lambda+iy\mu+\mu)^d}\right| \\
&\le C\sup_{\lambda\in\mathbb{R}} \left|\frac{p_R(-i\lambda-\mu)\mathcal{F}f(i\lambda+\mu)\eta}{(1+i\lambda+\mu)^d}\right| \le M\|f\|_{L^1}\|\eta\|.
\end{aligned}$$

Let $z = iy$. Then by (2-16) and using the fact that on $i\mathbb{R}$ there are no poles of $\mathcal{F}$,

$$\begin{aligned}\|T_{iy} f(\cdot)\eta\|^2_{L^2(i\mathbb{R})} &= \int_{\mathbb{R}} \left| \frac{p_R(-i\lambda - iy\mu)\mathcal{F}f(i\lambda + iy\mu)\eta}{(1 + i\lambda + iy\mu)^d} \right|^2 d\lambda \\ &\le C \int_{\mathbb{R}} |\mathcal{F}f(i\lambda + iy\mu)\eta|^2 \, d\lambda \\ &\le C \int_{\mathbb{R}} |\mathcal{F}f(i\lambda)\eta|^2 \, d\lambda \\ &\le C \|f\|^2_{L^2(X)} \|\eta\|.\end{aligned}$$

Now, consider the function

$$\phi(z) = \int_{\mathbb{R}} T_z f(i\lambda)(\eta)\overline{g(i\lambda)}\, d\lambda, \quad f \in C_c^\infty(K\backslash G/H), \quad g \in C_c^\infty(i\mathbb{R}).$$

By definition of $T_z f$, the function ϕ is holomorphic on $0 < \Re z < 1$ and continuous on $0 \le \Re z \le 1$. Using Lemma 5.3 we derive that $\|T_z f(\cdot)\eta\|_{L^\infty(i\mathbb{R})} \le M\|f\|_{L^1(X)}\|\eta\|$ and

$$\log|\phi(z)| \le M \log(\|f\|_{L^1(X)}\|\eta\|\|g\|_{L^1(i\mathbb{R})}).$$

Hence, $\{T_z\}$ is an admissible family of operators. Thus, by the complex interpolation theorem we have, for $\theta_r = (2/r - 1)$,

$$\|T_{\theta_r} f(\cdot)\eta\|_{L^{r'}(i\mathbb{R})} \le M\|f\|_{L^r(X)}\|\eta\| \quad \text{for all } \ f \in L^r(K\backslash X).$$

That is, for $\lambda_0 \in \mathbb{R}$ and $\lambda_0 \in a^*(R)$,

$$\left(\int_{\mathbb{R}} \left| \frac{p_R(-(\lambda_0 + i\lambda))\mathcal{F}f(\lambda_0 + i\lambda)(\eta)}{(1 + \lambda_0 + \lambda)^d} \right|^{r'} d\lambda \right)^{1/r'} \le M\|f\|_r\|\eta\|, \quad |\lambda_0| \le (2/r - 1)\rho.$$

We finally obtain that for a real number λ_0 with $\lambda_0 \in S_r \cap a^*(R)$, $w \in \mathcal{W}$, $\eta \in \mathbb{C}^{\mathcal{W}}$,

$$\left(\int_{\mathbb{R}} \left| \frac{p_R(-(\lambda_0 + i\lambda))\mathcal{F}f(\lambda_0 + i\lambda)\eta}{(1 + \lambda_0 + \lambda)^d} \right|^{r'} d\lambda \right)^{1/r'} \le M\|f\|_r\|\eta\|. \qquad \square$$

Remark 5.5. If $-\lambda_0 \in S_r \cap -a^*(R)$ is not a singularity of E°, that is, $p_R(-\lambda_0) \neq 0$, then $p_R(-\lambda_0 - i\lambda)/(1 + \lambda_0 + i\lambda)^d \asymp \text{const}$. Therefore, we can rewrite the above inequality as

$$\left(\int_{\mathbb{R}} |\mathcal{F}f(\lambda_0 + i\lambda)\eta|^{r'} d\lambda \right)^{1/r'} \le M\|f\|_r\|\eta\|.$$

Remark 5.6. Fix $\eta \in \mathbb{C}^{\mathcal{W}}$. Let $1 \le r < 2$, $f \in L^r(K\backslash G/H)$. Then $p_R(-\lambda)\mathcal{F}f(\lambda)(\eta)$ exists almost everywhere on the lines $|\Re\lambda| = (2/r - 1)\rho$ (see Proposition 5.1).

We now prove an analogue of the Riemann–Lebesgue lemma.

Theorem 5.7. *Let $1 \le r < 2$ and $\lambda_0 \in S_r^\circ \cap a^*(R)$ be a real number. Then for any $f \in L^r(K\backslash G/H)$ and for each fixed η,*

$$\lim_{\substack{\lambda\in\mathbb{R}\\ |\lambda|\to\infty}} |\mathcal{F}f(\lambda_0+i\lambda)\eta| = 0.$$

Proof. As a consequence of the proof of Proposition 5.1, for $\lambda_0 + i\lambda \in S_r^\circ \cap a^*(R)$,

$$|p_R(-\lambda_0 - i\lambda)\mathcal{F}f(\lambda_0+i\lambda)\eta| \le M'\|\eta\|(1+|\lambda_0+i\lambda|)^{\deg p_R}\|f\|_r.$$

For $\epsilon > 0$, choose a $\phi \in C_c^\infty(K\backslash G/H)$ such that $\|f-\phi\|_r < \epsilon$. From Proposition 2.5, for large enough $|i\lambda|$,

$$|p(-\lambda_0 - i\lambda)\mathcal{F}\phi(\lambda_0+i\lambda)\eta| \le M(1+|\lambda_0+i\lambda|)^{-n}\|\eta\| < \epsilon$$

for some $\epsilon > 0$. Thus,

$$\begin{aligned}
&|p_R(-\lambda_0-i\lambda)\mathcal{F}f(\lambda_0+i\lambda)\eta| \\
&\quad\le |p_R(-\lambda_0-i\lambda)(\mathcal{F}f(\lambda_0+i\lambda)\eta - \mathcal{F}\phi(\lambda_0+i\lambda)\eta)| + |p_R(-\lambda_0-i\lambda)\mathcal{F}\phi(\lambda_0+i\lambda)\eta| \\
&\quad\le \|\eta\|(1+|\lambda_0+i\lambda|)^{\deg p}\epsilon + \epsilon.
\end{aligned}$$

For large enough $|\lambda|$, $p_R(-\lambda_0 - i\lambda)$ has no zeroes, and thus

$$\lim_{\substack{|\lambda|>0\\ |\lambda|\to\infty}} |\mathcal{F}f(\lambda_0+i\lambda)\eta| = 0. \qquad \square$$

Remark 5.8. If $r = 1$, we can include the boundary point $|\lambda_0| = \rho$ in Theorem 5.7.

6. Fixed K-type

In this section we will study fixed left K-type normalized Eisenstein integrals on pseudo-Riemannian real hyperbolic spaces $\mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q)$. Our aim is to characterize bounded fixed left K-type normalized Eisenstein integrals and then we prove the Hausdorff–Young inequality in this case. To prove these results (among many things) we first need the explicit formula of the c-function. Since we are unable to obtain this in the general split rank-one/rank-one cases, we restrict ourselves to the pseudo-Riemannian real hyperbolic spaces. First, we give required preliminaries and then give an outline of the proof of the main theorems in this section.

6.1. ***Geometry and structure of pseudo-Riemannian real hyperbolic spaces.*** Here we give the required preliminaries of the pseudo-Riemannian real hyperbolic spaces. Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{p+q}$. Then $H = \mathrm{SO}_e(p-1,q)$ is identified as subgroup of G which is the stabilizer group of e_1. Let $K = \mathrm{SO}(p)\times\mathrm{SO}(q)$, $B = S^{p-1}\times S^{q-1}$ and

$$\rho = \frac{p+q-2}{2}.$$

Let us consider spherical coordinates, on X,

$$x(t, b) = \cosh tu + \sinh tv, \quad b = (u, v) \in B.$$

We have $\mathfrak{g}_+ \cong \mathfrak{so}(p-1) \times \mathfrak{so}(1, q)$. For $1 \le i, j \le p+q$, denote by E_{ij} the $(p+q)\times(p+q)$ matrix with 1 on the (i, j)-th entry and zero on all other entries. Let

$$Y = E_{p+q,1} + E_{1,p+q}.$$

Then $\mathfrak{a}_q = \mathbb{R}Y \cong \mathbb{R}$. Let $\Sigma(\mathfrak{a}_q, \mathfrak{g}) = \{\pm\alpha\}$, where $\alpha(Y) = 1$ and the Weyl group is $W \cong \{\pm 1\}$. When $q > 1$, $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+) = \Sigma(\mathfrak{a}_q, \mathfrak{g})$ and $W_{K\cap H} \cong \{\pm 1\}$. Therefore, $\mathcal{W} = \{1\}$ for $q > 1$. But if $q = 1$, $\Sigma(\mathfrak{a}_q, \mathfrak{g}_+) = \varnothing$ and $W_{K\cap H} \cong \{1\}$. Thus, if $q = 1$ we have that $\mathcal{W} = \{\pm 1\}$.

The σ-minimal parabolic subgroup P of $G = \mathrm{SO}_e(p, q)$ is the stabilizer of the line $\mathbb{R}(1, 0, \ldots, 0, 1)$. Moreover, $P = MAN$ where

$$A = \exp(\mathfrak{a}_q) = \left\{ a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & \mathrm{Id} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\},$$

$$N = \exp(\mathfrak{n}) = \left\{ \begin{pmatrix} 1+\frac{1}{2}\|v\|^2 & v^t & -\frac{1}{2}\|v\|^2 \\ v & \mathrm{Id} & -v \\ \frac{1}{2}\|v\|^2 & v^t & 1-\frac{1}{2}\|v\|^2 \end{pmatrix} : v \in \mathbb{R}^{p+q-2} \right\},$$

$$M = \left\{ \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \epsilon \end{pmatrix} : \epsilon = \pm 1,\ m \in \mathrm{SO}(p-1, q-1) \right\}$$

for $q > 1$, and $\epsilon = 1$ in the above description of M for $q = 1$. Thus, there is only one open P-orbit on X when $q > 1$, which is

$$\mathcal{O} = \{x \in X : -x_1 + x_{p+q} \neq 0\}.$$

For $q = 1$, there are two open P-orbits on X which are

$$\mathcal{O}_1 := P \cdot e_n = \{x \in X : x_1 - x_{n+1} > 0\},$$
$$\mathcal{O}_2 := P \cdot -e_n = \{x \in X : x_1 - x_{n+1} < 0\}.$$

Also, we have $m_1^+ = m_\alpha^+ = q-1$, $m_1^- = m_\alpha^- = p-1$, $m_2^+ = 0$ and $m_2^- = 0$. Then the Jacobian J (defined in (2-1)) associated with the decomposition (2-2) becomes

$$J(t) = \cosh^{p-1} t \sinh^{q-1} t. \tag{6-1}$$

Let Δ be the Laplace–Beltrami operator on X. Then the radial part of the Laplace–Beltrami operator is given by

$$L(\Delta) = \frac{1}{J(t)} \frac{d}{dt} \left(J(t) \frac{d}{dt} \right).$$

We will now consider the two cases when $q > 1$ and $q = 1$ separately, as their structure and analysis are different.

Case when $p \geq 1$ and $q > 1$: For $\eta \in \mathbb{C}$, we consider the function

$$j(\lambda, \eta)(x) = \begin{cases} |x_1 - x_{p+q}|^{\lambda-\rho}\eta & \text{if } x \in \mathcal{O}, \\ 0 & \text{if } x \notin \mathcal{O}. \end{cases}$$

Observe that we can write $j(\lambda, \eta)(x) = \eta j(\lambda, 1)(x)$. The Eisenstein integral is defined as

$$E(\lambda, \eta)(x) = \int_K j(\lambda, \eta)(k^{-1}x)\, dk.$$

By [Heckman and Schlichtkrull 1995, Example 6.5 and p. 162], $E(\lambda, \eta)(a_t \cdot e_1)$ may have singularities at $\lambda \in i\mathfrak{a}_q^*$ when $p > q + 2$. The normalized Eisenstein integral E° becomes (from (2-12) and (3-6))

$$E^0(\lambda, \eta)(a_t \cdot e_1) := \eta 2^{\lambda-\rho} \frac{\Gamma\left(\frac{\lambda+\rho}{2}\right)\Gamma\left(\frac{\lambda-\rho+q}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{q}{2}\right)} \times {}_2F_1\left(\frac{\lambda+\rho}{2}, \frac{-\lambda+\rho}{2}; \frac{q}{2}; -\sinh^2 t\right). \tag{6-2}$$

This does not have any singularities on $i\mathbb{R}$. In this case, we have, from Lemma 3.2,

$$c(\lambda) = 2^{2\lambda} \frac{\Gamma\left(\frac{\rho+\lambda}{2}\right)\Gamma(-\lambda)\Gamma\left(\frac{\lambda-\rho+q}{2}\right)}{\Gamma\left(\frac{\rho-\lambda}{2}\right)\Gamma(\lambda)\Gamma\left(\frac{-\lambda-\rho+q}{2}\right)}.$$

Case when $p > 1$, $q = 1$: In this case we consider the semisimple symmetric space $X = \mathrm{SO}_e(p, 1)/\mathrm{SO}_e(p-1, 1)$ with $p \geq 1$. Let $H = \mathrm{SO}_e(p-1, 1)$ and $K = \mathrm{SO}(p) \times \mathrm{SO}(1)$ and

$$\rho = \frac{p-1}{2}.$$

We have $m_1^+ = m_\alpha^+ = 0$ and $m_1^- = m_\alpha^- = p - 1$, $m_2^+ = 0$, $m_2^- = 0$. Then the Jacobian J (defined in (2-1)) associated with the decomposition (2-2) becomes

$$J(t) = \cosh^{p-1} t.$$

Let us define the function

$$j(\lambda, \eta)(x) = \begin{cases} |x_1 - x_{p+1}|^{\lambda-\rho}\eta_i & \text{if } x \in \mathcal{O}_i, \\ 0 & \text{if } x \notin \mathcal{O}_1 \cup \mathcal{O}_2 \end{cases}$$

for $\eta \in \mathbb{C}^2$. The Eisenstein integral is then defined as

$$E(\lambda, \eta)(x) = \int_K j(\lambda, \eta)(k^{-1}x)\, dk.$$

The normalized Eisenstein integral $E^\circ(\lambda, \eta)(x)$ is a regular function on X for every $\eta \in \mathbb{C}^2$. Let $e_1 = (1, \dots, 0, 0)$ and consider the disjoint orbits

$$\{a_t \cdot e_1 = \cosh te_1 + \sinh te_{p+1} : t > 0\}, \quad \{a_t \cdot -e_1 = -\cosh te_1 - \sinh te_{p+1} : t > 0\}.$$

We define

$$E^\circ_w(\lambda, \eta)(t) := E^\circ(\lambda, \eta)(a_t w \cdot e_1).$$

Thus, for $w = 1$, $E^\circ_1(\lambda, \eta)(t) = E^\circ(\lambda, \eta)(a_t \cdot e_1)$ and for $w = -1$, we have $E^\circ_{-1}(\lambda, \eta)(t) = E^\circ(\lambda, \eta)(a_t \cdot -e_1)$. The normalized Eisenstein integral E° becomes (from (2-12) and (3-6))

$$E^\circ_w(\lambda, \eta)(t) = \eta_w \frac{2^{\lambda-\rho}\Gamma\left(\frac{\lambda+\rho}{2}\right)\Gamma\left(\frac{\lambda-\rho+1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} \times {}_2F_1\left(\frac{\rho+\lambda}{2}, \frac{\rho-\lambda}{2}; \frac{1}{2}; -\sinh^2 t\right). \tag{6-3}$$

Also in this case, we have, from Lemma 3.2,

$$c(\lambda) = 2^{2\lambda} \frac{\Gamma\left(\frac{\rho+\lambda}{2}\right)\Gamma(-\lambda)\Gamma\left(\frac{(\lambda-\rho+1)}{2}\right)}{\Gamma\left(\frac{\rho-\lambda}{2}\right)\Gamma(\lambda)\Gamma\left(\frac{-\lambda-\rho+1}{2}\right)}.$$

Remark 6.1. We omit the case when $p = q = 1$ because in this case the group $G = \mathrm{SO}_e(1, 1)$ is abelian, and hence the general theory is not valid for this group.

Let δ_k be an irreducible representation of $\mathrm{SO}(p)$ on the space of spherical harmonics $\mathcal{H}_k$ of degree $|k|$. If $p = 1$, then $k = 0$; if $p = 2$, then $k \in \mathbb{Z}$; and if $p > 2$, then $k \in \mathbb{N}_0$. Let δ_l be the irreducible representation of $\mathrm{SO}(q)$ and write $\delta_{k,l} = \delta_k \otimes \delta_l$. Then $\delta_{k,l}$ is an irreducible representation of $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$ on $\mathcal{H}_k \otimes \mathcal{H}_l$. Let $\chi_{k,l}$ be the character of the representation $\delta_{k,l}$. We say that a function $f \in C_c^\infty(X)$ is of left (k, l)-type if

$$\chi_{k,l} *_K f = f,$$

that is,

$$\int_K \chi_{k,l}(k) f(k^{-1}x)\, dk = f(x), \quad x \in G.$$

We denote the space of all left (k, l)-type smooth functions with compact support and (k, l)-type integrable functions by $C_c^\infty(X)^{k,l}$ and $L^1(X)^{k,l}$, respectively.

The (k, l)-th component of the Laplace–Beltrami operator on X is given by

$$\Delta^{k,l} f(t) = \frac{1}{\sqrt{J(t)}}\left[\frac{\partial^2}{\partial t^2}\left(\sqrt{J(t)} f(t)\right) - \frac{\partial^2}{\partial t^2}\left(\sqrt{J(t)}\right) f(t)\right] + k(k+p-2)\frac{1}{\cosh^2 t} f(t) + l(l+q-2)\frac{1}{\sinh^2 t} f(t).$$

We consider the differential equation

$$\Delta^{k,l} f = (\lambda^2 - \rho^2) f.$$

Then there exists a unique formal series (see [Andersen 2001, p. 78] for $q > 1$ and [van den Ban and Schlichtkrull 1997a, p. 66] for the general case) given by

$$\Phi_\lambda^{k,l}(t) = e^{(\lambda-\rho)t} \sum_{m=0}^{\infty} \Gamma_m^{k,l}(\lambda) e^{-mt}$$

for $t > 0$, satisfying the above differential equation with $\Gamma_m^{k,l}(\lambda) \in \mathbb{C}$ and $\Gamma_0^{k,l}(\lambda) = 1$.

The normalized Eisenstein integral $E_{k,l}^\circ$ is an eigenfunction of the differential operator $\Delta^{k,l}$ that satisfies

$$\Delta^{k,l} E_{k,l}^\circ(\lambda, \eta)(t) = (\lambda^2 - \rho^2) E_{k,l}^\circ(\lambda, \eta)(t).$$

For $w \in \mathcal{W}$, we write

$$E_{k,l,w}^\circ(\lambda, \eta)(t) := E_{k,l}^\circ(\lambda, \eta)(a_t w \cdot e_1).$$

We then have $E_{k,l,w}^\circ$ satisfies

$$\Delta^{k,l} E_{k,l,w}^\circ(\lambda, \eta) = (\lambda^2 - \rho^2) E_{k,l,w}^\circ(\lambda, \eta) \tag{6-4}$$

for any $w \in \mathcal{W}$ and for almost all $\Re\lambda > 0$,

$$\lim_{t\to\infty} e^{(\rho-\lambda)t} E_{k,l,w}^\circ(\lambda, \eta)(t) = \eta_w.$$

Further, there exists an endomorphism $C^{k,l}(\lambda) : \mathbb{C}^{\mathcal{W}} \to \mathbb{C}^{\mathcal{W}}$ that is meromorphically dependent on λ such that

$$E_{k,l,w}^\circ(\lambda, \eta)(t) = \Phi_\lambda^{k,l}(t)\eta_w + \Phi_{-\lambda}^{k,l}(t)[C^{k,l}(\lambda)\eta]_w, \quad \lambda \notin \tfrac{1}{2}\mathbb{N}_0.$$

We also have

$$E_{k,l,w}^\circ(-\lambda, \eta) = E_{k,l,w}^\circ(\lambda, C^{k,l}(\lambda)\eta).$$

The above differential equation (6-4) can be transformed to a hypergeometric differential equation by substituting $z = -\sinh^2 t$. Thus, from the above asymptotics,

$$E_{k,l,w}^\circ(\lambda, \eta)(t) = \eta_w 2^{\lambda-\rho-|k|-|l|} \cosh^{|k|} t \sinh^{|l|} t \frac{\Gamma\left(\frac{\lambda+\rho+|k|+|l|}{2}\right)\Gamma\left(\frac{\lambda-\rho+q-|k|+|l|}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{q}{2}+|l|\right)}$$
$$\times {}_2F_1\left(\frac{\lambda+\rho+|k|+|l|}{2}, \frac{-\lambda+\rho+|k|+|l|}{2}; \frac{q}{2}+|l|; -\sinh^2 t\right).$$

By similar steps as in the proof of Lemma 3.2 we get, for $q > 1$,

$$C^{k,l}(\lambda) = 2^{2\lambda} \frac{\Gamma\left(\frac{1}{2}(\lambda+\rho+|k|+|l|)\right)\Gamma\left(\frac{1}{2}(\lambda-\rho+q-|k|+|l|)\right)\Gamma(-\lambda)}{\Gamma\left(\frac{1}{2}(-\lambda+\rho+|k|+|l|)\right)\Gamma\left(\frac{1}{2}(-\lambda-\rho+q-|k|+|l|)\right)\Gamma(\lambda)},$$

and for $q = 1$, $C^{k,l}(\lambda) = [c_{ij}(\lambda)]$ where

$$c_{11} = c_{22} = 2^{2\lambda} \frac{\Gamma\big(\frac{1}{2}(\lambda+\rho+|k|+|l|)\big)\Gamma\big(\frac{1}{2}(\lambda-\rho+q-|k|+|l|)\big)\Gamma(-\lambda)}{\Gamma\big(\frac{1}{2}(-\lambda+\rho+|k|+|l|)\big)\Gamma\big(\frac{1}{2}(-\lambda-\rho+q-|k|+|l|)\big)\Gamma(\lambda)},$$
$$c_{12} = c_{21} = 0.$$

Since $C^{k,l}$ is a diagonal matrix for $q = 1$, by combining the case $q > 1$ and $q = 1$ we can write

$$E^{\circ}_{k,l,w}(\lambda, \eta)(t) = \Phi^{k,l}_{\lambda}(t)\eta_w + c^{k,l}(\lambda)\eta_w \Phi^{k,l}_{-\lambda}(t),$$

where

$$c^{k,l}(\lambda) = 2^{2\lambda} \frac{\Gamma\big(\frac{1}{2}(\lambda+\rho+|k|+|l|)\big)\Gamma\big(\frac{1}{2}(\lambda-\rho+q-|k|+|l|)\big)\Gamma(-\lambda)}{\Gamma\big(\frac{1}{2}(-\lambda+\rho+|k|+|l|)\big)\Gamma\big(\frac{1}{2}(-\lambda-\rho+q-|k|+|l|)\big)\Gamma(\lambda)}.$$

The function $E^{\circ}_{k,l,w}(\lambda, \eta)$ is of (k, l)-type.

