Tunisian Journal of Mathematics
an international publication organized by the Tunisian Mathematical Society

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2019 vol. 1 no. 3

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TUNISIAN JOURNAL OF MATHEMATICS
Vol. 1, No. 3, 2019
dx.doi.org/10.2140/tunis.2019.1.321

Local estimates for Hörmander's operators with Gevrey coefficients and application to the regularity of their Gevrey vectors

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Given a general Hörmander's operator $P = \sum_{j=1}^{m} X_j^2 + Y + b$ in an open set $\Omega \subset \mathbb{R}^n$, where $Y, X_1, \dots, X_m$ are smooth real vector fields in Ω, $b \in C^\infty(\Omega)$, and given also an open, relatively compact set Ω_0 with $\overline{\Omega}_0 \subset \Omega$, and $s \in \mathbb{R}$, $s \geq 1$, such that the coefficients of P are in $G^s(\Omega_0)$ and P satisfies a $\frac{1}{p}$-Sobolev estimate in Ω_0, our aim is to establish local estimates reflecting local domination of ordinary derivatives by powers of P, in Ω_0. As an application, we give a direct proof of the $G^{2ps}(\Omega_0)$-regularity of any $G^s(\Omega_0)$-vector of P.

1. Introduction

The study of the regularity of analytic vectors of partial differential operators goes back to the work of T. Kotake and N. S. Narasimhan [1962] who proved the (local) analyticity of (local) analytic vectors of elliptic operators with analytic coefficients (see also [Nelson 1959] for another related context). This property, called the "iteration property" or even the "Kotake–Narasimhan property" was further studied in the following decades in more general situations (such as systems, or nonelliptic operators) and also in the Gevrey categories G^s, $s \geq 1$ ($s = 1$ corresponds to the analytic case). This was in particular the case for the class of differential operators of principal type with analytic coefficients and also, after the famous article by L. Hörmander on hypoelliptic operators of second order, the systems of real-analytic real vector fields satisfying the so-called Hörmander condition, and

MSC2010: 35B65, 35G99, 35J70, 35K65.
Keywords: Gevrey vectors, Degenerate elliptic-parabolic differential operators.

also for the Hörmander's operators themselves. A result of G. Métivier [1978] shows that the "iteration property" is not true for nonelliptic operators in the Gevrey category G^s, $s > 1$, and another one by M.S. Baouendi and Métivier [1982] gave the "iteration property" for hypoelliptic operators of principal type in the analytic setting. Concerning analytic real vector fields, satisfying Hörmander's condition, that property was shown by M. Damlakhi and B. Helffer [1980], followed by a more precise version by Helffer and C. Mattera [1980].

More recently, a series of papers studying the case of involutive systems of analytic complex vector fields, concerning analytic or more generally Gevrey vectors, have been published [Barostichi et al. 2011; Castellanos et al. 2013]. Even knowing that the "iteration property" is not true for nonelliptic operators, one can ask about the Gevrey regularity $G^{s'}$ of an s-Gevrey vector ($s \geq 1$) and even give the best s' one may obtain. Such a study is contained in the above mentioned papers. A more recent paper by N. Braun Rodrigues, G. Chinni, P. Cordaro and M. Jahnke [Braun Rodrigues et al. 2016] was partly devoted to the study of global analytic vectors for some sums of squares on a product of two tori. A little later, we studied in [Derridj $\geq$ 2019] the case of $G^k(\Omega)$-vectors of Hörmander's operators of the first kind (or degenerate elliptic) with $G^k(\Omega)$ coefficients, Ω an open set in $\mathbb{R}^n$ (see the definitions in Section 2), and $k \in \mathbb{N}^*$ (in particular analytic vectors for $k = 1$), in which we proved an optimal result (the optimality following from the work in the global case by the authors of [Braun Rodrigues et al. 2016]).

In this paper, we consider general Hörmander's operators P (of the second kind or degenerate elliptic parabolic) for which we study the existence of local estimates giving local domination of the ordinary derivatives by powers of P, when the coefficients of P are in $G^s(\Omega_0)$, $\overline{\Omega}_0 \subset \Omega$, and P satisfies a "$\frac{1}{p}$-Sobolev estimate" on Ω_0 (see Section 2). This, with our main result Theorem 4.2, is used to obtain $G^{2ps}(\Omega_0)$-regularity for $G^s(\Omega_0)$-vectors of P ($s \geq 1$), providing therefore, first a direct proof, without using the method of addition of an extra variable, and second the result, with any $s \in \mathbb{R}$, $s \geq 1$. Let us remark that, in our preceding result, as we used the above method, our result was obtained there for $s = k \in \mathbb{N}^*$.

In a forthcoming paper, we will consider operators of the first kind, for which the integer p (giving the $\frac{1}{p}$-Sobolev estimate in Ω_0) is intimately related to the vector fields $(X_1, \dots, X_m)$ and prove finer local estimates of domination by powers of P, giving, as an application, the optimal $G^{ps}(\Omega_0)$ regularity for any $G^s(\Omega_0)$-vector of P. A complete survey on results in this field until 1987 may be found in [Bolley et al. 1987] and a more very recent short one may be found in [Derridj 2017].

We recall in Section 2 some definitions and elementary facts about the operators P, Gevrey functions and Gevrey vectors. Section 3 will be devoted to preliminary lemmas and propositions as a preparation for the proof of our main theorem.

Our main theorem will be proved in Section 4 and the last section is devoted to the application of the main result to the regularity of Gevrey vectors of P.

2. Some notation and definitions

We consider a system $(Y, X_1, \dots, X_m)$ of real vector fields with smooth coefficients on an open set $\Omega \subset \mathbb{R}^n$, and

$$P = \sum_{j=1}^{m} X_j^2 + Y + b, \quad b \in C^\infty(\Omega), \tag{2-1}$$

see [Hörmander 1967; Kohn 1978; Rothschild and Stein 1976]. Let us just recall Hörmander's condition for hypoellipticity in Ω of the operator (2-1):

The Lie algebra, $\mathrm{Lie}(Y, X_1, \dots, X_m)$, generated by the vector fields $Y, X_1, \dots, X_m$, is of maximal rank in Ω. (2-2)

Concretely, the family of brackets of all lengths of the vector fields $Y, X_1, \dots, X_m$ span at any point $x \in \Omega$ the tangent space at x.

Under the condition (2-2), Hörmander proved the following a priori subelliptic estimate which we briefly describe. The L^2-norm and Sobolev H^σ-norm are denoted by $\|\cdot\|$ and $\|\cdot\|_\sigma$.

Let us recall, below, some norms introduced by Hörmander [1967]:

$$\begin{aligned} |||v||| &= \|v\| + \textstyle\sum_{j=1}^m \|X_j v\|, \quad v \in \mathcal{D}(\Omega), \\ \|v\|' &= \sup\big\{ \|(v, w)\| : |||w||| < 1,\ w \in \mathcal{D}(\Omega) \big\} \le \|v\|. \end{aligned} \tag{2-3}$$

Theorem 2.1 [Hörmander 1967]. *Let $\Omega_0 \Subset \Omega$ and assume* (2-2). *Then there exist $\sigma > 0$ and $C > 0$ such that*

$$\|v\|_\sigma \le C(|||v||| + \|Yv\|'), \quad \textit{for all } v \in \mathcal{D}(\Omega_0). \tag{2-4}$$

We call (2-4) *a subelliptic estimate for P.*

The constants σ and C depend on Ω_0 and $Y, X_1, \dots, X_m$. More specifically, σ depends on the length of the brackets needed in order to span the tangent space at every point of $\overline{\Omega}_0$.

Now, elementary technical manipulations give, with $C_0 > 0$,

$$\|v\|_\sigma \le C_0(\|Pv\|' + \|v\|), \quad \text{for all } v \in \mathcal{D}(\Omega_0). \tag{2-5}$$

A particular ingredient in order to get (2-5) and which we need in the sequel is the set of obvious inequalities:

$$\|X_j v\|' \le C\|v\|, \quad \forall v \in \mathcal{D}(\Omega), \text{ for some } C > 0,\ j = 1, \dots, m. \tag{2-6}$$

We want to say a word on the case $Y = 0$, or more generally the case where in (2-2) one considers the Lie algebra $\mathrm{Lie}(X_1, \dots, X_m)$ generated by $X_1, \dots, X_m$; in that case, one has a more precise estimate. We considered that case in our preceding work, obtaining an optimal result for the Gevrey regularity of k-Gevrey vectors of P (named in that case Hörmander's operator of the first kind), $k \in \mathbb{N}^*$.

Coming back to our general case (named operator of the second kind), we need to write a more precise a priori estimate than (2-5) which we need to consider in the sequel:

$$\|v\|_\sigma + \sum_{j=1}^{m} \|X_j v\| \le C_0(\|Pv\|' + \|v\|), \quad \text{for all } v \in \mathcal{D}(\Omega_0). \tag{2-7}$$

Again this is easily obtained, using (2-4).

Let us recall, in order to be complete, definitions of Gevrey functions and Gevrey vectors of a differential operator of order m (here it will be $m = 2$).

Definition 2.2. Let $s \ge 1$. The space of Gevrey functions of order s, $G^s(\Omega)$, is defined as

$$G^s(\Omega) := \Big\{ u \in C^\infty(\Omega) : \forall K \Subset \Omega,\ \exists C_K > 0 \text{ s.t. } |\partial^\alpha u|_K \le C_K^{|\alpha|+1} |\alpha|!^s \ \ \forall \alpha \in \mathbb{N}^n,\ |\alpha| = \textstyle\sum_{j=1}^n \alpha_j,\ \partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \Big\}. \tag{2-8}$$

Remark 2.3. It is known that for $s > 1$, one has the property of partition of unity in $G^s(\Omega)$: in particular, given two open sets Ω_1, Ω_2, with $\overline{\Omega}_1$ compact and $\overline{\Omega}_1 \subset \Omega_2$, there exists a function $\varphi \in \mathcal{D}(\Omega_2)$, $\varphi \equiv 1$ on $\overline{\Omega}_1$, $\varphi \in G^s(\Omega_2)$.

Definition 2.4. Let P be a differential operator of order m in Ω. The space $G^s(\Omega, P)$ of s-Gevrey vectors of P in Ω, $s \ge 1$, is defined as

$$G^s(\Omega, P) := \Big\{ u \in L^2_{\mathrm{loc}}(\Omega) : \forall K \Subset \Omega,\ \exists C_K > 0 \text{ s.t. } \forall k \in \mathbb{N},\ P^k u \in L^2(K) \text{ and } \|P^k u\|_{L^2(K)} \le C_K^{k+1} (mk)!^s \Big\}. \tag{2-9}$$

As in our case, P is of order 2 and hypoelliptic, with a subelliptic estimate (2-5) or (2-7), (2-9) reduces to

$$G^s(\Omega, P) := \Big\{ u \in C^\infty(\Omega) : \forall K \Subset \Omega,\ \exists C_K > 0 \text{ s.t. } \forall k \in \mathbb{N},\ \|P^k u\|_{L^2(K)} \le C_K^{k+1} (2k)!^s \Big\}. \tag{2-10}$$

Remark 2.5. We used in our definitions (2-9), (2-10), the commonly used L^2-norm, but in some specific situations, such as for systems of complex vectors, other norms are used [Barostichi et al. 2011; Castellanos et al. 2013].

