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SECOND-ORDER ESTIMATES FOR χ-SEMICONVEX SOLUTIONS OF HESSIAN EQUATIONS ON HERMITIAN MANIFOLDS

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We establish the modified concavity inequality for complex Hessian equations under the semiconvexity assumption inspired by Lu (2023) and Zhang (2025) for the real case. Then second-order estimates for admissible solutions of complex Hessian equations on compact Hermitian manifolds with both sides of equations depending on gradient terms are obtained by taking advantage of this crucial inequality.

1. Introduction

Let (M, ω) be a compact complex manifold of complex dimension $n \geq 2$ without boundary, and ω be the Hermitian metric on M. For any $u \in C^\infty(M)$, we consider a smooth real $(1, 1)$-form $\chi'(z, u)$ satisfying the positivity condition $\chi' \geq \varepsilon\omega$ for some $\varepsilon > 0$. Given any smooth $(1, 0)$-form a on M, we obtain a new real $(1, 1)$-form

$$\chi(z, u) = \chi'(z, u) + \sqrt{-1}a \wedge \bar{\partial}u - \sqrt{-1}\bar{a} \wedge \partial u + \sqrt{-1}\partial\bar{\partial}u. \tag{1-1}$$

We study second-order estimates for the complex Hessian equation

$$\chi^k \wedge \omega^{n-k} = \psi(z, Du, u)\omega^n \tag{1-2}$$

for $1 \leq k \leq n$ since the a priori estimates are crucial in the use of the continuity method, which is a natural approach to solve the equation (1-2). Here $\psi(z, v, u) \in C^\infty((T^{1,0}(M))^* \times \mathbb{R})$ is a given positive function, and D is the Chern connection with respect to the metric ω.

The complex Hessian equation (1-2) without gradient term on both sides (i.e., $\chi = \chi' + \sqrt{-1}\partial\bar{\partial}u$, $\psi = \psi(z, u)$) has been extensively studied due to its geometric applications such as J-flow (see [Song and Weinkove 2008]) and quaternionic geometry (see [Alesker and Verbitsky 2010]). Li [2004], and later Błocki [2005], considered the Dirichlet problem for complex Hessian equations on domain in $\mathbb{C}^n$. On compact Kähler manifolds, Hou [2009] proved the existence of a smooth admissible solution of (1-2) when $\chi' = \omega$, $a = 0$ and $\psi = \psi(z)$ by assuming the

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nonnegativity of the holomorphic bisectional curvature of ω. The curvature assumption was subsequently removed by Hou, Ma and Wu [Hou et al. 2010] and Dinew and Kołodziej [2014]. On compact Hermitian manifolds, the authors in [Cherrier 1987; Hanani 1996; Guan and Li 2009; Zhang 2010; Tosatti and Weinkove 2010; Zhang and Zhang 2011] studied complex Monge–Ampère equations. For complex Hessian equations or more general fully nonlinear elliptic equations, the related results can be found in [Sun 2017; Zhang 2017; Kołodziej and Nguyen 2016; Székelyhidi 2018].

Furthermore, complex Hessian equations with gradient terms on the left-hand side arise from the Gauduchon conjecture, which was studied by Székelyhidi, Tosatti and Weinkove [Székelyhidi et al. 2017], see also [Guan and Nie 2023] for related work. Complex Monge–Ampère equations with gradient terms inside the determinant can be found in the study of the Calabi–Yau equation on certain symplectic non-Kähler 4-manifolds, see Fino, Li, Salamon and Vezzoni's work [Fino et al. 2013]. See also [Tosatti and Weinkove 2021], where they solved complex Monge–Ampère equations with gradient terms left open by Yuan [2018]. More results for complex Hessian equations or general fully nonlinear elliptic equations with linear gradient terms on the left-hand side can be found in the works of Zhang [2010], Feng, Ge and Zheng [Feng et al. 2019] and Yuan [2018; 2020].

In the above works, the right-hand side function ψ is independent of Du. In fact, (1-2) with $\psi = \psi(z, Du, u)$ has been relatively less studied. An important case corresponding to $k = n = 2$ (i.e., complex Monge–Ampère equations in two dimensions) was solved in [Fu and Yau 2007; 2008] for a Strominger system on a toric fibration over a K3 surface. For general dimension n, the corresponding results on compact Kähler manifolds or Hermitian manifolds can be found in the works of Phong, Picard and Zhang [Phong et al. 2016; 2017; 2019; 2021] and Chu, Huang and Zhu [Chu et al. 2019a; 2019b]. Further, complex Hessian equations with gradient terms on both sides were studied in [Dong and Li 2021] on Hermitian manifolds, and second-order estimates were obtained for $\chi \in \Gamma_n(M)$. Subsequently, Dong [2021a; 2021b] proved second-order estimates under a weaker convexity assumption $\chi \in \Gamma_{k+1}(M)$ for (1-2) and a class of general Hessian equations on Hermitian manifolds, respectively.

From an analysis point of view, a natural problem is whether we can weaken the assumption $\chi \in \Gamma_{k+1}(M)$ in [Dong 2021b] to a more natural semiconvex assumption inspired by Lu [2023] and Zhang [2025]. Now we define χ-semiconvex function on Hermitian manifolds, which generalizes the definition of semiconvex function for the real case.

Definition 1.1. Let $\chi \in \mathcal{A}^{1,1}(M)$ be a smooth real $(1, 1)$-form defined by (1-1). If there exists a constant $A > 0$ such that for the Hermitian metric ω on M,

$$\chi + A\omega > 0,$$

then χ is semiconvex. We say the function u is χ-semiconvex if χ is semiconvex.

Definition 1.2. A function $u \in C^2(M)$ is called an admissible solution of (1-2) if $\chi \in \Gamma_k(M)$. The Gårding's cone on M is defined by

$$\Gamma_k(M) = \{\chi \in \mathcal{A}^{1,1}(M) : \sigma_i(\chi) > 0 \text{ for all } 1 \le i \le k\},$$

where $\mathcal{A}^{1,1}(M)$ denotes the space of real smooth $(1,1)$-forms on a Hermitian manifold (M, ω).

The main theorem is as follows.

Theorem 1.3. *Let (M, ω) be a compact Hermitian manifold of complex dimension n. Suppose $\chi'(z,u) \ge \varepsilon\omega$ and $u \in C^\infty(M)$ is an admissible, χ-semiconvex solution of* (1-2)*. Then we have uniform second-order estimates*

$$|D\bar{D}u| \le C,$$

where D is the Chern connection with respect to the metric ω, $|D\bar{D}u|$ is the second-order derivative of u with respect to the Chern connection of ω and C is a uniform constant depending on (M, ω), n, k, ε, χ', ψ, a, A, $\sup_M |u|$ and $\sup_M |Du|$.

It is easy to see that $\sigma_{k+1}(\chi) > -A$ for some positive constant A implies the χ-semiconvex condition according to Lemma 2.2 in [Zhang 2025].

Corollary 1.4. *Let (M, ω) be a compact Hermitian manifold of complex dimension n. Suppose $\chi'(z,u) \ge \varepsilon\omega$ and $u \in C^\infty(M)$ is an admissible solution of* (1-2) *with $\sigma_{k+1}(\chi) > -A$ for some positive constant A. Then we have uniform second-order estimates*

$$|D\bar{D}u| \le C,$$

where C is a uniform constant depending on (M, ω), n, k, ε, χ', ψ, a, A, $\sup_M |u|$ and $\sup_M |Du|$.

Due to the presence of Du in both sides of (1-2), the principal challenge in establishing second-order estimates is managing terms such as $|DDu|^2$, $|D\bar{D}u|^2$ and some bad third-order terms when differentiating the equation twice. However, we cannot directly control bad terms as in the real case owing to complex conjugacy. Furthermore, in comparison to calculations on Kähler manifolds, a greater number of undesirable third-order terms in the form of $T * D^3u$ are produced, here T is the torsion of ω on Hermitian manifolds. To surmount these obstacles, it is essential to establish a concavity inequality. Lu [2023] established curvature estimates for semiconvex solutions of real Hessian equations based on the concavity inequality

$$-\sum_{p\neq q} \sigma_k^{pp,qq}\xi_p\xi_q + \frac{\left(\sum_i \sigma_k^{ii}\xi_i\right)^2}{\sigma_k} + \delta_0 \sum_{i>l} \frac{\sigma_k^{ii}\xi_i^2}{\lambda_1} \ge (1-\epsilon)\frac{\sigma_k\xi_1^2}{\lambda_1^2}, \tag{1-3}$$

where $\xi = (\xi_1, \ldots, \xi_n)$ is an arbitrary vector in $\mathbb{R}^n$. Recently, Zhang [2025] improved the concavity inequality (1-3) and derived the following lemma.

