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Saturated morphisms of logarithmic schemes

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The notion of universally saturated morphisms between saturated log schemes was introduced by Kazuya Kato. In this paper, we study universally saturated morphisms systematically by introducing the notion of saturated morphisms between integral log schemes as a relative analogue of saturated log structures. We eventually show that a morphism of saturated log schemes is universally saturated if and only if it is saturated. We prove some fundamental properties and characterizations of universally saturated morphisms via this interpretation.

Introduction

The notion of universally saturated morphisms between saturated log schemes was introduced by Kazuya Kato. The purpose of this paper is to study universally saturated morphisms systematically.

We define saturated morphisms not only for saturated log schemes but also for integral log schemes (Definitions I.3.5, I.3.7, I.3.12, II.2.10). They are stable under compositions and base changes (in the category of log schemes) (Proposition II.2.11). The first important property of saturated morphisms is the following (Propositions I.3.9, II.2.12):

> For a saturated morphism of integral log schemes $(X, M_X) \to (Y, M_Y)$, if M_Y is saturated, then M_X is also saturated.

This and the stability of saturated morphisms under base changes imply that saturated morphisms of saturated log schemes are universally saturated. In fact, we see that the converse is also true (Proposition II.2.13).

Our main results concerning saturated morphisms are the following:

This paper was written in 1997, and has been circulated among some experts since then. The author made very minor revisions to the original keeping the reference numbers of theorems, propositions, etc., unchanged because the original had already been cited in some published papers.
Some of the results of this paper will be absorbed into the book on foundation of logarithmic algebraic geometry which Arthur Ogus has been writing for years.
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(1) For a prime p and a morphism $f : (X, M_X) \to (Y, M_Y)$ of fine saturated log schemes over $\mathbb{F}_p$, f is of Cartier type if and only if f is saturated (Proposition II.2.14, Theorem II.3.1). (This is an unpublished result of K. Kato.)

(2) Let $f : (X, M_X) \to (Y, M_Y)$ be an integral morphism of fine saturated log schemes and assume that we are given a chart $Q_Y \to M_Y$ with Q saturated. We regard (Y, M_Y) as a log scheme over $(S, M_S) = (\mathrm{Spec}(\mathbb{Z}[Q]), \text{can. log})$ by the chart. If X is quasi-compact, then there exists a positive integer n such that the base change $f' : (X', M_{X'}) \to (Y', M_{Y'})$ of f in the category of fine saturated log schemes by the morphism $(S, M_S) \to (S, M_S)$ induced by the multiplication by n on Q, is saturated (Theorem II.3.4).

(3) For a smooth integral morphism $f : (X, M_X) \to (Y, M_Y)$ of fine saturated log schemes, f is saturated if and only if every fiber of the underlying morphism of schemes of f is reduced (Theorem II.4.2).

This paper consists of two chapters. The first chapter is devoted to the study of saturated morphisms of monoids. In the second chapter, we deduce some results on saturated morphisms of log schemes from the results in the first chapter. We use freely the terminology introduced in [Kato 1989].

I. Saturated morphisms of monoids

I.1. ***Prime ideals of monoids.*** Throughout this paper, a monoid means a commutative monoid with a unit element, and its monoid law is written multiplicatively except for the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, which is regarded as a monoid by its additive law. A homomorphism of monoids always preserves the unit elements. For a monoid P, P^{gp} denotes the group associated to P (cf. [Kato 1989, §1]), and P^* denotes the group of invertible elements of P.

Definition I.1.1 [Kato 1994, (5.1) Definition]. A subset I of a monoid P is called an *ideal* of P if $PI \subset I$. An ideal I of P is called a *prime ideal* if its complement $P \backslash I$ is a submonoid of P. We denote by $\mathrm{Spec}(P)$ the set of all prime ideals of P.

A morphism of monoids $h : P \to Q$ induces a map

$$\mathrm{Spec}(Q) \to \mathrm{Spec}(P), \quad \mathfrak{q} \mapsto h^{-1}(\mathfrak{q}).$$

For a submonoid S of a monoid P, we define the monoid $S^{-1}P$ by $S^{-1}P = \{s^{-1}a \mid a \in P, s \in S\}/\sim$, where $s_1^{-1}a_1 \sim s_2^{-1}a_2$ if and only if there exists $t \in S$ such that $ts_1a_2 = ts_2a_1$ [Kato 1994, (5.2) Definition]. If P is integral (see [Kato 1989, (2.2)]), the last condition is equivalent to $s_1a_2 = s_2a_1$ and $S^{-1}P$ is canonically isomorphic to the submonoid of P^{gp} consisting of elements of the forms $s^{-1}a$ $(s \in S, a \in P)$.

For a prime ideal $\mathfrak{p}$ of P, we define $P_{\mathfrak{p}}$ to be $(P\backslash\mathfrak{p})^{-1}P$. If P is generated by $a_1, \dots, a_n \in P$, then $P\backslash\mathfrak{p}$ is generated by a_i contained in $P\backslash\mathfrak{p}$, and therefore $P_{\mathfrak{p}}$ is generated by $1^{-1}a_1, \dots, 1^{-1}a_n$ and $a_i^{-1}1$ ($a_i \in P\backslash\mathfrak{p}$). The set $\mathrm{Spec}(P_{\mathfrak{p}})$ is identified with the subset $\{\mathfrak{q} \in \mathrm{Spec}(P) \mid \mathfrak{q} \subset \mathfrak{p}\}$ of $\mathrm{Spec}(P)$ by the map $\mathrm{Spec}(P_{\mathfrak{p}}) \to \mathrm{Spec}(P)$ induced by the morphism $P \to P_{\mathfrak{p}}$, $a \mapsto 1^{-1}a$. For $\mathfrak{r}, \mathfrak{r}' \in \mathrm{Spec}(P_{\mathfrak{p}})$, and the corresponding prime ideals $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Spec}(P)$, we have $\mathfrak{r} \subset \mathfrak{r}'$ if and only if $\mathfrak{q} \subset \mathfrak{q}'$.

Definition I.1.2. (1) [Kato 1994, (5.4) Definition]. For a monoid P, we define the *dimension* $\dim(P)$ of P to be the maximal length of a sequence of prime ideals $\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_r$ of P. If the maximum does not exist, we define $\dim(P) = \infty$.

(2) For a prime ideal $\mathfrak{p}$ of a monoid P, we define the *height* $\mathrm{ht}(\mathfrak{p})$ of $\mathfrak{p}$ to be the maximal length of a sequence of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_r$ of P. If the maximum does not exist, we define $\mathrm{ht}(\mathfrak{p}) = \infty$.

By the above identification of $\mathrm{Spec}(P_{\mathfrak{p}})$ with a subset of $\mathrm{Spec}(P)$, we have $\mathrm{ht}(\mathfrak{p}) = \dim(P_{\mathfrak{p}})$.

Proposition I.1.3 [Kato 1994, (5.5) Proposition]. *Let P be a finitely generated integral monoid. Then*:

(1) $\mathrm{Spec}(P)$ *is a finite set.*

(2) $\dim(P) = \mathrm{rank}_{\mathbb{Z}}(P^{\mathrm{gp}}/P^*)$.

(3) *For* $\mathfrak{p} \in \mathrm{Spec}(P)$, $\mathrm{ht}(\mathfrak{p}) + \dim(P\backslash\mathfrak{p}) = \dim(P)$.

Proposition I.1.4. *Let $f : P \to Q$ be a morphism of monoids and assume that there exists a positive integer n such that, for any $b \in Q$, $b^n \in f(P)$ and, for any $a_1, a_2 \in P$, $f(a_1) = f(a_2)$ implies $a_1^n = a_2^n$. Then, the morphism* $\mathrm{Spec}(Q) \to \mathrm{Spec}(P)$, $\mathfrak{q} \mapsto f^{-1}(\mathfrak{q})$ *is bijective and, for* $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{Spec}(Q)$, $\mathfrak{q}_1 \subset \mathfrak{q}_2$ *if and only if* $f^{-1}(\mathfrak{q}_1) \subset f^{-1}(\mathfrak{q}_2)$. *Especially* $\dim(P) = \dim(Q)$ *and* $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(f^{-1}(\mathfrak{q}))$ *for* $\mathfrak{q} \in \mathrm{Spec}(Q)$.

Proof. For an element $b \in Q$, there exists $a \in P$ such that $b^n = f(a)$ and a^n is independent of the choice of a. Hence, we can define a map g from Q to P by associating a^n to b. We see easily that the map g is a morphism of monoids and $g \circ f = n^2$ and $f \circ g = n^2$. Now the claim follows from the fact that the multiplication by n^2 on P and on Q induces the identity maps on $\mathrm{Spec}(P)$ and on $\mathrm{Spec}(Q)$. □

Let P be a finitely generated saturated monoid (see [Kato 1994, (1.1)]) and let $\mathfrak{p}$ be a prime ideal of P of height 1. Then, since $\dim(P_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p}) = 1$, we have $\mathrm{rank}_{\mathbb{Z}}(P_{\mathfrak{p}}^{\mathrm{gp}}/P_{\mathfrak{p}}^*) = 1$. Since P is saturated by assumption, $P_{\mathfrak{p}}$ and $P_{\mathfrak{p}}/P_{\mathfrak{p}}^*$ are saturated. Hence $P_{\mathfrak{p}}/P_{\mathfrak{p}}^* \cong \mathbb{N}$. By taking the associated abelian groups and using

$P_{\mathfrak{p}}^{\mathrm{gp}} \cong P^{\mathrm{gp}}$, we get an isomorphism $P^{\mathrm{gp}}/P_{\mathfrak{p}}^* \cong \mathbb{Z}$. We define the valuation $v_{\mathfrak{p}}$ associated to $\mathfrak{p}$ to be the homomorphism $P^{\mathrm{gp}} \to P^{\mathrm{gp}}/P_{\mathfrak{p}}^* \cong \mathbb{Z}$.

Lemma I.1.5. *Let P be a finitely generated saturated monoid and let $\mathfrak{p}$ be a prime ideal of P of height* 1. *Then, we have*

$$P_{\mathfrak{p}} = \{x \in P^{\mathrm{gp}} \mid v_{\mathfrak{p}}(x) \geq 0\}.$$

Proof. By definition, it is trivial that $v_{\mathfrak{p}}(x) \geq 0$ for $x \in P_{\mathfrak{p}}$. Conversely, if $v_{\mathfrak{p}}(x) \geq 0$ for $x \in P^{\mathrm{gp}}$, then there exist $y \in P_{\mathfrak{p}}$ and $z \in P_{\mathfrak{p}}^*$ such that $x = yz$. Hence $x \in P_{\mathfrak{p}}$. □

Proposition I.1.6 [Kato 1994, (5.8) Proposition (1)]. *Let P be a finitely generated saturated monoid. Then, we have $P = \bigcap_{\mathfrak{p}} P_{\mathfrak{p}}$, where $\mathfrak{p}$ ranges over all prime ideals of P of height* 1.

Lemma I.1.7. *Let $f : P \to Q$ be a morphism of finitely generated saturated monoids. Let $\mathfrak{q}$ be a prime ideal of Q of height* 1 *such that the prime ideal $\mathfrak{p} = f^{-1}(\mathfrak{q})$ of P is of height* 1. *Then, there exists a positive integer n such that $v_{\mathfrak{q}} \circ f^{\mathrm{gp}} = n v_{\mathfrak{p}}$. We call the integer n the* ramification index *of f at $\mathfrak{q}$.*

Proof. Since $f(P\backslash\mathfrak{p}) \subset Q\backslash\mathfrak{q}$, the morphism f induces a morphism $P_{\mathfrak{p}} \to Q_{\mathfrak{q}}$ and hence a morphism $P_{\mathfrak{p}}/P_{\mathfrak{p}}^* \to Q_{\mathfrak{q}}/Q_{\mathfrak{q}}^*$. Furthermore, if $f^{\mathrm{gp}}(s^{-1}a) \in Q_{\mathfrak{q}}^*$ for $s \in P\backslash\mathfrak{p}$ and $a \in P$, then $f(a) \in Q_{\mathfrak{q}}^*$. This implies $f(a) \in Q\backslash\mathfrak{q}$, that is, $a \in P\backslash\mathfrak{p}$. Hence $s^{-1}a \in P_{\mathfrak{p}}^*$. Now we see the lemma easily. □

I.2. *Integral morphisms.*

Proposition I.2.1 [Kato 1989, Proposition (4.1)(1)]. *Let $f : P \to Q$ be a morphism of integral monoids. Then in the following conditions*, (i) *and* (iv) *are equivalent, and* (ii), (iii), *and* (v) *are equivalent.*

(i) *For any integral monoid P' and for any morphism $g : P \to P'$, the pushout of $Q \leftarrow P \to P'$ in the category of monoids is integral.*

(ii) *The homomorphism $\mathbb{Z}[P] \to \mathbb{Z}[Q]$ induced by f is flat.*

(iii) *For any field k, the homomorphism $k[P] \to k[Q]$ induced by f is flat.*

(iv) *If $a_1, a_2 \in P$, $b_1, b_2 \in Q$ and $f(a_1)b_1 = f(a_2)b_2$, there exist $a_3, a_4 \in P$ and $b \in Q$ such that $b_1 = f(a_3)b$ and $a_1a_3 = a_2a_4$ (which implies $b_2 = f(a_4)b$).*

(v) *The condition* (iv) *is satisfied and f is injective.*

Definition I.2.2. We say a morphism $f : P \to Q$ of integral monoids is *integral* if it satisfies the equivalent conditions (i) and (iv) in Proposition I.2.1.

Using the condition (i), we can easily verify the following.

Proposition I.2.3. (1) *Let $f : P \to Q$ and $g : Q \to R$ be morphisms of integral monoids. If f and g are integral, then $g \circ f$ is integral.*

(2) *Let* $f : P \to Q$ *and* $g : P \to P'$ *be morphisms of integral monoids and let* Q' *be the pushout of* $Q \leftarrow P \to P'$ *in the category of monoids. If* f *is integral, then the canonical morphism* $P' \to Q'$ *is integral.*

Lemma I.2.4. *Let* P *be a monoid and let* G *be a subgroup of* P*. If* P *is integral, then* P/G *is integral.*

Proof. Straightforward. □

Proposition I.2.5. *Let* $f : P \to Q$ *be a morphism of integral monoids and let* G *and* H *be subgroups of* P *and* Q *respectively such that* $f(G) \subset H$*. Let* $g : P/G \to Q/H$ *be the morphism induced by* f*. Then,* f *is integral if and only if* g *is integral.*

Proof. Note first that P/G, Q/H and $Q/f(G)$ are integral by Lemma I.2.4. If f is integral, then the base change $P/G \to Q/f(G)$ is also integral. The morphism $Q/f(G) \to Q/H \cong (Q/f(G))/(H/f(G))$ is always integral by Lemma I.2.4 and the condition (i) of Proposition I.2.1 for integral morphisms. Hence g is integral. Conversely, suppose g is integral. Since $P \to P/G$ is always integral by the same reason as above, the composite h of $P \xrightarrow{f} Q$ with $Q \to Q/H$ is integral. We will prove that f satisfies the condition (iv) of Proposition I.2.1. Let $a_1, a_2 \in P$, $b_1, b_2 \in Q$ such that $f(a_1)b_1 = f(a_2)b_2$. Since h is integral, there exist $a_3, a_4 \in P$, $b \in Q$, and $c \in H$ such that $b_1 = f(a_3)bc$ and $a_1a_3 = a_2a_4$. This completes the proof. □

Lemma I.2.6. *Let* P *be a monoid and let* S *be a submonoid of* P*. If* P *is integral, then* $S^{-1}P$ *is integral.*

Proof. Straightforward. □

Proposition I.2.7. *Let* $f : P \to Q$ *be a morphism of integral monoids and let* S *and* T *be submonoids of* P *and* Q *respectively such that* $f(S) \subset T$*. If* f *is integral, then the morphism* $S^{-1}P \to T^{-1}Q$ *induced by* f *is integral.*

Proof. Note first that $S^{-1}P$, $T^{-1}Q$ and $f(S)^{-1}Q$ are integral by Lemma I.2.6. If f is integral, the base change $S^{-1}P \to f(S)^{-1}Q$ is integral. The morphism $f(S)^{-1}Q \to T^{-1}(f(S)^{-1}Q) \cong T^{-1}Q$ is integral by Lemma I.2.6 and the condition (i) of Proposition I.2.1 for integral morphisms. Hence the morphism $S^{-1}P \to T^{-1}Q$ is integral. □

Proposition I.2.8. *Let* $f : P \to Q$ *be an integral morphism of integral monoids such that* $f^{-1}(Q^*) = P^*$*. Then* f *is exact* (see Definition I.3.1). *Furthermore, if* $P^* = \{1\}$, f^{gp} *is injective.*

Proof. Take $a_1, a_2 \in P$ such that $f^{\mathrm{gp}}((a_1)^{-1}a_2) \in Q$. Then there exists $b_1 \in Q$ such that $f(a_1)b_1 = f(a_2)$ in Q. By the condition (iv) of Proposition I.2.1 for integral morphisms, there exist $a_3, a_4 \in P$ and $b \in Q$ such that $b_1 = f(a_3)b$, $1 = f(a_4)b$ and $a_1a_3 = a_2a_4$. Since $f^{-1}(Q^*) = P^*$, $a_4 \in P^*$ and hence $(a_1)^{-1}a_2 = (a_4)^{-1}a_3 \in P$.