3. Preliminary lemmas and propositions

When trying to get estimates for derivatives $\partial^\alpha u$ of a function u, knowing Pu, we are faced in particular with the study of commutators $[P, \partial^\alpha]$, $\alpha \in \mathbb{N}^n$, so, to the study of commutators $[X_j^2, \partial^\alpha]$, $[Y+b, \partial^\alpha]$. Now, one has the following equality:

$$[X_j^2, \partial^\alpha] = 2X_j[X_j, \partial^\alpha] - [X_j, [X_j, \partial^\alpha]]. \tag{3-1}$$

So we have to look closely at the commutators $[X_j, \partial^\alpha]$, $[Y+b, \partial^\alpha]$ and the double commutators $[X_j, [X_j, \partial^\alpha]]$. In order to get an estimate for the coefficients of the differential operators above, it is sufficient to consider the following basic commutators

$$\begin{aligned} &[a_\ell \partial_\ell, \partial^\alpha], \quad \ell = 1, \dots, n, \qquad a_\ell \in G^s(\Omega), \\ &[a_\ell \partial_\ell, [a_k \partial_k, \partial^\alpha]], \quad \ell, k = 1, \dots, n, \quad a_\ell, a_k \in G^s(\Omega), \end{aligned} \tag{3-2}$$

as Y and X_j are linear combinations of the basic vector fields $a_\ell \partial_\ell$, $a_\ell \in G^s(\Omega)$.

Of course, the commutator $[b, \partial^\alpha]$ is elementary. Then

$$\begin{aligned} &[b, \partial^\alpha] = -\textstyle\sum_{\beta<\alpha} \binom{\alpha}{\beta} b^{(\alpha-\beta)} \partial^\beta = \sum_{\beta<\alpha} b_{\alpha\beta} \partial^\beta, \\ &\qquad\qquad \textstyle\binom{\alpha}{\beta} = \prod_j \binom{\alpha_j}{\beta_j}, \;\; \beta < \alpha \;\Leftrightarrow \beta_j \le \alpha_j, \;\; \beta \ne \alpha, \\ &[a_\ell \partial_\ell, \partial^\alpha] = -\textstyle\sum_{\beta<\alpha} \binom{\alpha}{\beta} a_\ell^{(\alpha-\beta)} \partial^{\beta+\ell}, \\ &\qquad\qquad \beta + \ell = \big\{(\beta+\ell)_i = \beta_i, \;\; i \ne \ell, \;\; (\beta+\ell)_\ell = \beta_\ell + 1\big\}. \end{aligned} \tag{3-3}$$

Let us, now, give the expression of $[a_\ell \partial_\ell, [a_k \partial_k, \partial^\alpha]]$ or, in view of (3-3),

$$\sum_{\beta<\alpha} \binom{\alpha}{\beta} [a_k^{(\alpha-\beta)}, \partial^{\beta+k}, a_\ell \partial_\ell].$$

But for $\beta < \alpha$,

$$[a_k^{(\alpha-\beta)} \partial^{\beta+k}, a_\ell \partial_\ell] = \sum_{\gamma<\beta+k} \binom{\beta+k}{\gamma} a_k^{(\alpha-\beta)} a_\ell^{(\beta+k-\gamma)} \partial^{\gamma+\ell} - a_\ell a_k^{(\alpha-\beta+\ell)} \partial^{\beta+k}.$$

Hence we get

$$\begin{aligned} &[a_\ell \partial_\ell, [a_k \partial_k, \partial^\alpha]] \\ &\quad = \sum_{\substack{\gamma<\beta+k, \\ \beta<\alpha}} \binom{\alpha}{\beta} \binom{\beta+k}{\gamma} a_k^{(\alpha-\beta)} a_\ell^{(\beta+k-\gamma)} \partial^{\gamma+\ell} - \sum_{\beta<\alpha} \binom{\alpha}{\beta} a_\ell a_k^{(\alpha-\beta+\ell)} \partial^{\beta+k}. \end{aligned} \tag{3-4}$$

In the first sum in the second member of (3-4), we distinguish two families of γ's such that $\gamma < \beta + k$:

(i) If $\gamma_k = 0$, then $\gamma \le \beta$, so $\gamma < \alpha$ (as $\beta < \alpha$). Hence in that case

$$\begin{aligned}\textstyle\sum_{\beta<\alpha,\,\gamma<\beta+k,\,\gamma_k=0}\binom{\alpha}{\beta}\binom{\beta+k}{\gamma}a_k^{(\alpha-\beta)}a_\ell^{(\beta+k-\gamma)}\partial^{\gamma+\ell} \quad \text{is a part of}\\ \textstyle\sum_{\beta<\alpha,\,\gamma\le\beta}\binom{\alpha}{\beta}\binom{\beta+k}{\gamma}a_k^{(\alpha-\beta)}a_\ell^{(\beta+k-\gamma)}\partial^{\gamma+\ell}.\end{aligned} \tag{3-5}$$

(ii) If $\gamma_k \ne 0$, we set $\delta = (\delta_1, \dots, \delta_n)$ with $\delta_\rho = \gamma_\rho$ if $\rho \ne k$ and $\delta_k = \gamma_k - 1$. So $\delta < \beta < \alpha$, in particular $|\delta| \le |\alpha| - 2$ (easy to see) and $\gamma = \delta + k$. Hence

$$\begin{aligned}\textstyle\sum_{\beta<\alpha,\,\gamma<\beta+k,\gamma_k\ne0}\binom{\alpha}{\beta}\binom{\beta+k}{\gamma}a_k^{(\alpha-\beta)}a_\ell^{(\beta+k-\gamma)}\partial^{\gamma+\ell} \quad \text{is a part of}\\ \textstyle\sum_{\beta<\alpha,\,\delta\le\beta}\binom{\alpha}{\beta}\binom{\beta+k}{\delta+k}a_k^{(\alpha-\beta)}a_\ell^{(\beta-\delta)}\partial^{\delta+\ell+k}.\end{aligned} \tag{3-6}$$

Setting $I_{(\ell,k)} = (I_1, \dots, I_n)$ with $I_\rho = 0$ if $\rho \notin (\ell, k)$ and $I_\rho = 1$ if $\rho \in \{\ell, k\}$, the sum in the second line of (3-6) can be written as

$$\sum_{\substack{\beta<\alpha\\ \delta<\beta}}\binom{\alpha}{\beta}\binom{\beta+k}{\delta+k}a_k^{(\alpha-\beta)}\partial^{\delta+I_{(\ell,k)}} \tag{3-7}$$

So, looking at (3-4) and in view of (3-5), (3-6) and taking as 0 the other coefficients in $\sum_{\beta<\alpha,\,\gamma\le\beta}$ (in (3-5)) and $\sum_{\beta<\alpha,\,\delta<\beta}$ (in (3-6)) which are not in (3-4), we may write

$$\begin{aligned}&[a_\ell\partial_\ell, [a_k\partial_k, \partial^\alpha]]\\ &\qquad = -\sum_{\beta<\alpha}a_{\ell k\alpha\beta}\partial^{\beta+k} + \sum_{\gamma<\alpha}b_{\ell k\alpha\gamma}\partial^{\gamma+\ell} + \sum_{\substack{\delta<\alpha\\ |\delta|\le|\alpha|-2}}c_{\ell k\alpha\delta}\partial^{\delta+I_{(\ell,k)}},\\ &\qquad\qquad \text{where}\quad a_{\ell k\alpha\beta} = \tbinom{\alpha}{\beta}a_\ell a_k^{(\alpha-\beta+\ell)},\\ &\qquad\qquad\qquad\quad b_{\ell k\alpha\gamma} = \textstyle\sum_{\gamma\le\beta<\alpha}\binom{\alpha}{\beta}\binom{\beta+k}{\gamma}a_k^{(\alpha-\beta)}a_\ell^{(\beta+k-\gamma)},\\ &\qquad\qquad\qquad\quad c_{\ell k\alpha\delta} = \textstyle\sum_{\delta<\beta<\delta}\binom{\alpha}{\beta}\binom{\beta+k}{\delta+k}a_k^{(\alpha-\beta)}a_\ell^{(\beta-\delta)}.\end{aligned} \tag{3-8}$$

Now considering a vector field X_j with smooth coefficients $a_{j\ell}$, i.e.,

$$X_j = \sum_{\ell=1}^n a_{j\ell}\partial_\ell = \sum_{\ell=1}^n Z_{j\ell},$$

we obtain from (3-3)

$$\begin{aligned}[][X_j, \partial^\alpha] &= \sum_{\ell=1}^n[Z_{j\ell}, \partial^\alpha] = -\sum_{\substack{\beta<\alpha\\ \ell=1,\dots,n}}\binom{\alpha}{\beta}a_{j\ell}^{(\alpha-\beta)}\partial^{\beta+\ell} = \sum_{\substack{\beta<\alpha\\ \ell=1,\dots,n}}a_{j\alpha\beta\ell}\partial^{\beta+\ell},\\ [Y, \partial^\alpha] &= \sum_{\ell=1}^n[b_\ell\partial_\ell, \partial^\alpha] = -\sum_{\substack{\beta<\alpha\\ \ell=1,\dots,n}}\binom{\alpha}{\beta}b_\ell^{(\alpha-\beta)}\partial^{\beta+\ell} = \sum_{\substack{\beta<\alpha\\ \ell=1,\dots,n}}b_{\alpha\beta\ell}\partial^{\beta\ell}.\end{aligned} \tag{3-9}$$

Concerning the double brackets, we obtain:

$$[X_j,[X_j\partial^\alpha]]=\textstyle\sum_{\ell,k}[Z_{j\ell},[Z_{jk},\partial^\alpha]],\quad \text{with } [Z_{j\ell},[Z_{k\ell},\partial^\alpha]] \text{ given in (3-8),}$$
$$\text{where } a_{\ell k\alpha\beta} \text{ is replaced by } a_{j\ell k\alpha\beta}, \text{ and so on.} \tag{3-10}$$

Now we assume that the coefficients of P are in $G^s(\Omega)$. So $a_{j\ell}$, b_ℓ and b are in $G^s(\Omega)$. So we have that for any compact K in Ω, there exists $C_K>0$ such that for all $\nu\in\mathbb{N}^n$,

$$|a^{(\nu)}_{j\ell}|_K+|b^{(\nu)}_\ell|_K+|b^{(\nu)}|_K\le C_K^{|\nu|+1}|\nu|!^s,\quad \ell\in\{1,\dots,n\},\ j\in\{1,\dots,m\}. \tag{3-11}$$

Writing (3-10) more concretely, we get using (3-8)

$$[X_j,[X_j,\partial^\alpha]]=\sum_{\substack{\beta<\alpha\\ k=1,\dots,n}} d_{jk\alpha\beta}\partial^{\beta+k}+\sum_{\substack{\beta<\alpha\\ k,\ell=1,\dots,n}} c_{j\ell k\alpha\beta}\partial^{\beta+\ell+k},$$
$$\text{where } d_{jk\alpha\beta}=-\textstyle\sum_{\ell=1}^n a_{j\ell k\alpha\beta}+\sum_{\ell=1}^n b_{jk\ell\alpha\beta}. \tag{3-12}$$

We want now to get estimates for the coefficients of the brackets and double brackets in (3-9) and (3-12) and the operators associated to these brackets, when the coefficients are in $G^s(\Omega)$.