Lemma 1.5 [Zhang 2025]. *Assume $\lambda=(\lambda_1,\dots,\lambda_n)\in\Gamma_k\subset\mathbb{R}^n$, $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$ and $\lambda_n>-A$ for a constant $A>0$. Then there exists some constant $C>0$ depending on n, k, σ_k, A and small constant $\delta_0>0$ depending on k such that if $\lambda_1\geq C$, then*

$$-\sum_{p\neq q}\sigma_k^{pp,qq}\xi_p\xi_q+K\frac{\left(\sum_i\sigma_k^{ii}\xi_i\right)^2}{\sigma_k}+2\sum_{i>1}\frac{\sigma_k^{ii}\xi_i^2}{(\lambda_1+A+1)}\geq(1+\delta_0)\frac{\sigma_k^{11}\xi_1^2}{\lambda_1} \tag{1-4}$$

for some sufficiently large $K>0$ (depending on δ_0), where $\xi=(\xi_1,\dots,\xi_n)$ is an arbitrary vector in $\mathbb{R}^n$.

By comparing (1-3) and (1-4), we see that each of these two inequalities has its own advantages. Specifically, we find that the term $\delta_0\sum_{i>l}\sigma_k^{ii}\xi_i^2/\lambda_1$ in (1-3) can be easier to control than the term $2\sum_{i>1}\sigma_k^{ii}\xi_i^2/(\lambda_1+A+1)$ in (1-4) since δ_0 can be chosen arbitrarily small. However, $\sigma_k\xi_1^2/\lambda_1^2$ in (1-3) will lead to the bad term $\sigma_k(\lambda|1)$, which is unmanageable term without the assumption $\chi\in\Gamma_{k+1}(M)$. It seems that the term $(1+\delta_0)\sigma_k^{11}\xi_1^2/\lambda_1\sigma_k$ in (1-4) is a higher choice. For complex Hessian equations, some good third-order terms will be lost due to complex conjugacy. Therefore, we will fully absorb the processing techniques of the above two inequalities, and optimize the coefficients in (1-4) from 2 to $1-\varepsilon_0$ for some small $\varepsilon_0>0$ while retaining the good terms. Then we establish the modified concavity inequality.

Lemma 1.6. *Let (M,ω) be a compact Hermitian manifold of complex dimension n, $W=(\omega_{p\bar{q}})$ be a Hermitian tensor and its eigenvalues $\lambda(W)=(\lambda_1,\dots,\lambda_n)\in\Gamma_k(M)$ satisfying $\lambda_1\geq\cdots\geq\lambda_n>-A$ for a constant $A>0$. For any $\varepsilon_0\in(0,1)$, if $\lambda_1\geq C(A)$ sufficiently large, then for $j=1,\dots,n$ and some positive constant K depending only on k,*

$$\begin{aligned}-\sum_{p\neq q}\frac{\sigma_k^{p\bar{p},q\bar{q}}\omega_{p\bar{p}j}\omega_{q\bar{q}\bar{j}}}{\sigma_k}+K\frac{|D_j\sigma_k|^2}{\sigma_k^2}+(1-\varepsilon_0)\sum_{i>1}\frac{\sigma_k^{i\bar{i}}|\omega_{i\bar{i}j}|^2}{\lambda_1\sigma_k}\\ \geq(1-\varepsilon_0)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{\lambda_1\sigma_k}.\end{aligned} \tag{1-5}$$

Remark 1.7. The selection of auxiliary functions is crucial in dealing with third-order terms. Dong [2021b] employed all eigenvalues of the solution to obtain additional good third-order terms to control bad terms. In our test function, we can choose the largest eigenvalue as the primary term with the help of the concavity inequality (1-5) and the techniques in [Chu 2021; Lu 2023].

The organization of the paper is as follows. In Section 2 we start with some properties for k-th elementary symmetric function and some basic calculations for complex Hessian equations on Hermitian manifolds. In Section 3 we prove the crucial concavity inequality. In Section 4 we give the proof of Theorem 1.3.

2. Preliminaries

For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, the k-th elementary symmetric function is defined by

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

We also set $\sigma_0 = 1$ and $\sigma_k = 0$ for $k > n$ or $k < 0$. The Gårding's cone is defined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_i(\lambda) > 0 \text{ for all } 1 \le i \le k\}.$$

Define $\sigma_k(\lambda|i)$ to be the k-th elementary function with $\lambda_i = 0$ and $\sigma_k(\lambda|ij)$ to be the k-th elementary function with $\lambda_i = \lambda_j = 0$. We list some properties of σ_k which will be used later.

Proposition 1. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $1 \le k \le n$. Then we have*

(1) $\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n$;

(2) $\sigma_{k-1}(\lambda|i) > 0$ *for* $\lambda \in \Gamma_k$ *and* $1 \le i \le n$;

(3) $\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i)$ *for* $1 \le i \le n$;

(4) $\sum_{i=1}^n \partial\left[\frac{\sigma_k}{\sigma_l}\right]^{1/(k-l)} / \partial\lambda_i \ge \left[\frac{C_n^k}{C_n^l}\right]^{1/(k-l)}$ *for* $\lambda \in \Gamma_k$ *and* $0 \le l < k$;

(5) $\left[\frac{\sigma_k}{\sigma_l}\right]^{1/(k-l)}$ *are concave in* Γ_k *for* $0 \le l < k$;

(6) *If* $\lambda \in \Gamma_k$, $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, *then* $\sigma_{k-1}(\lambda|1) \le \sigma_{k-1}(\lambda|2) \le \cdots \le \sigma_{k-1}(\lambda|n)$;

(7) $\sum_{i=1}^n \sigma_{k-1}(\lambda|i) = (n-k+1)\sigma_{k-1}(\lambda)$;

(8) *For* $\lambda \in \Gamma_k$ *and* $\lambda_1 \ge \cdots \ge \lambda_n$, $\lambda_1 \sigma_{k-1}(\lambda|1) \ge \frac{k}{n}\sigma_k(\lambda)$.

Proof. All the properties are well known. For example, see Chapter XV in [Lieberman 1996] or [Huisken and Sinestrari 1999] for (1), (2), (3), (6) and (7); see Lemma 2.2.19 in [Gerhardt 2006] for (4); see [Caffarelli et al. 1985; Lieberman 1996] for (5); see Lemma 2 in [Dong 2021b] for (8). □

The generalized Newton–Maclaurin inequality is as follows.

Proposition 2. *For $\lambda \in \Gamma_m$ and $m > l \ge 0$, $r > s \ge 0$, $m \ge r$, $l \ge s$, we have*

$$\left[\frac{\sigma_m(\lambda)/C_n^m}{\sigma_l(\lambda)/C_n^l}\right]^{1/(m-l)} \le \left[\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s}\right]^{1/(r-s)}.$$

Proof. See [Spruck 2005]. □

To be more clear, we shall rewrite (1-2) in a local coordinate chart. Let $\lambda\{a_{i\bar{j}}\}$ be the eigenvalues of a Hermitian matrix $\{a_{i\bar{j}}\}$. Define $\sigma_k(a_{i\bar{j}}) = \sigma_k(\lambda\{a_{i\bar{j}}\})$. This definition can be naturally extended to complex manifolds. Let $\mathcal{A}^{1,1}(M)$ be the space

of real smooth $(1,1)$-forms on Hermitian manifold (M,ω). For any $\chi\in\mathcal{A}^{1,1}(M)$, we write, in a local coordinate chart $(z^1,\dots,z^n)$,

$$\chi=\sqrt{-1}\chi_{i\bar{j}}dz^i\wedge d\bar{z}^j,$$

and define

$$\sigma_k(\chi)=C_n^k\frac{\chi^k\wedge\omega^{n-k}}{\omega^n}.$$

In particular, in a local normal coordinate system $\omega_{i\bar{j}}=\delta_{ij}$ with the Hermitian metric $\omega=\{\omega_{i\bar{j}}\}$, equation (1-2) can be rewritten by

$$\sigma_k(\chi)=\sigma_k(\chi'_{i\bar{j}}+u_{i\bar{j}}+a_iu_{\bar{j}}+a_{\bar{j}}u_i)=\psi(z,Du,u), \tag{2-1}$$

where a_i is the component of the smooth $(1,0)$-form a on M.