The exactness of f implies that $\mathrm{Ker}(f^{\mathrm{gp}}) \subset P^*$. Hence, if $P^* = \{1\}$, f^{gp} is injective. □

Corollary I.2.9. *Let $f : P \to Q$ be an integral morphism of finitely generated integral monoids. Let $\mathfrak{q}$ be a prime ideal of Q and let $\mathfrak{p}$ be the prime ideal $f^{-1}(\mathfrak{q})$ of P. Then* $\mathrm{ht}(\mathfrak{p}) \le \mathrm{ht}(\mathfrak{q})$.

Proof. The morphism $g : P_{\mathfrak{p}}/P_{\mathfrak{p}}^* \to Q_{\mathfrak{q}}/Q_{\mathfrak{q}}^*$ induced by f is integral by Propositions I.2.5 and I.2.7. By Proposition I.2.8, g^{gp} is injective. Hence $\mathrm{ht}(\mathfrak{q}) = \dim(Q_{\mathfrak{q}}) = \mathrm{rank}_{\mathbb{Z}}(Q_{\mathfrak{q}}^{\mathrm{gp}}/Q_{\mathfrak{q}}^*) \ge \mathrm{rank}_{\mathbb{Z}}(P_{\mathfrak{p}}^{\mathrm{gp}}/P_{\mathfrak{p}}^*) = \dim(P_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p})$. □

Proposition I.2.10. *Let $f : P \to Q$ and $g : Q \to R$ be morphisms of integral monoids. If $g \circ f$ is integral and g is exact, then f is integral.*

Proof. We will prove that f satisfies the condition (iv) of Proposition I.2.1. Take $a_1, a_2 \in P$ and $b_1, b_2 \in Q$ such that $f(a_1)b_1 = f(a_2)b_2$. Then $(g \circ f)(a_1)g(b_1) = (g \circ f)(a_2)g(b_2)$ and, since $g \circ f$ is integral, there exist $a_3, a_4 \in P$ and $c \in R$ such that $g(b_1) = (g \circ f)(a_3)c$ and $a_1a_3 = a_2a_4$. Since g is exact, $b = b_1 f(a_3)^{-1}$ is contained in Q. This completes the proof. □

Proposition I.2.11. *Let $f : P \to Q$ be a morphism of finitely generated integral monoids. Then the following two conditions are equivalent*:

(i) *f is integral and $f^{-1}(Q^*) = P^*$.*

(ii) *f is exact and, for any $b \in Q$, there exists $b' \in Q$ such that*

$$f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b'.$$

Lemma I.2.12. *Let $f : P \to Q$ be an exact morphism of finitely generated integral monoids. Let $b \in Q$ and define a subset I of P^{gp} to be $\{a \in P^{\mathrm{gp}} \mid f^{\mathrm{gp}}(a)b \in Q\}$. Then, there exists $c \in P$ such that $cI \subset P$.*

Proof. First we will prove the lemma assuming P and Q are saturated. For a prime ideal $\mathfrak{q}$ of Q of height 1, we define $c_{\mathfrak{q}} \in P$ as follows. If $v_{\mathfrak{q}}(f(P)) = 0$, we define $c_{\mathfrak{q}} = 1$. If $v_{\mathfrak{q}}(f(P)) \ne 0$, we define $c_{\mathfrak{q}}$ to be an element of P such that $v_{\mathfrak{q}}(f(c_{\mathfrak{q}})) \ge v_{\mathfrak{q}}(b)$. Set $c = \prod_{\mathfrak{q}} c_{\mathfrak{q}}$, where $\mathfrak{q}$ ranges over all prime ideals of Q of height 1. We assert $cI \subset P$. Since f is exact, it suffices to prove $f^{\mathrm{gp}}(cI) \subset Q$. Let $a \in P^{\mathrm{gp}}$ such that $f^{\mathrm{gp}}(a)b \in Q$. Let $\mathfrak{q}$ be a prime ideal of Q of height 1. By Lemma I.1.5 and Proposition I.1.6, it is enough to prove $v_{\mathfrak{q}}(f^{\mathrm{gp}}(ac)) \ge 0$. If $v_{\mathfrak{q}}(f(P)) = 0$, then $v_{\mathfrak{q}}(f^{\mathrm{gp}}(P^{\mathrm{gp}})) = 0$ and hence $v_{\mathfrak{q}}(f^{\mathrm{gp}}(ac)) = 0$. If $v_{\mathfrak{q}}(f(P)) \ne 0$, then

$$\begin{aligned} v_{\mathfrak{q}}(f^{\mathrm{gp}}(ac)) = v_{\mathfrak{q}}(f^{\mathrm{gp}}(a)) + v_{\mathfrak{q}}(f(c)) &\ge v_{\mathfrak{q}}(f^{\mathrm{gp}}(a)) + v_{\mathfrak{q}}(f(c_{\mathfrak{q}})) \\ &\ge v_{\mathfrak{q}}(f^{\mathrm{gp}}(a)) + v_{\mathfrak{q}}(b) = v_{\mathfrak{q}}(f^{\mathrm{gp}}(a)b) \ge 0. \end{aligned}$$

Next we will reduce the general case to the case where P and Q are saturated. Let P^{sat} and Q^{sat} be the saturated monoids associated to P and Q (see

Definition II.2.2), which are finitely generated by Proposition II.2.4, and let $f^{\mathrm{sat}}: P^{\mathrm{sat}} \to Q^{\mathrm{sat}}$ be the morphism induced by f. Then the morphism f^{sat} is exact. Define a subset J of $(P^{\mathrm{sat}})^{\mathrm{gp}} = P^{\mathrm{gp}}$ to be $\{a \in (P^{\mathrm{sat}})^{\mathrm{gp}} \mid f^{\mathrm{gp}}(a)b \in Q^{\mathrm{sat}}\}$. Then we have proven that there exists $c \in P^{\mathrm{sat}}$ such that $cJ \subset P^{\mathrm{sat}}$. By multiplying c by some element of P, we may assume $c \in P$. Let $a_1, \dots, a_r \in P^{\mathrm{sat}}$ be a system of generators and choose a positive integer n such that $a_i^n \in P$ for all $1 \le i \le r$. Then

$$P^{\mathrm{sat}} = \bigcup\nolimits_{0 \le n_i \le n-1} P \cdot \prod\nolimits_{1 \le i \le r} a_i^{n_i}.$$

Choose $d \in P$ such that $da_i \in P$ for all $1 \le i \le r$. Then $d^{r(n-1)} P^{\mathrm{sat}} \subset P$. Hence $d^{r(n-1)} cI \subset d^{r(n-1)} cJ \subset d^{r(n-1)} P^{\mathrm{sat}} \subset P$. □

Proof of Proposition I.2.11. (i) ⇒ (ii). By Proposition I.2.8, f is exact. Let I be the set $\{a \in P^{\mathrm{gp}} \mid f^{\mathrm{gp}}(a)b \in Q\}$. We have $PI \subset I$. By Lemma I.2.12, there exists $c \in P$ such that $cI \subset P$. By [Kato 1994, (5.6) Lemma], there exist $a_1, \dots, a_r \in I$ such that $I = \bigcup_{1 \le i \le r} Pa_i$. Set $b_i = f^{\mathrm{gp}}(a_i)b \in f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q$. Then $f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f^{\mathrm{gp}}(I)b = \bigcup_{1 \le i \le r} f(P)b_i$. If $r = 1$, we are done. Suppose $r \ge 2$. Since $b_1 b_2^{-1} \in f^{\mathrm{gp}}(P^{\mathrm{gp}})$ and $b_1, b_2 \in Q$, by the condition (iv) of Proposition I.2.1 for integral morphisms, there exist $d_1, d_2 \in P$, $b_1' \in Q$ such that $b_1 = f(d_1)b_1'$ and $b_2 = f(d_2)b_1'$. Then $b_1' \in f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q$ and we have

$$f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q \supset f(P)b_1' \cup \big(\bigcup\nolimits_{3 \le i \le r} f(P)b_i\big) \supset \bigcup\nolimits_{1 \le i \le r} f(P)b_i = f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q.$$

Hence $f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b_1' \cup \big(\bigcup_{3 \le i \le r} f(P)b_i\big)$. Repeating this procedure, we are reduced to the case $r = 1$.

(ii) ⇒ (i). It is trivial that the exactness of f implies $f^{-1}(Q^*) = P^*$. We will prove that f satisfies the condition (iv) of Proposition I.2.1. Take $a_1, a_2 \in P$ and $b_1, b_2 \in Q$ such that $f(a_1)b_1 = f(a_2)b_2$. Then, by assumption, there exists $b \in Q$ such that $f^{\mathrm{gp}}(P^{\mathrm{gp}})b_1 \cap Q = f^{\mathrm{gp}}(P^{\mathrm{gp}})b_2 \cap Q = f(P)b$. Choose $a_3, a_4 \in P$ such that $b_1 = f(a_3)b$ and $b_2 = f(a_4)b$. Then, by $f(a_1)b_1 = f(a_2)b_2$, we have $f(a_1a_3) = f(a_2a_4)$. The element $a := (a_1a_3)^{-1}a_2a_4$ belongs to $\mathrm{Ker}(f^{\mathrm{gp}})$, which is contained in P since f is exact. By replacing a_3 by aa_3, we obtain the desired elements $a_1, a_3 \in P$ and $b \in Q$. □

I.3. *p-saturated monoids and p-saturated morphisms.*

Definition I.3.1 [Kato 1989, Definition (4.6)(1)]. We say a morphism of integral monoids $f: P \to Q$ is *exact* if $(f^{\mathrm{gp}})^{-1}(Q) = P$.

Proposition I.3.2. (1) *Let $f: P \to Q$ and $g: Q \to R$ be morphisms of integral monoids. If f and g are exact, then $g \circ f$ is exact. If $g \circ f$ is exact, then f is exact.*

(2) *Let $f: P \to Q$ and $g: P \to P'$ be morphisms of integral monoids and define a morphism of integral monoids $f': P' \to Q'$ by the following cocartesian*

diagram in the category of integral monoids.

$$\begin{array}{ccc} Q' & \xleftarrow{\ h\ } & Q \\ {\scriptstyle f'}\uparrow & & \uparrow{\scriptstyle f} \\ P' & \xleftarrow{\ g\ } & P \end{array}$$

If f is exact, then f' is exact.

Proof. The claim (1) is trivial. We prove (2). Let a be an element of $(P')^{\mathrm{gp}}$ such that $(f')^{\mathrm{gp}}(a) \in Q'$. Since the diagram of abelian groups

$$\begin{array}{ccc} (Q')^{\mathrm{gp}} & \xleftarrow{\ h^{\mathrm{gp}}\ } & Q^{\mathrm{gp}} \\ {\scriptstyle (f')^{\mathrm{gp}}}\uparrow & & \uparrow{\scriptstyle f^{\mathrm{gp}}} \\ (P')^{\mathrm{gp}} & \xleftarrow{\ g^{\mathrm{gp}}\ } & P^{\mathrm{gp}} \end{array}$$

is cocartesian and $Q' = (f')^{\mathrm{gp}}(P')h^{\mathrm{gp}}(Q)$ in $(Q')^{\mathrm{gp}}$, there exist $b \in P'$, $c \in Q$ and $d \in P^{\mathrm{gp}}$ such that $a = b \cdot g^{\mathrm{gp}}(d)$ and $c = f^{\mathrm{gp}}(d)$. Since f is exact by assumption, $d \in P$ and hence $a \in P'$. □

Definition I.3.3. Let p be a prime. We say an integral monoid P is *p-saturated* if the multiplication by p on P is exact (or equivalently, for any $a \in P^{\mathrm{gp}}$, $a^p \in P$ implies $a \in P$).

It is easy to see that an integral monoid P is saturated if and only if P is p-saturated for every prime p.

Example I.3.4. Let n be a positive integer and let P be the submonoid $(\mathbb{N} \times \mathbb{N}_{>0}) \cup n\mathbb{N} \times \{0\}$ of $\mathbb{N} \oplus \mathbb{N}$, which is generated by $(n, 0)$ and $(m, 1)$ ($m \in \mathbb{N}, 0 \le m \le n-1$). Then, for a prime p, P is p-saturated if and only if $p \nmid n$.

Definition I.3.5. Let p be a prime and let $f : P \to Q$ be a morphism of integral monoids. Define Q', f' and g by the following cocartesian diagram in the category of integral monoids:

$$\begin{array}{ccc} Q' & \xleftarrow{\ g\ } & Q \\ {\scriptstyle f'}\uparrow & & \uparrow{\scriptstyle f} \\ P & \xleftarrow{\ p\ } & P \end{array}$$

Let h be the unique morphism $Q' \to Q$ such that $h \circ g = p$ and $h \circ f' = f$. We say the morphism f is *p-quasi-saturated* if h is exact.

Proposition I.3.6. *Let p be a prime.*

(1) *Let $f : P \to Q$ and $g : Q \to R$ be morphisms of integral monoids. If f and g are p-quasi-saturated, then $g \circ f$ is p-quasi-saturated.*

(2) *Let $f : P \to Q$ and $g : P \to R$ be morphisms of integral monoids and define a morphism of integral monoids $h : R \to S$ by the following cocartesian diagram in the category of integral monoids*:

$$\begin{array}{ccc} S & \xleftarrow{\ i\ } & Q \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle f} \\ R & \xleftarrow{\ g\ } & P \end{array}$$

If f is p-quasi-saturated, *then h is p-quasi-saturated.*

Proof. (1) We have the following commutative diagram of integral monoids in which the composite of each line is the multiplication by p, $j \circ g' = g$, $h \circ f' = f$ and each square is cocartesian in the category of integral monoids:

$$\begin{array}{ccccccc} R & \xleftarrow{\ j\ } & R' & \xleftarrow{\ i\ } & R'' & \longleftarrow & R \\ & & {\scriptstyle g'}\uparrow & & \uparrow & & \uparrow{\scriptstyle g} \\ & & Q & \xleftarrow{\ h\ } & Q' & \longleftarrow & Q \\ & & & & {\scriptstyle f'}\uparrow & & \uparrow{\scriptstyle f} \\ & & & & P & \xleftarrow{\ p\ } & P \end{array}$$

If f and g are p-quasi-saturated, h and j are exact. By Proposition I.3.2 (2), i is exact. Hence $j \circ i$ is exact and $g \circ f$ is p-quasi-saturated.

(2) Define morphisms $f' : P \to Q'$ and $j : Q' \to Q$ (resp. $h' : R \to S'$ and $k : S' \to S$) using $f : P \to Q$ (resp. $h : R \to S$) as in Definition I.3.5. Let $i' : Q' \to S'$ be the morphism induced by $g : P \to R$ and $i : Q \to S$. Then, we have a commutative diagram

$$\begin{array}{ccc} S & \xleftarrow{\ i\ } & Q \\ {\scriptstyle k}\uparrow & & \uparrow{\scriptstyle j} \\ S' & \xleftarrow{\ i'\ } & Q' \\ {\scriptstyle h'}\uparrow & & \uparrow{\scriptstyle f'} \\ R & \xleftarrow{\ g\ } & P. \end{array}$$

The outer big square and the lower square are cocartesian in the category of integral monoids. (For the first one, note $k \circ h' = h$ and $j \circ f' = f$.) Hence the upper square is also cocartesian. Therefore, by Proposition I.3.2 (2), if j is exact, then k is exact. □

Definition I.3.7. We say a morphism of integral monoids $f : P \to Q$ is *quasi-saturated* if it is p-quasi-saturated for every prime p.

Proposition I.3.8. *Let n be an integer ≥ 2. Let $f : P \to Q$ be a quasi-saturated morphism of integral monoids and define an integral monoid Q' by the cocartesian diagram in the category of integral monoids*

$$\begin{array}{ccc} Q' & \xleftarrow{\ g\ } & Q \\ {\scriptstyle f'}\uparrow & & {\scriptstyle f}\uparrow \\ P & \xleftarrow{\ n\ } & P. \end{array}$$

Let h be the unique morphism $Q' \to Q$ such that $h \circ g = n$ and $h \circ f' = f$. Then h is exact.

Proof. It suffices to prove that, if the proposition is true for integers $n_1 \geq 2$ and $n_2 \geq 2$, then it is true also for $n_3 = n_1 n_2$. Consider the following cocartesian diagrams in the category of integral monoids:

$$\begin{array}{ccccc} Q_2 & \xleftarrow{\ g_2\ } & Q_1 & \xleftarrow{\ g_1\ } & Q \\ {\scriptstyle f_2}\uparrow & & {\scriptstyle f_1}\uparrow & & {\scriptstyle f}\uparrow \\ P & \xleftarrow{\ n_2\ } & P & \xleftarrow{\ n_1\ } & P \end{array}$$

Let $h_1 : Q_1 \to Q$ (resp. $h_2 : Q_2 \to Q_1$) be the unique morphism such that $h_1 \circ f_1 = f$ and $h_1 \circ g_1 = n_1$ (resp. $h_2 \circ f_2 = f_1$ and $h_2 \circ g_2 = n_2$). Then $h_3 := h_1 \circ h_2 : Q_2 \to Q$ is the unique morphism such that $h_3 \circ f_2 = f$ and $h_3 \circ g_2 \circ g_1 = n_1 n_2$. Since f is quasi-saturated, h_1 is exact by assumption. Since f_1 is quasi-saturated by Proposition I.3.6 (2), h_2 is also exact by assumption. Hence h_3 is exact. □

Proposition I.3.9. *Let p be a prime and let $f : P \to Q$ be a morphism of integral monoids. If P is p-saturated* (*resp. saturated*) *and f is p-quasi-saturated* (*resp. quasi-saturated*), *then Q is p-saturated* (*resp. saturated*).

