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# LOCALIZATION AND TOEPLITZ OPERATORS ON DOUBLING FOCK SPACES

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**Let $F^p_\varphi$ be the weighted Fock space over the complex plane, where $\varphi$ is subharmonic and $\Delta\varphi\, dA$ is a doubling measure. Firstly we introduce the concept of weakly localized operators from $F^p_\varphi$ to $F^q_\varphi$ for $1 \le p, q < \infty$. Then we show that weakly localized operators form an algebra and characterize the compactness of weakly localized operators from one doubling Fock space to another. As applications, the boundedness and compactness of Toeplitz operators with $\mathrm{BMO}^s$ symbols from $F^p_\varphi$ to $F^q_\varphi$ for $1 < p \le q < \infty$ and $p = q = 1$, and with $\mathrm{IMO}^s$ symbols from $F^p_\varphi$ to $F^q_\varphi$ for $1 \le q < p < \infty$, are considered.**

## 1. Introduction

Let $\mathbb{C}$ be the complex plane, and let $dA$ be the Lebesgue area measure on $\mathbb{C}$. A Borel measure $\upsilon > 0$ on $\mathbb{C}$ is called a doubling measure if there is a constant $C > 0$ such that for $z \in \mathbb{C}$ and $r > 0$,

$$\upsilon(D(z, 2r)) \le C\upsilon(D(z, r)),$$

where $D(z, r) = \{w \in \mathbb{C} : |z - w| < r\}$ is the Euclidean disc with the center $z$ and radius $r$. Let $\varphi$ be a nonzero real subharmonic function on $\mathbb{C}$ with $\Delta\varphi\, dA$ a doubling measure, where $\Delta$ is the laplacian operator. For $z \in \mathbb{C}$, there exists a unique positive radius $\rho(z)$ such that $\upsilon\big(D(z, \rho(z))\big) = 1$. The function $\rho^{-2}$ can be thought of as a regularized version corresponding to $\Delta\varphi$; see [Marco et al. 2003; Christ 1991] for details.

For $0 < p < \infty$, the space $L^p_\varphi$ consists of all Lebesgue measurable functions $f$ on $\mathbb{C}$ such that

$$\|f\|_{L^p_\varphi} := \left\{\int_{\mathbb{C}} |f(z)e^{-\varphi(z)}|^p \, dA(z)\right\}^{1/p} < \infty.$$

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Its norm $\|\cdot\|_{F^p_\varphi}$ is inherited from $L^p_\varphi$. Let $H(\mathbb{C})$ denote the family of all analytic functions on $\mathbb{C}$. The doubling Fock space $F^p_\varphi$ is defined to be

$$F^p_\varphi = L^p_\varphi \cap H(\mathbb{C}).$$

The doubling Fock space $F^\infty_\varphi$ is the collection of all functions on $H(\mathbb{C})$ such that

$$\|f\|_{F^\infty_\varphi} = \sup_{z\in\mathbb{C}} |f(z)|e^{-\varphi(z)} < \infty.$$

For $0 < p < 1$, $F^p_\varphi$ is an $F$-space under $d(f,g) = \|f-g\|^p_{F^p_\varphi}$. For $1 \le p < \infty$, $F^p_\varphi$ is a Banach space. If $\varphi(z) = \frac{1}{2}\alpha|z|^2$ ($\alpha > 0$) then $F^p_\varphi$ is the classical Fock space. For some suitable number $A > 0$ and integer $m > 0$, if $\varphi(z) = -m\ln(A + |z|^2) + |z|^2$, then $F^2_\varphi$ is the Fock–Sobolev space first studied in [Cho and Zhu 2012]. For $\varphi(z) = |z|^m$, $F^2_\varphi$ is the generalized Fock space; see [Schneider and Schneider 2009; 2011]. Notice that when $n = 1$, the weights $\phi$ defined as in [Hu and Virtanen 2022b; 2023] satisfy $0 < c \le \Delta\phi \le C$, which implies that $\Delta\phi\, dA$ is a doubling measure and $\rho \simeq \frac{1}{\Delta\phi} \simeq 1$.

It is well known that $F^2_\varphi$ is a Hilbert space under the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} f(w)\overline{g(w)}e^{-2\varphi(w)}\, dA(w).$$

Let $P$ be the orthogonal projection from $L^2_\varphi$ onto $F^2_\varphi$. Then

$$P(g)(z) = \int_{\mathbb{C}} g(w)\overline{K_z(w)}e^{-2\varphi(w)}\, dA(w), \quad g \in L^2_\varphi,\ z \in \mathbb{C},$$

where $K_z(w) := K(w,z)$ is the reproducing kernel on $F^2_\varphi$. From [Oliver and Pascuas 2016, Corollary 3.7], we know that

$$\Gamma = \left\{\sum_{j=1}^N a_j K(\cdot, z_j) : N \in \mathbb{N}^+,\ a_j \in \mathbb{C},\ z_j \in \mathbb{C} \text{ for } 1 \le j \le N\right\}$$

is dense in $F^p_\varphi$ ($1 \le p \le \infty$). Then the domain of $P$ can be extended to $L^p_\varphi$ with $1 \le p < \infty$. Let

$$\mathcal{S} = \{f \text{ is measurable on } \mathbb{C} : fgK_z \in L^1(\mathbb{C}, e^{-2\varphi}dA) \text{ for } g \in \Gamma, z \in \mathbb{C}\}.$$

Given $s \in \mathbb{R}$, we say that $f$ belongs to the space $\mathrm{BMO}^s$ if $f \in \mathcal{S}$ and

$$\|f\|_{\mathrm{BMO}^s} := \sup_{z\in\mathbb{C}} \rho(z)^s\, \mathrm{MO}(f)(z) < \infty,$$

where

$$\mathrm{MO}(f)(z) = \int_{\mathbb{C}} |f(w) - \tilde{f}(z)|\, |k_z(w)|^2 e^{-2\varphi(w)}\, dA(w).$$

Here $k_z(\cdot) := K_z(\cdot)/\|K_z\|_{F^2_\varphi}$ and the Berezin transform $\tilde{f}$ of $f$ is defined as

$$\tilde{f}(z) = \int_{\mathbb{C}} f(w)|k_z(w)|^2 e^{-2\varphi(w)}\, dA(w).$$

The space $\mathrm{VMO}^s$ consists of all $f \in \mathrm{BMO}^s$ such that

$$\lim_{|z|\to\infty} \rho(z)^s \mathrm{MO}(f)(z) = 0.$$

For short, we write $\mathrm{BMO} = \mathrm{BMO}^0$ and $\mathrm{VMO} = \mathrm{VMO}^0$. It should be noted that the $\mathrm{BMO}^s$ in this paper is identical in spelling to the $\mathrm{BMO}^p$ in [Hu and Virtanen 2022a, p. 106], but there is an important difference in definition. For $f \in \mathcal{S}$, the Toeplitz operator $T_f$ on $\Gamma$ can be defined by

$$T_f g(z) = \int_{\mathbb{C}} f(w)g(w)K(z,w)e^{-2\varphi(w)}\, dA(w), \quad g \in \Gamma,\ z \in \mathbb{C}.$$

Then for $f \in \mathcal{S}$, $T_f$ is densely defined on $\Gamma$.

We are concerned with the localized operator from one doubling Fock space to another. As an application, we also consider the boundedness and the compactness of the Toeplitz operator $T_f$ with $\mathrm{BMO}^s$ and $\mathrm{IMO}^s$ (see Section 4.2 for more details) from one doubling Fock space to another.

It is well known that the boundedness and compactness of Toeplitz operators with some suitable symbols have received much attention in the academic community. Axler and Zheng [1998] first characterized the compactness of Toeplitz operators with bounded symbols on the Bergman space by using its Berezin transform. But for the unbounded symbol, Zorboska [2003], using the Berezin transform technique, characterized the boundedness and compactness of Toeplitz operators with BMO symbols on the Bergman space $A^2(\mathbb{D})$. Coburn, Isralowitz and Li [Coburn et al. 2011], using a different method than [Axler and Zheng 1998; Zorboska 2003], proved an analogue of this compactness criterion for Toeplitz operators with BMO symbols on the Segal–Bargmann space $F^2(\mathbb{C}^n)$. Wang [2020] characterized the boundedness (or compactness) of the Toeplitz operators $T_f$ with BMO symbols from the Fock pace $F^p$ to $F^q$ for $0 < p < q < \infty$. Given $f \in \mathrm{IMO}$ (the space of integrable mean oscillation functions), they proved that $T_f$ is bounded (or compact) from $F^p$ to $F^q$ for $0 < q < p < \infty$ if and only if its Berezin transform satisfies the some integral condition.

Recently, "sufficiently localized" operators and "weakly localized" operators have been the subject of much research attention. Xia and Zheng [2013] first introduced the concept of sufficiently localized operators on the Fock space $F^2$, and they showed that if every bounded operator $T$ from the $C^*$ algebra generated by a sufficiently localized operator satisfies the condition

$$\lim_{|z|\to\infty} \langle Tk_z, k_z\rangle_{F^2} = 0,$$

then $T$ is compact on $F^2$. After that, Isralowitz, Mitkovski and Wick [Isralowitz et al. 2015] gave the concept of weakly localized operators (weaker than the sufficiently localized operators) and they extended the results of Xia and Zheng to the cases of the Bergman space $A^p$ and the generalized Fock space $F^p_\phi$ (the weight function $\phi$ is subharmonic, real-valued and satisfies $dd^c\phi \simeq dd^c|z|^2$) for $1 < p < \infty$. For $0 < p \le 1$, Hu, Lv and Wick [Hu et al. 2018] characterized the compact operators $T$ in the algebra generated by weakly localized operators on $F^p_\phi$. They also showed that, for $0 < p < \infty$, an operator $T$ in the algebra generated by bounded Toeplitz operators with BMO symbols is compact on $F^p_\phi$ if and only if its Berezin transform vanishes at infinity. For more information about localized operators, please refer to [Liu and Wang 2021; Sadeghi and Zorboska 2020; Zorboska 2022].

