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JOINT DISTRIBUTION OF BUSEMANN FUNCTIONS IN THE EXACTLY SOLVABLE CORNER GROWTH MODEL

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The 1+1-dimensional corner growth model with exponential weights is a centrally important exactly solvable model in the Kardar–Parisi–Zhang class of statistical mechanical models. While significant progress has been made on the fluctuations of the growing random shape, understanding of the optimal paths, or geodesics, is less developed. The Busemann function is a useful analytical tool for studying geodesics. We describe the joint distribution of the Busemann functions, simultaneously in all directions of growth. As applications of this description we derive a marked point process representation for the Busemann function across a single lattice edge and calculate some marginal distributions of Busemann functions and semi-infinite geodesics.

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1. Introduction

The corner growth model in the Kardar–Parisi–Zhang class. The planar corner growth model (CGM) is a directed last-passage percolation (LPP) model on the planar integer lattice $\mathbb{Z}^2$ whose paths are allowed to take nearest-neighbor steps $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. In the exactly solvable case the random weights attached to the vertices of $\mathbb{Z}^2$ are i.i.d. exponentially or geometrically distributed random variables.

The exact solvability of the exponential and geometric CGM has been fundamental to the 20-year progress in the study of the 1+1-dimensional Kardar–Parisi–Zhang (KPZ) universality class. After the initial breakthrough by Baik, Deift and Johansson [Baik et al. 1999] on planar LPP on Poisson points,

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the Tracy–Widom limit of the geometric and exponential CGM followed in [Johansson 2000]. This work relied on techniques that today would be called *integrable probability*, a subject that applies ideas from representation theory and integrable systems to study stochastic models. A large literature has followed. Recent reviews appear in [Corwin 2018; 2016; 2012]. A different line of work was initiated in [Balázs et al. 2006] that gave a probabilistic proof of the KPZ exponents of the exponential CGM, following the seminal work [Cator and Groeneboom 2006] on the planar Poisson LPP. The proof utilized the tractable stationary version of the CGM and developed estimates by coupling perturbed versions of the CGM process. This opening led to the first proofs of KPZ exponents for the asymmetric simple exclusion process (ASEP) [Balázs and Seppäläinen 2010] and the KPZ equation [Balázs et al. 2011], to the discovery of the first exactly solvable positive-temperature lattice polymer model [Seppäläinen 2012], to a proof of KPZ exponents for a class of zero-range processes outside known exactly solvable models [Balázs et al. 2012], and most recently to Doob transforms and martingales in random walks in random environments (RWRE) that manifest KPZ behavior [Balázs et al. 2019b]. The estimates from [Balázs et al. 2006] have also been applied to coalescence times of geodesics [Pimentel 2016] and to the local behavior of Airy processes [Pimentel 2018].

Joint distribution of Busemann functions. The present article places the stationary CGM into a larger context by describing the natural coupling of all the stationary CGMs. This coupling arises from the joint distribution of the Busemann functions in all directions of growth.

Let $G_{x,y}$ denote the last-passage value between points x and y on the lattice $\mathbb{Z}^2$ (the precise definition follows in (2-1) in Section 2). The Busemann function $B^{\rho}_{x,y}$ is the limit of increments $G_{v_n,y} - G_{v_n,x}$ as v_n is taken to infinity in the direction parametrized by ρ. In a given direction this limit exists almost surely. These limits are extended to a process $B^{\bullet}$ by taking limits in the parameter ρ. Finite-dimensional distributions of $B^{\bullet}$ are identified as the unique invariant distributions of multiclass LPP processes. These distributions are conveniently described in terms of mappings that represent FIFO (first-in-first-out) queues. Key points of the development are (i) an intertwining between two types of multiclass processes, called the *multiline process* and the *coupled process*, and (ii) a triangular array representation of the intertwining mapping.

The results of this paper will have various applications in the study of the CGM, and they can be extended to other 1+1-dimensional growth and polymer models that have a tractable stationary version. A forthcoming work of the authors develops the joint distribution of Busemann functions for the positive-temperature log-gamma polymer model.

Two applications have been completed recently. Properties of the joint Busemann process $B^{\bullet}$ discovered here are applied in [Janjigian et al. 2019] to describe (i) the overall structure of the geodesics of the exponential corner growth model and (ii) the statistics of a new object termed the "instability graph" that captures the geometry of the jumps of the Busemann functions on the lattice. The joint Busemann distribution is necessary for a full picture of the geodesics because in a fixed direction semi-infinite geodesics are almost surely unique and coalesce (these facts are reviewed in Section 2B below) but there are random directions of nonuniqueness. The joint distribution captures the jumps of the Busemann function as the direction varies. These correspond to jumps in coalescence points and nonuniqueness of geodesics.

The article [Balázs et al. 2019a] gives a proof of the nonexistence of bi-infinite geodesics in the exponential CGM, based on couplings with the stationary version of the LPP process. The joint distribution described here is a critical ingredient of the proof.

An analogue of the Busemann function on the Airy sheet. An interesting similarity appears between our paper and recent work on the universal objects that arise from LPP. Basu, Ganguly and Hammond [Basu et al. 2019] study an analogue of the Busemann function in the Brownian last-passage model. Instead of the lattice scale and all spatial directions, they look at a difference of last-passage values on the scale $n^{2/3}$ into a fixed macroscopic direction, where universal objects such as Airy processes arise. Translated to the CGM, their object of interest is the weak limit $Z(z)$ of the scaled difference $n^{-1/3}[G_{(n^{2/3},0),(n+zn^{2/3},n)} - G_{(-n^{2/3},0),(n+zn^{2/3},n)} + 4n^{2/3}]$ that they call the *difference weight profile*. In terms of the Airy sheet $\{W(x, y) : x, y \in \mathbb{R}\}$ constructed recently by Dauvergne, Ortmann and Virág [Dauvergne et al. 2018], the limit $Z(z) = W(1, z) - W(-1, z)$.

The limit $Z(\cdot)$ is a continuous process, while the Busemann process we construct is a jump process. But like the Busemann process, the limit $Z(\cdot)$ is constant in a neighborhood of each point, except for a small set of exceptional points. In both settings this constancy reflects the same underlying phenomenon, namely the coalescence of geodesics. In our lattice setting, Theorem 3.4 gives a precise description of these exceptional directions in terms of an inhomogeneous Poisson process.

Past work. We mention related past work on queues, particle systems, and the CGM.

Queueing fixed points. We formulate a queueing operator as a mapping of bi-infinite sequences of interarrival times and service times into a bi-infinite sequence of interdeparture times (details in Section 2C). When the service times are i.i.d. exponential (memoryless, or $\cdot/M/1$ queue), it is classical that i.i.d. exponential times are preserved by the mapping from the interarrival process to the interdeparture process, subject to the stability condition that the mean interarrival time exceed the mean service time. Anantharam [1993] proved the uniqueness of this fixed point and Chang [1994] gave a shorter argument. (An unpublished manuscript of Liggett and Shiga is also cited in [Mountford and Prabhakar 1995].) Convergence to the fixed point was proved in [Mountford and Prabhakar 1995]. These results were partially extended to general $\cdot/G/1$ queues in [Mairesse and Prabhakar 2003; Prabhakar 2003].

We look at LPP processes with multiple classes of input, but this is *not* the same as a multiclass queue that serves customers in different priority classes. In queueing terms, the present paper describes the unique invariant distribution in a situation where a single memoryless queueing operator transforms a vector of interarrival processes into a vector of interdeparture processes. It is fairly evident a priori that this operation cannot preserve an independent collection of interarrival processes because they are correlated after passing through the same queueing operator. (For example, this operation preserves monotonicity.) It turns out that the queueing mappings themselves provide a way to describe the structure of the invariant distribution.

Multiclass measures for particle systems. In a series of remarkable papers, P. A. Ferrari and J. B. Martin [2006; 2007; 2009] developed queueing descriptions of the stationary distributions of the multiclass totally asymmetric simple exclusion process (TASEP) and the Aldous–Diaconis–Hammersley process.

The intertwining that establishes our Theorem 5.5 became possible after the discovery of a way to apply the ideas of Ferrari and Martin to the CGM. We use the terms multiline process and coupled process to highlight the analogy with their work.

Busemann functions and semi-infinite geodesics. Existence and properties of Busemann functions and semi-infinite geodesics are reviewed in Sections 2A and 2B. Two strategies exist for proving the existence of Busemann functions for the exponential CGM.

(i) Proofs by Ferrari and Pimentel [2005] and Coupier [2011] relied on C. Newman's approach to geodesics [Howard and Newman 2001; Licea and Newman 1996; Newman 1995]. This strategy is feasible because the exact solvability shows that the shape function (2-4) satisfies the required curvature hypotheses.

(ii) A direct argument from the stationary growth model to the Busemann limit was introduced in [Georgiou et al. 2015] for the log-gamma polymer, and applied to the exponential CGM in the lecture notes [Seppäläinen 2018]. An application of this strategy to the CGM with general i.i.d. weights appears in [Georgiou et al. 2017a; 2017b], where the role of the regularity of the shape function becomes explicit.

A sampling of other significant work on Busemann functions and geodesics can be found in [Cator and Pimentel 2012; 2013; Ferrari et al. 2009; Bakhtin et al. 2014; Hoffman 2005; 2008].

Organization of the paper. Section 2 collects preliminaries on the CGM and queues. The main results for Busemann functions and semi-infinite geodesics are stated in Section 3. Section 4 proves a key lemma for the queueing operator. Section 5 introduces the multiline process, the coupled process, and the multiclass LPP process, and then states and proves results on their invariant distributions. The key intertwining between the multiline process and the coupled process appears in (5-8) in the proof of Theorem 5.5 in Section 5D. Section 6 proves the results of Section 3. For the proof of Theorem 3.4, Section 6B introduces a triangular array representation for the intertwining mapping. Auxiliary matters on queues and exponential distributions are relegated to Appendices A and B.

Notation and conventions. Points $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ are ordered coordinatewise: $x \le y$ if and only if $x_1 \le y_1$ and $x_2 \le y_2$. The ℓ^1 norm is $|x|_1 = |x_1| + |x_2|$. Subscripts indicate restricted subsets of the reals and integers: for example $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$. Boldface notation for vectors: $\boldsymbol{e}_1 = (1, 0)$, $\boldsymbol{e}_2 = (0, 1)$, and members of the simplex $[\boldsymbol{e}_2, \boldsymbol{e}_1] = \{t\boldsymbol{e}_1 + (1-t)\boldsymbol{e}_2 : 0 \le t \le 1\}$ are denoted by $\boldsymbol{u}$.

For $n \in \mathbb{Z}_{>0}$, $[n] = \{1, 2, \dots, n\}$, with the convention that $[n] = \varnothing$ for $n \in \mathbb{Z}_{\le 0}$. A finite integer interval is denoted by $[\![m, n]\!] = \{m, m+1, \dots, n\}$, and $[\![m, \infty[\![= \{m, m+1, m+2, \dots\}$.

For $0 < \alpha < \infty$, $X \sim \mathrm{Exp}(\alpha)$ means that random variable X has exponential distribution with rate α; in other words $P(X > t) = e^{-\alpha t}$ for $t > 0$ and $E(X) = \alpha^{-1}$. In the discussion we parametrize exponential variables with their mean. For $0 < \rho < \infty$, ν^ρ is the probability distribution on the space $\mathbb{R}_{\ge 0}^{\mathbb{Z}}$ of bi-infinite sequences under which the coordinates are i.i.d. exponential variables with common mean ρ. Higher-dimensional product measures are denoted by $\nu^{(\rho_1, \rho_2, \dots, \rho_n)} = \nu^{\rho_1} \otimes \nu^{\rho_2} \otimes \cdots \otimes \nu^{\rho_n}$.

For $0 \le p \le 1$, $X \sim \mathrm{Ber}(p)$ means that random variable X has Bernoulli distribution with parameter p; in other words $P(X = 1) = p = 1 - P(X = 0)$.

In general, E^μ represents expectation under a measure μ.

2. Preliminaries

Section 2A introduces the main objects of discussion: the planar corner growth model (CGM), which is a special case of last-passage percolation (LPP), and Busemann functions. Section 2B explains the significance of Busemann functions in the description of directed semi-infinite geodesics and the asymptotic direction of the competition interface. The somewhat technical Section 2C defines FIFO (first-in-first-out) queueing mappings that are used in Section 3 to describe the joint distribution of the Busemann functions. To be sure, the distribution of the Busemann functions could be described by plain mathematical formulas without their queueing content. But the queueing context gives the mathematics meaning that can help comprehend the results.

2A. ***Busemann functions in the corner growth model.*** The setting for the exponential CGM is the following: $(\Omega, \mathfrak{S}, \mathbb{P})$ is a probability space with generic sample point ω. A group of measure-preserving measurable bijections $\{\theta_x\}_{x\in\mathbb{Z}^2}$ acts on $(\Omega, \mathfrak{S}, \mathbb{P})$. Measure preservation means that $\mathbb{P}(\theta_x A) = \mathbb{P}(A)$ for all sets $A \in \mathfrak{S}$ and $x \in \mathbb{Z}^2$. $Y = (Y_x)_{x\in\mathbb{Z}^2}$ is a random field of independent and identically distributed Exp(1) random weights defined on Ω that satisfies $Y_x(\theta_y\omega) = Y_{x+y}(\omega)$ for $x, y \in \mathbb{Z}^2$ and $\omega \in \Omega$.

The canonical choice for the sample space is the product space $\Omega = \mathbb{R}_{\geq 0}^{\mathbb{Z}^2}$ with its Borel σ-algebra $\mathfrak{S}$, generic sample point $\omega = (\omega_x)_{x\in\mathbb{Z}^2}$, translations $(\theta_x\omega)_y = \omega_{x+y}$, and coordinate random variables $Y_x(\omega) = \omega_x$. Then $\mathbb{P}$ is the i.i.d. product measure on Ω under which each Y_x is an Exp(1) random variable.

For $u \leq v$ on $\mathbb{Z}^2$ (coordinatewise ordering) let $\Pi_{u,v}$ denote the set of up-right paths $x_\bullet = (x_i)_{i=0}^{|v-u|_1}$ from $x_0 = u$ to $x_{|v-u|_1} = v$ with steps $x_i - x_{i-1} \in \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. (The left diagram of Figure 1 illustrates this.) Define the last-passage percolation (LPP) process

$$G_{u,v} = \max_{x_\bullet\in\Pi_{u,v}} \sum_{i=0}^{|v-u|_1} Y_{x_i} \quad \text{for } u \leq v \text{ on } \mathbb{Z}^2. \tag{2-1}$$

For $v \in u + \mathbb{Z}_{>0}^2$ we have the inductive equation

$$G_{u,v} = G_{u,v-\boldsymbol{e}_1} \vee G_{u,v-\boldsymbol{e}_2} + Y_v. \tag{2-2}$$

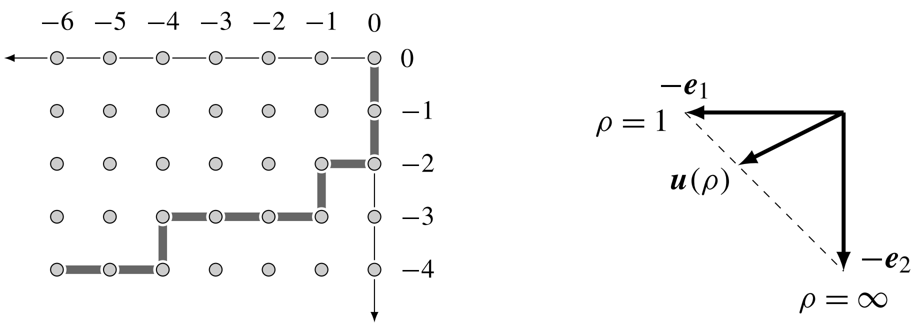

Figure 1. Left: The thickset line segments define an element of $\Pi_{(-6,-4),(0,0)}$. Right: As the parameter ρ increases from 1 to ∞, vector $\boldsymbol{u}(\rho)$ of (2-5) sweeps the directions from $-\boldsymbol{e}_1$ to $-\boldsymbol{e}_2$ in the third quadrant.

The convention of this paper is that growth proceeds in the *south-west* direction (into the *third quadrant* of the plane). Thus the well-known shape theorem (Theorem 5.1 in [Martin 2004], Theorem 3.5 in [Seppäläinen 2018]) of the CGM takes the following form. With probability one,

$$\lim_{r\to\infty} \sup_{x\in(\mathbb{Z}_{\le 0})^2:\,|x|_1\ge r} \frac{|G_{x,0}-g(x)|}{|x|_1}=0 \tag{2-3}$$

with the concave, continuous and one-homogeneous shape function (known since [Rost 1981])

$$g(x)=\big(\sqrt{|x_1|}+\sqrt{|x_2|}\,\big)^2 \quad \text{for } \ x=(x_1,x_2)\in\mathbb{R}^2_{\le 0}. \tag{2-4}$$

Busemann functions are limits of differences $G_{v,x}-G_{v,y}$ of last-passage values from two fixed points x and y to a common point v that is taken to infinity in a particular direction. These limits are described by relating the direction $\boldsymbol{u}$ that v takes to a real parameter ρ that specifies the distribution of the limits: a bijective mapping between directions $\boldsymbol{u}=(u_1,u_2)\in\,]-\boldsymbol{e}_1,-\boldsymbol{e}_2[$ in the open third quadrant of the plane and parameters $\rho\in(1,\infty)$ is defined by the equations

$$\boldsymbol{u}=\boldsymbol{u}(\rho)=-\left(\frac{1}{1+(\rho-1)^2},\,\frac{(\rho-1)^2}{1+(\rho-1)^2}\right) \iff \rho=\rho(\boldsymbol{u})=\frac{\sqrt{-u_1}+\sqrt{-u_2}}{\sqrt{-u_1}}. \tag{2-5}$$

(See the right diagram of Figure 1 for an illustration.)

The existence and properties of Busemann functions are summarized in the following theorem. By definition, a *down-right lattice path* $\{y_k\}$ satisfies $y_k-y_{k-1}\in\{\boldsymbol{e}_1,-\boldsymbol{e}_2\}$ for all k.

Theorem 2.1. *On the probability space* $(\Omega,\mathfrak{S},\mathbb{P})$ *there exists a cadlag process* $B^\rho=(B^\rho_{x,y})_{x,y\in\mathbb{Z}^2}$ *with state space* $\mathbb{R}^{\mathbb{Z}^2\times\mathbb{Z}^2}$, *indexed by* $\rho\in(1,\infty)$, *with the following properties.*

(i) Path properties. *There is a single event* Ω_0 *such that* $\mathbb{P}(\Omega_0)=1$ *and the following properties hold for all* $\omega\in\Omega_0$, *for all* $\lambda,\rho\in(1,\infty)$ *and* $x,y,z\in\mathbb{Z}^2$:

$$\text{If } \lambda<\rho \text{ then } B^\lambda_{x,x+\boldsymbol{e}_1}\le B^\rho_{x,x+\boldsymbol{e}_1} \text{ and } B^\lambda_{x,x+\boldsymbol{e}_2}\ge B^\rho_{x,x+\boldsymbol{e}_2}. \tag{2-6}$$

$$B^\rho_{x,y}+B^\rho_{y,z}=B^\rho_{x,z}. \tag{2-7}$$

$$Y_x=B^\rho_{x-\boldsymbol{e}_1,x}\wedge B^\rho_{x-\boldsymbol{e}_2,x}. \tag{2-8}$$

Cadlag property: the path $\rho\mapsto B^\rho_{x,y}$ *is right continuous and has left limits.*

(ii) Distributional properties. *Each process* B^ρ *is stationary under lattice shifts. The marginal distributions of nearest-neighbor increments are*

$$B^\rho_{x-\boldsymbol{e}_1,x}\sim \mathrm{Exp}(\rho^{-1}) \quad \text{and} \quad B^\rho_{x-\boldsymbol{e}_2,x}\sim \mathrm{Exp}(1-\rho^{-1}). \tag{2-9}$$

Along any down-right path $\{y_k\}_{k\in\mathbb{Z}}$ *on* $\mathbb{Z}^2$, *for fixed* $\rho\in(1,\infty)$ *the increments* $\{B^\rho_{y_k,y_{k+1}}\}_{k\in\mathbb{Z}}$ *are independent.*

(iii) Limits. *Fix* $\rho\in(1,\infty)$ *and let* $\boldsymbol{u}=\boldsymbol{u}(\rho)$ *be the vector determined by* (2-5). *Then there exists an event* $\Omega_0^{(\rho)}$ *such that* $\mathbb{P}(\Omega_0^{(\rho)})=1$ *and the following holds: for any sequence* $\{u_n\}$ *in* $\mathbb{Z}^2$ *such that* $|u_n|_1\to\infty$

and $u_n/n \to \boldsymbol{u}$ and for any $\omega \in \Omega_0^{(\rho)}$,

$$B^{\rho}_{x,y} = \lim_{n\to\infty} [G_{u_n,y} - G_{u_n,x}]. \tag{2-10}$$

Continuity from the left at a fixed $\rho \in (1,\infty)$ holds with probability one: $\lim_{\lambda\nearrow\rho} B^{\lambda}_{x,y} = B^{\rho}_{x,y}$ almost surely.

The theorem above is proved as Theorem 4.2 in lecture notes [Seppäläinen 2018]. The central point of the theorem is the limit (2-10), on account of which we call B^{ρ} the *Busemann function* in direction $\boldsymbol{u}$. We record some observations.

Additivity (2-7) implies that $B^{\rho}_{x,x} = 0$ and $B^{\rho}_{x,y} = -B^{\rho}_{y,x}$. The *weights recovery* property (2-8) can be seen from (2-2) and limits (2-10):

$$\begin{aligned} B^{\rho}_{x-\boldsymbol{e}_1,x} \wedge B^{\rho}_{x-\boldsymbol{e}_2,x} &= \lim_{n\to\infty} [G_{u_n,x} - G_{u_n,x-\boldsymbol{e}_1}] \wedge [G_{u_n,x} - G_{u_n,x-\boldsymbol{e}_2}] \\ &= \lim_{n\to\infty} [G_{u_n,x} - G_{u_n,x-\boldsymbol{e}_1} \vee G_{u_n,x-\boldsymbol{e}_2}] = Y_x. \end{aligned}$$

Lemma 2.2. *With probability one, for all $x \in \mathbb{Z}^2$ there exists a random parameter $\rho^*(x) \in (1,\infty)$ such that*

$$\begin{aligned} B^{\rho}_{x-\boldsymbol{e}_1,x} &= Y_x < B^{\rho}_{x-\boldsymbol{e}_2,x} \quad \text{for } \rho \in (1,\rho^*(x)), \\ B^{\rho}_{x-\boldsymbol{e}_2,x} &= Y_x < B^{\rho}_{x-\boldsymbol{e}_1,x} \quad \text{for } \rho \in (\rho^*(x),\infty). \end{aligned} \tag{2-11}$$

The distribution function of $\rho^*(x)$ is $\mathbb{P}\{\rho^*(x) \le \lambda\} = 1 - \lambda^{-1}$ for $1 \le \lambda < \infty$.

Proof. Monotonicity (2-6) and the exponential rates (2-9) force $B^{\rho}_{x-\boldsymbol{e}_2,x} \nearrow \infty$ almost surely as $\rho \searrow 1$ and $B^{\rho}_{x-\boldsymbol{e}_1,x} \nearrow \infty$ almost surely as $\rho \nearrow \infty$. Edges $\{x-\boldsymbol{e}_1, x\}$ and $\{x-\boldsymbol{e}_2, x\}$ are part of a down-right path, and hence $B^{\rho}_{x-\boldsymbol{e}_1,x}$ and $B^{\rho}_{x-\boldsymbol{e}_2,x}$ are independent exponential random variables for each fixed ρ. Consequently, with probability one, they are distinct for each rational $\rho > 1$. By monotonicity again there is a unique real $\rho^*(x) \in (1,\infty)$ such that for rational $\lambda \in (1,\infty)$,

$$\begin{aligned} \lambda < \rho^*(x) &\text{ implies } B^{\lambda}_{x-\boldsymbol{e}_1,x} < B^{\lambda}_{x-\boldsymbol{e}_2,x}, \\ \lambda > \rho^*(x) &\text{ implies } B^{\lambda}_{x-\boldsymbol{e}_1,x} > B^{\lambda}_{x-\boldsymbol{e}_2,x}. \end{aligned} \tag{2-12}$$

By monotonicity the same holds for real λ. Conditions (2-11) follow from weights recovery (2-8). The distribution function comes from (2-11), independence of $B^{\rho}_{x-\boldsymbol{e}_1,x}$ and $B^{\rho}_{x-\boldsymbol{e}_2,x}$, and (2-9). □

In particular, for a fixed x, the processes $\{B^{\rho}_{x-\boldsymbol{e}_1,x}\}_{1<\rho<\infty}$ and $\{B^{\rho}_{x-\boldsymbol{e}_2,x}\}_{1<\rho<\infty}$ are not independent of each other, even though for a fixed ρ, the random variables $B^{\rho}_{x-\boldsymbol{e}_1,x}$ and $B^{\rho}_{x-\boldsymbol{e}_2,x}$ are independent. Vector $\boldsymbol{u}(\rho^*(x))$ is the asymptotic direction of the competition interface emanating from x (see Remark 2.5).