Lemma 2.1. *If $F(A)=f(\lambda_1,\dots,\lambda_n)$ is a symmetric function of the eigenvalues of a Hermitian matrix $A=\{b_{i\bar{j}}\}$, then at a diagonal matrix A with distinct eigenvalues, we have*

$$F^{i\bar{j}}=\delta_{ij}f_i, \tag{2-2}$$

$$F^{i\bar{j},r\bar{s}}\xi_{i\bar{j}k}\xi_{r\bar{s}\bar{k}}=\sum_{i,j}f_{ij}\xi_{i\bar{i}k}\xi_{j\bar{j}\bar{k}}+\sum_{p\neq q}\frac{f_p-f_q}{\lambda_p-\lambda_q}|\xi_{p\bar{q}k}|^2, \tag{2-3}$$

where $F^{i\bar{j}}=\frac{\partial F}{\partial b_{i\bar{j}}}$, $F^{i\bar{j},r\bar{s}}=\frac{\partial^2F}{\partial b_{i\bar{j}}\partial b_{r\bar{s}}}$ and $\xi_{i\bar{j}k}$ is an arbitrary tensor.

Proof. See [Ball 1984; Dong 2021b]. □

In local complex coordinates $(z_1,\dots,z_n)$, the subscripts of a function u always denote the covariant derivatives of u with respect to the Chern connection of ω in the directions of the local frame $(\partial/\partial z^1,\dots,\partial/\partial z^n)$. Namely,

$$u_i=D_iu=D_{\partial/\partial z^i}u,\quad u_{i\bar{j}}=D_{\partial/\partial\bar{z}^j}D_{\partial/\partial z^i}u,\quad u_{i\bar{j}\ell}=D_{\partial/\partial z^\ell}D_{\partial/\partial\bar{z}^j}D_{\partial/\partial z^i}u.$$

Next, we list some well-known results on Hermitian manifolds.

Lemma 2.2. *We have*

$$\begin{aligned}&u_{pj}=u_{jp}+T^k_{pj}u_k,\qquad u_{i\bar{j}\ell}=u_{i\ell\bar{j}}-u_pR_{\ell\bar{j}i}{}^p,\\&u_{p\bar{j}\bar{m}}=u_{p\bar{m}\bar{j}}-\overline{T^q_{mj}}u_{p\bar{q}},\quad u_{i\bar{q}\ell}=u_{\ell\bar{q}i}-T^p_{\ell i}u_{p\bar{q}},\end{aligned} \tag{2-4}$$

and

$$u_{i\bar{j}\ell\bar{m}}=u_{\ell\bar{m}i\bar{j}}+u_{p\bar{j}}R_{\ell\bar{m}i}{}^p-u_{p\bar{m}}R_{i\bar{j}\ell}{}^p-T^p_{\ell i}u_{p\bar{m}\bar{j}}-\overline{T^q_{mj}}u_{\ell\bar{q}i}-T^p_{i\ell}\overline{T^q_{mj}}u_{p\bar{q}}, \tag{2-5}$$

where T^k_{pj} and $R_{\ell\bar{j}i}{}^p$ are the components of the torsion tensor and curvature tensor, respectively.

Proof. See [Tosatti and Weinkove 2019]. □

Define

$$\sigma_k^{p\bar{q}} = \frac{\partial \sigma_k}{\partial \chi_{p\bar{q}}}, \quad \sigma_k^{p\bar{q},r\bar{s}} = \frac{\partial^2 \sigma_k}{\partial \chi_{p\bar{q}} \partial \chi_{r\bar{s}}}, \quad \mathcal{F} = \sum_p \sigma_k^{p\bar{p}}. \tag{2-6}$$

We also use the notation, as in [Phong et al. 2019],

$$|DDu|^2_{\sigma\omega} = \sigma_k^{p\bar{q}} \omega^{m\bar{\ell}} u_{mp} u_{\bar{\ell}\bar{q}}, \quad |D\overline{D}u|^2_{\sigma\omega} = \sigma_k^{p\bar{q}} \omega^{m\bar{\ell}} u_{p\bar{\ell}} u_{m\bar{q}}, \tag{2-7}$$

and

$$|\eta|^2_\sigma = \sigma_k^{p\bar{q}} \eta_p \eta_{\bar{q}} \tag{2-8}$$

for any 1-form η, where $\{\omega^{m\bar{l}}\}$ is the inverse of the metric ω.

The following basic calculations are carried out at a point z on the manifold M, and we use coordinates such that at this point $\omega = \sqrt{-1}\sum_{k,\ell} \delta_{k\ell} dz^k \wedge d\bar{z}^\ell$ and $\{\chi_{i\bar{j}}\}$ is diagonal. Note that $\{\sigma_k^{i\bar{j}}\}$ is diagonal when $\{\chi_{i\bar{j}}\}$ is diagonal. Combining with (2-4) and (2-5), we derive the following two inequalities which will be used later.

Lemma 2.3. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of $\{\chi_{i\bar{j}}\}$ with the ordering $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then*

$$\begin{aligned} &\sigma_k^{p\bar{q}} D_{\bar{q}} D_p \chi_{i\bar{j}} \\ &\quad \geq -\sigma_k^{p\bar{q},r\bar{s}} D_{\bar{j}} \chi_{r\bar{s}} D_i \chi_{p\bar{q}} + \sum_\ell \psi_{v_\ell} D_\ell \chi_{i\bar{j}} + \sum_\ell \psi_{\bar{v}_\ell} D_{\bar{\ell}} \chi_{i\bar{j}} \\ &\qquad - \sigma_k^{p\bar{q}} (T^a_{pi} u_{a\bar{q}\bar{j}} + \overline{T^a_{qj}} u_{p\bar{a}i}) - C(1 + |DDu|^2 + |D\overline{D}u|^2 + \lambda_1 \mathcal{F} + \lambda_1) \\ &\qquad\qquad + \sigma_k^{p\bar{q}} (a_i u_{\bar{j}p\bar{q}} + a_{\bar{j}} u_{ip\bar{q}} - a_p u_{\bar{q}i\bar{j}} - a_{\bar{q}} u_{pi\bar{j}}) \\ &\qquad\qquad + \sigma_k^{p\bar{q}} (a_{ip} u_{\bar{j}\bar{q}} + a_{\bar{j}\bar{q}} u_{ip} - a_{pi} u_{\bar{q}\bar{j}} - a_{\bar{q}\bar{j}} u_{pi}), \end{aligned} \tag{2-9}$$

and

$$\begin{aligned} &\sigma_k^{p\bar{q}} |Du|^2_{p\bar{q}} \\ &\quad \geq 2\,\mathrm{Re}\left\{ \sum_{p,m} (u_{\bar{p}} u_{mp} + u_p u_{m\bar{p}}) \psi_{v_m} \right\} + \tfrac{1}{2} |DDu|^2_{\sigma\omega} + \tfrac{1}{2} |D\overline{D}u|^2_{\sigma\omega} - C - C\mathcal{F}, \end{aligned} \tag{2-10}$$

where a_{ip} is the covariant derivative of a_i.

Proof. See the proof of (19) and (23) in Section 2 of [Dong 2021b]. □

3. A concavity inequality

In this section, we provide a proof of the concavity inequality in the complex case. Since it is similar to the real case in [Zhang 2025], we mainly highlight the differences.

