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LOCAL SMOOTHING ESTIMATES FOR BILINEAR FOURIER INTEGRAL OPERATORS

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We formulate a local smoothing conjecture for bilinear Fourier integral operators in every dimension $d \geq 2$, derived from the celebrated linear case due to Sogge, which we refer to as the *bilinear smoothing conjecture*. We show that the linear local smoothing conjecture implies this bilinear version. As a consequence of our approach and due to the recent progress on the subject, we establish local smoothing estimates for bilinear Fourier integral operators in dimension $d = 2$, that is, on $\mathbb{R}^2_x \times \mathbb{R}_t$. Also, partial progress is presented for the high-dimensional case $d \geq 3$. In particular, our method allows us to deduce that the bilinear local smoothing conjecture holds for all odd dimensions d.

1. Introduction

We propose a conjecture on local smoothing estimates for bilinear Fourier integral operators (FIOs) derived from the linear case observed by Sogge [1991]. Recent progress in the field (see [Beltran et al. 2020; Gao et al. 2023]) allows our methods to fully prove the conjecture for $d = 2$ and to advance the problem partially for $d \geq 3$. In particular, we deduce that the bilinear local smoothing conjecture is true when d is odd. We first recall the linear local smoothing conjecture [Sogge 1991] and then introduce its bilinear form. Our main result is Theorem 2.6; our approach investigates the smoothing properties of high and of low frequencies of a bilinear Fourier integral operator. For the analysis of the high frequency portion, among

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other things, we make use of a boundedness theorem for maximal functions proved by Bourgain [1985].

According to Hörmander [1971], an FIO T of order $m \in \mathbb{R}$, with real phase function ϕ, is a continuous linear operator $T : C_0^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d \times \mathbb{R})$ of the form

$$T_a^\phi f(x,t) = \int_{\mathbb{R}^d} e^{i\phi(x,t,\xi)} a(x,t,\xi)\, \hat{f}(\xi)\, d\xi, \tag{1-1}$$

where $\hat{f}$ is the Fourier transform of $f \in C_0^\infty(\mathbb{R}^d)$. The symbol $a := a(x,t,\xi)$ is of Kohn–Nirenberg order $m \in \mathbb{R}$, that is, a satisfies the symbol inequalities

$$|\partial_x^\beta \partial_\xi^\alpha a(x,t,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|} \quad \text{for all } \alpha, \beta \in \mathbb{N}_0. \tag{1-2}$$

The symbol a has compact support in (x,t). We assume that in an open neighbourhood V of $\pi(\mathrm{supp}(a)) := \{(x,t) : (x,t,\xi) \in \mathrm{supp}(a)\}$ (with $\overline{V}$ being a compact set) the real-valued phase function ϕ is homogeneous of degree 1 in $\xi \neq 0$, smooth in $(x,t,\xi) \in V \times (\mathbb{R}^d \setminus \{0\})$, and satisfies the nondegeneracy condition

$$\det\big((\partial^2_{x,\xi}\phi(x,t,\eta))\big)_{1\le i,j\le d} \neq 0 \quad \text{for all } (x,t,\xi) \in \overline{V} \times (\mathbb{R}^d \setminus \{0\}). \tag{1-3}$$

The corresponding L^p-theory for these operators can be summarised as follows; see the works of Tao [2004] and Seeger, Sogge and Stein [Seeger et al. 1991].

Theorem 1.1. *Let $t \in \mathbb{R}$ be fixed in the support of the symbol $a := a(x,t,\xi)$. Let $T = T^{\phi(\cdot,t,\cdot)}_{a(\cdot,t,\cdot)}$ be a Fourier integral operator of order $m \in \mathbb{R}$. Then*:

(1) *If $m = -(d-1)/2$ and T is of weak $(1,1)$ type, then there is $C_t > 0$ such that*

$$\|Tf\|_{L^{1,\infty}(\mathbb{R}^d_x)} \le C_t \|f\|_{L^1(\mathbb{R}^d_x)} \quad \textit{for all } f \in L^1(\mathbb{R}^d). \tag{1-4}$$

(2) *Let $1 < p < \infty$. Let T be an FIO of order $m = -(d-1)\big|\frac{1}{2} - \frac{1}{p}\big|$. Then T is bounded from $L^p(\mathbb{R}^d_x)$ to $L^p(\mathbb{R}^d_x)$. If $p = 1$, then T is bounded from the Hardy space $H^1(\mathbb{R}^d_x)$ to $L^1(\mathbb{R}^d_x)$. As in* (1-4), *these operator norms depend on t.*

The case $p = 2$ in Theorem 1.1 was proved in [Duistermaat and Hörmander 1972]; see also [Èskin 1970]. Other references related to the global boundedness of Fourier integral operators are presented in Section 3. For now, we emphasise that, in terms of the L^p-regularity, Theorem 1.1 shows that Fourier integral operators lose $s = (d-1)\big|\frac{1}{2} - \frac{1}{p}\big|$ derivatives. In other words, on the scale of Sobolev spaces L^p_s, a Fourier integral operator $T = T^{\phi(\cdot,t,\cdot)}_{a(\cdot,t,\cdot)}$ of order $m \in \mathbb{R}$ satisfies the time-dependent estimate

$$\|Tf\|_{L^p_s(\mathbb{R}^d_x)} \lesssim_t \|f\|_{L^p_{s+m+(d-1)|1/2-1/p|}(\mathbb{R}^d_x)}, \quad 1 < p < \infty. \tag{1-5}$$

It was first observed by Sogge [1991], in the context of the wave equation, that averaging in time on the left-hand side of (1-5) allows one to improve the loss of

regularity. This observation led to the formulation of the (linear) *local smoothing conjecture* for the wave operator. If true, this conjecture would represent a remarkable breakthrough, as it formally implies several other major open problems in harmonic analysis, including the Bochner–Riesz, Fourier restriction, and Kakeya conjectures; see [Tao 1999].

In the setting of Fourier integral operators, the main assumption on the phase to formulate the linear local smoothing conjecture is the following *cinematic curvature condition*.

Definition 1.2 (cinematic curvature condition). We say that

$$\phi := \phi(x,t,\xi) \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}_t \times (\mathbb{R}^d \setminus \{0\}))$$

satisfies the *cinematic curvature condition* on a compact set $\mathcal{K} \subset \mathbb{R}^d \times \mathbb{R}_t$ if the following conditions are satisfied:

(i) The phase ϕ is nondegenerate, that is,

$$\operatorname{rank}(\partial^2_{x,\eta}\phi(x,t,\eta)) = d, \quad (x,t) \in \mathcal{K}, \quad \eta \in \mathbb{R}^d \setminus \{0\}. \tag{1-6}$$

(ii) Consider the Gauss mapping $G : \mathcal{K} \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{S}^d$, where

$$G(x,t,\eta) = \frac{G_0(x,t,\eta)}{|G_0(x,t,\eta)|} \quad \text{and} \quad G_0(x,t,\eta) := \bigwedge_{1\le j\le d} \partial_{\eta_j}\partial_{x,t}\phi(x,t,\eta).$$

The rank condition

$$\operatorname{rank}\big(\partial^2_{\eta\eta}\langle \partial_{x,t}\phi(x,t,\eta), G(z,\eta_0)\rangle|_{\eta=\eta_0}\big) = d-1 \tag{1-7}$$

holds for every $(x,t,\eta_0) \in \mathcal{K} \times \mathbb{R}^d$, $\eta_0 \neq 0$.

Adapted to Fourier integral operators, the linear *smoothing conjecture* can be stated as follows. In order to formulate it we use, for its critical index, the notation

$$\bar{p}_d := \begin{cases} \frac{2(d+1)}{d-1} & \text{if } d \text{ is odd,} \\ \frac{2(d+2)}{d} & \text{if } d \text{ is even.} \end{cases} \tag{1-8}$$

Conjecture 1.3 (linear smoothing conjecture for FIOs). Let $d \ge 2$. Let T^ϕ_a be a Fourier integral operator of order $m \in \mathbb{R}$, and with nondegenerate phase function satisfying the *cinematic curvature condition*; see Definition 1.2. For any $p \ge \bar{p}_d$, and $\sigma < \frac{1}{p}$,

$$\|T^\phi_a f\|_{L^p(\mathbb{R}^d\times I)} \lesssim \|f\|_{L^p_s(\mathbb{R}^d)}, \quad s = m + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma. \tag{1-9}$$

Remark 1.4. Examples constructed by Bourgain [1991; 1995] can be used to identify Fourier integral operators that satisfy the cinematic curvature condition but for which local smoothing fails for all $\sigma < \frac{1}{p}$, whenever $p < \bar{p}_d$. Compared

d	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\bar{p}_d$	4	4	3	3	$\frac{8}{3}$	$\frac{8}{3}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{12}{5}$	$\frac{12}{5}$	$\frac{7}{3}$	$\frac{7}{3}$	$\frac{16}{7}$	$\frac{16}{7}$	$\frac{9}{4}$	$\frac{9}{4}$	$\frac{20}{9}$	$\frac{20}{9}$	$\frac{11}{5}$

Table 1. Critical index $\bar{p}_d$ for dimensions $d = 2$ to 20.

with Theorem 1.1, Conjecture 1.3 claims a gain of regularity of order $\sigma < \frac{1}{p}$. We illustrate some values of the critical index $\bar{p}_d$ in Table 1.

Let us introduce the required notation to formulate the bilinear counterpart to Conjecture 1.3 by adopting the notion of bilinear Fourier integral operators as studied, e.g., by Grafakos and Peloso [2010], and for the form of the operators formulated here, see the work of Rodríguez-López, Rule, and Staubach [Israelsson et al. 2021].

Let $a \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}^d_\xi \times \mathbb{R}^d_\eta)$ be a *bilinear symbol*. We say that a has order $m \in \mathbb{R}$ if it satisfies the estimates

$$|\partial^\beta_{x,t} \partial^{\alpha_1}_\xi \partial^{\alpha_2}_\eta a(x,t,\xi,\eta)| \lesssim_{\beta,\alpha_1,\alpha_2} (1+|\xi|+|\eta|)^{m-|\alpha_1|-|\alpha_2|}. \tag{1-10}$$

The Kohn–Nirenberg bilinear class $S^m_{1,0} := S^m_{1,0}(\mathbb{R}^d_x \times \mathbb{R}_t \times \mathbb{R}^d_\xi \times \mathbb{R}^d_\eta)$ is the set formed by those bilinear symbols of order m.

Let us consider the *bilinear Fourier integral operator*

$$T^{\phi_1,\phi_2}_a(f,g)(x,t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i(\phi_1(x,t,\xi)+\phi_2(x,t,\eta))} a(x,t,\xi,\eta)\, \hat{f}(\xi)\hat{g}(\eta)\, d\xi\, d\eta,$$

where the symbol a is compactly supported in $(x,t) \in \mathbb{R}^d \times \mathbb{R}$, and each phase function $\phi_j(x,t,\xi) \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}_t \times (\mathbb{R}^d \setminus \{0\}))$ is positively homogeneous of order 1 in $\xi \in \mathbb{R}^d \setminus \{0\}$, and they satisfy the cinematic curvature condition; see Definition 1.2.

Here, we will denote by $\mathcal{I} = [a,b]$ an arbitrary nonempty finite interval and then $-\infty < a < b < \infty$. To slightly reduce our analysis, since the symbol $a := a(x,t,\xi,\eta)$ is assumed to be compactly supported in (x,t), and without loss of generality we will choose a compact set $K \subset \mathbb{R}^d$ such that

$$(x,t) - \operatorname{supp}(a) := \{(\tilde{x},\tilde{t}) \in \mathbb{R}^d \times \mathbb{R} : (\tilde{x},\tilde{t}) \in \operatorname{supp}(a)\} \subset K \times \mathcal{I}, \tag{1-11}$$

increasing the length of $\mathcal{I}$ if needed. It becomes clear that

$$\|T^{\phi_1,\phi_2}_a(f,g)\|_{L^p(\mathbb{R}^d_x \times \mathbb{R}_t)} = \|T^{\phi_1,\phi_2}_a(f,g)\|_{L^p(K \times \mathcal{I})}. \tag{1-12}$$

This fact will not be used in our formulation of the bilinear smoothing conjecture but will be crucial in our further analysis. Based on the linear smoothing conjecture for Fourier integral operators, it is natural to suppose that the following result holds in any dimension $d \geq 2$.