Definition 6.2. For $f \in C^{\infty}_c(X)^{k,l}$ we define the Fourier transform as

$$\begin{aligned}\mathcal{F}^{k,l} f(\lambda)(\eta) &= \int_X f(x) E^{\circ}_{k,l}(-\lambda, \eta)(x)\, dx \\ &= \sum_{w\in\mathcal{W}} \int_0^{\infty} f(a_t w.e_1) E^{\circ}_{k,l}(-\lambda, \eta)(a_t w \cdot e_1) J(t)\, dt,\end{aligned}$$

where $\lambda \in \mathfrak{a}^*_{q,\mathbb{C}}$.

Fix an $R > 0$. Then there exists a polynomial $p^{k,l}_R$ such that $p^{k,l}_R(\lambda) E^{\circ}_{k,l,w}(\lambda, \eta)$ is holomorphic in $-a^*(R)$. The polynomial $p^{k,l}_R(\lambda)$ is given by

$$p_R(\lambda) = \prod_{\substack{k_1\in\mathbb{N}_0 \\ \rho+|k|+|l|+2k_1\le R}} (\lambda+\rho+|k|+|l|+2k_1) \times \prod_{\substack{k_2\in\mathbb{N}_0 \\ -\rho+q-|k|+|l|+2k_2\le R}} (\lambda-\rho+q-|k|+|l|+2k_2).$$

Let

$$q_R(\lambda) = \big(\lambda - \tfrac{1}{2}\big)(\lambda-1)\cdots\big(\lambda - \tfrac{1}{2}\lfloor 2R\rfloor\big).$$

Again from [van den Ban and Schlichtkrull 1997a, Theorem 7.4], we have

$$|q_R(\lambda)\Gamma^{k,l}_m(\lambda)| \le M(1+m)^{\chi}(1+|\lambda|)^{\deg q_R}$$

for some $\chi > 0$, $\lambda \in \overline{a^*(R)}$ and for all $m \in \mathbb{N}_0$.

Let

$$\Psi_{\lambda}(t) = \sqrt{J(t)}\Phi^{k,l}_{\lambda}(t) = e^{\lambda t}\sum_{m=0}^{\infty} \widetilde{\Gamma}^{k,l}_m(\lambda) e^{-mt}.$$

Then Ψ_λ satisfies that

$$\Psi_\lambda''(t) + (k(k+p-2)c(t) + l(l+q-2)s(t) - d(t))\Psi_\lambda(t) = (\lambda^2 - \rho^2)\Psi_\lambda(t),$$

where

$$c(t) = \cosh^{-2} t = \sum_{m=0}^{\infty} c_m e^{-mt},$$

$$s(t) = \sinh^{-2} t = \sum_{m=0}^{\infty} s_m e^{-mt},$$

$$d(t) = \frac{J''}{\sqrt{J}} = \sum_{m=0}^{\infty} d_m e^{-mt}.$$

The constants are

$$\begin{aligned} &d_0 = \rho^2, \quad d_m = ((q-1)(q-3) + (-1)^m (p-1)(p-3))m, \\ &c_0 = 0, \quad c_{2m} = 4(-1)^m m, && c_{2m+1} = 0, \\ &s_0 = 0, \quad s_{2m} = 4m, && s_{2m+1} = 0. \end{aligned}$$

By substituting we obtain the recursion formula

$$\begin{aligned} (\lambda^2 - \rho^2)\widetilde{\Gamma}_m^{k,l} = (\lambda - k)^2 \widetilde{\Gamma}_m^{k,l} + k(k+p-2) \\ \times \sum_{i=0}^{m} \widetilde{\Gamma}_i^{k,l} c_{m-i} + l(l+q-2) \sum_{i=0}^{m} \widetilde{\Gamma}_i^{k,l} s_{m-i} - \sum_{i=0}^{m} \widetilde{\Gamma}_i^{k,l} d_{m-i}. \end{aligned}$$

That is,

$$\begin{aligned} m(2\lambda - m)\widetilde{\Gamma}_m^{k,l} = \rho^2 \widetilde{\Gamma}_m^{k,l} + k(k+p-2) \\ \times \sum_{i=0}^{m} \widetilde{\Gamma}_i^{k,l} c_{m-i} + l(l+q-2) \sum_{i=0}^{m} \widetilde{\Gamma}_i^{k,l} s_{m-i} - \sum_{i=0}^{m} \widetilde{\Gamma}_i^{k,l} d_{m-i}, \end{aligned}$$

which implies that

$$m(2\lambda - m)\widetilde{\Gamma}_m^{k,l} = k(k+p-2) \sum_{i=0}^{m-1} \widetilde{\Gamma}_i^{k,l} c_{m-i} + l(l+q-2) \sum_{i=0}^{m-1} \widetilde{\Gamma}_i^{k,l} s_{m-i} - \sum_{i=0}^{m-1} \widetilde{\Gamma}_i^{k,l} d_{m-i}.$$

Therefore, the poles of $\widetilde{\Gamma}_m^{k,l}$ lie in the set $\left\{\frac{1}{2}, 1, \ldots, \frac{1}{2}m\right\}$. With

$$\Gamma_m^{k,l}(\lambda) = \sum_{i=0}^{m} b_i \widetilde{\Gamma}_{m-i}^{k,l}(\lambda),$$

the poles of $\Gamma_m^{k,l}$ lie in $\left\{\frac{1}{2}, 1, \ldots, \frac{1}{2}m\right\}$.

Lemma 6.3. (1) *The singularities of* $\Gamma_m^{k,l}$ *are at most simple poles in* $\{\frac{1}{2}, 1, \ldots, \frac{1}{2}m\}$.

(2) *The singularities of* $\Phi_\lambda^{k,l}$ *lie in* $\frac{1}{2}\mathbb{N}$.

(3) *The poles of* $c^{k,l}(\lambda)$ *are*

$$\{-\rho - |k| - |l| - 2k_1 : k_1 \in \mathbb{N}_0\} \cup \{\rho - q + |k| - |l| - 2k_2 : k_2 \in \mathbb{N}_0\} \cup \{k_3 \in \mathbb{N}\}.$$

(4) *The poles of* $E^\circ_{k,l,w}(\cdot\,, \eta)(t)$ *are equal to*

$$\{-\rho - |k| - |l| - 2k_1 : k_1 \in \mathbb{N}_0\} \cup \{\rho - q + |k| - |l| - 2k_2 : k_2 \in \mathbb{N}_0 \text{ and } \rho - q + |k| - |l| - 2k_2 \neq 0\}.$$

(5) *The zeros of* $c^{k,l}(\lambda)$ *are*

$$\{\rho + |k| + |l| + 2k_1 : k_1 \in \mathbb{N}_0\} \cup \{-\rho + q - |k| + |l| + 2k_2 : k_2 \in \mathbb{N}_0\} \cup \{-k_3 \in \mathbb{N}\}.$$

(6) *The zeros of* $p_R^{k,l}(\lambda) E^\circ_{k,l,w}(\cdot\,, \eta)(t)$ *are* $\{-k_1 : k_1 \in \mathbb{N}_0\}$.

Remark 6.4. Suppose there exists $k_2 \in \mathbb{N}_0$ such that $\rho - q + |k| - |l| - 2k_2 = 0$. Then the numerator of

$$\frac{\Gamma\left(\frac{\lambda+\rho+|k|+|l|}{2}\right)\Gamma\left(\frac{\lambda-\rho+q-|k|+|l|}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{q}{2}+|l|\right)}$$

has a pole at $\lambda = 0$. On the other hand, it follows that $\lambda = 0$ is also a pole of the denominator. Hence, the singularity at $\lambda = 0$ is removable. Therefore, $E^\circ_{k,l,w}(\lambda, \eta)(t)$ is holomorphic at $\lambda = 0$, and consequently holomorphic on the line $i\mathbb{R}$.

If $\rho - q + |k| - |l| - 2k_2 \neq 0$ for any $k_2 \in \mathbb{N}_0$, then from the expression of $E^\circ_{k,l,w}$ it is holomorphic on $i\mathbb{R}$.

Then we have the following analogue of the Helgason–Johnson theorem for left (k, l)-type normalized Eisenstein integrals.

Theorem 6.5 (Helgason–Johnson). *Fix* $R > 0$, $w \in \mathcal{W}$ *and* $\eta \in \mathbb{C}^{\mathcal{W}}$.

(1) *Suppose* ρ *is not an integer.*

(a) *If* $\lambda_0 \in (S_1 \cap -a^*(R)) \cup -\mathbb{N}_0$, *then the function* $t \to p_R^{k,l}(\lambda) E^\circ_{k,l,w}(\lambda, \eta)(t)$ *is bounded on* $(0, \infty)$.

(b) *Conversely, for* $\lambda_0 \in -a^*(R)$ *and for nonzero* η_w, *if the function*

$$t \mapsto p_R^{k,l}(\lambda_0) E^\circ_{k,l,w}(\lambda_0, \eta)(t)$$

is bounded on $(0, \infty)$, *then* $\lambda_0 \in S_1 \cup -\mathbb{N}_0$.

(2) *Suppose* ρ *is an integer. Let* $A = \{\lambda \in \mathbb{Z} : \rho < \lambda \leq \rho - q + |l| - |m|\}$ *and let* $\lambda_0 \notin A$.

(a) *If* $\lambda_0 \in (S_1 \cap -a^*(R)) \cup -\mathbb{N}_0$, *then the function* $t \to p_R^{k,l}(\lambda) E^\circ_{k,l,w}(\lambda, \eta)(t)$ *is bounded on* $(0, \infty)$.

(b) *Conversely, for* $\lambda_0 \in -a^*(R)$ *and for nonzero* η_w, *if the function*

$$t \mapsto p_R^{k,l}(\lambda_0) E^\circ_{k,l,w}(\lambda_0, \eta)(t)$$

is bounded on $(0, \infty)$, *then* $\lambda_0 \in S_1 \cup -\mathbb{N}_0$.

Remark 6.6. The set A is nonempty if and only if $q < |l| - |m|$. For the case when $\lambda_0 \in A$ and ρ is an integer, the function $E^\circ_{k,l,w}(\lambda, \eta)$ has a simple pole at λ_0 but $c^{k,l}(\lambda)$ has a double pole at λ_0. Thus, the earlier techniques do not work in this case.

Proof of Theorem 6.5. The proof of this theorem is similar to the K-invariant case. Thus, we give only an outline. Fix an $R > 0$. Let $p_R^{k,l}(\lambda)$ be the polynomial such that the function $F_w^{k,l}(\lambda, \eta) := p_R^{k,l}(\lambda) E^\circ_{k,l,w}(\lambda, \eta)(t)$ is holomorphic in $-\mathfrak{a}_q^*(R)$. Then

$$F_w^{k,l}(\lambda_0, \eta)(t) = \frac{d}{d\lambda}\Big|_{\lambda=\lambda_0} \eta_w \sum_{m=0}^{\infty} \big[e^{(\lambda-\rho)t}(\lambda - \lambda_0) p_R^{k,l}(\lambda) \Gamma_m^{k,l}(\lambda) + e^{(-\lambda-\rho)t}(\lambda - \lambda_0) p_R^{k,l}(\lambda) c^{k,l}(\lambda) \Gamma_m^{k,l}(-\lambda) \big] e^{-mt}.$$

For $\lambda_0 \in -a^*(R)$, by following similar steps and estimates as in the case of K-invariant,

$$\begin{aligned} |F_w^{k,l}(\lambda_0, \eta) e^{(-\lambda_0+\rho)t} - \eta_w p_R^{k,l}(\lambda_0)| &\to 0, \quad \Re\lambda_0 > 0, \\ |F_w^{k,l}(\lambda_0, \eta) e^{(\lambda_0+\rho)t} - \eta_w p_R^{k,l}(\lambda_0) c^{k,l}(\lambda_0)| &\to 0, \quad \Re\lambda_0 < 0. \end{aligned} \tag{6-5}$$

Moreover, we obtain the following estimate analogous to the case of K-invariant:

$$|F_w^{k,l}(\lambda_0, \eta)| \le M \|\eta\| (1 + |\lambda|)^d (1 + t) e^{(|\Re\lambda_0| - \rho)t}. \tag{6-6}$$

Case 1. $-\rho < \Re\lambda_0 < \rho$: Using the above estimates we get that $F_w^{k,l}(\lambda_0, \eta)(t)$ is bounded in $t > 0$.

Case 2. $|\Re\lambda_0| = \rho$: From the asymptotics (6-5), we have

$$|F_w^{k,l}(\lambda_0, \eta)| \le M_{\lambda_0, \eta_w} e^{(|\Re\lambda_0| - \rho)} t.$$

Thus, if $|\Re\lambda_0| = \rho$, then $F_w^{k,l}(\lambda_0, \eta)$ is bounded for $t > 0$.

Case 3. $|\Re\lambda_0| > \rho$: Let $\lambda_0 \notin A$ and $|\Re\lambda_0| > \rho$ with $\lambda_0 \notin -\mathbb{N}_0$. Then, $p(\lambda_0) \neq 0$ and $p_R(\lambda_0) C^{k,l}(\lambda_0) \neq 0$. Thus,

$$\begin{aligned} |F_w^{k,l}(\lambda_0, \eta)(t)| &\asymp |\eta_w| e^{(|\Re\lambda_0| - \rho)t}, \\ |F_w^{k,l}(\lambda_0, \eta)(t)| &\asymp |\eta_w p_R^{k,l}(\lambda_0)| e^{(|\Re\lambda_0| - \rho)t}, \end{aligned}$$

implying that it is unbounded.

Case 4: $\lambda_0 \in -\mathbb{N}_0$. In this case, from Lemma 6.3, $p_R^{k,l}(\lambda_0)E_{k,l}^\circ(\lambda_0, \eta)$ is identically zero for $t > 0$ when $\lambda_0 \in -\mathbb{N}_0$. □

As a consequence of the Helgason–Johnson theorem and the estimate (6-6), the Fourier transform $\lambda \mapsto \mathcal{F}^{k,l} f(\lambda)\eta$ of $f \in L^1(X)^{k,l}$ extends meromorphically to S_1 and $p_R^{k,l}(\lambda)\mathcal{F}^{k,l} f(\lambda)\eta$ is holomorphic in the interior $S_1^\circ \cap a^*(R)$. It also follows that if $\phi \in C_c^\infty(X)^{k,l}$, then there exists a constant M such that, for any $n \in \mathbb{N}_0$,

$$|p(-\lambda)\mathcal{F}^{k,l}\phi(\lambda)(\eta)| \le M(1+|\lambda|)^{-n} e^{r|\Re\lambda|}\|\eta\|, \quad \lambda \in a^*(R),$$

where $r > 0$ depends on the support of ϕ. Furthermore, we have the following theorems and consequences:

Theorem 6.7. *Let $\lambda_0 \in S_r \cap a^*(R)$ be a real number. Then*

$$\left(\int_{\mathbb{R}} \left| \frac{p_R(-(\lambda_0 + i\lambda))\mathcal{F}^{k,l} f(\lambda_0 + i\lambda)\eta}{(1+\lambda_0+\lambda)^d} \right|^{r'} d\lambda \right)^{1/r'} \le M\|f\|_r\|\eta\|, \quad f \in L^r(X)^{k,l},$$

for some $M, d > 0$ and for any $\eta \in \mathbb{C}^{\mathcal{W}}$.

Theorem 6.8. *Let $1 \le r < 2$, $\theta = 2/r - 1$ and $\lambda_0 + i\lambda \in S_r^\circ \cap a^*(R)$. Then, for any $f \in L^r(X)^{k,l}$,*

$$\lim_{\substack{|\lambda|>0 \\ |\lambda|\to\infty}} |\mathcal{F}^{k,l} f(\lambda_0 + i\lambda)\eta| = 0.$$

The proof of the above theorems follows similarly as in the case of K-invariant functions.

Remark 6.9. It is known that, for $\alpha, \beta \ge -\frac{1}{2}$, the Jacobi function $\varphi_{-i\lambda}^{(\alpha,\beta)}(t)$ is bounded as a function of t whenever $\lambda \in S_1$. Moreover, when $-\frac{1}{2} \le \alpha \le \beta$, this condition is also necessary; that is, $\varphi_{-i\lambda}^{(\alpha,\beta)}(t)$ is bounded in t if and only if $\lambda \in S_1$. In the pseudo-Riemannian real hyperbolic case $\mathrm{SO}_e(p,q)/\mathrm{SO}_e(p-1,q)$, the left K-invariant Eisenstein integral is given by

$$E_w^\circ(\lambda, \eta)(t) = \eta_w 2^{\lambda-\rho} \frac{\Gamma\left(\frac{\lambda+\rho}{2}\right)\Gamma\left(\frac{\lambda-\rho+q}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{q}{2}\right)} \varphi_{-i\lambda}^{(\alpha,\beta)}(t),$$

where

$$\alpha = \frac{q}{2} - 1, \quad \beta = \frac{p}{2} - 1,$$

with $p > 1$, $q \ge 1$, or $p \ge 1$, $q > 1$. By Theorem 6.5, the Eisenstein integrals are unbounded in t whenever $\mathrm{Re}\,\lambda > \rho$. Since $E_w^\circ(\lambda, \eta)$ has no pole in this region, the above formula implies that the Jacobi function $\varphi_{-i\lambda}^{(\alpha,\beta)}(t)$ is also unbounded in t for $\Re\lambda > \rho$. Finally, since Jacobi functions are even in the spectral parameter λ, the same conclusion holds for $\Re\lambda < -\rho$. Consequently, for $\alpha \ge 0$, $\beta \ge -\frac{1}{2}$, or $\alpha \ge -\frac{1}{2}$, $\beta \ge 0$, the Jacobi function $\varphi_{-i\lambda}^{(\alpha,\beta)}(t)$ is bounded in t if and only if

$$|\Re\lambda| \le \rho.$$

This extends the known boundedness result of Flensted-Jensen and Koornwinder [1973] for Jacobi functions to the above parameter range for integers and half integers α and β.

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SANJOY PUSTI
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY
MUMBAI
INDIA
spusti@iitb.ac.in

ISWARYA SITIRAJU
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY, BOMBAY
MUMBAI
INDIA
iswarya@math.iitb.ac.in

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HOMOTOPY-THEORETIC PROPERTIES OF GYRATIONS

STEPHEN THERIAULT

Let M be a simply connected n-dimensional manifold with the property that the attaching map for the n-cell is inert, which means that the inclusion of the $(n-1)$-skeleton has a right homotopy inverse after looping. We show that a type of surgery on $M \times S^{k-1}$ producing a manifold called a gyration preserves the inertness property. The method generalises to Poincaré duality complexes and also recovers by different means a known homotopy equivalence for the based loops of the gyration. The underlying method is quite general and should have applications elsewhere.

1. Introduction

The algebraic topology of manifolds is a subject of long-standing interest. Recently, there has been a substantial amount of work using methods from homotopy theory to identify the homotopy types of the suspensions and loop spaces of manifolds. The methods apply more broadly to Poincaré duality complexes, which are CW-complexes whose cohomology satisfies Poincaré duality. We study homotopy-theoretic properties of manifolds called gyrations, which are obtained from a certain surgery, and their generalisation to Poincaré duality complexes.

Let M be a simply connected n-dimensional Poincaré duality complex. Then M has a CW-structure with a single n-cell. Fixing this, let $\overline{M}$ be the $(n-1)$-skeleton of M, and consider the homotopy cofibration

$$S^{n-1} \xrightarrow{f} \overline{M} \xrightarrow{i} M,$$

where f attaches the n-cell to M and i is the skeletal inclusion. Describing the effect of a cell attachment is a classical problem in homotopy theory. A special case is when f is *inert*, meaning that Ωi has a right homotopy inverse. Here, if X is a space then its based loop space $\Omega X = \mathrm{Map}_*(S^1, X)$ is the space of pointed continuous maps from a circle to X and if $X \xrightarrow{g} Y$ is a map then Ωg is the induced map of based loop spaces. A key property is that taking based loops shifts homotopy groups down by one: $\pi_{n-1}(\Omega X) \cong \pi_n(X)$. Thus the existence of a right homotopy inverse for Ωi implies that i induces an epimorphism on homotopy groups, which means

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that f annihilates some of the homotopy groups of $\overline{M}$ but does not introduce new ones for M. The definition is inspired by an analogous property in rational homotopy theory. Halperin and Lemaire [1987] showed that, rationally, f is always inert provided that the rational cohomology ring of M is not generated by a single element. There has been considerable work recently to identify those Poincaré duality complexes with f being inert either integrally or after localising away from a small number of primes [Chenery 2026; Huang 2024; 2025; Theriault 2024a; 2024b].

