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From compressible to incompressible inhomogeneous flows in the case of large data

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We are concerned with the mathematical derivation of the inhomogeneous incompressible Navier–Stokes equations (INS) from the compressible Navier–Stokes equations (CNS) in the large volume viscosity limit. We first prove a result of large-time existence of regular solutions for (CNS). Next, as a consequence, we establish that the solutions of (CNS) converge to those of (INS) when the volume viscosity tends to infinity. Analysis is performed in the two-dimensional torus $\mathbb{T}^2$ for general initial data. Compared to prior works, the main breakthrough is that we are able to handle *large* variations of density.

1. Introduction

We are concerned with the compressible Navier–Stokes system

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 & \text{in } (0,T)\times\mathbb{T}^2, \\ \rho v_t + \rho v\cdot\nabla v - \mu\Delta v - \nu\nabla \operatorname{div} v + \nabla P = 0 & \text{in } (0,T)\times\mathbb{T}^2. \end{cases} \tag{1-1}$$

Above, the unknown nonnegative function $\rho=\rho(t,x)$ and vector field $v=v(t,x)$ stand for the density and velocity of the fluid at (t,x). The two real numbers μ and ν denote the viscosity coefficients and are assumed to satisfy $\mu>0$ and $\nu+\mu>0$. We suppose that the pressure function $P=P(\rho)$ is C^1 with $P'>0$, and that $P(\bar\rho)=0$ for some positive constant reference density $\bar\rho$. Throughout, we set

$$e(\rho) := \rho\int_{\bar\rho}^{\rho}\frac{P(t)}{t^2}\,dt.$$

Note that $e(\bar\rho)=e'(\bar\rho)=0$ and $\rho e''(\rho)=P'(\rho)$. Hence e is a strictly convex function and, for any interval $[\rho_*,\rho^*]$, there exist two constants m_* and m^* such that

$$m_*(\rho-\bar\rho)^2 \le e(\rho) \le m^*(\rho-\bar\rho)^2. \tag{1-2}$$

The system is supplemented with the initial conditions

$$v|_{t=0} = v_0 \in \mathbb{R}^2 \quad\text{and}\quad \rho|_{t=0}=\rho_0\in\mathbb{R}_+. \tag{1-3}$$

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We aim at comparing the above compressible Navier–Stokes system with its incompressible but inhomogeneous version, namely

$$\begin{cases} \eta_t + u\cdot\nabla\eta = 0 & \text{in } (0,T)\times\mathbb{T}^2,\\ \eta u_t + \eta u\cdot\nabla u - \mu\Delta u + \nabla\Pi = 0 & \text{in } (0,T)\times\mathbb{T}^2,\\ \operatorname{div} u = 0 & \text{in } (0,T)\times\mathbb{T}^2. \end{cases} \tag{1-4}$$

At the formal level, one can expect the solutions of (1-1) to converge to those of (1-4) when ν goes to ∞. Indeed, the velocity equation of (1-1) may be rewritten

$$\nabla\operatorname{div} v = \frac{1}{\nu}(\rho v_t + \rho v\cdot\nabla v - \mu\Delta v + \nabla P)$$

and thus $\nabla\operatorname{div} v$ should tend to 0 when $\nu\to\infty$. This means that $\operatorname{div} v$ should tend to be independent of the space variable and, as it is the divergence of some periodic vector field, one must eventually have $\operatorname{div} v\to 0$. As, on the other side, one has for all values of ν,

$$\rho v_t + \rho v\cdot\nabla v - \mu\Delta v \quad \text{is a gradient,}$$

this means that if (ρ, v) tends to some pair (η, u) in a sufficiently strong manner, then necessarily (η, u) should satisfy (1-4).

The question of finding an appropriate framework for justifying that heuristic naturally arises. Let us first examine the *weak solution framework*, as it requires the fewest assumptions on the data. Regarding system (1-1) with a pressure law like $P(\rho) = a(\rho^\gamma - \bar\rho^\gamma)$ for some $a > 0$ and $\gamma > 1$, the state-of-the-art result for the weak solution theory is as follows (see [Lions 1998; Novotný and Straškraba 2004] for more details):

Theorem 1.1. *Assume that the initial data ρ_0 and v_0 satisfy $\sqrt{\rho_0}\, v_0 \in L_2(\mathbb{T}^2)$ and $\rho_0 \in L_\gamma(\mathbb{T}^2)$. Then there exists a global-in-time weak solution to* (1-1) *such that*

$$v \in L_\infty(\mathbb{R}_+; L_2(\mathbb{T}^2)) \cap L_2(\mathbb{R}_+; \dot H^1(\mathbb{T}^2)) \quad \textit{and} \quad e(\rho) \in L_\infty(\mathbb{R}_+; L_1(\mathbb{T}^2)) \tag{1-5}$$

and, *for all $T > 0$*,

$$\int_{\mathbb{T}^2} \big(\tfrac12\rho|v|^2 + e(\rho)\big)(T,\cdot)\,dx + \int_0^T \big(\mu\|\nabla v\|_2^2 + \nu\|\operatorname{div} v\|_2^2\big)\,dt \le \int_{\mathbb{T}^2} \big(\tfrac12\rho_0|v_0|^2 + e(\rho_0)\big)\,dx. \tag{1-6}$$

For system (1-4), there is a similar weak solution theory that was initiated by A. Kazhikhov [1974], then continued by J. Simon [1990] and completed by P.-L. Lions [1996]. However, to the best of our knowledge, it is not known how to connect system (1-1) to (1-4) in that framework. Justifying the convergence in that setting may be extremely difficult owing to the fact that the key extra estimate for the density that allows one to achieve the existence of weak solutions for

(1-1) strongly depends on the viscosity coefficient ν, and collapses when ν goes to infinity.

This thus motivates us to consider the problem for more regular solutions. Regarding system (1-1) in the multidimensional case, recall that the global existence issue of strong unique solutions has been answered just partially, and mostly in the small data case; see, e.g., [Danchin 2000; Kotschote 2014; Matsumura and Nishida 1980; Mucha 2003; Mucha and Zajączkowski 2002; 2004; Valli and Zajączkowski 1986]. For general large data (even if very smooth), only local-in-time solutions are available; see, e.g., [Danchin 2001; Nash 1962].

The theory of strong solutions for the inhomogeneous Navier–Stokes system (1-4) is more complete; see, e.g., [Danchin and Mucha 2012; Ladyzhenskaya and Solonnikov 1975; Huang et al. 2013; Li 2017]. In fact, the results are roughly the same as for the homogeneous (that is with constant density) incompressible Navier–Stokes system. In particular, we proved in [Danchin 2017] that, in the two-dimensional case, system (1-4) is uniquely and globally solvable in dimension two whenever the initial velocity is in H^1 and the initial density is nonnegative and bounded (initial data with vacuum may thus be considered).

It is tempting to study whether those better properties in dimension two for the (supposedly) limit system (1-4) may help us to improve our knowledge of system (1-1) in the case where the volume viscosity is very large. More precisely, we here want to address the following two questions:

- For regular data with no vacuum, given any fixed $T > 0$, can we find ν_0 so that the solution remains smooth (hence unique) until time T for all $\nu \geq \nu_0$?
- Considering a family (ρ_ν, v_ν) of solutions to (1-1) and letting $\nu \to \infty$, can we show strong convergence to some pair (η, u) satisfying (1-4) and, as the case may be, give an upper bound for the rate of convergence?

Those two issues have been considered recently in our paper [Danchin and Mucha 2017], in the particular case where the initial density is a perturbation of order $\nu^{-1/2}$ of some constant positive density (hence the limit system is just the classical incompressible Navier–Stokes equation). There, our results were based on Fourier analysis and involved so-called critical Besov norms. The cornerstone of the method was a refined analysis of the linearized system about the constant state $(\rho, v) = (\bar{\rho}, 0)$, thus precluding us from considering large density variations.

The present paper aims at shedding a new light on this issue, pointing out different results and techniques than in [Danchin and Mucha 2017]. In particular, we will go beyond the slightly inhomogeneous case, and will be able to consider large variations of density. Regarding the techniques, we here meet another motivation which is strictly mathematical; we want to advertise two tools that can be of some use in the analysis of systems of fluid mechanics:

- The first one is a nonstandard estimate with (limited) loss of integrability for solutions of the transport equation by a non-Lipschitz vector field that was first pointed out by B. Desjardins [1997] (see Section 3). Proving it requires a Moser–Trudinger inequality that holds true only in dimension two.[1]
- The second tool is an estimate for a parabolic system with just bounded coefficients in the maximal regularity framework of L_p spaces with p close to 2 (Section 4).

