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Construction of a stable blowup solution with a prescribed behavior for a non-scaling-invariant semilinear heat equation

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We consider the semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u \ln^{\alpha}(u^2+2)$$

in the whole space $\mathbb{R}^n$, where $p > 1$ and $\alpha \in \mathbb{R}$. Unlike the standard case $\alpha = 0$, this equation is not scaling invariant. We construct for this equation a solution which blows up in finite time T only at one blowup point a, according to the asymptotic dynamic

$$u(x,t) \sim \psi(t)\left(1 + \frac{(p-1)|x-a|^2}{4p(T-t)|\ln(T-t)|}\right)^{-1/(p-1)} \quad \text{as } t \to T,$$

where $\psi(t)$ is the unique positive solution of the ODE

$$\psi' = \psi^p \ln^{\alpha}(\psi^2+2), \quad \lim_{t\to T} \psi(t) = +\infty.$$

The construction relies on the reduction of the problem to a finite-dimensional one and a topological argument based on the index theory to get the conclusion. By the interpretation of the parameters of the finite-dimensional problem in terms of the blowup time and the blowup point, we show the stability of the constructed solution with respect to perturbations in initial data. To our knowledge, this is the first successful construction for a genuinely non-scale-invariant PDE of a stable blowup solution with the derivation of the blowup profile. From this point of view, we consider our result as a breakthrough.

1. Introduction

We are interested in the semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + F(u), \\ u(0) = u_0 \in L^{\infty}(\mathbb{R}^n), \end{cases} \tag{1-1}$$

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where $u(t) : \mathbb{R}^n \to \mathbb{R}$, Δ stands for the Laplacian in $\mathbb{R}^n$ and

$$F(u) = |u|^{p-1} u \ln^{\alpha}(u^2+2), \quad p > 1,\ \alpha \in \mathbb{R}. \tag{1-2}$$

By standard results the model (1-1) is well-posed in $L^\infty(\mathbb{R}^n)$ thanks to a fixed-point argument. More precisely, there is a unique maximal solution on $[0, T)$, with $T \le +\infty$. If $T < +\infty$, then the solution of (1-1) may develop singularities in finite time T, in the sense that

$$\|u(t)\|_{L^\infty} \to +\infty \quad \text{as } t \to T.$$

In this case, T is called the blowup time of u. Given $a \in \mathbb{R}^n$, we say that a is a blowup point of u if and only if there exists $(a_j, t_j) \to (a, T)$ as $j \to +\infty$ such that $|u(a_j, t_j)| \to +\infty$ as $j \to +\infty$.

In the special case $\alpha = 0$, (1-1) becomes the standard semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u. \tag{1-3}$$

This equation is invariant under the scaling transformation

$$u \mapsto u_\lambda(x, t) := \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t). \tag{1-4}$$

Extensive literature is devoted to (1-3) and no review can be exhaustive. Given our interest in the construction question with a prescribed blowup behavior, we only mention previous work in this direction.

Bricmont and Kupiainen [1994] showed the existence of a solution of (1-3) such that

$$\left\| (T-t)^{1/(p-1)} u\big(a + z\sqrt{(T-t)|\ln(T-t)|}, t\big) - \varphi_0(z) \right\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as } t \to T, \tag{1-5}$$

where

$$\varphi_0(z) = \left(p - 1 + \frac{(p-1)^2 z^2}{4p} \right)^{-1/(p-1)}$$

(note that Herrero and Velázquez [1992] proved the same result with a different method; note also that Bressan [1992] made a similar construction in the case of an exponential nonlinearity).

Later, Merle and Zaag [1997] (see also [Merle and Zaag 1996]) simplified the proof of [Bricmont and Kupiainen 1994] and proved the stability of the constructed solution satisfying the behavior (1-5). Their method relies on the linearization of the similarity variables version around the expected profile. In that setting, the linearized operator has two positive eigenvalues, a zero eigenvalue and then a negative spectrum. Then, they proceed in two steps:

- Reduction of an infinite-dimensional problem to finite-dimensional one: they show that controlling the similarity variable version around the profile reduces to the control of the components corresponding to the two positive eigenvalues.

- Then, they solve the finite-dimensional problem thanks to a topological argument based on index theory.

The method of [Merle and Zaag 1997] has proved to be successful in various situations, such as for the complex Ginzburg–Landau equation of [Masmoudi and Zaag 2008] (see also [Zaag 1998] for an earlier work) and for the case of a complex semilinear heat equation with no variational structure [Nouaili and Zaag 2015]. We also mention the work of Tayachi and Zaag [2015a; 2015b] and the work of Ghoul, Nguyen and Zaag [Ghoul et al. 2017a] dealing with a nonlinear heat equation with a double source depending on the solution and its gradient in a critical way. Ghoul, Nguyen and Zaag [Ghoul et al. 2016; 2017b] successfully adapted the method to construct a stable blowup solution for a nonvariational semilinear parabolic system.

In other words, the method of [Merle and Zaag 1997] has proved to be efficient even for the case of systems with nonvariational structure. However, all the previous examples enjoy a common scaling-invariant property like (1-4), which seemed at first to be a strong requirement for the method. In fact, this was proved to be untrue.

Ebde and Zaag [2011] were able to adapt the method to construct blowup solutions for the non-scaling-invariant equation

$$\partial_t u = \Delta u + |u|^{p-1}u + f(u, \nabla u), \tag{1-6}$$

where

$$|f(u, \nabla u)| \le C(1 + |u|^q + |\nabla u|^{q'}), \quad \text{with } q < p,\ q' < \frac{2p}{p+1}.$$

These conditions ensure that the perturbation $f(u, \nabla u)$ results in exponentially small coefficients in the similarity variables. Later, Nguyen and Zaag [2016] recorded a more spectacular achievement by addressing the case of stronger perturbation of (1-3), namely

$$\partial_t u = \Delta u + |u|^{p-1}u + \frac{\mu |u|^{p-1}u}{\ln^a(2+u^2)}, \tag{1-7}$$

where $\mu \in \mathbb{R}$ and $a > 0$. When moving to the similarity variables, the perturbation turns out to have a polynomial decay. Hence, when $a > 0$ is small, we are almost in the case of a critical perturbation.

In both cases addressed in [Ebde and Zaag 2011; Nguyen and Zaag 2016], the equations are indeed non-scaling-invariant, which shows the robustness of the method. However, since both papers proceed by perturbations around the standard case (1-3), it is as if we are still in the scaling-invariant case.

In this paper, we aim at trying the approach on a genuinely non-scaling-invariant case, namely (1-1). This is our main result.

Theorem 1.1 (blowup solutions for (1-1) with a prescribed behavior). *There exists an initial data $u_0 \in L^\infty(\mathbb{R}^n)$ such that the corresponding solution to* (1-1) *blows up in finite time $T = T(u_0) > 0$, only at the origin. Moreover, we have*:

(i) *For all $t \in [0, T)$, there exists a positive constant C_0 such that*

$$\left\| \psi^{-1}(t)u(x,t) - f_0\left(\frac{x}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right\|_{L^\infty(\mathbb{R}^n)} \le \frac{C_0}{\sqrt{|\ln(T-t)|}}, \tag{1-8}$$

where $\psi(t)$ is the unique positive solution of the ODE

$$\psi'(t) = \psi^p(t)\ln^\alpha(\psi^2(t)+2), \qquad \lim_{t\to T}\psi(t) = +\infty \tag{1-9}$$

(*see Lemma A.1 for the existence and uniqueness of ψ*), *and the profile f_0 is defined by*

$$f_0(z) = \left(1 + \frac{(p-1)}{4p}|z|^2\right)^{-1/(p-1)}. \tag{1-10}$$

(ii) *There exits $u^*(x) \in C^2(\mathbb{R}^n \setminus \{0\})$ such that $u(x,t) \to u^*(x)$ as $t \to T$ uniformly on compact sets of $\mathbb{R}^n \setminus \{0\}$, where*

$$u^*(x) \sim \left(\frac{(p-1)^2|x|^2}{8p\big|\ln|x|\big|}\right)^{-1/(p-1)} \left(\frac{4\big|\ln|x|\big|}{p-1}\right)^{-\alpha/(p-1)} \quad \text{as } x \to 0. \tag{1-11}$$

Remark 1.2. From (i), we see that $u(0,t) \sim \psi(t) \to +\infty$ as $t \to T$, which means that the solution blows up in finite time T at $x = 0$. From (ii), we deduce that the solution blows up only at the origin.

Remark 1.3. Note that the behavior in (1-8) is almost the same as in the standard case $\alpha = 0$ treated in [Bricmont and Kupiainen 1994; Merle and Zaag 1997]. However, the final profile u^* has a difference coming from the extra multiplication of the size $\big|\ln|x|\big|^{-\alpha/(p-1)}$, which shows that the nonlinear source in (1-1) has a strong effect on the dynamic of the solution in comparison with the standard case $\alpha = 0$.

Remark 1.4. Item (ii) is in fact a consequence of (1-8) and Lemma A.4. Therefore, the main goal of this paper is to construct for (1-1) a solution blowing up in finite time and satisfying the behavior (1-8).

Remark 1.5. By parabolic regularity, one can show that if the initial data $u_0 \in W^{2,\infty}(\mathbb{R}^n)$, then we have for $i = 0, 1, 2$,

$$\left\| \psi^{-1}(t)(T-t)^{i/2}\nabla_x^i u(x,t) - (T-t)^{i/2}\nabla_x^i f_0\left(\frac{x}{\sqrt{(T-t)|\ln(T-t)|}}\right)\right\|_{L^\infty} \le \frac{C}{\sqrt{|\ln(T-t)|}},$$

where f_0 is defined by (1-10).

From the technique of [Merle 1992a], we can prove the following result.

Corollary 1.6. *For an arbitrary set of m points $x_1, \dots, x_m$, there exists initial data u_0 such that the solution u of* (1-1) *with initial data u_0 blows up exactly at m points $x_1, \dots, x_m$. Moreover, the local behavior at each blowup point x_i is also given by* (1-8) *by replacing x by $x - x_i$.*

As a consequence of our technique, we prove the stability of the solution constructed in Theorem 1.1 under the perturbations of initial data. In particular, we have the following result.

Theorem 1.7 (stability of the solution constructed in Theorem 1.1). *Consider $\hat{u}$, the solution constructed in Theorem 1.1 and denote by $\widehat{T}$ its blowup time. Then there exists $\mathcal{U}_0 \subset L^\infty(\mathbb{R}^n)$ a neighborhood of $\hat{u}(0)$ such that for all $u_0 \in \mathcal{U}_0$,* (1-1) *with the initial data u_0 has a unique solution $u(t)$ blowing up in finite time $T(u_0)$ at a single point $a(u_0)$. Moreover, the statements* (i) *and* (ii) *in Theorem 1.1 are satisfied by $u(x - a(u_0), t)$, and*

$$(T(u_0), a(u_0)) \to (\widehat{T}, 0) \quad \text{as } \|u_0 - \hat{u}_0\|_{L^\infty(\mathbb{R}^n)} \to 0. \tag{1-12}$$

Remark 1.8. We will not give the proof of Theorem 1.7 because the stability result follows from the reduction to a finite-dimensional case as in [Merle and Zaag 1997] with the same proof. Here we only prove the existence and refer to that paper for the stability.

2. Formulation of the problem

We first use the matched asymptotic technique to formally derive the behavior (1-8). Then, we give the formulation of the problem in order to justify the formal result.

