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# Partial resolution by toroidal blow-ups

János Kollár

We give an alternate proof of a theorem of Tevelev about improving a nontoroidal ideal sheaf by a sequence of toroidal blow-ups.

**1.** ***Toroidal blow-up.*** Let $X$ be a smooth variety over a field and $\sum D_i$ a simple normal crossing (abbreviated as *snc*) divisor on $X$. A (closed) *stratum* of $(X, \sum D_i)$ is an irreducible component of an intersection $D_{i_1} \cap \cdots \cap D_{i_r}$. If $Z \subset X$ is a stratum (or a disjoint union of strata) and $\pi : B_Z X \to X$ the blow-up then $\bigl(B_Z X, \sum_i \pi_*^{-1} D_i + \sum_j E_j\bigr)$ is also an snc pair where the $E_j$ are the exceptional divisors of $\pi$. We call such blow-ups *toroidal.*

The following question was suggested by Keel.

**Question 2.** Let $(X, \sum D_i)$ be an snc pair over a field and $J \subset \mathcal{O}_X$ an ideal sheaf. How much can one improve $J$ by a sequence of toroidal blow-ups?

As a simple example, assume that $X$ is a surface. Then there are very few toroidal blow-ups: we can blow up either the curves $D_i \subset X$ (giving the identity map) or any of their intersection points. Thus if the cosupport of $J$ (that is, the support of $\mathcal{O}_X / J$) does not contain any strata then toroidal blow-ups have no effect on $J$. Similarly, one expects to be able to improve the singularities of $J$ along strata but not necessarily along other subvarieties. This leads to the following.

**Definition 3.** Let $(X, \Delta := \sum D_i)$ be an snc pair over a field and $J \subset \mathcal{O}_X$ an ideal sheaf. We say that $J$ is *toroidally resolved* if its cosupport does not contain any strata.

The key step of the proof is to show that each ideal sheaf $J \subset \mathcal{O}_X$ has a unique *toroidal hull* $J \subset J^t \subset \mathcal{O}_X$ such that the toroidal resolution problem for $J$ is equivalent to the ordinary resolution problem for $J^t$; see Definition–Theorem 17 and Proposition 20. The resolution of toroidal ideals is known over arbitrary fields by [Bierstone and Milman 2006], thus we get the following answer to Question 2.

*MSC2010:* 14E30, 14EXX.
*Keywords:* resolution, toroidal variety.

**Theorem 4.** *Let $(X, \Delta)$ be an snc pair over a field (of arbitrary characteristic) and $J \subset \mathcal{O}_X$ an ideal sheaf. Then there is a toroidal blow-up sequence*

$$(X_n, \Delta_n, J_n) \to \cdots \to (X_0, \Delta_0, J_0) := (X, \Delta, J)$$

*such that $J_n \subset \mathcal{O}_{X_n}$ is toroidally resolved.*

We state a more precise version in Theorem 10 and also explain how the ideals $J_i$ transform into each other, but first we apply Theorem 4 to the ideal sheaf of a divisor to get the following answer to the original question of Keel.

Tevelev pointed out that, using [De Concini and Procesi 1985], the methods of [Tevelev 2007] can easily be modified to obtain Corollary 5; see also [Popescu-Pampu 2004; Hacking 2008; Ulirsch 2015; Vogiannou 2015] for closely related variants. In fact, [Tevelev 2007] gives the stronger result that $\Pi_*^{-1} Y$ intersects each stratum in the expected codimension.

**Corollary 5** [Tevelev 2007]. *Let $(X, \Delta)$ be an snc pair over a field and $Y \subset X$ a closed subscheme that does not contain any of the irreducible components of $\Delta$. Then there is a sequence of toroidal blow-ups $\Pi : X_n \to \cdots \to X_0 := X$ such that the birational transform $\Pi_*^{-1} Y$ does not contain any strata of the pair*

$$\big(X_n, \Pi_*^{-1}\Delta + \operatorname{Ex}(\Pi)\big). \qquad \square$$

For another application, note that a divisor $B$ does not contain any strata of $(X, \Delta)$ if and only if $(X, \Delta + \epsilon B)$ is divisorial log terminal (abbreviated as *dlt*) for $0 < \epsilon \ll 1$, cf. [Kollár 2013, 2.8]. We can thus restate the divisorial case of Corollary 5 as follows.

**Corollary 6.** *Let $(X, \Delta)$ be an snc pair over a field and $B \subset X$ an effective divisor that does not contain any of the irreducible components of $\Delta$. Then there is a sequence of toroidal blow-ups $\Pi : X_n \to \cdots \to X_0 := X$ such that*

$$\big(X_n, \Pi_*^{-1}(\Delta + \epsilon B) + \operatorname{Ex}(\Pi)\big) \quad \text{is dlt for } 0 < \epsilon \ll 1. \qquad \square$$

The model obtained in Corollary 6 is related to the construction in [Odaka and Xu 2012] of dlt modifications of $(X, \Delta + \epsilon B)$ (in characteristic 0). Our models are smooth but the log canonical class need not be relatively nef. Nonetheless, this suggests that Corollary 6 might be approached using the minimal model program. A problem is that there are many different dlt modifications and most of them are singular. It is not clear to me how to guarantee smoothness using MMP.

**7.** ***Plan of the proof of Theorem 4.*** Assume for simplicity that $(X, \Delta)$ is toric with torus $T$. We assume that $\Delta$ consists of all $T$-invariant divisors. We show that Theorem 4 for $J$ is essentially equivalent to a special case of resolution, usually called monomialization, of the toric ideal $J^t := \sum_\tau \tau^* J$, where we sum over all $\tau \in T$. This is a combinatorial problem that is independent of the characteristic.

Note also that, at least in characteristic zero, $J^t$ is the ideal generated by logarithmic derivatives of all orders of elements of $J$; see Paragraph 14.4 and Proposition 15 for details.

In general, $(X, \Delta)$ is locally toric in the analytic or étale topology so we need to check that the local construction of $J^t$ gives a global ideal sheaf $J^t$. This is probably well known to experts. I do not know a reference that covers everything that we need, so we go through the details.

In the precise version of Theorem 4 we further restrict the blow-ups allowed in the sequence. For this we need some definitions first.

**8.** ***Toroidally equimultiple blow-ups.*** Let $X$ be a smooth variety and $J \subset \mathcal{O}_X$ an ideal sheaf. Let $Z \subset X$ be a smooth subvariety and $\pi : B_Z X \to X$ the blow-up of $Z$. Let $E \subset B_Z X$ denote the exceptional divisor.

Most resolution methods work with blow-up centers $Z \subset X$ such that $J$ is equimultiple along $Z$; that is, $\operatorname{mult}_z J = m$ for every $z \in Z$ for some fixed $m$. We then define the *birational transform* of $J$ by

$$\pi_*^{-1} J := \mathcal{O}_{B_Z X}(mE) \cdot \pi^* J. \tag{8-1}$$

(This is frequently called the "controlled" or "weak" transform.) This is an ideal sheaf on $B_Z X$. It has the pleasant property that $\operatorname{mult}_y \pi_*^{-1} J \le m$ for every $y \in E$.

Working toroidally, we would like $Z$ to be a stratum (or a disjoint union of strata). However, if the multiplicity of $J$ jumps at a single point that is not a stratum, then toroidal blow-ups are unlikely to change this. Thus, in a resolution procedure, the best one can hope for is that $J$ is *toroidally equimultiple* along $Z$, that is, $\operatorname{mult}_W J = \operatorname{mult}_Z J = m$ for every stratum $W \subset Z$ for some fixed $m$.

If this holds then we define the *birational transform* of $J$ by

$$\pi_*^{-1} J := \mathcal{O}_{B_Z X}(mE) \cdot \pi^* J. \tag{8-2}$$

As before, this is an ideal sheaf on $B_Z X$ and $\operatorname{mult}_V \pi_*^{-1} J \le m$ for every stratum $V \subset E$.

The resulting birational transform of $J$ then behaves as expected over generic points of strata $W \subset Z$ but can be rather badly behaved elsewhere. This is not a problem if we care only about generic points of strata.

Let us recall a somewhat detailed form of resolution (usually called monomialization) of ideal sheaves, as stated in [Kollár 2007, 3.68].

**Theorem 9.** *Let $(X, E)$ be an snc pair over a field of characteristic 0 and $J \subset \mathcal{O}_X$ an ideal sheaf. Then there is a blow-up sequence*

$$(X_n, J_n, E_n) \to \cdots \to (X_0, J_0, E_0) := (X, J, E)$$

*with the following properties*:

(1) *Each* $\pi_i : X_{i+1} \to X_i$ *is a blow-up with smooth center* $Z_i \subset X_i$ *and exceptional divisor* $E^{i+1}$.

(2) $J_i$ *is equimultiple along* $Z_i$.

(3) $J_{i+1} = (\pi_i)_*^{-1} J_i$ *as in* (8-1).

(4) $Z_i$ *has normal crossings with* $E_i$ *and* $E_{i+1} = (\pi_i)_*^{-1} E_i + E^{i+1}$.

(5) $(X_n, J_n, E_n)$ *is resolved*; *that is*, $J_n = \mathcal{O}_{X_n}$.

Now we can state the more precise form of Theorem 4 where we just add "toroidal" to the formulation of Theorem 9 in a few places.

**Theorem 10.** *Let* $(X, \Delta)$ *be an snc pair over a field of any characteristic and* $J \subset \mathcal{O}_X$ *an ideal sheaf. Then there is a toroidal blow-up sequence*

$$(X_n, \Delta_n, J_n) \to \cdots \to (X_0, \Delta_0, J_0) := (X, \Delta, J)$$

*with the following properties*:

(1) *Each* $\pi_i : X_{i+1} \to X_i$ *is a blow-up with smooth*, *toroidal center* $Z_i \subset X_i$ *and exceptional divisor* $E_{i+1}$.

(2) $J_i$ *is toroidally equimultiple along* $Z_i$.

(3) $J_{i+1} = (\pi_i)_*^{-1} J_i$ *as in* (8-2).

(4) $\Delta_{i+1} = (\pi_i)_*^{-1} \Delta_i + E_{i+1}$.

(5) $(X_n, \Delta_n, J_n)$ *is toroidally resolved.*

**Remark 11.** The role of the divisors $E$ and $\Delta$ is quite different in the two Theorems; the notation is changed to emphasize this. In Theorem 9 $E$ is but an auxiliary datum which gives very mild restrictions on the blow-up centers, whereas in Theorem 4 $\Delta$ gives extremely strong restrictions on the blow-up centers.

**Definition 12.** We call a blow-up sequence satisfying Theorem 9(1)–(4) *equimultiple* and a blow-up sequence satisfying Theorem 10(1)–(4) *toroidally equimultiple.*

Thus Theorem 9 says that, in characteristic 0, every ideal sheaf can be resolved by an equimultiple blow-up sequence.

**13.** ***Toroidal ideals.*** Let $X$ be a smooth variety and $\sum D_i$ an snc divisor. An ideal sheaf $I \subset \mathcal{O}_X$ is *toroidal* if $X$ is covered by open sets $U_j$ such that

$$I|_{U_j} = \sum_s \mathcal{O}_{U_j}\bigl(-\textstyle\sum_i m_{ijs} D_i|_{U_j}\bigr) \qquad (13\text{-}1)$$

for every $j$ and for suitable $m_{ijs} \in \mathbb{N}$.

Let $Z \subset X$ be a closed stratum and $Z^0 := Z \setminus \cup\{W : W \subsetneq Z \text{ is a stratum}\}$ the corresponding *open stratum*. For every $z \in Z^0 \cap U_j$ the $m_{ijs}$ give vectors

$$v_{js} := \bigl(m_{ijs} : D_i \supset Z\bigr) \in \sum_{i : D_i \supset Z} \mathbb{N}[D_i] \tag{13-2}$$

and these generate a subsemigroup

$$M_Z \subset \sum_{i : D_i \supset Z} \mathbb{N}[D_i] \tag{13-3}$$

which depends only on $Z$. For any inclusion of strata $W \subset Z$ we have the coordinate projection

$$p_{Z,W} : \sum_{i : D_i \supset W} \mathbb{N}[D_i] \to \sum_{i : D_i \supset Z} \mathbb{N}[D_i] \tag{13-4}$$

and the subsemigoups $M_Z$ satisfy the compatibility relation

$$p_{Z,W}(M_W) = M_Z. \tag{13-5}$$

This gives a one-to-one correspondence between toroidal ideals and collections of subsemigroups $\{M_Z\}$ satisfying the compatibility relations (13-5). In particular, we see that $I \mapsto I^{\mathrm{an}}$ gives a one-to-one correspondence

$$\{\text{toroidal ideals } I \subset \mathcal{O}_X\} \leftrightarrow \{\text{toroidal ideals } I^{\mathrm{an}} \subset \mathcal{O}_X^{\mathrm{an}}\}. \tag{13-6}$$

We claim that toroidal ideals are the only ones that can be "canonically" associated to the stratification of an snc pair.

**14.** ***Local stratified isomorphisms.*** Let $(X, \Delta)$ be an snc pair and $U_1, U_2 \subset X$ open sets. An isomorphism $\phi : U_1 \to U_2$ is called *stratification preserving* if $Z \cap U_1 = \phi^{-1}(Z \cap U_2)$ for every stratum $Z \subset X$. Note that our strata are the irreducible components of the intersections of the $D_i$, thus this is stronger than just assuming $D_i \cap U_1 = \phi^{-1}(D_i \cap U_2)$ for every $D_i$.

We say that an ideal sheaf $I \subset \mathcal{O}_X$ is invariant under stratification preserving local isomorphisms if $\phi^*\bigl(I|_{U_2}\bigr) = I|_{U_1}$ holds for every such $\phi : U_1 \to U_2$.

It is clear that a toroidal ideal is invariant under stratification preserving local isomorphisms and we would like to claim the converse. Unfortunately, if $X$ has no birational automorphisms then the identity map is the only stratification preserving local isomorphism. As usual, there are three ways to get more $U_i$.

**14.1.** *Complex analytic.* If $X$ is over $\mathbb{C}$, we use analytic open sets $U_1, U_2 \subset X^{\mathrm{an}}$.

**14.2.** *Étale local.* We use étale morphisms $\tau_i : U \to X$ and require that $\tau_1^{-1}(Z) = \tau_2^{-1}(Z)$ for every stratum $Z \subset X$.

**14.3.** *Formal local.* We use isomorphisms of complete local rings

$$\phi^* : \hat{\mathcal{O}}_{x_2,X} \to \hat{\mathcal{O}}_{x_1,X}.$$

(If the base field is not algebraically closed we also allow residue field extensions.)

**14.4.** *Micro local.* We assume the condition on the tangent space level. That is

$$\mathrm{Der}_X(-\log \Delta) \cdot I \subset I,$$

where $\mathrm{Der}_X(-\log \Delta)$ is the sheaf of logarithmic derivatives along $\Delta$; cf. [Kollár 2013, 3.87]. This works in characteristic 0 but not in positive characteristic. This shows that the concepts of toroidal ideal and toroidal hull (Definition–Theorem 17) are related to D-balanced ideals and well-tuned ideals used in resolution. See [Kollár 2007, Section 3.4] for the latter notions.

**Proposition 15.** *Let* $(X, \Delta)$ *be an snc pair and* $I \subset \mathcal{O}_X$ *an ideal sheaf that is invariant under all stratification preserving local isomorphisms in any of the settings of Paragraphs 14.1–14.3. Then* $I$ *is a toroidal ideal sheaf.*

*Proof.* We explain the complex analytic case and leave the details of the other settings to the reader. By (13-6) it is enough to show that $I^{\mathrm{an}}$ is toroidal.

Let $\mathbb{D} \subset \mathbb{C}$ denote the unit disc and $\mathbb{D}^*$ the punctured unit disc. We will view $\mathbb{D}^* \subset \mathbb{C}^*$ as a semigroup.

Let $Z^0 \subset X$ be an open stratum. After reindexing the $D_i$, for every $z \in Z^0$ we can choose a neighborhood of the form $(0 \in \mathbb{D}^n)$, where $D_i = (x_i = 0)$ for $i = 1, \dots, m$. We start with the natural $(\mathbb{D}^*)^m$ action on the first $m$ coordinates. This is a stratification preserving action.

Pick any $f = \sum_{i_1,\dots,i_m} f_{i_1,\dots,i_m}(x_{m+1}, \dots, x_n) \cdot x_1^{i_1} \cdots x_m^{i_m} \in I^{\mathrm{an}}$. Then

$$\tau^* f = \sum_{i_1,\dots,i_m} \chi_{i_1,\dots,i_m} \cdot f_{i_1,\dots,i_m}(x_{m+1}, \dots, x_n) \cdot x_1^{i_1} \cdots x_m^{i_m},$$

where $\chi_{i_1,\dots,i_m} : (\mathbb{D}^*)^m \to \mathbb{D}^*$ denotes the character $\lambda_1^{i_1} \cdots \lambda_m^{i_m}$. Since the characters of a group (in this case $(\mathbb{C}^*)^m$) are linearly independent we see that

$$f_{i_1,\dots,i_m}(x_{m+1}, \dots, x_n) \cdot x_1^{i_1} \cdots x_m^{i_m} \in I^{\mathrm{an}} + (x_1, \dots, x_m)^N$$

holds for every $N$. By Krull's intersection theorem this implies that

$$f_{i_1,\dots,i_m}(x_{m+1}, \dots, x_n) \cdot x_1^{i_1} \cdots x_m^{i_m} \in I^{\mathrm{an}}.$$

We next use translations by $(c_{m+1}, \dots, c_m)$ in the $x_{m+1}, \dots, x_n$ directions to achieve that $f_{i_1,\dots,i_m}(x_{m+1} + c_{m+1}, \dots, x_n + c_n)$ is nonzero at $(x_{m+1}, \dots, x_n) = (0, \dots, 0)$. Thus

$$x_1^{i_1} \cdots x_m^{i_m} \in I^{\mathrm{an}} \quad \text{provided } f_{i_1,\dots,i_m}(x_{m+1}, \dots, x_n) \not\equiv 0.$$

This shows that $I^{\mathrm{an}}$ is generated by monomials in $x_1, \dots, x_m$; hence it is toroidal. □

Note that $(X, \Delta)$ is toric with torus $T$ then we need only the $T$-action in the above proof. Thus we have showed the following elementary observation.

**Corollary 16.** *Let $(X, \Delta)$ be a smooth toric variety. Then an ideal is toric if and only if it is toroidal.* □

Now we come to the key definition, the toroidal hull of an ideal. The existence of the toroidal hull is a quite elementary observation which is at least implicit in several papers. See, for instance, the notion of the Newton polygon [Kouchnirenko 1976] and its connections with resolutions [Teissier 2004] or the D-balanced and well-tuned ideals discussed in [Włodarczyk 2005]; see also [Kollár 2007, Section 3.4] for more details on the latter.

**Definition–Theorem 17.** Let $(X, \Delta)$ be an snc pair over a field and $J \subset \mathcal{O}_X$ an ideal sheaf. There is a unique, smallest toroidal ideal sheaf $J^t \supset J$, called the *toroidal hull* of $J$.

Furthermore, if $W \subset X$ is a stratum then $\operatorname{mult}_W J^t = \operatorname{mult}_W J$. (A stronger version of this property is established in Lemma 19.)

*Proof.* As we noted in Paragraph 13, specifying $J^t$ is equivalent to specifying the semigroups $M_Z$ (13-3) and the latter can be done working in an analytic or formal neighborhood of a point $p_0 \in Z^0$ of an open stratum.

Then the recipe of constructing $J^t$ follows from the proof of Proposition 15:

(∗) Take all $f = \sum_{i_1,\dots,i_m} f_{i_1,\dots,i_m}(x_{m+1}, \dots, x_n) x_1^{i_1} \cdots x_m^{i_m} \in J$ and add the monomial $x_1^{i_1} \cdots x_m^{i_m}$ to $J^t$ whenever $f_{i_1,\dots,i_m} \not\equiv 0$.

This also shows that we have not decreased the multiplicity along $Z^0$ since

$$\operatorname{mult}_{p_0} x_1^{i_1} \cdots x_m^{i_m} = \inf_{p \in Z^0} \operatorname{mult}_p \big( f_{i_1,\dots,i_m} \cdot x_1^{i_1} \cdots x_m^{i_m} \big) \geq \inf_{p \in Z^0} \operatorname{mult}_p f. \qquad \square$$

**Corollary 18.** *Let $(X, \Delta)$ be an snc pair and $J \subset \mathcal{O}_X$ an ideal sheaf. Then $J$ is toroidally resolved if and only if $J^t = \mathcal{O}_X$.* □

The following result says that the toroidal hull commutes with toroidal blow-ups along toroidally equimultiple centers.

**Lemma 19.** *Assume that $J$ is toroidally equimultiple along $Z$. Then*

$$(\pi_*^{-1} J)^t = \pi_*^{-1}(J^t).$$

*Proof.* The question is local on $X$ and we can even replace $X$ by its completion $\hat{X}_x$. Thus we may assume that $(X, \Delta)$ is toric with torus $T$ acting on $X$. Then

$$J^t = \sum_\tau \tau^* J,$$

where we sum of all $\tau \in T$. If $J$ is toroidally equimultiple along $Z$ with multiplicity $m$ then the same holds for every $\tau^* J$. Thus

$$\pi_*^{-1}(J^t) = \mathcal{O}_{B_Z X}(mE) \cdot \pi^*\big(\textstyle\sum_\tau \tau^* J\big) = \textstyle\sum_\tau \big(\mathcal{O}_{B_Z X}(mE) \cdot \tau^* \pi^* J\big) = \big(\pi_*^{-1} J\big)^t. \ \square$$

The following observations transforms the toroidal resolution problem for $J$ to the usual resolution problem for its toroidal hull. Thus the toroidal hull is a variant of the concept of *tuning an ideal* used in resolution; see [Kollár 2007, 3.54].

**Proposition 20.** *Let $(X, \Delta)$ be an snc pair over a field and $J \subset \mathcal{O}_X$ an ideal sheaf. There is a natural equivalence between the following sets*:

(1) *toroidally equimultiple blow-up sequences for $J$,*

(2) *toroidally equimultiple blow-up sequences for $J^t$,*

(3) *equimultiple blow-up sequences for $J^t$.*

*Proof.* Definition–Theorem 17 shows that $J$ is toroidally equimultiple along a stratum $Z$ if and only if $J^t$ is toroidally equimultiple along $Z$. A toroidal ideal is toroidally equimultiple along a stratum $Z$ if and only if it is equimultiple along $Z$. Thus in all three settings the blow-ups allowed at the first step are the same.

Lemma 19 guarantees that this holds for all subsequent steps by induction. $\square$

**21.** ***Resolution of toroidal ideals.*** It has been long known that resolution of toric ideal sheaves is a combinatorial question that is independent of the characteristic [Kempf et al. 1973; Ash et al. 1975; Cox 2000; González Pérez and Teissier 2002]. However, we need a resolution that is obtained by an equimultiple blow-up sequence. The original toric references that I could find do not claim this and the methods do not seem to be designed for this purpose.

Resolution of toric and toroidal varieties and ideals using equimultiple blow-up sequences is proved in [Bierstone and Milman 2006]; see also [Blanco 2012a; 2012b]. Note that our setting is quite a bit easier since for us all strata are smooth. (This is also the reason why we do not need to worry about imperfect fields.)

One should also note that for toroidal ideals an étale-local resolution procedure is automatically combinatorial. So, although this is not stated, the resolution method discussed in [Włodarczyk 2005; Kollár 2007, Chapter 3] is combinatorial. Thus it yields the required resolution procedure for toroidal ideals over any field.

**22.** ***Proof of Theorem 10.*** By Theorem 9 (in characteristic $= 0$) and Paragraph 21 (in characteristic $\neq 0$) there is an equimultiple blow-up sequence

$$(X_n, \Delta_n, (J^t)_n) \to \cdots \to (X_0, \Delta_0, (J^t)_0) := (X, \Delta, J^t)$$

that resolves $J^t$. By Proposition 20 the same sequence gives a toroidally equimultiple blow-up sequence for $J$:

$$(X_n, \Delta_n, J_n) \to \cdots \to (X_0, \Delta_0, J_0) := (X, \Delta, J).$$

By Lemma 19 we know that $(J_n)^t = (J^t)_n$ and the latter is $\mathcal{O}_{X_n}$ by assumption. Thus $J_n$ is toroidally resolved by Corollary 18.

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# Construction of a stable blowup solution with a prescribed behavior for a non-scaling-invariant semilinear heat equation

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We consider the semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u \ln^{\alpha}(u^2 + 2)$$

in the whole space $\mathbb{R}^n$, where $p > 1$ and $\alpha \in \mathbb{R}$. Unlike the standard case $\alpha = 0$, this equation is not scaling invariant. We construct for this equation a solution which blows up in finite time $T$ only at one blowup point $a$, according to the asymptotic dynamic

$$u(x, t) \sim \psi(t)\left(1 + \frac{(p-1)|x-a|^2}{4p(T-t)|\ln(T-t)|}\right)^{-1/(p-1)} \quad \text{as } t \to T,$$

where $\psi(t)$ is the unique positive solution of the ODE

$$\psi' = \psi^p \ln^{\alpha}(\psi^2 + 2), \qquad \lim_{t\to T} \psi(t) = +\infty.$$

The construction relies on the reduction of the problem to a finite-dimensional one and a topological argument based on the index theory to get the conclusion. By the interpretation of the parameters of the finite-dimensional problem in terms of the blowup time and the blowup point, we show the stability of the constructed solution with respect to perturbations in initial data. To our knowledge, this is the first successful construction for a genuinely non-scale-invariant PDE of a stable blowup solution with the derivation of the blowup profile. From this point of view, we consider our result as a breakthrough.

## 1. Introduction

We are interested in the semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + F(u), \\ u(0) = u_0 \in L^{\infty}(\mathbb{R}^n), \end{cases} \tag{1-1}$$

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where $u(t) : \mathbb{R}^n \to \mathbb{R}$, $\Delta$ stands for the Laplacian in $\mathbb{R}^n$ and

$$F(u) = |u|^{p-1} u \ln^{\alpha}(u^2 + 2), \quad p > 1, \ \alpha \in \mathbb{R}. \tag{1-2}$$

By standard results the model (1-1) is well-posed in $L^{\infty}(\mathbb{R}^n)$ thanks to a fixed-point argument. More precisely, there is a unique maximal solution on $[0, T)$, with $T \le +\infty$. If $T < +\infty$, then the solution of (1-1) may develop singularities in finite time $T$, in the sense that

$$\|u(t)\|_{L^\infty} \to +\infty \quad \text{as } t \to T.$$

In this case, $T$ is called the blowup time of $u$. Given $a \in \mathbb{R}^n$, we say that $a$ is a blowup point of $u$ if and only if there exists $(a_j, t_j) \to (a, T)$ as $j \to +\infty$ such that $|u(a_j, t_j)| \to +\infty$ as $j \to +\infty$.

In the special case $\alpha = 0$, (1-1) becomes the standard semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u. \tag{1-3}$$

This equation is invariant under the scaling transformation

$$u \mapsto u_\lambda(x, t) := \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t). \tag{1-4}$$

Extensive literature is devoted to (1-3) and no review can be exhaustive. Given our interest in the construction question with a prescribed blowup behavior, we only mention previous work in this direction.

Bricmont and Kupiainen [1994] showed the existence of a solution of (1-3) such that

$$\left\| (T-t)^{1/(p-1)} u\big(a + z\sqrt{(T-t)|\ln(T-t)|}, t\big) - \varphi_0(z) \right\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as } t \to T, \tag{1-5}$$

where

$$\varphi_0(z) = \left( p - 1 + \frac{(p-1)^2 z^2}{4p} \right)^{-1/(p-1)}$$

(note that Herrero and Velázquez [1992] proved the same result with a different method; note also that Bressan [1992] made a similar construction in the case of an exponential nonlinearity).

Later, Merle and Zaag [1997] (see also [Merle and Zaag 1996]) simplified the proof of [Bricmont and Kupiainen 1994] and proved the stability of the constructed solution satisfying the behavior (1-5). Their method relies on the linearization of the similarity variables version around the expected profile. In that setting, the linearized operator has two positive eigenvalues, a zero eigenvalue and then a negative spectrum. Then, they proceed in two steps:

- Reduction of an infinite-dimensional problem to finite-dimensional one: they show that controlling the similarity variable version around the profile reduces to the control of the components corresponding to the two positive eigenvalues.

- Then, they solve the finite-dimensional problem thanks to a topological argument based on index theory.

The method of [Merle and Zaag 1997] has proved to be successful in various situations, such as for the complex Ginzburg–Landau equation of [Masmoudi and Zaag 2008] (see also [Zaag 1998] for an earlier work) and for the case of a complex semilinear heat equation with no variational structure [Nouaili and Zaag 2015]. We also mention the work of Tayachi and Zaag [2015a; 2015b] and the work of Ghoul, Nguyen and Zaag [Ghoul et al. 2017a] dealing with a nonlinear heat equation with a double source depending on the solution and its gradient in a critical way. Ghoul, Nguyen and Zaag [Ghoul et al. 2016; 2017b] successfully adapted the method to construct a stable blowup solution for a nonvariational semilinear parabolic system.

In other words, the method of [Merle and Zaag 1997] has proved to be efficient even for the case of systems with nonvariational structure. However, all the previous examples enjoy a common scaling-invariant property like (1-4), which seemed at first to be a strong requirement for the method. In fact, this was proved to be untrue.

Ebde and Zaag [2011] were able to adapt the method to construct blowup solutions for the non-scaling-invariant equation

$$\partial_t u = \Delta u + |u|^{p-1}u + f(u, \nabla u), \tag{1-6}$$

where

$$|f(u, \nabla u)| \le C(1 + |u|^q + |\nabla u|^{q'}), \quad \text{with } q < p,\ q' < \frac{2p}{p+1}.$$

These conditions ensure that the perturbation $f(u, \nabla u)$ results in exponentially small coefficients in the similarity variables. Later, Nguyen and Zaag [2016] recorded a more spectacular achievement by addressing the case of stronger perturbation of (1-3), namely

$$\partial_t u = \Delta u + |u|^{p-1}u + \frac{\mu |u|^{p-1}u}{\ln^a(2 + u^2)}, \tag{1-7}$$

where $\mu \in \mathbb{R}$ and $a > 0$. When moving to the similarity variables, the perturbation turns out to have a polynomial decay. Hence, when $a > 0$ is small, we are almost in the case of a critical perturbation.

In both cases addressed in [Ebde and Zaag 2011; Nguyen and Zaag 2016], the equations are indeed non-scaling-invariant, which shows the robustness of the method. However, since both papers proceed by perturbations around the standard case (1-3), it is as if we are still in the scaling-invariant case.

In this paper, we aim at trying the approach on a genuinely non-scaling-invariant case, namely (1-1). This is our main result.

**Theorem 1.1** (blowup solutions for (1-1) with a prescribed behavior). *There exists an initial data $u_0 \in L^\infty(\mathbb{R}^n)$ such that the corresponding solution to (1-1) blows up in finite time $T = T(u_0) > 0$, only at the origin. Moreover, we have:*

(i) *For all $t \in [0, T)$, there exists a positive constant $C_0$ such that*

$$\left\| \psi^{-1}(t) u(x,t) - f_0\left( \frac{x}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C_0}{\sqrt{|\ln(T-t)|}}, \tag{1-8}$$

*where $\psi(t)$ is the unique positive solution of the ODE*

$$\psi'(t) = \psi^p(t) \ln^\alpha(\psi^2(t) + 2), \qquad \lim_{t \to T} \psi(t) = +\infty \tag{1-9}$$

(*see Lemma A.1 for the existence and uniqueness of $\psi$*), *and the profile $f_0$ is defined by*

$$f_0(z) = \left( 1 + \frac{(p-1)}{4p} |z|^2 \right)^{-1/(p-1)}. \tag{1-10}$$

(ii) *There exits $u^*(x) \in C^2(\mathbb{R}^n \backslash \{0\})$ such that $u(x,t) \to u^*(x)$ as $t \to T$ uniformly on compact sets of $\mathbb{R}^n \setminus \{0\}$, where*

$$u^*(x) \sim \left( \frac{(p-1)^2 |x|^2}{8p \big| \ln|x| \big|} \right)^{-1/(p-1)} \left( \frac{4 \big| \ln|x| \big|}{p-1} \right)^{-\alpha/(p-1)} \quad \text{as } x \to 0. \tag{1-11}$$

**Remark 1.2.** From (i), we see that $u(0,t) \sim \psi(t) \to +\infty$ as $t \to T$, which means that the solution blows up in finite time $T$ at $x = 0$. From (ii), we deduce that the solution blows up only at the origin.

**Remark 1.3.** Note that the behavior in (1-8) is almost the same as in the standard case $\alpha = 0$ treated in [Bricmont and Kupiainen 1994; Merle and Zaag 1997]. However, the final profile $u^*$ has a difference coming from the extra multiplication of the size $\big| \ln|x| \big|^{-\alpha/(p-1)}$, which shows that the nonlinear source in (1-1) has a strong effect on the dynamic of the solution in comparison with the standard case $\alpha = 0$.

**Remark 1.4.** Item (ii) is in fact a consequence of (1-8) and Lemma A.4. Therefore, the main goal of this paper is to construct for (1-1) a solution blowing up in finite time and satisfying the behavior (1-8).

**Remark 1.5.** By parabolic regularity, one can show that if the initial data $u_0 \in W^{2,\infty}(\mathbb{R}^n)$, then we have for $i = 0, 1, 2$,

$$\left\| \psi^{-1}(t) (T-t)^{i/2} \nabla_x^i u(x,t) - (T-t)^{i/2} \nabla_x^i f_0\left( \frac{x}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty} \leq \frac{C}{\sqrt{|\ln(T-t)|}},$$

where $f_0$ is defined by (1-10).

From the technique of [Merle 1992a], we can prove the following result.

**Corollary 1.6.** *For an arbitrary set of $m$ points $x_1, \dots, x_m$, there exists initial data $u_0$ such that the solution $u$ of (1-1) with initial data $u_0$ blows up exactly at $m$ points $x_1, \dots, x_m$. Moreover, the local behavior at each blowup point $x_i$ is also given by (1-8) by replacing $x$ by $x - x_i$.*

As a consequence of our technique, we prove the stability of the solution constructed in Theorem 1.1 under the perturbations of initial data. In particular, we have the following result.

**Theorem 1.7** (stability of the solution constructed in Theorem 1.1). *Consider $\hat{u}$, the solution constructed in Theorem 1.1 and denote by $\widehat{T}$ its blowup time. Then there exists $\mathcal{U}_0 \subset L^\infty(\mathbb{R}^n)$ a neighborhood of $\hat{u}(0)$ such that for all $u_0 \in \mathcal{U}_0$, (1-1) with the initial data $u_0$ has a unique solution $u(t)$ blowing up in finite time $T(u_0)$ at a single point $a(u_0)$. Moreover, the statements* (i) *and* (ii) *in Theorem 1.1 are satisfied by $u(x - a(u_0), t)$, and*

$$(T(u_0), a(u_0)) \to (\widehat{T}, 0) \quad \text{as } \|u_0 - \hat{u}_0\|_{L^\infty(\mathbb{R}^n)} \to 0. \tag{1-12}$$

**Remark 1.8.** We will not give the proof of Theorem 1.7 because the stability result follows from the reduction to a finite-dimensional case as in [Merle and Zaag 1997] with the same proof. Here we only prove the existence and refer to that paper for the stability.

## 2. Formulation of the problem

We first use the matched asymptotic technique to formally derive the behavior (1-8). Then, we give the formulation of the problem in order to justify the formal result.

**2A.** ***A formal approach.*** We follow the approach of [Tayachi and Zaag 2015b] to formally explain how to derive the asymptotic behavior (1-8). To do so, we introduce the following self-similarity variables

$$u(x,t) = \psi(t) w(y,s), \quad y = \frac{x}{\sqrt{T-t}}, \quad s = -\ln(T-t), \tag{2-1}$$

where $\psi(t)$ is the unique positive solution of (1-9) and $\psi(t) \to +\infty$ as $t \to T$. Then, we see from (1-1) that $w(y,s)$ solves the following equation: for all $(y,s) \in \mathbb{R}^n \times [-\ln T, +\infty)$

$$\partial_s w = \Delta w - \tfrac{1}{2} y \cdot \nabla w - h(s) w + h(s) |w|^{p-1} w \frac{\ln^\alpha(\psi_1^2 w^2 + 2)}{\ln^\alpha(\psi_1^2 + 2)}, \tag{2-2}$$

where

$$h(s) = e^{-s} \psi_1^{p-1}(s) \ln^\alpha(\psi_1^2(s) + 2), \tag{2-3}$$

$$\psi_1(s) = \psi(T - e^{-s}). \tag{2-4}$$

Note that $h(s)$ admits the following asymptotic behavior as $s \to +\infty$:

$$h(s) = \frac{1}{p-1}\left(1 - \frac{\alpha}{s} - \frac{\alpha^2 \ln s}{s^2}\right) + O\left(\frac{1}{s^2}\right); \tag{2-5}$$

see (ii) of Lemma A.5 for the proof of (2-5). From (2-1), we see that the study of the asymptotic behavior of $u(x,t)$ as $t \to T$ is equivalent to the study of the long-time behavior of $w(y,s)$ as $s \to +\infty$. In other words, the construction of the solution $u(x,t)$, which blows up in finite time $T$ and satisfies the behavior (1-8), reduces to the construction of a global solution $w(y,s)$ for (2-2) satisfying

$$0 < \epsilon_0 \le \limsup_{s\to+\infty} \|w(s)\|_{L^\infty(\mathbb{R}^n)} \le \frac{1}{\epsilon_0}, \quad \epsilon_0 > 0, \tag{2-6}$$

and

$$\left\| w(y,s) - \left(1 + \frac{(p-1)y^2}{4ps}\right)^{-1/(p-1)} \right\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as } s \to +\infty. \tag{2-7}$$

In the following, we will formally explain how to derive the behavior (2-7).

*Inner expansion.* We remark that $0, \pm 1$ are the trivial constant solutions to (2-2). Since we are looking for a nonzero solution, let us consider the case when $w \to 1$ as $s \to +\infty$. We now introduce

$$w = 1 + \bar{w}; \tag{2-8}$$

then from (2-2), we see that $\bar{w}$ satisfies

$$\partial_s \bar{w} = \mathcal{L}(\bar{w}) + N(\bar{w}, s), \tag{2-9}$$

where

$$\mathcal{L} = \Delta - \tfrac{1}{2} y.\nabla + \mathrm{Id}, \tag{2-10}$$

$$N(\bar{w},s) = h(s)|\bar{w}+1|^{p-1}(\bar{w}+1)\frac{\ln^\alpha(\psi_1^2(\bar{w}+1)^2+2)}{\ln^\alpha(\psi_1^2+2)} - h(s)(\bar{w}+1) - \bar{w}, \tag{2-11}$$

$\psi_1(s)$ is defined in (2-4) and $h(s)$ behaves as in (2-5). Note that $N$ admits the asymptotic behavior

$$N(\bar{w},s) = \frac{p\bar{w}^2}{2} + O\left(\frac{|\bar{w}|\ln s}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right) + O(|\bar{w}|^3) \quad \text{as } (\bar{w},s) \to (0,+\infty), \tag{2-12}$$

(see Lemma A.6 for the proof of this statement).

Since $\bar{w}(s) \to 0$ as $s \to +\infty$ and the nonlinear term $N$ is quadratic in $\bar{w}$, we see from (2-9) that the linear part will play the main role in the analysis of our solution. Let us recall some properties of $\mathcal{L}$. The linear operator $\mathcal{L}$ is self-adjoint in $L^2_\rho(\mathbb{R}^n)$, where $L^2_\rho$ is the weighted space associated with the weight $\rho$ defined by

$$\rho(y) = \frac{e^{-|y|^2/4}}{(4\pi)^{n/2}},$$

and

$$\operatorname{spec}(\mathcal{L}) = \left\{1 - \frac{m}{2} : m \in \mathbb{N}\right\}.$$

More precisely, we have:

- When $n = 1$, all the eigenvalues of $\mathcal{L}$ are simple and the eigenfunction corresponding to the eigenvalue $1 - m/2$ is the Hermite polynomial defined by

$$h_m(y) = \sum_{j=0}^{[m/2]} \frac{(-1)^j m!\, y^{m-2j}}{j!\,(m-2j)!}. \tag{2-13}$$

In particular, we have the orthogonality

$$\int_{\mathbb{R}} h_i h_j \rho\, dy = i!\, 2^i \delta_{i,j} \quad \text{for all } (i, j) \in \mathbb{N}^2.$$

- When $n \geq 2$, the eigenspace corresponding to the eigenvalue $1-m/2$ is defined as

$$\mathcal{E}_m = \left\{h_\beta = h_{\beta_1} \cdots h_{\beta_n} : \text{for all } \beta \in \mathbb{N}^n,\ |\beta| = m,\ |\beta| = \beta_1 + \cdots + \beta_n\right\}. \tag{2-14}$$

Since the set of the eigenfunctions of $\mathcal{L}$ is a basis of $L^2_\rho$, we can expand $\bar{w}$ in this basis as

$$\bar{w}(y, s) = \sum_{\beta \in \mathbb{N}^n} \bar{w}_\beta(s) h_\beta(y).$$

For simplicity, let us assume that $\bar{w}$ is radially symmetric in $y$. Since $h_\beta$ with $|\beta| \geq 3$ corresponds to negative eigenvalues of $\mathcal{L}$, we may consider the solution $\bar{w}$ taking the form

$$\bar{w} = \bar{w}_0 + \bar{w}_2(s)(|y|^2 - 2n), \tag{2-15}$$

where $|\bar{w}_0(s)|$ and $|\bar{w}_2(s)|$ go to 0 as $s \to +\infty$. Injecting (2-15) and (2-12) into (2-9), then projecting (2-9) on the eigenspace $\mathcal{E}_m$ with $m = 0$ and $m = 2$, we obtain

$$\begin{aligned}
\bar{w}_0' &= \bar{w}_0 + \frac{p}{2}(\bar{w}_0^2 + 8n\bar{w}_2^2) + O\left(\frac{(|\bar{w}_0| + |\bar{w}_2|)\ln s}{s^2}\right) + O\left(\frac{|\bar{w}_0|^2 + |\bar{w}_2|^2}{s}\right) \\
&\qquad + O(|\bar{w}_0|^3 + |\bar{w}_2|^3), \\
\bar{w}_2' &= 4p\bar{w}_2^2 + p\bar{w}_0\bar{w}_2 + O\left(\frac{(|\bar{w}_0| + |\bar{w}_2|)\ln s}{s^2}\right) + O\left(\frac{|\bar{w}_0|^2 + |\bar{w}_2|^2}{s}\right) \\
&\qquad + O(|\bar{w}_0|^3 + |\bar{w}_2|^3)
\end{aligned} \tag{2-16}$$

as $s \to +\infty$. We now assume that $|\bar{w}_0(s)| \ll |\bar{w}_2(s)|$ as $s \to +\infty$; then (2-17) becomes

$$\begin{aligned} \bar{w}_0' &= \bar{w}_0 + O(|\bar{w}_2|^2) + O\left(\frac{|\bar{w}_2| \ln s}{s^2}\right), \\ \bar{w}_2' &= 4p\bar{w}_2^2 + o(|\bar{w}_2|^2) + O\left(\frac{|\bar{w}_2| \ln s}{s^2}\right) \end{aligned} \tag{2-17}$$

as $s \to +\infty$. We consider the following cases:

Case 1: Either $|\bar{w}_2| = O((\ln s)/s^2)$ or $|\bar{w}_2| \ll (\ln s)/s$ as $s \to +\infty$. Then the second equation in (2-17) becomes

$$\bar{w}_2' = O\left(\frac{|\bar{w}_2| \ln s}{s^2}\right) \quad \text{as } s \to +\infty,$$

which yields

$$\ln|\bar{w}_2| = O\left(\frac{\ln s}{s}\right) \quad \text{as } s \to +\infty,$$

which contradicts the condition $\bar{w}_2(s) \to 0$ as $s \to +\infty$.

Case 2: $|\bar{w}_2| \gg (\ln s)/s^2$ as $s \to +\infty$. Then (2-17) becomes

$$\bar{w}_0' = \bar{w}_0 + O(|\bar{w}_2|^2), \quad \bar{w}_2' = 4p\bar{w}_2^2 + o(|\bar{w}_2|^2)$$

as $s \to +\infty$. This yields

$$\bar{w}_0 = O\left(\frac{1}{s^2}\right), \quad \bar{w}_2 = -\frac{1}{4ps} + o\left(\frac{1}{s}\right) \tag{2-18}$$

as $s \to +\infty$. Substituting (2-18) into (2-17) yields

$$\bar{w}_0' = O\left(\frac{1}{s^2}\right), \quad \bar{w}_2' = 4p\bar{w}_2^2 + O\left(\frac{\ln s}{s^3}\right)$$

as $s \to +\infty$, from which we improve the error for $\bar{w}_2$ as

$$\bar{w}_0 = O\left(\frac{1}{s^2}\right), \quad \bar{w}_2 = -\frac{1}{4ps} + O\left(\frac{\ln^2 s}{s^2}\right) \tag{2-19}$$

as $s \to +\infty$. Hence, from (2-8), (2-15) and (2-19), we derive

$$w(y,s) = 1 - \frac{y^2}{4ps} + \frac{n}{2ps} + O\left(\frac{\ln^2 s}{s^2}\right) \tag{2-20}$$

in $L^2_\rho(\mathbb{R}^n)$ as $s \to +\infty$. Note that the asymptotic expansion (2-20) also holds for all $|y| \le K$, where $K$ is an arbitrary positive number.

*Outer expansion.* The asymptotic behavior of (2-20) suggests that the blowup profile depends on the variable

$$z = \frac{y}{\sqrt{s}}.$$

From (2-20), let us search for a regular solution of (2-2) of the form

$$w(y,s) = \phi_0(z) + \frac{n}{2ps} + o\Big(\frac{1}{s}\Big) \quad \text{in } L^\infty_{\text{loc}} \text{ as } s \to +\infty, \tag{2-21}$$

where $\phi_0$ is a bounded, smooth function to be determined. From (2-20), we impose the condition

$$\phi_0(0) = 1. \tag{2-22}$$

Since $w(y,s)$ is supposed to be bounded, we obtain from Lemma A.7 that

$$\left| h(s)|w|^{p-1} w \frac{\ln^\alpha(\psi_1^2 w^2 + 2)}{\ln^\alpha(\psi_1^2+2)} - \frac{|w|^{p-1}w}{p-1}\right| = O\Big(\frac{1}{s}\Big).$$

Note also that

$$\left|\left|\phi_0(z) + O\Big(\frac{1}{s}\Big)\right|^{p-1}\Big(\phi_0(z) + O\Big(\frac{1}{s}\Big)\Big) - |\phi_0(z)|^{p-1}\phi_0(z)\right| = O\Big(\frac{1}{s}\Big).$$

Hence, injecting (2-21) into (2-2) and comparing terms of order $O(1/s^i)$ for $j = 0, 1, \dots,$ we derive the following equation for $j = 0$:

$$-\tfrac{1}{2} z \cdot \nabla\phi_0(z) - \frac{\phi_0(z)}{p-1} + \frac{|\phi_0|^{p-1}\phi_0(z)}{p-1} = 0 \quad \text{for all } z \in \mathbb{R}^n. \tag{2-23}$$

Solving (2-23) with condition (2-22), we obtain

$$\phi_0(z) = (1 + c_0|z|^2)^{-1/(p-1)} \tag{2-24}$$

for some constant $c_0 \geq 0$ (since we want $\phi_0$ to be bounded for all $z \in \mathbb{R}^n$). From (2-21), (2-24) and a Taylor expansion, we obtain

$$w(y,s) = 1 - \frac{c_0 y^2}{(p-1)s} + \frac{n}{2ps} + o\Big(\frac{1}{s}\Big) \quad \text{for all } |y| \leq K \text{ as } s \to +\infty.$$

From this and the asymptotic behavior (2-20), we find that

$$c_0 = \frac{p-1}{4p}.$$

In conclusion, we have just derived the asymptotic profile

$$w(y,s) \sim \varphi(y,s) \quad \text{as } s \to +\infty, \tag{2-25}$$

where

$$\varphi(y,s) = \Big(1 + \frac{(p-1)y^2}{4ps}\Big)^{-1/(p-1)} + \frac{n}{2ps}. \tag{2-26}$$

**2B.** ***Formulation of the problem.*** We now set up the problem in order to justify the formal approach presented in Section 2A. In particular, we give a formulation to prove item (i) of Theorem 1.1. We aim at constructing for (1-1) a solution blowing up in finite time $T$ only at the origin and satisfying the behavior (1-8). In the similarity variables (2-1), this is equivalent to the construction of a solution $w(y, s)$ for (2-2) defined for all $(y, s) \in \mathbb{R}^n \times [s_0, +\infty)$ and satisfying (2-7). The formal approach given in Section 2A, see (2-25), suggests linearizing $w$ around the profile function $\varphi$ defined by (2-26). Let us introduce

$$q(y, s) = w(y, s) - \varphi(y, s), \tag{2-27}$$

where $\varphi$ is defined by (2-26). From (2-2), we see that $q$ satisfies the equation

$$\partial_s q = \mathcal{L}q + Vq + B(q) + R(y, s) + D(q, s), \tag{2-28}$$

where $\mathcal{L}$ is the linear operator defined by (2-10) and

$$V = \frac{p}{p-1}(\varphi^{p-1} - 1), \tag{2-29}$$

$$B(q) = \frac{|q+\varphi|^{p-1}(q+\varphi) - \varphi^p - p\varphi^{p-1}q}{p-1}, \tag{2-30}$$

$$R(y, s) = \Delta\varphi - \tfrac{1}{2}y\cdot\nabla\varphi - \frac{\varphi}{p-1} + \frac{\varphi^p}{p-1} - \partial_s\varphi, \tag{2-31}$$

$$D(q, s) = (q+\varphi)\left(\left(h(s) - \frac{1}{p-1}\right)(|q+\varphi|^{p-1} - 1) + h(s)|q+\varphi|^{p-1}(q+\varphi)L(q+\varphi, s)\right), \tag{2-32}$$

$$L(v, s) = \frac{2\alpha\psi_1^2}{\ln(\psi_1^2+2)(\psi_1^2+2)}(v-1) + \frac{1}{\ln^\alpha(\psi_1^2+2)}\int_1^v f''(u)(v-u)\,du, \tag{2-33}$$

with $h$, $\psi_1(s)$ and $\varphi$ being defined by (2-3), (2-4) and (2-26) respectively, and

$$f(z) = \ln^\alpha(\psi_1^2 z^2 + 2), \quad z \in \mathbb{R}.$$

Hence, proving (1-8) now reduces to constructing for (2-28) a solution $q$ such that

$$\lim_{s\to+\infty} \|q(s)\|_{L^\infty} \to 0.$$

Since we construct for (2-28) a solution $q$ satisfying $\|q(s)\|_{L^\infty} \to 0$ as $s \to +\infty$, and since

$$|B(q)| \le C|q|^{\min(2,p)}, \quad \|R(s)\|_{L^\infty} + \|D(q, s)\|_{L^\infty} \le \frac{C}{s},$$

(see Lemmas A.8, A.9 and A.10 for these estimates), we see that the linear part of (2-28) will play an important role in the analysis of the solution. The spectral property of the linear operator $\mathcal{L}$ is studied in the previous section (see page 19), and the potential $V$ has the following properties:

(i) Perturbation effect of $\mathcal{L}$ inside the blowup region $\{|y| \le K\sqrt{s}\}$:

$$\|V(s)\|_{L^2_\rho} \to 0 \quad \text{as } s \to +\infty.$$

(ii) For each $\epsilon > 0$, there exist $K_\epsilon > 0$ and $s_\epsilon > 0$ such that

$$\sup_{y/\sqrt{s} \ge K_\epsilon,\, s \ge s_\epsilon} \left| V(y,s) + \frac{p}{p-1} \right| \le \epsilon.$$

Since 1 is the biggest eigenvalue of $\mathcal{L}$, the operator $\mathcal{L} + V$ behaves as one with a fully negative spectrum outside the blowup region $\{|y| \ge K\sqrt{s}\}$, which makes the control of the solution in this region easy.

Since the behavior of the potential $V$ is different inside and outside the blowup region, we will consider the dynamics of the solution for $|y| \le 2K\sqrt{s}$ and for $|y| \ge K\sqrt{s}$ separately for some $K$ to be fixed large. We introduce the function

$$\chi(y,s) = \chi_0\left(\frac{|y|}{K\sqrt{s}}\right), \tag{2-34}$$

where $\chi_0 \in C_0^\infty[0,+\infty)$, $\|\chi_0\|_{L^\infty} \le 1$ and

$$\chi_0(x) = \begin{cases} 1 & \text{for } x \le 1, \\ 0 & \text{for } x \ge 2, \end{cases}$$

and $K$ is a positive constant to be fixed large later. We now decompose $q$ as

$$q = \chi q + (1-\chi) q = q_b + q_e. \tag{2-35}$$

(Note that $\operatorname{supp}(q_b) \subset \{|y| \le 2K\sqrt{s}\}$ and $\operatorname{supp}(q_e) \subset \{|y| \ge K\sqrt{s}\}$). Since the eigenfunctions of $\mathcal{L}$ span the whole space $L^2_\rho$, let us write

$$q_b(y,s) = q_0(s) + q_1(s)\cdot y + \tfrac{1}{2} y^T \cdot q_2(s) \cdot y - \operatorname{tr}(q_2(s)) + q_-(y,s), \tag{2-36}$$

where $q_m(s) = (q_\beta(s))_{\beta \in \mathbb{N}^n, |\beta| = m}$ and

$$\text{for all } \beta \in \mathbb{N}^n, \quad q_\beta(s) = \int_{\mathbb{R}^n} q_b(y,s)\tilde{h}_\beta(y)\rho\, dy, \quad \tilde{h}_\beta = \frac{h_\beta}{\|h_\beta\|^2_{L^2_\beta}}, \tag{2-37}$$

and

$$q_-(y,s) = \sum_{\beta \in \mathbb{N}^n, |\beta| \ge 3} q_\beta(s) h_\beta(y). \tag{2-38}$$

In particular, we set $q_1 = (q_{1,i})_{1 \le i \le n}$ and $q_2(s)$ is an $n \times n$ symmetric matrix defined explicitly by

$$q_2(s) = \int q_b \mathcal{M}(y)\rho\, dy = (q_{2,i,j})_{1 \le i,j \le n}, \tag{2-39}$$

with

$$\mathcal{M} = \left\{\tfrac{1}{4} y_i y_j - \tfrac{1}{2}\delta_{i,j}\right\}_{1\le i,j\le n}. \tag{2-40}$$

Hence, by (2-35) and (2-36), we can write

$$q(y,s) = q_0(s) + q_1(s)\cdot y + \tfrac{1}{2} y^T \cdot q_2(s)\cdot y - \operatorname{tr}(q_2(s)) + q_-(y,s) + q_e(y,s). \tag{2-41}$$

Note that $q_m (m = 0, 1, 2)$ and $q_-$ are the components of $q_b$, and not those of $q$.