Conjecture 1.5 (bilinear smoothing conjecture for FIOs). Let $d \geq 2$. For any $p_1, p_2 \geq \bar{p}_d$, let $\sigma_1 < \frac{1}{p_1}$, $\sigma_2 < \frac{1}{p_2}$. Let $T_a^{\phi_1,\phi_2}$ be a bilinear Fourier integral operator, with nondegenerate phase functions satisfying the cinematic curvature condition. Let $a \in S^m_{1,0}$ be its symbol, having order $m < 0$, and being compactly supported in $(x,t) \in \mathbb{R}^d \times \mathbb{R}$. Then

$$\|T_a^{\phi_1,\phi_2}(f,g)\|_{L^p(\mathbb{R}^d_x \times \mathbb{R}_t)} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d)}, \tag{1-13}$$

where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad s_i = m_i + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma_i, \quad m = m_1 + m_2, \quad m_1 \leq 0, m_2 < 0.$$

Remark 1.6. In the above display one has that $p \geq \frac{1}{2}\bar{p}_d$. Note that there is no loss of generality in assuming that $m < 0$ in Conjecture 1.3; see Remark 2.5. Moreover, if $a(x,t,\xi,\eta)$ has compact support in ξ, there is no restriction in assuming that $m_2 < 0$ has arbitrarily large absolute value. Conversely, if $a(x,t,\xi,\eta)$ has compact support in η, there is no restriction in assuming that $m_1 < 0$ has arbitrarily large absolute value.

Analysing the relationship between the two conjectures above is our main goal. Indeed, we prove the following result.

Theorem 1.7. *The linear smoothing Conjecture 1.3 implies the bilinear smoothing Conjecture 1.5.*

The proof of Theorem 1.7 will be deduced from a more qualitative version of it, see Theorem 2.6, which in particular will prove that any progress in the linear smoothing conjecture for FIOs implies a counterpart for bilinear FIOs in any dimension $d \geq 2$.

The local smoothing conjecture has been verified in dimension $d = 2$ by Guth, Wang, and Zhang [Guth et al. 2020] for solutions of the wave equation. Moreover, since the linear smoothing conjecture for FIOs has been proved in dimension $d = 2$ by Gao, Liu, Miao, and Xi [Gao et al. 2023], and for odd d by Beltran, Hickman, and Sogge [Beltran et al. 2020], their results, combined with Theorem 1.7, imply:

Corollary 1.8. *For $d = 2$ the bilinear local smoothing conjecture holds. In other words, for any $p_1, p_2 \geq 4$, let $\sigma_1 < \frac{1}{p_1}$, $\sigma_2 < \frac{1}{p_2}$. Let $a \in S^m_{1,0}$ be a symbol of order $m < 0$, being compactly supported in $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$. Then*

$$\|T_a^{\phi_1,\phi_2}(f,g)\|_{L^p(\mathbb{R}^2_x \times \mathbb{R}_t)} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^2)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^2)}, \tag{1-14}$$

where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad s_i = m_i + \frac{1}{2} - \frac{1}{p_i} - \sigma_i, \quad m = m_1 + m_2, \quad m_1 \leq 0, m_2 < 0. \tag{1-15}$$

On the other hand, if d is odd, the bilinear local smoothing conjecture holds. This means that for any

$$p_1, p_2 \geq \bar{p}_d := \frac{2(d+1)}{d-1}, \tag{1-16}$$

and for every $\sigma_1 < \frac{1}{p_1}$, $\sigma_2 < \frac{1}{p_2}$, an operator $T_a^{\phi_1,\phi_2}$ having order $m < 0$, with nondegenerate phase functions satisfying the cinematic curvature condition, satisfies

$$\|T_a^{\phi_1,\phi_2}(f,g)\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d)}\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d)}, \tag{1-17}$$

where

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}, \quad s_i = m_i + (d-1)\left(\frac{1}{2}-\frac{1}{p_i}\right)-\sigma_i, \quad m = m_1+m_2, \quad m_1 \leq 0, m_2 < 0.$$

There is no sense of thinking about a calculus of bilinear FIOs, however, each bilinear FIO has encoded its own *calculus* of linear FIOs when investigating its inner structure [Rodríguez-López et al. 2014]. We will adopt in our further analysis the structural properties introduced by Rodríguez-López, Rule, and Staubach [Rodríguez-López et al. 2014] for bilinear Fourier integral operators. These structural properties serve as a substitute for the calculus of Fourier integral operators due to Hörmander [1971] in this setting. A natural question is to what extent a bilinear FIO is approximately the composition (or the product) of two linear FIOs. Indeed, unless the bilinear symbol of an operator has compact support in one of the two frequency variables (ξ, η), a bilinear FIO cannot be written as the *composition* of two linear FIOs. Their method consists of making a careful continuous decomposition of the high frequencies of a bilinear FIO $T_a^{\phi_1,\phi_2}$. Roughly, it expresses the high frequency portion of the operator as a sum of two operators of the form (see (46) of [Rodríguez-López et al. 2014, page 18])

$$\begin{aligned}\widetilde{T}_{\tilde{a}}^{\phi_1,\phi_2}(\mu(D)f, \mu(D)g)(x,s) \\ = \int_0^1 \iint T^{\phi_1}_{\nu_1^{t,u}} f(x,s) T^{\phi_2}_{\mu_1^{t,v}} g(x,s) \frac{m(t,x,s,U)}{(1+|U|^2)^N}\, du\, dv\, \frac{dt}{t},\end{aligned} \tag{1-18}$$

where $|U|^2 = |u|^2 + |v|^2$. The operators $T^{\phi_1}_{\nu_1^{t,u}}$ and $T^{\phi_2}_{\mu_1^{t,v}}$ are approximations of linear Fourier integral operators. In particular, Rodríguez-López et al. [2014] have used this representation to establish a bilinear extension of the celebrated Seeger–Sogge–Stein theorem [Seeger et al. 1991].

In view of the linear smoothing property for FIOs proved in dimension $d = 2$ in [Gao et al. 2023], it is of interest for us to ask whether or not the methods in [Rodríguez-López et al. 2014] can be applied to smoothing estimates in the bilinear context considering the heavy machinery used in their analysis from various parts of harmonic analysis; see, e.g., the discussion in [Rodríguez-López et al. 2014, page 3].

One of the main differences of bilinear smoothing estimates compared with the analysis for time-dependent boundedness theorems as in [Rodríguez-López et al. 2014; Seeger et al. 1991] came from the fact that we cannot make use of endpoint estimates at infinity involving the BMO space, or at $p = 1$, and then involving the Hardy space H^1. In our case for the range of values $p_1, p_2 \geq \bar{p}_d$, we have to deal entirely with the $L^{p_1}_{s_1} \times L^{p_2}_{s_2} \to L^p$-estimate. Making use of square-maximal functions as in [Rodríguez-López et al. 2014] we deal with the complexity of working with the direct $L^{p_1}_{s_1} \times L^{p_2}_{s_2} \to L^p$-boundedness theorem by applying, in a certain portion of the continuous decomposition of the operator, a well-known theorem for estimations of maximal functions, due to Bourgain [1985; 1986]; see Theorem 4.2. In comparison with the previous methods in the literature this is the main novelty of this paper.

This paper is organised as follows. In Section 2 we present our main result (see Theorem 2.6) about smoothing estimates for bilinear FIOs. Section 3 is dedicated to providing preliminaries about the inner structure of these operators. Section 4 is dedicated to the proof of Theorem 2.6.

2. Smoothing estimates for bilinear FIOs

We formulate our main result in Theorem 2.6 presented below. First, we introduce two necessary definitions. Throughout, we denote by $\mathscr{S}(\mathbb{R}^d)$ the Schwartz class on $\mathbb{R}^d$.

Definition 2.1. Let $d \geq 2$, and let $2 \leq p^* \leq \infty$. Let T^{ϕ}_a be a Fourier integral operator of order $m \in \mathbb{R}$, and with nondegenerate phase function satisfying the cinematic curvature condition; see Definition 1.2. If for a fixed Lebesgue index p, with $p \geq p^*$, there exists $s = s(p, m, d)$ such that

$$\|T^{\phi}_a f\|_{L^p(\mathbb{R}^d \times \mathcal{I})} \lesssim \|f\|_{L^p_s(\mathbb{R}^d)} \quad \text{for all } \ f \in C^\infty_0(\mathbb{R}^d), \tag{2-1}$$

we say that the operator T satisfies the *linear local smoothing property* (LLSP) with parameters (p^*, p, s).

Remark 2.2. The local smoothing conjecture claims that any Fourier integral operator of order $m \in \mathbb{R}$ with phase satisfying the cinematic curvature condition should satisfy the linear local smoothing property (LLSP) with parameters $(\bar{p}_d, p, s)$, where

$$s = m + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma,$$

and $\sigma < \frac{1}{p}$.

Definition 2.3. Let $d \geq 2$, and let $2 \leq p^* \leq \infty$. Let T^{ϕ_1,ϕ_2}_a be a bilinear Fourier integral operator of order $m \in \mathbb{R}$, and with nondegenerate phase functions ϕ_1, ϕ_2,

satisfying the cinematic curvature condition. If for a fixed pair (p_1, p_2), with $p_1, p_2 \geq p^*$, there exist $s_1 = s(p_1, m, d)$ and $s_2 = s(p_2, m, d)$ such that

$$\|T_a^{\phi_1,\phi_2} f\|_{L^p(\mathbb{R}^d\times\mathcal{I})} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d)} \quad \text{for all } f, g \in C_0^\infty(\mathbb{R}^d), \tag{2-2}$$

with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we say that $T_a^{\phi_1,\phi_2}$ satisfies the *bilinear local smoothing property* (BLSP) with parameters $(p^*, p, p_1, p_2, s_1, s_2)$.

Remark 2.4. The bilinear local smoothing conjecture claims that any Fourier integral operator of order $m = m_1 + m_2 < 0$, $m_2 < 0$, with phases satisfying the cinematic curvature condition should satisfy the bilinear local smoothing property with parameters $(\bar{p}_d, p, p_1, p_2, s_1, s_2)$, where

$$s_i = m_i + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma_i,$$

where σ_i is any number satisfying $\sigma_i < \frac{1}{p_i}$.

Remark 2.5. There is no restriction in assuming that $m < 0$ in Conjecture 1.3. Indeed, if $m_0 \geq 0$ is the order of a Fourier integral operator T_a^ϕ, one can take s_0 such that $m = m_0 - s_0 < 0$, and to replace f by $f_{s_0} = (1-\Delta)^{s_0/2}$ to get for the FIO $\widetilde{T} = T(1-\Delta)^{-s_0/2}$ the smoothing property

$$\|\widetilde{T} f_{s_0}\|_{L^p(\mathbb{R}^d\times\mathcal{I})} \lesssim \|(1-\Delta)^{s_0/2} f\|_{L^p_{\tilde{s}}(\mathbb{R}^d)}, \quad \tilde{s} = m + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma.$$

The previous estimate is equivalent to

$$\|Tf\|_{L^p(\mathbb{R}^d\times\mathcal{I})} \lesssim \|f\|_{L^p_{\tilde{s}+s_0}(\mathbb{R}^d)}, \quad \tilde{s} + s_0 = m_0 + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma.$$

This fact and our evidence justify our choice of assuming that the order m is negative in Conjecture 1.5. Moreover, according to our methodology, we decompose the operator $T_a^{\phi_1,\phi_2}$ as

$$\begin{aligned} T_a^{\phi_1,\phi_2}(f,g) &= T_a^{\phi_1,\phi_2}(\mu(D)f, \mu(D)g) \\ &\quad + T_a^{\phi_1,\phi_2}(\mu(D)f, (1-\mu)(D)g) + T_a^{\phi_1,\phi_2}\big((1-\mu(D))f, g\big), \end{aligned}$$

where μ is supported outside of a ball centred at the origin. The second and the third terms are the low-frequency portions of the operator. Curiously, for the analysis of the low-frequencies we need the hypothesis $m_1 \leq 0$, while for the analysis of the high frequencies we need to assume that $m_2 < 0$.

