Generic colourful tori and inverse spectral transform
for Hankel operators

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This paper explores the regularity properties of an inverse spectral transform for Hilbert–Schmidt Hankel operators on the unit disc. This spectral transform plays the role of action-angle variables for an integrable infinite dimensional Hamiltonian system: the cubic Szegő equation. We investigate the regularity of functions on the tori supporting the dynamics of this system, in connection with some wave turbulence phenomenon, discovered in a previous work and due to relative small gaps between the actions. We revisit this phenomenon by proving that generic smooth functions and a G_δ dense set of irregular functions do coexist on the same torus. On the other hand, we establish some uniform analytic regularity for tori corresponding to rapidly decreasing actions which satisfy some specific property ruling out the phenomenon of small gaps.

1. Introduction

1.1. *The cubic Szegő equation.* This paper explores the properties of some inverse spectral transformation related to an integrable infinite dimensional Hamiltonian system. Introduced in [Gérard and Grellier 2010] as a model of nondispersive evolution equation, the cubic Szegő equation reads

$$i\,\partial_t u = \Pi(|u|^2 u), \tag{1}$$

where $u = u(t, x)$ is a function defined for $(t, x) \in \mathbb{R} \times \mathbb{T}$, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, such that, for every $t \in \mathbb{R}$, $u(t, \cdot)$ belongs to the Hardy space $L^2_+(\mathbb{T})$ of L^2 functions v on $\mathbb{T}$ with only nonnegative Fourier modes,

$$\text{for all } n < 0, \quad \hat{v}(n) = 0.$$

Here

$$\hat{v}(n) = \int_0^{2\pi} v(x) \mathrm{e}^{-inx}\, \frac{dx}{2\pi}, \quad n \in \mathbb{Z}$$

denotes the Fourier coefficient of $v \in L^2(\mathbb{T})$, and Π denotes the orthogonal projector from $L^2(\mathbb{T})$ onto $L^2_+(\mathbb{T})$:

$$\Pi\left(\sum_{n\in\mathbb{Z}} c_n e^{inx}\right) = \sum_{n=0}^{\infty} c_n e^{inx}.$$

It has been proved in [Gérard and Grellier 2010] that (1) is globally well posed on Sobolev spaces $H^s_+(\mathbb{T}) := H^s(\mathbb{T}) \cap L^2_+(\mathbb{T})$ for all $s \geq \frac{1}{2}$, with conservation of the $H^{\frac{1}{2}}$ norm. Recall that, for elements in $L^2_+(\mathbb{T})$, the H^s Sobolev norm reads

$$\|v\|^2_{H^s} = \sum_{n=0}^{\infty}(1+n)^{2s}|\hat{v}(n)|^2.$$

Furthermore, it turns out that (1) enjoys an unexpected Lax pair structure, discovered in [Gérard and Grellier 2010] and studied in [Gérard and Grellier 2012; 2015; 2017]. More precisely, consider, for every $u \in H^{\frac{1}{2}}_+(\mathbb{T})$, the Hankel operator $H_u : L^2_+(\mathbb{T}) \to L^2_+(\mathbb{T})$ defined as

$$H_u(h) = \Pi(u\bar{h}).$$

Notice that H_u is an antilinear realisation of the Hankel matrix $\Gamma_{\hat{u}}$, where, for every sequence $\alpha = (\alpha_n)_{n\geq 0}$ of complex numbers, Γ_α denotes the operator on $\ell^2(\mathbb{Z}_+)$ given by the infinite matrix $(\alpha_{n+p})_{n,p\geq 0}$. Indeed, if $\mathscr{F}$ denotes the Fourier transform $v \mapsto \hat{v}$ between $L^2_+(\mathbb{T})$ and $\ell^2(\mathbb{Z}_+)$, it easy to check that

$$\mathscr{F}H_u\mathscr{F}^{-1} = \Gamma_{\hat{u}} \circ \mathcal{C},$$

where $\mathcal{C}$ denotes the complex conjugation. The Lax pair identity then reads as follows, see [Gérard and Grellier 2010]. If $s > \frac{1}{2}$ and u is a H^s_+ solution of (1), then

$$\frac{dH_u}{dt} = [B_u, H_u],$$

where B_u is a linear anti-self-adjoint operator depending on u. As a consequence, there exists a one parameter family $U(t)$ of unitary operators on $L^2_+(\mathbb{T})$ such that

$$\text{for all } t \in \mathbb{R}, \quad H_{u(t)} = U(t)H_{u(0)}U(t)^*.$$

In particular, $H^2_{u(t)} = U(t)H^2_{u(0)}U(t)^*$. Notice that H^2_u is a linear positive operator on $L^2_+(\mathbb{T})$, and that

$$\mathscr{F}H^2_u\mathscr{F}^{-1} = \Gamma_{\hat{u}}\Gamma^*_{\hat{u}},$$

thus H^2_u is a trace class operator as soon as $u \in H^{\frac{1}{2}}_+(\mathbb{T})$, with

$$\text{Tr}(H^2_u) = \sum_{n=0}^{\infty}(1+n)|\hat{u}(n)|^2 = \|u\|^2_{H^{\frac{1}{2}}}.$$

Consequently, apart from 0, the spectrum of H_u^2 is made of eigenvalues, which are conservation laws of (1).

In fact, a second Lax pair for (1) holds [Gérard and Grellier 2012], which concerns the operator $K_u := S^* H_u = H_u S = H_{S^* u}$, where S denotes the shift operator on $L_+^2(\mathbb{T})$, namely multiplication by e^{ix}. Operator K_u is also a Hankel operator,

$$\mathscr{F} K_u \mathscr{F}^{-1} = \tilde{\Gamma}_{\hat{u}} \circ C,$$

where $\tilde{\Gamma}_\alpha$ denotes the shifted Hankel matrix $(\alpha_{n+p+1})_{n,p \geq 0}$. Again, it is possible to prove that

$$\text{for all } t \in \mathbb{R}, \quad K_{u(t)} = V(t) K_{u(0)} V(t)^*,$$

for some one parameter family $V(t)$ of unitary operators on $L_+^2(\mathbb{T})$, and consequently that the eigenvalues of K_u^2 are conservation laws of (1). Denote by $(\rho_j^2)_{j \geq 1}$ the positive eigenvalues of H_u^2 and by $(\sigma_k^2)_{k \geq 1}$ the positive eigenvalues of K_u^2, so that the ρ_j are the singular values of $\Gamma_{\hat{u}}$ and the σ_k are the singular values of $\tilde{\Gamma}_{\hat{u}}$. In view of the identity

$$K_u^2 = H_u^2 - (\cdot \mid u)_{L^2} u$$

and of the min-max theorem, the following interlacing property holds:

$$\rho_1 \geq \sigma_1 \geq \rho_2 \geq \sigma_2 \geq \cdots.$$

1.2. *The spectral transform.* If u belongs to a dense G_δ subset $H_{+,\text{gen}}^{\frac{1}{2}}(\mathbb{T})$ of $H_+^{\frac{1}{2}}(\mathbb{T})$, one can establish (see [Gérard and Grellier 2012]) that

$$\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots$$

We set

$$s_{2j-1} = \rho_j, \quad s_{2k} = \sigma_k, \qquad j, k \geq 1.$$

The s_r are called the singular values of the pair (H_u, K_u). Of course

$$\sum_{r=1}^\infty s_r^2 = \text{Tr}(H_u^2) + \text{Tr}(K_u^2) = \sum_{n=0}^\infty (1 + 2n) |\hat{u}(n)|^2 < \infty.$$