Theories of Toeplitz (or Hankel) operators on doubling Fock spaces have been studied extensively. For example, Oliver and Pascuas [2016] proved the boundedness, compactness and membership in the Schatten class of Toeplitz operators with positive Radon measure. In [Hu and Lv 2017; Lv 2019], the authors characterized the boundedness and compactness of Toeplitz operators with positive Borel measure from $F^p_\varphi$ to $F^q_\varphi$ for $0 < p, q \le \infty$. Recently, Lv and Wang [2024] studied the boundedness and compactness of single Hankel operators from a doubling Fock space $F^q_\varphi$ to a Lebesgue space $L^q_\varphi$ for $1 \le p, q < \infty$, extending results of [Hu and Virtanen 2023] for a special case $\rho \simeq 1$. The doubling Fock spaces $F^p_\varphi$ are different from the spaces in [Hu and Virtanen 2022b; 2023; Hu and Wang 2018] in some ways. For example, let $\varphi(z) = |z|^4$. Then $\Delta\varphi\, dA$ is a doubling measure, but

$$F^p_\varphi \setminus F^q_\varphi \neq \varnothing, \quad F^q_\varphi \setminus F^p_\varphi \neq \varnothing \tag{1-1}$$

for $p \neq q$; see [Constantin and Peláez 2016] for details. Hu and Virtanen [2022a] characterized the Fredholm properties of Toeplitz operators on $F^p_\varphi$ for $0 < p < \infty$.

In the light of the above, a number of problems have naturally arisen:

(1) When are Toeplitz operators induced by bounded symbols, from one doubling Fock space to another, compact?

(2) When are Toeplitz operators induced by $\mathrm{BMO}^s$ or $\mathrm{IMO}^s$ symbols, from one doubling Fock space to another, bounded and compact?

(3) Doubling Fock spaces do not have the property of (1-1), so it makes sense to define a weakly localized condition from $F^p_\varphi$ to $F^q_\varphi$ for $1 \le p, q < \infty$ and to study the case when an operator is compact if it satisfies this condition.

We study problems (2) and (3). If (2) holds, (1) naturally holds. The reproducing kernel of doubling Fock spaces does not have specific expressions, which makes the study of the problem difficult and requires the development of new techniques (a key Lemma 3.2). Unfortunately we only give partial solutions to (2) and (3); see Theorems 3.7 and 4.7. As far as the other cases are concerned, we have not

yet found a good way of dealing with them. In particular, we characterize the boundedness and compactness of Toeplitz operators with BMO symbols on $F^p_\varphi$ for $1 \le p < \infty$; see Corollary 4.8. But for Toeplitz operators with $\mathrm{IMO}^s$ symbols, we only characterize the boundedness and compactness of Toeplitz operators with $\mathrm{IMO}^s$ symbols from $F^p_\varphi$ to $F^p_\varphi$ for $1 \le q < p < \infty$; see Theorem 4.10.

The outline of this paper is as follows. In Section 2, we introduce some important lemmas that will be used frequently. In Section 3, we give the concept of weakly localized operators, obtain that weakly localized operators form an algebra and also characterize the compactness of weakly localized operators from $F^p_\varphi$ to $F^q_\varphi$ (on $F^1_\varphi$) for $1 < p \le q < \infty$. In Section 4, we obtain the boundedness and compactness of Toeplitz operators with $\mathrm{BMO}^s$ and $\mathrm{IMO}^s$ symbols from $F^p_\varphi$ to $F^p_\varphi$ for some suitable exponents $1 \le p, q < \infty$; see Sections 4.1 and 4.2, respectively.

For $a, b > 0$, the notation $a \lesssim b$ indicates that there exists some positive constant $C$ independent of the functions being considered such that $a \le Cb$ and $a \simeq b$ means that both $a \lesssim b$ and $b \lesssim a$ hold.

## 2. Preliminaries

In this section, we list some estimates and some integral inequalities related to the reproducing kernel. These will be used frequently in what follows. We start with function $\rho$. Let

$$D^r(z) := D(z, r\rho(z))$$

for $z \in \mathbb{C}$ and $r > 0$. Then by [Oliver and Pascuas 2016, Lemma 2.2], we have the following lemma.

**Lemma 2.1.** *For every $r > 0$:*

(1) *There exists a positive constant $C_r$ such that*

$$C_r^{-1}\rho(w) \le \rho(z) \le C_r\rho(w)$$

*for every $z \in \mathbb{C}$ and $w \in D^r(z)$.*

(2) *There exists a constant $B > 0$ such that*

$$D^r(z) \subset D^{Br}(w), \quad D^r(w) \subset D^{Br}(z)$$

*for every $z \in \mathbb{C}$ and $w \in D^r(z)$.*

For any $z, w \in \mathbb{C}$, the distance $d_\varphi(z, w)$ induced by the metric $\rho^{-2}\, dz \otimes d\bar{z}$ is defined by

$$d_\varphi(z, w) = \inf_\tau \int_0^1 \frac{|\tau'(t)|}{\rho(\tau(t))}\, dt,$$

where the infimum is taken over all piecewise $C^1$ curves $\tau:[0,1]\to\mathbb{C}$ with $\tau(0)=z$ and $\tau(1)=w$. It is known that the Bergman metric

$$\frac{\partial^2 \log K(z,z)}{\partial z\partial \bar{z}}\, dz\otimes d\bar{z}, \quad z\in\mathbb{C},$$

is comparable to the metric $\rho^{-2}\, dz\otimes d\bar{z}$; see [Constantin and Ortega-Cerdà 2011, page 355] for further details. Write $\beta(\cdot,\cdot)$ for the Bergman distance on $\mathbb{C}$. Therefore there exist two constants $C_1, C_2>0$ such that

$$C_1\, d_\varphi(z,w)\le \beta(z,w)\le C_2\, d_\varphi(z,w), \quad z,w\in\mathbb{C}. \tag{2-1}$$

For more information about the Bergman distance and metric, please refer to [Krantz 2013]. For $z\in\mathbb{C}$ and $r>0$, let $B(z,r)=\{w:\beta(z,w)<r\}$. By (2-1) and [Oliver and Pascuas 2016, Lemma 2.5(1)], we obtain that there exist two positive numbers $m$ and $M$ such that for every $z\in\mathbb{C}$,

$$D^m(z)\subseteq B(z,r)\subseteq D^M(z).$$

Therefore, we regard $B(z,r)$ and $D^x(z)$ as equivalent for numbers $r,x>0$.

Since $\beta$ is more suitable for the analysis of operators on doubling Fock spaces, we need the following estimates for the Bergman distance $\beta$ instead of $d_\varphi$. These estimates will be very useful in our analysis.

**Theorem 2.2.** *For $p>0$ and $k,l\in\mathbb{R}$,*

$$\int_{\mathbb{C}}(\beta(z,\xi)+1)^l|K(z,\xi)|^p e^{-p\varphi(\xi)}\rho(\xi)^k\, dA(\xi)\lesssim \rho(z)^{2(1-p)+k}e^{p\varphi(z)} \tag{2-2}$$

*for $z\in\mathbb{C}$*

*and*

$$\sup_{z\in\mathbb{C}}\rho(z)^{2(p-1)-k}e^{-p\varphi(z)} \times\int_{\mathbb{C}\setminus B(z,R)}(\beta(z,\xi)+1)^l|K(z,\xi)|^p e^{-p\varphi(\xi)}\rho(\xi)^k\, dA(\xi)\to 0 \tag{2-3}$$

*as $R\to\infty$.*

*Proof.* It is easy to see that these conclusions can be obtained by Lemma 2.2 and Corollary 2.4 in [Hu and Virtanen 2022a], so we will omit its proof here. □

For the convenience of the following applications, we obtained from [Oliver and Pascuas 2016, Proposition 2.9] the norm estimate for the reproducing kernel.

**Lemma 2.3.** *For any $1\le p\le\infty$,*

$$\|K_z\|_{F^p_\varphi}\simeq e^{\varphi(z)}\rho(z)^{2/p-2}, \quad z\in\mathbb{C}.$$

The following result comes from [Hu and Virtanen 2022a, Lemma 3.1], which implies that the point evaluation functional at $z \in \mathbb{C}$ is bounded.

**Lemma 2.4.** *Let $0 < p < \infty$. For any $r > 0$,*

$$|f(z)|^p e^{-p\varphi(z)} \lesssim \int_{D^r(z)} |f(w)|^p e^{-p\varphi(w)} \frac{dA(w)}{\rho^2(w)}$$

*for $z \in \mathbb{C}$ and $f \in H(\mathbb{C})$.*

As an application of the above lemma, we have the following lemma. It helps to study the Toeplitz operators with BMO symbols.

**Lemma 2.5.** *Let $\mu$ be a positive Borel measure on $\mathbb{C}$ and $0 < p < \infty$. Let $r > 0$ and $r \in \mathbb{R}$. Then for all $f \in H(\mathbb{C})$,*

$$\int_{\mathbb{C}} |f(z)|^p \rho(z)^k e^{-p\varphi(z)}\, d\mu(z) \lesssim \int_{\mathbb{C}} |f(w)|^p e^{-p\varphi(w)} \rho(w)^k \hat{\mu}_r(w)\, dA(w),$$

*where*

$$\hat{\mu}_r(w) := \frac{1}{\rho(w)^2} \int_{D^r(w)} d\mu.$$

*Proof.* It is easy to show that

$$|f(z)|^p \rho(z)^k e^{-p\varphi(z)} \lesssim \int_{D^r(z)} |f(w)|^p e^{-p\varphi(w)} \frac{dA(w)}{\rho^{2-k}(w)}$$

by Lemma 2.4. Then

$$\begin{aligned}\int_{\mathbb{C}} |f(z)|^p \rho(z)^k e^{-p\varphi(z)}\, d\mu(z) &\lesssim \int_{\mathbb{C}} \int_{D^r(z)} |f(w)|^p e^{-p\varphi(w)} \frac{dA(w)}{\rho^{2-k}(w)}\, d\mu(z) \\ &\lesssim \int_{\mathbb{C}} |f(w)|^p e^{-p\varphi(w)} \rho(w)^k \hat{\mu}_r(w)\, dA(w). \qquad \square\end{aligned}$$

## 3. Weakly localized operator

In this section we will introduce the concept of weakly localized operators from one doubling Fock space to another. We show that these operators form an algebra, and characterize their boundedness and compactness. We now proceed to define weakly localized operators. Let $k_{p,z}$ be the normalized reproducing kernel of $K_z$ with respect to the norm $\|\cdot\|_{F^p_\varphi}$, that is,

$$k_{p,z}(w) = \frac{K(w,z)}{\|K_z\|_{F^p_\varphi}}.$$

For a linear operator $T$ and for $f \in \Gamma$, we define

$$Tf(z) = \langle Tf, K_z \rangle, \quad z \in \mathbb{C}. \tag{3-1}$$

**Definition 3.1.** Let $1 \le p, q < \infty$ and $r > 0$. An operator $T$ defined by satisfying (3-1) is called a weakly localized operator from $F^p_\varphi$ to $F^q_\varphi$ (we denote the set of such operators simply by $\mathcal{W}(F^p_\varphi, F^q_\varphi)$) if

$$\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |\langle T k_{p,z}, k_{q',w}\rangle|\, d\sigma(w) < \infty; \tag{3-2}$$

$$\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |\langle T^* k_{q',z}, k_{p,w}\rangle|\, d\sigma(w) < \infty; \tag{3-3}$$

$$\lim_{r\to\infty} \sup_{z\in\mathbb{C}} \int_{D^r(z)^c} |\langle T k_{p,z}, k_{q',w}\rangle|\, d\sigma(w) = 0; \tag{3-4}$$

$$\lim_{r\to\infty} \sup_{z\in\mathbb{C}} \int_{D^r(z)^c} |\langle T^* k_{q',z}, k_{p,w}\rangle|\, d\sigma(w) = 0; \tag{3-5}$$

where $d\sigma(w) := dA(w)/\rho(w)^{-2}$ and $q'$ the conjugate exponent of $q$ for $1 < q < \infty$ and $q' = +\infty$ for $q = 1$.