## 3. Proof of the existence, assuming some technical results

We shall now describe the main argument behind the proof of Theorem 1.1. To avoid winding up with too many details, we shall postpone most of the technicalities involved to the next section. According to the transformations (2-1) and (2-27), proving (i) of Theorem 1.1 is equivalent to showing that there exists an initial data $q_0(y)$ at the time $s_0$ such that the corresponding solution $q$ of (2-28) satisfies

$$\|q(s)\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as } s \to +\infty.$$

In particular, we consider the function

$$\psi_{d_0,d_1}(y) = \frac{A}{s_0^2}(d_0 + d_1\cdot y)\chi(2y, s_0) \tag{3-1}$$

as the initial data for (2-28), where $(d_0, d_1) \in \mathbb{R}^{1+n}$ are the parameters to be determined, $s_0 > 1$ and $A > 1$ are constants to be fixed large enough, and $\chi$ is the function defined by (2-34).

We aim to prove that there exists $(d_0, d_1) \in \mathbb{R}\times\mathbb{R}^n$ such that the solution $q(y,s) = q_{d_0,d_1}(y,s)$ of (2-28) with initial data $\psi_{d_0,d_1}(y)$ satisfies

$$\|q_{d_0,d_1}(s)\|_{L^\infty} \to 0 \quad \text{as } s \to +\infty.$$

More precisely, we will show that there exists $(d_0, d_1) \in \mathbb{R}\times\mathbb{R}^n$ such that the solution $q_{d_0,d_1}(y,s)$ belongs to the shrinking set $S_A$ defined as follows:

**Definition 3.1** (a shrinking set to zero). For all $A \ge 1$, $s \ge 1$ we define $S_A(s)$ to be the set of all functions $q \in L^\infty(\mathbb{R}^n)$ such that

$$|q_0| \le \frac{A}{s^2}, \quad |q_{1,i}| \le \frac{A}{s^2}, \quad |q_{2,i,j}| \le \frac{A^2\ln^2 s}{s^2} \quad \text{for all } 1 \le i, j \le n,$$

$$\left\|\frac{q_-(y)}{1+|y|^3}\right\|_{L^\infty(\mathbb{R}^n)} \le \frac{A}{s^2}, \quad \|q_e(y)\|_{L^\infty(\mathbb{R}^n)} \le \frac{A^2}{\sqrt{s}},$$

where $q_0$, $q_1 = (q_{1,i})_{1\le i\le n}$, $q_2 = (q_{2,i,j})_{1\le i,j\le n}$, $q_-$ and $q_e$ are defined as in (2-41).

We also denote by $\widehat{S}_A(s)$ the set

$$\widehat{S}_A(s) = \left[-\frac{A}{s^2}, \frac{A}{s^2}\right] \times \left[-\frac{A}{s^2}, \frac{A}{s^2}\right]^n. \tag{3-2}$$

**Remark 3.2.** For each $A \geq 1$, $s \geq 1$, we have the following estimates for all $q(s) \in S_A(s)$:

$$|q(y,s)| \leq \frac{CA^2 \ln^2 s}{s^2}(1+|y|^3) \quad \text{for all } y \in \mathbb{R}^n, \tag{3-3}$$

$$\|q(s)\|_{L^\infty(\{|y|\leq 2K\sqrt{s}\})} \leq \frac{CA}{\sqrt{s}}, \tag{3-4}$$

$$\|q(s)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{CA^2}{\sqrt{s}}. \tag{3-5}$$

We aim to prove the following central proposition, which implies Theorem 1.1.

**Proposition 3.3** (existence of a solution trapped in $S_A(s)$). *There exists $A_1 \geq 1$ such that for all $A \geq A_1$ there exists $s_1(A) \geq 1$ such that for all $s_0 \geq s_1(A)$, there exists $(d_0, d_1) \in \mathbb{R}^{1+n}$ such that the solution $q(y,s) = q_{d_0,d_1}(y,s)$ of (2-28) with the initial data at the time $s_0$ given by $q(y,s_0) = \psi_{d_0,d_1}(y)$, where $\psi_{d_0,d_1}$ is defined as in (3-1), satisfies*

$$q(s) \in S_A(s) \quad \textit{for all } s \in [s_0, +\infty).$$

From (3-5), we see that once Proposition 3.3 is proved, item (i) of Theorem 1.1 directly follows. In the following, we shall give all the main arguments for the proof of this proposition assuming some technical results which are left to the next section.

As for the initial data at time $s_0$ defined as in (3-1), we have the following properties.

**Proposition 3.4.** *For each $A \geq 1$, there exists $s_2(A) > 1$ such that for all $s_0 \geq s_2(A)$ we have the following*:

(i) *There exists*

$$\mathbb{D}_{A,s_0} \subset [-2;2] \times [-2;2]^n$$

*such that the mapping*

$$\Phi_1 : \mathbb{R}^{1+n} \to \mathbb{R}^{1+n},$$
$$(d_0, d_1) \mapsto (\psi_0, \psi_1),$$

*is linear and one-to-one from $\mathbb{D}_{A,s_0}$ onto $\widehat{S}_A(s_0)$. Moreover,*

$$\Phi_1(\partial \mathbb{D}_{A,s_0}) \subset \partial \widehat{S}_A(s_0).$$

(ii) *For all* $(d_0, d_1) \in \mathbb{D}_{A,s_0}$ *we have* $\psi_{d_0,d_1} \in S_A(s_0)$ *with strict inequalities in the sense that*

$$|\psi_0| \le \frac{A}{s_0^2}, \quad |\psi_{1,i}| \le \frac{A}{s_0^2}, \quad |\psi_{2,i,j}| < \frac{A\ln^2 s_0}{s_0^2} \quad \text{for all } 1 \le i, j \le n,$$

$$\left\| \frac{\psi_-}{1+|y|^3} \right\|_{L^\infty(\mathbb{R})} < \frac{A}{s_0^2}, \quad \psi_e \equiv 0.$$

*Above*, $\chi(y, s_0)$ *is defined in* (2-34), $\psi_0$, $(\psi_{1,i})_{1\le i\le n}$, $(\psi_{2,i,j})_{1\le i,j\le 2}$, $\psi_-$ *and* $\psi_e$ *are the components of* $\psi_{d_0,d_1}$ *defined as in* (2-41), *and* $\psi_{d_0,d_1}$ *and* $\widehat{S}_A(s)$ *are defined by* (3-1) *and* (3-2).

*Proof.* See Proposition 4.5 of [Tayachi and Zaag 2015b] for a similar proof. □

From now on, we denote by $C$ the universal constant which only depends on $K$, where $K$ is introduced in (2-34). Let us now give the proof of Proposition 3.3 to complete the proof of item (i) of Theorem 1.1.

*Proof of Proposition 3.3.* We proceed into two steps to prove Proposition 3.3:

- In the first step, we reduce the problem of controlling $q(s)$ in $S_A(s)$ to controlling $(q_0, q_1)(s)$ in $\widehat{S}_A(s)$, where $q_0$ and $q_1$ are the components of $q$ corresponding to the positive modes defined as in (2-41) and $\widehat{S}_A$ is defined by (3-2). This means that we reduce the problem to a finite-dimensional one.

- In the second step, we argue by contradiction to solve the finite-dimensional problem thanks to a topological argument.

*Step 1: reduction to a finite-dimensional problem.* In this step, we show through an a priori estimate that the control of $q(s)$ in $S_A(s)$ reduces to the control of $(q_0, q_1)(s)$ in $\widehat{S}_A(s)$. This mainly follows from a good understanding of the properties of the linear part $\mathcal{L} + V$ of (2-28). In particular, we claim the following which is the heart of our analysis.

**Proposition 3.5** (control of $q(s)$ in $S_A(s)$ by $(q_0, q_1)(s)$ in $\widehat{S}_A(s)$). *There exists* $A_3 \ge 1$ *such that for all* $A \ge A_3$, *there exists* $s_3(A) \ge 1$ *such that for all* $s_0 \ge s_3(A)$, *the following holds*:

*If* $q(y, s)$ *is the solution of* (2-28) *with the initial data at time* $s_0$ *given by* (3-1) *with* $(d_0, d_1) \in \mathbb{D}_{A,s_0}$, *and* $q(s) \in S_A(s)$ *for all* $s \in [s_0, s_1]$ *for some* $s_1 \ge s_0$ *and* $q(s_1) \in \partial S_A(s_1)$, *then*:

(i) *Reduction to a finite-dimensional problem*: *we have* $(q_0, q_1)(s_1) \in \partial\widehat{S}_A(s_1)$.

(ii) *Transverse outgoing crossing*: *there exists* $\delta_0 > 0$ *such that*

$$\text{for all } \delta \in (0, \delta_0), \quad (q_0, q_1)(s_1 + \delta) \notin \widehat{S}_A(s_1 + \delta);$$

*hence*, $q(s_1 + \delta) \notin S_A(s_1 + \delta)$, *where* $\widehat{S}_A$ *is defined in* (3-2) *and* $\mathbb{D}_{A,s_0}$ *is introduced in* Proposition 3.4.

Let us suppose for the moment that Proposition 3.5 holds. Then we can take advantage of a topological argument quite similar to that already used in [Merle and Zaag 1997].

*Step 2: a basic topological argument.* From Proposition 3.5, we claim that there exists $(d_0, d_1) \in \mathbb{D}_{A,s_0}$ such that (2-28) with initial data (3-1) has a solution

$$q_{d_0,d_1}(s) \in S_A(s) \quad \text{for all } s \in [s_0, +\infty),$$

for suitable choice of the parameters $A$, $K$, $s_0$. Since the argument is analogous to that in [Merle and Zaag 1997], we only give the main ideas.

Let us consider $s_0$, $K$ and $A$ such that Propositions 3.4 and 3.5 hold. From Proposition 3.4, we have

$$\text{for all } (d_0, d_1) \in \mathbb{D}_{A,s_0}, \quad q_{d_0,d_1}(y, s_0) := \psi_{d_0,d_1} \in S_A(s_0),$$

where $\psi_{d_0,d_1}$ is defined by (3-1). Since the initial data belongs to $L^\infty$, we then deduce from the local existence theory for the Cauchy problem of (1-1) in $L^\infty$ that we can define for each $(d_0, d_1) \in \mathbb{D}_{A,s_0}$ a maximum time $s_*(d_0, d_1) \in [s_0, +\infty)$ such that

$$q_{d_0,d_1}(s) \in S_A(s) \quad \text{for all } s \in [s_0, s_*).$$

If $s_*(d_0, d_1) = +\infty$ for some $(d_0, d_1) \in \mathbb{D}_{A,s_0}$, then we are done. Otherwise, we argue by contradiction and assume that $s_*(d_0, d_1) < +\infty$ for all $(d_0, d_1) \in \mathbb{D}_{A,s_0}$. By continuity and the definition of $s_*$, we deduce that $q_{d_0,d_1}(s_*)$ is on the boundary of $S_A(s_*)$. From item (i) of Proposition 3.5, we have

$$(q_0, q_1)(s_*) \in \partial \widehat{S}_A(s_*).$$

Hence, we may define the rescaled function

$$\Gamma : \mathbb{D}_{A,s_0} \to \partial([-1, 1]^{1+n}),$$
$$(d_0, d_1) \mapsto \frac{s_*^2}{A}(q_0, q_1)(s_*).$$

From item (i) of Proposition 3.4, we see that if $(d_0, d_1) \in \partial\mathbb{D}_{A,s_0}$, then

$$q(s_0) \in S_A(s_0), \quad (q_0, q_1)(s_0) \in \partial \widehat{S}_A(s_0).$$

From item (ii) of Proposition 3.5, we see that $q(s)$ must leave $S_A(s)$ at $s = s_0$, thus, $s_*(d_0, d_1) = s_0$. Therefore, the restriction of $\Gamma$to $\partial\mathbb{D}_{A,s_0}$ is homeomorphic to the identity mapping, which is impossible thanks to index theorem, and the contradiction is obtained. This concludes the proof of Proposition 3.3 as well as item (i) of Theorem 1.1, assuming that Proposition 3.5 holds. □

*Proof of* (ii) *of Theorem 1.1.* The existence of $u^*$ in $C^2(\mathbb{R}^n \setminus \{0\})$ follows from the technique of [Merle 1992b]. Here, we want to find an equivalent formation for $u^*$

near the blowup point $x = 0$. The case $\alpha = 0$ was treated in [Zaag 1998]. When $\alpha \neq 0$, we follow the method of that paper, and no new idea is needed. Therefore, we just sketch the main steps for the sake of completeness.

We consider $K_0 > 0$ a constant to be fixed large enough, and $|x_0| \neq 0$ small enough. Then, we introduce the function

$$v(x_0, \xi, \tau) = \psi^{-1}(t_0(x_0))u(x, t), \tag{3-6}$$

where

$$(\xi, \tau) \in \mathbb{R}^n \times \left[-\frac{t_0(x_0)}{T - t_0(x_0)}, 1\right),$$

and

$$(x, t) = \left(x_0 + \xi\sqrt{T - t_0(x_0)},\, t_0(x_0) + \tau(T - t_0(x_0))\right), \tag{3-7}$$

with $t_0(x_0)$ being uniquely determined by

$$|x_0| = K_0\sqrt{(T - t_0(x_0))|\ln(T - t_0(x_0))|}. \tag{3-8}$$

From (3-6)–(3-8) and (1-8) we derive that

$$\sup_{|\xi|<2|\ln(T-t_0(x_0))|^{1/4}} |v(x_0, \xi, 0) - \varphi_0(K_0)| \le \frac{C}{1+(|\ln(T-t_0(x_0))|^{1/4})} \to 0 \quad \text{as } x_0 \to 0,$$

where

$$\varphi_0(x) = \left(1 + \frac{(p-1)x^2}{4p}\right)^{1/(p-1)}.$$

As in [Zaag 1998], we use the continuity with respect to initial data for (1-1) associated to a space-localization in the ball $B(0, |\xi| < |\ln(T - t_0(x_0))|^{1/4})$ to derive

$$\sup_{|\xi|<|\ln(T-t_0(x_0))|^{1/4},\ \tau\in[0,1)} |v(x_0, \xi, \tau) - \hat{v}_{K_0}(\tau)| \le \epsilon(x_0) \to 0 \quad \text{as } x_0 \to 0, \tag{3-9}$$

where

$$\hat{v}_{K_0}(\tau) = \left((1-\tau) + \frac{(p-1)K_0^2}{4p}\right)^{-1/(p-1)}.$$

From (3-7) and (3-9), we deduce

$$\begin{aligned} u^*(x_0) &= \lim_{t\to T} u(x_0, t) \\ &= \psi(t_0(x_0)) \lim_{\tau\to 1} v(x_0, 0, \tau) \sim \psi(t_0(x_0))\left(\frac{(p-1)K_0}{4p}\right)^{-1/(p-1)}. \end{aligned} \tag{3-10}$$

Using the relation (3-8), we find that

$$T - t_0 \sim \frac{|x_0|^2}{2K_0\big|\ln|x_0|\big|} \quad \text{and} \quad \ln(T - t_0(x_0)) \sim 2\ln(|x_0|) \quad \text{as } x_0 \to 0. \tag{3-11}$$

The formula (1-11) then follows from Lemma A.1, (3-10) and (3-11). This concludes the proof of Theorem 1.1, assuming that Proposition 3.5 holds. □

## 4. Proof of Proposition 3.5

This section is devoted to the proof of Proposition 3.5, which is the heart of our analysis. We proceed into two parts. In the first part, we derive a priori estimates on $q(s)$ in $S_A(s)$. In the second part, we show that the new bounds are better than those defined in $S_A(s)$, except for the first two components $(q_0, q_1)$. This means that the problem is reduced to the control of a finite-dimensional function $(q_0, q_1)$, which is the conclusion of item (i) of Proposition 3.5. Item (ii) of Proposition 3.5 is just direct consequence of the dynamics of the modes $q_0$ and $q_1$.

**4A.** ***A priori estimates on $q(s)$ in $S_A(s)$.*** We derive the a priori estimates on the components $q_2, q_-, q_e$ which imply the conclusion of Proposition 3.5. Firstly, let us give some dynamics of $q_0, q_1 = (q_{1,i})_{1\le i\le n}$ and $q_2 = (q_{2,i,j})_{1\le i,j\le n}$. More precisely, we claim the following.

**Proposition 4.1** (dynamics of (2-28)). *There exists $A_4 \ge 1$ such that for all $A \ge A_4$ there exists $s_4(A) \ge 1$ such that the following holds for all $s_0 \ge s_4(A)$: Assume that for all $s \in [\tau, s_1]$ for some $s_1 \ge \tau \ge s_0$, we have $q(s) \in S_A(s)$. Then the following holds for all $s \in [s_0, s_1]$:*

(i) *ODE satisfied by the positive and null modes*:

$$m = 0, 1, \quad \left| q_m'(s) - \left(1 - \frac{m}{2}\right) q_m(s)\right| \le \frac{C}{s^2}, \tag{4-1}$$

*and*

$$\left| q_2'(s) + \frac{2}{s} q_2(s)\right| \le \frac{C \ln s}{s^3}. \tag{4-2}$$

(ii) *control of the negative and outer modes*:

$$\left\| \frac{q_-(\cdot, s)}{1+|y|^3} \right\|_{L^\infty} \le C e^{-(s-\tau)/2} \left\| \frac{q_-(\cdot, \tau)}{1+|y|^3} \right\|_{L^\infty} + C \frac{e^{-(s-\tau)^2}}{s^{3/2}} \|q_e(\tau)\|_{L^\infty} + C \frac{(1+s-\tau)}{s^2}, \tag{4-3}$$

$$\|q_e(s)\|_{L^\infty} \le C e^{-(s-\tau)/p} \|q_e(\tau)\|_{L^\infty} + C e^{s-\tau} s^{3/2} \left\| \frac{q_-(\cdot, \tau)}{1+|y|^3} \right\|_{L^\infty} + C \frac{1+(s-\tau)e^{s-\tau}}{s^{1/2}}. \tag{4-4}$$

*Proof.* We proceed in two parts:

- In the first part we project (2-28) to write ODEs satisfied by $q_m$ for $m = 0, 1, 2$.
- In the second part we use the integral form of (2-28) and the dynamics of the linear operator $\mathcal{L} + V$ to derive a priori estimates on $q_-$ and $q_e$.

*Part 1: ODEs satisfying the positive and null modes.* We give the proof of (4-2); the same proof holds for (4-1). By formula (2-39) and (2-28), we write for each

$1 \le i, j \le n$,

$$\left| q'_{2,i,j}(s) - \int \big[\mathcal{L}q + Vq + B(q) + R(y,s) + D(q,s)\big]\chi\left(\frac{y_i y_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho\,dy \right| \le Ce^{-s}. \tag{4-5}$$

Using the assumption $q(s) \in S_A(s)$ for all $s \in [s_0, s_1]$, we derive the following estimates for all $s \in [s_0, s_1]$:

$$\left| \int \mathcal{L}(q)\chi\left(\frac{y_i y_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho\,dy \right| \le \frac{C}{s^3},$$

and from Lemmas A.8, A.9 and A.10

$$\left| \int Vq\chi\left(\frac{y_i y_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho\,dy + \frac{2}{s}q_{2,i,j}(s) \right| \le \frac{CA}{s^3},$$

$$\left| \int B(q)\chi\left(\frac{y_i y_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho\,dy \right| \le \frac{C}{s^3},$$

$$\left| \int R\chi\left(\frac{y_i y_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho\,dy \right| \le \frac{C}{s^3},$$

$$\left| \int D(q,s)\chi\left(\frac{y_i y_j}{4} - \frac{\delta_{i,j}}{2}\right)\rho\,dy \right| \le \frac{C\ln s}{s^3}.$$

Gathering all these above estimates in (4-5) yields

$$\left| q'_{2,i,j} + \frac{2}{s}q_{2,i,j} \right| \le \frac{C\ln s}{s^3},$$

which concludes the proof of (4-2).

*Part 2: control of the negative and outer modes.* We give the proofs of (4-3) and (4-4) in this part. The control of $q_-$ and $q_e$ is mainly based on the dynamics of the linear operator $\mathcal{L} + V$. In particular, we use the following integral form of (2-28): for each $s \ge \sigma \ge s_0$,

$$q(s) = \mathcal{K}(s,\sigma)q(\sigma) + \int_\sigma^s \mathcal{K}(s,\tau)\big[B(q)(\tau) + R(\tau) + D(q,\tau)\big]\,d\tau = \sum_{i=1}^4 \vartheta_i(s,\sigma), \tag{4-6}$$

where $\{\mathcal{K}(s,\sigma)\}_{s\ge\sigma}$ is defined by

$$\begin{cases} \partial_s \mathcal{K}(s,\sigma) = (\mathcal{L} + V)\mathcal{K}(s,\sigma), & s > \sigma, \\ \mathcal{K}(\sigma,\sigma) = \mathrm{Id}, \end{cases} \tag{4-7}$$

and

$$\vartheta_1(s,\sigma) = \mathcal{K}(s,\sigma)q(\sigma), \qquad \vartheta_2(s,\sigma) = \int_\sigma^s \mathcal{K}(s,\tau)B(q)(\tau)\,d\tau,$$

$$\vartheta_3(s,\sigma) = \int_\sigma^s \mathcal{K}(s,\tau)R(\cdot,\tau)\,d\tau, \quad \vartheta_4(s,\sigma) = \int_\sigma^s \mathcal{K}(s,\tau)D(q,\tau)\,d\tau.$$

From (4-6), it is clear to see the strong influence of the kernel $\mathcal{K}$ in this formula. It is therefore convenient to recall the following result on the dynamics of the linear operator $\mathcal{K} = \mathcal{L} + V$.

**Lemma 4.2** (a priori estimates of the linearized operator in the decomposition in (2-41)). *For all $\rho^* \geq 0$, there exists $s_5(\rho^*) \geq 1$ such that if $\sigma \geq s_5(\rho^*)$ and $v \in L^2_\rho$ satisfies*

$$\sum_{m=0}^{2} |v_m| + \left\| \frac{v_-}{1+|y|^3} \right\|_{L^\infty} + \|v_e\|_{L^\infty} < \infty, \tag{4-8}$$

*then, for all $s \in [\sigma, \sigma + \rho^*]$ the function $\theta(s) = \mathcal{K}(s, \sigma) v$ satisfies*

$$\left\| \frac{\theta_-(y,s)}{1+|y|^3} \right\|_{L^\infty} \leq \frac{C e^{s-\sigma}((s-\sigma)^2+1)}{s} (|v_0| + |v_1| + \sqrt{s}|v_2|) + C e^{-(s-\sigma)/2} \left\| \frac{v_-}{1+|y|^3} \right\|_{L^\infty} + C \frac{e^{-(s-\sigma)^2}}{s^{3/2}} \|v_e\|_{L^\infty}, \tag{4-9}$$

*and*

$$\|\theta_e(y,s)\|_{L^\infty} \leq C e^{s-\sigma} \left( \sum_{l=0}^{2} s^{l/2} |v_l| + s^{3/2} \left\| \frac{v_-}{1+|y|^3} \right\|_{L^\infty} \right) + C e^{-(s-\sigma)/p} \|v_e\|_{L^\infty}. \tag{4-10}$$

*Proof.* The proof of this result was given in [Bricmont and Kupiainen 1994] in the one-dimensional case. It was then extended to higher-dimensional cases in [Nguyen and Zaag 2017]. We kindly refer interested readers to Lemma 2.9 in that paper for the details of the proof. □

In view of formula (4-6), we see that Lemma 4.2 plays an important role in deriving the new bounds on the components $q_-$ and $q_e$. Indeed, given bounds on the components of $q$, $B(q)$, $D(q)$ and $R$, we directly apply Lemma 4.2 with $\mathcal{K}(s, \sigma)$ replaced by $\mathcal{K}(s, \tau)$ and then integrate over $\tau$ to obtain estimates on $q_-$ and $q_e$. In particular, we claim the following which immediately follows from (4-3) and (4-4) by addition.

**Lemma 4.3.** *For all $\tilde{A} \geq 1$, $A \geq 1$, $\rho^* \geq 0$, there exists $s_6(A, \rho^*) \geq 1$ such that for all $s_0 \geq s_6(A, \rho^*)$, if $q(s) \in S_A(s)$ for all $s \in [\sigma, \sigma + \rho^*]$ for some $\sigma \geq s_0$, then we have for all $s \in [\sigma, \sigma + \rho^*]$:*

(i) *The linear term $\vartheta_1(s, \sigma)$:*

$$\left\| \frac{(\vartheta_1(s,\sigma))_-}{1+|y|^3} \right\|_{L^\infty} \leq C e^{-(s-\sigma)/2} \left\| \frac{q_-(\cdot,\sigma)}{1+|y|^3} \right\|_{L^\infty} + \frac{C e^{-(s-\sigma)^2}}{s^{3/2}} \|q_e(\sigma)\| + \frac{C}{s^2},$$

$$\|(\vartheta_1(s,\sigma))_e\|_{L^\infty} \leq C e^{-(s-\sigma)/p} \|q_e(\sigma)\|_{L^\infty} + C e^{s-\sigma} s^{3/2} \left\| \frac{q_-(\cdot,\sigma)}{1+|y|^3} \right\|_{L^\infty} + \frac{C}{\sqrt{s}}.$$

(ii) *The quadratic term* $\vartheta_2(s,\sigma)$:
$$\left\|\frac{(\vartheta_2(s,\sigma))_-}{1+|y|^3}\right\|_{L^\infty} \le \frac{C(s-\sigma)}{s^{2+\epsilon}}, \quad \|(\vartheta_2(s,\sigma))_e\|_{L^\infty} \le \frac{C(s-\sigma)}{s^{1/2+\epsilon}},$$
*where* $\epsilon = \epsilon(p) > 0$.

(iii) *The correction term* $\vartheta_3(s,\sigma)$:
$$\left\|\frac{(\vartheta_3(s,\sigma))_-}{1+|y|^3}\right\|_{L^\infty} \le \frac{C(s-\sigma)}{s^{2}}, \quad \|(\vartheta_3(s,\sigma))_e\|_{L^\infty} \le \frac{C(s-\sigma)}{s^{3/4}}.$$

(iv) *The nonlinear term* $\vartheta_4(s,\sigma)$:
$$\left\|\frac{(\vartheta_4(s,\sigma))_-}{1+|y|^3}\right\|_{L^\infty} \le \frac{C(s-\sigma)}{s^{2}}, \quad \|(\vartheta_4(s,\sigma))_e\|_{L^\infty} \le \frac{C(s-\sigma)}{s^{3/4}}.$$

*Proof.* The proof simply follows from definition of the set $S_A$ and Lemma 4.2. In particular, we make use of Lemmas A.8 , A.9 and A.10 to derive the bounds on the components of the terms $B$, $D$ and $R$ as follows:
$$\sum_{m\in\mathbb{N}^n,\ |m|=0}^{2} |B(q)_m(s)| \le \frac{C}{s^3}, \quad \left\|\frac{B(q)_-(s)}{1+|y|^3}\right\|_{L^\infty} \le \frac{C}{s^{2+\epsilon}}, \quad \|B(q)_e(s)\|_{L^\infty} \le \frac{C}{s^{1/2+\epsilon}},$$
and
$$\sum_{m\in\mathbb{N}^n,\ |m|=0}^{2} |R_m(s)| \le \frac{C}{s^2}, \quad \left\|\frac{R_-(s)}{1+|y|^3}\right\|_{L^\infty} \le \frac{C}{s^{2+1/2}}, \quad \|R_e(s)\|_{L^\infty} \le \frac{C}{s^{3/4}},$$
and
$$\sum_{m\in\mathbb{N}^n,\ |m|=0}^{2} |D(q)_m(s)| + \left\|\frac{D(q)_-(s)}{1+|y|^3}\right\|_{L^\infty} \le \frac{C\ln s}{s^3}, \quad \|D(q)_e(s)\|_{L^\infty} \le \frac{C}{s^{3/4}},$$
where $\epsilon = \epsilon(p) > 0$. We simply inject these bounds into the a priori estimates given in Lemma 4.2 to obtain the bounds on $(\vartheta_m)_-$ and $(\vartheta_m)_e$ for $m = 2, 3, 4$. The estimate on $\vartheta_1$ directly follows from Lemma 4.2 and the assumption $q(s) \in S_A(s)$. This ends the proof of Lemma 4.3. □

By the formula (4-6), the estimates (4-3) and (4-4) simply follow from Lemma 4.3 by addition. This concludes the proof of Proposition 4.1. □

**4B. *Conclusion of Proposition 3.5.*** We now give the proof of Proposition 3.5, which is a consequence of the dynamics of (2-28) given in Proposition 4.1. Indeed, item (i) of Proposition 3.5 directly follows from the result below.

**Proposition 4.4** (control of $q(s)$ by $(q_0, q_1)(s)$ in $S_A(s)$). *There exists* $A_7 \ge 1$ *such that for all* $A \ge A_7$, *there exists* $s_7(A) \ge 1$ *such that for all* $s_0 \ge s_7(A)$ *if*

(a) $q(s_0) = \psi_{d_0,d_1,s_0}(y)$, *where* $(d_0, d_1) \in \mathbb{D}_{A,s_0}$,

(b) $q(s) \in S_A(s)$ *for all* $s \in [s_0, s_1]$, *then for all* $s \in [s_0, s_1]$, *we have*

$$\text{for all } i, j \in \{1, \dots, n\}, \quad |q_{2,i,j}(s)| < \frac{A^2 \ln^2 s}{s^2}, \tag{4-11}$$

$$\left\| \frac{q_-(y,s)}{1+|y|^3} \right\|_{L^\infty} \le \frac{A}{2s^2}, \quad \|q_e(s)\|_{L^\infty} \le \frac{A^2}{2\sqrt{s}}, \tag{4-12}$$

*where* $\mathbb{D}_{A,s_0}$ *is introduced in Proposition 3.4 and* $\psi_{d_0,d_1}$ *is defined as in (3-1).*

*Proof.* Since the proof of (4-12) is similar to the one written in [Merle and Zaag 1997], we only deal with the proof of (4-11) and refer to Proposition 3.7 in that paper for the proof of (4-12). We argue by contradiction to prove (4-11). Let $i, j \in \{1, \dots, n\}$ and assume that there is $s_* \in [s_0, s_1]$ such that

$$\text{for all } s \in [s_0, s_*), \quad |q_{2,i,j}(s)| < \frac{A^2 \ln^2(s)}{s^2} \quad \text{and} \quad |q_{2,i,j}(s_*)| = \frac{A^2 \ln^2(s_*)}{s_*^2}.$$

Assuming that $q_{2,i,j}(s_*) > 0$ (the negative case is similar), we have on the one hand

$$q_{2,i,j}'(s_*) \ge \frac{d}{ds}\left(\frac{A^2 \ln^2 s}{s^2}\right)_{s=s_*} = \frac{2A^2 \ln s_*}{s_*^3} - \frac{2A^2 \ln^2 s_*}{s_*^3}.$$

On the other hand, we have from (4-2),

$$q_{2,i,j}'(s_*) \le -\frac{2A^2 \ln^2 s_*}{s_*^3} + \frac{C \ln s_*}{s_*^3}.$$

Thus the contradiction then follows if $2A^2 > C$, and this concludes the proof of Proposition 4.4. □

From Proposition 4.4, we see that if $q(s) \in \partial S_A(s_1)$, the first two components $(q_0, q_1)(s_1)$ must be in $\partial \widehat{S}_A(s_1)$, which is the conclusion of item (i) of Proposition 3.5.

The proof of item (ii) of Proposition 3.5 follows from (4-1). Indeed, it is easy to see from (4-1) that for all $i \in \{1, \dots, n\}$ and for each $\epsilon_0, \epsilon_i = \pm 1$, if $q_0(s_1) = \epsilon_0 A / s_1^2$ and $q_{1,i}(s_1) = \epsilon_i A / s_1^2$, it follows that the signs of

$$\frac{dq_0}{ds}(s_1) \quad \text{and} \quad \frac{dq_{1,i}}{ds}(s_1)$$

are opposite the signs of

$$\frac{d}{ds}\left(\frac{\epsilon_0 A}{s^2}\right)(s_1) \quad \text{and} \quad \frac{d}{ds}\left(\frac{\epsilon_i A}{s^2}\right)(s_1)$$

respectively. Hence, $(q_0, q_1)(s)$ will actually leave $\widehat{S}_A(s)$ at $s_1 \ge s_0$ for $s_0$ large enough. This concludes the proof of Proposition 3.5.

## Appendix: Some elementary lemmas

**Lemma A.1.** *For each $T > 0$, there exists only one positive solution of* (1-9). *Moreover, the solution $\psi$ satisfies the asymptotic*

$$\psi(t) \sim \kappa_\alpha (T-t)^{-1/(p-1)} |\ln(T-t)|^{-\alpha/(p-1)} \quad \text{as } t \to T, \tag{A-1}$$

*where*

$$\kappa_\alpha = (p-1)^{-1/(p-1)} \left(\frac{p-1}{2}\right)^{\alpha/(p-1)}.$$

*Proof.* Consider the ODE

$$\psi' = \psi^p \ln^\alpha(\psi^2+2), \qquad \psi(0) > 0.$$

The uniqueness and local existence are derived by the Cauchy–Lipschitz property. Let $T_{\max}$, $T_{\min}$ be the maximum and minimum times of the existence of the positive solution; i.e., $\psi(t)$ exists for all $t \in (T_{\min}, T_{\max})$. We now prove that $T_{\max} < +\infty$ and $T_{\min} = -\infty$. By contradiction, we suppose that the solution exists on $[0, +\infty)$; we have

$$\lim_{t_1 \to +\infty} \int_0^{t_1} \frac{\psi'}{\psi^p \ln^\alpha(\psi^2+2)}\, dt = \lim_{t_1 \to +\infty} \int_0^{t_1} dt = +\infty.$$

Since $\int_0^{t_1} \psi'/(\psi^p \ln^\alpha(\psi^2+2))\, dt$ is bounded, the contradiction then follows. With a similar argument we can prove that $T_{\min} = -\infty$. Let us now prove (A-1). We deduce from (1-9) that

$$T - t = \int_{\psi(t)}^{+\infty} \frac{du}{u^p \ln^\alpha(u^2+2)}.$$

Thus, for all $\delta \in (0, p-1)$, there exist $t_\delta$ such that for all $t \in (t_\delta, T)$, we have

$$\int_{\psi(t)}^{+\infty} \frac{du}{u^{p+\delta}} \le T - t \le \int_{\psi(t)}^{+\infty} \frac{du}{u^{p-\delta}}.$$

For all $t \in (t_\delta, T)$ it follows that

$$\big((p-1+\delta)(T-t)\big)^{-1/(p-1+\delta)} \le \psi(t) \le \big((p-1-\delta)(T-t)\big)^{-1/(p-1-\delta)},$$

from which we have

$$\ln \psi(t) \sim -\frac{1}{p-1} \ln(T-t) \quad \text{as } t \to T,$$
$$\ln(\psi^2+2) \sim -\frac{2}{p-1} \ln(T-t) \quad \text{as } t \to T.$$

Hence, we obtain

$$\psi' = \psi^p \ln(\psi^2+2) \sim \psi^p \Big(-\frac{2}{p-1} \ln(T-t)\Big)^\alpha \quad \text{as } t \to T, \tag{A-2}$$

which yields

$$\frac{\psi'}{\psi^p} \sim \left(\frac{2}{p-1}\right)^{\alpha} |\ln(T-t)|^{\alpha} \quad \text{as } t \to T.$$

This implies

$$\frac{1}{p-1}\psi^{1-p} \sim \left(\frac{2}{p-1}\right)^{\alpha} \int_t^T |\ln(T-v)|^{\alpha}\,dv \sim \left(\frac{2}{p-1}\right)^{\alpha} (T-t)|\ln(T-t)|^{\alpha} \quad \text{as } t \to T,$$

which concludes the proof of (A-1). □

**Lemma A.2.** *For all* $\alpha \in (0, 1)$, $\theta > 0$ *and* $0 < h < 1$, *the integral*

$$I(h) = \int_h^1 (s-h)^{-\alpha} s^{-\theta}\,ds$$

*satisfies*:

(i) *If* $\alpha + \theta > 1$, *then*

$$I(h) \le \left(\frac{1}{1-\alpha} + \frac{1}{\alpha+\theta-1}\right) h^{1-\alpha-\theta}.$$

(ii) *If* $\alpha + \theta = 1$, *then*

$$I(h) \le \frac{1}{1-\alpha} + |\ln h|.$$

(iii) *If* $\alpha + \theta < 1$, *then*

$$I(h) \le \frac{1}{1-\alpha-\theta}.$$

*Proof.* See Lemma 2.2 of [Giga and Kohn 1989] □

**Lemma A.3** (a version of the Grönwall lemma). *If* $y(t)$, $r(t)$ *and* $q(t)$ *are continuous functions defined on* $[t_0, t_1]$ *such that*

$$y(t) \le y_0 + \int_{t_0}^t y(s) r(s)\,ds + \int_{t_0}^t h(s)\,ds \quad \textit{for all } t \in [t_0, t_1].$$

*Then*,

$$y(t) \le e^{\int_{t_0}^t r(s)\,ds} \left( y_0 + \int_{t_0}^t h(s) e^{-\int_{t_0}^s r(\tau)\,d\tau}\,ds \right).$$

*Proof.* See Lemma 2.3 of [Giga and Kohn 1989]. □

**Lemma A.4.** *For each* $T_2 < T$, $\delta > 0$. *There exists* $\epsilon = \epsilon(T, T_2, \delta, n, p) > 0$ *such that for each* $v(x, t)$ *satisfying*

$$|\partial_t v - \Delta v| \le C|v|^p \ln^{\alpha}(v^2 + 2) \quad \textit{for all } |x| \le \delta,\ t \in (T_2, T),\ \delta > 0, \tag{A-3}$$

*and*

$$|v(x, t)| \le \epsilon \psi(t) \quad \textit{for all } |x| \le \delta,\ t \in (T_2, T), \tag{A-4}$$

*where* $\psi(t)$ *is the unique positive solution of* (1-9). *Then*, $v(x, t)$ *does not blow up at* $(0, T)$.

*Proof.* Since the argument is almost the same as in [Giga and Kohn 1989] treated for the case $\alpha = 0$, we only sketch the main step for the sake of completeness. Let $\phi \in C^\infty(\mathbb{R}^n)$, $\phi = 1$ if $|x| \le \delta/2$, $\phi = 0$ if $|x| \ge \delta$, and consider $\omega = \phi v$ satisfying

$$\partial_t \omega - \Delta \omega = f\phi + g, \tag{A-5}$$

where

$$f = \partial_t v - \Delta v \quad \text{and} \quad g = v\Delta\phi - 2\nabla\cdot(v\nabla\phi).$$

By using Duhamel's formula, we write

$$\omega(t) = e^{(t-T_2)\Delta}(\omega(T_2)) + \int_{T_2}^t \big(e^{(t-\tau)\Delta}(\phi f) + e^{(t-\tau)\Delta}(g)\big)\,d\tau \quad \text{for all } t \in [T_2, T), \tag{A-6}$$

where $e^{t\Delta}$ is the heat semigroup satisfying the following properties: for all $h \in L^\infty$,

$$\|e^{t\Delta}h\|_{L^\infty} \le \|h\|_{L^\infty} \quad \text{and} \quad \|e^{t\Delta}\nabla h\|_{L^\infty} \le Ct^{-1/2}\|h\|_{L^\infty} \quad \text{for all } t > 0.$$

The formula (A-6) then yields

$$\|\omega(t)\|_{L^\infty} \le C + C\int_{T_2}^t \|\omega(\tau)\|_{L^\infty} \big\| |v|^{p-1}\ln^\alpha(v^2+2)(\tau)\big\|_{L^\infty(|x|\le\delta)} + C\int_{T_2}^t (t-\tau)^{-1/2}\|v(\tau)\|_{L^\infty(|x|\le\delta)}\,d\tau \tag{A-7}$$

for some constant $C = C(n, p, \phi, T, T_2, \delta) > 0$.

From (A-3), (A-4) and Lemma A.1, we find that for all $|x| \le \delta$, and $\tau \in [T_2, T)$,

$$|v(\tau)|^{p-1}\ln^\alpha(v^2(\tau)+2) \le C\psi^{p-1}(\tau)\ln^\alpha(\psi^2(\tau)+2) \le C(T-\tau)^{-1},$$

and

$$|v(\tau)| \le C(T-\tau)^{-1/(p-1)}|\ln(T-\tau)|^{-\alpha/(p-1)}.$$

The estimate (A-7) becomes

$$\|\omega(t)\|_{L^\infty} \le C + C\epsilon^{p-1}\int_{T_2}^t (T-\tau)^{-1}\|\omega(\tau)\|_{L^\infty}\,d\tau + C\epsilon\int_{T_2}^t (t-\tau)^{-1/2}(T-\tau)^{-1/(p-1)}|\ln(T-\tau)|^{-\alpha/(p-1)}\,d\tau. \tag{A-8}$$

In particular, we now consider $0 < \lambda \ll \frac{1}{2}$ fixed, then we have

$$(T-\tau)^{-1/(p-1)}|\ln(T-\tau)|^{-\alpha/(p-1)} \le C(\alpha,\lambda)(T-\tau)^{-(1/(p-1)+\lambda)} \quad \text{for all } \tau \in (T_2, T).$$

Hence, we rewrite (A-8) as

$$\|\omega(t)\|_{L^\infty} \le C + C\epsilon^{p-1}\int_{T_2}^{t}(T-\tau)^{-1}\|\omega(\tau)\|_{L^\infty}\,d\tau + C\epsilon\int_{T_2}^{t}(t-\tau)^{-1/2}(T-\tau)^{-(1/(p-1)+\lambda)}\,d\tau, \quad \text{(A-9)}$$

where $C(n, p, \phi, \alpha, \epsilon, \lambda, p)$. Beside that, by the change of variables $s = T-\tau$, $h = T-t$ we have

$$\int_{T_2}^{t}(t-\tau)^{-1/2}(T-\tau)^{-\theta(p,\lambda)}\,d\tau = \int_{h}^{T-T_2}(s-h)^{-1/2}(s)^{-\theta(p,\lambda)}\,ds, \quad \text{(A-10)}$$

where $\theta(p,\lambda) = (1/(p-1)+\lambda)$.

Case 1: If $\theta(p,\lambda) < \frac{1}{2}$, by using (iii) of Lemma A.2 we deduce from (A-9), (A-10) that

$$\|\omega(t)\|_{L^\infty} \le C + C\epsilon^{p-1}\int_{T_2}^{t}(T-s)^{-1}\|\omega(s)\|_{L^\infty}\,ds.$$

Therefore, by Lemma A.3,

$$\|\omega(t)\|_{L^\infty} \le C(T-t)^{-C\epsilon^{p-1}}. \quad \text{(A-11)}$$

Choose $\epsilon$ small enough such that $C\epsilon^{p-1} \le 1/(2(p-1))$. Then, we conclude from (A-11) that

$$|v(x,t)| \le C(T-t)^{-1/(2(p-1))} \quad \text{for } |x| \le \tfrac{1}{2},\ t \le T. \quad \text{(A-12)}$$

By using parabolic regularity theory and the same argument as in Lemma 3.3 of [Giga and Kohn 1987], we can prove that (A-12) actually prevents blowup.

Case 2: $\theta(\lambda, p) = \frac{1}{2}$ is similar to the first case. By using (ii) of Lemma A.2, (A-9) and (A-10) we get

$$\|\omega(t)\|_{L^\infty} \le C(1+|\ln(T-t)|) + C\epsilon^{p-1}\int_{T_2}^{t}(T-s)^{-1}\|\omega(s)\|_{L^\infty}\,ds.$$

However, we derive from Lemma A.3 that

$$\|\omega(t)\|_{L^\infty} \le C(T-t)^{-K\epsilon^{p-1}}, \quad \text{(A-13)}$$

where $C = C(n, p, \phi, T, T_2, \delta)$. We now take $\epsilon$ small enough such that $C\epsilon^{p-1} \le 1/(2(p-1))$, which follows (A-12).

Case 3: For $\theta(\lambda, p) > \frac{1}{2}$, by using Lemmas A.2, A.3 and similar arguments we obtain

$$|v(x,t)| \le C(T-t)^{1/2-\theta(p,\lambda)} \quad \text{for all } |x| \le \delta,\ t \in [T_2, T).$$

Repeating the step in finite steps would end up with (A-12). This concludes the proof of Lemma A.4. □

The following lemma gives the asymptotic behavior of $h(s)$ and $\psi_1(s)$.

**Lemma A.5.** *Let $h(s)$ and $\psi_1(s)$ be defined as in (2-3) and (2-4) respectively. Then we have*:

(i) $$\frac{1}{\ln(\psi_1^2(s)+2)} = \frac{p-1}{2s} + \frac{\alpha(p-1)\ln s}{2s^2} + O\left(\frac{1}{s^2}\right) \quad \textit{as } s \to +\infty. \tag{A-14}$$

(ii) $$h(s) = \frac{1}{p-1}\left(1 - \frac{\alpha}{s} - \frac{\alpha^2 \ln s}{s^2}\right) + O\left(\frac{1}{s^2}\right) \quad \textit{as } s \to +\infty. \tag{A-15}$$

*Proof.* (i) Consider $\psi(t)$ the unique positive solution of (1-9). We have

$$T - t = \int_{\psi(t)}^{+\infty} \frac{dx}{x^p \ln^\alpha(x^2+2)}. \tag{A-16}$$

An integration by parts yields

$$\begin{aligned} T - t = {} & \frac{1}{\psi^{p-1}(t)\ln^\alpha(\psi^2(t)+2)} \\ & \times \left(\frac{1}{p-1} - \frac{2\alpha}{(p-1)^2\ln(\psi^2(t)+2)} + O\left(\frac{1}{(\ln^2(\psi^2(t)+2))}\right)\right). \end{aligned} \tag{A-17}$$

Let us write $\psi(t) = \psi_1(s)$, where $s = -\log(T-t)$; then we have

$$\ln(\psi_1(s)) = \frac{s}{p-1} - \frac{\alpha}{(p-1)}\ln(\ln(\psi_1(s))) + O(1) \quad \text{as } s \to +\infty, \tag{A-18}$$

from which we deduce that

$$\ln(\psi_1(s)) = \frac{s}{p-1} - \frac{\alpha \ln(s)}{p-1} + O(1) \quad \text{as } s \to +\infty, \tag{A-19}$$

which is the conclusion (i).

(ii) From (2-3) and (A-17), we have

$$h(s) = \frac{1}{p-1} - \frac{2\alpha}{(p-1)^2\ln(\psi_1^2(s)+2)} + O\left(\frac{1}{\ln^2(\psi_1^2(s)+2)}\right). \tag{A-20}$$

Using (A-14), we conclude the proof of (A-15), as well as Lemma A.5. □

**Lemma A.6.** *Let $N$ be defined as in (2-11). We have*

$$N(\bar{w}, s) = \frac{p\bar{w}^2}{2} + O\left(\frac{|\bar{w}|\ln s}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right) + O(|\bar{w}|^3) \quad \textit{as } (\bar{w}, s) \to (0, +\infty). \tag{A-21}$$

*Proof.* From the definition (2-11) of $N$, let us write

$$N(\bar{w}, s) = N_1(\bar{w}, s) + N_2(\bar{w}, s),$$

where

$$N_1(\bar{w},s)=h(s)\big(|\bar{w}+1|^{p-1}(\bar{w}+1)-(\bar{w}+1)\big)-\bar{w},$$
$$N_2(\bar{w},s)=h(s)|\bar{w}+1|^{p-1}(\bar{w}+1)\left(\frac{\ln^\alpha(\psi_1^2(\bar{w}+1)^2+2)}{\ln^\alpha(\psi_1^2+2)}-1\right).$$

From (A-15) and a Taylor expansion, we find that

$$N_1(\bar{w},s)=\frac{p\bar{w}^2}{2}-\frac{\alpha\bar{w}}{s}+O\left(\frac{|\bar{w}|\ln s}{s^2}\right)+O\left(\frac{|\bar{w}|^2}{s}\right)+O(|\bar{w}|^3)\quad\text{as } (\bar{w},s)\to(0,+\infty).$$

We now claim the following

$$N_2(\bar{w},s)=\frac{\alpha\bar{w}}{s}+O\left(\frac{|\bar{w}|\ln s}{s^2}\right)+O\left(\frac{|\bar{w}|^2}{s}\right)\quad\text{as } (\bar{w},s)\to(0,+\infty).\qquad\text{(A-22)}$$

Then, the proof of (A-21) simply follows by addition.