Theorem 2.6. *Let $d \geq 2$, and let $2 \leq p^* \leq \infty$. Consider the following hypothesis*:

(H1) *Assume that a pair of real numbers (m_1, m_2) and a pair of Lebesgue indices (p_1, p_2) are such that $p_i \geq p^*$ and satisfy that any Fourier integral operator of order m_i satisfying the cinematic curvature condition also satisfies the LLSP with parameters (p^*, p_i, s_i) for some $s_i = s_i(p_i, m_i, d)$.*

Then, if $m := m_1 + m_2$, $m_1 \le 0$, *and* $m_2 < 0$, *every bilinear Fourier integral operator* $T_a^{\phi_1,\phi_2}$ *of order* m *with nondegenerate phase functions satisfying the cinematic curvature condition satisfies* (*BLSP*) *with parameters* $(p^*, p, p_1, p_2, s_1, s_2)$. *In other words*,

$$\|T_a^{\phi_1,\phi_2} f\|_{L^p(\mathbb{R}^d\times\mathcal{I})} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d)}, \tag{2-3}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 < p < \infty$.

Proof of Theorem 1.7. The theorem follows from Theorem 2.6 taking also into account Remarks 2.2 and 2.4. □

Proof of Corollary 1.8. The proof follows from Theorem 2.6 and the solution of the linear smoothing conjecture in dimension $d = 2$ due to [Gao et al. 2023]; also for d odd, this conjecture has been verified in [Beltran et al. 2020]. For the numerical details used here for this conjecture with d odd, see [Gao et al. 2023, page 1926]. This fact combined with Theorem 2.6 proves the statement in this case. □

3. Preliminaries

In this section, we present the structural properties of bilinear FIOs as introduced in [Rodríguez-López et al. 2014]. As mentioned before, these structural properties replace the calculus of Fourier integral operators due to Hörmander [1971] in this setting in the sense that we want to understand bilinear operators as twisted versions of compositions or products of linear FIOs. In other words, each bilinear FIO induces its own *calculus* of linear FIOs when investigating its inner structure [Rodríguez-López et al. 2014].

Standard notation such as $\psi(D)$ and $\psi(tD)$ will be used later for the corresponding multipliers with symbols $\psi := \psi(\eta)$ and $\psi(t\,\cdot) := \psi(t\eta)$, respectively.

3.1. ***Structural properties of bilinear FIOs.*** First, we will review the structure of the bilinear operator $T_a^{\phi_1,\phi_2}$ given by

$$T_a^{\phi_1,\phi_2}(f,g)(x,t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i(\phi_1(x,t,\xi)+\phi_2(x,t,\eta))} a(x,t,\xi,\eta)\,\hat{f}(\xi)\hat{g}(\eta)\,d\xi\,d\eta. \tag{3-1}$$

The first step will make a reduction of our analysis in low frequencies and in high frequencies. For this we follow the approach and notation in [Rodríguez-López et al. 2014, Sections 3–4]. Some remarks are in order.

Remark 3.1 (see [Rodríguez-López et al. 2014, page 12]). We will use the structure of the operator $T_a^{\phi_1,\phi_2}$ to decompose it in a suitable continuous form. The phase

functions ϕ_j are smooth, positively homogeneous of order one in the frequency variable and they are nondegenerate. By the mean value theorem

$$|\nabla_x \phi_j(x,t,\xi)| \lesssim |\xi| \quad \text{for all } (x,t,\xi) \in \mathcal{K}\times(\mathbb{R}^d\setminus\{0\}) = K\times\mathcal{I}\times(\mathbb{R}^d\setminus\{0\}). \tag{3-2}$$

In addition, the phases are nondegenerate and then one has the lower bound

$$|\xi| \lesssim |\nabla_x \phi_j(x,t,\xi)| \quad \text{for all } (x,t,\xi) \in \mathcal{K}\times(\mathbb{R}^d\setminus\{0\}) = K\times\mathcal{I}\times(\mathbb{R}^d\setminus\{0\}). \tag{3-3}$$

Therefore, there exists $\lambda > 0$ such that on $K\times\mathcal{I}\times(\mathbb{R}^d\setminus\{0\})$ one has the estimate

$$\lambda|\xi| \le |\nabla_x \phi_j(x,t,\xi)| \le \frac{|\xi|}{\lambda}. \tag{3-4}$$

We consider a smooth function μ such that $\mu \equiv 0$ on $\{\xi\in\mathbb{R}^d : |\xi| \le \frac{1}{4\lambda}\}$ and $\mu \equiv 1$ on $\{\xi\in\mathbb{R}^d : |\xi| \ge \frac{1}{3\lambda}\}$.

Remark 3.2. We decompose the operator $T_a^{\phi_1,\phi_2}$ as

$$T_a^{\phi_1,\phi_2}(f,g) = T_a^{\phi_1,\phi_2}(\mu(D)f, \mu(D)g) \tag{3-5}$$

$$+T_a^{\phi_1,\phi_2}(\mu(D)f,(1-\mu)(D)g)+T_a^{\phi_1,\phi_2}\big((1-\mu(D))f,g\big). \tag{3-6}$$

The second and the third term are the low-frequency portions of the operator. That these parts satisfy the $L^{p_1}_{s_1}\times L^{p_2}_{s_2}\to L^p$ estimate requires an analysis based on understanding the low frequency portion of the operator as an almost composition of linear FIOs; see Section 4.1.

Remark 3.3 (see [Rodríguez-López et al. 2014, Section 4]). For the analysis of the high-frequency portion $T_a^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g)$ one can introduce two smooth cut-off functions $\nu, \chi \in C_0^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ such that $\chi(\xi,\eta) = 1$ for $|\xi|+|\eta|\le 1$, $\chi(\xi,\eta) = 0$ for $|\xi|+|\eta|\ge 2$, $\nu(\xi,\eta) = 0$ for $\lambda^2|\xi| \le 16|\eta|$, and $\nu(\xi,\eta) = 1$ for $64|\eta| \le \lambda^2|\xi|$. Let us define

$$a_1 := (1-\chi)\nu a, \quad a_2 := (1-\chi)(1-\nu)a. \tag{3-7}$$

We have that a_1 and a_2 are bilinear symbols of order m, and

$$T_a^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g) = T_{a_1}^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g) + T_{a_2}^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g).$$

We have the representation (see equation (43) in [Rodríguez-López et al. 2014, page 16])

$$T_{a_1}^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g)(x,s) \tag{3-8}$$

$$= \int_0^1 \iint T^{\phi_1}_{\nu_1^{t,u}} f(x,s)\, T^{\phi_2}_{\mu_1^{t,v}} g(x,s) \frac{m(t,x,s,u,v)}{(1+|u|^2+|v|^2)^N}\, du\, dv\, \frac{dt}{t}.$$

The operator $T^{\phi_1}_{\nu_1^{t,u}}$ has the factorisation modulo an error term (see equation (54) in [Rodríguez-López et al. 2014, page 20])

$$T^{\phi_1}_{\nu_1^{t,u}} = t^{-m_2} Q_t^u T^{\phi_1}_{\nu_{m_1}} + t^{-m_2} E_t^1, \tag{3-9}$$

where:

• Q_t^u is a Fourier multiplier with symbol (see equation (54) in [Rodríguez-López et al. 2014, page 21])

$$\xi \mapsto |t\xi|^{m_2} \widehat{\psi}(t\xi) e^{itu\cdot\xi} \in \mathscr{S}(\mathbb{R}^n). \tag{3-10}$$

The function $\psi \in \mathscr{S}(\mathbb{R}^n)$ is chosen to be smooth, whose Fourier transform is supported on the annulus $\left\{\xi \in \mathbb{R}^d : \frac{1}{2} \le |\xi| \le 2\right\}$ and such that

$$\int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} = 1 \quad \text{for all } t > 0 \text{ and for all } \xi \neq 0.$$

• The operator $T^{\phi_1}_{\nu_{m_1}}$ is a Fourier integral operator of order m_1 with phase ϕ_1 and with symbol (see equation (55) in [Rodríguez-López et al. 2014, page 21])

$$(x, s, \xi) \mapsto \nu_{m_1}(x, \xi) = \mu(\xi)\chi(x)|\nabla_x \phi_1(x, s, \xi)|^{m_1} \in S^{m_1}_{1,0}(\mathbb{R}^d_x \times \mathbb{R}_s \times \mathbb{R}^d_\xi). \tag{3-11}$$

The operator E_t^1 has the same phase ϕ_1 of the operator $T^{\phi_1}_{\nu_{m_1}}$ and a symbol in the class $S^{m_1-(1/2-\varepsilon)}_{1,0}(\mathbb{R}^d_x \times \mathbb{R}_s \times \mathbb{R}^d_\xi)$, with seminorms bounded by the product of t^ε, $\varepsilon \in \left(0, \frac{1}{2}\right)$, with a polynomial expression in $|u|$.

On the other hand, $T^{\phi_2}_{\mu_1^{t,v}}$, modulo an error term, admits the factorisation (see equation (56) in [Rodríguez-López et al. 2014, page 21])

$$t^{-m_2} T^{\phi_2}_{\mu_1^{t,v}} = P_t^v T^{\phi_2}_{\mu_1^t} + E_t^2,$$

where:

(i) The symbol of P_t^v is given by

$$\eta \mapsto \widehat{\theta}(t\eta) e^{itv\cdot\eta} \in S^0_{1,0}(\mathbb{R}^d_\eta).$$

(ii) $T^{\phi_2}_{\mu_1^t}$ is an FIO of order $m_2 < 0$, with phase function ϕ_2, and with symbol

$$(x, s, \eta) \mapsto \mu_1^t(x, s, \eta) = \mu(\eta)\chi(x)\widehat{\theta}_1(t\nabla_x \phi_2(x, s, \eta))t^{-m_2} \in S^{m_2}_{1,0}(\mathbb{R}^d_x \times \mathbb{R}_s \times \mathbb{R}^d_\eta),$$

uniformly in $t \in (0, 1]$. Here, E_t^2 is a Fourier integral operator with phase ϕ_2 and a symbol in the class $S^{m_2-(1/2-\varepsilon)}_{1,0}(\mathbb{R}^d_x \times \mathbb{R}_s \times \mathbb{R}^d_\eta)$, with seminorms that behaves polynomially in $|v|$.

3.2. ***Decomposition à la Coifman–Meyer.*** Some remarks are in order.