For notational simplicity, *we assume from now on that the shear viscosity μ is equal to* 1 (which may always be achieved after a suitable rescaling). Our answer to the first question then reads as follows:

Theorem 1.2. *Fix some $T > 0$. Let ρ_* and ρ^* satisfy $0 < 4\rho_* \le \rho^*$, and assume*

$$2\rho_* \le \rho_0 \le \tfrac{1}{2}\rho^*. \tag{1-7}$$

There exists an exponent $q > 2$ depending only on ρ_* and ρ^* such that if $\nabla\rho_0 \in L_q(\mathbb{T}^2)$ then for any vector field v_0 in $W_q^{2-2/q}(\mathbb{T}^2)$ satisfying

$$\nu^{1/2}\|\operatorname{div} v_0\|_{L_2} \le 1, \tag{1-8}$$

there exists $\nu_0 = \nu_0(T, \rho_*, \rho^*, \|\nabla\rho_0\|_q, \|v_0\|_{W_q^{2-2/q}}, P, q)$ such that system (1-1) *with $\nu \ge \nu_0$ has a unique solution (ρ, v) on the time interval* $[0, T]$, *fulfilling*

$$\begin{gathered} v \in \mathcal{C}([0,T]; W_q^{2-2/q}(\mathbb{T}^2)), \quad v_t, \nabla^2 v \in L_q([0,T]\times\mathbb{T}^2),\\ \rho \in \mathcal{C}([0,T]; W_q^1(\mathbb{T}^2)), \end{gathered} \tag{1-9}$$

and

$$\rho_* \le \rho(t,x) \le \rho^* \quad \textit{for all } (t,x) \in [0,T]\times\mathbb{T}^2. \tag{1-10}$$

Furthermore, there exists a constant C_q depending only on q, a constant C_P depending only on P, and a universal constant C such that for all $t \in [0, T]$,

$$\begin{aligned} \|v(t)\|_{H^1} + \nu^{1/2}\|\operatorname{div} v(t)\|_{L_2} + \|\rho(t) - \bar\rho\|_{L_2} + \|\nabla v\|_{L_2([0,t];H^1)}\\ + \|v_t\|_{L_2(0,t\times\mathbb{R}^2)} + \nu^{1/2}\|\nabla \operatorname{div} v\|_{L_2(0,t\times\mathbb{R}^2)} \le Ce^{C\|v_0\|_2^4}E_0, \end{aligned} \tag{1-11}$$

$$\begin{aligned} \|v(t)\|_{W_q^{2-2/q}} + \|v_t, \nabla^2 v, \nu\nabla \operatorname{div} v\|_{L_q([0,t]\times\mathbb{T}^2)}\\ \le C_q\big(\|v_0\|_{W_q^{2-2/q}} + C_P t^{1/q}(1 + \|\nabla\rho_0\|_{L_q})\exp(t^{1/q'}I_0(t))\big), \end{aligned} \tag{1-12}$$

and

$$\|\nabla\rho(t)\|_{L_q} \le (1 + \|\nabla\rho_0\|_{L_q})\exp(t^{1/q'}I_0(t)), \tag{1-13}$$

with $E_0 := 1 + \|v_0\|_{H^1} + \|\rho_0 - \bar\rho\|_{L_2}$ and

$$I_0(t) := C_q\big(\|v_0\|_{W_q^{2-2/q}} + C_P t^{1/q}(1 + \|\nabla\rho_0\|_{L_q})e^{CE_0^2 t e^{C\|v_0\|_{L_2}^4}}\big).$$

[1]Consequently, we do not know how to adapt our approach to the higher-dimensional case.

As the data we here consider are regular and bounded away from zero, the short-time existence and uniqueness issues are clear (one may, e.g., adapt [Danchin 2010] to the case of periodic boundary conditions). In order to achieve large-time existence, we shall first take advantage of a rather standard higher-order energy estimate (at the H^1 level for the velocity) that will provide us with a control of ∇v in $L^2(0,T;H^1)$ in terms of the data and of the norm of $\nabla\rho$ in $L^\infty(0,T;L^2)$. The difficulty now is to control that latter norm, given that, at this stage, one has no bound for ∇v in $L^1(0,T;L^\infty)$. It may be overcome by adapting to our framework some estimates with loss of integrability for the transport equation, which were first pointed out in [Desjardins 1997]. However, this is not quite the end of the story since those estimates involve the quantity $\int_0^T \|\operatorname{div} v\|_{L^\infty}\,dt$. Then, the key observation is that the linear maximal regularity theory for the linearization of the momentum equation of (1-1) (neglecting the pressure term and taking $\rho\equiv 1$) provides, for all $1<q<\infty$, a control on $\nu\|\nabla\operatorname{div} v\|_{L_q(0,T;L_q(\mathbb{T}^2))}$ (not just $\|\nabla\operatorname{div} v\|_{L_q(0,T;L_q(\mathbb{T}^2))}$) in terms of $\|v_0\|_{W_q^{2-2/q}}$. In our framework where ρ is not constant, it turns out to be possible to recover a similar estimate if q is close enough to 2, and thus to eventually have, by Sobolev embedding, $\int_0^T \|\operatorname{div} v\|_{L^\infty}\,dt = \mathcal{O}(\nu^{-1})$. Then, putting all the arguments together and bootstrapping allows us to get all the estimates of Theorem 1.2, for large enough ν.

Regarding the asymptotics $\nu\to\infty$, it is clear that if one starts with fixed initial data, then uniform estimates are available from Theorem 1.2, only if we assume that $\operatorname{div} v_0\equiv 0$. Under that assumption, inequalities (1-11) and (1-12) already ensure that

$$
\begin{aligned}
\operatorname{div} v &= \mathcal{O}(\nu^{-1/2}) \quad && \text{in } L_\infty(0,T;L_2),\\
\nabla\operatorname{div} v &= \mathcal{O}(\nu^{-1}) && \text{in } L_q(0,T\times\mathbb{T}^2).
\end{aligned}
$$

Then, combining with the uniform bounds provided by (1-12) and (1-13), it is not difficult to pass to the weak limit in system (1-1) and to find that the limit solution fulfills system (1-4).

In the theorem below, we state a result that involves strong norms of all quantities at the level of energy norm, and exhibit an explicit rate of convergence.

Theorem 1.3. *Fix some $T>0$ and take initial data (ρ_0, v_0) fulfilling the assumptions of Theorem 1.2 with, in addition,* $\operatorname{div} v_0\equiv 0$. *Denote by (ρ_ν, v_ν) the corresponding solution of* (1-1) *with volume viscosity $\nu\ge\nu_0$. Finally, let (η,u) be the global solution of* (1-4) *supplemented with the same initial data (ρ_0,v_0). Then we have*

$$
\sup_{t\le T}\Big(\|\rho_\nu(t)-\eta(t)\|_{L_2}^2+\|\mathcal{P}v_\nu(t)-u(t)\|_{L_2}^2+\|\nabla\mathcal{Q}v_\nu(t)\|_{L_2}^2\Big) \\
+\int_0^T\Big(\|\nabla(\mathcal{P}v_\nu-u)\|_{L_2}^2+\|\nabla\mathcal{Q}v_\nu\|_{H^1}^2\Big)\,dt\le C_{0,T}\,\nu^{-1}, \quad (1\text{-}14)
$$

where $\mathcal{P}$ and $\mathcal{Q}$ are the Helmholtz projectors on divergence-free and potential vector fields, respectively,[2] *and where $C_{0,T}$ depends only on T and on the norms of the initial data.*

Compared to the question of low Mach number limit studied in, e.g., [Danchin 2002; Feireisl and Novotný 2013], there is an essential difference in the mechanism leading to convergence, as may be easily seen from a rough analysis of the linearized system (1-1). Indeed, in the case $\bar{\rho} = \mu = 1$ and $P'(1) = 1$ (for notational simplicity), that linearization (in the unforced case) is given by

$$\begin{cases} \eta_t + \operatorname{div} u = 0, \\ v_t - \Delta v - \nu \nabla \operatorname{div} v + \nabla \eta = 0. \end{cases}$$

Eliminating the velocity we obtain the damped wave equation

$$\eta_{tt} - (1+\nu)\Delta\eta_t - \Delta\eta = 0,$$

which can be solved explicitly at the level of the Fourier transform. We obtain two modes, one strongly parabolic, disappearing for $\nu \to \infty$, and the second one having the following form, in the high frequency regime:

$$\eta(t) \sim \eta(0)e^{-t/(1+\nu)} \to \eta(0).$$

This means that at the same time, we have that $\eta(t)$ tends strongly to 0 as $t \to \infty$ even for very large ν, but that for all $t > 0$ (even very large), $\eta(t) \to \eta(0)$ when ν tends to ∞.

The behavior corresponding to the low Mach number limit is of a different nature, as it corresponds to the linearization

$$\begin{cases} \eta_t + \dfrac{1}{\varepsilon} \operatorname{div} u = 0, \\ v_t - \Delta v - \nu \nabla \operatorname{div} v + \dfrac{1}{\varepsilon} \nabla \eta = 0, \end{cases}$$

which leads to the wave equation

$$\eta_{tt} - (1+\nu)\Delta\eta_t - \frac{1}{\varepsilon^2}\Delta\eta = 0.$$

Asymptotically for $\varepsilon \to 0$, the above damped wave equation behaves as a wave equation with propagation speed $1/\varepsilon$. Hence, in the periodic setting, we have huge oscillations that preclude any strong convergence result. However, after filtering by the wave operator, convergence becomes strong, which entails weak convergence, back to the original unknowns (see [Danchin 2002] for more details).