Proof. Define an integral monoid Q' and morphisms of monoids $f' : P \to Q'$, $g : Q \to Q'$ and $h : Q' \to Q$ as in Definition I.3.5 using p and $f : P \to Q$. If P is p-saturated, g is exact by Proposition I.3.2 (2). If f is p-quasi-saturated, h is exact. Hence $h \circ g = p : Q \to Q$ is exact, that is, Q is p-saturated. By considering all p, we obtain the claim in the case when P is saturated. □

Proposition I.3.10. *Let p be a prime and let $f : P \to Q$ be a morphism of p-saturated monoids. Then*, *the following three conditions are equivalent*:

(1) *The morphism f is p-quasi-saturated.*

(2) *For any p-saturated monoid P' and any morphism $g : P \to P'$, the pushout of the diagram $Q \xleftarrow{f} P \xrightarrow{g} P'$ in the category of integral monoids, is p-saturated.*

(3) *The pushout of the diagram $Q \xleftarrow{f} P \xrightarrow{p} P$ in the category of integral monoids, is p-saturated.*

Proof. The implication (1) ⇒ (2) follows from Propositions I.3.6 (2) and I.3.9 and (2) ⇒ (3) is trivial. We prove (3) ⇒ (1). Define an integral monoid Q' and a morphism of integral monoids $g : Q \to Q'$ and $h : Q' \to Q$ as in Definition I.3.5. Then, we see that $g \circ h : Q' \to Q'$ is the multiplication by p and hence it is exact by assumption. By Proposition I.3.2 (1), h is exact. □

Corollary I.3.11. *Let $f : P \to Q$ be a morphism of saturated monoids. Then, the following three conditions are equivalent:*

(1) *The morphism f is quasi-saturated.*

(2) *For any saturated monoid P' and any morphism $g : P \to P'$, the pushout of the diagram $Q \xleftarrow{f} P \xrightarrow{g} P'$ in the category of integral monoids, is saturated.*

(3) *For every prime p, the pushout of the diagram $Q \xleftarrow{f} P \xrightarrow{p} P$ in the category of integral monoids, is saturated.*

Definition I.3.12. Let p be a prime. We say a morphism of integral monoids $f : P \to Q$ is *p-saturated* (resp. *saturated*) if f is integral and p-quasi-saturated (resp. quasi-saturated).

Proposition I.3.13. *Let p be a prime and let $f : P \to Q$ be an integral morphism of p-saturated monoids. Then, the following three conditions are equivalent:*

(1) *The morphism f is p-saturated.*

(2) *For any p-saturated monoid P' and any morphism $g : P \to P'$, the pushout of the diagram $Q \xleftarrow{f} P \xrightarrow{g} P'$ in the category of monoids, is p-saturated.*

(3) *The pushout of the diagram $Q \xleftarrow{f} P \xrightarrow{p} P$ in the category of monoids, is p-saturated.*

Proposition I.3.14. *Let $f : P \to Q$ be an integral morphism of saturated monoids. Then, the following three conditions are equivalent:*

(1) *The morphism f is saturated.*

(2) *For any saturated monoid P' and any morphism $g : P \to P'$, the pushout of the diagram $Q \xleftarrow{f} P \xrightarrow{g} P'$ in the category of monoids, is saturated.*

(3) *For every prime p, the pushout of the diagram $Q \xleftarrow{f} P \xrightarrow{p} P$ in the category of monoids, is saturated.*

Lemma I.3.15. *Let P be an integral monoid and let G be a subgroup of P.*

(1) *The monoid P is saturated if and only if P/G is saturated.*

(2) *The morphism $P \to P/G$ is saturated.*

Proof. (1) Straightforward.

(2) The morphism $P \to P/G$ is integral by Proposition I.2.5. For any prime p, the base change of $P \to P/G$ by $p : P \to P$ in the category of monoids is given by the quotient $P \to P/G^p$. Hence $P \to P/G$ is p-quasi-saturated because the projection map $P/G^p \to P/G$ is exact. □

Proposition I.3.16. *Let $f : P \to Q$ be a morphism of integral monoids and let G and H be subgroups of P and Q respectively such that $f(G) \subset H$. Let $g : P/G \to Q/H$ be the morphism induced by f. Let p be a prime. Then f is p-saturated if and only if g is p-saturated. In particular, f is saturated if and only if g is saturated.*

Proof. By Proposition I.2.5, we may assume that f and g are integral. If f is p-saturated, the base change $P/G \to Q/f(G)$ of f by $P \to P/G$ in the category of monoids is p-saturated by Propositions I.2.3 (2) and I.3.6 (2). The morphism $Q/f(G) \to (Q/f(G))/(H/f(G)) \cong Q/H$ is p-saturated by Lemma I.3.15 (2). Hence $P/G \to Q/H$ is p-saturated by Propositions I.2.3 (1) and I.3.6 (1). Conversely, suppose that $P/G \to Q/H$ is p-saturated. Since $P \to P/G$ is p-saturated, $P \to Q/H$ is p-saturated. Put $\bar{Q} := Q/H$. Define a monoid Q' (resp. $\bar{Q}'$) and morphisms of monoids $k : Q \to Q'$ and $h : Q' \to Q$ (resp. $\bar{k} : \bar{Q} \to \bar{Q}'$ and $\bar{h} : \bar{Q}' \to \bar{Q}$) as in Definition I.3.5 using p and $f : P \to Q$ (resp. $P \xrightarrow{f} Q \to \bar{Q}$). Then the natural map $Q' \to \bar{Q}'$ is the quotient by $k(H)$. Therefore the morphisms $Q' \to \bar{Q}'$ and $Q \to \bar{Q}$ are exact. Since $\bar{h}$ is exact, we see that h is also exact by using Proposition I.3.2 (1). □

Lemma I.3.17. *Let P be an integral monoid and let S be a submonoid of P.*

(1) *If P is saturated, then $S^{-1}P$ is saturated.*

(2) *The morphism $P \to S^{-1}P$ is saturated.*

Proof. (1) Straightforward.

(2) The morphism $P \to S^{-1}P$ is integral by Proposition I.2.7. For a prime p, the following diagram is cocartesian in the category of monoids:

$$\begin{array}{ccc} S^{-1}P & \xleftarrow{p} & S^{-1}P \\ \uparrow & & \uparrow \\ P & \xleftarrow{p} & P \end{array}$$

Hence $P \to S^{-1}P$ is p-quasi-saturated. □

Proposition I.3.18. *Let $f : P \to Q$ be a morphism of integral monoids and let S and T be submonoids of P and Q respectively such that $f(S) \subset T$. Let p be a prime. If f is p-saturated (resp. saturated), then the morphism $S^{-1}P \to T^{-1}Q$ induced by f is p-saturated (resp. saturated).*

Proof. If f is p-saturated, then the base change $S^{-1}P \to f(S)^{-1}Q$ of f by $P \to S^{-1}P$ in the category of monoids is p-saturated by Propositions I.2.3 (2) and I.3.6 (2). The morphism $f(S)^{-1}Q \to T^{-1}(f(S)^{-1}Q) \cong T^{-1}Q$ is p-saturated by Lemma I.3.17 (2). Hence, the morphism $S^{-1}P \to T^{-1}Q$ is p-saturated by Propositions I.2.3 (1) and I.3.6 (1). □

Remark I.3.19. For an integral monoid P and a submonoid S of P, the natural morphism $P \to S^{-1}P$ induces an isomorphism $P/S \xrightarrow{\cong} S^{-1}P/S^{\mathrm{gp}}$. Therefore, Propositions I.2.5, I.2.7, I.3.16, and I.3.18 immediately imply the following claim: Let $f : P \to Q$ be a morphism of integral monoids, and let S and T be submonoids of P and Q respectively such that $f(S) \subset T$. Let p be a prime. If f is integral (resp. p-saturated, resp. saturated), then so is the morphism $P/S \to Q/T$ induced by f.

I.4. ***A criterion of p-saturated morphisms.*** In this section, we give a criterion for an integral morphism of finitely generated integral monoids to be p-saturated (Theorem I.4.2). As corollaries, we prove that, under certain conditions, p-saturated morphisms are always saturated (Corollaries I.4.5 and I.4.7).

Proposition I.4.1. *Let p be a prime and let $f : P \to Q$ be a morphism of integral monoids. We consider the following condition on f.*

- ($*$) *For any $a \in P$ and $b \in Q$ such that $f(a) \mid b^p$, there exists $c \in P$ such that $a \mid c^p$ and $f(c) \mid b$.*
- (1) *If P is p-saturated and f is p-quasi-saturated, then f satisfies ($*$).*
- (2) *If Q is p-saturated and f satisfies ($*$), then f is p-quasi-saturated.*

In particular, if P and Q are p-saturated, f is p-quasi-saturated if and only if f satisfies ($*$).

Proof. Consider the following cocartesian diagram in the category of integral monoids:

$$\begin{array}{ccc} Q' & \xleftarrow{\;g\;} & Q \\ {\scriptstyle f'}\uparrow & & \uparrow{\scriptstyle f} \\ P & \xleftarrow{\;p\;} & P \end{array}$$

Let $h : Q' \to Q$ be the unique morphism such that $h \circ g = p$ and $h \circ f' = f$. Then $(Q')^{\mathrm{gp}}$ is canonically identified with

$$(P^{\mathrm{gp}} \oplus Q^{\mathrm{gp}})/\{(a^p, f^{\mathrm{gp}}(a)^{-1}) \mid a \in P^{\mathrm{gp}}\}$$

and Q' corresponds to the image of $P \oplus Q$. For $(a, b) \in P^{\mathrm{gp}} \oplus Q^{\mathrm{gp}}$, we denote by $\overline{(a, b)}$ its image in $(Q')^{\mathrm{gp}}$. We have $h^{\mathrm{gp}}(\overline{(a, b)}) = f^{\mathrm{gp}}(a)b^p$.

(1) Suppose that P is p-saturated and that f is p-quasi-saturated, that is, h is exact. Let $a \in P$ and $b \in Q$ such that $f(a) \mid b^p$. Then $h^{\mathrm{gp}}(\overline{(a^{-1}, b)}) = f(a)^{-1}b^p \in Q$. Hence $\overline{(a^{-1}, b)} \in Q'$, that is, there exists $c \in P^{\mathrm{gp}}$ such that $a^{-1}c^p \in P$ and $bf^{\mathrm{gp}}(c)^{-1} \in Q$. Since P is p-saturated, $a^{-1}c^p \in P$ implies $c \in P$. Now we have $a \mid c^p$ and $f(c) \mid b$.

(2) Suppose that Q is p-saturated and that f satisfies $(*)$. Let $a \in P^{\mathrm{gp}}$ and $b \in Q^{\mathrm{gp}}$ be elements satisfying $h^{\mathrm{gp}}(\overline{(a, b)}) = f^{\mathrm{gp}}(a)b^p \in Q$. If a is of the form $a_2a_1^{-1}$ $(a_1, a_2 \in P)$, then $\overline{(a, b)} = \overline{((a_1a_2^{p-1})^{-1}, bf(a_2))}$. Hence we may assume that a is of the form a_2^{-1} $(a_2 \in P)$. Then, since Q is p-saturated, $f^{\mathrm{gp}}(a)b^p \in Q$ implies $b \in Q$ and $f(a_2) \mid b^p$. Hence, by $(*)$, there exists $c \in P$ such that $a_2 \mid c^p$ and $f(c) \mid b$, which implies $\overline{(a, b)} = \overline{((a_2)^{-1}c^p, f(c)^{-1}b)} \in Q'$. □

Theorem I.4.2. *Let p be a prime. Let $f : P \to Q$ be an integral morphism of finitely generated integral monoids such that $f^{-1}(Q^*) = P^*$. Then f is p-saturated if and only if f satisfies the following two conditions*:

(i) *For $b \in Q^{\mathrm{gp}}$, if there exists $a \in P$ such that $f(a)b^p \in Q$, then there exists $a' \in P$ such that $f(a')b \in Q$.*

(ii) *For $b \in Q$, if there exists $a \in P\backslash P^*$ such that $f(a) \mid b^p$, then there exists $a' \in P\backslash P^*$ such that $f(a') \mid b$.*

Lemma I.4.3. *Let $f : P \to Q$ be an integral morphism of finitely generated integral monoids such that $f^{-1}(Q^*) = P^*$. Then, for any $b \in Q$ and $b' \in f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q$, $f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b'$ if and only if $f(a) \nmid b'$ for all $a \in P\backslash P^*$.*

Proof. By Proposition I.2.11, there exists $b'' \in Q$ such that $f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b''$. Take $c \in P$ such that $b' = f(c)b''$. If $f(a) \nmid b'$ for all $a \in P\backslash P^*$, then $c \in P^*$ and hence $f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b'$. Conversely, assume $f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b'$. Then, for any $a \in P$ such that $f(a) \mid b'$, there exists $a' \in P$ such that $f(a)^{-1}b' = f(a')b'$ because $f(a)^{-1}b' \in f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b'$. Hence $f(a) \in Q^*$ and $a \in P^*$. □

Proof of Theorem I.4.2. We use the same notation as in the first paragraph of the proof of Proposition I.4.1.

First assume that f is p-saturated, that is, the morphism $h : Q' \to Q$ is exact. Take $b \in Q^{\mathrm{gp}}$ and $a \in P$ such that $f(a)b^p \in Q$. Then $h^{\mathrm{gp}}(\overline{(a, b)}) = f(a)b^p \in Q$. Since h is exact, $\overline{(a, b)} \in Q'$, that is, there exists $c \in P^{\mathrm{gp}}$ such that $ac^{-p} \in P$ and $bf^{\mathrm{gp}}(c) \in Q$. Hence f satisfies the condition (i). Take $b \in Q$ such that $f(a) \nmid b$ for all $a \in P\backslash P^*$. To prove that f satisfies the condition (ii), it suffices to prove that $f(a) \nmid b^p$ for all $a \in P\backslash P^*$. By Lemma I.4.3, $f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b$ and it is enough to prove $f^{\mathrm{gp}}(P^{\mathrm{gp}})b^p \cap Q = f(P)b^p$. Let $a \in P^{\mathrm{gp}}$ and suppose $f^{\mathrm{gp}}(a)b^p \in Q$.

Then $h^{\mathrm{gp}}(\overline{(a,b)}) = f^{\mathrm{gp}}(a)b^p \in Q$. Since h is exact, $\overline{(a,b)} \in Q'$, that is, there exists $c \in P^{\mathrm{gp}}$ such that $ac^{-p} \in P$ and $bf^{\mathrm{gp}}(c) \in Q$. By $f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b$, there exists $d \in P$ such that $f^{\mathrm{gp}}(c)b = f(d)b$, i.e., $f^{\mathrm{gp}}(c) = f(d)\,(\in Q)$. Since f is exact by Proposition I.2.8, $c \in P$ and hence $a \in c^p P \subset P$.

Next assume that f satisfies the conditions (i) and (ii). It suffices to prove that h is exact. Let $a \in P^{\mathrm{gp}}$, $b \in Q^{\mathrm{gp}}$ and suppose $h^{\mathrm{gp}}(\overline{(a,b)}) = f^{\mathrm{gp}}(a)b^p \in Q$. We will prove $\overline{(a,b)} \in Q'$. By the condition (i), there exists $c \in P$ such that $f(c)b \in Q$, and we have $\overline{(a,b)} = \overline{(ac^{-p}, f(c)b)}$. Hence we may assume $b \in Q$. By Proposition I.2.11, there exists $b' \in Q$ such that $f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b'$. Take $c \in P$ such that $b = f(c)b'$. Since $\overline{(a,b)} = \overline{(ac^p, b')}$, it is enough to prove $ac^p \in P$. By the condition (ii) and Lemma I.4.3, $f^{\mathrm{gp}}(P^{\mathrm{gp}})(b')^p \cap Q = f(P)(b')^p$. Since $f^{\mathrm{gp}}(ac^p)(b')^p = f^{\mathrm{gp}}(a)b^p \in Q$, we obtain $f^{\mathrm{gp}}(ac^p) \in f(P) \subset Q$. Since f is exact by Proposition I.2.8, $ac^p \in P$. □

Remark I.4.4. Let $f : P \to Q$ be an integral morphism of finitely generated integral monoids such that $f^{-1}(Q^*) = P^*$. Then, using Lemma I.4.3, we see easily that the condition (i) (resp. (ii)) in Theorem I.4.2 is equivalent to the condition (i′) (resp. (ii′)) below:

(i′) The image of Q in $Q^{\mathrm{gp}}/f^{\mathrm{gp}}(P^{\mathrm{gp}})$ is p-saturated.

(ii′) For $b \in Q$, if $f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b$, then $f^{\mathrm{gp}}(P^{\mathrm{gp}})b^p \cap Q = f(P)b^p$.

Corollary I.4.5. *Let p and q be two different primes and let $f : P \to Q$ be a morphism of finitely generated integral monoids. If Q is q-saturated and f is p-saturated, then f is q-saturated.* (*Thus, if Q is saturated and f is p-saturated, then f is saturated.*)

Proof. Let S be the submonoid $f^{-1}(Q^*)$ of P, which is the complement of the prime ideal $f^{-1}(Q \backslash Q^*)$ of P. Then the morphism f uniquely factors as $P \xrightarrow{g} S^{-1}P \xrightarrow{h} Q$. The monoid $S^{-1}P$ is finitely generated and integral. The morphism g is q-saturated by Lemma I.3.17 (2). Since $h : S^{-1}P \to Q$ is the base change of $f : P \to Q$ by $g : P \to S^{-1}P$, h is p-saturated. Thus, we are reduced to the case $f^{-1}(Q^*) = P^*$. By Theorem I.4.2, it suffices to prove that f satisfies the conditions (i) and (ii) in Theorem I.4.2 for the prime q. The condition (i) follows from the fact that Q is q-saturated. Indeed, if $f(a)b^q \in Q$ for $a \in P$ and $b \in Q^{\mathrm{gp}}$, then $(f(a)b)^q \in Q$ and hence $f(a)b \in Q$. Let $b \in Q$ and suppose that there exists $a \in P \backslash P^*$ such that $f(a) \mid b^q$. Choose a positive integer m such that $q \leq p^m$. Then, $f(a) \mid b^{p^m}$ and, by Theorem I.4.2, there exists $a' \in P \backslash P^*$ such that $f(a') \mid b$. □

Definition I.4.6. Let $f : P \to Q$ be a morphism of monoids. We say the morphism f is *vertical* if, for any $b \in Q$, there exists $a \in P$ such that $b \mid f(a)$, that is, $f(a) \in bQ$.

