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We introduce, and study examples of, a notion of the Artin shape for a motive related to a projective homogeneous variety. We apply it to the problem of finding the complete motivic decomposition of the variety. Our examples cover unitary involution varieties as well as some varieties given by a quadratic Weil transfer. Some of the decompositions obtained dispel prior expectations on how motivic decompositions of projective homogeneous varieties can look like.

1. The scope

We consider Grothendieck's Chow motives (defined as in [Elman et al. 2008, §64]) over a field with coefficients in $\mathbb{F} := \mathbb{Z}/p\mathbb{Z}$, where p is a prime number. Following a tradition, we also write $\mathbb{F}$ for the motive of the spectrum of the base field and call a *Tate motive* its (Tate) shift $\mathbb{F}\{i\}$ with any $i \in \mathbb{Z}$. Another important type of objects are direct summands in the spectra of étale algebras, called *Artin motives*.

Let M be a direct summand in the motive of a projective homogeneous variety under a reductive group G over a field F and let B be the variety of Borel subgroups in G. By [Chernousov et al. 2005, Theorem 7.5], over the function field $F(B)$, M becomes a direct sum of (Tate) shifts of indecomposable Artin motives which we call the *Artin shape* (or simply the *shape*) of M. As shown in [Chernousov and Merkurjev 2006] (see also [Karpenko 2013, Corollary 2.6]), the decomposition we use to define the Artin shape is unique. The motive M itself decomposes into a finite direct sum of indecomposables and such a decomposition, called *complete*, is also unique.

The shape of the entire motive $M(X)$ of a projective G-homogeneous variety X, which we also call the *motivic shape* of X, contains the Artin motive given by the Tate motive $\mathbb{F} := M(\mathrm{Spec}(F))$ with no shift (i.e., with the shift 0), whereas all remaining Artin motives in the shape have positive shifts. It follows that the

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complete motivic decomposition of X contains precisely one summand $U(X)$, called the *upper motive* of X, whose shape satisfies the same property.

By [Karpenko 2013, Corollary 2.15], the upper motives of projective homogeneous varieties satisfy a very efficient isomorphism criterion: $U(X) \simeq U(X')$ if and only if each of the varieties $X_{F(X')}$ and $X'_{F(X)}$ possesses a closed point of prime-to-p degree.

A description of summands that can appear in the complete decomposition of $M(X)$, already previously available in a different situation (see [Karpenko 2010, Theorem 1.1]), has been provided in [Clercq et al. 2025] for *p'-inner p-consistent* G, where G is *p'-inner* if it acquires inner type over a finite Galois field extension E/F of prime-to-p degree, and G is *p-consistent* if its *higher Tits p-indexes*, defined in [De Clercq and Garibaldi 2017], are stable under the action of the Galois group of E/F on the Dynkin diagram of G:

Theorem 1.1 [Clercq et al. 2025, Theorem 6.3 and Remark 5.14]. *Every summand in the complete motivic decomposition of a projective homogeneous variety X under a p'-inner p-consistent group G is a shift of the tensor product $U(Y) \otimes A$, where Y is a projective G-homogeneous variety with $X(F(Y)) \neq \varnothing$ and A is an Artin motive isomorphic to an indecomposable summand in the motive of the spectrum of the F-algebra L given by an intermediate field L in E/F.*

We work out several new applications for that result.

Assume that p is odd and fix a quadratic Galois field extension L/F. The motive of the spectrum of the F-algebra L splits off the Tate motive $\mathbb{F}$; we write A for the complementary summand. This A is an indecomposable Artin motive nonisomorphic to $\mathbb{F}$ over F but becoming isomorphic to $\mathbb{F}$ over L.