2A. ***A formal approach.*** We follow the approach of [Tayachi and Zaag 2015b] to formally explain how to derive the asymptotic behavior (1-8). To do so, we introduce the following self-similarity variables

$$u(x,t) = \psi(t) w(y,s), \qquad y = \frac{x}{\sqrt{T-t}}, \qquad s = -\ln(T-t), \tag{2-1}$$

where $\psi(t)$ is the unique positive solution of (1-9) and $\psi(t) \to +\infty$ as $t \to T$. Then, we see from (1-1) that $w(y,s)$ solves the following equation: for all $(y,s) \in \mathbb{R}^n \times [-\ln T, +\infty)$

$$\partial_s w = \Delta w - \tfrac{1}{2} y \cdot \nabla w - h(s) w + h(s)|w|^{p-1} w \frac{\ln^\alpha(\psi_1^2 w^2 + 2)}{\ln^\alpha(\psi_1^2 + 2)}, \tag{2-2}$$

where

$$h(s) = e^{-s} \psi_1^{p-1}(s) \ln^\alpha(\psi_1^2(s) + 2), \tag{2-3}$$

$$\psi_1(s) = \psi(T - e^{-s}). \tag{2-4}$$

Note that $h(s)$ admits the following asymptotic behavior as $s \to +\infty$:

$$h(s) = \frac{1}{p-1}\left(1 - \frac{\alpha}{s} - \frac{\alpha^2 \ln s}{s^2}\right) + O\left(\frac{1}{s^2}\right); \tag{2-5}$$

see (ii) of Lemma A.5 for the proof of (2-5). From (2-1), we see that the study of the asymptotic behavior of $u(x,t)$ as $t \to T$ is equivalent to the study of the long-time behavior of $w(y,s)$ as $s \to +\infty$. In other words, the construction of the solution $u(x,t)$, which blows up in finite time T and satisfies the behavior (1-8), reduces to the construction of a global solution $w(y,s)$ for (2-2) satisfying

$$0 < \epsilon_0 \le \limsup_{s \to +\infty} \|w(s)\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{\epsilon_0}, \quad \epsilon_0 > 0, \tag{2-6}$$

and

$$\left\| w(y,s) - \left(1 + \frac{(p-1)y^2}{4ps}\right)^{-1/(p-1)} \right\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as } s \to +\infty. \tag{2-7}$$

In the following, we will formally explain how to derive the behavior (2-7).

Inner expansion. We remark that $0, \pm 1$ are the trivial constant solutions to (2-2). Since we are looking for a nonzero solution, let us consider the case when $w \to 1$ as $s \to +\infty$. We now introduce

$$w = 1 + \bar{w}; \tag{2-8}$$

then from (2-2), we see that $\bar{w}$ satisfies

$$\partial_s \bar{w} = \mathcal{L}(\bar{w}) + N(\bar{w}, s), \tag{2-9}$$

where

$$\mathcal{L} = \Delta - \tfrac{1}{2} y.\nabla + \mathrm{Id}, \tag{2-10}$$

$$N(\bar{w}, s) = h(s)|\bar{w}+1|^{p-1}(\bar{w}+1)\frac{\ln^\alpha(\psi_1^2(\bar{w}+1)^2+2)}{\ln^\alpha(\psi_1^2+2)} - h(s)(\bar{w}+1) - \bar{w}, \tag{2-11}$$

$\psi_1(s)$ is defined in (2-4) and $h(s)$ behaves as in (2-5). Note that N admits the asymptotic behavior

$$N(\bar{w}, s) = \frac{p\bar{w}^2}{2} + O\left(\frac{|\bar{w}| \ln s}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right) + O(|\bar{w}|^3) \quad \text{as } (\bar{w}, s) \to (0, +\infty), \tag{2-12}$$

(see Lemma A.6 for the proof of this statement).

Since $\bar{w}(s) \to 0$ as $s \to +\infty$ and the nonlinear term N is quadratic in $\bar{w}$, we see from (2-9) that the linear part will play the main role in the analysis of our solution. Let us recall some properties of $\mathcal{L}$. The linear operator $\mathcal{L}$ is self-adjoint in $L^2_\rho(\mathbb{R}^n)$, where L^2_ρ is the weighted space associated with the weight ρ defined by

$$\rho(y) = \frac{e^{-|y|^2/4}}{(4\pi)^{n/2}},$$

and

$$\operatorname{spec}(\mathcal{L}) = \left\{1 - \frac{m}{2} : m \in \mathbb{N}\right\}.$$

More precisely, we have:

- When $n = 1$, all the eigenvalues of $\mathcal{L}$ are simple and the eigenfunction corresponding to the eigenvalue $1 - m/2$ is the Hermite polynomial defined by

$$h_m(y) = \sum_{j=0}^{[m/2]} \frac{(-1)^j m!\, y^{m-2j}}{j!\,(m-2j)!}. \tag{2-13}$$

In particular, we have the orthogonality

$$\int_{\mathbb{R}} h_i h_j \rho \, dy = i!\, 2^i \delta_{i,j} \quad \text{for all } (i, j) \in \mathbb{N}^2.$$

- When $n \geq 2$, the eigenspace corresponding to the eigenvalue $1-m/2$ is defined as

$$\mathcal{E}_m = \left\{h_\beta = h_{\beta_1} \cdots h_{\beta_n} : \text{for all } \beta \in \mathbb{N}^n,\ |\beta| = m,\ |\beta| = \beta_1 + \cdots + \beta_n\right\}. \tag{2-14}$$

Since the set of the eigenfunctions of $\mathcal{L}$ is a basis of L^2_ρ, we can expand $\bar{w}$ in this basis as

$$\bar{w}(y, s) = \sum_{\beta \in \mathbb{N}^n} \bar{w}_\beta(s) h_\beta(y).$$

For simplicity, let us assume that $\bar{w}$ is radially symmetric in y. Since h_β with $|\beta| \geq 3$ corresponds to negative eigenvalues of $\mathcal{L}$, we may consider the solution $\bar{w}$ taking the form

$$\bar{w} = \bar{w}_0 + \bar{w}_2(s)(|y|^2 - 2n), \tag{2-15}$$

where $|\bar{w}_0(s)|$ and $|\bar{w}_2(s)|$ go to 0 as $s \to +\infty$. Injecting (2-15) and (2-12) into (2-9), then projecting (2-9) on the eigenspace $\mathcal{E}_m$ with $m = 0$ and $m = 2$, we obtain

$$\begin{aligned} \bar{w}_0' &= \bar{w}_0 + \frac{p}{2}(\bar{w}_0^2 + 8n\bar{w}_2^2) + O\left(\frac{(|\bar{w}_0| + |\bar{w}_2|)\ln s}{s^2}\right) + O\left(\frac{|\bar{w}_0|^2 + |\bar{w}_2|^2}{s}\right) \\ &\qquad + O(|\bar{w}_0|^3 + |\bar{w}_2|^3), \\ \bar{w}_2' &= 4p\bar{w}_2^2 + p\bar{w}_0\bar{w}_2 + O\left(\frac{(|\bar{w}_0| + |\bar{w}_2|)\ln s}{s^2}\right) + O\left(\frac{|\bar{w}_0|^2 + |\bar{w}_2|^2}{s}\right) \\ &\qquad + O(|\bar{w}_0|^3 + |\bar{w}_2|^3) \end{aligned} \tag{2-16}$$

as $s \to +\infty$. We now assume that $|\bar{w}_0(s)| \ll |\bar{w}_2(s)|$ as $s \to +\infty$; then (2-17) becomes

$$\begin{aligned} \bar{w}_0' &= \bar{w}_0 + O(|\bar{w}_2|^2) + O\Big(\frac{|\bar{w}_2| \ln s}{s^2}\Big), \\ \bar{w}_2' &= 4p\bar{w}_2^2 + o(|\bar{w}_2|^2) + O\Big(\frac{|\bar{w}_2| \ln s}{s^2}\Big) \end{aligned} \tag{2-17}$$

as $s \to +\infty$. We consider the following cases:

Case 1: Either $|\bar{w}_2| = O((\ln s)/s^2)$ or $|\bar{w}_2| \ll (\ln s)/s$ as $s \to +\infty$. Then the second equation in (2-17) becomes

$$\bar{w}_2' = O\Big(\frac{|\bar{w}_2| \ln s}{s^2}\Big) \quad \text{as } s \to +\infty,$$

which yields

$$\ln|\bar{w}_2| = O\Big(\frac{\ln s}{s}\Big) \quad \text{as } s \to +\infty,$$

which contradicts the condition $\bar{w}_2(s) \to 0$ as $s \to +\infty$.

Case 2: $|\bar{w}_2| \gg (\ln s)/s^2$ as $s \to +\infty$. Then (2-17) becomes

$$\bar{w}_0' = \bar{w}_0 + O(|\bar{w}_2|^2), \quad \bar{w}_2' = 4p\bar{w}_2^2 + o(|\bar{w}_2|^2)$$

as $s \to +\infty$. This yields

$$\bar{w}_0 = O\Big(\frac{1}{s^2}\Big), \quad \bar{w}_2 = -\frac{1}{4ps} + o\Big(\frac{1}{s}\Big) \tag{2-18}$$

as $s \to +\infty$. Substituting (2-18) into (2-17) yields

$$\bar{w}_0' = O\Big(\frac{1}{s^2}\Big), \quad \bar{w}_2' = 4p\bar{w}_2^2 + O\Big(\frac{\ln s}{s^3}\Big)$$

as $s \to +\infty$, from which we improve the error for $\bar{w}_2$ as

$$\bar{w}_0 = O\Big(\frac{1}{s^2}\Big), \quad \bar{w}_2 = -\frac{1}{4ps} + O\Big(\frac{\ln^2 s}{s^2}\Big) \tag{2-19}$$

as $s \to +\infty$. Hence, from (2-8), (2-15) and (2-19), we derive

$$w(y,s) = 1 - \frac{y^2}{4ps} + \frac{n}{2ps} + O\Big(\frac{\ln^2 s}{s^2}\Big) \tag{2-20}$$

in $L^2_\rho(\mathbb{R}^n)$ as $s \to +\infty$. Note that the asymptotic expansion (2-20) also holds for all $|y| \le K$, where K is an arbitrary positive number.

Outer expansion. The asymptotic behavior of (2-20) suggests that the blowup profile depends on the variable

$$z = \frac{y}{\sqrt{s}}.$$

From (2-20), let us search for a regular solution of (2-2) of the form

$$w(y,s) = \phi_0(z) + \frac{n}{2ps} + o\Big(\frac{1}{s}\Big) \quad \text{in } L^\infty_{\text{loc}} \text{ as } s \to +\infty, \tag{2-21}$$

where ϕ_0 is a bounded, smooth function to be determined. From (2-20), we impose the condition

$$\phi_0(0) = 1. \tag{2-22}$$

Since $w(y,s)$ is supposed to be bounded, we obtain from Lemma A.7 that

$$\left| h(s)|w|^{p-1} w \frac{\ln^\alpha(\psi_1^2 w^2 + 2)}{\ln^\alpha(\psi_1^2 + 2)} - \frac{|w|^{p-1} w}{p-1} \right| = O\Big(\frac{1}{s}\Big).$$

Note also that

$$\left| \left|\phi_0(z) + O\Big(\frac{1}{s}\Big)\right|^{p-1} \Big(\phi_0(z) + O\Big(\frac{1}{s}\Big)\Big) - |\phi_0(z)|^{p-1}\phi_0(z) \right| = O\Big(\frac{1}{s}\Big).$$

Hence, injecting (2-21) into (2-2) and comparing terms of order $O(1/s^i)$ for $j = 0, 1, \dots$, we derive the following equation for $j = 0$:

$$-\tfrac{1}{2} z \cdot \nabla\phi_0(z) - \frac{\phi_0(z)}{p-1} + \frac{|\phi_0|^{p-1}\phi_0(z)}{p-1} = 0 \quad \text{for all } z \in \mathbb{R}^n. \tag{2-23}$$