Corollary I.4.7. *Let p be a prime and let $f : P \to Q$ be a morphism of finitely generated integral monoids. If f is vertical and p-saturated, then f is saturated.*

Proof. Let q be any prime different from p. We prove that f is q-saturated. By the same argument as in the proof of Corollary I.4.5, we may assume $f^{-1}(Q^*) = P^*$. Then, it suffices to prove that f satisfies the conditions (i) and (ii) in Theorem I.4.2 for the prime q. The assumption that f is vertical implies that, for any $b \in Q^{\mathrm{gp}}$, there exists $a \in P$ such that $f(a)b \in Q$. Hence f satisfies the condition (i). We can prove that f satisfies the condition (ii) exactly in the same way as in the proof of Corollary I.4.5. □

Remark I.4.8. (1) If P and Q are not finitely generated, Corollaries I.4.5 and I.4.7 are not true. We have the following counterexample. Let p be a prime and set $P = \{np^{-m} \mid n \in \mathbb{N}, m \in \mathbb{N}\} \subset \mathbb{Q}$. Let n be an integer≥ 2 prime to p and let $f : P \to P$ be the morphism defined by the multiplication by n. It is easy to see that P is saturated and the morphism f is integral and vertical. However, for a prime q, f is q-saturated if and only if q is prime to n. We prove it. If q is prime to n, then the following diagram is cocartesian in the category of monoids:

$$\begin{array}{ccc} P & \xleftarrow{\ q\ } & P \\ {\scriptstyle f}\uparrow & & {\scriptstyle f}\uparrow \\ P & \xleftarrow{\ q\ } & P \end{array}$$

Indeed, it is easy to see that this becomes cocartesian after taking the associated groups. On the other hand, we have $f(P)P^q = P$ because, for a sufficiently large integer m, there exist positive integers r and s such that $rn + sq = p^m$. Hence, by definition, f is q-saturated. If n is divisible by q, set

$$m = nq^{-1}, \quad G = P^{\mathrm{gp}}/(P^{\mathrm{gp}})^q \ (\cong \mathbb{Z}/q\mathbb{Z})$$

and define morphisms of monoids $g, h : P^{\mathrm{gp}} \to G \oplus P^{\mathrm{gp}}$ by

$$g(a) = (a \bmod (P^{\mathrm{gp}})^q, a^m), \quad h(a) = (0, a).$$

Then the following diagram is cocartesian:

$$\begin{array}{ccc} G \oplus P^{\mathrm{gp}} & \xleftarrow{\ h\ } & P^{\mathrm{gp}} \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f^{\mathrm{gp}}} \\ P^{\mathrm{gp}} & \xleftarrow{\ q\ } & P^{\mathrm{gp}} \end{array}$$

The pushout of the diagram $P \xleftarrow{f} P \xrightarrow{q} P$ is $g(P)h(P)$. On the other hand, we see easily that $g(P)h(P) \cap G = \{1\}$ and $G^q = \{1\}$. Hence $g(P)h(P)$ is not q-saturated. By Proposition I.3.13, f is not q-saturated.

(2) If f is not vertical, Corollary I.4.7 is not true. Indeed, for two integral monoids P and Q and a prime p, the morphism $P \to P \oplus Q$, $a \mapsto (a, 1)$ is integral and it is p-saturated if and only if Q is p-saturated. By taking the monoid in Example I.3.4 as Q, we obtain a counterexample.

I.5. ***A criterion of saturated morphisms, I.*** The purpose of this section is to prove Theorem I.5.1 below. This is an unpublished result of K. Kato. As a corollary, we will prove that every integral morphism of finitely generated saturated monoids is "potentially" saturated (Corollary I.5.4).

Theorem I.5.1. *Let $f : P \to Q$ be an integral morphism of finitely generated saturated monoids. Then, the morphism f is saturated if and only if, for every prime ideal $\mathfrak{q}$ of Q of height 1 such that the prime ideal $f^{-1}(\mathfrak{q})$ of P is of height 1, the ramification index of f at $\mathfrak{q}$* (Lemma I.1.7) *is* 1.

Lemma I.5.2. *Let n be a positive integer. Then the morphism $n : \mathbb{N} \to \mathbb{N}$ is saturated if and only if $n = 1$.*

Proof. Suppose $n \geq 2$. First note that the morphism $n : \mathbb{N} \to \mathbb{N}$ is integral. Let m be an integer ≥ 2 and set $d = \gcd(n, m)$, $m_0 = md^{-1}$ and $n_0 = nd^{-1}$. Choose integers r and s such that $s \cdot m_0 + r \cdot n_0 = 1$ and define morphisms $f, g : \mathbb{N} \to \mathbb{N} \oplus \mathbb{Z}/d\mathbb{Z}$ by $f(1) = (n_0, s)$ and $g(1) = (m_0, -r)$. Then, the diagram

$$\begin{array}{ccc} \mathbb{N} \oplus \mathbb{Z}/d\mathbb{Z} & \xleftarrow{\;g\;} & \mathbb{N} \\ {\scriptstyle f}\Big\uparrow & & \Big\uparrow{\scriptstyle n} \\ \mathbb{N} & \xleftarrow{\;m\;} & \mathbb{N} \end{array}$$

is cocartesian in the category of saturated monoids. Indeed, we see easily that the diagram becomes cocartesian in the category of abelian groups after taking the associated groups and that $\mathbb{N} \oplus \mathbb{Z}/d\mathbb{Z}$ is the saturation of $f(\mathbb{N}) + g(\mathbb{N})$. If $n_0 \geq 2$ and $m_0 \geq 2$, $\mathbb{N} \oplus \mathbb{Z}/d\mathbb{Z} \supsetneq f(\mathbb{N}) + g(\mathbb{N})$ because $(1, 0) \notin f(\mathbb{N}) + g(\mathbb{N})$. Otherwise, $d \geq 2$ and again $\mathbb{N} \oplus \mathbb{Z}/d\mathbb{Z} \supsetneq f(\mathbb{N}) + g(\mathbb{N})$ because $(0, 1 \bmod d) \notin f(\mathbb{N}) + g(\mathbb{N})$. Hence the pushout of $\mathbb{N} \xleftarrow{m} \mathbb{N} \xrightarrow{n} \mathbb{N}$ in the category of monoids is not saturated for integers $n, m \geq 2$. □

Proof of Theorem I.5.1. First let us prove the necessity. Let $\mathfrak{q}$ be a prime ideal of height 1 of Q such that $\mathfrak{p} = f^{-1}(\mathfrak{q})$ is a prime ideal of height 1 of P. Then, by Propositions I.3.16 and I.3.18, the morphism $\mathbb{N} \cong P_{\mathfrak{p}}/P^*_{\mathfrak{p}} \to Q_{\mathfrak{q}}/Q^*_{\mathfrak{q}} \cong \mathbb{N}$ induced by f is saturated. Hence, by Lemma I.5.2, the ramification index of f at $\mathfrak{q}$ is 1.

Next let us prove the sufficiency. First we prove it in the case $\dim(P) = 1$ and $f^{-1}(Q^*) = P^*$. The set $\mathrm{Spec}(P)$ consists of two elements $\varnothing$ and $P \backslash P^*$, and we have $P/P^* \cong \mathbb{N}$. If we choose a lifting $e \in P$ of the generator of P/P^*, the maximal ideal $P \backslash P^*$ is generated by e. Let p be a prime. We will prove that

f satisfies the conditions (i) and (ii) in Theorem I.4.2. The condition (i) follows from the fact that Q is saturated. Let $b \in Q$ and suppose that there exists $a \in P\backslash P^*$ such that $f(a) \mid b^p$, or equivalently, $f(e) \mid b^p$. Then, for any prime ideal $\mathfrak{q}$ of Q of height 1, $v_{\mathfrak{q}}(b^p) \geq v_{\mathfrak{q}}(f(e))$. By the assumption on f, $v_{\mathfrak{q}}(f(e)) = 1$ or 0. If $v_{\mathfrak{q}}(f(e)) = 1$, then $v_{\mathfrak{q}}(b) \geq 1$ and hence $v_{\mathfrak{q}}(bf(e)^{-1}) \geq 0$. If $v_{\mathfrak{q}}(f(e)) = 0$, then $v_{\mathfrak{q}}(bf(e)^{-1}) = v_{\mathfrak{q}}(b) \geq 0$. By Lemma I.1.5 and Proposition I.1.6, we see $bf(e)^{-1} \in Q$, that is, $f(e) \mid b$.

Now let us consider the general case. Let $\mathfrak{p}$ be the prime ideal $f^{-1}(Q\backslash Q^*)$ of P. Then the morphism f factors as $P \to P_{\mathfrak{p}} \to Q$. The first morphism is saturated and the second morphism is integral by Propositions I.3.18 and I.2.7. Hence we may assume $f^{-1}(Q^*) = P^*$. Let p be a prime. Since Q is saturated, f satisfies the condition (i) in Theorem I.4.2. It remains to prove that f satisfies the condition (ii′) in Remark I.4.4. Let $b \in Q$ and assume $f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b$. We will prove $f^{\mathrm{gp}}(P^{\mathrm{gp}})b^p \cap Q = f(P)b^p$. By Proposition I.2.11, there exists $b' \in Q$ such that $f^{\mathrm{gp}}(P^{\mathrm{gp}})b^p \cap Q = f(P)b'$. Choose $a \in P$ such that $b^p = f(a)b'$. By Proposition I.1.6, it suffices to prove $a \in P^*_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of P of height 1. Let $\mathfrak{p}$ be a prime ideal of height 1 of P and define an integral morphism of finitely generated saturated monoids $f_{\mathfrak{p}} : P_{\mathfrak{p}} \to Q_{\mathfrak{p}}$ by the following cocartesian diagram in the category of monoids:

$$\begin{array}{ccc} Q_{\mathfrak{p}} & \longleftarrow & Q \\ {\scriptstyle f_{\mathfrak{p}}}\uparrow & & {\scriptstyle f}\uparrow \\ P_{\mathfrak{p}} & \longleftarrow & P \end{array}$$

By Propositions I.2.11 and I.3.2 (2), $f_{\mathfrak{p}}$ is exact, and therefore $f_{\mathfrak{p}}^{-1}(Q^*_{\mathfrak{p}}) = P^*_{\mathfrak{p}}$. Using $Q_{\mathfrak{p}} = f(P\backslash\mathfrak{p})^{-1}Q$, we see that, for every prime ideal $\mathfrak{s}$ of $Q_{\mathfrak{p}}$ of height 1 such that the prime ideal $f_{\mathfrak{p}}^{-1}(\mathfrak{s})$ of $P_{\mathfrak{p}}$ is of height 1, the ramification index of $f_{\mathfrak{p}}$ at $\mathfrak{s}$ is 1. So, as we have proven above, the morphism $f_{\mathfrak{p}}$ is p-saturated. On the other hand, $f^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = f(P)b$ implies $(f_{\mathfrak{p}})^{\mathrm{gp}}((P_{\mathfrak{p}})^{\mathrm{gp}})b \cap Q_{\mathfrak{p}} = f_{\mathfrak{p}}(P_{\mathfrak{p}})b$. Hence, by Theorem I.4.2 and Remark I.4.4, we have $(f_{\mathfrak{p}})^{\mathrm{gp}}((P_{\mathfrak{p}})^{\mathrm{gp}})b^p \cap Q_{\mathfrak{p}} = f_{\mathfrak{p}}(P_{\mathfrak{p}})b^p$. Choose $c \in P_{\mathfrak{p}}$ such that $b' = f_{\mathfrak{p}}(c)b^p$ (in $(Q_{\mathfrak{p}})^{\mathrm{gp}} = Q^{\mathrm{gp}}$). Then $f_{\mathfrak{p}}(c)f(a) = 1$ and hence $f(a) \in Q^*_{\mathfrak{q}}$, which implies $a \in P^*_{\mathfrak{p}}$. □

Proposition I.5.3. *Let $f : P \to Q$ be an integral morphism of finitely generated saturated monoids. Let n be a positive integer and consider the following cocartesian diagram in the category of* saturated *monoids*:

$$\begin{array}{ccc} Q' & \xleftarrow{\;g\;} & Q \\ {\scriptstyle f'}\uparrow & & {\scriptstyle f}\uparrow \\ P & \xleftarrow{\;n\;} & P \end{array}$$

Then:

(1) *f' is integral.*

Let $\mathfrak{q}'$ be a prime ideal of height 1 *of Q' and let $\mathfrak{p}$ (resp. $\mathfrak{q}$) be the prime ideal $(f')^{-1}(\mathfrak{q}')$ (resp. $g^{-1}(\mathfrak{q}')$) of P (resp. Q). Then*:

(2) $\mathrm{ht}(\mathfrak{q}) = 1$ *and* $f^{-1}(\mathfrak{q}) = \mathfrak{p}$.

Let $n_{\mathfrak{q}'}$ be the ramification index of g at $\mathfrak{q}'$.

(3) *If* $\mathrm{ht}(\mathfrak{p}) = 0$, *then* $n_{\mathfrak{q}'} = 1$.

(4) *Suppose* $\mathrm{ht}(\mathfrak{p}) = 1$. *If we denote by $m_{\mathfrak{q}'}$ (resp. $m_{\mathfrak{q}}$) the ramification index of f' (resp. f) at $\mathfrak{q}'$ (resp. $\mathfrak{q}$), we have* $m_{\mathfrak{q}'} = m_{\mathfrak{q}} \gcd(n, m_{\mathfrak{q}})^{-1}$ *and* $n_{\mathfrak{q}'} = n \gcd(n, m_{\mathfrak{q}})^{-1}$.

Proof. (1) There exists a unique morphism $h : Q' \to Q$ such that $h \circ f' = f$ and $h \circ g = n$. The morphism $g \circ h$ is the multiplication by n on Q', which is exact. Hence h is exact, and the claim follows from Proposition I.2.10.

(2) The second claim follows from the fact that the inverse image of $\mathfrak{p}$ under $n : P \to P$ is $\mathfrak{p}$. For any $b \in Q'$, $b^n = g(h(b))$ and, for any $a_1, a_2 \in Q$, $g(a_1) = g(a_2)$ implies $a_1^n = h(g(a_1)) = h(g(a_2)) = a_2^n$. Hence, the first claim follows from Proposition I.1.4.

(3) The assumption $\mathrm{ht}(\mathfrak{p}) = 0$ implies $v_{\mathfrak{q}'}((f')^{\mathrm{gp}}(P^{\mathrm{gp}})) = 0$. Since Q'^{gp} is generated by $(f')^{\mathrm{gp}}(P^{\mathrm{gp}})$ and $g^{\mathrm{gp}}(Q^{\mathrm{gp}})$, we have

$$\mathbb{Z} = v_{\mathfrak{q}'}((Q')^{\mathrm{gp}}) = v_{\mathfrak{q}'}((f')^{\mathrm{gp}}(P^{\mathrm{gp}})) + v_{\mathfrak{q}'}(g^{\mathrm{gp}}(Q^{\mathrm{gp}})) = n_{\mathfrak{q}'} v_{\mathfrak{q}}(Q^{\mathrm{gp}}) = n_{\mathfrak{q}'}\mathbb{Z}.$$

Hence $n_{\mathfrak{q}'} = 1$.

(4) Since $(Q')^{\mathrm{gp}} = (f')^{\mathrm{gp}}(P^{\mathrm{gp}}) g^{\mathrm{gp}}(Q^{\mathrm{gp}})$, we have

$$\begin{aligned}\mathbb{Z} = v_{\mathfrak{q}'}((Q')^{\mathrm{gp}}) &= v_{\mathfrak{q}'}((f')^{\mathrm{gp}}(P^{\mathrm{gp}})) + v_{\mathfrak{q}'}(g^{\mathrm{gp}}(Q^{\mathrm{gp}})) \\ &= m_{\mathfrak{q}'} v_{\mathfrak{p}}(P^{\mathrm{gp}}) + n_{\mathfrak{q}'} v_{\mathfrak{q}}(Q^{\mathrm{gp}}) = m_{\mathfrak{q}'}\mathbb{Z} + n_{\mathfrak{q}'}\mathbb{Z}.\end{aligned}$$

Hence $(m_{\mathfrak{q}'}, n_{\mathfrak{q}'}) = 1$. On the other hand, since the ramification index of $n : P \to P$ at $\mathfrak{p}$ is n, we have $n_{\mathfrak{q}'} m_{\mathfrak{q}} = m_{\mathfrak{q}'} n$. The two equalities in (4) follow from these two facts. □

Corollary I.5.4. *Let $f : P \to Q$, n and $f' : P \to Q'$ be as in Proposition I.5.3. Then f' is saturated if and only if n is divisible by the least common multiple of the ramification indices of f at all prime ideals $\mathfrak{q}$ of Q of height* 1 *such that* $\mathrm{ht}(f^{-1}(\mathfrak{q})) = 1$.

Proof. This follows from Proposition I.5.3 and Theorem I.5.1. □

I.6. *A criterion of saturated morphisms, II.* In this section, we give several characterizations of saturated morphisms of finitely generated saturated monoids; see Theorem I.6.3.