**Lemma 3.2.** *Let $1 < p, q < \infty$ or $p = q = 1$, and let $\Omega_1$, $\Omega_2$ be fixed subsets of $\mathbb{C}$. If a linear operator $T$ satisfies (3-1), there is a positive constant $C$ such that*

$$\int_{\Omega_2}\left(\int_{\Omega_1} |f(w)\langle T^* K_z, K_w\rangle| e^{-2\varphi(w)}\, dA(w)\right)^q e^{-q\varphi(z)}\, dA(z) \le C C_1 C_2^{q/q'} \|f\|^q_{L^p_\varphi},$$

*where*

$$C_1 = \sup_{w\in\Omega_1} \int_{\Omega_2} |\langle T k_{p,w}, k_{q',z}\rangle|\, d\sigma(z),$$

$$C_2 = \sup_{z\in\Omega_2} \int_{\Omega_1} |\langle T^* k_{q',z}, k_{p,w}\rangle|\, d\sigma(w).$$

*Proof.* We first assume that $1 < p, q < \infty$. By Hölder's inequality, we obtain

$$\begin{aligned}
&\int_{\Omega_1} |f(w)\langle T^* K_z, K_w\rangle| e^{-2\varphi(w)}\, dA(w) \qquad (3\text{-}6)\\
&= \int_{\Omega_1} |f(w)\langle T k_{p,w}, k_{q',z}\rangle| \|K_w\|_{F^p_\varphi} \|K_z\|_{F^{q'}_\varphi} e^{-2\varphi(w)}\, dA(w)\\
&= \int_{\Omega_1} |f(w)|\, |\langle T k_{p,w}, k_{q',z}\rangle|^{\frac{1}{q}} \|K_w\|_{F^p_\varphi}^{-\frac{q-p}{(p-1)q}} \|K_z\|_{F^{q'}_\varphi} e^{(-1+\frac{q-p}{(p-1)q})\varphi(w)}\\
&\qquad\qquad \times |\langle T k_{p,w}, k_{q',z}\rangle|^{\frac{1}{q'}} \|K_w\|_{F^p_\varphi}^{1+\frac{q-p}{(p-1)q}} e^{(-1-\frac{q-p}{(p-1)q})\varphi(w)}\, dA(w)\\
&\le \left(\int_{\Omega_1} |f(w)|^q |\langle T k_{p,w}, k_{q',z}\rangle| \|K_w\|_{F^p_\varphi}^{-\frac{q-p}{p-1}} \|K_z\|^q_{F^{q'}_\varphi} e^{(-q+\frac{q-p}{p-1})\varphi(w)}\, dA(w)\right)^{\frac{1}{q}}\\
&\qquad\qquad \times \left(\int_{\Omega_1} |\langle T k_{p,w}, k_{q',z}\rangle| \|K_w\|^{p'}_{F^p_\varphi} e^{-p'\varphi(w)}\, dA(w)\right)^{\frac{1}{q'}}.
\end{aligned}$$

Lemma 2.4 shows that for $w \in \Omega_1$,

$$|f(w)|^{q-p} e^{(q-p)\varphi(w)} \lesssim \rho(w)^{-2(q-p)/p} \|f\|_{F^p_\varphi}^{q-p}. \tag{3-7}$$

Notice that

$$\|K_w\|_{F^p_\varphi}^{p'} e^{-p'\varphi(w)} \simeq \rho(w)^{-2}$$

by Lemma 2.3; therefore, by (3-6), (3-7) and Lemma 2.3 again, we obtain

$$\begin{aligned}
&\int_{\Omega_2}\bigg(\int_{\Omega_1} |f(w)\langle T^* K_z, K_w\rangle| e^{-2\varphi(w)}\, dA(w)\bigg)^q e^{-q\varphi(z)}\, dA(z)\\
&\quad\le \int_{\Omega_2}\int_{\Omega_1} |f(w)|^q |\langle T k_{p,w}, k_{q',z}\rangle| \|K_w\|_{F^p_\varphi}^{-\frac{q-p}{p-1}} \|K_z\|_{F^{q'}_\varphi}^q\\
&\qquad\qquad \times e^{(-q+\frac{q-p}{p-1})\varphi(w)}\, dA(w) e^{-q\varphi(z)}\, dA(z)\\
&\qquad\qquad \times \sup_{z\in\Omega_2}\bigg(\int_{\Omega_1} |\langle T k_{p,w}, k_{q',z}\rangle| \|K_w\|_{F^p_\varphi}^{p'} e^{-p'\varphi(w)}\, dA(w)\bigg)^{\frac{q}{q'}}\\
&\quad\lesssim C_2^{q/q'} \int_{\Omega_1} |f(w)|^q \|K_w\|_{F^p_\varphi}^{-\frac{q-p}{p-1}} e^{(-q+\frac{q-p}{p-1})\varphi(w)}\, dA(w)\\
&\qquad\qquad \times \sup_{w\in\Omega_1}\int_{\Omega_2} \langle T k_{p,w}, k_{q',z}\rangle \|K_z\|_{F^{q'}_\varphi}^q e^{-q\varphi(z)}\, dA(z)\\
&\quad= C_1 C_2^{q/q'} \int_{\Omega_1} |f(w)|^q \|K_w\|_{F^p_\varphi}^{-\frac{q-p}{p-1}} e^{(-q+\frac{q-p}{p-1})\varphi(w)}\, dA(w)\\
&\quad\lesssim C_1 C_2^{q/q'} \|f\|_{F^p_\varphi}^{q-p} \int_{\Omega_1} |f(w)|^p \rho(w)^{-2(q-p)/p} \|K_w\|_{F^p_\varphi}^{-\frac{q-p}{p-1}}\\
&\qquad\qquad \times e^{(-p+\frac{q-p}{p-1})\varphi(w)}\, dA(w)\\
&\quad\lesssim C_1 C_2^{q/q'} \|f\|_{F^p_\varphi}^{q-p} \int_{\Omega_1} |f(w)|^p e^{-p\varphi(w)}\, dA(w)\\
&\quad\le C_1 C_2^{q/q'} \|f\|_{F^p_\varphi}^q.
\end{aligned}$$

Next, we consider the case of $p = q = 1$. Fubini's theorem and Lemma 2.3 yield

$$\begin{aligned}
&\int_{\Omega_2}\int_{\Omega_1} |f(w)\langle T^* K_z, K_w\rangle| e^{-2\varphi(w)}\, dA(w) e^{-\varphi(z)}\, dA(z)\\
&\le \int_{\Omega_2}\int_{\Omega_1} |f(w)|\, |\langle T k_{1,w}, k_{\infty,z}\rangle| e^{-2\varphi(w)} \|K_w\|_{F^1_\varphi} \|K_z\|_{F^\infty_\varphi}\, dA(w) e^{-\varphi(z)}\, dA(z)\\
&\lesssim \int_{\Omega_1}\int_{\Omega_2} |\langle T k_{1,w}, k_{\infty,z}\rangle|\, d\sigma(z) |f(w)| e^{-\varphi(w)}\, dA(w)\\
&\lesssim C_1 \|f\|_{F^1_\varphi}.
\end{aligned}$$

□

**Theorem 3.3.** *Let $1 < p, q < \infty$ or $p = q = 1$. If $T \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$, then $T$ is bounded from $F^p_\varphi$ to $F^q_\varphi$.*

*Proof.* By the definition of the linear operator $T$, for $f \in \Gamma$,

$$|Tf(z)| = |\langle Tf, K_z\rangle| = \left|\int_{\mathbb{C}} f(w)\overline{\langle T^*K_z, K_w\rangle}e^{-2\varphi(w)}\,dA(w)\right|$$
$$\le \int_{\mathbb{C}} |f(w)\langle T^*K_z, K_w\rangle|e^{-2\varphi(w)}\,dA(w).$$

It is easy to show that

$$\|Tf\|^q_{F^q_\varphi} \lesssim C_1 C_2^{q/q'} \|f\|^q_{F^p_\varphi}$$

by the above inequality and Lemma 3.2, where

$$C_1 = \sup_{w\in\mathbb{C}} \int_{\mathbb{C}} |\langle Tk_{p,w}, k_{q',z}\rangle|\,d\sigma(z) \quad \text{and} \quad C_2 = \sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |\langle T^*k_{q',z}, k_{p,w}\rangle|\,d\sigma(w).$$

This shows that $T$ is bounded from $F^p_\varphi$ to $F^q_\varphi$. □

For $1 \le p, q < \infty$, let $\mathfrak{B}(F^p_\varphi, F^q_\varphi)$ be the family of all bounded operators from $F^p_\varphi$ to $F^q_\varphi$ and let $\mathcal{K}(F^p_\varphi, F^q_\varphi)$ be the set of all compact operators from $F^p_\varphi$ to $F^q_\varphi$. We write $\mathfrak{B}(F^p_\varphi) = \mathfrak{B}(F^p_\varphi, F^p_\varphi)$ and $\mathcal{K}(F^p_\varphi) = \mathcal{K}(F^p_\varphi, F^p_\varphi)$ for convenience.