The process $B = \{B^{\rho}_{x,y}\}$ is a Borel function of the weight configuration Y. Limits (2-10) define B^{ρ} as a function of Y for a countable dense set of ρ in $(1,\infty)$. The remaining ρ-values $B^{\rho}_{x,y}$ can then be defined as right limits. Shifts θ_u act on the weights by $(\theta_u Y)_x = Y_{x+u}$. The limits (2-10) give stationarity and ergodicity of B as stated in this lemma.

Lemma 2.3. *Fix $\rho_1,\dots,\rho_n \in (1,\infty)$ and $y_1,\dots,y_n \in \mathbb{Z}^2$. Let $A_x = (B^{\rho_1}_{x,x+y_1},\dots,B^{\rho_n}_{x,x+y_n})$ and let $0 \ne u \in \mathbb{Z}^2$. Then the $\mathbb{R}^n$-valued process $A = \{A_x\}_{x\in\mathbb{Z}^2}$ is stationary and ergodic under the shift θ_u.*

Proof. Since the i.i.d. process Y is stationary and ergodic under every shift, it suffices to show that $A_x = A_0 \circ \theta_x$ as functions of Y. Let $\boldsymbol{u}^i \in]-\boldsymbol{e}_1, -\boldsymbol{e}_2[$ be associated to ρ_i via (2-5) and fix sequences $\{u_m^1\}, \dots, \{u_m^n\}$ in $\mathbb{Z}^2$ such that, as $m \to \infty$, $|u_m^i|_1 \to \infty$ and $u_m^i/m \to \boldsymbol{u}^i$ for each $i \in [n]$. Then almost surely,

$$\begin{aligned} A_x &= (B^{\rho_1}_{x,x+y_1}, \dots, B^{\rho_n}_{x,x+y_n}) = \lim_{m\to\infty} \big([G_{u_m^1,x+y_1} - G_{u_m^1,x}], \dots, [G_{u_m^n,x+y_n} - G_{u_m^n,x}]\big) \\ &= \lim_{m\to\infty} \big([G_{u_m^1-x,y_1} - G_{u_m^1-x,0}], \dots, [G_{u_m^n-x,y_n} - G_{u_m^n-x,0}]\big) \circ \theta_x \\ &= (B^{\rho_1}_{0,y_1}, \dots, B^{\rho_n}_{0,y_n}) \circ \theta_x = A_0 \circ \theta_x. \end{aligned}$$

□

2B. ***Semi-infinite geodesics in the corner growth model.*** Let $x_\bullet = \{x_k\}$ be a finite or infinite south-west directed nearest-neighbor path on $\mathbb{Z}^2$ ($x_{k+1} \in \{x_k - \boldsymbol{e}_1, x_k - \boldsymbol{e}_2\}$). Then $x_\bullet$ is a *geodesic* if it gives a maximizing path between any two of its points: for any $k < \ell$ in the index set of $x_\bullet$,

$$G_{x_\ell,x_k} = \sum_{i=k}^{\ell} Y_{x_i}.$$

Given $\rho \in (1,\infty)$, define from each $x \in \mathbb{Z}^2$ the semi-infinite, south-west directed path $\boldsymbol{b}^{\rho,x} = \{\boldsymbol{b}_k^{\rho,x}\}_{k\in\mathbb{Z}_{\geq 0}}$ that starts at $x = \boldsymbol{b}_0^{\rho,x}$ and chooses a step from $\{-\boldsymbol{e}_1, -\boldsymbol{e}_2\}$ by following the minimal increment of B^ρ: for $k \geq 0$,

$$\boldsymbol{b}_{k+1}^{\rho,x} = \begin{cases} \boldsymbol{b}_k^{\rho,x} - \boldsymbol{e}_1, & \text{if } B^\rho_{\boldsymbol{b}_k^{\rho,x}-\boldsymbol{e}_1,\, \boldsymbol{b}_k^{\rho,x}} < B^\rho_{\boldsymbol{b}_k^{\rho,x}-\boldsymbol{e}_2,\, \boldsymbol{b}_k^{\rho,x}}, \\ \boldsymbol{b}_k^{\rho,x} - \boldsymbol{e}_2, & \text{if } B^\rho_{\boldsymbol{b}_k^{\rho,x}-\boldsymbol{e}_2,\, \boldsymbol{b}_k^{\rho,x}} \leq B^\rho_{\boldsymbol{b}_k^{\rho,x}-\boldsymbol{e}_1,\, \boldsymbol{b}_k^{\rho,x}}. \end{cases} \tag{2-13}$$

The tie-breaking rule in favor of $-\boldsymbol{e}_2$ is chosen simply to make $\boldsymbol{b}^{\rho,x}$ a cadlag function of ρ. For a given ρ, equality on the right-hand side happens with probability zero. Pictorially, to each point z attach the arrow that points from z to $\boldsymbol{b}_1^{\rho,z}$. For each x the path $\boldsymbol{b}^{\rho,x}$ is constructed by starting at x and following the arrows.

The additivity (2-7) and weights recovery (2-8) imply that $\boldsymbol{b}^{\rho,x}$ is a (semi-infinite) geodesic: let $\ell > k \geq 0$ and suppose $\{y_i\}_{i=k}^{\ell}$ is a south-west directed path from $y_k = \boldsymbol{b}_k^{\rho,x}$ to $y_\ell = \boldsymbol{b}_\ell^{\rho,x}$. Then

$$\sum_{i=k}^{\ell} Y_{y_i} \leq \sum_{i=k}^{\ell-1} B^\rho_{y_{i+1},y_i} + Y_{y_\ell} = B^\rho_{y_\ell,y_k} + Y_{y_\ell} = B^\rho_{\boldsymbol{b}_\ell^{\rho,x},\boldsymbol{b}_k^{\rho,x}} + Y_{\boldsymbol{b}_\ell^{\rho,x}} = \sum_{i=k}^{\ell-1} B^\rho_{\boldsymbol{b}_{i+1}^{\rho,x},\boldsymbol{b}_i^{\rho,x}} + Y_{\boldsymbol{b}_\ell^{\rho,x}} = \sum_{i=k}^{\ell} Y_{\boldsymbol{b}_i^{\rho,x}}.$$

Thus,

$$G_{\boldsymbol{b}_\ell^{\rho,x},\boldsymbol{b}_k^{\rho,x}} = \sum_{i=k}^{\ell} Y_{\boldsymbol{b}_i^{\rho,x}} = B^\rho_{\boldsymbol{b}_\ell^{\rho,x},\boldsymbol{b}_k^{\rho,x}} + Y_{\boldsymbol{b}_\ell^{\rho,x}}. \tag{2-14}$$

We call $\boldsymbol{b}^{\rho,x}$ a *Busemann geodesic*.

We state the key properties of semi-infinite geodesics in the next theorem.

Theorem 2.4. *Fix $\rho \in (1,\infty)$ and let $\boldsymbol{u} = \boldsymbol{u}(\rho)$ be the direction associated to ρ by (2-5). The following properties hold with probability one.*

(i) Directedness. *For all $x \in \mathbb{Z}^2$, $\lim_{k\to\infty} \boldsymbol{b}_k^{\rho,x}/k = \boldsymbol{u}$.*

(ii) Uniqueness. *Let* $x_\bullet = \{x_k\}_{k\in\mathbb{Z}_{\geq 0}}$ *be any semi-infinite geodesic that satisfies* $x_k/k \to \boldsymbol{u}$ *as* $k\to\infty$. *Then* $x_\bullet = \boldsymbol{b}^{\rho,x_0}$.

(iii) Coalescence. *For all* $x, y \in \mathbb{Z}^2$, *the paths* $\boldsymbol{b}^{\rho,x}$ *and* $\boldsymbol{b}^{\rho,y}$ *coalesce: there exists* $z = z^\rho(x, y) \in \mathbb{Z}^2$ *such that* $\boldsymbol{b}^{\rho,x} \cap \boldsymbol{b}^{\rho,y} = \boldsymbol{b}^{\rho,z}$.

It is clear from the construction (2-13) that once $\boldsymbol{b}^{\rho,x}$ and $\boldsymbol{b}^{\rho,y}$ come together, they stay together. We call $z^{\rho(\boldsymbol{u})}(x, y)$ the *coalescence point* of the unique $\boldsymbol{u}$-directed semi-infinite geodesics from x and y. The Busemann function satisfies

$$B^\rho_{x,y} = G_{z^\rho(x,y),y} - G_{z^\rho(x,y),x} \quad \text{a.s.} \tag{2-15}$$

It is important to note that parts (ii) and (iii) of Theorem 2.4 are true with probability one only for a given $\boldsymbol{u}$ and not simultaneously for all directions.

Theorem 2.4(i) follows from an ergodic theorem for Busemann functions and the shape equation (2-3) (see for example Theorem 4.3 in [Georgiou et al. 2017a]). Theorem 2.4(ii)–(iii) were established for the exponential CGM in [Coupier 2011; Ferrari and Pimentel 2005]. The article [Seppäläinen 2020] gives an alternative derivation of Theorem 2.4 based on the properties of the stationary exponential CGM. Versions of Theorem 2.4 for the CGM with general weights appear in [Georgiou et al. 2017a].

Remark 2.5 (competition interface). The *geodesic tree* emanating from x consists of all the geodesics between x and points $y \in x + \mathbb{Z}^2_{\leq 0}$ south and west of x. The semi-infinite geodesics $\boldsymbol{b}^{\rho,x}$ are infinite rays in this tree. Every geodesic to x comes through either $x - \boldsymbol{e}_1$ or $x - \boldsymbol{e}_2$. This dichotomy splits the tree into two subtrees. Between the two subtrees lies a unique path $\{\varphi^x_n\}_{n\in\mathbb{Z}_{\geq 0}}$ on the dual lattice $\left(\frac{1}{2}, \frac{1}{2}\right) + \mathbb{Z}^2$ that starts at $\varphi^x_0 = x - \left(\frac{1}{2}, \frac{1}{2}\right)$. φ^x_n is a.s. uniquely defined as the point in $x - \left(\frac{1}{2}, \frac{1}{2}\right) + \mathbb{Z}^2_{\leq 0}$ that satisfies $|x - \varphi^x_n|_1 = n + 1$ and

$$G_{\varphi^x_n + (-\frac{1}{2}, \frac{1}{2}),\, x-\boldsymbol{e}_1} - G_{\varphi^x_n + (-\frac{1}{2}, \frac{1}{2}),\, x-\boldsymbol{e}_2} > 0 > G_{\varphi^x_n + (\frac{1}{2}, -\frac{1}{2}),\, x-\boldsymbol{e}_1} - G_{\varphi^x_n + (\frac{1}{2}, -\frac{1}{2}),\, x-\boldsymbol{e}_2}.$$

(Use the convention $G_{x,y} = -\infty$ if $x \leq y$ fails.) The competition interface has a random asymptotic direction,

$$\lim_{n\to\infty} \frac{\varphi^x_n}{n} = \boldsymbol{u}(\rho^*(x)) \qquad \text{almost surely,} \tag{2-16}$$

where the limit is described in (2-5) and Lemma 2.2. This was first proved in [Ferrari and Pimentel 2005]. The limit came from the study of geodesics with Newman's approach. Identification of the limit came via a mapping of φ^x to a second class particle in the rarefaction fan of TASEP whose limit had been identified in [Ferrari and Kipnis 1995]. An alternative proof that relies on the stationary LPP processes was given in [Georgiou et al. 2017a].

2C. *Queues.* We begin with a standard formulation of a queue that obeys FIFO (first-in-first-out) discipline. This treatment goes back to classic works of Lindley [1952] and Loynes [1962]. Modern references that connect queues with LPP include [Glynn and Whitt 1991; Baccelli et al. 2000; Draief et al. 2005].

The inputs are two bi-infinite sequences: the *arrival process* $I = (I_k)_{k\in\mathbb{Z}}$ and the *service process* $\omega = (\omega_j)_{j\in\mathbb{Z}}$ in $\mathbb{R}^{\mathbb{Z}}_{\geq 0}$. They are assumed to satisfy

$$\lim_{m\to-\infty} \sum_{i=m}^{0} (\omega_i - I_{i+1}) = -\infty. \tag{2-17}$$

The interpretation is that I_j is the time between the arrivals of customers $j-1$ and j and ω_j is the service time of customer j. From these inputs three outputs $\tilde{I} = (\tilde{I}_k)_{k\in\mathbb{Z}}$, $J = (J_k)_{k\in\mathbb{Z}}$ and $\widetilde{\omega} = (\widetilde{\omega}_k)_{k\in\mathbb{Z}}$, also elements of $\mathbb{R}^{\mathbb{Z}}_{\geq 0}$, are constructed as follows.

Let $G = (G_k)_{k\in\mathbb{Z}}$ be any function on $\mathbb{Z}$ that satisfies $I_k = G_k - G_{k-1}$. Define the sequence $\widetilde{G} = (\widetilde{G}_\ell)_{\ell\in\mathbb{Z}}$ by

$$\widetilde{G}_\ell = \sup_{k:\, k\leq \ell} \left\{ G_k + \sum_{i=k}^{\ell} \omega_i \right\}, \quad \ell \in \mathbb{Z}. \tag{2-18}$$

Under assumption (2-17) the supremum in (2-18) is assumed at some finite k. The *interdeparture time* between customers $\ell - 1$ and ℓ is defined by

$$\tilde{I}_\ell = \widetilde{G}_\ell - \widetilde{G}_{\ell-1} \tag{2-19}$$

and the sequence $\tilde{I} = (\tilde{I}_k)_{k\in\mathbb{Z}}$ is the *departure process*. The *sojourn time* J_k of customer k is defined by

$$J_k = \widetilde{G}_k - G_k, \quad k \in \mathbb{Z}. \tag{2-20}$$

The third output,

$$\widetilde{\omega}_k = I_k \wedge J_{k-1}, \quad k \in \mathbb{Z}, \tag{2-21}$$

is the amount of time customer $k-1$ spends as the last customer in the queue.

$\tilde{I}$, J and $\widetilde{\omega}$ are well-defined nonnegative real sequences, and they do not depend on the choice of the function G as long as G has increments $I_k = G_k - G_{k-1}$. The three mappings are denoted by

$$\tilde{I} = D(I, \omega), \quad J = S(I, \omega), \quad \text{and} \quad \widetilde{\omega} = R(I, \omega). \tag{2-22}$$

The queueing story is good for imbuing the mathematics with meaning, but is not necessary for the sequel.

From

$$\widetilde{G}_k = \omega_k + G_k \vee \widetilde{G}_{k-1}, \tag{2-23}$$

follow the useful iterative equations

$$\tilde{I}_k = \omega_k + (I_k - J_{k-1})^+ \quad \text{and} \quad J_k = \omega_k + (J_{k-1} - I_k)^+. \tag{2-24}$$

The difference of the two equations above gives a "conservation law",

$$I_k + J_k = J_{k-1} + \tilde{I}_k. \tag{2-25}$$

We extend the queueing operator D to mappings

$$D^{(n)} : (\mathbb{R}^{\mathbb{Z}}_{\geq 0})^n \to \mathbb{R}^{\mathbb{Z}}_{\geq 0}$$

of multiple sequences into a single sequence. Let $\zeta, \zeta^1, \zeta^2, \ldots$ denote elements of $\mathbb{R}^{\mathbb{Z}}_{\geq 0}$. Then, as long as the actions below are well-defined, let

$$\begin{aligned} D^{(1)}(\zeta) &= D(\zeta, 0) = \zeta, \\ D^{(2)}(\zeta^1, \zeta^2) &= D\big(D^{(1)}(\zeta^1), \zeta^2\big) = D(\zeta^1, \zeta^2), \\ D^{(3)}(\zeta^1, \zeta^2, \zeta^3) &= D\big(D^{(2)}(\zeta^1, \zeta^2), \zeta^3\big) = D\big(D(\zeta^1, \zeta^2), \zeta^3\big), \\ \text{and, in general,} \quad D^{(n)}(\zeta^1, \zeta^2, \dots, \zeta^n) &= D\big(D^{(n-1)}(\zeta^1, \dots, \zeta^{n-1}), \zeta^n\big) \quad \text{for } n \geq 2. \end{aligned} \tag{2-26}$$

In queueing terms, $D^{(n)}(\zeta^1, \zeta^2, \dots, \zeta^n)$ is the departure process that results from feeding arrival process ζ^1 through a series of $n-1$ service stations labeled $i = 2, 3, \dots, n$. For $i = 2, 3, \dots, n$, ζ^i is the service process at station i. Departures from station $i-1$ are the arrivals at station i. The final output is the departure process from the last station whose service process is ζ^n.

We record some inequalities which are to be understood coordinatewise: for example, $I' \geq I$ means that $I'_k \geq I_k$ for all $k \in \mathbb{Z}$.

Lemma 2.6. *Assuming that the mappings below are well-defined, we have the following inequalities*:

$$D(I, \omega) \geq \omega. \tag{2-27}$$

$$\textit{If } I' \geq I \textit{ then } D(I', \omega) \geq D(I, \omega). \tag{2-28}$$

$$\textit{For } n \geq 2,\ D^{(n)}(\zeta^1, \zeta^2, \zeta^3, \dots, \zeta^n) \geq D^{(n-1)}(\zeta^2, \zeta^3, \dots, \zeta^n). \tag{2-29}$$

Proof. The first part of (2-24) implies (2-27). For (2-28) observe that

$$J_k = \sup_{j:\, j \leq k} \Big\{ G_j - G_k + \sum_{i=j}^{k} \omega_i \Big\} \geq \sup_{j:\, j \leq k} \Big\{ G'_j - G'_k + \sum_{i=j}^{k} \omega_i \Big\} = J'_k.$$

Now (2-24) gives $\tilde{I}'_k \geq \tilde{I}_k$.

Inequality (2-29) comes by induction on n. The case $n = 2$ is (2-27). Then, by induction and (2-28),

$$\begin{aligned} D^{(n)}(\zeta^1, \dots, \zeta^n) &= D\big(D^{(n-1)}(\zeta^1, \dots, \zeta^{n-1}), \zeta^n\big) \geq D\big(D^{(n-2)}(\zeta^2, \dots, \zeta^{n-1}), \zeta^n\big) \\ &= D^{(n-1)}(\zeta^2, \dots, \zeta^n). \end{aligned}$$ ☐

We record the most basic fact about M/M/1 queues. The following notation will be used in the sequel. Let

$$\lambda = (\lambda_1, \dots, \lambda_n) \in (0, \infty)^n$$

be an n-tuple of positive reals. Let $\zeta = (\zeta^1, \dots, \zeta^n) \in (\mathbb{R}_{\geq 0}^{\mathbb{Z}})^n$ with $\zeta^i = (\zeta^i_k)_{k \in \mathbb{Z}}$ denote an n-tuple of nonnegative bi-infinite random sequences. Then ζ has distribution ν^λ if all the coordinates ζ^i_k are mutually independent with marginal distributions $\zeta^i_k \sim \mathrm{Exp}(\lambda_i^{-1})$. In other words, ζ^i is a sequence of i.i.d. mean λ_i exponential variables, and the sequences are independent.

Lemma 2.7. *Let $n \geq 2$ and let $\lambda = (\lambda_1, \dots, \lambda_n)$ satisfy $\lambda_1 > \dots > \lambda_n > 0$. Let ζ have distribution ν^λ. Then $D^{(n)}(\zeta^1, \dots, \zeta^n)$ has distribution ν^{λ_1}, in other words, $D^{(n)}(\zeta^1, \dots, \zeta^n)$ is a sequence of i.i.d. mean λ_1 exponential random variables.*

Proof. The case $n = 2$ is in Lemma B.2. The general case follows by induction on n. ☐

3. Joint distribution of the Busemann functions

This section contains the main results on the joint distribution of the Busemann process

$$B^{\bullet} = \{B^{\rho} : 1 < \rho < \infty\}$$

defined in Theorem 2.1. Proofs are in Section 6. The distribution of the n-tuple

$$\{(B^{\rho_1}_{x-\boldsymbol{e}_1,x}, \dots, B^{\rho_n}_{x-\boldsymbol{e}_1,x})\}_{x\cdot \boldsymbol{e}_2=t}$$

on a given lattice level $t \in \mathbb{Z}$ comes through a mapping of a product of exponential distributions. This mapping is developed next.

3A. *Coupled exponential distributions.* Fix $n \in \mathbb{Z}_{>0}$ for the moment and define the following two spaces of n-tuples of nonnegative real sequences. The sequences themselves are denoted by $I^i = (I^i_k)_{k\in\mathbb{Z}}$ and $\eta^i = (\eta^i_k)_{k\in\mathbb{Z}}$ for $i \in [n]$:

$$\mathcal{Y}_n = \left\{ I = (I^1, I^2, \dots, I^n) \in (\mathbb{R}^{\mathbb{Z}}_{\geq 0})^n : \forall\, i \in [\![2, n]\!],\ \lim_{m\to-\infty} \frac{1}{|m|} \sum_{k=-m}^{0} I^i_k > \lim_{m\to-\infty} \frac{1}{|m|} \sum_{k=-m}^{0} I^{i-1}_k > 0 \right\}. \tag{3-1}$$

$$\mathcal{X}_n = \left\{ \eta = (\eta^1, \eta^2, \dots, \eta^n) \in (\mathbb{R}^{\mathbb{Z}}_{\geq 0})^n : \eta^i \geq \eta^{i-1}\ \ \forall\, i \in [\![2, n]\!] \text{ and } \lim_{m\to-\infty} \frac{1}{|m|} \sum_{k=-m}^{0} \eta^1_k > 0 \right\}. \tag{3-2}$$

The existence of the Cesàro limits as $m \to -\infty$ is part of the definitions. $\mathcal{Y}_n$ and $\mathcal{X}_n$ are Borel subsets of $(\mathbb{R}^{\mathbb{Z}}_{\geq 0})^n$ and thereby separable metric spaces in the product topology. We endow them with their Borel σ-algebras.

Define a mapping $\mathcal{D}^{(n)} : \mathcal{Y}_n \to \mathcal{X}_n$ in terms of the multiqueue mappings $D^{(k)}$ of (2-26) as follows: for $I = (I^1, I^2, \dots, I^n) \in \mathcal{Y}_n$, the image $\eta = (\eta^1, \eta^2, \dots, \eta^n) = \mathcal{D}^{(n)}(I)$ is defined by

$$\eta^i = D^{(i)}(I^i, I^{i-1}, \dots, I^1) \quad \text{for } \ i = 1, \dots, n. \tag{3-3}$$

In particular, the first sequence is just copied over: $\eta^1 = I^1$. Then $\eta^2 = D(I^2, I^1)$, $\eta^3 = D^{(3)}(I^3, I^2, I^1) = D(D(I^3, I^2), I^1)$, and so on. Iterated application of Lemma A.3 from Appendix A together with the assumption $I \in \mathcal{Y}_n$ ensures that the mappings $D^{(i)}(I^i, I^{i-1}, \dots, I^1)$ are well-defined. Furthermore, $\eta \in \mathcal{X}_n$ follows from inequalities (2-27) and (2-29). Lemma A.3 implies also that $\mathcal{D}^{(n)}$ maps $\mathcal{Y}_n$ into itself. We do not need this feature in the sequel, which is why we did not define $\mathcal{X}_n$ as a subspace of $\mathcal{Y}_n$.