Proposition I.6.1. *Let $f : P \to Q$ be a morphism of finitely generated saturated monoids. If f is saturated, $P^* = \{1\}$, $Q^* = \{1\}$, $f^{-1}(\{1\}) = \{1\}$ and* $\dim(P) = \dim(Q)$, *then f is an isomorphism.*

Proof. By Proposition I.2.8, the morphism $f^{\mathrm{gp}} : P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$ is injective. On the other hand, $\mathrm{rank}_{\mathbb{Z}}(P^{\mathrm{gp}}) = \dim(P) = \dim(Q) = \mathrm{rank}_{\mathbb{Z}}(Q^{\mathrm{gp}})$ by assumption. Hence $Q^{\mathrm{gp}}/f^{\mathrm{gp}}(P^{\mathrm{gp}})$ is a finite group. Set $G = Q^{\mathrm{gp}}/f^{\mathrm{gp}}(P^{\mathrm{gp}})$ and define morphisms of monoids $g, h : Q \to Q \oplus G$ by $g(b) = (b, 0)$ and $h(b) = (b, b \bmod f^{\mathrm{gp}}(P^{\mathrm{gp}}))$. Then the diagram of saturated monoids

$$\begin{array}{ccc} Q \oplus G & \xleftarrow{\ h\ } & Q \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f} \\ Q & \xleftarrow{\ f\ } & P \end{array}$$

is cocartesian in the category of saturated monoids. Indeed, we see easily that the diagram becomes cocartesian after taking the associated abelian groups and that $Q \oplus G$ is the saturated monoid associated to its submonoid $h(Q)g(Q)$ (see Definition II.2.2). Using $Q^* = \{1\}$, we see $h(Q)g(Q) \cap G = \{1\}$. On the other hand, since f is saturated, $h(Q)g(Q) = Q \oplus G$. Hence $G = \{1\}$ and $P^{\mathrm{gp}} = Q^{\mathrm{gp}}$. Since f is exact by Proposition I.2.8, $P = Q$. □

Corollary I.6.2. *Let $f : P \to Q$ be a saturated morphism of finitely generated saturated monoids. Let $\mathfrak{q}$ be a prime ideal of Q and let $\mathfrak{p}$ be the prime ideal $f^{-1}(\mathfrak{q})$ of P. If* $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p})$, *then the morphism $P_{\mathfrak{p}}/P_{\mathfrak{p}}^* \to Q_{\mathfrak{q}}/Q_{\mathfrak{q}}^*$ induced by f is an isomorphism.*

Proof. By Propositions I.3.16 and I.3.18, the morphism $P_{\mathfrak{p}}/P_{\mathfrak{p}}^* \to Q_{\mathfrak{q}}/Q_{\mathfrak{q}}^*$ is saturated. On the other hand, $\dim(P_{\mathfrak{p}}/P_{\mathfrak{p}}^*) = \mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{q}) = \dim(Q_{\mathfrak{q}}/Q_{\mathfrak{q}}^*)$. Hence the claim follows from Proposition I.6.1. □

Theorem I.6.3. *Let $f : P \to Q$ be an integral morphism of finitely generated saturated monoids such that $f^{-1}(Q^*) = P^*$. Set $\mathfrak{m}_P = P \backslash P^*$. Then the following conditions are equivalent*:

(1) *f is saturated.*

(2) *There exists a prime p such that f is p-saturated.*

(3) *For any $b \in Q$, if there exist a positive integer n and $a \in \mathfrak{m}_P$ such that $f(a) \mid b^n$, then there exists $a' \in \mathfrak{m}_P$ such that $f(a') \mid b$.*

(4) *For any $\mathfrak{q} \in$* $\mathrm{Spec}(Q)$ *and $\mathfrak{p} = f^{-1}(\mathfrak{q}) \in$* $\mathrm{Spec}(P)$ *such that* $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p})$, *the morphism $P_{\mathfrak{p}}/P_{\mathfrak{p}}^* \to Q_{\mathfrak{q}}/Q_{\mathfrak{q}}^*$ induced by f is an isomorphism.*

(5) *For any* $\mathfrak{q} \in \mathrm{Spec}(Q)$ *and* $\mathfrak{p} = f^{-1}(\mathfrak{q}) \in \mathrm{Spec}(P)$ *such that* $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p}) = 1$, *the morphism* $P_{\mathfrak{p}}/P^*_{\mathfrak{p}} \to Q_{\mathfrak{q}}/Q^*_{\mathfrak{q}}$ *induced by* f *is an isomorphism.*

(6) *For any* $\mathfrak{q} \in \mathrm{Spec}(Q)$ *such that* $f^{-1}(\mathfrak{q}) = \mathfrak{m}_P$ *and* $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{m}_P)$ $(= \dim(P))$, *the morphism* $P/P^* \to Q_{\mathfrak{q}}/Q^*_{\mathfrak{q}}$ *induced by* f *is an isomorphism.*

(7) *For any* $\mathfrak{q} \in \mathrm{Spec}(Q)$ *such that* $f^{-1}(\mathfrak{q}) = \mathfrak{m}_P$ *and* $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{m}_P)$ $(= \dim(P))$, $\mathfrak{q}(Q_{\mathfrak{q}}/Q^*_{\mathfrak{q}}) = f(\mathfrak{m}_P)(Q_{\mathfrak{q}}/Q^*_{\mathfrak{q}})$.

(8) *For any field* k *on which the order of the torsion part of* $Q^{\mathrm{gp}}/f(P^*)$ *is invertible,* $k[Q/f(P^*)]/f(\mathfrak{m}_P)k[Q/f(P^*)]$ *is reduced.*

(9) *There exists a field* k *such that* $k[Q/f(P^*)]/f(\mathfrak{m}_P)k[Q/f(P^*)]$ *satisfies* (R_0).

Theorem I.6.4 [Hochster 1972]. *For any finitely generated saturated monoid* P *and any field* k, *the ring* $k[P]$ *is Cohen–Macaulay.*

Proposition I.6.5 (A part of [EGA IV$_2$ 1965, Corollaire (6.3.5)]). *Let* $f : X \to Y$ *be a flat morphism of locally noetherian schemes. Let* $x \in X$ *and* $y = f(x)$. *If* $\mathcal{O}_{X,x}$ *is Cohen–Macaulay, then* $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} k(y)$ *and* $\mathcal{O}_{Y,y}$ *are Cohen–Macaulay.*

Corollary I.6.6. *Let* $f : P \to Q$ *be an integral morphism of finitely generated saturated monoids such that* $f^{-1}(Q^*) = P^* = \{1\}$. *Then, for any field* k, *the ring* $k[Q]/f(P\backslash\{1\})k[Q]$ *is Cohen–Macaulay.*

Proof. By Propositions I.2.8 and I.2.1, the homomorphism $k[P] \to k[Q]$ induced by f is flat. Hence, the claim follows from Theorem I.6.4 and Proposition I.6.5. □

Lemma I.6.7. *Let* P *be a finitely generated saturated monoid. Then, the surjective morphism* $P \to P/P^*$ *has a section* $s : P/P^* \to P$, *which induces an isomorphism* $(s, \iota) : P/P^* \oplus P^* \cong P$, *where* ι *is the inclusion* $P^* \hookrightarrow P$.

Proof. Since P is saturated, P^{gp}/P^* is torsion-free and the surjective homomorphism $P^{\mathrm{gp}} \to P^{\mathrm{gp}}/P^*$ has a section $t : P^{\mathrm{gp}}/P^* \to P^{\mathrm{gp}}$. It is easy to see that $t(P/P^*) \subset P$ and the restriction of t on P/P^* gives a desired morphism. □

Lemma I.6.8. *Let* P *be a finitely generated integral monoid. Then, for any field* k, *the dimension of every irreducible component of* $\mathrm{Spec}(k[P])$ *is* $\mathrm{rank}_{\mathbb{Z}}(P^{\mathrm{gp}})$.

Proof. Let $a_1, a_2, \dots, a_n$ be a set of generators of P. Then $\mathrm{Spec}(k[P^{\mathrm{gp}}]) = \mathrm{Spec}(k[P]_{a_1\cdots a_n})$. Since P is integral, $a_1 \cdots a_n$ is a nonzero divisor in $k[P]$. Hence the generic point of every irreducible component of $\mathrm{Spec}(k[P])$ is contained in $\mathrm{Spec}(k[P^{\mathrm{gp}}])$. Therefore we may assume $P = P^{\mathrm{gp}}$. Then $P \cong \mathbb{Z}^r \oplus C$ with C a finite group and $k[P]$ is isomorphic to $k[T_1^{\pm 1}, \dots, T_r^{\pm 1}] \otimes_k k[C]$, where $r = \mathrm{rank}_{\mathbb{Z}} P^{\mathrm{gp}}$. Hence $\mathrm{Spec}(k[P])_{\mathrm{red}}$ is a finite disjoint union of schemes of the form $\mathrm{Spec}(k'[T_1^{\pm 1}, \dots, T_r^{\pm 1}])$ with k' finite extensions of k. □

Proposition I.6.9. *Let* P *be a finitely generated saturated monoid and let* k *be a field. Set* $X = \mathrm{Spec}(k[P])$. *Let* $x \in X$ *and let* $\mathfrak{p}$ *be the inverse image of the* *maximal ideal of* $\mathcal{O}_{X,x}$ *in* P *(which is a prime ideal of* P*). Then* $\mathrm{ht}(\mathfrak{p}) \leq \dim(\mathcal{O}_{X,x})$*. The equality holds if and only if* x *is of codimension* 0 *in the closed subscheme* $Y = \mathrm{Spec}(k[P]/\mathfrak{p}k[P])$ *of* X*. Furthermore, if the order of the torsion part of* P^{gp} *is invertible in* k *and* $\mathrm{ht}(\mathfrak{p}) = \dim(\mathcal{O}_{X,x})$*, then the maximal ideal of* $\mathcal{O}_{X,x}$ *is generated by the image of* $\mathfrak{p}$*.*

Proof. Let U be the open subscheme $\mathrm{Spec}(k[P_{\mathfrak{p}}])$ of X. Then, the point x is contained in the closed subscheme $V = \mathrm{Spec}(k[P_{\mathfrak{p}}]/\mathfrak{p}k[P_{\mathfrak{p}}])$ of U. Note that the scheme V is an open subscheme of Y. By Lemma I.6.7, $P_{\mathfrak{p}} \cong P_{\mathfrak{p}}/P_{\mathfrak{p}}^{*} \oplus P_{\mathfrak{p}}^{*}$ and $\mathfrak{p}P_{\mathfrak{p}}$ corresponds to $\{(a,b) \in P_{\mathfrak{p}}/P_{\mathfrak{p}}^{*} \oplus P_{\mathfrak{p}}^{*} \mid a \neq 1\}$. Hence $V \cong \mathrm{Spec}(k[P_{\mathfrak{p}}^{*}])$. By Lemma I.6.8, $\dim(\mathcal{O}_{X,x}) \geq \mathrm{rank}_{\mathbb{Z}}(P^{\mathrm{gp}}) - \mathrm{rank}_{\mathbb{Z}}(P_{\mathfrak{p}}^{*}) = \mathrm{rank}_{\mathbb{Z}}(P_{\mathfrak{p}}^{\mathrm{gp}}/P_{\mathfrak{p}}^{*}) = \dim(P_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p})$, and the equality holds if and only if x is of codimension 0 in V, or, equivalently in Y. Suppose that the order of the torsion part of P^{gp} is invertible in k. Then V is a finite disjoint union of regular schemes. Hence, if x is of codimension 0 in V, the maximal ideal of $\mathcal{O}_{X,x}$ is generated by the image of $\mathfrak{p}$. □

Proof of Theorem I.6.3. The implications (1) ⇒ (2), (4) ⇒ (5), (4) ⇒ (6), (6) ⇒ (7) and (8) ⇒ (9) are trivial. It follows from Corollary I.4.5 that (2) ⇒ (1). Since Q is saturated, f satisfies the condition (i) in Theorem I.4.2. Hence the equivalence between (2) and (3) follows from Theorem I.4.2. It follows from Corollary I.6.2 that (1) ⇒ (4) and from Theorem I.5.1 that (5) ⇒ (1). Now it suffices to prove (7) ⇒ (8) and (9) ⇒ (3).

(7) ⇒ (8): Let k be a field satisfying the assumption in (8). The morphism $g: P/P^{*} \to Q/f(P^{*})$ induced by f is integral,

$$g^{-1}((Q/f(P^{*}))^{*}) = g^{-1}(Q^{*}/f(P^{*})) = \{1\} = (P/P^{*})^{*}$$

and $Q^{\mathrm{gp}}/f(P^{*}) \cong (Q/f(P^{*}))^{\mathrm{gp}}$. Furthermore, if f satisfies (7), then g also satisfies (7). Hence, we may assume $P^{*} = \{1\}$. By Corollary I.6.6, the ring $k[Q]/f(\mathfrak{m}_P)k[Q]$ is Cohen–Macaulay, in particular, it satisfies (S_1). Hence it suffices to prove that the ring $k[Q]/f(\mathfrak{m}_P)k[Q]$ satisfies (R_0). Set $Y = \mathrm{Spec}(k[Q])$, $X = \mathrm{Spec}(k[P])$, and $Z = \mathrm{Spec}(k[Q]/f(\mathfrak{m}_P)k[Q])$. By Propositions I.2.8 and I.2.1, the morphism $Y \to X$ induced by f is flat. Let y be a point of Z of codimension 0 and let x be the image of y in X, which is the closed point defined by the maximal ideal $\mathfrak{m}_P k[P]$ of $k[P]$. Since Y is flat over X and Z is the fiber over x, we have $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$. Let $\mathfrak{q}$ be the inverse image of the maximal ideal of $\mathcal{O}_{Y,y}$ in Q. Since the inverse image of the maximal ideal of $\mathcal{O}_{X,x}$ in P is $\mathfrak{m}_P$, we have $f^{-1}(\mathfrak{q}) = \mathfrak{m}_P$. By Proposition I.6.9, we have $\mathrm{ht}(\mathfrak{q}) \leq \dim(\mathcal{O}_{Y,y})$ and $\mathrm{ht}(\mathfrak{m}_P) = \dim(\mathcal{O}_{X,x})$. On the other hand, by Corollary I.2.9, we have $\mathrm{ht}(\mathfrak{m}_P) \leq \mathrm{ht}(\mathfrak{q})$. Hence $\mathrm{ht}(\mathfrak{m}_P) = \mathrm{ht}(\mathfrak{q}) = \dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$. Since the order of the torsion part of Q^{gp} is invertible in k, the maximal ideal of $\mathcal{O}_{Y,y}$ is generated by the image of $\mathfrak{q}$ by Proposition I.6.9. Therefore, if f satisfies the condition (7), then $\mathfrak{q}Q_{\mathfrak{q}} = f(\mathfrak{m}_P)Q_{\mathfrak{q}}$ and hence the maximal ideal of $\mathcal{O}_{Y,y}$ is generated by the image of $f(\mathfrak{m}_P)$, that is, Z is regular at y.

(9) $\Rightarrow$ (3): Let g be as in the proof of (7) $\Rightarrow$ (8). By Proposition I.3.16 and (1) $\Leftrightarrow$ (3), we may replace f by g and assume $P^* = \{1\}$. Then, by Corollary I.6.6, $k[Q]/f(\mathfrak{m}_P)k[Q]$ is Cohen–Macaulay, so it satisfies (S_1). Hence, if f satisfies (9), then $k[Q]/f(\mathfrak{m}_P)k[Q]$ is reduced. Let $b \in Q$, let n be a positive integer, and suppose that there exists $a \in \mathfrak{m}_P$ such that $f(a) \mid b^n$. Then $b^n \in f(\mathfrak{m}_P)k[Q]$. If $k[Q]/f(\mathfrak{m}_P)k[Q]$ is reduced, then $b \in f(\mathfrak{m}_P)k[Q]$. Hence, $b \in f(\mathfrak{m}_P)Q$. In other words, there exists $a' \in \mathfrak{m}_P$ such that $f(a') \mid b$. □

II. Saturated morphisms of log schemes

II.1. *Preliminaries on log schemes.* In this section, we prove some fundamental properties on log schemes.

Lemma II.1.1. *Let $f_1 : M_0 \to M_1$ and $f_2 : M_0 \to M_2$ be morphisms of log structures on a scheme X and let M_3 be the pushout of the diagram $M_1 \xleftarrow{f_1} M_0 \xrightarrow{f_2} M_2$ as sheaves of monoids.*

(1) *The sheaf of monoids M_3 endowed with the morphism $M_3 \to \mathcal{O}_X$ induced by the structure morphisms $M_0, M_1, M_2 \to \mathcal{O}_X$, is a log structure on X.*
(2) *The following diagram of sheaves of monoids is cocartesian*:

$$\begin{array}{ccc} M_3/\mathcal{O}_X^* & \longleftarrow & M_1/\mathcal{O}_X^* \\ \uparrow & & \uparrow \\ M_2/\mathcal{O}_X^* & \longleftarrow & M_0/\mathcal{O}_X^* \end{array}$$

Proof. Let α_i denote the structure morphism $M_i \to \mathcal{O}_X$ for $i = 0, 1, 2, 3$ and let g_1 and g_2 denote the canonical morphisms $M_1 \to M_3$ and $M_2 \to M_3$ respectively.