**Theorem 3.4.** *For $1 \le p < \infty$, $\mathcal{W}(F^p_\varphi, F^p_\varphi)$ forms a subalgebra of $\mathfrak{B}(F^p_\varphi)$. In particular, $\mathcal{W}(F^2_\varphi, F^2_\varphi)$ is a $*$-subalgebra of $\mathfrak{B}(F^2_\varphi)$.*

*Proof.* Suppose that $T, S \in \mathcal{W}(F^p_\varphi, F^p_\varphi)$. We show that $TS$ and $ST$ belong to $\mathcal{W}(F^p_\varphi, F^p_\varphi)$. By Theorem 3.3, we have $TS, ST \in \mathfrak{B}(F^p_\varphi)$. Next we prove that $TS$ satisfies condition (3-2). By Fubini's theorem and Lemma 2.3, for every $z \in \mathbb{C}$,

(3-8)
$$\int_{\mathbb{C}} |\langle (TS)k_{p,z}, k_{p',w}\rangle|\,d\sigma(w)$$
$$\le \int_{\mathbb{C}}\int_{\mathbb{C}} |\langle Sk_{p,z}, K_u\rangle|\,|\langle T^*k_{p',w}, K_u\rangle|e^{-2\varphi(u)}\,dA(u)\,d\sigma(w)$$
$$= \int_{\mathbb{C}}\int_{\mathbb{C}} |\langle Sk_{p,z}, k_{p',u}\rangle|\,|\langle T^*k_{p',w}, k_{p,u}\rangle| \times \|K_u\|_{F^p_\varphi}\|K_u\|_{F^{p'}_\varphi} e^{-2\varphi(u)}\,dA(u)\,d\sigma(w)$$
$$= \int_{\mathbb{C}}\int_{\mathbb{C}} |\langle Sk_{p,z}, k_{p',u}\rangle|\,|\langle T^*k_{p',w}, k_{p,u}\rangle|\,d\sigma(u)\,d\sigma(w)$$
$$\lesssim \int_{\mathbb{C}} \sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |\langle Sk_{p,z}, k_{p',u}\rangle|\,d\sigma(u)|\langle T^*k_{p',w}, k_{p,u}\rangle|\,d\sigma(w)$$
$$\lesssim \sup_{u\in\mathbb{C}} \int_{\mathbb{C}} |\langle T^*k_{p',w}, k_{p,u}\rangle|\,d\sigma(w) < \infty.$$

This shows that $TS$ satisfies condition (3-2).

We next show that $TS$ satisfies condition (3-4). Notice that $w \in D^r(u)^c$ when $u \in D^r(z)$ and $w \in D^{2r}(z)^c$. Similar to (3-8), we have

$$
\begin{aligned}
(3\text{-}9)\quad & \int_{D^{2r}(z)^c} |\langle (TS)k_{p,z}, k_{p',w}\rangle|\, d\sigma(w) \\
&\le \int_{D^{2r}(z)^c} \int_{\mathbb{C}} |\langle Sk_{p,z}, k_{p',u}\rangle|\, |\langle T^* k_{p',w}, k_{p,u}\rangle|\, d\sigma(u)\, d\sigma(w) \\
&= \int_{D^{2r}(z)^c} \left( \int_{D^r(z)+D^r(z)^c} \right) |\langle Sk_{p,z}, k_{p',u}\rangle|\, |\langle T^* k_{p',w}, k_{p,u}\rangle|\, d\sigma(u)\, d\sigma(w) \\
&\lesssim \int_{D^r(z)} |\langle Sk_{p,z}, k_{p',u}\rangle|\, d\sigma(u) \sup_{w\in D^r(z)} \int_{D^r(u)^c} |\langle T^* k_{p',w}, k_{p,u}\rangle|\, d\sigma(w) \\
&\quad + \int_{D^r(z)^c} |\langle Sk_{p,z}, k_{p',u}\rangle|\, d\sigma(u) \sup_{u\in D^r(z)^c} \int_{D^{2r}(u)^c} |\langle T^* k_{p',w}, k_{p,u}\rangle|\, d\sigma(w) \\
&\le \int_{\mathbb{C}} |\langle Sk_{p,z}, k_{p',u}\rangle|\, d\sigma(u) \sup_{w\in D^r(z)} \int_{D^r(u)^c} |\langle T^* k_{p',w}, k_{p,u}\rangle|\, d\sigma(w) \\
&\qquad + \int_{D^r(z)^c} |\langle Sk_{p,z}, k_{p',u}\rangle|\, d\sigma(u) \sup_{u\in\mathbb{C}} \int_{D^{2r}(u)^c} |\langle T^* k_{p',w}, k_{p,u}\rangle|\, d\sigma(w).
\end{aligned}
$$

This proves that $TS$ satisfies condition (3-4), because $T, S \in \mathcal{W}(F^p_\varphi, F^p_\varphi)$. Similar to the proof of (3-8) and (3-9), it is easy to prove that $TS$ satisfies the conditions (3-3) and (3-5). This shows that $\mathcal{W}(F^p_\varphi, F^p_\varphi)$ forms a subalgebra of $\mathfrak{B}(F^p_\varphi)$.

It is easy to show $\mathcal{W}(F^2_\varphi, F^2_\varphi)$ is a $*$-subalgebra of $\mathfrak{B}(F^2_\varphi)$, since $T \in \mathcal{W}(F^2_\varphi, F^2_\varphi)$ if and only if $T^* \in \mathcal{W}(F^2_\varphi, F^2_\varphi)$. □

Given some sequence $\{a_j\}_{j=1}^\infty \subset \mathbb{C}$ and $r > 0$, we call $\{a_j\}_{j=1}^\infty$ a $(\rho, r)$-lattice if

(a) $\{D^r(a_j)\}_{j=1}^\infty$ covers $\mathbb{C}$;

(b) the disks of $\{D^{r/5}(a_j)\}_{j=1}^\infty$ are pairwise disjoints.

The existence of a $(\rho, r)$-lattice follows from a standard covering lemma; see [Marco et al. 2003]. For a $(\rho, r)$-lattice and $m > 0$, there exists some integer $N$ such that $z \in \mathbb{C}$ belongs to at most $N$ disks of $\{D^{mr}(a_j)\}_{j=1}^\infty$. That is,

$$
(3\text{-}10)\qquad \sum_{j=1}^\infty \chi_{D^{mr}(a_j)}(z) \le N.
$$

Let $F_j = D^r(z_j) \setminus \bigcup_{i<j} D^r(z_i)$. It follows that $\{F_j\}_j$ is also a covering of $\mathbb{C}$ and $F_j \cap F_k = \varnothing$ $(j \ne k)$. We write $(F_j)^+_r = \bigcup_{x\in F_j} D^r(x)$, and let

$$
G_j = \{y : d(y, F_j) \le m^4 r\rho(z_j),\ m = m(r) > 1\}.
$$

There is some constant $N$ such that

$$
(3\text{-}11)\qquad \sum_{j=1}^\infty \chi_{F_j}(w) \le \sum_{j=1}^\infty \chi_{(F_j)^+_r}(w) \le N \quad \text{for any } w \in \mathbb{C}.
$$

**Lemma 3.5.** *For $1 < p, q < \infty$ or $p = q = 1$, if $T \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$, then for every $\varepsilon > 0$, there exists sufficiently large $r > 0$ such that for the covering $\{F_j\}$ depending only on $r$,*

$$\left\| T - \sum_{j=1} M_{\chi_{F_j}} T P M_{\chi_{G_j}} \right\|_{F^p_\varphi \to F^q_\varphi} < \varepsilon.$$

*Here the multiplier operator $M_f$ is defined by $M_f g = fg$ for $g \in \Gamma$.*

*Proof.* Let $S_r = T - \sum_{j=1} M_{\chi_{F_j}} T P M_{\chi_{G_j}}$ for some given $r$. For any fixed $z \in \mathbb{C}$, we assume that $z \in F_{j_0}$ for some integer $j_0$. For $f \in F^p_\varphi$, we have

$$\begin{aligned} S_r f(z) &= Tf(z) - \sum_{j=1} M_{\chi_{F_j}} T P M_{\chi_{G_j}} f(z) \\ &= \sum_{j=1} M_{\chi_{F_j}} Tf(z) - \sum_{j=1} M_{\chi_{F_j}} T P M_{\chi_{G_j}} f(z) \\ &= \sum_{j=1} M_{\chi_{F_j}}(z) T(P M_{\chi_{G_j^c}} f)(z) \\ &= \sum_{j=1} M_{\chi_{F_j}}(z) \langle T P M_{\chi_{G_j^c}} f, K_z \rangle \\ &= \sum_{j=1} M_{\chi_{F_j}}(z) \int_{\mathbb{C}} M_{\chi_{G_j^c}}(w) f(w) \overline{\langle T^* K_z, K_w \rangle} e^{-2\varphi(w)}\, dA(w). \end{aligned} \tag{3-12}$$

Notice that $D^r(z) \subseteq G_{j_0}$ for $z \in F_{j_0}$. From (3-12), we have

$$\begin{aligned} S_r f(z) &\le \sum_{j=1} M_{\chi_{F_j}}(z) \int_{\mathbb{C}} M_{\chi_{G_j^c}}(w) |f(w)|\, |\langle T^* K_z, K_w \rangle| e^{-2\varphi(w)}\, dA(w) \\ &= \int_{G^c_{j_0}} |f(w)|\, |\langle T^* K_z, K_w \rangle| e^{-2\varphi(w)}\, dA(w) \\ &\le \int_{D^r(z)^c} |f(w)|\, |\langle T^* K_z, K_w \rangle| e^{-2\varphi(w)}\, dA(w). \end{aligned}$$

Similarly to the proof of Lemma 3.2, if we combine the above inequality, we immediately obtain

$$\|S_r f\|^q_{L^q_\varphi} \lesssim C_1 C_2^{q/q'}(r) \|f\|^q_{F^p_\varphi},$$

where

$$C_1 = \sup_{w \in \mathbb{C}} \int_{\mathbb{C}} |\langle T k_{p,w}, k_{q',z} \rangle|\, d\sigma(z), \quad C_2(r) = \sup_{z \in \mathbb{C}} \int_{D^r(z)^c} |\langle T^* k_{q',z}, k_{p,w} \rangle|\, d\sigma(w).$$

Since $T \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$, we have $C_1$ is finite and $C_2(r) \to 0$ as $r \to \infty$. Then for any $\varepsilon > 0$, there exists a sufficiently large $r > 0$ such that

$$\|S_r\|_{F^p_\varphi \to F^q_\varphi} < \varepsilon. \qquad \square$$

We use $\|T\|_{e,F^p_\varphi\to F^q_\varphi}$ to stand for the essential norm of a given linear operator $T$ from $F^p_\varphi$ to $F^q_\varphi$. We write $\|T\|_e = \|T\|_{e,F^p_\varphi\to F^q_\varphi}$ for convenience.