Solving (2-23) with condition (2-22), we obtain

$$\phi_0(z) = (1 + c_0|z|^2)^{-1/(p-1)} \tag{2-24}$$

for some constant $c_0 \geq 0$ (since we want ϕ_0 to be bounded for all $z \in \mathbb{R}^n$). From (2-21), (2-24) and a Taylor expansion, we obtain

$$w(y,s) = 1 - \frac{c_0 y^2}{(p-1)s} + \frac{n}{2ps} + o\Big(\frac{1}{s}\Big) \quad \text{for all } |y| \leq K \text{ as } s \to +\infty.$$

From this and the asymptotic behavior (2-20), we find that

$$c_0 = \frac{p-1}{4p}.$$

In conclusion, we have just derived the asymptotic profile

$$w(y,s) \sim \varphi(y,s) \quad \text{as } s \to +\infty, \tag{2-25}$$

where

$$\varphi(y,s) = \left(1 + \frac{(p-1)y^2}{4ps}\right)^{-1/(p-1)} + \frac{n}{2ps}. \tag{2-26}$$

2B. *Formulation of the problem.* We now set up the problem in order to justify the formal approach presented in Section 2A. In particular, we give a formulation to prove item (i) of Theorem 1.1. We aim at constructing for (1-1) a solution blowing up in finite time T only at the origin and satisfying the behavior (1-8). In the similarity variables (2-1), this is equivalent to the construction of a solution $w(y,s)$ for (2-2) defined for all $(y,s) \in \mathbb{R}^n \times [s_0, +\infty)$ and satisfying (2-7). The formal approach given in Section 2A, see (2-25), suggests linearizing w around the profile function φ defined by (2-26). Let us introduce

$$q(y,s) = w(y,s) - \varphi(y,s), \tag{2-27}$$

where φ is defined by (2-26). From (2-2), we see that q satisfies the equation

$$\partial_s q = \mathcal{L}q + Vq + B(q) + R(y,s) + D(q,s), \tag{2-28}$$

where $\mathcal{L}$ is the linear operator defined by (2-10) and

$$V = \frac{p}{p-1}(\varphi^{p-1}-1), \tag{2-29}$$

$$B(q) = \frac{|q+\varphi|^{p-1}(q+\varphi)-\varphi^p-p\varphi^{p-1}q}{p-1}, \tag{2-30}$$

$$R(y,s) = \Delta\varphi-\tfrac{1}{2}y\cdot\nabla\varphi-\frac{\varphi}{p-1}+\frac{\varphi^p}{p-1}-\partial_s\varphi, \tag{2-31}$$

$$\begin{aligned} D(q,s) = (q+\varphi)\Bigg(\bigg(h(s)-\frac{1}{p-1}\bigg)&(|q+\varphi|^{p-1}-1) \\ &+h(s)|q+\varphi|^{p-1}(q+\varphi)L(q+\varphi,s)\Bigg), \end{aligned} \tag{2-32}$$

$$L(v,s) = \frac{2\alpha\psi_1^2}{\ln(\psi_1^2+2)(\psi_1^2+2)}(v-1)+\frac{1}{\ln^\alpha(\psi_1^2+2)}\int_1^v f''(u)(v-u)\,du, \tag{2-33}$$

with h, $\psi_1(s)$ and φ being defined by (2-3), (2-4) and (2-26) respectively, and

$$f(z) = \ln^\alpha(\psi_1^2 z^2 + 2), \quad z \in \mathbb{R}.$$

Hence, proving (1-8) now reduces to constructing for (2-28) a solution q such that

$$\lim_{s\to+\infty} \|q(s)\|_{L^\infty} \to 0.$$

Since we construct for (2-28) a solution q satisfying $\|q(s)\|_{L^\infty} \to 0$ as $s \to +\infty$, and since

$$|B(q)| \le C|q|^{\min(2,p)}, \quad \|R(s)\|_{L^\infty} + \|D(q,s)\|_{L^\infty} \le \frac{C}{s},$$

(see Lemmas A.8, A.9 and A.10 for these estimates), we see that the linear part of (2-28) will play an important role in the analysis of the solution. The spectral property of the linear operator $\mathcal{L}$ is studied in the previous section (see page 19), and the potential V has the following properties:

(i) Perturbation effect of $\mathcal{L}$ inside the blowup region $\{|y| \le K\sqrt{s}\}$:

$$\|V(s)\|_{L^2_\rho} \to 0 \quad \text{as } s \to +\infty.$$

(ii) For each $\epsilon > 0$, there exist $K_\epsilon > 0$ and $s_\epsilon > 0$ such that

$$\sup_{y/\sqrt{s} \ge K_\epsilon,\, s \ge s_\epsilon} \left| V(y,s) + \frac{p}{p-1} \right| \le \epsilon.$$

Since 1 is the biggest eigenvalue of $\mathcal{L}$, the operator $\mathcal{L} + V$ behaves as one with a fully negative spectrum outside the blowup region $\{|y| \ge K\sqrt{s}\}$, which makes the control of the solution in this region easy.

Since the behavior of the potential V is different inside and outside the blowup region, we will consider the dynamics of the solution for $|y| \le 2K\sqrt{s}$ and for $|y| \ge K\sqrt{s}$ separately for some K to be fixed large. We introduce the function

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K\sqrt{s}}\right), \tag{2-34}$$

where $\chi_0 \in C_0^\infty[0,+\infty)$, $\|\chi_0\|_{L^\infty} \le 1$ and

$$\chi_0(x) = \begin{cases} 1 & \text{for } x \le 1, \\ 0 & \text{for } x \ge 2, \end{cases}$$

and K is a positive constant to be fixed large later. We now decompose q as

$$q = \chi q + (1-\chi) q = q_b + q_e. \tag{2-35}$$

(Note that $\operatorname{supp}(q_b) \subset \{|y| \le 2K\sqrt{s}\}$ and $\operatorname{supp}(q_e) \subset \{|y| \ge K\sqrt{s}\}$). Since the eigenfunctions of $\mathcal{L}$ span the whole space L^2_ρ, let us write

$$q_b(y,s) = q_0(s) + q_1(s) \cdot y + \tfrac{1}{2} y^T \cdot q_2(s) \cdot y - \operatorname{tr}(q_2(s)) + q_-(y,s), \tag{2-36}$$

where $q_m(s) = (q_\beta(s))_{\beta \in \mathbb{N}^n, |\beta| = m}$ and

$$\text{for all } \beta \in \mathbb{N}^n, \quad q_\beta(s) = \int_{\mathbb{R}^n} q_b(y,s) \tilde{h}_\beta(y) \rho \, dy, \quad \tilde{h}_\beta = \frac{h_\beta}{\|h_\beta\|^2_{L^2_\beta}}, \tag{2-37}$$

and

$$q_-(y,s) = \sum_{\beta \in \mathbb{N}^n, |\beta| \ge 3} q_\beta(s) h_\beta(y). \tag{2-38}$$

In particular, we set $q_1 = (q_{1,i})_{1 \le i \le n}$ and $q_2(s)$ is an $n \times n$ symmetric matrix defined explicitly by

$$q_2(s) = \int q_b \mathcal{M}(y) \rho \, dy = (q_{2,i,j})_{1 \le i,j \le n}, \tag{2-39}$$

with

$$\mathcal{M} = \left\{\tfrac{1}{4}y_i y_j - \tfrac{1}{2}\delta_{i,j}\right\}_{1\le i,j\le n}. \tag{2-40}$$

Hence, by (2-35) and (2-36), we can write

$$q(y,s) = q_0(s) + q_1(s)\cdot y + \tfrac{1}{2}y^T\cdot q_2(s)\cdot y - \operatorname{tr}(q_2(s)) + q_-(y,s) + q_e(y,s). \tag{2-41}$$

Note that $q_m (m = 0, 1, 2)$ and q_- are the components of q_b, and not those of q.

3. Proof of the existence, assuming some technical results

We shall now describe the main argument behind the proof of Theorem 1.1. To avoid winding up with too many details, we shall postpone most of the technicalities involved to the next section. According to the transformations (2-1) and (2-27), proving (i) of Theorem 1.1 is equivalent to showing that there exists an initial data $q_0(y)$ at the time s_0 such that the corresponding solution q of (2-28) satisfies

$$\|q(s)\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as } s \to +\infty.$$

In particular, we consider the function

$$\psi_{d_0,d_1}(y) = \frac{A}{s_0^2}(d_0 + d_1\cdot y)\chi(2y, s_0) \tag{3-1}$$

as the initial data for (2-28), where $(d_0, d_1) \in \mathbb{R}^{1+n}$ are the parameters to be determined, $s_0 > 1$ and $A > 1$ are constants to be fixed large enough, and χ is the function defined by (2-34).

We aim to prove that there exists $(d_0, d_1) \in \mathbb{R}\times\mathbb{R}^n$ such that the solution $q(y,s) = q_{d_0,d_1}(y,s)$ of (2-28) with initial data $\psi_{d_0,d_1}(y)$ satisfies

$$\|q_{d_0,d_1}(s)\|_{L^\infty} \to 0 \quad \text{as } s \to +\infty.$$

More precisely, we will show that there exists $(d_0, d_1) \in \mathbb{R}\times\mathbb{R}^n$ such that the solution $q_{d_0,d_1}(y,s)$ belongs to the shrinking set S_A defined as follows:

Definition 3.1 (a shrinking set to zero). For all $A \ge 1$, $s \ge 1$ we define $S_A(s)$ to be the set of all functions $q \in L^\infty(\mathbb{R}^n)$ such that

$$|q_0| \le \frac{A}{s^2}, \quad |q_{1,i}| \le \frac{A}{s^2}, \quad |q_{2,i,j}| \le \frac{A^2\ln^2 s}{s^2} \quad \text{for all } 1 \le i, j \le n,$$

$$\left\|\frac{q_-(y)}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^n)} \le \frac{A}{s^2}, \quad \|q_e(y)\|_{L^\infty(\mathbb{R}^n)} \le \frac{A^2}{\sqrt{s}},$$

where q_0, $q_1 = (q_{1,i})_{1\le i\le n}$, $q_2 = (q_{2,i,j})_{1\le i,j\le n}$, q_- and q_e are defined as in (2-41).

We also denote by $\widehat{S}_A(s)$ the set

$$\widehat{S}_A(s) = \left[-\frac{A}{s^2}, \frac{A}{s^2}\right] \times \left[-\frac{A}{s^2}, \frac{A}{s^2}\right]^n. \tag{3-2}$$

Remark 3.2. For each $A \geq 1$, $s \geq 1$, we have the following estimates for all $q(s) \in S_A(s)$:

$$|q(y,s)| \leq \frac{CA^2 \ln^2 s}{s^2}(1+|y|^3) \quad \text{for all } y \in \mathbb{R}^n, \tag{3-3}$$

$$\|q(s)\|_{L^\infty(\{|y| \leq 2K\sqrt{s}\})} \leq \frac{CA}{\sqrt{s}}, \tag{3-4}$$

$$\|q(s)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{CA^2}{\sqrt{s}}. \tag{3-5}$$

We aim to prove the following central proposition, which implies Theorem 1.1.

Proposition 3.3 (existence of a solution trapped in $S_A(s)$). *There exists $A_1 \geq 1$ such that for all $A \geq A_1$ there exists $s_1(A) \geq 1$ such that for all $s_0 \geq s_1(A)$, there exists $(d_0, d_1) \in \mathbb{R}^{1+n}$ such that the solution $q(y,s) = q_{d_0,d_1}(y,s)$ of* (2-28) *with the initial data at the time s_0 given by $q(y,s_0) = \psi_{d_0,d_1}(y)$, where ψ_{d_0,d_1} is defined as in* (3-1), *satisfies*

$$q(s) \in S_A(s) \quad \textit{for all } s \in [s_0, +\infty).$$

From (3-5), we see that once Proposition 3.3 is proved, item (i) of Theorem 1.1 directly follows. In the following, we shall give all the main arguments for the proof of this proposition assuming some technical results which are left to the next section.

As for the initial data at time s_0 defined as in (3-1), we have the following properties.