If M is a manifold then $\overline{M}$ can be regarded as M with the interior of an n-disk removed and f as an embedding of the boundary of the disk. A surgery can then be performed on $M \times S^{k-1}$ for any $k \geq 2$ by cutting out the interior of $D^n \times S^{k-1}$ and gluing in $S^{n-1} \times D^k$ along the boundary $S^{n-1} \times S^{k-1}$. The manifold obtained after surgery is called a *gyration* and is written $\mathcal{G}^k(M)$. Gyrations in the case when $k = 2$ play an important role in toric topology and the intersections of quadrics [Gitler and López de Medrano 2013] and the classification of 6-manifolds with regular circle actions [Duan 2026]. This surgery can also be described by a pushout: $\mathcal{G}^k(M)$ is the pushout

$$\begin{array}{ccc} S^{n-1} \times S^{k-1} & \xrightarrow{1\times j} & S^{n-1} \times D^k \\ \downarrow{\scriptstyle f\times 1} & & \downarrow \\ \overline{M} \times S^{k-1} & \longrightarrow & \mathcal{G}^k(M) \end{array}$$

where j is the inclusion of the boundary. This pushout formulation works equally well if M is a Poincaré duality complex, in which case $\mathcal{G}^k(M)$ is also a Poincaré duality complex. In either case, $\mathcal{G}^k(M)$ has dimension $n + k - 1$.

Our main result is to show that if the attaching map for the n-cell of M is inert then so is the attaching map for the $(n+k-1)$-cell of $\mathcal{G}^k(M)$. As a consequence of the method, we also recover a homotopy equivalence for $\Omega\mathcal{G}^k(M)$ proved in [Huang and Theriault 2023, Theorem 1.1].

Theorem 1.1. *Let M be a simply connected n-dimensional Poincaré duality complex and let $S^{n-1} \xrightarrow{f} \overline{M}$ be the attaching map for the n-cell of M. If f is inert, then for any $k \geq 2$:*

(a) *The attaching map for the $(n+k-1)$-cell of $\mathcal{G}^k(M)$ is inert.*

(b) *There is a homotopy equivalence $\Omega\mathcal{G}^k(M) \simeq \Omega\overline{M} \times \Omega(\Sigma^k F)$, where F is the homotopy fibre of f.*

Special cases of Theorem 1.1(a) were known, all of which assume that the attaching map for the n-cell of M is inert. In [Huang 2024, Theorem 1.6] it was shown that if $k = 2$ and $n \equiv 0 \bmod 4$ with $n \geq 4$ then the attaching map for the $(n+k-1)$-cell of $\mathcal{G}^k(M)$ is inert, if $k = 2$ and $n \equiv 0 \bmod 2$ with $n \geq 4$ then that attaching map is inert after localisation away from 2, and if $k = 4$ and $n \equiv 0 \bmod 4$

with $n \geq 8$ then that attaching map is inert after localisation away from 2 and 3. In [Stanton and Theriault 2026, Theorem 6.8] it was shown that if M is $(m-1)$-connected and there is a map $S^m \to M$ with a left homotopy inverse then the attaching map of the $(n+k-1)$-cell of $\mathcal{G}^k(M)$ is inert. Our result holds with no extra conditions.

Both parts of Theorem 1.1 are special cases of a general result that is of interest in its own right and should have applications elsewhere. To state it requires some notation. If X and Y are spaces then their *right half-smash* is the quotient space

$$X \rtimes Y = (X \times Y)/\sim,$$

where $(*, y) \sim (*, *)$, that is, the copy of Y in $X \times Y$ is collapsed to the basepoint. This construction is natural for maps $X \to X'$ and $Y \to Y'$. Let $I = [0, 1]$ be the unit interval with 0 as the basepoint. The (reduced) *join* of X and Y is the quotient space

$$X * Y = (X \times I \times Y)/\sim,$$

where $(x, 0, y) \sim (x', 0, y)$, $(x, 1, y) \sim (x, 1, y')$ and $(*, t, *) \sim (*, 0, *)$ for all $x, x' \in X$, $y, y' \in Y$ and $t \in I$. There is a standard homotopy equivalence $X * Y \simeq \Sigma X \wedge Y$.

Theorem 1.2. *Let A, B and X be path-connected spaces. Suppose that there is a homotopy fibration $F \xrightarrow{g} A \xrightarrow{f} X$ where g is null homotopic and there is a homotopy pushout*

$$\begin{array}{ccc} A \times B & \xrightarrow{\pi_1} & A \\ \downarrow{\scriptstyle f\times 1} & & \downarrow \\ X \times B & \longrightarrow & Y \end{array}$$

Then there is a factorisation of $X \times B \to Y$ as

$$X \times B \to X \rtimes B \xrightarrow{s} Y$$

for some map s, the map Ωs has a right homotopy inverse and $\Omega Y \simeq \Omega X \times \Omega(F * B)$.

We go on to generalise Theorem 1.1 to the case of "twisted" gyrations, when the map from $S^{n-1} \times S^{k-1}$ to $\overline{M} \times S^{k-1}$ is twisted by the action of a map τ representing a homotopy class of $\pi_{k-1}(\mathrm{SO}(n))$ and we have localised away from primes dividing the order of $J(\tau)$, where J is the classical J-homomorphism. We also refine the homotopy equivalence for $\Omega\mathcal{G}^k(M)$ by showing that if ΩM is homotopy equivalent to a product of spheres and loops on spheres then $\Omega\mathcal{G}^k(M)$ is too, and we give an application of Theorem 1.1 to toric topology by describing the homotopy type of the loops on moment-angle manifolds associated to polytopes obtained by iterated vertex cuts on a cube.

2. A loop space decomposition

The purpose of this section is to prove Theorem 1.2. As in the statement of the theorem, let A and X be path-connected spaces and suppose there is a map $f : A \to X$. Define the space F and the map g by the homotopy fibration

$$F \xrightarrow{g} A \xrightarrow{f} X.$$

Let B be another path-connected space and define the space Y by the homotopy pushout

$$\begin{array}{ccc} A \times B & \xrightarrow{\pi_1} & A \\ \downarrow{\scriptstyle f\times 1} & & \downarrow \\ X \times B & \longrightarrow & Y \end{array} \tag{1}$$

where π_1 is the projection. Observe that the projection $A \times B \xrightarrow{\pi_1} A$ factors through the right half-smash, giving a map

$$\pi : A \rtimes B \to B.$$

Lemma 2.1. *There is an iterated homotopy pushout diagram*

$$\begin{array}{ccccc} A \times B & \longrightarrow & A \rtimes B & \xrightarrow{\pi} & A \\ \downarrow{\scriptstyle f\times 1} & & \downarrow{\scriptstyle f\rtimes 1} & & \downarrow \\ X \times B & \longrightarrow & X \rtimes B & \xrightarrow{s} & Y \end{array}$$

for some map s.

Proof. The left square homotopy commutes by the naturality of the half-smash. It is a homotopy pushout since the copy of B has been collapsed out of both $A \times B$ and $X \times B$. If Q is defined as the homotopy pushout of π and $f \rtimes 1$, then as the juxtaposition of homotopy pushouts is a homotopy pushout, Q is also the homotopy pushout of the outer rectangle, that is, Q is the homotopy pushout of π_1 and $f \times 1$. But this homotopy pushout is the definition of Y, so $Q \simeq Y$. □

Define $r : Y \longrightarrow X$ as the homotopy pushout map

$$\begin{array}{ccc} A \rtimes B & \xrightarrow{\pi} & A \\ \downarrow{\scriptstyle f\rtimes 1} & & \downarrow \\ X \rtimes B & \xrightarrow{s} & Y \end{array} \qquad \begin{array}{l} f : A \to X \\ r : Y \to X \\ \pi : X \rtimes B \to X \end{array} \tag{2}$$

Let R be the homotopy fibre of r.

Lemma 2.2. *The map r has a right homotopy inverse, implying that the homotopy fibration $R \to Y \xrightarrow{r} X$ splits after looping to give a homotopy equivalence $\Omega Y \simeq \Omega X \times \Omega R$.*

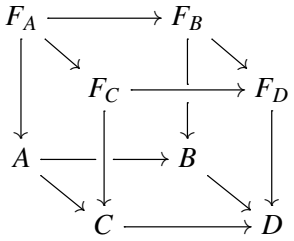
Proof. Let u be the composite $u : X \xrightarrow{i} X \rtimes B \to Y$, where i is the inclusion of the first factor. Then as $\pi \circ i$ equals the identity map, u is a right inverse for r. The splitting for ΩY immediately follows. □

Lemma 2.2 is a crude version of Theorem 1.2. To obtain the full version we have to much work harder to identify the homotopy type of R and show that s has a right homotopy inverse. To do so, we begin by stating a version of Mather's cube lemma [1976].

Lemma 2.3. *Suppose that there is a homotopy pushout*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

and a map $h : D \longrightarrow Z$. For X one of A, B, C or D, let F_X be the homotopy fibre of the composite $X \to D \xrightarrow{h} Z$. Then there is a homotopy commutative cube



where the four sides are homotopy pullbacks and the top face is a homotopy pushout. □

Mather proved a more general result without the hypothesis that the vertical maps are obtained by taking fibres over a common base space, for which he needed a different definition of a "homotopy commutative cube". In our case the stronger hypothesis lets one use [Panov and Theriault 2019, Lemma 3.1], for example, to establish Lemma 2.3.

Consider the inclusion $A \vee B \longrightarrow A \times B$ of the wedge into the product. Collapsing the copy of B to a point in both spaces gives a homotopy pushout

$$\begin{array}{ccc} A \vee B & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle i} \\ A \times B & \longrightarrow & A \rtimes B \end{array} \tag{3}$$

where i is the inclusion of the first wedge summand. Let r_A be the composite

$$r_A : A \rtimes B \xrightarrow{f\rtimes 1} X \rtimes B \xrightarrow{\pi} X.$$

We will compose all four corners of (3) with r_A, take homotopy fibres, and apply the cube lemma. Before doing so it will help to identify some of the homotopy fibres. Define the space G by the homotopy fibration

$$G \to A \rtimes B \xrightarrow{r_A} X.$$

The naturality of π implies that r_A is homotopic to the composite $A \rtimes B \xrightarrow{\pi} A \xrightarrow{f} X$. Thus the composite $A \times B \to A \rtimes B \xrightarrow{r_A} X$ is homotopic to the composite $A \times B \xrightarrow{\pi_1} A \xrightarrow{f} X$. Thus there is a homotopy fibration

$$F \times B \xrightarrow{g\times 1} A \times B \to X.$$

By definition of r_A, the composite $A \xrightarrow{i} A \rtimes B \xrightarrow{r_A} X$ is homotopic to f, so we obtain the homotopy fibration

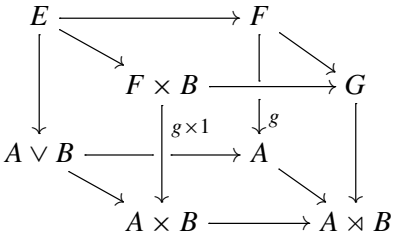
$$F \xrightarrow{g} A \xrightarrow{f} X.$$

Finally, composing the inclusion $A \vee B \longrightarrow A \times B$ with r_A, define the space E by the homotopy fibration

$$E \to A \vee B \to X.$$

The cube lemma (Lemma 2.3) then implies there is a homotopy commutative cube

(4)

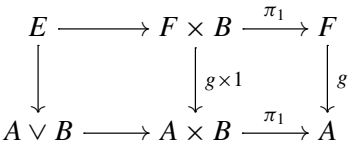


where the top face is a homotopy pushout. We begin by analysing the map $F \times B \to G$.

Lemma 2.4. *The map $F \times B \to G$ in* (4) *factors as the composite $F \times B \xrightarrow{\pi_1} F \to G$.*

Proof. The rear and left faces of the cube (4) will be considered simultaneously, using the fact that the pinch map $A \vee B \to A$ factors as the composite $A \vee B \to A \times B \xrightarrow{\pi_1} A$. This factorisation results in an iterated homotopy pullback diagram

(5)

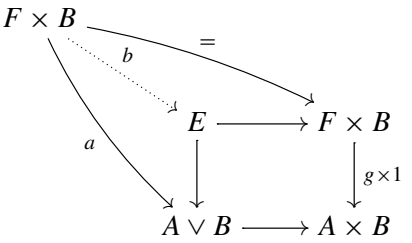


Notice that the left square in (5) is the left face in the cube (4) and the outer rectangle is the rear face in (4). Now assume in addition that g is null homotopic. Then if a is the composite

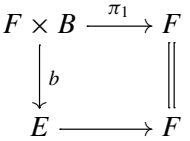
$$a : F \times B \xrightarrow{\pi_2} B \xrightarrow{i_2} A \vee B,$$

where i_2 is the inclusion of the second wedge summand, there is a homotopy pullback map

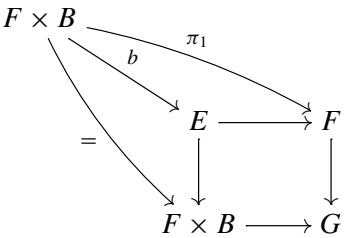
(6)



In particular, b is a right homotopy inverse for $E \to A \times B$. Combining this section with the upper row in (5) gives a homotopy commutative diagram

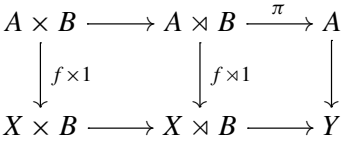


Thus in terms of the top face in the cube (4), there is a homotopy commutative diagram



showing that $F \times B \to G$ factors as the composite $F \times B \xrightarrow{\pi_1} F \to G$. □

Next, consider the iterated homotopy pushout



in Lemma 2.1. We will map from each space in the diagram to Y and compose with $Y \xrightarrow{r} X$, take homotopy fibres, and apply the cube lemma. Before doing so

we identify homotopy fibres. Recall that R is defined by the homotopy fibration

$$R \to Y \xrightarrow{r} X.$$

By the definition of r in (2), the composite $X \rtimes B \to Y \xrightarrow{r} X$ is homotopic to π. Define the space H by the homotopy fibration

$$H \to X \rtimes B \xrightarrow{\pi} X.$$

The composite $X \times B \to X \rtimes B \xrightarrow{\pi} X$ is homotopic to the projection to the first factor, so there is a homotopy fibration

$$B \xrightarrow{i_2} X \times B \xrightarrow{\pi_1} X,$$

where i_2 is the inclusion of the second factor. By the definition of r in (2), the composite $A \to Y \xrightarrow{r} X$ is homotopic to f, giving the homotopy fibration

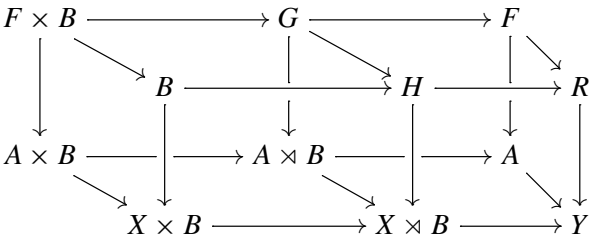
$$F \xrightarrow{g} A \xrightarrow{f} X.$$

Recall from the setup for (4) that the composite $A \rtimes B \xrightarrow{\pi} A \xrightarrow{f} X$ is homotopic to r_A, so there are homotopy fibrations

$$G \to A \rtimes B \xrightarrow{r_A} X \quad \text{and} \quad F \times B \xrightarrow{g \times 1} A \times B \to X.$$

Applying the cube lemma (Lemma 2.3) twice gives homotopy commutative cubes

(7)



in which all the vertical faces are homotopy pullbacks and the two top faces are homotopy pushouts.

The aim is to show that the map $H \to R$ has a right homotopy inverse. It is worth noting that the homotopy type of H is known: in general, the homotopy fibre of $C \rtimes D \xrightarrow{\pi} C$ is homotopy equivalent to $\Omega C * D$, so in our case, $H \simeq \Omega X * B$. However, this homotopy equivalence may not relate well to R, so we will produce an alternative homotopy equivalence that is compatible with the other spaces in the top two faces of (7).

To begin, we identify the homotopy class of the map $F \times B \to B$ in the leftmost face of (7).

Lemma 2.5. *The map $F \times B \to B$ in the leftmost face of* (7) *is homotopic to the projection to the second factor.*

Proof. The map $B \to X \times B$ in (7) is obtained from the homotopy fibration $B \xrightarrow{i_2} X \times B \xrightarrow{\pi_1} X$. The leftmost face in (7) is then obtained by pulling back $A \times B \xrightarrow{f\times 1} X \times B$ and $B \xrightarrow{i_2} X \times B$, implying that the map $F \times B \to B$ is canonically the projection π_2. □

Recall the map $F \to G$ from (4). Let G/F be its homotopy cofibre.

Proposition 2.6. *There is a homotopy equivalence $H \simeq (F * B) \vee (G/F)$ with the property that the composite $F * B \hookrightarrow (F * B) \vee (G/F) \simeq H \to R$ is also a homotopy equivalence.*

Proof. The homotopy pullback in the rear face of the left cube in (7) is the same as that in the front face of (4). Thus, by Lemma 2.4, the map $F \times B \to G$ factors as the composite $F \times B \xrightarrow{\pi_1} F \to G$. By Lemma 2.5, the map $F \times B \to B$ is homotopic to the projection to the second factor. Thus there is an iterated homotopy pushout diagram

$$\begin{array}{ccccc} F \times B & \xrightarrow{\pi_1} & F & \longrightarrow & G \\ \downarrow{\scriptstyle \pi_2} & & \downarrow & & \downarrow \\ B & \longrightarrow & F * B & \longrightarrow & H \end{array} \tag{8}$$

To be clear, it is well known that the homotopy pushout of π_1 and π_2 has the homotopy type of $F * B$, with the maps $F \to F * B$ and $B \to F * B$ both being null homotopic. The outer rectangle in an iterated homotopy pushout is itself a homotopy pushout, so this identifies the homotopy pushout in the right square as H.

On the other hand, the null homotopy for $F \to F * B$ implies that there is an iterated homotopy pushout diagram

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ * & \longrightarrow & G/F \\ \downarrow & & \downarrow \\ F * B & \longrightarrow & H \end{array}$$

But the homotopy pushout in the bottom square is $(F * B) \vee G/F$, so we obtain $H \simeq (F * B) \vee G/F$.

We now use this homotopy equivalence for H to produce one for R. First consider the two rear faces in (7),

$$\begin{array}{ccccc} F \times B & \longrightarrow & G & \longrightarrow & F \\ \downarrow{\scriptstyle g\times 1} & & \downarrow & & \downarrow{\scriptstyle g} \\ A \times B & \longrightarrow & A \rtimes B & \longrightarrow & A \end{array}$$

Both squares are homotopy pullbacks so the outer rectangle is as well. The bottom row is the projection to the first factor, so the map $F \times B \to F$ along the top row is canonically the projection to the first factor. On the other hand, the map $F \times B \to G$ factors as the composite $F \times B \xrightarrow{\pi_1} F \to G$, so the extended composite

$$F \times B \xrightarrow{\pi_1} F \to G \to F$$

is homotopic to π_1. This implies $F \to G \to F$ is homotopic to the identity map.

Next, consider the two top faces in (7), with the left top face refined as an iterated homotopy pushout by (8). We obtain an iterated homotopy pushout diagram

$$\begin{array}{ccccccc} F \times B & \xrightarrow{\pi_1} & F & \longrightarrow & G & \longrightarrow & F \\ \downarrow{\scriptstyle \pi_2} & & \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & F * B & \longrightarrow & H & \longrightarrow & R \end{array}$$

Focus on the middle and right squares, the outer rectangle of which is a homotopy pushout. Since the composite $F \to G \to F$ is homotopic to the identity map, the composite $F * B \to H \to R$ along the bottom must also be a homotopy equivalence. Thus the map $H \to R$ has a right homotopy inverse, and moreover, the homotopy equivalence $H \simeq (F * B) \vee G/F$ above was such that the map $F * B \to H$ is the inclusion of the left wedge summand, so $F * B \hookrightarrow (F * B) \vee (G/F) \simeq H \to R$ is a homotopy equivalence. □

Summarising the results in this section so far gives the following.

Proposition 2.7. *Let A, B and X be path-connected spaces. Suppose that there is a homotopy fibration $F \xrightarrow{g} A \xrightarrow{f} X$ where g is null homotopic and there is a homotopy pushout*

$$\begin{array}{ccc} A \times B & \xrightarrow{\pi_1} & A \\ \downarrow{\scriptstyle f \times 1} & & \downarrow \\ X \times B & \longrightarrow & Y \end{array}$$

Then $X \times B \to Y$ factors as $X \times B \to X \rtimes B \xrightarrow{s} Y$ for some map s, there is a map $Y \xrightarrow{r} X$ such that the composite $X \rtimes B \xrightarrow{s} Y \xrightarrow{r} X$ is homotopic to the projection, and there is a homotopy fibration diagram

$$\begin{array}{ccc} H & \longrightarrow & R \\ \downarrow & & \downarrow \\ X \rtimes B & \xrightarrow{s} & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle r} \\ X & = & X \end{array}$$

where $R \simeq F * B$ and $H \to R$ has a right homotopy inverse. □

Proposition 2.7 is used to produce the loop space decomposition in Theorem 1.2.