Conversely, given a square summable *strictly* decreasing sequence $(s_r)_{r \geq 1}$ of positive numbers, the set of $u \in H_+^{\frac{1}{2}}$ such that the s_r are the singular values of the pair (H_u, K_u), in the above sense, is an infinite dimensional torus $\mathcal{T}((s_r)_{r \geq 1})$ [Gérard and Grellier 2012] of $H_+^{\frac{1}{2}}(\mathbb{T})$. This torus is parametrised by the following explicit representation [Gérard and Grellier 2017], where we classically identify functions of $L_+^2(\mathbb{T})$ with holomorphic functions $u = u(z)$ on the unit disc such that

$$\sup_{r<1} \int_0^{2\pi} |u(re^{ix})|^2 \, dx < \infty.$$

The current element of the infinite dimensional torus $\mathcal{T}((s_r)_{r\geq 1})$ is then given by

$$u(z) = \lim_{N\to\infty} \langle \mathscr{C}_N(z)^{-1}(\mathbb{1}_N), \mathbb{1}_N \rangle, \qquad |z| < 1, \quad \mathbb{1}_N := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{C}^N, \quad (2)$$

where

$$\mathscr{C}_N(z) := \left(\frac{s_{2j-1}e^{i\psi_{2j-1}} - zs_{2k}e^{i\psi_{2k}}}{s_{2j-1}^2 - s_{2k}^2} \right)_{1\leq j,k\leq N} \tag{3}$$

and $(\psi_r)_{r\geq 1} \in \mathbb{T}^\infty$ is an arbitrary sequence of angles. Furthermore, the evolution of the new variables $(s_r, \psi_r)_{r\geq 1}$ through the dynamics of (1) is given by

$$\frac{ds_r}{dt} = 0, \quad \frac{d\psi_r}{dt} = s_r^2, \qquad r = 1, 2, \ldots.$$

A natural question is then the description of the regularity of u in these new variables. A first type of answer to this question is provided by results due to Peller and Semmes, see, e.g., [Peller 2003], which characterise the Schatten classes

$$\sum_{r\geq 1} s_r^p < \infty, \quad 0 < p < \infty,$$

in terms of the Besov spaces

$$\sum_{j=0}^\infty 2^j \int_{\mathbb{T}} |\Delta_j u|^p \, dx < \infty,$$

where $(\Delta_j u)_{j\geq 0}$ denotes the dyadic blocks of u. In particular, if u is smooth, then $(s_r)_{r\geq 1}$ satisfies

$$\sum_{r=1}^\infty s_r^p < \infty, \quad \text{for all } p < \infty.$$

However, the latter condition is far from being sufficient to control high regularity of u. In fact, Sobolev regularity H^s for $s > \frac{1}{2}$ cannot be easily described by the variables $(s_r, \psi_r)_{r\geq 1}$, as shown by the following result.

Theorem 1 [Gérard and Grellier 2017]. *There exists a dense G_δ subset of initial data in*

$$C_+^\infty(\mathbb{T}) := \bigcap_s H_+^s(\mathbb{T})$$

such that the corresponding solutions of (1) satisfies, for every $s > \frac{1}{2}$,

$$\text{for all } M \geq 1, \qquad \limsup_{t\to\infty} \frac{\|u(t)\|_{H^s}}{|t|^M} = +\infty, \qquad \liminf_{t\to\infty} \|u(t)\|_{H^s} < \infty.$$

In other words, in the $(s_r, \psi_r)_{r\geq 1}$ representation, the size of the high Sobolev norms may strongly depend on the angles $(\psi_r)_{r\geq 1}$. The goal of this paper is to investigate this phenomenon in more detail.

1.3. *Overview of the results.* Our first result claims that generic smooth functions u are located on a torus $\mathcal{T}((s_r)_{r\geq 1})$ containing also very singular functions.

Theorem 2. *There exists a dense G_δ subset $\mathcal{G}$ of $C_+^\infty(\mathbb{T})$ such that every element u of $\mathcal{G}$ belongs to $H_{+,\mathrm{gen}}^{\frac{1}{2}}(\mathbb{T})$, and the infinite dimensional torus $\mathcal{T}((s_r)_{r\geq 1})$ passing through u has a dense G_δ subset—for the $H^{\frac{1}{2}}$ topology—which is disjoint of H^s for every $s > \frac{1}{2}$.*

Theorem 2 states that, on the tori $\mathcal{T}((s_r)_{r\geq 1})$ passing through generic smooth functions, the regularity changes dramatically from C^∞ to the outside of H^s for every $s > \frac{1}{2}$. Of course, this result can be seen as a natural extension of Theorem 1 recalled above, of which we use the weaker form that tori $\mathcal{T}((s_r)_{r\geq 1})$ passing through generic smooth functions are unbounded in H^s for every $s > \frac{1}{2}$. However, in order to find singular functions on these tori, we combine it with a structure property of these tori, which we think has its own interest.

Lemma 3. *Let $s > \frac{1}{2}$ and let $(s_r)_{r\geq 1}$ be a square summable decreasing sequence of positive numbers such that the numbers s_r^2, $r \geq 1$ are linearly independent on $\mathbb{Q}$. Then we have the following alternatives:*

- *either $\mathcal{T}((s_r)_{r\geq 1})$ is a bounded subset of H^s,*
- *or $\mathcal{T}((s_r)_{r\geq 1}) \setminus H^s$ is a dense G_δ subset of $\mathcal{T}((s_r)_{r\geq 1})$ for the $H^{\frac{1}{2}}$ topology.*

The point of Theorem 2 is that, even for fast decaying singular values (s_r), the regularity of u may be spoiled by the relative smallness of the gaps $s_r - s_{r+1}$ with respect to s_r. In fact, if

$$u_N(z) = \langle \mathscr{C}_N(z)^{-1}(\mathbb{1}_N), \mathbb{1}_N \rangle, \quad \psi_r = 0, \ r = 1, 2, \ldots,$$

with the notation introduced above, then, using the positivity property of the Hankel matrices $\Gamma_{\hat{u}_N}$ and $\tilde{\Gamma}_{\hat{u}_N}$ equivalent to $\psi_r = 0$ for all r (see [Gérard and Grellier 2014; Gérard and Pushnitski 2015]), we prove in the Appendix that

$$\|u_N\|_{C^1(\mathbb{T})} \geq \sum_{j=1}^{N} \frac{s_{2j-1}s_{2j}}{s_{2j-1} - s_{2j}}. \tag{4}$$

It is then easy to find fast decaying sequences (s_r) such that the above right hand side tends to infinity as N goes to infinity, which implies that (u_N) is unbounded in $C^1(\mathbb{T})$. However, at this stage we do not know how to conclude that u is not in $C^1(\mathbb{T})$.