The main idea of Theorem 1.3 is just to compute the distance between the compressible and the incompressible solutions, by means of the standard energy norm

[2] They are defined by $\mathcal{Q}v := -\nabla(-\Delta)^{-1} \operatorname{div} v$ and $\mathcal{P}v := v - \mathcal{Q}v$.

(in sharp contrast with the approach in [Danchin and Mucha 2017] where critical Besov norms are used). In order to do so, it is convenient to decompose $\rho-\eta$ into two parts

$$\rho-\eta=(\rho-\tilde{\rho})+(\tilde{\rho}-\eta),$$

where the auxiliary density $\tilde{\rho}$ is the transport of ρ_0 by the flow of the divergence-free vector field $\mathcal{P}v$. As the bounds of Theorem 1.2 readily ensure that $\|\rho-\tilde{\rho}\|_q=\mathcal{O}(\nu^{-1})$, one may, somehow, perform the energy argument as if comparing $(\tilde{\rho}, v)$ and (η, u).

We end the introduction by presenting the main notation that is used throughout the paper. By ∇ we denote the gradient with respect to space variables, and by u_t the time derivative of the function u. By $\|\cdot\|_{L_p(Q)}$ (or sometimes just $\|\cdot\|_p$), we mean the p-power Lebesgue norm corresponding to the set Q, and $L_p(Q)$ is the corresponding Lebesgue space. We denote by W^s_p the Sobolev (Slobodeckij for s not integer) space on the torus $\mathbb{T}^2$, and put $H^s=W^s_2$. The homogeneous versions of those spaces (that is, the corresponding subspaces of functions with null mean) are denoted by $\dot{W}^s_p$ and $\dot{H}^s$.

Generic constants are denoted by C. By $A\lesssim B$ we mean that $A\le CB$, and $A\approx B$ stands for $C^{-1}A\le B\le CA$.

2. Energy estimates

The aim of this part is to provide bounds via energy-type estimates. We assume that the density is bounded from above and below. Let us first recall the basic energy identity.

Proposition 2.1. *For any $T>0$, sufficiently smooth solutions to* (1-1) *obey* (1-6).

Proof. That fundamental estimate follows from testing the momentum equation by v and integrating by parts in the diffusion and pressure terms. Indeed, using the definition of e and the mass equation, we get

$$\begin{aligned}\int_{\mathbb{T}^2}\nabla P\cdot v\,dx&=\int_{\mathbb{T}^2}\frac{P'(\rho)}{\rho}\nabla\rho\cdot(\rho v)\,dx=\int_{\mathbb{T}^2}\nabla(e'(\rho))\cdot(\rho v)\,dx\\&=-\int_{\mathbb{T}^2}e'(\rho)\ \mathrm{div}(\rho v)\,dx=\int_{\mathbb{T}^2}e'(\rho)\rho_t\,dx=\frac{d}{dt}\int_{\mathbb{T}^2}e(\rho)\,dx.\end{aligned}$$

Then integrating in time completes the proof. □

Let us next derive a higher-order energy estimate, pointing out the dependency with respect to the volume viscosity ν.

Proposition 2.2. *Assume that there exist positive constants $\rho_*<\rho^*$ such that*

$$\rho_*\le\rho(t,x)\le\rho^*\quad\textit{for all } (t,x)\in[0,T]\times\mathbb{T}^2. \tag{2-1}$$

Then solutions to (1-1) *with* $\mu = 1$ *fulfill the inequality*

$$\|v(T), \nabla v(T), \rho(T)-\bar\rho\|_2^2+\nu\|\operatorname{div} v(T)\|_2^2+\int_0^T\big(\|\nabla^2 v, \nabla v, v_t\|_2^2+\nu\|\operatorname{div} v\|_{H^1}^2\big)\,dt$$
$$\le C\exp(C\|v_0\|_2^4)\bigg(\|v_0, \nabla v_0, \rho_0-\bar\rho\|_2^2+\nu\|\operatorname{div} v_0\|_2^2 \\ +\nu^{-1}T\|v_0\|_2^2+\nu^{-1}\int_0^T\|\nabla\rho\|_2^2\,dt\bigg), \quad (2\text{-}2)$$

provided ν *is larger than some* $\nu_0 = \nu_0(\rho_*, \rho^*, P)$.

Proof. We take the $\mathbb{T}^2$ inner product of the momentum equation with v_t, getting

$$\int_{\mathbb{T}^2}\rho|v_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}(|\nabla v|^2+\nu(\operatorname{div} v)^2)\,dx + \int_{\mathbb{T}^2}\nabla P\cdot v_t\,dx \\ = -\int_{\mathbb{T}^2}(\rho v\cdot\nabla v)\cdot v_t\,dx. \quad (2\text{-}3)$$

Integrating by parts and using the mass equation yields

$$\begin{aligned}\int_{\mathbb{T}^2}\nabla P\cdot v_t\,dx &= -\int_{\mathbb{T}^2}P\ \operatorname{div} v_t\,dx\\ &= -\frac{d}{dt}\int_{\mathbb{T}^2}P\ \operatorname{div} v\,dx + \int_{\mathbb{T}^2}P'(\rho)\rho_t\operatorname{div} v\,dx\\ &= -\frac{d}{dt}\int_{\mathbb{T}^2}P\ \operatorname{div} v\,dx - \int_{\mathbb{T}^2}P'(\rho)\ \operatorname{div}(\rho v)\ \operatorname{div} v\,dx.\end{aligned}$$

Hence putting this together with (2-3), we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}(|\nabla v|^2+\nu(\operatorname{div} v)^2-2P\operatorname{div} v)\,dx + \int_{\mathbb{T}^2}\rho|v_t|^2\,dx \\ = \int_{\mathbb{T}^2}P'(\rho)\operatorname{div}(\rho v)\operatorname{div} v\,dx - \int_{\mathbb{T}^2}(\rho v\cdot\nabla v)\cdot v_t\,dx. \quad (2\text{-}4)$$

Now, setting $K(\rho) = \rho P'(\rho) - P(\rho)$, one can check that

$$\begin{aligned}\int_{\mathbb{T}^2}P'(\rho)\operatorname{div}(\rho v)\operatorname{div} v\,dx &= \int_{\mathbb{T}^2}(\operatorname{div} v)\ v\cdot\nabla(P(\rho))\,dx+\int_{\mathbb{T}^2}\rho\nabla P'(\rho)\ (\operatorname{div} v)^2\,dx\\ &= -\int_{\mathbb{T}^2}P(\rho)\ v\cdot\nabla\operatorname{div} v\,dx+\int_{\mathbb{T}^2}K(\rho)(\operatorname{div} v)^2\,dx.\end{aligned}$$

Hence, if (2-1) is fulfilled then we have

$$\frac{d}{dt}\int_{\mathbb{T}^2}\big(|\nabla v|^2+\nu(\operatorname{div} v)^2-2P(\rho)\operatorname{div} v\big)\,dx + \int_{\mathbb{T}^2}\rho|v_t|^2\,dx \\ \le C\int_{\mathbb{T}^2}\big(|v\cdot\nabla\operatorname{div} v| + (\operatorname{div} v)^2 + |v\cdot\nabla v|^2\big)\,dx. \quad (2\text{-}5)$$

Next, taking the L^2 scalar product of the momentum equation with Δv we get

$$\int_{\mathbb{T}^2}(|\Delta v|^2+\nu|\nabla \operatorname{div} v|^2)\,dx-\int_{\mathbb{T}^2}\rho v_t\cdot\Delta v\,dx-\int_{\mathbb{T}^2}\nabla P\cdot\Delta v\,dx\le\int_{\mathbb{T}^2}|\rho v\cdot\nabla v\Delta v|\,dx.$$

Note that

$$-\int_{\mathbb{T}^2}\nabla P\cdot\Delta v\,dx=-\int_{\mathbb{T}^2}\nabla P\cdot\nabla\operatorname{div} v\,dx\le C\int_{\mathbb{T}^2}|\nabla\rho|\,|\nabla\operatorname{div} v|\,dx.$$

Then, combining with the basic energy identity and with (2-5) and introducing

$$E(v,\rho):=\int_{\mathbb{T}^2}\big(\rho|v|^2+2e(\rho)+|\nabla v|^2+\nu\,(\operatorname{div} v)^2-2P(\rho)\operatorname{div} v\big)\,dx, \qquad (2\text{-}6)$$

we find,

$$\begin{aligned}\frac{d}{dt}E(v,\rho)+\int_{\mathbb{T}^2}\rho|v_t|^2\,dx+\frac{1}{\rho^*}\int_{\mathbb{T}^2}\big(|\nabla v|^2+|\nabla^2 v|^2+\nu(\operatorname{div} v)^2+\nu|\nabla\operatorname{div} v|^2\big)\,dx\\ \le\int_{\mathbb{T}^2}|v_t\cdot\Delta v|\,dx\\ +C\int_{\mathbb{T}^2}\Big((\operatorname{div} v)^2+|v\cdot\nabla\operatorname{div} v|+\rho|v\cdot\nabla v|^2+\frac{1}{\rho^*}|\nabla\rho|\,|\nabla\operatorname{div} v|\Big)\,dx. \qquad (2\text{-}7)\end{aligned}$$