(1) First note $M_3 = g_1(M_1)g_2(M_2)$. Take $a_1 \in M_1$ and $a_2 \in M_2$ and assume $\alpha_3(g_1(a_1)g_2(a_2)) \in \mathcal{O}_X^*$. Then

$$\alpha_1(a_1) = \alpha_3(g_1(a_1)) \in \mathcal{O}_X^* \quad \text{and} \quad \alpha_2(a_2) = \alpha_3(g_2(a_2)) \in \mathcal{O}_X^*,$$

that is, $a_1 \in \alpha_1^{-1}(\mathcal{O}_X^*)$ and $a_2 \in \alpha_2^{-1}(\mathcal{O}_X^*)$. Let a be the unique section of $\alpha_0^{-1}(\mathcal{O}_X^*)$ such that $\alpha_0(a) = \alpha_2(a_2)$. Then, since $f_2(a) = a_2$, we have $g_1(f_1(a)) = g_2(f_2(a)) = g_2(a_2)$. Hence $g_1(a_1)g_2(a_2) = g_1(a_1 f_1(a)) \in g_1(\alpha_1^{-1}(\mathcal{O}_X^*))$. Thus we obtain

$$\alpha_3^{-1}(\mathcal{O}_X^*) = g_1(\alpha_1^{-1}(\mathcal{O}_X^*)),$$

which implies that the morphism $\alpha_3^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ induced by α_3 is an isomorphism.

(2) Consider the following two diagrams of sheaves of monoids:

$$\begin{array}{ccccc} M_3/\mathcal{O}_X^* & \longleftarrow & M_3 & \xleftarrow{g_2} & M_2 \\ \uparrow & & {\scriptstyle g_1}\uparrow & & {\scriptstyle f_2}\uparrow \\ M_1/\mathcal{O}_X^* & \longleftarrow & M_1 & \xleftarrow{f_1} & M_0 \end{array} \qquad \begin{array}{ccccc} M_3/\mathcal{O}_X^* & \longleftarrow & M_2/\mathcal{O}_X^* & \longleftarrow & M_2 \\ \uparrow & & \uparrow & & {\scriptstyle f_2}\uparrow \\ M_1/\mathcal{O}_X^* & \longleftarrow & M_0/\mathcal{O}_X^* & \longleftarrow & M_0 \end{array}$$

The two squares of the first diagram are cocartesian. Hence the outer square of the second diagram is cocartesian. Since the left square of the second diagram is cocartesian, the right one is also cocartesian. □

Proposition II.1.2. *Consider a cartesian diagram in the category of log schemes*:

$$\begin{array}{ccc} (Z, M_Z) & \xrightarrow{h} & (Y, M_Y) \\ {\scriptstyle k}\downarrow & & {\scriptstyle g}\downarrow \\ (X, M_X) & \xrightarrow{f} & (S, M_S) \end{array}$$

Take $z \in Z$ and let x, y and s be the images of z in X, Y and S respectively. Then, the diagram of monoids

$$\begin{array}{ccc} (M_Z/\mathcal{O}_Z^*)_{\bar{z}} & \longleftarrow & (M_Y/\mathcal{O}_Y^*)_{\bar{y}} \\ \uparrow & & \uparrow \\ (M_X/\mathcal{O}_X^*)_{\bar{x}} & \longleftarrow & (M_S/\mathcal{O}_S^*)_{\bar{s}} \end{array}$$

induced by the above diagram of log schemes is cocartesian in the category of monoids.

Proof. Let M be the pushout of the diagram

$$k^*(M_X) \leftarrow (f \circ k)^*(M_S) = (g \circ h)^*(M_S) \rightarrow h^*(M_Y)$$

as sheaves of monoids. Then M endowed with the morphism $M \rightarrow \mathcal{O}_Z$ induced by the structure morphisms $k^*(M_X) \rightarrow \mathcal{O}_Z$ and $h^*(M_Y) \rightarrow \mathcal{O}_Z$ is a log structure by Lemma II.1.1 (1). One can verify that (Z, M) satisfies the universal property of fiber products. Hence $M_Z \cong M$. Now the claim follows from Lemma II.1.1 (2) and [Kato 1989, (1.4.1)]. □

Proposition II.1.3 [Kato 1989, Example (2.5)(2)]. *Let k be an algebraically closed field. Let M be an integral log structure on $s =$ Spec(k) and set $P = \Gamma(s, M/\mathcal{O}_s^*)$. Then, there exists a section α of the projection $\Gamma(s, M) \rightarrow P$. Furthermore, such a section induces an isomorphism of log structures $(P_s)^a \xrightarrow{\sim} M$.*

Proof. Since $\Gamma(s, M^{\mathrm{gp}})/k^* \cong P^{\mathrm{gp}}$ and k^* is divisible and hence injective as a $\mathbb{Z}$-module, the projection $\Gamma(s, M^{\mathrm{gp}}) \rightarrow P^{\mathrm{gp}}$ has a section α. One sees easily $\alpha(P) \subset \Gamma(s, M)$. One can also verify that the morphism $(1, \alpha) : k^* \oplus P \rightarrow \Gamma(s, M)$ is an isomorphism and the image of $\alpha(P\backslash\{1\})$ under $\Gamma(s, M) \to k$ is 0. These imply that α induces an isomorphism $(P_s)^a \xrightarrow{\sim} M$. □

Proposition II.1.4. *Let (X, M_X) be a fine log scheme and let $\alpha : P_X \to M_X$ be a chart of M_X. Let $x \in X$ and let $\mathfrak{p}$ be the inverse image of the maximal ideal of $\mathcal{O}_{X,\bar{x}}$ under the morphism $P \xrightarrow{\alpha_{\bar{x}}} (M_X)_{\bar{x}} \to \mathcal{O}_{X,\bar{x}}$, which is a prime ideal of P. Then:*

(1) *The morphism α induces an isomorphism $P_{\mathfrak{p}}/P_{\mathfrak{p}}^* \xrightarrow{\sim} (M_X/\mathcal{O}_X^*)_{\bar{x}}$.*

(2) *Let U be the maximal open subscheme of X on which the image of every element of $P\backslash\mathfrak{p}$ under $P \to \Gamma(X, M_X) \to \Gamma(X, \mathcal{O}_X)$ becomes invertible. (Note that $P\backslash\mathfrak{p}$ is finitely generated.) Then, the chart α induces a chart $(P_{\mathfrak{p}})_U \to M_U$ of the restriction M_U of M_X on U.*

Proof. (1) By the definition of associated log structures, the diagram of monoids

$$\begin{array}{ccc} P\backslash\mathfrak{p} & \longrightarrow & P \\ {\scriptstyle\alpha_{\bar{x}}}\downarrow & & {\scriptstyle\alpha_{\bar{x}}}\downarrow \\ \mathcal{O}_{X,\bar{x}}^* & \longrightarrow & M_{X,\bar{x}} \end{array}$$

is cocartesian. Hence $\alpha_{\bar{x}}$ induces an isomorphism $P/(P\backslash\mathfrak{p}) \xrightarrow{\sim} (M_X/\mathcal{O}_X^*)_{\bar{x}}$. Since the image of $P\backslash\mathfrak{p}$ in $M_{X,\bar{x}}$ is contained in $\mathcal{O}_{X,\bar{x}}^*$, this isomorphism factors as

$$P/(P\backslash\mathfrak{p}) \to P_{\mathfrak{p}}/P_{\mathfrak{p}}^* \to (M_X/\mathcal{O}_X^*)_{\bar{x}}.$$

One sees easily that the first morphism is an isomorphism and hence the second one is also an isomorphism.

(2) Since the image of $P\backslash\mathfrak{p}$ in $\Gamma(U, M_U)$ is contained in $\Gamma(U, \mathcal{O}_U^*)$, the morphism α induces a morphism $\beta : (P_{\mathfrak{p}})_U \to M_U$. Let $x \in U$ and let $\mathfrak{q}$ be the inverse image of the maximal ideal of $\mathcal{O}_{U,\bar{x}}$ in $P_{\mathfrak{p}}$ and set $\mathfrak{r} = P \cap \mathfrak{q}$. By (1) and the fact that M_X is integral, it suffices to prove that the morphism $(P_{\mathfrak{p}})_{\mathfrak{q}}/(P_{\mathfrak{p}})_{\mathfrak{q}}^* \to (M_U/\mathcal{O}_U^*)_{\bar{x}}$ induced by β is an isomorphism. This follows from $P_{\mathfrak{r}} = (P_{\mathfrak{p}})_{\mathfrak{q}}$ and the fact that α induces an isomorphism $P_{\mathfrak{r}}/P_{\mathfrak{r}}^* \cong (M_X/\mathcal{O}_X^*)_{\bar{x}}$ by (1). □

II.2. *p-saturated log schemes and p-saturated morphisms.*

Definition II.2.1. Let p be a prime. Let M_X be a log structure on a scheme X. We say the log structure M_X is *p-saturated* (resp. *saturated*) if $\Gamma(U, M_X)$ is p-saturated (resp. saturated) for every étale X-scheme U. We call a scheme with a p-saturated (resp. saturated) log structure a *p-saturated* (resp. *saturated*) *log scheme*.

Note that p-saturated (resp. saturated) log structures are integral. We see easily that a log scheme (X, M_X) is p-saturated (resp. saturated) if and only if $M_{X,\bar{x}}$ is p-saturated (resp. saturated) for every point $x \in X$ and that an integral log scheme (X, M_X) is p-saturated (resp. saturated) if and only if $(M_X/\mathcal{O}_X^*)_{\bar{x}} \cong M_{X,\bar{x}}/\mathcal{O}_{X,\bar{x}}^*$ is p-saturated (resp. saturated) for every point $x \in X$.

Definition II.2.2. Let p be a prime and let P be an integral monoid. We define $P^{p\text{-sat}}$, the *p-saturated monoid associated to P*, and P^{sat}, the *saturated monoid associated to P*, by

$$P^{p\text{-sat}} = \{a \in P^{\text{gp}} \mid \text{there exists an integer } m \geq 0 \text{ such that } a^{p^m} \in P\},$$
$$P^{\text{sat}} = \{a \in P^{\text{gp}} \mid \text{there exists an integer } n \geq 1 \text{ such that } a^n \in P\}.$$

Proposition II.2.3. *Let p be a prime. The functor from the category of integral monoids to the category of p-saturated monoids (resp. saturated monoids) associating $P^{p\text{-sat}}$ (resp. P^{sat}) to P is a left adjoint of the forgetful functor.*

Proof. Straightforward. □

Proposition II.2.4. *Let p be a prime and let P be a finitely generated integral monoid. Then $P^{p\text{-sat}}$ (resp. P^{sat}) is finitely generated.*

Proof. It suffices to prove that $\mathbb{Q}[P^{p\text{-sat}}]$ and $\mathbb{Q}[P^{\text{sat}}]$ are finitely generated $\mathbb{Q}[P]$-modules. Let $\bar{P}$ and $\bar{P}'$ be the image of P and P^{sat} in $P^{\text{gp}}/(P^{\text{gp}})_{\text{tor}}$, where $(P^{\text{gp}})_{\text{tor}}$ denotes the torsion part of P^{gp}. Then, $\mathbb{Q}[\bar{P}]$ is a noetherian integral domain and $\mathbb{Q}[\bar{P}']$ is contained in the integral closure of $\mathbb{Q}[\bar{P}]$, which is finite over $\mathbb{Q}[\bar{P}]$. Hence $\mathbb{Q}[\bar{P}']$ is a finitely generated $\mathbb{Q}[\bar{P}]$-module. Since $P^{\text{sat}} \supset (P^{\text{gp}})_{\text{tor}}$, this implies that $\mathbb{Q}[P^{\text{sat}}]$ is finitely generated over $\mathbb{Q}[P]$ and its submodule $\mathbb{Q}[P^{p\text{-sat}}]$ is also finitely generated. □

Proposition II.2.5. *Let p be a prime. Let Q be the pushout of a diagram of monoids $P \leftarrow S \rightarrow G$. Suppose P is integral (resp. p-saturated, resp. saturated) and G is a group. Then, Q is integral (resp. p-saturated, resp. saturated). In particular, if S is integral (resp. p-saturated, resp. saturated), then Q is also the pushout in the category of integral (resp. p-saturated, resp. saturated) monoids.*

Proof. Straightforward, using [Kato 1989, (1.3) Remark]. □

Corollary II.2.6. *Let p be a prime. Let $f : X \rightarrow Y$ be a morphism of schemes and let M_Y be a log structure on Y. If M_Y is p-saturated (resp. saturated), then $f^*(M_X)$ is p-saturated (resp. saturated).*

Proposition II.2.7. *Let p be a prime and let (X, M_X) be a fine p-saturated (resp. fine saturated) log scheme. Let $\alpha : P_X \rightarrow M_X$ be a chart of M_X. Then the morphism $\beta : P_X^{p\text{-sat}} \rightarrow M_X$ (resp. $\beta : P_X^{\text{sat}} \rightarrow M_X$) induced by α is also a chart of M_X.*

Proof. Let P' be $P^{p\text{-sat}}$ (resp. P^{sat}). Let α^a and β^a be the morphisms of log structures $(P_X)^a \rightarrow M_X$ and $(P'_X)^a \rightarrow M_X$ induced by α and β, respectively. The morphism α^a is an isomorphism by assumption, and we want to prove that β^a is an isomorphism. Let γ be the composition of $(\alpha^a)^{-1} : M_X \xrightarrow{\sim} (P_X)^a$ and the morphism of log structures $(P_X)^a \rightarrow (P'_X)^a$ induced by the canonical morphism $P \rightarrow P'$. Let δ be the composition $P_X \rightarrow P'_X \rightarrow (P'_X)^a$. Then we have $\gamma \circ \beta^a \circ \delta = \delta$

and $\beta^a \circ \gamma \circ \alpha = \alpha$. By using the universality of associated log structures and Proposition II.2.3, we see that $\gamma \circ \beta^a$ and $\beta^a \circ \gamma$ are the identity maps. □

Corollary II.2.8. *Let p be a prime and let (X, M_X) be a fine log scheme. Then M_X is p-saturated (resp. saturated) if and only if, étale locally on X, there exists a chart $P_X \to M_X$ such that P is p-saturated (resp. saturated).*

Proof. The necessity follows from Proposition II.2.7. The sufficiency follows from the definition of associated log structures and Proposition II.2.5. □

Proposition II.2.9. *Let (X, M_X) be a fine saturated log scheme. Then, for any $x \in X$, there exists a chart $P_U \to M_X|_U$ for an étale neighborhood U of x which induces an isomorphism $P \cong (M_X/\mathcal{O}_X^*)_{\bar{x}}$.*

Proof. Set $P = (M_X/\mathcal{O}_X^*)_{\bar{x}} \cong M_{X,\bar{x}}/\mathcal{O}_{X,\bar{x}}^*$. Since P is a finitely generated saturated monoid such that $P^* = \{1\}$, P^{gp} is a finitely generated free abelian group. Hence there exists a section $s : P \to M_{X,\bar{x}}$ of the projection $M_{X,\bar{x}} \to P$. We see easily that the inverse image of $M_{X,\bar{x}}$ under the morphism $s^{\mathrm{gp}} : P^{\mathrm{gp}} \to M_{X,\bar{x}}^{\mathrm{gp}}$ is P. Hence, by [Kato 1989, Lemma (2.10)], the section s is extended to a chart $P_U \to M_X|_U$ for an étale neighborhood U of x. □

Definition II.2.10. Let p be a prime. We say a morphism of integral log schemes $f : (X, M_X) \to (Y, M_Y)$ is *p-saturated* (resp. *saturated*) if, for every $x \in X$ and $y = f(x) \in Y$, the morphism $(M_Y/\mathcal{O}_Y^*)_{\bar{y}} \to (M_X/\mathcal{O}_X^*)_{\bar{x}}$ induced by f is p-saturated (resp. saturated).

Note that p-saturated morphisms and saturated morphisms are integral and that a morphism of integral log schemes is saturated if and only if it is p-saturated for every prime p.