Let D be a degree-p central division L-algebra, write X for its Severi–Brauer L-variety, and consider the F-variety $R(X)$ given by the Weil transfer of X. The variety $R(X)$ is projective homogeneous under the reductive group $G = R(\mathrm{Aut}(D))$ given by the Weil transfer of the automorphism group of D. This G acquires inner type over L. Since p is odd, the degree of the field extension L/F is prime to p. Therefore G is p'-inner. However, G is not necessarily p-consistent: it is p-consistent if and only if the algebra D is *balanced* in the sense of [Karpenko and Reichstein 2015, §4] meaning that the algebra $\check{D}$ conjugate to D is Brauer-equivalent to a tensor power of D. Note that if this is the r-th tensor power, then $r^2 \equiv 1 \bmod p$ implying that $r \equiv \pm 1 \bmod p$. It follows that $\check{D}$ is isomorphic either to D or to its opposite algebra. (And the two conditions are mutually exclusive.) Considering the norm algebra of D, defined in [Knus et al. 1998, §3B], one translates the first condition as $D \simeq D'_L$ for a degree-p central division F-algebra D'. The second condition means that the norm algebra is Brauer-trivial and translates by [Knus et al. 1998, Theorem 3.1(2)] as "D admits a unitary F-involution".

We are going to investigate the shape of the *upper motive* $U(R(X))$ as well as the complete motivic decompositions of the variety $R(X)$ and of another related variety in these two cases. Our main results are Theorems 2.3, 4.5, and 4.3.

2. The first case

Assume that we are in the first case, i.e., $D \simeq D'_L$, and let X' be the Severi–Brauer F-variety of D'. Since $R(X)_L \simeq X \times X$ and the degree of L/F is coprime with p, we have $U(R(X)) \simeq U(X')$ by the isomorphism criterion mentioned in Section 1.

By [Karpenko 2013, Corollary 2.22] (see also [Karpenko 1995, Theorem 2.2.1]), the motive $M(X')$ is indecomposable. Therefore $U(X') = M(X')$. The variety X' is projective homogeneous under the group $G' := \mathrm{Aut}(D')$. Over the function field of the corresponding variety of Borel subgroups, X' becomes isomorphic to a $(p-1)$-dimensional projective space $\mathbb{P}$ whose motivic shape is

$$\mathbb{F} \oplus \mathbb{F}\{1\} \oplus \cdots \oplus \mathbb{F}\{p-1\}.$$

So, this is also the shape of the upper motive $U(R(X))$ we started with. It can be represented "graphically" as

$$\begin{array}{cccccc} \{0\} & \{1\} & \cdots & \{p-1\} \\ \mathbb{F} & \mathbb{F} & \cdots & \mathbb{F} \end{array} \quad \text{or simply} \quad \mathbb{F}\ \mathbb{F}\ \cdots\ \mathbb{F}.$$

The graphical shape of $M(R(X))$ is

$$\begin{array}{ccccccccccccc} \mathbb{F} & & \mathbb{F} & & \cdots & & & & \mathbb{F} \\ & A & & A & & \cdots & & & & A \\ & & \mathbb{F} & & \mathbb{F} & & & \cdots & & & \mathbb{F} \\ & & & & A & & A & & \cdots & & & A \\ & & & & & \ddots & & & & & \ddots \\ & & & & & & \mathbb{F} & & \mathbb{F} & & \cdots & & \mathbb{F} \end{array} \tag{2.1}$$

(with p variously shifted lines, each of length p), where A is the Artin motive defined in Section 1. Indeed, this shape is given by the complete motivic decomposition of the Weil transfer $R(\mathbb{P})$. The latter, as explained in [Karpenko 2000, Proposition 5.6], is determined by the action of the Galois group of L/F on the Chow group of the product $R(\mathbb{P})_L = \mathbb{P} \times \mathbb{P}$, performing via exchange of the factors. The Chow group of each of the two (isomorphic) motives $\mathbb{F}_L$ and A_L is 1-dimensional, the first one has the trivial Galois action whereas the second has the nontrivial one. The standard basis of the codimension i Chow group of the product $\mathbb{P} \times \mathbb{P}$ is permuted by the action; each invariant basis element corresponds to $\mathbb{F}\{i\}$, whereas each pair of permuted basis elements corresponds to $\mathbb{F}\{i\} \oplus A\{i\}$.