**Theorem 3.6.** *For $1 < p \le q < \infty$ or $p = q = 1$, if $T \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$, there exist positive constants $r$ and $C = C(r)$ such that*

$$\|T\|_e \le C \limsup_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle Tk_{z,p}, k_{q',w}\rangle|^{1/q}.$$

*Proof.* The case of $p = q = 1$ comes from [Liu and Wang 2021, Theorem 25], so we only need to prove that the case of $1 < p \le q < \infty$. It is easy to prove that $\|T\|_e = \|TP\|_e$, because $P\mathcal{F}^p_\varphi = \mathcal{F}^p_\varphi$. We always assume that $\|TP\|_e > 0$ because $\|TP\|_e = 0$ is trivial. Lemma 3.5 shows there is some $r > 0$ satisfying

$$\left\| TP - \left( \sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}} \right) \right\|_{\mathcal{F}^p_\varphi\to\mathcal{F}^q_\varphi} < \tfrac{1}{2}\|TP\|_e.$$

Because $\sum_{j\le m} M_{\chi_{F_j}} TPM_{\chi_{G_j}}$ is a compact operator where $m \in \mathbb{N}$,

$$\begin{aligned}
\|TP\|_e &\le \left\| TP - \left( \sum_{j=1}^{m} M_{\chi_{F_j}} TPM_{\chi_{G_j}} \right) \right\|_{F^p_\varphi\to F^q_\varphi} \\
&\le \left\| TP - \left( \sum_{j=1}^{\infty} M_{\chi_{F_j}} TPM_{\chi_{G_j}} \right) \right\|_{F^p_\varphi\to F^q_\varphi} + \|T_m\|_{F^p_\varphi\to F^p_\varphi} \\
&< \tfrac{1}{2}\|TP\|_e + \|T_m\|_{F^p_\varphi\to F^q_\varphi},
\end{aligned}$$

where $T_m = \sum_{j>m} M_{\chi_{F_j}} TPM_{\chi_{G_j}}$. It remains to show that

$$\limsup_{m\to\infty} \|T_m\|_{F^p_\varphi\to F^q_\varphi} \lesssim \limsup_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle Tk_{p,z}, k_{q',w}\rangle|^{1/q} + \tfrac{1}{4}\|TP\|_e. \tag{3-13}$$

Let $f \in F^p_\varphi$ with $\|f\|_{F^p_\varphi} \le 1$. Notice that

$$F_j \cap F_k = \varnothing \quad (j \ne k).$$

Then

$$\begin{aligned}
\|T_m f\|^q_{F^q_\varphi} &= \sum_{j>m} \|M_{\chi_{F_j}} TPM_{\chi_{G_j}} f\|^q_{F^q_\varphi} = \sum_{j>m} \|M_{\chi_{F_j}} Tl_j\|^q_{F^q_\varphi} \|M_{\chi_{G_j}} f\|^q_{F^p_\varphi} \\
&\lesssim (\sup_{j>m} \|M_{\chi_{F_j}} Tl_j\|^q_{F^q_\varphi}) \sum_{j>m} \left( \int_{\mathbb{C}} \chi_{F_j}(z)|f(z)|^p e^{-p\varphi(z)}\, dA(z) \right)^{q/p} \\
&\le (\sup_{j>m} \|M_{\chi_{F_j}} Tl_j\|^q_{F^q_\varphi}) \left( \sum_{j>m} \int_{\mathbb{C}} \chi_{F_j}(z)|f(z)|^p e^{-p\varphi(z)}\, dA(z) \right)^{q/p} \\
&\lesssim \|f\|^q_{F^p_\varphi} (\sup_{j>m} \|M_{\chi_{F_j}} Tl_j\|^q_{F^q_\varphi}),
\end{aligned}$$

where $l_j := PM_{\chi_{G_j}} f / \|M_{\chi_{G_j}} f\|_{F^p_\varphi}$. It follows that

$$\|T_m\|_{F^p_\varphi \to F^q_\varphi} \lesssim \sup_{j>m} \sup_{\|f\|_{F^p_\varphi}\le 1} \left\{ \|M_{\chi_{F_j}} T l_j\|_{F^q_\varphi} : l_j = \frac{PM_{\chi_{G_j}} f}{\|M_{\chi_{G_j}} f\|_{F^p_\varphi}} \right\}.$$

Furthermore, we have

$$\begin{aligned} \text{(3-14)}\quad \limsup_{m\to\infty} \|T_m\|_{F^p_\varphi \to F^q_\varphi} &\\ \lesssim \limsup_{j\to\infty} \sup_{\|f\|_{F^p_\varphi}\le 1} &\left\{ \|M_{\chi_{F_j}} T l_j\|_{F^q_\varphi} : l_j = \frac{PM_{\chi_{G_j}} f}{\|M_{\chi_{G_j}} f\|_{F^p_\varphi}} \right\}. \end{aligned}$$

We can choose a sequence $\{f_j\}$ with $\|f_j\|_{F^p_\varphi} \le 1$ such that

$$\begin{aligned} \text{(3-15)}\quad \limsup_{j\to\infty} \sup_{\|f\|_{F^p_\varphi}\le 1} &\left\{ \|M_{\chi_{F_j}} T l_j\|_{F^p_\varphi} : l_j := \frac{PM_{\chi_{G_j}} f}{\|M_{\chi_{G_j}} f\|_{F^p_\varphi}} \right\} - \tfrac{1}{4}\|TP\|_e \\ &\le \limsup_{j\to\infty} \|M_{\chi_{F_j}} T g_j\|_{F^q_\varphi}, \end{aligned}$$

where $g_j := PM_{\chi_{G_j}} f_j / \|M_{\chi_{G_j}} f_j\|_{F^p_\varphi}$.

By Lemma 3.2, we deduce

$$\begin{aligned} \text{(3-16)}\quad &\|M_{\chi_{F_j}} T g_j\|^q_{F^q_\varphi} \\ &\le \int_{F_j} \left( \int_{G_j} \frac{|f_j(w)|\,|TK_w(z)| e^{-2\varphi(w)}}{\|\chi_{G_j} f_j\|_{F^p_\varphi}}\, dA(w) \right)^q e^{-q\varphi(z)}\, dA(z) \\ &\lesssim \frac{1}{\|\chi_{G_j} f_j\|^q_{F^p_\varphi}} \int_{F_j} \left( \int_{G_j} |f_j(w)| e^{-2\varphi(w)} |\langle TK_w, K_z\rangle|\, dA(w) \right)^q e^{-q\varphi(z)}\, dA(z) \\ &\lesssim C_1 C_2^{q/q'} \frac{\|\chi_{G_j} f_j\|^{q-p}_{F^p_\varphi} \|\chi_{G_j} f_j\|^p_{F^p_\varphi}}{\|\chi_{G_j} f_j\|^q_{F^p_\varphi}} \\ &= C_1 C_2^{q/q'}, \end{aligned}$$

where

$$\begin{aligned} C_1 &= \sup_{w\in G_j} \int_{F_j} |\langle T k_{p,w}, k_{q',z}\rangle|\, d\sigma(z), \\ C_2 &= \sup_{z\in F_j} \int_{G_j} |\langle T^* k_{q',z}, k_{p,w}\rangle|\, d\sigma(w) < \infty. \end{aligned}$$

For each $j$, there exists $z_j \in F_j \subset G_j$ such that $G_j \subset D^{br}(z_j)$ for some $b = b(r) > 0$ by Lemma 2.1(2). We can choose that $u_j, t_j \in D^{br}(z_j)$ such that

$$|\langle T k_{p,t_j}, k_{q',u_j}\rangle| = \sup_{z,w\in D^{br}(z_j)} |\langle T k_{p,w}, k_{q',z}\rangle|.$$

There is $k > 0$ such that $w \in D^{kbr}(u_j)$ for $w \in G_j$. It is easy to prove that

$$
\begin{aligned}
C_1 &= \sup_{w\in G_j} \int_{F_j} |\langle Tk_{p,w}, k_{q',z}\rangle|\, d\sigma(z) \\
&\le \sup_{w\in G_j} \int_{G_j} |\langle Tk_{p,w}, k_{q',z}\rangle|\, d\sigma(z) \\
&\lesssim |\langle Tk_{p,t_j}, k_{q',u_j}\rangle| \\
&\lesssim \sup_{w\in D^{kbr}(u_j)} |\langle Tk_{p,w}, k_{q',u_j}\rangle|.
\end{aligned} \tag{3-17}
$$

Since $F_j$ is a covering, there is a subsequence $\{u_{m_j}\}$ such that $u_{m_j} \to \infty$ as $j \to \infty$. Therefore, by (3-16) and (3-17), we immediately obtain

$$
\limsup_{j\to\infty} \|M_{\chi_{F_j}} Tg_j\|^q_{F^q_\varphi} \lesssim \limsup_{|z|\to\infty} \sup_{w\in D^{kbr}(z)} |\langle Tk_{p,w}, k_{q',z}\rangle|.
$$

By the above inequality, (3-14) and (3-15), we have (3-13) holds. □

**Theorem 3.7.** *For $1 < p \le q < \infty$ or $p = q = 1$, if $T \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$, then the following statements are equivalent*:

(1) $T \in \mathcal{K}(F^p_\varphi, F^q_\varphi)$.

(2) $\lim_{|z|\to\infty} \|Tk_{p,z}\|_{F^q_\varphi} = 0$.

(3) $\lim_{|z|\to\infty} \sup_{w\in\mathbb{C}} |\langle Tk_{p,z}, k_{q',w}\rangle| = 0$.

(4) $\lim_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle Tk_{p,z}, k_{q',w}\rangle| = 0$ *for any* $r > 0$.

*Proof.* Assume that (1) holds. Lv and Wang [2024, Lemma 2.1(4)] showed that if $k_{p,z} \to 0$ uniformly on compact subsets of $\mathbb{C}$ as $z \to \infty$, then

$$
\lim_{|z|\to\infty} \|Tk_{p,z}\|_{F^q_\varphi} = 0.
$$

This proves (1) ⇒ (2). Assume that (2) holds. If $1 < p \le q < \infty$, by Lemmas 2.3 and 2.4, we obtain

$$
\begin{aligned}
|\langle Tk_{p,z}, k_{q',w}\rangle|^q &= \frac{1}{\|K_w\|^q_{L^{q'}_\varphi}} |\langle Tk_{p,z}, K_w\rangle|^q \\
&\lesssim \rho(w)^2 e^{-q\varphi(w)} |\langle Tk_{p,z}, K_w\rangle|^q \\
&\lesssim \int_{D^r(w)} |\langle Tk_{p,z}, K_u\rangle|^q e^{-q\varphi(u)}\, dA(u) \\
&\le \|Tk_{p,z}\|^q_{F^q_\varphi}.
\end{aligned}
$$

If $p = q = 1$, it is easy to check that

$$
|\langle Tk_{1,z}, K_{\infty,w}\rangle| \le \|Tk_{1,z}\|_{F^1_\varphi}.
$$

This shows that (2) $\Rightarrow$ (3). It is trivial to show that (3) $\Rightarrow$ (4). It remains to show that (4) $\Rightarrow$ (1). Assume that (4) holds. By Theorem 3.6, there exist positive constants $r$ and $C = C(r)$ such that

$$\|T\|_e \le C \limsup_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle T k_{z,p}, k_{q',w}\rangle|^{1/q}.$$

The condition (4) implies that $\|T\|_e = 0$, or, equivalently, that $T \in \mathcal{K}(F^p_\varphi, F^q_\varphi)$. $\square$

## 4. Toeplitz operators with BMO$^s$ or IMO$^s$ symbols

In this section we give characterizations of Toeplitz operators with BMO$^s$ or IMO$^s$ symbols. This section is divided into two parts: the first considers Toeplitz operators with BMO$^s$ symbols and the second considers those with IMO$^s$ symbols.