Proof of Lemma 1.6. By (3.27) in [Zhang 2025], we have for $K \geq (k+1)^2$,

$$-\sum_{p\neq q}\frac{\sigma_k^{p\bar{p},q\bar{q}}\omega_{p\bar{p}j}\omega_{q\bar{q}\bar{j}}}{\sigma_k}+K\frac{|D_j\sigma_k|^2}{\sigma_k^2} \geq \frac{(k+2)^2}{(k+1)(k+3)}\frac{|\omega_{1\bar{1}j}|^2}{\lambda_1^2}-\frac{C}{\lambda_1^{1/(k-1)}}\sum_{i>1}\frac{\sigma_k^{i\bar{i}}|\omega_{i\bar{i}j}|^2}{\lambda_1\sigma_k}, \tag{3-1}$$

where C is a positive constant depending on n, k, σ_k, A. Recall that

$$\sigma_k = \lambda_1\sigma_k^{1\bar{1}}+\sigma_k(\lambda|1). \tag{3-2}$$

We note that if

$$\sigma_k(\lambda|1) \geqslant -\frac{\sigma_k}{2(k+2)^2-1}, \tag{3-3}$$

then (3-1) implies (1-5) by choosing $\lambda_1 \geq \left(\frac{C}{1-\varepsilon_0}\right)^{k-1}$.

Define $c_0 = \frac{1}{2(k+2)^2-1}$ and next we only need to consider the case

$$\sigma_k(\lambda|1) < -c_0\sigma_k. \tag{3-4}$$

Under this assumption, referring to the deduction of (3.73) in [Zhang 2025],

$$\begin{aligned}
&-\sum_{p\neq q}\frac{\sigma_k^{p\bar{p},q\bar{q}}\omega_{p\bar{p}j}\omega_{q\bar{q}\bar{j}}}{\sigma_k}+K\frac{|D_j\sigma_k|^2}{\sigma_k^2}+(1-\varepsilon_0)\sum_{i>1}\frac{\sigma_k^{i\bar{i}}|\omega_{i\bar{i}j}|^2}{\lambda_1\sigma_k}\\
&\geq (k+1-l)\left(1-\varepsilon_0-\frac{C_1}{M}\right)\left(1-\frac{C_1}{M}\right)(1+a)^2\frac{\lambda_1\sigma_k^{1\bar{1}}-\sigma_k}{\lambda_1^2\sigma_k}|\omega_{1\bar{1}j}|^2\\
&\qquad+\frac{(k+2)^2}{(k+1)(k+3)}\frac{|\omega_{1\bar{1}j}|^2}{\lambda_1^2}\\
&\quad+\frac{2a^2|\omega_{1\bar{1}j}|^2}{\lambda_1}\left(1-\frac{C_2}{M}\right)\frac{\lambda_1\sigma_k^{1\bar{1}}-\sigma_k}{\lambda_1\sigma_k}-C_3(a^2+(1-\varepsilon_0)(1+a)^2)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{M\lambda_1}\\
&\geq \frac{2(a^2+(1-\varepsilon_0)(1+a)^2)|\omega_{1\bar{1}j}|^2}{\lambda_1}\left(1-\frac{C_2}{(1-\varepsilon_0)M}\right)\frac{\lambda_1\sigma_k^{1\bar{1}}-\sigma_k}{\lambda_1\sigma_k}\\
&\qquad+\frac{(k+2)^2}{(k+1)(k+3)}\frac{|\omega_{1\bar{1}j}|^2}{\lambda_1^2}-C_3(a^2+(1-\varepsilon_0)(1+a)^2)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{M\lambda_1}\\
&\geq (2-\varepsilon_0)\frac{(a^2+(1-\varepsilon_0)(1+a)^2)|\omega_{1\bar{1}j}|^2}{\lambda_1}\frac{\lambda_1\sigma_k^{1\bar{1}}-\sigma_k}{\lambda_1\sigma_k}+\frac{(k+2)^2}{(k+1)(k+3)}\frac{|\omega_{1\bar{1}j}|^2}{\lambda_1^2}\\
&\qquad+\frac{(a^2+(1-\varepsilon_0)(1+a)^2)|\omega_{1\bar{1}j}|^2}{\lambda_1}\left[\frac{\varepsilon_0}{2}\frac{\lambda_1\sigma_k^{1\bar{1}}-\sigma_k}{\lambda_1\sigma_k}-\frac{C_3'\sigma_k^{1\bar{1}}}{\sigma_k M}\right],
\end{aligned}$$

by assuming $\lambda_1 > M^k$ and M large enough such that $1 - \frac{C_2}{(1-\varepsilon_0)M} \geq 1 - \frac{\varepsilon_0}{4}$. Here C_1, C_2, C_3, C_3' are positive constants depending on n, k, σ_k, A.

Under the assumption

$$\sigma_k(\lambda|1) < -c_0\sigma_k,$$

(3-2) implies that

$$\frac{\sigma_k^{1\bar{1}}}{\sigma_k} = \frac{1}{\lambda_1} - \frac{\sigma_k(\lambda|1)}{\lambda_1\sigma_k} > \frac{c_0+1}{\lambda_1}.$$

Then by choosing $M \geq \frac{2C_3'(c_0+1)}{c_0\varepsilon_0}$, we get

$$\begin{aligned}\frac{\varepsilon_0}{2}\frac{\lambda_1\sigma_k^{1\bar{1}} - \sigma_k}{\lambda_1\sigma_k} - \frac{C_3'\sigma_k^{1\bar{1}}}{\sigma_k M} &= \frac{\sigma_k^{1\bar{1}}}{\sigma_k}\left(\frac{\varepsilon_0}{2} - \frac{C_3'}{M}\right) - \frac{\varepsilon_0}{2\lambda_1} \\ &> \frac{1}{\lambda_1}\left[\left(\frac{\varepsilon_0}{2} - \frac{C_3'}{M}\right)(c_0+1) - \frac{\varepsilon_0}{2}\right] \geq 0.\end{aligned}$$

Thus

$$\begin{aligned}(3\text{-}5)\quad &-\sum_{p\neq q}\frac{\sigma_k^{p\bar{p},q\bar{q}}\omega_{p\bar{p}j}\omega_{q\bar{q}\bar{j}}}{\sigma_k} + K\frac{|D_j\sigma_k|^2}{\sigma_k^2} + (1-\varepsilon_0)\sum_{i>1}\frac{\sigma_k^{i\bar{i}}|\omega_{i\bar{i}j}|^2}{\lambda_1\sigma_k} \\ &\geq (2-\varepsilon_0)\frac{(a^2+(1-\varepsilon_0)(1+a)^2)|\omega_{1\bar{1}j}|^2}{\lambda_1}\frac{\lambda_1\sigma_k^{1\bar{1}}-\sigma_k}{\lambda_1\sigma_k} + \frac{(k+2)^2}{(k+1)(k+3)}\frac{|\omega_{1\bar{1}j}|^2}{\lambda_1^2} \\ &= (2-\varepsilon_0)(a^2+(1-\varepsilon_0)(1+a)^2)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{\lambda_1\sigma_k} \\ &\qquad + \left(\frac{(k+2)^2}{(k+1)(k+3)} - (2-\varepsilon_0)[a^2+(1-\varepsilon_0)(1+a)^2]\right)\frac{|\omega_{1\bar{1}j}|^2}{\lambda_1^2}.\end{aligned}$$

If

$$\frac{(k+2)^2}{(k+1)(k+3)} \geq (2-\varepsilon_0)[a^2+(1-\varepsilon_0)(1+a)^2],$$

(3-5) implies that

$$\begin{aligned}&-\sum_{p\neq q}\frac{\sigma_k^{p\bar{p},q\bar{q}}\omega_{p\bar{p}j}\omega_{q\bar{q}\bar{j}}}{\sigma_k} + K\frac{|D_j\sigma_k|^2}{\sigma_k^2} + (1-\varepsilon_0)\sum_{i>1}\frac{\sigma_k^{i\bar{i}}|\omega_{i\bar{i}j}|^2}{\lambda_1\sigma_k} \\ &\qquad \geq (2-\varepsilon_0)(a^2+(1-\varepsilon_0)(1+a)^2)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{\lambda_1\sigma_k} \\ &\qquad \geq (1-\varepsilon_0)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{\lambda_1\sigma_k}.\end{aligned}$$

In the last inequality, we use the fact that

$$a^2+(1-\varepsilon_0)(1+a)^2=(2-\varepsilon_0)\left[\left(a+\frac{1-\varepsilon_0}{1+\varepsilon_0}\right)^2-\frac{(1-\varepsilon_0)^2}{(2-\varepsilon_0)^2}+\frac{1-\varepsilon_0}{2-\varepsilon_0}\right]$$
$$\geq 1-\varepsilon_0-\frac{(1-\varepsilon_0)^2}{2-\varepsilon_0}=\frac{1-\varepsilon_0}{2-\varepsilon_0}.$$