Let us now give the proof of (A-22) to complete the proof of Lemma A.6 . We set

$$f(\bar{w})=\ln^\alpha(\psi_1^2(\bar{w}+1)^2+2),\quad |\bar{w}|\le\tfrac{1}{2}.$$

We apply Taylor expansion to $f(\bar{w})$ at $\bar{w}=0$ to find that

$$f(\bar{w})=\ln^\alpha(\psi_1^2+2)+2\alpha\ln^{\alpha-1}(\psi_1^2+2)\frac{\psi_1^2}{\psi_1^2+2}\bar{w}+\frac{f''(\theta)}{2}(\bar{w})^2,$$

where $\theta$ is between 0 and $\bar{w}$, and

$$f''(\theta)=\alpha(\alpha-1)\ln^{\alpha-2}(\psi_1^2(\theta+1)^2+2)\left(\frac{2(\theta+1)\psi_1^2}{\psi_1^2(\theta+1)^2+2}\right)^2$$
$$+\alpha\ln^{\alpha-1}(\psi_1^2(\theta+1)^2+2)\frac{(4\psi_1-2\psi_1^4(\theta+1)^2)}{(\psi_1^2(\theta+1)^2+2)^2}.$$

Since $|\theta|\le\frac{1}{2}$, one can show that

$$|f''(\theta)|\le C\ln^{\alpha-1}(\psi_1^2+2)\quad\text{for all } |\theta|\le\tfrac{1}{2}.$$

Thus, we have

$$f(\bar{w})=\ln^\alpha(\psi_1^2+2)+2\alpha\ln^{\alpha-1}(\psi_1^2+2)\bar{w}$$
$$+O\big(|\bar{w}|^2\ln^{\alpha-1}(\psi_1^2+2)\big)+O\left(\frac{|\bar{w}|\ln^{\alpha-1}(\psi_1^2+2)}{\psi_1^2}\right)$$

as $s\to+\infty$. This yields

$$\frac{\ln^\alpha(\psi_1^2(\bar{w}+1)^2+2)}{\ln^\alpha(\psi_1^2+2)}=1+\frac{2\alpha\bar{w}}{\ln(\psi_1^2+2)}+O\left(\frac{|\bar{w}|^2}{\ln(\psi_1^2+2)}\right)+O\left(\frac{|\bar{w}|}{\ln(\psi_1^2+2)\psi_1^2}\right)$$

as $(\bar{w}, s) \to (0, +\infty)$. From this and (A-14) we derive

$$\frac{\ln^\alpha(\psi_1^2(\bar{w}+1)^2+2)}{\ln^\alpha(\psi_1^2(s)+2)} - 1 = \frac{\alpha(p-1)\bar{w}}{s} + O\left(\frac{\ln s|\bar{w}|}{s^2}\right) + O\left(\frac{|\bar{w}|^2}{s}\right). \tag{A-23}$$

From the definition of $N_2$, (A-15), (A-23) and the fact that

$$|\bar{w}+1|^{p-1}(\bar{w}+1) = 1 + p\bar{w} + O(|\bar{w}|^2) \quad \text{as } \bar{w} \to 0,$$

we conclude the proof of (A-22) as well as Lemma A.6. □

**Lemma A.7.** *For all $|z| \le K_1$, there exists $C(K_1)$ such that for all $s \ge 1$ we have*

$$\left| h(s)|z|^{p-1} z \frac{\ln^\alpha(\psi_1^2 z^2+2)}{\ln^\alpha(\psi_1^2+2)} - \frac{|z|^{p-1}z}{p-1} \right| \le \frac{C(K_1)}{s}, \tag{A-24}$$

*where $h(s)$ satisfies the asymptotic (2-5).*

*Proof.* We consider $f(z) = \ln^\alpha(\psi_1^2 z^2 + 2)$ for all $z \in \mathbb{R}$; then we write

$$\ln^\alpha(\psi_1^2 z^2+2) = \ln^\alpha(\psi_1^2+2) + \int_1^{|z|} f'(v)\, dv.$$

Recalling from (2-5) that $h(s) = 1/(p-1) + O(1/s)$, we have

$$\left| h(s)|z|^{p-1} z \frac{\ln^\alpha(\psi_1^2 z^2+2)}{\ln^\alpha(\psi_1^2+2)} - \frac{|z|^{p-1}z}{p-1} \right| \\ \le \frac{C|z|^p}{\ln^\alpha(\psi_1^2+2)} \int_1^{|z|} |f'(v)|\, dv + \frac{C|z|^p}{s}. \tag{A-25}$$

From (i) of Lemma A.5 we have

$$\frac{1}{\ln(\psi_1^2+2)} \le \frac{C}{s}.$$

Thus it is sufficient to show that

$$A(z) := \frac{|z|^p}{\ln^{\alpha-1}(\psi_1^2+2)} \int_1^{|z|} |f'(v)|\, dv \le C(K_1) \quad \text{for all } |z| \le K_1,$$

where

$$f'(v) = \alpha \ln^{\alpha-1}(\psi_1^2 v^2 + 2) \frac{2v\psi_1^2}{\psi_1^2 v^2 + 2}.$$

For $1 \le |z| \le K_1$, it is trivial to see that $|A(z)| \le C(K_1)$. For $|z| < 1$, we consider two cases:

<u>Case 1</u>: $\alpha - 1 \ge 0$. Then

$$A(z) \le 2|\alpha||z|^p \int_{|z|}^1 \frac{1}{v}\, dv \le C(K_1).$$

Case 2: $\alpha - 1 < 0$. Then

$$A(z) \le 2|\alpha||z|^p \frac{\ln^{\alpha-1}(\psi_1^2 z^2 + 2)}{\ln^{\alpha-1}(\psi_1 + 2)} \int_{|z|}^1 \frac{1}{v}\, dv.$$

- If $\psi_1 z^2 \ge 1$ then

$$A(z) \le 2|\alpha| \frac{\ln^{1-\alpha}(\psi_1^2 + 2)}{\ln^{1-\alpha}(\psi_1 + 2)} |z|^p \int_{|z|}^1 \frac{1}{v}\, dv \le C(K_1).$$

- If $\psi_1 z^2 \le 1$ then $|z| \le v \le \psi_1^{-1/2}$ we deduce that

$$|A(z)| \le 2|\alpha| \psi_1^{(1-p)/2} \frac{\ln^{1-\alpha}(\psi_1^2 + 2)}{\ln^{1-\alpha}(2)} |z| \int_{|z|}^1 \frac{1}{v}\, dv \le C(K_1). \qquad \square$$

**Lemma A.8** (control of the nonlinear term $D$ in $S_A(s)$). *For all $A \ge 1$, there exists $\sigma_3(A) \ge 1$ such that for all $s \ge \sigma_3(A)$, $q(s) \in S_A(s)$ implies*

$$\text{for all } |y| \le 2K\sqrt{s}, \quad |D(q, s)| \le C(K) \frac{\ln s (1 + |y|)^4}{s^3}, \tag{A-26}$$

*and*

$$\|D(q, s)\|_{L^\infty(\mathbb{R}^n)} \le \frac{C}{s}. \tag{A-27}$$

*Proof.* From the definition (2-32) of $D$, we have the decomposition

$$D(q, s) = D_1(q, s) + D_2(q, s),$$

where

$$D_1(q, s) = \Big(h(s) - \frac{1}{p-1}\Big)\big(|q + \varphi|^{p-1}(q + \varphi) - (q + \varphi)\big),$$

$$D_2(q, s) = h(s)|q + \varphi|^{p-1}(q + \varphi) L(q + \varphi, s),$$

$h(s)$ admits the asymptotic behavior (A-15), and $L$ is defined in (2-33). The proof of (A-26) will follow once we show for all $|y| \le 2K\sqrt{s}$

$$\left| D_1 - \left( \frac{\alpha(|y|^2 - 2n)}{4ps^2} - \frac{\alpha}{s} q \right) \right| \le C \frac{(1 + |y|^4) \ln s}{s^3}, \tag{A-28}$$

$$\left| D_2 + \left( \frac{\alpha(|y|^2 - 2n)}{4ps^2} - \frac{\alpha}{s} q \right) \right| \le C \frac{(1 + |y|^4) \ln s}{s^3}. \tag{A-29}$$

Let us give a proof of (A-28). From the definition of $S_A(s)$, we note that if $q(s) \in S_A(s)$, then

$$\text{for all } y \in \mathbb{R}^n, \quad |q(y, s)| \le \frac{CA^2 \ln^2 s (1 + |y|^3)}{s^2}, \tag{A-30}$$

$$\|q(s)\|_{L^\infty(\mathbb{R}^n)} \le \frac{CA^2}{\sqrt{s}}. \tag{A-31}$$

From the definition (2-26) of $\varphi$ and (A-31), we see that for all $|y| \le 2K\sqrt{s}$, there exists a positive constant $C(K)$ such that

$$0 < \frac{1}{C(K)} \le (q+\varphi)(y,s) \le C(K). \tag{A-32}$$

Using Taylor expansion and the asymptotic (A-15), we write

$$D_1(q,s) = \left(-\frac{\alpha}{(p-1)s} + O\left(\frac{\ln s}{s^2}\right)\right)\left(\varphi^p - \varphi + (p\varphi^{p-1}-1)q\right) + O(q^2). \tag{A-33}$$

Using again the definition of $\varphi$ and a Taylor expansion, we derive

$$\varphi^p = 1 - \frac{(|y|^2-2n)}{4s} + O\left(\frac{1+|y|^4}{s^2}\right),$$

$$\varphi = 1 - \frac{(|y|^2-2n)}{4ps} + O\left(\frac{1+|y|^4}{s^2}\right),$$

$$p\varphi^{p-1} - 1 = p - 1 - \frac{(p-1)(|y|^2-2n)}{4ps} + O\left(\frac{1+|y|^4}{s^2}\right)$$

as $s \to +\infty$. Inserting (A-30) and these estimates into (A-33) yields (A-28).

We now turn to the proof of (A-29). Recall from (2-33) the definition of $L$,

$$L(q+\varphi,s) = \frac{2\alpha\psi_1^2}{\ln(\psi_1^2+2)(\psi_1^2+2)}(q+\varphi-1) + \frac{1}{\ln^\alpha(\psi_1^2+2)}\int_1^{q+\varphi} f''(v)(q+\varphi-v)\,dt,$$

where $f(v) = \ln^\alpha(\psi_1^2 v^2 + 2)$, $v \in \mathbb{R}$. From (A-32) and a direct computation, we estimate

$$\left|\frac{1}{\ln^\alpha(\psi_1^2+2)}\int_1^{q+\varphi} f''(v)(q+\varphi-v)\,dv\right| \le C(K)\frac{|q+\varphi-1|^2}{s},$$

which yields

$$\left|L(q+\varphi,s) - \frac{2\alpha\psi_1^2(q+\varphi-1)}{\ln(\psi_1^2+2)(\psi_1^2+2)}\right| \le C(K)\frac{|q+\varphi-1|^2}{s}. \tag{A-34}$$

From (A-14) and (A-34), we then have

$$\left|L(q+\varphi,s) - \frac{\alpha(p-1)(q+\varphi-1)}{s}\right| \le C(K)\left(\frac{|q+\varphi-1|^2}{s} + \frac{\ln s|q+\varphi-1|}{s^2}\right),$$

and additionally we have

$$|q+\varphi-1| \le \frac{C(1+|y|^2)}{s},$$

which implies

$$\left| L(q+\varphi, s) - \frac{\alpha(p-1)(q+\varphi-1)}{s} \right| \le C(K) \frac{\ln s(1+|y|^4)}{s^3}. \tag{A-35}$$

Moreover, from definition of $D_2$ and (A-35) we deduce that

$$\left| D_2(q,s) - \frac{\alpha}{s}\big(\varphi^{p+1} - \varphi^p + ((p+1)\varphi^p - p\varphi^{p-1})q\big) \right| \le C \frac{(1+|y|^4)\ln s}{s^3},$$

and

$$\varphi^{p+1} - \varphi^p = -\frac{(|y|^2-2)}{4ps} + O\left(\frac{1+|y|^4}{s^2}\right),$$

$$(p+1)\varphi^p - p\varphi^{p-1} = 1 - \frac{(|y|^2-2)}{2s} + O\left(\frac{1+|y|^4}{s^2}\right)$$

as $s \to +\infty$, which yields (A-29).

We now prove (A-27). From (A-15) and the boundedness of $q$ and $\varphi$, we have

$$|D_1(q,s)| \le \frac{C}{s}.$$

It is sufficient to prove that for all $y \in \mathbb{R}^n$,

$$|D_2(q,s)| \le \frac{C(K)}{s}.$$

Indeed, from definition (2-33) of $L$ we deduce that

$$D_2(q,s) = h(s)|q+\varphi|^{p-1}(q+\varphi)\frac{\ln^\alpha(\psi_1^2 z^2+2)}{\ln^\alpha(\psi^2+2)} - h(s)|q+\varphi|^{p-1}(q+\varphi).$$

Using Lemma A.7 we deduce

$$|D_2(q,s)| \le \frac{C(K)}{s}. \qquad \square$$

**Lemma A.9.** *For $s$ large enough, we have*:

(i) *estimates on $V$*:

$$|V(y,s)| \le \frac{C(1+|y|^2)}{s} \quad \textit{for all } y \in \mathbb{R}^n,$$

*and*

$$V = -\frac{(|y|^2-2n)}{4s} + \widetilde{V} \quad \textit{with } \widetilde{V} = O\left(\frac{1+|y|^4}{s^2}\right) \textit{ for all } |y| \le K\sqrt{s}.$$

(ii) *estimates on $R$*:

$$|R(y,s)| \le \frac{C}{s} \quad \textit{for all } y \in \mathbb{R}^n,$$

*and*

$$R(y,s) = \frac{c_p}{s^2} + \widetilde{R}(y,s) \quad \textit{with } \widetilde{R} = O\left(\frac{1+|y|^4}{s^3}\right) \textit{ for all } |y| \le K\sqrt{s}.$$

*Proof.* The proof simply follows from Taylor expansion. We refer to Lemmas B.1 and B.5 in [Zaag 1998] for similar proofs. ☐

**Lemma A.10** (estimates on $B(q)$). *For all $A > 0$ there exists $\sigma_5(A) > 0$ such that for all $s \ge \sigma_5(A)$, $q(s) \in S_A(s)$ implies*

$$|B(q(y,s))| \le C|q|^2, \tag{A-36}$$

*and*

$$|B(q)| \le C|q|^{\bar{p}}, \tag{A-37}$$

*with $\bar{p} = \min(p, 2)$.*

*Proof.* See Lemma 3.6 in [Merle and Zaag 1997] for the proof of this lemma. ☐

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# Troisième groupe de cohomologie non ramifiée des hypersurfaces de Fano

Jean-Louis Colliot-Thélène

Sur un corps algébriquement clos et sur un corps fini, on établit de nouveaux résultats d'annulation pour la cohomologie non ramifiée de degré 3 des hypersurfaces de Fano.

We establish the vanishing of degree three unramified cohomology for several new types of Fano hypersurfaces when the ground field is either finite or algebraically closed of arbitrary characteristic.

Soit $X$ une variété projective, lisse, géométriquement connexe sur un corps $k$ et $\ell \neq \mathrm{car}(k)$ un nombre premier. Pour tout couple d'entiers $i \geq 0$ et $j \in \mathbb{Z}$, le groupe de cohomologie non ramifiée $H^i_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ est par définition le groupe des sections globales du faisceau pour la topologie de Zariski sur $X$ associé au préfaisceau qui à un ouvert $U \subset X$ associe le groupe de cohomologie étale $H^i_{\text{ét}}(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ de $U$ à valeurs dans le groupe des racines $\ell$-primaires de l'unité tordues $j$ fois. Les propriétés générales de ces groupes sont décrites dans le rapport [Colliot-Thélène 1995]. Le groupe $H^2_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$ est la composante $\ell$-primaire du groupe de Brauer de $X$. Les groupes $H^i_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ sont des invariants $k$-birationnels des variétés projectives et lisses. On a une application naturelle du groupe de cohomologie galoisienne $H^i(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)) = H^i_{\text{ét}}(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$ dans le groupe $H^i_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$, application qui est un isomorphisme si $X$ est $k$-birationnelle à un espace projectif $\mathbb{P}^m_k$.

On s'intéresse ici au groupe $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$. Ce groupe joue un rôle important dans l'étude [Colliot-Thélène et Voisin 2012; Kahn 2012; Colliot-Thélène et Kahn 2013; Colliot-Thélène 2015] de l'application “cycle” sur le groupe de Chow des cycles de codimension 2

$$\mathrm{cyc}_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4_{\text{ét}}(X, \mathbb{Z}_\ell(2)).$$

Pour une hypersurface cubique lisse $X \subset \mathbb{P}^n_{\mathbb{C}}$ sur le corps des complexes, $n = 4$ et $n = 5$, on sait que l'on a $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ pour tout $\ell$. C'est une conséquence [Colliot-Thélène et Voisin 2012, théorème 1.1] de la conjecture de Hodge entière

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pour les cycles de codimension 2 sur ces hypersurfaces cubiques. Pour $n = 4$, cette conjecture est facile à établir (voir le théorème 2.1 ci-dessous). C'est aussi un cas très particulier d'un théorème général de Claire Voisin sur les solides uniréglés. Pour $n = 5$, cette conjecture fut démontrée dans [Voisin 2007, Theorem 18].

Dans [Colliot-Thélène 2015, §5.3], j'ai discuté des extensions de ce résultat aux hypersurfaces lisses de degré $d \leq n$ dans un espace projectif $\mathbb{P}^n_{\mathbb{C}}$ avec $n$ quelconque. Par la formule d'adjonction, ce sont exactement les hypersurfaces lisses de Fano, c'est-à-dire à fibré anticanonique ample.

Dans cet article, on considère la situation sur un corps algébriquement clos de caractéristique quelconque, et sur un corps fini.

Plus précisément, pour $X \subset \mathbb{P}^n_k$ une hypersurface lisse de degré $d \leq n$ sur un corps $k$ de caractéristique différente de $\ell$, on établit

$$H^3_{\text{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$$

dans chacun des cas suivants :

(i) $k$ algébriquement clos et $n \neq 5$ (théorème 2.1) ;

(ii) $k = \mathbb{F}$ fini et $n \neq 4, 5$ (théorème 3.1) ;

(iii) $k$ algébriquement clos (de caractéristique différente de 2 et 3), $d = 3$ et $n = 5$ (théorème 4.1) ;

(iv) $k = \mathbb{F}$ fini, $d = 3$ et $n = 4$ (théorème 5.1).

Le cas des hypersurfaces cubiques lisses dans $\mathbb{P}^5_{\mathbb{F}}$ reste ouvert.

La démonstration du cas (iii) repose sur un théorème de Charles et Pirutka [2015]. Dans le cas (iv), on offre deux démonstrations, utilisant toutes deux la théorie du corps de classes supérieur de K. Kato et S. Saito. L'une de ces démonstrations passe par un théorème de Parimala et Suresh [2016].

Pour $X$ une variété sur un corps $k$ et $\bar{k}$ une clôture séparable de $k$, on note $\bar{X} = X \times_k \bar{k}$.

## 1. Quelques rappels

**Lemme 1.1.** *Soit $\mathbb{F}$ un corps fini. Soit $X \subset \mathbb{P}^n_{\mathbb{F}}$, $n \geq 4$, une hypersurface cubique lisse. Le pgcd des degrés des extensions finies $L$ de $\mathbb{F}$ sur lesquelles $X_L$ possède une $L$-droite est égal à* 1.

*Démonstration.* D'après Fano, Altman et Kleiman [1977], sur tout corps $k$, la variété de Fano $F = F(X)$ des droites de $X \subset \mathbb{P}^n_k$, est non vide, projective et lisse [Altman et Kleiman 1977, Corollary 1.12] pour $n \geq 3$ et géométriquement connexe pour $n \geq 4$ [Altman et Kleiman 1977, Theorem 1.16(i)]. Sur un corps fini $\mathbb{F}$, les estimations de Lang–Weil donnent le résultat. □

**Remarque 1.2.** Des résultats précis sur l'existence de droites sur le corps fini $\mathbb{F}$ lui-même sont obtenus dans [Debarre et al. 2017].

**Proposition 1.3.** *Soit $X$ une surface projective, lisse, géométriquement connexe sur un corps $k$. Soit $\ell$ un nombre premier différent de la caractéristique de $k$. Si $k$ est algébriquement clos, ou si $k$ est fini, $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.*

*Démonstration.* On a $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \subset H^3(k(X), \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$. Ce dernier groupe est nul si $k$ est algébriquement clos, car la $\ell$-dimension cohomologique du corps des fonctions $k(X)$ est 2.

Pour toute surface $X$ projective, lisse, géométriquement connexe sur un corps fini et $\ell$ premier différent de la caractéristique de $k$, on a $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ (Sansuc, Soulé et l'auteur [Colliot-Thélène et al. 1983, remarque 2, p. 790] ; Kato [1986, Theorem 0.7 and Corollary]). □

**Proposition 1.4.** *Soit $n \geq 3$ un entier et soit $X \subset \mathbb{P}^n_k$ une hypersurface cubique lisse sur un corps $k$. Soit $\ell$ un nombre premier différent de la caractéristique de $k$.*

(i) *Si $X$ possède un zéro-cycle de degré* 1, *le quotient du groupe $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ par l'image de $H^3(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ est annulé par* 6.

(ii) *Si $X$ contient une droite $k$-rationnelle, le quotient du groupe $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ par l'image de $H^3(k, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ est annulé par* 2.

(iii) *Si $k$ est algébriquement clos, $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ est annulé par* 2.

(iv) *Si $k$ est fini, $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ est annulé par* 2.

*Démonstration.* Les énoncés [Auel et al. 2017, Theorem 1.4, Proposition 2.1] donnent que ce quotient est annulé par 6 si $X$ possède un zéro-cycle de degré 1, et par 2 si $X$ contient une droite $k$-rationnelle. Ceci établit (i), (ii) et (iii). Pour $k$ un corps fini, $X$ possède un zéro-cycle de degré 1, et même un point rationnel. L'énoncé (iv) pour $n = 3$ est un cas particulier de la proposition 1.3. Pour $n \geq 4$, l'énoncé (iv) résulte de la combinaison de l'énoncé (ii), du lemme 1.1 et d'un argument de corestriction-restriction. □

## 2. Hypersurfaces de Fano dans $\mathbb{P}^n_k$, $k$ algébriquement clos, $n \neq 5$

On étend en toute caractéristique des résultats de [Colliot-Thélène 2015]. On en profite pour rectifier la démonstration de [Colliot-Thélène 2015, théorème 5.6(vi)] pour une hypersurface dans $\mathbb{P}^4$.

**Théorème 2.1.** *Soit $n \geq 3$ un entier, et soit $X \subset \mathbb{P}^n_k$ une hypersurface lisse de degré $d$ sur un corps algébriquement clos $k$. Soit $\ell$ un nombre premier différent de la caractéristique de $k$.*

(i) *Pour $n = 3$ et $n \geq 6$, l'application cycle*

$$\mathrm{cyc}_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$$

*est surjective.*

(ii) *Pour $n = 4$ et $d \leq 4$, l'application cycle*

$$\mathrm{cyc}_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$$

*est surjective.*

(iii) *Pour $n \neq 5$ et $d \leq n$, on a $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.*

*Démonstration.* Établissons (i). Pour $n = 3$, la classe de tout $k$-point de $X$ engendre le $\mathbb{Z}_\ell$-module $H^4(X, \mathbb{Z}_\ell(2)) \simeq \mathbb{Z}_\ell$. L'énoncé (i) est donc clair pour $n = 3$.

Supposons $n \geq 4$. Soit $U = \mathbb{P}^n_k \setminus X$. Pour tout entier $m > 0$, on a la suite exacte de cohomologie étale à supports propres [Milne 1980, III.1.30] :

$$H^4_c(U, \mathbb{Z}/\ell^m(2)) \to H^4(\mathbb{P}^n, \mathbb{Z}/\ell^m(2)) \to H^4(X, \mathbb{Z}/\ell^m(2)) \to H^5_c(U, \mathbb{Z}/\ell^m(2)).$$

Les groupes finis $H^i_c(U, \mathbb{Z}/\ell^m(2))$ et $H^{2n-i}(U, \mathbb{Z}/\ell^m(2n-2))$ sont duaux (dualité de Poincaré [Milne 1980, VI.11.2]).

Pour $n \geq 6$, on a $2n-4 > 2n-5 > n$. Le théorème de Lefschetz affine [Milne 1980, VI.7.2] donne $H^{2n-4}(U, \mathbb{Z}/\ell^m(2n-2)) = 0$ et $H^{2n-5}(U, \mathbb{Z}/\ell^m(2n-2)) = 0$.

La flèche de restriction $H^4(\mathbb{P}^n, \mathbb{Z}/\ell^m(2)) \to H^4(X, \mathbb{Z}/\ell^m(2))$ est donc un isomorphisme de groupes finis pour tout $m$. La flèche de restriction

$$\mathbb{Z}_\ell = H^4(\mathbb{P}^n, \mathbb{Z}_\ell(2)) \to H^4(X, \mathbb{Z}_\ell(2))$$

est donc un isomorphisme. Ceci implique que l'application cycle

$$\mathrm{cyc}_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$$

est surjective. Ceci établit (i) pour $n \geq 6$.

Pour $n > 4$, la considération de la suite exacte

$$H^3(\mathbb{P}^n, \mathbb{Z}/\ell^m(2)) \to H^3(X, \mathbb{Z}/\ell^m(2)) \to H^4_c(U, \mathbb{Z}/\ell^m(2)),$$

la dualité de Poincaré et le théorème de Lefschetz affine donnent alors

$$H^3(X, \mathbb{Z}/\ell^m(2)) = 0$$

pour tout $m$ et donc $H^3(X, \mathbb{Z}_\ell(2)) = 0$. Ceci sera utilisé dans la démonstration du théorème 3.1 ci-après.

Établissons l'énoncé (ii). Soit donc $n = 4$. L'argument qui suit corrige celui donné dans [Colliot-Thélène 2015, théorème 5.6(vi)].

Pour tout degré $d$, et tout entier $m > 0$, la flèche de restriction $H^2(\mathbb{P}^4, \mathbb{Z}/\ell^m) \to H^2(X, \mathbb{Z}/\ell^m)$ est un isomorphisme, comme on voit en utilisant la suite exacte de cohomologie étale à supports, la dualité de Poincaré sur $U = \mathbb{P}^4 \setminus X$, et le théorème de Lefschetz affine. Ceci implique $H^2(X, \mathbb{Z}/\ell^m) \simeq \mathbb{Z}/\ell^m$, et ceci implique que l'application cycle $\mathrm{CH}^1(X)/\ell^m \to H^2(X, \mu_{\ell^m})$ définie via l'application de Kummer $\mathrm{Pic}(X)/\ell^m \to H^2(X, \mu_{\ell^m})$ est un isomorphisme.

Le cup-produit sur la cohomologie étale

$$H^4(X, \mathbb{Z}/\ell^m(2)) \times H^2(X, \mathbb{Z}/\ell^m(1)) \to H^6(X, \mathbb{Z}/\ell^m(3)) = \mathbb{Z}/\ell^m$$

est un accouplement non dégénéré de groupes finis (dualité de Poincaré). D'après ce qui précède, chacun des deux termes de cet accouplement est isomorphe à $\mathbb{Z}/\ell^m$. Considérons le diagramme

$$\begin{array}{ccccc} \mathrm{CH}^2(X)/\ell^m & \times & \mathrm{CH}^1(X)/\ell^m & \longrightarrow & \mathbb{Z}/\ell^m \\ \downarrow & & \downarrow \simeq & & \downarrow = \\ H^4(X, \mathbb{Z}/\ell^m(2)) & \times & H^2(X, \mathbb{Z}/\ell^m(1)) & \longrightarrow & \mathbb{Z}/\ell^m \end{array}$$

où l'accouplement supérieur est donné par l'intersection des cycles.

Pour $\ell \neq 2 = (\dim(X) - 1)!$, ce diagramme est commutatif [Milne 1980, Proposition VI.10.7], Pour tout premier $\ell$, il commute sur les couples de cycles $(Z_1, Z_2)$ transverses l'un à l'autre [Milne 1980, Proposition VI.9.5]. Soit $Y = H \cap X \subset X$ la trace d'un hyperplan $H \subset \mathbb{P}^4$. Sous l'hypothèse $d \leq 4$, l'hypersurface $X$ contient une droite $L \subset \mathbb{P}^4$. Ceci est bien connu pour $d = 3$ ; pour un énoncé général, voir [Debarre 2017, Theorem 2.1]. Dans l'accouplement supérieur, on a $(L, Y) = 1$. En appliquant [Milne 1980, Proposition VI.9.5], on voit que la classe de cycle de $L$ dans $H^4(X, \mathbb{Z}/\ell^m(2)) \simeq \mathbb{Z}/\ell^m$ engendre ce groupe. Ainsi l'application cycle

$$\mathrm{cyc}_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$$

est surjective. Ceci établit (ii) pour $n = 4$.

Montrons maintenant (iii). D'après [Kahn 2012, théorème 1.1] ou [Colliot-Thélène et Kahn 2013, théorème 2.2], la surjectivité de

$$\mathrm{cyc}_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2)) = \mathbb{Z}_\ell$$

implique que le groupe $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ est divisible.

D'après un théorème de Roitman [1980] (voir aussi [Chatzistamatiou et Levine 2017, §4]), l'hypothèse $d \leq n$ implique que sur tout corps algébriquement clos $K$ contenant $k$, l'application degré $\mathrm{CH}_0(X_K) \to \mathbb{Z}$ sur le groupe de Chow des zéro-cycles est un isomorphisme. D'après un argument général (voir [Colliot-Thélène et Kahn 2013, proposition 3.2]), ceci implique l'existence d'un entier $N > 0$ qui annule $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$.

Sous l'hypothèse $n \neq 5$ et $d \leq n$, on a donc établi que le groupe $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ est divisible et d'exposant fini. Il est donc nul. □

**Remarque 2.2.** Pour $k = \mathbb{C}$ et $X \subset \mathbb{P}^n_k$ comme ci-dessus avec $d \leq n$ et tout corps $F$ contenant $k$, et pour $n \geq 6$, on a établi dans [Colliot-Thélène 2015, théorème 5.6(vii)] que la flèche naturelle

$$H^3(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \to H^3_{\mathrm{nr}}(X_F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$$

est un isomorphisme. Il est très vraisemblable que ce résultat vaut sur tout corps $k$ algébriquement clos, avec $\ell$ distinct de la caractéristique de $k$.

## 3. Hypersurfaces de Fano dans $\mathbb{P}^n_{\mathbb{F}}$, $\mathbb{F}$ fini, $n = 3$ et $n \geq 6$

**Théorème 3.1.** *Soit $n \geq 3$ un entier et soit $X \subset \mathbb{P}^n_{\mathbb{F}}$ une hypersurface lisse de degré $d \leq n$ sur un corps fini $\mathbb{F}$. Soit $\ell$ un nombre premier différent de la caractéristique de $\mathbb{F}$. Pour $n = 3$ et pour $n \geq 6$, on a $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.*

*Démonstration.* D'après la proposition 1.3, on peut supposer $n \geq 6$.

Pour $n \geq 6$, on a établi dans la démonstration du théorème 2.1 que l'on a $H^3(\overline{X}, \mathbb{Z}_\ell(2)) = 0$ et que la restriction

$$\mathbb{Z}_\ell = H^4(\mathbb{P}^n_{\overline{\mathbb{F}}}, \mathbb{Z}_\ell(2)) \to H^4(\overline{X}, \mathbb{Z}_\ell(2))$$

est un isomorphisme. Pour toute $\mathbb{F}$-variété $Y$, on dispose de la suite exacte déduite de la suite spectrale de Leray

$$0 \to H^1(\mathbb{F}, H^3(\overline{Y}, \mathbb{Z}_\ell(2))) \to H^4(Y, \mathbb{Z}_\ell(2)) \to H^0(\mathbb{F}, H^4(\overline{Y}, \mathbb{Z}_\ell(2))) \to 0.$$

La comparaison de cette suite pour $Y = \mathbb{P}^n_{\mathbb{F}}$ et pour $Y = X$ donne que l'application cycle

$$\mathrm{cyc}_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2)) = \mathbb{Z}_\ell$$

est surjective.

D'après [Kahn 2012, théorème 1.1] ou [Colliot-Thélène et Kahn 2013, théorème 2.2], sur un corps fini $\mathbb{F}$, la surjectivité de

$$\mathrm{cyc}_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2)) = \mathbb{Z}_\ell$$

implique que le groupe $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ est divisible.

Comme rappelé dans la démonstration du théorème 2.1, l'hypothèse $d \leq n$, le théorème de Roitman [1980] et l'argument donné dans [Colliot-Thélène et Kahn 2013, proposition 3.2] impliquent que $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ est d'exposant fini.

Le groupe $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ est divisible et d'exposant fini, il est donc nul. □

## 4. Hypersurfaces cubiques dans $\mathbb{P}^5_k$, $k$ algébriquement clos

Déjà pour les hypersurfaces cubiques, le théorème 2.1, sur un corps algébriquement clos, laisse ouvert le cas $n = 5$. Pour $X \subset \mathbb{P}^5_{\mathbb{C}}$ une hypersurface cubique lisse sur le corps des complexes, Voisin [2007, Theorem 18] a établi la conjecture de Hodge entière dans ce contexte. D'après [Colliot-Thélène et Voisin 2012], ceci implique $H^3_{\mathrm{nr}}(X, \mathbb{Q}/\mathbb{Z}(2)) = 0$, et [Colliot-Thélène 1995, Theorem 4.4.1] montre alors que le résultat vaut pour toute hypersurface cubique lisse $X \subset \mathbb{P}^5_k$ sur un corps $k$ algébriquement clos de caractéristique zéro.

En utilisant [Charles et Pirutka 2015], on obtient l'analogue de ce résultat sur tout corps algébriquement clos, avec une restriction mineure sur la caractéristique.

**Théorème 4.1.** *Soit $k$ un corps algébriquement clos de caractéristique différente de 2 et 3. Soit $X \subset \mathbb{P}^5_k$ une hypersurface cubique lisse. Soit $\ell$ premier différent de la caractéristique de $k$. On a $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.*

*Démonstration.* D'après la proposition 1.4, le groupe $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ est d'exposant fini, en fait divisant 2. La proposition 1.4 donne donc déjà le résultat pour $\ell \neq 2$.

Par une variante du lemme de rigidité de Suslin [Colliot-Thélène 1995, Theorem 4.4.1], pour établir ce dernier énoncé $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$, on peut se limiter à considérer le cas où $k$ est une clôture algébrique d'un corps $F$ de type fini sur le corps premier, et où $X = X_0 \times_F k$ pour $X_0 \subset \mathbb{P}^5_F$ une hypersurface cubique lisse.

On considère l'application cycle $\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$. Elle respecte l'action du groupe de Galois $\mathrm{Gal}(k/F)$. Elle envoie donc le groupe des cycles dans le sous-groupe

$$H^4(X, \mathbb{Z}_\ell(2))^f \subset H^4(X, \mathbb{Z}_\ell(2))$$

des classes dont le stabilisateur est un sous-groupe ouvert.

Comme $H^4(X, \mathbb{Z}_\ell(2))$ est un $\mathbb{Z}_\ell$-module de type fini et l'action de $\mathrm{Gal}(k/F)$ est continue, le conoyau de

$$H^4(X, \mathbb{Z}_\ell(2))^f \to H^4(X, \mathbb{Z}_\ell(2))$$

est un groupe sans torsion [Colliot-Thélène et Kahn 2013, lemme 4.1].

Charles et Pirutka [2015, théorème 1.1] ont montré que l'application

$$\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))^f$$

est surjective. On conclut que le conoyau de

$$\mathrm{cyc}_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$$

est un groupe sans torsion. D'après [Kahn 2012, théorème 1.1] ou [Colliot-Thélène et Kahn 2013, théorème 2.2], le groupe fini donné par la torsion du conoyau de l'application cycle

$$\mathrm{cyc}_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$$

coïncide avec le groupe quotient de $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ par son sous-groupe divisible maximal. D'après la proposition 1.4, le groupe $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ est d'exposant fini. Ceci établit $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$. □

**Remarque 4.2.** Pour $X \subset \mathbb{P}^5_{\mathbb{C}}$ une hypersurface cubique lisse, la conjecture de Hodge rationnelle (à coefficients dans $\mathbb{Q}$) pour les cycles de codimension deux est connue depuis 1977 [Zucker 1977 ; Murre 1977]. La nullité de $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ établie ci-dessus et [Colliot-Thélène et Voisin 2012, théorème 1.1] redonnent donc la conjecture de Hodge entière pour les cycles de codimension deux sur ces hypersurfaces, c'est-à-dire le résultat établi en 2007 par Voisin [2007, Theorem 18 ; 2013, Theorem 3.11]. Il convient cependant d'observer que la démonstration ci-dessus repose de façon essentielle sur [Charles et Pirutka 2015], dont les méthodes géométriques sont inspirées de celles de [Voisin 2007] (qui cite [Zucker 1977]).

## 5. Hypersurfaces cubiques dans $\mathbb{P}^4_{\mathbb{F}}$, $\mathbb{F}$ corps fini

Pour les hypersurfaces cubiques lisses sur un corps fini, [Parimala et Suresh 2016] permet de compléter le théorème 3.1 pour $n = 4$.

**Théorème 5.1.** *Soit $X \subset \mathbb{P}^4_{\mathbb{F}}$ une hypersurface cubique lisse sur un corps fini $\mathbb{F}$ de caractéristique différente de* 2*.*

(i) *Pour tout $\ell$ premier différent de la caractéristique de $\mathbb{F}$, on a*

$$H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0.$$

(ii) *Soit $\overline{\mathbb{F}}$ une clôture algébrique de $\mathbb{F}$ et $G = \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$. L'application naturelle*

$$\mathrm{CH}^2(X) \to \mathrm{CH}^2(\overline{X})^G$$

*est un isomorphisme.*

(iii) *L'application cycle*

$$\mathrm{cyc}_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$$

*est surjective.*

*Démonstration.* (i) Le cas $\ell \neq 2$ résulte déjà de la proposition 1.4. Pour démontrer le théorème, par le lemme 1.1 et un argument de restriction-corestriction, on peut supposer que $X$ contient une droite $L \subset X$ définie sur le corps $\mathbb{F}$. En éclatant $X$ le long de $L$, on trouve une $\mathbb{F}$-variété projective et lisse $Y$ $\mathbb{F}$-birationnelle à $X$ et munie d'une structure de fibration en coniques sur $\mathbb{P}^2_{\mathbb{F}}$. Le théorème de Parimala et Suresh [2016, Corollary 5.6] donne alors $H^3_{\mathrm{nr}}(Y, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$, et donc $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$.

(ii) On sait (théorème de Lefschetz faible) que $H^3(\bar{X}, \mathbb{Z}_\ell)$ est sans torsion. Alors la nullité de $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ et [Colliot-Thélène et Kahn 2013, corollaire 6.9] donnent (ii).

(iii) Comme $X$ est géométriquement unirationnelle de dimension 3, le conoyau de l'application cycle $\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$ est un groupe fini [Colliot-Thélène et Kahn 2013, proposition 3.23]. D'après [Kahn 2012, théorème 1.1] ou [Colliot-Thélène et Kahn 2013, théorème 2.2], la torsion du conoyau de l'application cycle s'identifie au quotient de $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ par son sous-groupe divisible maximal. De (i) résulte donc (iii). □

**Remarque 5.2.** La démonstration du théorème de Parimala et Suresh [2016] utilise un résultat de théorie du corps de classes supérieur, à savoir la nullité de $H^3_{\mathrm{nr}}(S, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ pour $S$ une surface projective et lisse sur un corps fini [Colliot-Thélène et al. 1983, remarque 2, p. 790 ; Kato 1986, Theorem 0.7, Corollary]. Elle utilise aussi beaucoup d'autres arguments délicats.

En utilisant la théorie du corps de classes supérieur, et le lien entre la surface de Fano des droites de $X$ et le groupe des cycles de codimension 2 de $X$, on peut donner une démonstration alternative du théorème 5.1. Soit $Y/\mathbb{F}$ la surface de Fano de $X$, qui paramétrise les droites de $X$. C'est une surface projective, lisse, géométriquement connexe [Altman et Kleiman 1977, Corollary 1.12], qui possède donc un zéro-cycle de degré 1 sur le corps fini $\mathbb{F}$.

La famille universelle des droites de $X$ définit une correspondance entre $Y$ et $X$ qui induit un homomorphisme $\mathrm{CH}_0(Y) \to \mathrm{CH}^2(X)$, lequel induit une application $A_0(Y) \to \mathrm{CH}^2_0(X)$, où l'on a noté $A_0(Y) \subset \mathrm{CH}_0(Y)$ le sous-groupe des zéro-cycles de degré zéro, et $\mathrm{CH}^2_0(X) \subset \mathrm{CH}^2(X)$ le sous-groupe des 1-cycles d'intersection nulle avec une section hyperplane. Sur un corps de caractéristique différente de 2, on sait [Murre 1974, VI, VII] que l'application $A_0(\bar{Y}) \to \mathrm{CH}^2_0(\bar{X})$ se factorise comme

$$A_0(\bar{Y}) \to \mathrm{Alb}_Y(\bar{\mathbb{F}}) \xrightarrow{\simeq} \mathrm{CH}^2_0(\bar{X}).$$

D'après le théorème de Roitman, l'application d'Albanese $A_0(\bar{Y}) \to \mathrm{Alb}_Y(\bar{\mathbb{F}})$, qui est surjective, a son noyau uniquement divisible (en fait, pour $\mathbb{F}$ corps fini, cette flèche est un isomorphisme). Ceci assure que l'application $A_0(\bar{Y})^G \to \mathrm{CH}^2_0(\bar{X})^G$ est surjective. On a le diagramme commutatif

$$\begin{array}{ccc} A_0(Y) & \longrightarrow & \mathrm{CH}^2_0(X) \\ \downarrow & & \downarrow \\ A_0(\bar{Y})^G & \longrightarrow & \mathrm{CH}^2_0(\bar{X})^G \end{array}$$

La théorie du corps de classes supérieur [Kato et Saito 1983, Proposition 9.1] montre que, pour toute variété projective lisse $Y$ géométriquement connexe sur un corps fini, l'application $A_0(Y) \to A_0(\bar{Y})^G$ est surjective (pour $Y/\mathbb{F}$ une surface, voir aussi [Colliot-Thélène et Kahn 2013, §6.2]). On conclut donc que $\mathrm{CH}^2_0(X) \to \mathrm{CH}^2_0(\bar{X})^G$ est surjectif, puis que $\mathrm{CH}^2(X) \to \mathrm{CH}^2(\bar{X})^G$ est surjectif. Ceci donne l'énoncé (ii) du théorème 5.1. Comme on a $H^3_{\mathrm{nr}}(\bar{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$, l'énoncé (i) résulte alors de (ii) et de [Colliot-Thélène et Kahn 2013, corollaire 6.9]. L'application $\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$ a son conoyau fini. D'après [Kahn 2012] ou [Colliot-Thélène et Kahn 2013, théorème 2.2], ce conoyau s'identifie au quotient de $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ par son sous-groupe divisible maximal. Ainsi l'application $\mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2))$ est surjective.

**Remarque 5.3.** Sur un corps fini $\mathbb{F}$ et pour un nombre premier $\ell \neq \mathrm{car}(\mathbb{F})$, la question si l'on a $H^3_{\mathrm{nr}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = 0$ pour une hypersurface cubique lisse $X \subset \mathbb{P}^5_{\mathbb{F}}$ reste ouverte dans le cas crucial $\ell = 2$ (pour $\ell \neq 2$, voir la proposition 1.4(iv)). Elle est équivalente à la question de la surjectivité de l'application cycle

$$\mathrm{cyc}_X : \mathrm{CH}^2(X) \otimes \mathbb{Z}_\ell \to H^4(X, \mathbb{Z}_\ell(2)).$$

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# On the ultimate energy bound of solutions to some forced second-order evolution equations with a general nonlinear damping operator

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Under suitable growth and coercivity conditions on the nonlinear damping operator $g$ which ensure nonresonance, we estimate the ultimate bound of the energy of the general solution to the equation $\ddot{u}(t) + Au(t) + g(\dot{u}(t)) = h(t),\ t \in \mathbb{R}^+$, where $A$ is a positive selfadjoint operator on a Hilbert space $H$ and $h$ is a bounded forcing term with values in $H$. In general the bound is of the form $C(1 + \|h\|^4)$, where $\|h\|$ stands for the $L^\infty$ norm of $h$ with values in $H$ and the growth of $g$ does not seem to play any role. If $g$ behaves like a power for large values of the velocity, the ultimate bound has quadratic growth with respect to $\|h\|$ and this result is optimal. If $h$ is antiperiodic, we obtain a much lower growth bound and again the result is shown to be optimal even for scalar ODEs.

## 1. Introduction

We investigate a specific quantitative aspect of solutions to the equation

$$\ddot{u}(t) + Au(t) + g(\dot{u}(t)) = h(t),$$

where $V$ is a real Hilbert space, $A \in L(V, V')$ is a symmetric, positive, coercive operator, $g \in C(V, V')$ is monotone and $h$ is a forcing term. This equation has been intensively studied in the literature when $g$ is a local damping term, covering the following topics: existence of almost periodic solutions, asymptotic behavior of the general solution, rate of decay to 0 of the difference of two solutions in the energy space in the best cases; see, e.g., [Amerio and Prouse 1969; Prouse 1965a; 1965b; 1965c; 1965d; Biroli 1973; Biroli and Haraux 1980; Haraux 1981; 1982; 1985; 1987; 1991; Haraux and Zuazua 1988; Zuazua 1988]. In the more recent paper [Aloui et al. 2013], a result generalizing the theorems of [Haraux 1987] on boundedness and compactness has been proved for possibly nonlocal damping terms. However when looking at the arguments of those two papers and trying to extract an estimate of the solutions for $t$ large, we find immediately that

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the methodology cannot be adapted to that purpose. The present article aims at improving the situation. Actually we devise a new technique which allows us to “forget” the influence of the initial data from the very beginning of the estimates, thus dropping all unnecessary terms related to transient behavior. The plan of the paper is as follows: In Section 2, we introduce the basic tools used in the statements and proofs of the main results. Section 3 is devoted to a very general case. Section 4 covers a still rather general case where the damping operator behaves like a power for large values of the velocity, this for instance allows us to encompass any polynomial map, and we give a short list of examples in the field of PDEs of the second order in $t$ for which our result is optimal. In Section 5 we establish two partial results when the forcing is antiperiodic, a situation which is known (see, e.g., [Haraux 1989]) to prevent resonance under weaker conditions on $g$ than the general periodic case. We obtain a better estimate which is optimal in finite dimensions, but in the infinite-dimensional setting we can only slightly improve the general estimate and we do not reach what one might expect to be the optimal result.

## 2. Functional framework and the initial value problem

We now recall the exact functional framework that shall be used in the formulation as well as in the proofs of our new results. We follow the presentation from [Aloui et al. 2013] at the exception of a small difference for the approximation of weak solutions.

**2A.** ***Monotone operators.*** Let $\mathcal{H}$ be a real Hilbert space endowed with an inner product $(\cdot\,,\cdot)_{\mathcal{H}}$. We recall that a map $\mathcal{A}$ defined on a subset $\mathcal{D} = D(\mathcal{A})$ with values in $\mathcal{H}$ is monotone if

$$\forall\,(U, \widehat{U}) \in \mathcal{D}\times\mathcal{D}, \quad (\mathcal{A}U - \mathcal{A}\widehat{U}, U - \widehat{U})_{\mathcal{H}} \geq 0.$$

In addition $\mathcal{A}$ is called maximal monotone if

$$\forall\, F \in \mathcal{H},\ \exists\, U \in D(\mathcal{A}), \quad \mathcal{A}U + U = F.$$

The following result is well known; see [Brezis 1973].

**Proposition 2.1.** *If $\mathcal{A}$ is maximal monotone, for each $T > 0$, each $U_0 \in D(\mathcal{A})$ and $F = F(t) \in W^{1,1}(0, T; \mathcal{H})$ there is a unique function $U \in W^{1,1}(0, T; \mathcal{H})$ with $U(t) \in D(\mathcal{A})$ for almost all $t \in (0, T)$, $U(0) = U_0$ and such that for almost all $t \in (0, T)$*

$$U'(t) + \mathcal{A}U(t) = F(t). \tag{2-1}$$

*In addition if for some $\widehat{U}_0 \in D(\mathcal{A})$ and $\widehat{F} \in W^{1,1}(0, T; \mathcal{H})$ we consider the solution $\widehat{U} \in W^{1,1}(0, T; \mathcal{H})$ with $\widehat{U}(t) \in D(\mathcal{A})$ for almost all $t \in (0, T)$, $\widehat{U}(0) = \widehat{U}_0$ of*

$$\widehat{U}'(t) + \mathcal{A}\widehat{U}(t) = \widehat{F}(t),$$

*then the difference satisfies the inequality*

$$\forall\, t \in [0, T], \quad |U(t) - \widehat{U}(t)| \leq |U_0 - \widehat{U}_0| + \int_0^t |F(s) - \widehat{F}(s)|\, ds.$$

This proposition allows one to define by density, for any $U_0 \in \overline{D(\mathcal{A})}$ and $F = F(t) \in L^1(0, T; \mathcal{H})$, a weak solution of (2-1) such that $U(0) = U_0$ [Brezis 1973].

**2B.** ***Functional setting.*** Throughout this article we let $H$ and $V$ be two Hilbert spaces with norms respectively denoted by $\|\cdot\|$ and $|\cdot|$. We assume that $V$ is densely and continuously embedded into $H$. Identifying $H$ with its dual $H'$, we obtain $V \hookrightarrow H = H' \hookrightarrow V'$. We denote inner products by $(\cdot\,,\cdot)$ and duality products by $\langle\cdot\,,\cdot\rangle$; the spaces in question will be specified by subscripts. The notation $\langle f, u\rangle$ without any subscript will be used sometimes to denote $\langle f, u\rangle_{V',V}$. The duality map: $V \to V'$ will be denoted by $A$. We observe that $A$ is characterized by the property

$$\forall\,(u, v) \in V \times V, \quad \langle Au, v\rangle_{V',V} = (u, v)_V.$$

**2C.** ***Weak solutions.*** We consider the dissipative evolution equation

$$\ddot{u} + Au + g(\dot{u}) = h(t), \tag{2-2}$$

where $g \in C(V, V')$ satisfies

$$\forall\,(v, w) \in V \times V, \quad \langle g(v) - g(w), v - w\rangle \geq 0. \tag{2-3}$$

We consider the (generally unbounded) operator $\mathcal{A}$ defined on the Hilbert space $\mathcal{H} = V \times H$ by

$$D(\mathcal{A}) = \{(u, v) \in V \times V : Au + g(v) \in H\}$$

and

$$\forall\,(u, v) \in D(\mathcal{A}), \quad \mathcal{A}(u, v) = (-v, Au + g(v)).$$

**Lemma 2.2.** *The operator $\mathcal{A}$ is maximal monotone.*

*Proof.* Let $U = (u, v)$ and $\widehat{U} = (\hat{u}, \hat{v})$ be two elements of $D(\mathcal{A})$. We have

$$(\mathcal{A}U - \mathcal{A}\widehat{U}, U - \widehat{U})_{\mathcal{H}} = -(u-\hat{u}, v-\hat{v})_V + \big(Au+g(v)-A\hat{u}-g(\hat{v}), v-\hat{v}\big)_H$$
$$- (u-\hat{u}, v-\hat{v})_V + \big\langle Au+g(v)-A\hat{u}-g(\hat{v}), v-\hat{v}\big\rangle_{V',V}$$

since $Au + g(v) \in H$ and $A\hat{u} + g(\hat{v}) \in H$ while $v, \hat{v}$ are in $V$. This reduces to

$$(\mathcal{A}U - \mathcal{A}\widehat{U}, U - \widehat{U})_{\mathcal{H}} = \langle g(v) - g(\hat{v}), v - \hat{v}\rangle_{V',V} \geq 0.$$

Hence $\mathcal{A}$ is monotone. To prove that $\mathcal{A}$ is maximal monotone we are left to show that for any $F = (\varphi, \psi) \in \mathcal{H}$ the system

$$u - v = \varphi, \quad Au + g(v) + v = \psi$$

has a solution $U = (u, v) \in D(\mathcal{A})$. This is equivalent to finding a solution $v \in V$ of

$$Av + g(v) + v = \psi - A\varphi \in V'.$$

But now the operator $\mathcal{C} \in C(V, V')$, defined by

$$\forall\, v \in V, \quad \mathcal{C}v = Av + g(v) + v,$$

is continuous and coercive since $V \to V'$ is the sum of a monotone operator and the coercive duality map. Therefore by Corollary 14, p. 126 from [Brezis 1968], $\mathcal{C}$ is surjective. Finally $\mathcal{A}$ is maximal monotone as claimed. □

As a consequence of Proposition 2.1, for any $h \in L^1_{\text{loc}}(\mathbb{R}^+, H)$ and for each $(u_0, u_1) \in V \times H$ there is a unique weak solution

$$u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$$

of (2-2) such that $u(0) = u_0$ and $\dot{u}(0) = u_1$. This solution can be recovered on each compact interval $[0, T]$ by approximating the initial data by elements of the domain, approximating the forcing term $h$ by $C^1$ functions and passing to the limit; the limit is independent of the approximating elements so chosen. The next result shows that in fact the approximation can even be made uniform on $\mathbb{R}^+$.

## 2D. *Density of strong solutions.*

**Lemma 2.3.** *For any $h \in L^2_{\text{loc}}(\mathbb{R}^+, H)$, for each $(u_0, u_1) \in V \times H$ and for each $\delta > 0$ there exists $(w_0, w_1) \in D(\mathcal{A})$ and $k \in C^1(\mathbb{R}^+, H)$ for which the solution $w \in W^{1,1}_{\text{loc}}(\mathbb{R}^+, V) \cap W^{2,1}_{\text{loc}}(\mathbb{R}^+, H)$ of*

$$\ddot{w} + Aw + g(\dot{w}) = k(t), \quad w(0) = w_0, \ \ \dot{w}(0) = w_1,$$

*satisfies*

$$\forall\, t \geq 0, \quad \|u(t) - w(t)\| + |\dot{u}(t) - \dot{w}(t)| \leq \delta,$$

*and in addition*

$$\forall\, t \in \mathbb{R}^+, \quad \int_t^{t+1} |k(s) - h(s)|^2\, ds \leq 2\delta.$$

*Proof.* It suffices to use the last result of Proposition 2.1 by observing that for any $h \in L^2_{\text{loc}}(\mathbb{R}^+, H)$ we can find $k \in C^1(\mathbb{R}^+, H)$ such that

$$\forall\, n \in \mathbb{N}, \quad \int_n^{n+1} |k(s) - h(s)|^2\, ds \leq \delta 2^{-2n-2}.$$

Choosing $(w_0, w_1) \in D(\mathcal{A})$ such that

$$\|w_0 - u_0\| + |w_1 - v_1| \leq \delta 2^{-1}$$

the result follows immediately □

## 3. A general ultimate bound

We now give a quite different proof, in a slightly more general case, of a result stated in [Haraux 1985, Remark 1.2(b), p. 167]. We assume that $h \in S^2(\mathbb{R}^+, H)$ with

$$S^2(R^+, H) = \left\{ f \in L^2_{\text{loc}}(R^+, H) : \sup_{t \in R^+} \int_t^{t+1} |f(s)|^2 \, ds < \infty \right\}$$

and we set

$$\|h\|_{S^2(\mathbb{R}^+, H)} = \left( \sup_{t \in R^+} \int_t^{t+1} |h(s)|^2 \, ds \right)^{1/2}.$$

In particular if $h \in L^\infty(\mathbb{R}^+, H)$, then $h \in S^2(\mathbb{R}^+, H)$ and $\|h\|_{S^2(\mathbb{R}^+, H)} \leq \|h\|_{L^\infty(\mathbb{R}^+, H)}$.