Proposition 3.1. *Assume the coefficients of P are in $G^s(\Omega)$. Given any compact set K in Ω, there exists $B=B_K>0$ such that the coefficients of the operators $[b,\partial^\alpha]$ (given in* (3-3)*), $[X_j,\partial^\alpha]$, $[Y,\partial^\alpha]$ (given in* (3-9)*), $[X_j,[X_j,\partial^\alpha]]$ (given in* (3-12)*) satisfy the following estimates*:

$$\begin{aligned}|b_{\alpha\beta}|_K+|b_{\alpha\beta\ell}|_K+|a_{j\alpha\beta\ell}|_K+|\nabla b_{\alpha\beta}|_K&\\ +|\nabla b_{\alpha\beta\ell}|_K+|\nabla a_{j\alpha\beta\ell}|_K&\le B^{|\alpha-\beta|}\big(\tfrac{\alpha!}{\beta!}\big)^s,\\ |c_{j\ell\alpha\beta}|_K+|\nabla c_{j\ell k\alpha\beta}|_K&\le B^{|\alpha-\beta|}\big(\tfrac{(\alpha+k)!}{(\beta+k)!}\big)^s,\\ |d_{jk\alpha\beta}|_K+|\nabla d_{jk\alpha\beta}|_K&\le B^{|\alpha-\beta|}\big((|\alpha|+1)\tfrac{\alpha!}{\beta!}\big)^s.\end{aligned} \tag{3-13}$$

Proof. We recall first that $\beta<\alpha$ and $(\alpha+k)_\ell=\alpha_\ell$ for $\ell\ne k$ and $(\alpha+k)_k=\alpha_k+1$. The first line comes easily from the expression of the functions $b_{\alpha\beta}$, $b_{\alpha\beta\ell}$, $a_{j\alpha\beta\ell}$, and their derivatives. Note that we took $B^{|\alpha-\beta|}$ in place of $B^{|\alpha-\beta|+1}$ as $|\alpha-\beta|\ge1$ and used (3-11). The proof for the other functions needs more work. We first use the following estimate for $a^{(\alpha-\beta)}_{jk}a^{(\beta-\delta)}_{j\ell}$, which is in the expression of $c_{j\ell k\alpha\delta}$ (see last line in (3-8)):

$$\left|a^{(\alpha-\beta)}_{jk}a^{(\beta-\delta)}_\ell\binom{\alpha}{\beta}\binom{\beta+k}{\delta+k}\right|\le B^{|\alpha-\beta|}(\lambda B)^{|\beta-\delta|}\Big(\frac{(\alpha+k)!}{(\delta+k)!}\Big)^s,\quad \lambda\ge1. \tag{3-14}$$

For that we used

$$\frac{\alpha!}{\beta!}\frac{(\beta+k)!}{(\delta+k)!}=\frac{\alpha!(\beta_k+1)}{(\delta+k)!}\le\frac{(\alpha+k)!}{(\delta+k)!}.$$

Hence

$$|c_{j\ell k\alpha\delta}|_K \le \Big(\sum_{\delta<\beta<\alpha} B^{|\alpha-\beta|}(\lambda B)^{|\beta-\delta|}\Big)\Big(\frac{(\alpha+k)!}{(\delta+k)!}\Big)^s, \quad \lambda \ge 1.$$

As $(\lambda B)^{|\beta-\delta|} = (\lambda B)^{|\alpha-\delta|}(\lambda B)^{-|\alpha-\beta|}$, $(\delta < \beta < \alpha)$, we get:

$$|c_{j\ell k\alpha\delta}|_K \le \Big(\frac{(\alpha+k)!}{(\delta+k)!}\Big)^s (\lambda B)^{|\alpha-\delta|} \sum_{\beta<\alpha} \lambda^{-|\alpha-\beta|}.$$

Now we use the following easy lemma:

Lemma 3.2. *There exists $\epsilon_0 > 0$ (independent from α) such that if $0 < \epsilon \le \epsilon_0$ then $\sum_{\beta<\alpha} \epsilon^{|\alpha-\beta|} \le 1$.*

This comes from the fact that if $\beta_j \le \alpha_j$, $j = 1, \dots, n$, and $\alpha \ne \beta$ then $\sum_{j=1}^n \beta_j < \sum_{j=1}^n \alpha_j$. In fact, setting $\lambda_j = \alpha_j - \beta_j \in \mathbb{N}$, $\sum_{j=1}^n \lambda_j \ge 1$,

$$\begin{aligned}\sum \epsilon^{\sum \lambda_j} &\le \sum_{\lambda_1 \ge 1} \epsilon^{\sum \lambda_j} + \cdots + \sum_{\lambda_n \ge 1} \epsilon^{\sum \lambda_j} \\ &\le \epsilon \Big(\sum_{(\lambda_2,\dots,\lambda_n)\in\mathbb{N}^{n-1}} \epsilon^{\lambda_2+\dots+\lambda_n} + \cdots + \sum_{(\lambda_1,\dots,\lambda_{n-1})\in\mathbb{N}^{n-1}} \epsilon^{\lambda_1+\dots+\lambda_{n-1}} \Big) \\ &\le \epsilon n 2^{n-1} \quad \text{if } \epsilon \le \tfrac{1}{2}.\end{aligned}$$

The lemma follows by taking $\epsilon_0 = (n2^{n-1})^{-1}$.

Coming back to our proof, we consider $\lambda_0 = n2^{n-1} = \epsilon_0^{-1}$ and replace B by $\lambda_0 B$. We get the estimate for $c_{jk\alpha\beta}$. As $d_{jk\alpha\beta} = -\sum_{\ell=1}^n (a_{j\ell k\alpha\beta} - b_{jk\ell\alpha\beta})$, we just need to bound on K the functions $a_{j\ell k\alpha\beta}$ and $b_{jk\ell\alpha\beta}$ for $\ell = 1, \dots, n$. The worst term is $b_{jk\ell\alpha\gamma}$. As we did above, we use, with $\lambda \ge 1$,

$$\Big|\binom{\alpha}{\beta}\binom{\beta+\ell}{\gamma} a_\ell^{(\alpha-\beta)} a_k^{(\beta+\ell-\gamma)}\Big| \le B^{|\alpha-\beta|+1}(\lambda B)^{|\beta-\gamma|}\Big(\frac{\alpha!}{\gamma!}(\beta_\ell+1)\Big)^s.$$

So,

$$\begin{aligned}|b_{jk\ell\alpha\gamma}| &\le \Big(\sum_{\gamma\le\beta<\alpha} B^{|\alpha-\beta|+1}(\lambda B)^{|\beta-\gamma|}\Big)\Big(\frac{(\alpha+\ell)!}{\gamma!}\Big)^s \\ &\le \Big(\frac{(\alpha+\ell)!}{\gamma!}\Big)^s B(\lambda B)^{|\alpha-\gamma|} \sum_{\beta<\alpha} \lambda^{-|\alpha-\beta|} \\ &\le B\Big(\frac{\alpha!}{\gamma!}\Big)^s (|\alpha|+1)^s (\lambda B)^{|\alpha-\gamma|} \sum_{\beta<\alpha} \lambda^{-|\alpha-\beta|}.\end{aligned}$$

Now taking $\lambda \ge \lambda_0$, B large enough, we obtain what we want (more precisely we take $\tilde{B}$ such that (as $|\alpha-\gamma| \ge 1$), $B(\lambda B)^{|\alpha-\gamma|} \le (\lambda\tilde{B})^{|\alpha-\gamma|}$, and then choose the final B as $\lambda\tilde{B}$). Concerning the derivatives of first order, we just have to apply what we did, using bounds on K, not only for the coefficients of P, but also bounds of their derivatives of first order. In order to be rigorous and complete, let us bound a derivative of $b_{j\ell k\alpha\gamma}$, which we denote by $b'_{j\ell k\alpha\gamma}$. Then we have (see (3-8))

$$b'_{j\ell k\alpha\beta} = \sum_{\gamma\le\beta<\alpha} \binom{\alpha}{\beta}\binom{\beta+k}{\gamma}\big((a_{jk}^{(\alpha-\beta)})' a_{j\ell}^{(\beta+k-\gamma)} + a_{jk}^{(\alpha-\beta)}(a_{j\ell}^{(\beta+k-\gamma)})'\big).$$

So we just have to do the same as before to get the bounds on K for the functions a'_{jk}, $a_{j\ell}$, a_{jk} and $a'_{j\ell}$. Hence taking B, greater if necessary, we obtain (3-13). □

As a consequence, we obtain the following:

Proposition 3.3. *Assume that the coefficients of P are in $G^s(\Omega)$. Then for every relatively compact open set Ω_0 in Ω ($\Omega_0 \Subset \Omega$), there exists $B = B(\Omega_0, P) > 0$ such that, for every τ, $0 \le \tau \le 1$, one has for $j = 1, \dots, m$, $\beta < \alpha$, $\ell, k \in \{1, \dots, n\}$,*

$$\begin{aligned} &\|b_{\alpha\beta}v\|_\tau + \|b_{\alpha\beta\ell}v\|_\tau + \|a_{j\alpha\beta\ell}v\|_\tau \\ &\qquad\qquad \le B^{|\alpha-\beta|}\Big(\frac{\alpha!}{\beta!}\Big)^s \|v\|_\tau, \quad \beta<\alpha,\ \ell = 1,\dots,n, \\ &\|c_{j\ell k\alpha\beta}v\|_\tau \le B^{|\alpha-\beta|}\Big(\frac{(\alpha+k)!}{(\beta+k)!}\Big)^s \|v\|_\tau, \\ &\|d_{jk\alpha\beta}v\|_\tau \le B^{|\alpha-\beta|}\Big(\frac{\alpha!(|\alpha|+1)}{\beta!}\Big)^s \|v\|_\tau, \quad \forall v \in \mathcal{D}(\Omega_0). \end{aligned} \tag{3-15}$$

Proof. From inequalities (3-13), one obtains estimates (3-15) for $\tau = 0$ and $\tau = 1$. Then (3-15) follows from the cases $\tau = 0$ and $\tau = 1$ by interpolation between Sobolev spaces L^2 and H^1. □

Remark 3.4. Our Proposition 3.1 is a refinement of our Lemma 5.3 in [Derridj ≥ 2019]. We need this refinement in order to prove our local estimates of ordinary derivatives in terms of powers of P.

In order to begin to state what we need for our results, we make the following assumption:

$$\text{Estimate (2-4) is true with} \quad \sigma = \frac{1}{p}, \quad p \in \mathbb{N}^*. \tag{3-16}$$

Then our local estimates will use the equality $\sigma = \frac{1}{p}$. In this section, we want to give and prove the basic ones which we will use in another section in order to give a sequence of local estimates for P, assuming (2-7), (3-16) and P with coefficients in $G^s(\Omega)$, for some $s \ge 1$.