Hence, setting

$$D(v):=\|\nabla v\|_{H^1}^2+\|\sqrt{\rho}\,v_t\|_{L_2}^2+\nu\|\operatorname{div} v\|_{H^1}^2,$$

inequality (2-7) implies that for large enough ν,

$$\frac{d}{dt}E(v,\rho)+\frac{1}{\rho^*}D(v)\le C\int_{\mathbb{T}^2}\big(|v|^2\,|\nabla v|^2+(|v|+|\nabla\rho|)|\nabla\operatorname{div} v|\big)\,dx.$$

Of course, from the Ladyzhenskaya inequality, we have

$$\int_{\mathbb{T}^2}|v\cdot\nabla v|^2\,dx\le C\|v\|_2\,\|\nabla v\|_2^2\,\|\Delta v\|_2.$$

Therefore, we end up with

$$\frac{d}{dt}E(v,\rho)+\frac{1}{\rho^*}D(v)\le C\big(\|v\|_2^2\,\|\nabla v\|_2^2\,\|\nabla v\|_2^2+\nu^{-1}(\|v\|_2^2+\|\nabla\rho\|_2^2)\big).$$

Let us notice that if $\nu\ge\nu_0(\rho_*,\rho^*,P)$ then we have, according to (1-2),

$$E(v,\rho)\approx\|v\|_{H^1}^2+\|\rho-\bar\rho\|_{L_2}^2+\nu\|\operatorname{div} v\|_{L_2}^2. \qquad (2\text{-}8)$$

Hence the Gronwall inequality yields

$$\begin{aligned}E(v(T),\rho(T))+\frac{1}{\rho^*}\int_0^T D(t)\,dt\le\exp\Big(C\int_0^T\|v\|_2^2\,\|\nabla v\|_2^2\,dt\Big)\\ \times\Big(E(v_0,\rho_0)+\frac{C}{\nu}\int_0^T\exp\Big(-C\int_0^t\|v\|_2^2\,\|\nabla v\|_2^2\,dt\Big)(\|v\|_2^2+\|\nabla\rho\|_2^2)\,dt\Big).\end{aligned}$$

Remembering that the basic energy inequality implies

$$\int_0^T \|v\|_2^2\,\|\nabla v\|_2^2\,dt \le C\|v_0\|_2^4,$$

one may conclude that

$$E(v(T),\rho(T)) + \frac{1}{\rho^*}\int_0^T D(v)\,dt \\ \le \exp(C\|v_0\|_2^4)\left(E(v_0,\rho_0) + \frac{C}{\nu}\left(\|v_0\|_2^2\,T + \int_0^T \|\nabla\rho\|_2^2\,dt\right)\right),$$

which obviously yields (2-2). □

3. Estimates with loss of integrability for the transport equation

We are concerned with the proof of regularity estimates for the transport equation

$$\rho_t + v\cdot\nabla\rho + \rho\ \mathrm{div}\, v = 0 \tag{3-1}$$

in some endpoint case where the transport field v fails to be in $L_1(0,T;\mathrm{Lip})$ by a little.

More exactly, we aim at extending the results in [Desjardins 1997] to transport fields that are not divergence-free. Our main result is:

Proposition 3.1. *Let* $1 \le q \le \infty$ *and* $T > 0$. *Suppose* $\rho_0 \in W_q^1(\mathbb{T}^2)$ *and* $v \in L_2(0,T;H^2(\mathbb{T}^2))$ *are such that* $\mathrm{div}\, v \in L_1(0,T;L_\infty(\mathbb{T}^2)) \cap L_1(0,T;W_q^1(\mathbb{T}^2))$. *Then the solution to* (3-1) *fulfills for all* $1 \le p < q$,

$$\sup_{t<T}\|\nabla\rho(t)\|_p \le K\left(\|\nabla\rho_0\|_q + \|\rho_0\|_\infty \sup_{t<T}\left\|\int_0^t \nabla\,\mathrm{div}\, v\,d\tau\right\|_q\right) \\ \times \exp\left(CT\int_0^T \|\nabla^2 v\|_2^2\,dt\right)\exp\left(\int_0^T \|\,\mathrm{div}\, v\|_\infty\,dt\right),$$

where K *is an absolute constant, and the constant* C *depends only on* p *and* q.

Proof. We proceed by means of the standard characteristics method: our assumptions guarantee that v admits a unique (generalized) flow X, a solution to

$$X(t,y) = y + \int_0^t v(\tau, X(\tau,y))\,d\tau. \tag{3-2}$$

Then, setting

$$u(t,y) := v(t,X(t,y)) \quad \text{and} \quad a(t,y) = \rho(t,X(t,y)), \tag{3-3}$$

(3-1) can be rewritten as

$$\frac{da(t,y)}{dt} = -(\mathrm{div}\, v)(t,X(t,y))\cdot a(t,y), \tag{3-4}$$

the unique solution of which is given by

$$a(t,y)=\exp\biggl(-\int_0^t(\operatorname{div}v)(\tau,X(\tau,y))\,d\tau\biggr)a_0(y). \tag{3-5}$$

From the chain rule and the Leibniz formula, we thus infer

$$\begin{aligned}\nabla_y a(t,y)=\exp\biggl(-\int_0^t(\operatorname{div}v)(\tau,X(\tau,y))\,d\tau\biggr)\\ \times\biggl(\nabla_y a_0(y)-a_0(y)\int_0^t(\nabla\operatorname{div}v)(\tau,X(\tau,y))\cdot\nabla_y X(\tau,y)\,d\tau\biggr).\end{aligned}$$

Our goal is to estimate all these quantities in the Eulerian coordinates. Note that by (3-2) and the Gronwall lemma, we obtain pointwisely that, setting $Y(t,\cdot):=(X(t,\cdot))^{-1}$,

$$\begin{aligned}|\nabla_y X(t,y)|&\le\exp\biggl(\int_0^t|\nabla_x v(\tau,X(\tau,y))|\,d\tau\biggr),\\ |\nabla_x Y(t,x)|&\le\exp\biggl(\int_0^t|\nabla_y u(\tau,Y(\tau,x))|\,d\tau\biggr).\end{aligned} \tag{3-6}$$

As $\nabla_x\rho(t,x)=\nabla_y a(t,Y(t,x))\cdot\nabla_x Y(t,x)$, we get

$$\begin{aligned}|\nabla\rho(t,x)|\le\exp\biggl(3\int_0^t|\nabla v(\tau,X(\tau,Y(t,x)))|\,d\tau\biggr)\\ \times\biggl(\bigl|\nabla\rho_0(Y(t,x))\bigr|+\bigl|\rho_0(Y(t,x))\bigr|\biggl|\int_0^t\nabla\operatorname{div}v\bigl(\tau,X(\tau,Y(t,x))\bigr)\,d\tau\biggr|\biggr).\end{aligned}$$

Recall that the Jacobian of the change of coordinates $(t,y)\to(t,x)$ is given by

$$J_X(t,y)=\exp\biggl(\int_0^t\operatorname{div}v(\tau,X(\tau,y))\,d\tau\biggr)\le\exp\biggl(\int_0^t\|\operatorname{div}v\|_\infty\,d\tau\biggr). \tag{3-7}$$

Hence taking the $L_p(\mathbb{T}^2)$ norm and using the Hölder inequality with $1/p=1/q+1/m$, we get

$$\begin{aligned}\|\nabla\rho(t)\|_p\le\exp\biggl(\frac{1}{q}\int_0^t\|\operatorname{div}v\|_\infty\,d\tau\biggr)\\ \times\biggl(\|\nabla\rho_0\|_q+\|\rho_0\|_\infty\biggl\|\int_0^t\nabla\operatorname{div}v(\tau,X(\tau,\cdot))\,ds\biggr\|_q\biggr)\\ \times\biggl\|\exp\biggl(3\int_0^t|\nabla v(\tau,X(\tau,\cdot))|\,d\tau\biggr)\biggr\|_m.\end{aligned} \tag{3-8}$$

To bound the last term, we write that for all $\beta>0$,

$$\int_0^t|\nabla v(\tau,X(\tau,\cdot))|\,d\tau\le\beta\int_0^t\frac{|\nabla v(\tau,X(\tau,\cdot))|^2}{\|\nabla^2 v(\tau,\cdot)\|_2^2}\,d\tau+\frac{1}{4\beta}\int_0^t\|\nabla^2 v(\tau,\cdot)\|_2^2\,d\tau.$$