**Theorem 3.1.** *Assume that $g \in C(V, V')$ satisfies the condition (2-3) and*

$$\exists\, \gamma > 0\,,\ \exists\, C_1 \geq 0,\ \forall\, v \in V, \quad \langle g(v), v \rangle \geq \gamma |v|^2 - C_1, \tag{3-1}$$

$$\exists\, K > 0,\ \exists\, C_2 \geq 0,\ \forall\, v \in V, \quad \|g(v)\|_{V'} \leq C_2 + K \langle g(v), v \rangle. \tag{3-2}$$

*Then any solution $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$ of (2-2) is bounded on $\mathbb{R}^+$ in the sense that $u$ has bounded range in $V$ and $\dot{u}$ has bounded range in $H$. In addition we have for some constant $K$ depending only on $A$ and $g$*

$$\limsup_{t \to \infty} (|\dot{u}(t)|^2 + \|u(t)\|^2) \leq K(1 + \|h\|^4_{S^2(\mathbb{R}^+, H)}).$$

*Proof.* The boundedness result is known for local damping operators $g$, see the second case of Theorem IV.2.1.1 of [Haraux 1987], and in the general case it can be proved by adapting in this case the method from [Aloui et al. 2013]. However even in the local case these results cannot provide a reasonable estimate of the ultimate bound. We start by an estimate in the case of a strong solutions; i.e., we assume

$$u \in W^{1,1}_{\text{loc}}(\mathbb{R}^+, V) \cap W^{2,1}_{\text{loc}}(\mathbb{R}^+, H).$$

The general case will follow by density. Let

$$E(t) = \tfrac{1}{2}(|\dot{u}|^2 + \|u\|^2).$$

Under the regularity conditions $[u_0, v_0] \in V \times V,\ g(v_0) \in H$ and $h \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+, H)$, the function: $t \to E(t)$ is absolutely continuous and we have, for all $t \in \mathbb{R}^+$

$$\frac{d}{dt}E(t) = (h, \dot{u}) - \langle g(\dot{u}), \dot{u}\rangle. \tag{3-3}$$

In addition $t \to (u(t), \dot{u}(t))$ is absolutely continuous and

$$\frac{d}{dt}(u(t), \dot{u}(t)) = |\dot{u}|^2 - \|u\|^2 - \langle g(\dot{u}), u\rangle + (h, u).$$

By using (3-2), we obtain

$$\frac{d}{dt}(u(t), \dot{u}(t)) \le |\dot{u}|^2 - \|u\|^2 + \|u\|\big(P|h| + C_2 + K\langle g(\dot{u}), \dot{u}\rangle\big) \tag{3-4}$$

with

$$P = \sup\{|u| : u \in V,\ \|u\| = 1\}.$$

Introducing

$$\Phi(t) = 2E(t) \quad \forall t \ge 0,$$

we are reduced to estimating the upper limit bound for $\Phi(t)$. Let us introduce

$$M = \limsup_{t\to\infty} \Phi(t)$$

and let us consider a sequence of times $t_n$ tending to infinity for which

$$\Phi(t_n) \ge M - \frac{1}{n}.$$

In addition, for $n$ large enough we have $t_n \ge \tau$ and

$$\Phi(t_n - \tau) \le M + \frac{1}{n},$$

where $\tau$ is any fixed positive number to be chosen later. Therefore by integrating (3-3) on $[t_n - \tau, t_n]$ we find

$$\int_{t_n-\tau}^{t_n} \langle g(\dot{u}), \dot{u}\rangle\, dt \le \frac{1}{n} + \int_{t_n-\tau}^{t_n} \langle h, \dot{u}\rangle\, dt \le \frac{1}{n} + \frac{\gamma}{2}\int_{t_n-\tau}^{t_n} |\dot{u}|^2\, dt + \frac{1}{2\gamma}\int_{t_n-\tau}^{t_n} |h|^2\, dt.$$

As a consequence of (3-1) we deduce

$$\int_{t_n-\tau}^{t_n} \langle g(\dot{u}), \dot{u}\rangle\, dt \le \frac{2}{n} + \frac{1}{\gamma}\int_{t_n-\tau}^{t_n} |h|^2\, dt + C_3, \tag{3-5}$$

$$\int_{t_n-\tau}^{t_n} |\dot{u}|^2\, dt \le \frac{C_4}{n} + C_5\int_{t_n-\tau}^{t_n} |h|^2\, dt + C_6, \tag{3-6}$$

which provide an average bound of the kinetic part independent of the initial data and the transient behavior. This is remarkable since we only used the properties

$\Phi(t_n) \geq M - 1/n$ and $\Phi(t_n - \tau) \leq M + 1/n$ to express the fact that $t$ is large. By combining these two estimates we also find an estimate of the form

$$|\Phi(t) - \Phi(s)| \leq C_7\Big(1 + \int_{t_n-\tau}^{t_n} |h|^2\, dt\Big), \tag{3-7}$$

which is valid for all $s$, $t$ in $[t_n - \tau, t_n]$. As a consequence if we had an $L^1$ estimate of the total energy instead of the kinetic part, the proof would be completed with exponent 2 instead of 4. The difficulty in fact comes from the potential energy. From (3-4), by integrating on $[t_n - \tau, t_n]$ we find

$$\int_{t_n-\tau}^{t_n} \|u\|^2\, dt \leq \int_{t_n-\tau}^{t_n} |\dot u|^2\, dt + \int_{t_n-\tau}^{t_n} \big[\|u\|\big(P|h| + C_2 + K\langle g(\dot u), \dot u\rangle\big)\big]\, dt + \big|[(u(t), \dot u(t))]_{t_n-\tau}^{t_n}\big|$$

Recalling the notation $M = \limsup_{t\to\infty} \Phi(t)$ we find

$$\int_{t_n-\tau}^{t_n} \|u\|^2\, dt \leq \int_{t_n-\tau}^{t_n} |\dot u|^2\, dt + M^{1/2}\int_{t_n-\tau}^{t_n} \big(P|h| + C_2 + K\langle g(\dot u), \dot u\rangle\big)\, dt + C_8 M, \tag{3-8}$$

where $C_8$ does not depend on $\tau$. Combining (3-6) and (3-8) we obtain

$$\int_{t_n-\tau}^{t_n} \Phi(t)\, dt \leq C_5\int_{t_n-\tau}^{t_n} |h|^2\, dt + C_9 + M^{1/2}\int_{t_n-\tau}^{t_n} \big(P|h| + C_2 + K\langle g(\dot u), \dot u\rangle\big)\, dt + C_8 M \tag{3-9}$$

and by (3-5) this implies

$$\int_{t_n-\tau}^{t_n} \Phi(t)\, dt \leq C_{10}\Big(1 + \int_{t_n-\tau}^{t_n} |h|^2\, dt\Big)(1 + M^{1/2}) + C_8 M. \tag{3-10}$$

Finally by combining this last inequality with (3-7) we end up with

$$(\tau - C_8)M \leq C_{11}\Big(1 + \int_{t_n-\tau}^{t_n} |h|^2\, dt\Big)(1 + M^{1/2}). \tag{3-11}$$

Fixing $\tau \geq 1 + C_8$, the result now follows easily since

$$\int_{t_n-\tau}^{t_n} |h|^2\, dt \leq (1 + \tau)\|h\|_{S^2}^2.$$

The general case of weak solutions follows easily from density, relying on Lemma 2.3. □

**Remark 3.2.** This ultimate bound has been obtained under the most general known assumption ensuring boundedness of trajectories. It seems not to depend on the kind of damping operator as long as the coerciveness and growth conditions are satisfied. We have absolutely no idea whether it has a chance to be optimal in some cases. A more natural quadratic estimate is valid in many cases, as we shall see in the next section.

## 4. The case of a power-like damping term

For the main result of this section, we need to introduce an additional Banach space $Z$ such that

$$V \subset Z \subset H$$

with continuous embeddings. The norm in $Z$ of a vector $z \in Z$ will be denoted by $\|z\|_Z$.

### 4A. *Main result.*

**Theorem 4.1.** *Assume that $g \in C(V, V')$ satisfies the condition (2-3) and for some $\alpha \geq 0$ we have*

$$\exists\, \gamma > 0,\ \ \exists\, C_1 \geq 0,\ \forall\, v \in V, \quad \langle g(v), v\rangle \geq \gamma \|v\|_Z^{\alpha+2} - C_1, \tag{4-1}$$

$$\exists\, K > 0,\ \exists\, C_2 \geq 0,\ \forall\, v \in V, \quad \|g(v)\|_{V'} \leq C_2 + K\|v\|_Z^{\alpha+1}. \tag{4-2}$$

*Then any solution $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$ of (2-2) is bounded on $\mathbb{R}^+$ in the sense that $u$ has bounded range in $V$ and $\dot{u}$ has bounded range in $H$. In addition we have for some constant $K$ depending only on $A$ and $g$*

$$\limsup_{t\to\infty} (|\dot{u}(t)|^2 + \|u(t)\|^2) \leq K(1 + \|h\|^2_{S^2(\mathbb{R}^+, H)}).$$

*Proof.* We start as in the proof of Theorem 3.1; by integrating (3-3) on $[t_n - \tau, t_n]$ we find

$$\begin{aligned}\int_{t_n-\tau}^{t_n} \langle g(\dot{u}), \dot{u}\rangle\, dt &\leq \frac{1}{n} + \int_{t_n-\tau}^{t_n} \langle h, \dot{u}\rangle\, dt \\ &\leq \frac{1}{n} + \frac{\gamma}{2}\int_{t_n-\tau}^{t_n} \|\dot{u}\|_Z^{\alpha+2}\, dt + C(\gamma)\int_{t_n-\tau}^{t_n} |h|^{(\alpha+2)/(\alpha+1)}\, dt.\end{aligned}$$

As a consequence of (4-1) we deduce, since $n \geq 1$,

$$\int_{t_n-\tau}^{t_n} \langle g(\dot{u}), \dot{u}\rangle\, dt \leq C(\gamma, \tau)\biggl(\int_{t_n-\tau}^{t_n} |h|^2\, dt\biggr)^{(\alpha+2)/(2\alpha+2)} + C_3, \tag{4-3}$$

$$\int_{t_n-\tau}^{t_n} |\dot{u}|^2\, dt \leq C_4 + C_5(\gamma, \tau)\int_{t_n-\tau}^{t_n} |h|^2\, dt. \tag{4-4}$$

From (4-3) we also deduce, since $\frac{1}{2} \le (\alpha+2)/(2\alpha+2)$ the important new estimate

$$\int_{t_n-\tau}^{t_n} \|\dot{u}\|_Z^{\alpha+1}\,dt \le C_6 + C_7(\gamma,\tau)\left(\int_{t_n-\tau}^{t_n} |h|^2\,dt\right)^{1/2} \tag{4-5}$$

and by (4-2) this implies

$$\int_{t_n-\tau}^{t_n} \|g(\dot{u})\|_{V'}\,dt \le C_8 + C_9(\gamma,\tau)\left(\int_{t_n-\tau}^{t_n} |h|^2\,dt\right)^{1/2}. \tag{4-6}$$

Recalling the notation $M = \limsup_{t\to\infty} \Phi(t)$ we now find

$$\int_{t_n-\tau}^{t_n} \|u\|^2\,dt \le \int_{t_n-\tau}^{t_n} |\dot{u}|^2\,dt + C_9(\gamma,\tau)M^{1/2}\left(\int_{t_n-\tau}^{t_n} |h|^2\,dt\right)^{1/2} + C_{10}M + C_{11}, \tag{4-7}$$

where $C_{10}$ does not depend on $\tau$. Then by using Cauchy–Schwarz

$$\int_{t_n-\tau}^{t_n} \|u\|^2\,dt \le C_{12}(\gamma,\tau)\int_{t_n-\tau}^{t_n} |h|^2\,dt + (C_{10}+1)M + C_{11}. \tag{4-8}$$

By choosing $\tau$ large enough we obtain, as a consequence of (4-8) and (4-4), the inequality

$$\int_{t_n-\tau}^{t_n} \Phi(t)\,dt \le C_{12}(\gamma)\int_{t_n-\tau}^{t_n} |h|^2\,dt + C_{13}. \tag{4-9}$$

We conclude by using

$$|\Phi(t)-\Phi(s)| \le C_{14}(\gamma)\left(1 + \int_{t_n-\tau}^{t_n} |h|^2\,dt\right), \tag{4-10}$$

which is valid for all $s$, $t$ in $[t_n-\tau, t_n]$ and follows easily from (4-3) and (4-4). □

**Remark 4.2.** This result is optimal. For instance if we consider an eigenvector $\varphi$ of $A$ corresponding to the eigenvalue $\lambda > 0$, then for each $k > 0$, $k\varphi$ is a stationary solution of the equation with source term $h(t) \equiv k\lambda\varphi$ for any dissipative operator $g$. This shows that the ultimate bound of the energy is at least quadratic with respect to the size of the source term.

**4B. *Examples.*** In this section, $\Omega$ denotes a bounded open domain of $\mathbb{R}^N$ with $C^2$ boundary and $\alpha \ge 0$, $c > 0$. We consider four simple special cases.

**Example 1** (the wave equation with local damping).

$$\begin{cases} u_{tt} + c|u_t|^\alpha u_t - \Delta u = h(t,x) & \text{in } \mathbb{R}_+ \times \Omega, \\ \qquad\qquad\qquad u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{4-11}$$

Here $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $Z = L^{\alpha+2}(\Omega)$. We assume $(N-2)\alpha \le 2$.

**Example 2** (the wave equation with nonlinear averaged damping).

$$\begin{cases} u_{tt} + c\big(\int_\Omega {u_t}^2(t,x)\,dx\big)^{\alpha/2} u_t - \Delta u = h(t,x) & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{4-12}$$

Here $V = H_0^1(\Omega)$ and $H = L^2(\Omega) = Z$.

**Example 3** (a clamped plate equation with nonlinear structural averaged damping).

$$\begin{cases} u_{tt} - c\big(\int_\Omega |\nabla u_t|^2\,dx\big)^{\alpha/2} \Delta u_t + \Delta^2 u = h(t,x) & \text{in } \mathbb{R}_+ \times \Omega, \\ u = |\nabla u| = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{4-13}$$

Here $V = H_0^2(\Omega),\ H = L^2(\Omega)$ and $Z = H_0^1(\Omega)$.

**Example 4** (a simply supported plate equation with nonlinear structural averaged damping).

$$\begin{cases} u_{tt} - c\big(\int_\Omega |\nabla u_t|^2\,dx\big)^{\alpha/2} \Delta u_t + \Delta^2 u = h(t,x) & \text{in } \mathbb{R}_+ \times \Omega, \\ u = \Delta u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \tag{4-14}$$

Here $V = H^2 \cap H_0^1(\Omega),\ H = L^2(\Omega)$and $Z = H_0^1(\Omega)$.

As a consequence of Theorem 4.1 we obtain immediately:

**Corollary 4.3.** *In all four examples, let $h \in S^2(\mathbb{R}^+, H)$. Then any solution $u \in C(\mathbb{R}^+, V) \cap C^1(\mathbb{R}^+, H)$ of (2-2) is bounded on $\mathbb{R}^+$ in the sense that $u$ has bounded range in $V$ and $\dot{u}$ has bounded range in $H$. In addition we have for some constant $K$ independent of $h$ and the initial data*

$$\limsup_{t\to\infty} (|\dot{u}(t)|^2 + \|u(t)\|^2) \le K(1 + \|h\|^2_{S^2(\mathbb{R}^+, H)}).$$

**Remark 4.4.** In [Aloui et al. 2013], for the four previous examples, the authors proved the existence of a unique almost periodic solution when $h$ is an $S^2$-almost periodic source. In this case (in particular if $h$ is periodic), the ultimate bound coincides with the supremum of the energy of the almost periodic solution. Actually, if we try to estimate directly the periodic solution, some boundary (in time) term disappears but the main part of the estimate is not much simpler. In addition we know that the estimate is essentially optimal, only the multiplicative constants might be worked out if one wants a more precise inequality.

## 5. Partial results in the antiperiodic case

If $g$ is odd and $h$ is $\tau$-antiperiodic, i.e., if we have

$$h(t+\tau) = -h(t),$$

the interesting solutions are the antiperiodic ones; see, e.g., [Haraux 1989] for existence results. Since such solutions have mean value 0, the solution can be estimated through its time-derivative, and because the estimate of the derivative is generally much better, we can expect an improvement on the energy bound.

This idea is perfectly valid if $H$ is finite-dimensional, since then $u$ and $\dot{u}$ belong to the same space, but otherwise we have a problem to reach the norm of $u$ in $V$. At the present time we do not know what happens if $\dim H = \infty$. For the time being we can only prove the following partial results.

**Proposition 5.1.** *Assume that* $V = H$, $h \in C(\mathbb{R}, H)$ *is* $\tau$*-antiperiodic, that* $g \in C(H, H)$ *satisfies the condition* (2-3) *and for some* $\alpha \geq 0$ *we have*

$$\exists\, \gamma > 0,\ \exists\, C_1 \geq 0,\ \forall\, v \in H, \quad \langle g(v), v\rangle \geq \gamma |v|^{\alpha+2} - C_1, \tag{5-1}$$

$$\exists\, K > 0,\ \exists\, C_2 \geq 0,\ \forall\, v \in H, \quad |g(v)| \leq C_2 + K|v|^{\alpha+1}. \tag{5-2}$$

*Then any* $\tau$ *-antiperiodic solution* $u$ *of* (2-2) *is such that*

$$\sup_{t\in\mathbb{R}} (|\dot{u}(t)|^2 + |u(t)|^2) \leq C\big(1 + \|h\|_{L^\infty(\mathbb{R},H)}^{2/(\alpha+1)}\big),$$

*where* $C$ *is independent of* $h$.

*Proof.* The starting point is the same as for the proof of Theorem 4.1. From

$$\int_0^{2\tau} \langle g(\dot{u}), \dot{u}\rangle\, dt \leq C(\gamma, \tau)\bigg(\int_0^{2\tau} |h|^2\, dt\bigg)^{(\alpha+2)/(2\alpha+2)} + C_3 \tag{5-3}$$

we deduce, taking account of property (5-1), the more precise estimate

$$\int_0^{2\tau} |\dot{u}|^2\, dt \leq C_4 + C_5(\gamma, \tau)\bigg(\int_0^{2\tau} |h|^2\, dt\bigg)^{1/(\alpha+1)}, \tag{5-4}$$

which implies, since $u$ has mean value 0,

$$\sup_{t\in[0,2\tau]} |u(t)|^2 \leq C\big(1 + \|h\|_{L^\infty(\mathbb{R},H)}^{2/(\alpha+1)}\big). \tag{5-5}$$

To obtain the uniform bound on $\dot{u}$, the trick now consists in evaluating the maximum of $\Phi(t) = |\dot{u}|^2 + |A^{1/2}u|^2$. At a maximum point $\theta$ the derivative vanishes, which gives

$$\langle g(\dot{u}), \dot{u}\rangle = (h, \dot{u});$$

hence

$$|\dot{u}(\theta)|^2 \leq C'(1 + |h|^{2/(\alpha+1)}).$$

This implies

$$\max_{t\in[0,2\tau]} \Phi(t) = \Phi(\theta) \leq C''\big(1 + \|h\|_{L^\infty(\mathbb{R},H)}^{2/(\alpha+1)}\big)$$

and the conclusion follows immediately. □

**Remark 5.2.** This result is optimal. For instance if we consider an eigenvector $\varphi$ of $A$ corresponding to the eigenvalue $\lambda > 0$, then for each $k > 0$, we have $u_k(t) = k\cos(\lambda^{1/2}t)\varphi$ is a solution of the equation

$$\ddot{u} + Au + g(\dot{u}) = g(-k\lambda^{1/2}\sin(\lambda^{1/2}t)\varphi) =: h(t)$$

and the $L^\infty$ norm of the source term is less than a constant times $k^{\alpha+1}$ for $k$ large. Both $u$ and $h$ are antiperiodic.

We have a weaker result (intermediate between Theorem 4.1 and Proposition 5.1) which is also valid in the infinite-dimensional setting and can be stated as follows:

**Proposition 5.3.** *Assume that the conditions of Theorem 4.1 are satisfied with (4-2) reinforced into*

$$\exists\, K > 0,\ \exists\, C_2 \geq 0,\ \forall\, v \in V, \quad \|g(v)\|_{Z'} \leq C_2 + K\|v\|_Z^{\alpha+1}. \tag{5-6}$$

*Then any $\tau$ -antiperiodic solution $u \in C^1(\mathbb{R}, V) \cap C^2(\mathbb{R}, H)$ of (2-2) is such that*

$$\sup_{t\in\mathbb{R}} (|\dot{u}(t)|^2 + \|u(t)\|^2) \leq C\big(1 + \|h\|_{L^2([0,\tau],H)}^{(\alpha+2)/(\alpha+1)}\big),$$

*where $C$ is independent of $h$.*

*Proof.* The starting point is the same as for the proof of Theorem 4.1. From the inequality

$$\int_0^{2\tau} \langle g(\dot{u}), \dot{u}\rangle\, dt \leq C_3\bigg(1 + \int_0^{2\tau} |h|^2\, dt\bigg)^{(\alpha+2)/(2\alpha+2)} \tag{5-7}$$

we deduce the estimate

$$\int_0^{2\tau} |\dot{u}|^2\, dt \leq C_4\bigg(1 + \bigg(\int_0^{2\tau} |h|^2\, dt\bigg)^{1/(\alpha+1)}\bigg), \tag{5-8}$$

but also

$$\int_0^{2\tau} \|\dot{u}\|_Z^{\alpha+1}\, dt \leq C_5\bigg(1 + \bigg(\int_0^{2\tau} |h|^2\, dt\bigg)^{1/2}\bigg) \tag{5-9}$$

and by (5-6) this implies

$$\int_0^{2\tau} \|g(\dot{u})\|_{Z'}\, dt \leq C_6\bigg(1 + \bigg(\int_0^{2\tau} |h|^2\, dt\bigg)^{1/2}\bigg). \tag{5-10}$$

From (5-9), since $u$ has mean value 0, we deduce

$$\sup_{t\in[0,2\tau]} \|u(t)\|_Z \leq C_7\big(1 + \|h\|_{L^2([0,2\tau],H)}^{1/(\alpha+1)}\big). \tag{5-11}$$

The two last inequalities imply immediately

$$\bigg|\int_0^{2\tau} \langle g(\dot{u}), u\rangle\, dt\bigg| \leq C_8\big(1 + \|h\|_{L^2([0,2\tau],H)}^{(\alpha+2)/(\alpha+1)}\big).$$

Now, multiplying the equation by $u$ and integrating on the period we find easily after combining with (5-8)

$$\int_0^{2\tau} \Phi(t)\,dt \le C_9\bigl(1+\|h\|_{L^2([0,2\tau],H)}^{(\alpha+2)/(\alpha+1)}\bigr),$$

with $\Phi(t)=|\dot u|^2+\|u\|^2$. Since

$$\Phi'(t)=(h,\dot u)-\langle g(\dot u),\dot u\rangle$$

by $2\tau$-periodicity and the inequality $\Phi'\le |h|\,|\dot u|+C_1$, we find as a consequence of (5-8)

$$\Phi(t)\le \int_{t-\tau}^{t}\Phi(s)\,ds+C_{10}\bigl(1+\|h\|_{L^2([0,2\tau],H)}^{(\alpha+2)/(\alpha+1)}\bigr)$$

and the conclusion follows easily by using $\tau$-antiperiodicity. □

**Remark 5.4.** This result is certainly not optimal but it is all we can prove for the moment even in the most basic examples. Our result requires additional regularity on $u$; this is usually achieved by assuming some regularity on $h$. When $g$ is monotone, usually the antiperiodic solution is unique and depends continuously on $h$ in $L^2$, so that the estimate will be easy to transfer to the general case in the examples. This is important since we cannot derive strong estimates on solutions which are not antiperiodic and therefore approximation by strong solutions has to be performed within the antiperiodic class.

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# On the irreducibility of some induced representations of real reductive Lie groups

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We prove the irreducibility of some standard modules of the metaplectic group $\mathrm{Mp}_{2n}(\mathbb{R})$ and some nonstandard modules of the split odd special orthogonal group $\mathrm{SO}_{2n+1}(\mathbb{R})$.

## 1. Introduction

This article is a supplement to [Gan and Ichino 2017], in which we establish the Shimura–Waldspurger correspondence for the metaplectic group $\mathrm{Mp}_{2n}$ of higher rank. Namely, we describe the tempered part of the automorphic discrete spectrum of $\mathrm{Mp}_{2n}$ in terms of that of $\mathrm{SO}_{2n+1}$ via theta lifts. In the course of the proof, we use the inductive property of local $L$- and $A$-packets and need to show that some induced representations are irreducible. The purpose of this article is to prove this irreducibility in the real case.

We now describe our results. Let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$. We say that an irreducible representation $\phi$ of $W_{\mathbb{R}}$ is almost tempered if the image of $\phi|\cdot|^{-s}$ is bounded for some $s \in \mathbb{R}$ with $|s| < \frac{1}{2}$. We consider two cases and give the details in turn.

In Section 2, we consider some standard modules of the metaplectic group $\mathrm{Mp}_{2n}(\mathbb{R})$ (which is a nonlinear two-fold cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ of rank $n$). Let $\psi : W_{\mathbb{R}} \to \mathrm{Sp}_{2n}(\mathbb{C})$ be an $L$-parameter, which we may regard as a $2n$-dimensional symplectic representation of $W_{\mathbb{R}}$. We assume

$$\psi = \phi \oplus \phi^\vee \oplus \psi_0,$$

where

- $\phi$ is a $k$-dimensional representation of $W_{\mathbb{R}}$ whose irreducible summands are all nonsymplectic and almost tempered;
- $\psi_0$ is a $2n_0$-dimensional representation of $W_{\mathbb{R}}$ whose irreducible summands are all symplectic;
- $k + n_0 = n$.

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Let $P$ be a parabolic subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ with Levi component $\mathrm{GL}_k(\mathbb{R}) \times \mathrm{Sp}_{2n_0}(\mathbb{R})$ and $\widetilde{P}$ the preimage of $P$ in $\mathrm{Mp}_{2n}(\mathbb{R})$. Let $\tau$ be the irreducible representation of $\mathrm{GL}_k(\mathbb{R})$ associated to $\phi$ and $\tilde{\tau} = \tau \otimes \chi$ its twist by a fixed genuine quartic character $\chi$ of the two-fold cover of $\mathrm{GL}_k(\mathbb{R})$, as in [Gan and Ichino 2014, §2.5 and §2.6]. Then the $L$-packet $\Pi_\psi(\mathrm{Mp}_{2n}(\mathbb{R}))$ consists of the unique irreducible quotients of

$$\mathrm{Ind}_{\widetilde{P}}^{\mathrm{Mp}_{2n}(\mathbb{R})}(\tilde{\tau} \otimes \pi_0)$$

for all $\pi_0 \in \Pi_{\psi_0}(\mathrm{Mp}_{2n_0}(\mathbb{R}))$, and as stated in [Gan and Ichino 2017, Lemma 5.2], the irreducibility of this induced representation is required. We should mention that the irreducibility of standard modules of real reductive linear Lie groups was studied in [Speh and Vogan 1980] and their result was extended to the nonlinear case in [Miličić 1991] (via the Beilinson–Bernstein localization theorem). Nevertheless, for the convenience of the reader, we give a more direct proof of this irreducibility, following the argument in [Speh and Vogan 1980] but using the machinery of cohomological induction [Knapp and Vogan 1995].

In Section 3, we consider some nonstandard modules of the split odd special orthogonal group $\mathrm{SO}_{2n+1}(\mathbb{R})$ of rank $n$. Let

$$\psi : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})$$

be an $A$-parameter, which we may regard as a $2n$-dimensional symplectic representation of $W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$. We assume

$$\psi = \phi \oplus \phi^\vee \oplus \psi_0,$$

where

- $\phi$ is a $k$-dimensional representation of $W_{\mathbb{R}}$ whose irreducible summands are all nonsymplectic and almost tempered;
- $\psi_0$ is a $2n_0$-dimensional representation of $W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$ whose irreducible summands are all symplectic;
- $k + n_0 = n$.

Let $Q$ be a parabolic subgroup of $\mathrm{SO}_{2n+1}(\mathbb{R})$ with Levi component $\mathrm{GL}_k(\mathbb{R}) \times \mathrm{SO}_{2n_0+1}(\mathbb{R})$. Let $\tau$ be the irreducible representation of $\mathrm{GL}_k(\mathbb{R})$ associated to $\phi$. Then the $A$-packet $\Pi_\psi(\mathrm{SO}_{2n+1}(\mathbb{R}))$ consists of the semisimplifications of

$$\mathrm{Ind}_Q^{\mathrm{SO}_{2n+1}(\mathbb{R})}(\tau \otimes \sigma_0)$$

for all $\sigma_0 \in \Pi_{\psi_0}(\mathrm{SO}_{2n_0+1}(\mathbb{R}))$, and as stated in [Gan and Ichino 2017, Lemma 5.5], the irreducibility of this induced representation is required. To prove this irreducibility, we reduce it to the irreducibility of a standard module

$$\mathrm{Ind}_{Q'}^{\mathrm{SO}_{2k+1}(\mathbb{R}) \times \mathrm{SO}_{2n_0+1}(\mathbb{R})}(\tau \otimes \sigma_0)$$

of an endoscopic group of $SO_{2n+1}(\mathbb{R})$, where $Q'$ is a parabolic subgroup of $SO_{2k+1}(\mathbb{R}) \times SO_{2n_0+1}(\mathbb{R})$ with Levi component $GL_k(\mathbb{R}) \times SO_{2n_0+1}(\mathbb{R})$. This reduction relies on the Kazhdan–Lusztig algorithm and is essentially due to Matumoto [2004, §4], but we include it here for the sake of completeness. We also include a more direct proof of this irreducibility given to us by the referee, using normalized intertwining operators and the irreducibility result of [Speh and Vogan 1980].

## 2. Irreducibility of some standard modules of $\mathrm{Mp}_{2n}(\mathbb{R})$

In this section, we show that some standard modules of $\mathrm{Mp}_{2n}(\mathbb{R})$ are irreducible (see Proposition 2.3 below), which finishes the proof of [Gan and Ichino 2017, Lemma 5.2] in the real case.

**2A.** ***Notation.*** Let $G = \mathrm{Mp}_{2n}(\mathbb{R})$ be the metaplectic two-fold cover of $\mathrm{Sp}_{2n}(\mathbb{R})$, which we realize as

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \left\{ g \in \mathrm{GL}_{2n}(\mathbb{R}) \;\middle|\; g \begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix} {}^t g = \begin{pmatrix} & \mathbf{1}_n \\ -\mathbf{1}_n & \end{pmatrix} \right\}.$$

We define a maximal compact subgroup $K$ of $G$ as the preimage in $G$ of

$$\{ g \in \mathrm{Sp}_{2n}(\mathbb{R}) \mid {}^t g^{-1} = g \}.$$

Let $\theta$ be the Cartan involution of $G$ corresponding to $K$. Let $\mathfrak{g}_0 = \mathrm{Lie}\, G$ be the Lie algebra of $G$ and $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ its complexification; analogous notation is used for other groups.

For any nonnegative integers $k, l, m$ such that $k+2l+m=n$, we define a $\theta$-stable Cartan subalgebra $\mathfrak{h}_0^{k,l,m}$ of $\mathfrak{g}_0$ as follows. For $a = (a_1, \dots, a_k) \in \mathbb{R}^k$, put

$$h^{k,0,0}(a) = \begin{pmatrix} \boldsymbol{a} & \\ & -\boldsymbol{a} \end{pmatrix} \in \mathfrak{sp}_{2k}(\mathbb{R}),$$

where $\boldsymbol{a} = \mathrm{diag}(a_1, \dots, a_k)$. For $z = (z_1, \dots, z_l) \in \mathbb{C}^l$ with $z_i = x_i + \sqrt{-1}y_i$, put

$$h^{0,l,0}(z) = \begin{pmatrix} & \boldsymbol{x} & \boldsymbol{y} & \\ \boldsymbol{x} & & & -\boldsymbol{y} \\ -\boldsymbol{y} & & & -\boldsymbol{x} \\ & \boldsymbol{y} & -\boldsymbol{x} & \end{pmatrix} \in \mathfrak{sp}_{4l}(\mathbb{R}),$$

where $\boldsymbol{x} = \mathrm{diag}(x_1, \dots, x_l)$ and $\boldsymbol{y} = \mathrm{diag}(y_1, \dots, y_l)$. For $\vartheta = (\vartheta_1, \dots, \vartheta_m) \in \mathbb{R}^m$, put

$$h^{0,0,m}(\vartheta) = \begin{pmatrix} & \boldsymbol{\vartheta} \\ -\boldsymbol{\vartheta} & \end{pmatrix} \in \mathfrak{sp}_{2m}(\mathbb{R}),$$

where $\boldsymbol{\vartheta} = \mathrm{diag}(\vartheta_1, \dots, \vartheta_m)$. Let $h^{k,l,m}(a, z, \vartheta)$ be the image of

$$(h^{k,0,0}(a), h^{0,l,0}(z), h^{0,0,m}(\vartheta))$$

under the natural embedding

$$\mathfrak{sp}_{2k}(\mathbb{R}) \oplus \mathfrak{sp}_{4l}(\mathbb{R}) \oplus \mathfrak{sp}_{2m}(\mathbb{R}) \hookrightarrow \mathfrak{sp}_{2n}(\mathbb{R}).$$

Then we set

$$\mathfrak{h}_0^{k,l,m} = \{h^{k,l,m}(a, z, \vartheta) \mid a \in \mathbb{R}^k,\ z \in \mathbb{C}^l,\ \vartheta \in \mathbb{R}^m\}.$$

These $\mathfrak{h}_0^{k,l,m}$ with $k + 2l + m = n$ form a set of representatives for the $G$-conjugacy classes of Cartan subalgebras of $\mathfrak{g}_0$.

Fix such $k, l, m$ and write $\mathfrak{h}_0 = \mathfrak{h}_0^{k,l,m}$. We define a basis $e_1, \dots, e_n$ of $\mathfrak{h}^*$ by

$$\begin{aligned} e_i(h) &= a_i && (1 \le i \le k),\\ e_{k+2i-1}(h) &= x_i + \sqrt{-1}y_i && (1 \le i \le l),\\ e_{k+2i}(h) &= x_i - \sqrt{-1}y_i && (1 \le i \le l),\\ e_{k+2l+i}(h) &= \sqrt{-1}\vartheta_i && (1 \le i \le m) \end{aligned}$$

for $h = h^{k,l,m}(a, z, \vartheta)$. Note that

$$\begin{aligned} \theta(e_i) &= -e_i && (1 \le i \le k),\\ \theta(e_{k+2i-1}) &= -e_{k+2i} && (1 \le i \le l),\\ \theta(e_{k+2l+i}) &= e_{k+2l+i} && (1 \le i \le m). \end{aligned}$$

Using the above basis, we identify $\mathfrak{h}^*$ with $\mathbb{C}^n$. Let $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ be the standard bilinear form:

$$\langle \lambda, \mu \rangle = \lambda_1\mu_1 + \cdots + \lambda_n\mu_n$$

for $\lambda = (\lambda_1, \dots, \lambda_n)$, $\mu = (\mu_1, \dots, \mu_n) \in \mathfrak{h}^* \cong \mathbb{C}^n$. We denote by $\Delta$ the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$:

$$\Delta = \{\pm e_i \pm e_j \mid 1 \le i < j \le n\} \cup \{\pm 2e_i \mid 1 \le i \le n\}.$$

For any subspace $\mathfrak{f}$ of $\mathfrak{g}$ stable under the adjoint action of $\mathfrak{h}$, we denote by $\Delta(\mathfrak{f})$ the set of roots of $\mathfrak{h}$ in $\mathfrak{f}$ and put

$$\rho(\mathfrak{f}) = \frac{1}{2} \sum_{\alpha \in \Delta(\mathfrak{f})} \alpha.$$

**2B.** ***Discrete series.*** As in [Adams and Barbasch 1998, §3], genuine (limits of) discrete series representations of $G$ are classified as follows. Suppose $\mathfrak{h}_0 = \mathfrak{h}_0^{0,0,n}$. Let $\Delta_c$ be the set of compact roots and take the positive system

$$\Delta_c^+ = \{e_i - e_j \mid 1 \le i < j \le n\}.$$

Then a genuine discrete series representation of $G$ is parametrized by its Harish-Chandra parameter $\lambda \in \sqrt{-1}\mathfrak{h}_0^*$ of the form

$$\lambda = (a_1, \dots, a_r, -b_1, \dots, -b_s),$$

where

- $a_i, b_j \in \mathbb{Z} + \frac{1}{2}$;
- $a_1 > \cdots > a_r > 0$ and $0 < b_1 < \cdots < b_s$;
- $a_i \neq b_j$ for all $i$, $j$.

More generally, a genuine limit of discrete series representation of $G$ is parametrized by a pair $(\lambda, \Psi)$ consisting of $\lambda \in \sqrt{-1}\mathfrak{h}_0^*$ of the form

$$\lambda = (\underbrace{a_1, \dots, a_1}_{m_1}, \dots, \underbrace{a_r, \dots, a_r}_{m_r}, \underbrace{-a_r, \dots, -a_r}_{n_r}, \dots, \underbrace{-a_1, \dots, -a_1}_{n_1}),$$

where

- $a_i \in \mathbb{Z} + \frac{1}{2}$;
- $a_1 > \cdots > a_r > 0$;
- $m_i, n_j \geq 0$;
- $m_i + n_i > 0$ and $|m_i - n_i| \leq 1$ for all $i$,

and a positive system $\Psi$ of $\Delta$ such that

- $\Delta_c^+ \subset \Psi$;
- $\langle \alpha, \lambda \rangle \geq 0$ for all $\alpha \in \Psi$;
- if $\alpha$ is a simple root in $\Psi$ such that $\langle \alpha, \lambda \rangle = 0$, then $\alpha$ is noncompact; see the condition (F-1) in [Vogan 1984].

Note that, given such $\lambda$, there are precisely $2^t$ positive systems $\Psi$ satisfying the above conditions, where $t$ is the number of indices $i$ such that $m_i = n_i > 0$.

**Remark 2.1.** The $L$-parameter of the representation associated to $(\lambda, \Psi)$ is

$$\bigoplus_{i=1}^{r} (m_i + n_i)\mathcal{D}_{a_i},$$

where for $a \in \frac{1}{2}\mathbb{Z}$, we denote by $\mathcal{D}_a$ the 2-dimensional representation of $W_\mathbb{R}$ induced from the character $z \mapsto (z/\bar{z})^a$ of $W_\mathbb{C} = \mathbb{C}^\times$. Note that

- $\mathcal{D}_{-a} = \mathcal{D}_a$;
- $\mathcal{D}_a$ is irreducible if and only if $a \neq 0$;
- $\mathcal{D}_a$ is symplectic if and only if $a \in \mathbb{Z} + \frac{1}{2}$.

In particular, any irreducible summand of the above $L$-parameter is symplectic and the associated $L$-packet consists of $2^r$ representations.

**2C. *Standard modules.*** We will use Vogan's version [1984] of the Langlands classification for real reductive Lie groups in Harish-Chandra's class. Suppose again that $\mathfrak{h}_0 = \mathfrak{h}_0^{k,l,m}$ is arbitrary. Let $H$ be the centralizer of $\mathfrak{h}_0$ in $G$. Then $H$ is the preimage in $G$ of a Cartan subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ isomorphic to

$$(\mathbb{R}^\times)^k \times (\mathbb{C}^\times)^l \times (S^1)^m.$$

Let $\mathfrak{t}_0$ and $\mathfrak{a}_0$ be the $+1$ and $-1$ eigenspaces of $\theta$ in $\mathfrak{h}_0$, respectively. Put $T = H \cap K$ and $A = \exp(\mathfrak{a}_0)$, so that

$$H = T \times A.$$

Let $M$ be the centralizer of $\mathfrak{a}_0$ in $G$. Then $M$ is the preimage in $G$ of a Levi subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ isomorphic to

$$\mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times \mathrm{Sp}_{2m}(\mathbb{R}).$$

For the inducing data of a standard module, we take an irreducible representation of $M$ as follows. Let $\widetilde{\mathrm{GL}}_d(\mathbb{R})$ be the two-fold cover of $\mathrm{GL}_d(\mathbb{R})$ given in [Gan and Ichino 2014, §2.5]. Let $\chi_\psi$ be the genuine quartic character of $\widetilde{\mathrm{GL}}_1(\mathbb{R})$ given in §2.6 of the same paper, relative to a fixed nontrivial additive character $\psi$ of $\mathbb{R}$. For $1 \le i \le k$, let $\chi_i$ be a character of $\widetilde{\mathrm{GL}}_1(\mathbb{R})$ of the form

$$\chi_i = \mathrm{sgn}^{\delta_i} \otimes \chi_\psi \otimes |\cdot|^{\nu_i}$$

for some $\delta_i \in \{0, 1\}$ and some $\nu_i \in \mathbb{C}$. For $1 \le i \le l$, let $\tau_i$ be an irreducible representation of $\widetilde{\mathrm{GL}}_2(\mathbb{R})$ of the form

$$\tau_i = D_{\kappa_i} \otimes (\chi_\psi \circ \widetilde{\det}) \otimes |\det|^{\nu_i'}$$

for some $\kappa_i \in \frac{1}{2}\mathbb{Z}$ and some $\nu_i' \in \mathbb{C}$, where $D_{\kappa_i}$ is the relative (limit of) discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight $2|\kappa_i| + 1$ with central character trivial on $\mathbb{R}_+^\times$ and $\widetilde{\det}$ is the natural lift of the determinant map given in [Gan and Ichino 2014, §2.6]:

$$\begin{array}{ccc} \widetilde{\mathrm{GL}}_2(\mathbb{R}) & \xrightarrow{\widetilde{\det}} & \widetilde{\mathrm{GL}}_1(\mathbb{R}) \\ \downarrow & & \downarrow \\ \mathrm{GL}_2(\mathbb{R}) & \xrightarrow{\det} & \mathrm{GL}_1(\mathbb{R}) \end{array}$$

Note that $\tau_i$ does not depend on the choice of $\psi$ since $D_{\kappa_i} \otimes (\mathrm{sgn} \circ \det) \cong D_{\kappa_i}$. Let $\pi'$ be a genuine (limit of) discrete series representation of $\mathrm{Mp}_{2m}(\mathbb{R})$ associated to $(\lambda', \Psi')$ as in Section 2B. Then

$$\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi'$$

descends to an irreducible representation of $M$.

Put

$$\gamma = (\lambda, \nu) \in \mathfrak{h}^* = \mathfrak{t}^* \oplus \mathfrak{a}^*, \tag{2-1}$$

where

$$\lambda = (\underbrace{0, \dots, 0}_{k}, \kappa_1, -\kappa_1, \dots, \kappa_l, -\kappa_l, \lambda'_1, \dots, \lambda'_m),$$
$$\nu = (\nu_1, \dots, \nu_k, \nu'_1, \nu'_1, \dots, \nu'_l, \nu'_l, \underbrace{0, \dots, 0}_{m}).$$

Assume that the condition (F-2) in [Vogan 1984], which is explicated in [Adams and Barbasch 1998, Lemma 4.3], holds:

(i) If $\nu_i = \pm\nu_j$, then $\delta_i = \delta_j$.

(ii) If $\nu'_i = 0$, then $\kappa_i \in \mathbb{Z}$.

Choose a parabolic subgroup $P = MN$ of $G$ with Levi component $M$ and unipotent radical $N$ such that

$$\operatorname{Re} \langle \alpha, \nu \rangle \geq 0$$

for all roots $\alpha$ of $\mathfrak{a}$ in $\mathfrak{n}$. Then, by [Vogan 1984, Proposition 2.6], the normalized parabolic induction

$$\operatorname{Ind}_P^G(\pi)$$

has a unique irreducible quotient $\mathrm{J}_P^G(\pi)$. Note that $\mathrm{J}_P^G(\pi)$ is tempered if and only if $\operatorname{Re} \nu_i = \operatorname{Re} \nu'_j = 0$ for all $i, j$, in which case $\operatorname{Ind}_P^G(\pi)$ is irreducible. Moreover, every irreducible genuine representation of $G$ arises in this way; see [Vogan 1984, Theorem 2.9].

**Remark 2.2.** The $L$-parameter of $\mathrm{J}_P^G(\pi)$ is

$$\phi \oplus \phi^\vee \oplus \phi',$$

where $\phi$ is given by

$$\phi = \left( \bigoplus_{i=1}^{k} \operatorname{sgn}^{\delta_i} |\cdot|^{\nu_i} \right) \oplus \left( \bigoplus_{j=1}^{l} \mathcal{D}_{\kappa_j} |\cdot|^{\nu'_j} \right)$$

and $\phi'$ is the $L$-parameter of $\pi'$ (see Remark 2.1). Note that any irreducible summand of $\phi$ is nonsymplectic by the above conditions (i), (ii).

Finally, for any real root $\alpha \in \Delta$, we consider the following "parity conditions":

- If $\alpha = \pm(e_i - e_j)$ with $1 \leq i < j \leq k$, then either

$$\begin{cases} \delta_i = \delta_j \text{ and } \nu_i - \nu_j \in 2\mathbb{Z} + 1; \text{ or} \\ \delta_i \neq \delta_j \text{ and } \nu_i - \nu_j \in 2\mathbb{Z}. \end{cases}$$

- If $\alpha = \pm(e_i + e_j)$ with $1 \leq i < j \leq k$, then either

$$\begin{cases} \delta_i = \delta_j \text{ and } \nu_i + \nu_j \in 2\mathbb{Z} + 1; \text{ or} \\ \delta_i \neq \delta_j \text{ and } \nu_i + \nu_j \in 2\mathbb{Z}. \end{cases}$$

- If $\alpha = \pm 2e_i$ with $1 \le i \le k$, then $\nu_i \in \mathbb{Z} + \frac{1}{2}$.
- If $\alpha = \pm(e_{k+2i-1} + e_{k+2i})$ with $1 \le i \le l$, then either
$$\begin{cases} \kappa_i \in \mathbb{Z} \text{ and } \nu_i' \in \mathbb{Z} + \frac{1}{2}; \text{ or} \\ \kappa_i \in \mathbb{Z} + \frac{1}{2} \text{ and } \nu_i' \in \mathbb{Z}. \end{cases}$$

With the above notation, we now state the main result of this section.

**Proposition 2.3.** *Assume that there exists no root $\alpha \in \Delta$ such that either*

(i) *$\alpha$ is complex and satisfies $2\langle \alpha, \gamma\rangle / \langle \alpha, \alpha\rangle \in \mathbb{Z}$, $\langle \alpha, \gamma\rangle > 0$, and $\langle \theta\alpha, \gamma\rangle < 0$; or*

(ii) *$\alpha$ is real and satisfies the parity condition.*

*Then $\mathrm{Ind}_P^G(\pi)$ is irreducible. In particular, if $|\mathrm{Re}\, \nu_i|, |\mathrm{Re}\, \nu_j'| < \frac{1}{2}$ for all $i, j$, then $\mathrm{Ind}_P^G(\pi)$ is irreducible.*

**2D. *Proof of Proposition 2.3.*** We first express the standard module $\mathrm{Ind}_P^G(\pi)$ as cohomological induction from a principal series representation. By [Knapp and Vogan 1995, §XI.8], combined with Lemma 11.202 of the same work, we may write
$$\pi = ({}^{\mathrm{u}}\mathcal{R}_{\mathfrak{b},T}^{\mathfrak{m},M\cap K})^{\dim \mathfrak{v}\cap\mathfrak{k}}(\zeta \otimes \chi_{\rho(\mathfrak{v})}),$$
where

- $({}^{\mathrm{u}}\mathcal{R}_{\mathfrak{b},T}^{\mathfrak{m},M\cap K})^i$ is the functor defined by [Knapp and Vogan 1995, (11.71d)];
- $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{v}$ is a $\theta$-stable Borel subalgebra of $\mathfrak{m}$ with Levi component $\mathfrak{h}$ and nilpotent radical $\mathfrak{v}$ such that
$$\langle \alpha, \lambda\rangle \ge 0$$
for all $\alpha \in \Delta(\mathfrak{v})$;
- $\zeta$ is the character of $H$ given by
$$\zeta = \chi_1 \otimes \cdots \otimes \chi_k \otimes \xi_1 \otimes \cdots \otimes \xi_l \otimes \eta_1 \otimes \cdots \otimes \eta_m,$$
where
  - $\xi_i$ is the character of $\mathbb{C}^\times \times \{\pm 1\}$ given by $\xi_i(z, \epsilon) = \epsilon \cdot (z/\bar{z})^{\kappa_i} \cdot (z\bar{z})^{\nu_i'}$;
  - $\eta_i$ is the genuine character of the nonsplit two-fold cover of $S^1$ whose square descends to the character $z \mapsto z^{2\lambda_i'}$ of $S^1$;
- $\chi_{\rho(\mathfrak{v})}$ is the character of $H$ such that
  - $\chi_{\rho(\mathfrak{v})}$ factors through the image of $H$ in $\mathrm{Sp}_{2n}(\mathbb{R})$;
  - $\chi_{\rho(\mathfrak{v})}$ is trivial on $(\mathbb{R}^\times)^k$;
  - the differential of $\chi_{\rho(\mathfrak{v})}$ is $\rho(\mathfrak{v})$ (which is analytically integral).

Let $L$ be the centralizer of $\mathfrak{t}_0$ in $G$. Then $L$ is the preimage in $G$ of a Levi subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ isomorphic to
$$\mathrm{Sp}_{2k}(\mathbb{R}) \times \mathrm{U}(1,1)^l \times \mathrm{U}(1)^m.$$

Choose a $\theta$-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ with Levi component $\mathfrak{l}$ and nilpotent radical $\mathfrak{u}$ such that $\mathfrak{v} \subset \mathfrak{u}$ and such that

$$\langle \alpha, \lambda \rangle \geq 0$$

for all $\alpha \in \Delta(\mathfrak{u})$. Then, by [Knapp and Vogan 1995, Theorem 11.225], we have

$$\mathrm{Ind}_P^G(\pi) \cong \mathcal{L}_{\dim \mathfrak{u}\cap\mathfrak{k}}(\mathrm{Ind}_{P\cap L}^L(\zeta \otimes \chi_{\rho(\mathfrak{u})}^{-1})),$$

where

- $\mathcal{L}_i$ is the functor defined by [Knapp and Vogan 1995, (5.3a)];
- $P \cap L = H(N \cap L)$ is a Borel subgroup of $L$ with Levi component $H$ and unipotent radical $N \cap L$;
- $\chi_{\rho(\mathfrak{u})}$ is the character of $H$ such that
  - $\chi_{\rho(\mathfrak{u})}$ factors through the image of $H$ in $\mathrm{Sp}_{2n}(\mathbb{R})$;
  - $\chi_{\rho(\mathfrak{u})}$ is trivial on $(\mathbb{R}^\times)^k$;
  - the differential of $\chi_{\rho(\mathfrak{u})}$ is $\rho(\mathfrak{u})$ (which is analytically integral).

Assume for a moment that

$$|\mathrm{Re}\,\langle \alpha, \nu \rangle| \leq \langle \alpha, \lambda \rangle \tag{2-2}$$

for all $\alpha \in \Delta(\mathfrak{u})$. Then, by [Knapp and Vogan 1995, Corollary 11.227], $\mathrm{Ind}_P^G(\pi)$ is irreducible if $\mathrm{Ind}_{P\cap L}^L(\zeta \otimes \chi_{\rho(\mathfrak{u})}^{-1})$ is irreducible. Hence, noting that

$$\chi_{\rho(\mathfrak{u})} = \chi_1' \otimes \cdots \otimes \chi_k' \otimes \xi_1' \otimes \cdots \otimes \xi_l' \otimes \eta_1' \otimes \cdots \otimes \eta_m',$$

where

- $\chi_i'$ is the trivial character of $\mathbb{R}^\times$;
- $\xi_i'$ is a character of $\mathbb{C}^\times$ of the form $\xi_i'(z) = (z/\bar{z})^{a_i}$ for some $a_i \in \mathbb{Z}$;
- $\eta_i'$ is a character of $S^1$ of the form $\eta_i'(z) = z^{b_i}$ for some $b_i \in \mathbb{Z}$,

we are reduced to the following irreducibility:

- The principal series representation of $\mathrm{Mp}_{2k}(\mathbb{R})$ induced from $\chi_1 \otimes \cdots \otimes \chi_k$ is irreducible. Indeed, as in [Vogan 1981, Theorem 4.2.25], this can be deduced from the following:
  - The principal series representation of $\mathrm{GL}_d(\mathbb{R})$ induced from any unitary character is irreducible; see, e.g., [Mœglin 1997].
  - The principal series representation of $\mathrm{Mp}_{2d}(\mathbb{R})$ induced from any genuine unitary character is irreducible; see the proof of [Gan and Ichino 2017, Lemma 5.2].
  - For $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and $s_1, s_2 \in \mathbb{C}$, the principal series representation of $\mathrm{GL}_2(\mathbb{R})$ induced from $\mathrm{sgn}^{\epsilon_1}|\cdot|^{s_1} \otimes \mathrm{sgn}^{\epsilon_2}|\cdot|^{s_2}$ is irreducible if and only if either

$$\begin{cases} \epsilon_1 = \epsilon_2 \text{ and } s_1 - s_2 \notin 2\mathbb{Z}+1; \text{ or} \\ \epsilon_1 \neq \epsilon_2 \text{ and } s_1 - s_2 \notin 2\mathbb{Z} \smallsetminus \{0\}. \end{cases}$$

- For $\epsilon \in \{0, 1\}$ and $s \in \mathbb{C}$, the principal series representation of $\mathrm{Mp}_2(\mathbb{R})$ induced from $\mathrm{sgn}^\epsilon \chi_\psi |\cdot|^s$ is irreducible if and only if $s \notin \mathbb{Z} + \frac{1}{2}$.

- For $\kappa \in \frac{1}{2}\mathbb{Z}$ and $s \in \mathbb{C}$, the principal series representation of $\mathrm{U}(1, 1)$ induced from the character $z \mapsto (z/\bar{z})^\kappa \cdot (z\bar{z})^s$ of $\mathbb{C}^\times$ is irreducible if and only if either
$$\begin{cases} \kappa \in \mathbb{Z} \text{ and } s \notin \mathbb{Z} + \frac{1}{2}; \text{ or} \\ \kappa \in \mathbb{Z} + \frac{1}{2} \text{ and } s \notin \mathbb{Z}. \end{cases}$$

Thus, in view of condition (ii) in Proposition 2.3, we have shown that $\mathrm{Ind}_P^G(\pi)$ is irreducible under the assumption (2-2).

We now consider the general case. We reduce it to the case where $\gamma$ as in (2-1) satisfies the condition (2-2) by using the translation functor. Fix a positive system $\Delta^+(\mathfrak{l})$ of $\Delta(\mathfrak{l})$ such that
$$\mathrm{Re}\,\langle \alpha, \gamma \rangle \geq 0$$
for all $\alpha \in \Delta^+(\mathfrak{l})$. Then $\Delta^+ = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{u})$ is a positive system of $\Delta$. We denote by $\Delta(\gamma)$ the set of integral roots defined by $\gamma$:
$$\Delta(\gamma) = \left\{ \alpha \in \Delta \;\middle|\; 2\frac{\langle \alpha, \gamma \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \right\}.$$
Put $\Delta^+(\gamma) = \Delta(\gamma) \cap \Delta^+$. Then we have
$$\langle \alpha, \gamma \rangle \geq 0$$
for all $\alpha \in \Delta^+(\gamma)$. Indeed, if $\langle \alpha, \gamma \rangle < 0$ for some $\alpha \in \Delta(\gamma) \cap \Delta(\mathfrak{u})$, then since $\langle \alpha, \lambda \rangle \geq 0$, we have $\langle \alpha, \nu \rangle < 0$ and hence
$$\langle \theta\alpha, \gamma \rangle = \langle \alpha, \lambda \rangle - \langle \alpha, \nu \rangle > 0.$$
Namely, $-\alpha$ satisfies condition (i) in Proposition 2.3, which contradicts the assumption. Let $\mu \in \mathfrak{h}^*$ be an integral weight; i.e., $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_i \in \mathbb{Z}$. Then we have $\Delta(\gamma + \mu) = \Delta(\gamma)$. Recall that the translation functor $\psi^\gamma_{\gamma+\mu}$ for $G$ is defined by
$$\psi^\gamma_{\gamma+\mu}(X) = P_\gamma(P_{\gamma+\mu}(X) \otimes F_{-\mu})$$
for any $(\mathfrak{g}, K)$-module $X$ of finite length, where $P_\gamma$ is the projection to the $\gamma$-primary component and $F_{-\mu}$ is the (nongenuine) finite-dimensional irreducible $(\mathfrak{g}, K)$-module with extreme weight $-\mu$. The translation functor for $M$ is defined similarly and is also denoted by $\psi^\gamma_{\gamma+\mu}$; see [Knapp 1986, §XIV.12]. We now take $\mu$ of the form $\mu = (t\rho(\mathfrak{u}), \mu')$ for some positive integer $t$ and some integral weight $\mu' \in \mathfrak{a}^*$ such that

- $\langle \alpha, \mu' \rangle > 0$ for all $\alpha \in \Delta^+(\mathfrak{l})$;
- $|\mathrm{Re}\,\langle \alpha, \nu + \mu' \rangle| < \langle \alpha, \lambda + t\rho(\mathfrak{u}) \rangle$ for all $\alpha \in \Delta(\mathfrak{u})$.