**4.1.** ***Toeplitz operators with* BMO$^s$ *symbols.*** As an application of the technique of weakly localized operators, we characterize the boundedness and compactness of Toeplitz operators with BMO$^s$ symbols from $F^p_\varphi$ to $F^q_\varphi$ when $1 < p \le q < \infty$ or $p = q = 1$. In particular, we show that Toeplitz operators with BA symbols form a subalgebra of $\mathfrak{B}(F^p_\varphi)$ for $1 \le p < \infty$. We also give a characterization of the compactness of operators in this subalgebra. In order to do this, we will start with some notation and auxiliary function spaces.

Recall that for a given $s \in \mathbb{R}$, we say that $f \in \mathrm{BMO}^s$ if $f \in \mathcal{S}$ and

$$\|f\|_{\mathrm{BMO}^s} := \sup_{z\in\mathbb{C}} \rho(z)^s \,\mathrm{MO}(f)(z) < \infty,$$

where

$$\mathrm{MO}(f)(z) = \int_{\mathbb{C}} |f(w) - \tilde{f}(z)|\, |k_z(w)|^2 e^{-2\varphi(w)}\, dA(w).$$

For a continuous function $f$ on $\mathbb{C}$, we define the oscillation of $f$ at $z \in \mathbb{C}$ in the Bergman metric by

$$\omega_r(f)(z) = \sup\{|f(z) - f(w)| : w \in D^r(z)\}.$$

We say that $f \in \mathrm{BO}^s_r$ if the function $\rho^s \omega_r(f)$ is bounded. We now define the mean oscillation of $f$ at $z \in \mathbb{C}$ by

$$\mathrm{MO}_r(f)(z) = \frac{1}{|D^r(z)|} \int_{D^r(z)} |f(w) - \hat{f}_r(z)|\, dA(w),$$

where

$$\hat{f}_r(z) = \frac{1}{|D^r(z)|} \int_{D^r(z)} f(w)\, dA(w),$$

and $|D^r(z)|$ is the Euclidean area of $D^r(z)$. We say that $f \in \mathrm{BMO}^s_r$ if $f$ is a locally Lebesgue integrable function and $\rho(\cdot)^s \,\mathrm{MO}_r(f)$ is bounded on $\mathbb{C}$. Let $\mathrm{BA}^s_r$ denote the space of locally Lebesgue integrable functions $f$ on $\mathbb{C}$ such that $\rho(\cdot)^s \widehat{|f|}_r$

is bounded. We know that $\mathrm{BO}_r^s$ and $\mathrm{BA}_r^s$ are independent of $r$ when $s \le 0$; see [Lv and Wang 2024, Theorem 4.1]. We write $\mathrm{BO}^s$ for $\mathrm{BO}_r^s$ and $\mathrm{BA}^s$ for $\mathrm{BA}_r^s$ for convenience. In particular, when $s = 0$, we write BO for $\mathrm{BO}^0$ and BA for $\mathrm{BA}^0$.

**Lemma 4.1.** *Let $r > 0$, $s \le 0$ and $f \in \Gamma$. Then the following statements are equivalent*:

(1) $f \in \mathrm{BMO}_r^s$ *for some* (*or any*) $r > 0$.

(2) $f = f_1 + f_2$, *where* $f_1 \in \mathrm{BO}^s \cap C^2(\mathbb{C})$ *and* $f_2 \in \mathrm{BA}^s$.

(3) $f \in \mathrm{BMO}^s$.

*Moreover*,

$$\|f\|_{\mathrm{BMO}^s} \simeq (\|f_1\|_{\mathrm{BO}^s} + \|f_1\|_{\mathrm{BA}^s}) \simeq \|f\|_{\mathrm{BMO}_r^s}.$$

*Proof.* The equivalence of (1) and (2) comes from [Lv and Wang 2024, Theorem 4.1]. We only need to show that (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). Assume that (2) holds. Assume $f \in \mathrm{BA}^s$, that is, $\rho^s|\widehat{f}|$ is bounded on $\mathbb{C}$. Through appropriate modifications to [Hu and Lv 2017, Lemma 2.5], it is easy to show that $\rho^s|\widehat{f}|$ is bounded if and only if $\rho^s|\widetilde{f}|$ is bounded. Therefore, for every $z \in \mathbb{C}$,

$$\rho(z)^s\left(\int_{\mathbb{C}} |f(w) - \tilde{f}(z)|\,|k_{2,z}(w)|^2 e^{-2\varphi(w)}\,dA(w)\right) \le 2\rho(z)^s|\widetilde{f}|(z) < \infty.$$

This implies that $f \in \mathrm{BMO}^s$ when $f \in \mathrm{BA}^s$. We assume that $f \in \mathrm{BO}^s$, that is, the function $\rho^s\omega_r(f)$ is bounded. For any given $z \in \mathbb{C}$, similar to the proof of [Zhu 2007, Lemma 8.2], we have

$$\rho(z)^s(|f(z) - f(w)|) \lesssim \|f\|_{\mathrm{BO}^s}(\beta(z,w) + 1). \tag{4-1}$$

From this and (2-2), we obtain

$$\begin{aligned}
&\rho(z)^s\left(\int_{\mathbb{C}} |f(w) - \tilde{f}(z)|\,|k_{2,z}(w)|^2 e^{-2\varphi(w)}\,dA(w)\right)\\
&\quad\le \rho(z)^s\left(\int_{\mathbb{C}} |f(w) - f(z)|\,|k_{2,z}(w)|^2 e^{-2\varphi(w)}\,dA(w) + |f(z) - \tilde{f}(z)|\right)\\
&\quad\le 2\int_{\mathbb{C}} \rho(z)^s|f(w) - f(z)|\,|k_{2,z}(w)|^2 e^{-2\varphi(w)}\,dA(w)\\
&\quad\lesssim \int_{\mathbb{C}} (\beta(z,w) + 1)|k_{2,z}(w)|^2 e^{-2\varphi(w)}\,dA(w)\\
&\quad\lesssim 1.
\end{aligned}$$

This shows that $f \in \mathrm{BMO}^s$ when $f \in \mathrm{BO}^s$. This proves that (2) $\Rightarrow$ (3).

We now suppose that (3) holds. For any fixed $z \in \mathbb{C}$,

$$\begin{aligned}
&\rho(z)^s \,\mathrm{MO}_r(f)(z)\\
&\quad= \rho(z)^s \bigg( \frac{1}{|D^r(z)|} \int_{D^r(z)} |f(w) - \hat{f}_r(z)|\, dA(w) \bigg)\\
&\quad\lesssim \rho(z)^s \bigg( \int_{D^r(z)} |f(w) - \hat{f}_r(z)|\, |k_{2,z}(w)|^2 e^{-2\varphi(w)}\, dA(w) \bigg)\\
&\quad\lesssim \rho(z)^s \bigg( \int_{\mathbb{C}} |f(w) - \hat{f}_r(z)|\, |k_{2,z}(w)|^2 e^{-2\varphi(w)}\, dA(w) \bigg)\\
&\quad\lesssim \rho(z)^s \bigg( \int_{\mathbb{C}} |f(w) - \tilde{f}(z)|\, |k_{2,z}(w)|^2 e^{-2\varphi(w)}\, dA(w) + |\tilde{f}(z) - \widehat{f_r}(z)| \bigg)\\
&\quad\le 2\rho(z)^s \bigg( \int_{\mathbb{C}} |f(w) - \tilde{f}(z)|\, |k_{2,z}(w)|^2 e^{-2\varphi(w)}\, dA(w) \bigg),
\end{aligned}$$

which implies that $f \in \mathrm{BMO}^s_r$, when $f \in \mathrm{BMO}^s$. □

In order to characterize the boundedness and compactness of Toeplitz operators with $\mathrm{BMO}^s$ symbols from $F^p_\varphi$ to $F^q_\varphi$, we first prove that Toeplitz operators with $\mathrm{BMO}^s$ symbols are well defined.

**Lemma 4.2.** *For $s \le 0$ and $f \in \mathcal{S} \cap \mathrm{BMO}^s$, the Toeplitz operator $T_f$ is well defined on $\Gamma$. Moreover,*

$$T_f g(w) = \langle T_f g, K_w \rangle = \int_{\mathbb{C}} f(z) g(z) K(w,z) e^{-2\varphi(z)}\, dA(z), \quad g \in \Gamma,\ w \in \mathbb{C}.$$

*Proof.* For $f \in \mathcal{S}$, we only need to prove that for every $z, w \in \mathbb{C}$, $|\langle T_f K_z, K_w \rangle|$ is finite and $T_f K_z(w) = \langle T_f K_z, K_w \rangle$. Notice that

$$\begin{aligned}
\langle T_f K_z, K_w \rangle &= \int_{\mathbb{C}} T_f K_z(u) \overline{K_w(u)} e^{-2\varphi(u)}\, dA(u)\\
&= \int_{\mathbb{C}} \int_{\mathbb{C}} f(\xi) K_z(\xi) \overline{K_u(\xi)} e^{-2\varphi(\xi)}\, dA(\xi) \overline{K_w(u)} e^{-2\varphi(u)}\, dA(u).
\end{aligned} \tag{4-2}$$

We now check that

$$\begin{aligned}
&|\langle T_f K_z, K_w \rangle|\\
&\quad\le \int_{\mathbb{C}} \int_{\mathbb{C}} |f(\xi) K_z(\xi) K_u(\xi) e^{-2\varphi(\xi)} K_w(u) e^{-2\varphi(u)}|\, dA(u)\, dA(\xi) < \infty.
\end{aligned} \tag{4-3}$$

To this end, Lemma 2.3 and the definition of $F^\infty_\varphi$ show that

$$\int_{\mathbb{C}} |K_u(\xi) e^{-2\varphi(u)} K_w(u)|\, dA(u) \lesssim \|K_w\|_{F^\infty_\varphi} e^{\varphi(\xi)}. \tag{4-4}$$