Remark 3.4. The core of the proof of the structural properties for bilinear FIOs lies in the representation in (3-8) for $T_{a_1}^{\phi_1,\phi_2}$,[1] that is (see [Rodríguez-López et al. 2014, page 21]),

$$T_{a_1}^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g)(x,s) = \iiint_0^1 T_{\nu_1^{t,u}}^{\phi_1} f(x,s) T_{\mu_1^{t,v}}^{\phi_2} g(x,s) \frac{m(t,x,s,u,v)}{(1+|u|^2+|v|^2)^N}\, du\, dv\, \frac{dt}{t}. \tag{3-12}$$

About the $\frac{dt}{t}$-integral in (3-12), by writing $m(t,x,s) := m(t,x,s,u,v)$, one has

$$\int_0^1 T_{\nu_1^{t,u}}^{\phi_1} f(x,s) T_{\mu_1^{t,v}}^{\phi_2} g(x,s) m(t,x,s)\, \frac{dt}{t} = I_1 + I_2 + I_3,$$

with

$$I_1 = \int_0^1 Q_t^u T_{\nu_{m_1}}^{\phi_1} f(x,s) P_t^v T_{\mu_1^t}^{\phi_2} g(x,s) m(t,x,s)\, \frac{dt}{t},$$

$$I_2 = \int_0^1 Q_t^u T_{\nu_{m_1}}^{\phi_1} f(x,s) E_t^2 g(x,s) m(t,x,s)\, \frac{dt}{t},$$

$$I_3 = \int_0^1 E_t^1 f(x,s) t^{-m_2} T_{\mu_1^{t,v}}^{\phi_2} g(x,s) m(t,x,s)\, \frac{dt}{t}.$$

Remark 3.5. For $m < 0$, and with $m_2 < 0$, we have the representation (see [Rodríguez-López et al. 2014, Case II on page 24, and page 26])

$$I_1 = \sum_{j=1}^3 \int_\kappa^\infty \int_0^1 \chi_{(0,1)}(rt)(Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(P_t^v S_{r,t}^j g)(x,s) m(t,x,s)\, \frac{dt\, dr}{tr},$$

where $S_{r,t}^1 = r^{m_2} P_{1,t} Q_{tr} T_\gamma^{\phi_2}$, with $T_\gamma^{\phi_2}$ being a Fourier integral operator of order m_2 with symbol

$$\gamma(x,s,\eta) = \chi(x)\mu(\eta)|\nabla_x \phi_2(x,s,\eta)|^{m_2} \in S_{1,0}^{m_2}(\mathbb{R}_x^d \times \mathbb{R}_s \times \mathbb{R}_\eta^d), \tag{3-13}$$

and $P_{1,t}$ is the Fourier multiplier with symbol $\widehat{\theta}_1(t\eta)$. The operators $S_{r,t}^2$, $S_{r,t}^3$, have the form

$$S_{r,t}^2 g(x,s) = s^{m_2} P_{1,t}\bigg(\int e^{i\phi_2(\cdot,s,\eta)} R_{tr}^2(\cdot,s,\eta)\hat{g}(\eta)\, d\eta \bigg)(x),$$

$$S_{r,t}^3 g(x,s) = s^{m_2} \bigg(\int e^{i\phi_2(x,s,\eta)} R_t^1(x,s,\eta)\hat{g}(\eta)\, d\eta \bigg),$$

[1] A similar representation holds for $T_{a_2}^{\phi_1,\phi_2}$. So, we will just analyse one of these two terms in the proof of our main theorem.

where $R_t^1(x,s,\eta), R_t^1(x,s,\eta) \in S^{m_2-(1/2-\varepsilon)}_{1,0}(\mathbb{R}^d_x \times \mathbb{R}_s \times \mathbb{R}^d_\eta)$. Also, these symbols have seminorms that are bounded by t^ε and by $(tr)^\varepsilon$, with $\varepsilon \in \left(0, \frac{1}{2}\right)$, respectively.

Remark 3.6. The operators P_t^v and Q_t^u are uniformly $L^p(\mathbb{R}^d_x)$-bounded with operators norms growing polynomially in $|u|$ and $|v|$, respectively; see, e.g., [Stein 1993, page 161].

Having understood the *calculus* of Fourier integral operators inside a bilinear FIO we will start the analysis of low frequencies in the next section.

4. Proof of Theorem 2.6

In this section we prove our main Theorem 2.6. We split our analysis in two parts. First, we investigate the smoothing properties of the low frequencies of a bilinear FIO. This portion of the operator is approximately the composition of two linear Fourier integral operators. As for the portion of the operator formed by the high frequencies, the idea is to express it in terms of *paraproducts* à la [Coifman and Meyer 1978].

4.1. ***Analysis of the low frequencies.*** For the analysis of low frequencies it suffices to prove that a Fourier integral operator

$$T_a^{\phi_1,\phi_2}(f,g)(x,t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i(\phi_1(x,t,\xi)+\phi_2(x,t,\eta))} a(x,t,\xi,\eta)\, \hat{f}(\xi)\hat{g}(\eta)\, d\xi\, d\eta,$$

with a symbol $a := a(x,t,\xi,\eta)$ compactly supported in ξ or in η, satisfies the estimate

$$\|T_{a_1}^{\phi_1,\phi_2}(f,g)\|_{L^p(K\times\mathcal{I})} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}. \tag{4-1}$$

Here

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad s_i = m_i + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma_i, \quad m = m_1 + m_2, \quad m_1 \le 0,\ m_2 < 0.$$

Without loss of generality, let us assume that $f, g \in C_0^\infty(\mathbb{R}^n)$, and that the symbol a is compactly supported in (x,t,ξ). The following remark will be useful in our further analysis.

Remark 4.1. Assume that the parameters s_i are of the form

$$s_i = m_i + \tilde{s}_i(p_i, d). \tag{4-2}$$

Let N be an arbitrary integer. Observe that if the symbol $a(s,t,\xi,\eta)$ has compact support in ξ, then there is no restriction in Theorem 2.6 in assuming that $m_2 < 0$ has arbitrarily large absolute value; in particular, we may assume that $|m_2| > N$.

Similarly, if $a(s,t,\xi,\eta)$ has compact support in η, there is no restriction in assuming that $m_1 < 0$ has arbitrarily large absolute value.

To justify this claim, namely that it is enough to consider the conclusion of Theorem 2.6 when $|m_2|$ is large, we consider the first case, where $a(s,t,\xi,\eta)$ has compact support in ξ.

The idea is to consider $m_2 \leq 0$ with small absolute value, and reduce the proof to the case where $m_2 \leq 0$ has large absolute value.

Let $m = m_1 + m_2$, $m_2 < 0$ be the order of a bilinear FIO $T_a^{\phi_1,\phi_2}$. One can take s_0 such that $m_2' = m_2 - s_0 < -N$, and can replace g by $g_{s_0} = (1-\Delta)^{s_0/2} g$ to get for the bilinear FIO

$$\widetilde{T}_a^{\phi_1,\phi_2}(\cdot,\cdot) = T_a^{\phi_1,\phi_2}(\cdot, (1-\Delta_x)^{-s_0/2}\cdot) \tag{4-3}$$

of order $m - s = (m_1 - s_0) + m_2 = m_1 + m_2 = m_1 + (m_2 - s_0) = m_1 + m_2'$ that

$$\|T_{a_1}^{\phi_1,\phi_2}(f,g)\|_{L^p(K\times\mathcal{I})} = \|\widetilde{T}_{a_1}^{\phi_1,\phi_2}(f,g_{s_0})\|_{L^p(K\times\mathcal{I})} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g_{s_0}\|_{L^{p_2}_{s_2'}(\mathbb{R}^d_x)},$$

where

$$s_1 = m_1 + \tilde{s}_1(p_1,d), \quad s_2' = m_2 - s_0 + \tilde{s}_2(p_1,d).$$

Observe that

$$s_2 := s_2' + s_0 = m_2 + \tilde{s}_2(p_1,d) \tag{4-4}$$

does not depend on s_0. Note also that in the case of the linear smoothing conjecture one has

$$\tilde{s}_1(p_i,d) = (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma_i, \quad \sigma_i < \frac{1}{p_i}.$$

It is clear that

$$\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g_{s_0}\|_{L^{p_2}_{s_2'}(\mathbb{R}^d_x)} = \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2'+s_0}(\mathbb{R}^d_x)} = \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.$$

Observe that we have used that the order of the FIO $T_a^{\phi_1,\phi_2}(\cdot,(1-\Delta_x)^{-s_0/2}\cdot)$ is $m - s = m_1 - s + m_2$ since its symbol is given by

$$a(x,t,\xi,\eta)\langle\eta\rangle^{-s_0} \sim a(x,t,\xi,\eta)\langle\xi,\eta\rangle^{-s_0},$$

and is compactly supported in ξ. The previous analysis shows that, under the condition of the symbol to be compactly supported in one of the two frequency variables, if the bilinear smoothing estimate holds for m_2 with $|m_2|$ large enough, then the same property holds with any $m_2 < 0$.

We return our attention to the analysis of the low frequencies. For the proof of (4-1) let us combine the proof of the L^2-boundedness of pseudodifferential operators of order zero [Stein 1993, page 234] and the proof of Lemma 3.1 in [Rodríguez-López et al. 2014, page 12]. Let ψ be a smooth cut-off function that

is equal to 1 on the support of a in ξ. The support of the symbol a in ξ will be denoted by $\mathfrak{S}$. Then

$$T_{a_1}^{\phi_1,\phi_2}(f,g)(x,t) = \int e^{i\phi_1(x,t,\xi)}\mathfrak{a}_g(x,t,\xi)\hat{f}(\xi)\,d\xi = T_{\mathfrak{a}_g}^{\phi_1}(f)(x,t), \tag{4-5}$$
$$(x,t,\xi) \in K\times\mathcal{I}\times\mathfrak{S},$$

where

$$\mathfrak{a}_g(x,t,\xi) = \int e^{i\phi_2(x,t,\eta)}\psi(\xi)a(x,t,\xi,\eta)\hat{g}(\eta)\,d\eta = T^{\phi_2}_{\psi(\xi)a(\cdot,\cdot,\xi,\cdot)}g(x,t). \tag{4-6}$$

Observe that the support of $\mathfrak{a}_g(x,t,\xi)$ in (x,t) coincides with the support of a in (x,t) and then it is contained in $K\times\mathcal{I}$. Using the Fourier inversion formula,

$$\begin{aligned}
T_{\mathfrak{a}_g}^{\phi_1}(f)(x,t) &= \int e^{i\phi_1(x,t,\xi)}\mathfrak{a}_g(x,t,\xi)\zeta(\xi)\hat{f}(\xi)\,d\xi\\
&= C_d\int e^{i\phi_1(x,t,\xi)}\int \hat{\mathfrak{a}}_g(x,t,\lambda)e^{i\xi\cdot\lambda}\,d\lambda\zeta(\xi)\hat{f}(\xi)\,d\xi\\
&= C_d\int \hat{\mathfrak{a}}_g(x,t,\lambda)\int e^{i\phi_1(x,t,\xi)}\zeta(\xi)\widehat{f(\cdot-\lambda)}(\xi)\,d\xi\,d\lambda\\
&= C_d\int \hat{\mathfrak{a}}_g(x,t,\lambda)T_\zeta^{\phi_1}(f(\cdot-\lambda))(x,t)\,d\lambda.
\end{aligned}$$

Note that $\hat{\mathfrak{a}}_g(x,t,\lambda)$ has the same compact support in x as $\mathfrak{a}_g(x,t,\lambda)$. Let us consider a function $\chi\in C_0^\infty$ such that $\chi=1$ on $\operatorname{supp}_x(\hat{\mathfrak{a}}_g)$. Then,

$$\begin{aligned}
T_{\mathfrak{a}_g}^{\phi_1}(f)(x,t) &= C_d\int \hat{\mathfrak{a}}_g(x,t,\lambda)\chi(x)T_\zeta^{\phi_1}(f(\cdot-\lambda))(x,t)\,d\lambda\\
&= C_d\int \hat{\mathfrak{a}}_g(x,t,\lambda)T_{\chi\zeta}^{\phi_1}(f(\cdot-\lambda))(x,t)\,d\lambda.
\end{aligned}$$