Our two other results state some uniform analytic regularity for tori $\mathcal{T}((s_r)_{r\geq 1})$ where the sequence $(s_r)_{r\geq 1}$ satisfies some specific property ruling out the phenomenon of small gaps.

Theorem 4. *For every $\rho > 0$, there exists $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0)$, if*

$$\text{for all } r \geq 1, \quad s_{r+1} \leq \delta s_r,$$

all functions $u \in \mathcal{T}((s_r)_{r\geq 1})$ are holomorphic and uniformly bounded in the disc $|z| < 1 + \rho$. Consequently, for any initial datum corresponding to some of these functions, the solution of the cubic Szegő equation (1) *is analytic in the disc of radius $1 + \rho$ for all time, and is uniformly bounded in this disc. In particular, the trajectory is bounded in $C^\infty(\mathbb{T})$.*

Theorem 4 applies in particular to geometric sequences $s_r = e^{-rh}$ for $h > 0$ large enough. Our last result explores in more detail the case of geometric sequences $s_r = e^{-rh}$, where $h > 0$ is arbitrary. In this case, we still obtain some uniform analytic regularity, but with a constraint on the angles ψ_r.

Theorem 5. *Let $h > 0$ and $\theta \in \mathbb{R}$. Assume (s_r) is given by $s_r = e^{-rh}$ and (ψ_r) by $\psi_r = r\theta h$. Then there exists $\rho > 0$ such that the corresponding elements of $\mathcal{T}((s_r)_{r\geq 1})$ are holomorphic and uniformly bounded in the disc $|z| < 1 + \rho$.*

We do not know whether or not geometric tori are embedded into the space of analytic functions on $\mathbb{T}$. What we are able to prove is that, for transcendental γ, we have the following alternatives:

- either there exists $\rho > 0$ such that every element of $\mathcal{T}((\gamma^r)_{r\geq 1})$ is holomorphic on the disc $|z| < 1 + \rho$, with a uniform bound,

- or the nonanalytic elements of $\mathcal{T}((\gamma^r)_{r\geq 1})$ form a dense G_δ subset of $\mathcal{T}((\gamma^r)_{r\geq 1})$ for the $H^{\frac{1}{2}}$ topology.

This is a special case of an extension of Lemma 3 to analytic regularity (see Lemma 8).

1.4. *Open problems.* In view of the above theorems, the most natural open question is certainly to decide whether Theorem 4 can be generalised to any parameter $\delta < 1$. In particular, as we questioned above, if $0 < \gamma < 1$, is it true that the infinite dimensional torus $\mathcal{T}((\gamma^r)_{r\geq 1})$ is included in the space of analytic functions on $\mathbb{T}$?

Another question connected to Theorem 2 relies on estimate (4). Assuming that

$$\sum_{j=1}^{\infty} \frac{s_{2j-1} s_{2j}}{s_{2j-1} - s_{2j}} = \infty,$$

can one infer that the function $u \in \mathcal{T}((s_r)_{r \geq 1})$ characterised by $\psi_r = 0$, $r = 1, 2, \ldots$, is not C^1 on $\mathbb{T}$? In view of Lemma 3, this would imply, if moreover the s_r^2 are linearly independent on $\mathbb{Q}$, that most of the points on this torus would be singular — say, not in H^2. Then it would be interesting to draw the consequences of this property for long term behaviour of solutions of the cubic Szegő equation on this torus.

1.5. *Organisation of the paper.* The proof of Theorem 2 is provided in Section 2 after reducing to Lemma 3 and Theorem 1. The proof of Lemma 3 combines a Baire category argument and some elementary ergodic argument for the cubic Szegő flow. Section 3 is devoted to the proof of Theorem 4, which is based on brute force estimates on matrices $\mathscr{C}_N(z)$. In Section 4, we prove Theorem 5 by a different approach relying on the theory of Toeplitz operators and a theorem by Baxter which reduces our analysis to proving that the restriction to $\mathbb{T}$ of a meromorphic function given by an explicit series, has no zero and has index 0, which can be realised using some elementary complex analysis and the Poisson summation formula. Finally, the estimate (4) is derived in the Appendix from an explicit calculation using Cauchy matrices, in the spirit of [Gérard and Grellier 2017; Gérard and Pushnitski 2018].

2. The melting pot property

In this section, we prove Theorem 2. First we reduce the proof to Lemma 3 by the following classical argument.

Lemma 6. *The set of $u \in C_+^\infty(\mathbb{T}) \cap H_{+,\mathrm{gen}}^{\frac{1}{2}}(\mathbb{T})$ such that the squares $s_r(u)^2$, $r \geq 1$ of the singular values $s_r(u)$ are linearly independent on $\mathbb{Q}$, is a dense G_δ subset of $C_+^\infty(\mathbb{T})$.*

Proof. From the proof of [Gérard and Grellier 2012, Lemma 7], we already know that $C_+^\infty(\mathbb{T}) \cap H_{+,\mathrm{gen}}^{\frac{1}{2}}(\mathbb{T})$ is a dense G_δ subset of $C_+^\infty(\mathbb{T})$. In fact, we can slightly modify the proof as follows. For every N, consider the open subset $\mathcal{O}_N$ made of functions $u \in C_+^\infty(\mathbb{T})$ such that the first singular values of H_u and K_u satisfy

$$\rho_1(u) > \sigma_1(u) > \rho_2(u) > \sigma_2(u) > \cdots \rho_N(u) > \sigma_N(u),$$

and such that any nontrivial linear combination of

$$\rho_1(u)^2, \sigma_1(u)^2, \rho_2(u)^2, \sigma_2(u)^2, \ldots, \rho_N(u)^2, \sigma_N(u)^2$$

with integer coefficients in $[-N, N]$, is not zero. Approximating elements of $C_+^\infty(\mathbb{T})$ by rational functions, and using the inverse spectral theorem of [Gérard and Grellier 2012] for rational functions, we easily obtain that $\mathcal{O}_N$ is dense. The conclusion follows from Baire's theorem. $\square$

Intersecting the dense G_δ subset of $C_+^\infty(\mathbb{T})$ provided by this lemma with the one provided by Theorem 1 — or its weaker form, saying that the corresponding Szegő trajectories are unbounded in every H^s, $s > \frac{1}{2}$ — we observe that Theorem 2 is a consequence of Lemma 3, which we restate for the convenience of the reader.