From Lemma 4.1 we have the decomposition $\mathrm{BMO}^s = \mathrm{BA}^s + \mathrm{BO}^s$. If $f \in \mathrm{BA}^s$, then by Lemma 2.5, (4-4) and (2-2), we get

$$\begin{aligned}(4\text{-}5)\quad &\int_{\mathbb{C}}\int_{\mathbb{C}} |f(\xi)K_z(\xi)K_u(\xi)e^{-2\varphi(\xi)}K_w(u)e^{-2\varphi(u)}|\,dA(u)\,dA(\xi)\\ &\lesssim \|K_w\|_{F^\infty_\varphi}\int_{\mathbb{C}} |f(\xi)K_z(\xi)e^{-\varphi(\xi)}|\,dA(\xi)\\ &\lesssim \|K_w\|_{F^\infty_\varphi}\int_{\mathbb{C}} |K_z(\xi)e^{-\varphi(\xi)}\rho(\xi)^{-s}\rho(\xi)^s\widehat{|f|}_r(\xi)\,dA(\xi)\\ &\lesssim \rho(z)^{-s}e^{\varphi(z)}\|K_w\|_{F^\infty_\varphi}\|f\|_{\mathrm{BA}^s}<\infty.\end{aligned}$$

If $f \in \mathrm{BO}^s$, then by (4-1), (4-4) and (2-2) again, we have

$$\begin{aligned}(4\text{-}6)\quad &\int_{\mathbb{C}}\int_{\mathbb{C}} |f(\xi)K_z(\xi)K_u(\xi)e^{-2\varphi(\xi)}K_w(u)e^{-2\varphi(u)}|\,dA(u)\,dA(\xi)\\ &\lesssim \|K_w\|_{F^\infty_\varphi}\int_{\mathbb{C}} |f(\xi)K_z(\xi)e^{-\varphi(\xi)}|\,dA(\xi)\\ &\le \|K_w\|_{F^\infty_\varphi}\int_{\mathbb{C}} (|f(z)|+|f(z)-f(\xi)|)|K_z(\xi)e^{-\varphi(\xi)}\,dA(\xi)\\ &\lesssim |f(z)|\|K_w\|_{F^\infty_\varphi}\|K_z\|_{F^1_\varphi}+\rho(z)^{-s}\|K_w\|_{F^\infty_\varphi}\|f\|_{\mathrm{BO}^s}\\ &\qquad\qquad\times\int_{\mathbb{C}}(\beta(z,\xi)+1)|K_z(\xi)e^{-\varphi(\xi)}\,dA(\xi)\\ &\lesssim |f(z)|\|K_w\|_{F^\infty_\varphi}\|K_z\|_{F^1_\varphi}+\rho(z)^{-s}\|K_w\|_{F^\infty_\varphi}\|f\|_{\mathrm{BO}^s}e^{\varphi(z)}<\infty.\end{aligned}$$

We immediately get (4-3) by combining the conditions (4-5) and (4-6).

Finally, by Fubini's theorem and the property of the reproducing kernel in (4-2), we have $\langle T_f K_z, K_w\rangle = T_f K_z(w)$. □

Let $1 \le p, q < \infty$ and let $s = \frac{1}{q} - \frac{1}{p}$ in this section.

**Lemma 4.3.** *Let $1 < p \le q < \infty$ or $p = q = 1$. For $f \in \mathcal{S}$, if $f \in \mathrm{BMO}^{2s}$ with*

$$\sup_{z\in\mathbb{C}} \rho(z)^{2s}|\tilde{f}(z)| < \infty,$$

*then $T_f \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$.*

*Proof.* First note that by Lemma 4.2, $T_f$ is well defined on $\Gamma$. It can easily be shown that

$$\widetilde{|f|}(z) - |\tilde{f}(z)| = \int_{\mathbb{C}} (|f(w)| - |\tilde{f}(z)|)|k_{2,z}(w)|^2 e^{-2\varphi(w)}\,dA(w) \le \mathrm{MO}(f)(z)$$

since $k_{2,z}$ is a normalized reproducing kernel. Therefore, by our assumption,

$$\sup_{z\in\mathbb{C}} \rho(z)^{2s}|\tilde{f}|(z) < \infty.$$

From this, Lemma 2.5 and $|\widehat{f}|_r(u) \le |\widetilde{f}|_r(u)$ for all $u \in \mathbb{C}$, $r > 0$, we deduce that

$$\begin{aligned}|\langle T_f k_{p,z}, k_{q',w}\rangle| &\le \int_{\mathbb{C}} |k_{p,z}(u)|\,|k_{q',w}(u)|\,|f(u)|e^{-2\varphi(u)}\,dA(u)\\ &= \int_{\mathbb{C}} |k_{p,z}(u)|\,|k_{q',w}(u)|\,|\widetilde{f}|(u)e^{-2\varphi(u)}\,dA(u).\end{aligned}$$

Therefore, by the above inequality, Lemma 2.3, Fubini's theorem and (2-2), we have

$$\begin{aligned}&\int_{\mathbb{C}} |\langle T_f k_{p,z}, k_{q',w}\rangle|\,dA\sigma(w)\\ &\quad\le \int_{\mathbb{C}}\int_{\mathbb{C}} |k_{p,z}(u)|\,|k_{q',w}(u)|\,|\widetilde{f}|(u)e^{-2\varphi(u)}\,dA(u)\,dA\sigma(w)\\ &\quad= \int_{\mathbb{C}} |k_{p,z}(u)|\,|\widetilde{f}|(u)e^{-2\varphi(u)}\,dA(u)\int_{\mathbb{C}} |k_{q',w}(u)|\,dA\sigma(w)\\ &\quad= \int_{\mathbb{C}} |k_{p,z}(u)|\,|\widetilde{f}|(u)e^{-\varphi(u)}\rho(u)^{-2/q'}\,dA(u)\\ &\quad\lesssim \sup_{u\in\mathbb{C}} \rho(u)^{2s}\,|\widetilde{f}|(u)\int_{\mathbb{C}} |k_{p,z}(u)|e^{-\varphi(u)}\rho(u)^{2/p-2}\,dA(u)\\ &\quad\lesssim \sup_{u\in\mathbb{C}} \rho(u)^{2s}|\widetilde{f}|(u) < \infty.\end{aligned}$$

Therefore,

$$\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |\langle T_f k_{p,z}, k_{q',w}\rangle|\,dA\sigma(w) < \infty.$$

This shows that $T_f$ satisfies the condition (3-2).

To prove $T_f$ satisfies the condition (3-4), we only need to replace (2-2) by (2-3). The conditions (3-3) and (3-5) for the Toeplitz operator $T_f$ can be shown to hold in the same way. Hence, $T_f \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$. □

We immediately get the following corollary, whose proof is similar to that in [Hu et al. 2018, Theorem 3.2].

**Corollary 4.4.** *For $p \ge 1$ and $f \in \mathcal{S}$, if $f \in$ BMO with $\sup_{z\in\mathbb{C}} |\tilde{f}(z)| < \infty$, then $T_f \in \mathcal{W}(F^p_\varphi, F^p_\varphi)$.*

**Corollary 4.5.** *Let $1 \le p < \infty$ and let $f \in \mathcal{S}$. If $\sup_{z\in\mathbb{C}} |\widetilde{f}|(z) < \infty$, then $T_f \in \mathcal{W}(F^p_\varphi, F^p_\varphi)$. Furthermore,*

$$\{T_f : f \in \mathcal{S} \text{ and } \sup_{z\in\mathbb{C}} |\widetilde{f}|(z) < \infty\}$$

*forms a subalgebra of $\mathfrak{B}(F^p_\varphi)$.*

*Proof.* As an application of [Hu and Lv 2017, Lemma 2.5], it is easy to show that $\sup_{z\in\mathbb{C}} |\widetilde{f}|(z) < \infty$ if and only if $f \in$ BA. Lemma 4.1 implies that BA $\subset$ BMO and

Lemma 4.2 shows that $T_f$ is well defined on $\Gamma$. It follows that $T_f \in \mathcal{W}(F^p_\varphi, F^p_\varphi)$ by Lemma 4.3 when $\sup_{z\in\mathbb{C}} |\widetilde{f}|(z) < \infty$.

Therefore,

$$\left\{T_f : f \in \mathcal{S} \text{ and } \sup_{z\in\mathbb{C}} |\widetilde{f}|(z) < \infty\right\}$$

forms a subalgebra of $\mathfrak{B}(F^p_\varphi)$ by Theorem 3.4. □

As an application of Corollary 4.5 and Theorem 3.7, we immediately obtain the following corollary.

**Corollary 4.6.** *Let $1 \le p < \infty$ and $f \in \mathcal{S}$. If*

$$T \in \{T_f : f \in \mathcal{S} \text{ and } \sup_{z\in\mathbb{C}} |\widetilde{f}|(z) < \infty\},$$

*then the following conditions are equivalent*:

(1) $T \in \mathcal{K}(F^p_\varphi)$.

(2) $\lim_{|z|\to\infty} \|Tk_{p,z}\|_{F^p_\varphi} = 0$.

(3) $\lim_{|z|\to\infty} \sup_{w\in\mathbb{C}} |\langle Tk_{p,z}, k_{p',w}\rangle| = 0$.

(4) $\lim_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle Tk_{p,z}, k_{p',w}\rangle| = 0$ *for any* $r > 0$.

We are going to give the most important theorem of this section.

**Theorem 4.7.** *Let $1 < p \le q < \infty$ or $p = q = 1$, and $f \in \mathcal{S}$. If $f \in \mathrm{BMO}^{2s}$, then*:

(1) $T_f \in \mathfrak{B}(F^p_\varphi, F^q_\varphi)$ *if and only if*

$$\sup_{z\in\mathbb{C}} \rho(z)^{2s} |\tilde{f}(z)| < \infty. \tag{4-7}$$

(2) $T_f \in \mathcal{K}(F^p_\varphi, F^q_\varphi)$ *if and only if for any* $r > 0$,

$$\lim_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle Tk_{p,z}, k_{q',w}\rangle| = 0. \tag{4-8}$$

*Proof.* (1) Suppose that (4-7) holds. From Lemma 4.3 and Theorem 3.3, it follows that $T_f \in \mathfrak{B}(F^p_\varphi, F^q_\varphi)$.

