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**OPTIMAL LOWER BOUND ON THE LEAST SINGULAR VALUE OF
THE SHIFTED GINIBRE ENSEMBLE**

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We consider the least singular value of a large random matrix with real or complex i.i.d. Gaussian entries shifted by a constant $z \in \mathbb{C}$. We prove an optimal lower tail estimate on this singular value in the critical regime where z is around the spectral edge, thus improving the classical bound of Sankar, Spielman and Teng (*SIAM J. Matrix Anal. Appl.* **28**:2 (2006), 446–476) for the particular shift-perturbation in the edge regime. Lacking Brézin–Hikami formulas in the real case, we rely on the superbosonization formula (*Comm. Math. Phys.* **283**:2 (2008), 343–395).

1. Introduction

The effective numerical solvability of a large system of linear equations $Ax = b$ is determined by the condition number of the matrix A . In many practical applications the norm of A is bounded and thus the condition number critically depends on the smallest singular value $\sigma_1(A)$ of A . When the matrix elements of A come from noisy measured data, then the lower tail probability of $\sigma_1(A)$ tends to exhibit a universal scaling behavior, depending on the variance of the noise. In the simplest case A can be decomposed as

$$A = A_0 + X, \quad (1)$$

where A_0 is a deterministic square matrix and X is drawn from the *Ginibre ensemble*, i.e., X has i.i.d. centered Gaussian matrix elements with variance $E|x_{ij}|^2 = N^{-1}$, where N is the dimension.

The randomness in X smoothen out possible singular behavior of A^{-1} . In particular Sankar, Spielman and Teng [Sankar et al. 2006] showed that the smallest singular value $\sigma_1(A)$ lives on a scale not smaller than N^{-1} , equivalently, the smallest eigenvalue $\lambda_1(AA^*)$ of AA^* lives on a scale $\leq N^{-2}$, i.e.,

$$P(\lambda_1(AA^*) = [\sigma_1(A)]^2 \leq xN^{-2}) \lesssim \sqrt{x}, \quad \text{for any } x > 0, \quad (2)$$

up to logarithmic corrections, uniformly in A_0 . If X is a complex Ginibre matrix, then the $\sqrt{x}$ bound improves to x .

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The special case $A_0 = 0$ shows that the bound (2) is essentially optimal. Indeed, the tail probability of $\lambda_1(XX^*)$ of real and complex Ginibre ensembles has been explicitly computed by Edelman [1988] as

$$\lim_{N \rightarrow \infty} \mathbf{P}(\lambda_1(XX^*) \leq xN^{-2}) = \begin{cases} 1 - e^{-x/2 - \sqrt{x}} = \sqrt{x} + \mathcal{O}(x) & \text{in the real case,} \\ 1 - e^{-x} = x + \mathcal{O}(x^2) & \text{in the complex case.} \end{cases} \quad (3)$$

The complex Ginibre ensemble has a stronger smoothing effect in (3) due to the additional degrees of freedom. This observation is analogous to the different strength of the level repulsion in real symmetric and complex Hermitian random matrices.

The support of the spectrum of such *information plus noise matrices* AA^* becomes deterministic as $N \rightarrow \infty$ and it can be computed from the solution of a certain self-consistent equation [Dozier and Silverstein 2007]. Almost surely no eigenvalues lie outside the support of the limiting measure [Bai and Silverstein 2012]. Thus $\lambda_1(AA^*)$ has a simple N -independent positive lower bound if 0 is away from this support. However, when 0 is well inside the limiting spectrum, the smoothing mechanism becomes important yielding that $\lambda_1(AA^*)$ is of order N^{-2} with a lower tail given in (2). The regime where 0 is near the edge of this support is yet unexplored.

The goal of this paper is to study this transitional regime for $A = X - z$, i.e., for the important special case where $A_0 = -z\mathbb{1}$ is a constant multiple of the identity matrix, as the spectral parameter $z \in \mathbb{C}$ is varied. The limiting density of states of $Y^z := (X - z)(X - z)^*$ is supported in the interval $[0, \epsilon_+]$ for $|z| \leq 1$ and the interval $[\epsilon_-, \epsilon_+]$ with $\epsilon_- > 0$ for $|z| > 1$, where $\epsilon_{\pm}$ are explicit functions of $|z|$ given in (18a). As noted above, the problem is relatively simple if $|z| \geq 1 + \epsilon$ with some N -independent ϵ as in this case [Bai and Silverstein 2012] implies that almost surely $\lambda_1(Y^z) \geq C(\epsilon) > 0$ is bounded away from zero. In the opposite regime, when $|z| \leq 1 - \epsilon$, then typically $\lambda_1(Y^z) \sim N^{-2}$, and in fact (2) provides the correct corresponding upper bound (modulo logs).

Our main result on the tail probability of $\lambda_1(Y^z)$ is that for $|z| \leq 1 + CN^{-1/2}$

$$\mathbf{P}(\lambda_1(Y^z) \leq x \cdot c(N, z)) \lesssim \begin{cases} x + \sqrt{x}e^{-\frac{1}{2}N(\Im z)^2} & \text{in the real case,} \\ x & \text{in the complex case,} \end{cases} \quad (4a)$$

where

$$c(N, z) := \min \left\{ \frac{1}{N^{3/2}}, \frac{1}{N^2|1 - |z|^2|} \right\}. \quad (5)$$

Our bound is sharp up to logarithmic corrections; see Corollary 2.4 for the precise statement. Notice the transition between the x and $\sqrt{x}$ behavior in the real case of (4a): near the real axis, $|\Im z| \ll N^{-1/2}$, the result is analogous to the real case (3) at $z = 0$, otherwise the complex behavior (3) dominates at the edges even for real X ; see Figure 1. These results reveal how the robust bound (2) improves near the spectral edge in the transition regime $-CN^{-1/2} \leq 1 - |z| \ll 1$ in both symmetry classes. The transition to the Tracy–Widom scaling in the regime well outside of the spectrum $|z| - 1 \gg N^{-1/2}$ is deferred to our future work.

One motivation for studying $X - z$ is the classical ODE model $du/dt = (X - z)u$ on the stability of large biological networks by May [1972]. For example, the matrix elements x_{ij} may express random

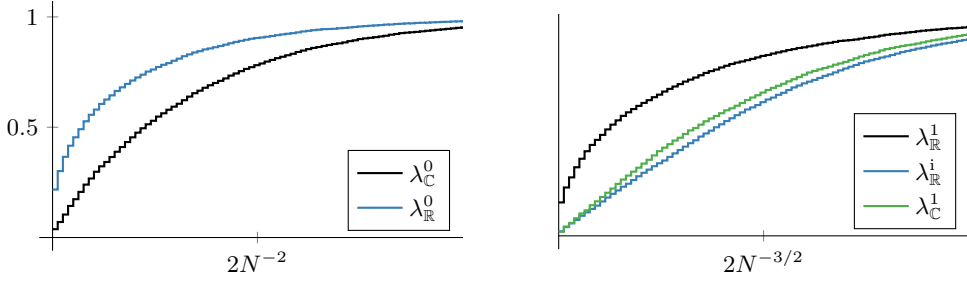


Figure 1. Plots of the cumulative histograms of the smallest eigenvalue $\lambda_{\mathbb{R},\mathbb{C}}^z$ of the matrix $(X - z)(X - z)^*$, where $\mathbb{R}, \mathbb{C}$ indicates whether X is distributed according to the real or complex Ginibre ensemble. The data was generated by sampling 5000 matrices of size 200×200 . The first plot confirms the difference between the x - and $\sqrt{x}$ -scaling close to 0; see (3). The second plot shows that this difference is also observable for shifted Ginibre matrices at the edge $|z| = 1$, but only for real spectral parameters $z = \pm 1$. When the complex parameter z is away from the real axis, then the real case behaves similarly to the complex case.

connectivity rates between neurons and z is the overall decay rate of neuron activation [Sompolinsky et al. 1988]. As $\Re z$ crosses 1, there is a fine phase transition in the large time behavior of u that depends on whether X is real or complex Ginibre matrix; see [Chalker and Mehlig 1998] and [Erdős et al. 2018] for the recent mathematical results, as well as for further references. Another important motivation is that an effective lower tail bound on the least singular value of $X - z$ is essential for the proof of the circular law via Girko’s formula; see [Bordenave and Chafaï 2012] for a detailed survey. In fact, this is the most delicate ingredient in any proof concerning eigenvalue distribution of large non-Hermitian matrices. In particular, relying on the main result of the current paper, we proved [Cipolloni et al. 2019b] that the local eigenvalue statistics for random matrices with centered i.i.d. entries near the spectral edge asymptotically coincide with those for the corresponding Ginibre ensemble as $N \rightarrow \infty$. This is the non-Hermitian analogue of the celebrated Tracy–Widom edge universality for Wigner matrices [Soshnikov 1999; Bourgade et al. 2014]. Similarly, the singular value bound from the present paper is also an important ingredient for the recent CLTs for complex and real i.i.d. matrices [Cipolloni et al. 2019a; 2020].

We now give a brief history of related results. In the $z = 0$ case, tail estimates for $\lambda_1(XX^*)$ beyond the Gaussian distribution have been the subject of intensive research [Rudelson 2008; Tao and Vu 2009] eventually obtaining (3) with an additive $\mathcal{O}(e^{-cN})$ error term for any X with i.i.d. entries with subgaussian tails in [Rudelson and Vershynin 2008]. The precise distribution of $\lambda_1(XX^*)$ was shown in [Tao and Vu 2010a] to coincide with the Gaussian case (3) under a bounded high moment condition and with an $\mathcal{O}(N^{-c})$ error term; see also [Che and Lopatto 2019a; 2019b] for more general ensembles. In the case of general A_0 , lower bounds on $\lambda_1(AA^*)$ in the non-Gaussian setting have been obtained in [Tao and Vu 2007; 2010b], albeit not uniformly in A_0 ; see also [Cook 2018; Tikhomirov 2020] beyond the i.i.d. case. We are not aware of any previous results improving (2) in the transitional regime (4a).

Since we consider Ginibre (i.e., purely Gaussian) ensembles, one might think that everything is explicitly computable from the well understood spectrum of X . The eigenvalue density of X converges to

the uniform distribution on the unit disk and the spectral radius of X converges to 1 (these results have also been established for the general non-Gaussian case; cf. Girko's circular law [Girko 1984; Bai 1997; Tao and Vu 2008; Geman 1986; Bai and Yin 1986; Bordenave et al. 2018]). Also the joint probability density function of all Ginibre eigenvalues, as well as their local correlation functions are explicitly known; see [Ginibre 1965] and [Mehta 1967] for the relatively simple complex case, and [Lehmann and Sommers 1991; Edelman 1997; Borodin and Sinclair 2009; Forrester and Nagao 2007] for the more involved real case, where the appearance of $\sim N^{1/2}$ real eigenvalues causes a singularity in the local density. However, eigenvalues of X give no direct information on the singular values of $X - z$ and the extensive literature on the Ginibre spectrum is not applicable. Notice that any intuition based upon the eigenvalues of X is misleading: the nearest eigenvalue to z is at a distance of order $N^{-1/2}$ for any $|z| \leq 1$. However, $\|(X - z)\|^{-1} \sim \max\{N^{3/4}, N|1 - |z|^2|^{1/2}\}$ for $|z| \leq 1 + CN^{-1/2}$, as a consequence of our result (4a). This is an indication that typically X is highly nonnormal (another indication is that the largest singular value of X is 2, while its spectral radius is only 1).

Regarding our strategy, in this paper we use supersymmetric methods to express the resolvent of Y^z . In particular, we use a multiple Grassmann integral formula for

$$\varrho_N^z(E) := \frac{1}{N\pi} \Im \mathbf{E} \operatorname{Tr} \frac{1}{Y^z - E + i0}, \quad (6)$$

the averaged density of states (or one-point function) of Y^z at energy $E \in \mathbb{R}$. For $|E| \lesssim c(N, z)$ a sizeable contribution to (6) comes from the lowest eigenvalue $\lambda_1(Y^z)$, hence a good upper estimate on (6) translates into a lower tail bound on $\lambda_1(Y^z)$.

With the help of the superbosonization formula by Littelmann, Sommers and Zirnbauer [Littelmann et al. 2008], we can drastically reduce the number of integration variables: instead of N bosonic and N fermionic variables we will have an explicit expression for (6) involving merely two contour integration variables in complex case and three in the real case. The remaining integrals are still highly oscillatory, but contour deformation allows us to estimate them optimally. In fact, saddle point analysis identifies the leading term as long as $|E| \gg c(N, z)$. However, in the critical regime, $|E| \lesssim c(N, z)$, the saddle point analysis breaks down. The leading term is extracted as a specific rescaling of a universal function given by a double integral. We work out the precise answer for (6) in the complex case and we provide optimal bounds in the real case, deferring the precise asymptotics to further work.

Lower tail estimates require delicate knowledge about individual eigenvalues, i.e., about the density of states below the scale of eigenvalue spacing, and it is crucial to exploit the Gaussianity of X via explicit formulas. There are essentially three methods: (i) orthogonal polynomials, (ii) Brézin–Hikami contour integration formula [1998] and (iii) supersymmetric formalism. We are not aware of any orthogonal polynomial approach to analyze $Y^z = (X - z)(X - z)^*$ in the real case (see [Desrosiers and Forrester 2006] in the complex case and [Mo 2012] for rank-1 perturbation of real X). In the complex case, the ensemble Y^z has also been extensively investigated by the Brézin–Hikami formula in [Ben Arous and Pécché 2005], where even the determinantal correlation kernel was computed as a double integral involving the Bessel kernel; see also [Guhr and Wettig 1996; Jackson et al. 1996] for a derivation via the supersymmetric version of the Itzykson–Zuber formula. Although the paper [Ben Arous and Pécché 2005]

did not analyze the resulting one point function, well known asymptotics for the Bessel function may be used to rederive our bounds and asymptotics on (6), as well as (4a), from [Ben Arous and P     2005, Theorem 7.1]; see Appendix C for more details. For the real case, however, there is no analogue of the Br  zin–Hikami formula.

Therefore, in this paper we explore the last option, the supersymmetric approach, that is available for both symmetry classes, although the real case is considerably more involved. Our main tool is the powerful superbosonization formula [Littellmann et al. 2008] followed by a delicate multivariable contour integral analysis. We remark that, alternatively, one may also use the Hubbard–Stratonovich transformation; see, e.g., [Afanasiev 2019, Proposition 1] where correlation functions, i.e., expectations of *products* of characteristic polynomials of X were computed in this way. Note, however, that the density of states (6) requires one to analyze *ratios* of determinants, a technically much more demanding task. While explicit formulas can be obtained with both methods (see [Recher et al. 2012] and especially [Kieburg et al. 2009] for an explicit comparison), the subsequent analysis seems to be more feasible with the formula obtained from the superbosonization approach, as our work demonstrates.

Supersymmetry is a compelling method originated in physics [Efetov 1997; Guhr 2011; Verbaarschot et al. 1985] to produce surprising identities related to random matrices, whose potential has not yet been fully exploited in mathematics. It has been especially successful in deriving rigorous results on Gaussian random band matrices [Bao and Erd  s 2017; Disertori et al. 2002; 2017; 2018; Shcherbina 2014; 2020; Shcherbina and Shcherbina 2016; 2017; 2019], sometimes even beyond the Gaussian case [Shcherbina 2011; 2013; Spencer 2011], as well as on overlaps of non-Hermitian Ginibre eigenvectors [Fyodorov 2018]. We also mention the recent results in [Fyodorov 2018] and [Fyodorov et al. 2018] as examples of a remarkable interplay between supersymmetric and orthogonal polynomial techniques in the theory of Ginibre and related matrices.

The main object of our work, the Hermitian block random matrix

$$H = H^z := \begin{pmatrix} 0 & X - z \\ X^* - \bar{z} & 0 \end{pmatrix} \quad (7)$$

arose in the physics literature as a chiral random matrix model for massless Dirac operator, introduced by Stephanov [1996]. Typically, instead of z and $\bar{z}$, both shift parameters are chosen equal to z (interpreted as i times the chemical potential) so that the corresponding H is not self-adjoint; this model has been extensively investigated by both supersymmetric and orthogonal polynomial techniques; see, e.g., [Verbaarschot and Zahed 1993; Akemann et al. 2005; Wilke et al. 1998; Guhr and Wettig 1997; Osborn et al. 1999]. However, in the special case when z is real, our H^z as given in (7) coincides with Stephanov’s model where z can be interpreted as temperature (or Matsubara frequency); see [Verbaarschot and Wettig 2000, Section 6.1].

2. Model and main results

We consider the model $Y = Y^z = (X - z)(X^* - \bar{z})$ with a fixed complex parameter $z \in \mathbb{C}$ and with a random matrix $X \in \mathbb{C}^{N \times N}$ having independent real or complex Gaussian entries $x_{ab} \sim \mathcal{N}(0, N^{-1})$, where

in the complex case we additionally assume $E x_{ab}^2 = 0$. Note that Y is related to the block matrix (7) through its resolvent via

$$\frac{\text{Tr}(H - \sqrt{w})^{-1}}{2\sqrt{w}} = \text{Tr}(Y - w)^{-1}, \quad \Re w > 0, \Im w > 0, \quad (8)$$

where the branch of $\sqrt{w}$ is chosen such that $\Im \sqrt{w} > 0$. It is well known that in the large N limit the normalized trace of the resolvent of many random matrix ensembles becomes deterministic and it satisfies an algebraic equation, the matrix Dyson equation (MDE) [Ajanki et al. 2019]. In the current case of i.i.d. entries, the MDE reduces to a simple cubic scalar equation

$$\frac{1}{m_{H^z}} + (w + m_{H^z}) - \frac{|z|^2}{w + m_{H^z}} = 0, \quad \Im m_{H^z}(w) > 0, \quad \Im w > 0, \quad (9)$$

that has a unique solution, denoted by m_{H^z} . The local law from [Alt et al. 2019] asserts that

$$\frac{1}{2N} \text{Tr}(H^z - w)^{-1} = m_{H^z}(w) + \mathcal{O}_{\prec}((N \Im w)^{-1}), \quad (10)$$

where $\mathcal{O}_{\prec}$ denotes a suitable concept of high-probability error term. Together with (8) it follows that the normalized trace of the resolvent $(Y^z - w)^{-1}$ of Y^z is well approximated as

$$\frac{1}{N} \text{Tr}(Y^z - w)^{-1} \approx m^z(w)$$

by the unique solution $m = m^z = m_{Y^z}$ to the equation

$$\frac{1}{m^z} + w(1 + m^z) - \frac{|z|^2}{1 + m^z} = 0, \quad \Im m^z(w) > 0, \quad \Re w > 0, \Im w > 0, \quad (11)$$

which is given by $m^z(w) = m_{H^z}(\sqrt{w})/\sqrt{w}$. Since m approximates the trace of the resolvent, the density of states is obtained as the imaginary part of the continuous extension of m to the real line, i.e., $\varrho_{\#}(E) = \pi^{-1} \lim_{\epsilon \rightarrow 0+} \Im m_{\#}(E + i\epsilon)$ for both choices $\# = H^z, Y^z$. For $\delta := 1 - |z|^2 \approx 0$ the Stieltjes transform m_{H^z} and its density of states exhibit a cusp formation at $w = 0$ as δ crosses the value 0. This cusp formation in H^z implies an analogous transition for m^z ; the corresponding density of states are depicted in Figure 2.

Complex case. The main result of this paper in the complex case is an asymptotic double-integral formula for $E \text{Tr}(Y - w)^{-1}$ at $w = E + i0$, $E \geq 0$. In the transitional regime it is convenient to introduce the rescaled variables

$$\lambda := E/c(N), \quad \tilde{\delta} := N^{1/2}\delta, \quad \text{where } \delta := 1 - |z|^2, \quad (12)$$

recalling that $c(N) = c(N, z)$ was defined in (5). For $r \geq 0$, let $\Psi = \Psi(r)$ be the unique solution to the cubic equation $1 + r\Psi + \Psi^3 = 0$ with $\Re \Psi, \Im \Psi > 0$. It is easy to see that $\Psi(r)$ satisfies $\Psi(0) = e^{i\pi/3}$ and $\Psi(r) \sim i\sqrt{r}$ for $r \gg 1$. We also introduce the notation $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$ for real numbers a, b .

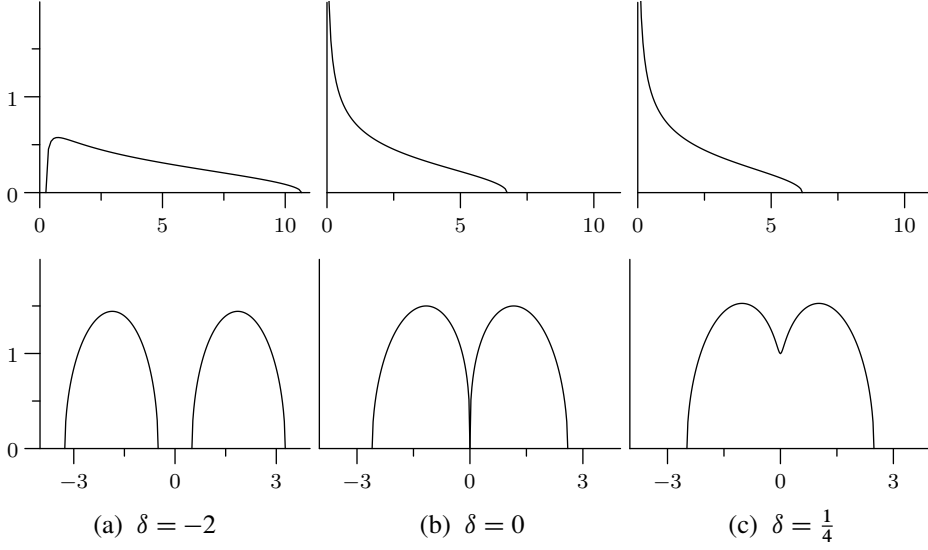


Figure 2. Density of states of Y^Z and H^Z around the cusp formation. The top and bottom figures show a plot of the boundary value of $\Im m^Z$ equal to $\Im m_{Y^Z}$ and $\Im m_{H^Z}$, respectively, on the real line.