Lemma 7. *Let $(s_r)_{r\geq 1}$ be a square-summable decreasing sequence of positive numbers such that the numbers s_r^2, $r \geq 1$ are linearly independent on $\mathbb{Q}$ and let $s > \frac{1}{2}$. Then we have the following alternatives*:

- *either $\mathcal{T}((s_r)_{r\geq 1})$ is a bounded subset of H^s,*
- *or $\mathcal{T}((s_r)_{r\geq 1}) \setminus H^s$ is a dense G_δ subset of $\mathcal{T}((s_r)_{r\geq 1})$ for the $H^{\frac{1}{2}}$ topology.*

Proof. Recall [Gérard and Grellier 2012; 2017] that, for the $H^{\frac{1}{2}}$ topology, $\mathcal{T}((s_r)_{r\geq 1})$ is homeomorphic to the infinite dimensional torus $\mathbb{T}^\infty$, endowed with the product topology, through the parametrisation given by (2) and (3). In particular, it is a compact metrisable space. For every $s > \frac{1}{2}$, the function

$$\|v\|_{H^s} = \left(\sum_{n=0}^\infty (1+n)^{2s}|\hat{v}(n)|^2\right)^{\frac{1}{2}}$$

is lower semicontinuous on $\mathcal{T}((s_r)_{r\geq 1})$. For every positive integer ℓ, consider

$$F_\ell = \{v \in \mathcal{T}((s_r)_{r\geq 1}) : \|v\|_{H^s} \leq \ell\}.$$

F_ℓ is a closed subset of $\mathcal{T}((s_r)_{r\geq 1})$, and the complement of the union of the F_ℓ is precisely $\mathcal{T}((s_r)_{r\geq 1}) \setminus H^s$. Hence, by the Baire theorem, either this set is a dense G_δ subset of $\mathcal{T}((s_r)_{r\geq 1})$, or there exists $\ell \geq 1$ such that F_ℓ has a nonempty interior. Assume that some F_ℓ has a nonempty interior, and let us show that $\mathcal{T}((s_r)_{r\geq 1})$ is a bounded subset of H^s. Let $(\psi_r^0)_{r\geq 1} \in \mathbb{T}^\infty$ such that the corresponding point v^0 in $\mathcal{T}((s_r)_{r\geq 1})$ lies in the interior of F_ℓ. In view of the product topology on $\mathbb{T}^\infty$, there exists some integer $N \geq 1$ and some $\varepsilon > 0$ such that all the elements of $\mathcal{T}((s_r)_{r\geq 1})$ corresponding to

$$\psi_r \in]\psi_r^0 - \varepsilon, \psi_r^0 + \varepsilon[, \quad r = 1, \dots, N,$$

form an open set U contained in F_ℓ. At this stage we appeal to the number theoretic assumption on the s_r^2, which we use classically under the form that the trajectory

$$\{(\psi_r^0 + ts_r^2)_{r=1,\dots,N} : t \in \mathbb{R}\}$$

is dense into the torus $\mathbb{T}^N$. Since, as recalled in the introduction, this trajectory is precisely the projection of the trajectory of the cubic Szegő flow Φ_t on the first N components, we infer that every element of $\mathcal{T}((s_r)_{r\geq 1})$ is contained in some open set $\Phi_t(U)$. Since the cubic Szegő equation is well-posed on H^s [Gérard and Grellier 2010], we infer that $\mathcal{T}((s_r)_{r\geq 1})$ is covered by the union of the interiors of

the F_m for $m \geq 1$. By compactness, it is covered by a finite union, which precisely means that $\mathcal{T}((s_r)_{r\geq 1})$ is bounded in H^s. $\square$

As stated in the introduction for geometric sequences, the following analogous result holds in the analytic setting.

Lemma 8. *Let $(s_r)_{r\geq 1}$ be a square summable decreasing sequence of positive numbers such that the numbers s_r^2, $r \geq 1$ are linearly independent on $\mathbb{Q}$. Then we have the following alternatives*:

- *either there exists $\rho > 0$ such that every element of $\mathcal{T}((s_r)_{r\geq 1})$ is holomorphic on the disc $|z| < 1 + \rho$, with a uniform bound,*
- *or the nonanalytic elements of $\mathcal{T}((s_r)_{r\geq 1})$ form a dense G_δ subset of $\mathcal{T}((s_r)_{r\geq 1})$ for the $H^{\frac{1}{2}}$ topology.*

Proof. The proof is an adaptation of the preceding one (Lemma 3) to the analytic setting. As, from [Gérard et al. 2015], the cubic Szegő equation propagates analyticity, the result follows from the Baire theorem applied to the closed sets

$$F_\ell := \left\{ v \in \mathcal{T}((s_r)_{r\geq 1}) \ : \ \sum_{n=0}^{\infty} \mathrm{e}^{\frac{n}{\ell}} |\hat{v}(n)| \leq \ell \right\}$$

for $\ell \geq 1$. $\square$

3. Example of bounded analytic tori

In this section, we prove Theorem 4.

Let $u \in \mathcal{T}((s_r)_{r\geq 1})$. Recall that

$$u = \lim_{N\to\infty} u_N, \quad \text{where } u_N(z) := \langle \mathscr{C}_N(z)^{-1} \mathbb{1}_N, \mathbb{1}_N \rangle,$$

$$\mathscr{C}_N(z) := \left(\frac{s_{2j-1}\mathrm{e}^{i\psi_{2j-1}} - z s_{2k}\mathrm{e}^{i\psi_{2k}}}{s_{2j-1}^2 - s_{2k}^2} \right)_{1\leq j,k\leq N},$$

and

$$\mathbb{1}_N = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{C}^N.$$

Our assumption is

$$s_{r+1} = \varepsilon_r s_r, \quad r \geq 1,$$

where the sequence $(\varepsilon_r)_{r\geq 1}$ satisfies

$$0 < \varepsilon_r \leq \delta \quad \text{for some } \delta < 1.$$

Our aim is to prove that, for δ sufficiently small, the functions u_N are holomorphic and uniformly bounded in some disc of radius $1 + \rho$, where $\rho > 0$, independently

of N. Our strategy is to use that $\mathscr{C}_N(0)$ is related to a Cauchy matrix, and hence, that an explicit formula for its inverse is known. We write

$$\mathscr{C}_N(z) = \mathscr{C}_N(0) - z\dot{\mathscr{C}}_N = \mathscr{C}_N(0)(I - z\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N),$$

where

$$\dot{\mathscr{C}}_N := \left(\frac{s_{2k}e^{i\psi_{2k}}}{s_{2j-1}^2 - s_{2k}^2}\right)_{1 \leq j,k \leq N},$$

and we establish the following lemma.