Then we have:

- $\gamma+\mu$ is regular;
- $\gamma+\mu$ satisfies (2-2);
- $\Delta^+(\gamma)=\{\alpha\in\Delta(\gamma)\mid\langle\alpha,\gamma+\mu\rangle>0\}$.

Moreover, if $\tilde{\pi}$ is the irreducible representation of $M$ associated to

$$\begin{aligned}
&\tilde{\delta}_i\equiv\delta_i+\mu_i \mod 2, && \tilde{\nu}_i=\nu_i+\mu_i,\\
&\tilde{\kappa}_i=\kappa_i+\tfrac{1}{2}(\mu_{k+2i-1}-\mu_{k+2i}), && \tilde{\nu}'_i=\nu'_i+\tfrac{1}{2}(\mu_{k+2i-1}+\mu_{k+2i}),\\
&\tilde{\lambda}'_i=\lambda'_i+\mu_{k+2l+i}, && \widetilde{\Psi}'=\Psi',
\end{aligned}$$

then we have shown that $\operatorname{Ind}_P^G(\tilde{\pi})$ is irreducible. On the other hand, by [Knapp and Vogan 1995, Theorem 7.237], we have

$$\psi^{\gamma}_{\gamma+\mu}(\tilde{\pi})=\pi.$$

Hence it follows from the argument in the proof of [Knapp 1986, Theorem 14.67] combined with [Vogan 1981, Lemma 7.2.18] that

$$\psi^{\gamma}_{\gamma+\mu}(\operatorname{Ind}_P^G(\tilde{\pi}))=\operatorname{Ind}_P^G(\psi^{\gamma}_{\gamma+\mu}(\tilde{\pi}))=\operatorname{Ind}_P^G(\pi).$$

From this and [Knapp and Vogan 1995, Theorem 7.229] (which asserts that under the integral dominance condition, the translation functor sends an irreducible $(\mathfrak{g},K)$-module to either an irreducible $(\mathfrak{g},K)$-module or zero), we deduce that $\operatorname{Ind}_P^G(\pi)$ is irreducible. This completes the proof.

## 3. Irreducibility of some nonstandard modules of $\mathbf{SO}_{2n+1}(\mathbb{R})$

In this section, we show that some nonstandard modules of $\mathrm{SO}_{2n+1}(\mathbb{R})$ are irreducible (see Proposition 3.4 below), which finishes the proof of [Gan and Ichino 2017, Lemma 5.5] in the real case.

**3A.** ***Notation.*** Let $G$ be a real reductive linear Lie group with abelian Cartan subgroups. Let $\mathfrak{g}_0=\operatorname{Lie}G$ be the Lie algebra of $G$ and fix a Cartan involution $\theta$ of $\mathfrak{g}_0$. We denote by $K$ the maximal compact subgroup of $G$ associated to $\theta$. Then we have a Cartan decomposition $\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{p}_0$, where $\mathfrak{k}_0=\operatorname{Lie}K$ and $\mathfrak{p}_0$ are the $+1$ and $-1$ eigenspaces of $\theta$ in $\mathfrak{g}_0$, respectively. Fix a nondegenerate invariant symmetric bilinear form

$$\langle\cdot,\cdot\rangle:\mathfrak{g}_0\times\mathfrak{g}_0\to\mathbb{R} \tag{3-1}$$

such that

- $\langle\cdot,\cdot\rangle$ is preserved by $\theta$;
- $\langle\cdot,\cdot\rangle$ is negative definite on $\mathfrak{k}_0$ and positive definite on $\mathfrak{p}_0$.

Let $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}_0$ and $Z(\mathfrak{g})$ the center of the universal enveloping algebra of $\mathfrak{g}$. Let $\mathrm{Ad}(\mathfrak{g})$ be the identity component of the automorphism group of $\mathfrak{g}$.

Let $H$ be a $\theta$-stable Cartan subgroup of $G$. Let $\mathfrak{h}_0 = \mathrm{Lie}\, H$ be the corresponding Cartan subalgebra of $\mathfrak{g}_0$ (so that $H$ is the centralizer of $\mathfrak{h}_0$ in $G$) and $\mathfrak{h} = \mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$ the complexification of $\mathfrak{h}_0$. Let $\langle \cdot , \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ be the bilinear form induced by (3-1). We denote by $\Delta(\mathfrak{g}, \mathfrak{h})$ the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$. Let $W(\mathfrak{g}, \mathfrak{h}) = W(\Delta(\mathfrak{g}, \mathfrak{h}))$ be the associated Weyl group and put $W(G,H) = N(G,H)/H$, where $N(G,H)$ is the normalizer of $H$ in $G$. Then we may regard $W(G,H)$ as a subgroup of $W(\mathfrak{g}, \mathfrak{h})$. For any regular element $\gamma \in \mathfrak{h}^*$, we denote by $\Delta(\gamma)$ the set of integral roots defined by $\gamma$:

$$\Delta(\gamma) = \left\{ \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \;\middle|\; 2\frac{\langle \alpha, \gamma \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \right\}.$$

Then $\Delta(\gamma)$ is a root system. Let $W(\gamma) = W(\Delta(\gamma))$ be the associated Weyl group. We may define a positive system $\Delta^+(\gamma)$ of $\Delta(\gamma)$ by

$$\Delta^+(\gamma) = \{\alpha \in \Delta(\gamma) \mid \langle \alpha, \gamma \rangle > 0\}.$$

Let $\Pi(\gamma)$ be the set of simple roots in $\Delta^+(\gamma)$. We define a homomorphism $\chi_\gamma : Z(\mathfrak{g}) \to \mathbb{C}$ as the composition of the Harish-Chandra isomorphism $Z(\mathfrak{g}) \cong S(\mathfrak{h})^{W(\mathfrak{g},\mathfrak{h})}$ with evaluation at $\gamma$.

Fix a $\theta$-stable maximally split Cartan subgroup $H^s$ of $G$ and write $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^s)$. Fix a regular element $\xi \in (\mathfrak{h}^s)^*$. For any $\gamma \in \mathfrak{h}^*$ such that $\chi_\gamma = \chi_\xi$, there exists an isomorphism $i_\gamma : (\mathfrak{h}^s)^* \to \mathfrak{h}^*$ such that

- $i_\gamma(\xi) = \gamma$;
- $i_\gamma$ is induced by some element $g \in \mathrm{Ad}(\mathfrak{g})$.

Since $\xi$ is regular, $i_\gamma$ does not depend on the choice of $g$. We define an automorphism $\theta_\gamma$ of $(\mathfrak{h}^s)^*$ by

$$\theta_\gamma = i_\gamma^{-1} \circ \theta \circ i_\gamma,$$

which depends only on the $K$-conjugacy class of $\gamma$. For $\alpha \in \Delta(\xi)$ and $w \in W(\xi)$, put

$$\alpha_\gamma = i_\gamma(\alpha) \in \Delta(\gamma), \tag{3-2}$$

$$w_\gamma = i_\gamma(w) \in W(\gamma). \tag{3-3}$$

Let $\Lambda = \Lambda^G$ be the subgroup of $\widehat{H^s}$ (where $\widehat{H^s}$ is the group of continuous characters of $H^s$) consisting of weights of finite-dimensional representations of $G$. For any $\lambda \in \Lambda$, we denote by $\bar{\lambda} \in (\mathfrak{h}^s)^*$ the differential of $\lambda$. Then the homomorphism $\lambda \mapsto \bar{\lambda}$ splits over the root lattice $\mathbb{Z}\Delta$ canonically; see [Vogan 1981, Lemma 0.4.5]. For any $\xi \in (\mathfrak{h}^s)^*$, we denote by $\xi + \Lambda$ the set of formal symbols $\xi + \lambda$ with $\lambda \in \Lambda$. Note that $W(\xi)$ acts on $\xi + \Lambda$; see [Vogan 1981, Definition 7.2.21].

We denote by $\mathrm{R}(\mathfrak{g}, K)$ the Grothendieck group of the category of $(\mathfrak{g}, K)$-modules of finite length. For any $(\mathfrak{g}, K)$-module $X$ of finite length, we denote by $[X]$ the image of $X$ in $\mathrm{R}(\mathfrak{g}, K)$.

**3B.** ***Regular characters.*** Following [Vogan 1984, Definition 2.2], we call a triple $\gamma = (H, \Gamma, \bar{\gamma})$ a regular character for $G$ if

- $H$ is a $\theta$-stable Cartan subgroup of $G$;
- $\Gamma$ is a continuous character of $H$;
- $\bar{\gamma} \in \mathfrak{h}^*$ is an element such that
  - if $\alpha \in \Delta(\mathfrak{g}, \mathfrak{h})$ is an imaginary root, then $\langle \alpha, \bar{\gamma} \rangle$ is a nonzero real number;
  - the differential of $\Gamma$ is
  $$\bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi),$$
  where $\Psi$ is the positive system of imaginary roots such that
  $$\langle \alpha, \bar{\gamma} \rangle > 0$$
  for all $\alpha \in \Psi$, $\rho(\Psi)$ is half the sum of the roots in $\Psi$, and $\rho_c(\Psi)$ is half the sum of the compact roots in $\Psi$.

If further $\bar{\gamma}$ is regular, we define the length $\ell(\gamma) = \ell^G(\gamma)$ of $\gamma$ by

$$\ell(\gamma) = \tfrac{1}{2}\left|\{\alpha \in \Delta^+(\bar{\gamma}) \mid \theta\alpha \notin \Delta^+(\bar{\gamma})\}\right| + \tfrac{1}{2}\dim \mathfrak{a}_0 \in \tfrac{1}{2}\mathbb{Z},$$

where $\mathfrak{a}_0$ is the $-1$ eigenspace of $\theta$ in $\mathfrak{h}_0$.

To any regular character $\gamma = (H, \Gamma, \bar{\gamma})$ for $G$ such that $\bar{\gamma}$ is regular, we may associate a $(\mathfrak{g}, K)$-module $X(\gamma) = X^G(\gamma)$ of finite length with infinitesimal character $\bar{\gamma}$ as follows; see [Vogan 1984, Definition 2.3]. Let $M$ be the centralizer of $\mathfrak{a}_0$ in $G$. Then there exists a unique relative discrete series $(\mathfrak{m}, M \cap K)$-module $X^M(\gamma)$ such that

- $X^M(\gamma)$ has infinitesimal character $\bar{\gamma}$;
- $X^M(\gamma)$ has a lowest $(M \cap K)$-type of highest weight $\Gamma|_{H \cap K}$.

Choose a parabolic subgroup $P = MN$ of $G$ with Levi component $M$ and unipotent radical $N$ such that

$$\operatorname{Re} \langle \alpha, \bar{\gamma} \rangle \le 0$$

for all roots $\alpha$ of $\mathfrak{a}$ in $\mathfrak{n}$. Then $X(\gamma)$ is given by

$$X(\gamma) = \operatorname{Ind}_P^G(X^M(\gamma)).$$

We recall some properties of $X(\gamma)$:

- $[X(\gamma)]$ depends only on the $K$-conjugacy class of $\gamma$.

- $X(\gamma)$ has a unique irreducible $(\mathfrak{g}, K)$-submodule $\overline{X}(\gamma)$.
- $\overline{X}(\gamma)$ depends only on the $K$-conjugacy class of $\gamma$.
- For any irreducible $(\mathfrak{g}, K)$-module $X$ with regular infinitesimal character, we have $X \cong \overline{X}(\gamma)$ for some $\gamma$.

For any $\theta$-stable Cartan subgroup $H$ of $G$ and any regular element $\xi \in (\mathfrak{h}^s)^*$, we denote by $\mathcal{R}^G(H, \xi)$ the set of regular characters $\gamma = (H, \Gamma, \bar{\gamma})$ for $G$ such that $\chi_{\bar{\gamma}} = \chi_\xi$. Put

$$\mathcal{R}^G(\xi) = \bigcup_H \mathcal{R}^G(H, \xi),$$

where the union runs over $\theta$-stable Cartan subgroups $H$ of $G$. Later, we also need the following notion.

**Definition 3.1.** We say that $H$ is $\xi$-integral if $\mathcal{R}^G(H, \xi) \neq \varnothing$.

**3C. *Coherent families.*** In this subsection, we recall some properties of coherent families.

Fix a regular element $\xi \in (\mathfrak{h}^s)^*$. Following [Vogan 1981, Definition 7.2.5], we call a map

$$\Theta : \xi + \Lambda \to \mathrm{R}(\mathfrak{g}, K)$$

a coherent family on $\xi + \Lambda$ if

- $\Theta(\xi + \lambda)$ has infinitesimal character $\xi + \bar{\lambda}$;
- for any finite-dimensional representation $F$ of $G$, we have

$$\Theta(\xi + \lambda) \otimes F = \sum_{\mu \in \Delta(F)} \Theta(\xi + \lambda + \mu),$$

  where $\Delta(F)$ is the multiset of weights of $H^s$ in $F$ (counted with multiplicity).

Then the following properties hold:

- For any coherent family $\Theta$ on $\xi + \Lambda$ and any $\lambda \in \Lambda$ such that $\xi + \bar{\lambda}$ is dominant for $\Delta^+(\xi)$ (but possibly singular), we have

$$\Theta(\xi + \lambda) = \psi_\xi^{\xi+\lambda}(\Theta(\xi)) \tag{3-4}$$

  by [Vogan 1981, Proposition 7.2.22], where $\psi_\xi^{\xi+\lambda}$ is the translation functor; see Definition 4.5.7 of the same work.
- For any $(\mathfrak{g}, K)$-module $X$ of finite length with infinitesimal character $\xi$, there exists a unique coherent family $\Theta_X$ on $\xi + \Lambda$ such that

$$\Theta_X(\xi) = [X]$$

  by [Vogan 1981, Theorem 7.2.7 and Corollary 7.2.27].

We denote by $\mathcal{C}(\xi+\Lambda)$ the free $\mathbb{Z}$-module of coherent families on $\xi+\Lambda$. Then we may define a representation $W(\xi)$ on $\mathcal{C}(\xi+\Lambda)$ by

$$(w\Theta)(\xi+\lambda)=\Theta(w^{-1}(\xi+\lambda))$$

for $w\in W(\xi)$ and $\Theta\in\mathcal{C}(\xi+\Lambda)$, which we call the coherent continuation representation; see [Vogan 1981, Definition 7.2.28].

For any $\gamma\in\mathcal{R}^G(\xi)$, we define coherent families $\Theta_\gamma=\Theta_\gamma^G$ and $\overline{\Theta}_\gamma=\overline{\Theta}_\gamma^G$ on $\xi+\Lambda$ by

$$\Theta_\gamma=\Theta_{X(\gamma)},\quad \overline{\Theta}_\gamma=\Theta_{\bar{X}(\gamma)}.$$

Put

$$\operatorname{Std}(G,\xi)=\{\Theta_\gamma\mid\gamma\in\mathcal{R}^G(\xi)\},\quad \operatorname{Irr}(G,\xi)=\{\overline{\Theta}_\gamma\mid\gamma\in\mathcal{R}^G(\xi)\}.$$

Then both $\operatorname{Std}(G,\xi)$ and $\operatorname{Irr}(G,\xi)$ are bases of $\mathcal{C}(\xi+\Lambda)$, so that we may define a bijection $\Theta\mapsto\overline{\Theta}$ from $\operatorname{Std}(G,\xi)$ to $\operatorname{Irr}(G,\xi)$ by $\Theta_\gamma\mapsto\overline{\Theta}_\gamma$ for $\gamma\in\mathcal{R}^G(\xi)$. Moreover, we may write

$$\overline{\Theta}_\gamma=\sum_{\Theta\in\operatorname{Std}(G,\xi)}M(\Theta,\overline{\Theta}_\gamma)\Theta \tag{3-5}$$

for some $M(\Theta,\overline{\Theta}_\gamma)\in\mathbb{Z}$.

Let $P$ be a parabolic subgroup of $G$ with Levi component $M$ such that $H^s\subset M$. In particular, $M$ is $\theta$-stable and $\Lambda^G\subset\Lambda^M$. Also, the parabolic induction functor $\operatorname{Ind}_P^G$ induces a homomorphism

$$\operatorname{Ind}_M^G:\mathrm{R}(\mathfrak{m},M\cap K)\to\mathrm{R}(\mathfrak{g},K),$$

which depends only on $M$. For any coherent family $\Theta^M$ on $\xi+\Lambda^M$, we may define a coherent family $\operatorname{Ind}_M^G(\Theta^M)$ on $\xi+\Lambda^G$ by

$$\operatorname{Ind}_M^G(\Theta^M)(\xi+\lambda)=\operatorname{Ind}_M^G(\Theta^M(\xi+\lambda))$$

for $\lambda\in\Lambda^G$; see [Speh and Vogan 1980, Lemma 5.8]. Then we have

$$\operatorname{Ind}_M^G(\Theta_\gamma^M)=\Theta_\gamma^G$$

for $\gamma\in\mathcal{R}^M(\xi)$, noting that $\mathcal{R}^M(\xi)\subset\mathcal{R}^G(\xi)$.

**3D.** ***The Kazhdan–Lusztig algorithm.*** In this subsection, we recall the Kazhdan–Lusztig algorithm for real reductive Lie groups, which determines the coefficients $M(\Theta,\overline{\Theta}_\gamma)$ in (3-5).

Fix a regular element $\xi\in(\mathfrak{h}^s)^*$. Recall the cross action of $W(\xi)$ on $\mathcal{R}^G(\xi)$:

$$w\times\gamma=(H,w_{\bar{\gamma}}^{-1}\times\Gamma,w_{\bar{\gamma}}^{-1}\bar{\gamma})$$

for $w\in W(\xi)$ and $\gamma=(H,\Gamma,\bar{\gamma})\in\mathcal{R}^G(\xi)$, where $w_{\bar{\gamma}}$ is as in (3-3) and $w_{\bar{\gamma}}^{-1}\times\Gamma$ is the cross product given in [Vogan 1981, Definition 8.3.1]. This descends to an action of $W(\xi)$ on $\mathrm{Std}(G,\xi)$ such that $w \times \Theta_\gamma = \Theta_{w\times\gamma}$ for $w \in W(\xi)$ and $\gamma \in \mathcal{R}^G(\xi)$.

Let $\alpha \in \Pi(\xi)$ and $\gamma = (H, \Gamma, \bar{\gamma}) \in \mathcal{R}^G(\xi)$. If the root $\alpha_{\bar{\gamma}}$ as in (3-2) either is noncompact imaginary, or is real and satisfies the parity condition [Vogan 1981, Definition 8.3.11], then we have the Cayley transform of $\Theta_\gamma$ through $\alpha$ (which is a subset of $\mathrm{Std}(G,\xi)$). We recall some details in turn.

• Suppose first that $\alpha_{\bar{\gamma}}$ is noncompact imaginary. Following [Vogan 1981, Definition 8.3.4], we say that $\alpha_{\bar{\gamma}}$ is type I (resp. type II) if the reflection in $W(\mathfrak{g},\mathfrak{h})$ with respect to $\alpha_{\bar{\gamma}}$ does not belong to (resp. belongs to) $W(G,H)$. Let $c^\alpha(\gamma)$ be the Cayley transform of $\gamma$ through $\alpha$; i.e., $c^\alpha(\gamma)$ is the subset of $\mathcal{R}^G(\xi)$ given in [Vogan 1981, Definition 8.3.6] of the form

$$c^\alpha(\gamma) = \{\gamma^\alpha\}, \quad \gamma^\alpha = (H^\alpha, \Gamma^\alpha, \bar{\gamma}^\alpha)$$

if $\alpha_{\bar{\gamma}}$ is type I, and

$$c^\alpha(\gamma) = \{\gamma^\alpha_+, \gamma^\alpha_-\}, \quad \gamma^\alpha_\pm = (H^\alpha, \Gamma^\alpha_\pm, \bar{\gamma}^\alpha)$$

if $\alpha_{\bar{\gamma}}$ is type II, where $H^\alpha$ is the $\theta$-stable Cartan subgroup of $G$ given in [Vogan 1981, Definition 8.3.4]. Then the subset

$$c^\alpha(\Theta_\gamma) = \{\Theta_{\gamma'} \mid \gamma' \in c^\alpha(\gamma)\}$$

of $\mathrm{Std}(G,\xi)$ depends only on the $K$-conjugacy class of $\gamma$.

• Suppose next that $\alpha_{\bar{\gamma}}$ is real and satisfies the parity condition [Vogan 1981, Definition 8.3.11]. Following Definition 8.3.8 of the same work, we say that $\alpha_{\bar{\gamma}}$ is type I (resp. type II) if $\alpha_{\bar{\gamma}} : H \cap K \to \{\pm 1\}$ is not surjective (resp. is surjective). Let $c_\alpha(\gamma)$ be the Cayley transform of $\gamma$ through $\alpha$; i.e., $c_\alpha(\gamma)$ is the subset of $\mathcal{R}^G(\xi)$ given in [Vogan 1981, Definitions 8.3.14 and 8.3.16] of the form

$$c_\alpha(\gamma) = \{\gamma_\alpha^+, \gamma_\alpha^-\}, \quad \gamma_\alpha^\pm = (H_\alpha, \Gamma_\alpha^\pm, \bar{\gamma}_\alpha^\pm)$$

if $\alpha_{\bar{\gamma}}$ is type I, and

$$c_\alpha(\gamma) = \{\gamma_\alpha\}, \quad \gamma_\alpha = (H_\alpha, \Gamma_\alpha, \bar{\gamma}_\alpha)$$

if $\alpha_{\bar{\gamma}}$ is type II, where $H_\alpha$ is the $\theta$-stable Cartan subgroup of $G$ given in [Vogan 1981, Definition 8.3.8]. Then the subset

$$c_\alpha(\Theta_\gamma) = \{\Theta_{\gamma'} \mid \gamma' \in c_\alpha(\gamma)\}$$

of $\mathrm{Std}(G,\xi)$ depends only on the $K$-conjugacy class of $\gamma$.

Let $\mathcal{H}(W(\xi))$ be the Hecke algebra of $W(\xi)$ over $\mathbb{Z}[q]$, where $q$ is an indeterminate. Note that the specialization at $q=1$ gives a surjection $\mathcal{H}(W(\xi)) \to \mathbb{Z}[W(\xi)]$.

Then, by [Vogan 1983b, Definition 5.2], see also [Vogan 1982, Definition 12.3 and Proposition 12.5], there exists an action of $\mathcal{H}(W(\xi))$ on

$$\mathcal{C}(\xi+\Lambda)_q = \mathcal{C}(\xi+\Lambda)\otimes_{\mathbb{Z}}\mathbb{Z}[q]$$

determined by the cross action and the Cayley transforms. Moreover, by [Vogan 1982, Lemma 14.5], the specialization of $\mathcal{C}(\xi+\Lambda)_q$ at $q=1$ is isomorphic to the coherent continuation representation tensored with the sign representation of $W(\xi)$. More explicitly, this isomorphism is induced by the surjection

$$\epsilon : \mathcal{C}(\xi+\Lambda)_q \to \mathcal{C}(\xi+\Lambda)$$

given by

$$\epsilon(q^i\Theta_\gamma) = (-1)^{\ell^I(\gamma)}\Theta_\gamma \tag{3-6}$$

for $i \geq 0$ and $\gamma \in \mathcal{R}^G(\xi)$, where the integral length $\ell^I(\gamma)$ of $\gamma$ is given by

$$\ell^I(\gamma) = \ell(\gamma) - c_0(G) \tag{3-7}$$

for some choice of $c_0(G) \in \frac{1}{2}\mathbb{Z}$ such that $\ell^I(\gamma)\in\mathbb{Z}$ for all $\gamma \in \mathcal{R}^G(\xi)$; see [Vogan 1982, Definition 12.1].

Finally, we recall the Kazhdan–Lusztig algorithm for real reductive Lie groups.

**Theorem 3.2** [Vogan 1983a; Adams et al. 1992, Theorem 16.22]. *For any $\gamma,\delta\in\mathcal{R}^G(\xi)$, we have*

$$M(\Theta_\gamma,\overline{\Theta}_\delta) = (-1)^{\ell^I(\gamma)-\ell^I(\delta)}P_{\gamma,\delta}(1),$$

*where $M(\Theta_\gamma,\overline{\Theta}_\delta)$ is the integer defined by (3-5) and $P_{\gamma,\delta}(q)$ is the Kazhdan–Lusztig–Vogan polynomial defined in terms of the $\mathcal{H}(W(\xi))$-module $\mathcal{C}(\xi+\Lambda)_q$. In particular, $M(\Theta_\gamma,\overline{\Theta}_\delta)$ can be computed by an algorithm which depends only on the $\mathcal{H}(W(\xi))$-module structure on $\mathcal{C}(\xi+\Lambda)_q$.*

**3E.** ***Comparison of Hecke algebra module structures.*** Let $G_1$ and $G_2$ be two real reductive linear Lie groups with abelian Cartan subgroups. For $i=1,2$, fix a Cartan involution $\theta_i$ of $(\mathfrak{g}_i)_0 = \operatorname{Lie} G_i$ and let $K_i$ be the maximal compact subgroup of $G_i$ associated to $\theta_i$. Fix a $\theta_i$-stable maximally split Cartan subgroup $H_i^s$ of $G_i$ and a regular element $\xi_1 \in (\mathfrak{h}_1^s)^*$.

We now assume that the following conditions hold:

(i) There exists an isomorphism

$$H_1^s \cong H_2^s.$$

(ii) Let $f:\widehat{H}_1^s \to \widehat{H}_2^s$ be the isomorphism induced by the isomorphism in (i). Then we have

$$f(\Lambda^{G_1}) \subset \Lambda^{G_2}.$$

(iii) Let $f : (\mathfrak{h}_1^s)^* \to (\mathfrak{h}_2^s)^*$ be the isomorphism induced by the isomorphism in (i) and put $\xi_2 = f(\xi_1)$. Then $\xi_2$ is regular.

(iv) The isomorphism in (iii) induces an isomorphism

$$f : \Delta(\xi_1) \to \Delta(\xi_2)$$

of root systems. This induces an isomorphism

$$f : W(\xi_1) \to W(\xi_2)$$

of the associated Weyl groups.

(v) There exists a bijection

$$\varphi : \mathrm{Std}(G_1, \xi_1) \to \mathrm{Std}(G_2, \xi_2).$$

(vi) Let $\gamma_i \in \mathcal{R}^{G_i}(\xi_i)$ be such that $\varphi(\Theta_{\gamma_1}) = \Theta_{\gamma_2}$. Then we have

$$f \circ \theta_{\bar{\gamma}_1} = \theta_{\bar{\gamma}_2} \circ f.$$

This implies that

$$\ell^{G_1}(\gamma_1) = \ell^{G_2}(\gamma_2),$$

and that for any $\alpha \in \Delta(\xi_1)$, $\alpha_{\bar{\gamma}_1}$ is imaginary (resp. real, resp. complex) if and only if $f(\alpha)_{\bar{\gamma}_2}$ is imaginary (resp. real, resp. complex).

(vii) Let $\gamma_i \in \mathcal{R}^{G_i}(\xi_i)$ and $\alpha \in \Delta(\xi_1)$ be such that $\varphi(\Theta_{\gamma_1}) = \Theta_{\gamma_2}$ and such that $\alpha_{\bar{\gamma}_1}$ is imaginary (and hence so is $f(\alpha)_{\bar{\gamma}_2}$). Then $\alpha_{\bar{\gamma}_1}$ is noncompact if and only if $f(\alpha)_{\bar{\gamma}_2}$ is noncompact, in which case $\alpha_{\bar{\gamma}_1}$ is type I (resp. type II) if and only if $f(\alpha)_{\bar{\gamma}_2}$ is type I (resp. type II).

(viii) Let $\gamma_i \in \mathcal{R}^{G_i}(\xi_i)$ and $\alpha \in \Delta(\xi_1)$ be such that $\varphi(\Theta_{\gamma_1}) = \Theta_{\gamma_2}$ and such that $\alpha_{\bar{\gamma}_1}$ is real (and hence so is $f(\alpha)_{\bar{\gamma}_2}$). Then $\alpha_{\bar{\gamma}_1}$ satisfies the parity condition if and only if $f(\alpha)_{\bar{\gamma}_2}$ satisfies the parity condition, in which case $\alpha_{\bar{\gamma}_1}$ is type I (resp. type II) if and only if $f(\alpha)_{\bar{\gamma}_2}$ is type I (resp. type II).

(ix) The bijection in (v) is compatible with the cross action: for $w \in W(\xi_1)$ and $\gamma \in \mathcal{R}^{G_1}(\xi_1)$, we have

$$\varphi(w \times \Theta_\gamma) = f(w) \times \varphi(\Theta_\gamma).$$

(x) The bijection in (v) is compatible with the Cayley transforms: for $\alpha \in \Pi(\xi_1)$ and $\gamma \in \mathcal{R}^{G_1}(\xi_1)$, we have

$$\varphi(c^\alpha(\Theta_\gamma)) = c^{f(\alpha)}(\varphi(\Theta_\gamma))$$

if $\alpha_{\bar{\gamma}}$ is noncompact imaginary, and

$$\varphi(c_\alpha(\Theta_\gamma)) = c_{f(\alpha)}(\varphi(\Theta_\gamma))$$

if $\alpha_{\bar{\gamma}}$ is real and satisfies the parity condition.

The bijection in (v) induces isomorphisms

$$\varphi : \mathcal{C}(\xi_1 + \Lambda^{G_1}) \to \mathcal{C}(\xi_2 + \Lambda^{G_2}),$$
$$\varphi_q : \mathcal{C}(\xi_1 + \Lambda^{G_1})_q \to \mathcal{C}(\xi_2 + \Lambda^{G_2})_q$$

of $\mathbb{Z}$-modules and $\mathbb{Z}[q]$-modules, respectively. By the definition of the $\mathcal{H}(W(\xi_i))$-module structure on $\mathcal{C}(\xi_i + \Lambda^{G_i})_q$, the above conditions imply $\varphi_q$ is equivariant under the action of $\mathcal{H}(W(\xi_1)) \cong \mathcal{H}(W(\xi_2))$. From this and the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(\xi_1 + \Lambda^{G_1})_q & \xrightarrow{\varphi_q} & \mathcal{C}(\xi_2 + \Lambda^{G_2})_q \\ \downarrow & & \downarrow \\ \mathcal{C}(\xi_1 + \Lambda^{G_1}) & \xrightarrow{\varphi} & \mathcal{C}(\xi_2 + \Lambda^{G_2}) \end{array}$$

induced by the specialization at $q = 1$ defined by (3-6) (with a suitable choice of $c_0(G_i)$ in the definition of the integral length; see (3-7)), we can deduce that $\varphi$ is an isomorphism of the coherent continuation representations of $W(\xi_1) \cong W(\xi_2)$. Moreover, by Theorem 3.2, we have

$$M\big(\varphi(\Theta_\gamma), \overline{\varphi(\Theta_\delta)}\big) = M(\Theta_\gamma, \overline{\Theta}_\delta)$$

for all $\gamma, \delta \in \mathcal{R}^{G_1}(\xi_1)$ and hence

$$\overline{\varphi(\Theta)} = \varphi(\overline{\Theta})$$

for all $\Theta \in \mathrm{Std}(G_1, \xi_1)$. In particular, $\varphi$ induces a bijection from $\mathrm{Irr}(G_1, \xi_1)$ to $\mathrm{Irr}(G_2, \xi_2)$.

**Lemma 3.3.** *For $i = 1, 2$, let $\Xi_i \in \mathcal{C}(\xi_i + \Lambda^{G_i})$ and $\lambda_i \in \Lambda^{G_i}$ be such that $\varphi(\Xi_1) = \Xi_2$ and $f(\lambda_1) = \lambda_2$. Assume there exists an irreducible $(\mathfrak{g}_2, K_2)$-module $X_2$ such that*

$$\Xi_2(\xi_2 + \lambda_2) = [X_2].$$

*Then there exists an irreducible $(\mathfrak{g}_1, K_1)$-module $X_1$ such that*

$$\Xi_1(\xi_1 + \lambda_1) = [X_1].$$

*Proof.* The assertion was proved by Matumoto [2004, Lemma 4.1.3] when Cartan subgroups of $G_i$ are all connected, but the argument works in the general case. We include the proof for the convenience of the reader.

Choose $w_1 \in W(\xi_1)$ such that $w_1(\xi_1 + \bar{\lambda}_1)$ is dominant for $\Delta^+(\xi_1)$ and write

$$w_1 \Xi_1 = \sum_{\overline{\Theta} \in \mathrm{Irr}(G_1, \xi_1)} a_{\overline{\Theta}} \overline{\Theta}$$

for some $a_{\overline{\Theta}} \in \mathbb{Z}$. Put $w_2 = f(w_1) \in W(\xi_2)$, so that $\varphi(w_1 \Xi_1) = w_2 \Xi_2$. Then

$$\begin{aligned} \sum_{\overline{\Theta} \in \mathrm{Irr}(G_1, \xi_1)} a_{\overline{\Theta}} \varphi(\overline{\Theta})(w_2(\xi_2 + \lambda_2)) &= \varphi(w_1 \Xi_1)(w_2(\xi_2 + \lambda_2)) \\ &= (w_2 \Xi_2)(w_2(\xi_2 + \lambda_2)) = \Xi_2(\xi_2 + \lambda_2) = [X_2]. \end{aligned}$$

On the other hand, since $w_2(\xi_2+\bar{\lambda}_2)$ is dominant for $\Delta^+(\xi_2)$, we deduce from (3-4) and [Vogan 1983b, Theorem 7.6], see also [Speh and Vogan 1980, Theorem 6.18], that for any $\overline{\Upsilon} \in \mathrm{Irr}(G_2, \xi_2)$, $\overline{\Upsilon}(w_2(\xi_2+\lambda_2))$ is either $[X]$ for some irreducible $(\mathfrak{g}_2, K_2)$-module $X$ or zero, and that there exists a unique $\overline{\Upsilon}_0 \in \mathrm{Irr}(G_2, \xi_2)$ such that

$$\overline{\Upsilon}_0(w_2(\xi_2+\lambda_2)) = [X_2].$$

Hence, noting that $\varphi(\overline{\Theta}) \in \mathrm{Irr}(G_2, \xi_2)$ for $\overline{\Theta} \in \mathrm{Irr}(G_1, \xi_1)$, we have

$$a_{\overline{\Theta}_0} = 1$$

for $\overline{\Theta}_0 = \varphi^{-1}(\overline{\Upsilon}_0)$, and either $a_{\overline{\Theta}} = 0$ or $\varphi(\overline{\Theta})(w_2(\xi_2+\lambda_2)) = 0$ for $\overline{\Theta} \neq \overline{\Theta}_0$. Moreover, recalling the definition of $\tau$-invariants, see [Vogan 1983b, Definition 5.3], we can also deduce from (3-4) and [Vogan 1983b, Theorem 7.6] that

$$\varphi(\overline{\Theta})(w_2(\xi_2+\lambda_2)) = 0 \iff \overline{\Theta}(w_1(\xi_1+\lambda_1)) = 0$$

for all $\overline{\Theta} \in \mathrm{Irr}(G_1, \xi_1)$. Thus, we obtain

$$\begin{aligned}\Xi_1(\xi_1+\lambda_1) &= (w_1\Xi_1)(w_1(\xi_1+\lambda_1)) \\ &= \sum_{\overline{\Theta}\in\mathrm{Irr}(G_1,\xi_1)} a_{\overline{\Theta}}\overline{\Theta}(w_1(\xi_1+\lambda_1)) = \overline{\Theta}_0(w_1(\xi_1+\lambda_1)) = [X_1]\end{aligned}$$

for some irreducible $(\mathfrak{g}_1, K_1)$-module $X_1$. □

**3F.** ***Some nonstandard modules of*** $\mathbf{SO_{2n+1}}(\mathbb{R})$**.** Let $G = \mathrm{SO}_{2n+1}(\mathbb{R})$ be the split odd special orthogonal group, which we realize as

$$\mathrm{SO}_{2n+1}(\mathbb{R}) = \left\{ g \in \mathrm{SL}_{2n+1}(\mathbb{R}) \;\middle|\; {}^t g \begin{pmatrix} \mathbf{1}_{n+1} & \\ & -\mathbf{1}_n \end{pmatrix} g = \begin{pmatrix} \mathbf{1}_{n+1} & \\ & -\mathbf{1}_n \end{pmatrix} \right\}.$$

We define a Cartan involution $\theta$ of $G$ by

$$\theta(g) = {}^t g^{-1}.$$

Let $K$ be the maximal compact subgroup of $G$ associated to $\theta$. We define the bilinear form

$$\langle\cdot,\cdot\rangle : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbb{R}$$

as in (3-1) by

$$\langle X, Y\rangle = \tfrac{1}{2}\,\mathrm{tr}(XY).$$

For any nonnegative integers $k, l, m$ such that $k+2l+m = n$, we define a $\theta$-stable Cartan subalgebra $\mathfrak{h}_0^{k,l,m}$ of $\mathfrak{g}_0$ as follows. For $a = (a_1, \dots, a_k) \in \mathbb{R}^k$, put

$$h^{k,0,0}(a) = \begin{pmatrix} & \boldsymbol{a} \\ \boldsymbol{a} & \end{pmatrix} \in \mathfrak{so}_{2k}(\mathbb{R}),$$

where $\boldsymbol{a} = \operatorname{diag}(a_1, \dots, a_k)$. For $z = (z_1, \dots, z_l) \in \mathbb{C}^l$ with $z_i = x_i + \sqrt{-1}y_i$, put

$$h^{0,l,0}(z) = \begin{pmatrix} & \boldsymbol{y} & & \boldsymbol{x} \\ -\boldsymbol{y} & & \boldsymbol{x} & \\ & \boldsymbol{x} & & -\boldsymbol{y} \\ \boldsymbol{x} & & \boldsymbol{y} & \end{pmatrix} \in \mathfrak{so}_{4l}(\mathbb{R}),$$

where $\boldsymbol{x} = \operatorname{diag}(x_1, \dots, x_l)$ and $\boldsymbol{y} = \operatorname{diag}(y_1, \dots, y_l)$. For $\vartheta = (\vartheta_1, \dots, \vartheta_m) \in \mathbb{R}^m$, put

$$h^{0,0,m}(\vartheta) = \operatorname{diag}(\boldsymbol{\vartheta}_1, \dots, \boldsymbol{\vartheta}_{m_1}, 0, -\boldsymbol{\vartheta}_{m_1+1}, \dots, -\boldsymbol{\vartheta}_m) \in \mathfrak{so}_{2m+1}(\mathbb{R}),$$

where

$$\boldsymbol{\vartheta}_i = \begin{pmatrix} & \vartheta_i \\ -\vartheta_i & \end{pmatrix}$$

and $m_1 = [(m+1)/2]$. Let $h^{k,l,m}(a, z, \vartheta)$ be the image of

$$(h^{k,0,0}(a), h^{0,l,0}(z), h^{0,0,m}(\vartheta))$$

under the natural embedding

$$\mathfrak{so}_{2k}(\mathbb{R}) \oplus \mathfrak{so}_{4l}(\mathbb{R}) \oplus \mathfrak{so}_{2m+1}(\mathbb{R}) \hookrightarrow \mathfrak{so}_{2n+1}(\mathbb{R}).$$

Then we set

$$\mathfrak{h}_0^{k,l,m} = \{h^{k,l,m}(a, z, \vartheta) \mid a \in \mathbb{R}^k,\ z \in \mathbb{C}^l,\ \vartheta \in \mathbb{R}^m\}.$$

These $\mathfrak{h}_0^{k,l,m}$ with $k + 2l + m = n$ form a set of representatives for the $G$-conjugacy classes of Cartan subalgebras of $\mathfrak{g}_0$. Let $H^{k,l,m}$ be the centralizer of $\mathfrak{h}_0^{k,l,m}$ in $G$. Then $H^{k,l,m}$ is a $\theta$-stable Cartan subgroup of $G$ isomorphic to

$$(\mathbb{R}^\times)^k \times (\mathbb{C}^\times)^l \times (S^1)^m.$$

Note that $W(\mathfrak{g}, \mathfrak{h}^{k,l,m}) \cong W(B_n)$ and

$$W(G, H^{k,l,m}) \cong W(B_k) \times (\mathfrak{S}_l \ltimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^l) \times W(B_{m_1}) \times W(B_{m_2}), \quad (3\text{-}8)$$

where $\mathfrak{S}_d$ is the symmetric group of degree $d$, $W(B_d) = \mathfrak{S}_d \ltimes (\mathbb{Z}/2\mathbb{Z})^d$ is the Weyl group of type $B_d$, $m_1 = [(m+1)/2]$, and $m_2 = [m/2]$; see, e.g., [Vogan 1982, Proposition 4.16].

Fix nonnegative integers $k, l, m$ such that $k + 2l + m = n$ and write $\mathfrak{h}_0 = \mathfrak{h}_0^{k,l,m}$. Let $M$ be the centralizer of $\mathfrak{a}_0$ in $G$, where $\mathfrak{a}_0$ is the $-1$ eigenspace of $\theta$ in $\mathfrak{h}_0$. Then $M$ is a Levi subgroup of $G$ isomorphic to

$$\mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times \mathrm{SO}_{2m+1}(\mathbb{R}).$$

We consider an irreducible representation $\pi$ of $M$ of the form

$$\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi',$$

where

- $\chi_i$ is a character of $\mathrm{GL}_1(\mathbb{R})$ of the form
$$\chi_i = \mathrm{sgn}^{\delta_i} \otimes |\cdot|^{\nu_i}$$
for some $\delta_i \in \{0, 1\}$ and some $\nu_i \in \mathbb{C}$;
- $\tau_i$ is an irreducible representation of $\mathrm{GL}_2(\mathbb{R})$ of the form
$$\tau_i = D_{\kappa_i} \otimes |\det|^{\nu'_i}$$
for some $\kappa_i \in \frac{1}{2}\mathbb{Z}$ and some $\nu'_i \in \mathbb{C}$, where $D_{\kappa_i}$ is the relative (limit of) discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight $2|\kappa_i| + 1$ with central character trivial on $\mathbb{R}_+^\times$;
- $\pi'$ is an irreducible representation of $\mathrm{SO}_{2m+1}(\mathbb{R})$ with infinitesimal character
$$\lambda' = (\lambda'_1, \ldots, \lambda'_m) \in (\mathfrak{h}^{m,0,0})^* \cong \mathbb{C}^m$$
(with the identification given in Section 3G below).

Choose a parabolic subgroup $P$ of $G$ with Levi component $M$.

We now state the main result of this section.

**Proposition 3.4.** *Assume that*

- *if* $\nu_i = \pm\nu_j$, *then* $\delta_i = \delta_j$;
- *if* $\nu'_i = 0$, *then* $\kappa_i \in \mathbb{Z}$;
- $|\mathrm{Re}\,\nu_i|, |\mathrm{Re}\,\nu'_j| < \frac{1}{2}$ *for all* $i, j$;
- $\lambda'_i \in \mathbb{Z} + \frac{1}{2}$ *for all* $i$.

*Then the normalized parabolic induction* $\mathrm{Ind}_P^G(\pi)$ *is irreducible.*

**3G.** ***Proof of Proposition 3.4.*** Put
$$G_1 = \mathrm{SO}_{2n+1}(\mathbb{R}), \quad G_2 = \mathrm{SO}_{2(n-m)+1}(\mathbb{R}) \times \mathrm{SO}_{2m+1}(\mathbb{R}).$$
We define embeddings $\iota : \mathrm{SO}_{2(n-m)+1}(\mathbb{R}) \hookrightarrow G_1$ and $\iota' : \mathrm{SO}_{2m+1}(\mathbb{R}) \hookrightarrow G_1$ by
$$\iota\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & & b & \\ & \mathbf{1}_m & & \\ c & & d & \\ & & & \mathbf{1}_m \end{pmatrix}, \qquad \begin{array}{ll} a \in \mathrm{M}_{n-m,n-m}(\mathbb{R}), & b \in \mathrm{M}_{n-m,n-m+1}(\mathbb{R}), \\ c \in \mathrm{M}_{n-m+1,n-m}(\mathbb{R}), & d \in \mathrm{M}_{n-m+1,n-m+1}(\mathbb{R}), \end{array}$$
$$\iota'\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{n-m} & & & \\ & a' & & b' \\ & & \mathbf{1}_{n-m} & \\ & c' & & d' \end{pmatrix}, \qquad \begin{array}{ll} a' \in \mathrm{M}_{m+1,m+1}(\mathbb{R}), & b' \in \mathrm{M}_{m+1,m}(\mathbb{R}), \\ c' \in \mathrm{M}_{m,m+1}(\mathbb{R}), & d' \in \mathrm{M}_{m,m}(\mathbb{R}). \end{array}$$
For $i = 1, 2$, let $\theta_i$ be the Cartan involution of $G_i$ as in Section 3F and $K_i$ the maximal compact subgroup of $G_i$ associated to $\theta_i$. We take a $\theta_i$-stable maximally

split Cartan subgroup $H_i^s$ of $G_i$ given by

$$H_1^s = H^{n,0,0}, \quad H_2^s = H^{n-m,0,0} \times H^{m,0,0}.$$

Then we have an isomorphism $H_2^s \to H_1^s$ given by

$$(h, h') \mapsto \iota(h)\iota'(h'). \tag{3-9}$$

This induces an isomorphism $f : \widehat{H}_1^s \to \widehat{H}_2^s$.

**Lemma 3.5.** *We have* $f(\Lambda^{G_1}) \subset \Lambda^{G_2}$.

*Proof.* Let $\mu \in \Lambda^{G_1}$, so that $\mu$ occurs in some finite-dimensional representation $F$ of $G_1$. Then $f(\mu)$ occurs in the representation $\iota^* F \otimes (\iota')^* F$ of $G_2$. Hence $f(\mu) \in \Lambda^{G_2}$. □

Also, the isomorphism (3-9) induces an isomorphism

$$f : (\mathfrak{h}_1^s)^* \to (\mathfrak{h}_2^s)^*. \tag{3-10}$$

We define a basis $e_1^s, \dots, e_n^s$ of $(\mathfrak{h}_1^s)^* = (\mathfrak{h}^{n,0,0})^*$ by

$$e_i^s(h^{n,0,0}(a)) = a_i.$$

Fix a regular element $\xi_1 = (x_1, \dots, x_n) \in (\mathfrak{h}_1^s)^* \cong \mathbb{C}^n$ (with the identification using the above basis) such that

$$\begin{aligned} x_i &\notin \mathbb{Z} + \tfrac{1}{2} \quad (1 \le i \le n-m), \\ x_i &\in \mathbb{Z} + \tfrac{1}{2} \quad (n-m < i \le n). \end{aligned} \tag{3-11}$$

Put $\xi_2 = f(\xi_1)$. Since $f(\Delta(\mathfrak{g}_1, \mathfrak{h}_1^s)) \supset \Delta(\mathfrak{g}_2, \mathfrak{h}_2^s)$, we know $\xi_2$ is regular.

**Lemma 3.6.** *The isomorphism* (3-10) *induces an isomorphism* $f : \Delta(\xi_1) \to \Delta(\xi_2)$ *of root systems.*

*Proof.* Since

$$\Delta(\mathfrak{g}_1, \mathfrak{h}_1^s) \setminus f^{-1}(\Delta(\mathfrak{g}_2, \mathfrak{h}_2^s)) = \{\pm e_i^s \pm e_j^s \mid 1 \le i \le n-m < j \le n\},$$

it follows from (3-11) that

$$2\frac{\langle \alpha, \xi_1 \rangle}{\langle \alpha, \alpha \rangle} \notin \mathbb{Z}$$

for all $\alpha \in \Delta(\mathfrak{g}_1, \mathfrak{h}_1^s) \setminus f^{-1}(\Delta(\mathfrak{g}_2, \mathfrak{h}_2^s))$. This implies the assertion. □

Recall that

$$\begin{aligned} &H^{k',l',m'} && (k' + 2l' + m' = n), \\ &H^{p,q,r} \times H^{p',q',r'} && (p + 2q + r = n-m,\ p' + 2q' + r' = m) \end{aligned}$$

form a set of representatives for the $K_i$-conjugacy classes of $\theta_i$-stable Cartan subgroups of $G_i$ for $i = 1, 2$, respectively.

**Lemma 3.7.** (i) *If the $\theta_1$-stable Cartan subgroup $H^{k',l',m'}$ of $G_1$ is $\xi_1$-integral, then $m' \le m$.*

(ii) *If the $\theta_2$-stable Cartan subgroup $H^{p,q,r} \times H^{p',q',r'}$ of $G_2$ is $\xi_2$-integral, then $r = 0$.*

*Proof.* We only prove (i); the proof of (ii) is similar. Put $H_1 = H^{k',l',m'}$ and $\mathfrak{h}_1 = \mathfrak{h}^{k',l',m'}$. We define a basis $e_1, \dots, e_n$ of $\mathfrak{h}_1^*$ by

$$\begin{aligned} e_i(h) &= a_i && (1 \le i \le k'),\\ e_{k'+2i-1}(h) &= x_i + \sqrt{-1}y_i && (1 \le i \le l'),\\ e_{k'+2i}(h) &= -x_i + \sqrt{-1}y_i && (1 \le i \le l'),\\ e_{k'+2l'+i}(h) &= \sqrt{-1}\vartheta_i && (1 \le i \le m') \end{aligned}$$

for $h = h^{k',l',m'}(a, z, \vartheta)$. Note that

$$\begin{aligned} \theta(e_i) &= -e_i && (1 \le i \le k'),\\ \theta(e_{k'+2i-1}) &= e_{k'+2i} && (1 \le i \le l'),\\ \theta(e_{k'+2l'+i}) &= e_{k'+2l'+i} && (1 \le i \le m'). \end{aligned}$$

Then there exists a unique isomorphism $j : (\mathfrak{h}_1^s)^* \to \mathfrak{h}_1^*$ such that

- $j(e_i^s) = e_i$ for all $i$;
- $j$ is induced by some element in $\mathrm{Ad}(\mathfrak{g}_1)$.

Let $\gamma_1 = (H_1, \Gamma_1, \bar{\gamma}_1) \in \mathcal{R}^{G_1}(\xi_1)$. Then $j$ is $W(\mathfrak{g}_1, \mathfrak{h}_1)$-conjugate to $i_{\bar{\gamma}_1}$. Under the identification $\mathfrak{h}_1^* \cong \mathbb{C}^n$ using the above basis, we write

$$\bar{\gamma}_1 = (u_1, \dots, u_n), \quad \rho(\Psi) - 2\rho_c(\Psi) = (v_1, \dots, v_n),$$

where $\Psi$ is the positive system of imaginary roots as in Section 3B. Then we have $v_i \in \mathbb{Z} + \frac{1}{2}$ for all $k' < i \le n$. Since $\bar{\gamma}_1 + \rho(\Psi) - 2\rho_c(\Psi)$ is the differential of a character of $H_1 \cong (\mathbb{R}^\times)^{k'} \times (\mathbb{C}^\times)^{l'} \times (S^1)^{m'}$, we must have

$$\begin{aligned} u_{k'+2i-1} + v_{k'+2i-1} + u_{k'+2i} + v_{k'+2i} &\in \mathbb{Z} && (1 \le i \le l'),\\ u_{k'+2l'+i} + v_{k'+2l'+i} &\in \mathbb{Z} && (1 \le i \le m'), \end{aligned}$$

so that

$$\begin{aligned} u_{k'+2i-1} + u_{k'+2i} &\in \mathbb{Z} && (1 \le i \le l'),\\ u_{k'+2l'+i} &\in \mathbb{Z} + \tfrac{1}{2} && (1 \le i \le m'). \end{aligned}$$

Hence, noting that $j(\xi_1)$ is $W(\mathfrak{g}_1, \mathfrak{h}_1)$-conjugate to $i_{\bar{\gamma}_1}(\xi_1) = \bar{\gamma}_1$, we deduce from (3-11) that $m' \le m$. ☐

We now define the map

$$\varphi' : \mathrm{Std}(G_2, \xi_2) \to \mathrm{Std}(G_1, \xi_1)$$

as follows. Let $\gamma_2 = (H_2, \Gamma_2, \bar{\gamma}_2) \in \mathcal{R}^{G_2}(\xi_2)$. Replacing $\gamma_2$ by a $K_2$-conjugate if necessary, we may assume that

$$H_2 = H^{p,q,r} \times H^{p',q',r'}$$

with $p+2q+r = n-m$ and $p'+2q'+r' = m$. By Lemma 3.7, we have $r = 0$. Put

$$H_1 = \{\iota(h)\iota'(h') \mid h \in H^{p,q,0},\ h' \in H^{p',q',r'}\}.$$

Then $H_1$ is a $\theta_1$-stable Cartan subgroup of $G_1$ and is $K_1$-conjugate to $H^{p+p',q+q',r'}$. Moreover, we have an isomorphism $H_2 \to H_1$ given by $(h, h') \mapsto \iota(h)\iota'(h')$. This induces isomorphisms $\phi : \widehat{H}_1 \to \widehat{H}_2$ and $\phi : \mathfrak{h}_1^* \to \mathfrak{h}_2^*$, which in turn induces an embedding

$$W(\mathfrak{g}_2, \mathfrak{h}_2) \hookrightarrow W(\mathfrak{g}_1, \mathfrak{h}_1).$$

We identify $W(\mathfrak{g}_2, \mathfrak{h}_2)$ with its image in $W(\mathfrak{g}_1, \mathfrak{h}_1)$.