Theorem 2.1 (asymptotic 1-point function in the complex case). *Uniformly in $\tilde{\delta} \geq -C$ and $0 \leq \lambda \leq C$ for some fixed large constant $C > 0$, we have*

$$\begin{aligned} & E \operatorname{Tr}(Y - \lambda \cdot c(N, \tilde{\delta}) - i0)^{-1} \\ &= \frac{1}{2\pi i} \frac{N^{3/2}}{\tilde{z}_*} \int dx \oint dy e^{h(y)-h(x)} \tilde{H}(x, y) + \mathcal{O}(N(1 \vee \tilde{\delta})(1 + |\log \lambda|)), \end{aligned} \quad (13a)$$

where

$$\begin{aligned} \tilde{H}(x, y) &:= \frac{1}{x^3} + \frac{1}{x^2 y} + \frac{1}{x y^2} + \frac{\tilde{\delta} \tilde{z}_*}{x y} + \frac{\tilde{\delta} \tilde{z}_*}{x^2}, \\ h(x) &:= -(1 \wedge \tilde{\delta}^{-1}) \lambda \tilde{z}_* x + \frac{\tilde{\delta}}{x \tilde{z}_*} + \frac{1}{2x^2 \tilde{z}_*^2}, \\ \tilde{z}_* &:= \lambda^{-1/3} (1 \vee \tilde{\delta}^{1/3}) |\Psi(\tilde{\delta} \lambda^{-1/3} (1 \vee \tilde{\delta}^{1/3}))|, \\ c(N, \tilde{\delta}) &= N^{-3/2} \cdot (1 \wedge \tilde{\delta}^{-1}), \end{aligned} \quad (13b)$$

and where the x -integration is over any contour from 0 to $e^{3i\pi/4}\infty$, going out from 0 in the direction of the positive real axis, and the y -integration is over any contour around 0 in a counterclockwise direction. Moreover, in the regime $\lambda \ll 1$, we have the bound

$$\left| \frac{1 \wedge \tilde{\delta}^{-1}}{\tilde{z}_*} \int dx \oint dy e^{h(y)-h(x)} \tilde{H}(x, y) \right| \lesssim \begin{cases} |\log \lambda|, & \lambda \geq \tilde{\delta}^3, \\ |\log \lambda \tilde{\delta}|, & \lambda < \tilde{\delta}^3. \end{cases} \quad (13c)$$

In (13c), the eigenvalue scaling, i.e., for $\lambda \gg 1$, the analogue of Theorem 2.1 reduces essentially to the local law asymptotics (10), albeit with a better error term due to the presence of the expectation.

Proposition 2.2. *Let $Y^z = (X - z)(X - z)^*$ where X is a complex Ginibre ensemble. Then, uniformly in $\delta := 1 - |z|^2$ and $E \in \mathbb{R}$, we have the asymptotic expansion in $E \pm := E - \epsilon_\pm$,*

$$E \operatorname{Tr}(Y^z - E - i0)^{-1} = Nm^z(E + i0) \times \left(1 + \mathcal{O}\left(\frac{1}{N|E_+|^{3/2}} + \frac{1}{NE^{2/3}} \wedge \left(\frac{\mathbb{1}_{\delta \geq 0}}{NE^{1/2}\delta^{1/2}} + \frac{\mathbb{1}_{\delta < 0}}{N|E_-|^{3/2}|\delta|^{5/2}} \right) \right) \right), \quad (14)$$

where the edges $\epsilon_\pm$ of $\Im m^z$ are explicit functions of δ given in (18a).

Real case. In the real case our main result is the following optimal bound on $E \operatorname{Tr}(Y + E)^{-1}$ for $E > 0$. Recall the notation $\delta := 1 - |z|^2$.

Theorem 2.3 (optimal bound on the resolvent trace in the real case). *Let $\rho > 0$ be any small constant. Then uniformly in $E \geq 0$ and $\delta \geq -CN^{-1/2}$ for some fixed large constant $C > 0$ we have that*

$$|E \operatorname{Tr}(Y + E)^{-1}| \lesssim \frac{e^{-\frac{1}{2}N(\Im z)^2} [N^{3/4} \vee N\sqrt{|\delta|}]}{\sqrt{E}} + (N^{3/2} \vee N^2|\delta|)[1 + |\log(NE^{2/3})|] \quad (15)$$

Finally, we present our bound on the tail asymptotics for both real and complex cases; for most applications, this can be viewed as the main result of this paper. Since a sizeable contribution to $\Im \operatorname{Tr}(Y - E + i0)^{-1}$ and $\Im \operatorname{Tr}(Y + E)^{-1}$ comes from the smallest eigenvalue $\lambda_1(Y^z)$, by a straightforward Markov inequality we immediately obtain the following corollary on the tail asymptotics of $\lambda_1(Y^z)$ as an easy consequence of Theorems 2.1 and 2.3.

Corollary 2.4 (tail asymptotics of $\lambda_1(Y^z)$). *For any $C > 0$, uniformly in $x \in (0, C]$ and $1 - |z|^2 > -CN^{-1/2}$, we have the bound*

$$P(\lambda_1(Y^z) \leq c(N, z)x) \lesssim (1 + |\log x|)x \quad (16)$$

in the complex case, and

$$P(\lambda_1(Y^z) \leq c(N, z)x) \lesssim e^{-\frac{1}{2}N(\Im z)^2} \sqrt{x} + (1 + |\log x|)x \quad (17)$$

in the real case, where we recall the definition of the scaling factor $c(N, z)$ from (5).

Properties of the asymptotic Stieltjes transform m^z . We now record some information on the deterministic Stieltjes transform m^z which will be useful later. The endpoints of the support of the density of states $\pi^{-1}\Im m^z$ are the zeros of the discriminant of the cubic equation (11) since passing through these points with the real parameter $E = \Re w$ creates solutions with nonzero imaginary part. Elementary calculations show that the support of $\Im m_{Y^z}$ is $[0, \epsilon_+]$ if $0 \leq \delta \leq 1$ and it is $[\epsilon_-, \epsilon_+]$ if $\delta < 0$, where

$$\epsilon_\pm := \frac{8\delta^2 \pm (9 - 8\delta)^{3/2} - 36\delta + 27}{8(1 - \delta)}, \quad (18a)$$

and ϵ_- is only considered if $\delta < 0$. Note that while $\epsilon_+ \sim 1$, the edge ϵ_- may be close to 0; more precisely $0 < \epsilon_- = -4\delta^3/27(1 + \mathcal{O}(|\delta|))$. The slope coefficient of the square-root density at the edge in $\epsilon_\pm$ is

given by

$$\Im m(\mathfrak{e}_\pm \mp \lambda) = \begin{cases} \gamma_\pm \sqrt{\lambda}(1 + \mathcal{O}(\sqrt{\lambda})), & \lambda \geq 0, \\ 0, & \lambda \leq 0, \end{cases} \quad \gamma_\pm := \frac{2\sqrt{2}(\sqrt{9-8\delta} \pm 1)^{3/2}}{(\sqrt{9-8\delta} \pm 3)^{5/2} \sqrt[4]{9-8\delta}}. \quad (18b)$$

Note that while the square-root edge at $\mathfrak{e}_+$ is nonsingular in the sense $\gamma_+ \sim 1$, the square-root edge in $\mathfrak{e}_-$ becomes singular for small $|\delta|$ as

$$\gamma_- = \frac{9}{4|\delta|^{5/2}}(1 + \mathcal{O}(|\delta|)).$$

3. Supersymmetric method

Let $\chi_1, \overline{\chi}_1, \dots, \chi_N, \overline{\chi}_N$ denote Grassmannian variables satisfying the commutation rules

$$\chi_i \chi_j = -\chi_j \chi_i, \quad \chi_i \overline{\chi}_j = -\overline{\chi}_j \chi_i, \quad \overline{\chi}_i \overline{\chi}_j = -\overline{\chi}_j \overline{\chi}_i,$$

from which it follows that $\chi_i^2 = \overline{\chi}_i^2 = 0$. As a convention we set $\overline{\overline{\chi}_i} := -\chi_i$. The power series of any function of Grassmannian variables is multilinear and it suffices to define the integral in the sense of Berezin [1987] over Grassmannian variables as the derivatives

$$\partial_{\chi_k} \chi_k = \partial_{\overline{\chi}_k} \overline{\chi}_k = 1, \quad \partial_{\chi_k} 1 = \partial_{\overline{\chi}_k} 1 = 0, \quad \partial_\chi := \partial_{\chi_1} \partial_{\overline{\chi}_1} \cdots \partial_{\chi_N} \partial_{\overline{\chi}_N}$$

and extend them multilinearly to all finite combinations of monomials in Grassmannians. We denote the column vectors with entries $\chi_1, \dots, \chi_N$ and $\overline{\chi}_1, \dots, \overline{\chi}_N$ by χ and $\overline{\chi}$, respectively. The conjugate transposes of those vectors, i.e., the row vectors with entries $\overline{\chi}_1, \dots, \overline{\chi}_N$ and $-\chi_1, \dots, -\chi_N$ will be denoted by χ^* and $\overline{\chi}^*$, respectively. Note that $(\chi^*)^* = -\chi$, $(\overline{\chi}^*)^* = -\overline{\chi}$. We now define the inner product of Grassmannian vectors χ, ϕ by

$$\langle \chi, \phi \rangle := \sum_i \overline{\chi}_i \phi_i,$$

so that the quadratic form $\sum_{i,j} \overline{\chi}_i A_{ij} \chi_j$ can be written as

$$\langle \chi, A\chi \rangle = \sum_{i,j} \overline{\chi}_i A_{ij} \chi_j,$$

where the matrix-vector product is understood in its usual sense. Similarly, s and $\overline{s}$ denote the column vectors with complex entries $s_1, \dots, s_N$ and their complex conjugates $\overline{s}_1, \dots, \overline{s}_N$, respectively, and for the conjugate transpose we have $(s^*)^* = s$ as usual. We have

$$\langle s, \phi \rangle := \sum_i \overline{s}_i \phi_i, \quad \langle \chi, s \rangle := \sum_i \overline{\chi}_i s_i,$$

and similarly for quadratic forms. The commutation rules naturally also apply to linear functions of the Grassmannians, and therefore also, for example, $\langle \chi, \chi \rangle^2 = \langle \chi, s \rangle^2 = 0$ for any vector s of complex numbers. The complex numbers s_i and often called *bosonic* variables, while Grassmannians are called *fermions*, motivated by the basic (anti)commutativity of the bosonic/fermionic field operators in physics.

3A. Determinant identities. The backbone of the supersymmetric method are the determinant identities

$$\frac{1}{i^N \det(H-w)} = \int_{\mathbb{C}^N} \exp(-i \operatorname{sgn}(\Im w) \langle s, (H-w)s \rangle) ds, \quad ds := \frac{d\Re s_1 d\Im s_1}{\pi} \dots \frac{d\Re s_N d\Im s_N}{\pi},$$

$$i^N \det(H-w) = \partial_\chi \exp(i \langle \chi, (H-w)\chi \rangle), \quad \partial_\chi := \partial_{\chi_1} \partial_{\bar{\chi}_1} \dots \partial_{\chi_N} \partial_{\bar{\chi}_N},$$

where the exponential is defined by its (terminating) Taylor series. Consequently we can conveniently express the generating function as

$$Z(w, w_1) := E \frac{\det(H-w_1)}{\det(H-w)} = E \int_{\mathbb{C}^N} ds \partial_\chi \exp(i \langle \chi, (H-w_1)\chi \rangle - i \langle s, (H-w)s \rangle),$$

for $w \in \mathbb{H}$ and $w_1 \in \mathbb{C}$, where choice of w with $\Im w > 0$ guarantees the convergence of the integral. By taking the w_1 derivative and setting $w = w_1$ it follows that

$$\operatorname{Tr}(H-w)^{-1} = - \frac{\partial}{\partial w_1} \frac{\det(H-w_1)}{\det(H-w)} \Big|_{w_1=w} = i \int \langle \chi, \chi \rangle e^{-i \operatorname{Tr}(H-w)[ss^* + \chi\chi^*]},$$

$$\int := \int_{\mathbb{C}^N} ds \partial_\chi. \quad (19)$$

3B. Superbosonization identity. After taking expectations, i.e., performing the Gaussian integration for the entries of $Y = Y^z = (X-z)(X-z)^*$, the resolvent identity (19) will depend on the complex vector s and the Grassmannian vector χ only via certain inner products. More specifically, after defining the $N \times 2$ and $N \times 4$ matrices $\Phi := (s, \chi)$ and $\Psi := (s, \bar{s}, \chi, \bar{\chi})$, the expectation of the resolvent can be expressed as an integral over the 2×2 or 4×4 supermatrices $\Phi^* \Phi$ or $\Psi^* \Psi$ in the complex and real case, respectively. *Supermatrices* are 2×2 block matrices whose diagonal blocks are commonly referred to as the boson-boson and the fermion-fermion block, while the off-diagonal blocks are the boson-fermion and fermion-boson block. For supermatrices Q the *supertrace* and *superdeterminant*, the natural generalizations of trace and determinant, are given by

$$\operatorname{STr} \begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix} := \operatorname{Tr}(x) - \operatorname{Tr}(y), \quad \operatorname{SDet} \begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix} := \frac{\det(x)}{\det(y - \tau x^{-1} \sigma)}, \quad (20)$$

and the inverse of a supermatrix is

$$\begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix}^{-1} = \begin{pmatrix} (x - \tau y^{-1} \sigma)^{-1} & -x^{-1} \sigma (y - \sigma x^{-1} \tau)^{-1} \\ -y^{-1} \tau (x - \tau y^{-1} \sigma)^{-1} & (y - \sigma x^{-1} \tau)^{-1} \end{pmatrix}. \quad (21)$$

The integral over the remaining degrees of freedom in Φ, Ψ other than the inner products in $\Phi^* \Phi, \Psi^* \Psi$ can conveniently be performed using the well known *superbosonization formula* which we now recall. It basically identifies the integration volume of the irrelevant degrees of freedom with the high power of the superdeterminant of the supermatrix containing the relevant inner products (collected in a 2×2 supermatrix Q in the complex case and a 4×4 supermatrix Q in the real case).

3B1. Complex superbosonization. For any analytic function F with sufficiently fast decay at $+\infty$ in the boson-boson sector (in the variable x) the complex superbosonization identity from [Littellmann et al. 2008, Eq. (1.10)] implies

$$\int F(\Phi^* \Phi) = \int_Q \text{SDet}^N(Q) F(Q), \quad \int_Q := \frac{1}{2\pi i} \int dx \oint dy \partial_\sigma \partial_\tau, \quad Q := \begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix}, \quad (22)$$

where $\int dx$ denotes the Lebesgue integral on $[0, \infty)$, $\oint dy$ denotes the counterclockwise complex line integral on $\{z \in \mathbb{C} \mid |z| = 1\}$ and σ, τ denote independent scalar Grassmannian variables. The key point is that while the integral on the left-hand side is performed over N complex numbers and $2N$ Grassmannians, the integral on the right is simply over a 2×2 supermatrix, i.e., two complex variables and two Grassmannians. Note that the identity in [Littellmann et al. 2008] is more general than (22) in the sense that it allows for bosonic and fermionic sectors of unequal sizes. For the case of equal sizes, which concerns us, the formula gets simplified, the measure DQ in [Littellmann et al. 2008, Eq. (1.8)] becomes the flat Lebesgue measure since two determinants cancel each other as

$$\begin{aligned} \det(1 - x^{-1} \sigma y^{-1} \tau) \det(1 - y^{-1} \tau x^{-1} \sigma) &= e^{\text{Tr}(\log(1 - x^{-1} \sigma y^{-1} \tau) + \log(1 - y^{-1} \tau x^{-1} \sigma))} \\ &= e^{-\sum_{k \geq 1} \frac{1}{k} \text{Tr}((x^{-1} \sigma y^{-1} \tau)^k + (y^{-1} \tau x^{-1} \sigma)^k)} = 1, \end{aligned} \quad (23)$$

where the sum is finite and the last equality followed using the commutation rules.

3B2. Real superbosonization. In the real case we similarly have the real superbosonization identity from [Littellmann et al. 2008, Eq. (1.13)]:

$$\begin{aligned} \int F(\Psi^* \Psi) &= \int_Q \text{SDet}^{N/2}(Q) F(Q), \\ \int_Q &:= \frac{1}{(2\pi)^2 i} \int dx \oint dy \partial_\sigma \left(\frac{\det(y)}{\det(x)} \right)^{1/2} \det\left(1 - \frac{x^{-1}}{y} \sigma \tau\right)^{-1/2}. \end{aligned} \quad (24)$$

The supermatrix Q has 2×2 blocks: x is nonnegative Hermitian, y is a scalar multiple of the identity matrix. The off-diagonal blocks σ, τ are related by

$$\tau := -t_a \sigma^t t_s^{-1}, \quad t_s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_a := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Here the $\int dx$ integral is the Lebesgue measure on nonnegative Hermitian 2×2 matrices x satisfying $x_{11} = x_{22}$, i.e.,

$$\int dx := \int_0^\infty dx_{11} \int_{|x_{12}| \leq x_{11}} d\Re x_{12} d\Im x_{12},$$

and the fermionic integral is defined as $\partial_\sigma := \partial_{\sigma_{11}} \partial_{\sigma_{22}} \partial_{\sigma_{21}} \partial_{\sigma_{12}}$. Furthermore, under the slight abuse of notation of identifying the 2×2 matrix y , which is a scalar multiple of the identity matrix, with the corresponding scalar, $\oint dy$ is the complex line integral over $|y| = 1$ in a counterclockwise direction. Unlike in the complex case, the matrix elements of the 4×4 supermatrix Q are not independent; there are only 4 (real) bosonic and 4 fermionic degrees of freedom. These identities among the elements of Q stem

from natural relations between the scalar product of the column vectors of Ψ . For example the identity $\langle s, s \rangle = \langle \bar{s}, \bar{s} \rangle$ from the first two diagonal elements of $(\Psi^* \Psi)$ corresponds to $x_{11} = x_{22}$, while $\langle \bar{\chi}, \bar{\chi} \rangle = 0$ is responsible for $y_{12} = 0$. The relation

$$\tau = \begin{pmatrix} \sigma_{22} & \sigma_{12} \\ -\sigma_{21} & -\sigma_{11} \end{pmatrix}$$

encoded in the last line of (24) corresponds to relations between scalar products of bosonic and fermionic vectors and their complex conjugates; for example $\tau_{21} = -\sigma_{21}$ comes from the identity $(\Psi^* \Psi)_{41} = \langle -\bar{\chi}, s \rangle = -\langle \bar{s}, \chi \rangle = -(\Psi^* \Psi)_{23}$, etc.

3C. Application to Y^z in the complex case. Our goal is to evaluate $E \operatorname{Tr}(H - w)^{-1}$ asymptotically on the scale where E is comparable with the eigenvalue spacing. We now use the identity

$$\begin{aligned} \operatorname{Tr}(Y - w)^{-1} &= i \int \langle \chi, \chi \rangle e^{-i \operatorname{Tr}[(X - z)(X^* - \bar{z}) - w](ss^* + \chi\chi^*)} \\ &= i \int \langle \chi, \chi \rangle e^{iw \langle s, s \rangle - iw \langle \chi, \chi \rangle - i \operatorname{Tr}(X - z)(X^* - \bar{z}) \Phi \Phi^*} \end{aligned} \quad (25)$$

for $w = E + i\epsilon$ with $|E| \gg \epsilon > 0$. We now compute the ensemble average as

$$\begin{aligned} E e^{-i \operatorname{Tr}(X - z)(X^* - \bar{z}) \Phi \Phi^*} &= \left(\frac{N}{\pi} \right)^{N^2} \int \exp \left[-N \operatorname{Tr} X^* \left(1 + i \frac{\Phi \Phi^*}{N} \right) X + i \bar{z} \operatorname{Tr} \Phi \Phi^* X + i z \operatorname{Tr} X^* \Phi \Phi^* - i |z|^2 \operatorname{Tr} \Phi \Phi^* \right] \\ &= \operatorname{SDet} \left(1 + i \frac{\Phi^* \Phi}{N} \right)^{-N} \exp \left(-i |z|^2 \left[\operatorname{Tr} \Phi \Phi^* - \frac{i}{N} \operatorname{Tr} \Phi \Phi^* \left(1 + i \frac{\Phi \Phi^*}{N} \right)^{-1} \Phi \Phi^* \right] \right), \\ &= \operatorname{SDet} \left(1 + i \frac{\Phi^* \Phi}{N} \right)^{-N} \exp \left(-i |z|^2 \operatorname{Tr} \Phi \left(1 + \frac{i}{N} \Phi^* \Phi \right)^{-1} \Phi^* \right), \\ &= \operatorname{SDet} \left(1 + i \frac{\Phi^* \Phi}{N} \right)^{-N} \exp \left(-N |z|^2 \operatorname{STr} \left(1 + \frac{i}{N} \Phi^* \Phi \right)^{-1} \frac{i}{N} \Phi^* \Phi \right). \end{aligned} \quad (26)$$

To perform the $\int = \int_{\mathbb{C}^N} ds d\chi$ integration in (25) we use the superbosonization formula (22) for the function

$$F(\Phi^* \Phi) := \langle \chi, \chi \rangle \operatorname{SDet} \left(1 + i \frac{\Phi^* \Phi}{N} \right)^{-N} \exp \left(-N |z|^2 \operatorname{STr} \left(1 + \frac{i}{N} \Phi^* \Phi \right)^{-1} \frac{i}{N} \Phi^* \Phi + iw \operatorname{STr} \Phi^* \Phi \right). \quad (27)$$

We view F as a function of the four independent variables collected in the entries of the 2×2 matrix $\Phi^* \Phi$. Strictly speaking the function F is only meromorphic but not entire in these four variables, but since the integration regimes on both sides of the superbosonization formula are well separated away from the poles of F , a simple approximation argument outlined in Appendix A justifies its usage. Together with the change of variables $\frac{i}{N} Q \mapsto Q$ it now implies that

$$E \operatorname{Tr}(Y - w)^{-1} = N \int_{Q'} y e^{Nw \operatorname{STr}(Q) + N \log \operatorname{SDet}(Q) - N \log \operatorname{SDet}(1 + Q) - N |z|^2 \operatorname{STr}(1 + Q)^{-1} Q},$$

where $\int_{Q'}$ indicates the changed integration regime due to the change of variables, more specifically,

under Q' the x -integration is over $[0, i\infty)$ and the y -integration is over a small circle $\{u \in \mathbb{C} \mid |u| = N^{-1}\}$. Note that the change of variables through scaling does not contribute an additional factor, since superdeterminants are scale invariant. It remains to perform the Berezinian integral. To do so we split

$$Q = q + \mu, \quad q = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad \mu = \begin{pmatrix} 0 & \sigma \\ \tau & 0 \end{pmatrix}$$

and compute

$$\begin{aligned} \exp(-N \log \text{SDet}(1 + Q)) &= \exp(-N \text{STr} \log(1 + q + \mu)) \\ &= \exp\left(-N \text{STr} \log(1 + q) + \frac{N}{2} \text{STr}(1 + q)^{-1} \mu (1 + q)^{-1} \mu\right) \\ &= \exp(-N \log(1 + x) + N \log(1 + y)) \left(1 + N \frac{\sigma \tau}{(1 + x)(1 + y)}\right), \\ \exp(N \log \text{SDet}(Q)) &= \exp(N \log(x) - N \log(y)) \left(1 - N \frac{\sigma \tau}{xy}\right) \end{aligned}$$

and

$$\begin{aligned} \exp(-N |z|^2 \text{STr}(1 + Q)^{-1} Q) &= \exp(-N |z|^2 \text{STr}[(1 + q)^{-1} q - (1 + q)^{-1} \mu (1 + q)^{-1} \mu (1 + q)^{-1}]) \\ &= \exp\left(-N |z|^2 \frac{x}{1 + x} + N |z|^2 \frac{y}{1 + y}\right) \left(1 + N |z|^2 \frac{\sigma \tau}{(1 + x)(1 + y)} \left(\frac{1}{1 + x} + \frac{1}{1 + y}\right)\right). \end{aligned}$$

By combining these identities we arrive at the final result:¹

$$\begin{aligned} E \text{Tr}(Y - w)^{-1} &= \frac{N^2}{2\pi i} \int_0^\infty dx \oint dy e^{-Nf(x) + Nf(y)} y \cdot G(x, y), \\ G(x, y) &:= \frac{1}{xy} - \frac{1}{(1 + x)(1 + y)} \left[1 + \frac{|z|^2}{1 + x} + \frac{|z|^2}{1 + y}\right], \\ f(x) &:= \log \frac{1 + x}{x} - \frac{|z|^2}{1 + x} - wx, \end{aligned} \tag{28}$$

where the x -integration is over $(0, i\infty)$ and the y -integration is over a circle of radius N^{-1} around the origin.