Hölder's inequality implies that

$$\begin{aligned}
&\|T_{\mathfrak{a}_g}^{\phi_1}(f)(x,t)\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)}\\
&\qquad\lesssim \int \|\hat{\mathfrak{a}}_g(x,t,\lambda)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\|T_{\chi\zeta}^{\phi_1}(f(\cdot-\lambda))(x,t)\|_{L^{p_1}(\mathbb{R}^d_x\times\mathbb{R}_t)}\,d\lambda\\
&\qquad\lesssim \int \|\hat{\mathfrak{a}}_g(x,t,\lambda)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\|f(\cdot-\lambda)\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\,d\lambda.
\end{aligned}$$

Observe that, as remarked by the referee, since $T_{\chi\zeta}^{\phi_1}$ has a symbol of order $-\infty$,

$$\|T_{\chi\zeta}^{\phi_1}(f(\cdot-\lambda))(x,t)\|_{L^{p_1}(\mathbb{R}^d_x\times\mathbb{R}_t)}\lesssim\|f(\cdot-\lambda)\|_{L^{p_1}_{s_1}} = \|((1-\Delta_x)^{s_1/2}f)(\cdot-\lambda)\|_{L^{p_1}}.$$

Indeed, the operator $(1-\Delta_x)^{s_1/2}$ and the translation operator $\tau_\lambda : f\mapsto f(\cdot-\lambda)$ commute. Since the Lebesgue norms are also invariant by the action of τ_λ,

$$\|T_{\chi\zeta}^{\phi_1}(f(\cdot-\lambda))(x,t)\|_{L^{p_1}(\mathbb{R}^d_x\times\mathbb{R}_t)}\lesssim\|f\|_{L^{p_1}_{s_1}}.$$

In order to estimate the norm $\|\hat{\mathfrak{a}}_g(x,t,\lambda)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}$, observe that

$$\hat{\mathfrak{a}}_g(x,t,\lambda)=\int e^{i\lambda\xi}\mathfrak{a}_g(x,t,\lambda)\,dx=\langle\lambda\rangle^{-2N}\int e^{i\lambda\xi}(I-\Delta_\xi)^N\mathfrak{a}_g(x,t,\xi)\,d\xi.$$

Then

$$\|\hat{\mathfrak{a}}_g(x,t,\lambda)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\lesssim\langle\lambda\rangle^{-2N}\int_{\mathfrak{S}}\|(I-\Delta_\xi)^N\mathfrak{a}_g(x,t,\xi)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\,d\xi,$$

where we have denoted the support of the symbol a in ξ by $\mathfrak{S}$. We can compute

$$(I-\Delta_\xi)^N\mathfrak{a}_g(x,t,\xi)=(I-\Delta_\xi)^N(\psi(\xi)T^{\phi_2}_{a(\cdot,\cdot,\xi,\cdot)}g(x,t))=T^{\phi_2}_{\varkappa(\xi)}g(x,t).$$

Note that the symbol of $T^{\phi_2}_{\varkappa(\xi)}$ is given by

$$\varkappa(\xi):=(1-\Delta_\xi)^N[\psi(\xi)a(\cdot,\cdot,\xi,\cdot)],$$

and is a linear combination of symbols of order $\leq m=m_1+m_2\leq m_2$, where we have used that $m_1\leq 0$. Thus, $\varkappa$ is a symbol of order m_2 and we have the smoothing estimate

$$\|T^{\phi_2}_{\varkappa(\xi)}g(x,t)\|_{L^{p_2}}\lesssim\|g\|_{L^{p_2}_{s_2}}.$$

By taking $N>d/2$,

$$\|T^{\phi_1}_{\mathfrak{a}_g}(f)(x,t)\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)}\lesssim\int\|\hat{\mathfrak{a}}_g(x,t,\lambda)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\,d\lambda.$$

Summarising, we have proved that

$$\begin{aligned}
\|T^{\phi_1,\phi_2}_{a_1}(f,g)\|_{L^p(K\times\mathcal{I})}&\\
&\lesssim\int\langle\lambda\rangle^{-2N}\int_{\mathfrak{S}}\|(I-\Delta_\xi)^N\mathfrak{a}_g(x,t,\xi)\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\,d\xi\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\,d\lambda\\
&=\int\langle\lambda\rangle^{-2N}\int_{\mathfrak{S}}\|T^{\phi_2}_{\varkappa(\xi)}g\|_{L^{p_2}(\mathbb{R}^d_x\times\mathbb{R}_t)}\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\,d\lambda\\
&\lesssim\int\langle\lambda\rangle^{-2N}\,d\lambda\sup_{\xi\in\mathfrak{S}}\|T_{\varkappa(\xi)}\|_{\mathscr{B}(L^{p_2}_{s_2},L^{p_2})}|\mathfrak{S}|\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\\
&\lesssim_{K\times\mathcal{I}\times\mathfrak{S}}\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.
\end{aligned}$$

Thus, the previous estimate proves (4-1).

4.2. ***An estimate due to Bourgain.*** In our further analysis for high-frequencies we will make a continuous decomposition of the operator $T^{\phi_1,\phi_2}_{a_1}$ and we will estimate its norm using certain maximal functions. Of crucial interest for us will be the next estimate due to Bourgain.

Theorem 4.2 [Bourgain 1986]. *Let $K \in L^1(\mathbb{R}^d)$ be such that its Fourier transform $\widehat{K}$ is differentiable. Define*

$$\alpha_j = \sup_{|\xi|\sim 2^j} |\widehat{K}(\xi)|, \quad \beta_j = \sup_{|\xi|\sim 2^j} |\langle \nabla \widehat{K}(\xi), \xi\rangle|, \tag{4-7}$$

where $\nabla K = (\partial_{x_i} K)_{1\le i\le d}$ denotes the gradient of K, and $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$ denotes the inner product on $\mathbb{R}^d$. Then, we have the estimate, on the maximal operator associated to K,

$$\Big\| \sup_{t>0} |f * K_t| \Big\|_{L^2(\mathbb{R}^d)} \le C\,\Gamma(K)\|f\|_{L^2(\mathbb{R}^d)} \tag{4-8}$$

for every $f \in \mathscr{S}(\mathbb{R}^d)$, where

$$\Gamma(K) = \sum_{j\in\mathbb{Z}} \alpha_j^{1/2}(\alpha_j + \beta_j)^{1/2}. \tag{4-9}$$

There are several reasons Theorem 4.2 is interesting by itself. First, the constant $\Gamma(K)$ in (4-9) is sharp in some sense; see [Bourgain 1986, page 1475]. On the other hand, it has been useful in the analysis of the L^2-theory of maximal functions, for convex bodies and for the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$; see, e.g., the introduction of [Cardona 2024].

Remark 4.3 [Stein 1993, pages 27–29]. Recall that if $\psi \in \mathscr{S}(\mathbb{R}^d)$ is such that its Fourier transform $\widehat{\psi}$ is supported in an annulus isolating the origin (and then $\widehat{\psi}(0) = \int \psi(x)\,dx = 0$), one has the L^p-boundedness property

$$\left\| \left(\int_{\mathbb{R}^d_x} |u * \psi_t(x)|^2\,dx \right)^{1/2} \right\|_{L^p(\mathbb{R}^d_x)} \lesssim \|u\|_{L^p(\mathbb{R}^d_x)}, \quad \psi_t := t^{-d}\psi(t^{-1}\cdot). \tag{4-10}$$

This fact will be used later.

4.3. ***Analysis of the high frequencies.*** The analysis of the high frequencies consists in proving that

$$\|T_a^{\phi_1,\phi_2}(\mu(D)f, \mu(D)g)\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d)}\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d)}, \tag{4-11}$$

where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad s_i = m_i + (d-1)\left(\frac{1}{2} - \frac{1}{p_i}\right) - \sigma_i, \quad m = m_1 + m_2, \quad m_i < 0.$$

This is the remaining main step in the proof of Theorem 2.6.

Proof of Theorem 2.6. Since the analysis of the low frequencies has been carried out in Section 4.1, according to the notation above, we need to prove the estimate

$$\|I_1\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)} + \|I_2\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)} + \|I_3\|_{L^p(\mathbb{R}^d_x\times\mathbb{R}_t)} \lesssim \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.$$

We recall that the first term I_1 can be decomposed as (see Remark 3.5)

$$I_1 = \sum_{j=1}^{3} \int_{\kappa}^{\infty} \int_{0}^{1} \chi_{(0,1)}(rt)(Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(P_t^v S_{r,t}^j g)(x,s) m(t,x,s) \frac{dt\,dr}{tr}.$$

The operators P_t^v and Q_t^u are uniformly $L^p(\mathbb{R}_x^d)$-bounded with operators norms growing polynomially in $|u|$ and $|v|$, respectively (see Remark 3.6). Consequently, the following $L^{p_2}(\mathbb{R}_x^d)$-estimates are satisfied:

- $\|P_t^v S_{r,t}^3 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d)} \lesssim_v \|S_{r,t}^3 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d)}$,
- $\|P_t^v S_{r,t}^2 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d)} \lesssim_v \|S_{r,t}^2 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d)}$,
- $\|Q_t^u T_{\nu_{m_1}}^{\phi_1} f(x,s)\|_{L^{p_1}(\mathbb{R}_x^d)} \lesssim_u \|T_{\nu_{m_1}}^{\phi_1} f(x,s)\|_{L^{p_1}(\mathbb{R}_x^d)}$.

As a consequence, the following $L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)$-estimates hold:

- $\|P_t^v S_{r,t}^3 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim_v \|S_{r,t}^3 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)}$,
- $\|P_t^v S_{r,t}^2 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim_v \|S_{r,t}^2 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)}$,
- $\|Q_t^u T_{\nu_{m_1}}^{\phi_1} f(x,s)\|_{L^{p_1}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim_u \|T_{\nu_{m_1}}^{\phi_1} f(x,s)\|_{L^{p_1}(\mathbb{R}_x^d \times \mathbb{R}_s)}$.

Since the operators $S_{r,t}^1$, $S_{r,t}^2$ and $S_{r,t}^3$ are Fourier integral operators with phase function $\phi_2(x,s,\eta)$ and of order $\leq m_2$, and the order of the Fourier integral operator $T_{\nu_{m_1}}^{\phi_1}$ equals m_1, in view of the hypothesis (H1) of Theorem 2.6, we have that

- $\|S_{r,t}^3 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim r^{m_2} t^{\varepsilon} \|g\|_{L_{s_2}^{p_2}(\mathbb{R}_x^d)}$,
- $\|S_{r,t}^2 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim r^{m_2+\varepsilon} t^{\varepsilon} \|g\|_{L_{s_2}^{p_2}(\mathbb{R}_x^d)}$,
- $\|T_{\nu_{m_1}}^{\phi_1} f(x,s)\|_{L^{p_1}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim \|f\|_{L_{s_1}^{p_1}(\mathbb{R}_x^d)}$.

Consequently, we have proved that, for $\varepsilon \in \left(0, \frac{1}{2}\right)$,

- $\|P_t^v S_{r,t}^3 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim r^{m_2} t^{\varepsilon} \|g\|_{L_{s_2}^{p_2}(\mathbb{R}_x^d)}$,
- $\|P_t^v S_{r,t}^2 g(x,s)\|_{L^{p_2}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim r^{m_2+\varepsilon} t^{\varepsilon} \|g\|_{L_{s_2}^{p_2}(\mathbb{R}_x^d)}$,
- $\|Q_t^u T_{\nu_{m_1}}^{\phi_1} f(x,s)\|_{L^{p_1}(\mathbb{R}_x^d \times \mathbb{R}_s)} \lesssim \|f\|_{L_{s_1}^{p_1}(\mathbb{R}_x^d)}$.