Proposition II.2.11. *Let p be a prime.*

(1) *Let $f : (X, M_X) \to (Y, M_Y)$ and $g : (Y, M_Y) \to (Z, M_Z)$ be morphisms of integral log schemes. If f and g are p-saturated (resp. saturated), then $g \circ f$ is also p-saturated (resp. saturated).*

(2) *Let $f : (X, M_X) \to (Y, M_Y)$ and $g : (Y', M_{Y'}) \to (Y, M_Y)$ be morphisms of integral log schemes. If f is p-saturated (resp. saturated), then the base change $f' : (X', M_{X'}) \to (Y', M_{Y'})$ of f by g in the category of log schemes, is also p-saturated (resp. saturated).*

Proof. The claim (1) follows from Propositions I.2.3 (1) and I.3.6 (1), and (2) follows from Propositions II.1.2, I.2.3 (2) and I.3.6 (2). □

Proposition II.2.12. *Let p be a prime. Let $f : (X, M_X) \to (Y, M_Y)$ be a morphism of integral log schemes. If f is p-saturated (resp. saturated) and M_Y is p-saturated (resp. saturated), then M_X is p-saturated (resp. saturated).*

Proof. Take $x \in X$ and $y = f(x) \in Y$. If M_Y is p-saturated (resp. saturated), then $(M_Y/\mathcal{O}_Y^*)_{\bar{y}}$ is p-saturated (resp. saturated). Hence, by Proposition I.3.9, if f is p-saturated (resp. saturated), then $(M_X/\mathcal{O}_X^*)_{\bar{x}}$ is p-saturated (resp. saturated). Since M_X is integral, this implies that $(M_X)_{\bar{x}}$ is p-saturated (resp. saturated). □

Proposition II.2.13. *Let p be a prime.*

(1) *Let $f : (X, M_X) \to (Y, M_Y)$ be an integral morphism of p-saturated (resp. saturated) log schemes. Then f is p-saturated (resp. saturated) if and only if, for any p-saturated (resp. saturated) log scheme $(Y', M_{Y'})$ and any morphism $g : (Y', M_{Y'}) \to (Y, M_Y)$, the base change $(X', M_{X'})$ of (X, M_X) by g in the category of log schemes is p-saturated (resp. saturated).*

(2) *Let $f : (X, M_X) \to (Y, M_Y)$ be an integral morphism of fine and p-saturated (resp. saturated) log schemes. Then f is p-saturated (resp. saturated) if and only if, for any fine and p-saturated (resp. saturated) log scheme $(Y', M_{Y'})$ and any morphism $g : (Y', M_{Y'}) \to (Y, M_Y)$, the base change $(X', M_{X'})$ of (X, M_X) by g in the category of log schemes is p-saturated (resp. saturated).*

Proof. The necessity follows from Propositions II.2.11 (2) and II.2.12. Let us prove the sufficiency. Let $f : (X, M_X) \to (Y, M_Y)$ be an integral morphism of integral log schemes. Take $x \in X$ and $y = f(x) \in Y$. Let k be an algebraic closure of the residue field of Y at y and set $\bar{y} := \mathrm{Spec}(k)$. Let N be the inverse image of M_Y under the canonical morphism $i_{\bar{y}} : \bar{y} \to Y$. Then, by Proposition II.1.3, there exists a section α of the projection $\Gamma(\bar{y}, N) \to \Gamma(\bar{y}, N)/k^* =: P$, which induces an isomorphism $(P_{\bar{y}})^a \cong N$. Let n be a positive integer and define a morphism $g : (\bar{y}, N) \to (\bar{y}, N)$ by the multiplication by n on P and the identity on k. Let $(X', M_{X'})$ be the base change of (X, M_X) by the morphism $i_{\bar{y}} \circ g : (\bar{y}, N) \to (Y, M_Y)$. Let x' be a point on X' whose image in X is x. Then, by Proposition II.1.2, the following diagram of monoids is cocartesian:

$$\begin{array}{ccc}
(M_{X'}/\mathcal{O}_{X'}^*)_{\bar{x}'} & \longleftarrow & (M_X/\mathcal{O}_X^*)_{\bar{x}} \\
\uparrow & & \uparrow \\
P = N_{\bar{y}}/k^* & \xleftarrow{\ n\ } & P = N_{\bar{y}}/k^* = (M_Y/\mathcal{O}_Y^*)_{\bar{y}}
\end{array}$$

If M_X, M_Y and $M_{X'}$ are p-saturated (resp. saturated), then the log structure N is p-saturated (resp. saturated) and the monoids $(M_X/\mathcal{O}_X^*)_{\bar{x}}$, $(M_Y/\mathcal{O}_Y^*)_{\bar{y}}$, P and $(M_{X'}/\mathcal{O}_{X'}^*)_{\bar{x}'}$ are p-saturated (resp. saturated). Now the claim (1) follows from Propositions I.3.13 and I.3.14. The claim (2) follows from the same propositions and the fact that N is fine if M_Y is fine. □

Proposition II.2.14. *Let p be a prime and let $f : (X, M_X) \to (Y, M_Y)$ be a morphism of integral log schemes over $\mathbb{F}_p$. Then f is p-saturated if and only if f is of Cartier type.*

Proof. This follows from Proposition II.1.2 and the fact that, for $x \in X$ and $y = f(x) \in Y$, the absolute Frobenius of (X, M_X) (resp. (Y, M_Y)) induces the multiplication by p on $(M_X/\mathcal{O}_X^*)_{\bar{x}}$ (resp. $(M_Y/\mathcal{O}_Y^*)_{\bar{y}}$). □

II.3. *Some properties of saturated morphisms.*

Theorem II.3.1. *Let p and q be two different primes. Let $f : (X, M_X) \to (Y, M_Y)$ be a morphism of fine log schemes. If f is p-saturated and M_X is q-saturated, then f is q-saturated.*

Proof. This follows from Corollary I.4.5. □

By Proposition II.2.14, this theorem implies that a morphism of fine saturated log schemes over $\mathbb{F}_p$ is of Cartier type if and only if it is saturated. This is an unpublished result of K. Kato.

Definition II.3.2. We say a morphism of log schemes $f : (X, M_X) \to (Y, M_Y)$ is *vertical* if, for every $x \in X$ and $y = f(x) \in Y$, the morphism $M_{Y,\bar{y}} \to M_{X,\bar{x}}$ induced by f is vertical (Definition I.4.6), or equivalently, the morphism $(M_Y/\mathcal{O}_Y^*)_{\bar{y}} \to (M_X/\mathcal{O}_X^*)_{\bar{x}}$ induced by f is vertical.

Theorem II.3.3. *Let p be a prime and let $f : (X, M_X) \to (Y, M_Y)$ be a morphism of fine log schemes. If f is p-saturated and vertical, then f is saturated.*

Proof. This follows from Corollary I.4.7. □

Theorem II.3.4. *Let $f : (X, M_X) \to (Y, M_Y)$ be an integral morphism of fine saturated log schemes and assume that we are given a chart $\beta : Q_Y \to M_Y$ of M_Y with Q saturated. If X is quasi-compact, then there exists a positive integer n satisfying the following property: Define a fine saturated log scheme $(Y', M_{Y'})$ to be*

$$(Y, M_Y) \times_{(\mathrm{Spec}(\mathbb{Z}[Q]),\text{can. log}),g} (\mathrm{Spec}(\mathbb{Z}[Q]), \text{can. log}),$$

where $g : (\mathrm{Spec}(\mathbb{Z}[Q]), \text{can. log}) \to (\mathrm{Spec}(\mathbb{Z}[Q]), \text{can. log})$ denotes the morphism induced by the multiplication by n on Q. Then, the base change $f' : (X', M_{X'}) \to (Y', M_{Y'})$ of f by the projection $(Y', M_{Y'}) \to (Y, M_Y)$ in the category of fine saturated log schemes is saturated.

Proof. Note first that the question is étale local on Y and on X. So we may assume that there exists a chart $(\alpha : P_X \to M_X, \beta, h : Q \to P)$ of the morphism f. Take $x \in X$ and $y = f(x) \in Y$. Let $\mathfrak{p}$ (resp. $\mathfrak{q}$) be the inverse image of the maximal ideal of $\mathcal{O}_{X,\bar{x}}$ (resp. $\mathcal{O}_{Y,\bar{y}}$) in P (resp. Q), which is a prime ideal. Since $((Q\backslash\mathfrak{q})^n)^{-1}Q \cong Q_{\mathfrak{q}}$, we may replace P, Q by $P_{\mathfrak{p}}$, $Q_{\mathfrak{q}}$ using Proposition II.1.4, and assume that α (resp. β) induces an isomorphism $P/P^* \cong (M_X/\mathcal{O}_X^*)_{\bar{x}}$ (resp. $Q/Q^* \cong (M_Y/\mathcal{O}_Y^*)_{\bar{y}}$). Since M_X is saturated and P is integral, P is saturated by Lemma I.3.15. Furthermore, since the morphism $(M_Y/\mathcal{O}_Y^*)_{\bar{y}} \to (M_X/\mathcal{O}_X^*)_{\bar{x}}$ is integral by assumption, the morphism $h : Q \to P$ is integral by Proposition I.2.5. Now we can apply Corollary I.5.4

to $h : Q \to P$ and find a positive integer n satisfying the following property: If we denote by P' the pushout of the diagram $Q \xleftarrow{n} Q \xrightarrow{h} P$ in the category of monoids, then the canonical morphism $h' : Q \to P' \to (P')^{\mathrm{sat}}$ is saturated. We assert that this n is the desired integer. Let $(X'', M_{X''})$ be the base change of (X, M) by $(Y', M_{Y'}) \to (Y, M_Y)$ in the category of log schemes. Then, since the diagram of log schemes

$$\begin{array}{ccc} (\mathrm{Spec}(\mathbb{Z}[P']), \text{can. log}) & \longrightarrow & (\mathrm{Spec}(\mathbb{Z}[P]), \text{can. log}) \\ \downarrow & & \downarrow \\ (\mathrm{Spec}(\mathbb{Z}[Q]), \text{can. log}) & \xrightarrow{g} & (\mathrm{Spec}(\mathbb{Z}[Q]), \text{can. log}) \end{array}$$

is cartesian, the three strict morphisms

$$(X, M_X) \to (\mathrm{Spec}(\mathbb{Z}[P]), \text{can. log}), \quad (Y, M_Y) \to (\mathrm{Spec}(\mathbb{Z}[Q]), \text{can. log}),$$
$$(Y', M_{Y'}) \to (\mathrm{Spec}(\mathbb{Z}[Q]), \text{can. log})$$

induce a strict morphism $(X'', M_{X''}) \to (\mathrm{Spec}(\mathbb{Z}[P']), \text{can. log})$ and hence a strict morphism $(X', M_{X'}) \to (\mathrm{Spec}(\mathbb{Z}[(P')^{\mathrm{sat}}], \text{can. log}))$. (Recall that we say a morphism of log schemes $\varphi : (S, M_S) \to (T, M_T)$ is strict if the morphism $\varphi^*(M_T) \to M_S$ is an isomorphism.) Thus, we obtain a chart

$$\big(((P')^{\mathrm{sat}})_{X'} \to M_{X'},\ Q_{Y'} \to M_{Y'},\ h' : Q \to (P')^{\mathrm{sat}}\big)$$

of f' such that h' is saturated. Now the claim follows from Lemma II.3.5 below. □

Lemma II.3.5. *Let $f : (X, M_X) \to (Y, M_Y)$ be a morphism of fine saturated log schemes. Suppose that there exists a chart $(\alpha : P_X \to M_X, \beta : Q_Y \to M_Y, h : Q \to P)$ of f such that P and Q are saturated and h is saturated. Then, the morphism f is saturated.*

Proof. Take $x \in X$ and $y = f(x) \in Y$. Let $\mathfrak{p}$ (resp. $\mathfrak{q}$) be the inverse image of the maximal ideal of $\mathcal{O}_{X,\bar{x}}$ (resp. $\mathcal{O}_{Y,\bar{y}}$) in P (resp. Q), which is a prime ideal. Then, by Proposition II.1.4 (1), the morphism α (resp. β) induces an isomorphism $P_{\mathfrak{p}}/P_{\mathfrak{p}}^* \cong (M_X/\mathcal{O}_X^*)_{\bar{x}}$ (resp. $Q_{\mathfrak{q}}/Q_{\mathfrak{q}}^* \cong (M_Y/\mathcal{O}_Y^*)_{\bar{y}}$). Since the morphism $Q_{\mathfrak{q}}/Q_{\mathfrak{q}}^* \to P_{\mathfrak{p}}/P_{\mathfrak{p}}^*$ induced by h is saturated by Propositions I.3.16 and I.3.18, the morphism $(M_Y/\mathcal{O}_Y^*)_{\bar{y}} \to (M_X/\mathcal{O}_X^*)_{\bar{x}}$ induced by f is saturated. □

II.4. *Criteria of saturated morphisms.*

Proposition II.4.1. *Let $f : (X, M_X) \to (Y, M_Y)$ be a smooth integral morphism of fine saturated log schemes. Then, every fiber of the underlying morphism of schemes of f is Cohen–Macaulay.*

Proof. First note that the question is étale local on X and on Y. Take $x \in X$ and $y = f(x) \in Y$. By Proposition II.2.9, we may assume that we have a chart

$\beta : Q_Y \to M_Y$ which induces an isomorphism $Q \cong (M_Y/\mathcal{O}_Y^*)_{\bar{y}}$. By [Kato 1989, Theorem (3.5)], we may assume that, there exists a chart of f,

$$(\alpha : P_X \to M_X,\ \beta : Q_Y \to M_Y,\ h : Q \to P),$$

such that h is injective, the order of the torsion part of the cokernel of $h^{\mathrm{gp}} : Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is invertible on X and the morphism $X \to Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}(\mathbb{Z}[P])$ induced by the chart is étale. Let $\mathfrak{p}$ be the inverse image of the maximal ideal of $\mathcal{O}_{X,\bar{x}}$ in P. By Proposition II.1.4, we may replace P by $P_{\mathfrak{p}}$ and assume that α induces an isomorphism $P/P^* \cong (M_X/\mathcal{O}_X^*)_{\bar{x}}$. Since P is integral and $P/P^* \cong (M_X/\mathcal{O}_X^*)_{\bar{x}}$ is saturated, P is saturated by Lemma I.3.15. Since the morphism $(M_Y/\mathcal{O}_Y^*)_{\bar{y}} \to (M_X/\mathcal{O}_X^*)_{\bar{x}}$ induced by f is integral by assumption, the morphism h is integral by Proposition I.2.5. Furthermore, we have $h^{-1}(P^*) = Q^* = \{1\}$. Hence, by Corollary I.6.6, the ring $k[P]/h(Q\backslash\{1\})k[P]$ is Cohen–Macaulay for any field k. If we choose the residue field of Y at y as k, then we have an étale morphism $f^{-1}(y) \to \mathrm{Spec}(k[P]/h(Q\backslash\{1\})k[P])$ by the choice of the chart. Hence $f^{-1}(y)$ is Cohen–Macaulay. □

Theorem II.4.2. *Let $f : (X, M_X) \to (Y, M_Y)$ be a smooth integral morphism of fine saturated log schemes. Then the following conditions are equivalent*:

(1) *f is saturated.*

(2) *There exists a prime p such that f is p-saturated.*

(3) *Every fiber of the underlying morphism of schemes of f is reduced.*

(4) *Every fiber of the underlying morphism of schemes of f satisfies (R_0).*

Proof. By Theorem II.3.1, (1) and (2) are equivalent. By Proposition II.4.1, (3) and (4) are equivalent. We will prove that (1) and (3) are equivalent.

Take $x \in X$ and $y = f(x) \in Y$. As in the proof of Proposition II.4.1, we may assume that there exists a chart $(\alpha : P_X \to M_X, \beta : Q_Y \to M_Y, h : Q \to P)$ of the morphism f such that $h^{\mathrm{gp}} : Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is injective, the order of the torsion part of the cokernel of h^{gp} is invertible on X, the morphism $X \to Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}(\mathbb{Z}[P])$ induced by the chart is étale, and the morphism α (resp. β) induces an isomorphism $P/P^* \cong (M_X/\mathcal{O}_X^*)_{\bar{x}}$ (resp. $Q \cong (M_Y/\mathcal{O}_Y^*)_{\bar{y}}$). Furthermore, as in the proof of Proposition II.4.1, these properties imply that P and Q are saturated, the morphism h is integral, and $h^{-1}(P^*) = Q^* = \{1\}$. We also see that Q^{gp} is torsion-free, and the order of the torsion part of P^{gp} is invertible on X.

(1) ⇒ (3): We will prove that $f^{-1}(y)$ is reduced. Since $(M_Y/\mathcal{O}_Y^*)_{\bar{y}} \to (M_X/\mathcal{O}_X^*)_{\bar{x}}$ is saturated, the morphism $h : Q \to P$ is saturated by Proposition I.3.16. Let k be the residue field of Y at y. Then the scheme $\mathrm{Spec}(k[P]/h(Q\backslash\{1\})k[P])$ is reduced, by Theorem I.6.3. Since $f^{-1}(y)$ is étale over the last scheme by the choice of the chart, the scheme $f^{-1}(y)$ is also reduced.