3D. Application to Y^z in the real case. We now consider the real case and introduce the $N \times 4$ matrix $\Psi := (s, \bar{s}, \chi, \bar{\chi})$, the 2×2 matrix $Z := \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$ and the 4×4 matrix $Z_2 := \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}$, and use that

$$i\bar{z} \text{Tr} \Phi \Phi^* X + iz \text{Tr} X^t \Phi \Phi^* = \frac{i}{2} \text{Tr} \Psi Z_2^* \Psi^* X + \frac{i}{2} \text{Tr} X^t \Psi Z_2 \Psi^*, \quad (\Psi Z_2 \Psi^*)^t = \Psi Z_2^* \Psi^*,$$

¹Essentially the same formula, obtained by direct computations, was presented by M. Shcherbina in her seminar talk on Jan 11, 2016 in Bonn. Our derivation of the same formula via superbosonization is merely a pedagogical preparation to the much more involved real case in [Section 3D](#).

to compute

$$\begin{aligned}
& \mathbf{E} e^{-i\mathrm{Tr}(X-z)(X'-\bar{z})\Phi\Phi^*} \\
&= \mathbf{E} \left(\frac{N}{2\pi} \right)^{\frac{N^2}{2}} \int \exp \left(-\frac{N}{2} \mathrm{Tr} X'^t \left(1 + \frac{2i}{N} \Phi\Phi^* \right) X + i\bar{z} \mathrm{Tr} \Phi\Phi^* X + iz \mathrm{Tr} X'^t \Phi\Phi^* - i|z|^2 \mathrm{Tr} \Phi\Phi^* \right) \\
&= \mathbf{E} \left(\frac{N}{2\pi} \right)^{\frac{N^2}{2}} \int \exp \left(-\frac{N}{2} \mathrm{Tr} X'^t \left(1 + \frac{i}{N} \Psi\Psi^* \right) X + \frac{i}{2} \mathrm{Tr} \Psi Z_2^* \Psi^* X + \frac{i}{2} \mathrm{Tr} X'^t \Psi Z_2 \Psi^* - \frac{i|z|^2}{2} \mathrm{Tr} \Psi\Psi^* \right) \\
&= \mathrm{SDet} \left(1 + \frac{i}{N} \Psi^* \Psi \right)^{-N/2} \exp \left(-\frac{i}{2} \mathrm{Tr} \Psi Z_2^* \left(1 - \frac{i}{N} \Psi^* \left(1 + \frac{i}{N} \Psi\Psi^* \right)^{-1} \Psi \right) Z_2 \Psi^* \right) \\
&= \mathrm{SDet} \left(1 + \frac{i}{N} \Psi^* \Psi \right)^{-N/2} \exp \left(-\frac{i}{2} \mathrm{Tr} \Psi Z_2^* \left(1 + \frac{i}{N} \Psi^* \Psi \right)^{-1} Z_2 \Psi^* \right) \\
&= \mathrm{SDet} \left(1 + \frac{i}{N} \Psi^* \Psi \right)^{-N/2} \exp \left(-\frac{N}{2} \mathrm{STr} \left(1 + \frac{i}{N} \Psi^* \Psi \right)^{-1} Z_2 \frac{i}{N} \Psi^* \Psi Z_2^* \right), \tag{29}
\end{aligned}$$

where we used that X is real and $\Psi\Psi^*$ is symmetric. The superbosonization formula thus implies

$$\begin{aligned}
\mathbf{E} \mathrm{Tr}(H-w)^{-1} &= \frac{N}{2(2\pi)^2 i} \int \frac{\mathrm{Tr}(y) \det(y)^{1/2}}{\det(x)^{1/2}} \exp \left(-\frac{1}{2} \log \det(1 - x^{-1} \sigma y^{-1} \tau) \right) \\
&\quad \times \exp \left(\frac{N}{2} [\mathrm{STr}(wQ) - \log \mathrm{SDet}(1+Q) + \log \mathrm{SDet}(Q) - \mathrm{STr}(1+Q)^{-1} Z_2 Q Z_2^*] \right) \\
&= \frac{N}{2(2\pi)^2} \int \frac{\mathrm{Tr}(y) \det(y)^{1/2}}{\det(x)^{1/2}} \exp \left(-\frac{1}{2} \mathrm{Tr} \log(1 - x^{-1} \sigma y^{-1} \tau) \right) \\
&\quad \times \exp \left(\frac{N}{2} [\mathrm{STr}(wQ) - \mathrm{STr} \log(1+Q) + \mathrm{STr} \log(Q) - \mathrm{STr}(1+Q)^{-1} Z_2 Q Z_2^*] \right),
\end{aligned}$$

where

$$Q = \begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix} \equiv \frac{i}{N} \Psi^* \Psi, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In order to expand the exponential terms to fourth order in σ we introduce the shorthand notation

$$Q = q + \mu, \quad q = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad \mu = \begin{pmatrix} 0 & \sigma \\ \tau & 0 \end{pmatrix}. \tag{30}$$

We compute

$$\begin{aligned}
\mathrm{STr}(1+Q)^{-1} Z_2 Q Z_2^* &= \mathrm{STr}(1+q)^{-1} Z_2 q Z_2^* \\
&\quad - \mathrm{STr}(1+q)^{-1} \mu (1+q)^{-1} (Z_2 \mu Z_2^* - \mu (1+q)^{-1} Z_2 q Z_2^*) \\
&\quad - ((1+q)^{-1})^3 (1+q)^{-1} (Z_2 \mu Z_2^* - \mu (1+q)^{-1} Z_2 q Z_2^*), \\
&= \mathrm{Tr}(1+x)^{-1} Z x Z^* - |z|^2 \mathrm{Tr} \frac{y}{1+y} - \mathrm{Tr}(\sigma Z \tau Z^* A + Z \sigma Z^* \tau A) \\
&\quad + \mathrm{Tr} \sigma \tau A C' - \mathrm{Tr} \sigma \tau A (\sigma Z \tau Z^* A + Z \sigma Z^* \tau A) + \mathrm{Tr} \sigma \tau A \sigma \tau A C',
\end{aligned}$$

where we introduced matrices A, C' as in

$$\begin{aligned} A &:= \frac{(1+x)^{-1}}{1+y}, \\ B &:= \frac{x^{-1}}{y}, \\ C' &:= ZXZ^*(1+x)^{-1} + |z|^2 \frac{y}{1+y}, \end{aligned}$$

as well as B , which will be used in the sequel. In deriving these formulas we used that q and Z_2 have zero off-diagonal blocks and μ has zero diagonal blocks, to eliminate terms with odd powers of μ after taking the supertrace, and that y is a scalar multiple of the identity. Similarly we find for the logarithmic terms

$$\begin{aligned} & \text{STr}(\log(q+\mu) - \log(1+q+\mu)) \\ &= \text{STr} \log(q(1+q)^{-1}) - \frac{1}{2} \text{STr}(q^{-1}\mu)^2 - \frac{1}{4} \text{STr}(q^{-1}\mu)^4 + \frac{1}{2} \text{STr}((1+q)^{-1}\mu)^2 + \frac{1}{4} \text{STr}((1+q)^{-1}\mu)^4 \\ &= \log \frac{\det(x)}{\det(1+x)} - \log \frac{\det(y)}{\det(1+y)} + \text{Tr} \sigma \tau (A-B) + \frac{1}{2} \text{Tr}(\sigma \tau A)^2 - \frac{1}{2} \text{Tr}(\sigma \tau B)^2 \end{aligned}$$

and

$$-\frac{1}{2} \text{Tr} \log(1 - x^{-1} \sigma y^{-1} \tau) = \frac{1}{2} \text{Tr} \sigma \tau B + \frac{1}{4} \text{Tr}(\sigma \tau B)^2.$$

Hence

$$\begin{aligned} E \text{Tr}(H-w)^{-1} &= \frac{N}{2(2\pi)^2 i} \int dx \oint dy \frac{\text{Tr}(y) \det(y)^{1/2}}{\det(x)^{1/2}} \exp\left(-\frac{N}{2} f(x) + \frac{N}{2} f(y)\right) G(x, y, z), \\ f(x) &:= -w \text{Tr} x + \log \frac{\det(1+x)}{\det(x)} + \text{Tr} ZXZ^*(1+x)^{-1} - 2|z|^2, \\ G(x, y, z) &:= \partial_\sigma \exp \left[\frac{N}{2} (\text{Tr} \sigma \tau (A(1-C') - (1 - \frac{1}{N})B) + \text{Tr}(\sigma Z \tau Z^* A + Z \sigma Z^* \tau A) \right. \\ &\quad \left. + \text{Tr} \sigma \tau A (\sigma \tau A (\frac{1}{2} - C') + \sigma Z \tau Z^* A + Z \sigma Z^* \tau A) - \frac{1}{2} \left(1 - \frac{1}{N}\right) \text{Tr}(\sigma \tau B)^2) \right], \quad (31) \end{aligned}$$

where $\int dx = \int dx_{11} d\Re x_{12} d\Im x_{12}$ is the integral over matrices of the form

$$x = \begin{pmatrix} ix_{11} & ix_{12} \\ i\bar{x}_{12} & ix_{11} \end{pmatrix}$$

with $x_{11} \in [0, \infty)$ and $x_{12} \in \mathbb{C}$ with $|x_{12}| \leq x_{11}$. The integral $\oint dy = \oint dy_{11}$ is the integral over scalar matrices $y = y_{11}I$ with dy_{11} being the complex line integral over $|y_{11}| = N^{-1}$ in a counterclockwise direction.

To integrate out the Grassmannians we expand the exponential to second order, use the relation (30) between σ and τ , and use that for 2×2 matrices X and Y which are constant on the diagonal

($x_{11} = x_{22}$, $y_{11} = y_{22}$) we have the identities

$$\begin{aligned}
\frac{N^2}{8} \partial_\sigma \operatorname{Tr}^2(\sigma Z \tau Z^* X + Z \sigma Z^* \tau X) &= -4N^2 |z|^2 (\Re z)^2 \det(X) + N^2 |z|^2 (\Im z)^2 \operatorname{Tr}^2(X), \\
\frac{N^2}{8} \partial_\sigma \operatorname{Tr}^2(\sigma \tau X) &= -N^2 \det(X), \\
\frac{N^2}{4} \partial_\sigma \operatorname{Tr}(\sigma \tau X) \operatorname{Tr}(\sigma Z \tau Z^* Y + Z \sigma Z^* \tau Y) &= 2N^2 (\Re z)^2 (\operatorname{Tr}(XY) - \operatorname{Tr}(X) \operatorname{Tr}(Y)) \\
&\quad + 2iN^2 (\Im z)(\Re z) (X_{12} Y_{21} - X_{21} Y_{12}), \\
\frac{N}{2} \partial_\sigma \operatorname{Tr}(\sigma \tau X \sigma \tau Y) &= N(\operatorname{Tr}(X) \operatorname{Tr}(Y) - \operatorname{Tr}(XY)), \\
\frac{N}{2} \partial_\sigma \operatorname{Tr} \sigma \tau X (\sigma Z \tau Z^* X + Z \sigma Z^* \tau X) &= 4N (\Re z)^2 \det(X).
\end{aligned}$$

Hence we finally have the expression

$$\begin{aligned}
G = -N^2 &\left[\det(A(1 - C') - \left(1 - \frac{1}{N}\right) B) + (4|z|^2 (\Re z)^2 + 2(\Re z)^2 (2 - \operatorname{Tr} C')) \det A \right. \\
&\quad - |z|^2 (\Im z)^2 \operatorname{Tr}^2 A - 2(\Re z)^2 \left(1 - \frac{1}{n}\right) (\operatorname{Tr} A \operatorname{Tr} B - \operatorname{Tr} AB) \\
&\quad \left. - 2(\Re z)^2 (\Im z)^2 \det A^2 \det(1 + y) (4 \det(x) - \operatorname{Tr}^2 x) \right] \\
&\quad + N \left(\det(A) (1 + 4(\Re z)^2 - \operatorname{Tr} C') - \left(1 - \frac{1}{N}\right) \det(B) \right). \tag{32}
\end{aligned}$$

We now rewrite (31) by using the parametrizations

$$x = \begin{pmatrix} a & a\sqrt{1-\tau}e^{i\varphi} \\ a\sqrt{1-\tau}e^{-i\varphi} & a \end{pmatrix}, \quad y = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}, \tag{33}$$

with $a \in i\mathbb{R}_+$, $\tau \in [0, 1]$, $\varphi \in [0, 2\pi]$ and $|\xi| = N^{-1}$. Since the integral over $\varphi \in [0, 2\pi]$ is equal to 2π as a consequence of the fact that the functions f, g, G_N defined below do not depend on φ , we have that

$$E \operatorname{Tr}[Y - w]^{-1} = \frac{N}{4\pi i} \oint d\xi \int_0^{i\infty} da \int_0^1 d\tau \frac{\xi^2 a}{\tau^{1/2}} e^{N[f(\xi) - g(a, \tau, \eta)]} G_N(a, \tau, \xi, z), \tag{34}$$

where, using the notation $\eta := \Im z$, the functions f and g are defined by

$$f(\xi) := -w\xi + \log(1 + \xi) - \log \xi - \frac{|z|^2}{1 + \xi}, \tag{35}$$

$$g(a, \tau, \eta) := -wa + \frac{1}{2} \log[1 + 2a + a^2 \tau] - \log a - \frac{1}{2} \log \tau - \frac{|z|^2(1 + a) - 2\eta^2 a^2(1 - \tau)}{1 + 2a + a^2 \tau}. \tag{36}$$

Note that $g(a, 1, \eta) = f(a)$; in particular, we remark that $g(a, 1, \eta)$ is independent of η for any $a \in \mathbb{C}$.

Furthermore, using the parametrizations in (33) the function $G_N := G_{1,N} + G_{2,N}$ is given by

$$G_{1,N} = \left(N^2 \frac{p_{2,0,0}}{a^2 \xi^2 (\xi+1)^2 \tau} - N \frac{p_{1,0,0}}{a^2 \xi^2 (\xi+1) \tau} + \delta N^2 \frac{p_{2,0,1}}{a \xi (\xi+1)^2 \tau} - N \delta \frac{p_{1,0,1}}{a \xi (\xi+1) \tau} + N^2 \delta^2 \frac{p_{2,0,2}}{(\xi+1)^2} \right) \times ((a^2 \tau + 2a + 1)^2 (\xi+1)^2)^{-1}, \quad (37)$$

$$G_{2,N} = \left(N^2 \eta^2 \frac{p_{2,2,0}}{a \xi (\xi+1)^3 \tau} - N \eta^2 \frac{p_{1,2,0}}{a \xi \tau} + N^2 \eta^2 \delta \frac{p_{2,2,1}}{(\xi+1)} \right) \times ((a^2 \tau + 2a + 1)^2 (\xi+1)^2)^{-1},$$

where $p_{i,j,k} = p_{i,j,k}(a, \tau, \xi)$ are explicit polynomials in a, τ, ξ which we defer to [Appendix B](#), $\eta := \Im z$ and $\delta := 1 - |z|^2$. The indices i, j, k in the definition of $p_{i,j,k}$ denote the N , η and δ power, respectively. We split G_N as the sum of $G_{1,N}$ and $G_{2,N}$ since $G_{1,N}$ depends only on $|z|$, whilst $G_{2,N}$ depends explicitly by $\eta = \Im[z]$, hence $G_{2,N} = 0$ if $z \in \mathbb{R}$.

4. Asymptotic analysis in the complex case for the saddle point regime

For the density of states ϱ_{Y^z} on the positive semiaxis $E > 0$ we expect a singular behavior for $E \approx 0$ and a square-root edge for $E \approx \epsilon_+$. The singularity at $E = 0$ exhibits a phase transition in δ at 0; for $\delta > 0$ the transition is between an $E^{-1/3}$ -singularity for $\delta = 0$ and a $\delta^{1/2} E^{-1/2}$ -singularity for $0 < \delta \leq 1$, while for $\delta < 0$ the transition is between the $E^{-1/3}$ -singularity for $\delta = 0$ and square-root edge in $\epsilon_- \sim |\delta|^3$ of slope $|\delta|^{-5/2}$. We now analyze the location of the critical point(s) x_* , i.e., the solutions to $f'(x_*) = 0$, as well as the asymptotics of the phase function f around them precisely in all of the above regimes. For the saddlepoint approximation the second derivative $f''(x_*)$ is of particular importance and we find that it can only vanish in the vicinity of $E \approx \epsilon_+$ and $E \approx \epsilon_- \vee 0$, and otherwise satisfies $|f''(x_*)| \gtrsim 1$.

The saddle point equation $f'(x_*) = 0$ leads to the simple cubic equation

$$wx_*^3 + 2wx_*^2 + wx_* + \delta x_* + 1 = 0,$$

which is precisely the MDE equation from (11), whose explicit solution via Cardano's formula reveals that for $E \in (\epsilon_-, \epsilon_+)$ there are two relevant critical points $x_*, \bar{x}_*$ with $\Re f(x_*) = \Re f(\bar{x}_*)$, while for $E \geq \epsilon_+$ or $0 \leq E \leq \epsilon_-$ there is one relevant critical point x_* , where x_* is given by

$$x_* = \begin{cases} e^{-2i\pi/3} \sqrt[3]{q + \sqrt{q^2 + p^3}} + e^{2i\pi/3} \sqrt[3]{q - \sqrt{q^2 + p^3}} - \frac{2}{3}, & E \leq \epsilon_-, \\ e^{2i\pi/3} \sqrt[3]{q + \sqrt{q^2 + p^3}} + e^{-2i\pi/3} \sqrt[3]{q - \sqrt{q^2 + p^3}} - \frac{2}{3}, & \epsilon_- \leq E \leq \epsilon_+, \\ \sqrt[3]{q + \sqrt{q^2 + p^3}} + \sqrt[3]{q - \sqrt{q^2 + p^3}} - \frac{2}{3}, & E \geq \epsilon_+, \end{cases} \quad (38)$$

$$q := \frac{\delta}{3E} + \frac{1}{27} - \frac{1}{2E}, \quad p := \frac{\delta}{3E} - \frac{1}{9},$$

where $q^2 + p^3 > 0$ as long as $E \in (\epsilon_-, \epsilon_+)$ and $q^2 + p^3 < 0$ for $E > \epsilon_+$ or $E < \epsilon_-$. Here we chose the branch of the cubic root such that $\sqrt[3]{\mathbb{R}} = \mathbb{R}$ and that $\sqrt[3]{z}$ for $z \in \mathbb{C} \setminus \mathbb{R}$ is the cubic root with the maximal real part. Note that the choice of the cubic root implies $\Im x_* \geq 0$ and $x_* = x_*(E) = m^z(E + i0)$, where m^z has been defined in (11).