Below,

$$\|m\|_{L^\infty} := \|m\|_{L_U^\infty}$$

denotes the L^∞-norm of m as a function of (t,x,s). So m is bounded polynomially

as a function of $U = 1 + |u| + |v|$. Observe that

$$\begin{aligned}
&\Big\| \int_\kappa^\infty \int_0^1 \chi_{(0,1)}(rt)(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^v S^3_{r,t} g)(x,s) m(t,x,s) \frac{dt\,dr}{tr} \Big\|_{L^p(K\times\mathcal{I})} \\
&\qquad \le \int_\kappa^\infty \int_0^{1/r} \|(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^v S^3_{r,t} g)(x,s) m(t,x,s)\|_{L^p(K\times\mathcal{I})} \frac{dt\,dr}{tr} \\
&\qquad \lesssim \int_\kappa^\infty \int_0^{1/r} \|(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)\|_{L^{p_1}(K\times\mathcal{I})} \\
&\qquad\qquad \times \|(P_t^v S^3_{r,t} g)(x,s)\|_{L^{p_2}(K\times\mathcal{I})} \|m\|_{L^\infty} \frac{dt\,dr}{tr} \\
&\qquad \lesssim \int_\kappa^\infty \int_0^{1/r} r^{m_2} t^\varepsilon \frac{dt\,dr}{tr} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.
\end{aligned}$$

We have used that

$$\int_\kappa^\infty \int_0^{1/r} r^{m_2} t^\varepsilon \frac{dt\,dr}{tr} < \infty, \tag{4-12}$$

since $m_2 < 0$. Also, we have the estimations

$$\begin{aligned}
&\Big\| \int_\kappa^\infty \int_0^1 \chi_{(0,1)}(rt)(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^v S^2_{r,t} g)(x,s) m(t,x,s) \frac{dt\,dr}{tr} \Big\|_{L^p(K\times\mathcal{I})} \\
&\qquad \le \int_\kappa^\infty \int_0^{1/r} \|(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^v S^2_{r,t} g)(x,s) m(t,x,s)\|_{L^p(K\times\mathcal{I})} \frac{dt\,dr}{tr} \\
&\qquad \lesssim \int_\kappa^\infty \int_0^{1/r} \|(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)\|_{L^{p_1}(K\times\mathcal{I})} \\
&\qquad\qquad \times \|(P_t^v S^2_{r,t} g)(x,s)\|_{L^{p_2}(K\times\mathcal{I})} \|m\|_{L^\infty} \frac{dt\,dr}{tr} \\
&\qquad \lesssim \int_\kappa^\infty \int_0^{1/r} r^{m_2+\varepsilon} t^\varepsilon \frac{dt\,dr}{tr} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.
\end{aligned}$$

Now, since $S^1_{r,t} = r^{m_2} P_{1,t} Q_{tr} T^{\phi_2}_\gamma$, in order to finish the estimate of I_1 we need to deal with the norm

$$\begin{aligned}
&\Big\| \int_\kappa^\infty r^{m_2} \int_0^1 \chi_{(0,1)}(rt)(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s) \\
&\qquad\qquad \times (P_t^v [P_{1,t} Q_{tr} T^{\phi_2}_\gamma] g)(x,s) m(t,x,s) \frac{dt\,dr}{tr} \Big\|_{L^p(K\times\mathcal{I})}.
\end{aligned}$$

To estimate the previous norm let us take an arbitrary $h \in L^{p'}(K\times\mathcal{I})$ assumed to be supported in $K\times\mathcal{I}$, when considering its extension by zero to $\mathbb{R}^d \times \mathbb{R}$. It suffices

to prove that

$$|\mathscr{I}| = \left| \int_{K\times\mathcal{I}} \int_0^1 h(x,s)(Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(\widetilde{P}_t^v [R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s) \frac{dt}{t}\, dx\, ds \right|$$
$$\lesssim \|h\|_{L^{p'}(K\times\mathcal{I})} \|f\|_{L_{s_1}^{p_1}(\mathbb{R}_x^d)} \|g\|_{L_{s_2}^{p_2}(\mathbb{R}_x^d)},$$

where

$$R_t(G)(x) = \int_\kappa^{1/t} r^{m_2} Q_{tr}(G)(x) \frac{dr}{r} \quad \text{and} \quad \widetilde{P}_t^v := P_t^v P_{1,t}.$$

Let us write

$$\mathscr{I} = \int_{K\times\mathcal{I}} \int_0^1 h(x,s)(Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(\widetilde{P}_t^v [R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s) \frac{dt}{t}\, dx\, ds$$
$$= I + II,$$

where I and II will be defined later but the idea will be to transfer a little bit of the information of the multipliers Q_t^u and $\widetilde{P}_t^u$ to the product hm. First, we assume without loss of generality that the x-support of h is contained in the x-support of $m(t,x,s)$. We write

$$\widetilde{P}_t^v = P_t^{v,2} + Q_t^{v,2},$$

where the symbol of $P_t^{v,2}$ is given by $\widehat{\theta}_2(t\xi)e^{it\xi\cdot v}$. The support of $\widehat{\theta}_2$ is chosen small enough to assure that, by choosing a suitable radial function $\widehat{\psi}_2$ supported in an annulus, the identity

$$(Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s)$$
$$= \widetilde{Q}_t (Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s)$$

holds. We have denoted by $\widetilde{Q}_t$ the multiplier with symbol $\widehat{\psi}(t\xi)$. In a similar way, we can choose a radial function $\widehat{\theta}_3$ supported in a ball such that

$$(Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s)$$
$$= \widetilde{P}_t (Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s),$$

with $\widetilde{P}_t$ denoting the multiplier with symbol $\widehat{\theta}_3(t\xi)$. We have that

$$|\mathscr{I}| = \left| \int_{K\times\mathcal{I}} \int_0^1 h(x,s)(Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(\widetilde{P}_t^v [R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s) \frac{dt}{t}\, dx\, ds \right|$$
$$\leq \left| \int_{K\times\mathcal{I}} \int_0^1 h(x,s)(Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s) \frac{dt}{t}\, dx\, ds \right|$$
$$+ \left| \int_{K\times\mathcal{I}} \int_0^1 h(x,s)(Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \right.$$
$$\left. \times m(t,x,s) \frac{dt}{t}\, dx\, ds \right|$$

$$= \left| \int_{K\times\mathcal{I}} \int_0^1 \widetilde{Q}_t (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times h(x,s)m(t,x,s)\frac{dt}{t}\,dx\,ds \right|$$

$$+ \left| \int_{K\times\mathcal{I}} \int_0^1 \widetilde{P}_t (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times h(x,s)m(t,x,s)\frac{dt}{t}\,dx\,ds \right|$$

$$= \left| \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times \widetilde{Q}_t(h(\cdot,s)m(t,\cdot,s))(x)\frac{dt}{t}\,dx\,ds \right|$$

$$+ \left| \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times \widetilde{P}_t(h(\cdot,s)m(t,\cdot,s))(x)\frac{dt}{t}\,dx\,ds \right|.$$

Let us define

$$I = \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times \widetilde{Q}_t(h(\cdot,s)m(t,\cdot,s))(x)\frac{dt}{t}\,dx\,ds,$$

$$II = \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times \widetilde{P}_t(h(\cdot,s)m(t,\cdot,s))(x)\frac{dt}{t}\,dx\,ds.$$

Now, consider the multiplication operator

$$M_m G \equiv M_m^{t,s} G := m(t,x,s)G. \tag{4-13}$$

The idea will be to estimate I and II by carrying out an analysis over the commutators $[\widetilde{P}_t, M_m]$ and $[\widetilde{Q}_t, M_m]$.

Starting with I,

$$I = \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s)m(t,x,s) \times (\widetilde{Q}_t h(\cdot,s))(x)\frac{dt}{t}\,dx\,ds \tag{4-14}$$

$$+ \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times ([\widetilde{Q}_t, M_m]h(\cdot,s))(x)\frac{dt}{t}\,dx\,ds$$

$$= I_4 + I_5,$$

with

$$I_5 = \int_{K\times\mathcal{I}}\int_0^1 (Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times [\widetilde{Q}_t, M_m](h(\cdot,s))(x)\,\frac{dt}{t}\,dx\,ds.$$

Similarly,

$$\begin{aligned} II &= \int_{K\times\mathcal{I}}\int_0^1 (Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s)m(t,x,s) \times (\widetilde{P}_t h(\cdot,s))(x)\,\frac{dt}{t}\,dx\,ds \\ &\quad + \int_{K\times\mathcal{I}}\int_0^1 (Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times [\widetilde{P}_t, M_m](h(\cdot,s))(x)\,\frac{dt}{t}\,dx\,ds \\ &= I_6 + I_7, \end{aligned}$$

with

$$I_7 = \int_{K\times\mathcal{I}}\int_0^1 (Q_t^u T_{\nu_{m_1}}^{\phi_1} f)(x,s)(Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \times [\widetilde{P}_t, M_m]h(\cdot,s)(x)\,\frac{dt}{t}\,dx\,ds.$$

The terms involving the commutators, namely I_5 and I_7, can be estimated due to the $L^{p'}$-boundedness of $[\widetilde{P}_t, M_m]$ and of $[\widetilde{Q}_t, M_m]$. Indeed, they satisfy the boundedness properties

$$\begin{aligned} \|[\widetilde{P}_t, M_m]h(\cdot,s)\|_{L^{p'}(\mathbb{R}^d_x)} &\lesssim t\|h(\cdot,s)\|_{L^{p'}(\mathbb{R}^2_x)}, \\ \|[\widetilde{Q}_t, M_m]h(\cdot,s)\|_{L^{p'}(\mathbb{R}^d_x)} &\lesssim t\|h(\cdot,s)\|_{L^{p'}(\mathbb{R}^d_x)}. \end{aligned}$$

In view of the hypothesis (H1) in Theorem 2.6,

$$\|P_t^{v,2}[R_t T_\gamma^{\phi_2}]g\|_{L^{p_2}(K\times\mathcal{I})} \lesssim C(v)\|T_\gamma^{\phi_2} g\|_{L^{p_2}(\mathbb{R}^d\times\mathbb{R})} \lesssim C(v)\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}, \tag{4-15}$$

and also

$$\|Q_t^{v,2}[R_t T_\gamma^{\phi_2}]g\|_{L^{p_2}(K\times\mathcal{I})} \lesssim C(v)\|T_\gamma^{\phi_2} g\|_{L^{p_2}(\mathbb{R}^d\times\mathbb{R})} \lesssim C(v)\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}. \tag{4-16}$$

Thus, the analysis above implies that

$$|I_5| + |I_7| \lesssim C(u,v,\varepsilon)\|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}\|h\|_{L^{p'}(\mathbb{R}^d_x\times\mathbb{R}_s)}.$$

In order to finish our analysis of $\mathscr{I}$ let us estimate the terms I_4 and I_6. By the

Cauchy–Schwarz inequality,

$$
\begin{aligned}
|I_4| &= \left| \int_{K\times\mathcal{I}} \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \right. \\
&\qquad \left. \times m(t,x,s)(\widetilde{Q}_t h(\cdot,s))(x)\,\frac{dt}{t}\,dx\,ds \right| \\
&= \left| \int_{\mathcal{I}} \int_K \int_0^1 (Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) \right. \\
&\qquad \left. \times m(t,x,s)(\widetilde{Q}_t h(\cdot,s))(x)\,\frac{dt}{t}\,dx\,ds \right| \\
&\le \left(\int_{\mathcal{I}} \int_K \int_0^1 |(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)|^2\,\frac{dt}{t}\,dx\,ds \right)^{1/2} \\
&\quad \times \left(\int_{\mathcal{I}} \int_K \int_0^1 |(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s)(\widetilde{Q}_t h(\cdot,s))(x)|^2\,\frac{dt}{t}\,dx\,ds \right)^{1/2}.
\end{aligned}
$$