Proof of Theorem 1.2. By Proposition 2.7 there is a homotopy fibration diagram

$$\begin{array}{ccc} H & \longrightarrow & R \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\ X \rtimes B & \xrightarrow{s} & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle r} \\ X & = & X \end{array}$$

where h and k are just names for maps, $R \simeq F * B$ and the map $H \to R$ has a right homotopy inverse $t : R \to H$. The right homotopy inverse t implies that $k \simeq s \circ h \circ t$, in particular, k factors through s. Let $j : X \to X \rtimes B$ be the inclusion of the left factor. Then $r \circ s \circ j$ is homotopic to the identity map on X, implying that r has a right homotopy inverse that factors through s. The existence of a right homotopy inverse for r implies that the homotopy fibration $R \to Y \xrightarrow{r} X$ splits after looping; precisely, the composite

$$\Omega X \times \Omega R \xrightarrow{\Omega(s\circ j)\times\Omega k} \Omega Y \times \Omega Y \xrightarrow{\mu} \Omega Y$$

is a homotopy equivalence, where μ is the standard loop multiplication. Both $s \circ j$ and k factor through s, so there is in fact a diagram

$$\begin{array}{ccccc} \Omega X \times \Omega R & \xrightarrow{\Omega j \times (\Omega h \circ \Omega t)} & \Omega(X \rtimes B) \times \Omega(X \rtimes B) & \xrightarrow{\mu} & \Omega(X \rtimes B) \\ & \searrow^{\Omega(s\circ j)\times\Omega k} & \downarrow{\scriptstyle \Omega s \times \Omega s} & & \downarrow{\scriptstyle \Omega s} \\ & & \Omega Y \times \Omega Y & \xrightarrow{\mu} & \Omega Y \end{array}$$

where the triangle homotopy commutes since $k \simeq s \circ h \circ t$ and the square homotopy commutes since Ωs is an H-map. The lower direction around the diagram is the homotopy equivalence for ΩY, so the upper direction around the diagram implies that Ωs has a right homotopy inverse.

Finally, as $R \simeq F * B$, the homotopy equivalence for the space ΩY refines to $\Omega Y \simeq \Omega X \times \Omega(F * B)$. □

Remark 2.8. It is reasonable to ask whether Theorem 1.2 is natural. This would require compatible choices of maps at three key points. First, in (2) the homotopy pushout map $Y \xrightarrow{r} X$ would need to be natural. This is not automatic since a homotopy pushout map need not even be unique up to homotopy. Second, the applications of the cube lemma in both (4) and (7) would need to be natural. Third, the homotopy pullback map b in (6) would need to be natural, and this would involve ensuring the null homotopy for $F \xrightarrow{g} A$ is also natural. Whether all this can be done in practice is unclear, unless circumstances are constrained.

3. Gyrations

This section applies Theorem 1.2 to certain Poincaré duality complexes. Let M be a simply connected n-dimensional Poincaré duality complex. Since any simply connected CW-complex can be given a cell-structure with one cell corresponding to each $\mathbb{Z}$-summand in homology, it may be assumed that M has a single n-cell. Let $\overline{M}$ be the $(n-1)$-skeleton of M. There is a homotopy cofibration

$$S^{n-1} \xrightarrow{f} \overline{M} \xrightarrow{i} M,$$

where f attaches the n-cell to M and i is the inclusion of the $(n-1)$-skeleton.

Definition 3.1. For $k \geq 2$, define the *k-gyration* $\mathcal{G}^k(M)$ of M by the pushout

$$\begin{array}{ccc} S^{n-1} \times S^{k-1} & \xrightarrow{1\times j} & S^{n-1} \times D^k \\ \downarrow{\scriptstyle f\times 1} & & \downarrow \\ \overline{M} \times S^{k-1} & \longrightarrow & \mathcal{G}^k(M) \end{array} \tag{9}$$

where j is the inclusion of the boundary.

As noted in the introduction, if M is a smooth manifold then $\mathcal{G}^k(M)$ is the output from a surgery on $M \times S^{k-1}$. This pushout formulation holds more generally for M being a Poincaré duality complex, and the output is a Poincaré duality complex $\mathcal{G}^k(M)$ of dimension $n+k-1$.

There is a refinement of the pushout (9). As D^k is contractible, collapsing it to a point in (9) gives a homotopy pushout

$$\begin{array}{ccc} S^{n-1} \times S^{k-1} & \xrightarrow{\pi_1} & S^{n-1} \\ \downarrow{\scriptstyle f\times 1} & & \downarrow \\ \overline{M} \times S^{k-1} & \longrightarrow & \mathcal{G}^k(M) \end{array} \tag{10}$$

where π_1 is the projection to the first factor. This is the type of homotopy pushout that appears in Theorem 1.2. To apply the theorem we would need to know that if the space F and the map g are defined by the homotopy fibration $F \xrightarrow{g} S^{n-1} \xrightarrow{f} \overline{M}$ then g is null homotopic. This is related to the notion of an inert attaching map, as follows.

Definition 3.2. Suppose that there is a homotopy cofibration $S^{n-1} \xrightarrow{f} X \xrightarrow{i} Y$ where X is a simply connected CW-complex, f attaches an n-cell to X for some $n \geq 2$, and i is the inclusion into $Y = X \cup_f e^n$. The attaching map f is *inert* if Ωi has a right homotopy inverse.

Observe that Ωi having a right homotopy inverse implies that i induces an epimorphism on homotopy groups. Thus the cell attachment f annihilates some homotopy groups of X but does not introduce new ones in Y. Inert attaching maps were first studied in rational homotopy theory by Halperin and Lemaire [1987], but it is worth noting that we are working in a context that does not require localisation.

The inert condition allows for the homotopy fibre of i to be identified.

Theorem 3.3 [Theriault 2024b]. *Suppose that there is a homotopy cofibration $S^{n-1} \xrightarrow{f} X \xrightarrow{i} Y$ where f is inert. Then there is a homotopy fibration*

$$S^{n-1} \rtimes \Omega Y \xrightarrow{\varphi} X \xrightarrow{i} Y,$$

where the restriction of φ to S^{n-1} is f and the map Ωi has a right homotopy inverse. This right homotopy inverse implies that there is a homotopy equivalence

$$\Omega X \simeq \Omega Y \times \Omega(S^{n-1} \rtimes \Omega M). \qquad \square$$

One consequence of Theorem 3.3 that seems not to have been noticed before, or at least is underutilised, is the following.

Corollary 3.4. *Suppose that there is a homotopy cofibration $S^{n-1} \xrightarrow{f} X \xrightarrow{i} Y$ where the map f is inert. Then Ωf has a left homotopy inverse. Consequently, if $F \xrightarrow{g} S^{n-1} \xrightarrow{f} X$ is a homotopy fibration, then g is null homotopic.*

Proof. By Theorem 3.3, the homotopy fibration $S^{n-1} \rtimes \Omega Y \xrightarrow{\varphi} X \xrightarrow{i} Y$ splits after looping. Therefore $\Omega\varphi$ has a left homotopy inverse. As the restriction of φ to S^{n-1} is f, and S^{n-1} retracts off $S^{n-1} \rtimes \Omega M$, it follows that Ωf has a left homotopy inverse. The consequence in the statement of the corollary follows immediately. $\square$

Using this in the context of gyrations results in a proof of our main result.

Proof of Theorem 1.1. Consider the homotopy fibration $F \xrightarrow{g} S^{n-1} \xrightarrow{f} \overline{M}$. By hypothesis, the map f is inert, so Corollary 3.4 implies that g is null homotopic. Theorem 1.2 therefore applies to the homotopy pushout (10), and concludes that the map $\overline{M} \rtimes S^{k-1} \xrightarrow{s} \mathcal{G}^k(M)$ has the property that Ωs has a right homotopy inverse. By [Basu and Ghosh 2024, Proposition 6.9] or [Chenery and Theriault 2025, Lemma 3.2], there is a homotopy cofibration $S^{n+k-2} \xrightarrow{g} \overline{M} \rtimes S^{k-1} \xrightarrow{s} \mathcal{G}^k(M)$ where g attaches the $(n+k-1)$-cell. Thus, by definition of inertness, the right homotopy inverse for Ωs implies that g is inert. $\square$

4. Twisted gyrations

There is a generalisation of a gyration in which the attaching map f is twisted by an action of $\mathrm{SO}(n)$ on S^{n-1}. Let τ represent a class in $\pi_{k-1}(\mathrm{SO}(n))$. Define $\phi_\tau : S^{n-1} \times S^{k-1} \to S^{n-1} \times S^{k-1}$ by $\phi_\tau(x, y) = (\tau(y) \cdot x, y)$. Note that if τ is homotopic to τ' then ϕ_τ is homotopic to $\phi_{\tau'}$.

Definition 4.1. For $k \geq 2$ and τ representing a class in $\pi_n(\mathrm{SO}(n))$, define the *twisted k-gyration* $\mathcal{G}^k_\tau(M)$ of M by the pushout

$$\begin{array}{ccc} S^{n-1} \times S^{k-1} & \xrightarrow{1\times j} & S^{n-1} \times D^k \\ \downarrow{\scriptstyle (f\times 1)\circ\phi_\tau} & & \downarrow \\ \overline{M} \times S^{k-1} & \longrightarrow & \mathcal{G}^k_\tau(M) \end{array} \tag{11}$$

As with ordinary gyrations, the pushout (11) refines to a homotopy pushout

$$\begin{array}{ccc} S^{n-1} \times S^{k-1} & \xrightarrow{\pi_1} & S^{n-1} \\ \downarrow{\scriptstyle (f\times 1)\circ\phi_\tau} & & \downarrow \\ \overline{M} \times S^{k-1} & \longrightarrow & \mathcal{G}^k_\tau(M) \end{array} \tag{12}$$

where π_1 is the projection to the first factor.

Consider the map $\phi_\tau : S^{n-1} \times S^{k-1} \to S^{n-1} \times S^{k-1}$. If τ is null homotopic then ϕ_τ is homotopic to the product of the identity map on S^{n-1} and the identity map on S^{k-1}, in which case $\mathcal{G}^k_\tau(M) \simeq \mathcal{G}^k(M)$. If τ is not null homotopic then the S^{k-1} factor is acting nontrivially on the S^{n-1} factor, implying that the composite $(f \times 1) \circ \phi_\tau$ is likely not a product map, in which case Theorem 1.2 does not apply. However, it is shown in [Huang and Theriault 2026, Proposition 3.7] that if $J : \pi_k(\mathrm{SO}(n)) \to \pi_{n+k-1}(S^n)$ is the classical J-homomorphism and $J(\tau)$ is null homotopic then $\mathcal{G}^k_\tau$ is the homotopy pushout of $f \times 1$ and π_1, implying that $\mathcal{G}^k_\tau(M) \simeq \mathcal{G}^k(M)$. Moreover, if $J(\tau)$ is nontrivial, then after localising away from the primes diving the order of $J(\tau)$ the homotopy pushout of $f \times 1$ and π_1 is $\mathcal{G}^k_\tau(M)$, implying that $\mathcal{G}^k_\tau(M) \simeq \mathcal{G}^k(M)$. Theorem 1.1 may now be applied to obtain the following.

Theorem 4.2. *Let M be a simply connected n-dimensional Poincaré duality complex and let $S^{n-1} \xrightarrow{f} \overline{M}$ be the attaching map for the n-cell of M. If f is inert then for any $k \geq 2$ and $\tau \in \pi_{k-1}(\mathrm{SO}(n))$, after localisation away from the primes dividing the order of $J(\tau)$:*

(a) *The attaching map for the $(n+k-1)$-cell of $\mathcal{G}^k_\tau(M)$ is inert.*

(b) *There is a homotopy equivalence $\Omega\mathcal{G}^k_\tau(M) \simeq \Omega\overline{M} \times \Omega(F * S^{k-1})$, where F is the homotopy fibre of f.* □

The image of the J-homomorphism is known in the stable range by work of Adams [1966] and Quillen [1971]. They show that when $n \geq k+1$ the image of J is

$$\mathrm{Im}\, J \cong \begin{cases} 0 & \text{if } k \equiv 3, 5, 6, 7 \bmod 8, \\ \mathbb{Z}/2 & \text{if } k \equiv 1, 2 \bmod 8,\ k \neq 1, \\ \mathbb{Z}/d_s & \text{if } k = 4s, \end{cases} \tag{13}$$

where d_s is the denominator of $B_s/4s$ and B_s is the s-th Bernoulli number. In particular, if $n \geq k+1$ and $k \equiv 3, 5, 6, 7 \bmod 8$ then $J(\tau) = 0$ so Theorem 4.2 holds without localisation, and if $n \geq k+1$ and $k \equiv 1, 2 \bmod 8, k \neq 1$ then Theorem 4.2 holds after localising only away from 2.

5. A refinement and an application

A refined loop space decomposition. By Theorem 1.1, if the attaching map for the n-cell of M is inert then there is a homotopy equivalence

$$\Omega\mathcal{G}^k(M) \simeq \Omega\overline{M} \times \Omega(F * S^{k-1}).$$

We consider a context in which this homotopy decomposition can be refined by showing it decomposes further as a product of spheres and loops on spheres.

Definition 5.1. Let $\mathcal{P}$ be the class of path-connected finite-type spaces Y that are homotopy equivalent to a product of spheres and loops on spheres. Let $\mathcal{W}$ be the class of simply connected finite-type spaces Z that are homotopy equivalent to a wedge of spheres.

As in [Stanton 2024], there are three properties of the classes $\mathcal{P}$ and $\mathcal{W}$ that will be used:

- Property (a): if $Y \in \mathcal{P}$ then $\Sigma Y \in \mathcal{W}$.
- Property (b): if $Z \in \mathcal{W}$ then $\Omega Z \in \mathcal{P}$.
- Property (c): if $Y \in \mathcal{P}$ and Y' retracts off Y then $Y' \in \mathcal{P}$.

Theorem 5.2. *Let M be a simply connected n-dimensional Poincaré duality complex. If the attaching map for the n-cell of M is inert and $\Omega M \in \mathcal{P}$ then the attaching map for the $(n+k-1)$-cell of $\mathcal{G}^k(M)$ is inert and $\Omega\mathcal{G}^k(M) \in \mathcal{P}$.*

Proof. Since the attaching map for the n-cell of M is inert, the attaching map for the $(n+k-1)$-cell of $\mathcal{G}^k_\tau(M)$ is inert by Theorem 1.1.

The inertness of the attaching map for the n-cell of M also implies by Theorem 3.3 that there is a homotopy equivalence

$$\Omega\overline{M} \simeq \Omega M \times \Omega(S^{n-1} \rtimes \Omega M).$$

In general, there is a homotopy equivalence $\Sigma X \rtimes Y \simeq \Sigma X \vee (\Sigma X \wedge Y)$. In our case, $S^{n-1} \rtimes \Omega M \simeq S^{n-1} \vee (S^{n-1} \wedge \Omega M)$. As $\Omega M \in \mathcal{P}$, Property (a) implies that $\Sigma\Omega M \in \mathcal{W}$, so we obtain $S^{n-1} \wedge \Omega M \in \mathcal{W}$, and hence $S^{n-1} \rtimes \Omega M \in \mathcal{W}$. Thus Property (b) implies that $\Omega(S^{n-1} \rtimes \Omega M) \in \mathcal{P}$. The homotopy equivalence for $\Omega\overline{M}$ therefore implies that $\Omega\overline{M} \in \mathcal{P}$.

Next consider the homotopy fibration $F \to S^{n-1} \xrightarrow{f} \overline{M}$, where f is the attaching map for the n-cell of M. Since f is inert, by Corollary 3.4 the map $F \to S^{n-1}$ is

null homotopic. Therefore there is a homotopy equivalence $\Omega\overline{M} \simeq \Omega S^{n-1} \times F$. Now, as F is a retract of $\Omega\overline{M}$ and $\Omega\overline{M} \in \mathcal{P}$, Property (c) implies that $F \in \mathcal{P}$. Thus $F * S^{k-1} \simeq \Sigma^k F \in \mathcal{W}$ by Property (a), and hence $\Omega(F * S^{k-1}) \in \mathcal{P}$ by Property (b). The homotopy equivalence $\Omega\mathcal{G}^k(M) \simeq \Omega\overline{M} \times \Omega(F * S^{k-1})$ in Theorem 1.1 therefore implies that $\Omega\mathcal{G}^k(M) \in \mathcal{P}$. □

Note that Theorem 5.2 allows for iterations: the conclusions about $\mathcal{G}^k(M)$ allow the theorem to now be applied to it to show the same properties hold for a gyration of a gyration.

Connected sums in toric topology. Suppose that M and N are n-dimensional Poincaré duality complexes. Let $M \# N$ be their connected sum, obtained by deleting the interior of an n-disk in each of M and N and then gluing together what remains along the boundary of the n-disk. Collapsing the collar along which the gluing occurs to a point gives a map $M \# N \to M \vee N$, and this may be composed with the inclusion into the product to give a map

$$\alpha : M \# N \to M \times N.$$

In [Theriault 2025] it was shown that if M and N are simply connected, $n \geq 3$ and the attaching map of the n-cell for both M and N is inert, then $\Omega\alpha$ has a right homotopy inverse and there is a homotopy equivalence

$$\Omega(M \# N) \simeq \Omega M \times \Omega N \times \Omega(S^{n-1} \rtimes (\Omega M \times \Omega N) \vee (\Sigma\Omega M \wedge \Omega N)). \tag{14}$$

We apply this in a case of interest to toric topology. Let P be a simple polytope of dimension n with m facets. As in [Buchstaber and Panov 2015, Theorem 4.1.4], there is a *moment-angle manifold* $\mathcal{Z}_P$ associated to P, which is a central object in toric topology. Suppose that P is obtained by cutting a vertex off a simple polytope P' of dimension n with $m-1$ facets, that is, intersect P' with a hyperplane that contains no vertex of P' and has a half-space containing only one vertex of P', then let P be the intersection of P' and the other half-space. In [Gitler and López de Medrano 2013; Chen et al. 2020] it was shown that there is a diffeomorphism

$$\mathcal{Z}_P \cong \mathcal{G}^2(\mathcal{Z}_{P'}) \# \mathop{\#}_{i=1}^{m-n} (S^{i+2} \times S^{m+n-i-1})^{\#\binom{m-n}{i}},$$

where $N^{\#s}$ is the iterated connected sum of s copies of N. By [Theriault 2024b, Examples 5.3 and 5.4], the attaching map for the top cell of $\mathcal{Z}_{P'}$ is inert, so Theorem 1.1 implies the same is true of the attaching map for the top cell of $\mathcal{G}^2(\mathcal{Z}_{P'})$. The Hilton–Milnor theorem implies that the attaching map for the top cell of $S^a \times S^b$ is inert for any $a, b \geq 2$. Thus, iterating (14) implies there is a homotopy equivalence

$$\Omega\mathcal{Z}_P \simeq \Omega\mathcal{G}^2(\mathcal{Z}_{P'}) \times \prod_{i=1}^{m-n} (\Omega S^{i+2} \times \Omega S^{m+n-i-1})^{\times\binom{m-n}{i}} \times Z,$$

where $N^{\times s}$ is the iterated product of s copies of N and Z is a complementary factor which could, in principle, be explicitly identified.

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STEPHEN THERIAULT
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF SOUTHAMPTON
SOUTHAMPTON
UNITED KINGDOM
s.d.theriault@soton.ac.uk

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LOCALIZATION AND TOEPLITZ OPERATORS ON DOUBLING FOCK SPACES

CHUNXU XU AND HONG ZOU

Let F^p_φ be the weighted Fock space over the complex plane, where φ is subharmonic and $\Delta\varphi\, dA$ is a doubling measure. Firstly we introduce the concept of weakly localized operators from F^p_φ to F^q_φ for $1 \le p, q < \infty$. Then we show that weakly localized operators form an algebra and characterize the compactness of weakly localized operators from one doubling Fock space to another. As applications, the boundedness and compactness of Toeplitz operators with BMO^s symbols from F^p_φ to F^q_φ for $1 < p \le q < \infty$ and $p = q = 1$, and with IMO^s symbols from F^p_φ to F^q_φ for $1 \le q < p < \infty$, are considered.

1. Introduction

Let $\mathbb{C}$ be the complex plane, and let dA be the Lebesgue area measure on $\mathbb{C}$. A Borel measure $\upsilon > 0$ on $\mathbb{C}$ is called a doubling measure if there is a constant $C > 0$ such that for $z \in \mathbb{C}$ and $r > 0$,

$$\upsilon(D(z, 2r)) \le C\upsilon(D(z, r)),$$

where $D(z, r) = \{w \in \mathbb{C} : |z - w| < r\}$ is the Euclidean disc with the center z and radius r. Let φ be a nonzero real subharmonic function on $\mathbb{C}$ with $\Delta\varphi\, dA$ a doubling measure, where Δ is the laplacian operator. For $z \in \mathbb{C}$, there exists a unique positive radius $\rho(z)$ such that $\upsilon\big(D(z, \rho(z))\big) = 1$. The function ρ^{-2} can be thought of as a regularized version corresponding to $\Delta\varphi$; see [Marco et al. 2003; Christ 1991] for details.