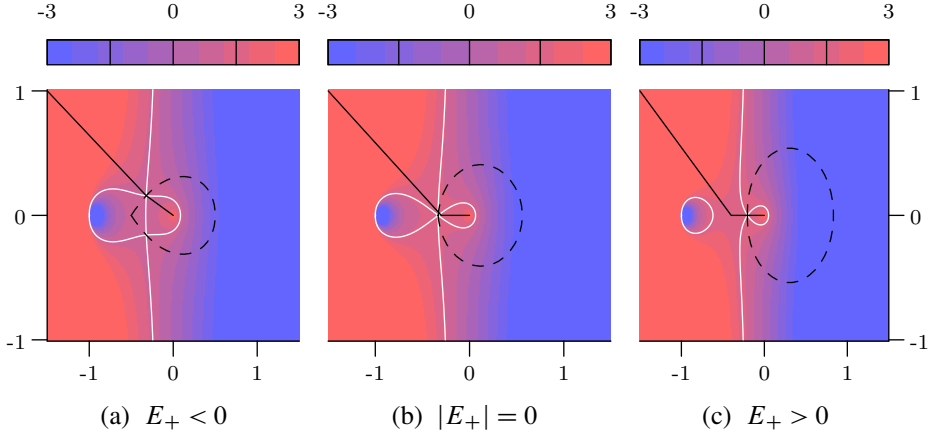


Figure 3. Contour plot of $\Re f(x)$ in the regime $E \approx \epsilon_+$. The solid white lines represent the level set $\Re f(x) = \Re f(x_*)$, while the solid and dashed black lines represent the chosen contours for the x - and y -integrations, respectively.

Before concluding this section with the proof of [Proposition 2.2](#), we collect certain asymptotics of the critical point x_* and the phase function f in its vicinity, which will also be used in the main estimates of the present paper in [Sections 5–6](#). In the edges $\epsilon_{\pm}$ the critical points have the simple expressions

$$x_*(\epsilon_{\pm}) = -\frac{2}{3 \pm \sqrt{9-8\delta}}$$

and satisfy $x_*(\epsilon_+) \sim -1$ and $x_*(\epsilon_-) = -3/(2\delta)[1 + \mathcal{O}(|\delta|)]$. Elementary expansions of [\(38\)](#) for E near the edges reveal the following asymptotics of x_* in the various regimes.

Regime $E \approx \epsilon_+$. Close to the spectral edge $E \approx \epsilon_+$ we have the asymptotic expansion

$$x_* = x_*(\epsilon_+) + \gamma_+ \sqrt{E_+} (1 + \mathcal{O}(|E_+^{1/2}|)) \quad (39a)$$

in $E_+ := E - \epsilon_+$, where γ_+ was defined in [\(18b\)](#). The location of saddle point(s) in the regime $E \approx \epsilon_+$ is depicted in [Figure 3](#). The second derivative of f is asymptotically given by

$$f''(x_*) = \frac{2\sqrt{E_+}}{\gamma_+} (1 + \mathcal{O}(|E_+^{1/2}|)). \quad (39b)$$

Regime $E \approx 0$ in the case $\delta \geq 0$. For $E \approx 0$ we have the asymptotic expansions

$$x_* = E^{-1/3} \Psi\left(\frac{\delta}{E^{1/3}}\right) [1 + \mathcal{O}(\delta + E^{1/3})], \quad (40a)$$

where $\Psi(\lambda)$ is the unique solution to the cubic equation

$$1 + \lambda \Psi(\lambda) + \Psi(\lambda)^3 = 0, \quad \Re \Psi(\lambda) > 0, \quad \Im \Psi(\lambda) > 0, \quad \lambda \geq 0.$$

The explicit function $\Psi(\lambda)$ has the asymptotics

$$\lim_{\lambda \searrow 0} \Psi(\lambda) = \Psi(0) = e^{i\pi/3}, \quad \lim_{\lambda \rightarrow \infty} \frac{\Psi(\lambda)}{\sqrt{\lambda}} = i.$$

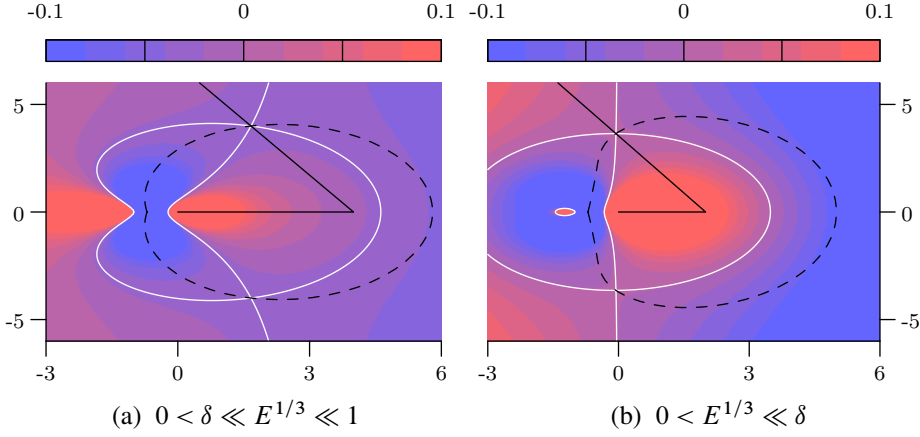


Figure 4. Contour plot of $\Re f(x)$ for $\delta > 0$ in the regime $E \approx 0$. The solid white lines represent the level set $\Re f(x) = \Re f(x_*)$, while the solid and dashed black lines represent the chosen contours for the x - and y -integrations, respectively.

Thus it follows that

$$\begin{aligned} x_* &= \left(i\sqrt{\frac{\delta}{E}} - 1 + \frac{1}{2\delta} \right) \left(1 + \mathcal{O}\left(\frac{E}{\delta^3}\right) \right) \\ &= \left(\frac{e^{i\pi/3}}{E^{1/3}} - \frac{2}{3} \right) \left(1 + \mathcal{O}\left(E^{2/3} + \frac{\delta}{E^{1/3}}\right) \right), \end{aligned} \quad (40b)$$

where the first expansion is informative in the $E \ll \delta^3$, and the second one in the $E \gg \delta^3$ regime. The location of saddle point(s) in the regime $E \approx 0$ is depicted in Figure 4. For the second derivative we have the expansions

$$\begin{aligned} f''(x_*) &= 3e^{2i\pi/3} E^{4/3} \left(1 + \mathcal{O}\left(E + \frac{\delta}{E^{1/3}}\right) \right) \\ &= 2i \frac{E^{3/2}}{\delta^{1/2}} \left(1 + \mathcal{O}\left(\frac{E}{\delta^3}\right) \right) \end{aligned} \quad (40c)$$

and similarly for higher derivatives, $|f^{(k)}(x_*)| \sim E^{(2+k)/3} \wedge E^{(k+1)/2} \delta^{-(k-1)/2}$ for $k \geq 3$.

Regime $E \approx \epsilon_-$ in the case $\delta < 0$. Around the spectral edge ϵ_- the critical point admits the asymptotic expansion

$$\begin{aligned} x_* &= x_*(\epsilon_-) + \gamma_- \left(1 + \mathcal{O}\left(\frac{|E|^{1/2}}{\delta^{3/2}}\right) \right) \begin{cases} i\sqrt{|E_-|}, & E_- \geq 0, \\ -\sqrt{|E_-|}, & E_- \leq 0, \end{cases} \\ &= \frac{1}{E^{1/3}} \left(e^{i\pi/3} + \frac{i}{3} e^{i\pi/3} \frac{\delta}{E^{1/3}} (1 + \mathcal{O}(|E|^{1/3})) + \mathcal{O}\left(\frac{|\delta|^2}{E^{2/3}}\right) \right), \end{aligned} \quad (41a)$$

where $E_- := E - \epsilon_-$, and these separate expansions are relevant in the $|E| \ll |\delta|^3$ and $|E| \gg |\delta|^3$ regimes, respectively. The location of saddle point(s) in the regime $E \approx \epsilon_-$ is depicted in Figure 5. The second

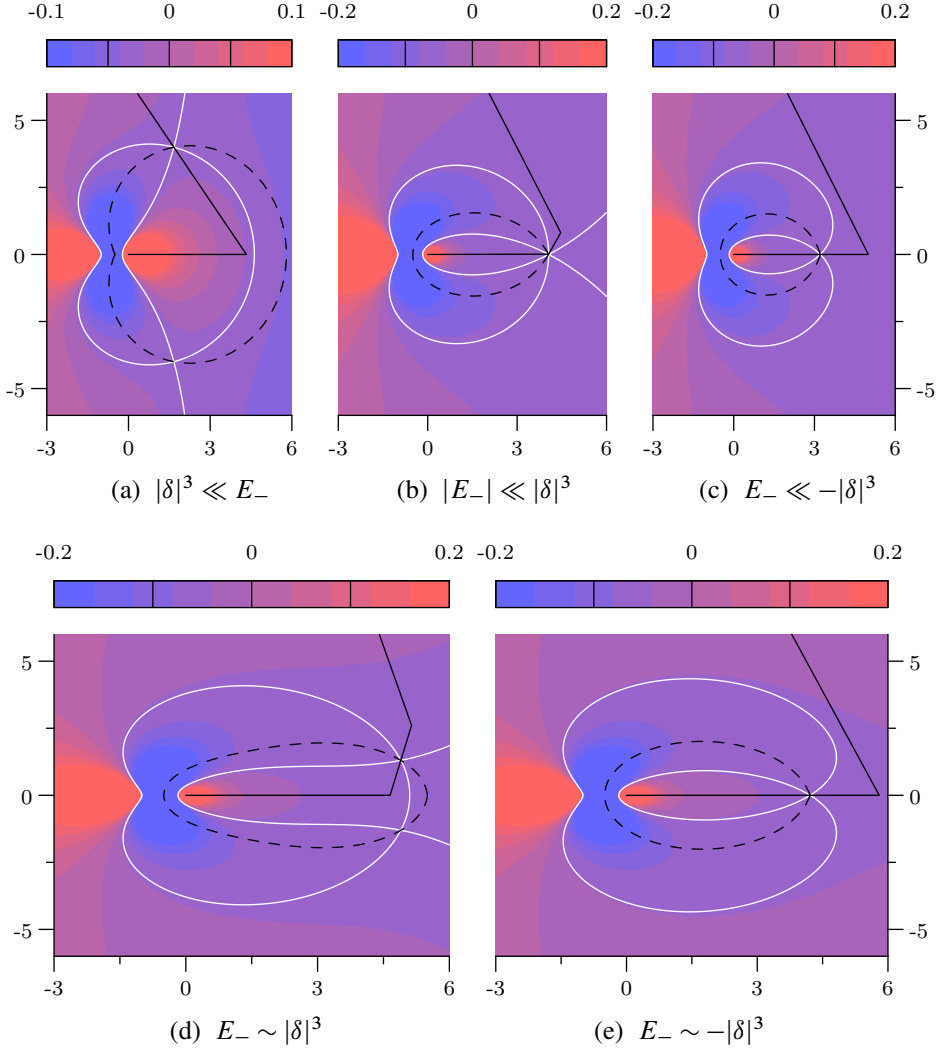


Figure 5. Contour plot of $\Re f(x)$ in the regime $E \approx \epsilon_-$. The solid white lines represent the level set $\Re f(x) = \Re f(x_*)$, while the solid and dashed black lines represent the chosen contours for the x - and y -integrations, respectively.

derivative around x_* is given by

$$\begin{aligned}
 f''(x_*) &= \frac{2}{\gamma_-} (1 + \mathcal{O}(|E_-|^{1/2} |\delta|^{-3/2})) \times \begin{cases} \sqrt{|E_-|}, & E_- \leq 0, \\ -i\sqrt{|E_-|}, & E_- \geq 0, \end{cases} \\
 &= 3e^{2i\pi/3} E^{4/3} \left(1 + \mathcal{O}\left(E + \frac{|\delta|}{E^{1/3}}\right) \right), \tag{41b}
 \end{aligned}$$

with $\gamma_- \sim |\delta|^{-5/2}$.

Proof of Proposition 2.2. As the functions f and G in (28) are meromorphic we are free to deform the contours for the x - and y -integrals as long as we are not crossing 0 or -1 and the x -contour goes

out from 0 in the “right” direction (in the region $\Re[x] < 0$, $\Im[x] > 0$ in Figure 3, and in the region $\Re[x] > |\Im[x]|$ in Figures 4–5). It is easy to see that the contours can always be deformed in such a way that $\Re f(x) > \Re f(x_*) = \Re f(\bar{x}_*)$ and $\Re f(y) < \Re f(x_*)$ for all $x, y \neq x_*, \bar{x}_*$; see Figures 3–5 for an illustration of the chosen contours.

We now compute the integral (28) in the large N limit when E is near the edges. In certain regimes of the parameters N , E and δ a saddle point analysis is applicable after a suitable contour deformation. In most cases, the result is a point evaluation of the integrand at the saddle points. In some transition regimes of the parameters the saddle point analysis only allows us to explicitly scale out some combination of the parameters and leaving an integral depending only on a reduced set of rescaled parameters.

We recall the classical quadratic saddle point approximation for holomorphic functions $f(z)$, $g(z)$ such that $f(z)$ has a unique critical point in some z_* , and that γ can be deformed to go through z_* in such a way that $\Re f(z) < \Re f(z_*)$ for all $\gamma \ni z \neq z_*$. Then for large $\lambda \gg 1$ the saddle point approximation is given by

$$\int_{\gamma} g(z) e^{\lambda f(z)} dz = \pm g(z_*) e^{\lambda f(z_*)} \sqrt{\frac{2\pi}{\lambda |f''(z_*)|}} i e^{-\frac{i}{2} \arg f''(z_*)} \left(1 + \mathcal{O}\left(\frac{1}{\lambda}\right)\right), \quad (42)$$

where $\pm$ is determined by the direction of γ through z_* with $+$ corresponding to the direction parallel to $i \exp(-\frac{i}{2} \arg f''(z_*))$. This formula is applicable, i.e., we can use point evaluation in the *saddle point regime*, whenever the lengthscale $\ell_f \sim (N |f''(x_*)|)^{-1/2}$ of the exponential decay from the quadratic approximation of the phase function is much smaller than the scale $\ell_g \sim |g(x_*)|/|\nabla g(x_*)|$ on which g is essentially unchanged. For our integral (28) we thus need to check the condition

$$\frac{1}{\sqrt{N |f''(x_*)|}} \ll \left| \frac{\nabla(yG(x, y))}{yG(x, y)} \right|_{(x,y)=(x_*,x_*)}^{-1} \quad (43)$$

in all regimes separately.

In the regime $E \approx \epsilon_+$, using the asymptotics for x_* from (39a) and (39b), the quadratic saddle point approximation is valid if

$$\frac{1}{\sqrt{N |E_+|^{1/2}}} \ll |E_+|^{1/2},$$

i.e., if $|E_+| \gg N^{-2/3}$. Here the length-scale $|E_+|^{1/2}$ represents the length-scale on which $(x, y) \mapsto yG(x, y)$ is essentially constant which can be obtained by explicitly computing the log-derivative

$$\left| \frac{\nabla(yG(x, y))}{yG(x, y)} \right|_{(x,y)=(x_*,x_*)} \sim |E_+|^{-1/2}.$$

Similar calculations yield that for $E \approx 0$ and $\delta \geq 0$ the quadratic saddle point approximation is valid if

$$\frac{1}{\sqrt{N(E^{4/3} \wedge E^{3/2} \delta^{-1/2})}} \ll E^{-1/3} \vee \delta^{1/2} E^{-1/2}, \quad \text{i.e., } E \gg N^{-3/2} \wedge N^{-2} \delta^{-1},$$

while for $E \approx \epsilon_-$ and $\delta < 0$ the condition (43) reads

$$\frac{1}{\sqrt{N(E^{4/3} \vee |E_-|^{1/2} |\delta|^{5/2})}} \ll E^{-1/3} \wedge |E_-|^{1/2} |\delta|^{-5/2}, \quad \text{i.e., } |E_-| \gg N^{-2/3} |\delta|^{5/3},$$

recalling that $E = E_- + \epsilon_-$ and $\epsilon_- \sim \delta^3$ from (18b).

In these regimes we can thus apply (42) to (28) and using that

$$G(x_*, x_*) = f''(x_*), \quad G(x_*, \bar{x}_*) = 0,$$

as follows from explicit computations, we thus finally conclude (14). Here the error terms in (14) follow from (42) by choosing $\lambda \sim (\ell_g/\ell_f)^2$ according to asymptotics of the second derivatives and log-derivatives above. More precisely, for example in the second case $E \approx 0$ and $E^{1/3} \gg \delta > 0$, the phase function f is approximately given by

$$f(x) \approx E^{2/3} \left[\frac{1}{2(E^{1/3}x)} - E^{1/3}x \right],$$

while G can asymptotically be written as

$$yG(x, y) \approx x_*G(x_*, x_*) + 3E^{4/3}e^{2i\pi/3}(2(x - x_*) + (y - x_*)) + \mathcal{O}(E^{5/3}(|x - x_*|^2 + |y - x_*|^2)).$$

Thus we make the change of variables $x = x_* + E^{-1/3}x'$, $y = x_* + E^{-1/3}y'$ to find

$$\begin{aligned} & \frac{N^2}{2\pi i} E^{-2/3} \int dx' \int dy' e^{-NE^{-2/3}f''(x_*)(\frac{x'^2}{2} - \frac{y'^2}{2}) - NE^{-1}f'''(x_*)(\frac{x'^2}{6} - \frac{y'^2}{6}) + \mathcal{O}(NE^{2/3}(|x'|^4 + |y'|^4))} \\ & \quad \times (x_*G(x_*, x_*) + 3Ee^{2i\pi/3}(y' + 2x') + \mathcal{O}(E(|x'|^2 + |y'|^2))) \\ & = x_* \left(1 + \mathcal{O}\left(\frac{1}{NE^{2/3}}\right) \right), \end{aligned}$$

where we used that $G(x_*, x_*) \sim E^{4/3}$. The other cases in (14) can be checked similarly. $\square$

5. Derivation of the 1-point function in the critical regime for the complex case

In this section we prove [Theorem 2.1](#), i.e., we study $E \operatorname{Tr}[Y - w]^{-1}$, with $w = E + i\epsilon$, $1 \gg |E| \gg \epsilon > 0$, for E so close to 0 such that $|E|$ is smaller or comparable with the eigenvalues scaling around 0. We will first consider the case $\delta \geq 0$ and afterwards explain the necessary changes in the regime $-CN^{-1/2} \leq \delta < 0$.

5A. Case $0 \leq \delta \leq 1$. In the following of this section we assume that $E > 0$, since we are interested in the computations of (28) for $E = \Re[w]$ inside the spectrum of Y . In order to study the transition between the local law regime, that is considered in [Section 4](#), and the regime when the main contribution to (28) comes from the smallest eigenvalue of Y , we define the parameter

$$c(N) = c(N, \delta) := \frac{1}{N^{3/2}} \wedge \frac{1}{\delta N^2}. \quad (44)$$

In particular, in the regime $E \gg c(N)$ the double integral in (28) is computed by saddle point analysis in (14), i.e., the main contribution comes from the regime around the stationary point x_* of f , with x_*

defined in (40b), whilst for $E \lesssim c(N)$ the main contribution to (28) comes from a larger regime around the stationary point x_* . From now on we assume that $E \lesssim c(N)$. In the following we denote the leading order of the stationary point x_* by

$$z_* = z_*(E, \delta) := E^{-1/3} \Psi(\delta E^{-1/3}), \quad (45)$$

where $\Psi(\lambda)$ was defined in (40a) and has the asymptotics $\Psi(0) = e^{i\pi/3}$ and $\Psi(\lambda)/\sqrt{\lambda} \rightarrow i$ as $\lambda \rightarrow \infty$. Note that $|z_*| \gg 1$ for any $E \ll 1$, $0 \leq \delta \leq 1$. For this reason, we expect that the main contribution to the double integral in (28) comes from the regime when $|x|$ and $|y|$ are both large, say $|x|, |y| \geq N^\rho$, for some small fixed $0 < \rho < 1/2$. Later on in this section, see Lemma 5.4, we prove that the contribution to (28) in the regime when either $|x|$ or $|y|$ are smaller than N^ρ is exponentially small. In order to get the asymptotics in (13a), is not affordable to estimate the error terms in the Taylor expansion by absolute value. In particular, it is not affordable to estimate the integral of $e^{Nf(y)}$ over Γ by absolute value, hence the improved bound in (54) is needed. To make our writing easier, for any $R \in \mathbb{N}$, $R \geq 2$, we introduce the notation

$$\mathcal{O}^\#(x^{-R}) := \{g \in \mathcal{P}_R\}, \quad \mathcal{O}^\#((x, y)^{-R}) := \{g \in \mathcal{Q}_R\}, \quad (46)$$

where $\mathcal{P}_R$ and $\mathcal{Q}_R$ are defined as Laurent series of order at least R around infinity, i.e.,

$$\begin{aligned} \mathcal{P}_R &:= \left\{ g: \mathbb{C} \rightarrow \mathbb{C} \mid g(x) = \sum_{\alpha \geq R} \frac{\tilde{c}_\alpha}{x^\alpha}, \text{ with } |\tilde{c}_\alpha| \leq C^\alpha, \text{ if } |x| \geq 2C \right\}, \\ \mathcal{Q}_R &:= \left\{ \tilde{g}: \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \mid \tilde{g}(x, y) = \sum_{\alpha, \beta \geq 1, \alpha + \beta \geq R} \frac{c_{\alpha, \beta}}{x^\alpha y^\beta}, \text{ with } |c_{\alpha, \beta}| \leq C^{\alpha + \beta}, \text{ if } |x|, |y| \geq 2C \right\}, \end{aligned}$$

for some constant $C > 0$ that is implicit in the $\mathcal{O}^\#$ notation. Here α, β are integer exponents. Note that

$$\mathcal{O}^\#(|x|^{-R}) = \mathcal{O}(|x|^{-R})$$

for any $x \in \mathbb{C}$. Then, we expand the phase function f for large argument as follows

$$f(x) = g(x) + \mathcal{O}^\#(x^{-3} + \delta x^{-2}), \quad g(x) := -(E + i\epsilon)x + \frac{\delta}{x} + \frac{1}{2x^2}. \quad (47a)$$

and for large x and y we expand G as

$$G(x, y) = H(x, y) + \mathcal{O}^\#((x, y)^{-5} + \delta(x, y)^{-4}), \quad H(x, y) := \frac{1}{x^3 y} + \frac{1}{x^2 y^2} + \frac{1}{x y^3} + \frac{\delta}{x y^2} + \frac{\delta}{x^2 y}. \quad (47b)$$