For the first term, Minkowski's integral inequality and Remark 4.3 imply that

$$
\begin{aligned}
&\left(\int_{\mathcal{I}} \int_K \int_0^1 |(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)|^2\,\frac{dt}{t}\,dx\,ds \right)^{1/2} \\
&\qquad \le \int_{\mathcal{I}} \int_K \left(\int_0^1 |(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)|^2\,\frac{dt}{t} \right)^{1/2} dx\,ds \\
&\qquad \le \int_{\mathcal{I}} \left(\int_K \left(\int_0^1 |(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)|^2\,\frac{dt}{t} \right)^{p_1/2} dx \right)^{1/p_1} \left(\int_K dx \right)^{1/p_1'} ds \\
&\qquad \lesssim \int_{\mathcal{I}} \left\| \left(\int_0^1 |(Q_t^u T^{\phi_1}_{\nu_{m_1}} f)(x,s)|^2\,\frac{dt}{t} \right)^{1/2} \right\|_{L^{p_1}(K)} ds \\
&\qquad \lesssim \int_{\mathcal{I}} \|T^{\phi_1}_{\nu_{m_1}} f\|_{L^{p_1}}\,ds \\
&\qquad \lesssim \left(\int_{\mathcal{I}} \|T^{\phi_1}_{\nu_{m_1}} f\|_{L^{p_1}}^{p_1}\,ds \right)^{1/p_1} \left(\int_{\mathcal{I}} ds \right)^{1/p_1'} ds.
\end{aligned}
$$

For the second factor, observe that

$$
\begin{aligned}
&\left(\int_{\mathcal{I}} \int_K \int_0^1 |(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s) m(t,x,s)(\widetilde{Q}_t h(\cdot,s))(x)|^2\,\frac{dt}{t}\,dx\,ds \right)^{1/2} \\
&\quad \le \|m\|_{L^\infty} \left(\int_{\mathcal{I}} \int_K \int_0^1 |(P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s)(\widetilde{Q}_t h(\cdot,s))(x)|^2\,\frac{dt}{t}\,dx\,ds \right)^{1/2}.
\end{aligned}
$$

Let

$$
A_t = (P_t^{v,2}[R_t T_\gamma^{\phi_2}]g)(x,s), \quad B_t = (\widetilde{Q}_t h(\cdot,s))(x).
$$

We can estimate the integral involving these two terms using repeatedly Hölder's inequality after using Minkowski's inequality. Indeed,

$$\begin{aligned}
&\Big(\int_{\mathcal{I}}\int_K\int_0^1 |A_t B_t|^2\,\frac{dt}{t}\,dx\,ds\Big)^{1/2}\\
&\quad\le \int_{\mathcal{I}}\int_K\Big(\int_0^1 |A_t B_t|^2\,\frac{dt}{t}\Big)^{1/2}\,dx\,ds\\
&\quad\le \int_{\mathcal{I}}\int_K \sup_{0<t<1}|A_t|\Big(\int_0^1 |B_t|^2\,\frac{dt}{t}\Big)^{1/2}\,dx\,ds\\
&\quad\le \Big(\int_{\mathcal{I}}\int_K \sup_{0<t<1}|A_t|^p\,dx\,ds\Big)^{1/p}\Big(\int_{\mathcal{I}}\int_K\Big(\int_0^1 |B_t|^2\,\frac{dt}{t}\Big)^{p'/2}\,dx\,ds\Big)^{1/p'}\\
&\quad\le \Big(\Big(\int_{\mathcal{I}}\int_K \sup_{0<t<1}|A_t|^{pp_2/p}\,dx\,ds\Big)^{p/p_2}\Big(\int_{K\times\mathcal{I}} dx\,ds\Big)^{1/(p_2/p)'}\Big)^{1/p}\\
&\qquad\qquad\times\Big\|\Big(\int_0^1 |B_t|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^{p'}}\\
&\quad\lesssim \big\|\sup_{0<t<1}|A_t|\big\|_{L^{p_2}}\Big\|\Big(\int_0^1 |B_t|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^{p'}}.
\end{aligned}$$

Note that the use of Hölder's inequality before is justified by the fact that $\frac{p_2}{p}\ge 1$, since $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}\ge\frac{1}{p_2}$. In view of the hypothesis (H1) in Theorem 2.6 and of Remark 4.3,

$$\begin{aligned}
\big\|\sup_{0<t<1}|A_t|\big\|_{L^{p_2}}\Big\|\Big(\int_0^1 |B_t|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^{p'}} &\lesssim C(v)\|T^{\phi_2}_\gamma g\|_{L^{p_2}(\mathbb{R}^d\times\mathbb{R})}\|h\|_{L^{p'}(\mathbb{R}^d\times\mathbb{R})}\\
&\lesssim C(v)\|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}\|h\|_{L^{p'}(\mathbb{R}^d_x\times\mathbb{R}_s)}.
\end{aligned}$$

Observe that we have used that the estimate

$$\big\|\sup_{0<t<1}|A_t|\big\|_{L^{p_2}}\lesssim C(v)\|T^{\phi_2}_\gamma g\|_{L^{p_2}(\mathbb{R}^d\times\mathbb{R})} \tag{4-17}$$

holds. Indeed, in order to prove this fact observe that

$$A_t=(P_t^{v,2}[R_tT^{\phi_2}_\gamma]g)(x,s)=R_t(P_t^{v,2}[T^{\phi_2}_\gamma]g)(x,s).$$

Now, the action of the multiplier $R_tP_t^{v,2}$ on $G=G_s=T^{\phi_2}_\gamma g(\cdot,s)$ is given by

$$\begin{aligned}
R_t(P_t^{v,2}G)(x)&=\int_\kappa^{1/t} s^{m_2}Q_{ts}P_t^{v,2}G(x)\,\frac{ds}{s}\\
&=\int_\kappa^{1/t} s^{m_2}\int e^{ix\cdot\xi}\widehat{\psi}(ts\xi)\widehat{\theta}_2(t\xi)e^{it\xi\cdot v}\widehat{G}(\xi)\,d\xi\,\frac{ds}{s}.
\end{aligned}$$

Since the supports of $\widehat{\psi}$ and of $\widehat{\theta}_2$ are contained in small annuli, on the support of these functions,

$$|\xi| \asymp \frac{1}{ts} \asymp \frac{1}{t},$$

which implies that $s \asymp 1$, and the integral in s is supported on an interval around $s = 1$ and on an annulus isolating $s = 0$. So

$$\begin{aligned}
|R_t(P_t^{v,2}G)(x)| &\lesssim \int_{s\asymp 1} s^{m_2} \left| \int e^{ix\cdot\xi} \widehat{\psi}(ts\xi)\widehat{\theta}_2(t\xi) e^{it\xi\cdot v} \widehat{G}(\xi) d\xi \right| \frac{ds}{s} \\
&\lesssim \int_{s\asymp 1} \left| \int e^{ix\cdot\xi} \widehat{\psi}(ts\xi)\widehat{\theta}_2(t\xi) e^{it\xi\cdot v} \widehat{G}(\xi) d\xi \right| ds \\
&= \int_{s\asymp 1} |\widehat{\psi}(tsD) P_t^{v,2} G(x)|\, ds.
\end{aligned}$$

Let θ_2^v be the kernel with Fourier transform $\widehat{\theta}_2^v(\cdot)$, i.e., it is the symbol of $P_t^{v,2}$, that is,

$$\widehat{\theta}_2^v(t\xi) := \widehat{\theta}_2(t\xi) e^{it\xi\cdot v}.$$

Then

$$\begin{aligned}
&\int_{s\asymp 1} |\widehat{\psi}(tsD) P_t^{v,2} G(x)|\, ds \\
&\qquad = \int_{s\asymp 1} |G * (s^{-n} t^{-n} (\psi(s^{-1}\cdot) * (\theta_2^v)(t^{-1}\cdot)))(x)|\, ds \\
&\qquad = \int_{s\asymp 1} \left| \int G(x-y) s^{-n} t^{-n} (\psi(s^{-1}\cdot) * (\theta_2^v))(t^{-1}y)) dy \right| ds \\
&\qquad \asymp \int_{s\asymp 1} \left| \int G(x-y) t^{-n} (\psi(s^{-1}\cdot) * (\theta_2^v))(t^{-1}y))\, dy \right| ds \\
&\qquad \le \|G\|_{L^\infty} \int_{s\asymp 1} t^{-n} \left| \int (\psi(s^{-1}\cdot) * (\theta_2^v))(t^{-1}y))\, dy \right| ds \\
&\qquad \le \|G\|_{L^\infty} \int_{s\asymp 1} \left| \int (\psi(s^{-1}\cdot) * (\theta_2^v)(y))\, dy \right| ds.
\end{aligned}$$

As a consequence,

$$\Big\| \sup_{0<t<1} |R_t(P_t^{v,2}G)| \Big\|_{L^\infty} \lesssim \|G\|_{L^\infty}. \tag{4-18}$$

Now, let us prove the estimate

$$\Big\| \sup_{0<t<1} |R_t(P_t^{v,2}G)| \Big\|_{L^2} \lesssim \|G\|_{L^2}. \tag{4-19}$$

For this we will use Bourgain's lemma (see Theorem 4.2). We proceed with the proof of (4-19) in the following remark.

Remark 4.4. Note that

$$\begin{aligned} R_t(P_t^{v,2}G)(x) &= \int_{s\asymp 1} s^{m_2} \int e^{ix\cdot\xi}\widehat{\psi}(ts\xi)\widehat{\theta}_2(t\xi)e^{it\xi\cdot v}\widehat{G}(\xi)d\xi\,\frac{ds}{s} \\ &= \int_{s\asymp 1} s^{m_2}\widehat{\psi}(tsD)P_t^{v,2}G(x)\,\frac{ds}{s} \\ &= \int_{s\asymp 1} s^{m_2} G * \big(s^{-n}t^{-n}(\psi(s^{-1}\cdot) * (\theta_2^v)(t^{-1}\cdot))\big)(x)\,\frac{ds}{s} \\ &= \left(G * \int_{s\asymp 1} s^{m_2}s^{-n}t^{-n}(\psi(s^{-1}\cdot) * (\theta_2^v)(t^{-1}\cdot))\,\frac{ds}{s}\right)(x). \end{aligned}$$

We have that the kernel K_t of $R_t P_t^{v,2}$ is given by

$$K_t(x) = \int_{s\asymp 1} s^{m_2}\big(s^{-n}t^{-n}(\psi(s^{-1}\cdot) * (\theta_2^v)(t^{-1}x))\big)\,\frac{ds}{s}. \tag{4-20}$$

Let us write

$$K(x) = \int_{s\asymp 1} s^{m_2}s^{-n}(\psi(s^{-1}\cdot) * (\theta_2^v))(x)\,\frac{ds}{s}.$$

Its Fourier transform is

$$\widehat{K}(\xi) = \int_{s\asymp 1} s^{m_2-n}\mathscr{F}(\psi(s^{-1}\cdot) * (\theta_2^v))(\xi))\,\frac{ds}{s}, \tag{4-21}$$

and

$$\begin{aligned} \alpha_j := \sup_{|\xi|\sim 2^j} |\widehat{K}(\xi)| &\le \int_{s\asymp 1} \|\mathscr{F}(\psi(s^{-1}\cdot) * (\theta_2^v))\|_{L^\infty} s^{m_2-n}\,\frac{ds}{s} \\ &\lesssim \int_{s\asymp 1} \|\psi(s^{-1}\cdot) * (\theta_2^v)\|_{L^1}\,\frac{ds}{s}. \end{aligned}$$