For $0 < p < \infty$, the space L^p_φ consists of all Lebesgue measurable functions f on $\mathbb{C}$ such that

$$\|f\|_{L^p_\varphi} := \left\{\int_{\mathbb{C}} |f(z)e^{-\varphi(z)}|^p \, dA(z)\right\}^{1/p} < \infty.$$

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Its norm $\|\cdot\|_{F^p_\varphi}$ is inherited from L^p_φ. Let $H(\mathbb{C})$ denote the family of all analytic functions on $\mathbb{C}$. The doubling Fock space F^p_φ is defined to be

$$F^p_\varphi = L^p_\varphi \cap H(\mathbb{C}).$$

The doubling Fock space F^∞_φ is the collection of all functions on $H(\mathbb{C})$ such that

$$\|f\|_{F^\infty_\varphi} = \sup_{z\in\mathbb{C}} |f(z)|e^{-\varphi(z)} < \infty.$$

For $0 < p < 1$, F^p_φ is an F-space under $d(f,g) = \|f-g\|_{F^p_\varphi}$. For $1 \le p < \infty$, F^p_φ is a Banach space. If $\varphi(z) = \frac{1}{2}\alpha|z|^2$ ($\alpha > 0$) then F^p_φ is the classical Fock space. For some suitable number $A > 0$ and integer $m > 0$, if $\varphi(z) = -m\ln(A + |z|^2) + |z|^2$, then F^2_φ is the Fock–Sobolev space first studied in [Cho and Zhu 2012]. For $\varphi(z) = |z|^m$, F^2_φ is the generalized Fock space; see [Schneider and Schneider 2009; 2011]. Notice that when $n = 1$, the weights ϕ defined as in [Hu and Virtanen 2022b; 2023] satisfy $0 < c \le \Delta\phi \le C$, which implies that $\Delta\phi\, dA$ is a doubling measure and $\rho \simeq \frac{1}{\Delta\phi} \simeq 1$.

It is well known that F^2_φ is a Hilbert space under the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} f(w)\overline{g(w)}e^{-2\varphi(w)}\, dA(w).$$

Let P be the orthogonal projection from L^2_φ onto F^2_φ. Then

$$P(g)(z) = \int_{\mathbb{C}} g(w)\overline{K_z(w)}e^{-2\varphi(w)}\, dA(w), \quad g \in L^2_\varphi,\ z \in \mathbb{C},$$

where $K_z(w) := K(w,z)$ is the reproducing kernel on F^2_φ. From [Oliver and Pascuas 2016, Corollary 3.7], we know that

$$\Gamma = \left\{\sum_{j=1}^N a_j K(\cdot, z_j) : N \in \mathbb{N}^+,\ a_j \in \mathbb{C},\ z_j \in \mathbb{C} \text{ for } 1 \le j \le N\right\}$$

is dense in F^p_φ ($1 \le p \le \infty$). Then the domain of P can be extended to L^p_φ with $1 \le p < \infty$. Let

$$\mathcal{S} = \{f \text{ is measurable on } \mathbb{C} : fgK_z \in L^1(\mathbb{C}, e^{-2\varphi}dA) \text{ for } g \in \Gamma, z \in \mathbb{C}\}.$$

Given $s \in \mathbb{R}$, we say that f belongs to the space BMO^s if $f \in \mathcal{S}$ and

$$\|f\|_{\mathrm{BMO}^s} := \sup_{z\in\mathbb{C}} \rho(z)^s\, \mathrm{MO}(f)(z) < \infty,$$

where

$$\mathrm{MO}(f)(z) = \int_{\mathbb{C}} |f(w) - \tilde{f}(z)|\, |k_z(w)|^2 e^{-2\varphi(w)}\, dA(w).$$

Here $k_z(\cdot) := K_z(\cdot)/\|K_z\|_{F^2_\varphi}$ and the Berezin transform $\tilde{f}$ of f is defined as

$$\tilde{f}(z) = \int_{\mathbb{C}} f(w)|k_z(w)|^2 e^{-2\varphi(w)}\, dA(w).$$

The space VMO^s consists of all $f \in \mathrm{BMO}^s$ such that

$$\lim_{|z|\to\infty} \rho(z)^s \mathrm{MO}(f)(z) = 0.$$

For short, we write $\mathrm{BMO} = \mathrm{BMO}^0$ and $\mathrm{VMO} = \mathrm{VMO}^0$. It should be noted that the BMO^s in this paper is identical in spelling to the BMO^p in [Hu and Virtanen 2022a, p. 106], but there is an important difference in definition. For $f \in \mathcal{S}$, the Toeplitz operator T_f on Γ can be defined by

$$T_f g(z) = \int_{\mathbb{C}} f(w)g(w)K(z,w)e^{-2\varphi(w)}\, dA(w), \quad g \in \Gamma,\ z \in \mathbb{C}.$$

Then for $f \in \mathcal{S}$, T_f is densely defined on Γ.

We are concerned with the localized operator from one doubling Fock space to another. As an application, we also consider the boundedness and the compactness of the Toeplitz operator T_f with BMO^s and IMO^s (see Section 4.2 for more details) from one doubling Fock space to another.

It is well known that the boundedness and compactness of Toeplitz operators with some suitable symbols have received much attention in the academic community. Axler and Zheng [1998] first characterized the compactness of Toeplitz operators with bounded symbols on the Bergman space by using its Berezin transform. But for the unbounded symbol, Zorboska [2003], using the Berezin transform technique, characterized the boundedness and compactness of Toeplitz operators with BMO symbols on the Bergman space $A^2(\mathbb{D})$. Coburn, Isralowitz and Li [Coburn et al. 2011], using a different method than [Axler and Zheng 1998; Zorboska 2003], proved an analogue of this compactness criterion for Toeplitz operators with BMO symbols on the Segal–Bargmann space $F^2(\mathbb{C}^n)$. Wang [2020] characterized the boundedness (or compactness) of the Toeplitz operators T_f with BMO symbols from the Fock pace F^p to F^q for $0 < p < q < \infty$. Given $f \in \mathrm{IMO}$ (the space of integrable mean oscillation functions), they proved that T_f is bounded (or compact) from F^p to F^q for $0 < q < p < \infty$ if and only if its Berezin transform satisfies the some integral condition.

Recently, "sufficiently localized" operators and "weakly localized" operators have been the subject of much research attention. Xia and Zheng [2013] first introduced the concept of sufficiently localized operators on the Fock space F^2, and they showed that if every bounded operator T from the C^* algebra generated by a sufficiently localized operator satisfies the condition

$$\lim_{|z|\to\infty} \langle Tk_z, k_z\rangle_{F^2} = 0,$$

then T is compact on F^2. After that, Isralowitz, Mitkovski and Wick [Isralowitz et al. 2015] gave the concept of weakly localized operators (weaker than the sufficiently localized operators) and they extended the results of Xia and Zheng to the cases of the Bergman space A^p and the generalized Fock space F^p_ϕ (the weight function ϕ is subharmonic, real-valued and satisfies $dd^c\phi \simeq dd^c|z|^2$) for $1 < p < \infty$. For $0 < p \le 1$, Hu, Lv and Wick [Hu et al. 2018] characterized the compact operators T in the algebra generated by weakly localized operators on F^p_ϕ. They also showed that, for $0 < p < \infty$, an operator T in the algebra generated by bounded Toeplitz operators with BMO symbols is compact on F^p_ϕ if and only if its Berezin transform vanishes at infinity. For more information about localized operators, please refer to [Liu and Wang 2021; Sadeghi and Zorboska 2020; Zorboska 2022].

Theories of Toeplitz (or Hankel) operators on doubling Fock spaces have been studied extensively. For example, Oliver and Pascuas [2016] proved the boundedness, compactness and membership in the Schatten class of Toeplitz operators with positive Radon measure. In [Hu and Lv 2017; Lv 2019], the authors characterized the boundedness and compactness of Toeplitz operators with positive Borel measure from F^p_φ to F^q_φ for $0 < p, q \le \infty$. Recently, Lv and Wang [2024] studied the boundedness and compactness of single Hankel operators from a doubling Fock space F^q_φ to a Lebesgue space L^q_φ for $1 \le p, q < \infty$, extending results of [Hu and Virtanen 2023] for a special case $\rho \simeq 1$. The doubling Fock spaces F^p_φ are different from the spaces in [Hu and Virtanen 2022b; 2023; Hu and Wang 2018] in some ways. For example, let $\varphi(z) = |z|^4$. Then $\Delta\varphi\, dA$ is a doubling measure, but

$$F^p_\varphi \setminus F^q_\varphi \neq \varnothing, \quad F^q_\varphi \setminus F^p_\varphi \neq \varnothing \tag{1-1}$$

for $p \neq q$; see [Constantin and Peláez 2016] for details. Hu and Virtanen [2022a] characterized the Fredholm properties of Toeplitz operators on F^p_φ for $0 < p < \infty$.

In the light of the above, a number of problems have naturally arisen:

(1) When are Toeplitz operators induced by bounded symbols, from one doubling Fock space to another, compact?

(2) When are Toeplitz operators induced by BMO^s or IMO^s symbols, from one doubling Fock space to another, bounded and compact?

(3) Doubling Fock spaces do not have the property of (1-1), so it makes sense to define a weakly localized condition from F^p_φ to F^q_φ for $1 \le p, q < \infty$ and to study the case when an operator is compact if it satisfies this condition.

We study problems (2) and (3). If (2) holds, (1) naturally holds. The reproducing kernel of doubling Fock spaces does not have specific expressions, which makes the study of the problem difficult and requires the development of new techniques (a key Lemma 3.2). Unfortunately we only give partial solutions to (2) and (3); see Theorems 3.7 and 4.7. As far as the other cases are concerned, we have not

yet found a good way of dealing with them. In particular, we characterize the boundedness and compactness of Toeplitz operators with BMO symbols on F^p_φ for $1 \le p < \infty$; see Corollary 4.8. But for Toeplitz operators with IMO^s symbols, we only characterize the boundedness and compactness of Toeplitz operators with IMO^s symbols from F^p_φ to F^p_φ for $1 \le q < p < \infty$; see Theorem 4.10.

The outline of this paper is as follows. In Section 2, we introduce some important lemmas that will be used frequently. In Section 3, we give the concept of weakly localized operators, obtain that weakly localized operators form an algebra and also characterize the compactness of weakly localized operators from F^p_φ to F^q_φ (on F^1_φ) for $1 < p \le q < \infty$. In Section 4, we obtain the boundedness and compactness of Toeplitz operators with BMO^s and IMO^s symbols from F^p_φ to F^p_φ for some suitable exponents $1 \le p, q < \infty$; see Sections 4.1 and 4.2, respectively.

For $a, b > 0$, the notation $a \lesssim b$ indicates that there exists some positive constant C independent of the functions being considered such that $a \le Cb$ and $a \simeq b$ means that both $a \lesssim b$ and $b \lesssim a$ hold.

2. Preliminaries

In this section, we list some estimates and some integral inequalities related to the reproducing kernel. These will be used frequently in what follows. We start with function ρ. Let

$$D^r(z) := D(z, r\rho(z))$$

for $z \in \mathbb{C}$ and $r > 0$. Then by [Oliver and Pascuas 2016, Lemma 2.2], we have the following lemma.

Lemma 2.1. *For every $r > 0$:*

(1) *There exists a positive constant C_r such that*

$$C_r^{-1}\rho(w) \le \rho(z) \le C_r\rho(w)$$

for every $z \in \mathbb{C}$ and $w \in D^r(z)$.

(2) *There exists a constant $B > 0$ such that*

$$D^r(z) \subset D^{Br}(w), \quad D^r(w) \subset D^{Br}(z)$$

for every $z \in \mathbb{C}$ and $w \in D^r(z)$.

For any $z, w \in \mathbb{C}$, the distance $d_\varphi(z, w)$ induced by the metric $\rho^{-2}\, dz \otimes d\bar{z}$ is defined by

$$d_\varphi(z, w) = \inf_\tau \int_0^1 \frac{|\tau'(t)|}{\rho(\tau(t))}\, dt,$$

where the infimum is taken over all piecewise C^1 curves $\tau : [0, 1] \to \mathbb{C}$ with $\tau(0) = z$ and $\tau(1) = w$. It is known that the Bergman metric

$$\frac{\partial^2 \log K(z, z)}{\partial z \partial \bar{z}}\, dz \otimes d\bar{z}, \quad z \in \mathbb{C},$$

is comparable to the metric $\rho^{-2}\, dz \otimes d\bar{z}$; see [Constantin and Ortega-Cerdà 2011, page 355] for further details. Write $\beta(\cdot, \cdot)$ for the Bergman distance on $\mathbb{C}$. Therefore there exist two constants $C_1, C_2 > 0$ such that

$$C_1\, d_\varphi(z, w) \le \beta(z, w) \le C_2\, d_\varphi(z, w), \quad z, w \in \mathbb{C}. \tag{2-1}$$

For more information about the Bergman distance and metric, please refer to [Krantz 2013]. For $z \in \mathbb{C}$ and $r > 0$, let $B(z, r) = \{w : \beta(z, w) < r\}$. By (2-1) and [Oliver and Pascuas 2016, Lemma 2.5(1)], we obtain that there exist two positive numbers m and M such that for every $z \in \mathbb{C}$,

$$D^m(z) \subseteq B(z, r) \subseteq D^M(z).$$

Therefore, we regard $B(z, r)$ and $D^x(z)$ as equivalent for numbers $r, x > 0$.

Since β is more suitable for the analysis of operators on doubling Fock spaces, we need the following estimates for the Bergman distance β instead of d_φ. These estimates will be very useful in our analysis.

Theorem 2.2. *For $p > 0$ and $k, l \in \mathbb{R}$,*

$$\int_{\mathbb{C}} (\beta(z, \xi) + 1)^l |K(z, \xi)|^p e^{-p\varphi(\xi)} \rho(\xi)^k\, dA(\xi) \lesssim \rho(z)^{2(1-p)+k} e^{p\varphi(z)} \quad \text{for } z \in \mathbb{C} \tag{2-2}$$

and

$$\sup_{z \in \mathbb{C}} \rho(z)^{2(p-1)-k} e^{-p\varphi(z)} \times \int_{\mathbb{C} \setminus B(z, R)} (\beta(z, \xi) + 1)^l |K(z, \xi)|^p e^{-p\varphi(\xi)} \rho(\xi)^k\, dA(\xi) \to 0 \tag{2-3}$$

as $R \to \infty$.

Proof. It is easy to see that these conclusions can be obtained by Lemma 2.2 and Corollary 2.4 in [Hu and Virtanen 2022a], so we will omit its proof here. □

For the convenience of the following applications, we obtained from [Oliver and Pascuas 2016, Proposition 2.9] the norm estimate for the reproducing kernel.

Lemma 2.3. *For any $1 \le p \le \infty$,*

$$\|K_z\|_{F^p_\varphi} \simeq e^{\varphi(z)} \rho(z)^{2/p-2}, \quad z \in \mathbb{C}.$$

The following result comes from [Hu and Virtanen 2022a, Lemma 3.1], which implies that the point evaluation functional at $z \in \mathbb{C}$ is bounded.

Lemma 2.4. *Let $0 < p < \infty$. For any $r > 0$,*

$$|f(z)|^p e^{-p\varphi(z)} \lesssim \int_{D^r(z)} |f(w)|^p e^{-p\varphi(w)} \frac{dA(w)}{\rho^2(w)}$$

for $z \in \mathbb{C}$ and $f \in H(\mathbb{C})$.

As an application of the above lemma, we have the following lemma. It helps to study the Toeplitz operators with BMO symbols.

Lemma 2.5. *Let μ be a positive Borel measure on $\mathbb{C}$ and $0 < p < \infty$. Let $r > 0$ and $r \in \mathbb{R}$. Then for all $f \in H(\mathbb{C})$,*

$$\int_{\mathbb{C}} |f(z)|^p \rho(z)^k e^{-p\varphi(z)}\, d\mu(z) \lesssim \int_{\mathbb{C}} |f(w)|^p e^{-p\varphi(w)} \rho(w)^k \hat{\mu}_r(w)\, dA(w),$$

where

$$\hat{\mu}_r(w) := \frac{1}{\rho(w)^2} \int_{D^r(w)} d\mu.$$

Proof. It is easy to show that

$$|f(z)|^p \rho(z)^k e^{-p\varphi(z)} \lesssim \int_{D^r(z)} |f(w)|^p e^{-p\varphi(w)} \frac{dA(w)}{\rho^{2-k}(w)}$$

by Lemma 2.4. Then

$$\begin{aligned}\int_{\mathbb{C}} |f(z)|^p \rho(z)^k e^{-p\varphi(z)}\, d\mu(z) &\lesssim \int_{\mathbb{C}} \int_{D^r(z)} |f(w)|^p e^{-p\varphi(w)} \frac{dA(w)}{\rho^{2-k}(w)}\, d\mu(z) \\ &\lesssim \int_{\mathbb{C}} |f(w)|^p e^{-p\varphi(w)} \rho(w)^k \hat{\mu}_r(w)\, dA(w). \qquad \square\end{aligned}$$

3. Weakly localized operator

In this section we will introduce the concept of weakly localized operators from one doubling Fock space to another. We show that these operators form an algebra, and characterize their boundedness and compactness. We now proceed to define weakly localized operators. Let $k_{p,z}$ be the normalized reproducing kernel of K_z with respect to the norm $\|\cdot\|_{F^p_\varphi}$, that is,

$$k_{p,z}(w) = \frac{K(w,z)}{\|K_z\|_{F^p_\varphi}}.$$

For a linear operator T and for $f \in \Gamma$, we define

$$Tf(z) = \langle Tf, K_z \rangle, \quad z \in \mathbb{C}. \tag{3-1}$$

Definition 3.1. Let $1 \le p, q < \infty$ and $r > 0$. An operator T defined by satisfying (3-1) is called a weakly localized operator from F^p_φ to F^q_φ (we denote the set of such operators simply by $\mathcal{W}(F^p_\varphi, F^q_\varphi)$) if

$$\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |\langle T k_{p,z}, k_{q',w}\rangle| \, d\sigma(w) < \infty; \tag{3-2}$$

$$\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |\langle T^* k_{q',z}, k_{p,w}\rangle| \, d\sigma(w) < \infty; \tag{3-3}$$

$$\lim_{r\to\infty} \sup_{z\in\mathbb{C}} \int_{D^r(z)^c} |\langle T k_{p,z}, k_{q',w}\rangle| \, d\sigma(w) = 0; \tag{3-4}$$

$$\lim_{r\to\infty} \sup_{z\in\mathbb{C}} \int_{D^r(z)^c} |\langle T^* k_{q',z}, k_{p,w}\rangle| \, d\sigma(w) = 0; \tag{3-5}$$

where $d\sigma(w) := dA(w)/\rho(w)^{-2}$ and q' the conjugate exponent of q for $1 < q < \infty$ and $q' = +\infty$ for $q = 1$.

Lemma 3.2. *Let $1 < p, q < \infty$ or $p = q = 1$, and let Ω_1, Ω_2 be fixed subsets of $\mathbb{C}$. If a linear operator T satisfies* (3-1), *there is a positive constant C such that*

$$\int_{\Omega_2} \left(\int_{\Omega_1} |f(w)\langle T^* K_z, K_w\rangle| e^{-2\varphi(w)} \, dA(w) \right)^q e^{-q\varphi(z)} \, dA(z) \le C C_1 C_2^{q/q'} \|f\|^q_{L^p_\varphi},$$

where

$$C_1 = \sup_{w\in\Omega_1} \int_{\Omega_2} |\langle T k_{p,w}, k_{q',z}\rangle| \, d\sigma(z),$$

$$C_2 = \sup_{z\in\Omega_2} \int_{\Omega_1} |\langle T^* k_{q',z}, k_{p,w}\rangle| \, d\sigma(w).$$

Proof. We first assume that $1 < p, q < \infty$. By Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{\Omega_1} |f(w)\langle T^* K_z, K_w\rangle| e^{-2\varphi(w)} \, dA(w) \qquad (3\text{-}6)\\
&= \int_{\Omega_1} |f(w)\langle T k_{p,w}, k_{q',z}\rangle| \|K_w\|_{F^p_\varphi} \|K_z\|_{F^{q'}_\varphi} e^{-2\varphi(w)} \, dA(w)\\
&= \int_{\Omega_1} |f(w)| \, |\langle T k_{p,w}, k_{q',z}\rangle|^{\frac{1}{q}} \|K_w\|^{-\frac{q-p}{(p-1)q}}_{F^p_\varphi} \|K_z\|_{F^{q'}_\varphi} e^{(-1+\frac{q-p}{(p-1)q})\varphi(w)}\\
&\qquad\qquad \times |\langle T k_{p,w}, k_{q',z}\rangle|^{\frac{1}{q'}} \|K_w\|^{1+\frac{q-p}{(p-1)q}}_{F^p_\varphi} e^{(-1-\frac{q-p}{(p-1)q})\varphi(w)} \, dA(w)\\
&\le \left(\int_{\Omega_1} |f(w)|^q |\langle T k_{p,w}, k_{q',z}\rangle| \|K_w\|^{-\frac{q-p}{p-1}}_{F^p_\varphi} \|K_z\|^q_{F^{q'}_\varphi} e^{(-q+\frac{q-p}{p-1})\varphi(w)} \, dA(w) \right)^{\frac{1}{q}}\\
&\qquad\qquad \times \left(\int_{\Omega_1} |\langle T k_{p,w}, k_{q',z}\rangle| \|K_w\|^{p'}_{F^p_\varphi} e^{-p'\varphi(w)} \, dA(w) \right)^{\frac{1}{q'}}.
\end{aligned}$$

Lemma 2.4 shows that for $w \in \Omega_1$,

$$|f(w)|^{q-p} e^{(q-p)\varphi(w)} \lesssim \rho(w)^{-2(q-p)/p} \|f\|_{F^p_\varphi}^{q-p}. \tag{3-7}$$