In order to compute the integral in (28) we deform the contours Λ and Γ through z_* , with z_* defined in (45). In particular, we are allowed to deform the contours as long as the x -contour goes out from zero in region $\Re[x] > |\Im[x]|$, it ends in the region $\Re[x] < 0$, $\Im[x] > 0$, and it does not cross 0 and -1 along the deformation; the y -contour, instead, can be freely deformed as long as it does not cross 0 and -1 . Hence, we can deform the y -contour as $\Gamma = \Gamma_{z_*} := \Gamma_{1, z_*} \cup \Gamma_{2, z_*}$, where

$$\Gamma_{1, z_*} := \left\{ -\frac{2}{3} + it : 0 \leq |t| \leq \sqrt{|z_*|^2 - \frac{4}{9}} \right\}, \quad \Gamma_{2, z_*} := \{ |z_*^*| e^{i\psi} : \psi \in [-\psi_{z_*}, \psi_{z_*}] \}, \quad (48a)$$

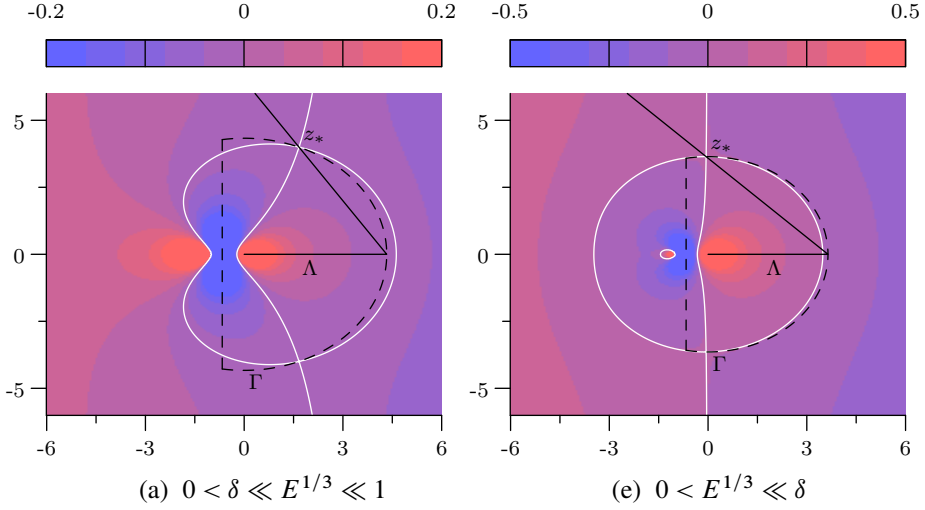


Figure 6. Illustration of the contours (48a)–(48b) together with the phase diagram of $\Re f$, where the white line represents the level set $\Re f(x) = \Re f(z_*)$. Note that the precise choice of the contours is only important close to 0 and for very large $|x|$ as otherwise the phase function is small.

with $\psi_{z_*} = \arccos[-2/(3|z_*|)]$, and the x -contour as $\Lambda = \Lambda_{z_*} := \Lambda_{1,z_*} \cup \Lambda_{2,z_*}$, with

$$\Lambda_{1,z_*} := [0, |z_*|), \quad \Lambda_{2,z_*} := \{|z_*| - qs + is : s \in [0, +\infty)\}, \quad (48b)$$

where

$$q = q_{z_*} := \Im[z_*]^{-1}(|z_*| - \Re[z_*]). \quad (48c)$$

Note that $q \sim 1$ uniformly in N , E and δ , since $\Re[z_*] \lesssim \Im[z_*]$ for any $E \ll 1$, $0 \leq \delta \leq 1$. We assume the convention that the orientation of Γ is counter clockwise. See Figure 6 for an illustration of Γ and Λ .

Before proceeding with the computation of the leading term of (28), in the following lemma we state some properties of the function f on the contours Γ , Λ . Using that $\epsilon \ll E$, the proof of the lemma below follows by easy computations.

Lemma 5.1. *Let f be the phase function defined in (28), then the following properties hold true:*

(i) *For any $y = -2/3 + it \in \Gamma_{1,z_*}$, we have that*

$$\Re[f(-2/3 + it)] = \frac{2E}{3} + \epsilon t - \frac{1}{2t^2} + \mathcal{O}(|t|^{-3} + \delta|t|^{-2}), \quad (49)$$

and

$$\Im[f(-2/3 + it)] = \frac{2\epsilon}{3} - Et - \frac{\delta}{t} + \mathcal{O}(|t|^{-3}). \quad (50)$$

(ii) *For $\epsilon = 0$, the function $t \mapsto \Re[f(-2/3 + it)]$ on Γ_{1,z_*} is strictly increasing if $t > 0$ and strictly decreasing if $t < 0$.*

(iii) *The function $x \mapsto \Re[f(x)]$ is strictly decreasing on Λ_{1,z_*} .*

(iv) Let $x \in \Lambda_{2,z_*}$ be parametrized as $x = |z_*| - qs + is$, for $s \in [0, +\infty)$, with q defined in (48c), then

$$\Re[f(x)] = -E(|z_*| - qs) + \epsilon s + \frac{\delta(|z_*| - qs)}{s^2 + (qs - |z_*|)^2} + \frac{(1-2\delta)[(|z_*| - qs)^2 - s^2]}{(s^2 + (qs - |z_*|)^2)^2} + \mathcal{O}([s^2 + |z_*|^2]^{-3/2}). \quad (51)$$

Despite the fact that saddle point analysis is not useful anymore in this regime, we expect that the main contribution to (28) comes from the regime in the double integral when both x and y are large, i.e., $|x|, |y| \geq N^\rho$. For this purpose we define

$$\tilde{\Lambda} := \{x \in \Lambda_{1,z_*} : |x| \leq N^\rho\}, \quad \tilde{\Gamma} := \{y \in \Gamma_{1,z_*} : |y| \leq N^\rho\}. \quad (52)$$

In the following part of this section we will first prove that the contribution to (28) in the regime when either $x \in \tilde{\Lambda}$ or $y \in \tilde{\Gamma}$ is exponentially small and then we explicitly compute the leading term of (28) in the regime

$$(x, y) \in (\Lambda \setminus \tilde{\Lambda}) \times (\Gamma \setminus \tilde{\Gamma}).$$

For this purpose, we first prove a bound for the double integral in the regime $y \in \Gamma \setminus \tilde{\Gamma}$ or $x \in \Lambda \setminus \tilde{\Lambda}$ in Lemma 5.2 and Lemma 5.3, respectively, and then we conclude the estimate for $x \in \tilde{\Lambda}$ or $y \in \tilde{\Gamma}$ in Lemma 5.4. Finally, in Theorem 2.1 we consider the regime $(x, y) \in (\Lambda \setminus \tilde{\Lambda}) \times (\Gamma \setminus \tilde{\Gamma})$ and compute the leading term of (28).

Lemma 5.2. Let $c(N)$ be defined in (44), $E \lesssim c(N)$, $b \in \mathbb{N}$, and let f be defined in (28), then

$$\left| \int_{\Gamma \setminus \tilde{\Gamma}} \frac{e^{Nf(y)}}{y^b} dy \right| \lesssim |z_*|^{1-b} + \begin{cases} |z_*|, & b = 0, \\ 1 + |\log(N|z_*|^{-2})|, & b = 1, \delta < |z_*|^{-1}, \\ 1 + |\log(N\delta|z_*|^{-1})|, & b = 1, \delta \geq |z_*|^{-1}, \\ N^{\frac{1-b}{2}} \wedge (N\delta)^{1-b}, & b \geq 2, \end{cases} \quad (53)$$

where $\Gamma, \tilde{\Gamma}$ are defined in (48a) and (52), respectively. Furthermore, we have that

$$\int_{\Gamma \setminus \tilde{\Gamma}} e^{Nf(y)} dy = \mathcal{O}(N^{1/2} \vee (N\delta)), \quad \int_{\Gamma \setminus \tilde{\Gamma}} \frac{e^{Nf(y)}}{y} dy = \mathcal{O}(1). \quad (54)$$

Proof. First, we notice that if $y \in \Gamma \setminus \tilde{\Gamma}$ then $|y| \geq N^\rho$, hence we expand f as in (47a), i.e.,

$$f(y) = -(E + i\epsilon)y + \frac{\delta}{y} + \frac{1}{2y^2} + \mathcal{O}(|y|^{-3} + \delta|y|^{-2}). \quad (55)$$

Moreover, by (45) it follows that $|z_*| \sim E^{-1/3} \vee \sqrt{\delta E^{-1}}$, and so that

$$|Nf(y)| \lesssim NE^{2/3} + N\sqrt{\delta E} \quad (56)$$

for any $|y| \sim |z_*|$, $E \lesssim c(N)$. Note that $\Gamma \setminus \tilde{\Gamma} = (\Gamma_{1,z_*} \setminus \tilde{\Gamma}) \cup \Gamma_{2,z_*}$, with $\Gamma_{1,z_*}, \Gamma_{2,z_*}$ defined in (48a). By (56) it easily follows that $|Nf(y)| \lesssim 1$ for any $y \in \Gamma_{2,z_*}$, which clearly implies that

$$\int_{\Gamma_{2,z_*}} \left| \frac{e^{Nf(y)}}{y^b} \right| |dy| \lesssim |z_*|^{1-b}. \quad (57)$$

To conclude the proof of (53) we bound the integral on $\Gamma_{1,z_*} \setminus \tilde{\Gamma}$. Let $w = E + i\epsilon$, then in this regime, by (49)–(50), we have

$$\begin{aligned} & \int_{\Gamma_{1,z_*} \setminus \tilde{\Gamma}} \frac{e^{Nf(y)}}{y^b} dy \\ &= -i \int_{N^\rho}^{\sqrt{|z_*|^2 - 4/9}} e^{-N[1/(2t^2) + \mathcal{O}(|t|^{-3} + \delta|t|^{-2})]} \times \left(\frac{e^{-N[iwt + i\frac{\delta}{t}]} }{(-2/3 + it)^b} + \frac{e^{N[iwt + i\frac{\delta}{t}]} }{(-2/3 - it)^b} \right) (1 + \mathcal{O}(NE)) dt. \end{aligned} \quad (58)$$

For any $b \in \mathbb{N}$, we estimate the integral above as follows

$$\left| \int_{\Gamma_{1,z_*} \setminus \tilde{\Gamma}} \frac{e^{Nf(y)}}{y^b} dy \right| \lesssim \left| \int_{N^\rho}^{|z_*|} \frac{e^{-\frac{N}{2t^2} + i\frac{\delta N}{t}}}{t^b} dt \right| \lesssim \begin{cases} |z_*|, & b = 0, \\ 1 + |\log(N|z_*|^{-2})|, & b = 1, \delta < |z_*|^{-1}, \\ 1 + |\log(N\delta|z_*|^{-1})|, & b = 1, \delta \geq |z_*|^{-1}, \\ N^{\frac{1-b}{2}} \wedge (N\delta)^{1-b}, & b \geq 2. \end{cases} \quad (59)$$

Note that in (59) for $b = 0, 1$ we get the bound in the r.h.s. bringing the absolute value inside the integral, whilst this is not affordable to get the bound for $b \geq 2$, since the term $e^{i\delta N/t}$ has to be used. Indeed, we would get a bound $N^{(1-b)/2}$ for $b \geq 2$ if we estimate the integral in (59) moving the absolute value inside. In the following part of the proof we compute the integral (58) for $b = 0, 1$ without estimating it by absolute value.

In particular, for $b = 1$, we prove that the leading term of the r.h.s. of (58) is $\mathcal{O}(1)$, instead of the overestimate $1 + |\log(N|z_*|^{-2})|$ in (59), as a consequence of the symmetry of Γ_{1,z_*} with respect to 0. For this computation we have to distinguish the cases $\delta \gg N^{-1/2}$ and $\delta \lesssim N^{-1/2}$. If $E \sim c(N)$ and $\delta \lesssim N^{-1/2}$, then $N|z_*|^{-2} \sim 1$, hence the bound in (54) directly follows by (57) and (59). We are left with the cases $E \ll c(N)$ and $E \sim c(N)$, $\delta \gg N^{-1/2}$. For $\delta \gg N^{-1/2}$, we have $|z_*| \sim \sqrt{\delta/E}$ and using $|Nwt| \leq NE|z_*| \ll 1$, if $E \ll c(N)$, and $|Nwt| \sim 1$, if $E \sim c(N)$, we conclude

$$\int_{N^\rho}^{\sqrt{|z_*|^2 - 4/9}} \frac{e^{-N[1/(2t^2) \pm i\frac{\delta}{t} \pm itw]}}{-2/3 + it} dt = \int_{N^\rho}^{|z_*|} \frac{e^{-N/(2t^2) \pm i\frac{\delta}{t}}}{t} dt + \mathcal{O}(1) = |\log(N\delta|z_*|^{-1})| + \mathcal{O}(1).$$

Similarly we prove that the integral in the l.h.s. of the above equalities is equal to $|\log(N|z_*|^{-2})| + \mathcal{O}(1)$ if $\delta \ll N^{-1/2}$. Similar calculation holds if the denominator is $(-\frac{2}{3} - it)$ instead of $(-\frac{2}{3} + it)$, just an overall sign changes. Thus the leading terms from the two parts of the integral in (58) cancel each other. We thus conclude the second bound in (54) combining the above computations with (57) and (58).

Next, we compute the integral of $e^{Nf(y)}$ on $\Gamma \setminus \tilde{\Gamma}$, i.e., we prove the first bound in (54). We consider only the regime $E \ll c(N)$, since in the regime $E \sim c(N)$ the bound in (54) follows directly by (57), (59), and the definition of $c(N)$ in (44), since $|z_*| \sim N^{1/2} \vee N\delta$. On Γ_{2,z_*} , using the parametrization $y = |z_*|e^{i\psi}$, and that by (56) we have $|Nf(y)| \ll 1$ for $E \ll c(N)$, we Taylor expand $e^{Nf(y)}$ and conclude that

$$\int_{\Gamma_{2,z_*}} e^{Nf(y)} dy = 2|z_*|i \cdot [1 + \mathcal{O}(NE|z_*| + \delta N|z_*|^{-1} + N|z_*|^{-2})]. \quad (60)$$

Furthermore, by (58) for $b = 0$, using that $E \ll c(N)$ and so that $|Nwt| \ll 1$ on Γ_{1,z_*} , we have

$$\int_{\Gamma_{1,z_*} \setminus \tilde{\Gamma}} e^{Nf(y)} dy = -2i|z_*| + \mathcal{O}(N^{1/2} \vee N\delta).$$

The minus sign is due to the counter clockwise orientation of Γ , i.e., the vertical line Γ_{1,z_*} is parametrized from the top to the bottom. Combining this computation with (60) and using that

$$NE|z_*|^2 + N\delta + N|z_*|^{-1} \ll N^{1/2} + N\delta,$$

since $|z_*| \sim E^{-1/3} + \sqrt{\delta E^{-1}}$ by (45), we conclude the proof of this lemma. $\square$

Lemma 5.3. *Let $c(N)$ be defined in (44), $E \lesssim c(N)$ and let f be defined in (28) and $a \in \mathbb{R}$, then the following bound holds true:*

$$\int_{\Lambda \setminus \tilde{\Lambda}} \left| \frac{e^{-Nf(x)}}{x^a} \right| |dx| \lesssim \begin{cases} |z_*|^{1-a} + (NE)^{a-1}, & a < 1, \\ 1 + |\log(N|z_*|^{-2})|, & a = 1, \delta < |z_*|^{-1}, \\ 1 + |\log(N\delta|z_*|^{-1})|, & a = 1, \delta \geq |z_*|^{-1}, \\ N^{\frac{1-a}{2}} \wedge (N\delta)^{1-a}, & a > 1, \end{cases} \quad (61)$$

where $\Lambda, \tilde{\Lambda}$ are defined in (48b) and (52).

Proof. We split the computation of the integral of $e^{Nf(x)}x^{-a}$ as the sum of the integral over $\Lambda_{1,z_*} \setminus \tilde{\Lambda}$ and Λ_{2,z_*} . Using the parametrization $x = |z_*| - qs + is$, with $s \in [0, +\infty)$ and q defined in (48c), by (51), we estimate the integral over Λ_{2,z_*} as follows:

$$\int_{\Lambda_{2,z_*}} \left| \frac{e^{-Nf(x)}}{x^a} \right| |dx| \lesssim \int_0^{+\infty} \frac{e^{-N \left[-E(|z_*| - qs) + \frac{\delta(|z_*| - qs)}{(|z_*| - qs)^2 + s^2} + \frac{(|z_*| - qs)^2 - s^2}{2[(|z_*| - qs)^2 + s^2]^2} \right]}}{[(|z_*| - qs)^2 + s^2]^{a/2}} ds. \quad (62)$$

We split the computation of the integral in the r.h.s. of (62) into two parts:

$$|s| \in [0, |z_*|) \quad \text{and} \quad s \in [|z_*|, +\infty).$$

Since $q \sim 1$ and $NE|z_*| \lesssim 1$, in the regime $|s| \in [0, |z_*|)$ we estimate the integral in the r.h.s. of (62) as

$$e^{NE|z_*|} \int_0^{|z_*|} \frac{e^{-\frac{N\delta}{|z_*|} - \frac{N}{|z_*|^2}}}{|z_*|^a} ds \lesssim |z_*|^{1-a}. \quad (63)$$

In the regime $s \in [|z_*|, +\infty)$, instead, we have

$$e^{NE|z_*|} \int_{|z_*|}^{+\infty} \frac{e^{-NEqs}}{s^a} ds \lesssim \begin{cases} (NE)^{a-1}, & a < 1, \\ 1 + |\log(NE|z_*|)|, & a = 1, \\ |z_*|^{1-a}, & a > 1. \end{cases} \quad (64)$$

We are left with the estimate of the integral over $\Lambda_{1,z_*} \setminus \tilde{\Lambda}$. As for the bound in (59), using that $\Lambda_{1,z_*} \setminus \tilde{\Lambda} = [N^\rho, |z_*|)$ and that $NE|z_*| \lesssim 1$, we have that

$$\int_{\Lambda_{1,z_*} \setminus \tilde{\Lambda}} \left| \frac{e^{-Nf(x)}}{x^a} \right| |dx| \lesssim \int_{N^\rho}^{|z_*|} \frac{e^{-\frac{N\delta}{s} - \frac{N}{2s^2}}}{s^a} ds \lesssim \begin{cases} |z_*|^{1-a}, & a < 1, \\ 1 + |\log(N|z_*|^{-2})|, & a = 1, \delta < |z_*|^{-1}, \\ 1 + |\log(N\delta|z_*|^{-1})|, & a = 1, \delta \geq |z_*|^{-1}, \\ N^{\frac{1-a}{2}} \wedge (N\delta)^{1-a}, & a > 1. \end{cases} \quad (65)$$

Combining (62)–(65) we conclude the proof of (61). $\square$

Using Lemmas 5.2 and 5.3 in the following lemma we prove that the contribution to (28) in the regime where either $x \in \tilde{\Lambda}$ or $y \in \tilde{\Gamma}$ is exponentially small.

Lemma 5.4. *Let $c(N)$ be defined in (44), and let f, G be defined in (28), then, as $\epsilon \rightarrow 0^+$, for any $E \lesssim c(N)$ we have that*

$$\left| \left(\int_{\Lambda} dx \int_{\Gamma} dy - \int_{\Lambda \setminus \tilde{\Lambda}} dx \int_{\Gamma \setminus \tilde{\Gamma}} dy \right) [e^{N[f(y)-f(x)]} y G(x, y)] \right| \lesssim N^\rho (N^{1/2} + N\delta + |\log(NE^{2/3})|) e^{-\frac{1}{2}N^{1-2\rho}}. \quad (66)$$

Proof. We split the estimate of the integral over $(\Lambda \times \Gamma) \setminus [(\Lambda \setminus \tilde{\Lambda}) \times (\Gamma \setminus \tilde{\Gamma})]$ into three regimes: $(x, y) \in \tilde{\Lambda} \times \tilde{\Gamma}$, $(x, y) \in \tilde{\Lambda} \times (\Gamma \setminus \tilde{\Gamma})$, $(x, y) \in (\Lambda \setminus \tilde{\Lambda}) \times \tilde{\Gamma}$. By (ii), (iii) of Lemma 5.1, in the regimes $y \in \tilde{\Gamma}$ and $x \in \tilde{\Lambda}$, respectively, it follows that the function f attains its maximum at $y = -2/3 \pm iN^\rho$ on $\tilde{\Gamma}$ and f attains its minimum at $x = N^\rho$ on $\tilde{\Lambda}$. Hence, by the expansion in (47a) it follows that

$$\sup_{y \in \tilde{\Gamma}} |e^{Nf(y)}| + \sup_{x \in \tilde{\Lambda}} |e^{-Nf(x)}| \lesssim e^{-Nf(N^\rho)}, \quad (67)$$

with

$$f(N^\rho) = \frac{\delta}{N^\rho} + \frac{1}{2N^{2\rho}} + \mathcal{O}(N^{-3\rho} + \delta N^{-2\rho}). \quad (68)$$

Then, by (67) and (68), it follows that the integral over $(x, y) \in \tilde{\Lambda} \times \tilde{\Gamma}$ is bounded by $N^{2\rho} e^{-N^{1-2\rho}}$. Note that in the regimes $(x, y) \in \tilde{\Lambda} \times (\Gamma \setminus \tilde{\Gamma})$ and $(x, y) \in (\Lambda \setminus \tilde{\Lambda}) \times \tilde{\Gamma}$ one among $|x|$ and $|y|$ is bigger than N^ρ . Hence, expanding (28) for large x or y argument, using Lemmas 5.2 and 5.3 to estimate the regimes $x \in \tilde{\Lambda}$ and $y \in \tilde{\Gamma}$, respectively, by (67)–(68), we conclude that the integral over $(x, y) \in \tilde{\Lambda} \times (\Gamma \setminus \tilde{\Gamma})$ is bounded by $N^\rho (N^{1/2} + (N\delta)) e^{-N^{1-2\rho}/2}$, and that the one over $(x, y) \in (\Lambda \setminus \tilde{\Lambda}) \times \tilde{\Gamma}$ is bounded by $N^\rho (1 + |\log(NE^{2/3})|) e^{-N^{1-2\rho}/2}$. $\square$

Next, we compute the leading term of (28). We define $\tilde{z}_*$ as

$$\tilde{z}_*(\lambda, \tilde{\delta}) := N^{-1/2} |z_*(\lambda c(N), N^{-1/2} \tilde{\delta})|, \quad \lambda = \frac{E}{c(N)}, \quad \tilde{\delta} = N^{1/2} \delta, \quad (69)$$

where we also recalled the rescaled parameters λ and $\tilde{\delta}$ from (12). Note that $\tilde{z}_*(\lambda, \tilde{\delta})$ is N -independent, indeed all N factors scale out by using the definition of $z_*(E, \delta)$ from (45). Since $E \ll 1$ and $\delta \in [0, 1]$, by (69) it follows that the range of the new parameters is $\tilde{\delta} \leq N^{1/2}$ and $\lambda \ll c(N)^{-1}$.