Recall:

For $\rho = (\rho_1, \dots, \rho_n) \in (0, \infty)^n$, $I = (I^1, I^2, \dots, I^n)$ has distribution ν^{ρ} if

$$\text{all coordinates } I^i_k \text{ are independent and } I^i_k \sim \mathrm{Exp}(\rho_i^{-1}) \text{ for each } k \in \mathbb{Z} \text{ and } i \in [n]. \tag{3-4}$$

If ρ satisfies $0 < \rho_1 < \rho_2 < \cdots < \rho_n$ then ν^{ρ} is supported on $\mathcal{Y}_n$. For these ρ define the probability measure μ^{ρ} on $\mathcal{X}_n$ as the image of ν^{ρ} under $\mathcal{D}^{(n)}$:

$$\mu^{\rho} = \nu^{\rho} \circ (\mathcal{D}^{(n)})^{-1} \qquad \text{for } \rho = (\rho_1, \rho_2, \dots, \rho_n) \text{ such that } 0 < \rho_1 < \rho_2 < \cdots < \rho_n. \tag{3-5}$$

By Lemma 2.7, if η has distribution μ^ρ with $0 < \rho_1 < \rho_2 < \cdots < \rho_n$, then for each $i \in [n]$, $\eta^i = (\eta^i_k)_{k\in\mathbb{Z}}$ is a sequence of i.i.d. mean ρ_i exponential variables. The mapping $\mathcal{D}^{(n)}$ couples the variables η^i_k together so that $\eta^{i-1}_k \le \eta^i_k$ for all $i \in [\![2, n]\!]$ and $k \in \mathbb{Z}$.

Translations $\{\theta_\ell\}_{\ell\in\mathbb{Z}}$ act on n-tuples of sequences by $(\theta_\ell\eta)^i_k = \eta^i_{k+\ell}$ for $i \in [n]$ and $k, \ell \in \mathbb{Z}$. A *translation-ergodic* probability measure Q on $\mathcal{X}_n$ is invariant under $\{\theta_\ell\}$ and satisfies $Q(A) \in \{0, 1\}$ for any Borel set $A \subset \mathcal{X}_n$ that is invariant under $\{\theta_\ell\}$ (and similarly for any other sequence space).

Theorem 3.1. *The probability measures μ^ρ are translation-ergodic and have the following properties*:

(i) Continuity. *The probability measure μ^ρ is weakly continuous as a function of ρ on the set of vectors that satisfy* $0 < \rho_1 < \rho_2 < \cdots < \rho_n$.

(ii) Consistency. *If* $(\eta^1, \dots, \eta^n) \sim \mu^{(\rho_1,\dots,\rho_n)}$, *then* $(\eta^1, \dots, \eta^{j-1}, \eta^{j+1}, \dots, \eta^n) \sim \mu^{(\rho_1,\dots,\rho_{j-1},\rho_{j+1},\dots,\rho_n)}$ *for all* $j \in [n]$.

Continuity of $\rho \mapsto \mu^\rho$ is proved in Section 6. Translation-covariance of the queueing mappings $(D(\theta_\ell I, \theta_\ell\omega) = \theta_\ell D(I, \omega))$ implies that μ^ρ inherits the translation-ergodicity of ν^ρ. We omit the proof of consistency. Consistency will be an indirect consequence of the uniqueness of μ^ρ as the translation-ergodic invariant distribution of the so-called coupled process (Theorem 5.3).

3B. ***Distribution of Busemann functions.*** Return to the Busemann functions $B^\bullet$ defined in Theorem 2.1. For each level $t \in \mathbb{Z}$ define the level-t sequence of weights $\overline{Y}_t = (Y_{(k,t)})_{k\in\mathbb{Z}}$ and for a given $\rho \in (1, \infty)$, sequences of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ Busemann variables at level t:

$$\overline{B}^{\rho,\boldsymbol{e}_1}_t = (B^\rho_{(k-1,t),(k,t)})_{k\in\mathbb{Z}} \quad \text{and} \quad \overline{B}^{\rho,\boldsymbol{e}_2}_t = (B^\rho_{(k,t-1),(k,t)})_{k\in\mathbb{Z}}.$$

The next main result characterizes uniquely the distribution of the joint process $(Y, B^\bullet)$ of weights and Busemann functions.

Theorem 3.2. *Let $Y = (Y_x)_{x\in\mathbb{Z}^2}$ be i.i.d.* Exp(1) *variables as in Section 2A. Let $1 < \rho_1 < \cdots < \rho_n$. Then at each level $t \in \mathbb{Z}$, the $(n+1)$-tuple of sequences $(\overline{Y}_t, \overline{B}^{\rho_1,\boldsymbol{e}_1}_t, \dots, \overline{B}^{\rho_n,\boldsymbol{e}_1}_t)$ has distribution $\mu^{(1,\rho_1,\dots,\rho_n)}$.*

Once the process $\{\overline{B}^{\rho,\boldsymbol{e}_1}_{t-1}\}_{1<\rho<\infty}$ on a single level $t-1$ is given, the variables $B^\rho_{x-\boldsymbol{e}_i,x}$ at higher levels $t, t+1, t+2, \dots$ can be deduced by drawing independent weights $\overline{Y}_t, \overline{Y}_{t+1}, \dots$ and by applying queueing mappings. By stationarity, the full distribution will then have been determined. The next lemma describes the single step of computing the $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ Busemann increments on level t from the process $\{\overline{B}^{\rho,\boldsymbol{e}_1}_{t-1}\}_{1<\rho<\infty}$ and independent level-t weights $\overline{Y}_t$. The mappings D and S were specified in (2-22).

Lemma 3.3. *There exists an event of full probability on which*

$$\overline{B}^{\rho,\boldsymbol{e}_1}_t = D(\overline{B}^{\rho,\boldsymbol{e}_1}_{t-1}, \overline{Y}_t) \quad \textit{and} \quad \overline{B}^{\rho,\boldsymbol{e}_2}_t = S(\overline{B}^{\rho,\boldsymbol{e}_1}_{t-1}, \overline{Y}_t) \quad \textit{for all } \rho \in (1, \infty) \textit{ and } t \in \mathbb{Z}.$$

The remainder of this section describes some distributional properties of $B^\bullet$ restricted to horizontal edges and lines on $\mathbb{Z}^2$. The corresponding statements for vertical edges and lines are obtained by replacing ρ with $\rho/(\rho-1)$. This is due to the distributional equality $\{B^\rho_{x-\boldsymbol{e}_2,x}\}_{x\in\mathbb{Z}^2} \overset{d}{=} \{B^{\rho/(\rho-1)}_{Rx-\boldsymbol{e}_1,Rx}\}_{x\in\mathbb{Z}^2}$ where $R(x_1, x_2) = (x_2, x_1)$. This follows from (2-5) and the limits (2-10), by reflecting the lattice across the diagonal.

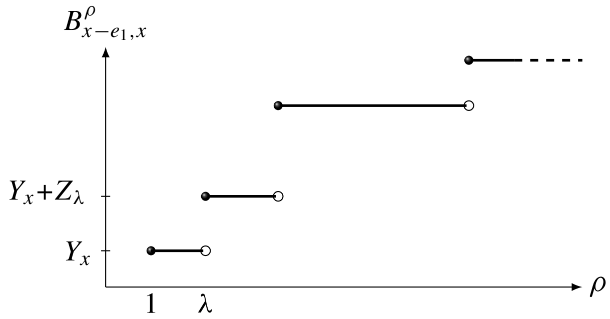

Figure 2. A sample path of the pure jump process $\{B^{\rho}_{x-e_1,x}\}_{\rho\in[1,\infty)}$, with initial value $B^1_{x-e_1,x} = Y_x$. The jump times are a Poisson point process on $(1, \infty)$ with intensity $s^{-1}ds$. Given that there is a jump at λ, the jump size is an independent $\mathrm{Exp}(\lambda^{-1})$ variable Z_λ.

3C. ***Marginal distribution on a single edge.*** Lemma 2.2 implies that for a fixed horizontal edge $(x-\boldsymbol{e}_1, x)$ we can extend $\{B^{\rho}_{x-\boldsymbol{e}_1,x} : 1 < \rho < \infty\}$ to a cadlag process

$$B^{\bullet}_{x-\boldsymbol{e}_1,x} = \{B^{\rho}_{x-\boldsymbol{e}_1,x} : 1 \le \rho < \infty\}$$

by setting $B^1_{x-\boldsymbol{e}_1,x} = Y_x$. We describe the distribution of this process in terms of a marked point process. Figure 2 illustrates a sample path of this process.

Let N be the simple point process on the interval $\{s : 1 \le s < \infty\}$ that has a point at $s = 1$ with probability one, and on the open interval $(1, \infty)$ N is a Poisson point process with parameter measure $s^{-1}\,ds$. (We use N to denote both the random discrete set of locations and the resulting random point measure.) Let N be the ground process of the marked point process $\sum_{t\in N} \delta_{(t,\,Z_t)}$ where the mark Z_t at location $t \in N$ is $\mathrm{Exp}(t^{-1})$-distributed and independent of the other marks. Define the nondecreasing cadlag process $X(\cdot) = \{X(\rho) : \rho \in [1, \infty)\}$ as

$$X(\rho) = \sum_{t\,\in\, N\cap[1,\rho]} Z_t; \tag{3-6}$$

namely, $X(\rho)$ is the total weight of the marks in $[1, \rho]$. The Laplace transform of $X(\rho)$ is given in (6-12).

Theorem 3.4. *Fix $x \in \mathbb{Z}^2$. The nondecreasing cadlag processes $B^{\bullet}_{x-\boldsymbol{e}_1,x}$ and $X(\cdot)$ indexed by $[1, \infty)$ are equal in distribution.*

A qualitative consequence of Theorem 3.4 is that for any given $\lambda \in (1, \infty)\setminus N$, $\rho \mapsto B^{\rho}_{x-\boldsymbol{e}_1,x}$ is constant in an interval around λ. From identity (2-15), it is evident that this is due to the fact that the coalescence point function $\rho \mapsto z^{\rho}(x-\boldsymbol{e}_1, x)$ is constant in an interval. This is an analogue of the local constancy of the difference weight profile in Theorem 1.1 of [Basu et al. 2019]. The implications of Theorem 3.4 for the coalescence structure of the geodesics of the CGM are explored in [Janjigian et al. 2019].

Theorem 3.4 is proved by establishing that $B^{\bullet}_{x-\boldsymbol{e}_1,x}$ has independent increments and by deducing the distribution of an increment. Independent increments means that for $1 = \rho_0 < \rho_1 < \cdots < \rho_n$, the random variables $Y_x = B^{\rho_0}_{x-\boldsymbol{e}_1,x}$, $B^{\rho_1}_{x-\boldsymbol{e}_1,x} - B^{\rho_0}_{x-\boldsymbol{e}_1,x}, \ldots, B^{\rho_n}_{x-\boldsymbol{e}_1,x} - B^{\rho_{n-1}}_{x-\boldsymbol{e}_1,x}$ are independent. For $1 \le \lambda < \rho < \infty$, the distribution of the increment is

$$\begin{aligned}\mathbb{P}\{B^{\rho}_{x-\boldsymbol{e}_1,x}-B^{\lambda}_{x-\boldsymbol{e}_1,x}=0\}&=\mathbb{P}\{N(\lambda,\rho]=0\}=\frac{\lambda}{\rho},\\ \mathbb{P}\{B^{\rho}_{x-\boldsymbol{e}_1,x}-B^{\lambda}_{x-\boldsymbol{e}_1,x}>s\}&=\Big(1-\frac{\lambda}{\rho}\Big)e^{-s/\rho}\quad\text{for }\ s>0.\end{aligned}\tag{3-7}$$

For the process $B^{\bullet}_{x-\boldsymbol{e}_2,x}$ on a vertical edge, the result of Theorem 3.4 is that

$$\{B^{\rho}_{x-\boldsymbol{e}_2,x}:1<\rho<\infty\}\stackrel{d}{=}\Big\{X\Big(\Big(\frac{\rho}{\rho-1}\Big)+\Big):1<\rho<\infty\Big\}.\tag{3-8}$$

3D. ***Marginal distribution on a level of the lattice.*** A striking and useful property of the Busemann process $\{B^{\rho}_{(k-1,t),(k,t)}\}_{k\in\mathbb{Z}}$ along a horizontal line in $\mathbb{Z}^2$ for a *fixed* value $\rho\in(1,\infty)$ is that the variables $\{B^{\rho}_{(k-1,t),(k,t)}\}_{k\in\mathbb{Z}}$ are i.i.d. (part (ii) of Theorem 2.1). For example, [Balázs et al. 2006] used this feature heavily to deduce the KPZ fluctuation exponents of the corner growth model. The next theorem shows that this property breaks down totally already for the joint process $\{(B^{\lambda}_{(k-1,t),(k,t)},B^{\rho}_{(k-1,t),(k,t)})\}_{k\in\mathbb{Z}}$ for two parameter values $\lambda<\rho$. Namely, this pair process is not even a Markov chain and not reversible. However, if we restrict attention to the differences $B^{\rho}_{(k-1,t),(k,t)}-B^{\lambda}_{(k-1,t),(k,t)}$, we can recover the reversibility. The differences are of interest because they indicate a jump in the coalescence point $z^{\bullet}((k-1,t),(k,t))$ in (2-15) as a function of the direction.

For the statement of the theorem below, the negative part of a real number is $x^-=(-x)\vee 0$. The Markov chain X_k in part (a) below has a queueing interpretation as the difference between the sojourn time of customer $k-1$ and the waiting time till the arrival of customer k. The details are in the proof in Lemma 6.5.

Theorem 3.5. *Let* $1\le\lambda<\rho<\infty$.

(a) *The sequence of differences* $\{B^{\rho}_{(k-1,t),(k,t)}-B^{\lambda}_{(k-1,t),(k,t)}\}_{k\in\mathbb{Z}}$ *is not a Markov chain, but there exists a stationary reversible Markov chain* $\{X_k\}_{k\in\mathbb{Z}}$ *such that this distributional equality of processes holds*:

$$\{B^{\rho}_{(k-1,t),(k,t)}-B^{\lambda}_{(k-1,t),(k,t)}\}_{k\in\mathbb{Z}}\stackrel{d}{=}\{X_k^-\}_{k\in\mathbb{Z}}.$$

In particular, the process of differences is reversible:

$$\{B^{\rho}_{(k-1,t),(k,t)}-B^{\lambda}_{(k-1,t),(k,t)}\}_{k\in\mathbb{Z}}\stackrel{d}{=}\{B^{\rho}_{(-k-1,t),(-k,t)}-B^{\lambda}_{(-k-1,t),(-k,t)}\}_{k\in\mathbb{Z}}.$$

(b) *The sequence of pairs* $\{(B^{\lambda}_{(k-1,t),(k,t)},B^{\rho}_{(k-1,t),(k,t)})\}_{k\in\mathbb{Z}}$ *is not a Markov chain. The joint distribution of two successive pairs*

$$\big((B^{\lambda}_{(k-1,t),(k,t)},B^{\rho}_{(k-1,t),(k,t)}),(B^{\lambda}_{(k,t),(k+1,t)},B^{\rho}_{(k,t),(k+1,t)})\big)$$

is ***not*** *the same as the joint distribution of its transpose*

$$\big((B^{\lambda}_{(k,t),(k+1,t)},B^{\rho}_{(k,t),(k+1,t)}),(B^{\lambda}_{(k-1,t),(k,t)},B^{\rho}_{(k-1,t),(k,t)})\big).$$

In particular, the process of pairs $\{(B^{\lambda}_{(k-1,t),(k,t)},B^{\rho}_{(k-1,t),(k,t)})\}_{k\in\mathbb{Z}}$ *is* ***not*** *reversible.*

3E. ***The initial segment of the Busemann geodesic.*** As the last application of Theorem 3.2 we calculate the probability distribution of the length of the initial horizontal run of a semi-infinite geodesic.

Let $\boldsymbol{a}^{\rho}_x = \boldsymbol{b}^{\rho,x}_1 - x$ be the first step of the B^ρ Busemann geodesic (2-13) started at x. $\{\boldsymbol{a}^{\rho}_x\}_{x\in\mathbb{Z}^2}$ is a random configuration with values in $\{-\boldsymbol{e}_1, -\boldsymbol{e}_2\}$. By weight recovery (2-8), $\boldsymbol{a}^{\rho}_x = -\boldsymbol{e}_1$ if and only if $B^{\rho}_{x-\boldsymbol{e}_1,x} - Y_x = 0$. Hence by Theorem 3.5(a) with $\lambda = 1$, reversibility holds along a line: $\{\boldsymbol{a}^{\rho}_{k\boldsymbol{e}_i}\}_{k\in\mathbb{Z}} \overset{d}{=} \{\boldsymbol{a}^{\rho}_{-k\boldsymbol{e}_i}\}_{k\in\mathbb{Z}}$.

The first part of the theorem below gives a queueing characterization for the process $\{\boldsymbol{a}^{\rho}_{k\boldsymbol{e}_1}\}_{k\in\mathbb{Z}}$. To that end, for the queueing mapping $\tilde{I} = D(I, \omega)$ of (2-22) define the indicator variables

$$\eta_k = \mathbf{1}_{\tilde{I}_k=\omega_k} = \mathbf{1}\{\text{customer } k \text{ has to wait before entering service}\}. \tag{3-9}$$

Let

$$\xi_x = \inf\{k \in \mathbb{Z}_{\geq 0} : \boldsymbol{a}^{\rho}_{x-k\boldsymbol{e}_1} = -\boldsymbol{e}_2\}$$

denote the number of consecutive $-\boldsymbol{e}_1$ steps that $\boldsymbol{b}^{\rho,x}$ takes from a deterministic starting point x. Part (b) of the theorem gives the distribution of ξ_x. The *Catalan triangle* $\{C(n,k) : 0 \leq k \leq n\}$ is given by

$$C(n,k) = \frac{(n+k)!(n-k+1)}{k!(n+1)!}. \tag{3-10}$$

Information about $C(n,k)$ is given above Lemma B.3 in Appendix B.

Theorem 3.6. *Let* $1 < \rho < \infty$.

(a) *Let the service and arrival processes satisfy* $(\omega, I) \sim \nu^{(1,\rho)}$ *and define* η_k *by* (3-9). *Then we have the distributional equality*

$$\{\mathbf{1}\{\boldsymbol{a}^{\rho}_{k\boldsymbol{e}_1} = -\boldsymbol{e}_1\}\}_{k\in\mathbb{Z}} \overset{d}{=} \{\eta_k\}_{k\in\mathbb{Z}}.$$

(b) *Let* $x \in \mathbb{Z}^2$. *Then* $\mathbb{P}\{\xi_x = 0\} = 1 - \rho^{-1}$ *and for* $n \in \mathbb{Z}_{>0}$,

$$\mathbb{P}\{\xi_x = n\} = (1-\rho^{-1}) \sum_{k=0}^{n-1} C(n-1,k) \frac{\rho^k}{(\rho+1)^{n+k}}. \tag{3-11}$$

The distribution in (3-11) is proper; that is, $\sum_{n\in\mathbb{Z}_{\geq 0}} \mathbb{P}\{\xi_x = n\} = 1$. This follows for example from Theorem 2.4(i) according to which the Busemann geodesic has direction strictly off the axes.

Remark 3.7. If we take $(\omega, I) \sim \nu^{(\lambda,\rho)}$ for $1 < \lambda < \rho$ in Theorem 3.6 and define η_k again by (3-9), we get the distributional equality $\{\mathbf{1}\{B^{\rho}_{(k-1,t),(k,t)} = B^{\lambda}_{(k-1,t),(k,t)}\}\}_{k\in\mathbb{Z}} \overset{d}{=} \{\eta_k\}_{k\in\mathbb{Z}}$. The calculation that produced part (b) gives the distribution $\mathbb{P}\{\xi^{\lambda,\rho}_x = 0\} = (\rho-\lambda)/\rho$ and

$$\mathbb{P}\{\xi^{\lambda,\rho}_x = n\} = \frac{\rho-\lambda}{\rho} \sum_{k=0}^{n-1} C(n-1,k) \frac{\rho^k \lambda^n}{(\lambda+\rho)^{n+k}} \qquad \text{for } n \in \mathbb{Z}_{>0},$$

for the random variable

$$\xi^{\lambda,\rho}_x = \inf\{k \in \mathbb{Z}_{\geq 0} : B^{\rho}_{x-(k+1)\boldsymbol{e}_1, x-k\boldsymbol{e}_1} > B^{\lambda}_{x-(k+1)\boldsymbol{e}_1, x-k\boldsymbol{e}_1}\}.$$

Note that $B^{\rho}_{x-\boldsymbol{e}_1,x} = B^{\lambda}_{x-\boldsymbol{e}_1,x}$ tells us that $\boldsymbol{b}^{\rho,x}_1 = \boldsymbol{b}^{\lambda,x}_1$ but not which step is chosen.

4. Properties of queueing mappings

This section proves a property of the queueing mapping D (Lemma 4.4) on which the intertwining property that comes in Section 5D rests. To prove Lemma 4.4 we develop a duality in the queueing setting of Section 2C: namely, an LPP process defined in terms of weights (I, ω) can be equivalently described in terms of weights $(\tilde{I}, \widetilde{\omega})$ defined by (2-22). Routine facts about the queueing mappings are collected in Appendix A.

Fix an origin $m \in \mathbb{Z}$. Assume given nonnegative real weights

$$J_m, \quad (I_i)_{i \geq m+1}, \quad \text{and} \quad (\omega_i)_{i \geq m+1}. \tag{4-1}$$

From these define iteratively for $k = m+1, m+2, \ldots$

$$\tilde{I}_k = \omega_k + (I_k - J_{k-1})^+, \quad J_k = \omega_k + (J_{k-1} - I_k)^+, \quad \text{and} \quad \widetilde{\omega}_k = I_k \wedge J_{k-1}. \tag{4-2}$$

There is a duality or reversibility of sorts here. For a fixed k, equations (4-2) are equivalent to

$$I_k = \widetilde{\omega}_k + (\tilde{I}_k - J_k)^+, \quad J_{k-1} = \widetilde{\omega}_k + (J_k - \tilde{I}_k)^+, \quad \text{and} \quad \omega_k = \tilde{I}_k \wedge J_k. \tag{4-3}$$

We turn this reversibility into a lemma as follows. Restrict the given J, I and ω weights in (4-1) to the interval $[\![m, n]\!]$. Then on the interval $[\![-n, -m]\!]$ define the given weights J'_{-n}, $(I'_i)_{-n+1 \leq i \leq -m}$ and $(\omega'_i)_{-n+1 \leq i \leq -m}$ as

$$I'_i = \tilde{I}_{-i+1}, \quad J'_{-n} = J_n, \quad \text{and} \quad \omega'_i = \widetilde{\omega}_{-i+1}. \tag{4-4}$$

Now apply (4-2) to these given weights to compute $(\widetilde{I'_k}, J'_k, \widetilde{\omega}'_k)$ for $k \in [\![-n+1, -m]\!]$. First assume by induction that $J'_{k-1} = J_{-k+1}$. The base case $k-1 = -n$ is covered by the definition in (4-4). Then

$$\begin{aligned} J'_k &= \omega'_k + (J'_{k-1} - I'_k)^+ = \widetilde{\omega}_{-k+1} + (J_{-k+1} - \tilde{I}_{-k+1})^+ \\ &= I_{-k+1} \wedge J_{-k} + (J_{-k} - I_{-k+1})^+ = J_{-k}. \end{aligned}$$

The third equality above used the definition of $\widetilde{\omega}$ in (4-2) and the conservation law

$$I_k + J_k = J_{k-1} + \tilde{I}_k \tag{4-5}$$

that follows from (4-2). Thus $J'_k = J_{-k}$ for all $k \in [\![-n, -m]\!]$. Next

$$\begin{aligned} \widetilde{I'_k} &= \omega'_k + (I'_k - J'_{k-1})^+ = \widetilde{\omega}_{-k+1} + (\widetilde{I}_{-k+1} - J_{-k+1})^+ \\ &= I_{-k+1} \wedge J_{-k} + (I_{-k+1} - J_{-k})^+ = I_{-k+1}. \end{aligned}$$

Finally,

$$\widetilde{\omega}'_k = I'_k \wedge J'_{k-1} = \tilde{I}_{-k+1} \wedge J_{-k+1} = \omega_{-k+1}$$

as follows again from (4-2). We summarize this finding as follows.