Notice that

$$\|K_w\|_{F^p_\varphi}^{p'} e^{-p'\varphi(w)} \simeq \rho(w)^{-2}$$

by Lemma 2.3; therefore, by (3-6), (3-7) and Lemma 2.3 again, we obtain

$$\begin{aligned}
&\int_{\Omega_2}\bigg(\int_{\Omega_1} |f(w)\langle T^* K_z, K_w\rangle| e^{-2\varphi(w)}\, dA(w)\bigg)^q e^{-q\varphi(z)}\, dA(z)\\
&\quad\le \int_{\Omega_2}\int_{\Omega_1} |f(w)|^q |\langle T k_{p,w}, k_{q',z}\rangle| \|K_w\|_{F^p_\varphi}^{-\frac{q-p}{p-1}} \|K_z\|_{F^{q'}_\varphi}^q\\
&\qquad\qquad \times e^{(-q+\frac{q-p}{p-1})\varphi(w)}\, dA(w) e^{-q\varphi(z)}\, dA(z)\\
&\qquad\qquad \times \sup_{z\in\Omega_2}\bigg(\int_{\Omega_1} |\langle T k_{p,w}, k_{q',z}\rangle| \|K_w\|_{F^p_\varphi}^{p'} e^{-p'\varphi(w)}\, dA(w)\bigg)^{\frac{q}{q'}}\\
&\quad\lesssim C_2^{q/q'} \int_{\Omega_1} |f(w)|^q \|K_w\|_{F^p_\varphi}^{-\frac{q-p}{p-1}} e^{(-q+\frac{q-p}{p-1})\varphi(w)}\, dA(w)\\
&\qquad\qquad \times \sup_{w\in\Omega_1}\int_{\Omega_2} \langle T k_{p,w}, k_{q',z}\rangle \|K_z\|_{F^{q'}_\varphi}^q e^{-q\varphi(z)}\, dA(z)\\
&\quad= C_1 C_2^{q/q'} \int_{\Omega_1} |f(w)|^q \|K_w\|_{F^p_\varphi}^{-\frac{q-p}{p-1}} e^{(-q+\frac{q-p}{p-1})\varphi(w)}\, dA(w)\\
&\quad\lesssim C_1 C_2^{q/q'} \|f\|_{F^p_\varphi}^{q-p} \int_{\Omega_1} |f(w)|^p \rho(w)^{-2(q-p)/p} \|K_w\|_{F^p_\varphi}^{-\frac{q-p}{p-1}}\\
&\qquad\qquad \times e^{(-p+\frac{q-p}{p-1})\varphi(w)}\, dA(w)\\
&\quad\lesssim C_1 C_2^{q/q'} \|f\|_{F^p_\varphi}^{q-p} \int_{\Omega_1} |f(w)|^p e^{-p\varphi(w)}\, dA(w)\\
&\quad\le C_1 C_2^{q/q'} \|f\|_{F^p_\varphi}^q.
\end{aligned}$$

Next, we consider the case of $p = q = 1$. Fubini's theorem and Lemma 2.3 yield

$$\begin{aligned}
&\int_{\Omega_2}\int_{\Omega_1} |f(w)\langle T^* K_z, K_w\rangle| e^{-2\varphi(w)}\, dA(w) e^{-\varphi(z)}\, dA(z)\\
&\le \int_{\Omega_2}\int_{\Omega_1} |f(w)|\, |\langle T k_{1,w}, k_{\infty,z}\rangle| e^{-2\varphi(w)} \|K_w\|_{F^1_\varphi} \|K_z\|_{F^\infty_\varphi}\, dA(w) e^{-\varphi(z)}\, dA(z)\\
&\lesssim \int_{\Omega_1}\int_{\Omega_2} |\langle T k_{1,w}, k_{\infty,z}\rangle|\, d\sigma(z) |f(w)| e^{-\varphi(w)}\, dA(w)\\
&\lesssim C_1 \|f\|_{F^1_\varphi}.
\end{aligned}$$

□

Theorem 3.3. *Let $1 < p, q < \infty$ or $p = q = 1$. If $T \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$, then T is bounded from F^p_φ to F^q_φ.*

Proof. By the definition of the linear operator T, for $f \in \Gamma$,

$$|Tf(z)| = |\langle Tf, K_z\rangle| = \left|\int_{\mathbb{C}} f(w)\overline{\langle T^*K_z, K_w\rangle} e^{-2\varphi(w)}\, dA(w)\right|$$
$$\le \int_{\mathbb{C}} |f(w)\langle T^*K_z, K_w\rangle| e^{-2\varphi(w)}\, dA(w).$$

It is easy to show that

$$\|Tf\|^q_{F^q_\varphi} \lesssim C_1 C_2^{q/q'} \|f\|^q_{F^p_\varphi}$$

by the above inequality and Lemma 3.2, where

$$C_1 = \sup_{w\in\mathbb{C}} \int_{\mathbb{C}} |\langle Tk_{p,w}, k_{q',z}\rangle|\, d\sigma(z) \quad \text{and} \quad C_2 = \sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |\langle T^*k_{q',z}, k_{p,w}\rangle|\, d\sigma(w).$$

This shows that T is bounded from F^p_φ to F^q_φ. □

For $1 \le p, q < \infty$, let $\mathfrak{B}(F^p_\varphi, F^q_\varphi)$ be the family of all bounded operators from F^p_φ to F^q_φ and let $\mathcal{K}(F^p_\varphi, F^q_\varphi)$ be the set of all compact operators from F^p_φ to F^q_φ. We write $\mathfrak{B}(F^p_\varphi) = \mathfrak{B}(F^p_\varphi, F^p_\varphi)$ and $\mathcal{K}(F^p_\varphi) = \mathcal{K}(F^p_\varphi, F^p_\varphi)$ for convenience.

Theorem 3.4. *For $1 \le p < \infty$, $\mathcal{W}(F^p_\varphi, F^p_\varphi)$ forms a subalgebra of $\mathfrak{B}(F^p_\varphi)$. In particular, $\mathcal{W}(F^2_\varphi, F^2_\varphi)$ is a $*$-subalgebra of $\mathfrak{B}(F^2_\varphi)$.*

Proof. Suppose that $T, S \in \mathcal{W}(F^p_\varphi, F^p_\varphi)$. We show that TS and ST belong to $\mathcal{W}(F^p_\varphi, F^p_\varphi)$. By Theorem 3.3, we have $TS, ST \in \mathfrak{B}(F^p_\varphi)$. Next we prove that TS satisfies condition (3-2). By Fubini's theorem and Lemma 2.3, for every $z \in \mathbb{C}$,

$$\text{(3-8)}\quad \int_{\mathbb{C}} |\langle (TS)k_{p,z}, k_{p',w}\rangle|\, d\sigma(w)$$
$$\le \int_{\mathbb{C}}\int_{\mathbb{C}} |\langle Sk_{p,z}, K_u\rangle|\, |\langle T^*k_{p',w}, K_u\rangle| e^{-2\varphi(u)}\, dA(u)\, d\sigma(w)$$
$$= \int_{\mathbb{C}}\int_{\mathbb{C}} |\langle Sk_{p,z}, k_{p',u}\rangle|\, |\langle T^*k_{p',w}, k_{p,u}\rangle| \times \|K_u\|_{F^p_\varphi} \|K_u\|_{F^{p'}_\varphi} e^{-2\varphi(u)}\, dA(u)\, d\sigma(w)$$
$$= \int_{\mathbb{C}}\int_{\mathbb{C}} |\langle Sk_{p,z}, k_{p',u}\rangle|\, |\langle T^*k_{p',w}, k_{p,u}\rangle|\, d\sigma(u)\, d\sigma(w)$$
$$\lesssim \int_{\mathbb{C}} \sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |\langle Sk_{p,z}, k_{p',u}\rangle|\, d\sigma(u) |\langle T^*k_{p',w}, k_{p,u}\rangle|\, d\sigma(w)$$
$$\lesssim \sup_{u\in\mathbb{C}} \int_{\mathbb{C}} |\langle T^*k_{p',w}, k_{p,u}\rangle|\, d\sigma(w) < \infty.$$

This shows that TS satisfies condition (3-2).

We next show that TS satisfies condition (3-4). Notice that $w \in D^r(u)^c$ when $u \in D^r(z)$ and $w \in D^{2r}(z)^c$. Similar to (3-8), we have

$$
\begin{aligned}
(3\text{-}9)\quad & \int_{D^{2r}(z)^c} |\langle (TS)k_{p,z}, k_{p',w}\rangle|\, d\sigma(w) \\
&\le \int_{D^{2r}(z)^c} \int_{\mathbb{C}} |\langle Sk_{p,z}, k_{p',u}\rangle|\, |\langle T^* k_{p',w}, k_{p,u}\rangle|\, d\sigma(u)\, d\sigma(w) \\
&= \int_{D^{2r}(z)^c} \left(\int_{D^r(z)+D^r(z)^c} \right) |\langle Sk_{p,z}, k_{p',u}\rangle|\, |\langle T^* k_{p',w}, k_{p,u}\rangle|\, d\sigma(u)\, d\sigma(w) \\
&\lesssim \int_{D^r(z)} |\langle Sk_{p,z}, k_{p',u}\rangle|\, d\sigma(u) \sup_{w\in D^r(z)} \int_{D^r(u)^c} |\langle T^* k_{p',w}, k_{p,u}\rangle|\, d\sigma(w) \\
&\quad + \int_{D^r(z)^c} |\langle Sk_{p,z}, k_{p',u}\rangle|\, d\sigma(u) \sup_{u\in D^r(z)^c} \int_{D^{2r}(u)^c} |\langle T^* k_{p',w}, k_{p,u}\rangle|\, d\sigma(w) \\
&\le \int_{\mathbb{C}} |\langle Sk_{p,z}, k_{p',u}\rangle|\, d\sigma(u) \sup_{w\in D^r(z)} \int_{D^r(u)^c} |\langle T^* k_{p',w}, k_{p,u}\rangle|\, d\sigma(w) \\
&\qquad + \int_{D^r(z)^c} |\langle Sk_{p,z}, k_{p',u}\rangle|\, d\sigma(u) \sup_{u\in\mathbb{C}} \int_{D^{2r}(u)^c} |\langle T^* k_{p',w}, k_{p,u}\rangle|\, d\sigma(w).
\end{aligned}
$$

This proves that TS satisfies condition (3-4), because $T, S \in \mathcal{W}(F^p_\varphi, F^p_\varphi)$. Similar to the proof of (3-8) and (3-9), it is easy to prove that TS satisfies the conditions (3-3) and (3-5). This shows that $\mathcal{W}(F^p_\varphi, F^p_\varphi)$ forms a subalgebra of $\mathfrak{B}(F^p_\varphi)$.

It is easy to show $\mathcal{W}(F^2_\varphi, F^2_\varphi)$ is a $*$-subalgebra of $\mathfrak{B}(F^2_\varphi)$, since $T \in \mathcal{W}(F^2_\varphi, F^2_\varphi)$ if and only if $T^* \in \mathcal{W}(F^2_\varphi, F^2_\varphi)$. □

Given some sequence $\{a_j\}_{j=1}^\infty \subset \mathbb{C}$ and $r > 0$, we call $\{a_j\}_{j=1}^\infty$ a (ρ, r)-lattice if

(a) $\{D^r(a_j)\}_{j=1}^\infty$ covers $\mathbb{C}$;

(b) the disks of $\{D^{r/5}(a_j)\}_{j=1}^\infty$ are pairwise disjoints.

The existence of a (ρ, r)-lattice follows from a standard covering lemma; see [Marco et al. 2003]. For a (ρ, r)-lattice and $m > 0$, there exists some integer N such that $z \in \mathbb{C}$ belongs to at most N disks of $\{D^{mr}(a_j)\}_{j=1}^\infty$. That is,

$$
\sum_{j=1}^\infty \chi_{D^{mr}(a_j)}(z) \le N. \tag{3-10}
$$

Let $F_j = D^r(z_j) \setminus \bigcup_{i<j} D^r(z_i)$. It follows that $\{F_j\}_j$ is also a covering of $\mathbb{C}$ and $F_j \cap F_k = \varnothing$ $(j \ne k)$. We write $(F_j)^+_r = \bigcup_{x\in F_j} D^r(x)$, and let

$$
G_j = \{y : d(y, F_j) \le m^4 r\rho(z_j),\ m = m(r) > 1\}.
$$

There is some constant N such that

$$
\sum_{j=1}^\infty \chi_{F_j}(w) \le \sum_{j=1}^\infty \chi_{(F_j)^+_r}(w) \le N \quad \text{for any } w \in \mathbb{C}. \tag{3-11}
$$

Lemma 3.5. *For $1 < p, q < \infty$ or $p = q = 1$, if $T \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$, then for every $\varepsilon > 0$, there exists sufficiently large $r > 0$ such that for the covering $\{F_j\}$ depending only on r,*

$$\left\| T - \sum_{j=1} M_{\chi_{F_j}} T P M_{\chi_{G_j}} \right\|_{F^p_\varphi \to F^q_\varphi} < \varepsilon.$$

Here the multiplier operator M_f is defined by $M_f g = fg$ for $g \in \Gamma$.

Proof. Let $S_r = T - \sum_{j=1} M_{\chi_{F_j}} T P M_{\chi_{G_j}}$ for some given r. For any fixed $z \in \mathbb{C}$, we assume that $z \in F_{j_0}$ for some integer j_0. For $f \in F^p_\varphi$, we have

$$\begin{aligned}
S_r f(z) &= Tf(z) - \sum_{j=1} M_{\chi_{F_j}} T P M_{\chi_{G_j}} f(z) \\
&= \sum_{j=1} M_{\chi_{F_j}} Tf(z) - \sum_{j=1} M_{\chi_{F_j}} T P M_{\chi_{G_j}} f(z) \\
&= \sum_{j=1} M_{\chi_{F_j}}(z) T(P M_{\chi_{G_j^c}} f)(z) \\
&= \sum_{j=1} M_{\chi_{F_j}}(z) \langle T P M_{\chi_{G_j^c}} f, K_z \rangle \\
&= \sum_{j=1} M_{\chi_{F_j}}(z) \int_{\mathbb{C}} M_{\chi_{G_j^c}}(w) f(w) \overline{\langle T^* K_z, K_w \rangle} e^{-2\varphi(w)}\, dA(w).
\end{aligned} \tag{3-12}$$

Notice that $D^r(z) \subseteq G_{j_0}$ for $z \in F_{j_0}$. From (3-12), we have

$$\begin{aligned}
S_r f(z) &\le \sum_{j=1} M_{\chi_{F_j}}(z) \int_{\mathbb{C}} M_{\chi_{G_j^c}}(w) |f(w)|\, |\langle T^* K_z, K_w \rangle| e^{-2\varphi(w)}\, dA(w) \\
&= \int_{G^c_{j_0}} |f(w)|\, |\langle T^* K_z, K_w \rangle| e^{-2\varphi(w)}\, dA(w) \\
&\le \int_{D^r(z)^c} |f(w)|\, |\langle T^* K_z, K_w \rangle| e^{-2\varphi(w)}\, dA(w).
\end{aligned}$$

Similarly to the proof of Lemma 3.2, if we combine the above inequality, we immediately obtain

$$\|S_r f\|^q_{L^q_\varphi} \lesssim C_1 C_2^{q/q'}(r) \|f\|^q_{F^p_\varphi},$$

where

$$C_1 = \sup_{w \in \mathbb{C}} \int_{\mathbb{C}} |\langle T k_{p,w}, k_{q',z} \rangle|\, d\sigma(z), \quad C_2(r) = \sup_{z \in \mathbb{C}} \int_{D^r(z)^c} |\langle T^* k_{q',z}, k_{p,w} \rangle|\, d\sigma(w).$$

Since $T \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$, we have C_1 is finite and $C_2(r) \to 0$ as $r \to \infty$. Then for any $\varepsilon > 0$, there exists a sufficiently large $r > 0$ such that

$$\|S_r\|_{F^p_\varphi \to F^q_\varphi} < \varepsilon. \qquad \square$$

We use $\|T\|_{e,F^p_\varphi\to F^q_\varphi}$ to stand for the essential norm of a given linear operator T from F^p_φ to F^q_φ. We write $\|T\|_e = \|T\|_{e,F^p_\varphi\to F^q_\varphi}$ for convenience.

Theorem 3.6. *For $1 < p \le q < \infty$ or $p = q = 1$, if $T \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$, there exist positive constants r and $C = C(r)$ such that*

$$\|T\|_e \le C \limsup_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle Tk_{z,p}, k_{q',w}\rangle|^{1/q}.$$

Proof. The case of $p = q = 1$ comes from [Liu and Wang 2021, Theorem 25], so we only need to prove that the case of $1 < p \le q < \infty$. It is easy to prove that $\|T\|_e = \|TP\|_e$, because $P\mathcal{F}^p_\varphi = \mathcal{F}^p_\varphi$. We always assume that $\|TP\|_e > 0$ because $\|TP\|_e = 0$ is trivial. Lemma 3.5 shows there is some $r > 0$ satisfying

$$\left\| TP - \left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}} \right) \right\|_{\mathcal{F}^p_\varphi\to\mathcal{F}^q_\varphi} < \tfrac{1}{2}\|TP\|_e.$$

Because $\sum_{j\le m} M_{\chi_{F_j}} TPM_{\chi_{G_j}}$ is a compact operator where $m \in \mathbb{N}$,

$$\begin{aligned}
\|TP\|_e &\le \left\| TP - \left(\sum_{j=1}^{m} M_{\chi_{F_j}} TPM_{\chi_{G_j}} \right) \right\|_{F^p_\varphi\to F^q_\varphi} \\
&\le \left\| TP - \left(\sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}} \right) \right\|_{F^p_\varphi\to F^q_\varphi} + \|T_m\|_{F^p_\varphi\to F^p_\varphi} \\
&< \tfrac{1}{2}\|TP\|_e + \|T_m\|_{F^p_\varphi\to F^q_\varphi},
\end{aligned}$$

where $T_m = \sum_{j>m} M_{\chi_{F_j}} TPM_{\chi_{G_j}}$. It remains to show that

$$\limsup_{m\to\infty} \|T_m\|_{F^p_\varphi\to F^q_\varphi} \lesssim \limsup_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle Tk_{p,z}, k_{q',w}\rangle|^{1/q} + \tfrac{1}{4}\|TP\|_e. \tag{3-13}$$

Let $f \in F^p_\varphi$ with $\|f\|_{F^p_\varphi} \le 1$. Notice that

$$F_j \cap F_k = \varnothing \quad (j \ne k).$$

Then

$$\begin{aligned}
\|T_m f\|^q_{F^q_\varphi} &= \sum_{j>m} \|M_{\chi_{F_j}} TPM_{\chi_{G_j}} f\|^q_{F^q_\varphi} = \sum_{j>m} \|M_{\chi_{F_j}} Tl_j\|^q_{F^q_\varphi} \|M_{\chi_{G_j}} f\|^q_{F^p_\varphi} \\
&\lesssim (\sup_{j>m} \|M_{\chi_{F_j}} Tl_j\|^q_{F^q_\varphi}) \sum_{j>m} \left(\int_{\mathbb{C}} \chi_{F_j}(z)|f(z)|^p e^{-p\varphi(z)}\, dA(z) \right)^{q/p} \\
&\le (\sup_{j>m} \|M_{\chi_{F_j}} Tl_j\|^q_{F^q_\varphi}) \left(\sum_{j>m} \int_{\mathbb{C}} \chi_{F_j}(z)|f(z)|^p e^{-p\varphi(z)}\, dA(z) \right)^{q/p} \\
&\lesssim \|f\|^q_{F^p_\varphi} (\sup_{j>m} \|M_{\chi_{F_j}} Tl_j\|^q_{F^q_\varphi}),
\end{aligned}$$

where $l_j := PM_{\chi_{G_j}} f / \|M_{\chi_{G_j}} f\|_{F^p_\varphi}$. It follows that

$$\|T_m\|_{F^p_\varphi \to F^q_\varphi} \lesssim \sup_{j>m} \sup_{\|f\|_{F^p_\varphi}\le 1} \left\{ \|M_{\chi_{F_j}} T l_j\|_{F^q_\varphi} : l_j = \frac{PM_{\chi_{G_j}} f}{\|M_{\chi_{G_j}} f\|_{F^p_\varphi}} \right\}.$$

Furthermore, we have

$$\limsup_{m\to\infty} \|T_m\|_{F^p_\varphi \to F^q_\varphi} \lesssim \limsup_{j\to\infty} \sup_{\|f\|_{F^p_\varphi}\le 1} \left\{ \|M_{\chi_{F_j}} T l_j\|_{F^q_\varphi} : l_j = \frac{PM_{\chi_{G_j}} f}{\|M_{\chi_{G_j}} f\|_{F^p_\varphi}} \right\}. \tag{3-14}$$

We can choose a sequence $\{f_j\}$ with $\|f_j\|_{F^p_\varphi} \le 1$ such that

$$\limsup_{j\to\infty} \sup_{\|f\|_{F^p_\varphi}\le 1} \left\{ \|M_{\chi_{F_j}} T l_j\|_{F^p_\varphi} : l_j := \frac{PM_{\chi_{G_j}} f}{\|M_{\chi_{G_j}} f\|_{F^p_\varphi}} \right\} - \tfrac{1}{4}\|TP\|_e \le \limsup_{j\to\infty} \|M_{\chi_{F_j}} T g_j\|_{F^q_\varphi}, \tag{3-15}$$

where $g_j := PM_{\chi_{G_j}} f_j / \|M_{\chi_{G_j}} f_j\|_{F^p_\varphi}$.