Proposition 3.4. *For each $A \geq 1$, there exists $s_2(A) > 1$ such that for all $s_0 \geq s_2(A)$ we have the following*:

(i) *There exists*

$$\mathbb{D}_{A,s_0} \subset [-2; 2] \times [-2; 2]^n$$

such that the mapping

$$\Phi_1 : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n},$$
$$(d_0, d_1) \mapsto (\psi_0, \psi_1),$$

is linear and one-to-one from $\mathbb{D}_{A,s_0}$ onto $\widehat{S}_A(s_0)$. Moreover,

$$\Phi_1(\partial \mathbb{D}_{A,s_0}) \subset \partial \widehat{S}_A(s_0).$$

(ii) *For all* $(d_0, d_1) \in \mathbb{D}_{A,s_0}$ *we have* $\psi_{d_0,d_1} \in S_A(s_0)$ *with strict inequalities in the sense that*

$$|\psi_0| \le \frac{A}{s_0^2}, \quad |\psi_{1,i}| \le \frac{A}{s_0^2}, \quad |\psi_{2,i,j}| < \frac{A\ln^2 s_0}{s_0^2} \quad \textit{for all } 1 \le i, j \le n,$$

$$\left\| \frac{\psi_-}{1+|y|^3} \right\|_{L^\infty(\mathbb{R})} < \frac{A}{s_0^2}, \quad \psi_e \equiv 0.$$

Above, $\chi(y, s_0)$ *is defined in* (2-34), ψ_0, $(\psi_{1,i})_{1\le i\le n}$, $(\psi_{2,i,j})_{1\le i,j\le 2}$, ψ_- *and* ψ_e *are the components of* ψ_{d_0,d_1} *defined as in* (2-41), *and* ψ_{d_0,d_1} *and* $\widehat{S}_A(s)$ *are defined by* (3-1) *and* (3-2).

Proof. See Proposition 4.5 of [Tayachi and Zaag 2015b] for a similar proof. □

From now on, we denote by C the universal constant which only depends on K, where K is introduced in (2-34). Let us now give the proof of Proposition 3.3 to complete the proof of item (i) of Theorem 1.1.

Proof of Proposition 3.3. We proceed into two steps to prove Proposition 3.3:

- In the first step, we reduce the problem of controlling $q(s)$ in $S_A(s)$ to controlling $(q_0, q_1)(s)$ in $\widehat{S}_A(s)$, where q_0 and q_1 are the components of q corresponding to the positive modes defined as in (2-41) and $\widehat{S}_A$ is defined by (3-2). This means that we reduce the problem to a finite-dimensional one.

- In the second step, we argue by contradiction to solve the finite-dimensional problem thanks to a topological argument.

Step 1: reduction to a finite-dimensional problem. In this step, we show through an a priori estimate that the control of $q(s)$ in $S_A(s)$ reduces to the control of $(q_0, q_1)(s)$ in $\widehat{S}_A(s)$. This mainly follows from a good understanding of the properties of the linear part $\mathcal{L} + V$ of (2-28). In particular, we claim the following which is the heart of our analysis.

Proposition 3.5 (control of $q(s)$ in $S_A(s)$ by $(q_0, q_1)(s)$ in $\widehat{S}_A(s)$). *There exists* $A_3 \ge 1$ *such that for all* $A \ge A_3$, *there exists* $s_3(A) \ge 1$ *such that for all* $s_0 \ge s_3(A)$, *the following holds*:

If $q(y, s)$ *is the solution of* (2-28) *with the initial data at time* s_0 *given by* (3-1) *with* $(d_0, d_1) \in \mathbb{D}_{A,s_0}$, *and* $q(s) \in S_A(s)$ *for all* $s \in [s_0, s_1]$ *for some* $s_1 \ge s_0$ *and* $q(s_1) \in \partial S_A(s_1)$, *then*:

(i) *Reduction to a finite-dimensional problem*: *we have* $(q_0, q_1)(s_1) \in \partial\widehat{S}_A(s_1)$.

(ii) *Transverse outgoing crossing*: *there exists* $\delta_0 > 0$ *such that*

$$\textit{for all } \delta \in (0, \delta_0), \quad (q_0, q_1)(s_1 + \delta) \notin \widehat{S}_A(s_1 + \delta);$$

hence, $q(s_1 + \delta) \notin S_A(s_1 + \delta)$, *where* $\widehat{S}_A$ *is defined in* (3-2) *and* $\mathbb{D}_{A,s_0}$ *is introduced in Proposition 3.4.*

Let us suppose for the moment that Proposition 3.5 holds. Then we can take advantage of a topological argument quite similar to that already used in [Merle and Zaag 1997].

Step 2: a basic topological argument. From Proposition 3.5, we claim that there exists $(d_0, d_1) \in \mathbb{D}_{A,s_0}$ such that (2-28) with initial data (3-1) has a solution

$$q_{d_0,d_1}(s) \in S_A(s) \quad \text{for all } s \in [s_0, +\infty),$$

for suitable choice of the parameters A, K, s_0. Since the argument is analogous to that in [Merle and Zaag 1997], we only give the main ideas.

Let us consider s_0, K and A such that Propositions 3.4 and 3.5 hold. From Proposition 3.4, we have

$$\text{for all } (d_0, d_1) \in \mathbb{D}_{A,s_0}, \quad q_{d_0,d_1}(y, s_0) := \psi_{d_0,d_1} \in S_A(s_0),$$

where ψ_{d_0,d_1} is defined by (3-1). Since the initial data belongs to L^∞, we then deduce from the local existence theory for the Cauchy problem of (1-1) in L^∞ that we can define for each $(d_0, d_1) \in \mathbb{D}_{A,s_0}$ a maximum time $s_*(d_0, d_1) \in [s_0, +\infty)$ such that

$$q_{d_0,d_1}(s) \in S_A(s) \quad \text{for all } s \in [s_0, s_*).$$

If $s_*(d_0, d_1) = +\infty$ for some $(d_0, d_1) \in \mathbb{D}_{A,s_0}$, then we are done. Otherwise, we argue by contradiction and assume that $s_*(d_0, d_1) < +\infty$ for all $(d_0, d_1) \in \mathbb{D}_{A,s_0}$. By continuity and the definition of s_*, we deduce that $q_{d_0,d_1}(s_*)$ is on the boundary of $S_A(s_*)$. From item (i) of Proposition 3.5, we have

$$(q_0, q_1)(s_*) \in \partial \widehat{S}_A(s_*).$$

Hence, we may define the rescaled function

$$\Gamma : \mathbb{D}_{A,s_0} \to \partial([-1, 1]^{1+n}),$$
$$(d_0, d_1) \mapsto \frac{s_*^2}{A}(q_0, q_1)(s_*).$$

From item (i) of Proposition 3.4, we see that if $(d_0, d_1) \in \partial\mathbb{D}_{A,s_0}$, then

$$q(s_0) \in S_A(s_0), \quad (q_0, q_1)(s_0) \in \partial \widehat{S}_A(s_0).$$

From item (ii) of Proposition 3.5, we see that $q(s)$ must leave $S_A(s)$ at $s = s_0$, thus, $s_*(d_0, d_1) = s_0$. Therefore, the restriction of $\Gamma$to $\partial\mathbb{D}_{A,s_0}$ is homeomorphic to the identity mapping, which is impossible thanks to index theorem, and the contradiction is obtained. This concludes the proof of Proposition 3.3 as well as item (i) of Theorem 1.1, assuming that Proposition 3.5 holds. □

Proof of (ii) *of Theorem 1.1.* The existence of u^* in $C^2(\mathbb{R}^n \setminus \{0\})$ follows from the technique of [Merle 1992b]. Here, we want to find an equivalent formation for u^*

near the blowup point $x = 0$. The case $\alpha = 0$ was treated in [Zaag 1998]. When $\alpha \neq 0$, we follow the method of that paper, and no new idea is needed. Therefore, we just sketch the main steps for the sake of completeness.

We consider $K_0 > 0$ a constant to be fixed large enough, and $|x_0| \neq 0$ small enough. Then, we introduce the function

$$\upsilon(x_0, \xi, \tau) = \psi^{-1}(t_0(x_0))u(x, t), \tag{3-6}$$

where

$$(\xi, \tau) \in \mathbb{R}^n \times \left[-\frac{t_0(x_0)}{T - t_0(x_0)}, 1\right),$$

and

$$(x, t) = \big(x_0 + \xi\sqrt{T - t_0(x_0)}, t_0(x_0) + \tau(T - t_0(x_0))\big), \tag{3-7}$$

with $t_0(x_0)$ being uniquely determined by

$$|x_0| = K_0\sqrt{(T - t_0(x_0))|\ln(T - t_0(x_0))|}. \tag{3-8}$$

From (3-6)–(3-8) and (1-8) we derive that

$$\sup_{|\xi|<2|\ln(T-t_0(x_0))|^{1/4}} |v(x_0, \xi, 0) - \varphi_0(K_0)| \leq \frac{C}{1+(|\ln(T-t_0(x_0))|^{1/4})} \to 0 \quad \text{as } x_0 \to 0,$$

where

$$\varphi_0(x) = \left(1 + \frac{(p-1)x^2}{4p}\right)^{1/(p-1)}.$$

As in [Zaag 1998], we use the continuity with respect to initial data for (1-1) associated to a space-localization in the ball $B(0, |\xi| < |\ln(T - t_0(x_0))|^{1/4})$ to derive

$$\sup_{|\xi|<|\ln(T-t_0(x_0))|^{1/4},\ \tau\in[0,1)} |v(x_0, \xi, \tau) - \hat{v}_{K_0}(\tau)| \leq \epsilon(x_0) \to 0 \quad \text{as } x_0 \to 0, \tag{3-9}$$

where

$$\hat{v}_{K_0}(\tau) = \left((1 - \tau) + \frac{(p-1)K_0^2}{4p}\right)^{-1/(p-1)}.$$

From (3-7) and (3-9), we deduce

$$\begin{aligned} u^*(x_0) &= \lim_{t\to T} u(x_0, t) \\ &= \psi(t_0(x_0)) \lim_{\tau\to 1} v(x_0, 0, \tau) \sim \psi(t_0(x_0))\left(\frac{(p-1)K_0}{4p}\right)^{-1/(p-1)}. \end{aligned} \tag{3-10}$$

Using the relation (3-8), we find that

$$T - t_0 \sim \frac{|x_0|^2}{2K_0\big|\ln|x_0|\big|} \quad \text{and} \quad \ln(T - t_0(x_0)) \sim 2\ln(|x_0|) \quad \text{as } x_0 \to 0. \tag{3-11}$$

The formula (1-11) then follows from Lemma A.1, (3-10) and (3-11). This concludes the proof of Theorem 1.1, assuming that Proposition 3.5 holds. □

4. Proof of Proposition 3.5

This section is devoted to the proof of Proposition 3.5, which is the heart of our analysis. We proceed into two parts. In the first part, we derive a priori estimates on $q(s)$ in $S_A(s)$. In the second part, we show that the new bounds are better than those defined in $S_A(s)$, except for the first two components (q_0, q_1). This means that the problem is reduced to the control of a finite-dimensional function (q_0, q_1), which is the conclusion of item (i) of Proposition 3.5. Item (ii) of Proposition 3.5 is just direct consequence of the dynamics of the modes q_0 and q_1.

4A. ***A priori estimates on $q(s)$ in $S_A(s)$.*** We derive the a priori estimates on the components q_2, q_-, q_e which imply the conclusion of Proposition 3.5. Firstly, let us give some dynamics of $q_0, q_1 = (q_{1,i})_{1\le i\le n}$ and $q_2 = (q_{2,i,j})_{1\le i,j\le n}$. More precisely, we claim the following.