We are now ready to prove our main result on the leading term of (28), denoted by $q_{\tilde{\delta}}(\lambda)$, in the complex case, Theorem 2.1. Then, the one point function of Y is asymptotically given by $p_{\tilde{\delta}}(\lambda) := \Im[q_{\tilde{\delta}}(\lambda)]$. The main inputs for the proof are the bounds in (53) and (61) that will be used to estimate the error terms in the expansions for large arguments of f and G in (47a) and (47b).

Proof of Theorem 2.1 in the case $\delta \geq 0$. By (28) and Lemma 5.4 it follows that

$$E \operatorname{Tr}[Y - w]^{-1} = \frac{N^2}{2\pi i} \int_{\Lambda \setminus \tilde{\Lambda}} dx \int_{\Gamma \setminus \tilde{\Gamma}} dy e^{-Nf(x) + Nf(y)} y \cdot G(x, y) + \mathcal{O}(N^\rho(N^{1/2} + N\delta + |\log(NE^{2/3})|)e^{-\frac{1}{2}N^{1-2\rho}}). \quad (70)$$

Note that $|x|, |y| \geq N^\rho$ for any $(x, y) \in (\Lambda \setminus \tilde{\Lambda}) \times (\Gamma \setminus \tilde{\Gamma})$. In order to prove (13a) we first estimate the error terms in the expansions of f and G in (47a)–(47b) and then in order to get an N -independent double integral we rescale the phase function by $|z_*|$. By Lemma 5.2 and Lemma 5.3, using that $|\log(N|z_*|^{-2})| + |\log(N\delta|z_*|^{-1})| \lesssim |\log(NE^{2/3})|$ by the definition of z_* in (45), it follows that

$$\left| \int_{\Lambda \setminus \tilde{\Lambda}} dx \int_{\Gamma \setminus \tilde{\Gamma}} dy \frac{e^{-Nf(x) + Nf(y)}}{x^a y^b} \right| \lesssim \begin{cases} N^{\frac{2-d}{2}} (1 \wedge \tilde{\delta}^{1-d})(1 \vee \tilde{\delta}), & b = 0, \\ N^{\frac{1-b}{2}} (1 \wedge \tilde{\delta}^{1-b})(1 + |\log(NE^{2/3})|), & a = 1, b \geq 1, \\ N^{\frac{2-d}{2}} (1 \wedge \tilde{\delta}^{2-d}), & a > 1, b \geq 1, \end{cases} \quad (71)$$

for any $a \geq 1$, $b \in \mathbb{N}$, where $d := a + b$. In order to get the bound in the r.h.s. of (71) we estimated the terms with $b = 0$ and $b = 1$ using the improved bound in (54), all the other terms are estimated by absolute value. Note that for $\lambda \ll 1$ the bound in (71) and the definition of λ in (69) imply that

$$\lim_{\epsilon \rightarrow 0^+} |E \operatorname{Tr}[Y - (E + i\epsilon)]^{-1}| \lesssim N^{3/2} (1 \vee \tilde{\delta}) \begin{cases} |\log \lambda|, & \lambda \geq \tilde{\delta}^3, \\ |\log \lambda \tilde{\delta}|, & \lambda < \tilde{\delta}^3, \end{cases}$$

if $\lambda \ll 1$, since the leading term in the expansion of $yG(x, y)$ in (47b) consists of monomials of the form $x^{-a}y^{-b}$, with $a + b = 3$, and $\delta x^{-a}y^{-b}$, with $a + b = 2$. This concludes the proof of (13c).

Now we prove the more precise asymptotics (13a). We will replace the functions f and G in (70) by their leading order approximations, denoted by g and H from (47a) and (47b). The error of this replacement in the phase function f is estimated by the Taylor expanding the exponent $e^{\mathcal{O}^\#(x^{-3} + \delta x^{-2})}$, with $\mathcal{O}^\#(x^{-3} + \delta x^{-2})$ defined in (46). Hence, by (47a) and (47b) and the bound in (71), as $\epsilon \rightarrow 0^+$, we conclude that

$$E \operatorname{Tr}[Y - w]^{-1} = \frac{N^2}{2\pi i} \int_{\Lambda \setminus \tilde{\Lambda}} dx \int_{\Gamma \setminus \tilde{\Gamma}} dy e^{-Ng(x) + Ng(y)} H(x, y) + \mathcal{O}((N + N^{3/2}\delta)[1 + |\log(NE^{2/3})|]). \quad (72)$$

The error estimates in (72) come from terms with $d \geq 4$ or terms with $d \geq 3$ multiplied by δ in (71).

We recall that $\lambda = Ec(N)^{-1}$, $\tilde{\delta} = \delta N^{1/2}$, and that $|z_*| = N^{1/2}\tilde{z}_*(\lambda, \tilde{\delta})$. Then, defining the contours

$$\hat{\Gamma} := |z_*|^{-1}\Gamma, \quad \hat{\Lambda} := |z_*|^{-1}\Lambda, \quad (73)$$

and using the change of variables $x \rightarrow x|z_*|$, $y \rightarrow y|z_*|$ in the leading term of (72) we conclude that

$$E \operatorname{Tr}[Y - w]^{-1} = \frac{N^{3/2} \tilde{z}_*(\lambda, \tilde{\delta})^{-1}}{2\pi i} \int_{\hat{\Gamma}} dy \int_{\hat{\Lambda}} dx e^{h_{\lambda, \tilde{\delta}}(y) - h_{\lambda, \tilde{\delta}}(x)} \tilde{H}_{\lambda, \tilde{\delta}}(x, y) + \mathcal{O}(N(1 \vee \tilde{\delta})[1 + |\log \lambda|]) \quad (74)$$

with $h_{\lambda, \tilde{\delta}}(x)$ and $\tilde{H}_{\lambda, \tilde{\delta}}(x, y)$ defined in (13b). Note that in order to get (74) we used that the integral in the regime when either

$$x \in [0, N^\rho |z_*|^{-1}] \text{ or } y \in [-2|z_*|^{-1}/3, -2|z_*|^{-1}/3 + iN^\rho |z_*|^{-1}]$$

is exponentially small. Moreover, since by holomorphicity we can deform the contour $\hat{\Lambda}$ to any contour, which does not cross -1 , from 0 to $e^{3i\pi/4}\infty$ and we can deform the contour $\hat{\Gamma}$ as long as it does not cross 0 , (74) concludes the proof of Theorem 2.1. $\square$

5B. Case $\delta < 0$, $|\delta| \lesssim N^{-1/2}$. We now explain the necessary changes in the case $\delta < 0$. Throughout this section we assume that $E \lesssim N^{-3/2}$. Let x_* be the stationary point of f defined in (41a); that is, the point around where the main contribution to (28) comes from in the saddle point regime for $\delta < 0$. Then, at leading order, x_* is given by $3|\delta|^{-1}/2$ if $E \ll |\delta|^3$, by $e^{\pi i/3} E^{-1/3}$ if $E \gg |\delta|^3$, and by $\mu(c)e^{\pi i/3} E^{-1/3}$ if $E = c|\delta|^3$, for some function $\mu(c) > 0$ for any fixed constant $c > 0$ independent of N , E and δ .

This regime can be treated similarly to the regime $0 \leq \delta \leq 1$, since for $|\delta| \lesssim N^{-1/2}$ the term δx^{-1} in the expansion of f , for $|x| \gg 1$, does not play any role in the bounds of Lemma 5.2, Lemma 5.3. Indeed, instead of deforming the contours Γ and Λ through the leading term of the stationary point x_* , we deform Γ and Λ through $z_* := e^{\pi i/3} E^{-1/3}$ as

$$\Gamma = \Gamma_{z_*} := \Gamma_{1, z_*} \cup \Gamma_{2, z_*} \quad \text{and} \quad \Lambda = \Lambda_{z_*} := \Lambda_{1, z_*} \cup \Lambda_{2, z_*},$$

where Γ_{1, z_*} , Γ_{2, z_*} and Λ_{1, z_*} , Λ_{2, z_*} are defined in (48a) and (48b), respectively. We could have made the same choice in the case $0 \leq \delta \lesssim N^{-1/2}$, but not for the regime $N^{-1/2} \ll \delta \leq 1$, hence, to treat both the regimes in the same way, in Section 5A we deformed the contours through (45). Note that z_* defined here is not the analogue of (45), since in all cases

$$z_* = e^{\pi i/3} E^{-1/3}.$$

The fact that $E \ll 1$ implies that $|z_*| \gg 1$, hence, like in the case $0 \leq \delta \leq 1$, we expect that the main contribution to (28) comes from the regime when $|x|, |y| \geq N^\rho$, for some small $0 < \rho < 1/2$. Hence, in order to compute the leading term of (28) we expand f and G for large arguments as in (47a) and (47b).

The phase function f defined in (28) satisfies the properties (i), (ii) and (iv) of Lemma 5.1, but (iii) does not hold true for $\delta < 0$ if $E \ll |\delta|^3$. Instead, it is easy to see that the following lemma holds true.

Lemma 5.5. *Let f be the phase function defined in (28), then, as $\epsilon \rightarrow 0^+$, the function $x \mapsto \Re[f(x)]$ has a unique global minimum on Λ_{1, z_*} at $x = 3|\delta|^{-1}/2$ if $E \ll |\delta|^3$.*

Note that, since $|\delta| \lesssim N^{-1/2}$, by Lemma 5.5 it follows that the function $x \mapsto \Re[f(x)]$ is strictly decreasing for $0 \leq x \ll N^{-1/2}$.

Proof of Theorem 2.1 for $-CN^{-1/2} \leq \delta < 0$. Let $\tilde{\Gamma}$, $\tilde{\Lambda}$ be defined in (52), then using that

$$e^{-N\left[\frac{\delta}{s} + \frac{1}{2s^2}\right]} \lesssim e^{-\frac{N}{4s^2}},$$

for $s \in [N^\rho, |z_*|]$ and $|\delta| \lesssim N^{-1/2}$, Lemma 5.3 and the improved bounds in (54) of Lemma 5.2, for $b = 0, 1$, we conclude the bound in the following lemma exactly as in (71) without the improvement involving $\tilde{\delta}$.

Lemma 5.6. *Let $E \lesssim N^{-3/2}$, and let f be defined in (28). Then, the bound*

$$\left| \int_{\Lambda \setminus \tilde{\Lambda}} dx \int_{\Gamma \setminus \tilde{\Gamma}} dy \frac{e^{-Nf(x) + Nf(y)}}{x^a y^b} \right| \lesssim \begin{cases} N^{\frac{2-d}{2}}, & b = 0, \\ N^{\frac{1-b}{2}} (1 + |\log(NE^{2/3})|), & a = 1, b \geq 1, \\ N^{\frac{2-d}{2}}, & a > 1, b \geq 1, \end{cases}$$

holds for any $a \geq 1$ and $b \in \mathbb{N}$, where $d := a + b$.

Then, much as in the case $0 \leq \delta \leq 1$, by Lemma 5.6 we conclude that the contribution to (28) of the regime when either $x \in \tilde{\Lambda}$ or $y \in \tilde{\Gamma}$ is exponentially small, i.e., Lemma 5.4 holds true. Hence, by (28), Lemma 5.6 and the expansion of G in (47b), we easily conclude Theorem 2.1 also in the regime $-CN^{-1/2} \leq \delta < 0$. $\square$

6. The real case below the saddle point regime

In this section we prove Theorem 2.3. Throughout this section we always assume that $\Re w < 0$, hence to make our notation easier we define $w = -E + i\epsilon$ with some $E > 0$ and $\epsilon > 0$. Moreover, we always assume that $E \lesssim c(N)$, with $c(N)$ defined in (44). We are interested in estimating (34) in the transitional regime of $|z|$ around one. For this purpose we introduce the parameter $\delta = \delta_z := 1 - |z|^2$. In order to have an optimal estimate of the leading order term of (34) it is not affordable to estimate the error terms in the expansions, for large a and ξ , of f , $g(\cdot, 1, \eta)$ and $G_{1,N}, G_{2,N}$ by absolute value. For this reason, to compute the error terms in the expansions of f , $g(\cdot, 0, \eta)$, $G_{1,N}, G_{2,N}$ we use a notation $\mathcal{O}^\#(\cdot)$ similar to the one introduced in (46). In order to keep track of the power of τ in the expansion of $G_{1,N}$ and $G_{2,N}$, we define the set of functions

$$\mathcal{O}^\#((a, \tau, \xi)^{-1}) :=$$

$$\left\{ h : \mathbb{C} \times [0, 1] \times \mathbb{C} \rightarrow \mathbb{C} : h(a, \tau, \xi) = \sum_{\substack{\alpha + \beta \geq 1, \\ 0 \leq \gamma \leq \alpha}} \frac{c_{\alpha, \beta, \gamma}}{a^\alpha \tau^\gamma \xi^\beta} \text{ with } |c_{\alpha, \beta, \gamma}| \leq C^{\alpha + \beta} \text{ if } |a|, |a\tau|, |\xi| \geq 2C \right\},$$

for some constant $C > 0$ implicit in the $\mathcal{O}^\#(\cdot)$ notation. The exponents α, β are nonnegative integers.

We expand the functions $f, g(\cdot, 1, \eta), G_{1,N}, G_{2,N}$ for large a and ξ arguments as

$$\begin{aligned} f(\xi) &= (E - i\epsilon)\xi + \frac{\delta}{\xi} + \frac{1}{2\xi^2} + \mathcal{O}^\#(\xi^{-3} + \delta\xi^{-2}), \\ g(a, 1, \eta) &= (E - i\epsilon)a + \frac{\delta}{a} + \frac{1}{2a^2} + \mathcal{O}(a^{-3} + \delta a^{-2}), \end{aligned} \tag{75}$$

with $\mathcal{O}^\#(\cdot)$ defined as in (46), and

$$G_{1,N}(a, \tau, \xi, |z|) = \left[\sum_{\substack{\alpha, \beta \geq 2, \alpha + \beta = 8, \\ \gamma = \min\{\alpha - 1, 3\}}} \frac{c_{\alpha, \beta, \gamma} N^2}{a^\alpha \tau^\gamma \xi^\beta} + \sum_{\substack{\alpha, \beta \geq 2, \alpha + \beta = 7, \\ \gamma = \min\{\alpha - 1, 3\}}} \frac{c_{\alpha, \beta, \gamma} N^2 \delta}{a^\alpha \tau^\gamma \xi^\beta} \right. \\ \left. + \sum_{\substack{\alpha, \beta \geq 2, \alpha + \beta = 6, \\ \gamma = \min\{\alpha - 1, 2\}}} \frac{c_{\alpha, \beta, \gamma} N}{a^\alpha \tau^\gamma \xi^\beta} + \sum_{\substack{\alpha, \beta \geq 2, \alpha + \beta = 6 \\ \gamma = \min\{\alpha - 1, 2\}}} \frac{c_{\alpha, \beta, \gamma} N^2 \delta^2}{a^\alpha \tau^\gamma \xi^\beta} \right] \\ \times [1 + \mathcal{O}^\#((a, \tau, \xi)^{-1})], \quad (76)$$

$$G_{2,N}(a, \tau, \xi, z) = \left[\sum_{\substack{\alpha, \beta \geq 2, \alpha + \beta = 6, \\ \gamma = \max\{\alpha - 1, 2\}}} \frac{c_{\alpha, \beta, \gamma} N^2 \eta^2}{a^\alpha \tau^\gamma \xi^\beta} \right. \\ \left. + \sum_{\substack{\alpha, \beta \geq 2, \alpha + \beta = 5, \\ \gamma = \max\{\alpha - 1, 2\}}} \frac{c_{\alpha, \beta, \gamma} N^2 \eta^2 \delta}{a^\alpha \tau^\gamma \xi^\beta} + \sum_{\substack{\alpha, \beta = 2, 3, \alpha + \beta = 5 \\ \gamma = \max\{\alpha - 1, 2\}}} \frac{c_{\alpha, \beta, \gamma} N \eta^2}{a^\alpha \tau^\gamma \xi^\beta} \right] \\ \times [1 + \mathcal{O}^\#((a, \tau, \xi)^{-1})], \quad (77)$$

where $c_{\alpha, \beta, \gamma} \in \mathbb{R}$ is a constant that may change term by term. To make our notation easier in (76)–(77) we used the convention to write a common multiplicative error for all the terms, even if in principle the constants in the series expansion of the error terms differ term by term.

In the following we deform the integration contours in (34) with the following constraints: the ξ contour can be freely deformed as long as it does not cross 0 and -1 , the a -contour can be deformed as long as it goes out from zero in the region $\Re[a] > |\Im[a]|$, it ends in the region $\Re[a] > 0$, and it does not cross 0 and -1 along the deformation. The τ -contour will not be deformed.

Specifically, we deform the a -contour in (34) to $\Lambda = [0, +\infty)$, and we can deform the ξ -contour to any contour around 0 not encircling -1 (this contour is denoted by Γ in (78)). Moreover, since for $a \in \mathbb{R}_+$ we have

$$|e^{-N(E - i\epsilon)a}| = e^{-NEa}$$

and since the factor e^{-NEa} makes the integral convergent, we may pass to the limit $\epsilon \rightarrow 0^+$. Hence, for any $E > 0$ we conclude that

$$E \operatorname{Tr}[Y + E]^{-1} = \frac{N}{4\pi i} \int_{\Gamma} d\xi \int_0^{+\infty} da \int_0^1 ds \frac{\xi^2 a}{\tau^{1/2}} e^{N[f(\xi) - g(a, \tau, \eta)]} G_N(a, \tau, \xi, z). \quad (78)$$

We split the computation of the leading order term of (78) into the cases $-CN^{-1/2} \leq \delta < 0$ and $\delta \geq 0$.

6A. Case $0 \leq \delta \leq 1$. In order to estimate the leading term of (78) we compute the ξ -integral and the (a, τ) -integral separately. In particular, we compute the (a, τ) -integral first, performing the τ -integral for any fixed a , and then we compute the a -integral. Note that, since $E' = -E$ is negative, the relevant

stationary point of $f(\xi)$ is real and its leading order ξ_* is given by

$$\xi_* := \begin{cases} \sqrt{\frac{\delta}{E}}, & E \ll \delta^3, \\ \mu(c)E^{-1/3}, & E = c\delta^3, \\ E^{-1/3}, & E \gg \delta^3, \end{cases} \quad (79)$$

for some $\mu(c) > 0$ and any fixed constant $c > 0$ independent of N , E and δ . Note that $\xi_* \gg 1$ for any $E \lesssim c(N)$, $0 \leq \delta \leq 1$. We will show that in the regime $a \in [N^\rho, +\infty)$ the τ -integral is concentrated around 1 as long as $|e^{-N[g(a, \tau, \eta) - g(a, 1, \eta)]}|$ is effective, i.e., as long as $N|g(a, \tau, \eta) - g(a, 1, \eta)| \gg 1$, and that it is concentrated around 0 if $N|g(a, \tau, \eta) - g(a, 1, \eta)| \lesssim 1$. For this purpose, using that $g(a, 1, \eta) = f(a)$ for any $a \in \mathbb{C}$, we rewrite (78) as

$$E \operatorname{Tr}[Y + E]^{-1} = \frac{N}{4\pi i} \int_{\Gamma} d\xi e^{Nf(\xi)} \xi^2 \int_0^{+\infty} da e^{-Nf(a)} a \int_0^1 d\tau \frac{e^{-N[g(a, \tau, \eta) - g(a, 1, \eta)]}}{\tau^{1/2}} G_N, \quad (80)$$

where we used that by holomorphicity, we can deform the contour Γ as $\Gamma = \Gamma_{\xi_*} := \Gamma_{1, \xi_*} \cup \Gamma_{2, \xi_*}$, with Γ_{1, ξ_*} , Γ_{2, ξ_*} defined in (48a) replacing z_* by ξ_* . We will show that the contribution to (80) of the integrals in the regime when either $|a| \leq N^\rho$ or $|\xi| \leq N^\rho$, for some small fixed $0 < \rho < 1/2$, is exponentially small. Moreover, we will show that also the τ -integral is exponentially small for τ very close to 0 because of the term $\log \tau$ in the phase function $g(a, \tau, \eta)$. Hence, we define $\tilde{\Gamma}$, $\tilde{\Lambda}$ as in (52), and $I \subset [0, 1]$ as $I = I_a := [0, N^{\rho/2}a^{-1}]$, for any $a \in [0, +\infty)$.