By Theorem 1.1, the complete decomposition of $M(R(X))$ consists of shifts of $U(R(X))$ and of $U(R(X)) \otimes A$. It follows that the complete decomposition

of $M(R(X))$ (over F) is

$$\begin{aligned}(2.2)\quad M(R(X)) &\simeq U(R(X))\oplus\big(U(R(X))\otimes A\big)\{1\}\\ &\quad\oplus U(R(X))\{2\}\oplus\big(U(R(X))\otimes A\big)\{3\}\oplus\cdots\oplus U(R(X))\{p-1\}\\ &=\Big(\bigoplus_{i=0}^{(p-1)/2} U(R(X))\{2i\}\Big)\oplus\Big(\bigoplus_{i=1}^{(p-1)/2}\big(U(R(X))\otimes A\big)\{2i-1\}\Big).\end{aligned}$$

We put together the main result of this section dispelling some prior expectations on how motivic decompositions of projective homogeneous varieties can look like. Indeed, for the very first time a projective homogeneous variety is found whose motive is not a direct sum of shifts of *corestrictions* (in the sense of [Clercq et al. 2025, §2]) of upper motives.

Theorem 2.3. *For an odd prime p, let X' be the Severi–Brauer variety of a degree-p central division algebra over a field F. For a quadratic Galois field extension L/F, formula* (2.2) *provides the complete motivic decomposition for the Weil transfer $R(X)$ of the L-variety $X := X'_L$. The decomposition holds in the category of Chow motives with coefficients in $\mathbb{Z}/p\mathbb{Z}$.*

3. The second case

Now let us assume that D admits a unitary F-involution. First of all, let us note that the shape of $M(R(X))$ is still given by (2.1): the proof remains unchanged. What we want to figure out is the shape of $U(R(X))$ and the complete decomposition of $M(R(X))$. Concerning the latter, by Theorem 1.1 once again we still know a priori that $M(R(X))$ is a direct sum of shifts of $U(R(X))$ and of $U(R(X))\otimes A$: indeed, $Y(L(X))\neq\varnothing$ for any projective G-homogeneous variety Y (note that the group $G=R(\mathrm{Aut}(D))$ becomes $\mathrm{Aut}(D)\times\mathrm{Aut}(\check{D})$ over L whereas $\check{D}$ is now the opposite algebra of D), so that the upper motive of Y is isomorphic to $U(X)$ as long as the condition $X(F(Y))\neq\varnothing$ of Theorem 1.1 is satisfied.

By [Clercq et al. 2025, Lemma 5.10], the motive $U(R(X))$ is still indecomposable over L so that $U(R(X))_L\simeq U(R(X)_L)$. Since $U(R(X)_L)\simeq U(X)=M(X)$ has the shape $\mathbb{F}\ \mathbb{F}\cdots\mathbb{F}$, we conclude that the shape of $U(R(X))$ looks like

$$\mathbb{F}\ *\ *\cdots*,$$

where each $*$ is either $\mathbb{F}$ or A. We will be able to precisely determine the shape in the next section based on the following preliminary result:

Lemma 3.1. *The shape of the upper motive $U(R(X))$ is either $\mathbb{F}\ \mathbb{F}\cdots\mathbb{F}$ (as in Section 2) or $\mathbb{F}\ A\ \mathbb{F}\ A\cdots\mathbb{F}\ A\ \mathbb{F}$ (i.e., $(p+1)/2$ exemplars of $\mathbb{F}$ alternated with $(p-1)/2$ exemplars of A). The complete decomposition of $M(R(X))$ in the first*

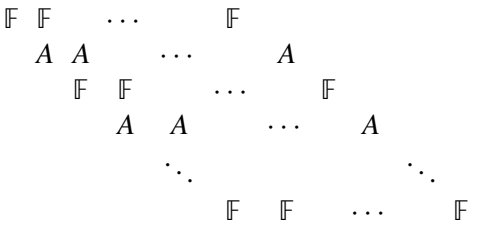
case is (2.2); *in the second case it is given by*

$$M(R(X)) \simeq U(R(X)) \oplus U(R(X))\{1\} \oplus \cdots \oplus U(R(X))\{p-1\} = \bigoplus_{i=0}^{p-1} U(R(X))\{i\}. \tag{3.2}$$