Lemma 4.1. *Fix $m < n$. Assume given J_m, $(I_i)_{m+1 \leq i \leq n}$ and $(\omega_i)_{m+1 \leq i \leq n}$. Compute $(\tilde{I}_k, J_k, \widetilde{\omega}_k)_{m+1 \leq k \leq n}$ from (4-2). Then define J'_{-n}, $(I'_i)_{-n+1 \leq i \leq -m}$ and $(\omega'_i)_{-n+1 \leq i \leq -m}$ by (4-4) and apply (4-2) to compute $(\widetilde{I'_k}, J'_k, \widetilde{\omega}'_k)_{-n+1 \leq k \leq -m}$. The conclusion is that $(\widetilde{I'_k}, J'_k, \widetilde{\omega}'_k) = (I_{-k+1}, J_{-k}, \omega_{-k+1})$ for $k \in [\![-n+1, -m]\!]$.*

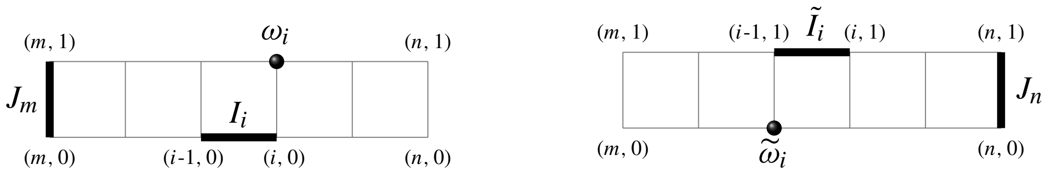

Figure 3. Illustration of the weights (I, J, ω) on the left and weights $(\tilde{I}, J, \widetilde{\omega})$ on the right. Pairs $(k, a) \in [\![m, n]\!] \times [\![0, 1]\!]$ mark vertices of the two-level strip.

Next we use the weights given in (4-1) to construct a last-passage process on the two-level strip $[\![m, \infty[\![\times [\![0, 1]\!]$ in $\mathbb{Z}^2$. In this construction, I_i serves as a weight on the horizontal edge $((i-1, 0), (i, 0))$ on the lower 0-level, J_m is a weight on the vertical edge $((m, 0), (m, 1))$, and ω_i is a weight at vertex $(i, 1)$ on the upper 1-level. (The left diagram of Figure 3 illustrates this.) The last-passage values $H_{(m,0),(n,a)}$ are defined for $(n, a) \in [\![m, \infty[\![\times [\![0, 1]\!]$ as follows:

$$\begin{aligned} &H_{(m,0),(m,0)} = 0 \quad \text{and} \quad H_{(m,0),(n,0)} = \sum_{i=m+1}^{n} I_i \quad \text{for } n > m, \\ &H_{(m,0),(m,1)} = J_m\,, \\ &H_{(m,0),(n,1)} = \Bigl\{ J_m + \sum_{i=m+1}^{n} \omega_i \Bigr\} \vee \max_{m+1 \le j \le n} \Bigl\{ \sum_{i=m+1}^{j} I_i + \sum_{i=j}^{n} \omega_i \Bigr\}, \qquad n > m. \end{aligned} \tag{4-6}$$

If the given weights (4-1) come from the queueing setting of Section 2C, then $H_{(m,0),(n,1)} = \widetilde{G}_n - G_m$. But this connection is not needed for the present.

The next lemma gives alternative formulas for H in terms of the weights calculated in (4-2). Pictorially, imagine $\tilde{I}_i$ as a weight on the edge $((i-1, 1), (i, 1))$ and $\widetilde{\omega}_i$ as a weight on the vertex $(i-1, 0)$. (The right diagram of Figure 3 illustrates this.) In (4-7), a sum expression of the form $a_j + \cdots + a_{j-1}$ is interpreted as zero. The equation (4-8) makes sense also for $\ell = n$ in which case the right-hand side simplifies to J_n.

Lemma 4.2. *Let $m \le n$. Then*

$$H_{(m,0),(n,1)} = I_{m+1} + \cdots + I_k + J_k + \tilde{I}_{k+1} + \cdots + \tilde{I}_n \quad \textit{for each } k \in [\![m, n]\!]. \tag{4-7}$$

For each $\ell \in [\![m, n-1]\!]$,

$$H_{(m,0),(n,1)} - H_{(m,0),(\ell,0)} = \max_{\ell+1 \le j \le n} \Bigl\{ \sum_{i=\ell+1}^{j} \widetilde{\omega}_i + \sum_{i=j}^{n} \tilde{I}_i \Bigr\} \vee \Bigl\{ \sum_{i=\ell+1}^{n} \widetilde{\omega}_i + J_n \Bigr\}. \tag{4-8}$$

We make some observations before the proof. By the two top lines of (4-6), equivalent to (4-7) are the increment formulas (for all $n > m$)

$$\tilde{I}_n = H_{(m,0),(n,1)} - H_{(m,0),(n-1,1)} \quad \text{and} \quad J_n = H_{(m,0),(n,1)} - H_{(m,0),(n,0)}. \tag{4-9}$$

Taking $\ell = m$ in (4-8) gives this dual representation for H:

$$H_{(m,0),(n,1)} = \max_{m+1 \le j \le n} \Bigl\{ \sum_{i=m+1}^{j} \widetilde{\omega}_i + \sum_{i=j}^{n} \tilde{I}_i \Bigr\} \vee \Bigl\{ \sum_{i=m+1}^{n} \widetilde{\omega}_i + J_n \Bigr\}. \tag{4-10}$$

Proof of Lemma 4.2. Let $m < n$ and develop the definition (4-6). As in (2-2),

$$\begin{aligned} H_{(m,0),(n,1)} &= \Big\{ J_m + \sum_{i=m+1}^{n-1} \omega_i \Big\} \vee \max_{m+1 \le j \le n-1} \Big\{ \sum_{i=m+1}^{j} I_i + \sum_{i=j}^{n-1} \omega_i \Big\} \vee \Big\{ \sum_{i=m+1}^{n} I_i \Big\} + \omega_n \\ &= H_{(m,0),(n-1,1)} \vee H_{(m,0),(n,0)} + \omega_n. \end{aligned} \tag{4-11}$$

Set temporarily

$$A_n = H_{(m,0),(n,1)} - H_{(m,0),(n-1,1)} \quad \text{and} \quad B_n = H_{(m,0),(n,1)} - H_{(m,0),(n,0)}.$$

Then (4-11) gives the iterative equations

$$A_n = \omega_n + (I_n - B_{n-1})^+ \quad \text{and} \quad B_n = \omega_n + (B_{n-1} - I_n)^+.$$

Definition (4-6) gives $B_m = J_m$. This starts an induction. Apply the equations above together with (4-2) to obtain $A_n = \tilde{I}_n$ and $B_n = J_n$ for all $n \ge m+1$. This establishes (4-9), and (4-7) follows.

We prove (4-8) by induction as ℓ decreases. The base case $\ell = n$ comes from the just proved $B_n = J_n$. Assume (4-8) for $\ell+1$. Then for ℓ the right-hand side of (4-8) equals

$$\begin{aligned} &\widetilde{\omega}_{\ell+1} + \Big\{ \sum_{i=\ell+1}^{n} \tilde{I}_i \Big\} \vee \max_{\ell+2 \le j \le n} \Big\{ \sum_{i=\ell+2}^{j} \widetilde{\omega}_i + \sum_{i=j}^{n} \tilde{I}_i \Big\} \vee \Big\{ \sum_{i=\ell+2}^{n} \widetilde{\omega}_i + J_n \Big\} \\ &\quad = H_{(m,0),(\ell+1,0)} \wedge H_{(m,0),(\ell,1)} - H_{(m,0),(\ell,0)} + \big\{ H_{(m,0),(n,1)} - H_{(m,0),(\ell,1)} \big\} \vee \big\{ H_{(m,0),(n,1)} - H_{(m,0),(\ell+1,0)} \big\} \\ &\quad = H_{(m,0),(n,1)} - H_{(m,0),(\ell,0)}. \end{aligned}$$

In the first equality we used $\widetilde{\omega}_{\ell+1} = I_{\ell+1} \wedge J_\ell$, (4-9) and the induction assumption. □

The last line of (4-6) and formula (4-10) give dual representations of the quantity $H_{(m,0),(n,1)}$. The next lemma shows that equality persists if we drop the terms that involve J from both formulas. This statement is the crucial ingredient of Lemma 4.4 below.

Lemma 4.3. *Let $m \le n$ in $\mathbb{Z}$. Assume given nonnegative weights J_{m-1}, $(I_i)_{m \le i \le n}$ and $(\omega_i)_{m \le i \le n}$. Compute $(\tilde{I}_k, J_k, \widetilde{\omega}_k)_{m \le k \le n}$ from (4-2). Define*

$$T_{m,n} = \max_{m \le j \le n} \Big\{ \sum_{i=m}^{j} I_i + \sum_{i=j}^{n} \omega_i \Big\} \quad \textit{and} \quad \widetilde{T}_{m,n} = \max_{m \le j \le n} \Big\{ \sum_{i=m}^{j} \widetilde{\omega}_i + \sum_{i=j}^{n} \tilde{I}_i \Big\}. \tag{4-12}$$

Then $T_{m,n} = \widetilde{T}_{m,n}$.

Proof. The case $m = n$ is the identity $I_n + \omega_n = \widetilde{\omega}_n + \tilde{I}_n$ that follows from (4-2).

Let $n \ge m+1$ and assume by induction that $\widetilde{T}_{m,n-1} \le T_{m,n-1}$. Develop the definitions.

$$T_{m,n} = \max_{m \le j \le n-1} \Big\{ \sum_{i=m}^{j} I_i + \sum_{i=j}^{n-1} \omega_i \Big\} \vee \Big\{ \sum_{i=m}^{n} I_i \Big\} + \omega_n = T_{m,n-1} \vee \Big\{ \sum_{i=m}^{n} I_i \Big\} + \omega_n. \tag{4-13}$$

Similarly,

$$\begin{aligned}\widetilde{T}_{m,n} &= \widetilde{T}_{m,n-1} \vee \left\{\sum_{i=m}^{n} \widetilde{\omega}_i\right\} + \tilde{I}_n = \widetilde{T}_{m,n-1} \vee \left\{\sum_{i=m}^{n} (I_i \wedge J_{i-1})\right\} + \omega_n + (I_n - J_{n-1})^+ \\ &\le T_{m,n-1} \vee \left\{\sum_{i=m}^{n} I_i\right\} + \omega_n + (I_n - J_{n-1})^+ .\end{aligned} \tag{4-14}$$

The induction assumption was used in the last step.

Case 1: $I_n \le J_{n-1}$. This assumption kills the last term of (4-14) and gives

$$\widetilde{T}_{m,n} \le T_{m,n-1} \vee \left\{\sum_{i=m}^{n} I_i\right\} + \omega_n = T_{m,n}.$$

Case 2: $I_n > J_{n-1}$. For this case induction is not needed. We use the last-passage process $H_{(m-1,0),(\bullet,\bullet)}$. Conservation law (4-5) and (4-9) imply

$$I_n > J_{n-1} \iff \tilde{I}_n > J_n \iff H_{(m-1,0),(n-1,1)} < H_{(m-1,0),(n,0)}.$$

Then by (4-11),

$$H_{(m-1,0),(n,1)} = H_{(m-1,0),(n,0)} + \omega_n = \sum_{i=m}^{n} I_i + \omega_n \le T_{m,n}.$$

On the other hand, by definition (4-6),

$$H_{(m-1,0),(n,1)} = \left\{J_{m-1} + \sum_{i=m}^{n} \omega_i\right\} \vee T_{m,n}.$$

Hence $H_{(m-1,0),(n,1)} = T_{m,n}$. By the dual formula (4-10),

$$H_{(m-1,0),(n,1)} = \widetilde{T}_{m,n} \vee \left\{\sum_{i=m}^{n} \widetilde{\omega}_i + J_n\right\} \ge \widetilde{T}_{m,n}.$$

We conclude that in Case 2, $\widetilde{T}_{m,n} \le T_{m,n}$.

We have shown that $\widetilde{T}_{m,n} \le T_{m,n}$. This suffices for the proof by the duality in Lemma 4.1 because the roles of $T_{m,n}$ and $\widetilde{T}_{m,n}$ can be switched around. □

The next lemma is the key property of the queueing mapping D that underlies our results. Its proof relies on Lemma 4.3. Lemma 4.3 applies to the queueing setting described in Section 2C because equations (2-21) and (2-24) ensure that the assumptions of Lemma 4.3 are satisfied.

Lemma 4.4. *Assume given three sequences $I^2, I^1, \omega^1 \in \mathbb{R}_{\ge 0}^{\mathbb{Z}}$ such that the queueing operations below are well-defined. Let $\omega^2 = R(I^1, \omega^1)$ as defined in (2-21). Then we have the identity*

$$D\big(D(I^2, \omega^2), D(I^1, \omega^1)\big) = D\big(D(I^2, I^1), \omega^1\big). \tag{4-15}$$

Proof. Choose G^1 and G^2 so that $I^t_k = G^t_k - G^t_{k-1}$ for $t = 1, 2$. Let

$$H_j = \sup_{\ell:\, \ell \le j} \Big\{ G^2_\ell + \sum_{i=\ell}^{j} I^1_i \Big\}$$

and then

$$\widetilde{H}_k = \sup_{j:\, j \le k} \Big\{ H_j + \sum_{i=j}^{k} \omega^1_i \Big\} = \sup_{\ell:\, \ell \le k} \Big\{ G^2_\ell + \max_{j:\ell \le j \le k} \Big[\sum_{i=\ell}^{j} I^1_i + \sum_{i=j}^{k} \omega^1_i \Big] \Big\}. \tag{4-16}$$

The sequence $(\widetilde{H}_k - \widetilde{H}_{k-1})_{k\in\mathbb{Z}}$ is the output $D(D(I^2, I^1), \omega^1)$.

For the left-hand side of (4-15) define first for $D(I^t, \omega^t)$ the sequence

$$\widetilde{G}^t_j = \sup_{\ell:\, \ell \le j} \Big\{ G^t_\ell + \sum_{i=\ell}^{j} \omega^t_i \Big\}, \quad t \in \{1, 2\}.$$

Set $\tilde{I}^1_k = \widetilde{G}^1_k - \widetilde{G}^1_{k-1}$. The output $D(D(I^2, \omega^2), D(I^1, \omega^1))$ is given by the increments of the sequence

$$\widehat{H}_k = \sup_{j:\, j \le k} \Big\{ \widetilde{G}^2_j + \sum_{i=j}^{k} \tilde{I}^1_i \Big\} = \sup_{\ell:\, \ell \le k} \Big\{ G^2_\ell + \max_{j:\ell \le j \le k} \Big[\sum_{i=\ell}^{j} \omega^2_i + \sum_{i=j}^{k} \tilde{I}^1_i \Big] \Big\}. \tag{4-17}$$

The rightmost members of lines (4-16) and (4-17) are equal because the innermost maxima over the quantities in square brackets $[\cdots]$ agree, by Lemma 4.3. We have shown that $\widetilde{H} = \widehat{H}$ and thereby proved the lemma. ☐

We extend Lemma 4.4 inductively.

Lemma 4.5. *Let $n \ge 2$ and assume given $n+1$ sequences $I^1, I^2, \dots, I^n, \omega^1 \in \mathbb{R}^{\mathbb{Z}}_{\ge 0}$ such that all the queueing operations below are well-defined. Define iteratively*

$$\omega^j = R(I^{j-1}, \omega^{j-1}) \quad \textit{for } j = 2, \dots, n. \tag{4-18}$$

Then we have these identities for $1 \le k \le n-1$:

$$D^{(n+1)}(I^n, I^{n-1}, \dots, I^1, \omega^1) = D^{(k+1)}\big(D^{(n-k+1)}[I^n, \dots, I^{k+1}, \omega^{k+1}], D(I^k, \omega^k), \dots, D(I^1, \omega^1)\big). \tag{4-19}$$

Proof. The case $n = 2$ is Lemma 4.4.

Let $n \ge 3$ and assume that the claim of the lemma holds when n is replaced by $n-1$, for $1 \le k \le n-2$. We prove the claim for n.

First the case $k = 1$, beginning with the right-hand side of (4-19):

$$\begin{aligned} D^{(2)}\big(D^{(n)}(I^n, \dots, I^2, \omega^2), D(I^1, \omega^1)\big) &= D\big(D\big[D^{(n-1)}(I^n, \dots, I^2), \omega^2\big], D(I^1, \omega^1)\big) \\ &= D\big(D\big[D^{(n-1)}(I^n, \dots, I^2), I^1\big], \omega^1\big) \\ &= D\big(D^{(n)}(I^n, \dots, I^1), \omega^1\big) = D^{(n+1)}(I^n, \dots, I^1, \omega^1). \end{aligned}$$

The first and last two equalities above are from definition (2-26) of $D^{(n)}$, and the middle equality is Lemma 4.4.

Now let $2 \le k \le n-1$. The first equality below is definition (2-26) for $D^{(k+1)}$. The second equality is the induction assumption:

$$\begin{aligned} D^{(k+1)}\big(D^{(n-k+1)}(I^n, \dots, I^{k+1}, \omega^{k+1}), D(I^k, \omega^k), \dots, D(I^1, \omega^1)\big) \\ = D\big(D^{(k)}\big[D^{(n-k+1)}(I^n, \dots, I^{k+1}, \omega^{k+1}), D(I^k, \omega^k), \dots, D(I^2, \omega^2)\big], D(I^1, \omega^1)\big) \\ = D\big(D^{(n)}[I^n, \dots, I^2, \omega^2], D(I^1, \omega^1)\big). \end{aligned}$$

The last line above is the same as the left-hand side of the previous display. The calculation is completed as was done there. $\square$

In particular, for $k = n-1$, (4-19) gives

$$D^{(n+1)}(I^n, \dots, I^1, \omega^1) = D^{(n)}\big(D(I^n, \omega^n), \dots, D(I^1, \omega^1)\big) \tag{4-20}$$

and for $k = 1$,

$$D^{(n+1)}(I^n, \dots, I^1, \omega^1) = D\big(D^{(n)}\big[I^n, \dots, I^2, \omega^2\big], D(I^1, \omega^1)\big). \tag{4-21}$$

5. Multiclass processes

The distribution $\mu^{(1,\rho_1,\dots,\rho_n)}$ of the $(n+1)$-tuple $(\overline{Y}_t, \bar{B}_t^{\rho_1,\boldsymbol{e}_1}, \dots, \bar{B}_t^{\rho_n,\boldsymbol{e}_1})$ given in Theorem 3.2 is deduced through studying two multiclass LPP processes. Fix a positive integer n, the number of levels or classes. We define two discrete-time Markov processes on n-tuples of sequences, the *multiline process* and the *coupled process*. Their state space is

$$\mathcal{A}_n = \left\{ I = (I^1, I^2, \dots, I^n) \in (\mathbb{R}_{\ge 0}^{\mathbb{Z}})^n : \forall\, i \in [n],\ \lim_{m \to -\infty} \frac{1}{|m|} \sum_{k=-m}^{0} I_k^i > 1 \right\}. \tag{5-1}$$

At each step their evolution is driven by an independent sequence of i.i.d. exponential weights, so assume that

$$\omega = (\omega_k)_{k \in \mathbb{Z}} \text{ is a sequence of i.i.d. variables } \omega_k \sim \mathrm{Exp}(1). \tag{5-2}$$

5A. *Multiline process.* At time $t \in \mathbb{Z}_{\ge 0}$, the state of the *multiline process* is denoted by

$$I(t) = (I^1(t), \dots, I^n(t)) \in \mathcal{A}_n.$$

The one-step evolution from time t to $t+1$ is defined as follows in terms of the mappings (2-22). Given the time t configuration $I(t) = I = (I^1, I^2, \dots, I^n)$ in the space $\mathcal{A}_n$ and independent driving weights ω, define the time $t+1$ configuration $I(t+1) = \bar{I} = (\bar{I}^1, \bar{I}^2, \dots, \bar{I}^n)$ iteratively as follows:

$$\begin{aligned} &\text{set } \omega^1 = \omega \text{ and } \bar{I}^1 = D(I^1, \omega^1); \\ &\text{for } i = 2, 3, \dots, n: \\ &\qquad \text{set } \omega^i = R(I^{i-1}, \omega^{i-1}) \text{ and } \bar{I}^i = D(I^i, \omega^i). \end{aligned} \tag{5-3}$$

Thus the driving sequence ω acts on the first line I^1 directly, and is then transformed at each stage before it is passed to the next line. Lemma A.3 guarantees that, for almost every ω from (5-2), the Cesàro limit $\lim_{m\to-\infty}|m|^{-1}\sum_{k=m}^{0}\omega_k^i = 1$ holds for each $i \in [n]$ and the new state $\bar{I}$ lies in $\mathcal{A}_n$.

Theorem 5.1. *Assume (5-2). Then for each $\rho = (\rho_1, \dots, \rho_n) \in (1, \infty)^n$, the product measure ν^ρ defined in (3-4) is invariant for the multiline process $(I(t))_{t\in\mathbb{Z}_{\geq 0}}$.*

Theorem 5.1 follows from Lemma B.2 in Appendix B: induction on k shows that $\bar{I}^1, \dots, \bar{I}^k$, ω^{k+1}, $I^{k+1}, \dots, I^n$ are independent with $\bar{I}^i \sim \nu^{\rho_i}$, $\omega^{k+1} \sim \nu^1$, $I^j \sim \nu^{\rho_j}$. We do not have proof that ν^ρ is the unique translation-ergodic stationary distribution with mean vector ρ, but have no reason to doubt this either.

5B. *Coupled process.* At time $t \in \mathbb{Z}_{\geq 0}$, the state of the *coupled process* is denoted by

$$\eta(t) = (\eta^1(t), \dots, \eta^n(t)) \in \mathcal{A}_n$$

where again $\eta^i(t) = (\eta^i_k(t))_{k\in\mathbb{Z}}$. The evolution is simple: the queueing operator D acts on each sequence η^i with service times ω:

$$\eta(t+1) = \big(D(\eta^1(t), \omega), D(\eta^2(t), \omega), \dots, D(\eta^n(t), \omega)\big). \tag{5-4}$$

We call $\eta(t)$ the coupled process because it lives also on the smaller state space $\mathcal{X}_n \cap \mathcal{A}_n$ (recall (3-2)) where the sequences η^i are coupled so that $\eta^{i-1} \leq \eta^i$. This is the case relevant for the Busemann processes because the latter are monotone (recall (2-6)). Inequality (2-28) and Lemma A.1 ensure that the Markovian evolution $\eta(\cdot)$ is well-defined on $\mathcal{X}_n \cap \mathcal{A}_n$. However, since the mapping (5-4) is well-defined for more general states, we consider it on the larger state space $\mathcal{A}_n$ of (5-1).

To state an invariance and uniqueness theorem for all parameter vectors $\rho \in (1, \infty)^n$ we extend μ^ρ of (3-5), by ordering ρ and by requiring that $\eta^i = \eta^{i+1}$ if $\rho_i = \rho_{i+1}$. This is necessary because the mapping $\mathcal{D}^{(n)}$ in (3-5) cannot be applied if some $\rho_i = \rho_{i+1}$. For if I and ω are both i.i.d. $\mathrm{Exp}(\rho^{-1})$ sequences, then $\widetilde{G}$ in (2-18) is identically infinite because it equals a random constant plus the supremum of a symmetric random walk.

Definition 5.2. Let $\rho = (\rho_1, \rho_2, \dots, \rho_n) \in (0, \infty)^n$. The probability measure μ^ρ on the space $(\mathbb{R}^{\mathbb{Z}}_{\geq 0})^n$ is defined as follows.

(i) If $0 < \rho_1 < \rho_2 < \cdots < \rho_n$ then apply (3-5).