If

$$\frac{(k+2)^2}{(k+1)(k+3)}<(2-\varepsilon_0)[a^2+(1-\varepsilon_0)(1+a)^2],$$

(3-5) implies that

$$-\sum_{p\neq q}\frac{\sigma_k^{p\bar{p},q\bar{q}}\omega_{p\bar{p}j}\omega_{q\bar{q}\bar{j}}}{\sigma_k}+K\frac{|D_j\sigma_k|^2}{\sigma_k^2}+(1-\varepsilon_0)\sum_{i>1}\frac{\sigma_k^{i\bar{i}}|\omega_{i\bar{i}j}|^2}{\lambda_1\sigma_k}$$
$$>\left(1+\frac{1}{(k+1)(k+3)}\right)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{\lambda_1\sigma_k}>(1-\varepsilon_0)\frac{\sigma_k^{1\bar{1}}|\omega_{1\bar{1}j}|^2}{\lambda_1\sigma_k}.$$

Hence we obtain (1-5) and complete the proof. □

4. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. Consider the auxiliary function

$$Q=\log\lambda_1+\varphi(|Du|^2)+\phi(u),$$

where λ_1 is the largest eigenvalue of the matrix $\{\chi_{i\bar{j}}\}$ with respect to ω.

Define

$$\varphi(s)=e^{Ns},\quad 0\leq s\leq S-1,$$

and

$$\phi(t)=e^{\Lambda(-t+T)},\quad -T+1\leq t\leq T-1.$$

Here we set

$$S=\sup_M|Du|^2+1,\quad T=\sup_M|u|+1,$$

and Λ, $N>1$ are large constants to be determined later. Clearly, φ, ϕ satisfy

$$\varphi'=N\varphi>0,\quad \varphi''=\frac{(\varphi')^2}{\varphi},\qquad \phi'=-\Lambda\phi<0,\quad \phi''=\frac{(\phi')^2}{\phi},$$

and

$$\varphi''-2\phi''\frac{(\varphi')^2}{(\phi')^2}=\varphi''-2\frac{(\varphi')^2}{\phi}=N^2e^{Ns}-\frac{2N^2e^{2Ns}}{e^{\Lambda(-t+T)}}>0, \tag{4-1}$$

when $\Lambda\gg N>1$. Suppose Q attains its maximum at $z_0\in M$. Denote by $(\lambda_1,\lambda_2,\ldots,\lambda_n)$ the eigenvalues of the matrix $\{\chi_{i\bar{j}}\}$ with respect to ω. Without

loss of generality, assume λ_1 has multiplicity m, we choose the coordinate system centered at z_0 such that $\omega = \sqrt{-1}\delta_{kl}\, dz^k \wedge d\bar{z}^l$ and $\{\chi_{i\bar{j}}\}$ is diagonal with

$$\lambda_1 = \lambda_2 = \cdots = \lambda_m \geq \lambda_{m+1} \geq \cdots \geq \lambda_n.$$

We now apply a perturbation argument (see [Székelyhidi 2018]). Let B be a diagonal matrix such that $B_{1\bar{1}} = 0$ and $0 < B_{2\bar{2}} < \cdots < B_{n\bar{n}}$ are small, satisfying $B_{n\bar{n}} < 2B_{2\bar{2}}$. Define $\widetilde{\chi}_{i\bar{j}} = \chi_{i\bar{j}} - B_{i\bar{j}}$ and denote its eigenvalues by $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n)$. It follows that $\lambda_1 \geq \tilde{\lambda}_1$ near z_0 and, at z_0,

$$\tilde{\lambda}_1 = \lambda_1, \qquad \tilde{\lambda}_i = \lambda_i - B_{i\bar{i}} \quad \text{for } i > 1.$$

Thus $\tilde{\lambda}_1 > \tilde{\lambda}_2$ at z_0, which implies $\tilde{\lambda}_1$ is smooth at z_0. We consider the new function

$$\widetilde{Q} = \log \tilde{\lambda}_1 + \varphi(|Du|^2) + \phi(u).$$

It still achieves a local maximum at z_0. Then at z_0, we have

$$0 = \widetilde{Q}_i = \frac{\tilde{\lambda}_{1,i}}{\lambda_1} + \varphi' D_i(|Du|^2) + \phi' u_i, \tag{4-2}$$

and

$$\begin{aligned} 0 \geq \sigma_k^{i\bar{i}} \widetilde{Q}_{i\bar{i}} = {} & \sigma_k^{i\bar{i}} (\log \tilde{\lambda}_1)_{i\bar{i}} + \varphi'' \sigma_k^{i\bar{i}} |D_i(|Du|^2)|^2 \\ & + \varphi' \sigma_k^{i\bar{i}} |Du|^2_{i\bar{i}} + \phi'' \sigma_k^{i\bar{i}} |u_i|^2 + \phi' \sigma_k^{i\bar{i}} u_{i\bar{i}}. \end{aligned} \tag{4-3}$$

The following calculations are all at z_0. For convenience, we will use a unified notation C to denote a constant depending on (M, ω), n, k, ε, χ', ψ, a, A, $\sup_M |u|$ and $\sup_M |Du|$.

First, we deal with the term $\sigma_k^{i\bar{i}} (\log \tilde{\lambda}_1)_{i\bar{i}}$ in (4-3). Thus

$$\tilde{\lambda}_{1,i} = \frac{\partial \tilde{\lambda}_1}{\partial \widetilde{\chi}_{p\bar{q}}} D_i \widetilde{\chi}_{p\bar{q}} = \delta_{1p}\delta_{1q} D_i \widetilde{\chi}_{p\bar{q}} = D_i \widetilde{\chi}_{1\bar{1}} = D_i \chi_{1\bar{1}} - D_i B_{1\bar{1}},$$

and

$$\begin{aligned} \tilde{\lambda}_{1,i\bar{i}} &= \frac{\partial \tilde{\lambda}_1}{\partial \widetilde{\chi}_{p\bar{q}}} D_{\bar{i}} D_i \widetilde{\chi}_{p\bar{q}} + \frac{\partial^2 \tilde{\lambda}_1}{\partial \widetilde{\chi}_{p\bar{q}} \partial \widetilde{\chi}_{r\bar{s}}} D_i \widetilde{\chi}_{p\bar{q}} D_{\bar{i}} \widetilde{\chi}_{r\bar{s}} \\ &= \delta_{1p}\delta_{1q} D_{\bar{i}} D_i \widetilde{\chi}_{p\bar{q}} + \left[(1-\delta_{1p}) \frac{\delta_{1q}\delta_{1r}\delta_{ps}}{\tilde{\lambda}_1 - \tilde{\lambda}_p} + (1-\delta_{1r}) \frac{\delta_{1s}\delta_{1p}\delta_{qr}}{\tilde{\lambda}_1 - \tilde{\lambda}_r} \right] D_i \widetilde{\chi}_{p\bar{q}} D_{\bar{i}} \widetilde{\chi}_{r\bar{s}} \\ &= D_{\bar{i}} D_i \chi_{1\bar{1}} + D_{\bar{i}} D_i B_{1\bar{1}} + \sum_{p>1} \frac{|D_i \chi_{p\bar{1}}|^2 + |D_i \chi_{1\bar{p}}|^2}{\lambda_1 - \tilde{\lambda}_p} \\ &\quad + \frac{\partial^2 \tilde{\lambda}_1}{\partial \widetilde{\chi}_{p\bar{q}} \partial \widetilde{\chi}_{r\bar{s}}} D_i B_{p\bar{q}} D_{\bar{i}} B_{r\bar{s}} - 2 \operatorname{Re} \sum_{p>1} \frac{D_i \chi_{p\bar{1}} D_{\bar{i}} B_{1\bar{p}} + D_i \chi_{1\bar{p}} D_{\bar{i}} B_{p\bar{1}}}{\lambda_1 - \tilde{\lambda}_p}. \end{aligned}$$

We can absorb the terms $D_i\chi_{p\bar{1}}D_{\bar{i}}B_{1\bar{p}}$ by

$$|D_i\chi_{p\bar{1}}D_{\bar{i}}B_{1\bar{p}}| \le \frac{\beta}{4}|D_i\chi_{p\bar{1}}|^2 + \frac{C}{\beta}|B_{n\bar{n}}|^2,$$

using that the derivative of B is controlled by $B_{n\bar{n}}$. Here β is a positive constant to be determined later.