In order to compute the leading term of (80) we first bound the integral in the regime

$$(a, \tau, \xi) \in (\Lambda \setminus \tilde{\Lambda}) \times ([0, 1] \setminus I) \times (\Gamma \setminus \tilde{\Gamma}),$$

with $\tilde{\Lambda}$ and $\tilde{\Gamma}$ defined in (52), that is the regime where we expect that the main contribution comes from, and then we use these bounds to first prove that the integral in the regime when either $|\xi| \leq N^\rho$ or $|a| \leq N^\rho$ is exponentially small for any $\tau \in [0, 1]$, and then prove that also the τ -integral on I is exponentially small if $|a| \geq N^\rho$. The bounds for the ξ -integral over $\Gamma \setminus \tilde{\Gamma}$ are exactly the same as in Lemma 5.2, since the phase function $f(\xi)$ and the Γ -contour are exactly the same as the complex case. In order to estimate the integral over $(a, \tau) \in (\Lambda \setminus \tilde{\Lambda}) \times ([0, 1] \setminus I)$, we start with the estimate of the τ -integral over $[0, 1] \setminus I$ in Lemma 6.2 and then we will conclude the computation of the a -integral over $[N^\rho, +\infty)$ in Lemma 6.3.

Before proceeding with the bounds for large $|a|$, $|\xi|$, in the following lemma we state some properties of the functions f and g . The proof of this lemma follows by elementary computations. From now on, for simplicity, we assume that $\eta \geq 0$; the case $\eta < 0$ is completely analogous since the functions g and G_N in (78) depend only on η^2 and $|z|^2$.

Lemma 6.1. *Let f and g be the phase functions defined in (35) and (36), respectively, then the following properties hold true:*

(i) *For any $\xi = -2/3 + it \in \Gamma_{1, \xi_*}$, we have that*

$$\Re[f(-2/3 + it)] = \frac{2E}{3} + \epsilon t - \frac{1}{2t^2} + \mathcal{O}(|t|^{-3} + \delta|t|^{-2}), \quad (81)$$

and

$$\Im[f(-2/3 + it)] = \frac{2\epsilon}{3} - Et - \frac{\delta}{t} + \mathcal{O}(|t|^{-3}). \quad (82)$$

- (ii) For $\epsilon = 0$, the function $t \mapsto \Re[f(-2/3 + it)]$ on Γ_{1, ξ_*} is strictly increasing if $t > 0$ and strictly decreasing if $t < 0$.
- (iii) For any $a \in [0, +\infty)$, we have that $g(a, \tau, \eta) \geq g(a, \tau, 0)$ and the function $\tau \mapsto g(a, \tau, 0)$ is strictly decreasing on $[0, 1]$.
- (iv) The function $a \mapsto g(a, 1, \eta)$ is strictly decreasing on $[0, \xi_*/2]$.

Lemma 6.2. Let $\rho > 0$ be sufficiently small, $I = I_a = [0, N^{\rho/2}a^{-1}]$, $\gamma \in \mathbb{N}$, $\gamma \geq 1$, $c(N)$ be defined in (44), $E \lesssim c(N)$, $0 \leq \delta \leq 1$, and let g be defined as in (36). Then, for any $a \in [N^\rho, +\infty)$, we have

$$\begin{aligned} & \int_{[0,1] \setminus I_a} \frac{|e^{-N[g(a,\tau,\eta)-g(a,1,\eta)]}|}{\tau^{\gamma+1/2}} d\tau \\ & \lesssim F(a) := \begin{cases} a^2 N^{-1} \wedge 1, & N^\rho \leq a \leq \delta^{-1} \wedge \eta^{-1}, \\ a(N\delta)^{-1} \wedge 1, & \delta^{-1} \vee N^\rho \leq a \leq \delta\eta^{-2}, \\ (N\eta^2)^{-1} \wedge 1, & a \geq (\delta\eta^{-1} \vee 1)\eta^{-1}, \end{cases} \\ & + e^{-\frac{1}{2}N\eta^2} \times \begin{cases} e^{-\frac{N(\delta \vee N^{-1/2})}{a}}, & N^\rho \leq a \leq N(\delta \vee N^{-1/2}), \\ (a(N\delta \vee \sqrt{N})^{-1})^{\gamma-1/2}, & N(\delta \vee N^{-1/2}) \leq a \leq (\delta \vee N^{-1/2})\eta^{-2}, \\ (N\eta^2)^{1/2-\gamma}, & a \geq (\delta \vee N^{-1/2})\eta^{-2}, \end{cases} \quad (83) \end{aligned}$$

where some regimes in (83) might be empty for certain values of δ and η .

Proof. In order to estimate the integral in the l.h.s. of (83) we first compute the expansion

$$\begin{aligned} g(a, \tau, \eta) - g(a, 1, \eta) &= \frac{(1-2\delta)(1-\tau) - (1-\tau)^2}{a^2\tau^2} + \frac{2\eta^2(1-\tau)}{\tau} + \frac{(1-\tau)\delta}{a\tau} \\ &+ \mathcal{O}\left(\frac{1-\tau}{a^2\tau} + \frac{(1-\tau)(\delta+a^{-1})}{a^2\tau^2} + \frac{\eta^2(1-\tau)}{a\tau^2}\right), \quad (84) \end{aligned}$$

which holds true for any $\tau \in [0, 1] \setminus I = [N^{\rho/2}a^{-1}, 1]$. Note that by (84) it follows that for any $(a, \tau) \in [N^\rho, +\infty) \times [N^{\rho/2}a^{-1}, 1]$,

$$g(a, \tau, \eta) - g(a, 1, \eta) \geq \frac{1-\tau}{2} \left[\frac{1}{a^2\tau^2} + \frac{\delta}{a\tau} + \frac{\eta^2}{\tau} \right].$$

Then, by (84) it follows that

$$\int_{[0,1] \setminus I} \frac{|e^{-N[g(a,\tau,\eta)-g(a,1,\eta)]}|}{\tau^{\gamma+1/2}} d\tau \lesssim \int_{N^{\rho/2}a^{-1}}^1 \frac{e^{-(1-\tau)\frac{N}{2}\left[\frac{1}{a^2\tau^2} + \frac{\delta}{a\tau} + \frac{\eta^2}{\tau}\right]}}{\tau^{\gamma+1/2}} d\tau. \quad (85)$$

In order to bound the r.h.s. of (85) we split the computations into two cases: $\delta \leq N^{-1/2}$ and $\delta > N^{-1/2}$. We first consider the case $\delta > N^{-1/2}$. In order to prove the bound in the r.h.s. of (83) we further split the

computation of the τ -integral into the regimes $\tau \in [N^{\rho/2}a^{-1}, 1/2]$ and $\tau \in [1/2, 1]$. We start estimating the integral over $[1/2, 1]$ as follows:

$$\begin{aligned} \int_{1/2}^1 \frac{e^{-(1-\tau)\frac{N}{2}\left[\frac{1}{a^2\tau^2} + \frac{\delta}{a\tau} + \frac{\eta^2}{\tau}\right]}}{\tau^{\gamma+1/2}} d\tau &\lesssim \int_{1/2}^1 e^{-(1-\tau)\frac{N}{2}\left[\frac{1}{a^2} + \frac{\delta}{a} + \eta^2\right]} d\tau \\ &\lesssim \begin{cases} a^2 N^{-1} \wedge 1, & N^\rho \leq a \leq \delta^{-1} \wedge \eta^{-1}, \\ a(N\delta)^{-1} \wedge 1, & N^\rho \vee \delta^{-1} \leq a \leq \delta\eta^{-2}, \\ (N\eta^2)^{-1} \wedge 1, & a \geq \delta\eta^{-2} \vee \eta^{-1}. \end{cases} \end{aligned} \quad (86)$$

For the integral over $\tau \in [N^{\rho/2}a^{-1}, 1/2]$, instead, we bound the r.h.s. of (85) as

$$\int_{N^{\rho/2}a^{-1}}^{1/2} \frac{e^{-\frac{N}{2}\left[\frac{\delta}{a\tau} + \frac{\eta^2}{\tau}\right]}}{\tau^{\gamma+1/2}} d\tau \lesssim e^{-\frac{1}{2}N\eta^2} \times \begin{cases} e^{-\frac{1}{2}N\delta a^{-1}}(a(N\delta)^{-1})^{\gamma-1/2}, & N^\rho \leq a \leq \delta\eta^{-2}, \\ (N\eta^2)^{1/2-\gamma}, & a \geq \delta\eta^{-2}. \end{cases} \quad (87)$$

Then, combining (86)–(87) we conclude the bound in (83) for $\delta > N^{-1/2}$. Using similar computations for $\delta \leq N^{-1/2}$, we conclude the bound in (83). $\square$

In the following lemma we conclude the bound for the double integral (80) in the regime $(a, \tau) \in ([N^\rho, +\infty) \times ([0, 1] \setminus I))$ using the bound in (83) as an input.

Lemma 6.3. *Let $\rho > 0$ be sufficiently small, $I = [0, N^{\rho/2}a^{-1}]$, $0 \leq \delta \leq 1$, let $c(N)$ be defined in (44), $E \lesssim c(N)$, and let g be defined in (36). Then, for any integers $\alpha \geq 2$, $1 \leq \gamma \leq \alpha$, we have*

$$\int_{\Lambda \setminus \tilde{\Lambda}} \int_{[0,1] \setminus I}^1 \left| \frac{e^{-Ng(a,\tau,\eta)}}{a^{\alpha-1}\tau^{\gamma+1/2}} \right| d\tau da \lesssim C_1 + e^{-\frac{1}{2}N\eta^2} (N\eta^2)^{1/2-\gamma} C_2 + e^{-\frac{1}{2}N\eta^2} (N\delta \vee \sqrt{N})^{1/2-\gamma} C_3, \quad (88)$$

where

$$\begin{aligned} C_1 &:= \begin{cases} 1 + |\log[N(\delta \vee N^{-1/2})\xi_*^{-1}]|, & \alpha = 2, [\delta\eta^{-1} \vee 1]\eta^{-1} > \xi_*, \\ ((N\eta^2)^{-1} \wedge 1)(1 + |\log[N(\delta \vee N^{-1/2})\xi_*^{-1}]|), & \alpha = 2, [\delta\eta^{-1} \vee 1]\eta^{-1} \leq \xi_*, \\ [N(\delta \vee N^{-1/2})]^{2-\alpha}, & \alpha \geq 3, [\delta\eta^{-1} \vee 1]\eta^{-1} > \xi_* \\ ((N\eta^2)^{-1} \wedge 1)[N(\delta \vee N^{-1/2})]^{2-\alpha}, & \alpha \geq 3, [\delta\eta^{-1} \vee 1]\eta^{-1} \leq \xi_* \end{cases} \\ C_2 &:= \begin{cases} (1 + |\log[N(\delta \vee N^{-1/2})\xi_*^{-1}]|)e^{-\frac{NE}{2\eta^2}(\delta \vee N^{-1/2})}, & \alpha = 2, \\ [(\delta^{-1} \wedge \sqrt{N})\eta^2]^{\alpha-2}e^{-\frac{NE}{2\eta^2}(\delta \vee N^{-1/2})}, & \alpha \geq 3, (\delta^{-1} \wedge \sqrt{N})\eta^2 \leq NE, \\ [(\delta^{-1} \wedge \sqrt{N})\eta^2]^{\alpha-2}, & \alpha \geq 3, (\delta^{-1} \wedge \sqrt{N})\eta^2 \geq NE, \end{cases} \\ C_3 &:= \begin{cases} (NE)^{\alpha-\gamma-3/2}, & \gamma = \alpha, \alpha-1, (\delta^{-1} \wedge \sqrt{N})\eta^2 \leq NE, \\ [(\delta^{-1} \wedge N^{1/2})\eta^2]^{\alpha-\gamma-3/2}, & \gamma = \alpha, \alpha-1, (\delta^{-1} \wedge \sqrt{N})\eta^2 \geq NE, \\ (N\delta \vee \sqrt{N})^{3/2+\gamma-\alpha}, & \gamma \leq \alpha-2. \end{cases} \end{aligned}$$

Proof. First, we add and subtract $Ng(a, 1, \eta) = Nf(a)$ to the phase function in the exponent and conclude, by Lemma 6.2, that

$$\int_{N^\rho}^{+\infty} \int_{[0,1] \setminus I}^1 \left| \frac{e^{-Ng(a,\tau,\eta)}}{a^{\alpha-1} \tau^{\gamma+1/2}} \right| d\tau da \lesssim \int_{N^\rho}^{+\infty} \frac{|e^{-Ng(a,1,\eta)}| \cdot F(a)}{a^{\alpha-1}} da, \quad (89)$$

with $F(a)$ defined in (83). In the rest of the proof we often use that $NE\xi_* \lesssim 1$ by the definition of ξ_* in (79), which implies $e^{NE\xi_*} \lesssim 1$. We split the computation of the integral in the r.h.s. of (89) as the sum of the integrals over $[N^\rho, \xi_*]$ and $[\xi_*, +\infty)$. From now on we consider only the case $\delta > N^{-1/2}$, since the case $\delta \leq N^{-1/2}$ is completely analogous. In the regime $\delta > N^{-1/2}$ we have

$$E \lesssim c(N) = \delta^{-1} N^{-2} \lesssim \delta^3,$$

therefore

$$\xi_* \sim \sqrt{\delta E^{-1}}$$

from (79).

Then, using the expansion for large a -argument of $g(a, 1, \eta) = f(a)$ in (75), we start estimating the integral over $[N^\rho, +\infty)$ as follows:

$$\begin{aligned} e^{NE\xi_*} \int_{N^\rho}^{\xi_*} \frac{e^{-N\left[\frac{\delta}{a} + \frac{1}{2a^2}\right]} F(a)}{a^{\alpha-1}} da \\ \lesssim \chi(\delta\eta^{-2} \leq \xi_*) e^{-\frac{1}{2}N\eta^2} (N\eta^2)^{1/2-\gamma} \times \begin{cases} 1 + |\log(N\delta\xi_*^{-1})|, & \alpha = 2, \\ (\delta^{-1}\eta^2)^{\alpha-2}, & \alpha \geq 3, \end{cases} \\ + e^{-\frac{1}{2}N\eta^2} \times \begin{cases} N^{1/2-\gamma} \delta^{2-\alpha} \eta^{2\alpha-2\gamma-3}, & \gamma = \alpha, \alpha-1, \delta\eta^{-2} \leq \xi_*, \\ (N\delta)^{1/2-\gamma} \xi_*^{3/2+\gamma-\alpha}, & \gamma = \alpha, \alpha-1, \delta\eta^{-2} \geq \xi_*, \\ (N\delta)^{2-\alpha}, & \gamma \leq \alpha-2, \end{cases} \\ + \begin{cases} 1 + |\log(N\delta\xi_*^{-1})|, & \alpha = 2, [\delta\eta^{-1} \vee 1]\eta^{-1} > \xi_*, \\ ((N\eta^2)^{-1} \wedge 1)(1 + |\log(N\delta\xi_*^{-1})|), & \alpha = 2, [\delta\eta^{-1} \vee 1]\eta^{-1} \leq \xi_*, \\ (N\delta)^{2-\alpha}, & \alpha \geq 3 [\delta\eta^{-1} \vee 1]\eta^{-1} > \xi_*, \\ ((N\eta^2)^{-1} \wedge 1)(N\delta)^{2-\alpha} & \alpha \geq 3 [\delta\eta^{-1} \vee 1]\eta^{-1} \leq \xi_*. \end{cases} \quad (90) \end{aligned}$$

To conclude the proof we are left with the estimate of the a -integral on $[\xi_*, +\infty)$. In this regime we bound the r.h.s. of (89) as follows:

$$\begin{aligned} e^{NE\xi_*} \int_{\xi_*}^{+\infty} \frac{e^{-NEa} F(a)}{a^{\alpha-1}} da \\ \lesssim e^{-\frac{1}{2}N\eta^2} (N\eta^2)^{1/2-\gamma} \times \begin{cases} (1 + |\log(NE\xi_*)|) e^{-NE\delta\eta^{-2}/2}, & \alpha = 2, \\ e^{-NE\delta\eta^{-2}} (\delta^{-1}\eta^2)^{\alpha-2}, & \alpha \geq 3, \delta^{-1}\eta^2 \leq NE \wedge \xi_*^{-1}, \\ (\delta^{-1}\eta^2)^{\alpha-2}, & \alpha \geq 3, NE \leq \delta^{-1}\eta^2 \leq \xi_*^{-1}, \\ \xi_*^{2-\alpha}, & \alpha \geq 3, \delta^{-1}\eta^2 \geq \xi_*^{-1}, \end{cases} \end{aligned}$$

$$\begin{aligned}
& + \chi(\xi_* \leq \delta \eta^{-2}) e^{-\frac{1}{2} N \eta^2} (N \delta)^{1/2-\gamma} \times \begin{cases} (NE)^{\alpha-\gamma-3/2}, & \gamma = \alpha, \alpha-1, \delta^{-1} \eta^2 \leq NE, \\ (\delta^{-1} \eta^2)^{\alpha-\gamma-3/2}, & \gamma = \alpha, \alpha-1, \delta^{-1} \eta^2 \geq NE, \\ \xi_*^{3/2+\gamma-\alpha}, & \gamma \leq \alpha-2, \end{cases} \\
& + \begin{cases} 1 + |\log(N \delta \xi_*^{-1})|, & \alpha = 2, [\delta \eta^{-1} \vee 1] \eta^{-1} > \xi_*, \\ ((N \eta^2)^{-1} \wedge 1)(1 + |\log(N \delta \xi_*^{-1})|), & \alpha = 2, [\delta \eta^{-1} \vee 1] \eta^{-1} \leq \xi_*, \\ (\xi_*)^{2-\alpha}, & \alpha \geq 3, [\delta \eta^{-1} \vee 1] \eta^{-1} > \xi_*, \\ ((N \eta^2)^{-1} \wedge 1)(\xi_*)^{2-\alpha}, & \alpha \geq 3, [\delta \eta^{-1} \vee 1] \eta^{-1} \leq \xi_*. \end{cases} \quad (91)
\end{aligned}$$

Finally, combining (90) and (91) we conclude the bound in (88). $\square$

In order to conclude the estimate of the leading order term of (80), in the following lemma, using the bounds in Lemma 5.2 for the ξ -integral and the ones in Lemma 6.3 for the (a, τ) -integral, we prove that the contribution to (80) in the regime when either $a \in [0, N^\rho]$ or $\xi \in \tilde{\Gamma}$ and in the regime $\tau \in I$ is exponentially small.

Lemma 6.4. *Let $c(N)$ be defined in (44), $0 \leq \delta \leq 1$, $I = I_a = [0, N^{\rho/2} a^{-1}]$, and let f, g and G_N be defined in (35)–(37), then, for any $E \lesssim c(N)$, we have that*

$$\begin{aligned}
& \left| \left(\int_{\Gamma} d\xi \int_{\Lambda} da \int_0^1 d\tau - \int_{\Gamma \setminus \tilde{\Gamma}} d\xi \int_{\Lambda \setminus \tilde{\Lambda}} da \int_{[0,1] \setminus I} d\tau \right) \left[e^{N[f(\xi) - g(a, \tau, \eta)]} \frac{a \xi^2}{\tau^{1/2}} G_N(a, \tau, \xi, z) \right] \right| \\
& \lesssim \frac{N^{5/2+5\rho} (N^{1/2} + N\delta)}{E^{1/2}} e^{-\frac{1}{2} N^{1-2\rho}}. \quad (92)
\end{aligned}$$

Proof. We split the proof into three parts, we first prove that the contribution to (80) in the regime $a \in \tilde{\Lambda} = [0, N^\rho]$ is exponentially small uniformly in $\tau \in [0, 1]$ and $\xi \in \Gamma$, then we prove that for $a \geq N^\rho$ the contribution to (80) in the regime $\tau \in I$ is exponentially small uniformly in $\xi \in \Gamma$, and finally we conclude that also the contribution for $\xi \in \tilde{\Gamma}$ is negligible.

Note that for any $a \in [0, +\infty)$, $\tau \in [0, 1]$ we have that the map $\tau \mapsto g(a, \tau, 0)$ is strictly decreasing by (iii) of Lemma 6.1, hence, using that $g(a, \tau, \eta) \geq g(a, \tau, 0)$ and (ii)–(iv) of Lemma 6.1, it follows that

$$\sup_{\xi \in \tilde{\Gamma}} |e^{Nf(\xi)}| + \sup_{a \in \tilde{\Lambda}} |e^{-Ng(a, \tau, \eta)}| \leq \sup_{\xi \in \tilde{\Gamma}} |e^{Nf(\xi)}| + \sup_{a \in \tilde{\Lambda}} |e^{-Ng(a, 1, 0)}| \lesssim e^{-Nf(N^\rho)}, \quad (93)$$

with

$$f(N^\rho) = \frac{\delta}{N^\rho} + \frac{1}{2N^{2\rho}} + \mathcal{O}(N^{-3\rho} + \delta N^{-2\rho}). \quad (94)$$

In order to estimate the regime $a \in \tilde{\Lambda}$, we split the computation into two cases: $(a, \xi) \in \tilde{\Lambda} \times \tilde{\Gamma}$ and $(a, \xi) \in \tilde{\Lambda} \times (\Gamma \setminus \tilde{\Gamma})$. Then, by (93)–(94) it follows that the integral in the regime $(a, \tau, \xi) \in \tilde{\Lambda} \times [0, 1] \times \tilde{\Gamma}$ is bounded by $N^{2\rho} e^{-N^{1-2\rho}/2}$. Note that in the regime $(a, \tau, \xi) \in \tilde{\Lambda} \times [0, 1] \times (\Gamma \setminus \tilde{\Gamma})$ we have $|\xi| \geq N^\rho$. Hence, by the explicit form of $G_{1,N}, G_{2,N}$ in (37), using the bound in (93) for $e^{-Ng(a, \tau, \eta)}$, that $|\xi| \geq N^\rho$ and so Lemma 5.2 to bound the regime $\Gamma \setminus \tilde{\Gamma}$, we conclude that the integral over $(a, \tau, \xi) \in \tilde{\Lambda} \times [0, 1] \times (\Gamma \setminus \tilde{\Gamma})$ is bounded by $N^{3+\rho} (N^{1/2} + (N\delta)) e^{-N^{1-2\rho}/2}$.