Lemma 9. *For any $0 < \delta < 1$, there exists some constant $C_\delta > 0$ such that, for any $N \geq 1$,*

$$\sum_{j,k} |(\mathscr{C}_N(0)^{-1})_{j,k}| \leq C_\delta s_1. \tag{5}$$

There exists a universal constant $A > 0$ such that, for $\delta \in \left(0, \frac{1}{2}\right)$ and for any $N \geq 1$,

$$\|\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N\|_{\ell^1 \to \ell^1} \leq A\,\delta. \tag{6}$$

Let us assume Lemma 9 proved. Take $\rho > 0$, and choose δ_0 such that $A\delta_0(1+\rho) \leq \frac{1}{2}$. Hence, for any $\delta \in (0, \delta_0)$, from estimate (6),

$$(I - z\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N)$$

is invertible for any z with $|z| < 1 + \rho$ and its inverse $R_N(z)$ is analytic and has uniformly bounded norm for any z with $|z| < 1 + \rho$. Indeed, for any $N \geq 1$, and any z with $|z| < 1 + \rho$, by the Neumann series identity,

$$\|R_N(z)\|_{\ell^1 \to \ell^1} \leq \sum_{k=0}^{\infty} |z|^k \|\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N\|_{\ell^1 \to \ell^1}^k \leq \sum_{k=0}^{\infty} 2^{-k} \leq 2. \tag{7}$$

Writing

$$\mathscr{C}_N(z)^{-1} = (I - z(\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N))^{-1}\mathscr{C}_N(0)^{-1} = R_N(z)(\mathscr{C}_N(0))^{-1}$$

we get

$$u_N(z) = \langle R_N(z)\mathscr{C}_N(0)^{-1}(\mathbb{1}_N), \mathbb{1}_N \rangle.$$

Using (5) and (7), we conclude that the series defining u_N converges uniformly for $|z| < 1 + \rho$. Hence u_N is analytic and uniformly bounded in the disc of radius $1 + \rho$. We infer that u is as well analytic in the disc of radius $1 + \rho$ and bounded on this disc.

This completes the proof of Theorem 4, modulo Lemma 9.

3.1. *Proof of Lemma 9.* Notice that

$$\mathscr{C}_N(0) = \operatorname{diag}(s_{2j-1}e^{i\psi_{2j-1}})\mathscr{T}, \quad \mathscr{T} := \left(\frac{1}{s_{2j-1}^2 - s_{2k}^2}\right)_{1 \le j,k \le N}.$$

Since $\mathscr{T}$ is a Cauchy matrix, its inverse is explicitly known, so the inverse of $\mathscr{C}_N(0)$ is given by

$$\mathscr{C}_N(0)^{-1} = \left(\frac{(-1)^{j+k+N}\alpha_j^{(N)}\beta_k^{(N)}}{s_{2j-1}^2 - s_{2k}^2}\frac{1}{s_{2j-1}e^{i\psi_{2j-1}}}\right)_{1 \le k,j \le N},$$

where

$$\alpha_j^{(N)} := \frac{\prod_\ell (s_{2j-1}^2 - s_{2\ell}^2)}{\prod_{\ell<j}(s_{2\ell-1}^2 - s_{2j-1}^2)\prod_{\ell>j}(s_{2j-1}^2 - s_{2\ell-1}^2)},$$

$$\beta_k^{(N)} := \frac{\prod_\ell (s_{2\ell-1}^2 - s_{2k}^2)}{\prod_{\ell<k}(s_{2\ell}^2 - s_{2k}^2)\prod_{\ell>k}(s_{2k}^2 - s_{2\ell}^2)}.$$

In particular,

$$|\alpha_j^{(N)}| = \prod_{\ell<j}\frac{s_{2\ell}^2}{s_{2\ell-1}^2}\prod_{\ell<j}\left(\frac{1 - \prod_{r=2\ell}^{2j-2}\varepsilon_r^2}{1 - \prod_{r=2\ell-1}^{2j-2}\varepsilon_r^2}\right)\prod_{\ell>j}\frac{(1 - \prod_{r=2j-1}^{2\ell-1}\varepsilon_r^2)}{(1 - \prod_{r=2j-1}^{2\ell-2}\varepsilon_r^2)}s_{2j-1}^2(1 - \varepsilon_{2j-1}^2)$$

$$\le \prod_{\ell<j}\varepsilon_{2\ell-1}^2\frac{s_{2j-1}^2}{\prod_{m=1}^\infty(1 - \delta^{4m})}.$$

Indeed, in the first line above, the factors in the second product are bounded by 1, while, in the third product, the ℓ-factor is bounded by $\frac{1}{1-\delta^{4(\ell-j)}}$. Similarly, we have

$$|\beta_k^{(N)}| = \prod_{\ell<k}\frac{s_{2\ell-1}^2}{s_{2\ell}^2}\prod_{\ell>k}\left(\frac{1 - \prod_{r=2k}^{2\ell-2}\varepsilon_r^2}{1 - \prod_{r=2k}^{2\ell-1}\varepsilon_r^2}\right)\prod_{\ell<k}\frac{(1 - \prod_{r=2\ell-1}^{2k-1}\varepsilon_r^2)}{(1 - \prod_{r=2\ell}^{2k-1}\varepsilon_r^2)}s_{2k-1}^2(1 - \varepsilon_{2k-1}^2)$$

$$\le \prod_{\ell<k}\frac{1}{\varepsilon_{2\ell-1}^2}\frac{s_{2k-1}^2}{\prod_{m=1}^\infty(1 - \delta^{4m})}.$$

Setting

$$B_\delta = \frac{1}{\prod_{m=1}^\infty(1 - \delta^{4m})^2},$$

we obtain

$$|(\mathscr{C}_N(0)^{-1})_{kj}| \le B_\delta \frac{s_{2j-1}s_{2k-1}^2}{|s_{2j-1}^2 - s_{2k}^2|} \prod_{\ell < j} \varepsilon_{2\ell-1}^2 \prod_{\ell < k} \frac{1}{\varepsilon_{2\ell-1}^2}$$

$$\le B_\delta \begin{cases} \dfrac{1}{1 - \delta^{4(k-j)+2}} \dfrac{s_{2k-1}^2}{s_{2j-1}} \prod_{j \le \ell < k} \dfrac{1}{\varepsilon_{2\ell-1}^2} & \text{if } j < k, \\[3mm] \dfrac{1}{1 - \delta^2} s_{2j-1} & \text{if } j = k, \\[3mm] \dfrac{1}{1 - \delta^{4(j-k-1)+2}} s_{2j-1} \dfrac{s_{2k-1}^2}{s_{2k}^2} \prod_{k \le \ell < j} \varepsilon_{2\ell-1}^2 & \text{if } j > k. \end{cases}$$

To summarise,

$$|(\mathscr{C}_N(0)^{-1})_{kj}| \le \frac{B_\delta}{1 - \delta^2} s_{2j-1} \begin{cases} \delta^{2(k-j)} & \text{if } j < k, \\ 1 & \text{if } j = k, k+1, \\ \delta^{2(j-k-1)} & \text{if } j > k+1. \end{cases} \tag{8}$$

In particular, it gives

$$\sum_{k,j} |(\mathscr{C}_N(0)^{-1})_{jk}| \le \frac{2B_\delta}{(1 - \delta^2)^2} \sum_{j \le N} s_{2j-1} \le \frac{2B_\delta s_1}{(1 - \delta^2)^3}.$$

This proves estimate (5).