**Lemma 3.8.** *We have*

$$W(G_2, H_2) = W(\mathfrak{g}_2, \mathfrak{h}_2) \cap W(G_1, H_1).$$

*Proof.* The assertion follows from (3-8). □

Put $\gamma_1 = (H_1, \Gamma_1, \bar{\gamma}_1)$, where

$$\Gamma_1 = \phi^{-1}(\Gamma_2), \qquad \bar{\gamma}_1 = \phi^{-1}(\bar{\gamma}_2).$$

Then we have $\gamma_1 \in \mathcal{R}^{G_1}(\xi_1)$, and by Lemma 3.8, the $K_1$-conjugacy class of $\gamma_1$ is uniquely determined by the $K_2$-conjugacy class of $\gamma_2$. Hence we may define $\varphi'$ by

$$\varphi'(\Theta_{\gamma_2}) = \Theta_{\gamma_1}.$$

We also define the map

$$\varphi : \mathrm{Std}(G_1, \xi_1) \to \mathrm{Std}(G_2, \xi_2)$$

as follows. Let $\gamma_1 = (H_1, \Gamma_1, \bar{\gamma}_1) \in \mathcal{R}^{G_1}(\xi_1)$. Replacing $\gamma_1$ by a $K_1$-conjugate if necessary, we may assume that

$$H_1 = H^{k',l',m'}$$

with $k'+2l'+m' = n$. Write $\bar{\gamma}_1 = (u_1, \dots, u_n)$ as in the proof of Lemma 3.7 and put

$$p' = \left|\left\{1 \le i \le k' \mid u_i \in \mathbb{Z} + \tfrac{1}{2}\right\}\right|,$$
$$q' = \tfrac{1}{2}\left|\left\{1 \le i \le 2l' \mid u_{k'+i} \in \mathbb{Z} + \tfrac{1}{2}\right\}\right|,$$
$$r' = \left|\left\{1 \le i \le m' \mid u_{k'+2l'+i} \in \mathbb{Z} + \tfrac{1}{2}\right\}\right|.$$

Then it follows from the proof of Lemma 3.7 that

$$q' \in \mathbb{Z}, \quad r' = m', \quad p' + 2q' + r' = m.$$

Put

$$H_2 = H^{p,q,0} \times H^{p',q',r'},$$

where $p = k' - p'$ and $q = l' - q'$. Then $H_2$ is a $\theta_2$-stable Cartan subgroup of $G_2$. Replacing $\gamma_1$ by a $K_1$-conjugate again, we may now assume that

$$H_1 = \{\iota(h)\iota'(h') \mid h \in H^{p,q,0},\ h' \in H^{p',q',r'}\}.$$

Let $\phi : \widehat{H}_1 \to \widehat{H}_2$ and $\phi : \mathfrak{h}_1^* \to \mathfrak{h}_2^*$ be the isomorphisms induced by the isomorphism $H_2 \to H_1$ given by $(h, h') \mapsto \iota(h)\iota'(h')$. Put $\gamma_2 = (H_2, \Gamma_2, \bar{\gamma}_2)$, where

$$\Gamma_2 = \phi(\Gamma_1), \quad \bar{\gamma}_2 = \phi(\bar{\gamma}_1).$$

Replacing $\gamma_1$ by a $W(G_1, H_1)$-conjugate if necessary, we may further assume that $\chi_{\bar{\gamma}_2} = \chi_{\xi_2}$. Then we have $\gamma_2 \in \mathcal{R}^{G_2}(\xi_2)$, and by Lemma 3.8, the $K_2$-conjugacy class of $\gamma_2$ is uniquely determined by the $K_1$-conjugacy class of $\gamma_1$. Hence we may define $\varphi$ by

$$\varphi(\Theta_{\gamma_1}) = \Theta_{\gamma_2}.$$

By construction, we have:

**Lemma 3.9.** *The two maps $\varphi$ and $\varphi'$ are inverses of each other. Moreover, the conditions (i)–(x) in Section 3E hold.*

Finally, as in Section 3F, we define a Levi subgroup $M_i$ of $G_i$ with respect to the $\theta_i$-stable Cartan subgroup

$$H^{k,l,m}, \quad H^{k,l,0} \times H^{0,0,m}$$

of $G_i$ for $i = 1, 2$, respectively. Then we have $H_i^s \subset M_i$ and

$$M_i \cong \mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times \mathrm{SO}_{2m+1}(\mathbb{R}).$$

Since $M_2 = M_3 \times \mathrm{SO}_{2m+1}(\mathbb{R})$ for some Levi subgroup $M_3 \cong \mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l$ of $\mathrm{SO}_{2(n-m)+1}(\mathbb{R})$, we may identify $M_2$ with $M_1$ via the isomorphism $M_2 \to M_1$ given by $(h, h') \mapsto \iota(h)\iota'(h')$. Let $P_i$ be a parabolic subgroup of $G_i$ with Levi component $M_i$. Note that $P_2 = P_3 \times \mathrm{SO}_{2m+1}(\mathbb{R})$ for some parabolic subgroup $P_3$ of $\mathrm{SO}_{2(n-m)+1}(\mathbb{R})$ with Levi component $M_3$. Recall the irreducible representation

$$\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi'$$

of $M_1$ as in Section 3F. Put

$$\xi_1' = (\nu_1, \ldots, \nu_k, \kappa_1 + \nu_1', \kappa_1 - \nu_1', \ldots, \kappa_l + \nu_l', \kappa_l - \nu_l', \lambda_1', \ldots, \lambda_m') \in (\mathfrak{h}_1^s)^* \cong \mathbb{C}^n,$$

so that $\mathrm{Ind}_{P_1}^{G_1}(\pi)$ has infinitesimal character $\xi_1'$. Fix a positive system $\Delta^+$ of $\Delta(\mathfrak{g}_1, \mathfrak{h}_1^s)$ such that

$$\mathrm{Re}\,\langle \alpha, \xi_1' \rangle \geq 0$$

for all $\alpha \in \Delta^+$ and let $\rho(\Delta^+)$ be half the sum of the roots in $\Delta^+$. Choose a sufficiently large positive integer $t$ such that

$$\xi_1 = \xi_1' + 2t\rho(\Delta^+)$$

is regular. Then we have $\Delta^+(\xi_1) = \Delta(\xi_1) \cap \Delta^+$, and by the assumption on $\pi$, $\xi_1$ satisfies (3-11). By construction, we have

$$\varphi(\Theta_\gamma^{G_1}) = \Theta_\gamma^{G_2}$$

for all $\gamma \in \mathcal{R}^{M_1}(\xi_1) = \mathcal{R}^{M_2}(\xi_2)$. Since $\Theta_\gamma^{G_i} = \mathrm{Ind}_{M_i}^{G_i}(\Theta_\gamma^{M_i})$ and $\mathrm{Ind}_{M_i}^{G_i}$ is additive, we have

$$\varphi(\mathrm{Ind}_{M_1}^{G_1}(\overline{\Theta})) = \mathrm{Ind}_{M_2}^{G_2}(\overline{\Theta})$$

for all $\overline{\Theta} \in \mathrm{Irr}(M_1, \xi_1) = \mathrm{Irr}(M_2, \xi_2)$. On the other hand, by (3-4) and [Vogan 1983b, Theorem 7.6], there exists $\overline{\Theta} \in \mathrm{Irr}(M_1, \xi_1)$ such that

$$\overline{\Theta}(\xi_1 + \lambda_1) = [\pi],$$

where $\lambda_1 \in \Lambda^{G_1}$ with $\bar{\lambda}_1 = -2t\rho(\Delta^+)$. Put $\Xi_i = \mathrm{Ind}_{M_i}^{G_i}(\overline{\Theta})$ and $\lambda_2 = f(\lambda_1)$, so that

$$\Xi_i(\xi_i + \lambda_i) = [\mathrm{Ind}_{P_i}^{G_i}(\pi)].$$

Then, applying Lemma 3.3 to $\Xi_i$ and $\lambda_i$, we can reduce the irreducibility of $\mathrm{Ind}_{P_1}^{G_1}(\pi)$ to that of

$$\mathrm{Ind}_{P_2}^{G_2}(\pi) = \mathrm{Ind}_{P_3}^{\mathrm{SO}_{2(n-m)+1}(\mathbb{R})}(\chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l) \otimes \pi'.$$

Since $\mathrm{Ind}_{P_2}^{G_2}(\pi)$ is a standard module (with a suitable choice of $P_2$), its irreducibility follows from [Speh and Vogan 1980, Theorem 6.19] (see also Section 3J below) and the assumption on $\pi$. This completes the proof.

**3H.** ***Normalized intertwining operators.*** In the rest of this section, we will give another proof of Proposition 3.4 given to us by the referee, using normalized intertwining operators and the irreducibility result of [Speh and Vogan 1980].

We need to introduce more notation. Let $G$ be a connected reductive linear algebraic group over $\mathbb{R}$. We confuse $G$ with the group of $\mathbb{R}$-rational points of $G$. Let $P = MN$ be a parabolic subgroup of $G$ with Levi component $M$ and unipotent radical $N$. We denote by $\overline{P} = M\overline{N}$ the parabolic subgroup of $G$ opposite to $P$. Let $A_M$ be the split component of the center of $M$ and put

$$\mathfrak{a}_M^* = \mathrm{Rat}(A_M) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \mathfrak{a}_M = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(A_M), \mathbb{R}),$$

where $\mathrm{Rat}(A_M)$ is the group of algebraic characters of $A_M$ defined over $\mathbb{R}$. Let $\mathfrak{a}^*_{M,\mathbb{C}} = \mathfrak{a}^*_M \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{a}^*_M$. Put

$$\mathfrak{a}^{*+}_P = \{\lambda \in \mathfrak{a}^*_M \mid \langle \lambda, \check{\alpha} \rangle > 0 \text{ for all } \alpha \in \Sigma(P)\},$$
$$\bar{\mathfrak{a}}^{*+}_P = \{\lambda \in \mathfrak{a}^*_M \mid \langle \lambda, \check{\alpha} \rangle \geq 0 \text{ for all } \alpha \in \Sigma(P)\},$$

where $\langle \cdot, \cdot \rangle : \mathfrak{a}^*_M \times \mathfrak{a}_M \to \mathbb{R}$ is the natural pairing, $\Sigma(P) \subset \mathfrak{a}^*_M$ is the set of reduced roots of $A_M$ in $P$, and $\check{\alpha} \in \mathfrak{a}_M$ is the coroot corresponding to $\alpha$. Noting that $\mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{Rat}(A_M) \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathfrak{a}^*_M$, we may define a homomorphism $H_M : M \to \mathfrak{a}_M$ by

$$\langle \chi, H_M(m) \rangle = \log |\chi(m)|$$

for $\chi \in \mathrm{Rat}(M)$ and $m \in M$. For any continuous character $\omega$ of $A_M$, we define $\mathrm{Re}\,\omega \in \mathfrak{a}^*_M$ by

$$\langle \mathrm{Re}\,\omega, H_M(a) \rangle = \log |\omega(a)|$$

for $a \in A_M$.

Let $M$ be a Levi subgroup of $G$. Let $\pi$ be an irreducible representation of $M$ with central character $\omega_\pi$ on $A_M$. Put $\pi_\lambda(m) = \pi(m) e^{\langle \lambda, H_M(m) \rangle}$ for $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ and $m \in M$. Let $P$ and $P'$ be two parabolic subgroups of $G$ with common Levi component $M$. Then we define an intertwining operator

$$J_{P'|P}(\pi_\lambda) : \mathrm{Ind}^G_P(\pi_\lambda) \to \mathrm{Ind}^G_{P'}(\pi_\lambda)$$

by

$$(J_{P'|P}(\pi_\lambda) f)(g) = \int_{N \cap N' \backslash N'} f(n'g)\, dn'$$

for $f \in \mathrm{Ind}^G_P(\pi_\lambda)$ and $g \in G$, where $N$ and $N'$ are the unipotent radicals of $P$ and $P'$, respectively. Note that this integral converges absolutely if $\mathrm{Re}\,\lambda$ lies in some cone and admits a meromorphic continuation to $\mathfrak{a}^*_{M,\mathbb{C}}$. Moreover, by [Arthur 1989], there exists a meromorphic function $r_{P'|P}(\pi_\lambda)$ on $\mathfrak{a}^*_{M,\mathbb{C}}$ such that the normalized intertwining operator

$$R_{P'|P}(\pi_\lambda) = r_{P'|P}(\pi_\lambda)^{-1} J_{P'|P}(\pi_\lambda)$$

satisfies the following properties:

- If $\pi$ is tempered, then $R_{P'|P}(\pi_\lambda)$ is holomorphic for $\mathrm{Re}\,\lambda \in \bar{\mathfrak{a}}^{*+}_P$.
- If $P$, $P'$, and $P''$ are three parabolic subgroups of $G$ with common Levi component $M$, then
$$R_{P''|P}(\pi_\lambda) = R_{P''|P'}(\pi_\lambda) R_{P'|P}(\pi_\lambda).$$
- Let $L$ be a Levi subgroup of $G$ containing $M$. Let $Q$ and $Q'$ be two parabolic subgroups of $G$ with common Levi component $L$. Let $S$ and $S'$ be two parabolic

subgroups of $L$ with common Levi component $M$. Let $Q(S)$, $Q'(S)$, and $Q(S')$ be the unique parabolic subgroups of $G$ with common Levi component $M$ such that

$$\begin{aligned} Q(S) &\subset Q, & Q(S)\cap L &= S, \\ Q'(S) &\subset Q', & Q'(S)\cap L &= S, \\ Q(S') &\subset Q, & Q(S')\cap L &= S', \end{aligned}$$

respectively. Then we have

$$\begin{aligned} R_{Q'(S)|Q(S)}(\pi_\lambda) &= R_{Q'|Q}(\mathrm{Ind}_S^L(\pi_\lambda)), \\ R_{Q(S')|Q(S)}(\pi_\lambda) &= \mathrm{Ind}_Q^G(R_{S'|S}(\pi_\lambda)). \end{aligned}$$

**3I.** ***Another proof of Proposition 3.4.*** We now return to the setting of Section 3F, so that $G = \mathrm{SO}_{2n+1}(\mathbb{R})$. Recall that $P$ is a parabolic subgroup of $G$ with Levi component

$$M \cong \mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times \mathrm{SO}_{2m+1}(\mathbb{R}),$$

where $k + 2l + m = n$. Recall also that $\pi$ is an irreducible representation of $M$ of the form

$$\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi',$$

where

- $\chi_i = \mathrm{sgn}^{\delta_i} \otimes |\cdot|^{\nu_i}$ for some $\delta_i \in \{0, 1\}$ and some $\nu_i \in \mathbb{C}$ with $|\mathrm{Re}\,\nu_i| < \frac{1}{2}$ such that if $\nu_i = \pm\nu_j$, then $\delta_i = \delta_j$;
- $\tau_i = D_{\kappa_i} \otimes |\det|^{\nu_i'}$ for some $\kappa_i \in \frac{1}{2}\mathbb{Z}$ and some $\nu_i' \in \mathbb{C}$ with $|\mathrm{Re}\,\nu_i'| < \frac{1}{2}$ such that if $\nu_i' = 0$, then $\kappa_i \in \mathbb{Z}$;
- $\pi'$ is an irreducible representation of $\mathrm{SO}_{2m+1}(\mathbb{R})$ with infinitesimal character $\lambda' = (\lambda_1', \ldots, \lambda_m')$ such that $\lambda_i' \in \mathbb{Z} + \frac{1}{2}$ for all $i$.

We will show that $\mathrm{Ind}_P^G(\pi)$ is irreducible.

By the Langlands classification and the condition on $\pi'$, there exist a parabolic subgroup $P'$ of $\mathrm{SO}_{2m+1}(\mathbb{R})$ with Levi component

$$M' \cong \mathrm{GL}_1(\mathbb{R})^p \times \mathrm{GL}_2(\mathbb{R})^q \times \mathrm{SO}_{2r+1}(\mathbb{R}),$$

where $p + 2q + r = m$, and an irreducible representation $\pi_0'$ of $M'$ of the form

$$\pi_0' = \chi_1' \otimes \cdots \otimes \chi_p' \otimes \tau_1' \otimes \cdots \otimes \tau_q' \otimes \pi'',$$

where

- $\chi_i' = \mathrm{sgn}^{\delta_i'} \otimes |\cdot|^{\mu_i}$ for some $\delta_i' \in \{0, 1\}$ and some $\mu_i \in \mathbb{Z} + \frac{1}{2}$ such that if $\mu_i = \pm\mu_j$, then $\delta_i' = \delta_j'$;
- $\tau_i' = D_{\kappa_i'} \otimes |\det|^{\mu_i'}$ for some $\kappa_i' \in \frac{1}{2}\mathbb{Z}$ and some $\mu_i' \in \mathbb{Z} + \kappa_i' + \frac{1}{2}$ with $\mu_i' \neq 0$;
- $\pi''$ is a (limit of) discrete series representation of $\mathrm{SO}_{2r+1}(\mathbb{R})$

such that $\pi'$ is a unique irreducible quotient of $\mathrm{Ind}_{P'}^{\mathrm{SO}_{2m+1}(\mathbb{R})}(\pi_0')$. Then $\pi'$ is the image of $R_{\bar{P}'|P'}(\pi_0')$. Let $S_0$ be a parabolic subgroup of $M$ with Levi component $M_0$ such that

$$S_0 \cong \mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times P',$$
$$M_0 \cong \mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times M'.$$

We define an irreducible representation $\pi_0$ of $M_0$ by

$$\pi_0 = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi_0'.$$

Then $\pi$ is the image of $R_{\bar{S}_0|S_0}(\pi_0)$.

Let $P_1$ and $P_2$ be the unique parabolic subgroups of $G$ with common Levi component $M_0$ such that

$$P_1 \subset P, \quad P_1 \cap M = S_0,$$
$$P_2 \subset P, \quad P_2 \cap M = \bar{S}_0,$$

respectively. Then we have

$$R_{P_2|P_1}(\pi_0) = \mathrm{Ind}_P^G(R_{\bar{S}_0|S_0}(\pi_0)),$$

so that $\mathrm{Ind}_P^G(\pi)$ is the image of $R_{P_2|P_1}(\pi_0)$. On the other hand, if we take a parabolic subgroup $P_0$ of $G$ with Levi component $M_0$ such that

$$P_0 \cap M = S_0,$$
$$\mathrm{Re}\,\omega_{\pi_0} \in \bar{\mathfrak{a}}_{P_0}^{*+},$$

then

$$R_{P_2|P_1}(\pi_0) = R_{P_2|\bar{P}_0}(\pi_0) R_{\bar{P}_0|P_0}(\pi_0) R_{P_0|P_1}(\pi_0).$$

**Lemma 3.10.** *The normalized intertwining operators $R_{P_0|P_1}(\pi_0)$ and $R_{P_2|\bar{P}_0}(\pi_0)$ are isomorphisms.*

*Proof.* We only prove the assertion for $R_{P_0|P_1}(\pi_0)$; the proof for $R_{P_2|\bar{P}_0}(\pi_0)$ is similar. Put $R_0 = P_1$ and write

$$\Sigma(P_1) \cap \Sigma(\bar{P}_0) = \{\alpha_1, \ldots, \alpha_t\}.$$

For $1 \le i \le t$, let $R_i$ be the parabolic subgroup of $G$ with Levi component $M_0$ such that

$$\Sigma(R_{i-1}) \cap \Sigma(\bar{R}_i) = \{\alpha_i\}.$$

Then we have $R_t = P_0$ and hence

$$R_{P_0|P_1}(\pi_0) = R_{R_t|R_{t-1}}(\pi_0) \cdots R_{R_1|R_0}(\pi_0).$$

Thus, it remains to show that $R_{R_i|R_{i-1}}(\pi_0)$ is an isomorphism for all $1 \le i \le t$.

Let $L_i$ be the centralizer of $A_{\alpha_i}$ in $G$, where $A_{\alpha_i}$ is the identity component of the kernel of $\alpha_i$ in $A_{M_0}$. Put $S_i = R_{i-1} \cap L_i$. Then $L_i$ is a Levi subgroup of $G$

containing $M_0$ and $S_i$ is a maximal parabolic subgroup of $L_i$ with Levi component $M_0$. Moreover, we have $\bar{S}_i = R_i \cap L_i$ and hence

$$R_{R_i|R_{i-1}}(\pi_0) = \mathrm{Ind}_{Q_i}^{G}(R_{\bar{S}_i|S_i}(\pi_0)),$$

where $Q_i$ is the parabolic subgroup of $G$ with Levi component $L_i$ such that $R_{i-1} \subset Q_i$. Since $\alpha_i$ is not a root in $M$, it follows from [Speh and Vogan 1980, Theorem 6.19] (see also Section 3J below) and the condition on $\pi_0$ that $\mathrm{Ind}_{S_i}^{L_i}(\pi_0)$ is irreducible. Hence $R_{\bar{S}_i|S_i}(\pi_0)$ is an isomorphism, and so is $R_{R_i|R_{i-1}}(\pi_0)$. □

Hence, to prove the irreducibility of $\mathrm{Ind}_P^G(\pi)$, it suffices to show that the image of $R_{\bar{P}_0|P_0}(\pi_0)$ is irreducible. There exists a unique parabolic subgroup $Q$ of $G$ with Levi component $L$ such that

$$P_0 \subset Q, \quad M_0 \subset L, \quad \mathrm{Re}\,\omega_{\pi_0} \in \mathfrak{a}_Q^{*+}.$$

Put $S = P_0 \cap L$, so that $S$ is a parabolic subgroup of $L$ with Levi component $M_0$. Then we have $P_0 = Q(S)$ and hence

$$R_{\bar{P}_0|P_0}(\pi_0) = R_{\bar{Q}(\bar{S})|Q(S)}(\pi_0) = R_{\bar{Q}(\bar{S})|Q(\bar{S})}(\pi_0) R_{Q(\bar{S})|Q(S)}(\pi_0).$$

Since $R_{\bar{S}|S}(\pi_0)$ is an isomorphism, so is

$$R_{Q(\bar{S})|Q(S)}(\pi_0) = \mathrm{Ind}_Q^G(R_{\bar{S}|S}(\pi_0)).$$

Also, we have

$$R_{\bar{Q}(\bar{S})|Q(\bar{S})}(\pi_0) = R_{\bar{Q}|Q}(\mathrm{Ind}_{\bar{S}}^{L}(\pi_0)).$$

By [Speh and Vogan 1980, Theorem 6.19] (see also Section 3J below) and the condition on $\pi_0$, $\mathrm{Ind}_{\bar{S}}^{L}(\pi_0)$ is irreducible, so that $\mathrm{Ind}_Q^G(\mathrm{Ind}_{\bar{S}}^{L}(\pi_0))$ is a standard module. (We remark that the irreducibility of $\mathrm{Ind}_{\bar{S}}^{L}(\pi_0)$ also follows from a result of Knapp and Zuckerman [1982a; 1982b].) Hence the image of $R_{\bar{Q}(\bar{S})|Q(\bar{S})}(\pi_0)$ is irreducible, and so is of $R_{\bar{P}_0|P_0}(\pi_0)$. This completes the proof.

**3J.** ***Explicit form of the irreducibility results.*** Finally, for the convenience of the reader, we explicate the irreducibility results which are used in the proof of Proposition 3.4. For $\kappa \in \frac{1}{2}\mathbb{Z}$, we have denoted by $D_\kappa$ the relative (limit of) discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of weight $2|\kappa| + 1$ with central character trivial on $\mathbb{R}_+^\times$.

We first recall the following irreducibility criterion due to Speh; see [Mœglin 1997, Theorem 10b]:

- For $\epsilon_1, \epsilon_2 \in \{0, 1\}$ and $s_1, s_2 \in \mathbb{C}$, the representation of $\mathrm{GL}_2(\mathbb{R})$ parabolically induced from

$$\mathrm{sgn}^{\epsilon_1}|\cdot|^{s_1} \otimes \mathrm{sgn}^{\epsilon_2}|\cdot|^{s_2}$$

is irreducible if and only if either

$$\begin{cases} \epsilon_1 = \epsilon_2 \text{ and } s_1 - s_2 \notin 2\mathbb{Z}+1; \text{ or} \\ \epsilon_1 \neq \epsilon_2 \text{ and } s_1 - s_2 \notin 2\mathbb{Z} \smallsetminus \{0\}. \end{cases}$$

- For $\epsilon \in \{0, 1\}$, $\kappa \in \frac{1}{2}\mathbb{Z}$, and $s_1, s_2 \in \mathbb{C}$, the representation of $\mathrm{GL}_3(\mathbb{R})$ parabolically induced from

$$\mathrm{sgn}^{\epsilon} |\cdot|^{s_1} \otimes D_{\kappa} |\det|^{s_2}$$

is irreducible if and only if either

$$\begin{cases} s_1 - s_2 \notin \mathbb{Z}+\kappa; \text{ or} \\ s_1 - s_2 \in \mathbb{Z}+\kappa \text{ and } |s_1 - s_2| \leq |\kappa|. \end{cases}$$

- For $\kappa_1, \kappa_2 \in \frac{1}{2}\mathbb{Z}$ and $s_1, s_2 \in \mathbb{C}$, the representation of $\mathrm{GL}_4(\mathbb{R})$ parabolically induced from

$$D_{\kappa_1} |\det|^{s_1} \otimes D_{\kappa_2} |\det|^{s_2}$$

is irreducible if and only if either

$$\begin{cases} s_1 - s_2 \notin \mathbb{Z}+\kappa_1+\kappa_2; \text{ or} \\ s_1 - s_2 \in \mathbb{Z}+\kappa_1+\kappa_2 \text{ and } |s_1 - s_2| \leq \min(|\kappa_1+\kappa_2|, |\kappa_1-\kappa_2|). \end{cases}$$

We next recall the irreducibility result of [Speh and Vogan 1980] for $G = \mathrm{SO}_{2n+1}(\mathbb{R})$. We retain the notation of Section 3F, so that $P$ is a parabolic subgroup of $G$ with Levi component

$$M \cong \mathrm{GL}_1(\mathbb{R})^k \times \mathrm{GL}_2(\mathbb{R})^l \times \mathrm{SO}_{2m+1}(\mathbb{R}),$$

where $k + 2l + m = n$. Put $\mathfrak{h} = \mathfrak{h}^{k,l,m}$. We define a basis $e_1, \ldots, e_n$ of $\mathfrak{h}^*$ by

$$\begin{aligned} e_i(h) &= a_i && (1 \leq i \leq k), \\ e_{k+2i-1}(h) &= x_i + \sqrt{-1} y_i && (1 \leq i \leq l), \\ e_{k+2i}(h) &= -x_i + \sqrt{-1} y_i && (1 \leq i \leq l), \\ e_{k+2l+i}(h) &= \sqrt{-1} \vartheta_i && (1 \leq i \leq m) \end{aligned}$$

for $h = h^{k,l,m}(a, z, \vartheta)$. Using the above basis, we identify $\mathfrak{h}^*$ with $\mathbb{C}^n$. We denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$:

$$\Delta = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}.$$

We also denote by $\Delta_i$, $\Delta_r$, and $\Delta_{cx}$ the sets of imaginary, real, and complex roots, respectively:

$$\begin{aligned} \Delta_i &= \{\alpha \in \Delta \mid \theta\alpha = \alpha\}, \\ \Delta_r &= \{\alpha \in \Delta \mid \theta\alpha = -\alpha\}, \\ \Delta_{cx} &= \{\alpha \in \Delta \mid \theta\alpha \neq \pm\alpha\}, \end{aligned}$$

where $\theta$ is the Cartan involution of $\mathfrak{g}$. Let $\pi$ be an irreducible representation of $M$ of the form

$$\pi = \chi_1 \otimes \cdots \otimes \chi_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \pi',$$

where

- $\chi_i = \mathrm{sgn}^{\delta_i} \otimes |\cdot|^{\nu_i}$ for some $\delta_i \in \{0, 1\}$ and some $\nu_i \in \mathbb{C}$;
- $\tau_i = D_{\kappa_i} \otimes |\det|^{\nu'_i}$ for some $\kappa_i \in \frac{1}{2}\mathbb{Z}$ and some $\nu'_i \in \mathbb{C}$;
- $\pi'$ is a (limit of) discrete series representation of $\mathrm{SO}_{2m+1}(\mathbb{R})$ with Harish-Chandra parameter
$$\lambda' = (\lambda'_1, \dots, \lambda'_m) \in (\mathfrak{h}^{0,0,m})^* \cong \mathbb{C}^m,$$
where $\lambda'_i \in \mathbb{Z} + \frac{1}{2}$ for all $i$.

Put

$$\gamma = (\nu_1, \dots, \nu_k, \kappa_1 + \nu'_1, \kappa_1 - \nu'_1, \dots, \kappa_l + \nu'_l, \kappa_l - \nu'_l, \lambda'_1, \dots, \lambda'_m) \in \mathfrak{h}^* \cong \mathbb{C}^n.$$

Then, by [Speh and Vogan 1980, Theorem 6.19], $\mathrm{Ind}_P^G(\pi)$ is irreducible if

(i) there exists no complex root $\alpha \in \Delta_{cx}$ satisfying $2\langle \alpha, \gamma \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$, $\langle \alpha, \gamma \rangle > 0$, and $\langle \theta\alpha, \gamma \rangle < 0$; and

(ii) there exists no real root $\alpha \in \Delta_r$ satisfying the parity condition [Vogan 1981, Definition 8.3.11].

We now explicate the conditions (i) and (ii). We start with the following special cases:

- Suppose that $k = 1$, $l = 0$, and $m = n - 1$. In this case, we have

$$\begin{aligned} \Delta_i &= \{\pm e_i \pm e_j \mid 2 \le i < j \le n\} \cup \{\pm e_i \mid 2 \le i \le n\}, \\ \Delta_r &= \{\pm e_1\}, \\ \Delta_{cx} &= \{\pm e_1 \pm e_j \mid 2 \le j \le n\}. \end{aligned}$$

Hence the conditions (i) and (ii) are equivalent to the following conditions, respectively:

$$\text{(i}'\text{)} \begin{cases} \nu_1 \notin \mathbb{Z} + \frac{1}{2}; \text{ or} \\ \nu_1 \in \mathbb{Z} + \frac{1}{2} \text{ and } |\nu_1| \le |\lambda'_i| \text{ for all } 1 \le i \le m; \end{cases}$$

(ii′) $\nu_1 \notin \mathbb{Z} + \frac{1}{2}$.

- Suppose that $k = 0$, $l = 1$, and $m = n - 2$. In this case, we have

$$\begin{aligned} \Delta_i &= \{\pm(e_1 + e_2)\} \cup \{\pm e_i \pm e_j \mid 3 \le i < j \le n\} \cup \{\pm e_i \mid 3 \le i \le n\}, \\ \Delta_r &= \{\pm(e_1 - e_2)\}, \\ \Delta_{cx} &= \{\pm e_i \pm e_j \mid 1 \le i \le 2 < j \le n\} \cup \{\pm e_i \mid 1 \le i \le 2\}. \end{aligned}$$

Hence the conditions (i) and (ii) are equivalent to the following conditions, respectively:

(i′) $\begin{cases} \nu_1' \notin \frac{1}{2}\mathbb{Z}; \text{ or} \\ \nu_1' \in \mathbb{Z}+\kappa_1 \text{ and } |\nu_1'| \le |\kappa_1|; \text{ or} \\ \nu_1' \in \mathbb{Z}+\kappa_1+\frac{1}{2},\ |\nu_1'| \le |\kappa_1|, \text{ and } |\nu_1'| \le \min(|\kappa_1+\lambda_i'|, |\kappa_1-\lambda_i'|) \text{ for all } 1 \le i \le m; \end{cases}$

(ii′) $\nu_1' \notin \mathbb{Z}+\kappa_1+\frac{1}{2}$.

Similarly, in the general case, we can show that the conditions (i) and (ii) hold if and only if

- for $1 \le i < j \le k$, $\delta_i = \delta_j$ and $\nu_i - \nu_j \notin 2\mathbb{Z}+1$, or $\delta_i \ne \delta_j$ and $\nu_i - \nu_j \notin 2\mathbb{Z}$;
- for $1 \le i < j \le k$, $\delta_i = \delta_j$ and $\nu_i + \nu_j \notin 2\mathbb{Z}+1$, or $\delta_i \ne \delta_j$ and $\nu_i + \nu_j \notin 2\mathbb{Z}$;
- for $1 \le i \le k$ and $1 \le j \le l$, $\nu_i - \nu_j' \notin \mathbb{Z}+\kappa_j$, or $\nu_i - \nu_j' \in \mathbb{Z}+\kappa_j$ and $|\nu_i - \nu_j'| \le |\kappa_j|$;
- for $1 \le i \le k$ and $1 \le j \le l$, $\nu_i + \nu_j' \notin \mathbb{Z}+\kappa_j$, or $\nu_i + \nu_j' \in \mathbb{Z}+\kappa_j$ and $|\nu_i + \nu_j'| \le |\kappa_j|$;
- for $1 \le i < j \le l$, $\nu_i' - \nu_j' \notin \mathbb{Z}+\kappa_i+\kappa_j$, or $\nu_i' - \nu_j' \in \mathbb{Z}+\kappa_i+\kappa_j$ and $|\nu_i' - \nu_j'| \le \min(|\kappa_i+\kappa_j|, |\kappa_i-\kappa_j|)$;
- for $1 \le i < j \le l$, $\nu_i' + \nu_j' \notin \mathbb{Z}+\kappa_i+\kappa_j$, or $\nu_i' + \nu_j' \in \mathbb{Z}+\kappa_i+\kappa_j$ and $|\nu_i' + \nu_j'| \le \min(|\kappa_i+\kappa_j|, |\kappa_i-\kappa_j|)$;
- for $1 \le i \le k$, $\nu_i \notin \mathbb{Z}+\frac{1}{2}$;
- for $1 \le i \le l$, $\nu_i' \notin \frac{1}{2}\mathbb{Z}$, or $\nu_i' \in \mathbb{Z}+\kappa_i$ and $|\nu_i'| \le |\kappa_i|$.

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# Truncated operads and simplicial spaces

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It was shown by Boavida de Brito and Weiss (*J. Topol.* **11**:1 (2018), 65–143) that a well-known construction which to a (monochromatic, symmetric) topological operad associates a topological category and a functor from it to the category of finite sets is homotopically fully faithful, under mild conditions on the operads. The main result here is a generalization of that statement to $k$-truncated topological operads. A $k$-truncated operad is a weaker version of operad where all operations have arity $\le k$.

## 1. Introduction

There is a well-known construction which to a (monochromatic, symmetric) topological operad $P$ associates a so-called PROP, a small topological category with a symmetric monoidal product. The set of objects of the PROP is identified with the set of natural numbers and the monoidal product corresponds to addition. See [Voronov 2005] for more details and explanations. Forgetting the monoidal product in the PROP and passing to the comma category of *objects over 1* produces a topological category (category object in the category of spaces) called $\mathcal{C}_P$ in [Boavida de Brito and Weiss 2018, §7]; this paper is henceforth abbreviated [BW]. This comes with a distinguished functor to $\mathsf{Fin}$, essentially the category of all finite sets. It is well known that $P$ can be reconstructed from the associated PROP with the monoidal structure. By contrast, the construction $P \mapsto \mathcal{C}_P$ has a forgetful character, even if we include the reference functor $\mathcal{C}_P \to \mathsf{Fin}$, and there are elementary examples to illustrate that it really does forget essential features [BW, Remark 7.3]. Write $N$ for the nerve construction (from small topological categories to simplicial spaces). The main result of [BW, §7] is that for topological operads $P$ and $Q$, the map

$$\mathbb{R}\mathrm{map}(P, Q) \to \mathbb{R}\mathrm{map}_{N\mathsf{Fin}}(N\mathcal{C}_P, N\mathcal{C}_Q)$$

of derived mapping spaces induced by the construction $P \mapsto N\mathcal{C}_P$ is nevertheless a weak equivalence under a reasonable condition on $P$ and $Q$. The condition is that

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the spaces of 0-ary and 1-ary operations for both $P$ and $Q$ are weakly contractible. (Beware that topological operads with an *empty space* of 0-ary operations do not qualify, although they are popular.) More details and an example are given in Section 2.

This paper here relies on [BW, §7], but in doing so develops a slightly different proof of the same result. The advantage of the new proof is that it carries over without essential change to the setting of *k-truncated* topological operads. (A $k$-truncated operad is a weaker version of operad which has operations of arity $\leq k$ only.) This extension is used in [Weiss 2016, Theorem 2.3.1]. In more detail, that paper is about nontrivial nonvanishing phenomena for the rational Pontryagin classes of fiber bundles with fiber $\mathbb{R}^n$. It relies on a description of certain spaces of smooth embeddings along manifold calculus lines (in order to construct exotic families of homeomorphisms). In that description of spaces of smooth embeddings, following [BW, §5], spaces of the form $\mathbb{R}\mathrm{map}_{N\mathsf{Fin}}(N\mathcal{C}_P, N\mathcal{C}_Q)$ (where $P$ and $Q$ are little disk operads) and truncated versions are essential ingredients. But computational tools are more readily available for $\mathbb{R}\mathrm{map}(P, Q)$ and truncated versions. Specifically, [Weiss 2016] refers to a forthcoming PhD thesis by F. Göppl for some computational tools. This should be available very soon.

## 2. Operads, dendroidal spaces and simplicial spaces

This section is a review of the main definitions and results of [BW, §7].

Let $P$ be an operad in the symmetric monoidal category of spaces. For convenience, the *category of spaces* is understood to be the category of simplicial sets. The following paragraph in quotation marks is quoted verbatim from [BW, §7].

"We think of this in the following terms: $P$ is a functor from the category of finite sets and bijections to spaces, and for every map $f : T \to S$ of finite sets there is an operation

$$\lambda_f : P(S) \times \prod_{i \in S} P(T_i) \to P(T),$$

where $T_i = f^{-1}(i) \subset T$. Also $P(S)$ contains a distinguished unit element when $S$ is a singleton. Sensible naturality, associativity and unital properties are satisfied. Note in particular that any permutation $f : S \to S$ induces a map $P(S) \to P(S)$ in two ways: firstly because $P$ is a functor from the category of finite sets and bijections to spaces, and secondly by

$$P(S) \ni x \mapsto \lambda_f(x, 1, 1, \ldots, 1) \in P(S).$$

These two maps agree as per definition.

What we have described is also called a *plain* operad in the category of spaces ..."

Note that *plain* is synonymous with *monochromatic*. It means that the operad has only one object. (We may think of $P(S)$ as the space of $S$-ary operations from that object to itself.) The fact that we allow arbitrary finite sets without a total ordering means that we are dealing with a *symmetric* operad. In the following we just write *operad* to mean *monochromatic symmetric operad*.

There is a construction $P \mapsto \mathcal{C}_P$ taking an operad $P$ as above to a topological category $\mathcal{C}_P$ (category object in the category of spaces). The category $\mathcal{C}_P$ comes with a forgetful functor to the category of finite sets; more precisely we use a skeleton $\mathsf{Fin}$ of the category of finite sets. The objects of $\mathsf{Fin}$ are the sets $\underline{k} = \{1, 2, \dots, k\}$, where $k \geq 0$. (Note in particular that $\underline{0}$ is the empty set.) The morphisms in $\mathsf{Fin}$ from $\underline{k}$ to $\underline{\ell}$ are the maps from $\underline{k}$ to $\underline{\ell}$.

The space of objects of $\mathcal{C}_P$ is

$$\coprod_{k \geq 0} P(\underline{k}).$$

The space of morphisms in $\mathcal{C}_P$ lifting a morphism $f : \underline{k} \to \underline{\ell}$ in $\mathsf{Fin}$ is

$$P(\underline{\ell}) \times \prod_{i \in \underline{\ell}} P(\underline{k_i})$$

where $k_i$ is the cardinality of $f^{-1}(i)$. Source and target of an element in that space are determined by applying to it $\lambda_f$ and the projection to $P(\ell)$, respectively. Composition and identity morphisms are obvious. See [BW, Remark 7.3] for the relationship between the construction $P \mapsto \mathcal{C}_P$ and a better-known construction [Voronov 2005] which turns an operad into a PROP.

The main result of [BW, §7] states that the construction $P \mapsto \mathcal{C}_P$ is homotopically fully faithful as long as it is only used with operads $P$ for which the spaces $P(\underline{0})$ and $P(\underline{1})$ are weakly contractible. Making this precise requires a few decisions. Clearly $P \mapsto \mathcal{C}_P$ is a functor. But we need to reason about derived spaces of morphisms between two topological operads, or between two small topological categories. Therefore it is convenient to work with a preferred model category of topological operads, and with a preferred model category of small topological categories.

Rezk has introduced a model category setting for the *category of small topological categories*. The underlying category is simply the category of simplicial spaces (and here *space* still means *simplicial set* for us). A simplicial space $X$ is said to be a *Segal space* if it satisfies the following condition $(\sigma)$. Let $u_i : \{0, 1\} \to \{0, 1, 2, \dots, n\}$ be the order-preserving map defined by $u_i(0) = i - 1$ and $u_i(1) = i$.

$(\sigma)$ For each $n \geq 2$ the map $(u_1^*, u_2^*, \dots, u_n^*)$ from $X_n$ to the homotopy limit of

$$X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1 \xrightarrow{d_0} \cdots \qquad \cdots \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1$$

is a weak homotopy equivalence.

**Example.** The nerve $N\mathcal{C}$ of a small category $\mathcal{C}$, which is a simplicial set and as such a simplicial space, is a Segal space. (Note in passing that the set of $r$-simplices of $N\mathcal{C}$ is the set of *contravariant* functors from $I_r$ to $\mathcal{C}$, where $I_r$ is $\{0, 1, \dots, r\}$ viewed as a poset with the usual ordering.) The nerve of a small topological category $\mathcal{C}$ is a Segal space provided that at least one of the maps *source, target* from the space of morphisms of $\mathcal{C}$ to the space of objects of $\mathcal{C}$ is a (Kan) fibration.

For a Segal space $X$ it is sensible to view $X_0$ as the space of objects of something slightly more general than a topological category, and to view $X_1$ as the space of morphisms, and $d_0, d_1 : X_1 \to X_0$ as the operators *source* and *target*, respectively. In the same spirit, let

$$\mathrm{mor}^h_X(a, b) := \mathrm{hofiber}_{(a,b)}[(d_0, d_1) : X_1 \to X_0 \times X_0]$$

for $a, b \in X_0$. There is a composition map $\mathrm{mor}^h_X(b, c) \times \mathrm{mor}^h_X(a, b) \to \mathrm{mor}^h_X(a, c)$, well-defined at least up to homotopy.

A simplicial map $f : X \to Y$ between simplicial spaces which satisfy $(\sigma)$ is a *Dwyer–Kan* equivalence if

- for every $a, b \in X_0$ the map
$$\mathrm{mor}^h_X(a, b) \to \mathrm{mor}^h_Y(f(a), f(b))$$
induced by $f$ is a weak equivalence;
- *essential surjectivity*: for every $c \in Y_0$ there exist $b \in X_0$ and an element of $\mathrm{mor}^h_Y(f(b), c)$ which is weakly invertible.

A simplicial space $X$ is a *complete Segal space* if in addition to $(\sigma)$ it satisfies another property $(\kappa)$, completeness, for which the reader can consult [BW, §2]. The point of this additional condition is the following. Firstly, every Segal space admits a Dwyer–Kan equivalence to a complete Segal space. Secondly, a Dwyer–Kan equivalence between two Segal spaces which are both complete is a degreewise weak equivalence.

Rezk's model category structure on the category $s\mathcal{S}$ of simplicial spaces can be described roughly as follows. Start with the standard model category structure on $\mathcal{S}$, the category of spaces (= simplicial sets). Use it to define a model category structure on $s\mathcal{S}$ where either the weak equivalences and the cofibrations are defined levelwise, or the weak equivalences and the fibrations are defined levelwise (yes, there are two options). Write $s\mathcal{S}_1$ for $s\mathcal{S}$ with this model category structure. There is a unique model category structure on $s\mathcal{S}$ which has the same cofibrations as $s\mathcal{S}_1$ and which for the fibrant objects has the complete Segal spaces which are also fibrant as objects of $s\mathcal{S}_1$. See [BW, §B.2]. This is the Rezk model category structure. Write $s\mathcal{S}_2$ for $s\mathcal{S}$ with that model category structure. A map $f : X \to Y$

of simplicial spaces is a weak equivalence in $s\mathcal{S}_2$ if and only if, for every complete Segal space $Z$ which is fibrant in $s\mathcal{S}_1$, the induced map

$$\mathbb{R}\mathrm{map}_{s\mathcal{S}_1}(Y, Z) \to \mathbb{R}\mathrm{map}_{s\mathcal{S}_1}(X, Z)$$

(of derived mapping spaces formed in $s\mathcal{S}_1$) is a weak equivalence. This characterization of the weak equivalences means that $s\mathcal{S}_2$ can also be constructed from $s\mathcal{S}_1$ by a left localization.

We will also need a model category structure on the over category $s\mathcal{S}/Z$ where $Z$ is an object of $s\mathcal{S}$, for us typically a Segal space but not a complete one. There is a standard way by which a model category structure on some category $\mathcal{C}$ determines a model category structure on each of the over categories $\mathcal{C}/c$, for objects $c$ of $\mathcal{C}$. See [Goerss and Schemmerhorn 2007, Example 1.7]. We apply this with $s\mathcal{S}_2$ and we denote the result by $s\mathcal{S}_2/Z$. By definition the fibrant objects in $s\mathcal{S}_2/Z$ are simply the fibrations with target $Z$ in $s\mathcal{S}_2$. But they also have a characterization as the morphisms $Y \to Z$ in $s\mathcal{S}$ which are fibrations in $s\mathcal{S}_1$ and make $Y$ into a *fiberwise complete Segal space over* $Z$; see [BW, §B] for more details.

Cisinski and Moerdijk [2013] have an analogous framework for topological operads; see also related earlier papers by Cisinski and Moerdijk [2011], and by Moerdijk and I. Weiss [2007]. The starting point for this is a small category Tree (in their notation, $\Omega$) whose objects are certain finite trees. In more detail, an object $T$ of Tree is a finite nonempty set $\epsilon(T)$ with a partial order $\le$ and a distinguished subset $\lambda(T)$ of the set of maximal elements of $\epsilon(T)$ such that the following conditions are satisfied:

- $\epsilon(T)$ has a minimal element (called the *root*).
- For each element $e$ of $\epsilon(T)$, the set $\{y \in \epsilon(T) \mid y \le e\}$ with the restricted ordering is linearly ordered.

The elements of $\epsilon(T)$ are also called *edges* of the tree $T$. The elements of $\lambda(T)$ are called the *leaves* of $T$. (The elements of $\epsilon(T) \setminus \lambda(T)$ are sometimes called *vertices*, but I have found this confusing and I prefer to call them *nonleaf edges*.) An object $T$ of Tree generates a finite (colored!) operad with color set $\epsilon(T)$; for each nonleaf edge $a$ in $T$ there is a generating operation with target $a$ and with multisource equal to the set of edges which are just above $a$ in the ordering. A morphism $S \to T$ in Tree is by definition a morphism of the associated finite operads. See [BW, §7.2] for more details and examples. The category Tree contains a copy of the category $\Delta$ (with objects $[k] = \{0, 1, \ldots, k\}$ for $k \ge 0$ and with monotone maps as morphisms). A *dendroidal space* is a functor from $\mathsf{Tree}^{\mathrm{op}}$ to spaces. Therefore a dendroidal space determines a simplicial space by restriction from $\mathsf{Tree}^{\mathrm{op}}$ to $\Delta^{\mathrm{op}}$.

There is a concept of *Segal dendroidal space* analogous to the concept of Segal space, and a corresponding model category structure on the category $d\mathcal{S}$ of dendroidal spaces. Again, this can be obtained by starting with a standard model category structure on $d\mathcal{S}$, for which we write $d\mathcal{S}_1$, and then applying a localization process. The result is $d\mathcal{S}_2$, which has the same underlying category and the same cofibrations as $d\mathcal{S}_1$, but the fibrant objects in $d\mathcal{S}_2$ are the Segal dendroidal spaces which are also fibrant in $d\mathcal{S}_1$.

A topological operad $Q$ has a dendroidal nerve $N_d Q$, which is a dendroidal space such that $(N_d Q)_T$ is the space of operad maps from the operad associated with the tree $T$ to $Q$. If $Q$ is monochromatic, then $(N_d Q)_T$ is a point for $T = \eta$, the tree with a single edge. (The notation $N_d Q$ for the dendroidal nerve of an operad $Q$ is widely used. It clashes with the notation $N_d \mathcal{C}$ for the set of $d$-simplices in the nerve of a category $\mathcal{C}$. Notation like $(N\mathcal{C})_d$ can be used in such cases.) The dendroidal nerve $N_d Q$ is then a Segal dendroidal space. (It does not make sense to insist on a completeness property as in [Rezk 2001], or to claim such a property, if we want to work with monochromatic operads. I am indebted to the referee for drawing my attention to this point. By analogy, the bar construction alias nerve of a topological monoid $M$ is a simplicial space which qualifies as a Segal space, but it is a *complete* Segal space only if the subspace of weakly invertible elements in $M$ is weakly contractible. See also [BW, Example 7.7]. Similarly, if the monochromatic operad $Q$ satisfies $Q(1) \simeq *$, then $N_d Q$ can call itself a *complete* Segal dendroidal space, but we are not imposing that condition yet.) A key result of the dendroidal theory is that the dendroidal nerve functor induces a weak equivalence

$$\mathbb{R}\mathrm{map}(P, Q) \xrightarrow{\simeq} \mathbb{R}\mathrm{map}(N_d P, N_d Q) \tag{2-1}$$

when $P$ and $Q$ are (monochromatic) operads. In the left-hand side, the derived mapping space $\mathbb{R}\mathrm{map}(P, Q)$ can be interpreted using a standard model structure with levelwise weak equivalences and fibrations. In the right-hand side, use either $d\mathcal{S}_1$ or $d\mathcal{S}_2$; it does not matter which because $N_d Q$ is a Segal dendroidal space.

For a simplicial set $Y$ let $\mathsf{simp}(Y)$ be the category whose objects are pairs $(m, y)$ with $y \in X_m$; a morphism from $(m, y)$ to $(n, z)$ is a morphism $f : [m] \to [n]$ in $\Delta$ such that $f^*(z) = y$. There is a functor

$$\varphi : \mathsf{simp}(N\mathsf{Fin}) \to \mathsf{Tree}$$

defined as follows. To an object $(p, S_*)$ of $\mathsf{simp}(N\mathsf{Fin})$, where

$$S_* = (S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow \cdots \leftarrow S_p),$$

associate the tree $T$ where $\epsilon(T)$ is the disjoint union of the $S_i$ and an additional element $r$, with $\lambda(T)$ corresponding to $S_p$. The partial order on $\epsilon(T)$ is the obvious one: $r$ is the minimal element and $x \in S_i$ is $\leq y \in S_j$ if $i \leq j$ and the composite map from $S_j$ to $S_i$ in the string $S_*$ takes $y$ to $x$. The functor $\varphi$ establishes a close relationship between dendroidal spaces and simplicial spaces over $N\mathsf{Fin}$. Indeed, a simplicial space over $N\mathsf{Fin}$ is the same thing as a contravariant functor from $\mathsf{simp}(N\mathsf{Fin})$ to spaces. Therefore precomposition with $\varphi$ is a functor $\varphi^*$ which takes us from dendroidal spaces to simplicial spaces over $N\mathsf{Fin}$. *Important example*: for a topological operad there is an isomorphism of $\varphi^*(N_d P)$ with $N\mathcal{C}_P$, both viewed as simplicial spaces over $N\mathsf{Fin}$.

The main result of [BW, §7] is that for topological operads $P$ and $Q$ the functor $\varphi^*$ induces a weak equivalence

$$\begin{array}{rl}\mathbb{R}\mathrm{map}(N_d P, N_d Q) \longrightarrow & \mathbb{R}\mathrm{map}_{N\mathsf{Fin}}(\varphi^* N_d P, \varphi^* N_d Q) \\ & \| \\ & \mathbb{R}\mathrm{map}_{N\mathsf{Fin}}(N\mathcal{C}_P, N\mathcal{C}_Q)\end{array} \tag{2-2}$$

provided $P(\underline{0})$, $P(\underline{1})$, $Q(\underline{0})$ and $Q(\underline{1})$ are weakly contractible. In the right-hand side we can use the model category structure $s\mathcal{S}_1/N\mathsf{Fin}$ or $s\mathcal{S}_2/N\mathsf{Fin}$; it does not matter because $N\mathcal{C}_Q$ is a fiberwise complete Segal space over $N\mathsf{Fin}$. In view of (2-1) this implies that the nerve functor determines a weak equivalence

$$\mathbb{R}\mathrm{map}(P, Q) \xrightarrow{\simeq} \mathbb{R}\mathrm{map}_{N\mathsf{Fin}}(N\mathcal{C}_P, N\mathcal{C}_Q) \tag{2-3}$$

under the same conditions on $P$ and $Q$.