(ii) If $0 < \rho_1 \leq \rho_2 \leq \cdots \leq \rho_n$, there exist $m \in [n]$, a vector $\sigma = (\sigma_1, \dots, \sigma_m)$ such that $0 < \sigma_1 < \cdots < \sigma_m$, and indices $1 = i_1 < i_2 < \cdots < i_m < i_{m+1} = n+1$ such that $\rho_{i_\ell} = \cdots = \rho_{i_{\ell+1}-1} = \sigma_\ell$ for $\ell = 1, \dots, m$. Let $I \sim \nu^\sigma$, $\zeta = \mathcal{D}^{(m)}(I)$, and then define $\eta = (\eta^1, \dots, \eta^n) \in \mathcal{X}_n$ by $\eta^{i_\ell} = \cdots = \eta^{i_{\ell+1}-1} = \zeta^\ell$ for $\ell = 1, \dots, m$. Define μ^ρ to be the distribution of η.

(iii) For general $\rho = (\rho_1, \dots, \rho_n) \in (0, \infty)^n$, choose a permutation π such that $\pi\rho = (\rho_{\pi(1)}, \dots, \rho_{\pi(n)})$ satisfies $\rho_{\pi(1)} \leq \rho_{\pi(2)} \leq \cdots \leq \rho_{\pi(n)}$. Let π act on weight configurations $\eta = (\eta^1, \dots, \eta^n)$ by $\pi\eta = (\eta^{\pi(1)}, \dots, \eta^{\pi(n)})$. Define $\mu^\rho = \mu^{\pi\rho} \circ \pi^{-1}$, or more explicitly

$$E^{\mu^\rho}[f] = E^{\mu^{\pi\rho}}[f(\pi\eta)] = E^{\mu^{\pi\rho}\circ\pi^{-1}}[f]$$

for bounded Borel functions f on $(\mathbb{R}^{\mathbb{Z}}_{\geq 0})^n$, where the measure $\mu^{\pi\rho}$ is the one defined in step (ii).

If there is more than one ordering permutation in step (iii), there are identical sequences whose ordering among themselves is immaterial. If $\rho \in (1, \infty)^n$ then μ^ρ is supported on the space $\mathcal{A}_n$ of (5-1). The next existence and uniqueness theorem is proved in Section 5D.

Theorem 5.3. *Assume* (5-2).

(i) Invariance. *Let $\rho = (\rho_1, \rho_2, \dots, \rho_n) \in (1, \infty)^n$. Then the probability measure μ^ρ of Definition 5.2 is invariant for the Markov chain $(\eta(t))_{t \in \mathbb{Z}_{\geq 0}}$ defined by* (5-4).

(ii) Uniqueness. *Let $\widetilde{\mu}$ be a translation-ergodic probability measure on $\mathcal{A}_n$ under which coordinates η^i_k have finite means $\rho_i = E^{\widetilde{\mu}}[\eta^i_k] > 1$. If $\widetilde{\mu}$ is invariant for the process $\eta(t)$, then $\widetilde{\mu} = \mu^\rho$ for $\rho = (\rho_1, \dots, \rho_n) \in (1, \infty)^n$.*

5C. *Stationary multiclass LPP on the upper half-plane.* We reformulate the coupled process as a multiclass CGM on the upper half-plane. Fix the number n of classes. Assume given i.i.d. Exp(1) random weights $\{\omega_x\}_{x \in \mathbb{Z} \times \mathbb{Z}_{>0}}$, and an initial configuration $\eta(0) = (\eta^1(0), \dots, \eta^n(0)) \in \mathcal{A}_n$ independent of ω. Define a vector of LPP processes $G_x = (G^1_x, \dots, G^n_x)$ for $x \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$ as follows. First choose initial functions $\{G^i_{(k,0)}\}_{k \in \mathbb{Z}}$ with the property $\eta^i_k(0) = G^i_{(k,0)} - G^i_{(k-1,0)}$. Then for $(k, t) \in \mathbb{Z} \times \mathbb{Z}_{>0}$ define

$$G^i_{(k,t)} = \sup_{j:\, j \leq k} \{G^i_{(j,0)} + G_{(j,1),(k,t)}\}, \tag{5-5}$$

where $G_{x,y}$ is the usual LPP process of (2-1) with weights $Y_x(\omega) = \omega_x$. Then lastly define the process $\eta(t) = (\eta^1(t), \dots, \eta^n(t))$ for $t \in \mathbb{Z}_{>0}$ as the increments

$$\eta^i_k(t) = G^i_{(k,t)} - G^i_{(k-1,t)} \qquad \text{for } i \in [n] \text{ and } k \in \mathbb{Z}. \tag{5-6}$$

Theorem 5.4. *Let $\rho = (\rho_1, \rho_2, \dots, \rho_n) \in (1, \infty)^n$. Then μ^ρ of Definition 5.2 is an invariant measure of the increment process $\eta(\cdot)$ defined above by* (5-6) *in the multiclass exponential corner growth model. Measure μ^ρ is the unique invariant measure for $\eta(\cdot)$ among translation-ergodic probability measures on $\mathcal{A}_n$ with means given by ρ.*

This follows from Theorem 5.3 simply by noting that (5-6) can be reformulated inductively as

$$\eta(t) = \big(D(\eta^1(t-1), \overline{\omega}_t), D(\eta^2(t-1), \overline{\omega}_t), \dots, D(\eta^n(t-1), \overline{\omega}_t)\big), \quad t \in \mathbb{Z}_{>0}, \tag{5-7}$$

where $\overline{\omega}_t = \{\omega_{(k,t)}\}_{k \in \mathbb{Z}}$ is the sequence of weights on level t.

5D. *Invariant distribution for the coupled process.* This section proves Theorem 5.3. We separate the invariance of μ^ρ and the uniqueness in Theorems 5.5 and 5.6 below. Their combination establishes Theorem 5.3. The proof of the next theorem shows how the invariance of μ^ρ for $\eta(t)$ follows from the invariance of ν^ρ for $I(t)$ and the fact that the mapping $\mathcal{D}^{(n)}$ intertwines the evolutions of $I(t)$ and $\eta(t)$.

Theorem 5.5. *Let $\rho = (\rho_1, \rho_2, \dots, \rho_n) \in (1, \infty)^n$. Then μ^ρ of Definition 5.2 is an invariant distribution for the $(\mathbb{R}^{\mathbb{Z}}_{\geq 0})^n$-valued Markov chain $\eta(t)$ defined by* (5-4).

Proof. The general claim follows from the case $1 < \rho_1 < \rho_2 < \cdots < \rho_n$ because permuting the $\{\eta^i\}$ or setting $\eta^i = \eta^j$ produces the exact same change in the image of the mapping in (5-4).

So assume $1 < \rho_1 < \rho_2 < \cdots < \rho_n$. Given a driving sequence ω, denote by $\mathcal{S}^\omega$ and $\mathcal{T}^\omega$ the mappings on the state spaces that encode a single temporal evolution step of the processes $I(\cdot)$ and $\eta(\cdot)$. In other words, the mapping from time t to $t+1$ defined by (5-3) for the multiline process is encoded as $I(t+1) = \mathcal{S}^\omega(I(t))$. For the coupled process the step in (5-4) is encoded as $\eta(t+1) = \mathcal{T}^\omega(\eta(t))$. Let $\mathcal{D} = \mathcal{D}^{(n)}$ denote the mapping (3-3) that constructs the coupled configuration from the multiline configuration. Let $\mathcal{D}_k$, $\mathcal{S}^\omega_k$ and $\mathcal{T}^\omega_k$ denote the k-th $\mathbb{R}^{\mathbb{Z}}_{\ge 0}$-valued coordinates of the images of these mappings.

Let $I \sim \nu^\rho$ be a multiline configuration with product exponential distribution ν^ρ. We need to show that if η has the distribution μ^ρ of $\mathcal{D}(I)$, then so does $\mathcal{T}^\omega(\eta)$ when ω is an independent sequence of i.i.d. Exp(1) weights. For the argument we can assume that $\eta = \mathcal{D}(I)$. As before let $\omega^1 = \omega$ and iteratively $\omega^j = R(I^{j-1}, \omega^{j-1})$ for $j = 2, 3, \ldots, n$. The fourth equality below is (4-20). The other equalities are consequences of definitions:

$$\begin{aligned}
\mathcal{T}^\omega_k(\eta) &= D(\eta^k, \omega) = D\big(D^{(k)}(I^k, \ldots, I^1), \omega^1\big) = D^{(k+1)}(I^k, \ldots, I^1, \omega^1) \\
&\overset{(4\text{-}20)}{=} D^{(k)}\big(D(I^k, \omega^k), D(I^{k-1}, \omega^{k-1}), \ldots, D(I^1, \omega^1)\big) \\
&= D^{(k)}\big(\mathcal{S}^\omega_k(I), \mathcal{S}^\omega_{k-1}(I), \ldots, \mathcal{S}^\omega_1(I)\big) = \mathcal{D}_k(\mathcal{S}^\omega(I)).
\end{aligned}$$

Since the above works for all coordinates $k \in [n]$, we have $\mathcal{T}^\omega(\eta) = \mathcal{D}(\mathcal{S}^\omega(I))$. Since $\eta = \mathcal{D}(I)$, we have verified the intertwining

$$\mathcal{T}^\omega(\mathcal{D}(I)) = \mathcal{D}(\mathcal{S}^\omega(I)). \tag{5-8}$$

By Theorem 5.1, $\mathcal{S}^\omega(I) \overset{d}{=} I \sim \nu^\rho$. Consequently $\mathcal{T}^\omega(\eta) \overset{d}{=} \mathcal{D}(I) \sim \mu^\rho$. □

Theorem 5.6. *Assume (5-2). Let $\widetilde{\mu}$ be a translation-ergodic probability measure on $\mathcal{X}_n$ under which each coordinate η^i_k has a finite mean. If $\widetilde{\mu}$ is invariant for the coupled process $\eta(t)$, then $\widetilde{\mu} = \mu^\rho$ for the mean vector ρ of $\widetilde{\mu}$.*

We prove Theorem 5.6 following [Chang 1994], by showing that the evolution contracts the $\bar{\rho}$ distance between stationary and ergodic sequences. Let $\eta = (\eta_k)_{k\in\mathbb{Z}}$ and $\xi = (\xi_k)_{k\in\mathbb{Z}}$ be stationary processes taking values in $\mathbb{R}^n_{\ge 0}$. Their $\bar{\rho}$ distance is defined by

$$\bar{\rho}(\eta, \xi) = \inf_{(X,Y)\in\mathcal{M}} \boldsymbol{E}[\,|X_0 - Y_0|_1], \tag{5-9}$$

where $\mathcal{M}$ is the set of jointly defined stationary sequences $(X, Y) = (X_k, Y_k)_{k\in\mathbb{Z}}$ such that $X \overset{d}{=} \eta$ and $Y \overset{d}{=} \xi$, $\boldsymbol{E}$ is the expectation on the probability space on which the coupling (X, Y) is defined, and $|\cdot|_1$ is the ℓ^1 distance on $\mathbb{R}^n_{\ge 0}$.

From [Gray 2009, Theorem 9.2], we know that (i) $\bar{\rho}$ induces a metric on the space of translation-invariant distributions and (ii) if η and ξ are both ergodic, there exists a jointly stationary and ergodic pair (X, Y) at which the infimum in (5-9) is attained.

The following is a straightforward generalization of Theorem 2.4 of [Chang 1994] to $\mathbb{R}^n_{\ge 0}$-valued stationary and ergodic sequences $\eta = (\eta^1, \ldots, \eta^n)$ and $\xi = (\xi^1, \ldots, \xi^n)$ where $\eta^i = (\eta^i_k)_{k\in\mathbb{Z}}$ and $\xi^i = (\xi^i_k)_{k\in\mathbb{Z}}$

are random elements of $\mathbb{R}^{\mathbb{Z}}_{\geq 0}$. Let

$$\widetilde{\eta} = (\widetilde{\eta}^1, \dots, \widetilde{\eta}^n) = \big(D(\eta^1, \omega), \dots, D(\eta^n, \omega)\big)$$

and similarly $\widetilde{\xi} = (\widetilde{\xi}^1, \dots, \widetilde{\xi}^n)$ denote the outcome of applying the queueing map $D(\cdot, \omega)$ to each sequence-valued coordinate.

Proposition 5.7. *Let ω satisfy (5-2). Let the $\mathbb{R}^n_{\geq 0}$-valued stationary and ergodic processes η and ξ be independent of ω and have finite means that satisfy $\mathbb{E}[\eta^i_k] = \mathbb{E}[\xi^i_k] = \lambda_i > 1$ for $i \in [n]$ and $k \in \mathbb{Z}$. Then*

$$\bar{\rho}(\widetilde{\eta}, \widetilde{\xi}) \leq \bar{\rho}(\eta, \xi). \tag{5-10}$$

If η and ξ have different distributions the inequality in (5-10) is strict.

Before the proof we complete the proof of Theorem 5.6. Let $\rho = E^{\widetilde{\mu}}[\eta_0]$ be the mean vector of $\widetilde{\mu}$. Let $\eta \sim \mu^\rho$ and $\xi \sim \widetilde{\mu}$. By the known invariance of μ^ρ and the assumed invariance of $\widetilde{\mu}$, $\widetilde{\eta} \overset{d}{=} \eta$ and $\widetilde{\xi} \overset{d}{=} \xi$. Hence $\bar{\rho}(\widetilde{\eta}, \widetilde{\xi}) = \bar{\rho}(\eta, \xi)$. The last statement of Proposition 5.7 forces $\widetilde{\mu} = \mu^\rho$.

Proof of Proposition 5.7. Let $(X, Y) = ((X^1, \dots, X^n), (Y^1, \dots, Y^n))$ be an arbitrary $\mathbb{R}^{2n}_{\geq 0}$-valued jointly stationary and ergodic process with marginals $X \overset{d}{=} \eta$ and $Y \overset{d}{=} \xi$, independent of the weights ω, with (X, Y, ω) coupled together under a probability measure $\boldsymbol{P}$ with expectation $\boldsymbol{E}$. As above, write $\widetilde{X}^i = (\widetilde{X}^i_k)_{k\in\mathbb{Z}} = D(X^i, \omega)$ and $\widetilde{Y}^i = (\widetilde{Y}^i_k)_{k\in\mathbb{Z}} = D(Y^i, \omega)$ for the action of the queueing operator on the individual sequences $X^i = (X^i_k)_{k\in\mathbb{Z}}$ and $Y^i = (Y^i_k)_{k\in\mathbb{Z}}$. Inequality (5-10) follows from showing

$$\boldsymbol{E}[|\widetilde{X}_0 - \widetilde{Y}_0|_1] \leq \boldsymbol{E}[|X_0 - Y_0|_1]. \tag{5-11}$$

Define the process Z by $Z^i_k = X^i_k \vee Y^i_k$. Then

$$|X_0 - Y_0|_1 = \sum_{i=1}^n |X^i_0 - Y^i_0| = \sum_{i=1}^n (2Z^i_0 - X^i_0 - Y^i_0). \tag{5-12}$$

Let $\widetilde{Z}^i = D(Z^i, \omega)$. Then $\widetilde{Z}^i \geq \widetilde{X}^i \vee \widetilde{Y}^i$ by monotonicity (2-28). Hence

$$|\widetilde{X}_0 - \widetilde{Y}_0|_1 = \sum_{i=1}^n |\widetilde{X}^i_0 - \widetilde{Y}^i_0| = \sum_{i=1}^n \big(2(\widetilde{X}^i_0 \vee \widetilde{Y}^i_0) - \widetilde{X}^i_0 - \widetilde{Y}^i_0\big) \leq \sum_{i=1}^n (2\widetilde{Z}^i_0 - \widetilde{X}^i_0 - \widetilde{Y}^i_0). \tag{5-13}$$

The triple (X, Y, ω) is jointly stationary and ergodic because ω is an i.i.d. process independent of the ergodic process (X, Y). Consequently, as translation-respecting mappings of ergodic processes, both (X, Y, Z, ω) and $(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$ are jointly stationary and ergodic. The queueing stability condition $\boldsymbol{E}(X^i_0) > \boldsymbol{E}(\omega_0)$ implies $\boldsymbol{E}(\widetilde{X}^i_0) = \boldsymbol{E}(X^i_0)$, and by the same token $\boldsymbol{E}(\widetilde{Y}^i_0) = \boldsymbol{E}(Y^i_0)$ and $\boldsymbol{E}(\widetilde{Z}^i_0) = \boldsymbol{E}(Z^i_0)$. This goes back to Loynes [1962] and follows also from Lemma A.3 in Appendix A. Taking expectations on both sides of (5-12) and (5-13) gives (5-11).

For the strict inequality assume that η and ζ are not equal in distribution and let (X, Y) be a jointly ergodic pair that gives the minimum in (5-9). To deduce the strict inequality

$$\sum_{i=1}^n \boldsymbol{E}[\widetilde{X}^i_0 \vee \widetilde{Y}^i_0] < \sum_{i=1}^n \boldsymbol{E}(\widetilde{Z}^i_0). \tag{5-14}$$

we can tap directly into the proof of part (ii) of Theorem 2.4 in [Chang 1994], once we show that X^i and Y^i must cross for some $i \in [n]$. X^i and Y^i *cross* if with probability one there exist $k, \ell \in \mathbb{Z}$ such that $X^i_k > Y^i_k$ and $X^i_\ell < Y^i_\ell$.

Suppose X^i and Y^i do not cross. Then $\boldsymbol{P}(\{X^i \geq Y^i\} \cup \{X^i \leq Y^i\}) = 1$. We show that this implies $X^i = Y^i$ a.s. This gives us the contradiction needed, since $X^i = Y^i$ for all $i \in [n]$ implies that $\eta \overset{d}{=} \xi$.

To show $X^i = Y^i$ a.s., write $\{X^i = Y^i\}^c = A^+ \cup A^-$ a.s. for

$$A^+ = \{X^i \geq Y^i \text{ and } X^i_k > Y^i_k \text{ for some } k \in \mathbb{Z}\},$$
$$A^- = \{X^i \leq Y^i \text{ and } X^i_k < Y^i_k \text{ for some } k \in \mathbb{Z}\}.$$

A^+ is a shift-invariant event. By the joint ergodicity of (X, Y) and $\boldsymbol{E}(X^i_0 - Y^i_0) = 0$,

$$\begin{aligned} 0 &= \mathbf{1}_{A^+} \cdot \lim_{n\to\infty} \frac{1}{2n+1} \sum_{-n\leq k\leq n} (X^i_k - Y^i_k) \\ &= \lim_{n\to\infty} \frac{1}{2n+1} \sum_{-n\leq k\leq n} (X^i_k - Y^i_k) \cdot \mathbf{1}_{\theta_{-k}A^+} = \boldsymbol{E}[(X^i_0 - Y^i_0) \cdot \mathbf{1}_{A^+}] \quad \text{a.s.} \end{aligned}$$

Thus $X^i_0 = Y^i_0$ a.s. on A^+. By the shift-invariance of A^+, $X^i_k = Y^i_k$ a.s. on A^+ for all $k \in \mathbb{Z}$. But then it must be that $\boldsymbol{P}(A^+) = 0$. Similarly $\boldsymbol{P}(A^-) = 0$.

To summarize, we have shown that some X^i and Y^i must cross. Following the proof on page 1131–1132 of [Chang 1994] gives the strict inequality (5-14). The connection between the notation of [Chang 1994] and ours is $S_k = \omega_k$, $(T^1_{1,k-1}, T^1_{2,k-1}) = (X^i_k, \widetilde{X}^i_k)$ and $(T^2_{1,k-1}, T^2_{2,k-1}) = (Y^i_k, \widetilde{Y}^i_k)$. □

6. Proofs of the results for Busemann functions

We prove the theorems of Section 3 in the order in which they were stated.

6A. *Continuity of μ^ρ and distribution of the Busemann process.*

Proof of the continuity claim of Theorem 3.1. Fix $\rho = (\rho_1, \dots, \rho_n)$ such that $0 < \rho_1 < \cdots < \rho_n$. Let $\{\rho^h\}_{h\in\mathbb{Z}_{>0}}$ be a sequence of parameter vectors such that $\rho^h = (\rho^h_1, \dots, \rho^h_n) \to (\rho_1, \dots, \rho_n)$ as $h \to \infty$. We construct variables $\eta^h \sim \mu^{\rho^h}$ and $\eta \sim \mu^\rho$ such that $\eta^h \to \eta$ coordinatewise almost surely.

Let $I = (I^1, \dots, I^n) \sim \nu^\rho$ and define $I^{h,i}_k = (\rho^h_i/\rho_i) I^i_k$. Then $I^h = (I^{h,1}, \dots, I^{h,n}) \sim \nu^{\rho^h}$ and we have the pointwise limits $I^{h,i}_k \to I^i_k$ for all $i \in [n]$ and $k \in \mathbb{Z}$ as $h \to \infty$. Furthermore, the assumption in (A-2) holds:

$$\varlimsup_{\substack{m\to-\infty \\ h\to\infty}} \left| \frac{1}{|m|} \sum_{j=m}^{0} I^{h,i}_j - \rho_i \right| = 0 \quad \text{almost surely for all } i \in [n]. \tag{6-1}$$

Let $\eta^h = \mathcal{D}^{(n)}(I^h)$ and $\eta = \mathcal{D}^{(n)}(I)$. Apply Lemma A.2 repeatedly to show that $\eta^h \to \eta$ coordinatewise almost surely:

(1) $\eta^{h,1} = I^{h,1} \to I^1 = \eta^1$ needs no proof.

(2) Lemma A.2 gives the limit $\eta^{h,2} = D(I^{h,2}, I^{h,1}) \to D(I^2, I^1) = \eta^2$ and that $D(I^{h,2}, I^{h,1})$ satisfies the hypotheses of the lemma.

(3) For $\eta^{h,3} = D^{(3)}(I^{h,3}, I^{h,2}, I^{h,1}) = D(D(I^{h,3}, I^{h,2}), I^{h,1})$, by case (2), $D(I^{h,3}, I^{h,2})$ satisfies the hypotheses of Lemma A.2. Then Lemma A.2 gives $D(D(I^{h,3}, I^{h,2}), I^{h,1}) \to D(D(I^3, I^2), I^1)$ and that $D^{(3)}(I^{h,3}, I^{h,2}, I^{h,1})$ satisfies the hypotheses of Lemma A.2.

(4) Proceed by induction. From the case of $i-1$ sequences, $D^{(i-1)}(I^{h,i}, I^{h,i-1}, \dots, I^{h,2})$ satisfies the hypotheses of Lemma A.2. Apply the Lemma to conclude that the mapping for i sequences obeys the limit

$$\eta^{h,i} = D^{(i)}(I^{h,i}, \dots, I^{h,2}, I^{h,1}) = D(D^{(i-1)}(I^{h,i}, I^{h,i-1}, \dots, I^{h,2}), I^{h,1}) \to \eta^i$$

and also satisfies the assumptions of Lemma A.2. This is then passed on to be used for the case of $i+1$ sequences.

This completes the proof of $\eta^h \to \eta$. □

Proof of Theorem 3.2. Introduce an $(n+1)$-st parameter value $\rho_0 \in (1, \rho_1)$. By Lemma 2.3, the $\mathbb{R}^{n+1}_{\geq 0}$-valued $\mathbb{Z}$-indexed process

$$\bar{B}_t^{\rho_0,\dots,\rho_n,\boldsymbol{e}_1} = \{(B^{\rho_0}_{(k-1,t),(k,t)}, B^{\rho_1}_{(k-1,t),(k,t)}, \dots, B^{\rho_n}_{(k-1,t),(k,t)})\}_{k\in\mathbb{Z}} \tag{6-2}$$

is stationary and ergodic under translation of the k-index and furthermore $\bar{B}_t^{\rho_0,\dots,\rho_n,\boldsymbol{e}_1}$ has the same distribution as the sequence $\bar{B}_{t-1}^{\rho_0,\dots,\rho_n,\boldsymbol{e}_1}$ on the previous level $t-1$. Lemma 3.3 gives $\bar{B}_t^{\rho_0,\dots,\rho_n,\boldsymbol{e}_1} = D(\bar{B}_{t-1}^{\rho_0,\dots,\rho_n,\boldsymbol{e}_1}, \bar{Y}_t)$. By the uniqueness given in Theorem 5.3, the distribution of $\bar{B}_t^{\rho_0,\dots,\rho_n,\boldsymbol{e}_1}$ must be the invariant distribution $\mu^{(\rho_0,\dots,\rho_n)}$.