By Lemma 3.2, we deduce

$$\begin{aligned} &\|M_{\chi_{F_j}} T g_j\|^q_{F^q_\varphi} \\ &\le \int_{F_j} \left(\int_{G_j} \frac{|f_j(w)|\,|TK_w(z)| e^{-2\varphi(w)}}{\|\chi_{G_j} f_j\|_{F^p_\varphi}}\, dA(w) \right)^q e^{-q\varphi(z)}\, dA(z) \\ &\lesssim \frac{1}{\|\chi_{G_j} f_j\|^q_{F^p_\varphi}} \int_{F_j} \left(\int_{G_j} |f_j(w)| e^{-2\varphi(w)} |\langle TK_w, K_z\rangle|\, dA(w) \right)^q e^{-q\varphi(z)}\, dA(z) \\ &\lesssim C_1 C_2^{q/q'} \frac{\|\chi_{G_j} f_j\|^{q-p}_{F^p_\varphi} \|\chi_{G_j} f_j\|^p_{F^p_\varphi}}{\|\chi_{G_j} f_j\|^q_{F^p_\varphi}} \\ &= C_1 C_2^{q/q'}, \end{aligned} \tag{3-16}$$

where

$$C_1 = \sup_{w\in G_j} \int_{F_j} |\langle Tk_{p,w}, k_{q',z}\rangle|\, d\sigma(z),$$

$$C_2 = \sup_{z\in F_j} \int_{G_j} |\langle T^* k_{q',z}, k_{p,w}\rangle|\, d\sigma(w) < \infty.$$

For each j, there exists $z_j \in F_j \subset G_j$ such that $G_j \subset D^{br}(z_j)$ for some $b = b(r) > 0$ by Lemma 2.1(2). We can choose that $u_j, t_j \in D^{br}(z_j)$ such that

$$|\langle Tk_{p,t_j}, k_{q',u_j}\rangle| = \sup_{z,w\in D^{br}(z_j)} |\langle Tk_{p,w}, k_{q',z}\rangle|.$$

There is $k > 0$ such that $w \in D^{kbr}(u_j)$ for $w \in G_j$. It is easy to prove that

$$
\begin{aligned}
C_1 &= \sup_{w\in G_j} \int_{F_j} |\langle T k_{p,w}, k_{q',z}\rangle|\, d\sigma(z) \\
&\le \sup_{w\in G_j} \int_{G_j} |\langle T k_{p,w}, k_{q',z}\rangle|\, d\sigma(z) \\
&\lesssim |\langle T k_{p,t_j}, k_{q',u_j}\rangle| \\
&\lesssim \sup_{w\in D^{kbr}(u_j)} |\langle T k_{p,w}, k_{q',u_j}\rangle|.
\end{aligned} \tag{3-17}
$$

Since F_j is a covering, there is a subsequence $\{u_{m_j}\}$ such that $u_{m_j} \to \infty$ as $j \to \infty$. Therefore, by (3-16) and (3-17), we immediately obtain

$$
\limsup_{j\to\infty} \|M_{\chi_{F_j}} T g_j\|^q_{F^q_\varphi} \lesssim \limsup_{|z|\to\infty} \sup_{w\in D^{kbr}(z)} |\langle T k_{p,w}, k_{q',z}\rangle|.
$$

By the above inequality, (3-14) and (3-15), we have (3-13) holds. □

Theorem 3.7. *For $1 < p \le q < \infty$ or $p = q = 1$, if $T \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$, then the following statements are equivalent*:

(1) $T \in \mathcal{K}(F^p_\varphi, F^q_\varphi)$.

(2) $\lim_{|z|\to\infty} \|T k_{p,z}\|_{F^q_\varphi} = 0$.

(3) $\lim_{|z|\to\infty} \sup_{w\in\mathbb{C}} |\langle T k_{p,z}, k_{q',w}\rangle| = 0$.

(4) $\lim_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle T k_{p,z}, k_{q',w}\rangle| = 0$ *for any* $r > 0$.

Proof. Assume that (1) holds. Lv and Wang [2024, Lemma 2.1(4)] showed that if $k_{p,z} \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $z \to \infty$, then

$$
\lim_{|z|\to\infty} \|T k_{p,z}\|_{F^q_\varphi} = 0.
$$

This proves (1) $\Rightarrow$ (2). Assume that (2) holds. If $1 < p \le q < \infty$, by Lemmas 2.3 and 2.4, we obtain

$$
\begin{aligned}
|\langle T k_{p,z}, k_{q',w}\rangle|^q &= \frac{1}{\|K_w\|^q_{L^{q'}_\varphi}} |\langle T k_{p,z}, K_w\rangle|^q \\
&\lesssim \rho(w)^2 e^{-q\varphi(w)} |\langle T k_{p,z}, K_w\rangle|^q \\
&\lesssim \int_{D^r(w)} |\langle T k_{p,z}, K_u\rangle|^q e^{-q\varphi(u)}\, dA(u) \\
&\le \|T k_{p,z}\|^q_{F^q_\varphi}.
\end{aligned}
$$

If $p = q = 1$, it is easy to check that

$$
|\langle T k_{1,z}, K_{\infty,w}\rangle| \le \|T k_{1,z}\|_{F^1_\varphi}.
$$

This shows that (2) $\Rightarrow$ (3). It is trivial to show that (3) $\Rightarrow$ (4). It remains to show that (4) $\Rightarrow$ (1). Assume that (4) holds. By Theorem 3.6, there exist positive constants r and $C = C(r)$ such that

$$\|T\|_e \le C \limsup_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle T k_{z,p}, k_{q',w}\rangle|^{1/q}.$$

The condition (4) implies that $\|T\|_e = 0$, or, equivalently, that $T \in \mathcal{K}(F^p_\varphi, F^q_\varphi)$. $\square$

4. Toeplitz operators with BMO^s or IMO^s symbols

In this section we give characterizations of Toeplitz operators with BMO^s or IMO^s symbols. This section is divided into two parts: the first considers Toeplitz operators with BMO^s symbols and the second considers those with IMO^s symbols.

4.1. ***Toeplitz operators with* BMO^s *symbols.*** As an application of the technique of weakly localized operators, we characterize the boundedness and compactness of Toeplitz operators with BMO^s symbols from F^p_φ to F^q_φ when $1 < p \le q < \infty$ or $p = q = 1$. In particular, we show that Toeplitz operators with BA symbols form a subalgebra of $\mathfrak{B}(F^p_\varphi)$ for $1 \le p < \infty$. We also give a characterization of the compactness of operators in this subalgebra. In order to do this, we will start with some notation and auxiliary function spaces.

Recall that for a given $s \in \mathbb{R}$, we say that $f \in \mathrm{BMO}^s$ if $f \in \mathcal{S}$ and

$$\|f\|_{\mathrm{BMO}^s} := \sup_{z\in\mathbb{C}} \rho(z)^s \,\mathrm{MO}(f)(z) < \infty,$$

where

$$\mathrm{MO}(f)(z) = \int_{\mathbb{C}} |f(w) - \tilde{f}(z)|\,|k_z(w)|^2 e^{-2\varphi(w)}\,dA(w).$$

For a continuous function f on $\mathbb{C}$, we define the oscillation of f at $z \in \mathbb{C}$ in the Bergman metric by

$$\omega_r(f)(z) = \sup\{|f(z) - f(w)| : w \in D^r(z)\}.$$

We say that $f \in \mathrm{BO}^s_r$ if the function $\rho^s \omega_r(f)$ is bounded. We now define the mean oscillation of f at $z \in \mathbb{C}$ by

$$\mathrm{MO}_r(f)(z) = \frac{1}{|D^r(z)|}\int_{D^r(z)} |f(w) - \hat{f}_r(z)|\,dA(w),$$

where

$$\hat{f}_r(z) = \frac{1}{|D^r(z)|}\int_{D^r(z)} f(w)\,dA(w),$$

and $|D^r(z)|$ is the Euclidean area of $D^r(z)$. We say that $f \in \mathrm{BMO}^s_r$ if f is a locally Lebesgue integrable function and $\rho(\cdot)^s\,\mathrm{MO}_r(f)$ is bounded on $\mathbb{C}$. Let BA^s_r denote the space of locally Lebesgue integrable functions f on $\mathbb{C}$ such that $\rho(\cdot)^s \widehat{|f|}_r$

is bounded. We know that BO_r^s and BA_r^s are independent of r when $s \le 0$; see [Lv and Wang 2024, Theorem 4.1]. We write BO^s for BO_r^s and BA^s for BA_r^s for convenience. In particular, when $s = 0$, we write BO for BO^0 and BA for BA^0.

Lemma 4.1. *Let $r > 0$, $s \le 0$ and $f \in \Gamma$. Then the following statements are equivalent*:

(1) $f \in \mathrm{BMO}_r^s$ *for some (or any)* $r > 0$.

(2) $f = f_1 + f_2$, *where* $f_1 \in \mathrm{BO}^s \cap C^2(\mathbb{C})$ *and* $f_2 \in \mathrm{BA}^s$.

(3) $f \in \mathrm{BMO}^s$.

Moreover,

$$\|f\|_{\mathrm{BMO}^s} \simeq (\|f_1\|_{\mathrm{BO}^s} + \|f_1\|_{\mathrm{BA}^s}) \simeq \|f\|_{\mathrm{BMO}_r^s}.$$

Proof. The equivalence of (1) and (2) comes from [Lv and Wang 2024, Theorem 4.1]. We only need to show that (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). Assume that (2) holds. Assume $f \in \mathrm{BA}^s$, that is, $\rho^s|\widehat{f}|$ is bounded on $\mathbb{C}$. Through appropriate modifications to [Hu and Lv 2017, Lemma 2.5], it is easy to show that $\rho^s|\widehat{f}|$ is bounded if and only if $\rho^s|\widetilde{f}|$ is bounded. Therefore, for every $z \in \mathbb{C}$,

$$\rho(z)^s\bigg(\int_{\mathbb{C}} |f(w) - \tilde{f}(z)|\,|k_{2,z}(w)|^2 e^{-2\varphi(w)}\,dA(w)\bigg) \le 2\rho(z)^s|\widetilde{f}|(z) < \infty.$$

This implies that $f \in \mathrm{BMO}^s$ when $f \in \mathrm{BA}^s$. We assume that $f \in \mathrm{BO}^s$, that is, the function $\rho^s\omega_r(f)$ is bounded. For any given $z \in \mathbb{C}$, similar to the proof of [Zhu 2007, Lemma 8.2], we have

$$\rho(z)^s(|f(z) - f(w)|) \lesssim \|f\|_{\mathrm{BO}^s}(\beta(z,w) + 1). \tag{4-1}$$

From this and (2-2), we obtain

$$\begin{aligned}
&\rho(z)^s\bigg(\int_{\mathbb{C}} |f(w) - \tilde{f}(z)|\,|k_{2,z}(w)|^2 e^{-2\varphi(w)}\,dA(w)\bigg)\\
&\quad\le \rho(z)^s\bigg(\int_{\mathbb{C}} |f(w) - f(z)|\,|k_{2,z}(w)|^2 e^{-2\varphi(w)}\,dA(w) + |f(z) - \tilde{f}(z)|\bigg)\\
&\quad\le 2\int_{\mathbb{C}} \rho(z)^s|f(w) - f(z)|\,|k_{2,z}(w)|^2 e^{-2\varphi(w)}\,dA(w)\\
&\quad\lesssim \int_{\mathbb{C}} (\beta(z,w) + 1)|k_{2,z}(w)|^2 e^{-2\varphi(w)}\,dA(w)\\
&\quad\lesssim 1.
\end{aligned}$$

This shows that $f \in \mathrm{BMO}^s$ when $f \in \mathrm{BO}^s$. This proves that (2) $\Rightarrow$ (3).

We now suppose that (3) holds. For any fixed $z \in \mathbb{C}$,

$$\begin{aligned}
&\rho(z)^s \,\mathrm{MO}_r(f)(z) \\
&\quad = \rho(z)^s \left(\frac{1}{|D^r(z)|} \int_{D^r(z)} |f(w) - \hat{f}_r(z)| \, dA(w) \right) \\
&\quad \lesssim \rho(z)^s \left(\int_{D^r(z)} |f(w) - \hat{f}_r(z)| \, |k_{2,z}(w)|^2 e^{-2\varphi(w)} \, dA(w) \right) \\
&\quad \lesssim \rho(z)^s \left(\int_{\mathbb{C}} |f(w) - \hat{f}_r(z)| \, |k_{2,z}(w)|^2 e^{-2\varphi(w)} \, dA(w) \right) \\
&\quad \lesssim \rho(z)^s \left(\int_{\mathbb{C}} |f(w) - \tilde{f}(z)| \, |k_{2,z}(w)|^2 e^{-2\varphi(w)} \, dA(w) + |\tilde{f}(z) - \widehat{f_r}(z)| \right) \\
&\quad \le 2\rho(z)^s \left(\int_{\mathbb{C}} |f(w) - \tilde{f}(z)| \, |k_{2,z}(w)|^2 e^{-2\varphi(w)} \, dA(w) \right),
\end{aligned}$$

which implies that $f \in \mathrm{BMO}^s_r$, when $f \in \mathrm{BMO}^s$. □

In order to characterize the boundedness and compactness of Toeplitz operators with BMO^s symbols from F^p_φ to F^q_φ, we first prove that Toeplitz operators with BMO^s symbols are well defined.

Lemma 4.2. *For $s \le 0$ and $f \in \mathcal{S} \cap \mathrm{BMO}^s$, the Toeplitz operator T_f is well defined on Γ. Moreover,*

$$T_f g(w) = \langle T_f g, K_w \rangle = \int_{\mathbb{C}} f(z) g(z) K(w,z) e^{-2\varphi(z)} \, dA(z), \quad g \in \Gamma, \, w \in \mathbb{C}.$$

Proof. For $f \in \mathcal{S}$, we only need to prove that for every $z, w \in \mathbb{C}$, $|\langle T_f K_z, K_w \rangle|$ is finite and $T_f K_z(w) = \langle T_f K_z, K_w \rangle$. Notice that

$$\begin{aligned}
\langle T_f K_z, K_w \rangle &= \int_{\mathbb{C}} T_f K_z(u) \overline{K_w(u)} e^{-2\varphi(u)} \, dA(u) \\
&= \int_{\mathbb{C}} \int_{\mathbb{C}} f(\xi) K_z(\xi) \overline{K_u(\xi)} e^{-2\varphi(\xi)} \, dA(\xi) \overline{K_w(u)} e^{-2\varphi(u)} \, dA(u).
\end{aligned} \tag{4-2}$$

We now check that

$$\begin{aligned}
&|\langle T_f K_z, K_w \rangle| \\
&\quad \le \int_{\mathbb{C}} \int_{\mathbb{C}} |f(\xi) K_z(\xi) K_u(\xi) e^{-2\varphi(\xi)} K_w(u) e^{-2\varphi(u)}| \, dA(u) \, dA(\xi) < \infty.
\end{aligned} \tag{4-3}$$

To this end, Lemma 2.3 and the definition of F^∞_φ show that

$$\int_{\mathbb{C}} |K_u(\xi) e^{-2\varphi(u)} K_w(u)| \, dA(u) \lesssim \|K_w\|_{F^\infty_\varphi} e^{\varphi(\xi)}. \tag{4-4}$$

From Lemma 4.1 we have the decomposition $\mathrm{BMO}^s = \mathrm{BA}^s + \mathrm{BO}^s$. If $f \in \mathrm{BA}^s$, then by Lemma 2.5, (4-4) and (2-2), we get

$$\begin{aligned}(4\text{-}5)\quad \int_{\mathbb{C}}\int_{\mathbb{C}} |f(\xi)K_z(\xi)K_u(\xi)e^{-2\varphi(\xi)}K_w(u)e^{-2\varphi(u)}|\,dA(u)\,dA(\xi)&\\ \lesssim \|K_w\|_{F^\infty_\varphi}\int_{\mathbb{C}} |f(\xi)K_z(\xi)e^{-\varphi(\xi)}|\,dA(\xi)&\\ \lesssim \|K_w\|_{F^\infty_\varphi}\int_{\mathbb{C}} |K_z(\xi)e^{-\varphi(\xi)}\rho(\xi)^{-s}\rho(\xi)^s\widehat{|f|}_r(\xi)\,dA(\xi)&\\ \lesssim \rho(z)^{-s}e^{\varphi(z)}\|K_w\|_{F^\infty_\varphi}\|f\|_{\mathrm{BA}^s} < \infty.&\end{aligned}$$

If $f \in \mathrm{BO}^s$, then by (4-1), (4-4) and (2-2) again, we have

$$\begin{aligned}(4\text{-}6)\quad \int_{\mathbb{C}}\int_{\mathbb{C}} |f(\xi)K_z(\xi)K_u(\xi)e^{-2\varphi(\xi)}K_w(u)e^{-2\varphi(u)}|\,dA(u)\,dA(\xi)&\\ \lesssim \|K_w\|_{F^\infty_\varphi}\int_{\mathbb{C}} |f(\xi)K_z(\xi)e^{-\varphi(\xi)}|\,dA(\xi)&\\ \le \|K_w\|_{F^\infty_\varphi}\int_{\mathbb{C}} (|f(z)| + |f(z)-f(\xi)|)|K_z(\xi)e^{-\varphi(\xi)}\,dA(\xi)&\\ \lesssim |f(z)|\|K_w\|_{F^\infty_\varphi}\|K_z\|_{F^1_\varphi} + \rho(z)^{-s}\|K_w\|_{F^\infty_\varphi}\|f\|_{\mathrm{BO}^s}&\\ \times\int_{\mathbb{C}}(\beta(z,\xi)+1)|K_z(\xi)e^{-\varphi(\xi)}\,dA(\xi)&\\ \lesssim |f(z)|\|K_w\|_{F^\infty_\varphi}\|K_z\|_{F^1_\varphi} + \rho(z)^{-s}\|K_w\|_{F^\infty_\varphi}\|f\|_{\mathrm{BO}^s}e^{\varphi(z)} < \infty.&\end{aligned}$$

We immediately get (4-3) by combining the conditions (4-5) and (4-6).

Finally, by Fubini's theorem and the property of the reproducing kernel in (4-2), we have $\langle T_f K_z, K_w\rangle = T_f K_z(w)$. □

Let $1 \le p, q < \infty$ and let $s = \frac{1}{q} - \frac{1}{p}$ in this section.

Lemma 4.3. *Let $1 < p \le q < \infty$ or $p = q = 1$. For $f \in \mathcal{S}$, if $f \in \mathrm{BMO}^{2s}$ with*

$$\sup_{z\in\mathbb{C}} \rho(z)^{2s}|\tilde{f}(z)| < \infty,$$

then $T_f \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$.

Proof. First note that by Lemma 4.2, T_f is well defined on Γ. It can easily be shown that

$$\widehat{|f|}(z) - |\tilde{f}(z)| = \int_{\mathbb{C}} (|f(w)| - |\tilde{f}(z)|)|k_{2,z}(w)|^2 e^{-2\varphi(w)}\,dA(w) \le \mathrm{MO}(f)(z)$$

since $k_{2,z}$ is a normalized reproducing kernel. Therefore, by our assumption,

$$\sup_{z\in\mathbb{C}} \rho(z)^{2s}|\tilde{f}|(z) < \infty.$$

From this, Lemma 2.5 and $|\widehat{f}|_r(u) \le |\widetilde{f}|_r(u)$ for all $u \in \mathbb{C}$, $r > 0$, we deduce that

$$\begin{aligned}|\langle T_f k_{p,z}, k_{q',w}\rangle| &\le \int_{\mathbb{C}} |k_{p,z}(u)|\,|k_{q',w}(u)|\,|f(u)|e^{-2\varphi(u)}\,dA(u)\\ &= \int_{\mathbb{C}} |k_{p,z}(u)|\,|k_{q',w}(u)|\,|\widetilde{f}|(u)e^{-2\varphi(u)}\,dA(u).\end{aligned}$$

Therefore, by the above inequality, Lemma 2.3, Fubini's theorem and (2-2), we have

$$\begin{aligned}&\int_{\mathbb{C}} |\langle T_f k_{p,z}, k_{q',w}\rangle|\,dA\sigma(w)\\ &\qquad\le \int_{\mathbb{C}}\int_{\mathbb{C}} |k_{p,z}(u)|\,|k_{q',w}(u)|\,|\widetilde{f}|(u)e^{-2\varphi(u)}\,dA(u)\,dA\sigma(w)\\ &\qquad= \int_{\mathbb{C}} |k_{p,z}(u)|\,|\widetilde{f}|(u)e^{-2\varphi(u)}\,dA(u)\int_{\mathbb{C}} |k_{q',w}(u)|\,dA\sigma(w)\\ &\qquad= \int_{\mathbb{C}} |k_{p,z}(u)|\,|\widetilde{f}|(u)e^{-\varphi(u)}\rho(u)^{-2/q'}\,dA(u)\\ &\qquad\lesssim \sup_{u\in\mathbb{C}} \rho(u)^{2s}\,|\widetilde{f}|(u)\int_{\mathbb{C}} |k_{p,z}(u)|e^{-\varphi(u)}\rho(u)^{2/p-2}\,dA(u)\\ &\qquad\lesssim \sup_{u\in\mathbb{C}} \rho(u)^{2s}|\widetilde{f}|(u) < \infty.\end{aligned}$$

Therefore,

$$\sup_{z\in\mathbb{C}}\int_{\mathbb{C}} |\langle T_f k_{p,z}, k_{q',w}\rangle|\,dA\sigma(w) < \infty.$$

This shows that T_f satisfies the condition (3-2).

To prove T_f satisfies the condition (3-4), we only need to replace (2-2) by (2-3). The conditions (3-3) and (3-5) for the Toeplitz operator T_f can be shown to hold in the same way. Hence, $T_f \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$. □

We immediately get the following corollary, whose proof is similar to that in [Hu et al. 2018, Theorem 3.2].

Corollary 4.4. *For $p \ge 1$ and $f \in \mathcal{S}$, if $f \in$ BMO with $\sup_{z\in\mathbb{C}} |\tilde{f}(z)| < \infty$, then $T_f \in \mathcal{W}(F^p_\varphi, F^p_\varphi)$.*

Corollary 4.5. *Let $1 \le p < \infty$ and let $f \in \mathcal{S}$. If $\sup_{z\in\mathbb{C}} |\widetilde{f}|(z) < \infty$, then $T_f \in \mathcal{W}(F^p_\varphi, F^p_\varphi)$. Furthermore,*

$$\Big\{T_f : f \in \mathcal{S} \text{ and } \sup_{z\in\mathbb{C}} |\widetilde{f}|(z) < \infty\Big\}$$

forms a subalgebra of $\mathfrak{B}(F^p_\varphi)$.