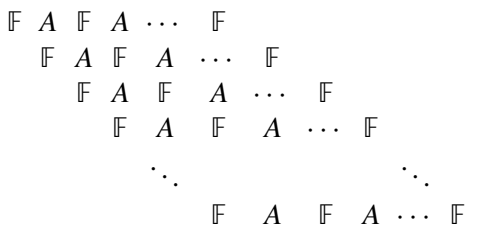
Proof. Note that, by (2.1), the shape of $U := U(R(X))$ takes the first (or second) indicated value if and only if the complete decomposition of $M := M(R(X))$ does.

Let a be the number of A in the graphical shape of U and let b be the number of shifts of $U \otimes A$ in the complete decomposition of M. Then $ab + (p-a)(p-b)$ is the number of $\mathbb{F}$ in the graphical shape of M. Since this number is divisible by p whereas both a and b are smaller than p, we conclude that $a = 0$ or $b = 0$. □

At this point, we do not know yet if decomposition (3.2) actually occurs, but, as already mentioned, we will prove that it does in the next section. Let us illustrate this result by noticing that decompositions (2.2) and (3.2) correspond to the two different graphical representations



and



of the motivic shape of $R(X)$.

Let us also note that the ratio

$$\frac{\text{number of } \mathbb{F}}{\text{number of } A} \tag{3.3}$$

in the second possible graphical shape of $U(R(X))$ equals $(p+1)/(p-1)$ and is the same as in the graphical shape of $M(R(X))$. Note that the similar ratio for $U(R(X)) \otimes A$ equals $(p-1)/(p+1)$ which is lower. This gives another proof of (3.2) in the second case of Lemma 3.1: we already know that $M(R(X))$ is a direct sum of shifts of $U(R(X))$ and of $U(R(X)) \otimes A$, but due to the ratio considerations, the tensor product with A cannot appear.

4. The unitary involution variety

In the situation of Section 3, let us fix a unitary F-involution τ on the L-algebra D. The reductive F-group $G := \mathrm{Aut}_L(D,\tau)$ (the projective unitary group of (D,τ)) is p'-inner p-consistent; over L it acquires inner type becoming isomorphic to the group $\mathrm{Aut}(D)$. Let us determine the complete motivic decomposition of the corresponding *involution variety* Y — the variety of rank-1 (over L) right isotropic ideals in D. We start with some information on its motivic shape:

Proposition 4.1. *Given any $i = 1, 2, \dots, (p-1)/2$, the motivic shape of the variety Y_i of rank-i (over L) right isotropic ideals in the algebra D contains $(b_i + a_i)/2$ shifts of the Tate motive $\mathbb{F}$ and of $(b_i - a_i)/2$ shifts of the Artin motive A, where*

$$a_i = 2^i \cdot \binom{(p-1)/2}{i} \quad \text{and} \quad b_i = \binom{p}{i} \cdot \binom{p-i}{i}.$$

Proof. Here is the meaning of the integers a_i and b_i appearing in the statement of Proposition 4.1. The integer b_i is the *rank* of $M(Y_i)$ defined as the number of Tate motives in the complete decomposition of $M(Y_i)$ over an algebraic closure. (Since the shape of $M(Y_i)$ consists of shifts of motives $\mathbb{F}$ and A which become Tate motives over an algebraic closure, b_i is just the number of motives in the shape.) To get the formula for b_i, note that over a separable closure of the base field, Y_i becomes the variety of flags in a p-dimensional vector space, where the flags are given by an i-plane contained in a $(p-i)$-plane.