Let $\rho_0 \searrow 1$. By Lemma 2.2, almost surely,

$$\lim_{\rho_0\searrow 1} \bar{B}_t^{\rho_0,\rho_1,\dots,\rho_n,\boldsymbol{e}_1} = \{(Y_{(k,t)}, B^{\rho_1}_{(k-1,t),(k,t)}, \dots, B^{\rho_n}_{(k-1,t),(k,t)})\}_{k\in\mathbb{Z}},$$

while Theorem 3.1 gives the weak convergence $\mu^{(\rho_0,\rho_1,\dots,\rho_n)} \to \mu^{(1,\rho_1,\dots,\rho_n)}$ as $\rho_0 \searrow 1$. □

The proof of Lemma 3.3 below relies on the iterative equations (2-24). Since these equations can have solutions other than the one coming from the queuing mapping, additional conditions are needed as specified in Lemma A.4 in Appendix A.

Proof of Lemma 3.3. We show that there is an event Ω_0 of full probability on which the assumptions of Lemma A.4 hold for the sequences $(\widetilde{I}, J, I, \omega) = (\bar{B}_t^{\rho,\boldsymbol{e}_1}, \bar{B}_t^{\rho,\boldsymbol{e}_2}, \bar{B}_{t-1}^{\rho,\boldsymbol{e}_1}, \bar{Y}_t)$ simultaneously for *all* uncountably many $\rho \in (1,\infty)$ and $t \in \mathbb{Z}$.

Assumption (A-14) requires

$$\lim_{m\to-\infty} \sum_{k=m}^{0} (Y_{(k,t)} - B^{\rho}_{(k,t-1),(k+1,t-1)}) = -\infty \quad \text{for all } t \in \mathbb{Z}.$$

This holds almost surely simultaneously for all ρ in a dense countable subset of $(1,\infty)$. By the monotonicity (2-6) this extends to all $\rho \in (1,\infty)$ on a single event of full probability.

Utilizing the recovery property (2-8) and additivity (2-7),

$$\begin{aligned}
Y_{(k,t)} &+ (B^{\rho}_{(k-1,t-1),(k,t-1)} - B^{\rho}_{(k-1,t-1),(k-1,t)})^{+} \\
&= B^{\rho}_{(k-1,t),(k,t)} \wedge B^{\rho}_{(k,t-1),(k,t)} + (B^{\rho}_{(k-1,t),(k,t)} - B^{\rho}_{(k,t-1),(k,t)})^{+} \\
&= B^{\rho}_{(k-1,t),(k,t)}
\end{aligned}$$

and

$$\begin{aligned}
Y_{(k,t)} &+ (B^{\rho}_{(k-1,t-1),(k-1,t)} - B^{\rho}_{(k-1,t-1),(k,t-1)})^{+} \\
&= B^{\rho}_{(k-1,t),(k,t)} \wedge B^{\rho}_{(k,t-1),(k,t)} + (B^{\rho}_{(k,t-1),(k,t)} - B^{\rho}_{(k-1,t),(k,t)})^{+} \\
&= B^{\rho}_{(k,t-1),(k,t)}.
\end{aligned}$$

These equations are valid for all ρ and all (k,t) on a single event of full probability because this is true of properties (2-8) and (2-7). Assumption (A-15) has been verified.

Lemma A.5 implies that with probability one, for all ρ in a dense countable subset of $(1,\infty)$, $Y_{(k,t)} = B^{\rho}_{(k,t-1),(k,t)}$ for infinitely many $k<0$. Monotonicity (2-6) and recovery (2-8) extend this property to all $\rho \in (1,\infty)$ on the same event. □

6B. *Triangular arrays and independent increments.* To extract further properties of the distribution μ^{ρ}, we develop an alternative representation for $\eta = \mathcal{D}^{(n)}(I)$ of (3-3). Assume given $I = (I^1, \dots, I^n) \in \mathcal{Y}_n$. Define arrays $\{\eta^{i,j} : 1 \le j \le i \le n\}$ and $\{\xi^{i,j} : 1 \le j \le i \le n\}$ of elements of $\mathbb{R}^{\mathbb{Z}}_{\ge 0}$ as follows. The ξ variables are passed from one i level to the next.

(i) For $i=1$, set $\eta^{1,1} = I^1 = \xi^{1,1}$.

(ii) For $i = 2, 3, \dots, n$,

$$\begin{aligned}
\eta^{i,1} &= I^i, & & \\
\eta^{i,j} &= D(\eta^{i,j-1}, \xi^{i-1,j-1}) & &\text{for } j = 2,3,\dots,i, \\
\xi^{i,j-1} &= R(\eta^{i,j-1}, \xi^{i-1,j-1}) & &\text{for } j = 2,3,\dots,i, \\
\xi^{i,i} &= \eta^{i,i}.
\end{aligned} \tag{6-3}$$

Step i takes inputs from two sources: from the outside it takes I^i, and from step $i-1$ it takes the configuration $\xi^{i-1,\bullet} = (\xi^{i-1,1}, \xi^{i-1,2}, \dots, \xi^{i-1,i-2}, \xi^{i-1,i-1} = \eta^{i-1,i-1})$.

Lemma A.3 ensures that the arrays are well-defined for $I \in \mathcal{Y}_n$. The inputs $I^1, \dots, I^n$ enter the algorithm one by one in order. If the process is stopped after the step $i = m$ is completed for some $m < n$, it produces the arrays for $(I^1, \dots, I^m) \in \mathcal{Y}_m$.

The arrays are illustrated in Figure 4. The following properties of the arrays come from Lemmas 6.1 and 6.2 and their proofs.

(i) The input of the $\mathcal{D}^{(n)}$-mapping lies on the left edge of the η-array: $(\eta^{1,1}, \dots, \eta^{n,1}) = (I^1, \dots, I^n)$. The output of the $\mathcal{D}^{(n)}$-mapping lies on the right-hand diagonal edges of both arrays:

$$(\eta^{1,1}, \eta^{2,2}, \dots, \eta^{n,n}) = (\xi^{1,1}, \xi^{2,2}, \dots, \xi^{n,n}) = \mathcal{D}^{(n)}(I^1, \dots, I^n) \sim \mu^{(\rho_1, \rho_2, \dots, \rho_n)}.$$

$$\begin{array}{lllll} \eta^{1,1} & & & & \\ \eta^{2,1} & \eta^{2,2} & & & \\ \eta^{3,1} & \eta^{3,2} & \eta^{3,3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \eta^{n,1} & \eta^{n,2} & \eta^{n,3} & \cdots & \eta^{n,n} \end{array} \qquad \begin{array}{lllll} \xi^{1,1} & & & & \\ \xi^{2,1} & \xi^{2,2} & & & \\ \xi^{3,1} & \xi^{3,2} & \xi^{3,3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \xi^{n,1} & \xi^{n,2} & \xi^{n,3} & \cdots & \xi^{n,n} \end{array}$$

Figure 4. Arrays $\{\eta^{i,j} : 1 \le j \le i \le n\}$ and $\{\xi^{i,j} : 1 \le j \le i \le n\}$.

(ii) The j-th column $(\eta^{j,j}, \eta^{j+1,j}, \dots, \eta^{n,j})$ of the η-array has the product distribution $\nu^{(\rho_j, \rho_{j+1}, \dots, \rho_n)}$. It is obtained from the $(j-1)$-st column $(\eta^{j,j-1}, \eta^{j+1,j-1}, \dots, \eta^{n,j-1})$ by the mapping (5-3) with $\eta^{j-1,j-1} = \xi^{j-1,j-1}$ as the external driving weights.

(iii) Row $(\xi^{i,1}, \xi^{i,2}, \dots, \xi^{i,i})$ of the ξ-array has the product distribution $\nu^{(\rho_1, \rho_2, \dots, \rho_i)}$.

Lemma 6.1. *Let $I = (I^1, \dots, I^n) \in \mathcal{Y}_n$. Let $(\widetilde{\eta}^1, \dots, \widetilde{\eta}^n) = \mathcal{D}^{(n)}(I^1, \dots, I^n)$ be given by the mapping (3-3). Let $\{\eta^{i,j}\}$ be the array defined above. Then $\widetilde{\eta}^i = \eta^{i,i}$ for $i = 1, \dots, n$.*

Proof. It suffices to prove $\tilde{\eta}^n = \eta^{n,n}$ because the same proof applies to all i. The construction of the array can be reimagined as follows. Start with $(\eta^{1,1}, \eta^{2,1}, \dots, \eta^{n,1}) = (I^1, I^2, \dots, I^n)$. Then for $\ell = 2, 3, \dots, n-1$ iterate the following step that maps the $(n-\ell+2)$-vector

$$(\eta^{n,\ell-1}, \eta^{n-1,\ell-1}, \dots, \eta^{\ell,\ell-1}, \eta^{\ell-1,\ell-1})$$

to the $(n-\ell+1)$-vector

$$\begin{aligned}&(\eta^{n,\ell}, \eta^{n-1,\ell}, \dots, \eta^{\ell+1,\ell}, \eta^{\ell,\ell}) \\ &\qquad = \big(D(\eta^{n,\ell-1}, \xi^{n-1,\ell-1}), D(\eta^{n-1,\ell-1}, \xi^{n-2,\ell-1}), \dots, D(\eta^{\ell+1,\ell-1}, \xi^{\ell,\ell-1}), D(\eta^{\ell,\ell-1}, \eta^{\ell-1,\ell-1})\big).\end{aligned}$$

The ξ-variables above satisfy

$$\begin{aligned}\xi^{\ell,\ell-1} &= R(\eta^{\ell,\ell-1}, \xi^{\ell-1,\ell-1}) = R(\eta^{\ell,\ell-1}, \eta^{\ell-1,\ell-1}) \\ \xi^{\ell+1,\ell-1} &= R(\eta^{\ell+1,\ell-1}, \xi^{\ell,\ell-1}) \\ &\ \vdots \\ \xi^{n-1,\ell-1} &= R(\eta^{n-1,\ell-1}, \xi^{n-2,\ell-1}).\end{aligned}$$

Thus (4-20) implies that

$$D^{(n-\ell+2)}(\eta^{n,\ell-1}, \eta^{n-1,\ell-1}, \dots, \eta^{\ell,\ell-1}, \eta^{\ell-1,\ell-1}) = D^{(n-\ell+1)}(\eta^{n,\ell}, \eta^{n-1,\ell}, \dots, \eta^{\ell+1,\ell}, \eta^{\ell,\ell}). \tag{6-4}$$

In the derivation below, use the first line of (6-3) to replace each I^i with $\eta^{i,1}$. Then iterate (6-4) from $\ell = 2$ to $\ell = n-1$ to obtain

$$\begin{aligned}\widetilde{\eta}^n &= D^{(n)}(I^n, I^{n-1}, \dots, I^3, I^2, I^1) = D^{(n)}(\eta^{n,1}, \eta^{n-1,1}, \dots, \eta^{3,1}, \eta^{2,1}, \eta^{1,1}) \\ &= D^{(n-1)}\big(\eta^{n,2}, \eta^{n-1,2}, \dots, \eta^{3,2}, \eta^{2,2}\big) \\ &= \dots = D^{(3)}(\eta^{n,n-2}, \eta^{n-1,n-2}, \eta^{n-2,n-2}) = D(\eta^{n,n-1}, \eta^{n-1,n-1}) = \eta^{n,n}.\end{aligned}$$

□

The next two lemmas describe the distributions of the arrays.

Lemma 6.2. *Fix $0 < \rho_1 < \cdots < \rho_n$ and let the multiline configuration $I = (I^1, \dots, I^n)$ have distribution $\nu^{(\rho_1,\dots,\rho_n)}$. Let $\{\eta^{i,j}\}_{1\le j\le i\le n}$ and $\{\xi^{i,j}\}_{1\le j\le i\le n}$ be the arrays defined above. Then for each $1 \le i, j \le n$, configuration $(\eta^{j,j}, \eta^{j+1,j}, \dots, \eta^{n,j})$ has distribution $\nu^{(\rho_j,\rho_{j+1},\dots,\rho_n)}$ and configuration $(\xi^{i,1}, \xi^{i,2}, \dots, \xi^{i,i})$ has distribution $\nu^{(\rho_1,\rho_2,\dots,\rho_i)}$. In particular, each $\eta^{i,j}$ has distribution ν^{ρ_i} and each $\xi^{i,j}$ has distribution ν^{ρ_j}.*

Proof. First we prove the claim for $(\eta^{j,j}, \eta^{j+1,j}, \dots, \eta^{n,j})$. Recall that by definition $\xi^{j,j} = \eta^{j,j}$.

For $j = 1$, the definitions give $(\xi^{1,1} = \eta^{1,1}, \eta^{2,1}, \dots, \eta^{n,1}) = (I^1, I^2, \dots, I^n) \sim \nu^{(\rho_1,\rho_2,\dots,\rho_n)}$.

Let $j \in [\![2, n]\!]$. Assume inductively that

$$(\xi^{j-1,j-1} = \eta^{j-1,j-1}, \eta^{j,j-1}, \dots, \eta^{n,j-1}) \sim \nu^{(\rho_{j-1},\rho_j,\dots,\rho_n)}.$$

The mapping from $(\eta^{j,j-1}, \dots, \eta^{n,j-1})$ to $(\eta^{j,j}, \dots, \eta^{n,j})$ is the mapping (5-3) of the multiline process, with $\xi^{j-1,j-1}$ as the external driving weights ω. Namely, this mapping is carried out by iterating

$$\eta^{j+k,j} = D(\eta^{j+k,j-1}, \xi^{j+k-1,j-1}), \qquad \xi^{j+k,j-1} = R(\eta^{j+k,j-1}, \xi^{j+k-1,j-1})$$

for $k = 0, 1, \dots, n - j$. Then $(\eta^{j,j}, \eta^{j+1,j}, \dots, \eta^{n,j}) \sim \nu^{(\rho_j,\rho_{j+1},\dots,\rho_n)}$ follows from the invariance in Theorem 5.1.

Next the proof for $(\xi^{i,1}, \xi^{i,2}, \dots, \xi^{i,i})$. The claim is immediate for $i = 1$ because there is just one sequence $\eta^{1,1} = I^1 = \xi^{1,1} \sim \nu^{\rho_1}$. Let $i \in [\![2, n]\!]$ and assume inductively that $(\xi^{i-1,1}, \xi^{i-1,2}, \dots, \xi^{i-1,i-1}) \sim \nu^{(\rho_1,\rho_2,\dots,\rho_{i-1})}$. By construction, $\eta^{i,1} = I^i \sim \nu^{\rho_i}$ is independent of $\xi^{i-1,\bullet}$, and hence

$$(\eta^{i,1}, \xi^{i-1,1}, \xi^{i-1,2}, \dots, \xi^{i-1,i-1}) \sim \nu^{(\rho_i,\rho_1,\rho_2,\dots,\rho_{i-1})}.$$

Now we transform the sequence above by repeated application of the mapping $(\eta^{i,\ell}, \xi^{i-1,\ell}) \mapsto (\xi^{i,\ell}, \eta^{i,\ell+1})$ defined by (6-3):

$$\eta^{i,\ell+1} = D(\eta^{i,\ell}, \xi^{i-1,\ell}),$$
$$\xi^{i,\ell} = R(\eta^{i,\ell}, \xi^{i-1,\ell})$$

for $\ell = 1, \dots, i - 1$. The pair to be transformed next slides successively to the right. The succession of sequences produced by this process is displayed below, beginning with the first one from above. The pair to which the mapping is applied next is enclosed in the box. The distribution follows from Lemma B.2:

$$\begin{aligned}
&(\boxed{\eta^{i,1}, \xi^{i-1,1}}, \xi^{i-1,2}, \xi^{i-1,3}, \dots, \xi^{i-1,i-1}) \sim \nu^{(\rho_i,\rho_1,\rho_2,\rho_3,\dots,\rho_{i-1})}\\
&(\xi^{i,1}, \boxed{\eta^{i,2}, \xi^{i-1,2}}, \xi^{i-1,3}, \dots, \xi^{i-1,i-1}) \sim \nu^{(\rho_1,\rho_i,\rho_2,\rho_3,\dots,\rho_{i-1})}\\
&\qquad\qquad\ddots\\
&(\xi^{i,1}, \dots, \xi^{i,\ell-1}, \boxed{\eta^{i,\ell}, \xi^{i-1,\ell}}, \xi^{i-1,\ell+1}, \dots, \xi^{i-1,i-1}) \sim \nu^{(\rho_1,\dots,\rho_{\ell-1},\rho_i,\rho_\ell,\rho_{\ell+1}\dots,\rho_{i-1})}\\
&\qquad\qquad\ddots\\
&(\xi^{i,1}, \dots, \xi^{i,i-1}, \eta^{i,i}) \sim \nu^{(\rho_1,\dots,\rho_{i-1},\rho_i)}.
\end{aligned}$$

To complete the induction from $i - 1$ to i, set $\xi^{i,i} = \eta^{i,i}$. $\square$

Remark 6.3 (notation). To keep track of the inputs when processes are constructed by queueing mappings (2-22), superscripts indicate the arrival and service processes used in the construction. This works as follows when the arrival process is I and the service process is ω:

- G^I denotes a function that satisfies $I_k = G^I_k - G^I_{k-1}$.
- $\widetilde{G}^{I,\omega}$ is the process defined by (2-18) whose increments are the output $\tilde{I}^{I,\omega}_k = \widetilde{G}^{I,\omega}_k - \widetilde{G}^{I,\omega}_{k-1}$, and so $\tilde{I}^{I,\omega} = D(I,\omega)$.
- $J^{I,\omega} = S(I,\omega)$ is the process defined by (2-20) as $J^{I,\omega}_k = \widetilde{G}^{I,\omega}_k - G^I_k$.
- $\widetilde{\omega}^{I,\omega} = R(I,\omega)$.

Lemma 6.4. *Fix $0 < \rho_1 < \cdots < \rho_n$ and let the multiline configuration $I = (I^1, \dots, I^n)$ have distribution $\nu^{(\rho_1,\dots,\rho_n)}$. Let $\eta = (\eta^1, \dots, \eta^n) = \mathcal{D}^{(n)}(I)$ and let $\{\eta^{i,j}\}$ and $\{\xi^{i,j}\}$ be the arrays constructed above. Then for each $m \in [\![2, n]\!]$ and $k \in \mathbb{Z}$, the following random variables are independent*:

$$\{\xi^{m,1}_i\}_{i\le k},\ \{\xi^{m,2}_i\}_{i\le k}, \dots, \{\xi^{m,m-1}_i\}_{i\le k},\ \{\eta^m_i\}_{i\le k-1},\ \eta^m_k - \eta^{m-1}_k,\ \eta^{m-1}_k - \eta^{m-2}_k, \dots, \eta^2_k - \eta^1_k,\ \eta^1_k.$$

Proof. Index k is fixed throughout the proof. We begin with the case $m = 2$.

By the definitions, $\eta^1 = I^1$,

$$\xi^{2,1} = R(\eta^{2,1}, \xi^{1,1}) = R(I^2, I^1) = \widetilde{\omega}^{I^2,I^1} \quad\text{and}\quad \eta^2 = D(I^2, I^1) = \tilde{I}^{I^2,I^1}.$$

Hence $\xi^{2,1}_i = I^2_i \wedge J^{I^2,I^1}_{i-1}$ and $\eta^2_k - \eta^1_k = (I^2_k - J^{I^2,I^1}_{k-1})^+$. By Lemma B.2(a), $\{\tilde{I}^{I^2,I^1}_i\}_{i\le k-1}$, J^{I^2,I^1}_{k-1}, $\{\widetilde{\omega}^{I^2,I^1}_i\}_{i\le k-1}$, I^2_k, I^1_k are independent. To be precise, Lemma B.2(a) gives the independence of $\{\widetilde{\omega}^{I^2,I^1}_i\}_{i\le k-1}$, $\{\tilde{I}^{I^2,I^1}_i\}_{i\le k-1}$, and J^{I^2,I^1}_{k-1}. These are functions of $\{I^2_i, I^1_i\}_{i\le k-1}$, and thereby independent of I^2_k, I^1_k. Properties of independent exponentials (Lemma B.1(i)) imply that

$$\xi^{2,1}_k = I^2_k \wedge J^{I^2,I^1}_{k-1} \text{ and } \eta^2_k - \eta^1_k = (I^2_k - J^{I^2,I^1}_{k-1})^+ \text{ are mutually independent.} \tag{6-5}$$

Altogether we have that $\{\xi^{2,1}_i\}_{i\le k}$, $\{\eta^2_i\}_{i\le k-1}$, $\eta^2_k - \eta^1_k$, η^1_k are independent.

Let $m \ge 3$ and make an induction assumption:

$$\{\xi^{m-1,1}_i\}_{i\le k}, \dots, \{\xi^{m-1,m-2}_i\}_{i\le k},\ \{\eta^{m-1}_i\}_{i\le k-1},\ \eta^{m-1}_k - \eta^{m-2}_k, \dots, \eta^2_k - \eta^1_k, \eta^1_k \text{ are independent.} \tag{6-6}$$

The previous paragraph verified this assumption for $m = 3$.

Since $\eta^{m,1} = I^m$ is independent of all the variables in (6-6), apply Lemma B.2(a) to the pair $\xi^{m,1} = R(\eta^{m,1}, \xi^{m-1,1})$, $\eta^{m,2} = D(\eta^{m,1}, \xi^{m-1,1})$ to conclude the independence of

$$\begin{gathered}\{\xi^{m,1}_i\}_{i\le k},\ \{\eta^{m,2}_i\}_{i\le k},\ \{\xi^{m-1,2}_i\}_{i\le k}, \dots, \{\xi^{m-1,m-2}_i\}_{i\le k},\\ \{\eta^{m-1}_i\}_{i\le k-1},\ \eta^{m-1}_k - \eta^{m-2}_k, \dots, \eta^2_k - \eta^1_k, \eta^1_k.\end{gathered} \tag{6-7}$$

This starts an induction on $j = 2, 3, \dots, m-1$, whose induction assumption is the independence of

$$\begin{gathered}\{\xi^{m,1}_i\}_{i\le k}, \dots, \{\xi^{m,j-1}_i\}_{i\le k}, \{\eta^{m,j}_i\}_{i\le k}, \{\xi^{m-1,j}_i\}_{i\le k}, \dots, \{\xi^{m-1,m-2}_i\}_{i\le k},\\ \{\eta^{m-1}_i\}_{i\le k-1}, \eta^{m-1}_k - \eta^{m-2}_k, \dots, \eta^2_k - \eta^1_k, \eta^1_k.\end{gathered} \tag{6-8}$$

The induction step is the application of Lemma B.2(a) to the pair $\xi^{m,j} = R(\eta^{m,j}, \xi^{m-1,j})$, $\eta^{m,j+1} = D(\eta^{m,j}, \xi^{m-1,j})$ to conclude the independence of

$$\begin{gathered}\{\xi_i^{m,1}\}_{i\le k}, \dots, \{\xi_i^{m,j-1}\}_{i\le k},\ \{\xi_i^{m,j}\}_{i\le k},\ \{\eta_i^{m,j+1}\}_{i\le k},\\ \{\xi_i^{m-1,j+1}\}_{i\le k}, \dots, \{\xi_i^{m-1,m-1}\}_{i\le k},\ \{\eta_i^{m-1}\}_{i\le k-1},\ \eta_k^{m-1}-\eta_k^{m-2}, \dots, \eta_k^2-\eta_k^1,\ \eta_k^1.\end{gathered} \tag{6-9}$$

Thus the induction assumption (6-8) for j has been advanced to $j+1$ in (6-9).