For the second estimate, one has to consider

$$\|\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N\|_{\ell^1 \to \ell^1} \le \sup_\ell \sum_k |(\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N)_{k\ell}|.$$

Recall that

$$\dot{\mathscr{C}}_N := \left(\frac{s_{2\ell} e^{i\psi_{2\ell}}}{s_{2j-1}^2 - s_{2\ell}^2} \right)_{1 \le j, \ell \le N}.$$

In particular,

$$|(\dot{\mathscr{C}}_N)_{j\ell}| \le \frac{1}{1 - \delta^2} \begin{cases} \frac{s_{2\ell}}{s_{2j-1}^2} & \text{if } j \le \ell, \\ \frac{1}{s_{2\ell}} & \text{if } j \ge \ell + 1. \end{cases}$$

As $(\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N)_{k\ell} = \sum_j (\mathscr{C}_N(0)^{-1})_{kj}(\dot{\mathscr{C}}_N)_{j\ell}$, we get from the preceding estimate (8) on $|(\mathscr{C}_N(0)^{-1})_{kj}|$ that:

• If $k > \ell$,

$$|(\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N)_{k\ell}|$$
$$\le \frac{B_\delta}{(1 - \delta^2)^2} \left(\sum_{j \le \ell} \frac{s_{2\ell}}{s_{2j-1}} \prod_{j \le r \le k-1} \varepsilon_{2r}^2 + \sum_{\ell+1 \le j \le k} \frac{s_{2j-1}}{s_{2\ell}} \prod_{j \le r \le k-1} \varepsilon_{2r}^2 \right.$$
$$\left. + \sum_{j \ge k+1} \frac{s_{2j-1}}{s_{2\ell}} \prod_{k+1 \le r \le j-1} \varepsilon_{2r-1}^2 \right)$$

$$\leq \frac{B_\delta}{(1-\delta^2)^2}\left(\sum_{j\leq \ell}\delta^{2(\ell-j)+1}\delta^{2(k-j)} + \sum_{\ell+1\leq j\leq k}\delta^{2(j-\ell-1)+1}\delta^{2(k-j)}\right.$$
$$\left.+ \sum_{j\geq k+1}\delta^{2(j-\ell-1)+1}\delta^{2(j-k-1)}\right)$$
$$\leq \delta\frac{B_\delta}{(1-\delta^2)^2}\left(2\delta^{2(k-\ell)}\sum_{s\geq 0}\delta^{4s} + \sum_{\ell+1\leq j\leq k}\delta^{2(j-\ell-1)}\delta^{2(k-j)}\right),$$

then since

$$\sum_{k:k\geq \ell+1}\sum_{\ell+1\leq j\leq k}\delta^{2(j-\ell-1)}\delta^{2(k-j)} = \sum_{j\geq \ell+1}\sum_{k\geq j}\delta^{2(j-\ell-1)}\delta^{2(k-j)} = \frac{1}{(1-\delta^2)^2},$$

one gets

$$\sum_{k:k>\ell}|(\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N)_{k\ell}| \leq \frac{\delta B_\delta}{(1-\delta^2)^4}\left(\frac{1+3\delta^2}{1+\delta^2}\right).$$

• If $k < \ell$,

$$|(\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N)_{k\ell}|$$
$$\leq \frac{B_\delta}{(1-\delta^2)^2}\left(\sum_{j\leq k}\frac{s_{2\ell}}{s_{2j-1}}\prod_{j\leq r\leq k-1}\varepsilon_{2r}^2 + \sum_{k+1\leq j\leq \ell}\frac{s_{2\ell}}{s_{2j-1}}\prod_{k+1\leq r\leq j-1}\varepsilon_{2r-1}^2\right.$$
$$\left.+ \sum_{j\geq \ell+1}\frac{s_{2j-1}}{s_{2\ell}}\prod_{k+1\leq r\leq j-1}\varepsilon_{2r-1}^2\right)$$
$$\leq \frac{B_\delta}{(1-\delta^2)^2}\left(\sum_{j\leq k}\delta^{2(\ell-j)+1}\delta^{2(k-j)} + \sum_{k+1\leq j\leq \ell}\delta^{2(\ell-j)+1}\delta^{2(j-k-1)}\right.$$
$$\left.+ \sum_{j\geq \ell+1}\delta^{2(j-\ell-1)+1}\delta^{2(j-k-1)}\right)$$
$$\leq \frac{\delta B_\delta}{(1-\delta^2)^2}\left(2\delta^{2(\ell-k)}\sum_{s\geq 0}\delta^{4s} + \sum_{k+1\leq j\leq \ell}\delta^{2(\ell-j)}\delta^{2(j-k-1)}\right)$$

and, as before,

$$\sum_{k:k<\ell}|(\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N)_{k\ell}| \leq \frac{\delta B_\delta}{(1-\delta^2)^4}\left(\frac{1+3\delta^2}{1+\delta^2}\right).$$

• For $k = \ell$,

$$|(\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N)_{kk}|$$
$$\leq \frac{B_\delta}{(1-\delta^2)^2}\left(\sum_{j\leq \ell}\frac{s_{2\ell}}{s_{2j-1}}\prod_{j\leq r\leq k-1}\varepsilon_{2r}^2 + \sum_{j\geq k+1}\frac{s_{2j-1}}{s_{2\ell}}\prod_{k+1\leq r\leq j-1}\varepsilon_{2r-1}^2\right)$$

$$\leq \frac{B_\delta}{(1-\delta^2)^2}\left(\sum_{j\leq\ell}\delta^{2(\ell-j)+1}\delta^{2(k-j)} + \sum_{j\geq k+1}\delta^{2(j-\ell-1)+1}\delta^{2(j-k-1)}\right)$$

$$\leq \frac{B_\delta}{(1-\delta^2)^2}\left(\sum_{j\leq k}\delta^{4(k-j)+1} + \sum_{j\geq k+1}\delta^{4(j-k-1)+1}\right)$$

$$= 2\frac{\delta B_\delta}{(1-\delta^2)^2(1-\delta^4)}.$$

Eventually, if $\delta \leq \frac{1}{2}$, say, we obtain, with a universal constant A,

$$\|(\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N)\|_{\ell^1\to\ell^1} \leq \sup_\ell \sum_k |(\mathscr{C}_N(0)^{-1}\dot{\mathscr{C}}_N)_{k\ell}| \leq A\delta.$$

This completes the proof of Lemma 9.