Moreover, for $p > 1$ there is

$$\frac{1}{\lambda_1 - \tilde{\lambda}_p} \le \frac{1}{B_{2\bar{2}}} < \frac{2}{B_{n\bar{n}}},$$

and so

$$\frac{|B_{n\bar{n}}|^2}{\lambda_1 - \tilde{\lambda}_p} + \frac{\partial^2\tilde{\lambda}_1}{\partial\widetilde{\chi}_{p\bar{q}}\partial\widetilde{\chi}_{r\bar{s}}}D_iB_{p\bar{q}}D_{\bar{i}}B_{r\bar{s}} = O(B_{n\bar{n}}).$$

Then by choosing B small enough,

$$\tilde{\lambda}_{1,i\bar{i}} \ge D_{\bar{i}}D_i\chi_{1\bar{1}} + \left(1-\frac{\beta}{2}\right)\sum_{p>1}\frac{|D_i\chi_{p\bar{1}}|^2 + |D_i\chi_{1\bar{p}}|^2}{\lambda_1 - \tilde{\lambda}_p} - \frac{C}{\beta}.$$

Using Cauchy–Schwarz inequality, we have

$$|\sigma_k^{p\bar{q}}(a_{ip}u_{\bar{j}\bar{q}} + a_{\bar{j}\bar{q}}u_{ip} - a_{pi}u_{\bar{q}\bar{j}} - a_{\bar{q}\bar{j}}u_{pi})| \le \tfrac{1}{4}|D\bar{D}u|^2_{\sigma\omega} + \tfrac{1}{4}|DDu|^2_{\sigma\omega} + C\mathcal{F}. \tag{4-4}$$

Without loss of generality, we assume $\lambda_1 \ge 1$. Then combining with (2-9) and (4-4),

$$\begin{aligned}
\sigma_k^{i\bar{i}}(\log\tilde{\lambda}_1)_{i\bar{i}} &= \frac{\sigma_k^{i\bar{i}}\tilde{\lambda}_{1,i\bar{i}}}{\lambda_1} - \frac{\sigma_k^{i\bar{i}}|\tilde{\lambda}_{1,i}|^2}{\lambda_1^2} \\
&\ge \frac{\sigma_k^{i\bar{i}}D_{\bar{i}}D_i\chi_{1\bar{1}}}{\lambda_1} + \left(1-\frac{\beta}{2}\right)\sum_{p>1}\sigma_k^{i\bar{i}}\frac{|D_i\chi_{p\bar{1}}|^2 + |D_i\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1 - \tilde{\lambda}_p)} \\
&\qquad - \left(1+\frac{\beta}{2}\right)\frac{\sigma_k^{i\bar{i}}|D_i\chi_{1\bar{1}}|^2}{\lambda_1^2} - \frac{C}{\beta}\mathcal{F} \\
&\ge -\frac{\sigma_k^{i\bar{j},r\bar{s}}D_1\chi_{i\bar{j}}D_{\bar{1}}\chi_{r\bar{s}}}{\lambda_1} + \frac{1}{\lambda_1}\left(\sum_\ell \psi_{v_\ell}D_\ell\chi_{1\bar{1}} + \sum_\ell \psi_{\bar{v}_\ell}D_{\bar{\ell}}\chi_{1\bar{1}}\right) \\
&\qquad - \frac{1}{\lambda_1}\sigma_k^{i\bar{i}}(T^a_{i1}u_{a\bar{i}\bar{1}} + \overline{T^a_{i1}}u_{i\bar{a}1}) - \frac{1}{\lambda_1}\left(\tfrac{1}{4}|DDu|^2_{\sigma\omega} + \tfrac{1}{4}|D\bar{D}u|^2_{\sigma\omega}\right) \\
&\qquad + \frac{1}{\lambda_1}\sigma_k^{i\bar{i}}(a_1u_{\bar{1}i\bar{i}} + a_{\bar{1}}u_{1i\bar{i}} - a_iu_{\bar{i}1\bar{1}} - a_{\bar{i}}u_{i1\bar{1}}) \\
&\qquad + \left(1-\frac{\beta}{2}\right)\sum_{p>1}\sigma_k^{i\bar{i}}\frac{|D_i\chi_{p\bar{1}}|^2 + |D_i\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1 - \tilde{\lambda}_p)} \\
&\qquad - \left(1+\frac{\beta}{2}\right)\frac{\sigma_k^{i\bar{i}}|D_i\chi_{1\bar{1}}|^2}{\lambda_1^2} - \frac{C}{\lambda_1}(1 + |DDu|^2 + |D\bar{D}u|^2) - C - \frac{C}{\beta}\mathcal{F}.
\end{aligned}$$

Using (2-10) and (4-2), we get

$$
\begin{aligned}
(4\text{-}5)\quad & \frac{1}{\lambda_1}\bigg(\sum_\ell \psi_{v_\ell} D_\ell \chi_{1\bar{1}} + \sum_\ell \psi_{\bar{v}_\ell} D_{\bar{\ell}} \chi_{1\bar{1}}\bigg) + \varphi' \sigma_k^{i\bar{i}} |Du|^2_{i\bar{i}} \\
&\geq \frac{1}{\lambda_1}\bigg(\sum_\ell \psi_{v_\ell} D_\ell \chi_{1\bar{1}} + \sum_\ell \psi_{\bar{v}_\ell} D_{\bar{\ell}} \chi_{1\bar{1}}\bigg) + \frac{\varphi'}{2}|DDu|^2_{\sigma\omega} + \frac{\varphi'}{2}|D\overline{D}u|^2_{\sigma\omega} \\
&\qquad + 2\varphi' \operatorname{Re}\bigg\{\sum_{p,m} \psi_{v_m}(u_{\bar{p}} u_{mp} + u_p u_{m\bar{p}})\bigg\} - C\varphi' - C\varphi'\mathcal{F} \\
&\geq -\phi' \sum_m (\psi_{v_m} u_m + \psi_{\bar{v}_m} u_{\bar{m}}) + \frac{\varphi'}{2}|DDu|^2_{\sigma\omega} + \frac{\varphi'}{2}|D\overline{D}u|^2_{\sigma\omega} - C\varphi' - C\varphi'\mathcal{F} \\
&\geq \frac{\varphi'}{2}|DDu|^2_{\sigma\omega} + \frac{\varphi'}{2}|D\overline{D}u|^2_{\sigma\omega} + C\phi' - C\varphi' - C\varphi'\mathcal{F}.
\end{aligned}
$$

Since $\lambda_i \geq -A$ for any $i = 1, \ldots, n$, and if we further assume that $\lambda_1 \geq \max\{1, A\}$ large enough,

$$
(4\text{-}6)\qquad \frac{1}{\lambda_1 - \tilde{\lambda}_i} = \frac{1}{\lambda_1 - \lambda_i + 1} \geq \frac{1}{3\lambda_1}.
$$

Then by the Cauchy–Schwarz inequality, for any $\beta \in (0, 1)$,

$$
\begin{aligned}
(4\text{-}7)\quad & \frac{1}{\lambda_1}\sigma_k^{i\bar{i}} |T^a_{i1} u_{a\bar{i}\bar{1}} + \overline{T^a_{i1}} u_{i\bar{a}1}| \\
&= \frac{2}{\lambda_1}\sigma_k^{i\bar{i}} |\operatorname{Re}\{\overline{T^p_{i1}} u_{i\bar{p}1}\}| \leq \frac{2}{\lambda_1}\sigma_k^{i\bar{i}} |\overline{T^p_{i1}}(u_{1\bar{p}i} + T^q_{i1} u_{q\bar{p}})| \\
&\leq \frac{2}{\lambda_1}\sigma_k^{i\bar{i}} |\overline{T^p_{i1}} D_i \chi_{1\bar{p}}| + \frac{C}{\lambda_1}(|DDu|^2_{\sigma\omega} + |D\overline{D}u|^2_{\sigma\omega}) + C\mathcal{F} \\
&\leq \frac{\beta}{12}\sum_p \frac{\sigma_k^{i\bar{i}} |D_i \chi_{1\bar{p}}|^2}{\lambda_1^2} + \frac{C}{\beta}\mathcal{F} + \frac{C}{\lambda_1}(|DDu|^2_{\sigma\omega} + |D\overline{D}u|^2_{\sigma\omega}) \\
&\leq \frac{\beta}{4}\frac{\sigma_k^{i\bar{i}} |D_i \chi_{1\bar{1}}|^2}{\lambda_1^2} + \frac{\beta}{4}\sum_{p>1} \frac{\sigma_k^{i\bar{i}} |D_i \chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1 - \tilde{\lambda}_p)} + \frac{C}{\beta}\mathcal{F} + \frac{C}{\lambda_1}(|DDu|^2_{\sigma\omega} + |D\overline{D}u|^2_{\sigma\omega}).
\end{aligned}
$$