Proposition 3.5. *Assume* (2-4), (3-16) *and that P has smooth coefficients. Let Ω_1 be a relatively compact open subset of Ω_0. Then there exists a constant $C = C(\Omega_0, P) > 0$ such that for all $(u, \varphi) \in C^\infty(\bar{\Omega}_1) \cap \mathcal{D}(\Omega_1)$,*

$$\|\varphi u\|_\sigma \le C\Big(\|\varphi P u\|' + \sum_{|\beta|\le 2} \|\varphi^{(\beta)} u\|\Big). \tag{3-17}$$

Proof. This proposition has a simple proof. Taking $v=\varphi u$ in (2-7), we get

$$\|\varphi u\|_\sigma+\sum_{j=1}^m\|X_j\varphi u\|\le C_0(\|P\varphi u\|'+\|\varphi u\|),$$

taking $\Omega_0=\Omega_1$ and C_0 as in (2-7) related to $\overline{\Omega}_0\Subset\Omega$, we have

$$\|P\varphi u\|'\le\|\varphi Pu\|'+\|[P,\varphi]u\|', \tag{3-18}$$

with

$$\begin{aligned}\|[P,\varphi]u\|'&\le 2\sum_{j=1}^m\bigl(\|X_j[X_j,\varphi]u\|'+\|X_j^2(\varphi)u\|'\bigr)+\|Y(\varphi)u\|'\\&\le 2\sum_{j=1}^m\bigl(\|X_j(\varphi)u\|+\|X_j^2(\varphi)u\|\bigr)+\|Y(\varphi)u\|,\quad\text{from (2-3), (2-6).}\end{aligned}$$

Now as the X_j, Y are smooth in Ω and $\overline{\Omega}_0\subset\Omega$,

$$\|X_j(\varphi)u\|\le C_1\sum_{|\beta|\le1}\|\varphi^{(\beta)}u\|,\quad \|X_j^2(\varphi)u\|\le C_1\sum_{|\beta|\le2}\|\varphi^{(\beta)}u\|. \tag{3-19}$$

So, with a suitable C, (3-17) is obtained from (3-18), (3-19). □

In order to reach estimates for ordinary derivatives, one way is to try to obtain estimates for φu in the Sobolev spaces $H^{\ell\sigma}$, $\ell=1,\dots,p$, (3-17) corresponding to $\ell=1$. As the basic estimate (2-6) is with H^σ (i.e., $\ell=1$), it is natural to use the following, recalling some notation and definitions in [Derridj ≥ 2019, Section 2]: let Ω_0, be such that $\overline{\Omega}_1\subset\Omega_0\subset\overline{\Omega}_0\Subset\Omega$. So we consider $\psi\in\mathcal{D}(\Omega_0)$, $\psi=1$ on $\overline{\Omega}_1$; one way to estimate $\|v\|_{\ell\sigma}$, knowing (2-7) is to consider $\psi T_{\ell\sigma}v$, $v\in\mathcal{D}(\Omega_1)$; in fact, for all $v\in\mathcal{D}(\Omega_1)$,

$$\|v\|_{(\ell+1)\sigma}=\|\psi v\|_{(\ell+1)\sigma}=\|T_{(\ell+1)\sigma}\psi v\|=\|T_{\ell\sigma}\psi v\|_\sigma,$$

(see [Derridj ≥ 2019, 2.11-2.13]),

where $\widehat{T_\rho w}(\xi)=(1+\|\xi\|^2)^{\rho/2}\widehat{w}(\xi)$, for all $w\in\mathcal{D}(\mathbb{R}^n)$. Hence

$$\|v\|_{(\ell+1)\sigma}\le\|[T_{\ell\sigma},\psi]v\|_\sigma+\|\psi T_{\ell\sigma}v\|_\sigma\le C_2\|v\|_{(\ell+1)\sigma-1}+\|\psi T_{\ell\sigma}v\|_\sigma.$$

Now for $\ell=1,\dots,p-1$, $(\ell+1)\sigma-1\le0$. So

$$\|v\|_{(\ell+1)\sigma}\le\tilde C\|v\|+\|\psi T_{\ell\sigma}v\|_\sigma,\quad\text{for all } v\in\mathcal{D}(\Omega_1). \tag{3-20}$$

So this boils down to applying (2-7) to $\psi T_{\ell\sigma}v\in\mathcal{D}(\Omega_0)$ using the constant C_0. Then we have

$$\|v\|_{(\ell+1)\sigma}\le\tilde C(\|v\|+\|\psi T_{\ell\sigma}v\|_\sigma)\le\tilde C\biggl(\|\psi T_{\ell\sigma}v\|+\sum_j\|X_j\psi T_{\ell\sigma}v\|+\|v\|\biggr).$$

So, we get:

$$\|\varphi u\|_{(\ell+1)\sigma} \le C_0\tilde{C}\big(\|P\psi T_{\ell\sigma}\varphi u\|' + \|\psi T_{\ell\sigma}\varphi u\| + \|\varphi u\|\big),$$

yielding

$$\|\varphi u\|_{(\ell+1)\sigma} \le C_3\big(\|P\psi T_{\ell\sigma}\varphi u\|' + \|\varphi u\|_{\ell\sigma}\big), \qquad \text{for all } (u,\varphi) \in C^\infty(\overline{\Omega}_1) \times \mathfrak{D}(\Omega_1). \tag{3-21}$$

Now we provide a suitable bound for $\|P\psi T_{\ell\sigma}\varphi u\|'$ as follows:

$$\begin{aligned}
&\|P\psi T_{\ell\sigma}\varphi u\|' \\
&\quad \le \|[P,\psi T_{\ell\sigma}]\varphi u\|' + \|\psi T_{\ell\sigma}[P,\varphi]u\|' + \|\psi T_{\ell\sigma}\varphi P u\|' \\
&\quad \le \sum_{j=1}^{m}\big(2\|X_j[X_j,\psi T_{\ell\sigma}]\varphi u\|' + \|[X_j,[X_j,\psi T_{\ell\sigma}]]\varphi u\|\big) + \|[Y+b,\psi T_{\ell\sigma}]\varphi u\| \\
&\qquad + \sum_{j=1}^{m}\big(2\|\psi T_{\ell\sigma}X_j[X_j,\varphi]u\|' + \|\psi T_{\ell\sigma}[X_j[X_j,\varphi]]u\|'\big) \\
&\qquad\qquad + \|\psi T_{\ell\sigma}[Y,\varphi]u\|' + \|\psi T_{\ell\sigma}\varphi P u\|'.
\end{aligned}$$

Using again inequalities in (2-3), (2-6), we get, with some constant C which may vary from line to another,

$$\begin{aligned}
&\|P\psi T_{\ell\sigma}\varphi u\|' \\
&\quad \le C\bigg(\|\varphi u\|_{\ell\sigma} + \sum_{j=1}^{m}\Big(2\|[\psi T_{\ell\sigma},X_j][X_j,\varphi]u\|' + \|X_j\psi T_{\ell\sigma}[X_j,\varphi]u\|' \\
&\qquad\qquad + \|X_j^2(\varphi)u\|_{\ell\sigma}\Big) + \|Y(\varphi)u\|_{\ell\sigma}\bigg) + \|\psi T_{\ell\sigma}\varphi P u\|' \\
&\quad \le C\bigg(\|\psi T_{\ell\sigma}\varphi P u\|' + \sum_{j=1}^{m}\big(\|X_j(\varphi)u\|_{\ell\sigma} + \|X_j^2(\varphi)u\|_{\ell\sigma} + \|\varphi u\|_{\ell\sigma} \\
&\qquad\qquad + \|Y(\varphi)u\|_{\ell\sigma}\big)\bigg).
\end{aligned} \tag{3-22}$$

Hence, in fact, we proved the following with Ω_0 as above:

Proposition 3.6. *Under the hypotheses in Proposition 3.5, there exists a constant $C = C(\Omega_0, P) > 0$ such that*

$$\|\varphi u\|_{(\ell+1)\sigma} \le C\bigg(\|\varphi P u\|_{\ell\sigma} + \sum_{|\beta|\le 2}\|\varphi^{(\beta)}u\|_{\ell\sigma}\bigg), \quad \ell = 0,\dots,p-1. \tag{3-23}$$

The next step is to obtain such estimates for couples $(\partial^\alpha u, \varphi)$ for $(u,\varphi) \in C^\infty(\overline{\Omega}_1) \times \mathfrak{D}(\Omega_1)$, $\alpha \in \mathbb{N}^n$. In order to get such an estimate, let us first consider the case $\ell = 0$. We use (3-17). So

$$\|\varphi\partial^\alpha u\|_\sigma \le C\|\varphi P\partial^\alpha u\|' + \sum_{|\beta|\le 2}\|\varphi^{(\beta)}\partial^\alpha u\|.$$

Here, what is new is to use $\varphi P\partial^\alpha u = \varphi[P, \partial^\alpha]u + \varphi\partial^\alpha Pu$:

$$\begin{aligned}\|\varphi P\partial^\alpha u\|' &\le \|\varphi[P, \partial^\alpha]u\|' + \|\varphi\partial^\alpha Pu\|' \\ &\le \sum_{j=1}^m \big(2\|\varphi X_j[X_j, \partial^\alpha]u\|' + \|\varphi[X_j, [X_j, \partial^\alpha]]u\|'\big) \\ &\qquad\qquad + \|\varphi[Y+b, \partial^\alpha]u\|' + \|\varphi\partial^\alpha Pu\|'. \quad (3\text{-}24)\end{aligned}$$

Writing $\varphi X_j[X_j, \partial^\alpha]u = [\varphi, X_j][X_j, \partial^\alpha]u + X_j\varphi[X_j, \partial^\alpha]u$ and using again (2-3) and (2-6), we get

$$\begin{aligned}\|\varphi P\partial^\alpha u\| \le \sum_{j=1}^m \big(2\|X_j(\varphi)[X_j, \partial^\alpha]u\| + \|\varphi[X_j, [X_j, \partial^\alpha]]u\|\big) \\ + \|\varphi[Y+b, \partial^\alpha]u\| + \|\varphi\partial^\alpha Pu\|,\end{aligned}$$

and then we have to use expressions in (3-9) and (3-12). As we have to do that in order to bound $\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma}$, we will write the step after the use of (3-9), (3-12) just in this general case; replacing u by $\partial^\alpha u$ in (3-22), we get, with some constant $C > 0$, which may vary from line to line,

$$\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} \le C\bigg(\|\psi T_{\ell\sigma}\varphi P\partial^\alpha u\|' + \sum_{|\beta|\le 2} \|\varphi^{(\beta)}\partial^\alpha u\|_{\ell\sigma}\bigg). \quad (3\text{-}25)$$

Now, as we did above with (3-24), we get

$$\begin{aligned}&\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} \\ &\le C\bigg(\|\psi T_{\ell\sigma}\varphi\partial^\alpha Pu\|' + \sum_{j=1}^m \big(\|X_j(\varphi)[X_j, \partial^\alpha]u\|_{\ell\sigma} + \|\varphi[X_j[X_j, \partial^\alpha]]u\|_{\ell\sigma}\big) \\ &\qquad\qquad + \|\varphi[Y+b, \partial^\alpha]u\|_{\ell\sigma} + \sum_{|\beta|\le 2} \|\varphi^{(\beta)}\partial^\alpha u\|_{\ell\sigma}\bigg).\end{aligned}$$

As $\ell\sigma \le 1$, ($\ell = 0, \dots, p-1$), we finally obtain:

Proposition 3.7. *There exists a constant $C > 0$ such that*

$$\text{for all } (u, \varphi) \in C^\infty(\overline{\Omega}_1) \times \mathcal{D}(\Omega_1), \quad \text{and all } \alpha \in \mathbb{N}^n,$$

$$\begin{aligned}&\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} \\ &\le C\Bigg(\|\varphi\partial^\alpha Pu\|_{\ell\sigma} + \sum_{|\beta|\le 2} \|\varphi^{(\beta)}\partial^\alpha u\|_{\ell\sigma} \\ &\qquad + \sum_{j=1}^m \bigg(\sum_{|\beta|=1} \|\varphi^{(\beta)}[X_j, \partial^\alpha]u\|_{\ell\sigma} + \|\varphi[X_j, [X_j, \partial^\alpha]]u\|_{\ell\sigma}\bigg) \\ &\qquad\qquad + \|\varphi[Y+b, \partial^\alpha]u\|_{\ell\sigma}\Bigg). \quad (3\text{-}26)\end{aligned}$$

In what follows, s is given and $s \ge 1$.