Hence using the Jensen inequality

$$\exp\left(\int_0^t \phi(s)\,ds\right) \le \frac{1}{t}\int_0^t e^{t\phi(s)}\,ds,$$

we discover that

$$\int_{\mathbb{T}^2} \exp\left(3m\int_0^t |\nabla v(\tau, X(\tau,x))|\,d\tau\right)dx \\ \le \exp\left(\frac{m}{4\beta}\int_0^t \|\nabla^2 v\|_2^2\,d\tau\right)\frac{1}{t}\int_0^t\int_{\mathbb{T}^2}\exp\left(9m\beta t\frac{|\nabla v(\tau, X(\tau,x))|^2}{\|\nabla^2 v\|_2^2}\right)dx\,d\tau.$$

In the last integral we change coordinates and get

$$\int_{\mathbb{T}^2} \exp\left(3m\int_0^t |\nabla v(\tau, X(\tau,x))|\,d\tau\right)dx \\ \le \frac{1}{t}\exp\left(\frac{m}{4\beta}\int_0^t \|\nabla^2 v\|_2^2\,d\tau\right) \\ \times\left(\int_0^t\int_{\mathbb{T}^2}\exp\left(9m\beta t\frac{|\nabla v(\tau,x)|^2}{\|\nabla^2 v\|_2^2}\right)dx\,d\tau\right)\exp\left(\int_0^t \|\operatorname{div} v\|_\infty\,d\tau\right).$$

At this stage, to complete the proof, it suffices to apply the following Trudinger inequality, see for example [Adams 1975], to $f = \nabla v$: there exist constants δ_0 and K such that for all f in $H^1(\mathbb{T}^2)$,

$$\int_{\mathbb{T}^2}\exp\left(\delta_0\frac{|f(x)-\bar f|^2}{\|\nabla f\|_2^2}\right)dx \le K \quad \text{with } \bar f := \frac{1}{|\mathbb{T}^2|}\int_{\mathbb{T}^2} f\,dx. \tag{3-9}$$

Then, taking β so small that $9m\beta t = \delta_0$, we end up with

$$\int_{\mathbb{T}^2} \exp\left(3m\int_0^t |\nabla v(\tau, X(\tau,x))|\,d\tau\right)dx \\ \le C\exp\left(\frac{9mt}{4\delta_0}\int_0^t \|\nabla^2 v\|_2^2\,d\tau\right)\exp\left(\int_0^t \|\operatorname{div} v\|_\infty\,d\tau\right). \tag{3-10}$$

Combining with (3-8) completes the proof of the proposition. □

4. Linear systems with variable coefficients

Here we are concerned with the proof of maximal regularity estimates for the linear system

$$\begin{cases} \rho u_t - \Delta u - \nu\nabla \operatorname{div} u = f & \text{in } (0,T)\times\mathbb{T}^N, \\ u|_{t=0} = u_0 & \text{in } \mathbb{T}^N, \end{cases} \tag{4-1}$$

assuming only that $\rho = \rho(t,x)$ is bounded by above and from below (no time or space regularity whatsoever).

In contrast with the previous section, we do not need the space dimension to be 2. As we want to keep track of the dependency with respect to ν for $\nu \to \infty$, we shall assume throughout that $\nu \geq 0$ for simplicity.

Theorem 4.1. *Let $T > 0$. Assume that $\nu \geq 0$ and that*

$$0 < \rho_* \leq \rho(t, x) \leq \rho^* \quad \text{for } (t, x) \in [0, T] \times \mathbb{T}^N. \tag{4-2}$$

There exist positive constants 2_*, 2^* depending only on ρ_* and ρ^*, with $2_* < 2 < 2^*$, such that for all $r \in (2_*, 2^*)$ we have

$$\begin{aligned}\|u_t, \nabla^2 u, \nu\nabla \operatorname{div} u\|_{L_r((0,T)\times\mathbb{T}^N)} \\ \leq C(r, \rho_*, \rho^*)\big(\|f\|_{L_r((0,T)\times\mathbb{T}^N)} + \|u_0\|_{W_r^{2-2/r}(\mathbb{T}^N)}\big).\end{aligned} \tag{4-3}$$

Proof. First, we reduce the problem to the one with null initial data, solving

$$\begin{cases}\rho^*\bar{u}_t - \Delta\bar{u} - \nu\nabla \operatorname{div}\bar{u} = 0 & \text{in } (0, T) \times \mathbb{T}^N, \\ \bar{u}|_{t=0} = u_0 & \text{in } \mathbb{T}^N.\end{cases} \tag{4-4}$$

Applying the divergence operator to the equation yields

$$\rho^*(\operatorname{div}\bar{u})_t - (1+\nu)\Delta \operatorname{div}\bar{u} = 0.$$

Hence the basic maximal regularity theory for the heat equation in the torus gives

$$(1+\nu)\|\nabla \operatorname{div}\bar{u}\|_{L_p((0,T)\times\mathbb{T}^N)} \leq C\|\operatorname{div} u_0\|_{W_p^{1-2/p}(\mathbb{T}^N)}. \tag{4-5}$$

Then we restate system (4-4) in the form

$$\rho^*\bar{u}_t - \Delta\bar{u} = \nu\nabla \operatorname{div}\bar{u}, \tag{4-6}$$

and get

$$\begin{aligned}\|\bar{u}_t, \nabla^2\bar{u}\|_{L_p(\mathbb{T}^N\times(0,T))} &\leq K_p\big(\nu\|\nabla \operatorname{div}\bar{u}\|_{L_p((0,T)\times\mathbb{T}^N)} + \|u_0\|_{W_p^{2-2/p}(\mathbb{T}^N)}\big) \\ &\leq K_p\left(\frac{\nu}{1+\nu}\right)\|u_0\|_{W_p^{2-2/p}(\mathbb{T}^N)}.\end{aligned}$$

Therefore, as $\nu \geq 0$, we end up with

$$\|\bar{u}_t, \nabla^2\bar{u}, \nu\nabla \operatorname{div}\bar{u}\|_{L_p((0,T)\times\mathbb{T}^N)} \leq K_p\|u_0\|_{W_p^{2-2/p}(\mathbb{T}^N)}. \tag{4-7}$$

Next we look for u in the form

$$u = w + \bar{u}, \tag{4-8}$$

where w fulfills

$$\rho w_t - \Delta w - \nu\nabla \operatorname{div} w = f + (\rho^* - \rho)\bar{u}_t =: g, \quad w|_{t=0} = 0. \tag{4-9}$$

Thanks to (4-2) and (4-9), we have

$$\|g\|_{L_p((0,T)\times\mathbb{T}^N)} \leq \|f\|_{L_p((0,T)\times\mathbb{T}^N)} + K_p(\rho^* - \rho_*)\|u_0\|_{W_p^{2-2/p}(\mathbb{T}^N)}. \tag{4-10}$$

Now, setting $h := g + (\rho^* - \rho)w_t$, system (4-9) reduces to

$$\begin{cases} \rho^* w_t - \Delta w - \nu \nabla \operatorname{div} w = h & \text{in } (0,T) \times \mathbb{T}^N, \\ w|_{t=0} = 0 & \text{in } \mathbb{T}^N. \end{cases} \tag{4-11}$$

We claim that for all $p \in (1, \infty)$ we have

$$\|\rho^* w_t\|_{L_p((0,T)\times\mathbb{T}^N)} \le C_p \|h\|_{L_p((0,T)\times\mathbb{T}^N)} \tag{4-12}$$

with $C_p \to 1$ for $p \to 2$.

Indeed, to see that $C_2 = 1$, we just take the L^2 scalar product of (4-11) with w_t, which yields

$$\rho^* \|w_t\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\big(\|\nabla w\|_{L^2}^2 + \nu \|\operatorname{div} v\|_{L^2}^2\big) = \int_{\mathbb{T}^N} h\, w_t\, dx \le \frac{1}{2}\rho^* \|w_t\|_{L^2}^2 + \frac{1}{2\rho^*}\|h\|_{L^2}^2.$$

Then for any fixed $p_0 \in (1, \infty) \setminus \{2\}$, the standard maximal regularity estimate is

$$\|\rho^* w_t\|_{L_{p_0}((0,T)\times\mathbb{T}^N)} \le K_{p_0} \|h\|_{L_{p_0}((0,T)\times\mathbb{T}^N)},$$

and the Hölder inequality gives us for all $\theta \in [0, 1]$,

$$\|z\|_{L_r((0,T)\times\mathbb{T}^N)} \le \|z\|_{L_2((0,T)\times\mathbb{T}^N)}^{1-\theta} \|z\|_{L_{p_0}((0,T)\times\mathbb{T}^N)}^{\theta} \quad \text{with } \frac{1}{r} = \frac{1-\theta}{2} + \frac{\theta}{p_0}.$$

Therefore $C_p \le C_{p_0}^{\theta}$, whence $\limsup C_p \le 1$ for $p \to 2$ (as $\theta \to 0$).