Proposition 4.1 (dynamics of (2-28)). *There exists $A_4 \ge 1$ such that for all $A \ge A_4$ there exists $s_4(A) \ge 1$ such that the following holds for all $s_0 \ge s_4(A)$: Assume that for all $s \in [\tau, s_1]$ for some $s_1 \ge \tau \ge s_0$, we have $q(s) \in S_A(s)$. Then the following holds for all $s \in [s_0, s_1]$:*

(i) *ODE satisfied by the positive and null modes*:

$$m = 0, 1, \qquad \left| q_m'(s) - \left(1 - \frac{m}{2}\right) q_m(s) \right| \le \frac{C}{s^2}, \tag{4-1}$$

and

$$\left| q_2'(s) + \frac{2}{s} q_2(s) \right| \le \frac{C \ln s}{s^3}. \tag{4-2}$$

(ii) *control of the negative and outer modes*:

$$\left\| \frac{q_-(\cdot, s)}{1+|y|^3} \right\|_{L^\infty} \le C e^{-(s-\tau)/2} \left\| \frac{q_-(\cdot, \tau)}{1+|y|^3} \right\|_{L^\infty} + C \frac{e^{-(s-\tau)^2}}{s^{3/2}} \|q_e(\tau)\|_{L^\infty} + C \frac{(1+s-\tau)}{s^2}, \tag{4-3}$$

$$\|q_e(s)\|_{L^\infty} \le C e^{-(s-\tau)/p} \|q_e(\tau)\|_{L^\infty} + C e^{s-\tau} s^{3/2} \left\| \frac{q_-(\cdot, \tau)}{1+|y|^3} \right\|_{L^\infty} + C \frac{1+(s-\tau)e^{s-\tau}}{s^{1/2}}. \tag{4-4}$$

Proof. We proceed in two parts:

- In the first part we project (2-28) to write ODEs satisfied by q_m for $m = 0, 1, 2$.
- In the second part we use the integral form of (2-28) and the dynamics of the linear operator $\mathcal{L} + V$ to derive a priori estimates on q_- and q_e.

Part 1: ODEs satisfying the positive and null modes. We give the proof of (4-2); the same proof holds for (4-1). By formula (2-39) and (2-28), we write for each

$1 \le i, j \le n$,

$$\left| q'_{2,i,j}(s) - \int \big[\mathcal{L}q + Vq + B(q) + R(y,s) + D(q,s)\big]\chi\left(\frac{y_i y_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho\, dy\right| \le Ce^{-s}. \tag{4-5}$$

Using the assumption $q(s) \in S_A(s)$ for all $s \in [s_0, s_1]$, we derive the following estimates for all $s \in [s_0, s_1]$:

$$\left|\int \mathcal{L}(q)\chi\left(\frac{y_i y_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho\, dy\right| \le \frac{C}{s^3},$$

and from Lemmas A.8, A.9 and A.10

$$\left|\int Vq\chi\left(\frac{y_i y_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho\, dy + \frac{2}{s}q_{2,i,j}(s)\right| \le \frac{CA}{s^3},$$

$$\left|\int B(q)\chi\left(\frac{y_i y_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho\, dy\right| \le \frac{C}{s^3},$$

$$\left|\int R\chi\left(\frac{y_i y_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho\, dy\right| \le \frac{C}{s^3},$$

$$\left|\int D(q,s)\chi\left(\frac{y_i y_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho\, dy\right| \le \frac{C\ln s}{s^3}.$$

Gathering all these above estimates in (4-5) yields

$$\left|q'_{2,i,j} + \frac{2}{s}q_{2,i,j}\right| \le \frac{C\ln s}{s^3},$$

which concludes the proof of (4-2).

Part 2: control of the negative and outer modes. We give the proofs of (4-3) and (4-4) in this part. The control of q_- and q_e is mainly based on the dynamics of the linear operator $\mathcal{L} + V$. In particular, we use the following integral form of (2-28): for each $s \ge \sigma \ge s_0$,

$$q(s) = \mathcal{K}(s,\sigma)q(\sigma) + \int_\sigma^s \mathcal{K}(s,\tau)\big[B(q)(\tau) + R(\tau) + D(q,\tau)\big]\,d\tau = \sum_{i=1}^4 \vartheta_i(s,\sigma), \tag{4-6}$$

where $\{\mathcal{K}(s,\sigma)\}_{s\ge\sigma}$ is defined by

$$\begin{cases} \partial_s \mathcal{K}(s,\sigma) = (\mathcal{L} + V)\mathcal{K}(s,\sigma), & s > \sigma, \\ \mathcal{K}(\sigma,\sigma) = \mathrm{Id}, \end{cases} \tag{4-7}$$

and

$$\vartheta_1(s,\sigma) = \mathcal{K}(s,\sigma)q(\sigma), \qquad \vartheta_2(s,\sigma) = \int_\sigma^s \mathcal{K}(s,\tau)B(q)(\tau)\,d\tau,$$

$$\vartheta_3(s,\sigma) = \int_\sigma^s \mathcal{K}(s,\tau)R(\cdot,\tau)\,d\tau, \quad \vartheta_4(s,\sigma) = \int_\sigma^s \mathcal{K}(s,\tau)D(q,\tau)\,d\tau.$$

From (4-6), it is clear to see the strong influence of the kernel $\mathcal{K}$ in this formula. It is therefore convenient to recall the following result on the dynamics of the linear operator $\mathcal{K} = \mathcal{L} + V$.

Lemma 4.2 (a priori estimates of the linearized operator in the decomposition in (2-41)). *For all $\rho^* \geq 0$, there exists $s_5(\rho^*) \geq 1$ such that if $\sigma \geq s_5(\rho^*)$ and $v \in L^2_\rho$ satisfies*

$$\sum_{m=0}^{2} |v_m| + \left\| \frac{v_-}{1+|y|^3} \right\|_{L^\infty} + \|v_e\|_{L^\infty} < \infty, \tag{4-8}$$

then, for all $s \in [\sigma, \sigma + \rho^*]$ the function $\theta(s) = \mathcal{K}(s, \sigma)v$ satisfies

$$\begin{aligned} \left\| \frac{\theta_-(y,s)}{1+|y|^3} \right\|_{L^\infty} \leq \frac{Ce^{s-\sigma}((s-\sigma)^2+1)}{s}(|v_0| + |v_1| + \sqrt{s}|v_2|) \\ + Ce^{-(s-\sigma)/2} \left\| \frac{v_-}{1+|y|^3} \right\|_{L^\infty} + C\frac{e^{-(s-\sigma)^2}}{s^{3/2}} \|v_e\|_{L^\infty}, \end{aligned} \tag{4-9}$$

and

$$\begin{aligned} &\|\theta_e(y,s)\|_{L^\infty} \\ &\quad \leq Ce^{s-\sigma}\left(\sum_{l=0}^{2} s^{l/2}|v_l| + s^{3/2} \left\| \frac{v_-}{1+|y|^3} \right\|_{L^\infty} \right) + Ce^{-(s-\sigma)/p}\|v_e\|_{L^\infty}. \end{aligned} \tag{4-10}$$

Proof. The proof of this result was given in [Bricmont and Kupiainen 1994] in the one-dimensional case. It was then extended to higher-dimensional cases in [Nguyen and Zaag 2017]. We kindly refer interested readers to Lemma 2.9 in that paper for the details of the proof. □

In view of formula (4-6), we see that Lemma 4.2 plays an important role in deriving the new bounds on the components q_- and q_e. Indeed, given bounds on the components of q, $B(q)$, $D(q)$ and R, we directly apply Lemma 4.2 with $\mathcal{K}(s, \sigma)$ replaced by $\mathcal{K}(s, \tau)$ and then integrate over τ to obtain estimates on q_- and q_e. In particular, we claim the following which immediately follows from (4-3) and (4-4) by addition.

Lemma 4.3. *For all $\tilde{A} \geq 1$, $A \geq 1$, $\rho^* \geq 0$, there exists $s_6(A, \rho^*) \geq 1$ such that for all $s_0 \geq s_6(A, \rho^*)$, if $q(s) \in S_A(s)$ for all $s \in [\sigma, \sigma + \rho^*]$ for some $\sigma \geq s_0$, then we have for all $s \in [\sigma, \sigma + \rho^*]$:*

(i) *The linear term $\vartheta_1(s, \sigma)$:*

$$\left\| \frac{(\vartheta_1(s,\sigma))_-}{1+|y|^3} \right\|_{L^\infty} \leq Ce^{-(s-\sigma)/2} \left\| \frac{q_-(\,\cdot\,,\sigma)}{1+|y|^3} \right\|_{L^\infty} + \frac{Ce^{-(s-\sigma)^2}}{s^{3/2}} \|q_e(\sigma)\| + \frac{C}{s^2},$$

$$\|(\vartheta_1(s,\sigma))_e\|_{L^\infty} \leq Ce^{-(s-\sigma)/p}\|q_e(\sigma)\|_{L^\infty} + Ce^{s-\sigma}s^{3/2} \left\| \frac{q_-(\,\cdot\,,\sigma)}{1+|y|^3} \right\|_{L^\infty} + \frac{C}{\sqrt{s}}.$$

(ii) *The quadratic term* $\vartheta_2(s,\sigma)$:

$$\left\|\frac{(\vartheta_2(s,\sigma))_-}{1+|y|^3}\right\|_{L^\infty} \le \frac{C(s-\sigma)}{s^{2+\epsilon}}, \quad \|(\vartheta_2(s,\sigma))_e\|_{L^\infty} \le \frac{C(s-\sigma)}{s^{1/2+\epsilon}},$$

where $\epsilon=\epsilon(p)>0$.

(iii) *The correction term* $\vartheta_3(s,\sigma)$:

$$\left\|\frac{(\vartheta_3(s,\sigma))_-}{1+|y|^3}\right\|_{L^\infty} \le \frac{C(s-\sigma)}{s^{2}}, \quad \|(\vartheta_3(s,\sigma))_e\|_{L^\infty} \le \frac{C(s-\sigma)}{s^{3/4}}.$$

(iv) *The nonlinear term* $\vartheta_4(s,\sigma)$:

$$\left\|\frac{(\vartheta_4(s,\sigma))_-}{1+|y|^3}\right\|_{L^\infty} \le \frac{C(s-\sigma)}{s^{2}}, \quad \|(\vartheta_4(s,\sigma))_e\|_{L^\infty} \le \frac{C(s-\sigma)}{s^{3/4}}.$$

Proof. The proof simply follows from definition of the set S_A and Lemma 4.2. In particular, we make use of Lemmas A.8 , A.9 and A.10 to derive the bounds on the components of the terms B, D and R as follows:

$$\sum_{m\in\mathbb{N}^n,\ |m|=0}^{2} |B(q)_m(s)| \le \frac{C}{s^3}, \quad \left\|\frac{B(q)_-(s)}{1+|y|^3}\right\|_{L^\infty} \le \frac{C}{s^{2+\epsilon}}, \quad \|B(q)_e(s)\|_{L^\infty} \le \frac{C}{s^{1/2+\epsilon}},$$

and

$$\sum_{m\in\mathbb{N}^n,\ |m|=0}^{2} |R_m(s)| \le \frac{C}{s^2}, \quad \left\|\frac{R_-(s)}{1+|y|^3}\right\|_{L^\infty} \le \frac{C}{s^{2+1/2}}, \quad \|R_e(s)\|_{L^\infty} \le \frac{C}{s^{3/4}},$$

and

$$\sum_{m\in\mathbb{N}^n,\ |m|=0}^{2} |D(q)_m(s)| + \left\|\frac{D(q)_-(s)}{1+|y|^3}\right\|_{L^\infty} \le \frac{C\ln s}{s^3}, \quad \|D(q)_e(s)\|_{L^\infty} \le \frac{C}{s^{3/4}},$$

where $\epsilon=\epsilon(p)>0$. We simply inject these bounds into the a priori estimates given in Lemma 4.2 to obtain the bounds on $(\vartheta_m)_-$ and $(\vartheta_m)_e$ for $m=2,3,4$. The estimate on ϑ_1 directly follows from Lemma 4.2 and the assumption $q(s)\in S_A(s)$. This ends the proof of Lemma 4.3. □

By the formula (4-6), the estimates (4-3) and (4-4) simply follow from Lemma 4.3 by addition. This concludes the proof of Proposition 4.1. □

4B. ***Conclusion of Proposition 3.5.*** We now give the proof of Proposition 3.5, which is a consequence of the dynamics of (2-28) given in Proposition 4.1. Indeed, item (i) of Proposition 3.5 directly follows from the result below.