This shows that $\sup_j \sup_{|\xi|\sim 2^j} |\widehat{K}(\xi)|$ is bounded. However, due to the properties of the support of ψ and θ_2^v we can improve this bound as

$$\alpha_j \lesssim \sup_{|\xi|\sim 2^j} \int_{s\asymp 1} |\widehat{\psi}(s\xi)\widehat{\theta}_2^v(\xi)|\,\frac{ds}{s} \lesssim \sup_{|\xi|\sim 2^j} \min\{|\xi|, |\xi|^{-1}\} = 2^{-|j|}.$$

On the other hand

$$\nabla\widehat{K}(\xi) = \left(\int_{s\asymp 1} s^{m_2-n}\mathscr{F}((\psi(s^{-1}\cdot) * (\theta_2^v))ix_j)(\xi))\,\frac{ds}{s}\right)_{1\le j\le d}.$$

Therefore, we have the estimate

$$\sup_{|\xi|\sim 2^j} |\langle\nabla\widehat{K}(\xi),\xi\rangle| \le \sum_{r=1}^{d}\int_{s\asymp 1} s^{m_2-n}|\mathscr{F}((\psi(s^{-1}\cdot) * (\theta_2^v))ix_r)(\xi))|\,|\xi_r|\,\frac{ds}{s}.$$

Observe that, using integration by parts,

$$\begin{aligned}\mathscr{F}\big((\psi(s^{-1}\cdot)*(\theta_2^v)ix_r)(\xi)\big)\xi_r &= \int e^{ix\xi}(\psi(s^{-1}\cdot)*(\theta_2^v))(x)ix_r\xi_r\,dx\\ &= \int \partial_{x_r}(e^{ix\xi})(\psi(s^{-1}\cdot)*(\theta_2^v))(x)x_r\,dx\\ &= -\int e^{ix\xi}\partial_{x_r}[(\psi(s^{-1}\cdot)*(\theta_2^v)(x)]x_r\,dx.\end{aligned}$$

We have proved that

$$\begin{aligned}\sup_{|\xi|\sim 2^j}|\langle\nabla\widehat{K}(\xi),\xi\rangle| &\le \sum_{r=1}^d\int_{s\asymp 1} s^{m_2-n}\big\|\mathscr{F}\big(\partial_{x_r}[(\psi(s^{-1}\cdot)*(\theta_2^v))(x)]x_r\big)\big\|_{L^\infty}\frac{ds}{s}\\ &\le \sum_{r=1}^d\int_{s\asymp 1}\|\partial_{x_r}[(\psi(s^{-1}\cdot)*(\theta_2^v)(x)]x_r\|_{L^1}\frac{ds}{s}<\infty.\end{aligned}$$

We have included the previous computation to emphasise the cancellation of the quantity $|\xi_r|\lesssim 2^j$ after using integration by parts which we found interesting. Nevertheless, one can improve this bound. Indeed, due to the properties of the support of ψ and θ_2^v,

$$Z_r := (\psi(s^{-1}\cdot)*(\theta_2^v))ix_r\in\mathscr{S}(\mathbb{R}^d),$$

and

$$\begin{aligned}\beta_j &\lesssim \sup_{|\xi|\sim 2^j}\sum_{r=1}^d\int_{s\asymp 1}|\mathscr{F}(Z_r)(\xi)|\,|\xi_r|\,ds\\ &\asymp \sup_{|\xi|\sim 2^j}\sum_{r=1}^d\int_{s\asymp 1}|\mathscr{F}(\partial_{x_r}Z_r)(\xi)|\,ds\lesssim \sup_{|\xi|\sim 2^j}\min\{|\xi|,|\xi|^{-1}\}=2^{-|j|}.\end{aligned}$$

As a consequence,

$$\Gamma(K) := \sum_{j\in\mathbb{Z}}\alpha_j^{1/2}(\alpha_j+\beta_j)^{1/2}<\infty. \tag{4-22}$$

In view of Theorem 4.2 we have proved that

$$G\mapsto S^*G := \sup_{0<t<1}|R_t(P_t^{v,2}G)| : L^2\to L^2$$

is bounded.

So, interpolating the L^p-estimates (4-18) and (4-19) for the sublinear operator S^*, defined via

$$S^*G := \sup_{0<t<1}|R_t(P_t^{v,2}G)|,$$

in view of the Marcinkiewicz interpolation theorem we have proved that for any $2 \leq \kappa \leq \infty$,

$$\Big\| \sup_{0<t<1} |R_t(P_t^{v,2}G)| \Big\|_{L^\kappa} \lesssim C_\kappa(v) \|G\|_{L^\kappa}, \tag{4-23}$$

with a polynomial constant $C_\varkappa(v)$ in v. With $\kappa = p_2$ we have proved (4-17) once we take the L^{p_2}-norm in the s-variable on both sides of (4-23). As a consequence,

$$|I_4| \lesssim \|h\|_{L^{p'}(K\times\mathcal{I})} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)},$$

since h is assumed to be supported in $K \times I$. Since the same analysis done before for I_4 also works for I_6,

$$|I_4| + |I_6| \lesssim \|h\|_{L^{p'}(K\times\mathcal{I})} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.$$

We have completed the proof of the estimate

$$|\mathscr{I}| \lesssim \|h\|_{L^{p'}(K\times\mathcal{I})} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.$$

It follows that

$$\|I_1\|_{L^p(K\times\mathcal{I})} \lesssim_{u,v} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.$$

To estimate I_2, note that the hypothesis (H1) in Theorem 2.6 implies that

$$\|Q_t^u T^{\phi_1}_{\nu_{m_1}} f(x,s)\|_{L^{p_1}(K\times\mathcal{I})} \lesssim_u \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)}.$$

Then Hölder's inequality gives

$$\begin{aligned}
\|I_2\|_{L^p(K\times\mathcal{I})} &\leq \int_0^1 \|Q_t^u T^{\phi_1}_{\nu_{m_1}} f(x,s) E_t^2 g(x,s) m(t,x,s)\|_{L^p(K\times\mathcal{I})} \frac{dt}{t} \\
&\leq \int_0^1 \|Q_t^u T^{\phi_1}_{\nu_{m_1}} f(x,s)\|_{L^{p_1}(K\times\mathcal{I})} \|E_t^2 g(x,s) m(t,x,s)\|_{L^{p_2}(K\times\mathcal{I})} \frac{dt}{t} \\
&\leq \int_0^1 \|Q_t^u T^{\phi_1}_{\nu_{m_1}} f(x,s)\|_{L^{p_1}(K\times\mathcal{I})} \|E_t^2 g(x,s)\|_{L^{p_2}(K\times\mathcal{I})} \|m\|_{L^\infty} \frac{dt}{t} \\
&\leq \int_0^1 \|E_t^2 g(x,s)\|_{L^{p_2}(K\times\mathcal{I})} \|m\|_{L^\infty} \frac{dt}{t} \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \\
&\lesssim_u \int_0^1 t^{\varepsilon-1} \|m\|_{L^\infty} dt \|f\|_{L^{p_1}_{s_1}(\mathbb{R}^d_x)} \|g\|_{L^{p_2}_{s_2}(\mathbb{R}^d_x)}.
\end{aligned}$$

The same analysis above applies for I_3. Remember that $\|m\|_{L^\infty} := \|m\|_{L^\infty_U}$ denotes the L^∞-norm of m as a function of (t,x,s) and then it is bounded polynomially as a function of $U = 1 + |u| + |v|$. As a consequence, taking N large enough, we

have from (3-8) that

$$\begin{aligned}\|T_{a_1}^{\phi_1,\phi_2}(\mu(D)f,\mu(D)g)(x,s)\|_{L^p(K\times\mathcal{I})}&\\ &\leq \iint \frac{\|m\|_{L_U^\infty}}{(1+|u|^2+|v|^2)^N}\,du\,dv\|f\|_{L_{s_1}^{p_1}(\mathbb{R}_x^d)}\|g\|_{L_{s_2}^{p_2}(\mathbb{R}_x^d)}\\ &\lesssim_N \|f\|_{L_{s_1}^{p_1}(\mathbb{R}_x^d)}\|g\|_{L_{s_2}^{p_2}(\mathbb{R}_x^d)}.\end{aligned}$$

The proof of Theorem 2.6 is complete. □

Appendix: Bibliographic discussion

The study of the continuity properties for FIOs has attracted considerable attention due to their applications to hyperbolic PDEs. Initial enquiries into global L^p boundedness results can be traced to the pioneering works [Asada and Fujiwara 1978; Peral 1980; Miyachi 1980]. Subsequent research has significantly expanded these results to a wide classes of symbols [Ruzhansky and Sugimoto 2006; 2011; Coriasco and Ruzhansky 2014; Coriasco et al. 2016]. A notable line of enquiry involves establishing global criteria for FIOs within Hörmander symbol classes $S^m_{\rho,\delta}$, specifically by relaxing the traditional $\delta < \rho$ condition in the setting of pseudo-differential operators to allow broader parameter ranges $0 \leq \delta < 1$ and $0 < \rho \leq 1$ [Dos Santos Ferreira and Staubach 2014; Kenig and Staubach 2007; Israelsson et al. 2021; 2023; Rodríguez-López et al. 2014; 2015]. This development was partly inspired by analogous results in the theory of pseudodifferential operators, which demonstrated boundedness without the $\delta < \rho$ restriction [Delgado 2006; Álvarez and Hounie 1990]. For general pseudodifferential operators on graded Lie groups we refer to the work of D. Cardona with M. Ruzhansky and J. Delgado [Cardona et al. 2021]. The question of global parametrisation for Lagrangian distributions has also been explored through alternative geometric and analytical approaches [Laptev et al. 1994; Ruzhansky 2001]. Additionally, global definitions of FIOs have been formulated in specialised settings, such as on compact Lie groups using the representation theory of these groups [Cardona and Ruzhansky 2026].

The systematic theory of FIOs with complex-phase functions originated from [Melin and Sjöstrand 1974], motivated by constructing parametrices for partial differential operators with complex principal symbols [Melin and Sjöstrand 1976]. This framework has found significant application in several complex variables, where the singular structure of the Bergman kernel can be described via asymptotic expansions involving complex-phase FIOs [Boutet de Monvel and Sjöstrand 1976]. The comprehensive analysis of the Bergman kernel itself represents another major area of study [Fefferman 1974; 1976].

A distinct and highly active research direction concerns smoothing estimates for FIOs, particularly following the resolution of the decoupling conjecture by Bourgain

and Demeter [2015]. The conjecture regarding the inherent smoothing properties of FIOs was formally posed by Sogge [1991]. While a complete survey of this extensive field is beyond our scope, the literature includes influential early results [Bourgain 1991] and numerous recent advances that explore various aspects of this smoothing phenomenon, employing diverse methods from harmonic analysis; see [Minicozzi and Sogge 1997; Guth et al. 2020; Beltran et al. 2020; Gao et al. 2023].

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Volume 344 No. 2 October 2026

Local smoothing estimates for bilinear Fourier integral operators 241
DUVÁN CARDONA

Second-order estimates for χ-semiconvex solutions of Hessian equations on Hermitian manifolds 273
XIAOJUAN CHEN, QIANG TU and NI XIANG

Higher-rank instantons sheaves on Fano threefolds 293
GAIA COMASCHI and DANIELE FAENZI

Artin shapes 359
NIKITA KARPENKO and GUANGZHAO ZHU

L-packets and the generic Arthur packet conjectures for even unitary similitude groups 367
YEANSU KIM, MUTHU KRISHNAMURTHY and FREYDOON SHAHIDI

Boundedness of Eisenstein integrals on split rank-one semisimple symmetric spaces 395
SANJOY PUSTI and ISWARYA SITIRAJU

Homotopy-theoretic properties of gyrations 431
STEPHEN THERIAULT

Localization and Toeplitz operators on doubling Fock spaces 449
CHUNXU XU and HONG ZOU

Correction to the article Topological invariance of quantum quaternion spheres 475
BIPUL SAURABH