Now, remembering the definition of h, we write for all $p \in (1, \infty)$,

$$\begin{aligned} \|\rho^* w_t\|_{L_p((0,T)\times\mathbb{T}^N)} &\le C_p\big(\|g\|_{L_p((0,T)\times\mathbb{T}^N)} + \|(\rho^* - \rho)w_t\|_{L_p((0,T)\times\mathbb{T}^N)}\big) \\ &\le C_p \|g\|_{L_p((0,T)\times\mathbb{T}^N)} + C_p\left(1 - \frac{\rho_*}{\rho^*}\right)\|\rho^* w_t\|_{L_p((0,T)\times\mathbb{T}^N)}. \end{aligned}$$

Therefore, if[3]

$$1 - C_p\left(1 - \frac{\rho_*}{\rho^*}\right) \ge \frac{1}{2}\frac{\rho_*}{\rho^*}, \tag{4-13}$$

then we end up with

$$\|\rho^* w_t\|_{L_p((0,T)\times\mathbb{T}^N)} \le \frac{2\rho^* C_p}{\rho_*} \|g\|_{L_p((0,T)\times\mathbb{T}^N)}. \tag{4-14}$$

Let us emphasize that (4-13) is fulfilled for p close enough to 2, due to $C_p \to 1$ for $p \to 2$.

It is now easy to complete the proof: We rewrite (4-11) in the form

$$\begin{cases} -\Delta w - \nu \nabla \operatorname{div} w = g - \rho w_t & \text{in } (0,T) \times \mathbb{T}^N, \\ w|_{t=0} = 0 & \text{in } \mathbb{T}^N. \end{cases}$$

[3]Clearly, we just need that $1 - C_p(1 - \rho_*/\rho^*) > 0$. However taking that slightly stronger condition allows us to get a more explicit inequality.

Then one concludes as before that

$$\begin{aligned}\|\nabla^2 w,\, \nu\nabla \operatorname{div} w\|_{L_p((0,T)\times\mathbb{T}^N)} &\le K_p\|g-\rho w_t\|_{L_p((0,T)\times\mathbb{T}^N)}\\ &\le K_p\big(\|g\|_{L_p((0,T)\times\mathbb{T}^N)}+\rho^*\|w_t\|_{L_p((0,T)\times\mathbb{T}^N)}\big).\end{aligned}$$

Hence, putting together with (4-14) and assuming that p is close enough to 2,

$$\|w_t,\, \nabla^2 w,\, \nu\nabla \operatorname{div} w\|_{L_p((0,T)\times\mathbb{T}^N)} \le C_{\rho_*,\rho^*}\|g\|_{L_p((0,T)\times\mathbb{T}^N)}. \tag{4-15}$$

Then combining with (4-10) and (4-7) completes the proof. □

5. Final bootstrap argument

In what follows, we fix some $0<\rho_*<\rho^*$ and denote by 2_* and 2^* the corresponding Lebesgue exponents provided by Theorem 4.1. We assume that the initial data (ρ_0, v_0) satisfies all the requirements of Theorem 1.2

Take some time T such that $1\le T\le\nu$ (stronger conditions will appear below), and assume that we have a solution (ρ, v) to (1-1) on $[0,T]\times\mathbb{T}^2$, fulfilling the regularity properties of Theorem 1.2 for some $2<q<\min(2^*,4)$, and

$$\exp\left(\int_0^T \|\operatorname{div} v\|_\infty\,dt\right)\le 2. \tag{5-1}$$

Then it is clear that ρ obeys

$$\rho_*\le\rho\le\rho^* \quad \text{on } [0,T]\times\mathbb{T}^2. \tag{5-2}$$

For all $p\in[2,q]$, define $A_p(T):=\|\nabla\operatorname{div} v\|_{L_1(0,T;L_p(\mathbb{T}^2))}$ and assume that, for some small enough constant $c_0>0$, we have

$$A_q(T)\le c_0. \tag{5-3}$$

Clearly, if $Kc_0\le\log 2$, where K stands for the norm of the embedding $\dot W^1_q(\mathbb{T}^2)\hookrightarrow L_\infty(\mathbb{T}^2)$, then (5-1) is fulfilled. We shall assume in addition that $c_0\rho^*\le 1$.

We are going to show that if (5-3) is fulfilled then, for sufficiently large ν, all the norms of the solution are under control. Then, bootstrapping, this will justify (5-3) a posteriori.

Step 1: high-order energy estimate for v. Let $E_0^2:=1+\|v_0\|_{H^1}^2+\|\rho_0-\bar\rho\|_2^2$. By (2-2) we easily get, remembering that $\nu^{-1}T\le 1$,

$$\begin{aligned}&\|v\|^2_{L_\infty(0,T;H^1)}+\nu\|\operatorname{div} v\|^2_{L_\infty(0,T;L_2)}+\|\rho-\bar\rho\|^2_{L_\infty(0,T;L_2)}\\ &\qquad\qquad+\int_0^T\big(\|\nabla v\|^2_{H^1}+\|v_t\|_2^2+\nu\|\nabla\operatorname{div} v\|_2^2\big)\,dt\\ &\quad\le Ce^{C\|v_0\|_2^4}\big(E_0^2+\nu^{-1}T\|\nabla\rho\|^2_{L_\infty(0,T;L_2)}\big).\end{aligned} \tag{5-4}$$

Step 2: regularity estimates at L_p level for the density. From Proposition 3.1, we find that there exists an absolute constant K such that for all $r \in [2, q)$, there exists some constant $C_r > 0$ such that

$$\sup_{t\in[0,T]} \|\nabla\rho(t)\|_r \le K\biggl(\bigl(\|\nabla\rho_0\|_q + \rho^* A_q(T)\bigr)\exp\biggl(C_r T\int_0^T \|\nabla^2 v\|_2^2\,dt\biggr)\biggr).$$

Hence, bounding the last term according to (5-4), and using (5-3) and the definition of E_0,

$$\sup_{t\in[0,T]} \|\nabla\rho(t)\|_r \le K(\|\nabla\rho_0\|_q + 1)\exp(C_r E_0^2 T e^{C\|v_0\|_2^4}) \\ \times \exp\bigl(C_r \nu^{-1} T^2 e^{C\|v_0\|_2^4}\|\nabla\rho\|^2_{L_\infty(0,T;L_2)}\bigr). \quad (5\text{-}5)$$

Taking $r = 2$, we deduce that if

$$C_2\nu^{-1}T^2 e^{C\|v_0\|_2^4}\|\nabla\rho\|^2_{L_\infty(0,T;L_2)} \le \log 2,$$

then we have

$$\sup_{t\in[0,T]} \|\nabla\rho(t)\|_2 \le 2K(\|\nabla\rho_0\|_q + 1)\exp(C_2 E_0^2 T e^{C\|v_0\|_2^4}). \quad (5\text{-}6)$$

Using an obvious connectivity argument, we conclude that (5-6) holds whenever

$$\nu > \frac{4K^2C_2}{\log 2}(\|\nabla\rho_0\|_q + 1)^2\exp(2C_2E_0^2 T e^{C\|v_0\|_2^4})T^2 e^{C\|v_0\|_2^4}. \quad (5\text{-}7)$$

Reverting to (5-4), we readily get, taking a larger constant C if need be,

$$\|v\|^2_{L_\infty(0,T;H^1)} + \nu\|\operatorname{div} v\|^2_{L_\infty(0,T;L_2)} + \|\rho - \bar\rho\|^2_{L_\infty(0,T;L_2)} \\ + \int_0^T \bigl(\|\nabla v\|^2_{H^1} + \|v_t\|^2_{L_2} + \nu\|\nabla\operatorname{div} v\|^2_{L_2}\bigr)\,dt \le C e^{C\|v_0\|_2^4} E_0^2. \quad (5\text{-}8)$$

Of course, combining (5-6) with (5-5) ensures that for all $r \in [2, q)$, we have

$$\sup_{t\in[0,T]} \|\nabla\rho(t)\|_{L_r} \le K(\|\nabla\rho_0\|_q + 1)\,\exp(C_r E_0^2 T e^{C\|v_0\|_2^4}). \quad (5\text{-}9)$$

Step 3: maximal regularity at L_p level for the velocity. We rewrite the velocity equation as

$$\rho\partial_t v - \Delta v - \nu\nabla\operatorname{div} v = -\nabla P - \rho v\cdot\nabla v.$$

Then Theorem 4.1 ensures that for all $p \in [2, q)$,

$$V_p(T) \le C_p\bigl(\|v_0\|_{W_p^{2-2/p}} + \|\nabla P + \rho v\cdot\nabla v\|_{L_p(0,T\times\mathbb{T}^2)}\bigr) \quad (5\text{-}10)$$

with $V_p(T) := \|v\|_{L_\infty(0,T;W_p^{2-2/p})} + \|v_t, \nabla^2 v, \nu\nabla\operatorname{div} v\|_{L_p(0,T\times\mathbb{T}^2)}$.