We also have

$$
\begin{aligned}
(4\text{-}8)\quad & \frac{1}{\lambda_1}\sigma_k^{i\bar{i}} |a_1 u_{\bar{1}i\bar{i}} + a_{\bar{1}} u_{1i\bar{i}} - a_i u_{\bar{i}1\bar{1}} - a_{\bar{i}} u_{i1\bar{1}}| \\
&\leq \frac{\beta}{12}\bigg(\frac{\sigma_k^{i\bar{i}} |D_i \chi_{1\bar{i}}|^2}{\lambda_1^2} + \frac{\sigma_k^{i\bar{i}} |D_i \chi_{1\bar{1}}|^2}{\lambda_1^2}\bigg) + \frac{C}{\lambda_1}(|DDu|^2_{\sigma\omega} + |D\overline{D}u|^2_{\sigma\omega}) + \frac{C}{\beta}\mathcal{F} \\
&\leq \frac{\beta}{12}\sum_p \frac{\sigma_k^{i\bar{i}} |D_i \chi_{1\bar{p}}|^2}{\lambda_1^2} + \frac{C}{\beta}\mathcal{F} + \frac{C}{\lambda_1}(|DDu|^2_{\sigma\omega} + |D\overline{D}u|^2_{\sigma\omega}) \\
&\leq \frac{\beta}{4}\frac{\sigma_k^{i\bar{i}} |D_i \chi_{1\bar{1}}|^2}{\lambda_1^2} + \frac{\beta}{4}\sum_{p>1} \frac{\sigma_k^{i\bar{i}} |D_i \chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1 - \tilde{\lambda}_p)} + \frac{C}{\beta}\mathcal{F} + \frac{C}{\lambda_1}(|DDu|^2_{\sigma\omega} + |D\overline{D}u|^2_{\sigma\omega}).
\end{aligned}
$$

Using the critical equation (4-2), we obtain

$$\phi''\sigma_k^{i\bar{i}}|u_i|^2 \geq \frac{\phi''}{2(\phi')^2}\frac{\sigma_k^{i\bar{i}}|D_i\chi_{1\bar{1}}|^2}{\lambda_1^2} - \phi''\left(\frac{\varphi'}{\phi'}\right)^2\sigma_k^{i\bar{i}}|D_i(|Du|^2)|^2, \tag{4-9}$$

and

$$\begin{aligned}
&|\phi'\sigma_k^{i\bar{i}}(a_iu_{\bar{i}}+a_{\bar{i}}u_i)| \\
&\quad = 2|\phi'\sigma_k^{i\bar{i}}\,\mathrm{Re}\{a_iu_{\bar{i}}\}| = 2\left|\sigma_k^{i\bar{i}}\,\mathrm{Re}\left\{a_i\left(\frac{D_{\bar{i}}\chi_{1\bar{1}}}{\lambda_1}+\varphi' D_{\bar{i}}(|Du|^2)\right)\right\}\right| \\
&\quad \leq \frac{\phi''}{8(\phi')^2}\sigma_k^{i\bar{i}}\left|\frac{D_{\bar{i}}\chi_{1\bar{1}}}{\lambda_1}+\varphi' D_{\bar{i}}(|Du|^2)\right|^2 + \frac{C(\phi')^2}{\phi''}\mathcal{F} \\
&\quad \leq \frac{\phi''}{4(\phi')^2}\frac{\sigma_k^{i\bar{i}}|D_{\bar{i}}\chi_{1\bar{1}}|^2}{\lambda_1^2} + (\varphi')^2\frac{\phi''}{4(\phi')^2}\sigma_k^{i\bar{i}}|D_i(|Du|^2)|^2 + \frac{C(\phi')^2}{\phi''}\mathcal{F}.
\end{aligned} \tag{4-10}$$

Since $\chi'(z,u)\geq\varepsilon\omega$,

$$\begin{aligned}
-\sigma_k^{p\bar{q}}u_{p\bar{q}} &= \sigma_k^{p\bar{q}}(\chi'_{p\bar{q}}+a_pu_{\bar{q}}+a_{\bar{q}}u_p-\chi_{p\bar{q}}) \\
&\geq \varepsilon\mathcal{F}+\sigma_k^{p\bar{q}}(a_pu_{\bar{q}}+a_{\bar{q}}u_p)-k\psi.
\end{aligned} \tag{4-11}$$

Combining (4-3)–(4-11), we get

$$\begin{aligned}
0 \geq\ & -\frac{1}{\lambda_1}\sigma_k^{i\bar{j},r\bar{s}}D_1\chi_{i\bar{j}}D_{\bar{1}}\chi_{r\bar{s}} + \left(\frac{\varphi'}{2}-\frac{C}{\lambda_1}\right)(|DDu|^2_{\sigma\omega}+|D\bar{D}u|^2_{\sigma\omega}) \\
&+(1-\beta)\sum_{p>1}\sigma_k^{i\bar{i}}\frac{|D_i\chi_{p\bar{1}}|^2+|D_i\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)} - \left(1+\beta-\frac{\phi''}{4(\phi')^2}\right)\frac{\sigma_k^{i\bar{i}}|D_i\chi_{1\bar{1}}|^2}{\lambda_1^2} \\
&+\left(\varphi''-2\phi''\frac{(\varphi')^2}{(\phi')^2}\right)\sigma_k^{i\bar{i}}|D_i(|Du|^2)|^2 - \frac{C}{\lambda_1}(|DDu|^2+|D\bar{D}u|^2) \\
&+\left(-\varepsilon\phi'-C\varphi'-\frac{C(\phi')^2}{\phi''}-\frac{C}{\beta}\right)\mathcal{F}+C\phi'-C\varphi'-C.
\end{aligned} \tag{4-12}$$

According to (2-3),

$$\begin{aligned}
&-\frac{1}{\lambda_1}\sigma_k^{i\bar{j},r\bar{s}}D_1\chi_{i\bar{j}}D_{\bar{1}}\chi_{r\bar{s}} + (1-\beta)\sum_{p>1}\sigma_k^{i\bar{i}}\frac{|D_i\chi_{p\bar{1}}|^2+|D_i\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)} \\
&\quad \geq -\frac{1}{\lambda_1}\sum_{p\neq q}\sigma_k^{p\bar{p},q\bar{q}}D_1\chi_{p\bar{p}}D_{\bar{1}}\chi_{q\bar{q}} + \frac{1}{\lambda_1}\sum_{p\neq q}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{q\bar{q}}}{\lambda_q-\lambda_p}|D_1\chi_{p\bar{q}}|^2 \\
&\qquad + (1-\beta)\sum_{p>1}\sigma_k^{1\bar{1}}\frac{|D_1\chi_{p\bar{1}}|^2+|D_1\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)} \\
&\qquad + (1-\beta)\sum_{p>1}\sigma_k^{p\bar{p}}\frac{|D_p\chi_{p\bar{1}}|^2+|D_p\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}
\end{aligned} \tag{4-13}$$

$$\geq -\frac{1}{\lambda_1}\sum_{p\neq q}\sigma_k^{p\bar{p},q\bar{q}}D_1\chi_{p\bar{p}}D_{\bar{1}}\chi_{q\bar{q}}+\sum_{p>m}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{1\bar{1}}}{\lambda_1(\lambda_1-\lambda_p)}|D_1\chi_{p\bar{1}}|^2$$
$$+(1-\beta)\sum_{p>1}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}+(1-\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}.$$