To explain the meaning of a_i, we first note that over $F(B)$ the motive $M(Y_i)$ is also a direct sum of shifts of the Tate motive $\mathbb{F}$ and of the motive of spectrum of the $F(B)$-algebra $L(B)$; the integer a_i is the number of shifts of $\mathbb{F}$. It follows from [Karpenko 2012b, Lemma 7.1] for unitary grassmannians and the similar formula [Karpenko 2012a, Formula 2.6] for orthogonal grassmannians that a_i coincides with the rank of the motive of an orthogonal grassmannian of totally isotropic i-planes in a p-dimensional nondegenerate quadratic form, indicated, e.g., in [Buch et al. 2009, Proof of Theorem 2.2] (see also [Buch et al. 2009, Proof of Theorem 1.2]). □

Corollary 4.2. *The number of $\mathbb{F}$ in the motivic shape of $Y = Y_1$ is $(p+1)(p-1)/2$ whereas the number of A is $(p-1)^2/2$. In particular, ratio* (3.3) *for the motivic shape of $Y = Y_1$ is equal to $(p+1)/(p-1)$.* □

The involution variety Y is a subvariety of X. By the isomorphism criterion mentioned in Section 1, $U(Y)$ is isomorphic to the upper motive $U(R(X))$ because the algebra D splits over the function field of the L-variety $R(X)_L$. Therefore, the two possible shapes of $U(Y)$ are listed in Lemma 3.1.

Theorem 4.3. *Let p be an odd prime, L/F a quadratic Galois field extension, D a degree-p central division L-algebra endowed with a unitary F-involution τ, and Y*

the corresponding involution variety (*over* F). *Then the shape of* $U(Y)$ *is as in the second case of Lemma 3.1 whereas the complete motivic decomposition of* Y *is*

$$M(Y) \simeq \bigoplus_{i=0}^{p-2} U(Y)\{i\}. \tag{4.4}$$

In contrast to (3.2), i in (4.4) ranges up to $p-2$ only.

Proof of Theorem 4.3. By Theorem 1.1, every summand in the complete motivic decomposition of Y is a shift of $U(Y)$ or of $U(Y)\otimes A$: since the algebra D splits over the function field of the L-variety Y_L, every projective G-homogeneous variety acquires a closed point of prime-to-p degree over $F(Y)$. Therefore, if $U(Y)$ has the first shape of Lemma 3.1, the number of $\mathbb{F}$ in the shape of $M(Y)$ is divisible by p. On the other hand, it is not by Corollary 4.2. This contradiction proves the statement of Theorem 4.3 on the shape of $U(Y)$.

Now, by the same argument as in Section 3, the complete decomposition of $M(Y)$ consists of shifts of $U(Y)$ only. Over L, the variety Y becomes a rank-$(p-2)$ projective bundle over the Severi–Brauer variety X, so the motive of Y decomposes over L as in (4.4). It follows that the same decomposition holds already over F. □

For illustrative purposes, we draw the shape of the motive of Y that reflects its complete decomposition as

$$\begin{array}{ccccccccccccc}
\mathbb{F} & A & \mathbb{F} & A & \cdots & \mathbb{F} & & & & & & & \\
 & \mathbb{F} & A & \mathbb{F} & A & \cdots & \mathbb{F} & & & & & & \\
 & & \mathbb{F} & A & \mathbb{F} & A & \cdots & \mathbb{F} & & & & & \\
 & & & \mathbb{F} & A & \mathbb{F} & A & \cdots & \mathbb{F} & & & & \\
 & & & & \ddots & & & & & & \ddots & & \\
 & & & & & & \mathbb{F} & A & \mathbb{F} & A & \cdots & \mathbb{F}
\end{array}$$

where there are $p-1$ variously shifted lines, each of length p.

With Theorem 4.3, we also proved our last main result:

Theorem 4.5. *In the situation of Theorem 4.3, let* X *be the Severi–Brauer variety of* D *and let* $R(X)$ *be its Weil transfer to* F. *Then its upper motive* $U(R(X))$ *has the second shape of Lemma 3.1 whereas its total motive* $M(R(X))$ *has the complete decomposition* (3.2). □

Decompositions of Theorem 4.5 and 4.3 put together yield

$$M(R(X)) \simeq M(Y) \oplus U(Y)\{p-1\}.$$

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