By the Hölder inequality

$$\|v\cdot\nabla v\|_{L_p(0,T\times\mathbb{T}^2)}\leq T^{1/s}\|v\|_{L_\infty(0,T;L_s)}\|\nabla v\|_{L_4(0,T;L_4)}\quad\text{with }\frac{1}{s}+\frac{1}{4}=\frac{1}{p}.$$

Hence using embedding and inequality (5-8),

$$\|v\cdot\nabla v\|_{L_p(0,T\times\mathbb{T}^2)}\leq CT^{1/p-1/4}E_0^2e^{C\|v_0\|_2^4},$$

and reverting to (5-10) and using (5-9) thus yields for some constant C_P depending only on the pressure law,

$$V_p(T)\leq C_p\big(\|v_0\|_{W_p^{2-2/p}}+C_PT^{1/p}(\|\nabla\rho_0\|_q+1)e^{CE_0^2Te^{C\|v_0\|_2^4}}\\ +T^{1/p-1/4}E_0^2e^{C\|v_0\|_2^4}\big).\quad(5\text{-}11)$$

Step 4: regularity estimate at L_q level for the density. The standard estimate for the transport equation with Lipschitz velocity field yields

$$\sup_{t\leq T}\|\nabla\rho(t)\|_q\leq(\|\nabla\rho_0\|_q+\rho^*A_q(T))\exp(\|\nabla v\|_{L_1(0,T;L_\infty)}).$$

Hence, remembering (5-3) and using the embedding $\dot W_p^1(\mathbb{T}^2)\hookrightarrow L_\infty(\mathbb{T}^2)$ to handle the last term, we get

$$\sup_{t\leq T}\|\nabla\rho(t)\|_q\leq(\|\nabla\rho_0\|_q+1)\exp(CT^{1/p'}V_p(T)).$$

Then one can bound $V_p(T)$ according to (5-11) and eventually get

$$\sup_{t\leq T}\|\nabla\rho(t)\|_q\leq(\|\nabla\rho_0\|_q+1)\exp(T^{1/p'}I_0^p(T)),\quad(5\text{-}12)$$

with $I_0^p(T):=C_p\big(\|v_0\|_{W_p^{2-2/p}}+C_PT^{1/p}(\|\nabla\rho_0\|_q+1)e^{CE_0^2Te^{C\|v_0\|_2^4}}\big)$.

Step 5: maximal regularity at L_q level for the velocity. Let us use again Theorem 4.1, but with Lebesgue exponent q. We have

$$V_q(T)\leq C_q\big(\|v_0\|_{W_q^{2-2/q}}+\|\nabla P\|_{L_q(0,T\times\mathbb{T}^2)}+\|\rho v\cdot\nabla v\|_{L_q(0,T\times\mathbb{T}^2)}\big).\quad(5\text{-}13)$$

The last term may be bounded as in (5-11) (with q instead of p), and the pressure term may be handled thanks to (5-12). In the end we get

$$V_q(T)\leq C_q\big(\|v_0\|_{W_q^{2-2/q}}+C_PT^{1/q}(\|\nabla\rho_0\|_{L_q}+1)\exp(T^{1/q'}I_0^q(T))\big).$$

Step 6: final bootstrap. In order to complete the proof, it suffices to check that if ν is large enough then we do have (5-3). This is just a consequence of the fact that

$$A_q(T)\leq T^{1/q'}\|\nabla\operatorname{div}v\|_{L_q(0,T\times\mathbb{T}^2)}\leq\frac{1}{\nu}T^{1/q'}V_q(T).$$

Hence it suffices to choose ν fulfilling (5-7) and

$$\nu\geq T^{1/q'}C_q\big(\|v_0\|_{W_q^{2-2/q}}+C_PT^{1/q}(\|\nabla\rho_0\|_{L_q}+1)\exp(T^{1/p'}I_0^q(T))\big).$$

6. The incompressible limit issue

The aim of this section is to prove Theorem 1.3. In what follows the time T is fixed, and ν is larger than the threshold viscosity ν_0 given by Theorem 1.2. Throughout, we shall agree that $C_{0,T}$ denotes a "constant" depending only on T and on the norms of the initial data appearing in Theorem 1.2. Let us consider the corresponding solution (ρ, v). Then inequality (1-11) already ensures that all the terms with $\mathcal{Q}v$ in (1-14) are bounded as required.

In order to bound the other terms of (1-14), it is convenient to restate system (1-1) in terms of the divergence-free part $\mathcal{P}v$ and potential part $\mathcal{Q}v$ of the velocity field v, and in terms of the discrepancy $r := \rho - \tilde{\rho}$ between ρ and the "incompressible" density $\tilde{\rho}$ defined as the unique solution of the transport equation

$$\tilde{\rho}_t + \mathcal{P}v \cdot \nabla\tilde{\rho} = 0, \quad \tilde{\rho}|_{t=0} = \rho_0. \tag{6-1}$$

As r fulfills

$$r_t + \mathcal{P}v \cdot \nabla r = -\operatorname{div}(\rho\mathcal{Q}v), \quad r|_{t=0} = 0, \tag{6-2}$$

we have for all $t \in [0, T]$,

$$\|r(t)\|_q \le \int_0^t \big(\|\rho \operatorname{div} \mathcal{Q}v\|_q + \|\mathcal{Q}v \cdot \nabla\rho\|_q\big)\, d\tau. \tag{6-3}$$

Now, we have

$$\|\mathcal{Q}v \cdot \nabla\rho\|_{L_q(0,T\times\mathbb{T}^2)} \le \|\mathcal{Q}v\|_{L_q(0,T;L_\infty)}\|\nabla\rho\|_{L_\infty(0,T;L_q)}$$

and, by virtue of the Poincaré inequality,

$$\|\rho \operatorname{div} \mathcal{Q}v\|_{L_q(0,T\times\mathbb{T}^2)} \le C\rho^*\|\nabla \operatorname{div} \mathcal{Q}v\|_{L_q(0,T\times\mathbb{T}^2)}.$$

Therefore, taking advantage of Sobolev embedding and of inequality (1-12), we end up with

$$\sup_{0\le t\le T} \|r(t)\|_q \le C_{0,T}\nu^{-1}. \tag{6-4}$$

Next, we restate the second equation in (1-1) as

$$\tilde{\rho}\mathcal{P}v_t + \tilde{\rho}\mathcal{P}v \cdot \nabla\mathcal{P}v - \Delta\mathcal{P}v + \nabla Q + K = 0, \tag{6-5}$$

where $Q := P - (1+\nu)\operatorname{div} v$ and $K = K_1 + K_2 + K_3 + K_4$, with

$$K_1 := r\mathcal{P}v_t, \quad K_2 := \rho\mathcal{Q}v_t, \quad K_3 := r\mathcal{P}v\cdot\nabla\mathcal{P}v, \quad K_4 := \rho(\mathcal{Q}v\cdot\nabla\mathcal{P}v + v\cdot\nabla\mathcal{Q}v).$$

Subtracting (1-4) from (6-5) yields

$$\eta(\mathcal{P}v - u)_t + \eta u \cdot \nabla(\mathcal{P}v - u) - \Delta(\mathcal{P}v - u) + \nabla(Q - \Pi) + K + L = 0 \tag{6-6}$$

with

$$L := (\tilde{\rho} - \eta)\mathcal{P}v_t + (\tilde{\rho} - \eta)\mathcal{P}v \cdot \nabla\mathcal{P}v + \eta(\mathcal{P}v - u) \cdot \nabla\mathcal{P}v.$$

Of course, initially, we have

$$\mathcal{P}v - u|_{t=0} = 0, \quad \tilde{\rho} - \eta|_{t=0} = 0.$$

Now, we take the L^2 scalar product of (6-6) with $\mathcal{P}v - u$ getting, since $\operatorname{div} u = 0$,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} \eta|\mathcal{P}v - u|^2\,dx + \int_{\mathbb{T}^2} |\nabla(\mathcal{P}v - u)|^2\,dx = \int_{\mathbb{T}^2} K\cdot(u - \mathcal{P}v)\,dx + \int_{\mathbb{T}^2} L\cdot(u - \mathcal{P}v)\,dx. \quad (6\text{-}7)$$

To analyze the terms of the right-hand side, we need some information coming from the continuity equations. The difference of $\tilde{\rho}$ and η fulfills

$$(\tilde{\rho} - \eta)_t + u\cdot\nabla(\tilde{\rho} - \eta) = -(\mathcal{P}v - u)\cdot\nabla\tilde{\rho}.$$

Testing it by $(\tilde{\rho} - \eta)$ and defining q^* by $1/q^* + 1/q = \frac{1}{2}$, we find that

$$\sup_{t\le T}\|(\tilde{\rho} - \eta)(t)\|_2 \le \int_0^T \|\mathcal{P}v - u\|_{q^*}\ \|\nabla\tilde{\rho}\|_q\,dt.$$

As $\tilde{\rho}$ satisfies (6-1), we have for all $t \in [0, T]$,

$$\|\nabla\tilde{\rho}(t)\|_q \le \|\nabla\tilde{\rho}_0\|_q\, e^{\int_0^t \|\nabla\mathcal{P}v\|_\infty\,d\tau}.$$