(3) $\Rightarrow$ (1): We will prove that the morphism $(M_Y/\mathcal{O}_Y^*)_{\bar{y}} \to (M_X/\mathcal{O}_X^*)_{\bar{x}}$ induced by f is saturated. Let k be the residue field of Y at y. Then, by the choice of the chart, the morphism $f^{-1}(y) \to \mathrm{Spec}(k[P]/h(Q\backslash\{1\})k[P])$ induced by the chart is étale. Let $\bar{P}$ be the quotient P/P^* and let $\bar{h}: Q \to \bar{P}$ be the morphism induced by $h: Q \to P$. Then, $\bar{P}^* = \{1\}$, $\bar{h}^{-1}(\bar{P}^*) = Q^*(=\{1\})$, and the morphism $\bar{h}$ is integral (Proposition I.2.5). By the choice of the chart of f, we have the following commutative diagram whose horizontal arrows are isomorphisms:

$$\begin{array}{ccc} (M_X/\mathcal{O}_X^*)_{\bar{x}} & \xleftarrow{\sim} & \bar{P} \\ \uparrow f_{\bar{x}}^* & & \uparrow \bar{h} \\ (M_Y/\mathcal{O}_Y^*)_{\bar{y}} & \xleftarrow{\sim} & Q \end{array}$$

Hence, by Theorem I.6.3, it suffices to prove that the scheme

$$\bar{Z} := \mathrm{Spec}(k[\bar{P}]/\bar{h}(Q\backslash\{1\})k[\bar{P}])$$

satisfies (R_0). Choose a section $s: \bar{P} \to P$ of the projection $P \to \bar{P}$. Such a section exists and it induces an isomorphism $(\iota, s): P^* \oplus \bar{P} \xrightarrow{\sim} P$ by Lemma I.6.7. We have $h(Q\backslash\{1\})k[P] = s\circ\bar{h}(Q\backslash\{1\})k[P]$ because, for any $b \in Q$, there exists $a \in P^*$ such that $h(b) = s\circ\bar{h}(b)a$. On the other hand, we see that the morphism $\mathrm{Spec}(k[P]/s\circ\bar{h}(Q\backslash\{1\})k[P]) \to \bar{Z} = \mathrm{Spec}(k[\bar{P}]/\bar{h}(Q\backslash\{1\})k[\bar{P}])$ induced by s is smooth as follows. Since the order of the torsion part of $P^*(\subset P^{\mathrm{gp}})$ is invertible on k, $\mathrm{Spec}(k[P^*])$ is smooth over k and hence $\mathrm{Spec}(k[P]) \cong \mathrm{Spec}(k[P^*]\otimes_k k[\bar{P}])$ is smooth over $\mathrm{Spec}(k[\bar{P}])$. Now we have smooth morphisms

$$\begin{aligned} f^{-1}(y) \longrightarrow \mathrm{Spec}(k[P]/h(Q\backslash\{1\})k[P]) &= \mathrm{Spec}(k[P]/s\circ\bar{h}(Q\backslash\{1\})k[P]) \\ &\longrightarrow \bar{Z} = \mathrm{Spec}(k[\bar{P}]/\bar{h}(Q\backslash\{1\})k[\bar{P}]). \end{aligned}$$

Since $f^{-1}(y)$ is reduced by assumption, $\bar{Z}$ is reduced on an open neighborhood of the image x_0 of x. In fact, x_0 is the closed point defined by the ideal generated by $\bar{P}\backslash\{1\}$ because we have $s(\bar{P}\backslash\{1\}) \subset P\backslash P^*$ and the image of $P\backslash P^*$ in $\mathcal{O}_{X,\bar{x}}$ under $\alpha_{\bar{x}}: P \to \mathcal{O}_{X,\bar{x}}$ is contained in the maximal ideal. Hence, by Lemma II.4.3 below, the scheme $\bar{Z}$ satisfies (R_0). □

Lemma II.4.3. *Let P be a finitely generated integral monoid such that P^{gp} is torsion-free, and let I be an ideal of P. Let k be a field. Then, for any point z of codimension* 0 *of the scheme $Z :=$* $\mathrm{Spec}(k[P]/Ik[P])$, $\overline{\{z\}}$ *contains the underlying set of the closed subscheme* $\mathrm{Spec}(k[P]/(P\backslash P^*)k[P])$ *of Z.*

Proof. Let $\mathfrak{p}$ be the inverse image of the maximal ideal of $\mathcal{O}_{Z,z}$ in P, which obviously contains I. Then we have $\mathfrak{p} \subset P\backslash P^*$, and z is of codimension 0 in the closed subscheme $\mathrm{Spec}(k[P]/\mathfrak{p}k[P])$ of Z. Hence the claim follows from Sublemma II.4.4 below. □

Sublemma II.4.4. *Let P be a finitely generated integral monoid such that P^{gp} is torsion-free. Then, for any prime ideal $\mathfrak{p}$ of P and any field k, the ring $k[P]/\mathfrak{p}k[P]$ is an integral domain.*

Proof. The morphism of monoids $P\backslash\mathfrak{p} \to P$ induces an isomorphism $k[P\backslash\mathfrak{p}] \xrightarrow{\sim} k[P]/\mathfrak{p}k[P]$. The ring $k[P\backslash\mathfrak{p}]$ is a subring of $k[(P\backslash\mathfrak{p})^{\mathrm{gp}}]$, and $k[(P\backslash\mathfrak{p})^{\mathrm{gp}}]$ is an integral domain because $(P\backslash\mathfrak{p})^{\mathrm{gp}}$ is torsion-free by assumption. □

Definition II.4.5 (cf. [Kato 1994, (2.1) Definition]). Let (X, M_X) be a fine saturated log scheme such that X is locally noetherian. We say (X, M_X) is *regular* at $x \in X$ if $\mathcal{O}_{X,\bar{x}}/I_{\bar{x}}\mathcal{O}_{X,\bar{x}}$ is regular and

$$\dim(\mathcal{O}_{X,\bar{x}}) = \dim(\mathcal{O}_{X,\bar{x}}/I_{\bar{x}}\mathcal{O}_{X,\bar{x}}) + \mathrm{rank}_{\mathbb{Z}}((M_X^{\mathrm{gp}}/\mathcal{O}_X^*)_{\bar{x}}),$$

where $I_{\bar{x}} = M_{X,\bar{x}}\backslash\mathcal{O}^*_{X,\bar{x}}$ and $I_{\bar{x}}\mathcal{O}_{X,\bar{x}}$ denotes the ideal of $\mathcal{O}_{X,\bar{x}}$ generated by the image of $I_{\bar{x}}$. We say (X, M_X) is *regular* if (X, M_X) is regular at every point $x \in X$.

Lemma II.4.6. *Let (X, M_X) be a fine saturated log scheme such that X is locally noetherian, and assume that we are given a chart $P_X \to M_X$ with P saturated. Let M_X^{Zar} be the log structure on the Zariski site* [Kato 1994, §1] *associated to $P \to \Gamma(X, M_X) \to \Gamma(X, \mathcal{O}_X)$. Then, for any $x \in X$, (X, M_X) is regular at x if and only if (X, M_X^{Zar}) is regular* [Kato 1994, (2.1) Definition] *at x.*

Proof. Let $\mathfrak{p}$ be the inverse image of the maximal ideal of $\mathcal{O}_{X,\bar{x}}$ in P. Then $\mathfrak{p}$ is also the inverse image of the maximal ideal of $\mathcal{O}_{X,x}$ because the morphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\bar{x}}$ is local. Hence, by Proposition II.1.4 and the corresponding fact for log structures in Zariski topology, the canonical morphisms $P_X \to M_X$ on $X_{\text{ét}}$ and $P_X \to M_X^{\mathrm{Zar}}$ on X_{Zar} induce isomorphisms $P_{\mathfrak{p}}/P_{\mathfrak{p}}^* \cong M_{X,\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$ and $P_{\mathfrak{p}}/P_{\mathfrak{p}}^* \cong M^{\mathrm{Zar}}_{X,x}/\mathcal{O}^*_{X,x}$. In particular $\mathrm{rank}_{\mathbb{Z}}((M_X^{\mathrm{gp}}/\mathcal{O}_X^*)_{\bar{x}}) = \mathrm{rank}_{\mathbb{Z}}(((M_X^{\mathrm{Zar}})^{\mathrm{gp}}/\mathcal{O}_X^*)_x)$. On the other hand, if we set $I_{\bar{x}} = M_{X,\bar{x}}\backslash\mathcal{O}^*_{X,\bar{x}}$ and $I_x = (M_X^{\mathrm{Zar}})_x\backslash\mathcal{O}^*_{X,x}$, we have $I_{\bar{x}}\mathcal{O}_{X,\bar{x}} = \mathfrak{p}\mathcal{O}_{X,\bar{x}}$ and $I_x\mathcal{O}_{X,x} = \mathfrak{p}\mathcal{O}_{X,x}$. Hence $\mathcal{O}_{X,\bar{x}}/I_{\bar{x}}\mathcal{O}_{X,\bar{x}}$ is the strict henselization of $\mathcal{O}_{X,x}/I_x\mathcal{O}_{X,x}$. So $\mathcal{O}_{X,\bar{x}}/I_{\bar{x}}\mathcal{O}_{X,\bar{x}}$ is regular if and only if $\mathcal{O}_{X,x}/I_x\mathcal{O}_{X,x}$ is regular. Now the lemma is a direct consequence of the definition. □

Theorem II.4.7 (cf. [Kato 1994, (4.1) Theorem]). *Let (X, M_X) be a fine saturated log scheme. If (X, M_X) is regular, then X is Cohen–Macaulay and normal.*

Proof. Since the question is étale local on X, the proposition follows from [Kato 1994, (4.1) Theorem], Lemma II.4.6 and Corollary II.2.8. □

Proposition II.4.8 (cf. [Kato 1994, (8.2) Theorem]). *Let $f : (X, M_X) \to (Y, M_Y)$ be a smooth morphism of fine saturated log schemes. If (Y, M_Y) is regular, then (X, M_X) is regular.*

Proof. Since the question is étale local on X and on Y, as in the proof of Proposition II.4.1, we may assume that there exists a chart

$$(\alpha : P_X \to M_X,\ \beta : Q_Y \to M_Y,\ h : Q \to P)$$

of the morphism f such that P and Q are saturated, h is injective, the order of the torsion part of the cokernel of $h^{\mathrm{gp}} : Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is invertible on X and the morphism $X \to Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}(\mathbb{Z}[P])$ induced by the chart is étale. Then the proposition follows from Lemma II.4.6 and [Kato 1994, (8.1) and (8.2) Theorem]. □

Lemma II.4.9 (cf. [Kato 1994, (7.3) Corollary]). *Let (X, M_X) be a regular fine saturated log scheme. Let $x \in X$ and assume that we are given a chart $\alpha : P_X \to M_X$ with P saturated such that the inverse image of the maximal ideal of $\mathcal{O}_{X,\bar{x}}$ under $P \xrightarrow{\alpha_{\bar{x}}} M_{X,\bar{x}} \to \mathcal{O}_{X,\bar{x}}$ is $P \backslash P^*$. Then, for any prime ideal $\mathfrak{p}$ of P, there exists a point $y \in X$ which satisfies the following conditions*:

(1) $x \in \overline{\{y\}}$.

(2) *The inverse image of the maximal ideal of $\mathcal{O}_{X,\bar{y}}$ in P is $\mathfrak{p}$.*

(3) *The image of $\mathfrak{p}$ in $\mathcal{O}_{X,\bar{y}}$ generates the maximal ideal.*

(4) $\dim(\mathcal{O}_{X,y}) = \mathrm{ht}(\mathfrak{p})$.

Proof. Let M_X^{Zar} be the log structure on the Zariski site associated to

$$P \to \Gamma(X, M) \to \Gamma(X, \mathcal{O}_X).$$

By Lemma II.4.6, (X, M_X^{Zar}) is regular. Since the homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\bar{x}}$ is local, the inverse image of the maximal ideal of $\mathcal{O}_{X,x}$ in P is also $P \backslash P^*$. Hence the canonical morphism $P/P^* \to M_{X,x}^{\mathrm{Zar}}/\mathcal{O}_{X,x}^*$ is an isomorphism. Thus the map $\mathrm{Spec}(M_{X,x}^{\mathrm{Zar}}) \to \mathrm{Spec}(P)$ induced by $P \to M_{X,x}^{\mathrm{Zar}}$ is bijective. Let $\mathfrak{q}$ be the prime ideal of $M_{X,x}^{\mathrm{Zar}}$ corresponding to $\mathfrak{p}$ under this bijection. Then, $\mathfrak{q}$ is generated by the image of $\mathfrak{p}$ and $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{p})$. Hence, by [Kato 1994, (7.3) Corollary], the ideal $\mathfrak{p}\mathcal{O}_{X,x} (= \mathfrak{q}\mathcal{O}_{X,x})$ is a prime ideal of height $\mathrm{ht}(\mathfrak{p}) (= \mathrm{ht}(\mathfrak{q}))$. Let $y \in \mathrm{Spec}(\mathcal{O}_{X,x}) \subset X$ be the point corresponding to the prime ideal. We assert that y satisfies the required conditions. The conditions (1), (3) and (4) are trivial. For (2), it suffices to prove that the inverse image of the maximal ideal of $\mathcal{O}_{X,y}$ in P is $\mathfrak{p}$. Let $\mathfrak{p}'$ be the inverse image. By the analogue of Proposition II.1.4 (1) for log structures in Zariski topology, we have $P_{\mathfrak{p}'}/P_{\mathfrak{p}'}^* \cong (M_X^{\mathrm{Zar}}/\mathcal{O}_X^*)_y$. Since the image of $\mathfrak{p}\,(\subset P)$ in $\mathcal{O}_{X,y}$ generates the maximal ideal and (X, M_X^{Zar}) is regular, we have $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,y}) = \mathrm{rank}_{\mathbb{Z}}((M_X^{\mathrm{Zar}}/\mathcal{O}_X^*)_y) = \mathrm{rank}_{\mathbb{Z}}(P_{\mathfrak{p}'}/P_{\mathfrak{p}'}^*) = \mathrm{ht}(\mathfrak{p}')$. (The last equality follows from Proposition I.1.3 (2).) Since $\mathfrak{p} \subset \mathfrak{p}'$, this implies $\mathfrak{p} = \mathfrak{p}'$. □

Lemma II.4.10. *Let (X, M_X) be a regular fine saturated log scheme and assume that we are given a chart $P_X \to M_X$ with P saturated. Let x be a point of X of codimension 1 such that $M_{X,\bar{x}} \neq \mathcal{O}_{X,\bar{x}}^*$ and let $\mathfrak{p}$ denote the inverse image of the* *maximal ideal of* $\mathcal{O}_{X,\bar{x}}$ *in* P. *Then,* $\mathfrak{p}$ *is a prime ideal of height* 1 *and the composite* $P \to \mathcal{O}_{X,\bar{x}} \xrightarrow{v_{\bar{x}}} \mathbb{Z}$ *coincides with the valuation* $v_{\mathfrak{p}}$ *associated to* $\mathfrak{p}$, *where* $v_{\bar{x}}$ *denotes the discrete valuation of* $\mathcal{O}_{X,\bar{x}}$.

Proof. Note first that $\mathcal{O}_{X,\bar{x}}$ is a discrete valuation ring by Theorem II.4.7. By Proposition II.1.4 (1), we have $P_{\mathfrak{p}}/P^*_{\mathfrak{p}} \cong M_{X,\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$. Since (X, M_X) is regular and $M_{X,\bar{x}}/\mathcal{O}^*_{X,\bar{x}} \neq \{1\}$, the group $M^{\mathrm{gp}}_{X,\bar{x}}/\mathcal{O}^*_{X,\bar{x}}$ has rank 1 and the maximal ideal of $\mathcal{O}_{X,\bar{x}}$ is generated by the image of $I_{\bar{x}} := M_{X,\bar{x}} \backslash \mathcal{O}^*_{X,\bar{x}}$. By Proposition I.1.3 (2), $\mathrm{ht}(\mathfrak{p}) = 1$. On the other hand, we see easily that $I_{\bar{x}}$ is generated by the image of $\mathfrak{p}$ and hence the maximal ideal of $\mathcal{O}_{X,\bar{x}}$ is generated by the image of $\mathfrak{p}$. Since the inverse image of $\mathcal{O}^*_{X,\bar{x}}$ under the morphism $P_{\mathfrak{p}} \to \mathcal{O}_{X,\bar{x}}$ is $P^*_{\mathfrak{p}}$, this implies that the composite $P \to \mathcal{O}_{X,\bar{x}} \xrightarrow{v_{\bar{x}}} \mathbb{Z}$ coincides with $v_{\mathfrak{p}}$. □

Theorem II.4.11. *Let* $f : (X, M_X) \to (Y, M_Y)$ *be a smooth integral morphism of fine saturated log schemes and assume that* (Y, M_Y) *is regular. Then* f *is saturated if and only if, for every point* y *of* Y *of codimension* 1 *such that* $M_{Y,\bar{y}} \neq \mathcal{O}^*_{Y,\bar{y}}$, *the fiber of the underlying morphism of schemes of* f *over* y *satisfies* (R_0).

Proof. The necessity follows from Theorem II.4.2. We will prove the sufficiency. By Proposition II.4.8, (X, M_X) is regular. Take $x \in X$ and $y = f(x) \in Y$. We will prove that the morphism $(M_Y/\mathcal{O}^*_Y)_{\bar{y}} \to (M_X/\mathcal{O}^*_X)_{\bar{x}}$ induced by f is saturated. By Proposition II.1.4 (2), we may assume that we have a chart of f,

$$(\alpha : P_X \to M_X,\ \beta : Q_Y \to M_Y,\ h : Q \to P),$$

such that α (resp. β) induces an isomorphism $P/P^* \cong (M_X/\mathcal{O}^*_X)_{\bar{x}}$ (resp. $Q/Q^* \cong (M_Y/\mathcal{O}^*_Y)_{\bar{y}}$). By Lemma I.3.15, P and Q are saturated. By Proposition I.2.5, h is integral. Let $\mathfrak{p}$ be a prime ideal of P of height 1 such that the prime ideal $\mathfrak{q} := h^{-1}(\mathfrak{p})$ of Q is also of height 1. By Theorem I.5.1, it suffices to prove that the ramification index of h at $\mathfrak{p}$ is 1. By Lemma II.4.9, there exists a point $x' \in X$ of codimension 1 such that the inverse image of the maximal ideal of $\mathcal{O}_{X,\bar{x}'}$ in P is $\mathfrak{p}$ and the maximal ideal is generated by the image of $\mathfrak{p}$. Set $y' = f(x')$. Then, since the homomorphism $\mathcal{O}_{Y,\bar{y}'} \to \mathcal{O}_{X,\bar{x}'}$ is local, the inverse image of the maximal ideal of $\mathcal{O}_{Y,\bar{y}'}$ in Q is $\mathfrak{q}$ and hence β induces an isomorphism $Q_{\mathfrak{q}}/Q^*_{\mathfrak{q}} \cong (M_Y/\mathcal{O}^*_Y)_{\bar{y}'}$ by Proposition II.1.4 (1). Since $(Q_{\mathfrak{q}})^{\mathrm{gp}}/Q^*_{\mathfrak{q}}$ is of rank 1 and (Y, M_Y) is regular, we have $\dim(\mathcal{O}_{Y,\bar{y}'}) \geq 1$. On the other hand, the underlying morphism of schemes of f is flat by [Kato 1989, Corollary (4.5)]. Hence

$$\dim(\mathcal{O}_{Y,\bar{y}'}) = \dim(\mathcal{O}_{X,\bar{x}'}) = 1$$

and the codimension of x' in $f^{-1}(y')$ is 0. Since $M_{Y,\bar{y}'} \neq \mathcal{O}^*_{Y,\bar{y}'}$, the maximal ideal of $\mathcal{O}_{X,\bar{x}'}$ is generated by the image of the maximal ideal of $\mathcal{O}_{Y,\bar{y}'}$ by the assumption on f. By Lemma II.4.10, it follows that the ramification index of h at $\mathfrak{p}$ is 1. □

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