4. Local relations of domination by powers of P

We need to introduce, in this section, further notation:

For $\epsilon > 0$, $j \in \mathbb{N}$, $\gamma \in \mathbb{N}^n$, and $(u, \varphi) \in C^\infty(\bar{\Omega}_1) \times \mathfrak{D}(\Omega_1)$, we set

$$N^\epsilon_{j,\gamma}(u, \varphi) = \epsilon^{|\gamma|+2j} |\gamma|!^{-s} (2j)!^{-s} \|\varphi^{(\gamma)} P^j u\|. \quad (4\text{-}1)$$

Once ϵ is fixed, we often delete ϵ and write $N^\epsilon_{j,\gamma} = N_{j,\gamma}$.

Before stating our main theorem, we give a simple useful lemma, which we will apply many times.

Lemma 4.1. *Let* $(k, \beta) \in \mathbb{N} \times \mathbb{N}^n$, *and* $\rho \in \mathbb{N}$. *Then*

$$\rho!^s N^\epsilon_{j,\gamma}(P^k u, \varphi^{(\beta)}) \le \epsilon^{-(|\beta|+2k)} (\rho + |\beta| + 2k)!^s N^\epsilon_{j+k,\gamma+\beta}(u, \varphi), \quad \textit{if } |\gamma| + 2j \le \rho. \quad (4\text{-}2)$$

Proof. From the definition in (4-1), we see that

$$N^\epsilon_{j,\gamma}(P^k u, \varphi^{(\beta)}) = \epsilon^{-(|\beta|+2k)} (|\gamma|+1)^s \cdots (|\gamma|+|\beta|)^s \big((2j+1)\cdots(2j+2k)\big)^s N_{j+k,\gamma+\beta}(u, \varphi).$$

Then, observe that, for $|\gamma| + 2j \le \rho$,

$$\rho!(|\gamma|+1)\cdots(|\gamma|+|\beta|)(2j+1)\cdots(2(j+k)) \le (\rho + |\beta| + 2k)!. \quad (4\text{-}3)$$

The proof is then finished by taking (4-3) to the power $s \ge 1$. □

Theorem 4.2. *Let $s \ge 1$ be given. Assume that the coefficients of P are in $G^s(\Omega)$ and properties* (2-4) *and* (3-16) *hold on Ω_0, $\bar{\Omega}_0 \subset \Omega$. Let Ω_1 be an open set with $\bar{\Omega}_1 \subset \Omega_0$. For every $0 < \epsilon \le 1$, there exists $M = M(\epsilon, \Omega_0, P) \ge 1$ such that for all $\alpha \in \mathbb{N}^n$, $\ell = 0, \ldots, p$, $(u, \varphi) \in C^\infty(\bar{\Omega}_1) \times \mathfrak{D}(\Omega_1)$,*

$$\|\varphi \partial^\alpha u\|_{\ell\sigma} \le M^{2p|\alpha|+\ell+1} (2(p|\alpha|+\ell))!^s \sum_{|\beta|+2j \le 2(p|\alpha|+\ell)} N_{j,\beta}(u, \varphi). \quad (4\text{-}4)$$

Proof. It consists of a double induction on $|\alpha| = r$ and on ℓ. More precisely, in a first step, we prove the estimates (4-2) for $\alpha = 0$. In all the proof, we will specify $(4\text{-}4)_{\alpha,\ell}$ for (4-4) when we consider the couple $(\alpha, \ell) \in \mathbb{N} \times \{0, \ldots, p\}$. So we want, in this first step, to prove $(4\text{-}4)_{0,\ell}$, $\ell \in \{0, \ldots, p\}$.

(A) Proof of $(4\text{-}4)_{0,\ell}$, $\ell \in \{0, \ldots, p\}$: As $(4\text{-}4)_{0,0}$ is trivial, all we have to do is to make an induction on ℓ. So assume that $(4\text{-}4)_{0,i}$ is true for $i \le \ell$, $\ell \in \{0, \ldots, p-1\}$. Then we want to prove $(4\text{-}4)_{0,\ell+1}$. For that we use (3-23) in Proposition 3.6 for $(u, \varphi) \in C^\infty(\bar{\Omega}_1) \times \mathfrak{D}(\Omega_1)$. So in order to bound $\|\varphi u\|_{(\ell+1)\sigma}$, we just have to suitably bound $\|\varphi P u\|_{\ell\sigma}$ and $\sum_{|\beta|\le 2} \|\varphi^{(\beta)} u\|$. Hence this reduces to applying $(4\text{-}4)_{0,\ell}$

respectively to the couples (Pu, φ) and $(u, \varphi^{(\beta)})$ in $C^\infty(\overline{\Omega}_1) \times \mathscr{D}(\Omega_1)$. So

$$\|\varphi Pu\|_{\ell\sigma} \le M^{\ell+1}(2\ell)!^s \sum_{|\beta|+2j\le 2\ell} N_{j,\beta}(Pu, \varphi), \quad (u, \varphi) \in C^\infty(\overline{\Omega}_1) \times \mathscr{D}(\Omega_1). \tag{4-5}$$

Now we apply Lemma 4.1 with $\rho = 2\ell$ in (4-2). Then

$$\begin{aligned}\|\varphi Pu\|_{\ell\sigma} &\le \epsilon^{-2}(2(\ell+1))!^s M^{\ell+1} \sum_{|\beta|+2j\le 2(\ell+1)} N_{j,\beta}(u, \varphi) \\ &\le \epsilon^{-2} M^{-1} M^{(\ell+1)+1}(2(\ell+1))!^s \sum_{|\beta|+2j\le 2(\ell+1)} N_{j,\beta}(u, \varphi).\end{aligned}$$

We do exactly the same for $\|\varphi^{(\beta)}u\|_{\ell\sigma}$, as $|\beta| \le 2$:

$$\|\varphi^{(\beta)}u\|_{\ell\sigma} \le \epsilon^{-2} M^{-1} M^{(\ell+1)+1}(2(\ell+1))!^s \sum_{|\gamma|+2j\le 2(\ell+1)} N_{j,\gamma}(u, \varphi). \tag{4-6}$$

So from (3-23), (4-5) and (4-6), we get

$$\|\varphi u\|_{(\ell+1)\sigma} \le C(2+n+n^2)\epsilon^{-2} M^{-1} M^{(\ell+1)+1}(2(\ell+1))!^s \sum_{|\beta|+2j\le 2(\ell+1)} N_{j,\beta}(u, \varphi). \tag{4-7}$$

So we see that under the condition

$$C(2+n+n^2)\epsilon^{-2}M^{-1} \le 1 \quad \big(\text{equivalently, } M \ge C(2+n+n^2)\epsilon^{-2}\big), \tag{4-8}$$

Equation $(4\text{-}4)_{0,\ell+1}$ is satisfied for all $(u, \varphi) \in C^\infty(\overline{\Omega}_1) \times \mathscr{D}(\Omega_1)$.

(B) Proof of $(4\text{-}4)_{\alpha,\ell}$ for all $(\alpha, \ell) \in \mathbb{N}^n \times \{0, \dots, p\}$: All we have to do, as $(4\text{-}4)_{0,\ell}$ is true for $\ell = 0, \dots, p$, is to make an induction on $|\alpha| = r$. More precisely, if $(4\text{-}4)_{\alpha,\ell}$ is true for $|\alpha| \le r$, $\ell \in \{0, \dots, p\}$, then it is true for $|\alpha| = r+1$, $\ell = 0, \dots, p$. We use the same kind of proof as before: given $\alpha + i$, with $|\alpha| = r$, $i \in \{1, \dots, n\}$, we want to suitably bound $\|\varphi\partial_i\partial^\alpha u\|_{\ell\sigma}$ for $\ell \in \{0, \dots, p\}$.

(1) $\ell = 0$: We use $\|\varphi\partial_i\partial^\alpha u\| \le \|\partial_i(\varphi)\partial^\alpha u\| + \|\varphi\partial^\alpha u\|_1$. Hence we just have to apply $(4\text{-}4)_{\alpha,0}$ with $(u, \varphi^{(i)})$ and $(4\text{-}4)_{\alpha,p}$ with (u, φ). We will obtain, directly, or applying also Lemma 4.1,

$$\|\varphi^{(i)}\partial^\alpha u\| \le \epsilon^{-1} M^{-2p} M^{2p(|\alpha|+1)+1}(2p(|\alpha|+1))!^s \sum_{|\beta|+2j\le 2p(|\alpha|+1)} N_{j,\beta}(u, \varphi), \tag{4-9}$$

$$\|\varphi\partial^\alpha u\|_{p\sigma} \le M^{-p} M^{2p(|\alpha|+1)+1}(2p(|\alpha|+1))!^s \sum_{|\beta|+2j\le 2p(|\alpha|+1)} N_{j,\beta}(u, \varphi). \tag{4-10}$$

So, summing (4-9) and (4-10), we get that if

$$M^{-p}(1+\epsilon^{-1}M^{-p}) \le 1, \tag{4-11}$$

then $(4\text{-}4)_{\alpha+i,0}$ is satisfied, for $i = 1, \dots, n$.

(2) Proof of $(4\text{-}4)_{\alpha,\ell}$, for $|\alpha| = r+1$, $\ell > 0$, i.e., $\ell \in \{1, \dots, p\}$: So assuming that $(4\text{-}4)_{\alpha,\ell}$ is true for $|\alpha| = r$, $\ell = 0, \dots, p$, and $(4\text{-}4)_{\alpha,\rho}$ is true for $|\alpha| = r+1$, $\rho \in \{1, \dots, \ell\}$, then, if $\ell < p$, we want to prove that $(4\text{-}4)_{\alpha,\ell+1}$ is true, $|\alpha| = r+1$. Now we have to use (3-26) in Proposition 3.7. So in order to suitably bound $\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma}$, we are led to bound $\|\varphi\partial^\alpha Pu\|_{\ell\sigma}$, $\|\varphi^{(\beta)}\partial^\alpha u\|_{\ell\sigma}$, $|\beta| \le 2$, but also much more terms like simple brackets of X_j's and Y with ∂^α and double brackets of X_j's with ∂^α.

The proof will follow the lines of our proof for $\alpha = 0$, but here with more terms, and some are more difficult to handle than others, namely those coming from the brackets of the X_j's with ∂^α (simple and double brackets). The term $\|\varphi[Y+b, \partial^\alpha]u\|_{\ell\sigma}$ is similar to the terms $\|\varphi^{(\beta)}[X_j, \partial^\alpha]u\|$, $\beta = 0$.