At the end of the j-induction we have the independence of

$$\begin{gathered}\{\xi_i^{m,1}\}_{i\le k}, \dots, \{\xi_i^{m,m-2}\}_{i\le k}, \{\eta_i^{m,m-1}\}_{i\le k},\\ \{\eta_i^{m-1}\}_{i\le k-1},\ \eta_k^{m-1}-\eta_k^{m-2}, \dots, \eta_k^2-\eta_k^1, \eta_k^1.\end{gathered} \tag{6-10}$$

Split $\{\eta_i^{m,m-1}\}_{i\le k}$ into the independent pieces $\{\eta_i^{m,m-1}\}_{i\le k-1}$ and $\eta_k^{m,m-1}$. Combine the former with $\{\eta_i^{m-1}\}_{i\le k-1}$, Lemma B.2(a), and the transformations $\xi^{m,m-1} = R(\eta^{m,m-1}, \eta^{m-1})$, $\eta^m = D(\eta^{m,m-1}, \eta^{m-1})$ to form the independent variables $\{\xi_i^{m,m-1}\}_{i\le k-1}$, $\{\eta_i^m\}_{i\le k-1}$ and $J_{k-1}^{\eta^{m,m-1},\eta^{m-1}}$. Transform the independent pair $(\eta_k^{m,m-1}, J_{k-1}^{\eta^{m,m-1},\eta^{m-1}})$ into the independent pair of $\xi_k^{m,m-1} = \eta_k^{m,m-1} \wedge J_{k-1}^{\eta^{m,m-1},\eta^{m-1}}$ and $\eta_k^m - \eta_k^{m-1} = (\eta_k^{m,m-1} - J_{k-1}^{\eta^{m,m-1},\eta^{m-1}})^+$. Attach $\xi_k^{m,m-1}$ to the sequence $\{\xi_i^{m,m-1}\}_{i\le k-1}$. After these steps, we have the independence of

$$\begin{gathered}\{\xi_i^{m,1}\}_{i\le k}, \dots, \{\xi_i^{m,m-2}\}_{i\le k}, \{\xi_i^{m,m-1}\}_{i\le k},\\ \{\eta_i^{m}\}_{i\le k-1},\ \eta_k^{m}-\eta_k^{m-1}, \eta_k^{m-1}-\eta_k^{m-2}, \dots, \eta_k^2-\eta_k^1, \eta_k^1.\end{gathered} \tag{6-11}$$

Thus the induction assumption (6-6) has been advanced from $m-1$ to m. □

Proof of Theorem 3.4. Fix $1 < \rho_1 < \cdots < \rho_n$ and let the multiline configuration $I = (I^0, \dots, I^n)$ have distribution $\nu^{(1,\rho_1,\dots,\rho_n)}$. Let $\eta = (\eta^0, \dots, \eta^n) = \mathcal{D}^{(n+1)}(I)$. By Theorem 3.2 proved above, $(\overline{Y}_t, \overline{B}_t^{\rho_1,\boldsymbol{e}_1}, \dots, \overline{B}_t^{\rho_n,\boldsymbol{e}_1}) \overset{d}{=} \eta \sim \mu^{(1,\rho_1,\dots,\rho_n)}$. Lemma 6.4 gives the independence of the components of the vector

$$(\eta_k^1, \eta_k^2-\eta_k^1, \dots, \eta_k^n-\eta_k^{n-1}) \overset{d}{=} (Y_x, B_{x-\boldsymbol{e}_1,x}^{\rho_1} - Y_x, B_{x-\boldsymbol{e}_1,x}^{\rho_2} - B_{x-\boldsymbol{e}_1,x}^{\rho_1}, \dots, B_{x-\boldsymbol{e}_1,x}^{\rho_n} - B_{x-\boldsymbol{e}_1,x}^{\rho_{n-1}}).$$

(Above $k \in \mathbb{Z}$ and $x \in \mathbb{Z}^2$ are arbitrary.)

The distribution of an increment $\eta_k^m - \eta_k^{m-1}$ can be computed from the 2-component mapping $(\eta^{m-1}, \eta^m) = \mathcal{D}^{(2)}(I^{m-1}, I^m) = (I^{m-1}, D(I^m, I^{m-1}))$ where $(I^{m-1}, I^m) \sim \nu^{\rho_{m-1},\rho_m}$. The first equation of (2-24) gives

$$\eta_k^m - \eta_k^{m-1} = \eta_k^m - I_k^{m-1} = (I_k^m - J_{k-1}^{I^m,I^{m-1}})^+.$$

The right-hand side has the distribution in (3-7) with $(\lambda, \rho) = (\rho_{m-1}, \rho_m)$ because, by the structure of the queueing mapping, I_k^m and $J_{k-1}^{I^m,I^{m-1}}$ are independent exponentials with parameters ρ_m^{-1} and $\rho_{m-1}^{-1} - \rho_m^{-1}$.

A computation of the Laplace transform of the increment $X(\rho) - X(\lambda)$ of the process defined by (3-6) gives, for $\rho > \lambda \ge 1$ and $\alpha > 0$,

$$E[e^{-\alpha(X(\rho)-X(\lambda))}] = \frac{1+\lambda\alpha}{1+\rho\alpha}. \tag{6-12}$$

This is the Laplace transform of the distribution in (3-7). Thus $\eta_k^m - \eta_k^{m-1}$ has the same distribution as $X(\rho_m) - X(\rho_{m-1})$.

To summarize, the nondecreasing cadlag processes $B^{\bullet}_{x-\boldsymbol{e}_1,x}$ and $X(\cdot)$ have identically distributed initial values (both $B^1_{x-\boldsymbol{e}_1,x} = Y_x$ and $X(1)$ are Exp(1)-distributed) and identically distributed independent increments. Hence the processes are equal in distribution. □

6C. *Bivariate Busemann process on a line.* The remainder of this section proves statements for the sequence $\{(B^{\lambda}_{(k-1,t),(k,t)}, B^{\rho}_{(k-1,t),(k,t)})\}_{k\in\mathbb{Z}}$ that has distribution $\mu^{(\lambda,\rho)}$. We use the following notation: Let $\rho > \lambda > 0$, $(I^1, I^2) \sim \nu^{(\lambda,\rho)}$ and $(\eta^1, \eta^2) = \mathcal{D}^{(2)}(I^1, I^2) = (I^1, D(I^2, I^1))$. Then $(\eta^1, \eta^2) \sim \mu^{(\lambda,\rho)}$. Let $J = J^{I^2,I^1} = S(I^2, I^1)$.

Proof of Theorem 3.5. The next auxiliary lemma identifies a reversible Markov chain.

Lemma 6.5. *Let $X_i = J_{i-1} - I_i^2$. Then $\{X_i\}_{i\in\mathbb{Z}}$ and $\{X_i^+\}_{i\in\mathbb{Z}}$ are stationary reversible Markov chains. $\{X_i^-\}_{i\in\mathbb{Z}}$ is not a Markov chain.*

Proof. From the second equation of (2-24),

$$X_{i+1} = J_i - I_{i+1}^2 = I_i^1 + (J_{i-1} - I_i^2)^+ - I_{i+1}^2 = X_i^+ + I_i^1 - I_{i+1}^2.$$

Since J_{i-1} is a function of $(I_k^1, I_k^2)_{k\le i-1}$, X_i is independent of (I_i^1, I_{i+1}^2). Schematically, we can express the transition probability as $X_{i+1} = X_i^+ + \mathrm{Exp}(\lambda^{-1}) - \mathrm{Exp}(\rho^{-1})$, where the three terms on the right-hand side are independent.

Similarly, using conservation (2-25) and the dual equations (4-3),

$$X_i = J_{i-1} - I_i^2 = J_i - \eta_i^2 = \widetilde{\omega}_{i+1} + (J_{i+1} - \eta_{i+1}^2)^+ - \eta_i^2 = X_{i+1}^+ + \widetilde{\omega}_{i+1} - \eta_i^2. \tag{6-13}$$

J_i and η_i^2 are independent by Lemma B.2(a), and hence the triple $(J_i, \eta_i^2, I_{i+1}^2)$ is independent. Consequently so is the triple

$$(X_{i+1}, \widetilde{\omega}_{i+1}, \eta_i^2) = (J_i - I_{i+1}^2, J_i \wedge I_{i+1}^2, \eta_i^2)$$

and we can express (6-13) as $X_i = X_{i+1}^+ + \mathrm{Exp}(\lambda^{-1}) - \mathrm{Exp}(\rho^{-1})$ where again the three terms on the right-hand side are independent. The transitions from X_i to X_{i+1} and back are the same.

From the equations above we obtain equations that show X_i^+ as a reversible Markov chain.

Writing temporarily $U_i = I_{i-1}^1 - I_i^2$, we get these equations for X_{i+1}^-:

$$X_{i+1}^- = (X_i^+ + U_{i+1})^- = \big((X_{i-1}^+ + U_i)^+ + U_{i+1}\big)^-.$$

Conditioned on $X_i \ge 0$, $X_i \sim \mathrm{Exp}(\lambda^{-1} - \rho^{-1})$. Thus

$$\begin{aligned} P(X_{i+1}^- = 0 \mid X_i^- = 0) &= P(X_i^+ + U_{i+1} \ge 0 \mid X_i \ge 0) \\ &= P\{\mathrm{Exp}(\lambda^{-1} - \rho^{-1}) + U_{i+1} \ge 0\}. \end{aligned} \tag{6-14}$$

For the next calculation, note that $X_{i-1} < 0$ implies $X_i = X_{i-1}^+ + U_i = U_i$ and then $X_{i+1} = U_i^+ + U_{i+1}$.

$$\begin{aligned} P(X_{i+1}^- = 0 \mid X_i^- = 0, X_{i-1}^- > 0) &= \frac{P(X_{i+1}^- = 0, X_i^- = 0, X_{i-1}^- > 0)}{P(X_i^- = 0, X_{i-1}^- > 0)} \\ &= \frac{P(U_i + U_{i+1} \geq 0, U_i \geq 0, X_{i-1} < 0)}{P(U_i \geq 0, X_{i-1} < 0)} \\ &= P(U_i + U_{i+1} \geq 0 \mid U_i \geq 0) \\ &= P\{\mathrm{Exp}(\lambda^{-1}) + U_{i+1} \geq 0\}. \end{aligned} \tag{6-15}$$

We used above the independence of X_{i-1} from (U_i, U_{i+1}) and then the conditional distribution $U_i \sim \mathrm{Exp}(\lambda^{-1})$, given that $U_i \geq 0$. The conditional distributions in (6-14) and (6-15) do not agree, and consequently X_i^- is not a Markov chain. □

Since $\eta_k^2 - \eta_k^1 = \eta_k^2 - I_k^1 = (I_k^2 - J_{k-1})^+ = X_k^-$, we conclude that $\eta_k^2 - \eta_k^1$ is not a Markov chain, but it is a function of a reversible Markov chain. Part (a) of Theorem 3.5 has been proved.

We give here two more auxiliary lemmas.

Lemma 6.6. *The process $(\eta_k^1, \eta_k^2)_{k\in\mathbb{Z}}$ is not a Markov chain.*

Proof. The construction gives $\eta_{k+1}^2 = I_{k+1}^1 + (I_{k+1}^2 - J_k)^+$. On the right, the variables I_{k+1}^1 and I_{k+1}^2 are independent and independent of J_k and $(\eta_j^1, \eta_j^2)_{j\leq k}$. The conclusion of the lemma follows from showing that conditioning on $\eta_k^1 = \eta_k^2$ gives J_k an unbounded distribution, while conditioning on $\eta_{k-1}^1 < \eta_{k-1}^2$ and $\eta_k^1 = \eta_k^2$ implies $J_k \leq \eta_{k-1}^1 + \eta_k^1$. Thus conditioning on (η_k^1, η_k^2) does not completely decouple η_{k+1}^2 from the earlier past.

From the three independent variables (J_{k-1}, I_k^1, I_k^2) the queueing formulas define

$$\eta_k^1 = I_k^1, \quad \eta_k^2 = I_k^1 + (I_k^2 - J_{k-1})^+ \quad \text{and} \quad J_k = I_k^1 + (J_{k-1} - I_k^2)^+. \tag{6-16}$$

The condition $\eta_k^1 = \eta_k^2$ is equivalent to $J_{k-1} \geq I_k^2$, and conditioning on this implies

$$J_{k-1} - I_k^2 \sim \mathrm{Exp}(\lambda^{-1} - \rho^{-1}).$$

Thus J_k is unbounded.

For the second scenario consider the five independent variables $(J_{k-2}, I_{k-1}^1, I_{k-1}^2, I_k^1, I_k^2)$ and augment (6-16) with the equations of the prior step:

$$\eta_{k-1}^1 = I_{k-1}^1, \quad \eta_{k-1}^2 = I_{k-1}^1 + (I_{k-1}^2 - J_{k-2})^+ \quad \text{and} \quad J_{k-1} = I_{k-1}^1 + (J_{k-2} - I_{k-1}^2)^+. \tag{6-17}$$

Now $\eta_{k-1}^1 < \eta_{k-1}^2$ implies $J_{k-1} = I_{k-1}^1$ and then $\eta_k^1 = \eta_k^2$ implies $J_k = I_k^1 + J_{k-1} - I_k^2 = I_k^1 + I_{k-1}^1 - I_k^2$. Hence $J_k \leq I_k^1 + I_{k-1}^1 = \eta_k^1 + \eta_{k-1}^1$. The lemma is proved.

With service process $I^1 = \eta^1$, arrival process I^2 and departure process η^2, the queueing explanation of the proof is that $\eta_k^1 = \eta_k^2$ implies that customer k had to wait before entering service, and hence delays from the past can influence the next interdeparture time η_{k+1}^2. □

Lemma 6.7. *The pair $((\eta_k^1, \eta_k^2), (\eta_{k+1}^1, \eta_{k+1}^2))$ and its transpose $((\eta_{k+1}^1, \eta_{k+1}^2), (\eta_k^1, \eta_k^2))$ are not equal in distribution.*

Proof. By the queueing construction, $\eta^1_{k+1} = I^1_{k+1}$ is independent of (η^1_k, η^2_k) because the latter pair is a function of $(I^1_i, I^2_i)_{i\le k}$. To see that $\eta^1_k = I^1_k$ is not independent of $(\eta^1_{k+1}, \eta^2_{k+1})$, write

$$\eta^2_{k+1} - \eta^1_{k+1} = (I^2_{k+1} - J_k)^+ = (I^2_{k+1} - I^1_k - [J_{k-1} - I^2_k]^+)^+$$

where all four variables in the last expression are independent. □

Part (b) of Theorem 3.5 follows from the two lemmas above. □

Proof of Theorem 3.6 and Remark 3.7. Part (a) comes from translating the condition $B^\rho_{(k-1)e_1,ke_1} = Y_{ke_1}$ into a statement about the queueing mapping $\tilde{I} = D(I, \omega)$.

By (2-13),

$$\mathbb{P}\{\xi_x = 0\} = \mathbb{P}\{B^\rho_{x-e_2,x} < B^\rho_{x-e_1,x}\} = 1 - \rho^{-1}$$

from the independence and exponential distributions in Theorem 2.1(ii). From (3-7),

$$\mathbb{P}\{\xi^{\lambda,\rho}_x = 0\} = \mathbb{P}\{B^\rho_{x-e_1,x} > B^\lambda_{x-e_1,x}\} = \frac{\rho-\lambda}{\rho}.$$

To calculate $\mathbb{P}\{\xi^{\lambda,\rho}_x = n\}$ for $n \ge 1$ we put x on the x-axis and use the distribution $(\bar{B}^{\lambda,e_1}_0, \bar{B}^{\rho,e_1}_0) \overset{d}{=} (\eta^1, \eta^2) \sim \mu^{(\lambda,\rho)}$ given by Theorem 3.2, with the notation from the start of Section 6C. By setting $\lambda = 1$ the same calculation gives $\mathbb{P}\{\xi_x = n\}$ because $\overline{Y}_0 = \bar{B}^{1,e_1}_0$.

$$\begin{aligned}
\mathbb{P}\{\xi^{\lambda,\rho}_{ne_1} = n\} &= \mathbb{P}\{B^\rho_{-e_1,0} > B^\lambda_{-e_1,0},\, B^\rho_{0,e_1} = B^\lambda_{0,e_1},\, B^\rho_{e_1,2e_1} = B^\lambda_{e_1,2e_1}, \dots, B^\rho_{(n-1)e_1,ne_1} = B^\lambda_{(n-1)e_1,ne_1}\} \\
&= P\{\eta^2_0 > I^1_0,\, \eta^2_1 = I^1_1,\, \eta^2_2 = I^1_2, \dots, \eta^2_n = I^1_n\} \\
&= P\{I^2_0 > J_{-1},\, I^2_1 \le J_0,\, I^2_2 \le J_1, \dots, I^2_n \le J_{n-1}\}
\end{aligned} \tag{6-18}$$

The last equality used $\eta^2_i = I^1_i + (I^2_i - J_{i-1})^+$ repeatedly: $\eta^2_i > I^1_i$ is equivalent to $I^2_i > J_{i-1}$.

Next apply repeatedly the equation $J_i = I^1_i + (J_{i-1} - I^2_i)^+$ inside the last probability in (6-18). $I^2_0 > J_{-1}$ implies $J_0 = I^1_0$. Then $I^2_1 \le J_0$ implies $J_1 = I^1_1 + J_0 - I^2_1 = I^1_1 + I^1_0 - I^2_1$. Assume inductively that

$$J_i = I^1_i + \dots + I^1_0 - I^2_1 - \dots - I^2_i. \tag{6-19}$$

Then $I^2_{i+1} \le J_i$ implies

$$J_{i+1} = I^1_{i+1} + J_i - I^2_{i+1} = I^1_{i+1} + (I^1_i + \dots + I^1_0 - I^2_1 - \dots - I^2_i) - I^2_{i+1}$$

and the induction goes from i to $i+1$. Substitute (6-19) for $J_0, \dots, J_{n-1}$ in the last probability in (6-18). Use the independence of the variables $J_{-1}, \{I^1_i, I^2_i\}_{i\ge 0}$. Let S^α_m denote the sum of m i.i.d. Exp(α) random variables, with S and $\widetilde{S}$ denoting independent sums.

$$\begin{aligned}
\mathbb{P}\{\xi^{\lambda,\rho}_{ne_1} = n\} &= P\{I^2_0 > J_{-1},\, I^2_1 \le I^1_0,\, I^2_2 + I^2_1 \le I^1_1 + I^1_0, \dots, I^2_n + \dots + I^2_1 \le I^1_{n-1} + \dots + I^1_0\} \\
&= P\{I^2_0 > J_{-1}\}\, P\{S^{\rho^{-1}}_m \le \widetilde{S}^{\lambda^{-1}}_m\ \forall m \in [n]\} = \frac{\rho-\lambda}{\rho} \sum_{k=0}^{n-1} C(n-1,k) \frac{\rho^k \lambda^n}{(\lambda+\rho)^{n+k}}.
\end{aligned} \tag{6-20}$$

The last line comes from the independence of I^2_0 and J_{-1}, their distributions $I^2_0 \sim \text{Exp}(\rho^{-1})$ and $J_{-1} \sim \text{Exp}(\lambda^{-1} - \rho^{-1})$, and Lemma B.3. □

Appendix A. Queues

We prove elementary lemmas about the queueing mappings. Unless otherwise stated, the weights are real numbers without any probability distributions.

Lemma A.1. *Fix $0 \le a < b$. Let $I = (I_k)_{k\in\mathbb{Z}}$ and $\omega = (\omega_j)_{j\in\mathbb{Z}}$ in $\mathbb{R}^{\mathbb{Z}}_{\ge 0}$ satisfy*

$$\varliminf_{m\to-\infty} \frac{1}{|m|}\sum_{i=m}^{0} I_i \ge b \quad \textit{and} \quad \lim_{m\to-\infty} \frac{1}{|m|}\sum_{i=m}^{0} \omega_i = a. \tag{A-1}$$

Then $\tilde I = D(I,\omega)$ is well-defined and satisfies $\varliminf_{m\to-\infty} |m|^{-1}\sum_{i=m}^{0} \tilde I_i \ge b$.

Proof. Assumption (2-17) is obviously satisfied. Without loss of generality assume $G_0 = 0$. Let $0 < \varepsilon < (b-a)/3$. Then for large enough n,

$$\begin{aligned}\widetilde G_{-n} &= \sup_{k:\,k\le -n}\Big\{G_k + \sum_{i=k}^{-n}\omega_i\Big\} = \sup_{k:\,k\le -n}\Big\{-\sum_{i=k+1}^{0} I_i + \sum_{i=k}^{0}\omega_i\Big\} - \sum_{i=-n+1}^{0}\omega_i \\ &\le \sup_{k:\,k\le -n}\big\{-|k|(b-\varepsilon) + |k|(a+\varepsilon)\big\} - n(a-\varepsilon) = n(-b+3\varepsilon).\end{aligned}$$

Since $\sum_{i=m+1}^{0}\tilde I_i = \widetilde G_0 - \widetilde G_m$, this proves $\varliminf_{m\to-\infty}|m|^{-1}\sum_{i=m}^{0}\tilde I_i \ge b$. $\square$

Lemma A.2. *Fix $0 \le a < b$. Assume given nonnegative real sequences $I = (I_i)_{i\in\mathbb{Z}}$, $\omega = (\omega_i)_{i\in\mathbb{Z}}$, $I^{(h)} = (I_i^{(h)})_{i\in\mathbb{Z}}$ and $\omega^{(h)} = (\omega_i^{(h)})_{i\in\mathbb{Z}}$ where $h \in \mathbb{Z}_{>0}$ is an index. Assume $I_i^{(h)} \to I_i$ and $\omega_i^{(h)} \to \omega_i$ as $h\to\infty$ for all $i \in \mathbb{Z}$, and furthermore,*

$$\varlimsup_{\substack{m\to-\infty\\ h\to\infty}} \left|\frac{1}{|m|}\sum_{i=m}^{0} I_i^{(h)} - b\right| = 0 \quad \textit{and} \quad \varlimsup_{\substack{m\to-\infty\\ h\to\infty}} \left|\frac{1}{|m|}\sum_{i=m}^{0} \omega_i^{(h)} - a\right| = 0. \tag{A-2}$$

Then $\tilde I = D(I,\omega)$ and $\widetilde\omega = R(I,\omega)$ are well-defined, as are $\tilde I^{(h)} = D(I^{(h)},\omega^{(h)})$ and $\widetilde\omega^{(h)} = R(I^{(h)},\omega^{(h)})$ for large enough h. We have the limits

$$\lim_{h\to\infty} \tilde I_i^{(h)} = \tilde I_i \quad \textit{and} \quad \lim_{h\to\infty} \widetilde\omega_i^{(h)} = \widetilde\omega_i \qquad \textit{for all } i\in\mathbb{Z} \tag{A-3}$$

and

$$\varlimsup_{\substack{m\to-\infty\\ h\to\infty}} \left|\frac{1}{|m|}\sum_{i=m}^{0} \tilde I_i^{(h)} - b\right| = 0 \quad \textit{and} \quad \varlimsup_{\substack{m\to-\infty\\ h\to\infty}} \left|\frac{1}{|m|}\sum_{i=m}^{0} \widetilde\omega_i^{(h)} - a\right| = 0. \tag{A-4}$$

Proof. Assumption (2-17) is satisfied to make $\tilde I^{(h)} = D(I^{(h)},\omega^{(h)})$ well-defined for large enough h.

We can assume $G_0^{(h)} = 0$. Compute $\tilde I^{(h)} = D(I^{(h)},\omega^{(h)})$ as the increments of the function

$$\widetilde G_\ell^{(h)} = \sup_{k\le\ell}\Big\{G_k^{(h)} + \sum_{i=k}^{\ell}\omega_i^{(h)}\Big\}. \tag{A-5}$$

Let k_0 be a maximizer in (2-18) for $\widetilde{G}_\ell$. Then

$$\varliminf_{h\to\infty} \widetilde{G}_\ell^{(h)} \geq \varliminf_{h\to\infty} \Big\{ G_{k_0}^{(h)} + \sum_{i=k_0}^{\ell} \omega_i^{(h)} \Big\} = G_{k_0} + \sum_{i=k_0}^{\ell} \omega_i = \widetilde{G}_\ell. \tag{A-6}$$

Let $k(h)$ be a maximizer in (A-5). If $\varlimsup_{h\to\infty} \widetilde{G}_\ell^{(h)} \leq \widetilde{G}_\ell$ fails then it must be that $k(h) \to -\infty$ along a subsequence. But we can write

$$\begin{aligned} \widetilde{G}_\ell^{(h)} &= G_{k(h)}^{(h)} + \sum_{i=k(h)}^{\ell} \omega_i^{(h)} \\ &= -\sum_{i=k(h)+1}^{0} I_i^{(h)} + \sum_{i=k(h)}^{0} \omega_i^{(h)} + \Big(\mathbf{1}_{\ell>0} \sum_{i=1}^{\ell} \omega_i^{(h)} - \mathbf{1}_{\ell<0} \sum_{i=\ell+1}^{0} \omega_i^{(h)} \Big) \end{aligned} \tag{A-7}$$

which converges to $-\infty$ as $k(h) \to -\infty$ by the assumptions and thereby contradicts (A-6). We have now proved that

$$\lim_{h\to\infty} \widetilde{G}_\ell^{(h)} = \widetilde{G}_\ell \qquad \text{for all } \ell \in \mathbb{Z} \tag{A-8}$$

and thereby verified (A-3) for $\tilde{I}^{(h)}$.