Next, we consider the integral over $(a, \tau, \xi) \in (\Lambda \setminus \tilde{\Lambda}) \times [0, N^{\rho/2}a^{-1}] \times \Gamma$. Note that in this regime $a \geq N^\rho$. Since $g(a, \tau, \eta) \geq g(a, \tau, 0)$ and $\tau \mapsto g(a, \tau, 0)$ is strictly decreasing by (iii) of Lemma 6.1, we have that

$$e^{-Ng(a, \tau, \eta)} \leq e^{-Ng(a, N^{\rho/2}a^{-1}, 0)}, \quad (95)$$

where

$$g(a, N^{\rho/2}a^{-1}, \eta) = Ea + \frac{\delta}{N^{\rho/2}} + \frac{1}{N^\rho} + \mathcal{O}(N^{-3\rho/2} + \delta N^{-\rho}). \quad (96)$$

Additionally, using the explicit expression of G_N in (37), the bound (93) on $e^{Nf(\xi)}$ for the regime $\xi \in \tilde{\Gamma}$, and Lemma 5.2 for $\xi \in \Gamma \setminus \tilde{\Gamma}$, we get

$$\left| \int_{\Gamma} d\xi G_N(a, \tau, \xi, z) \xi^2 e^{Nf(\xi)} \right| \lesssim C(N, \eta, a, \tau), \quad (97)$$

where

$$C(a, \tau) = C(N, \eta, a, \tau) := (N^{1/2} + N\delta) \left(N^2 \eta^2 + N^2 \tau + \frac{N^2}{a} + \frac{N^2}{a^2 \tau} \right).$$

Thus, using (97) and the explicit form of $g(a, \tau, \eta)$ in (36), for $\tau \in [0, N^{\rho/2}a^{-1}]$ we have

$$\begin{aligned} \left| \frac{a}{\tau^{1/2}} e^{-Ng(a, \tau, \eta)} \int_{\Gamma} d\xi G_N(a, \tau, \xi, z) \xi^2 e^{Nf(\xi)} \right| &\lesssim C(a, \tau) e^{-(N-2)g(a, \tau, \eta)} \frac{a^3 \tau}{\tau^{1/2}(1 + 2a + a^2 \tau)} \\ &\lesssim C(a, \tau) a^2 \tau^{1/2} e^{-(N-2)g(a, \tau, 0)} e^{-\frac{(N-2)\eta^2 a^2}{1+2a+a^2 \tau}} \\ &\lesssim C(a, \tau) a^2 \tau^{1/2} e^{-\frac{1}{2}N^{1-2\rho}} e^{-NEa - \frac{(N-2)\eta^2 a^2}{1+2a+a^2 \tau}}. \end{aligned} \quad (98)$$

Hence, integrating (98) with respect to (a, τ) , and using that in the regime $\tau \in [0, N^{\rho/2}a^{-1}]$,

$$N^{-\rho/2}a \lesssim \frac{a^2}{1 + 2a + a^2 \tau} \lesssim a,$$

we conclude that the integral over $(a, \tau, \xi) \in (\Lambda \setminus \tilde{\Lambda}) \times [0, N^{\rho/2}a^{-1}] \times \Gamma$ is bounded by

$$N^{5/2+5\rho}(N^{1/2} + N\delta)E^{-1/2}e^{-N^{1-2\rho}}.$$

Finally, in order to conclude the bound in (92), we are left with the estimate of the integral over $(a, \tau, \xi) \in (\Lambda \setminus \tilde{\Lambda}) \times [N^{\rho/2}a^{-1}, 1] \times \tilde{\Gamma}$. In this regime, using the bound in (93) on $e^{Nf(\xi)}$ for $\xi \in \tilde{\Gamma}$, and Lemma 6.3 to estimate the integral over $(a, \tau) \in (\Lambda \setminus \tilde{\Lambda}) \times [N^{\rho/2}a^{-1}, 1]$, we get the bound $N^{5/2+\rho}E^{-1/2}e^{-N^{1-2\rho/2}}$. This concludes the proof of (92). $\square$

Proof of Theorem 2.3 in the case $\delta \geq 0$. Using Lemma 6.4 we remove the regime $a \leq N^\rho$, $|\xi| \leq N^\rho$ or $\tau \in [0, N^{\rho/2}a^{-1}]$ in (80). Then, using the expansion for $G_{1,N}$ and $G_{2,N}$ in (76)–(77) in the remaining regime of (80), combining Lemmas 5.2 and 6.3 we conclude Theorem 2.3. $\square$

6B. Case $-CN^{-1/2} \leq \delta < 0$. Now we summarize the necessary changes for the case $\delta < 0$. As in the case $0 \leq \delta \leq 1$, throughout this section we assume that $E' < 0$ in (35)–(36), i.e., $E' = -E$ with $0 \leq E \lesssim N^{-3/2}$.

Let x_* be the real stationary point of f , i.e., x_* at leading order is given by

$$x_* \approx \begin{cases} 3|\delta|^{-1/2} & \text{if } E \ll |\delta|^3, \\ \mu(c)E^{-1/3} & \text{if } E = c|\delta|^3, \\ E^{-1/3} & \text{if } E \gg |\delta|^3, \end{cases}$$

for some function $\mu(c) > 0$ and any fixed constant $c > 0$ independent of N , E , and δ . As in the complex case, we can treat the regime $0 < -\delta \lesssim N^{-1/2}$ similarly to the regime $0 \leq \delta \lesssim N^{-1/2}$, since for $|\delta| \lesssim N^{-1/2}$ the only δ -dependent terms, i.e., the term δa^{-1} in the expansion of $g(a, 1, \eta) = f(a)$ in (75) and the term $(1 - \tau)\delta(a\tau)^{-1}$ in the expansion of $g(a, \tau, \eta) - g(a, 1, \eta)$, do not play any role in the estimates of the (a, τ) integral in the regime $a \geq N^\rho$, $\tau \in [N^{\rho/2}a^{-1}, 1]$. Note that this is also the case for $0 \leq \delta \leq N^{-1/2}$, when the estimates (83) and (88) were derived. For this reason, unlike the case $0 \leq \delta \leq 1$, in the present case, $\delta < 0$, $|\delta| \lesssim N^{-1/2}$ and $E \lesssim c(N)$, we do not deform the ξ -contour through the leading order of the saddle x_* , but we always deform it as $\Gamma = \Gamma_{\xi_*} := \Gamma_{1, \xi_*} \cup \Gamma_{2, \xi_*}$, where $\xi_* := E^{-1/3}$, with $\Gamma_{1, \xi_*}, \Gamma_{2, \xi_*}$ defined in (48a) replacing z_* by ξ_* . Note that we could have done the same choice in the case $0 \leq \delta \lesssim N^{-1/2}$, but not for the regime $N^{-1/2} \ll \delta \leq 1$, hence, in order to treat the regime $0 \leq \delta \leq 1$ in the same way for any δ , in Section 6A we deformed the contour Γ through (79). For any $E \lesssim c(N)$ and $0 < -\delta \lesssim N^{-1/2}$ we have $\xi_* \gg 1$, hence we prove that the main contribution to (80) comes from the regime when $a, |\xi| \geq N^\rho$. Moreover, similarly to the case $0 \leq \delta \leq 1$, the contribution to (80) in the regime $(a, \tau, \xi) \in (\Lambda \setminus \tilde{\Lambda}) \times [0, N^{\rho/2}a^{-1}] \times (\Gamma \setminus \tilde{\Gamma})$ will be exponentially small. Hence, in order to estimate the leading order of (80), we expand f , $g(\cdot, 1, \eta)$ and G_N for large a and $|\xi|$ arguments as in (75)–(77).

The phase functions f and g , defined in (35) and (36), respectively, satisfy the properties (i) and (ii) of Lemma 6.1, but not the ones in (iii) and (iv). Instead, it is easy to see that the following lemma holds true.

Lemma 6.5. *Let f and g be the phase functions defined in (35) and (36), respectively. Then, the following properties hold true:*

(iii') *For any $a \in [0, +\infty)$, we have that $g(a, \tau, \eta) \geq g(a, \tau, 0)$ and that*

$$e^{-Ng(a, \tau, 0)} \lesssim e^{-Ng(a, \tau_0, 0)},$$

for any fixed $\tau_0 \in [0, 1]$ and any $\tau \in [0, \tau_0]$.

(iv') *The function $a \mapsto g(a, 1, 0)$ is strictly decreasing on $[0, |\delta|^{-1/2}]$.*

Since $|\delta| \lesssim N^{-1/2}$, by (iv') of Lemma 6.5 it clearly follows that the function $a \mapsto g(a, 1, \eta)$ is strictly decreasing on $[0, N^\rho]$. Note that for $|\delta| \lesssim N^{-1/2}$ we have

$$e^{-N(1-\tau)\left[\frac{1}{a^2\tau^2} + \frac{\delta}{a\tau}\right]} \lesssim e^{-\frac{(1-\tau)N}{2a^2\tau^2}}, \quad e^{-N\left[\frac{\delta}{a} + \frac{1}{2a^2}\right]} \lesssim e^{-\frac{N}{4a^2}}, \quad (99)$$

for any $a \in [N^\rho, +\infty)$ and $\tau \in [N^{\rho/2}a^{-1}, 1]$. Using (99), inspecting the proofs of (83) and (88) in Lemma 6.2 and Lemma 6.3, respectively, and noticing that in the regime $0 \leq \delta \leq N^{-1/2}$ the sign of δ did

not play any role we conclude that the bounds (83), (88) hold true for the case $\delta < 0$, $|\delta| \lesssim N^{-1/2}$ as well. Then, much as in the case $0 \leq \delta \leq 1$, by (i)–(ii) of Lemma 6.1 and (iii')–(iv') of Lemma 6.5, using the bound in (88) to estimate the (a, τ) -integral in the regime $(a, \tau) \in [N^\rho, +\infty) \times [N^{\rho/2}a^{-1}, 1]$ and the ones in Lemma 5.2 to estimate the ξ -integral in the regime $|\xi| \geq N^\rho$ we conclude that the contribution to (80) in the regime when either $a \in [0, N^\rho]$ or $|\xi| \leq N^\rho$ and in the regime $[0, N^{\rho/2}a^{-1}]$ is exponentially small, i.e., Lemma 6.4 holds true once δ is replaced by $|\delta|$ everywhere.

Proof of Theorem 2.3 in the case $-CN^{-1/2} \leq \delta < 0$. By combining (88), Lemmas 5.2 and 6.4, using the expansion of G_N in (76)–(77), we conclude the proof of Theorem 2.3 also in the case $-CN^{-1/2} \leq \delta < 0$. $\square$

Appendix A. Superbosonization formula for meromorphic functions

The superbosonization formulas [Littellmann et al. 2008, Eq. (1.10) and (1.13)] (see also [Alldridge and Shaikh 2014, Corollary 2.6] for more precise conditions) are stated under the condition that

$$F(\Phi^* \Phi) = F \begin{pmatrix} \langle s, s \rangle & \langle s, \chi \rangle \\ \langle \chi, s \rangle & \langle \chi, \chi \rangle \end{pmatrix},$$

viewed as a function of four independent variables, is holomorphic and decays faster than any inverse power at real $+\infty$ in the $\langle s, s \rangle$ variable (for definiteness, we discuss the complex case; the argument for the real case is analogous). Our function F defined in (26) has a pole at $\langle s, s \rangle = iN$ and $\langle \chi, \chi \rangle = iN$ using the definitions (20)–(21) after expanding the inverse of the matrix $1 + \frac{i}{N} \Phi^* \Phi$ in the Grassmannian variables but this pole is far away from the integration domain on both sides of (22). We now outline a standard approximation procedure to verify the superbosonization formula for such meromorphic functions; for simplicity we consider only our concrete function from (26).

In the first step notice that the integration at infinity on the noncompact domain for the boson-boson variable is absolutely convergent on both sides as guaranteed by the $\exp(iw \text{STr } \Phi^* \Phi)$ regularization, since $\Im w > 0$.

Second, in the l.h.s. of (22) using Taylor expansions, we expand F into a finite polynomial in the Grassmannian variables with meromorphic coefficient functions in the variable $\langle s, s \rangle$. Algebraically, we perform exactly the same expansion in the r.h.s. of (22). For the fermionic variables σ, τ these expansions naturally terminate after finitely many terms. From (28) it is clear that only the geometric expansion

$$(1 + y)^{-1} = 1 - y + y^2 - \dots$$

may result in an infinite power series instead of a finite polynomial. However, owing to the contour integral in y and that the integrand has a pole of at most finite order ($\approx N$) at zero, we may replace this power series with its finite truncation without changing the value of the r.h.s. of (22). We choose the order of truncation sufficiently large that the remaining formula contains all nonzero terms on both sides. We denote this new truncated function by $\tilde{F}$.

Now we are in the situation where on both sides of (22), with $\tilde{F}$ replacing F , we have the same finite polynomial in the variables $\langle s, \chi \rangle, \langle \chi, s \rangle$ and $\langle \chi, \chi \rangle$ in the l.h.s. as in the variables σ, τ, y in the r.h.s., with coefficients that are meromorphic in $\langle s, s \rangle$, resp. in x . All coefficient functions $h_k(x)$ are analytic in

a neighborhood of the positive real axis (their possible pole is at -1) and they have an exponential decay $\approx \exp(-(\Im w)\langle s, s \rangle)$ in the l.h.s., resp. $\exp(-(\Im w)x)$ in the r.h.s., at infinity from the regularization observed in the first step.

Finally, in the third step, dropping the k index temporarily, we write each coefficient function as $h(x) = g(x)e^{-\alpha x}$ with $\alpha = \frac{1}{2}\Im w$. For any given $\epsilon > 0$ we approximate $g(x)$ via classical (rescaled) Laguerre polynomials $p_n(x)$ of degree n with weight function $e^{-\alpha x}$ such that $\int_0^\infty |g(x) - p_n(x)|^2 e^{-\alpha x} dx \leq (\epsilon\alpha)^2$, where n depends on ϵ and $\Im w$. By completeness of the Laguerre polynomials in $L^2(\mathbb{R}_+, e^{-\alpha x} dx)$ and by $\int |g(x)|^2 e^{-\alpha x} dx = \int |h(x)|^2 e^{\alpha x} dx < \infty$ such an approximating polynomial exists. Therefore, with a Schwarz inequality, we have

$$\int_0^\infty |h(x) - p_n(x)e^{-\alpha x}| dx = \int_0^\infty |g(x) - p_n(x)| e^{-\alpha x} dx \leq \epsilon.$$

Since there are only finitely many coefficient functions $h(x) = h_k(x)$ in $\tilde{F}$, we can replace each of them with an entire function (namely with a polynomial times $e^{-\alpha x}$) with at most an ϵ error in the r.h.s. of (22). The same estimates hold on the r.h.s. But for these replacements the superbosonization formula [Littellmann et al. 2008, Eq. (1.10)] is applicable since the new functions are entire. The error is at most ϵ on both sides, but this argument is valid for arbitrary $\epsilon > 0$. This proves the superbosonization formula for the function (26).

Appendix B. Explicit formulas for the real symmetric integral representation

Here we collect the explicit formulas for the polynomials of a, ξ, τ in the definition of G_N in (34).

$$\begin{aligned} p_{2,0,0} &:= a^4 \tau^2 + 2a^3 \xi \tau + 4a^3 \tau - a^2 \xi^2 \tau + 4a^2 \xi^2 + 8a^2 \xi + 2a^2 \tau \\ &\quad + 4a^2 + 2a \xi^3 + 8a \xi^2 + 10a \xi + 4a + \xi^4 + 4\xi^3 + 6\xi^2 + 4\xi + 1, \\ p_{1,0,0} &:= -a^4 \xi \tau^2 + a^4 \tau^2 - 2a^3 \xi^2 \tau - 2a^3 \xi \tau + 4a^3 \tau - a^2 \xi^3 \tau - 3a^2 \xi^2 \tau \\ &\quad - 2a^2 \xi \tau + 4a^2 \xi + 2a^2 \tau + 4a^2 + 2a \xi^2 + 6a \xi + 4a + \xi^3 + 3\xi^2 + 3\xi + 1, \\ p_{2,2,0} &:= 4(a+1)(a^2 \tau + a \xi \tau + 2a \tau + \xi^2 + 2\xi + 1), \\ p_{1,2,0} &:= 4(a+1)(a^2 \tau + a \xi \tau + 2a \tau + \xi + 1), \\ p_{2,0,1} &:= 2(a^3 \tau^2 + 2a^2 \xi \tau + 4a^2 \tau + 2a \xi^2 + 2a \xi \tau + 4a \xi + 3a \tau + 2a + \xi^3 + 4\xi^2 + 5\xi + 2), \\ p_{1,0,1} &:= 2(a^3 \tau^2 + 2a^2 \xi \tau + 4a^2 \tau + a \xi^2 \tau + 3a \xi \tau + 2a \xi + 3a \tau + 2a + \xi^2 + 3\xi + 2), \\ p_{2,2,1} &:= 4(a+1)(a + \xi + 2), \\ p_{2,0,2} &:= a^2 \tau + 2a \xi + 4a + \xi^2 + 4\xi + 4. \end{aligned}$$

Appendix C. Comparison with the contour-integral derivation

In [Ben Arous and P     2005] the correlation kernel of $(X - z)(X - z)^*$ for complex Ginibre matrices X has been derived using contour-integral methods. Earlier, the joint eigenvalue density for the general *Laguerre ensemble* had been obtained in the physics literature [Guhr and Wettig 1996; Jackson et al.

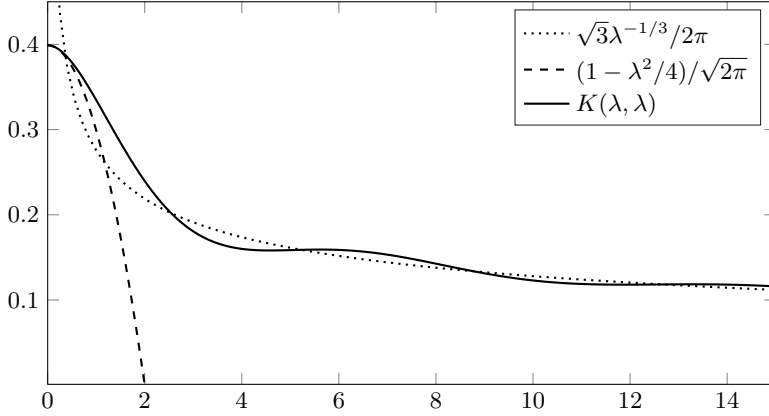


Figure 7. Plot of the 1-point function $K(\lambda, \lambda) = \pi^{-1} \Im q_0(\lambda)$ in the complex case with $|z| = 1$. The dotted and dashed lines show the large λ and small λ asymptotes, respectively.

1996] via supersymmetric methods; see also [Verbaarschot and Zahed 1993] with orthogonal polynomials. Adapting [Ben Arous and P      2005] to our scaling, and choosing $y_i = \pm 1$, it follows from [Ben Arous and P      2005, Theorem 7.1] that for $|z| = 1$ the rescaled kernel $K_N(N^{-3/2}\lambda, N^{-3/2}\mu)$ is given by

$$\begin{aligned} & \frac{N^3}{i\pi} \int_{\Gamma} dx \int_{\gamma} dy K_B(2N^{1/4}x\sqrt{\lambda}, 2N^{1/4}y\sqrt{\mu}) e^{N(h(y)-h(x))} \left(1 - \frac{1}{1-x^2} \frac{1}{1-y^2}\right) xy, \\ K_B(x, y) &:= \frac{xI_1(x)I_0(y) - yI_0(x)I_1(y)}{x^2 - y^2}, \quad h(x) = x^2 + \log(1-x^2) = -\frac{x^4}{2} + \mathcal{O}(x^5), \end{aligned} \quad (100)$$

where I_0, I_1 are the modified Bessel function of 0-th and 1-st kind. The contour Γ is any contour encircling $[-1, 1]$ in a counterclockwise direction (in contradiction to the contours depicted in [Ben Arous and P      2005, Figure 8.1]) and the contour γ is composed of two straight half-lines $[0, i\infty)$ and $[0, -i\infty)$. The main contribution in (100) comes from the $|x| \lesssim N^{-1/4}$ and $|y| \lesssim N^{-1/4}$ regime which motivates the change of variables $x \mapsto N^{-1/4}x$, $y \mapsto N^{-1/4}y$. Together with the expansion of $1 - (1-x^2)^{-1}(1-y^2)^{-1} = -x^2 - y^2 + \mathcal{O}(x^4 + y^4)$ it follows that

$$K_N(N^{-3/2}\lambda, N^{-3/2}\mu) \approx N^{3/2} K(\lambda, \mu),$$

where

$$K(\lambda, \mu) := \frac{i}{\pi} \int_{\Gamma'} dx \int_{\gamma} dy K_B(2x\sqrt{\lambda}, 2y\sqrt{\mu}) e^{x^4/2 - y^4/2} xy(x^2 + y^2)$$

and Γ' consists of four straight half-lines $(e^{i\pi/4}\infty, 0]$, $[0, e^{3i\pi/4}\infty)$, $(e^{5i\pi/4}\infty, 0]$, $[0, e^{7i\pi/4}\infty)$.

We now compare the limiting 1-point function $K(\lambda, \lambda)$ with the asymptotic expansion we derived in Theorem 2.1, which in the case $|z| = 1$, i.e., $\delta = 0$, simplifies to

$$q_0(\lambda) = \frac{\lambda^{1/3}}{2\pi i} \int dx \oint dy e^{\lambda^{2/3}(-y+1/(2y^2)+x-1/(2x^2))} \left(\frac{1}{x^3} + \frac{1}{x^2 y} + \frac{1}{x y^2} \right).$$

The resulting 1-point function, given by $\pi^{-1} \Im q_0(\lambda)$, coincides precisely with $K(\lambda, \lambda)$ and is plotted in Figure 7.

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