Conversely, assume that $T_f \in \mathfrak{B}(F^p_\varphi, F^q_\varphi)$. If $1 < p \le q < \infty$, then by Lemma 2.3,

$$\begin{aligned} \rho(z)^{2s} \tilde{f}(z) &= \rho(z)^{2s} \langle T_f k_{2,z}, k_{2,z}\rangle \\ &= \rho(z)^{2s} \|K_z\|^{-2}_{F^2_\varphi} \int_{\mathbb{C}} f(u) |K_z(u)|^2 e^{-2\varphi(u)}\, dA(u) \\ &\simeq \rho(z)^{2s} \rho(z)^{-2s} \|K_z\|^{-1}_{F^p_\varphi} \|K_z\|^{-1}_{F^{q'}_\varphi} \langle T_f K_z, K_z\rangle \\ &\simeq \langle T_f k_{p,z}, k_{q',z}\rangle. \end{aligned} \tag{4-9}$$

From this and Hölder's inequality, we have

$$\rho(z)^{2s}|\tilde{f}(z)| \simeq |\langle T_f k_{p,z}, k_{q',z}\rangle| \lesssim \|T_f k_{p,z}\|_{F^p_\varphi} \|k_{q',z}\|_{F^{q'}_\varphi} < \infty.$$

If $p = q = 1$, then by Lemma 2.3 again, we obtain

$$\begin{aligned}\tilde{f}(z) &= \langle T_f k_{2,z}, k_{2,z}\rangle \\ &= \|K_z\|^{-2}_{F^2_\varphi} \int_{\mathbb{C}} f(u)|K_z(u)|^2 e^{-2\varphi(u)}\, dA(u) \\ &\simeq \|K_z\|^{-1}_{F^1_\varphi} \|K_z\|^{-1}_{F^\infty_\varphi} \langle T_f K_z, K_z\rangle \\ &= \langle T_f k_{1,z}, k_{\infty,z}\rangle.\end{aligned}$$

It follows that

$$|\tilde{f}(z)| \simeq |\langle T_f k_{1,z}, k_{\infty,z}\rangle| \le \|T_f k_{1,z}\|_{F^1_\varphi} < \infty.$$

Hence, $T_f \in \mathfrak{B}(F^p_\varphi, F^q_\varphi)$ implies that (4-7) holds.

(2) Suppose that $T_f \in \mathcal{K}(F^p_\varphi, F^q_\varphi)$. By (4-7) and Lemma 4.3, we obtain

$$T_f \in \mathcal{W}(F^p_\varphi, F^q_\varphi).$$

Furthermore,

$$\lim_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle T k_{p,z}, k_{q',w}\rangle| = 0$$

by Theorem 3.7.

Assume that (4-8) holds. By Theorem 3.7, we have

$$\lim_{|z|\to\infty} \sup_{w\in\mathbb{C}} |\langle T k_{p,z}, k_{q',w}\rangle| = 0.$$

In particular,

$$\lim_{|z|\to\infty} |\langle T k_{p,z}, k_{q',z}\rangle| = 0.$$

From this and (4-9), we have $\rho(z)^{2s}\tilde{f}(z)$ is bounded and by Lemma 4.3, we have $T_f \in \mathcal{W}(F^p_\varphi, F^q_\varphi)$. By Theorem 3.7 again and our assumption, we obtain $T_f \in \mathcal{K}(F^p_\varphi, F^q_\varphi)$. □

**Corollary 4.8.** *Let $1 \le p < \infty$ and $f \in \mathrm{BMO} \cap \mathcal{S}$. Then*:

(1) *$T_f \in \mathfrak{B}(F^p_\varphi)$ if and only if the Berezin transform $\tilde{f}$ is bounded.*

(2) *$T_f \in \mathcal{K}(F^p_\varphi)$ if and only if for any $r > 0$,*

$$\lim_{|z|\to\infty} \sup_{w\in D^r(z)} |\langle T k_{p,z}, k_{q',w}\rangle| = 0.$$

*Proof.* It is easy to get this conclusion by using Theorem 4.7 and Corollary 4.4. □

**4.2.** ***Toeplitz operators with*** $\mathbf{IMO}^s$ ***symbols.*** For $s > 0$, the space $\mathrm{IMO}^s$ consists of $f \in \Gamma$ such that

$$\|f\|_{\mathrm{IMO}^s} = \left( \int_{\mathbb{C}} |\mathrm{MO}(f)(z)|^s \, dA(z) \right)^{1/s} < \infty.$$

Let $1 \le q < p < \infty$. We will characterize the boundedness and compactness of Toeplitz operators $T_f$ with $\mathrm{IMO}^s$ from $F^p_\varphi$ to $F^q_\varphi$ in this section. We need to use Luecking's approach [1993] based on Khinchine's inequality. Recall that Rademacher functions $R_n$ are defined by

$$R_0(t) = \begin{cases} 1 & \text{if } 0 \le t - [t] < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \le t - [t] < 1; \end{cases} \qquad R_n(z^n t), \quad n \ge 1,$$

where $[t]$ denotes the largest integer not exceeding $t$.

**Lemma 4.9** (Khinchine's inequality). *Let $0 < p < \infty$. There exists a positive constant $C_p$ such that*

$$C_p^{-1} \left( \sum_{k=1}^n |\lambda_k|^2 \right)^{p/2} \lesssim \int_0^1 \left| \sum_{k=1}^n \lambda_k R_k(t) \right|^p dt \lesssim C_p \left( \sum_{k=1}^n |\lambda_k|^2 \right)^{p/2}$$

*for all $n \in \mathbb{N}$ and sequences $\{\lambda_k\}$ in $\mathbb{C}$.*

We will give the main result of this section.

**Theorem 4.10.** *Let $1 \le q < p < \infty$ and let $s = \frac{pq}{p-q}$. Then for $f \in \mathrm{IMO}^s$, the following statements are equivalent*:

(1) *$T_f$ is bounded from $F^p_\varphi$ to $F^q_\varphi$.*

(2) *$T_f$ is compact from $F^p_\varphi$ to $F^q_\varphi$.*

(3) *The Berezin transform $\tilde{f} \in L^s(\mathbb{C}, dA)$.*

*Proof.* (2) $\Rightarrow$ (1) is trivial. Suppose that (1) holds. Let $r > 0$ and $\{z_j\}_{j=1}^\infty$ be a sequence such that $\{D^r(z_j)\}_{j=1}^\infty$ is a cover of $\mathbb{C}$. Then for $1 < p < \infty$, by [Hu and Lv 2017, Lemma 2.6], we get the function

$$g_t(z) := \sum_{j=1}^\infty \gamma_j r_j(t) k_{2,z_j}(z) \rho(z_j)^{1-2/p} = \sum_{j=1}^\infty \gamma_j r_j(t) k_{p,z_j}(z)$$

belongs to $F^p_\varphi$ and

$$\|g_t\|_{F^p_\varphi} \lesssim \|\{\gamma_j\}_{j=1}^\infty\|_{\ell^p}.$$

Thus, by our assumption, we have

$$\|T_f g_t\|_{F^q_\varphi} \lesssim \|\{\gamma_j\}_{j=1}^\infty\|_{\ell^p}. \tag{4-10}$$

Fubini's theorem and Khinchine's inequality yield that

$$\begin{aligned}\int_0^1 \|T_f g_t\|_{F_\varphi^q}^q \, dt &\gtrsim \int_{\mathbb{C}} \Big( \sum_{j=1}^{\infty} |\gamma_j T_f k_{p,z_j}(z)|^2 \Big)^{q/2} e^{-q\varphi(z)} \, dA(z) \\ &\gtrsim \sum_{i=1}^{\infty} \int_{D^r(z_i)} \Big( \sum_{j=1}^{\infty} |\gamma_j T_f k_{p,z_j}(z)|^2 \Big)^{q/2} e^{-q\varphi(z)} \, dA(z) \\ &\gtrsim \sum_{i=1}^{\infty} \gamma_i^q \int_{D^r(z_i)} |T_f k_{p,z_i}(z)|^q e^{-q\varphi(z)} \, dA(z) \\ &\gtrsim \sum_{i=1}^{\infty} \gamma_i^q |T_f k_{p,z_i}(z_i)|^q e^{-q\varphi(z)} \rho(z_i)^2,\end{aligned} \tag{4-11}$$

where the last inequality comes from Lemma 2.4. Using the property of reproducing kernel and Lemma 2.3, we obtain

$$|T_f k_{p,z_i}(z_i)| e^{-\varphi(z)} \simeq |\tilde{f}(z_i)| \rho(z_i)^{-2/p}. \tag{4-12}$$

Combining with (4-10), (4-11) and (4-12), we have

$$\sum_{i=1}^{\infty} \gamma_i^q |\tilde{f}(z_i)|^q \rho(z_i)^{2-2q/p} \lesssim \int_0^1 \|T_f g_t\|_{F_\varphi^q}^q \, dt \lesssim \|\{\gamma_j\}_{j=1}^{\infty}\|_{\ell^p}^q = \|\{|\gamma_j|^q\}_{j=1}^{\infty}\|_{\ell^{q/p}}.$$

Since the duality of $\ell^{p/q}$ is $\ell^{p/(p-q)}$, we have

$$|\tilde{f}(z_i)|^q \rho(z_i)^{2-2q/p} \in \ell^{p/(p-q)},$$

that is,

$$\sum_{i=1}^{\infty} |\tilde{f}(z_i)|^s \rho(z_i)^2 < \infty.$$

Choose a point $a_i \in \overline{D^r(z_i)}$ so that the condition

$$|\tilde{f}(a_i)| = \sup_{z \in \overline{D^r(z_i)}} |\tilde{f}(z)|$$

is satisfied and there is a constant $b > 0$ such that $\{D^{br}(a_i)\}$ is a $(\rho, br)$-lattice. Similar to proving the above, we obtain

$$\sum_{i=1}^{\infty} |\tilde{f}(a_i)|^s \rho(a_i)^2 < \infty. \tag{4-13}$$

Therefore, by (4-13),

$$\int_{\mathbb{C}} |\tilde{f}(z)|^s \, dA(z) \le \sum_{i=1}^{\infty} \int_{D^r(z_i)} |\tilde{f}(z)|^s \, dA(z) \lesssim \sum_{i=1}^{\infty} |\tilde{f}(a_i)|^s \rho(a_i)^2 < \infty.$$

This proves (1) $\Rightarrow$ (3).

We suppose that (3) holds. It is easy to show that

$$\widetilde{|f|}(z) - |\tilde{f}(z)| = \int_{\mathbb{C}} (|f(w)| - |\tilde{f}(z)|)|k_{2,z}(w)|^2 e^{-2\varphi(w)}\, dA(w) \le \mathrm{MO}(f)(z),$$

which implies that $\widetilde{|f|} \in L^s(\mathbb{C}, dA)$. As an application of [Hu and Lv 2017, Thoerem 3.4], $T_{|f|}$ is compact from $F^p_\varphi$ to $F^q_\varphi$ if and only if $\widetilde{|f|} \in L^s(\mathbb{C}, dA)$. Hence, $T_f$ is also compact from $F^p_\varphi$ to $F^q_\varphi$. □

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