Proposition 4.4 (control of $q(s)$ by $(q_0,q_1)(s)$ in $S_A(s)$). *There exists* $A_7\ge 1$ *such that for all* $A\ge A_7$, *there exists* $s_7(A)\ge 1$ *such that for all* $s_0\ge s_7(A)$ *if*

(a) $q(s_0)=\psi_{d_0,d_1,s_0}(y)$, *where* $(d_0,d_1)\in\mathbb{D}_{A,s_0}$,

(b) $q(s)\in S_A(s)$ *for all* $s\in[s_0,s_1]$, *then for all* $s \in [s_0, s_1]$, *we have*

$$\textit{for all } i, j \in \{1, \dots, n\}, \quad |q_{2,i,j}(s)| < \frac{A^2 \ln^2 s}{s^2}, \tag{4-11}$$

$$\left\| \frac{q_-(y,s)}{1+|y|^3} \right\|_{L^\infty} \le \frac{A}{2s^2}, \quad \|q_e(s)\|_{L^\infty} \le \frac{A^2}{2\sqrt{s}}, \tag{4-12}$$

where $\mathbb{D}_{A,s_0}$ *is introduced in Proposition 3.4 and* ψ_{d_0,d_1} *is defined as in* (3-1).

Proof. Since the proof of (4-12) is similar to the one written in [Merle and Zaag 1997], we only deal with the proof of (4-11) and refer to Proposition 3.7 in that paper for the proof of (4-12). We argue by contradiction to prove (4-11). Let $i, j \in \{1, \dots, n\}$ and assume that there is $s_* \in [s_0, s_1]$ such that

$$\text{for all } s \in [s_0, s_*), \quad |q_{2,i,j}(s)| < \frac{A^2 \ln^2(s)}{s^2} \quad \text{and} \quad |q_{2,i,j}(s_*)| = \frac{A^2 \ln^2(s_*)}{s_*^2}.$$

Assuming that $q_{2,i,j}(s_*) > 0$ (the negative case is similar), we have on the one hand

$$q_{2,i,j}'(s_*) \ge \frac{d}{ds}\left(\frac{A^2 \ln^2 s}{s^2}\right)_{s=s_*} = \frac{2A^2 \ln s_*}{s_*^3} - \frac{2A^2 \ln^2 s_*}{s_*^3}.$$

On the other hand, we have from (4-2),

$$q_{2,i,j}'(s_*) \le -\frac{2A^2 \ln^2 s_*}{s_*^3} + \frac{C \ln s_*}{s_*^3}.$$

Thus the contradiction then follows if $2A^2 > C$, and this concludes the proof of Proposition 4.4. □

From Proposition 4.4, we see that if $q(s) \in \partial S_A(s_1)$, the first two components $(q_0, q_1)(s_1)$ must be in $\partial \widehat{S}_A(s_1)$, which is the conclusion of item (i) of Proposition 3.5.

The proof of item (ii) of Proposition 3.5 follows from (4-1). Indeed, it is easy to see from (4-1) that for all $i \in \{1, \dots, n\}$ and for each $\epsilon_0, \epsilon_i = \pm 1$, if $q_0(s_1) = \epsilon_0 A/s_1^2$ and $q_{1,i}(s_1) = \epsilon_i A/s_1^2$, it follows that the signs of

$$\frac{dq_0}{ds}(s_1) \quad \text{and} \quad \frac{dq_{1,i}}{ds}(s_1)$$

are opposite the signs of

$$\frac{d}{ds}\left(\frac{\epsilon_0 A}{s^2}\right)(s_1) \quad \text{and} \quad \frac{d}{ds}\left(\frac{\epsilon_i A}{s^2}\right)(s_1)$$

respectively. Hence, $(q_0, q_1)(s)$ will actually leave $\widehat{S}_A(s)$ at $s_1 \ge s_0$ for s_0 large enough. This concludes the proof of Proposition 3.5.

Appendix: Some elementary lemmas

Lemma A.1. *For each $T > 0$, there exists only one positive solution of* (1-9). *Moreover, the solution ψ satisfies the asymptotic*

$$\psi(t) \sim \kappa_\alpha (T-t)^{-1/(p-1)} |\ln(T-t)|^{-\alpha/(p-1)} \quad \text{as } t \to T, \tag{A-1}$$

where

$$\kappa_\alpha = (p-1)^{-1/(p-1)} \left(\frac{p-1}{2}\right)^{\alpha/(p-1)}.$$

Proof. Consider the ODE

$$\psi' = \psi^p \ln^\alpha(\psi^2+2), \quad \psi(0) > 0.$$

The uniqueness and local existence are derived by the Cauchy–Lipschitz property. Let $T_{\max}$, $T_{\min}$ be the maximum and minimum times of the existence of the positive solution; i.e., $\psi(t)$ exists for all $t \in (T_{\min}, T_{\max})$. We now prove that $T_{\max} < +\infty$ and $T_{\min} = -\infty$. By contradiction, we suppose that the solution exists on $[0, +\infty)$; we have

$$\lim_{t_1 \to +\infty} \int_0^{t_1} \frac{\psi'}{\psi^p \ln^\alpha(\psi^2+2)}\, dt = \lim_{t_1 \to +\infty} \int_0^{t_1} dt = +\infty.$$

Since $\int_0^{t_1} \psi'/(\psi^p \ln^\alpha(\psi^2+2))\, dt$ is bounded, the contradiction then follows. With a similar argument we can prove that $T_{\min} = -\infty$. Let us now prove (A-1). We deduce from (1-9) that

$$T - t = \int_{\psi(t)}^{+\infty} \frac{du}{u^p \ln^\alpha(u^2+2)}.$$

Thus, for all $\delta \in (0, p-1)$, there exist t_δ such that for all $t \in (t_\delta, T)$, we have

$$\int_{\psi(t)}^{+\infty} \frac{du}{u^{p+\delta}} \le T - t \le \int_{\psi(t)}^{+\infty} \frac{du}{u^{p-\delta}}.$$

For all $t \in (t_\delta, T)$ it follows that

$$\big((p-1+\delta)(T-t)\big)^{-1/(p-1+\delta)} \le \psi(t) \le \big((p-1-\delta)(T-t)\big)^{-1/(p-1-\delta)},$$

from which we have

$$\ln \psi(t) \sim -\frac{1}{p-1} \ln(T-t) \quad \text{as } t \to T,$$
$$\ln(\psi^2+2) \sim -\frac{2}{p-1} \ln(T-t) \quad \text{as } t \to T.$$

Hence, we obtain

$$\psi' = \psi^p \ln(\psi^2+2) \sim \psi^p \Big(-\frac{2}{p-1} \ln(T-t)\Big)^\alpha \quad \text{as } t \to T, \tag{A-2}$$

which yields

$$\frac{\psi'}{\psi^p} \sim \left(\frac{2}{p-1}\right)^{\alpha} |\ln(T-t)|^{\alpha} \quad \text{as } t \to T.$$

This implies

$$\frac{1}{p-1}\psi^{1-p} \sim \left(\frac{2}{p-1}\right)^{\alpha} \int_t^T |\ln(T-v)|^{\alpha}\, dv \sim \left(\frac{2}{p-1}\right)^{\alpha} (T-t)|\ln(T-t)|^{\alpha} \quad \text{as } t \to T,$$

which concludes the proof of (A-1). □

Lemma A.2. *For all* $\alpha \in (0, 1)$, $\theta > 0$ *and* $0 < h < 1$, *the integral*

$$I(h) = \int_h^1 (s-h)^{-\alpha} s^{-\theta}\, ds$$

satisfies:

(i) *If* $\alpha + \theta > 1$, *then*

$$I(h) \le \left(\frac{1}{1-\alpha} + \frac{1}{\alpha+\theta-1}\right) h^{1-\alpha-\theta}.$$

(ii) *If* $\alpha + \theta = 1$, *then*

$$I(h) \le \frac{1}{1-\alpha} + |\ln h|.$$

(iii) *If* $\alpha + \theta < 1$, *then*

$$I(h) \le \frac{1}{1-\alpha-\theta}.$$

Proof. See Lemma 2.2 of [Giga and Kohn 1989] □

Lemma A.3 (a version of the Grönwall lemma). *If* $y(t)$, $r(t)$ *and* $q(t)$ *are continuous functions defined on* $[t_0, t_1]$ *such that*

$$y(t) \le y_0 + \int_{t_0}^t y(s) r(s)\, ds + \int_{t_0}^t h(s)\, ds \quad \textit{for all } t \in [t_0, t_1].$$

Then,

$$y(t) \le e^{\int_{t_0}^t r(s)\, ds}\left(y_0 + \int_{t_0}^t h(s) e^{-\int_{t_0}^s r(\tau)\, d\tau}\, ds\right).$$

Proof. See Lemma 2.3 of [Giga and Kohn 1989]. □

Lemma A.4. *For each* $T_2 < T$, $\delta > 0$. *There exists* $\epsilon = \epsilon(T, T_2, \delta, n, p) > 0$ *such that for each* $v(x, t)$ *satisfying*

$$|\partial_t v - \Delta v| \le C|v|^p \ln^{\alpha}(v^2+2) \quad \textit{for all } |x| \le \delta,\ t \in (T_2, T),\ \delta > 0, \tag{A-3}$$

and

$$|v(x, t)| \le \epsilon \psi(t) \quad \textit{for all } |x| \le \delta,\ t \in (T_2, T), \tag{A-4}$$

where $\psi(t)$ *is the unique positive solution of* (1-9). *Then*, $v(x, t)$ *does not blow up at* $(0, T)$.

Proof. Since the argument is almost the same as in [Giga and Kohn 1989] treated for the case $\alpha = 0$, we only sketch the main step for the sake of completeness. Let $\phi \in C^\infty(\mathbb{R}^n)$, $\phi = 1$ if $|x| \le \delta/2$, $\phi = 0$ if $|x| \ge \delta$, and consider $\omega = \phi v$ satisfying

$$\partial_t \omega - \Delta \omega = f\phi + g, \tag{A-5}$$

where

$$f = \partial_t v - \Delta v \quad \text{and} \quad g = v\Delta\phi - 2\nabla \cdot (v\nabla\phi).$$

By using Duhamel's formula, we write

$$\omega(t) = e^{(t-T_2)\Delta}(\omega(T_2)) + \int_{T_2}^{t} \bigl(e^{(t-\tau)\Delta}(\phi f) + e^{(t-\tau)\Delta}(g)\bigr)\, d\tau \quad \text{for all } t \in [T_2, T), \tag{A-6}$$

where $e^{t\Delta}$ is the heat semigroup satisfying the following properties: for all $h \in L^\infty$,

$$\|e^{t\Delta} h\|_{L^\infty} \le \|h\|_{L^\infty} \quad \text{and} \quad \|e^{t\Delta}\nabla h\|_{L^\infty} \le C t^{-1/2} \|h\|_{L^\infty} \quad \text{for all } t > 0.$$

The formula (A-6) then yields

$$\|\omega(t)\|_{L^\infty} \le C + C\int_{T_2}^{t} \|\omega(\tau)\|_{L^\infty} \bigl\| |v|^{p-1} \ln^\alpha(v^2+2)(\tau)\bigr\|_{L^\infty(|x|\le\delta)} + C\int_{T_2}^{t} (t-\tau)^{-1/2} \|v(\tau)\|_{L^\infty(|x|\le\delta)}\, d\tau \tag{A-7}$$

for some constant $C = C(n, p, \phi, T, T_2, \delta) > 0$.