On the one hand, by Cauchy–Schwarz inequality, for any $\beta\in(0,1)$,

$$\sum_{p>m}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{1\bar{1}}}{\lambda_1(\lambda_1-\lambda_p)}|D_1\chi_{p\bar{1}}|^2+(1-\beta)\sum_{p>1}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}-(1-3\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{1}}|^2}{\lambda_1^2}$$
$$\geq(1-\beta)\sum_{p>m}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{1\bar{1}}}{\lambda_1(\lambda_1-\lambda_p)}|D_1\chi_{p\bar{1}}|^2+(1-\beta)\sum_{p>1}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}$$
$$-(1-3\beta)(1+\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1^2}-C(1-3\beta)\left(1+\frac{1}{\beta}\right)\mathcal{F}$$
$$\geq(1-\beta)\Bigg[\sum_{p>m}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{1\bar{1}}}{\lambda_1(\lambda_1-\lambda_p)}|D_1\chi_{p\bar{1}}|^2$$
$$+\sum_{p>1}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}-(1-\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1^2}\Bigg]-\frac{C}{\beta}\mathcal{F}$$
$$\geq-\frac{C}{\beta}\mathcal{F}.$$

In the last inequality, we use the fact

$$\sum_{p>m}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{1\bar{1}}}{\lambda_1(\lambda_1-\lambda_p)}|D_1\chi_{p\bar{1}}|^2+\sum_{p>1}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}-(1-\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1^2}$$
$$=\sum_{p>m}\frac{\sigma_k^{p\bar{p}}-\sigma_k^{1\bar{1}}}{\lambda_1(\lambda_1-\lambda_p)}|D_1\chi_{p\bar{1}}|^2+\sum_{1<i\leq m}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{i\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_i)}+\sum_{p>m}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}$$
$$-(1-\beta)\sum_{1<i\leq m}\frac{\sigma_k^{1\bar{1}}|D_1\chi_{i\bar{1}}|^2}{\lambda_1^2}-(1-\beta)\sum_{p>m}\frac{\sigma_k^{p\bar{p}}|D_1\chi_{p\bar{1}}|^2}{\lambda_1^2}$$
$$=\sum_{1<i\leq m}\frac{\beta\lambda_1+(1-\beta)\tilde{\lambda}_i}{\lambda_1^2(\lambda_1-\tilde{\lambda}_i)}\sigma_k^{1\bar{1}}|D_1\chi_{i\bar{1}}|^2+\sum_{p>m}\frac{\beta\lambda_1+(1-\beta)\lambda_p}{\lambda_1^2(\lambda_1-\lambda_p)}\sigma_k^{p\bar{p}}|D_1\chi_{p\bar{1}}|^2$$
$$+\sum_{p>m}\frac{\tilde{\lambda}_p-\lambda_p}{\lambda_1(\lambda_1-\lambda_p)(\lambda_1-\tilde{\lambda}_p)}\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2$$

$$\geq \sum_{1<i\leq m} \frac{\beta\lambda_1+(1-\beta)\tilde{\lambda}_i}{\lambda_1^2(\lambda_1-\tilde{\lambda}_i)}\sigma_k^{1\bar{1}}|D_1\chi_{i\bar{1}}|^2+\sum_{p>m}\frac{\beta\lambda_1+(1-\beta)\tilde{\lambda}_p}{\lambda_1^2(\lambda_1-\tilde{\lambda}_p)}\sigma_k^{1\bar{1}}|D_1\chi_{p\bar{1}}|^2$$
$$\geq 0,$$

by choosing $\lambda_1 \geq \frac{1-\beta}{\beta}(A+1)$ large enough such that $\beta\lambda_1+(1-\beta)\tilde{\lambda}_i \geq 0$ for any i.

On the other hand, by Lemma 1.6 and choosing $\varepsilon_0 = 3\beta$ and $\lambda_1 \geq \frac{1-2\beta}{\beta}(A+1)$ large enough, we derive

$$\begin{aligned}
&-\frac{1}{\lambda_1}\sum_{p\neq q}\sigma_k^{p\bar{p},q\bar{q}}D_1\chi_{p\bar{p}}D_{\bar{1}}\chi_{q\bar{q}}+(1-\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}-(1-3\beta)\frac{\sigma_k^{1\bar{1}}|D_1\chi_{1\bar{1}}|^2}{\lambda_1^2}\\
&\quad\geq -K\frac{|D_1\psi|^2}{\lambda_1\sigma_k}+(1-\varepsilon_0)\frac{\sigma_k^{1\bar{1}}|D_1\chi_{1\bar{1}}|^2}{\lambda_1^2}-(1-\varepsilon_0)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_1\chi_{p\bar{p}}|^2}{\lambda_1^2}\\
&\qquad+(1-\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}-(1-3\beta)\frac{\sigma_k^{1\bar{1}}|D_1\chi_{1\bar{1}}|^2}{\lambda_1^2}\\
&\quad\geq -\frac{CK}{\lambda_1}(|DDu|^2+|D\bar{D}u|^2)+(1-\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{p}}|^2}{\lambda_1(\lambda_1-\tilde{\lambda}_p)}\\
&\qquad-(1-3\beta)(1+\beta)\sum_{p>1}\frac{\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{p}}|^2}{\lambda_1^2}-\frac{C}{\beta}\mathcal{F}\\
&\quad\geq \sum_{p>1}\frac{\beta\lambda_1+(1-2\beta)\tilde{\lambda}_p}{\lambda_1^2(\lambda_1-\tilde{\lambda}_p)}\sigma_k^{p\bar{p}}|D_p\chi_{1\bar{p}}|^2-\frac{CK}{\lambda_1}(|DDu|^2+|D\bar{D}u|^2)-\frac{C}{\beta}\mathcal{F}\\
&\quad\geq -\frac{CK}{\lambda_1}(|DDu|^2+|D\bar{D}u|^2)-\frac{C}{\beta}\mathcal{F}.
\end{aligned}$$

Since $\sigma_k^{i\bar{i}} \geq \sigma_k^{1\bar{1}} \geq \frac{k\sigma_k}{n\lambda_1} \geq \frac{C}{\lambda_1}$ for any i,

$$|DDu|^2_{\sigma\omega}+|D\bar{D}u|^2_{\sigma\omega} \geq \frac{C}{\lambda_1}(|DDu|^2+|D\bar{D}u|^2) \geq \frac{C}{\lambda_1}|DDu|^2+C\lambda_1. \tag{4-14}$$

Combining (4-1) and (4-12)–(4-14), we obtain

$$\begin{aligned}
0 \geq{}& \left(\frac{\phi''}{4(\phi')^2}-4\beta\right)\frac{\sigma_k^{i\bar{i}}|D_i\chi_{1\bar{1}}|^2}{\lambda_1^2}+\left(\frac{\varphi'}{2}-\frac{C}{\lambda_1}\right)(|DDu|^2_{\sigma\omega}+|D\bar{D}u|^2_{\sigma\omega})\\
&-\frac{C(K+1)}{\lambda_1}(|DDu|^2+|D\bar{D}u|^2)+\left(-\varepsilon\phi'-C\varphi'-\frac{C(\phi')^2}{\phi''}-\frac{C}{\beta}\right)\mathcal{F}\\
&+C\phi'-C\varphi'-C\\
\geq{}& \left(\frac{C\varphi'}{\lambda_1}-\frac{C(K+1)}{\lambda_1}\right)|DDu|^2+(C\varphi'-C(K+1))\lambda_1\\
&+(-\varepsilon\phi'-C\varphi'-C\phi)\mathcal{F}+C\phi'-C\varphi'-C,
\end{aligned} \tag{4-15}$$

by choosing $\varepsilon_0 = 3\beta = \frac{3}{16e^{\Lambda(-u(z_0)+T)}} = \frac{3\phi''}{16(\phi')^2}(z_0)$ and $\lambda_1 \geq \frac{4C}{N}$ large enough such that $\frac{\varphi'}{2} - \frac{C}{\lambda_1} \geq \frac{\varphi'}{4}$. Then choosing N large enough such that $C\varphi' - C(K+1) \geq 1$ and $\Lambda \gg N$ such that $-\varepsilon\phi' - C\varphi' - C\phi = \varepsilon\Lambda\phi - CN\varphi - C\phi > 0$. Thus (4-15) implies that

$$\lambda_1 \leq -C\phi' + C\varphi' + C.$$

The proof of Theorem 1.3 is completed.

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