Proof. As an application of [Hu and Lv 2017, Lemma 2.5], it is easy to show that $\sup_{z\in\mathbb{C}} |\widetilde{f}|(z) < \infty$ if and only if $f \in$ BA. Lemma 4.1 implies that BA $\subset$ BMO and

Lemma 4.2 shows that T_f is well defined on Γ. It follows that $T_f \in \mathcal{W}(F^p_\varphi, F^p_\varphi)$ by Lemma 4.3 when $\sup_{z\in\mathbb{C}} |\widetilde{f}|(z) < \infty$.

Therefore,

$$\left\{T_f : f \in \mathcal{S} \text{ and } \sup_{z\in\mathbb{C}} |\widetilde{f}|(z) < \infty\right\}$$

forms a subalgebra of $\mathfrak{B}(F^p_\varphi)$ by Theorem 3.4. □

As an application of Corollary 4.5 and Theorem 3.7, we immediately obtain the following corollary.

Corollary 4.6. *Let $1 \le p < \infty$ and $f \in \mathcal{S}$. If*

$$T \in \{T_f : f \in \mathcal{S} \text{ and } \sup_{z\in\mathbb{C}} |\widetilde{f}|(z) < \infty\},$$

then the following conditions are equivalent:

(1) $T \in \mathcal{K}(F^p_\varphi)$.

(2) $\lim_{|z|\to\infty} \|Tk_{p,z}\|_{F^p_\varphi} = 0$.

(3) $\lim_{|z|\to\infty} \sup_{w\in\mathbb{C}} |\langle Tk_{p,z}, k_{p',w}\rangle| = 0$.

(4) $\lim_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle Tk_{p,z}, k_{p',w}\rangle| = 0$ *for any* $r > 0$.

We are going to give the most important theorem of this section.

Theorem 4.7. *Let $1 < p \le q < \infty$ or $p = q = 1$, and $f \in \mathcal{S}$. If $f \in \mathrm{BMO}^{2s}$, then*:

(1) $T_f \in \mathfrak{B}(F^p_\varphi, F^q_\varphi)$ *if and only if*

$$\sup_{z\in\mathbb{C}} \rho(z)^{2s} |\tilde{f}(z)| < \infty. \tag{4-7}$$

(2) $T_f \in \mathcal{K}(F^p_\varphi, F^q_\varphi)$ *if and only if for any* $r > 0$,

$$\lim_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle Tk_{p,z}, k_{q',w}\rangle| = 0. \tag{4-8}$$

Proof. (1) Suppose that (4-7) holds. From Lemma 4.3 and Theorem 3.3, it follows that $T_f \in \mathfrak{B}(F^p_\varphi, F^q_\varphi)$.

Conversely, assume that $T_f \in \mathfrak{B}(F^p_\varphi, F^q_\varphi)$. If $1 < p \le q < \infty$, then by Lemma 2.3,

$$\begin{aligned} \rho(z)^{2s} \tilde{f}(z) &= \rho(z)^{2s} \langle T_f k_{2,z}, k_{2,z}\rangle \\ &= \rho(z)^{2s} \|K_z\|^{-2}_{F^2_\varphi} \int_{\mathbb{C}} f(u) |K_z(u)|^2 e^{-2\varphi(u)}\, dA(u) \\ &\simeq \rho(z)^{2s} \rho(z)^{-2s} \|K_z\|^{-1}_{F^p_\varphi} \|K_z\|^{-1}_{F^{q'}_\varphi} \langle T_f K_z, K_z\rangle \\ &\simeq \langle T_f k_{p,z}, k_{q',z}\rangle. \end{aligned} \tag{4-9}$$

From this and Hölder's inequality, we have

$$\rho(z)^{2s}|\tilde{f}(z)| \simeq |\langle T_f k_{p,z}, k_{q',z}\rangle| \lesssim \|T_f k_{p,z}\|_{F^p_\varphi}\|k_{q',z}\|_{F^{q'}_\varphi} < \infty.$$

If $p = q = 1$, then by Lemma 2.3 again, we obtain

$$\begin{aligned}\tilde{f}(z) &= \langle T_f k_{2,z}, k_{2,z}\rangle \\ &= \|K_z\|^{-2}_{F^2_\varphi}\int_{\mathbb{C}} f(u)|K_z(u)|^2 e^{-2\varphi(u)}\,dA(u) \\ &\simeq \|K_z\|^{-1}_{F^1_\varphi}\|K_z\|^{-1}_{F^\infty_\varphi}\langle T_f K_z, K_z\rangle \\ &= \langle T_f k_{1,z}, k_{\infty,z}\rangle.\end{aligned}$$

It follows that

$$|\tilde{f}(z)| \simeq |\langle T_f k_{1,z}, k_{\infty,z}\rangle| \le \|T_f k_{1,z}\|_{F^1_\varphi} < \infty.$$

Hence, $T_f \in \mathfrak{B}(F^p_\varphi, F^q_\varphi)$ implies that (4-7) holds.

(2) Suppose that $T_f \in \mathcal{K}(F^p_\varphi, F^q_\varphi)$. By (4-7) and Lemma 4.3, we obtain

$$T_f \in \mathcal{W}(F^p_\varphi, F^q_\varphi).$$

Furthermore,

$$\lim_{|z|\to\infty}\sup_{w\in D^r(z)}|\langle T k_{p,z}, k_{q',w}\rangle| = 0$$

by Theorem 3.7.

Assume that (4-8) holds. By Theorem 3.7, we have

$$\lim_{|z|\to\infty}\sup_{w\in\mathbb{C}}|\langle T k_{p,z}, k_{q',w}\rangle| = 0.$$

In particular,

$$\lim_{|z|\to\infty}|\langle T k_{p,z}, k_{q',z}\rangle| = 0.$$

From this and (4-9), we have $\rho(z)^{2s}\tilde{f}(z)$ is bounded and by Lemma 4.3, we have $T_f \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$. By Theorem 3.7 again and our assumption, we obtain $T_f \in \mathcal{K}(F^p_\varphi, F^q_\varphi)$. □

Corollary 4.8. *Let $1 \le p < \infty$ and $f \in \mathrm{BMO} \cap \mathcal{S}$. Then*:

(1) *$T_f \in \mathfrak{B}(F^p_\varphi)$ if and only if the Berezin transform $\tilde{f}$ is bounded.*

(2) *$T_f \in \mathcal{K}(F^p_\varphi)$ if and only if for any $r > 0$,*

$$\lim_{|z|\to\infty}\sup_{w\in D^r(z)}|\langle T k_{p,z}, k_{q',w}\rangle| = 0.$$

Proof. It is easy to get this conclusion by using Theorem 4.7 and Corollary 4.4. □

4.2. ***Toeplitz operators with* IMO^s *symbols.*** For $s > 0$, the space IMO^s consists of $f \in \Gamma$ such that

$$\|f\|_{\mathrm{IMO}^s} = \left(\int_{\mathbb{C}} |\mathrm{MO}(f)(z)|^s \, dA(z) \right)^{1/s} < \infty.$$

Let $1 \le q < p < \infty$. We will characterize the boundedness and compactness of Toeplitz operators T_f with IMO^s from F^p_φ to F^q_φ in this section. We need to use Luecking's approach [1993] based on Khinchine's inequality. Recall that Rademacher functions R_n are defined by

$$R_0(t) = \begin{cases} 1 & \text{if } 0 \le t - [t] < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \le t - [t] < 1; \end{cases} \qquad R_n(z^n t), \quad n \ge 1,$$

where $[t]$ denotes the largest integer not exceeding t.

Lemma 4.9 (Khinchine's inequality). *Let $0 < p < \infty$. There exists a positive constant C_p such that*

$$C_p^{-1} \left(\sum_{k=1}^n |\lambda_k|^2 \right)^{p/2} \lesssim \int_0^1 \left| \sum_{k=1}^n \lambda_k R_k(t) \right|^p dt \lesssim C_p \left(\sum_{k=1}^n |\lambda_k|^2 \right)^{p/2}$$

for all $n \in \mathbb{N}$ and sequences $\{\lambda_k\}$ in $\mathbb{C}$.

We will give the main result of this section.

Theorem 4.10. *Let $1 \le q < p < \infty$ and let $s = \frac{pq}{p-q}$. Then for $f \in \mathrm{IMO}^s$, the following statements are equivalent*:

(1) *T_f is bounded from F^p_φ to F^q_φ.*

(2) *T_f is compact from F^p_φ to F^q_φ.*

(3) *The Berezin transform $\tilde{f} \in L^s(\mathbb{C}, dA)$.*

Proof. (2) $\Rightarrow$ (1) is trivial. Suppose that (1) holds. Let $r > 0$ and $\{z_j\}_{j=1}^\infty$ be a sequence such that $\{D^r(z_j)\}_{j=1}^\infty$ is a cover of $\mathbb{C}$. Then for $1 < p < \infty$, by [Hu and Lv 2017, Lemma 2.6], we get the function

$$g_t(z) := \sum_{j=1}^\infty \gamma_j r_j(t) k_{2,z_j}(z) \rho(z_j)^{1-2/p} = \sum_{j=1}^\infty \gamma_j r_j(t) k_{p,z_j}(z)$$

belongs to F^p_φ and

$$\|g_t\|_{F^p_\varphi} \lesssim \|\{\gamma_j\}_{j=1}^\infty\|_{\ell^p}.$$

Thus, by our assumption, we have

$$\|T_f g_t\|_{F^q_\varphi} \lesssim \|\{\gamma_j\}_{j=1}^\infty\|_{\ell^p}. \tag{4-10}$$

Fubini's theorem and Khinchine's inequality yield that

$$\begin{aligned}(4\text{-}11)\quad \int_0^1 \|T_f g_t\|_{F^q_\varphi}^q\, dt &\gtrsim \int_{\mathbb{C}} \Big(\sum_{j=1}^{\infty} |\gamma_j T_f k_{p,z_j}(z)|^2\Big)^{q/2} e^{-q\varphi(z)}\, dA(z)\\ &\gtrsim \sum_{i=1}^{\infty} \int_{D^r(z_i)} \Big(\sum_{j=1}^{\infty} |\gamma_j T_f k_{p,z_j}(z)|^2\Big)^{q/2} e^{-q\varphi(z)}\, dA(z)\\ &\gtrsim \sum_{i=1}^{\infty} \gamma_i^q \int_{D^r(z_i)} |T_f k_{p,z_i}(z)|^q e^{-q\varphi(z)}\, dA(z)\\ &\gtrsim \sum_{i=1}^{\infty} \gamma_i^q |T_f k_{p,z_i}(z_i)|^q e^{-q\varphi(z)} \rho(z_i)^2,\end{aligned}$$

where the last inequality comes from Lemma 2.4. Using the property of reproducing kernel and Lemma 2.3, we obtain

$$(4\text{-}12)\qquad |T_f k_{p,z_i}(z_i)| e^{-\varphi(z)} \simeq |\tilde{f}(z_i)| \rho(z_i)^{-2/p}.$$

Combining with (4-10), (4-11) and (4-12), we have

$$\sum_{i=1}^{\infty} \gamma_i^q |\tilde{f}(z_i)|^q \rho(z_i)^{2-2q/p} \lesssim \int_0^1 \|T_f g_t\|_{F^q_\varphi}^q\, dt \lesssim \|\{\gamma_j\}_{j=1}^{\infty}\|_{\ell^p}^q = \|\{|\gamma_j|^q\}_{j=1}^{\infty}\|_{\ell^{q/p}}.$$

Since the duality of $\ell^{p/q}$ is $\ell^{p/(p-q)}$, we have

$$|\tilde{f}(z_i)|^q \rho(z_i)^{2-2q/p} \in \ell^{p/(p-q)},$$

that is,

$$\sum_{i=1}^{\infty} |\tilde{f}(z_i)|^s \rho(z_i)^2 < \infty.$$

Choose a point $a_i \in \overline{D^r(z_i)}$ so that the condition

$$|\tilde{f}(a_i)| = \sup_{z \in \overline{D^r(z_i)}} |\tilde{f}(z)|$$

is satisfied and there is a constant $b > 0$ such that $\{D^{br}(a_i)\}$ is a (ρ, br)-lattice. Similar to proving the above, we obtain

$$(4\text{-}13)\qquad \sum_{i=1}^{\infty} |\tilde{f}(a_i)|^s \rho(a_i)^2 < \infty.$$

Therefore, by (4-13),

$$\int_{\mathbb{C}} |\tilde{f}(z)|^s\, dA(z) \le \sum_{i=1}^{\infty} \int_{D^r(z_i)} |\tilde{f}(z)|^s\, dA(z) \lesssim \sum_{i=1}^{\infty} |\tilde{f}(a_i)|^s \rho(a_i)^2 < \infty.$$

This proves (1) $\Rightarrow$ (3).

We suppose that (3) holds. It is easy to show that

$$\widetilde{|f|}(z) - |\tilde{f}(z)| = \int_{\mathbb{C}} (|f(w)| - |\tilde{f}(z)|)|k_{2,z}(w)|^2 e^{-2\varphi(w)}\, dA(w) \le \mathrm{MO}(f)(z),$$

which implies that $\widetilde{|f|} \in L^s(\mathbb{C}, dA)$. As an application of [Hu and Lv 2017, Thoerem 3.4], $T_{|f|}$ is compact from F^p_φ to F^q_φ if and only if $\widetilde{|f|} \in L^s(\mathbb{C}, dA)$. Hence, T_f is also compact from F^p_φ to F^q_φ. □

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CHUNXU XU
SCHOOL OF SCIENCE
NANJING FORESTRY UNIVERSITY
NANJING
CHINA
1968385450@qq.com

HONG ZOU
SCHOOL OF MATHEMATICS AND COMPUTATIONAL SCIENCE
XIANGTAN UNIVERSITY
XIANGTAN
CHINA
hongzou@xtu.edu.cn

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CORRECTION TO THE ARTICLE TOPOLOGICAL INVARIANCE OF QUANTUM QUATERNION SPHERES

BIPUL SAURABH

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We previously explicitly described all homogeneous extensions ϕ_m, $m \in \mathbb{Z}$, of $C(S_0^{2\ell+1})$ by $C(\mathbb{T}) \otimes \mathcal{K}$ (*Pacific J. Math.* 288:2 (2017), 435–452). However, for $m = 0$, the given extension is not homogeneous. In this corrigendum, we correct this mistake by constructing a trivial homogeneous extension of $C(S_0^{2\ell+1})$ by $C(\mathbb{T}) \otimes \mathcal{K}$. The K_0-groups of the middle C^*-algebra turn out to be the same as those mentioned in the original article. Consequently, the proofs of all results in that paper remain valid.

1. Correction

For $m \in \mathbb{Z}$, define the $*$-homomorphisms ϕ_m as

$$\begin{aligned}
\phi_m : C(S_0^{2\ell+1}) &\to Q\big(\mathcal{K}(\ell^2(\mathbb{N})^{\otimes(\ell+1)} \otimes C(\mathbb{T}))\big),\\
S^* \otimes 1 \otimes \cdots \otimes 1 &\mapsto S^* \otimes 1 \otimes \cdots \otimes 1,\\
p \otimes S^* \otimes 1 \otimes \cdots \otimes 1 &\mapsto p \otimes S^* \otimes 1 \otimes \cdots \otimes 1,\\
&\vdots\\
p \otimes p \otimes \cdots \otimes p \otimes S^* \otimes 1 &\mapsto p \otimes p \otimes \cdots \otimes S^* \otimes 1 \otimes 1,\\
p \otimes p \otimes \cdots \otimes t &\mapsto p \otimes p \otimes \cdots \otimes p \otimes (S^*)^m \otimes 1.
\end{aligned}$$

In [Saurabh 2017], it is claimed that the ϕ_m's are essential unital extensions. Moreover, since the last component is 1, these extensions are also homogeneous. For $m \neq 0$, the claim is indeed true. However, for $m = 0$, the claim does not hold. To see this, note that

$$\phi_0(p^{\otimes\ell} \otimes t) = p^{\otimes\ell} \otimes 1 \otimes 1 = \phi_0(p^{\otimes\ell} \otimes 1).$$

This proves that ϕ_0 is not injective; hence, it is not an essential extension and, consequently, not a homogeneous extension.

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To rectify this, we need to construct a trivial homogeneous extension of $C(S_0^{2\ell+1})$ by $C(\mathbb{T})\otimes\mathcal{K}$. First, recall from [Saurabh 2017], the construction of a homogeneous extension $\Sigma^2\tau$ of $\Sigma^2 A$ induced from a homogeneous extension τ of A. Let $\mathcal{H}=\ell^2(\mathbb{N})$. Let τ be a unital homogeneous extension of A by $C(Y)\otimes\mathcal{K}(\mathcal{H})$. Define $\tilde{\tau}: A\to Q(C(Y)\otimes\mathcal{K}(\mathcal{H})\otimes\mathcal{K}(\mathcal{H}))$ by $\tilde{\tau}(a)=[\tau_a\otimes p]$, where $[\tau_a]=\tau(a)$. By the universal property of quantum double suspension (see [Hong and Szymański 2008, Proposition 2.2]), we have a homomorphism

$$\Sigma^2\tau : \Sigma^2 A \to Q(C(Y)\otimes\mathcal{K}(\mathcal{H})\otimes\mathcal{K}(\mathcal{H})) \tag{1-1}$$

such that

$$\Sigma^2\tau(a\otimes p)=\tilde{\tau}(a)=[\tau_a\otimes p] \quad \text{and} \quad \Sigma^2\tau(1\otimes S)=[1\otimes 1\otimes S].$$

One can show that $\Sigma^2\tau$ is a unital homogeneous extension. This induces a map

$$\beta : \mathrm{Ext}_{\mathrm{PPV}}(Y, A)\to \mathrm{Ext}_{\mathrm{PPV}}(Y, \Sigma^2 A), \quad [\tau]_u\mapsto[\Sigma^2\tau]_u.$$

From Proposition 2.4 and Lemma 2.8 in [Saurabh 2017], one concludes that if $Y=\mathbb{T}$, then β is an isomorphism between $\mathrm{Ext}_{\mathrm{PPV}}(\mathbb{T}, A)$ and $\mathrm{Ext}_{\mathrm{PPV}}(\mathbb{T}, \Sigma^2 A)$.

Let $\Upsilon := \{q\in\mathbb{C} : 0<|q|<1,\ \theta=\frac{1}{\pi}\arg(q) \text{ is irrational}\}$. Fix $q\in\Upsilon$. From Theorem 3.9 in [Guin and Saurabh 2022], it follows that the extension

$$\mathcal{E}_q : 0\to\mathcal{K}(\ell^2(\mathbb{N}))\otimes C(\mathbb{T})\to C(U_q(2)/_\psi\mathbb{T})\xrightarrow{\vartheta} C(\mathbb{T})\to 0$$

is a trivial homogeneous extension of $C(\mathbb{T})$ by $C(\mathbb{T})\otimes\mathcal{K}$. Therefore the equivalence class $[\phi_q]$ of the Busby invariant ϕ_q of $\mathcal{E}_q$ is a zero element in $\mathrm{Ext}_{\mathrm{PPV}}(\mathbb{T}, C(\mathbb{T}))$. Then $(\Sigma^2\Sigma^2\cdots\Sigma^2)^\ell\phi_q$ is a homogeneous extension of $(\Sigma^2\Sigma^2\cdots\Sigma^2)^\ell C(\mathbb{T})$ by $C(\mathbb{T})\otimes\mathcal{K}$. Hence its equivalence class $[(\Sigma^2\Sigma^2\cdots\Sigma^2)^\ell\phi_q]$ gives a zero element of $\mathrm{Ext}_{\mathrm{PPV}}(\mathbb{T}, C(S_0^{2\ell+1}))$. Further, note that the middle C^*-algebra of $(\Sigma^2\Sigma^2\cdots\Sigma^2)^\ell\mathcal{E}_q$ is $(\Sigma^2\Sigma^2\cdots\Sigma^2)^\ell C(U_q(2)/_\psi\mathbb{T})$. Using Theorem 3.4 of [Guin and Saurabh 2022] and the short term exact sequence

$$0\to A\otimes\mathcal{K}\to\Sigma^2 A\to C(\mathbb{T})\to 0$$

for any unital C^*-algebra A, it is easy to verify that

$$K_0((\Sigma^2\Sigma^2\cdots\Sigma^2)^\ell C(U_q(2)/_\psi\mathbb{T}))\cong\mathbb{Z}\oplus\mathbb{Z}.$$

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BIPUL SAURABH
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY GANDHINAGAR
GANDHINAGAR
INDIA
saurabhbipul2@gmail.com

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PACIFIC JOURNAL OF MATHEMATICS

Volume 344 No. 2 October 2026

Local smoothing estimates for bilinear Fourier integral operators 241
DUVÁN CARDONA

Second-order estimates for χ-semiconvex solutions of Hessian equations on Hermitian manifolds 273
XIAOJUAN CHEN, QIANG TU and NI XIANG

Higher-rank instantons sheaves on Fano threefolds 293
GAIA COMASCHI and DANIELE FAENZI

Artin shapes 359
NIKITA KARPENKO and GUANGZHAO ZHU

L-packets and the generic Arthur packet conjectures for even unitary similitude groups 367
YEANSU KIM, MUTHU KRISHNAMURTHY and FREYDOON SHAHIDI

Boundedness of Eisenstein integrals on split rank-one semisimple symmetric spaces 395
SANJOY PUSTI and ISWARYA SITIRAJU

Homotopy-theoretic properties of gyrations 431
STEPHEN THERIAULT

Localization and Toeplitz operators on doubling Fock spaces 449
CHUNXU XU and HONG ZOU

Correction to the article Topological invariance of quantum quaternion spheres 475
BIPUL SAURABH