Let $0 < \varepsilon < (b-a)/3$. By assumption (A-2) there exist finite $n_1(\varepsilon)$ and $h_1(\varepsilon)$ such that, when $n \geq n_1(\varepsilon)$ and $h \geq h_1(\varepsilon)$,

$$\begin{aligned} \widetilde{G}_{-n}^{(h)} &= \sup_{k:\,k\leq -n} \Big\{ G_k^{(h)} + \sum_{i=k}^{-n} \omega_i^{(h)} \Big\} = \sup_{k:\,k\leq -n} \Big\{ -\sum_{i=k+1}^{0} I_i^{(h)} + \sum_{i=k}^{0} \omega_i^{(h)} \Big\} - \sum_{i=-n+1}^{0} \omega_i^{(h)} \\ &\leq \sup_{k:\,k\leq -n} \big\{ -|k|(b-\varepsilon) + |k|(a+\varepsilon) \big\} - n(a-\varepsilon) = n(-b+3\varepsilon). \end{aligned}$$

From this,

$$\varlimsup_{m\to-\infty} \sup_{h\geq h_1(\varepsilon)} \frac{\widetilde{G}_m^{(h)}}{|m|} \leq -b + 3\varepsilon. \tag{A-9}$$

Since $\sum_{i=m}^{0} \tilde{I}_i^{(h)} = \widetilde{G}_0^{(h)} - \widetilde{G}_{m-1}^{(h)}$ and $\widetilde{G}_0^{(h)} \geq \omega_0^{(h)} \geq 0$, this proves

$$\varliminf_{m\to-\infty} \inf_{h\geq h_1(\varepsilon)} \frac{1}{|m|} \sum_{i=m}^{0} \tilde{I}_i^{(h)} \geq b - 3\varepsilon. \tag{A-10}$$

For the complementary upper bound, get a lower bound for $\widetilde{G}_{m-1}^{(h)}$ by taking $k = \ell$ in (A-5).

$$\sum_{i=m}^{0} \tilde{I}_i^{(h)} = \widetilde{G}_0^{(h)} - \widetilde{G}_{m-1}^{(h)} \leq \widetilde{G}_0^{(h)} - G_{m-1}^{(h)} = \widetilde{G}_0^{(h)} + \sum_{i=m}^{0} I_i^{(h)}.$$

Apply limit (A-8) and assumption (A-2). Limit (A-4) has been proved for $\tilde{I}^{(h)}$.

The limits for $\widetilde{\omega}^{(h)}$ follow from the other limits and the generally valid identity

$$\omega_k + I_k = \widetilde{\omega}_k + \tilde{I}_k \tag{A-11}$$

that comes from equations (2-21) and (2-24). ☐

For reference elsewhere in the paper we state the simple consequence of Lemma A.2 where the sequences are constant functions of h.

Lemma A.3. *Fix $0 \le a < b$. Let $I = (I_k)_{k\in\mathbb{Z}}$ and $\omega = (\omega_j)_{j\in\mathbb{Z}}$ in $\mathbb{R}^{\mathbb{Z}}_{\ge 0}$ satisfy*

$$\lim_{m\to-\infty} \frac{1}{|m|} \sum_{i=m}^{0} I_i = b \quad \text{and} \quad \lim_{m\to-\infty} \frac{1}{|m|} \sum_{i=m}^{0} \omega_i = a. \tag{A-12}$$

Then $\tilde{I} = D(I,\omega)$ and $\widetilde{\omega} = R(I,\omega)$ are well-defined and satisfy

$$\lim_{m\to-\infty} \frac{1}{|m|} \sum_{i=m}^{0} \tilde{I}_i = b \quad \text{and} \quad \lim_{m\to-\infty} \frac{1}{|m|} \sum_{i=m}^{0} \widetilde{\omega}_i = a. \tag{A-13}$$

For the purpose of verifying that Busemann functions obey the queueing operation $\tilde{I} = D(I,\omega)$, it is convenient to have a lemma that deduces this from assuming the iterative equations (2-24). The first lemma below makes a statement without randomness.

Lemma A.4. *Let $\{\tilde{I}_k, J_k, I_k, \omega_k\}_{k\in\mathbb{Z}}$ be nonnegative real numbers that satisfy the three assumptions below*:

$$\lim_{m\to-\infty} \sum_{i=m}^{0} (\omega_i - I_{i+1}) = -\infty. \tag{A-14}$$

$$\tilde{I}_k = \omega_k + (I_k - J_{k-1})^+ \quad \text{and} \quad J_k = \omega_k + (J_{k-1} - I_k)^+ \quad \text{for all } k \in \mathbb{Z}. \tag{A-15}$$

$$J_k = \omega_k \quad \text{for infinitely many } k < 0. \tag{A-16}$$

Then $\tilde{I} = D(I,\omega)$ and $J = S(I,\omega)$.

Proof. Rewrite the second equation of (A-15) as follows. Let $W_k = J_k - \omega_k$ and $U_k = \omega_k - I_{k+1}$. Then

$$W_k = (W_{k-1} + U_{k-1})^+. \tag{A-17}$$

This is Lindley's recursion from queueing theory and W_k is the *waiting time* of customer k. Equation (A-17) iterates inductively to give

$$W_k = \biggl\{ \Bigl(W_\ell + \sum_{i=\ell}^{k-1} U_i \Bigr) \bigvee \Bigl(\max_{m:\, \ell+1 \le m \le k-1} \sum_{i=m}^{k-1} U_i \Bigr) \biggr\}^+ \qquad \text{for all } \ell < k. \tag{A-18}$$

We claim that

$$W_k = \Bigl(\sup_{m:\, m \le k-1} \sum_{i=m}^{k-1} U_i \Bigr)^+ \qquad \text{for all } k \in \mathbb{Z}. \tag{A-19}$$

Dropping the first term on the right in (A-18) and letting $\ell \to -\infty$ gives $\ge$ in (A-19). By assumption (A-16) $W_\ell = 0$ for some $\ell < k$. Then (A-18) gives also $\le$ in (A-19).

The proof is completed by making explicit the content of (A-19). Let G and $\widetilde{G}$ be as defined in the definition of the mappings D and S. Then from (A-19) and (2-23) deduce

$$\begin{aligned} J_k &= \omega_k + \Big(\sup_{m:\, m \le k-1} \sum_{i=m}^{k-1} (\omega_i - I_{i+1}) \Big)^+ = \omega_k + \Big(\sup_{m:\, m \le k-1} \Big\{ G_m - G_k + \sum_{i=m}^{k-1} \omega_i \Big\} \Big)^+ \\ &= \omega_k + (\widetilde{G}_{k-1} - G_k)^+ = \omega_k + \widetilde{G}_{k-1} \vee G_k - G_k = \widetilde{G}_k - G_k. \end{aligned}$$

Thus $J = S(I, \omega)$. Finally, from the first equation of (A-15) and $I_k = G_k - G_{k-1}$,

$$\begin{aligned} \tilde{I}_k &= \omega_k + (I_k - J_{k-1})^+ = I_k - J_{k-1} + \omega_k + (J_{k-1} - I_k)^+ = I_k + J_k - J_{k-1} \\ &= I_k + (\widetilde{G}_k - G_k) - (\widetilde{G}_{k-1} - G_{k-1}) = \widetilde{G}_k - \widetilde{G}_{k-1}. \end{aligned}$$ □

Here is a version for a random sequence.

Lemma A.5. *Let $\{\tilde{I}_k, J_k, I_k, \omega_k\}_{k \in \mathbb{Z}}$ be finite nonnegative random variables that satisfy assumptions* (i)–(iii) *below*:

(i) $\lim_{m \to -\infty} \sum_{i=m}^{0} (\omega_i - I_{i+1}) = -\infty$ *almost surely.*

(ii) $\{J_k, \omega_k\}_{k \in \mathbb{Z}}$ *is a stationary process.*

(iii) *Equations*

$$\tilde{I}_k = \omega_k + (I_k - J_{k-1})^+ \quad \textit{and} \quad J_k = \omega_k + (J_{k-1} - I_k)^+ \tag{A-20}$$

are valid for all $k \in \mathbb{Z}$, almost surely.

Then $J_k = \omega_k$ for infinitely many $k < 0$ with probability one, and $\tilde{I} = D(I, \omega)$ and $J = S(I, \omega)$ almost surely.

Proof. Lemma A.4 gives the conclusion once we verify that assumption (A-16) holds almost surely. Using the waiting time notation W from the previous proof, it suffices to show that

$$\mathbb{P}\{W_\ell = 0 \text{ for infinitely many } \ell < 0\} = 1 \tag{A-21}$$

The complementary event is $B = \{\text{there exists } m < 0 \text{ such that } W_k > 0 \text{ for all } k \le m\}$. B is a shift-invariant event. On the event B, the right-hand side of (A-17) is strictly positive for all $k \le m$ (for a random m). This implies, for all $k < m$,

$$0 < W_m = W_{m-1} + U_{m-1} = W_{m-2} + U_{m-2} + U_{m-1} = \cdots = W_k + \sum_{i=k}^{m-1} U_i = W_k + \sum_{i=k}^{m-1} (\omega_i - I_{i+1}).$$

By assumption (i) of the lemma, $W_k \to \infty$ a.s. on the event B as $k \to -\infty$. Let $c < \infty$. By the shift-invariance of B and the stationarity of the process $\{W_k = J_k - \omega_k\}_{k \in \mathbb{Z}}$,

$$\mathbb{P}(W_0 \ge c, B) = \mathbb{P}(W_k \ge c, B) \to \mathbb{P}(B) \quad \text{as } k \to -\infty.$$

We conclude that $W_0 = \infty$ a.s. on the event B, and hence $\mathbb{P}(B) = 0$. Claim (A-21) has been verified. □

Remark A.6 (nonstationary solution to Lindley's recursion). Some result such as Lemma A.5 is needed, for there can be another solution to Lindley's recursion that blows up as $n \to -\infty$. Suppose $\{U_k\}$ is ergodic and $\mathbb{E}U_k < 0$. Pick any random N such that $\sum_{k=m}^{N} U_k < 0$ for all $m \le N$. Set

$$\begin{aligned} W_n &= -\sum_{k=n}^{N} U_k && \text{for } n \le N, \\ W_{N+1} &= 0, \\ W_n &= (W_{n-1} + U_{n-1})^+ && \text{for } n \ge N+2. \end{aligned}$$

One can check that $W_n = (W_{n-1} + U_{n-1})^+$ holds for all $n \in \mathbb{Z}$.

Appendix B. Exponential distributions

The next lemma is elementary. The mapping $(I, J, W) \mapsto (I', J', W')$ in the lemma is an involution, that is, its own inverse.

Lemma B.1. *Let $\alpha, \beta > 0$. Assume given independent variables $W \sim \mathrm{Exp}(\alpha+\beta)$, $I \sim \mathrm{Exp}(\alpha)$, and $J \sim \mathrm{Exp}(\beta)$. Define*

$$\begin{aligned} I' &= W + (I - J)^+, \\ J' &= W + (I - J)^-, \\ W' &= I \wedge J. \end{aligned} \tag{B-1}$$

(i) *$I - J$ and $I \wedge J$ are independent.*

(ii) *$(I - J)^+ \sim \mathrm{Ber}(\beta/(\alpha+\beta)) \cdot \mathrm{Exp}(\alpha)$, that is, the product of a Bernoulli with success probability $\beta/(\alpha+\beta)$ and an independent rate α exponential.*

(iii) *The triple (I', J', W') has the same distribution as (I, J, W).*

We use the previous lemma to establish some facts about the queueing operators. To be consistent with the queueing discussion we parametrize exponentials with their means τ and ρ.

Lemma B.2. *Let $0 < \tau < \rho$. Let $(I_k)_{k\in\mathbb{Z}}$ and $(\omega_j)_{j\in\mathbb{Z}}$ be mutually independent random variables such that $I_k \sim \mathrm{Exp}(\rho^{-1})$ and $\omega_j \sim \mathrm{Exp}(\tau^{-1})$. Let $\tilde{I} = D(I, \omega)$ as defined by (2-18) and (2-19), $\widetilde{\omega} = R(I, \omega)$ as defined by (2-21), and $J_k = \widetilde{G}_k - G_k$ as in (2-20). Let $\Lambda_k = (\{\tilde{I}_j\}_{j\le k},\ J_k,\ \{\widetilde{\omega}_j\}_{j\le k})$.*

(a) *$\{\Lambda_k\}_{k\in\mathbb{Z}}$ is a stationary, ergodic process. For each $k \in \mathbb{Z}$, the random variables $\{\tilde{I}_j\}_{j\le k},\ J_k,\ \{\widetilde{\omega}_j\}_{j\le k}$ are mutually independent with marginal distributions*

$$\tilde{I}_j \sim \mathrm{Exp}(\rho^{-1}),\ \ \widetilde{\omega}_j \sim \mathrm{Exp}(\tau^{-1}) \text{ and } J_k \sim \mathrm{Exp}(\tau^{-1} - \rho^{-1}).$$

(b) *$\tilde{I}$ and $\widetilde{\omega}$ are independent sequences of i.i.d. variables.*

Proof. Part (b) follows from part (a) by dropping the J_k coordinate and letting $k \to \infty$. Stationarity and ergodicity of $\{\Lambda_k\}$ follow from its construction as a mapping applied to the independent i.i.d. sequences I and ω.

The distributional claims in part (a) are proved by coupling $(\tilde I_k, J_{k-1}, \widetilde\omega_k)_{k\in\mathbb{Z}}$ with another sequence whose distribution we know. Construct a process $(\widehat I_k, \widehat J_{k-1}, \widehat\omega_k)_{k\ge 1}$ as follows. First let $\widehat J_0$ be an $\text{Exp}(\tau^{-1}-\rho^{-1})$ variable that is independent of (I,ω). Then for $k=1,2,3,\dots$, iterate the steps

$$\begin{aligned}\widehat I_k &= \omega_k + (I_k - \widehat J_{k-1})^+,\\ \widehat J_k &= \omega_k + (\widehat J_{k-1} - I_k)^+,\\ \widehat\omega_k &= I_k \wedge \widehat J_{k-1}.\end{aligned} \tag{B-2}$$

We prove the following claim by induction for each $m \ge 1$:

$$\begin{aligned}&\text{The variables } \widehat I_1,\dots,\widehat I_m, \widehat J_m, \widehat\omega_1,\dots,\widehat\omega_m \text{ are mutually independent,}\\ &\qquad\text{with marginal distributions } \widehat I_k \sim \text{Exp}(\rho^{-1}),\ \widehat J_m \sim \text{Exp}(\tau^{-1}-\rho^{-1}) \text{ and } \widehat\omega_j \sim \text{Exp}(\tau^{-1}).\end{aligned} \tag{B-3}$$

By construction, the variables $(I_1, \widehat J_0, \omega_1)$ are independent with distributions

$$(\text{Exp}(\rho^{-1}),\ \text{Exp}(\tau^{-1}-\rho^{-1}),\ \text{Exp}(\tau^{-1})).$$

The base case $m=1$ of (B-3) comes by applying Lemma B.1 to the mapping (B-2) with $k=1$. Now assume (B-3) holds for m. Then $(I_{m+1}, \widehat J_m, \omega_{m+1})$ are independent with distributions

$$(\text{Exp}(\rho^{-1}),\ \text{Exp}(\tau^{-1}-\rho^{-1}),\ \text{Exp}(\tau^{-1}))$$

because, by construction, $\widehat J_m$ is a function of $(I_1,\dots,I_m,\widehat J_0,\omega_1,\dots,\omega_m)$ and thereby independent of (I_{m+1},ω_{m+1}). By Lemma B.1, mapping (B-2) turns the triple $(I_{m+1},\widehat J_m,\omega_{m+1})$ into the triple $(\widehat I_{m+1},\widehat J_{m+1},\widehat\omega_{m+1})$ of independent variables, which is also independent of $\widehat I_1,\dots,\widehat I_m,\widehat\omega_1,\dots,\widehat\omega_m$. Statement (B-3) has been extended to $m+1$.

Our next claim is as follows:

$$\text{There exists (almost surely a random index) } m_0 \ge 0 \text{ such that } J_{m_0} = \widehat J_{m_0}. \tag{B-4}$$

Suppose first that $J_0 \ge \widehat J_0$. Then (2-24) and (B-2) imply that $J_k \ge \widehat J_k$ for all $k \ge 0$. If (B-4) fails then $J_k > \widehat J_k$ for all $k\ge 0$. But then for all $k>0$,

$$J_k = J_{k-1} + \omega_k - I_k = \cdots = J_0 + \sum_{j=1}^{k}(\omega_j - I_j) \to -\infty \quad \text{almost surely, as } k\to\infty,$$

which contradicts the fact that $J_k \ge 0$ for all k. Thus in this case (B-4) happens. The case $J_0 \le \widehat J_0$ is symmetric.

Through equations (2-24) and (B-2), (B-4) implies that $\tilde I_k = \widehat I_k$, $J_k = \widehat J_k$, and $\widetilde\omega_k = \widehat\omega_k$ for all $k > m_0$. Part (a) follows from (B-3), because for any ℓ, $(\tilde I_{\ell-n},\dots,\tilde I_\ell, J_\ell, \widetilde\omega_{\ell-n},\dots,\widetilde\omega_\ell)$ has the same distribution as $(\tilde I_{k-n},\dots,\tilde I_k, J_k,\widetilde\omega_{k-n},\dots,\widetilde\omega_k)$ which agrees with $(\widehat I_{k-n},\dots,\widehat I_k,\widehat J_k,\widehat\omega_{k-n},\dots,\widehat\omega_k)$ with probability tending to one as $k\to\infty$. $\square$

Next we compute a competition probability for two independent homogeneous Poisson processes on $[0,\infty)$ with rates α and β. Let $\{\sigma_i\}_{i\ge1}$ be the jump times of the rate α Poisson process and $\{\tau_i\}_{i\ge1}$ the jump times of the rate β Poisson process. For $n \geq 1$ define the events

$$A_n = \{\sigma_i < \tau_i \text{ for all } i \in [n]\},$$
$$B_n = \{\sigma_i < \tau_i \text{ for all } i \in [n-1],\ \sigma_n > \tau_n\}.$$

The *Catalan numbers* $\{C_n : n \geq 0\}$ are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{B-5}$$

The following properties of the Catalan triangle $\{C(n,k) : 0 \leq k \leq n\}$ given in (3-10) can be deduced with elementary arguments. $C(n,0)=1$, $C(n,k)=\binom{n+k}{k}-\binom{n+k}{k-1}$ for $k>0$, $C(n,k)=C(n,k-1)+C(n-1,k)$,

$$\sum_{k=0}^{i} C(n,k) = C(n+1,i) \quad \text{for } 0 \leq i \leq n, \tag{B-6}$$

and

$$\sum_{k=0}^{n} C(n,k) = C(n+1,n) = C(n+1,n+1) = C_{n+1}. \tag{B-7}$$

Lemma B.3. *For $n \geq 1$,*

$$P(A_n) = \sum_{k=0}^{n-1} C(n-1,k)\,\frac{\alpha^n\beta^k}{(\alpha+\beta)^{n+k}}, \tag{B-8}$$

$$P(B_n) = C_{n-1}\,\frac{\alpha^{n-1}\beta^n}{(\alpha+\beta)^{2n-1}}. \tag{B-9}$$

Remark B.4. The generating function of the Catalan numbers is

$$f(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x} \qquad \text{for } |x| \leq \tfrac{1}{4}.$$

Hence from (B-9),

$$\sum_{n=1}^{\infty} P(B_n) = \frac{\beta}{\alpha+\beta}\, f\!\left(\frac{\alpha\beta}{(\alpha+\beta)^2}\right) = \begin{cases} 1 & \text{if } \beta \geq \alpha, \\ \frac{\beta}{\alpha} & \text{if } \beta < \alpha. \end{cases}$$

In other words, the rate α process stays forever ahead of the rate β process with probability $(1-\beta/\alpha)^+$.

Proof. We compute $P(B_n)$ first and then obtain $P(A_n)$ by inclusion-exclusion.

Since $C_0 = 1$, (B-9) holds for $n = 1$. For $n \geq 2$ condition on (σ_n, τ_n):

$$P(B_n) = \int_{a>b>0} P_{(a,b)}\big\{U_i \leq V_i \text{ for } i \in [n-1]\big\}\, P((\sigma_n,\tau_n) \in d(a,b)), \tag{B-10}$$

where under $P_{(a,b)}$, $0 < U_1 < \cdots < U_{n-1}$ are the order statistics of $n-1$ i.i.d. uniform random variables on $[0,a]$ and $0 < V_1 < \cdots < V_{n-1}$ are the same on $[0,b]$, independent of the $\{U_i\}$. We calculate the probability inside the integral.

Below, first use the equal probability of the permutations of $\{x_i\}$ among themselves and $\{y_j\}$ among themselves. Note that $a > b$ and the conditions $x_i < y_i$ force all $\{x_i, y_j\}$ to lie in $[0, b]$. Then use the equal probability of all permutations of $\{x_i, y_j\}$ together. The Catalan number C_k is the number of permutations of $\{x_1, \dots, x_k, y_1, \dots, y_k\}$ such that $x_1 < \cdots < x_k,\ y_1 < \cdots < y_k$ and $x_i < y_i$ for all i (see Corollary 6.2.3 and item **dd** on page 223 of [Stanley 1999]).

$$P_{(a,b)}(U_i \le V_i \text{ for } i \in [n-1]) = \frac{((n-1)!)^2}{(ab)^{n-1}} \int\limits_{\substack{x_1<\cdots<x_{n-1}<a \\ y_1<\cdots<y_{n-1}<b}} \mathbf{1}_{x_i<y_i\ \forall i\in[n-1]}\, d\boldsymbol{x}\, d\boldsymbol{y}$$

$$= C_{n-1} \frac{((n-1)!)^2}{(ab)^{n-1}} \int\limits_{x_1<\cdots<x_{n-1}<y_1<\cdots<y_{n-1}<b} d\boldsymbol{x}\, d\boldsymbol{y} = C_{n-1} \frac{((n-1)!)^2}{(ab)^{n-1}} \cdot \frac{b^{2(n-1)}}{(2n-2)!}.$$

Substitute this back into (B-10). Use the gamma densities of σ_n and τ_n.

$$P(B_n) = C_{n-1} \frac{((n-1)!)^2}{(2n-2)!} \int_{0<b<a<\infty} \frac{b^{n-1}}{a^{n-1}} \cdot \frac{(\alpha a)^{n-1}}{\Gamma(n)} \alpha e^{-\alpha a} \cdot \frac{(\beta b)^{n-1}}{\Gamma(n)} \beta e^{-\beta b}\, da\, db$$

$$= C_{n-1} \frac{\alpha^n \beta^n}{(2n-2)!} \int_{0<b<a<\infty} b^{2n-2}\, e^{-\alpha a - \beta b}\, da\, db \;=\; C_{n-1} \frac{\alpha^{n-1}\, \beta^n}{(\beta+\alpha)^{2n-1}}.$$

We prove (B-8). The case $n = 1$ is elementary. Let $n \ge 2$ and assume (B-8) for $n - 1$. Abbreviate $p = \beta/(\alpha+\beta)$ and $q = \alpha/(\alpha+\beta)$. Use (B-6) and (B-7) below.

$$P(A_n) = P(A_{n-1}) - P(B_n) = q^{n-1} \sum_{k=0}^{n-2} C(n-2, k)\, p^k - C_{n-1} q^{n-1}\, p^n$$

$$= q^{n-1} \sum_{k=0}^{n-2} C(n-2, k)\, (p^k - p^n) = q^n \sum_{k=0}^{n-2} \sum_{j=k}^{n-1} C(n-2, k)\, p^j$$

$$= q^n \sum_{j=0}^{n-1} \sum_{k=0}^{j\wedge(n-2)} C(n-2, k)\, p^j = q^n \sum_{j=0}^{n-1} C(n-1, j\wedge(n-2))\, p^j = q^n \sum_{j=0}^{n-1} C(n-1, j) p^j. \quad \square$$

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