(a) A bound on $\|\varphi\partial^\alpha Pu\|_{\ell\sigma}$: We apply $(4\text{-}4)_{\alpha,\ell}$ for the couple (Pu, φ):

$$\|\varphi\partial^\alpha Pu\|_{\ell\sigma} \le M^{2p|\alpha|+\ell+1}(2(p|\alpha|+\ell))!^s \sum_{|\beta|+2j\le 2(p|\alpha|+\ell)} N_{j,\beta}(Pu, \varphi).$$

Applying Lemma 4.1, we get (here $\rho = 2(p|\alpha|+\ell)$)

$$\begin{aligned}
&\|\varphi\partial^\alpha Pu\|_{\ell\sigma} \\
&\quad \le \epsilon^{-2}M^{2p|\alpha|+\ell+1}(2(p|\alpha|+\ell+1))!^s \sum_{|\beta|+2j\le 2(p|\alpha|+\ell+1)} N_{j,\beta}(u, \varphi) \\
&\quad \le (\epsilon^{-2}M^{-1})M^{2p|\alpha|+(\ell+1)+1}(2(p|\alpha|+\ell+1))!^s \\
&\qquad\qquad \cdot \sum_{|\beta|+2j\le 2(p|\alpha|+\ell+1)} N_{j,\beta}(u, \varphi).
\end{aligned} \tag{4-12}$$

(b) A bound on $\|\varphi^{(\beta)}\partial^\alpha u\|_{\ell\sigma}$, $|\beta| \le 2$: We apply $(4\text{-}4)_{\alpha,\ell}$ to the couple $(u, \varphi^{(\beta)})$:

$$\|\varphi^{(\beta)}\partial^\alpha u\|_{\ell\sigma} M^{2p|\alpha|+\ell+1}(2(p|\alpha|+\ell))!^s \sum_{|\beta|+2j\le 2(p|\alpha|+\ell)} N_{j,\beta}(u, \varphi^{(\beta)}).$$

Applying again Lemma 4.1 (as $|\beta| \le 2$), we get

$$\begin{aligned}
&\|\varphi^{(\beta)}\partial^\alpha u\|_{\ell\sigma} \\
&\quad \le \epsilon^{-2}M^{2p|\alpha|+\ell+1}(2(p|\alpha|+\ell+1))!^s \sum_{|\beta|+2j\le 2(p|\alpha|+\ell+1)} N_{j,\beta}(u, \varphi) \\
&\quad \le \epsilon^{-2}M^{-1}M^{2p|\alpha|+(\ell+1)+1}(2(p|\alpha|+\ell+1))!^s \\
&\qquad\qquad \cdot \sum_{|\beta|+2j\le 2(p|\alpha|+\ell+1)} N_{j,\beta}(u, \varphi).
\end{aligned} \tag{4-13}$$

(c) A bound on $\|\varphi^{(\beta)}[X_j, \partial^\alpha]u\|_{\ell\sigma}$, $|\beta| \le 1$, $j = 1, \dots, m$, $\|\varphi[Y + b, \partial^\alpha]u\|_{\ell\sigma}$: (It is easy to see that the term $\|\varphi[Y + b, \partial^\alpha]u\|_{\ell\sigma}$ is much simpler than the others, since it corresponds to $\beta = 0$.)

From expressions in (3-9) where we delete $j \in \{1, \dots, m\}$:

$$\|\varphi^{(\beta)}[X, \partial^\alpha]u\|_{\ell\sigma} \le \sum_{\substack{\gamma<\alpha\\ i=1,\dots,n}} \|a_{\alpha\gamma i}\varphi^{(\beta)}\partial^{\gamma+i}u\|_{\ell\sigma}. \tag{4-14}$$

Then applying estimates in (3-15) (recall that j is deleted):

$$\|\varphi^{(\beta)}[X, \partial^\alpha]u\|_{\ell\sigma} \le \sum_{\substack{\gamma<\alpha\\ i=1,\dots,n}} B^{|\alpha-\gamma|}\Big(\frac{\alpha!}{(\gamma+i)!}\Big)^s \|\varphi^{(\beta)}\partial^{\gamma+i}u\|_{\ell\sigma}. \tag{4-15}$$

As $|\gamma + i| \le |\alpha| = r + 1$, we can apply $(4\text{-}4)_{\gamma+i,\ell}$ to $(u, \varphi^{(\beta)})$. So we get

$$\begin{aligned}\|\varphi^{(\beta)}&[X, \partial^\alpha]u\|_{\ell\sigma}\\ &\le n \sum_{\substack{\gamma<\alpha\\ i=1,\dots,n}} \bigg(B^{|\alpha-\gamma|}M^{2p(|\gamma|+1)+\ell+1}(2(p(|\gamma|+1)+\ell))!^s\Big(\frac{\alpha!}{\gamma!}\Big)^s\\ &\qquad\qquad\cdot \sum_{|\delta|+2j\le 2(p(|\gamma|+1)+\ell)} N_{j,\delta}(u, \varphi^{(\beta)})\bigg).\end{aligned} \tag{4-16}$$

Using (4-2) in Lemma 4.1, with $\rho = 2(p(|\gamma| + 1) + \ell)$, we find

$$(2(p(|\gamma|+1)+\ell))!^s N_{j,\delta}(u, \varphi^{(\beta)}) \le (2(p(|\gamma|+1)+\ell)+1)!^s N_{j,\delta+\beta}(u, \varphi), \tag{4-17}$$

and hence

$$\begin{aligned}(2(p(|\gamma| + 1) + \ell))!^s\Big(\frac{\alpha!}{\gamma!}\Big)^s N_{j,\delta}(u, \varphi^{(\beta)})&\\ \le (2(p(|\alpha| + \ell + 1)!^s N_{j,\delta+\beta}(u, \varphi),&\end{aligned} \tag{4-18}$$

for all (j, δ) such that $|\delta| + 2j \le 2(p(|\gamma| + 1) + \ell) + 1$. So, coming back to the second member in (4-16),

$$\begin{aligned}\|\varphi^{(\beta)}&[X, \partial^\alpha]u\|_{\ell\sigma}\\ &\le n \sum_{\gamma<\alpha}\bigg(B^{|\alpha-\gamma|}M^{2p(|\gamma|+1)+\ell+1}(2(p|\alpha| + \ell + 1))!^s\\ &\qquad\qquad\cdot\epsilon^{-1} \sum_{|\delta|+2j\le 2(p|\alpha|+\ell+1)} N_{j,\delta}(u, \varphi)\bigg),\end{aligned} \tag{4-19}$$

or

$$\|\varphi^{(\beta)}[X, \partial^{\alpha}]u\|_{\ell\sigma} \le n(2(p|\alpha|+\ell+1))!^{s} \sum_{|\delta|+2j\le 2(p|\alpha|+\ell+1)} N_{j,\delta}(u,\varphi) \cdot \epsilon^{-1} \sum_{\gamma<\alpha} B^{|\alpha-\gamma|} M^{2p(|\gamma|+1)+\ell+1}. \tag{4-20}$$

The last sum is bounded as follows:

$$\sum_{\gamma<\alpha} B^{|\alpha-\beta|} M^{2p(|\gamma|+1)+\ell+1} = BM^{2p|\alpha|+\ell+1} \sum_{\gamma<\alpha} \Big(\frac{B}{M^{2p}}\Big)^{|\alpha-\gamma|-1}. \tag{4-21}$$

Now we have the following lemma:

Lemma 4.3. *There exists $\theta_0 > 0$, independent of α, such that*

$$\sum_{\gamma<\alpha} \lambda^{|\alpha-\gamma|-1} \le n+1, \quad \text{if } 0 \le \lambda < \theta_0. \tag{4-22}$$

The proof of Lemma 4.3 is similar to that of Lemma 3.2 after noticing that $\sum_{\gamma<\alpha,\, |\alpha-\gamma|=1} \lambda^{|\alpha-\gamma|-1} = n$.

Remark 4.4. Lemma 4.3 is not true when one works with the sum $\sum_{|\gamma|<|\alpha|} \lambda^{|\alpha-\gamma|-1}$ as the sum $\sum_{|\gamma|<|\alpha|,\, |\alpha-\gamma|=1} 1$ is not bounded by a constant independent of α.

Applying (4-22) we see that under the condition

$$M^{2p} \ge \theta_0^{-1} B, \tag{4-23}$$

we obtain from (4-20), (4-21) and (4-22)

$$\|\varphi^{(\beta)}[X, \partial^{\alpha}]u\|_{\ell\sigma} \le n(n+1)\epsilon^{-1} BM^{-1} M^{2p|\alpha|+(\ell+1)+1} (2p|\alpha|+\ell+1)!^{s} \cdot \sum_{|\delta|+2j\le 2(p|\alpha|+\ell+1)} N_{j,\delta}(u,\varphi). \tag{4-24}$$

(d) A bound on $\|\varphi[X_j, [X_j, \partial^{\alpha}]]u\|_{\ell\sigma}$, $j = 1, \dots, m$: As we did above we delete the index j and write $\|\varphi[X, [X, \partial^{\alpha}]]u\|$. Of course, we also delete j in the coefficients of the bracket. Looking at (3-12), we have two kinds of terms to study:

(i) A bound on $\sum_{\beta<\alpha,\, k=1,\dots,n} \|\varphi d_{k\alpha\beta} \partial^{\beta+k} u\|_{\ell\sigma} = E_1$: Using Equation (3-15) in Proposition 3.3, as $\ell\sigma \le 1$, we have

$$E_1 \le \sum_{\substack{\beta<\gamma \\ k=1,\dots,n}} B^{|\alpha-\beta|} \Big(\frac{\alpha!}{\beta!}(|\alpha|+1)\Big)^{s} M^{2\beta(|\beta|+1)+\ell+1} (2(p(|\beta|+1)+\ell))!^{s} \cdot \sum_{|\gamma|+2j\le 2(p(|\beta|+1)+\ell)} N_{j,\gamma}(u,\varphi). \tag{4-25}$$

We want to remark here that there is a factor $(|\alpha|+1)^s$ in (4-25), but it is compensated by the fact that one has φ, not $\varphi^{(\beta)}$, $|\beta|=1$. Precisely, we have

$$E_1 \le n \sum_{\beta<\alpha} B^{|\alpha-\beta|}\Big(\frac{(|\alpha|+1)!}{|\beta|!}\Big)^s (2p(|\beta|+1)+\ell)!^s M^{2p(|\beta|+1)+\ell+1} \cdot \sum_{|\gamma|+2j\le 2(p|\alpha|+\ell)} N_{j,\gamma}(u,\varphi), \quad \text{as } |\beta|+1\le|\alpha|. \tag{4-26}$$

Now we have the following inequality:

$$\Big(\frac{(|\alpha|+1)!}{|\beta|!}\Big)^s \big(2(p(|\beta|+1)+\ell)\big)!^s \le (2(p|\alpha|+\ell+1))!^s, \quad |\beta|+1\le|\alpha|. \tag{4-27}$$

So, from (4-26) and (4-27), we get

$$E_1 \le n(2(p|\alpha|+\ell+1))!^s \sum_{\beta<\alpha} B^{|\alpha-\beta|} M^{2p(|\beta|+1)+\ell+1} \cdot \sum_{|\gamma|+2j\le 2(p|\alpha|+\ell)} N_{j,\gamma}(u,\varphi). \tag{4-28}$$