**Example 2.1.** Suppose that $P$ is the operad of little $m$-disks and $Q$ is the operad of little $n$-disks. Then $N\mathcal{C}_P$ and $N\mathcal{C}_Q$ are weakly equivalent, over $N\mathsf{Fin}$, to certain simplicial spaces $\mathsf{con}(\mathbb{R}^m)$ and $\mathsf{con}(\mathbb{R}^n)$ defined as nerves of certain (topological) categories of configurations in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. See [BW, §3 and Example 7.2]. With Andrade's particle models for the configuration categories, $M \mapsto \mathsf{con}(M)$ is a continuous functor on the category of topological manifolds and injective continuous maps. It follows that there are compatible actions of the homeomorphism group of $\mathbb{R}^m$ (from the right) and the homeomorphism group of $\mathbb{R}^n$ (from the left) on

$$\mathbb{R}\mathrm{map}_{N\mathsf{Fin}}(\mathsf{con}(\mathbb{R}^m), \mathsf{con}(\mathbb{R}^n)) \simeq \mathbb{R}\mathrm{map}_{N\mathsf{Fin}}(N\mathcal{C}_P, N\mathcal{C}_Q).$$

Similarly, there is an interesting map from the space of injective continuous maps $\mathbb{R}^m \to \mathbb{R}^n$ to the space $\mathbb{R}\mathrm{map}_{N\mathsf{Fin}}(\mathsf{con}(\mathbb{R}^m), \mathsf{con}(\mathbb{R}^n))$. All these good features are not easy to see in the operadic description $\mathbb{R}\mathrm{map}(P, Q)$.

## 3. Truncated operads and truncated dendroidal spaces

For an integer $k \geq 1$, a $k$-truncated operad in the symmetric monoidal category of spaces is defined like an operad in spaces except for the following changes: all operations have arity $\leq k$ and composition of operations is only defined where it does not contradict this restriction. More specifically, we can describe a $k$-truncated (monochromatic, symmetric) operad in spaces in the following terms. It is a functor $P$ from the category of finite sets of cardinality $\leq k$ and bijections to spaces, and for every map $f : T \to S$ of finite sets of cardinality $\le k$ there is an operation

$$\lambda_f : P(S) \times \prod_{i \in S} P(T_i) \to P(T)$$

where $T_i = f^{-1}(i) \subset T$. Also $P(S)$ contains a distinguished unit element when $S$ is a singleton. Sensible naturality, associativity and unital properties are satisfied. In particular any permutation $f : S \to S$, where $|S| \le k$, induces a map $P(S) \to P(S)$ in two ways: firstly because $P$ is a functor, and secondly by

$$P(S) \ni x \mapsto \lambda_f(x, 1, 1, \ldots, 1) \in P(S).$$

These two maps agree by definition.

For an integer $k \ge 1$ let $\mathsf{Tree}_k \subset \mathsf{Tree}$ be the full subcategory whose objects are the trees $T$ such that for every $t \in \epsilon(T) \setminus \lambda(T)$ the set $\{s \in \epsilon(T) \mid s > t\}$ has at most $k$ minimal elements. (These minimal elements are often called the *incoming edges* of a fictional vertex associated with the nonleaf edge $t$.) Similarly let $\mathsf{Fin}_{\le k}$ be the full subcategory spanned by the objects $\underline{m}$ where $m \le k$. The functor $\varphi$ above restricts to a functor

$$\varphi_k : \mathsf{simp}(N\mathsf{Fin}_{\le k}) \to \mathsf{Tree}_k.$$

An object $T$ of $\mathsf{Tree}_k$ generates a finite $k$-truncated (colored) operad with color set $\epsilon(T)$; for each nonleaf edge $a$ in $T$ there is a generating operation with target $a$ and with multisource equal to the set of edges which are just above $a$ in the ordering. A $k$-truncated operad $P$ has a nerve $N_d P$ which we regard as a contravariant functor from $\mathsf{Tree}_k$ to spaces. It is defined in such a way that $(N_d P)_T$ is the space of $k$-truncated operad morphisms from the $k$-truncated operad associated with $T$ to $P$. The $k$-truncated operad $P$ also determines a topological category $\mathcal{C}_P$ (category in spaces) with a forgetful functor to $\mathsf{Fin}_{\le k}$. The space of objects of $\mathcal{C}_P$ is

$$\coprod_{m=0}^{k} P(\underline{m}).$$

For a morphism $f : \underline{\ell} \to \underline{m}$ in $\mathsf{Fin}_{\le k}$ the space of morphisms in $\mathcal{C}_P$ lifting $f$ is

$$P(\underline{\ell}) \times \prod_{j \in \underline{m}} P(\underline{\ell}_j)$$

where $\ell_j$ is the cardinality of $f^{-1}(j)$.

Keeping in mind that a simplicial space over $N\mathsf{Fin}_{\le k}$ is the same thing as a contravariant functor from $\mathsf{simp}(N\mathsf{Fin}_{\le k})$, we can say that composition with $\varphi_k$ is a functor $\varphi_k^*$ from $k$-truncated dendroidal spaces to simplicial spaces over $N\mathsf{Fin}_{\le k}$. In that sense there is an isomorphism

$$\varphi_k^*(N_d P) \cong N\mathcal{C}_P$$

of simplicial spaces over $N\mathsf{Fin}_{\le k}$, assuming that $P$ is a $k$-truncated operad in spaces.

**Theorem 3.1.** *Let $P$ and $Q$ be $k$-truncated operads (in spaces, $k \geq 1$) for which the spaces $P(\underline{0})$, $P(\underline{1})$, $Q(\underline{0})$ and $Q(\underline{1})$ are weakly contractible. Then composition with the functor $\varphi_k^*$ is a weak equivalence*:

$$\begin{array}{rcl} \mathbb{R}\mathrm{map}(N_d P, N_d Q) & \xrightarrow{\simeq} & \mathbb{R}\mathrm{map}_{N\mathsf{Fin}_{\leq k}}(\varphi_k^* N_d P, \varphi^* N_d Q) \\ & & \| \\ & & \mathbb{R}\mathrm{map}_{N\mathsf{Fin}_{\leq k}}(N\mathcal{C}_P, N\mathcal{C}_Q) \end{array}$$

This will be proved in the following sections. For clarification, we make sense of $\mathbb{R}\mathrm{map}(N_d P, N_d Q)$ using a model category structure with levelwise weak equivalences on the category of contravariant functors from $\mathsf{Tree}_k$ to spaces. We make sense of $\mathbb{R}\mathrm{map}_{N\mathsf{Fin}_{\leq k}}(N\mathcal{C}_P, N\mathcal{C}_Q)$ using a model category structure on the category of simplicial spaces over $N\mathsf{Fin}_{\leq k}$ with levelwise weak equivalences. (See [Dwyer and Kan 1980], where it is shown that the derived mapping spaces in a model category depend mainly on the subcategory of weak equivalences, and not much on the subcategories of cofibrations and fibrations, respectively.) The proof of Theorem 3.1 given here works equally well in the untruncated setting, $k = \infty$. It can be seen as another way to show that (2-2) is a weak equivalence which happens to generalize easily to the truncated situation.

## 4. Leaves are unnecessary

In [BW, §7] the weak equivalence (2-2) is established in essentially two steps. In the first step, which is the easier one, the category $\mathsf{Tree}$ gets replaced by a much more accessible subcategory $\mathsf{Tree}^{\mathsf{rc}}$. It is the subcategory of $\mathsf{Tree}$ obtained by allowing as objects only the trees $T$ with empty leaf set $\lambda(T)$ and as morphisms only the morphisms in $\mathsf{Tree}$ between trees with no leaves which take root to root.

An object $T$ of $\mathsf{Tree}^{\mathsf{rc}}$ can therefore be described as a finite set $T$ with an order relation (written $\leq$) such that

- $T$ has a (unique) minimal element, called the *root*;
- for every $s \in T$ the set $\{t \in T \mid t \leq s\}$ is linearly ordered (with the ordering induced from $T$).

(We need not distinguish anymore between $T$ and the edge set of $T$.) A morphism $S \to T$ in $\mathsf{Tree}^{\mathsf{rc}}$ is a map $f$ from $S$ to $T$ which respects the order relations, takes root to root and has the additional property that it preserves *independence*. That is, if for $u, v \in S$ we have neither $u \leq v$ nor $v \leq u$, then for $f(u), f(v) \in T$ we have neither $f(u) \leq f(v)$ nor $f(v) \leq f(u)$.

There is a functor $\psi : \mathsf{simp}(N\mathsf{Fin}) \to \mathsf{Tree}^{\mathsf{rc}}$ very similar to $\varphi : \mathsf{simp}(N\mathsf{Fin}) \to \mathsf{Tree}$. For an object $(p, S_*)$ of $\mathsf{simp}(N\mathsf{Fin})$, where

$$S_* = (S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow \cdots \leftarrow S_p),$$

let $\psi(p, S_*)$ be tree $T$ without leaves which, as a set, is the disjoint union of the $S_i$ and an additional element $r$. The partial order on $T$ is the obvious one: $r$ is the minimal element and $x \in S_i$ is $\le y \in S_j$ if $i \le j$ and the composite map from $S_j$ to $S_i$ in the string $S_*$ takes $y$ to $x$.

The formal relationship between $\varphi$ and $\psi$ is a little more complicated than one might expect. The inclusion $\iota : \mathsf{Tree}^{\mathsf{rc}} \to \mathsf{Tree}$ has a left adjoint $\kappa : \mathsf{Tree} \to \mathsf{Tree}^{\mathsf{rc}}$. Equations such as $\varphi = \iota\psi$ or $\psi = \kappa\varphi$ come to mind, but both are false. Instead we have $\varphi\beta = \iota\psi$, where $\beta : \mathsf{simp}(N\mathsf{Fin}) \to \mathsf{simp}(N\mathsf{Fin})$ is the endofunctor which takes the string

$$(S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow \cdots \leftarrow S_p)$$

to the string

$$(S_0 \leftarrow S_1 \leftarrow S_2 \leftarrow \cdots \leftarrow S_p \leftarrow \varnothing).$$

Now let $N_d^{rc}P$ and $N_d^{rc}Q$ be the restrictions of $N_dP$ and $N_dQ$, respectively, to $\mathsf{Tree}^{\mathsf{rc}}$. On the basis of the observations just above, showing that (2-2) is a weak equivalence reduces easily to showing that the map

$$\mathbb{R}\mathrm{map}(N_d^{rc}P, N_d^{rc}Q) \to \mathbb{R}\mathrm{map}(\psi^* N_d^{rc}P, \psi^* N_d^{rc}Q) \tag{4-1}$$

obtained by composition with $\psi^*$ is a weak equivalence. We are still assuming that $P(\underline{0})$, $P(\underline{1})$, $Q(\underline{0})$ and $Q(\underline{1})$ are weakly contractible. (There is a commutative diagram

$$\begin{array}{ccc} \mathbb{R}\mathrm{map}(\varphi^* N_dP, \varphi^* N_dQ) & \xleftarrow{(2\text{-}2)} & \mathbb{R}\mathrm{map}(N_dP, N_dQ) \\ \downarrow{\scriptstyle \beta^*} & & \downarrow{\scriptstyle \iota^*} \\ \mathbb{R}\mathrm{map}(\psi^* N_d^{rc}P, \psi^* N_d^{rc}Q) & \xleftarrow{(4\text{-}1)} & \mathbb{R}\mathrm{map}(N_d^{rc}P, N_d^{rc}Q) \end{array}$$

where the vertical arrows are weak equivalences; use $\varphi\beta = \iota\psi$ and $N_d^{rc} = \iota^* N_d$.)

The message of this short section is that the same argument applies in the truncated situation. There is a functor

$$\psi_k : \mathsf{simp}(N\mathsf{Fin}_{\le k}) \to \mathsf{Tree}_k \cap \mathsf{Tree}^{\mathsf{rc}}$$

obtained by restriction of $\psi$. Suppose that $P$ and $Q$ are $k$-truncated operads for which $P(\underline{0})$, $P(\underline{1})$, $Q(\underline{0})$ and $Q(\underline{1})$ are weakly contractible. Let $N_d^{rc}P$ and $N_d^{rc}Q$ be the contravariant functors from $\mathsf{Tree}_k \cap \mathsf{Tree}^{\mathsf{rc}}$ to spaces obtained by restricting $N_dP$ and $N_dQ$, respectively. Then for the proof of Theorem 3.1 it suffices to show that the map

$$\mathbb{R}\mathrm{map}(N_d^{rc}P, N_d^{rc}Q) \to \mathbb{R}\mathrm{map}(\psi_k^* N_d^{rc}P, \psi_k^* N_d^{rc}Q) \tag{4-2}$$

obtained by composing with $\psi_k^*$ is a weak equivalence.

## 5. Bridging the gap

Let $\mathbb{R}\psi_*$ be the homotopy right Kan extension along $\psi$. This is applicable to contravariant functors from $\mathsf{simp}(N\mathsf{Fin})$ to spaces and yields contravariant functors from $\mathsf{Tree}^{\mathsf{rc}}$ to spaces. It serves as a homotopy right adjoint to the functor $\psi^*$ given by precomposition with $\psi$. It is shown in [BW, §7] that, under conditions on $Q$ as in (4-1), the homotopy unit

$$N_d^{rc}Q \to \mathbb{R}\psi_*\psi^*(N_d^{rc}Q) \tag{5-1}$$

is a weak equivalence. This implies in a formal manner that the map (4-1) is a weak equivalence. See [BW, Lemma A.1].

This type of argument is also available in the truncated setting, but showing that the truncated analogue of (5-1) is a weak equivalence is harder than showing that (5-1) is a weak equivalence. Therefore we proceed in two steps by writing the functor $\psi$ and its truncated variant $\psi_k$ as a composition of two functors. In doing so we deviate from the line of reasoning developed in [BW, §7]. It amounts to additional work, but there is the surprising reward that we can avoid the use of a difficult lemma [BW, Lemma 7.14].

**Definition 5.1.** There is a category $\mathsf{Levtree}$ which is halfway between $\mathsf{simp}(N\mathsf{Fin})$ and $\mathsf{Tree}^{\mathsf{rc}}$. An object of $\mathsf{Levtree}$ is an object of $\mathsf{simp}(N\mathsf{Fin})$. A morphism in $\mathsf{Levtree}$ from $S_* = (S_0 \leftarrow S_1 \leftarrow \cdots \leftarrow S_k)$ to $R_* = (R_0 \leftarrow R_1 \leftarrow \cdots \leftarrow R_\ell)$ consists of a monotone map $u : [k] \to [\ell]$ and monotone injections $v_j : S_j \to R_{u(j)}$, one for every $j \in [k]$, such that the diagram

$$\begin{array}{ccc} S_{j-1} & \longleftarrow & S_j \\ \big\downarrow{\scriptstyle v_{j-1}} & & \big\downarrow{\scriptstyle v_j} \\ R_{u(j-1)} & \longleftarrow \cdots \longleftarrow & R_{u(j)} \end{array}$$

commutes for $j \in \{1, \ldots, k\}$. If the $v_j$ are bijective, then they are necessarily identity maps and the collection $(u, (v_j))$ is a morphism in $\mathsf{simp}(N\mathsf{Fin})$ from $S_*$ to $R_*$. Therefore $\mathsf{simp}(N\mathsf{Fin}) \subset \mathsf{Levtree}$.

Let us note that in the category $\mathsf{simp}(N\mathsf{Fin})$ or, for that matter, in any category of the form $\mathsf{simp}(Y)$ where $Y$ is a simplicial set, no object admits nontrivial automorphisms. The same can be said of $\mathsf{Levtree}$: no object admits nontrivial automorphisms.

Write $\alpha : \mathsf{simp}(N\mathsf{Fin}) \to \mathsf{Levtree}$ for the inclusion. The functor $\psi$ has an obvious extension to a functor $\xi : \mathsf{Levtree} \to \mathsf{Tree}^{\mathsf{rc}}$, so that there is a commutative triangle:

$$\begin{array}{ccc} \mathsf{simp}(N\mathsf{Fin}) & \xrightarrow{\quad\psi\quad} & \mathsf{Tree}^{\mathsf{rc}} \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\xi} & \\ & \mathsf{Levtree} & \end{array} \tag{5-2}$$

**Lemma 5.2.** *Let $Z = \xi^*(N_d^{rc}Q)$. The homotopy unit $Z \to \mathbb{R}\alpha_*\alpha^*Z$ is a weak equivalence.*

*Proof.* Let $R_* = (R_0 \leftarrow R_1 \leftarrow \cdots \leftarrow R_\ell)$ be an object of $\mathsf{Levtree}$. For $(\mathbb{R}\alpha_*\alpha^*Z)(R_*)$ we have the standard formula

$$\operatorname*{holim}_{S_* \to R_*} Z(S_*),$$

where the homotopy limit is taken over a certain comma category alias over-category $\mathcal{U}$ which depends on $R_*$. The objects of $\mathcal{U}$ are pairs consisting of an object $S_*$ of $\mathsf{simp}(N\mathsf{Fin})$ and a morphism $f : S_* \to R_*$ in $\mathsf{Levtree}$. A morphism in $\mathcal{U}$ from $(S_*, f)$ to $(S'_*, g)$ is a morphism $S_* \to S'_*$ in $\mathsf{simp}(N\mathsf{Fin})$ which is over $R_*$ when viewed as a morphism in $\mathsf{Levtree}$.

We introduce full subcategories

$$\mathcal{U}_{-1} \supset \mathcal{U}_0 \supset \mathcal{U}_1 \supset \mathcal{U}_2 \supset \cdots \supset \mathcal{U}_{\ell-1} \supset \mathcal{U}_\ell$$

of $\mathcal{U}$, where $\mathcal{U}_{-1} = \mathcal{U}$ and $\mathcal{U}_\ell$ is the comma category determined by the identity functor $\mathsf{simp}(N\mathsf{Fin}) \to \mathsf{simp}(N\mathsf{Fin})$ and the object $R_*$. (The integer $\ell$ is determined by $R_*$.) The details are as follows. An object of $\mathcal{U}$ given by $S_* \to R_*$, or more precisely, by the commutative diagram

$$\begin{array}{ccccccc}
\cdots \longleftarrow & S_{j-1} & \longleftarrow & S_j & \longleftarrow \cdots \\
& \Big\downarrow {\scriptstyle v_{j-1}} & & \Big\downarrow {\scriptstyle v_j} & \\
\longleftarrow \cdots \leftarrow & R_{u(j-1)} & \leftarrow \cdots \leftarrow & R_{u(j)} & \leftarrow \cdots \longleftarrow
\end{array} \tag{5-3}$$

belongs to $\mathcal{U}_r$ if and only if $v_j$ is bijective (hence an identity map) for all $j$ such that $u(j) \le r$. In particular that object $S_* \to R_*$ belongs to $\mathcal{U}_\ell$ if and only if $S_* \to R_*$ is a morphism in $\mathsf{simp}(N\mathsf{Fin})$. We use the abbreviation $\bar{Z}(S_* \to R_*) := Z(S_*)$ for an object $S_* \to R_*$ in $\mathcal{U}$. Then $\bar{Z}$ is a contravariant functor from $\mathcal{U}$ to spaces and

$$(\mathbb{R}\alpha_*\alpha^*Z)(R_*) = \operatorname{holim} \bar{Z} = \operatorname{holim} \bar{Z}|\mathcal{U}_{-1}.$$

There is a string of forgetful projections

$$\operatorname{holim} \bar{Z}|\mathcal{U}_{-1} \to \operatorname{holim} \bar{Z}|\mathcal{U}_0 \to \cdots \to \operatorname{holim} \bar{Z}|\mathcal{U}_\ell.$$

We have our unit map from $Z(R_*)$ to $(\mathbb{R}\alpha_*\alpha^*Z)(R_*) = \operatorname{holim} \bar{Z}|\mathcal{U}_{-1}$ such that the composition

$$Z(R_*) \to \operatorname{holim} \bar{Z}|\mathcal{U}_{-1} \to \operatorname{holim} \bar{Z}|\mathcal{U}_0 \to \cdots \to \operatorname{holim} \bar{Z}|\mathcal{U}_\ell$$

is a weak equivalence for rather trivial reasons. Therefore it suffices to show that each of the projection maps $\operatorname{holim} \bar{Z}|\mathcal{U}_r \to \operatorname{holim} \bar{Z}|\mathcal{U}_{r+1}$ admits a homotopy left inverse, making holim $\bar{Z}|\mathscr{U}_r$ a homotopy retract of holim $\bar{Z}|\mathscr{U}_{r+1}$. To achieve that, we shall construct two functors

$$V : \mathscr{U}_r \to \mathscr{U}_r, \quad W : \mathscr{U}_r \to \mathscr{U}_{r+1}$$

and natural transformations id $\Rightarrow V \Leftarrow W$, where the functor $V$ takes $\mathscr{U}_{r+1}$ to itself. The crucial property is that, for every object $S_* \to R_*$ in $\mathscr{U}_r$, the natural morphism $W(S_* \to R_*) \to V(S_* \to R_*)$ is taken to a weak equivalence by the functor $\bar{Z}$. Then we shall have the maps

$$\begin{array}{ccc} \operatorname{holim} \bar{Z}|\mathscr{U}_{r+1} \longrightarrow \operatorname{holim}(\bar{Z}|\mathscr{U}_{r+1}) \circ W & & \\ \simeq\Big\uparrow & & \\ \operatorname{holim}(\bar{Z}|\mathscr{U}_r) \circ V & \longrightarrow & \operatorname{holim} \bar{Z}|\mathscr{U}_r \end{array}$$

(the first by prolongation, the other two using the natural transformations), which give us the required homotopy class of maps from holim $\bar{Z}|\mathscr{U}_{r+1}$ to holim $\bar{Z}|\mathscr{U}_r$.

For the description of $V$, imagine an object $S_* \to R_*$ of $\mathscr{U}_r$ given by a diagram like (5-3), where $j$ runs through $\{0, 1, \ldots, k\}$. Determine the unique $t \in \{0, 1, \ldots, k, k+1\}$ such that $u(j) \le r$ whenever $j < t$ and $u(j) > r$ whenever $j \ge t$. Let $V(S_* \to R_*) = (S'_* \to R_*)$, where

$$S'_* = (S_0 \leftarrow S_1 \leftarrow \cdots \leftarrow S_{t-1} \leftarrow R_{r+1} \leftarrow S_t \leftarrow S_{t+1} \leftarrow \cdots \leftarrow S_k),$$

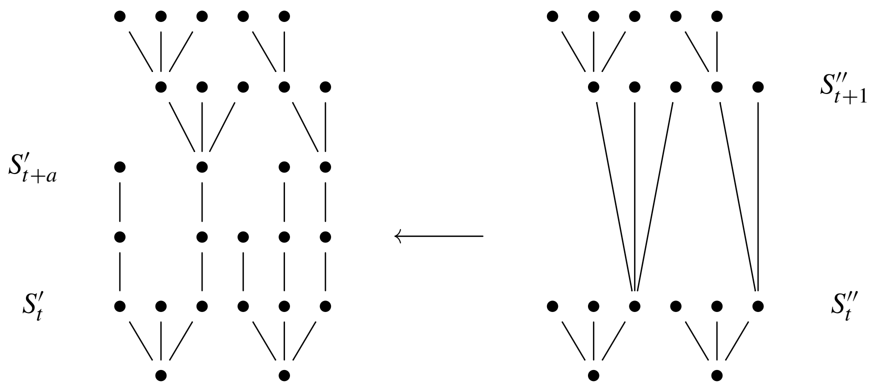
that is, $S'_j = S_j$ for $j < t$, $S'_t = R_{r+1}$ and $S'_j = S_{j-1}$ for $t < j \le k+1$. The arrow from $S'_{t+1} = S_t$ to $S'_t = R_{r+1}$ is $v_t$. The morphism (in Levtree) from $S'_*$ to $R_*$ is defined in such a way that there is a commutative triangle

$$\begin{array}{ccc} S_* & \longrightarrow & S'_* \\ & \searrow \quad \swarrow & \\ & R_* & \end{array}$$

in Levtree, where the horizontal arrow (a morphism in simp($N$Fin)) is defined by the monotone injection $[k] \to [k+1]$ which omits $t$. Of course, $S'_t$ should be mapped to $R_{r+1} = R_{u(t)}$ by the identity. This commutative triangle contributes to a sketchy description not only of $V$, but also of our preferred natural transformation from id to $V$. Now the functor $W$ is defined, on an object $S_* \to R_*$ of $\mathscr{U}$ as before, by starting from $V(S_* \to R_*) = (S'_* \to R_*)$ as described and erasing from $S'_*$ all the terms $S'_{j+1} = S_j$ where $j \ge t$ and $v(j) = r+1$. Call the result $S''_* \to R_*$. Again there is a commutative triangle

$$\begin{array}{ccc} S'_* & \longleftarrow & S''_* \\ & \searrow \quad \swarrow & \\ & R_* & \end{array}$$

where the horizontal arrow (a morphism in $\mathsf{simp}(N\mathsf{Fin})$) is defined by a monotone injection $[k+1-a] \to [k+1]$ which omits an interval of the form $\{t+1, \ldots, t+a\}$. This contributes to a sketchy description not only of $W$, but also of our preferred natural transformation from $W$ to $V$. The morphism in $\mathscr{U}_r$ defined by this triangle is taken to a homotopy equivalence by the functor $\bar{Z}$. To illustrate that, here is a picture describing $S'_*$ and $S''_*$ and the preferred morphism between them:




We are saying that the corresponding operator in $N_d^{rc}Q$ is a weak equivalence. This is based on the assumption that $Q(\underline{0})$ and $Q(\underline{1})$ are weakly contractible. □

**Lemma 5.3.** *Let $Z = N_d^{rc}Q$. The homotopy unit $Z \to \mathbb{R}\xi_*\xi^*Z$ is a weak equivalence.*

*Proof.* Let $T$ be an object of $\mathsf{Tree}^{\mathsf{rc}}$. The standard formula for $(\mathbb{R}\xi_*\xi^*Z)(T)$ is

$$\operatorname*{holim}_{\xi(S_*)\to T} Z(\xi(S_*)).$$

Here the homotopy inverse limit is taken over a comma category $\mathscr{V}(T)$. An object of $\mathscr{V}(T)$ is an object $S_*$ of $\mathsf{Levtree}$ together with a morphism $f : \xi(S_*) \to T$ in $\mathsf{Tree}^{\mathsf{rc}}$. A morphism in $\mathscr{V}(T)$ from $(S_*, f)$ to $(R_*, g)$ is a morphism $S_* \to R_*$ in $\mathsf{Levtree}$ which turns into a morphism over $T$ on applying $\xi$. Let $F_T$ from $\mathscr{V}(T)$ to spaces be the functor which takes $(S_*, f)$ to $Z(\xi(S_*))$. Then we can write

$$(\mathbb{R}\xi_*\xi^*Z)(T) = \operatorname{holim} F_T.$$

We proceed by induction on the number of nodes of $T$, where *node* means a vertex with more than one incoming edge. The induction beginning includes the case where $T$ has zero nodes. Then $T$ is linearly ordered. It is easy to see that $Z(T)$ is weakly contractible, since we are assuming weak contractibility of $Q(\underline{0})$ and $Q(\underline{1})$. Also, for each object $(S_*, f)$ in $\mathscr{V}(T)$, the space $Z(\xi(S_*)) = F_T(S_*, f)$ is weakly contractible since $\xi(S_*)$ is linearly ordered. Therefore, if $T$ has zero nodes, the unit map from $Z(T)$ to $(\mathbb{R}\xi_*\xi^*Z)(T)$ is a weak equivalence.

The induction beginning also includes the case where $T$ has exactly one node. This case is surprisingly hard. Let $\mathscr{V}_0(T)$ be the full subcategory of $\mathscr{V}(T)$ obtained by deleting all objects $(S_*, f)$ of $\mathscr{V}(T)$ where $\xi(S_*)$ has zero nodes, or equivalently, the sets $S_i$ all have cardinality $\leq 1$. It is easy to see that the restriction map

$$\operatorname{holim} F_T \to \operatorname{holim}(F_T|\mathscr{V}_0(T))$$

is a weak equivalence, since the value of $F_T$ on any object of $\mathscr{V}(T)$ not in $\mathscr{V}_0(T)$ is weakly contractible and since any morphism in $\mathscr{V}(T)$ with source in $\mathscr{V}_0(T)$ has target in $\mathscr{V}_0(T)$. Next, let $\mathscr{V}_1(T)$ be the full subcategory of $\mathscr{V}_0(T)$ consisting of those objects $(S_*, f)$ where the set $S_0$ has cardinality $> 1$. (We write

$$S_* = (S_0 \leftarrow S_1 \leftarrow \cdots \leftarrow S_{k-1} \leftarrow S_k)$$

as usual.) The inclusion functor $\mathscr{V}_1(T) \to \mathscr{V}_0(T)$ has a right adjoint. (If $(S_*, f)$ in $\mathscr{V}_0(T)$ has $|S_0| = |S_1| = \cdots = |S_j| = 1$ and $|S_{j+1}| > 1$, then the value of that right adjoint on $(S_*, f)$ is obtained by deleting the terms $S_0, S_1, S_2, \ldots, S_j$.) Moreover the counit morphisms of the adjunction are taken to a weak equivalence by $F_T$. It follows that the restriction map

$$\operatorname{holim}(F_T|\mathscr{V}_0(T)) \to \operatorname{holim}(F_T|\mathscr{V}_1(T))$$

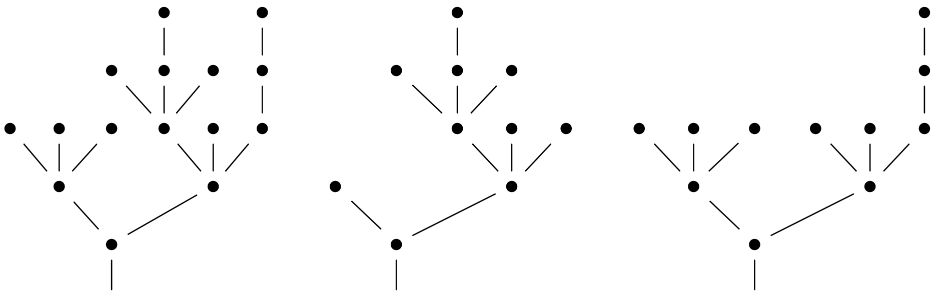
is a weak equivalence. Next, let $U \subset T$ be the set of incoming edges to the unique node of $T$. It has at least two elements by our assumption on $T$. Choose a total ordering on $U$. Let $\mathscr{V}_2(T) \subset \mathscr{V}_1(T)$ be the subcategory of $\mathscr{V}_1(T)$ defined as follows. An object $(S_*, f)$ of $\mathscr{V}_1(T)$ qualifies as an object of $\mathscr{V}_2(T)$ if $f$ takes the ordered set $S_0$ to $U$ by an order-preserving bijection. A morphism $(S_*, f) \to (R_*, g)$ in $\mathscr{V}_1(T)$ between such objects, given by a monotone map $u : [k] \to [\ell]$ and monotone injections $v_j : S_j \to R_{u(j)}$ for $j \in [k]$, qualifies as a morphism in $\mathscr{V}_2(T)$ precisely if $u(0) = 0$. (In that case $v_0 : S_0 \to R_0$ must be an order preserving bijection, hence an identity map.) The inclusion of $\mathscr{V}_2(T)$ in $\mathscr{V}_1(T)$ has a left adjoint. Therefore the restriction map

$$\operatorname{holim}(F_T|\mathscr{V}_1(T)) \to \operatorname{holim}(F_T|\mathscr{V}_2(T))$$

is a weak equivalence. Now $F_T$ takes every morphism in $\mathscr{V}_2(T)$ to a weak equivalence of spaces. Moreover $\mathscr{V}_2(T)$ clearly has an initial object $\omega$. Together these properties imply that the projection from $\operatorname{holim}(F_T|\mathscr{V}_2(T))$ to $F_T(\omega)$ is a weak equivalence. Putting all that together, it follows that the projection from $\operatorname{holim} F_T$ itself to $F_T(\omega)$ is a weak equivalence. Then it follows easily by inspection that the unit map $Z(T) \to (\mathbb{R}\xi_*\xi^*Z)(T) = \operatorname{holim} F_T \simeq F_T(\omega)$ is a weak equivalence. This completes our discussion of the case where $T$ has exactly one node.

We come to the induction step, which is rather formal and uses a mildly sheaf-theoretic argument. Suppose that $T$ is an object of $\mathsf{Tree}^{\mathsf{rc}}$ such that $T = T_0 \cup T_1$

where $T_0$ and $T_1$ are subtrees (by which, at this point, we simply mean subposets) of $T$. More precisely we require that if $e \in T_0$ and $e' \in T$ with $e' \le e$, we have $e' \in T_0$, and similarly for $T_1$. We also require that if $e, e' \in T$ and both are incoming edges for the same vertex, then either $e, e'$ are both in $T_0$ or both not in $T_0$; and similarly for $T_1$. The following picture is an example:



$T$ $\qquad$ $T_0$ $\qquad$ $T_1$

Then it is easy to see that the square of inclusion-induced maps

$$\begin{array}{ccc} Z(T) & \longrightarrow & Z(T_0) \\ \downarrow & & \downarrow \\ Z(T_1) & \longrightarrow & Z(T_0 \cap T_1) \end{array}$$

is a homotopy pullback square. If we can show that the square of inclusion-induced maps

$$\begin{array}{ccc} (\mathbb{R}\xi_*\xi^* Z)(T) & \longrightarrow & (\mathbb{R}\xi_*\xi^* Z)(T_0) \\ \downarrow & & \downarrow \\ (\mathbb{R}\xi_*\xi^* Z)(T_1) & \longrightarrow & (\mathbb{R}\xi_*\xi^* Z)(T_0 \cap T_1) \end{array}$$

is also a homotopy pullback square, then that can pass for the induction step. With the abbreviations or translations above, we have $(\mathbb{R}\xi_*\xi^* Z)(T) = \operatorname{holim} F_T$, where $F_T$ is defined on $\mathscr{V}_T$. Let $\mathscr{V}'_T$ be the full subcategory of $\mathscr{V}_T$ consisting of all objects

$$(S_*,\ f : \xi(S_*) \to T)$$

such that $\xi(S_*)$ lands in $T_0$ or in $T_1$ or even in $T_0 \cap T_1$. By inspection, the square of restriction maps

$$\begin{array}{ccc} \operatorname{holim}(F_T | \mathscr{V}'_T) & \longrightarrow & \operatorname{holim} F_{T_0} \\ \downarrow & & \downarrow \\ \operatorname{holim} F_{T_1} & \longrightarrow & \operatorname{holim} F_{T_0 \cap T_1} \end{array}$$

is a homotopy pullback square. Therefore it is enough to show that the restriction map from holim $F_T$ to holim$(F_T|\mathscr{V}'_T)$ is a weak equivalence. A formula of Dwyer and Kan [1984, 9.7] or Bousfield and Kan allows us to identify holim$(F_T|\mathscr{V}'_T)$ with the homotopy inverse limit of $\mathbb{R}\eta_*(F_T|\mathscr{V}'_T)$, where $\eta : \mathscr{V}'_T \to \mathscr{V}_T$ is the inclusion and $\mathbb{R}\eta_*$ denotes the homotopy right Kan extension along $\eta$. In the definition of $\mathbb{R}\eta_*$ we use, for each object $(S_*, f)$ in $\mathscr{V}_T$, the comma category $\eta/(S_*, f)$. In that comma category there is the diagram (of three objects)

$$\begin{array}{ccccc} (S_*, f)_{T_0} & \longleftarrow & (S_*, f)_{T_0\cap T_1} & \longrightarrow & (S_*, f)_{T_1} \\ & \searrow & \downarrow & \swarrow & \\ & & (S_*, f) & & \end{array}$$

where the subscripts indicate evident pullback operations; for example $(S_*, f)_{T_0}$ is made from the portion of $S_*$ taken to $T_0$ by $f$. That diagram can be viewed as a subcategory of $\eta/(S_*, f)$. It is a terminal subcategory in the sense that the inclusion functor has a left adjoint. Hence the value of $\mathbb{R}\eta_*(F_T|\mathscr{V}'_T)$ at the object $(S_*, f)$ of $\mathscr{V}_T$ can be identified with the homotopy pullback of

$$F_T((S_*, f)_{T_0}) \to F_T((S_*, f)_{T_0\cap T}) \leftarrow F_T((S_*, f)_{T_1})$$

which in turn can be identified with $F_T(S_*, f) = Z(\xi(S_*))$ by the sheaf property of $Z$. □

It is very easy to show (again) that the map (4-1) is a weak equivalence using Lemmas 5.2 and 5.3. Indeed, Lemma 5.2 implies that the standard map

$$\mathbb{R}\mathrm{map}(\xi^* N_d^{rc} P, \xi^* N_d^{rc} Q) \to \mathbb{R}\mathrm{map}(\alpha^*\xi^* N_d^{rc} P, \alpha^*\xi^* N_d^{rc} Q)$$

is a weak equivalence and Lemma 5.3 implies that the standard map

$$\mathbb{R}\mathrm{map}(N_d^{rc} P, N_d^{rc} Q) \to \mathbb{R}\mathrm{map}(\xi^* N_d^{rc} P, \xi^* N_d^{rc} Q)$$

is a weak equivalence. See [BW, Lemma A.1] and remember diagram (5-2). This new proof carries over, mutatis mutandis, to show that the map (4-2) is also weak equivalence. In this way we have completed the proof of Theorem 3.1, since that had already been reduced to showing that (4-2) is a weak equivalence.

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# From compressible to incompressible inhomogeneous flows in the case of large data

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We are concerned with the mathematical derivation of the inhomogeneous incompressible Navier–Stokes equations (INS) from the compressible Navier–Stokes equations (CNS) in the large volume viscosity limit. We first prove a result of large-time existence of regular solutions for (CNS). Next, as a consequence, we establish that the solutions of (CNS) converge to those of (INS) when the volume viscosity tends to infinity. Analysis is performed in the two-dimensional torus $\mathbb{T}^2$ for general initial data. Compared to prior works, the main breakthrough is that we are able to handle *large* variations of density.

## 1. Introduction

We are concerned with the compressible Navier–Stokes system

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0 & \text{in } (0,T)\times\mathbb{T}^2, \\ \rho v_t + \rho v\cdot\nabla v - \mu\Delta v - \nu\nabla\operatorname{div} v + \nabla P = 0 & \text{in } (0,T)\times\mathbb{T}^2. \end{cases} \tag{1-1}$$

Above, the unknown nonnegative function $\rho = \rho(t,x)$ and vector field $v = v(t,x)$ stand for the density and velocity of the fluid at $(t,x)$. The two real numbers $\mu$ and $\nu$ denote the viscosity coefficients and are assumed to satisfy $\mu > 0$ and $\nu + \mu > 0$. We suppose that the pressure function $P = P(\rho)$ is $C^1$ with $P' > 0$, and that $P(\bar\rho) = 0$ for some positive constant reference density $\bar\rho$. Throughout, we set

$$e(\rho) := \rho\int_{\bar\rho}^{\rho}\frac{P(t)}{t^2}\,dt.$$

Note that $e(\bar\rho) = e'(\bar\rho) = 0$ and $\rho e''(\rho) = P'(\rho)$. Hence $e$ is a strictly convex function and, for any interval $[\rho_*, \rho^*]$, there exist two constants $m_*$ and $m^*$ such that

$$m_*(\rho-\bar\rho)^2 \le e(\rho) \le m^*(\rho-\bar\rho)^2. \tag{1-2}$$

The system is supplemented with the initial conditions

$$v|_{t=0} = v_0 \in \mathbb{R}^2 \quad\text{and}\quad \rho|_{t=0} = \rho_0 \in \mathbb{R}_+. \tag{1-3}$$

*MSC2010:* 76N10.
*Keywords:* compressible Navier–Stokes equations, inhomogeneous fluids, large volume viscosity limit.

We aim at comparing the above compressible Navier–Stokes system with its incompressible but inhomogeneous version, namely

$$\begin{cases} \eta_t + u\cdot\nabla\eta = 0 & \text{in } (0,T)\times\mathbb{T}^2,\\ \eta u_t + \eta u\cdot\nabla u - \mu\Delta u + \nabla\Pi = 0 & \text{in } (0,T)\times\mathbb{T}^2,\\ \operatorname{div} u = 0 & \text{in } (0,T)\times\mathbb{T}^2. \end{cases} \tag{1-4}$$

At the formal level, one can expect the solutions of (1-1) to converge to those of (1-4) when $\nu$ goes to $\infty$. Indeed, the velocity equation of (1-1) may be rewritten

$$\nabla \operatorname{div} v = \frac{1}{\nu}(\rho v_t + \rho v\cdot\nabla v - \mu\Delta v + \nabla P)$$

and thus $\nabla \operatorname{div} v$ should tend to 0 when $\nu\to\infty$. This means that $\operatorname{div} v$ should tend to be independent of the space variable and, as it is the divergence of some periodic vector field, one must eventually have $\operatorname{div} v\to 0$. As, on the other side, one has for all values of $\nu$,

$$\rho v_t + \rho v\cdot\nabla v - \mu\Delta v \quad \text{is a gradient,}$$

this means that if $(\rho, v)$ tends to some pair $(\eta, u)$ in a sufficiently strong manner, then necessarily $(\eta, u)$ should satisfy (1-4).

The question of finding an appropriate framework for justifying that heuristic naturally arises. Let us first examine the *weak solution framework*, as it requires the fewest assumptions on the data. Regarding system (1-1) with a pressure law like $P(\rho) = a(\rho^\gamma - \bar\rho^\gamma)$ for some $a > 0$ and $\gamma > 1$, the state-of-the-art result for the weak solution theory is as follows (see [Lions 1998; Novotný and Straškraba 2004] for more details):

**Theorem 1.1.** *Assume that the initial data $\rho_0$ and $v_0$ satisfy $\sqrt{\rho_0}\, v_0 \in L_2(\mathbb{T}^2)$ and $\rho_0 \in L_\gamma(\mathbb{T}^2)$. Then there exists a global-in-time weak solution to (1-1) such that*

$$v \in L_\infty(\mathbb{R}_+; L_2(\mathbb{T}^2)) \cap L_2(\mathbb{R}_+; \dot H^1(\mathbb{T}^2)) \quad \textit{and} \quad e(\rho) \in L_\infty(\mathbb{R}_+; L_1(\mathbb{T}^2)) \tag{1-5}$$

*and, for all $T > 0$,*

$$\int_{\mathbb{T}^2} \bigl(\tfrac12\rho|v|^2 + e(\rho)\bigr)(T,\cdot)\,dx + \int_0^T \bigl(\mu\|\nabla v\|_2^2 + \nu\|\operatorname{div} v\|_2^2\bigr)\,dt \le \int_{\mathbb{T}^2}\bigl(\tfrac12\rho_0|v_0|^2 + e(\rho_0)\bigr)\,dx. \tag{1-6}$$

For system (1-4), there is a similar weak solution theory that was initiated by A. Kazhikhov [1974], then continued by J. Simon [1990] and completed by P.-L. Lions [1996]. However, to the best of our knowledge, it is not known how to connect system (1-1) to (1-4) in that framework. Justifying the convergence in that setting may be extremely difficult owing to the fact that the key extra estimate for the density that allows one to achieve the existence of weak solutions for

(1-1) strongly depends on the viscosity coefficient $\nu$, and collapses when $\nu$ goes to infinity.

This thus motivates us to consider the problem for more regular solutions. Regarding system (1-1) in the multidimensional case, recall that the global existence issue of strong unique solutions has been answered just partially, and mostly in the small data case; see, e.g., [Danchin 2000; Kotschote 2014; Matsumura and Nishida 1980; Mucha 2003; Mucha and Zając̨zkowski 2002; 2004; Valli and Zając̨zkowski 1986]. For general large data (even if very smooth), only local-in-time solutions are available; see, e.g., [Danchin 2001; Nash 1962].

The theory of strong solutions for the inhomogeneous Navier–Stokes system (1-4) is more complete; see, e.g., [Danchin and Mucha 2012; Ladyzhenskaya and Solonnikov 1975; Huang et al. 2013; Li 2017]. In fact, the results are roughly the same as for the homogeneous (that is with constant density) incompressible Navier–Stokes system. In particular, we proved in [Danchin 2017] that, in the two-dimensional case, system (1-4) is uniquely and globally solvable in dimension two whenever the initial velocity is in $H^1$ and the initial density is nonnegative and bounded (initial data with vacuum may thus be considered).

It is tempting to study whether those better properties in dimension two for the (supposedly) limit system (1-4) may help us to improve our knowledge of system (1-1) in the case where the volume viscosity is very large. More precisely, we here want to address the following two questions:

- For regular data with no vacuum, given any fixed $T > 0$, can we find $\nu_0$ so that the solution remains smooth (hence unique) until time $T$ for all $\nu \geq \nu_0$?
- Considering a family $(\rho_\nu, v_\nu)$ of solutions to (1-1) and letting $\nu \to \infty$, can we show strong convergence to some pair $(\eta, u)$ satisfying (1-4) and, as the case may be, give an upper bound for the rate of convergence?

Those two issues have been considered recently in our paper [Danchin and Mucha 2017], in the particular case where the initial density is a perturbation of order $\nu^{-1/2}$ of some constant positive density (hence the limit system is just the classical incompressible Navier–Stokes equation). There, our results were based on Fourier analysis and involved so-called critical Besov norms. The cornerstone of the method was a refined analysis of the linearized system about the constant state $(\rho, v) = (\bar{\rho}, 0)$, thus precluding us from considering large density variations.

The present paper aims at shedding a new light on this issue, pointing out different results and techniques than in [Danchin and Mucha 2017]. In particular, we will go beyond the slightly inhomogeneous case, and will be able to consider large variations of density. Regarding the techniques, we here meet another motivation which is strictly mathematical; we want to advertise two tools that can be of some use in the analysis of systems of fluid mechanics:

- The first one is a nonstandard estimate with (limited) loss of integrability for solutions of the transport equation by a non-Lipschitz vector field that was first pointed out by B. Desjardins [1997] (see Section 3). Proving it requires a Moser–Trudinger inequality that holds true only in dimension two.[1]
- The second tool is an estimate for a parabolic system with just bounded coefficients in the maximal regularity framework of $L_p$ spaces with $p$ close to 2 (Section 4).

For notational simplicity, *we assume from now on that the shear viscosity $\mu$ is equal to* 1 (which may always be achieved after a suitable rescaling). Our answer to the first question then reads as follows:

**Theorem 1.2.** *Fix some $T > 0$. Let $\rho_*$ and $\rho^*$ satisfy $0 < 4\rho_* \le \rho^*$, and assume*

$$2\rho_* \le \rho_0 \le \tfrac{1}{2}\rho^*. \tag{1-7}$$

*There exists an exponent $q > 2$ depending only on $\rho_*$ and $\rho^*$ such that if $\nabla\rho_0 \in L_q(\mathbb{T}^2)$ then for any vector field $v_0$ in $W_q^{2-2/q}(\mathbb{T}^2)$ satisfying*

$$\nu^{1/2}\|\operatorname{div} v_0\|_{L_2} \le 1, \tag{1-8}$$

*there exists $\nu_0 = \nu_0(T, \rho_*, \rho^*, \|\nabla\rho_0\|_q, \|v_0\|_{W_q^{2-2/q}}, P, q)$ such that system (1-1) with $\nu \ge \nu_0$ has a unique solution $(\rho, v)$ on the time interval $[0, T]$, fulfilling*

$$\begin{gathered} v \in \mathcal{C}([0,T]; W_q^{2-2/q}(\mathbb{T}^2)), \quad v_t, \nabla^2 v \in L_q([0,T]\times\mathbb{T}^2), \\ \rho \in \mathcal{C}([0,T]; W_q^1(\mathbb{T}^2)), \end{gathered} \tag{1-9}$$

*and*

$$\rho_* \le \rho(t,x) \le \rho^* \quad \textit{for all } (t,x) \in [0,T]\times\mathbb{T}^2. \tag{1-10}$$

*Furthermore, there exists a constant $C_q$ depending only on $q$, a constant $C_P$ depending only on $P$, and a universal constant $C$ such that for all $t \in [0, T]$,*

$$\begin{aligned} \|v(t)\|_{H^1} + \nu^{1/2}\|\operatorname{div} v(t)\|_{L_2} + \|\rho(t) - \bar\rho\|_{L_2} + \|\nabla v\|_{L_2([0,t];H^1)} \\ + \|v_t\|_{L_2(0,t\times\mathbb{R}^2)} + \nu^{1/2}\|\nabla\operatorname{div} v\|_{L_2(0,t\times\mathbb{R}^2)} \le Ce^{C\|v_0\|_2^4}E_0, \end{aligned} \tag{1-11}$$

$$\begin{aligned} \|v(t)\|_{W_q^{2-2/q}} + \|v_t, \nabla^2 v, \nu\nabla\operatorname{div} v\|_{L_q([0,t]\times\mathbb{T}^2)} \\ \le C_q\big(\|v_0\|_{W_q^{2-2/q}} + C_P t^{1/q}(1 + \|\nabla\rho_0\|_{L_q})\exp(t^{1/q'}I_0(t))\big), \end{aligned} \tag{1-12}$$

*and*

$$\|\nabla\rho(t)\|_{L_q} \le (1 + \|\nabla\rho_0\|_{L_q})\exp(t^{1/q'}I_0(t)), \tag{1-13}$$

*with $E_0 := 1 + \|v_0\|_{H^1} + \|\rho_0 - \bar\rho\|_{L_2}$ and*

$$I_0(t) := C_q\big(\|v_0\|_{W_q^{2-2/q}} + C_P t^{1/q}(1 + \|\nabla\rho_0\|_{L_q})e^{CE_0^2 t e^{C\|v_0\|_{L_2}^4}}\big).$$

[1] Consequently, we do not know how to adapt our approach to the higher-dimensional case.

As the data we here consider are regular and bounded away from zero, the short-time existence and uniqueness issues are clear (one may, e.g., adapt [Danchin 2010] to the case of periodic boundary conditions). In order to achieve large-time existence, we shall first take advantage of a rather standard higher-order energy estimate (at the $H^1$ level for the velocity) that will provide us with a control of $\nabla v$ in $L^2(0,T;H^1)$ in terms of the data and of the norm of $\nabla\rho$ in $L^\infty(0,T;L^2)$. The difficulty now is to control that latter norm, given that, at this stage, one has no bound for $\nabla v$ in $L^1(0,T;L^\infty)$. It may be overcome by adapting to our framework some estimates with loss of integrability for the transport equation, which were first pointed out in [Desjardins 1997]. However, this is not quite the end of the story since those estimates involve the quantity $\int_0^T \|\operatorname{div} v\|_{L^\infty}\,dt$. Then, the key observation is that the linear maximal regularity theory for the linearization of the momentum equation of (1-1) (neglecting the pressure term and taking $\rho\equiv 1$) provides, for all $1<q<\infty$, a control on $\nu\|\nabla \operatorname{div} v\|_{L_q(0,T;L_q(\mathbb{T}^2))}$ (not just $\|\nabla \operatorname{div} v\|_{L_q(0,T;L_q(\mathbb{T}^2))}$) in terms of $\|v_0\|_{W_q^{2-2/q}}$. In our framework where $\rho$ is not constant, it turns out to be possible to recover a similar estimate if $q$ is close enough to 2, and thus to eventually have, by Sobolev embedding, $\int_0^T \|\operatorname{div} v\|_{L^\infty}\,dt=\mathcal{O}(\nu^{-1})$. Then, putting all the arguments together and bootstrapping allows us to get all the estimates of Theorem 1.2, for large enough $\nu$.

Regarding the asymptotics $\nu\to\infty$, it is clear that if one starts with fixed initial data, then uniform estimates are available from Theorem 1.2, only if we assume that $\operatorname{div} v_0\equiv 0$. Under that assumption, inequalities (1-11) and (1-12) already ensure that

$$\begin{aligned}\operatorname{div} v &= \mathcal{O}(\nu^{-1/2}) &&\text{in } L_\infty(0,T;L_2),\\ \nabla \operatorname{div} v &= \mathcal{O}(\nu^{-1}) &&\text{in } L_q(0,T\times\mathbb{T}^2).\end{aligned}$$

Then, combining with the uniform bounds provided by (1-12) and (1-13), it is not difficult to pass to the weak limit in system (1-1) and to find that the limit solution fulfills system (1-4).

In the theorem below, we state a result that involves strong norms of all quantities at the level of energy norm, and exhibit an explicit rate of convergence.

**Theorem 1.3.** *Fix some $T>0$ and take initial data $(\rho_0, v_0)$ fulfilling the assumptions of Theorem 1.2 with, in addition,* $\operatorname{div} v_0\equiv 0$. *Denote by $(\rho_\nu, v_\nu)$ the corresponding solution of (1-1) with volume viscosity $\nu\geq\nu_0$. Finally, let $(\eta,u)$ be the global solution of (1-4) supplemented with the same initial data $(\rho_0,v_0)$. Then we have*

$$\sup_{t\leq T}\Big(\|\rho_\nu(t)-\eta(t)\|_{L_2}^2+\|\mathcal{P}v_\nu(t)-u(t)\|_{L_2}^2+\|\nabla\mathcal{Q}v_\nu(t)\|_{L_2}^2\Big) + \int_0^T\Big(\|\nabla(\mathcal{P}v_\nu-u)\|_{L_2}^2+\|\nabla\mathcal{Q}v_\nu\|_{H^1}^2\Big)\,dt\leq C_{0,T}\,\nu^{-1}, \quad (1\text{-}14)$$

*where $\mathcal{P}$ and $\mathcal{Q}$ are the Helmholtz projectors on divergence-free and potential vector fields, respectively,*[2] *and where $C_{0,T}$ depends only on $T$ and on the norms of the initial data.*

Compared to the question of low Mach number limit studied in, e.g., [Danchin 2002; Feireisl and Novotný 2013], there is an essential difference in the mechanism leading to convergence, as may be easily seen from a rough analysis of the linearized system (1-1). Indeed, in the case $\bar\rho = \mu = 1$ and $P'(1) = 1$ (for notational simplicity), that linearization (in the unforced case) is given by

$$\begin{cases} \eta_t + \operatorname{div} u = 0, \\ v_t - \Delta v - \nu \nabla \operatorname{div} v + \nabla \eta = 0. \end{cases}$$

Eliminating the velocity we obtain the damped wave equation

$$\eta_{tt} - (1+\nu)\Delta\eta_t - \Delta\eta = 0,$$

which can be solved explicitly at the level of the Fourier transform. We obtain two modes, one strongly parabolic, disappearing for $\nu \to \infty$, and the second one having the following form, in the high frequency regime:

$$\eta(t) \sim \eta(0) e^{-t/(1+\nu)} \to \eta(0).$$

This means that at the same time, we have that $\eta(t)$ tends strongly to 0 as $t \to \infty$ even for very large $\nu$, but that for all $t > 0$ (even very large), $\eta(t) \to \eta(0)$ when $\nu$ tends to $\infty$.