Therefore, thanks to (1-13) and Sobolev embedding,

$$\sup_{t\le T}\|(\tilde{\rho} - \eta)(t)\|_2 \le C_{0,T}\int_0^T \|\mathcal{P}v - u\|_{q^*}\,dt. \quad (6\text{-}8)$$

One can now estimate all the terms of the right-hand side of (6-7). Regarding the first term of L, we have

$$\begin{aligned}\int_0^T\!\!\int_{\mathbb{T}^2} (\tilde{\rho} - \eta)\mathcal{P}v_t\cdot(\mathcal{P}v - u)\,dx\,dt &\le \int_0^T \|\tilde{\rho} - \eta\|_2\ \|\mathcal{P}v_t\|_q\ \|\mathcal{P}v - u\|_{q^*}\,dt\\ &\le C_{0,T}\left(\int_0^T \|\mathcal{P}v - u\|_{q^*}\,dt\right)\left(\int_0^T \|\mathcal{P}v_t\|_q^2\,dt\right)^{1/2}\left(\int_0^T \|\mathcal{P}v - u\|_{q^*}^2\,dt\right)^{1/2}.\end{aligned}$$

Hence taking $\theta \in (0, 1)$ below according to the Gagliardo–Nirenberg inequality, and remembering that $q > 2$ and that $H^1(\mathbb{T}^2) \hookrightarrow L_m(\mathbb{T}^2)$ for all $m < \infty$, we get

$$\begin{aligned}\int_0^T\!\!\int_{\mathbb{T}^2} (\tilde{\rho} - \eta)\mathcal{P}v_t\cdot(\mathcal{P}v - u)\,dx\,dt &\le C_{0,T}\int_0^T \|\nabla(\mathcal{P}v - u)\|_2^{2\theta}\ \|\mathcal{P}v - u\|_2^{2-2\theta}\,dt\\ &\le \frac{1}{8}\int_0^T \|\nabla(\mathcal{P}v - u)\|_2^2\,dt + C_{0,T}\int_0^T \|\mathcal{P}v - u\|_2^2\,dt. \quad (6\text{-}9)\end{aligned}$$

Next, we write

$$\left|\int_{\mathbb{T}^2}(\tilde\rho-\eta)(\mathcal{P}v\cdot\nabla\mathcal{P}v)\cdot(\mathcal{P}v-u)\,dx\right|\leq\|\tilde\rho-\eta\|_2\;\|\mathcal{P}v\cdot\nabla\mathcal{P}v\|_q\;\|\mathcal{P}v-u\|_{q^*};$$

hence, arguing exactly as above,

$$\left|\int_0^T\!\!\int_{\mathbb{T}^2}(\tilde\rho-\eta)(\mathcal{P}v\cdot\nabla\mathcal{P}v)\cdot(\mathcal{P}v-u)\,dx\,dt\right|\\ \leq\frac18\int_0^T\|\nabla(\mathcal{P}v-u)\|_2^2\,dt+C_{0,T}\int_0^T\|\mathcal{P}v-u\|_2^2\,dt.$$

Similarly, we have

$$\left|\int_0^T\!\!\int_{\mathbb{T}^2}\eta((\mathcal{P}v-u)\cdot\nabla\mathcal{P}v)\cdot(\mathcal{P}v-u)\,dx\,dt\right|\leq\rho^*\int_0^T\|\nabla\mathcal{P}v\|_\infty\;\|\mathcal{P}v-u\|_2^2\,dt.$$

Regarding K_1, we have, defining $\tilde q$ by $2/q+1/\tilde q=1$,

$$\begin{aligned}\left|\int_0^T\!\!\int_{\mathbb{T}^2}r\mathcal{P}v_t\cdot(\mathcal{P}v-u)\,dx\,dt\right|&\leq\int_0^T\|r\|_q\;\|\mathcal{P}v\|_q\;\|\mathcal{P}v-u\|_{\tilde q}\,dt\\&\leq\frac18\int_0^T\|\nabla(\mathcal{P}v-u)\|_2^2\,dt+C_{0,T}\int_0^T\|\mathcal{P}v-u\|_2^2\,dt,\end{aligned}$$

and for K_2, one can write that

$$\int_{\mathbb{T}^2}\rho\mathcal{Q}v_t\cdot(\mathcal{P}v-u)\,dx=\frac{d}{dt}\int_{\mathbb{T}^2}\rho\mathcal{Q}v\cdot(\mathcal{P}v-u)\,dx-\int_{\mathbb{T}^2}(\rho(\mathcal{P}v-u))_t\cdot\mathcal{Q}v\,dx.$$

For the last term, we have, using that $\rho_t=-\operatorname{div}(\rho v)$ and integrating by parts,

$$\begin{aligned}\int_{\mathbb{T}^2}(\rho(\mathcal{P}v-u))_t\cdot\mathcal{Q}v\,dx&=\int_{\mathbb{T}^2}\rho(\mathcal{P}v-u)_t\cdot\mathcal{Q}v\,dx+\int_{\mathbb{T}^2}\rho_t(\mathcal{P}v-u)\cdot\mathcal{Q}v\,dx\\&=\int_{\mathbb{T}^2}\rho(\mathcal{P}v-u)_t\cdot\mathcal{Q}v\,dx+\int_{\mathbb{T}^2}(\rho v)\cdot(\nabla(\mathcal{P}v-u)\cdot\mathcal{Q}v)\,dx\\&\qquad+\int_{\mathbb{T}^2}(\rho v)\cdot((\mathcal{P}v-u)\cdot\nabla\mathcal{Q}v)\,dx.\end{aligned}$$

The first term is of order ν^{-1} after time integration on $[0,T]$, since it may be bounded by

$$\left|\int_{\mathbb{T}^2}(\rho(\mathcal{P}v-u))_t\cdot\mathcal{Q}v\,dx\right|\leq\rho^*\|\mathcal{Q}v\|_2(\|\mathcal{P}v_t\|_2+\|u_t\|_2).$$

For the second term, one may write

$$\left|\int_{\mathbb{T}^2}(\rho v)\cdot(\nabla(\mathcal{P}v-u)\cdot\mathcal{Q}v)\,dx\right|\leq\frac18\int_{\mathbb{T}^2}\|\nabla(\mathcal{P}v-u)\|_2^2\,dx+C(\rho^*)^2\|v\|_\infty^2\;\|\mathcal{Q}v\|_2^2,$$

and for the last one, we have

$$\left|\int_{\mathbb{T}^2}(\rho v)\cdot((\mathcal{P}v-u)\cdot\nabla\mathcal{Q}v)\,dx\right|\le\rho^*\|v\|_\infty\;\|\mathcal{P}v-u\|_2\;\|\nabla\mathcal{Q}v\|_2.$$

In the same way, we get

$$\left|\int_0^T\!\!\int_{\mathbb{T}^2}(K_3+K_4)\cdot(\mathcal{P}v-u)\,dx\,dt\right| \le\int_0^T\|\mathcal{P}v-u\|_{q^*}\big(\|\mathcal{Q}v\|_q\;\|\nabla\mathcal{P}v\|_2+\|\nabla\mathcal{Q}v\|_q\;\|v\|_2\big)\,dt,$$

whence using (1-12) and the Poincaré inequality to handle the terms with $\mathcal{Q}v$,

$$\left|\int_0^T\!\!\int_{\mathbb{T}^2}(K_3+K_4)\cdot(\mathcal{P}v-u)\,dx\,dt\right|\le\frac{1}{8}\int_0^T\|\mathcal{P}v-u\|_{H^1}^2\,dt+\nu^{-2}C_{0,T}.$$

Summing up, we return to (6-7) and integrate to find

$$\rho_*\sup_{t\le T}\|(\mathcal{P}v-u)(t)\|_2^2+\int_0^T\|\nabla(\mathcal{P}v-u)\|_2^2\,dt \le\sup_{t\le T}\left|\int_{\mathbb{T}^2}(\rho\mathcal{Q}v)(t)\cdot(\mathcal{P}v-u)(t)\,dx\right|+C_{0,T}\int_0^T\|\mathcal{P}v-u\|_2^2\,dt+C_{0,T}\nu^{-1}.$$

But we see that

$$\left|\int_{\mathbb{T}^2}\rho\mathcal{Q}v.(\mathcal{P}v-u)\,dx\right|\le\tfrac{1}{2}\rho_*\|\mathcal{P}v-u\|_2^2+C\|\mathcal{Q}v\|_2^2\le\tfrac{1}{2}\rho_*\|\mathcal{P}v-u\|_2^2+C_{0,T}\nu^{-1}.$$

So altogether, we get after using the Gronwall lemma,

$$\sup_{t\le T}\big(\|(\mathcal{P}v-u)(t)\|_2^2+\|(\tilde\rho-\eta)(t)\|_2^2\big)+\int_0^T\|\nabla(\mathcal{P}v-u)\|_2^2\,dt\le C_{0,T}\nu^{-1}.$$

Recalling (6-4) and $\tilde\rho-\eta=r+(\tilde\rho-\eta)$ completes the proof of Theorem 1.3. □

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