The sum in the second member in (4-28) is the same than the sum in (4-21), replacing γ by β. So from (4-22) in Lemma 4.3, we get, under condition (4-23),

$$E_1 \le n(n+1)BM^{-1}\, M^{2p|\alpha|+(\ell+1)+1}(2p|\alpha|+\ell+1)!^s \cdot \sum_{|\gamma|+2j\le 2(p|\alpha|+\ell)} N_{j,\gamma}(u,\varphi). \tag{4-29}$$

(ii) A bound on $\sum_{\beta<\alpha,\, i,k=1,\dots,n} \|\varphi c_{ik\alpha\beta}\partial^{\beta+i+k}u\|_{\ell\sigma} = E_2$: We write the proof, for completeness, noticing however that $\partial^{\beta+i+k}$ is of the form $\partial^{\beta+I}$ with $|I|=2$. Using estimates in (3-15), we get

$$E_2 \le \sum_{\substack{\beta<\alpha\\ i,k=1,\dots,n}} B^{|\alpha-\beta|}\Big(\frac{(\alpha+k)!}{(\beta+k)!}\Big)^s \|\varphi\partial^{\beta+i+k}u\|_{\ell\sigma}. \tag{4-30}$$

Hence,

$$E_2 \le \sum_{\substack{\beta<\alpha,\ |\beta|\le|\alpha|-2\\ i,k=1,\dots,n}} B^{|\alpha-\beta|} M^{2p(|\beta|+2)+\ell+1}\Big(\frac{(\alpha+k)!}{(\beta+k)!}\Big)^s (2(p(|\beta|+2)+\ell))!^s \cdot \sum_{|\gamma|+2j\le 2(p(|\beta|+2)+\ell)} N_{j,\gamma}(u,\varphi). \tag{4-31}$$

Now we have the following:

$$\Big(\frac{(\alpha+k)!}{(\beta+k)!}\Big)^s (2(p(|\beta|+2)+\ell))!^s \le (2(p|\alpha|+\ell+1))!^s. \tag{4-32}$$

(4-32) is a consequence of $(\alpha+k)!/(\beta+k)! \le (|\alpha|+1)!/(|\beta|+1)!$. Hence

$$E_2 \le n^2 (2(p|\alpha|+\ell+1))!^s \Biggl(\sum_{\substack{\beta<\alpha \\ |\beta|\le|\alpha|-2}} B^{|\alpha-\beta|} M^{2p(|\beta|+2)+\ell+1} \Biggr) \cdot \sum_{|\gamma|+2j\le 2(p|\alpha|+\ell+1)} N_{j,\gamma}(u,\varphi), \quad (4\text{-}33)$$

as $\mathrm{Card}\bigl\{(i,k)\in\{1,\dots,n\}\bigr\} = n^2$. Now, we bound

$$F = \textstyle\sum_{\beta<\alpha,\, |\beta|\le|\alpha|-2} B^{|\alpha-\beta|} M^{2p(|\beta|+2)+\ell+1}$$

with

$$F \le \sum_{\substack{\beta<\alpha \\ |\beta|\le|\alpha|-2}} \Bigl(\frac{B}{M^{2p}}\Bigr)^{|\alpha-\beta|-2} M^{2p|\alpha|+(\ell+1)+1} B^2 M^{-1}. \quad (4\text{-}34)$$

Now using Lemma 4.3 and the fact that $\sum_{\beta<\alpha,\, |\beta|=|\alpha|-2} 1 \le n^2$, we obtain

$$E_2 \le (n^2+1)^2 B^2 M^{-1}\, M^{2p|\alpha|+(\ell+1)+1} (2(p|\alpha|+\ell+1))!^s \cdot \sum_{|\delta|+2j\le 2(p|\alpha|+\ell+1)} N_{j,\delta}(u,\varphi). \quad (4\text{-}35)$$

We recall that we said in (c) that we do not consider the term

$$\|\varphi[Y+b, \partial^\alpha]\|_{\ell\sigma} \le \|\varphi[Y,\partial^\alpha]u\|_{\ell\sigma} + \|\varphi[b,\partial^\alpha]u\|_{\ell\sigma}$$

as $\|\varphi[Y,\partial^\alpha]u\|_{\ell\sigma}$ is like $\|\varphi[X_j,\partial^\alpha]u\|_{\ell\sigma}$ and $\|\varphi[b,\partial^\alpha]u\|_{\ell\sigma}$ much smaller. So in order to bound the term $\|\varphi[X_j,[X_j,\partial^\alpha]]u\|_{\ell\sigma}$, we have to collect (4-29) and (4-35), which are true under condition (4-23). We find

$$\|\varphi[X_j,[X_j,\partial^\alpha]]u\|_{\ell\sigma} \le 2(n^2+1)^2 B^2 M^{-1} M^{2p|\alpha|+(\ell+1)+1} (2(p|\alpha|+\ell+1))!^s \cdot \sum_{|\gamma|+2j\le 2(p|\alpha|+\ell)} N_{j,\gamma}(u,\varphi) \quad (4\text{-}36)$$

In order to give a bound for $\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma}$, we have from (3-26) in Proposition 3.7 to take C times the bound in (a) plus $C(n^2+n+1)$ times the bound in (b) plus $Cm(n+2)$ times the bound in (c) plus Cm times the bound in (d), under, of course, the conditions on M indicated in the proofs of (a), (b), (c) and (d).

Of course, we also have to take care of the conditions needed on M for the validity of these bounds. These may be summarized as follows:

(a) (4-12) is satisfied, just under the induction hypothesis,

(b) (4-13) is satisfied, under the induction hypothesis,

(c) (4-24) is satisfied, under the induction hypothesis and (4-23),

(d) (4-36) is satisfied, under the induction hypothesis and (4-23).

So adding all these estimates with the suitable factors yields

$$\|\varphi\partial^\alpha u\|_{(\ell+1)\sigma} \le CM^{-1}(2\epsilon^{-2}+n(n+1)\epsilon^{-1}B+2(n^2+1)^2B^2)\, M^{2p|\alpha|+(\ell+1)+1}(2(p|\alpha|+\ell+1))!^s \cdot \sum_{|\gamma|+2j\le 2(p|\alpha|+\ell)} N_{j,\gamma}(u,\varphi) \quad (4\text{-}37)$$

under the condition (4-23). From (4-37), we deduce that if M satisfies

$$M \ge \sup\bigl\{(\theta_0^{-1}B)^{1/2^p},\ 1,\ C\bigl(2\epsilon^{-2}+n(n+1)\epsilon^{-1}B+2(n^2+1)^2B^2\bigr)\bigr\}, \quad (4\text{-}38)$$

then $(4\text{-}4)_{\alpha,\ell+1}$ is true. So $(4\text{-}4)_{\alpha,\ell}$ is true, $|\alpha| = r+1$, for all ℓ.

Now, let us finish the proof of the theorem. Since we proved $(4\text{-}4)_{0,\ell}$, $\ell = 1, \dots, p$, and the induction

$$\bigl\{(4\text{-}4)_{\alpha,\ell},\ |\alpha| \le r,\ \ell \in \{1,\dots,p\}\bigr\} \Rightarrow \bigl\{(4\text{-}4)_{\alpha,\ell},\ |\alpha| = r+1,\ \ell \in \{1,\dots,p\}\bigr\},$$

under respectively condition ((4-8) and (4-11)) and condition (4-38), we have $M = M(\epsilon, \Omega_0, P) > 0$ so that the theorem is completely proved, when the conditions (4-8), (4-11) and (4-38) are satisfied. As $\epsilon \le 1$, we see that $M^p \ge 2\epsilon^{-1}$ implies that (4-11) holds. So, everything boils down to only the following condition:

$$\begin{aligned} M \ge \sup\Bigl\{&(\theta_0^{-1}B)^{1/2^p},\ C(2+n+n^2)\epsilon^{-2},\\ &C\bigl(2\epsilon^{-2}+n(n+1)\epsilon^{-1}B+2(n^2+1)^2B^2\bigr)\Bigr\}\\ = M(&\epsilon,\theta_0,B). \end{aligned} \quad (4\text{-}39)$$

B depends on P and Ω_0 and θ_0 depends on n. Hence $M(\epsilon, \theta_0, B)$ can be written, as n is fixed, $M(\epsilon, \Omega_0, P)$. The proof of Theorem 4.2 is now complete. □

5. Gevrey regularity for Gevrey vectors

We want to give in this section an application of Theorem 4.2. In fact, we shall just use the estimates (4-4) for $\ell = 0$, which we rewrite here, for $(u,\varphi) \in C^\infty(\overline{\Omega}_1) \times \mathcal{D}(\Omega_1)$:

$$\|\varphi\partial^\alpha u\| \le M_\epsilon^{2p|\alpha|+1}(2p|\alpha|)!^s \sum_{|\beta|+2j\le 2p|\alpha|} N^\epsilon_{j,\beta}(u,\varphi). \quad (5\text{-}1)$$

Moreover, we want to state a theorem for operators of order m, satisfying estimates similar to (5-1), but with 2 replaced by m and with coefficients in $G^s(\Omega)$. For that purpose, we clarify the notation as m replaces 2.

Firstly, Ω and Ω_1 are as in Section 4, $\overline{\Omega}_1 \Subset \Omega$ and $s \ge 1$. Then define

$$N^\epsilon_{j,\beta} = \epsilon^{|\beta|+mj}|\beta|!^{-s}(mj)!^{-s}\|\varphi^{(\beta)}P^j u\|, \quad (u,\varphi) \in C^\infty(\overline{\Omega}_1) \times \mathcal{D}(\Omega_1). \quad (5\text{-}2)$$

Now assume that P satisfies the following estimates:

$$\forall\epsilon,\ 0<\epsilon\le 1, \exists M_\epsilon\ge 1 \quad \text{such that} \quad \forall\alpha\in\mathbb{N}^n,\ \forall(u,\varphi)\in C^\infty(\overline{\Omega}_1)\times\mathscr{D}(\Omega_1)$$

$$\|\varphi\partial^\alpha u\| \le M_\epsilon^{r|\alpha|+1}(r|\alpha|)!^s \sum_{|\beta|+mj\le r|\alpha|} N_{j,\beta}(u,\varphi) \quad \text{for some } r\in\mathbb{N}^*. \tag{5-3}$$

Now, we provide a proposition on Gevrey regularity of Gevrey vectors of P satisfying (5-3).

Proposition 5.1. *Let P be a linear partial differential operator with $G^s(\Omega)$ coefficients, of order m, satisfying* (5-3) *in Ω_1 with $\overline{\Omega}_1\subset\Omega$. Then any $G^s(\Omega_1)$-vector of P which is $C^\infty(\Omega)$ is in $G^{rs}(\Omega_1)$, $s\ge 1$.*

Proof. We have to distinguish between the cases $s>1$ and $s=1$.