4. The totally geometric spectral data

In this section, we consider the totally geometric case and prove Theorem 5. For some fixed $h > 0$ and $\theta \in \mathbb{R}$, we consider the symbol u with spectral data (s_r, ψ_r) with $s_r = \mathrm{e}^{-rh}$ and $\psi_r = r\theta h$. In particular, $s_{r+1} = s_r\mathrm{e}^{-h}$ so that, for h sufficiently large, it becomes a particular case of subgeometric spectral data treated in Theorem 4. However, the result here does not require any smallness on e^{-h}.

Our strategy here is to use Toeplitz operators and a stability result from [Baxter 1963].

4.1. *Background on Toeplitz operators.* Let us first introduce some basic notation. For a continuous function Φ on $\mathbb{T}$, we denote by $T(\Phi)$ the Toeplitz operator of symbol Φ defined on $L_+^2(\mathbb{T})$ by

$$T(\Phi)(f) = \Pi(\Phi f)$$

or equivalently, the operator defined on $\ell^2(\mathbb{N})$ by

$$(T(\Phi)((a_k)))_j := \sum_{k=0}^\infty \hat{\Phi}(j-k)a_k, \quad j \in \mathbb{N}.$$

For any integer N, we denote by $T_N(\Phi)$ the truncated operator defined by

$$T_N(\Phi) := \Pi_N T(\Phi)\Pi_N.$$

Here

$$\Pi_N : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$

is the orthogonal projector:

$$(x_0, x_1, x_2, \dots) \mapsto (x_0, x_1, x_2, \dots, x_{N-1}).$$

The operator T_N corresponds to the $N \times N$ truncated Toeplitz matrix

$$(\hat{\Phi}(j - k))_{0 \leq j, k \leq N-1}.$$

Recall that a sequence of $N \times N$ matrices $(A_N)_{N \geq 1}$ is said to be stable if there is an N_0 such that the matrices A_N are invertible for all $N \geq N_0$ and

$$\sup_{N \geq N_0} \|A_N^{-1}\|_{\ell^2 \to \ell^2} < \infty.$$

Theorem 10 [Baxter 1963; Böttcher and Grudsky 2000]. *The sequence $(T_N(\Phi))_{N \geq 1}$ is stable if and only if $T(\Phi)$ is invertible.*

Let us emphasise that the operators are considered as operators acting on $\ell^2(\mathbb{N})$ or $L^2_+(\mathbb{T})$ so that the stability is evaluated in the $\ell^2(\mathbb{N})$ norm. The characterisation of the invertibility of Toeplitz operators is well known. We recall it for the convenience of the reader.

Theorem 11. *Let Φ be a continuous function on the unit circle. If Φ has index 0 and does not vanish on the circle, then T_Φ is invertible on $L^2_+(\mathbb{T})$. Under these hypotheses, $\Phi = e^\varphi = \Phi_+ \overline{\Phi_-}$ with*

$$\Phi_+ = e^{\Pi(\varphi)} \quad and \quad \Phi_- = e^{\overline{(I - \Pi)(\varphi)}}$$

and the inverse of T_Φ is given by $T_{\Phi_+^{-1}} T_{\overline{\Phi_-}^{-1}}$.

As an immediate consequence, one gets the following characterisation of the stability of truncated Toeplitz operators.

Corollary 12. *Let Φ be a continuous function on the unit circle. The sequence of truncated Toeplitz operators $(T_N(\Phi))$ is stable if and only if Φ has no zero on the unit circle and has index 0.*

We are going to use this argument to prove Theorem 5.

4.2. *Totally geometric spectral data and Toeplitz operators.*
We claim that in the case of totally geometric spectral data, the explicit formula giving u_N involves the inverse of a truncated Toeplitz operator. From direct computation, one has

$$\mathscr{C}_N(z) = \left(\frac{\omega^{2j-1} - z\omega^{2k}}{|\omega|^{4j-2} - |\omega|^{4k}} \right)_{1 \leq j, k \leq N} = \left(\frac{1}{\bar{\omega}^{2j-1}} \frac{1 - z\omega^{2(k-j)+1}}{1 - |\omega|^{4(k-j)+2}} \right)_{1 \leq j, k \leq N},$$

where $\omega = e^{-h(1-i\theta)}$. In that case, if $T_N(z)$ and $T_{N,r}(z)$ denote the matrices

$$T_N(z) = \left(\frac{1 - z\omega^{2(k-j)+1}}{1 - |\omega|^{4(k-j)+2}} \right)_{1 \leq j, k \leq N}$$

and

$$T_{N,r}(z) = \left(r^{k-j} \frac{1 - z\omega^{2(k-j)+1}}{1 - |\omega|^{4(k-j)+2}} \right)_{1 \leq j, k \leq N},$$

we get from our explicit formula, for any $r > 0$,

$$u_N(z) = \langle T_N(z)^{-1}(\bar{\omega}^{2j-1}), \mathbb{1} \rangle = \langle T_{N,r}(z)^{-1}(r^{-j}\bar{\omega}^{2j-1})_{1 \le j \le N}), \ (r^k)_{1 \le k \le N} \rangle.$$

We consider for $|\zeta| = r$, $|\omega|^2 < r < 1$, $z \in \mathbb{C}$, the symbol

$$\Phi(z, \zeta) := \sum_{\ell \in \mathbb{Z}} \frac{1 - z\omega^{2\ell+1}}{1 - |\omega|^{4\ell+2}} \zeta^\ell.$$

The transpose of the matrix

$$\left(r^{k-j} \frac{1 - z\omega^{2(k-j)+1}}{1 - |\omega|^{4(k-j)+2}} \right)_{j,k \ge 1}$$

corresponds to the matrix of the Toeplitz operator of symbol

$$\Phi(z, r\cdot) : \zeta \mapsto \Phi(z, r\zeta).$$

We are going to prove the following result.

Proposition 13. *There exist $|\omega|^2 < r < 1$ and $\rho > 0$ such that the function $\zeta \mapsto \Phi(z, r\zeta)$ has no zero and has index 0 on the unit circle, for every z such that $|z| < 1 + \rho$.*

Assuming this result proved, we obtain by Corollary 12 that, uniformly in z, $|z| < 1 + \rho$, $\|T_{N,r}(z)^{-1}\|_{\ell^2 \to \ell^2}$ is bounded (or more precisely the norm of its transpose is bounded). As $|\omega|^2 < r < 1$, we obtain that the sequence $(u_N(z))_N$ with

$$u_N(z) = \langle T_{N,r}(z)^{-1}(r^{-j}\bar{\omega}^{2j-1})_{1 \le j \le N}), \ (r^k)_{1 \le k \le N} \rangle$$

is uniformly bounded and converges to $u(z)$ for any z, $|z| < 1 + \rho$. We conclude as in the previous section. This ends the proof of Theorem 5.

It remains to prove Proposition 13, which is the objective of the next subsections. As a preliminary, observe that, for $|\omega|^2 < |\zeta| < 1$, $\gamma = |\omega|^2$,

$$\Phi(z, \zeta) = F_\gamma(\zeta) - z\omega F_\gamma(\zeta\omega^2),$$

where

$$F_\gamma(\zeta) = \Phi(0, \zeta) = \sum_{j \in \mathbb{Z}} \frac{\zeta^j}{1 - \gamma^{2j+1}}, \quad \gamma = |\omega|^2. \tag{9}$$

We collect some basic properties of function F_γ in the following lemma.