From (A-3), (A-4) and Lemma A.1, we find that for all $|x| \le \delta$, and $\tau \in [T_2, T)$,

$$|v(\tau)|^{p-1} \ln^\alpha(v^2(\tau)+2) \le C\psi^{p-1}(\tau) \ln^\alpha(\psi^2(\tau)+2) \le C(T-\tau)^{-1},$$

and

$$|v(\tau)| \le C(T-\tau)^{-1/(p-1)} |\ln(T-\tau)|^{-\alpha/(p-1)}.$$

The estimate (A-7) becomes

$$\|\omega(t)\|_{L^\infty} \le C + C\epsilon^{p-1} \int_{T_2}^{t} (T-\tau)^{-1} \|\omega(\tau)\|_{L^\infty}\, d\tau + C\epsilon \int_{T_2}^{t} (t-\tau)^{-1/2} (T-\tau)^{-1/(p-1)} |\ln(T-\tau)|^{-\alpha/(p-1)}\, d\tau. \tag{A-8}$$

In particular, we now consider $0 < \lambda \ll \frac{1}{2}$ fixed, then we have

$$(T-\tau)^{-1/(p-1)} |\ln(T-\tau)|^{-\alpha/(p-1)} \le C(\alpha, \lambda)(T-\tau)^{-(1/(p-1)+\lambda)} \quad \text{for all } \tau \in (T_2, T).$$

Hence, we rewrite (A-8) as

$$\|\omega(t)\|_{L^\infty} \le C + C\epsilon^{p-1}\int_{T_2}^{t}(T-\tau)^{-1}\|\omega(\tau)\|_{L^\infty}\,d\tau + C\epsilon\int_{T_2}^{t}(t-\tau)^{-1/2}(T-\tau)^{-(1/(p-1)+\lambda)}\,d\tau, \quad \text{(A-9)}$$

where $C(n, p, \phi, \alpha, \epsilon, \lambda, p)$. Beside that, by the change of variables $s = T - \tau$, $h = T - t$ we have

$$\int_{T_2}^{t}(t-\tau)^{-1/2}(T-\tau)^{-\theta(p,\lambda)}\,d\tau = \int_{h}^{T-T_2}(s-h)^{-1/2}(s)^{-\theta(p,\lambda)}\,ds, \quad \text{(A-10)}$$

where $\theta(p, \lambda) = (1/(p-1)+\lambda)$.

Case 1: If $\theta(p,\lambda) < \frac{1}{2}$, by using (iii) of Lemma A.2 we deduce from (A-9), (A-10) that

$$\|\omega(t)\|_{L^\infty} \le C + C\epsilon^{p-1}\int_{T_2}^{t}(T-s)^{-1}\|\omega(s)\|_{L^\infty}\,ds.$$

Therefore, by Lemma A.3,

$$\|\omega(t)\|_{L^\infty} \le C(T-t)^{-C\epsilon^{p-1}}. \quad \text{(A-11)}$$

Choose ϵ small enough such that $C\epsilon^{p-1} \le 1/(2(p-1))$. Then, we conclude from (A-11) that

$$|v(x,t)| \le C(T-t)^{-1/(2(p-1))} \quad \text{for } |x| \le \tfrac{1}{2},\ t \le T. \quad \text{(A-12)}$$

By using parabolic regularity theory and the same argument as in Lemma 3.3 of [Giga and Kohn 1987], we can prove that (A-12) actually prevents blowup.

Case 2: $\theta(\lambda, p) = \frac{1}{2}$ is similar to the first case. By using (ii) of Lemma A.2, (A-9) and (A-10) we get

$$\|\omega(t)\|_{L^\infty} \le C(1 + |\ln(T-t)|) + C\epsilon^{p-1}\int_{T_2}^{t}(T-s)^{-1}\|\omega(s)\|_{L^\infty}\,ds.$$

However, we derive from Lemma A.3 that

$$\|\omega(t)\|_{L^\infty} \le C(T-t)^{-K\epsilon^{p-1}}, \quad \text{(A-13)}$$

where $C = C(n, p, \phi, T, T_2, \delta)$. We now take ϵ small enough such that $C\epsilon^{p-1} \le 1/(2(p-1))$, which follows (A-12).

Case 3: For $\theta(\lambda, p) > \frac{1}{2}$, by using Lemmas A.2, A.3 and similar arguments we obtain

$$|v(x,t)| \le C(T-t)^{1/2-\theta(p,\lambda)} \quad \text{for all } |x| \le \delta,\ t \in [T_2, T).$$

Repeating the step in finite steps would end up with (A-12). This concludes the proof of Lemma A.4. □

The following lemma gives the asymptotic behavior of $h(s)$ and $\psi_1(s)$.

Lemma A.5. *Let $h(s)$ and $\psi_1(s)$ be defined as in* (2-3) *and* (2-4) *respectively. Then we have*:

(i) $$\frac{1}{\ln(\psi_1^2(s)+2)} = \frac{p-1}{2s} + \frac{\alpha(p-1)\ln s}{2s^2} + O\left(\frac{1}{s^2}\right) \quad \text{as } s \to +\infty. \tag{A-14}$$

(ii) $$h(s) = \frac{1}{p-1}\left(1 - \frac{\alpha}{s} - \frac{\alpha^2 \ln s}{s^2}\right) + O\left(\frac{1}{s^2}\right) \quad \text{as } s \to +\infty. \tag{A-15}$$

Proof. (i) Consider $\psi(t)$ the unique positive solution of (1-9). We have

$$T - t = \int_{\psi(t)}^{+\infty} \frac{dx}{x^p \ln^\alpha(x^2+2)}. \tag{A-16}$$

An integration by parts yields

$$\begin{aligned} T - t = {}& \frac{1}{\psi^{p-1}(t)\ln^\alpha(\psi^2(t)+2)} \\ & \times \left(\frac{1}{p-1} - \frac{2\alpha}{(p-1)^2\ln(\psi^2(t)+2)} + O\left(\frac{1}{(\ln^2(\psi^2(t)+2))}\right)\right). \end{aligned} \tag{A-17}$$

Let us write $\psi(t) = \psi_1(s)$, where $s = -\log(T-t)$; then we have

$$\ln(\psi_1(s)) = \frac{s}{p-1} - \frac{\alpha}{(p-1)} \ln(\ln(\psi_1(s))) + O(1) \quad \text{as } s \to +\infty, \tag{A-18}$$

from which we deduce that

$$\ln(\psi_1(s)) = \frac{s}{p-1} - \frac{\alpha \ln(s)}{p-1} + O(1) \quad \text{as } s \to +\infty, \tag{A-19}$$

which is the conclusion (i).

(ii) From (2-3) and (A-17), we have

$$h(s) = \frac{1}{p-1} - \frac{2\alpha}{(p-1)^2\ln(\psi_1^2(s)+2)} + O\left(\frac{1}{\ln^2(\psi_1^2(s)+2)}\right). \tag{A-20}$$

Using (A-14), we conclude the proof of (A-15), as well as Lemma A.5. □

Lemma A.6. *Let N be defined as in* (2-11)*. We have*

$$N(\bar{w}, s) = \frac{p\bar{w}^2}{2} + O\left(\frac{|\bar{w}|\ln s}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right) + O(|\bar{w}|^3) \quad \text{as } (\bar{w}, s) \to (0, +\infty). \tag{A-21}$$

Proof. From the definition (2-11) of N, let us write

$$N(\bar{w}, s) = N_1(\bar{w}, s) + N_2(\bar{w}, s),$$

where

$$N_1(\bar{w}, s) = h(s)\big(|\bar{w}+1|^{p-1}(\bar{w}+1) - (\bar{w}+1)\big) - \bar{w},$$
$$N_2(\bar{w}, s) = h(s)|\bar{w}+1|^{p-1}(\bar{w}+1)\left(\frac{\ln^\alpha(\psi_1^2(\bar{w}+1)^2+2)}{\ln^\alpha(\psi_1^2+2)} - 1\right).$$

From (A-15) and a Taylor expansion, we find that

$$N_1(\bar{w}, s) = \frac{p\bar{w}^2}{2} - \frac{\alpha\bar{w}}{s} + O\left(\frac{|\bar{w}|\ln s}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right) + O(|\bar{w}|^3) \quad \text{as } (\bar{w}, s) \to (0, +\infty).$$

We now claim the following

$$N_2(\bar{w}, s) = \frac{\alpha\bar{w}}{s} + O\left(\frac{|\bar{w}|\ln s}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right) \quad \text{as } (\bar{w}, s) \to (0, +\infty). \tag{A-22}$$

Then, the proof of (A-21) simply follows by addition.

Let us now give the proof of (A-22) to complete the proof of Lemma A.6 . We set

$$f(\bar{w}) = \ln^\alpha(\psi_1^2(\bar{w}+1)^2+2), \quad |\bar{w}| \le \tfrac{1}{2}.$$

We apply Taylor expansion to $f(\bar{w})$ at $\bar{w}=0$ to find that

$$f(\bar{w}) = \ln^\alpha(\psi_1^2+2) + 2\alpha\ln^{\alpha-1}(\psi_1^2+2)\frac{\psi_1^2}{\psi_1^2+2}\bar{w} + \frac{f''(\theta)}{2}(\bar{w})^2,$$

where θ is between 0 and $\bar{w}$, and

$$f''(\theta) = \alpha(\alpha-1)\ln^{\alpha-2}(\psi_1^2(\theta+1)^2+2)\left(\frac{2(\theta+1)\psi_1^2}{\psi_1^2(\theta+1)^2+2}\right)^2 + \alpha\ln^{\alpha-1}(\psi_1^2(\theta+1)^2+2)\frac{(4\psi_1 - 2\psi_1^4(\theta+1)^2)}{(\psi_1^2(\theta+1)^2+2)^2}.$$

Since $|\theta| \le \frac{1}{2}$, one can show that

$$|f''(\theta)| \le C\ln^{\alpha-1}(\psi_1^2+2) \quad \text{for all } |\theta| \le \tfrac{1}{2}.$$

Thus, we have

$$f(\bar{w}) = \ln^\alpha(\psi_1^2+2) + 2\alpha\ln^{\alpha-1}(\psi_1^2+2)\bar{w} + O\big(|\bar{w}|^2\ln^{\alpha-1}(\psi_1^2+2)\big) + O\left(\frac{|\bar{w}|\ln^{\alpha-1}(\psi_1^2+2)}{\psi_1^2}\right)$$

as $s \to +\infty$. This yields

$$\frac{\ln^\alpha(\psi_1^2(\bar{w}+1)^2+2)}{\ln^\alpha(\psi_1^2+2)} = 1 + \frac{2\alpha\bar{w}}{\ln(\psi_1^2+2)} + O\left(\frac{|\bar{w}|^2}{\ln(\psi_1^2+2)}\right) + O\left(\frac{|\bar{w}|}{\ln(\psi_1^2+2)\psi_1^2}\right)$$

as $(\bar{w}, s) \to (0, +\infty)$. From this and (A-14) we derive

$$\frac{\ln^\alpha(\psi_1^2(\bar{w}+1)^2+2)}{\ln^\alpha(\psi_1^2(s)+2)} - 1 = \frac{\alpha(p-1)\bar{w}}{s} + O\left(\frac{\ln s|\bar{w}|}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right). \quad \text{(A-23)}$$

From the definition of N_2, (A-15), (A-23) and the fact that

$$|\bar{w}+1|^{p-1}(\bar{w}+1) = 1 + p\bar{w} + O(|\bar{w}|^2) \quad \text{as } \bar{w} \to 0,$$

we conclude the proof of (A-22) as well as Lemma A.6. □

Lemma A.7. *For all $|z| \le K_1$, there exists $C(K_1)$ such that for all $s \ge 1$ we have*

$$\left| h(s)|z|^{p-1} z \frac{\ln^\alpha(\psi_1^2 z^2+2)}{\ln^\alpha(\psi_1^2+2)} - \frac{|z|^{p-1}z}{p-1} \right| \le \frac{C(K_1)}{s}, \quad \text{(A-24)}$$

where $h(s)$ satisfies the asymptotic (2-5).