(1) Case $s>1$: Let $u\in C^\infty(\Omega)\cap G^s(\Omega_1,P)$. In order to prove that $u\in G^{rs}(\Omega_1)$, we have to show that given any open set Ω_2 with $\overline{\Omega}_2\subset\Omega_1$, we have:

$$\exists C_{\Omega_2}>0 \text{ s.t. } \forall\alpha\in\mathbb{N}^n,\quad \|\partial^\alpha u\|_{L^2(\Omega_2)}\le C_{\Omega_2}^{|\alpha|+1}(r|\alpha|)!^s = C^{|\alpha|+1}(r|\alpha|)!^s. \tag{5-4}$$

First, we consider $\varphi\in G^s(\Omega_1)\cap\mathscr{D}(\Omega_1)$, $\varphi=1$ on Ω_2. Then

$$\|\partial^\alpha u\|_{L^2(\Omega_2)}\le\|\varphi\partial^\alpha u\|. \tag{5-5}$$

As $u\in C^\infty(\overline{\Omega}_1)$ and $\varphi\in\mathscr{D}(\Omega_1)$, we apply (5-3). So we get

$$\|\partial^\alpha u\|_{L^2(\Omega_2)}\le (CM_\epsilon)^{r|\alpha|+1}(r|\alpha|)!^s\sum_{|\beta|+mj\le r|\alpha|} N^\epsilon_{j,\beta}(u,\varphi). \tag{5-6}$$

Now as $\varphi\in G^s(\Omega_1)\cap\mathscr{D}(\Omega_1)$, there exists $A=A_\varphi$ such that

$$\sup|\varphi^{(\beta)}|\le A^{|\beta|+1}|\beta|!^s. \tag{5-7}$$

Also, from (5-2), we have

$$N^\epsilon_{j,\beta}=\epsilon^{|\beta|+mj}|\beta|!^{-s}(mj)!^{-s}\|\varphi^{(\beta)}P^ju\|. \tag{5-8}$$

Now as $u\in G^s(\Omega_1,P)$, taking $K=\operatorname{Supp}(\varphi)\subset\Omega_1$

$$\exists B=B_K>0,\quad \text{such that}\quad \|P^ju\|_{L^2(K)}\le B^{mj+1}(mj)!^s,\quad \forall j\in\mathbb{N}. \tag{5-9}$$

From (5-7), (5-8) and (5-9), we obtain

$$N^\epsilon_{j,\beta}(u,\varphi)\le\epsilon^{|\beta|+mj}A^{|\beta|+1}\,B^{mj+1}\le\epsilon^{|\beta|+mj}D^{|\beta|+mj+1} \tag{5-10}$$

for some constant D (for example $A+B$). So, with (5-6),

$$\|\partial^\alpha u\|_{L^2(\Omega_2)}\le D(CM_\epsilon)^{r|\alpha|+1}(r|\alpha|)!^s\sum_{|\beta|+mj\le r|\alpha|}(\epsilon D)^{|\beta|+mj}. \tag{5-11}$$

Now let us choose ϵ such that

$$\epsilon D = \tfrac{1}{2}. \tag{5-12}$$

Then with $M = M_{1/(2D)}$, we have

$$\|\partial^\alpha u\|_{L^2(\Omega_2)} \le D(CM)^{r|\alpha|+1}(r|\alpha|)!^s \sum_{|\beta|+mj\le r|\alpha|} \left(\tfrac{1}{2}\right)^{|\beta|+mj}. \tag{5-13}$$

We have to estimate the sum in the (5-13).

Lemma 5.2. *There exists a constant $C_0 > 0$ such that*

$$\sum_{|\beta|+mj\le r|\alpha|} \left(\tfrac{1}{2}\right)^{|\beta|+mj} \le C_0, \quad \forall \alpha \in \mathbb{N}^n. \tag{5-14}$$

Proof of Lemma 5.2. This simple lemma has an elementary proof. For completeness, let us give it. For $\alpha = 0$, it is trivial. So assume $|\alpha| \ge 1$. It suffices to see that

$$\sum_{|\beta|+mj\le r|\alpha|} \left(\tfrac{1}{2}\right)^{|\beta|+mj} = \sum_{k=0}^{r|\alpha|}\Bigg(\sum_{|\beta|+mj=k} 1\Bigg)\left(\tfrac{1}{2}\right)^k \le \sum_{k=0}^{r|\alpha|}(k+1)^n\left(\tfrac{1}{2}\right)^k$$
$$\le \sum_{k=0}^{\infty}(k+1)^n\left(\tfrac{1}{2}\right)^k = C_0 < +\infty. \qquad \square$$

Coming back to the proof of Proposition 5.1, we have, with some constant $C_1 > 0$, using (5-13) and (5-14), the following estimate:

$$\|\partial^\alpha u\|_{L^2(\Omega_2)} \le C_1(CM)^{r|\alpha|}(r|\alpha|)!^s \tag{5-15}$$

which shows that u is in $G^{rs}(\Omega_2)$, hence in $G^{rs}(\Omega_1)$ as Ω_2 is any relatively compact set in Ω_1.

(2) Case $s = 1$: In this case, as we have no $\varphi \in \mathcal{D}(\Omega_1)$ which is in $G^1(\Omega_1)$, we proceed by using a sequence of functions of L. Ehrenpreis associated to the couple (Ω_0, Ω_1) with $\overline{\Omega}_1 \subset \Omega_0$ and $\overline{\Omega}_0 \subset \Omega$. We state below a proposition due to Ehrenpreis, providing the precise details regarding the sequence.

Proposition 5.3 [Ehrenpreis 1960]. *Let Ω_0, Ω_1 be as above. Then there exists a constant $\tilde{C} > 0$ such that*:

$\forall N \in \mathbb{N},\ \exists \varphi_N \in \mathcal{D}(\Omega_0),\ \varphi_N|_{\Omega_1} = 1,$

$$\textit{such that} \quad |\varphi_N^{(\beta)}| \le \tilde{C}^{|\beta|+1}N^\beta, \quad \textit{for } |\beta| \le N. \tag{5-16}$$

In our proof below, in order to bound $\|\partial^\alpha u\|_{L^2(\Omega_2)}$, we use, in place of φ used in case (1), the function $\varphi_{r|\alpha|}$ given by taking $N = r|\alpha|$ in (5-16). So, as $\overline{\Omega}_2 \subset \Omega_1$, we have

$$\|\partial^\alpha u\|_{L^2(\Omega_2)} \le \|\varphi_{r|\alpha|}\partial^\alpha u\| \le M_\epsilon^{r|\alpha|+1}(r|\alpha|)! \sum_{|\beta|+mj\le r|\alpha|} N^\epsilon_{j,\beta}(u, \varphi_{r|\alpha|}). \tag{5-17}$$

Now, taking A given by:

$$A = \sup(B, \tilde{C}), \tag{5-18}$$

where B is given by (5-9) with $K = \overline{\Omega}_0$, we get

$$E_\alpha = \sum_{|\beta|+mj \le r|\alpha|} N^{\epsilon}_{j,\beta}(u, \varphi_{r|\alpha|}) \le A \sum_{|\beta|+mj \le r|\alpha|} (\epsilon A)^{|\beta|+mj} |\beta|!^{-1} (r|\alpha|)^{|\beta|}, \tag{5-19}$$

since $|\beta| \le r|\alpha|$ in the sum. Looking at the second member in (5-19), we may bound by

$$E_\alpha \le \sum_{mj \le r|\alpha|} (\epsilon A)^{mj} \sum_{|\beta| \le r|\alpha|} (\epsilon A)^{|\beta|} |\beta|!^{-1} (r|\alpha|)^{|\beta|}. \tag{5-20}$$

Hence

$$E_\alpha \le \sum_{mj \le r|\alpha|} (\epsilon A)^{mj} \sum_{k=0}^{r|\alpha|} \Big(\sum_{|\beta|=k} 1 \Big) (\epsilon A r|\alpha|)^k k!^{-1}. \tag{5-21}$$

If we choose ϵ_0 so that $\epsilon_0 A = \frac{1}{2}$, we get

$$E_\alpha \le 2 \sum_{k=0}^{r|\alpha|} (k+1)^n \Big(\frac{r|\alpha|}{2}\Big)^k k!^{-1} \le 2(r|\alpha|+1)^n \sum_k \Big(\frac{r|\alpha|}{2}\Big)^k k!^{-1} \tag{5-22}$$

So finally, with some constant $C_1 > 0$, $C_1 = C_1(n)$,

$$E_\alpha \le 2(r|\alpha|+1)^n \exp\Big(\frac{r|\alpha|}{2}\Big) \le C_1 \exp(r|\alpha|). \tag{5-23}$$

Then, coming back to (5-17), we obtain

$$\|\partial^\alpha u\|_{L^2(\Omega_2)} \le M_{\epsilon_0}^{r|\alpha|+1} (r|\alpha|)!\, C_1 \exp(r|\alpha|) = C_1 M_{\epsilon_0} (e M_{\epsilon_0})^{r|\alpha|} (r|\alpha|)!. \tag{5-24}$$

As we took any Ω_2 with $\overline{\Omega}_2 \subset \Omega_1$, $\overline{\Omega}_1 \subset \Omega_0$, we obtain that $u \in G^r(\Omega_1)$. The proof of Proposition 5.1 is complete. □

As a corollary of Theorem 4.2 and Proposition 5.1, we get:

Theorem 5.4. *Let P be a Hörmander's operator on an open set Ω in $\mathbb{R}^n$ and $s \in \mathbb{R}$, $s \ge 1$. Assume that P satisfies the estimate* (2-4) *in some open subset Ω_0 with $\overline{\Omega}_0 \subset \Omega$ with $\sigma = 1/p$, $p \in \mathbb{N}^*$ and that its coefficients are in $G^s(\Omega_0)$. Then $G^s(\Omega_0, P) \subset G^{2ps}(\Omega_0)$.*

We conclude this article with some final remarks.

(1) In the case $s = 1$, there is another proof, using the method of addition of an extra variable (see, for example, [Bolley et al. 1987] or [Lions and Magenes 1970]), by considering the operator $\partial_t^2 + P$ in $\mathbb{R} \times \Omega \subset \mathbb{R}^{n+1}$, which is also a Hörmander's operator in $\mathbb{R} \times \Omega$, with analytic coefficients (case $s = 1$), to which one can use the theorem of Gevrey hypoellipticity G^s for $s \ge 2p$, [Derridj and Zuily 1973].

(2) We know nothing on optimality of our result. In our preceding paper [Derridj ≥ 2019], in the case of operators of the first kind, the result was optimal: $G^k(\Omega_0, P) \subset G^{pk}(\Omega_0),\ k \in \mathbb{N}$.

(3) In a forthcoming paper, we will study the question of local relations of domination by powers of P, in the case where P is of the first kind, which will be finer, giving therefore the optimal result $G^s(\Omega_0, P) \subset G^{ps}(\Omega_0)$, p being the type of $\overline{\Omega}_0$.

Acknowledgements

I am greatly indebted to Nordine Mir for encouraging me to write the present paper when he knew about the result and for his help.

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Received 24 Nov 2017. Revised 11 Dec 2017.

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Tunisian Journal of Mathematics
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The Tunisian Journal of Mathematics is an international publication organized by the Tunisian Mathematical Society (http://www.tms.rnu.tn) and published in electronic and print formats by MSP in Berkeley.

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Tunisian Journal of Mathematics (ISSN 2576-7666 electronic, 2576-7658 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

TJM peer review and production are managed by EditFlow® from MSP.

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nonprofit scientific publishing
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Tunisian Journal of Mathematics

2019 vol. 1 no. 3