The behavior corresponding to the low Mach number limit is of a different nature, as it corresponds to the linearization

$$\begin{cases} \eta_t + \dfrac{1}{\varepsilon} \operatorname{div} u = 0, \\ v_t - \Delta v - \nu \nabla \operatorname{div} v + \dfrac{1}{\varepsilon} \nabla \eta = 0, \end{cases}$$

which leads to the wave equation

$$\eta_{tt} - (1+\nu)\Delta\eta_t - \frac{1}{\varepsilon^2}\Delta\eta = 0.$$

Asymptotically for $\varepsilon \to 0$, the above damped wave equation behaves as a wave equation with propagation speed $1/\varepsilon$. Hence, in the periodic setting, we have huge oscillations that preclude any strong convergence result. However, after filtering by the wave operator, convergence becomes strong, which entails weak convergence, back to the original unknowns (see [Danchin 2002] for more details).

The main idea of Theorem 1.3 is just to compute the distance between the compressible and the incompressible solutions, by means of the standard energy norm

[2] They are defined by $\mathcal{Q}v := -\nabla(-\Delta)^{-1} \operatorname{div} v$ and $\mathcal{P}v := v - \mathcal{Q}v$.

(in sharp contrast with the approach in [Danchin and Mucha 2017] where critical Besov norms are used). In order to do so, it is convenient to decompose $\rho - \eta$ into two parts

$$\rho - \eta = (\rho - \tilde{\rho}) + (\tilde{\rho} - \eta),$$

where the auxiliary density $\tilde{\rho}$ is the transport of $\rho_0$ by the flow of the divergence-free vector field $\mathcal{P}v$. As the bounds of Theorem 1.2 readily ensure that $\|\rho - \tilde{\rho}\|_q = \mathcal{O}(\nu^{-1})$, one may, somehow, perform the energy argument as if comparing $(\tilde{\rho}, v)$ and $(\eta, u)$.

We end the introduction by presenting the main notation that is used throughout the paper. By $\nabla$ we denote the gradient with respect to space variables, and by $u_t$ the time derivative of the function $u$. By $\|\cdot\|_{L_p(Q)}$ (or sometimes just $\|\cdot\|_p$), we mean the $p$-power Lebesgue norm corresponding to the set $Q$, and $L_p(Q)$ is the corresponding Lebesgue space. We denote by $W^s_p$ the Sobolev (Slobodeckij for $s$ not integer) space on the torus $\mathbb{T}^2$, and put $H^s = W^s_2$. The homogeneous versions of those spaces (that is, the corresponding subspaces of functions with null mean) are denoted by $\dot{W}^s_p$ and $\dot{H}^s$.

Generic constants are denoted by $C$. By $A \lesssim B$ we mean that $A \le CB$, and $A \approx B$ stands for $C^{-1}A \le B \le CA$.

## 2. Energy estimates

The aim of this part is to provide bounds via energy-type estimates. We assume that the density is bounded from above and below. Let us first recall the basic energy identity.

**Proposition 2.1.** *For any $T > 0$, sufficiently smooth solutions to (1-1) obey (1-6).*

*Proof.* That fundamental estimate follows from testing the momentum equation by $v$ and integrating by parts in the diffusion and pressure terms. Indeed, using the definition of $e$ and the mass equation, we get

$$\begin{aligned}\int_{\mathbb{T}^2} \nabla P \cdot v\,dx &= \int_{\mathbb{T}^2} \frac{P'(\rho)}{\rho} \nabla\rho \cdot (\rho v)\,dx = \int_{\mathbb{T}^2} \nabla(e'(\rho)) \cdot (\rho v)\,dx \\ &= -\int_{\mathbb{T}^2} e'(\rho)\,\operatorname{div}(\rho v)\,dx = \int_{\mathbb{T}^2} e'(\rho)\rho_t\,dx = \frac{d}{dt}\int_{\mathbb{T}^2} e(\rho)\,dx.\end{aligned}$$

Then integrating in time completes the proof. □

Let us next derive a higher-order energy estimate, pointing out the dependency with respect to the volume viscosity $\nu$.

**Proposition 2.2.** *Assume that there exist positive constants $\rho_* < \rho^*$ such that*

$$\rho_* \le \rho(t,x) \le \rho^* \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{T}^2. \tag{2-1}$$

*Then solutions to* (1-1) *with* $\mu = 1$ *fulfill the inequality*

$$\|v(T), \nabla v(T), \rho(T) - \bar\rho\|_2^2 + \nu\|\operatorname{div} v(T)\|_2^2 + \int_0^T \big(\|\nabla^2 v, \nabla v, v_t\|_2^2 + \nu\|\operatorname{div} v\|_{H^1}^2\big)\,dt$$
$$\le C\exp(C\|v_0\|_2^4)\Big(\|v_0, \nabla v_0, \rho_0 - \bar\rho\|_2^2 + \nu\|\operatorname{div} v_0\|_2^2 + \nu^{-1}T\|v_0\|_2^2 + \nu^{-1}\int_0^T \|\nabla\rho\|_2^2\,dt\Big), \quad (2\text{-}2)$$

*provided* $\nu$ *is larger than some* $\nu_0 = \nu_0(\rho_*, \rho^*, P)$.

*Proof.* We take the $\mathbb{T}^2$ inner product of the momentum equation with $v_t$, getting

$$\int_{\mathbb{T}^2} \rho|v_t|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}(|\nabla v|^2 + \nu(\operatorname{div} v)^2)\,dx + \int_{\mathbb{T}^2}\nabla P\cdot v_t\,dx = -\int_{\mathbb{T}^2}(\rho v\cdot\nabla v)\cdot v_t\,dx. \quad (2\text{-}3)$$

Integrating by parts and using the mass equation yields

$$\begin{aligned}\int_{\mathbb{T}^2}\nabla P\cdot v_t\,dx &= -\int_{\mathbb{T}^2} P\ \operatorname{div} v_t\,dx\\ &= -\frac{d}{dt}\int_{\mathbb{T}^2} P\ \operatorname{div} v\,dx + \int_{\mathbb{T}^2} P'(\rho)\rho_t \operatorname{div} v\,dx\\ &= -\frac{d}{dt}\int_{\mathbb{T}^2} P\ \operatorname{div} v\,dx - \int_{\mathbb{T}^2} P'(\rho)\ \operatorname{div}(\rho v)\ \operatorname{div} v\,dx.\end{aligned}$$

Hence putting this together with (2-3), we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}(|\nabla v|^2 + \nu(\operatorname{div} v)^2 - 2P\operatorname{div} v)\,dx + \int_{\mathbb{T}^2}\rho|v_t|^2\,dx = \int_{\mathbb{T}^2} P'(\rho)\operatorname{div}(\rho v)\operatorname{div} v\,dx - \int_{\mathbb{T}^2}(\rho v\cdot\nabla v)\cdot v_t\,dx. \quad (2\text{-}4)$$

Now, setting $K(\rho) = \rho P'(\rho) - P(\rho)$, one can check that

$$\begin{aligned}\int_{\mathbb{T}^2} P'(\rho)\operatorname{div}(\rho v)\operatorname{div} v\,dx &= \int_{\mathbb{T}^2}(\operatorname{div} v)\ v\cdot\nabla(P(\rho))\,dx + \int_{\mathbb{T}^2}\rho\nabla P'(\rho)\,(\operatorname{div} v)^2\,dx\\ &= -\int_{\mathbb{T}^2} P(\rho)\ v\cdot\nabla\operatorname{div} v\,dx + \int_{\mathbb{T}^2} K(\rho)(\operatorname{div} v)^2\,dx.\end{aligned}$$

Hence, if (2-1) is fulfilled then we have

$$\frac{d}{dt}\int_{\mathbb{T}^2}\big(|\nabla v|^2 + \nu(\operatorname{div} v)^2 - 2P(\rho)\operatorname{div} v\big)\,dx + \int_{\mathbb{T}^2}\rho|v_t|^2\,dx \le C\int_{\mathbb{T}^2}\big(|v\cdot\nabla\operatorname{div} v| + (\operatorname{div} v)^2 + |v\cdot\nabla v|^2\big)\,dx. \quad (2\text{-}5)$$

Next, taking the $L^2$ scalar product of the momentum equation with $\Delta v$ we get

$$\int_{\mathbb{T}^2}(|\Delta v|^2+\nu|\nabla\operatorname{div} v|^2)\,dx-\int_{\mathbb{T}^2}\rho v_t\cdot\Delta v\,dx-\int_{\mathbb{T}^2}\nabla P\cdot\Delta v\,dx\le\int_{\mathbb{T}^2}|\rho v\cdot\nabla v\Delta v|\,dx.$$

Note that

$$-\int_{\mathbb{T}^2}\nabla P\cdot\Delta v\,dx=-\int_{\mathbb{T}^2}\nabla P\cdot\nabla\operatorname{div} v\,dx\le C\int_{\mathbb{T}^2}|\nabla\rho|\,|\nabla\operatorname{div} v|\,dx.$$

Then, combining with the basic energy identity and with (2-5) and introducing

$$E(v,\rho):=\int_{\mathbb{T}^2}\big(\rho|v|^2+2e(\rho)+|\nabla v|^2+\nu\,(\operatorname{div} v)^2-2P(\rho)\operatorname{div} v\big)\,dx,\tag{2-6}$$

we find,

$$\begin{aligned}\frac{d}{dt}E(v,\rho)+\int_{\mathbb{T}^2}\rho|v_t|^2\,dx+\frac{1}{\rho^*}\int_{\mathbb{T}^2}\big(|\nabla v|^2+|\nabla^2 v|^2+\nu(\operatorname{div} v)^2+\nu|\nabla\operatorname{div} v|^2\big)\,dx\\ \le\int_{\mathbb{T}^2}|v_t\cdot\Delta v|\,dx\\ +C\int_{\mathbb{T}^2}\Big((\operatorname{div} v)^2+|v\cdot\nabla\operatorname{div} v|+\rho|v\cdot\nabla v|^2+\frac{1}{\rho^*}|\nabla\rho|\,|\nabla\operatorname{div} v|\Big)\,dx.\end{aligned}\tag{2-7}$$

Hence, setting

$$D(v):=\|\nabla v\|_{H^1}^2+\|\sqrt{\rho}\,v_t\|_{L_2}^2+\nu\|\operatorname{div} v\|_{H^1}^2,$$

inequality (2-7) implies that for large enough $\nu$,

$$\frac{d}{dt}E(v,\rho)+\frac{1}{\rho^*}D(v)\le C\int_{\mathbb{T}^2}\big(|v|^2\,|\nabla v|^2+(|v|+|\nabla\rho|)|\nabla\operatorname{div} v|\big)\,dx.$$

Of course, from the Ladyzhenskaya inequality, we have

$$\int_{\mathbb{T}^2}|v\cdot\nabla v|^2\,dx\le C\|v\|_2\,\|\nabla v\|_2^2\,\|\Delta v\|_2.$$

Therefore, we end up with

$$\frac{d}{dt}E(v,\rho)+\frac{1}{\rho^*}D(v)\le C\big(\|v\|_2^2\,\|\nabla v\|_2^2\,\|\nabla v\|_2^2+\nu^{-1}(\|v\|_2^2+\|\nabla\rho\|_2^2)\big).$$

Let us notice that if $\nu\ge\nu_0(\rho_*,\rho^*,P)$ then we have, according to (1-2),

$$E(v,\rho)\approx\|v\|_{H^1}^2+\|\rho-\bar\rho\|_{L_2}^2+\nu\|\operatorname{div} v\|_{L_2}^2.\tag{2-8}$$

Hence the Gronwall inequality yields

$$\begin{aligned}E(v(T),\rho(T))+\frac{1}{\rho^*}\int_0^T D(t)\,dt\le\exp\Big(C\int_0^T\|v\|_2^2\,\|\nabla v\|_2^2\,dt\Big)\\ \times\Big(E(v_0,\rho_0)+\frac{C}{\nu}\int_0^T\exp\Big(-C\int_0^t\|v\|_2^2\,\|\nabla v\|_2^2\,dt\Big)(\|v\|_2^2+\|\nabla\rho\|_2^2)\,dt\Big).\end{aligned}$$

Remembering that the basic energy inequality implies

$$\int_0^T \|v\|_2^2 \, \|\nabla v\|_2^2 \, dt \leq C\|v_0\|_2^4,$$

one may conclude that

$$E(v(T), \rho(T)) + \frac{1}{\rho^*}\int_0^T D(v)\, dt \\ \leq \exp(C\|v_0\|_2^4)\bigg(E(v_0, \rho_0) + \frac{C}{\nu}\bigg(\|v_0\|_2^2\, T + \int_0^T \|\nabla\rho\|_2^2 \, dt\bigg)\bigg),$$

which obviously yields (2-2). □

## 3. Estimates with loss of integrability for the transport equation

We are concerned with the proof of regularity estimates for the transport equation

$$\rho_t + v \cdot \nabla\rho + \rho \ \mathrm{div}\, v = 0 \tag{3-1}$$

in some endpoint case where the transport field $v$ fails to be in $L_1(0, T; \mathrm{Lip})$ by a little.

More exactly, we aim at extending the results in [Desjardins 1997] to transport fields that are not divergence-free. Our main result is:

**Proposition 3.1.** *Let* $1 \leq q \leq \infty$ *and* $T > 0$. *Suppose* $\rho_0 \in W_q^1(\mathbb{T}^2)$ *and* $v \in L_2(0, T; H^2(\mathbb{T}^2))$ *are such that* $\mathrm{div}\, v \in L_1(0, T; L_\infty(\mathbb{T}^2)) \cap L_1(0, T; W_q^1(\mathbb{T}^2))$. *Then the solution to* (3-1) *fulfills for all* $1 \leq p < q$,

$$\sup_{t<T} \|\nabla\rho(t)\|_p \leq K\bigg(\|\nabla\rho_0\|_q + \|\rho_0\|_\infty \sup_{t<T}\bigg\|\int_0^t \nabla\, \mathrm{div}\, v\, d\tau\bigg\|_q\bigg) \\ \times \exp\bigg(CT\int_0^T \|\nabla^2 v\|_2^2\, dt\bigg)\exp\bigg(\int_0^T \|\mathrm{div}\, v\|_\infty\, dt\bigg),$$

*where* $K$ *is an absolute constant, and the constant* $C$ *depends only on* $p$ *and* $q$.

*Proof.* We proceed by means of the standard characteristics method: our assumptions guarantee that $v$ admits a unique (generalized) flow $X$, a solution to

$$X(t, y) = y + \int_0^t v(\tau, X(\tau, y))\, d\tau. \tag{3-2}$$

Then, setting

$$u(t, y) := v(t, X(t, y)) \quad \text{and} \quad a(t, y) = \rho(t, X(t, y)), \tag{3-3}$$

(3-1) can be rewritten as

$$\frac{da(t, y)}{dt} = -(\mathrm{div}\, v)(t, X(t, y)) \cdot a(t, y), \tag{3-4}$$

the unique solution of which is given by

$$a(t, y) = \exp\left(-\int_0^t (\operatorname{div} v)(\tau, X(\tau, y))\, d\tau\right) a_0(y). \tag{3-5}$$

From the chain rule and the Leibniz formula, we thus infer

$$\nabla_y a(t, y) = \exp\left(-\int_0^t (\operatorname{div} v)(\tau, X(\tau, y))\, d\tau\right) \times \left(\nabla_y a_0(y) - a_0(y) \int_0^t (\nabla \operatorname{div} v)(\tau, X(\tau, y)) \cdot \nabla_y X(\tau, y)\, d\tau\right).$$

Our goal is to estimate all these quantities in the Eulerian coordinates. Note that by (3-2) and the Gronwall lemma, we obtain pointwisely that, setting $Y(t, \cdot) := (X(t, \cdot))^{-1}$,

$$\begin{aligned} |\nabla_y X(t, y)| &\le \exp\left(\int_0^t |\nabla_x v(\tau, X(\tau, y))|\, d\tau\right), \\ |\nabla_x Y(t, x)| &\le \exp\left(\int_0^t |\nabla_y u(\tau, Y(\tau, x))|\, d\tau\right). \end{aligned} \tag{3-6}$$

As $\nabla_x \rho(t, x) = \nabla_y a(t, Y(t, x)) \cdot \nabla_x Y(t, x)$, we get

$$|\nabla \rho(t,x)| \le \exp\left(3\int_0^t |\nabla v(\tau, X(\tau, Y(t,x)))|\, d\tau\right) \times \left(\big|\nabla \rho_0(Y(t,x))\big| + \big|\rho_0(Y(t,x))\big| \left|\int_0^t \nabla \operatorname{div} v\big(\tau, X(\tau, Y(t,x))\big)\, d\tau\right|\right).$$

Recall that the Jacobian of the change of coordinates $(t, y) \to (t, x)$ is given by

$$J_X(t, y) = \exp\left(\int_0^t \operatorname{div} v(\tau, X(\tau, y))\, d\tau\right) \le \exp\left(\int_0^t \|\operatorname{div} v\|_\infty\, d\tau\right). \tag{3-7}$$

Hence taking the $L_p(\mathbb{T}^2)$ norm and using the Hölder inequality with $1/p = 1/q + 1/m$, we get

$$\begin{aligned} \|\nabla \rho(t)\|_p \le \exp\left(\frac{1}{q}\int_0^t \|\operatorname{div} v\|_\infty\, d\tau\right) & \\ \times \left(\|\nabla \rho_0\|_q + \|\rho_0\|_\infty \left\|\int_0^t \nabla \operatorname{div} v(\tau, X(\tau, \cdot))\, ds\right\|_q\right) & \\ \times \left\|\exp\left(3\int_0^t |\nabla v(\tau, X(\tau, \cdot))|\, d\tau\right)\right\|_m. & \end{aligned} \tag{3-8}$$

To bound the last term, we write that for all $\beta > 0$,

$$\int_0^t |\nabla v(\tau, X(\tau, \cdot))|\, d\tau \le \beta \int_0^t \frac{|\nabla v(\tau, X(\tau, \cdot))|^2}{\|\nabla^2 v(\tau, \cdot)\|_2^2}\, d\tau + \frac{1}{4\beta}\int_0^t \|\nabla^2 v(\tau, \cdot)\|_2^2\, d\tau.$$

Hence using the Jensen inequality

$$\exp\left(\int_0^t \phi(s)\,ds\right) \le \frac{1}{t}\int_0^t e^{t\phi(s)}\,ds,$$

we discover that

$$\int_{\mathbb{T}^2} \exp\left(3m\int_0^t |\nabla v(\tau, X(\tau,x))|\,d\tau\right)dx \le \exp\left(\frac{m}{4\beta}\int_0^t \|\nabla^2 v\|_2^2\,d\tau\right)\frac{1}{t}\int_0^t\int_{\mathbb{T}^2}\exp\left(9m\beta t\frac{|\nabla v(\tau, X(\tau,x))|^2}{\|\nabla^2 v\|_2^2}\right)dx\,d\tau.$$

In the last integral we change coordinates and get

$$\int_{\mathbb{T}^2} \exp\left(3m\int_0^t |\nabla v(\tau, X(\tau,x))|\,d\tau\right)dx \le \frac{1}{t}\exp\left(\frac{m}{4\beta}\int_0^t \|\nabla^2 v\|_2^2\,d\tau\right) \times\left(\int_0^t\int_{\mathbb{T}^2}\exp\left(9m\beta t\frac{|\nabla v(\tau,x)|^2}{\|\nabla^2 v\|_2^2}\right)dx\,d\tau\right)\exp\left(\int_0^t \|\operatorname{div} v\|_\infty\,d\tau\right).$$

At this stage, to complete the proof, it suffices to apply the following Trudinger inequality, see for example [Adams 1975], to $f = \nabla v$: there exist constants $\delta_0$ and $K$ such that for all $f$ in $H^1(\mathbb{T}^2)$,

$$\int_{\mathbb{T}^2}\exp\left(\delta_0\frac{|f(x)-\bar f|^2}{\|\nabla f\|_2^2}\right)dx \le K \quad \text{with } \bar f := \frac{1}{|\mathbb{T}^2|}\int_{\mathbb{T}^2} f\,dx. \tag{3-9}$$

Then, taking $\beta$ so small that $9m\beta t = \delta_0$, we end up with

$$\int_{\mathbb{T}^2} \exp\left(3m\int_0^t |\nabla v(\tau, X(\tau,x))|\,d\tau\right)dx \le C\exp\left(\frac{9mt}{4\delta_0}\int_0^t \|\nabla^2 v\|_2^2\,d\tau\right)\exp\left(\int_0^t \|\operatorname{div} v\|_\infty\,d\tau\right). \tag{3-10}$$

Combining with (3-8) completes the proof of the proposition. □

## 4. Linear systems with variable coefficients

Here we are concerned with the proof of maximal regularity estimates for the linear system

$$\begin{cases} \rho u_t - \Delta u - \nu\nabla\operatorname{div} u = f & \text{in } (0,T)\times\mathbb{T}^N,\\ u|_{t=0} = u_0 & \text{in } \mathbb{T}^N,\end{cases} \tag{4-1}$$

assuming only that $\rho = \rho(t,x)$ is bounded by above and from below (no time or space regularity whatsoever).

In contrast with the previous section, we do not need the space dimension to be 2. As we want to keep track of the dependency with respect to $\nu$ for $\nu \to \infty$, we shall assume throughout that $\nu \geq 0$ for simplicity.

**Theorem 4.1.** *Let $T > 0$. Assume that $\nu \geq 0$ and that*

$$0 < \rho_* \leq \rho(t,x) \leq \rho^* \quad \text{for } (t,x) \in [0,T] \times \mathbb{T}^N. \tag{4-2}$$

*There exist positive constants $2_*$, $2^*$ depending only on $\rho_*$ and $\rho^*$, with $2_* < 2 < 2^*$, such that for all $r \in (2_*, 2^*)$ we have*

$$\|u_t, \nabla^2 u, \nu\nabla \operatorname{div} u\|_{L_r((0,T)\times\mathbb{T}^N)} \leq C(r,\rho_*,\rho^*)\big(\|f\|_{L_r((0,T)\times\mathbb{T}^N)} + \|u_0\|_{W_r^{2-2/r}(\mathbb{T}^N)}\big). \tag{4-3}$$

*Proof.* First, we reduce the problem to the one with null initial data, solving

$$\begin{cases} \rho^* \bar{u}_t - \Delta \bar{u} - \nu \nabla \operatorname{div} \bar{u} = 0 & \text{in } (0,T)\times\mathbb{T}^N, \\ \bar{u}|_{t=0} = u_0 & \text{in } \mathbb{T}^N. \end{cases} \tag{4-4}$$

Applying the divergence operator to the equation yields

$$\rho^*(\operatorname{div}\bar{u})_t - (1+\nu)\Delta \operatorname{div}\bar{u} = 0.$$

Hence the basic maximal regularity theory for the heat equation in the torus gives

$$(1+\nu)\|\nabla \operatorname{div}\bar{u}\|_{L_p((0,T)\times\mathbb{T}^N)} \leq C\|\operatorname{div} u_0\|_{W_p^{1-2/p}(\mathbb{T}^N)}. \tag{4-5}$$

Then we restate system (4-4) in the form

$$\rho^*\bar{u}_t - \Delta\bar{u} = \nu\nabla \operatorname{div}\bar{u}, \tag{4-6}$$

and get

$$\begin{aligned} \|\bar{u}_t, \nabla^2\bar{u}\|_{L_p(\mathbb{T}^N\times(0,T))} &\leq K_p\big(\nu\|\nabla\operatorname{div}\bar{u}\|_{L_p((0,T)\times\mathbb{T}^N)} + \|u_0\|_{W_p^{2-2/p}(\mathbb{T}^N)}\big) \\ &\leq K_p\left(\frac{\nu}{1+\nu}\right)\|u_0\|_{W_p^{2-2/p}(\mathbb{T}^N)}. \end{aligned}$$

Therefore, as $\nu \geq 0$, we end up with

$$\|\bar{u}_t, \nabla^2\bar{u}, \nu\nabla\operatorname{div}\bar{u}\|_{L_p((0,T)\times\mathbb{T}^N)} \leq K_p\|u_0\|_{W_p^{2-2/p}(\mathbb{T}^N)}. \tag{4-7}$$

Next we look for $u$ in the form

$$u = w + \bar{u}, \tag{4-8}$$

where $w$ fulfills

$$\rho w_t - \Delta w - \nu\nabla\operatorname{div} w = f + (\rho^* - \rho)\bar{u}_t =: g, \quad w|_{t=0} = 0. \tag{4-9}$$

Thanks to (4-2) and (4-9), we have

$$\|g\|_{L_p((0,T)\times\mathbb{T}^N)} \leq \|f\|_{L_p((0,T)\times\mathbb{T}^N)} + K_p(\rho^* - \rho_*)\|u_0\|_{W_p^{2-2/p}(\mathbb{T}^N)}. \tag{4-10}$$

Now, setting $h := g + (\rho^* - \rho)w_t$, system (4-9) reduces to

$$\begin{cases} \rho^* w_t - \Delta w - \nu \nabla \operatorname{div} w = h & \text{in } (0,T) \times \mathbb{T}^N, \\ w|_{t=0} = 0 & \text{in } \mathbb{T}^N. \end{cases} \tag{4-11}$$

We claim that for all $p \in (1, \infty)$ we have

$$\|\rho^* w_t\|_{L_p((0,T)\times\mathbb{T}^N)} \le C_p \|h\|_{L_p((0,T)\times\mathbb{T}^N)} \tag{4-12}$$

with $C_p \to 1$ for $p \to 2$.

Indeed, to see that $C_2 = 1$, we just take the $L^2$ scalar product of (4-11) with $w_t$, which yields

$$\rho^* \|w_t\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\big(\|\nabla w\|_{L^2}^2 + \nu \|\operatorname{div} v\|_{L^2}^2\big) = \int_{\mathbb{T}^N} h\, w_t\, dx \le \frac{1}{2}\rho^* \|w_t\|_{L^2}^2 + \frac{1}{2\rho^*}\|h\|_{L^2}^2.$$

Then for any fixed $p_0 \in (1, \infty) \setminus \{2\}$, the standard maximal regularity estimate is

$$\|\rho^* w_t\|_{L_{p_0}((0,T)\times\mathbb{T}^N)} \le K_{p_0} \|h\|_{L_{p_0}((0,T)\times\mathbb{T}^N)},$$

and the Hölder inequality gives us for all $\theta \in [0, 1]$,

$$\|z\|_{L_r((0,T)\times\mathbb{T}^N)} \le \|z\|_{L_2((0,T)\times\mathbb{T}^N)}^{1-\theta} \|z\|_{L_{p_0}((0,T)\times\mathbb{T}^N)}^{\theta} \quad \text{with } \frac{1}{r} = \frac{1-\theta}{2} + \frac{\theta}{p_0}.$$

Therefore $C_p \le C_{p_0}^{\theta}$, whence $\limsup C_p \le 1$ for $p \to 2$ (as $\theta \to 0$).

Now, remembering the definition of $h$, we write for all $p \in (1, \infty)$,

$$\begin{aligned} \|\rho^* w_t\|_{L_p((0,T)\times\mathbb{T}^N)} &\le C_p \big(\|g\|_{L_p((0,T)\times\mathbb{T}^N)} + \|(\rho^* - \rho) w_t\|_{L_p((0,T)\times\mathbb{T}^N)}\big) \\ &\le C_p \|g\|_{L_p((0,T)\times\mathbb{T}^N)} + C_p\left(1 - \frac{\rho_*}{\rho^*}\right) \|\rho^* w_t\|_{L_p((0,T)\times\mathbb{T}^N)}. \end{aligned}$$

Therefore, if[3]

$$1 - C_p\left(1 - \frac{\rho_*}{\rho^*}\right) \ge \frac{1}{2}\frac{\rho_*}{\rho^*}, \tag{4-13}$$

then we end up with

$$\|\rho^* w_t\|_{L_p((0,T)\times\mathbb{T}^N)} \le \frac{2\rho^* C_p}{\rho_*} \|g\|_{L_p((0,T)\times\mathbb{T}^N)}. \tag{4-14}$$

Let us emphasize that (4-13) is fulfilled for $p$ close enough to 2, due to $C_p \to 1$ for $p \to 2$.

It is now easy to complete the proof: We rewrite (4-11) in the form

$$\begin{cases} -\Delta w - \nu \nabla \operatorname{div} w = g - \rho w_t & \text{in } (0,T) \times \mathbb{T}^N, \\ w|_{t=0} = 0 & \text{in } \mathbb{T}^N. \end{cases}$$

[3]Clearly, we just need that $1 - C_p(1 - \rho_*/\rho^*) > 0$. However taking that slightly stronger condition allows us to get a more explicit inequality.

Then one concludes as before that

$$\begin{aligned}\|\nabla^2 w, \nu\nabla \operatorname{div} w\|_{L_p((0,T)\times\mathbb{T}^N)} &\le K_p\|g-\rho w_t\|_{L_p((0,T)\times\mathbb{T}^N)}\\ &\le K_p\big(\|g\|_{L_p((0,T)\times\mathbb{T}^N)}+\rho^*\|w_t\|_{L_p((0,T)\times\mathbb{T}^N)}\big).\end{aligned}$$

Hence, putting together with (4-14) and assuming that $p$ is close enough to 2,

$$\|w_t, \nabla^2 w, \nu\nabla \operatorname{div} w\|_{L_p((0,T)\times\mathbb{T}^N)} \le C_{\rho_*,\rho^*}\|g\|_{L_p((0,T)\times\mathbb{T}^N)}. \tag{4-15}$$

Then combining with (4-10) and (4-7) completes the proof. □

## 5. Final bootstrap argument

In what follows, we fix some $0<\rho_*<\rho^*$ and denote by $2_*$ and $2^*$ the corresponding Lebesgue exponents provided by Theorem 4.1. We assume that the initial data $(\rho_0, v_0)$ satisfies all the requirements of Theorem 1.2

Take some time $T$ such that $1\le T\le \nu$ (stronger conditions will appear below), and assume that we have a solution $(\rho, v)$ to (1-1) on $[0,T]\times\mathbb{T}^2$, fulfilling the regularity properties of Theorem 1.2 for some $2<q<\min(2^*,4)$, and

$$\exp\biggl(\int_0^T\|\operatorname{div} v\|_\infty\,dt\biggr)\le 2. \tag{5-1}$$

Then it is clear that $\rho$ obeys

$$\rho_*\le\rho\le\rho^* \quad \text{on } [0,T]\times\mathbb{T}^2. \tag{5-2}$$

For all $p\in[2,q]$, define $A_p(T):=\|\nabla \operatorname{div} v\|_{L_1(0,T;L_p(\mathbb{T}^2))}$ and assume that, for some small enough constant $c_0>0$, we have

$$A_q(T)\le c_0. \tag{5-3}$$

Clearly, if $Kc_0\le\log 2$, where $K$ stands for the norm of the embedding $\dot W^1_q(\mathbb{T}^2)\hookrightarrow L_\infty(\mathbb{T}^2)$, then (5-1) is fulfilled. We shall assume in addition that $c_0\rho^*\le 1$.

We are going to show that if (5-3) is fulfilled then, for sufficiently large $\nu$, all the norms of the solution are under control. Then, bootstrapping, this will justify (5-3) a posteriori.

*Step 1: high-order energy estimate for $v$.* Let $E_0^2:=1+\|v_0\|_{H^1}^2+\|\rho_0-\bar\rho\|_2^2$. By (2-2) we easily get, remembering that $\nu^{-1}T\le 1$,

$$\begin{aligned}\|v\|^2_{L_\infty(0,T;H^1)}+\nu\|\operatorname{div} v\|^2_{L_\infty(0,T;L_2)}+\|\rho-\bar\rho\|^2_{L_\infty(0,T;L_2)}&\\ +\int_0^T\big(\|\nabla v\|^2_{H^1}+\|v_t\|_2^2+\nu\|\nabla\operatorname{div} v\|_2^2\big)\,dt&\\ \le Ce^{C\|v_0\|_2^4}\big(E_0^2+\nu^{-1}T\|\nabla\rho\|^2_{L_\infty(0,T;L_2)}\big).&\end{aligned} \tag{5-4}$$

*Step 2: regularity estimates at $L_p$ level for the density.* From Proposition 3.1, we find that there exists an absolute constant $K$ such that for all $r \in [2, q)$, there exists some constant $C_r > 0$ such that

$$\sup_{t\in[0,T]} \|\nabla\rho(t)\|_r \le K\Big(\big(\|\nabla\rho_0\|_q + \rho^* A_q(T)\big)\exp\Big(C_r T\int_0^T \|\nabla^2 v\|_2^2\,dt\Big)\Big).$$

Hence, bounding the last term according to (5-4), and using (5-3) and the definition of $E_0$,

$$\sup_{t\in[0,T]} \|\nabla\rho(t)\|_r \le K(\|\nabla\rho_0\|_q + 1)\exp(C_r E_0^2 T e^{C\|v_0\|_2^4}) \times \exp\big(C_r \nu^{-1} T^2 e^{C\|v_0\|_2^4}\|\nabla\rho\|^2_{L_\infty(0,T;L_2)}\big). \tag{5-5}$$

Taking $r = 2$, we deduce that if

$$C_2\nu^{-1}T^2 e^{C\|v_0\|_2^4}\|\nabla\rho\|^2_{L_\infty(0,T;L_2)} \le \log 2,$$

then we have

$$\sup_{t\in[0,T]} \|\nabla\rho(t)\|_2 \le 2K(\|\nabla\rho_0\|_q + 1)\exp(C_2 E_0^2 T e^{C\|v_0\|_2^4}). \tag{5-6}$$

Using an obvious connectivity argument, we conclude that (5-6) holds whenever

$$\nu > \frac{4K^2C_2}{\log 2}(\|\nabla\rho_0\|_q + 1)^2\exp(2C_2E_0^2Te^{C\|v_0\|_2^4})T^2e^{C\|v_0\|_2^4}. \tag{5-7}$$

Reverting to (5-4), we readily get, taking a larger constant $C$ if need be,

$$\|v\|^2_{L_\infty(0,T;H^1)} + \nu\|\operatorname{div} v\|^2_{L_\infty(0,T;L_2)} + \|\rho-\bar\rho\|^2_{L_\infty(0,T;L_2)} + \int_0^T \big(\|\nabla v\|^2_{H^1} + \|v_t\|^2_{L_2} + \nu\|\nabla\operatorname{div} v\|^2_{L_2}\big)\,dt \le Ce^{C\|v_0\|_2^4}E_0^2. \tag{5-8}$$

Of course, combining (5-6) with (5-5) ensures that for all $r \in [2, q)$, we have

$$\sup_{t\in[0,T]} \|\nabla\rho(t)\|_{L_r} \le K(\|\nabla\rho_0\|_q + 1)\ \exp(C_r E_0^2 T e^{C\|v_0\|_2^4}). \tag{5-9}$$

*Step 3: maximal regularity at $L_p$ level for the velocity.* We rewrite the velocity equation as

$$\rho\partial_t v - \Delta v - \nu\nabla\operatorname{div} v = -\nabla P - \rho v\cdot\nabla v.$$

Then Theorem 4.1 ensures that for all $p \in [2, q)$,

$$V_p(T) \le C_p\big(\|v_0\|_{W_p^{2-2/p}} + \|\nabla P + \rho v\cdot\nabla v\|_{L_p(0,T\times\mathbb{T}^2)}\big) \tag{5-10}$$

with $V_p(T) := \|v\|_{L_\infty(0,T;W_p^{2-2/p})} + \|v_t,\ \nabla^2 v,\ \nu\nabla\operatorname{div} v\|_{L_p(0,T\times\mathbb{T}^2)}$.

By the Hölder inequality

$$\|v\cdot\nabla v\|_{L_p(0,T\times\mathbb{T}^2)}\leq T^{1/s}\|v\|_{L_\infty(0,T;L_s)}\|\nabla v\|_{L_4(0,T;L_4)}\quad\text{with } \frac{1}{s}+\frac{1}{4}=\frac{1}{p}.$$

Hence using embedding and inequality (5-8),

$$\|v\cdot\nabla v\|_{L_p(0,T\times\mathbb{T}^2)}\leq CT^{1/p-1/4}E_0^2e^{C\|v_0\|_2^4},$$

and reverting to (5-10) and using (5-9) thus yields for some constant $C_P$ depending only on the pressure law,

$$V_p(T)\leq C_p\Big(\|v_0\|_{W_p^{2-2/p}}+C_PT^{1/p}(\|\nabla\rho_0\|_q+1)e^{CE_0^2Te^{C\|v_0\|_2^4}}\\ +T^{1/p-1/4}E_0^2e^{C\|v_0\|_2^4}\Big).\tag{5-11}$$

*Step 4: regularity estimate at $L_q$ level for the density.* The standard estimate for the transport equation with Lipschitz velocity field yields

$$\sup_{t\leq T}\|\nabla\rho(t)\|_q\leq(\|\nabla\rho_0\|_q+\rho^*A_q(T))\exp(\|\nabla v\|_{L_1(0,T;L_\infty)}).$$

Hence, remembering (5-3) and using the embedding $\dot W_p^1(\mathbb{T}^2)\hookrightarrow L_\infty(\mathbb{T}^2)$ to handle the last term, we get

$$\sup_{t\leq T}\|\nabla\rho(t)\|_q\leq(\|\nabla\rho_0\|_q+1)\exp(CT^{1/p'}V_p(T)).$$

Then one can bound $V_p(T)$ according to (5-11) and eventually get

$$\sup_{t\leq T}\|\nabla\rho(t)\|_q\leq(\|\nabla\rho_0\|_q+1)\exp(T^{1/p'}I_0^p(T)),\tag{5-12}$$

with $I_0^p(T):=C_p\big(\|v_0\|_{W_p^{2-2/p}}+C_PT^{1/p}(\|\nabla\rho_0\|_q+1)e^{CE_0^2Te^{C\|v_0\|_2^4}}\big)$.

*Step 5: maximal regularity at $L_q$ level for the velocity.* Let us use again Theorem 4.1, but with Lebesgue exponent $q$. We have

$$V_q(T)\leq C_q\big(\|v_0\|_{W_q^{2-2/q}}+\|\nabla P\|_{L_q(0,T\times\mathbb{T}^2)}+\|\rho v\cdot\nabla v\|_{L_q(0,T\times\mathbb{T}^2)}\big).\tag{5-13}$$

The last term may be bounded as in (5-11) (with $q$ instead of $p$), and the pressure term may be handled thanks to (5-12). In the end we get

$$V_q(T)\leq C_q\big(\|v_0\|_{W_q^{2-2/q}}+C_PT^{1/q}(\|\nabla\rho_0\|_{L_q}+1)\exp(T^{1/q'}I_0^q(T))\big).$$

*Step 6: final bootstrap.* In order to complete the proof, it suffices to check that if $\nu$ is large enough then we do have (5-3). This is just a consequence of the fact that

$$A_q(T)\leq T^{1/q'}\|\nabla\operatorname{div}v\|_{L_q(0,T\times\mathbb{T}^2)}\leq\frac{1}{\nu}T^{1/q'}V_q(T).$$

Hence it suffices to choose $\nu$ fulfilling (5-7) and

$$\nu\geq T^{1/q'}C_q\big(\|v_0\|_{W_q^{2-2/q}}+C_PT^{1/q}(\|\nabla\rho_0\|_{L_q}+1)\exp(T^{1/p'}I_0^q(T))\big).$$

## 6. The incompressible limit issue

The aim of this section is to prove Theorem 1.3. In what follows the time $T$ is fixed, and $\nu$ is larger than the threshold viscosity $\nu_0$ given by Theorem 1.2. Throughout, we shall agree that $C_{0,T}$ denotes a "constant" depending only on $T$ and on the norms of the initial data appearing in Theorem 1.2. Let us consider the corresponding solution $(\rho, v)$. Then inequality (1-11) already ensures that all the terms with $\mathcal{Q}v$ in (1-14) are bounded as required.

In order to bound the other terms of (1-14), it is convenient to restate system (1-1) in terms of the divergence-free part $\mathcal{P}v$ and potential part $\mathcal{Q}v$ of the velocity field $v$, and in terms of the discrepancy $r := \rho - \tilde{\rho}$ between $\rho$ and the "incompressible" density $\tilde{\rho}$ defined as the unique solution of the transport equation

$$\tilde{\rho}_t + \mathcal{P}v \cdot \nabla \tilde{\rho} = 0, \qquad \tilde{\rho}|_{t=0} = \rho_0. \tag{6-1}$$

As $r$ fulfills

$$r_t + \mathcal{P}v \cdot \nabla r = -\operatorname{div}(\rho \mathcal{Q}v), \qquad r|_{t=0} = 0, \tag{6-2}$$

we have for all $t \in [0, T]$,

$$\|r(t)\|_q \leq \int_0^t \big(\|\rho \operatorname{div} \mathcal{Q}v\|_q + \|\mathcal{Q}v \cdot \nabla\rho\|_q\big)\, d\tau. \tag{6-3}$$

Now, we have

$$\|\mathcal{Q}v \cdot \nabla\rho\|_{L_q(0,T\times\mathbb{T}^2)} \leq \|\mathcal{Q}v\|_{L_q(0,T;L_\infty)} \|\nabla\rho\|_{L_\infty(0,T;L_q)}$$

and, by virtue of the Poincaré inequality,

$$\|\rho \operatorname{div} \mathcal{Q}v\|_{L_q(0,T\times\mathbb{T}^2)} \leq C\rho^* \|\nabla \operatorname{div} \mathcal{Q}v\|_{L_q(0,T\times\mathbb{T}^2)}.$$

Therefore, taking advantage of Sobolev embedding and of inequality (1-12), we end up with

$$\sup_{0\leq t\leq T} \|r(t)\|_q \leq C_{0,T}\,\nu^{-1}. \tag{6-4}$$

Next, we restate the second equation in (1-1) as

$$\tilde{\rho}\mathcal{P}v_t + \tilde{\rho}\mathcal{P}v \cdot \nabla\mathcal{P}v - \Delta\mathcal{P}v + \nabla Q + K = 0, \tag{6-5}$$

where $Q := P - (1+\nu)\operatorname{div} v$ and $K = K_1 + K_2 + K_3 + K_4$, with

$$K_1 := r\mathcal{P}v_t, \quad K_2 := \rho\mathcal{Q}v_t, \quad K_3 := r\mathcal{P}v\cdot\nabla\mathcal{P}v, \quad K_4 := \rho(\mathcal{Q}v\cdot\nabla\mathcal{P}v + v\cdot\nabla\mathcal{Q}v).$$

Subtracting (1-4) from (6-5) yields

$$\eta(\mathcal{P}v - u)_t + \eta u \cdot \nabla(\mathcal{P}v - u) - \Delta(\mathcal{P}v - u) + \nabla(Q - \Pi) + K + L = 0 \tag{6-6}$$

with

$$L := (\tilde{\rho} - \eta)\mathcal{P}v_t + (\tilde{\rho} - \eta)\mathcal{P}v \cdot \nabla\mathcal{P}v + \eta(\mathcal{P}v - u)\cdot\nabla\mathcal{P}v.$$

Of course, initially, we have

$$\mathcal{P}v - u|_{t=0} = 0, \quad \tilde{\rho} - \eta|_{t=0} = 0.$$

Now, we take the $L^2$ scalar product of (6-6) with $\mathcal{P}v - u$ getting, since $\operatorname{div} u = 0$,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} \eta|\mathcal{P}v - u|^2\,dx + \int_{\mathbb{T}^2} |\nabla(\mathcal{P}v - u)|^2\,dx = \int_{\mathbb{T}^2} K\cdot(u - \mathcal{P}v)\,dx + \int_{\mathbb{T}^2} L\cdot(u - \mathcal{P}v)\,dx. \tag{6-7}$$

To analyze the terms of the right-hand side, we need some information coming from the continuity equations. The difference of $\tilde{\rho}$ and $\eta$ fulfills

$$(\tilde{\rho} - \eta)_t + u\cdot\nabla(\tilde{\rho} - \eta) = -(\mathcal{P}v - u)\cdot\nabla\tilde{\rho}.$$

Testing it by $(\tilde{\rho} - \eta)$ and defining $q^*$ by $1/q^* + 1/q = \frac{1}{2}$, we find that

$$\sup_{t\le T} \|(\tilde{\rho} - \eta)(t)\|_2 \le \int_0^T \|\mathcal{P}v - u\|_{q^*}\,\|\nabla\tilde{\rho}\|_q\,dt.$$

As $\tilde{\rho}$ satisfies (6-1), we have for all $t \in [0, T]$,

$$\|\nabla\tilde{\rho}(t)\|_q \le \|\nabla\tilde{\rho}_0\|_q\, e^{\int_0^t \|\nabla\mathcal{P}v\|_\infty\,d\tau}.$$

Therefore, thanks to (1-13) and Sobolev embedding,

$$\sup_{t\le T} \|(\tilde{\rho} - \eta)(t)\|_2 \le C_{0,T}\int_0^T \|\mathcal{P}v - u\|_{q^*}\,dt. \tag{6-8}$$

One can now estimate all the terms of the right-hand side of (6-7). Regarding the first term of $L$, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^2} (\tilde{\rho} - \eta)\mathcal{P}v_t\cdot(\mathcal{P}v - u)\,dx\,dt &\le \int_0^T \|\tilde{\rho} - \eta\|_2\,\|\mathcal{P}v_t\|_q\,\|\mathcal{P}v - u\|_{q^*}\,dt\\ &\le C_{0,T}\left(\int_0^T \|\mathcal{P}v - u\|_{q^*}\,dt\right)\left(\int_0^T \|\mathcal{P}v_t\|_q^2\,dt\right)^{1/2}\left(\int_0^T \|\mathcal{P}v - u\|_{q^*}^2\,dt\right)^{1/2}.\end{aligned}$$

Hence taking $\theta \in (0, 1)$ below according to the Gagliardo–Nirenberg inequality, and remembering that $q > 2$ and that $H^1(\mathbb{T}^2) \hookrightarrow L_m(\mathbb{T}^2)$ for all $m < \infty$, we get

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^2} (\tilde{\rho} - \eta)\mathcal{P}v_t\cdot(\mathcal{P}v - u)\,dx\,dt &\le C_{0,T}\int_0^T \|\nabla(\mathcal{P}v - u)\|_2^{2\theta}\,\|\mathcal{P}v - u\|_2^{2-2\theta}\,dt\\ &\le \frac{1}{8}\int_0^T \|\nabla(\mathcal{P}v - u)\|_2^2\,dt + C_{0,T}\int_0^T \|\mathcal{P}v - u\|_2^2\,dt.\end{aligned} \tag{6-9}$$

Next, we write

$$\left|\int_{\mathbb{T}^2}(\tilde{\rho}-\eta)(\mathcal{P}v\cdot\nabla\mathcal{P}v)\cdot(\mathcal{P}v-u)\,dx\right| \leq \|\tilde{\rho}-\eta\|_2\;\|\mathcal{P}v\cdot\nabla\mathcal{P}v\|_q\;\|\mathcal{P}v-u\|_{q^*};$$

hence, arguing exactly as above,

$$\left|\int_0^T\!\!\int_{\mathbb{T}^2}(\tilde{\rho}-\eta)(\mathcal{P}v\cdot\nabla\mathcal{P}v)\cdot(\mathcal{P}v-u)\,dx\,dt\right| \leq \frac{1}{8}\int_0^T\|\nabla(\mathcal{P}v-u)\|_2^2\,dt + C_{0,T}\int_0^T\|\mathcal{P}v-u\|_2^2\,dt.$$

Similarly, we have

$$\left|\int_0^T\!\!\int_{\mathbb{T}^2}\eta((\mathcal{P}v-u)\cdot\nabla\mathcal{P}v)\cdot(\mathcal{P}v-u)\,dx\,dt\right| \leq \rho^*\int_0^T\|\nabla\mathcal{P}v\|_\infty\;\|\mathcal{P}v-u\|_2^2\,dt.$$

Regarding $K_1$, we have, defining $\tilde{q}$ by $2/q+1/\tilde{q}=1$,

$$\begin{aligned}\left|\int_0^T\!\!\int_{\mathbb{T}^2}r\mathcal{P}v_t\cdot(\mathcal{P}v-u)\,dx\,dt\right| &\leq \int_0^T\|r\|_q\;\|\mathcal{P}v\|_q\;\|\mathcal{P}v-u\|_{\tilde{q}}\,dt\\ &\leq \frac{1}{8}\int_0^T\|\nabla(\mathcal{P}v-u)\|_2^2\,dt + C_{0,T}\int_0^T\|\mathcal{P}v-u\|_2^2\,dt,\end{aligned}$$

and for $K_2$, one can write that

$$\int_{\mathbb{T}^2}\rho\mathcal{Q}v_t\cdot(\mathcal{P}v-u)\,dx = \frac{d}{dt}\int_{\mathbb{T}^2}\rho\mathcal{Q}v\cdot(\mathcal{P}v-u)\,dx - \int_{\mathbb{T}^2}(\rho(\mathcal{P}v-u))_t\cdot\mathcal{Q}v\,dx.$$

For the last term, we have, using that $\rho_t=-\operatorname{div}(\rho v)$ and integrating by parts,

$$\begin{aligned}\int_{\mathbb{T}^2}(\rho(\mathcal{P}v-u))_t\cdot\mathcal{Q}v\,dx &= \int_{\mathbb{T}^2}\rho(\mathcal{P}v-u)_t\cdot\mathcal{Q}v\,dx + \int_{\mathbb{T}^2}\rho_t(\mathcal{P}v-u)\cdot\mathcal{Q}v\,dx\\ &= \int_{\mathbb{T}^2}\rho(\mathcal{P}v-u)_t\cdot\mathcal{Q}v\,dx + \int_{\mathbb{T}^2}(\rho v)\cdot(\nabla(\mathcal{P}v-u)\cdot\mathcal{Q}v)\,dx\\ &\qquad + \int_{\mathbb{T}^2}(\rho v)\cdot((\mathcal{P}v-u)\cdot\nabla\mathcal{Q}v)\,dx.\end{aligned}$$

The first term is of order $\nu^{-1}$ after time integration on $[0,T]$, since it may be bounded by

$$\left|\int_{\mathbb{T}^2}(\rho(\mathcal{P}v-u))_t\cdot\mathcal{Q}v\,dx\right| \leq \rho^*\|\mathcal{Q}v\|_2(\|\mathcal{P}v_t\|_2+\|u_t\|_2).$$

For the second term, one may write

$$\left|\int_{\mathbb{T}^2}(\rho v)\cdot(\nabla(\mathcal{P}v-u)\cdot\mathcal{Q}v)\,dx\right| \leq \frac{1}{8}\int_{\mathbb{T}^2}\|\nabla(\mathcal{P}v-u)\|_2^2\,dx + C(\rho^*)^2\|v\|_\infty^2\;\|\mathcal{Q}v\|_2^2,$$

and for the last one, we have

$$\left|\int_{\mathbb{T}^2} (\rho v)\cdot((\mathcal{P}v-u)\cdot\nabla\mathcal{Q}v)\,dx\right| \le \rho^*\|v\|_\infty\,\|\mathcal{P}v-u\|_2\,\|\nabla\mathcal{Q}v\|_2.$$

In the same way, we get

$$\left|\int_0^T\!\!\int_{\mathbb{T}^2}(K_3+K_4)\cdot(\mathcal{P}v-u)\,dx\,dt\right| \le \int_0^T \|\mathcal{P}v-u\|_{q^*}\big(\|\mathcal{Q}v\|_q\,\|\nabla\mathcal{P}v\|_2+\|\nabla\mathcal{Q}v\|_q\,\|v\|_2\big)\,dt,$$

whence using (1-12) and the Poincaré inequality to handle the terms with $\mathcal{Q}v$,

$$\left|\int_0^T\!\!\int_{\mathbb{T}^2}(K_3+K_4)\cdot(\mathcal{P}v-u)\,dx\,dt\right| \le \frac{1}{8}\int_0^T\|\mathcal{P}v-u\|_{H^1}^2\,dt+\nu^{-2}C_{0,T}.$$

Summing up, we return to (6-7) and integrate to find

$$\rho_*\sup_{t\le T}\|(\mathcal{P}v-u)(t)\|_2^2+\int_0^T\|\nabla(\mathcal{P}v-u)\|_2^2\,dt \le \sup_{t\le T}\left|\int_{\mathbb{T}^2}(\rho\mathcal{Q}v)(t)\cdot(\mathcal{P}v-u)(t)\,dx\right|+C_{0,T}\int_0^T\|\mathcal{P}v-u\|_2^2\,dt+C_{0,T}\nu^{-1}.$$

But we see that

$$\left|\int_{\mathbb{T}^2}\rho\mathcal{Q}v.(\mathcal{P}v-u)\,dx\right| \le \tfrac{1}{2}\rho_*\|\mathcal{P}v-u\|_2^2+C\|\mathcal{Q}v\|_2^2 \le \tfrac{1}{2}\rho_*\|\mathcal{P}v-u\|_2^2+C_{0,T}\nu^{-1}.$$

So altogether, we get after using the Gronwall lemma,

$$\sup_{t\le T}\big(\|(\mathcal{P}v-u)(t)\|_2^2+\|(\tilde\rho-\eta)(t)\|_2^2\big)+\int_0^T\|\nabla(\mathcal{P}v-u)\|_2^2\,dt \le C_{0,T}\nu^{-1}.$$

Recalling (6-4) and $\tilde\rho-\eta=r+(\tilde\rho-\eta)$ completes the proof of Theorem 1.3. □

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