Proof. We consider $f(z) = \ln^\alpha(\psi_1^2 z^2+2)$ for all $z \in \mathbb{R}$; then we write

$$\ln^\alpha(\psi_1^2 z^2+2) = \ln^\alpha(\psi_1^2+2) + \int_1^{|z|} f'(v)\,dv.$$

Recalling from (2-5) that $h(s) = 1/(p-1) + O(1/s)$, we have

$$\left| h(s)|z|^{p-1} z \frac{\ln^\alpha(\psi_1^2 z^2+2)}{\ln^\alpha(\psi_1^2+2)} - \frac{|z|^{p-1}z}{p-1} \right| \le \frac{C|z|^p}{\ln^\alpha(\psi_1^2+2)} \int_1^{|z|} |f'(v)|\,dv + \frac{C|z|^p}{s}. \quad \text{(A-25)}$$

From (i) of Lemma A.5 we have

$$\frac{1}{\ln(\psi_1^2+2)} \le \frac{C}{s}.$$

Thus it is sufficient to show that

$$A(z) := \frac{|z|^p}{\ln^{\alpha-1}(\psi_1^2+2)} \int_1^{|z|} |f'(v)|\,dv \le C(K_1) \quad \text{for all } |z| \le K_1,$$

where

$$f'(v) = \alpha \ln^{\alpha-1}(\psi_1^2 v^2+2) \frac{2v\psi_1^2}{\psi_1^2 v^2+2}.$$

For $1 \le |z| \le K_1$, it is trivial to see that $|A(z)| \le C(K_1)$. For $|z| < 1$, we consider two cases:

<u>Case 1</u>: $\alpha - 1 \ge 0$. Then

$$A(z) \le 2|\alpha|\,|z|^p \int_{|z|}^1 \frac{1}{v}\,dv \le C(K_1).$$

Case 2: $\alpha - 1 < 0$. Then

$$A(z) \le 2|\alpha|\,|z|^p \frac{\ln^{\alpha-1}(\psi_1^2 z^2+2)}{\ln^{\alpha-1}(\psi_1+2)} \int_{|z|}^1 \frac{1}{v}\,dv.$$

- If $\psi_1 z^2 \ge 1$ then

$$A(z) \le 2|\alpha| \frac{\ln^{1-\alpha}(\psi_1^2+2)}{\ln^{1-\alpha}(\psi_1+2)} |z|^p \int_{|z|}^1 \frac{1}{v}\,dv \le C(K_1).$$

- If $\psi_1 z^2 \le 1$ then $|z| \le v \le \psi_1^{-1/2}$ we deduce that

$$|A(z)| \le 2|\alpha| \psi_1^{(1-p)/2} \frac{\ln^{1-\alpha}(\psi_1^2+2)}{\ln^{1-\alpha}(2)} |z| \int_{|z|}^1 \frac{1}{v}\,dv \le C(K_1). \qquad \square$$

Lemma A.8 (control of the nonlinear term D in $S_A(s)$). *For all $A \ge 1$, there exists $\sigma_3(A) \ge 1$ such that for all $s \ge \sigma_3(A)$, $q(s) \in S_A(s)$ implies*

$$\text{for all } |y| \le 2K\sqrt{s}, \quad |D(q,s)| \le C(K) \frac{\ln s (1+|y|)^4}{s^3}, \tag{A-26}$$

and

$$\|D(q,s)\|_{L^\infty(\mathbb{R}^n)} \le \frac{C}{s}. \tag{A-27}$$

Proof. From the definition (2-32) of D, we have the decomposition

$$D(q,s) = D_1(q,s) + D_2(q,s),$$

where

$$D_1(q,s) = \Big(h(s) - \frac{1}{p-1}\Big)\big(|q+\varphi|^{p-1}(q+\varphi) - (q+\varphi)\big),$$

$$D_2(q,s) = h(s)|q+\varphi|^{p-1}(q+\varphi)L(q+\varphi,s),$$

$h(s)$ admits the asymptotic behavior (A-15), and L is defined in (2-33). The proof of (A-26) will follow once we show for all $|y| \le 2K\sqrt{s}$

$$\left| D_1 - \left(\frac{\alpha(|y|^2-2n)}{4ps^2} - \frac{\alpha}{s} q \right) \right| \le C \frac{(1+|y|^4)\ln s}{s^3}, \tag{A-28}$$

$$\left| D_2 + \left(\frac{\alpha(|y|^2-2n)}{4ps^2} - \frac{\alpha}{s} q \right) \right| \le C \frac{(1+|y|^4)\ln s}{s^3}. \tag{A-29}$$

Let us give a proof of (A-28). From the definition of $S_A(s)$, we note that if $q(s) \in S_A(s)$, then

$$\text{for all } y \in \mathbb{R}^n, \quad |q(y,s)| \le \frac{CA^2 \ln^2 s (1+|y|^3)}{s^2}, \tag{A-30}$$

$$\|q(s)\|_{L^\infty(\mathbb{R}^n)} \le \frac{CA^2}{\sqrt{s}}. \tag{A-31}$$

From the definition (2-26) of φ and (A-31), we see that for all $|y| \le 2K\sqrt{s}$, there exists a positive constant $C(K)$ such that

$$0 < \frac{1}{C(K)} \le (q+\varphi)(y,s) \le C(K). \tag{A-32}$$

Using Taylor expansion and the asymptotic (A-15), we write

$$D_1(q,s) = \left(-\frac{\alpha}{(p-1)s} + O\left(\frac{\ln s}{s^2}\right)\right)\left(\varphi^p - \varphi + (p\varphi^{p-1}-1)q\right) + O(q^2). \tag{A-33}$$

Using again the definition of φ and a Taylor expansion, we derive

$$\varphi^p = 1 - \frac{(|y|^2-2n)}{4s} + O\left(\frac{1+|y|^4}{s^2}\right),$$

$$\varphi = 1 - \frac{(|y|^2-2n)}{4ps} + O\left(\frac{1+|y|^4}{s^2}\right),$$

$$p\varphi^{p-1} - 1 = p - 1 - \frac{(p-1)(|y|^2-2n)}{4ps} + O\left(\frac{1+|y|^4}{s^2}\right)$$

as $s \to +\infty$. Inserting (A-30) and these estimates into (A-33) yields (A-28).

We now turn to the proof of (A-29). Recall from (2-33) the definition of L,

$$L(q+\varphi,s) \\ = \frac{2\alpha\psi_1^2}{\ln(\psi_1^2+2)(\psi_1^2+2)}(q+\varphi-1) + \frac{1}{\ln^\alpha(\psi_1^2+2)}\int_1^{q+\varphi} f''(v)(q+\varphi-v)\,dt,$$

where $f(v) = \ln^\alpha(\psi_1^2 v^2+2),\ v \in \mathbb{R}$. From (A-32) and a direct computation, we estimate

$$\left|\frac{1}{\ln^\alpha(\psi_1^2+2)}\int_1^{q+\varphi} f''(v)(q+\varphi-v)\,dv\right| \le C(K)\frac{|q+\varphi-1|^2}{s},$$

which yields

$$\left|L(q+\varphi,s) - \frac{2\alpha\psi_1^2(q+\varphi-1)}{\ln(\psi_1^2+2)(\psi_1^2+2)}\right| \le C(K)\frac{|q+\varphi-1|^2}{s}. \tag{A-34}$$

From (A-14) and (A-34), we then have

$$\left|L(q+\varphi,s) - \frac{\alpha(p-1)(q+\varphi-1)}{s}\right| \le C(K)\left(\frac{|q+\varphi-1|^2}{s} + \frac{\ln s|q+\varphi-1|}{s^2}\right),$$

and additionally we have

$$|q+\varphi-1| \le \frac{C(1+|y|^2)}{s},$$

which implies

$$\left|L(q+\varphi,s)-\frac{\alpha(p-1)(q+\varphi-1)}{s}\right|\le C(K)\frac{\ln s(1+|y|^4)}{s^3}. \tag{A-35}$$

Moreover, from definition of D_2 and (A-35) we deduce that

$$\left|D_2(q,s)-\frac{\alpha}{s}\big(\varphi^{p+1}-\varphi^p+((p+1)\varphi^p-p\varphi^{p-1})q\big)\right|\le C\frac{(1+|y|^4)\ln s}{s^3},$$

and

$$\varphi^{p+1}-\varphi^p=-\frac{(|y|^2-2)}{4ps}+O\left(\frac{1+|y|^4}{s^2}\right),$$
$$(p+1)\varphi^p-p\varphi^{p-1}=1-\frac{(|y|^2-2)}{2s}+O\left(\frac{1+|y|^4}{s^2}\right)$$

as $s\to+\infty$, which yields (A-29).

We now prove (A-27). From (A-15) and the boundedness of q and φ, we have

$$|D_1(q,s)|\le\frac{C}{s}.$$

It is sufficient to prove that for all $y\in\mathbb{R}^n$,

$$|D_2(q,s)|\le\frac{C(K)}{s}.$$

Indeed, from definition (2-33) of L we deduce that

$$D_2(q,s)=h(s)|q+\varphi|^{p-1}(q+\varphi)\frac{\ln^\alpha(\psi_1^2z^2+2)}{\ln^\alpha(\psi^2+2)}-h(s)|q+\varphi|^{p-1}(q+\varphi).$$

Using Lemma A.7 we deduce

$$|D_2(q,s)|\le\frac{C(K)}{s}.$$ □

Lemma A.9. *For s large enough, we have*:

(i) *estimates on V*:

$$|V(y,s)|\le\frac{C(1+|y|^2)}{s}\quad\textit{for all } y\in\mathbb{R}^n,$$

and

$$V=-\frac{(|y|^2-2n)}{4s}+\widetilde{V}\quad\textit{with } \widetilde{V}=O\left(\frac{1+|y|^4}{s^2}\right)\textit{ for all } |y|\le K\sqrt{s}.$$

(ii) *estimates on R*:

$$|R(y,s)|\le\frac{C}{s}\quad\textit{for all } y\in\mathbb{R}^n,$$

and

$$R(y,s) = \frac{c_p}{s^2} + \widetilde{R}(y,s) \quad \text{with } \widetilde{R} = O\left(\frac{1+|y|^4}{s^3}\right) \text{ for all } |y| \le K\sqrt{s}.$$

Proof. The proof simply follows from Taylor expansion. We refer to Lemmas B.1 and B.5 in [Zaag 1998] for similar proofs. □

Lemma A.10 (estimates on $B(q)$). *For all $A > 0$ there exists $\sigma_5(A) > 0$ such that for all $s \ge \sigma_5(A)$, $q(s) \in S_A(s)$ implies*

$$|B(q(y,s))| \le C|q|^2, \tag{A-36}$$

and

$$|B(q)| \le C|q|^{\bar{p}}, \tag{A-37}$$

with $\bar{p} = \min(p, 2)$.

Proof. See Lemma 3.6 in [Merle and Zaag 1997] for the proof of this lemma. □

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GIAO KY DUONG:

duong@univ-paris13.fr
Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), Villetaneuse, France

VAN TIEN NGUYEN:

tien.nguyen@nyu.edu
Department of Mathematics, New York University Abu Dhabi, Abu Dhabi, United Arab Emirates

HATEM ZAAG:

hatem.zaag@univ-paris13.fr
Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), Villetaneuse, France

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