Lemma 14. *The function F_γ has a meromorphic extension in $\mathbb{C} \setminus \{0\}$ given by*

$$F_\gamma(\zeta) = \sum_{\ell \in \mathbb{Z}} \frac{\gamma^\ell}{1 - \zeta\gamma^{2\ell}}. \tag{10}$$

Its only poles in $\mathbb{C} \setminus \{0\}$ are the $\gamma^{2\ell}$, $\ell \in \mathbb{Z}$ and $F_\gamma(\gamma^{2\ell+1}) = 0$, $\ell \in \mathbb{Z}$. Furthermore

$$F_\gamma\left(\frac{1}{\zeta}\right) = -\zeta \, F_\gamma(\zeta), \quad F_\gamma\left(\frac{\zeta}{\gamma^2}\right) = \gamma \, F_\gamma(\zeta). \tag{11}$$

Proof. Let us give another expression of F_γ. By assumption,

$$\gamma < |\zeta| < 1,$$

hence, $|\zeta| > \gamma^2$ and

$$
\begin{aligned}
F_\gamma(\zeta) &= \sum_{j=0}^{\infty} \frac{\zeta^j}{1 - \gamma^{2j+1}} + \sum_{j=0}^{\infty} \frac{\zeta^{-j-1}}{1 - \gamma^{-2j-1}} \\
&= \sum_{j=0}^{\infty} \zeta^j \sum_{\ell=0}^{\infty} \gamma^{(2j+1)\ell} - \sum_{j=0}^{\infty} \frac{\zeta^{-j-1}\gamma^{2j+1}}{1 - \gamma^{2j+1}} \\
&= \sum_{\ell=0}^{\infty} \gamma^\ell \sum_{j=0}^{\infty} (\zeta \gamma^{2\ell})^j - \gamma \zeta^{-1} \sum_{\ell=0}^{\infty} \gamma^\ell \sum_{j=0}^{\infty} (\zeta^{-1} \gamma^{2\ell+2})^j \\
&= \sum_{\ell=0}^{\infty} \frac{\gamma^\ell}{1 - \zeta \gamma^{2\ell}} - \sum_{\ell=0}^{\infty} \frac{\gamma^{\ell+1}}{\zeta - \gamma^{2\ell+2}},
\end{aligned}
$$

and we obtain (10). The other properties are elementary consequences of this equality. $\qquad\square$

Remark 15. Set $\gamma = \mathrm{e}^{-\pi\tau}$, $\tau > 0$. From the second identity (11), we observe that the meromorphic function

$$G_\tau(w) = \mathrm{e}^{2i\pi w}(F_\gamma(\mathrm{e}^{2i\pi w}))^2$$

satisfies

$$\text{for all } \lambda \in \mathbb{Z} + i\tau\mathbb{Z}, \quad G_\tau(w + \lambda) = G_\tau(w),$$

which means that G_τ is an elliptic function relative to the lattice $\mathbb{Z} + i\tau\mathbb{Z}$. Since G_τ has only double poles at the lattice points, with singularity

$$\frac{1}{(\zeta - 1)^2} \sim -\frac{1}{4\pi^2 w^2}$$

at $w = 0$, and since it cancels at points $i\frac{\tau}{2} + \mathbb{Z} + i\tau\mathbb{Z}$, we infer that

$$G_\tau(w) = -\frac{1}{4\pi^2}\left(\mathfrak{P}_\tau(w) - \mathfrak{P}_\tau\left(i\frac{\tau}{2}\right)\right),$$

where

$$\mathfrak{P}_\tau(w) = \frac{1}{w^2} + \sum_{\lambda \neq 0}\left(\frac{1}{(w - \lambda)^2} - \frac{1}{\lambda^2}\right)$$

denotes the Weierstrass $\mathfrak{P}$ function relative to the lattice $\mathbb{Z} + i\tau\mathbb{Z}$. See, e.g., [Saks and Zygmund 1952].

4.3. *Ruling out the zeroes on the unit circle.* In this section, we prove the following lemma.

Lemma 16. *There exists $\rho > 0$ such that $\Phi(z, \zeta)$ does not vanish in a neighbourhood of the circle $|\zeta| = 1$ for any z such that $|z| \leq 1 + \rho$.*

Lemma 16 is a consequence of the following result.

Lemma 17. *For every $\gamma \in (0, 1)$,*

$$\gamma^{\frac{1}{2}} \max_{|\zeta| = \gamma} |F_\gamma(\zeta)| < \min_{|\zeta| = 1} |F_\gamma(\zeta)|.$$

Proof. First of all we rewrite both sides of the above inequality. If $\zeta = e^{i\theta}$,

$$
\begin{aligned}
F_\gamma(\zeta) &= \sum_{k=0}^{\infty} \frac{\gamma^k}{1 - \gamma^{2k} e^{i\theta}} + \sum_{\ell=1}^{\infty} \frac{\gamma^{-\ell}}{1 - \gamma^{-2\ell} e^{i\theta}} \\
&= \sum_{k=0}^{\infty} \frac{\gamma^k}{1 - \gamma^{2k} e^{i\theta}} + \sum_{\ell=1}^{\infty} \frac{\gamma^\ell}{\gamma^{2\ell} - e^{i\theta}} \\
&= \frac{1}{1 - e^{i\theta}} + \sum_{\ell=1}^{\infty} \frac{\gamma^\ell (1 + \gamma^{2\ell})(1 - e^{-i\theta})}{1 + \gamma^{4\ell} - 2\gamma^{2\ell} \cos\theta} \\
&= (1 - e^{-i\theta}) \left(\frac{1}{2(1 - \cos\theta)} + \sum_{\ell=1}^{\infty} \frac{\gamma^\ell (1 + \gamma^{2\ell})}{1 + \gamma^{4\ell} - 2\gamma^{2\ell} \cos\theta} \right),
\end{aligned}
$$

hence

$$
\begin{aligned}
|F_\gamma(\zeta)| &= \frac{1}{2|\sin(\theta/2)|} + 2|\sin(\theta/2)| \sum_{\ell=1}^{\infty} \frac{\gamma^\ell (1 + \gamma^{2\ell})}{1 + \gamma^{4\ell} - 2\gamma^{2\ell} \cos\theta} \\
&= |\sin(\theta/2)| \sum_{\ell \in \mathbb{Z}} \frac{\gamma^\ell (1 + \gamma^{2\ell})}{1 + \gamma^{4\ell} - 2\gamma^{2\ell} \cos\theta}.
\end{aligned}
$$

Similarly, if $\zeta = \gamma e^{i\varphi}$, we have

$$
\begin{aligned}
F_\gamma(\zeta) &= \sum_{k=0}^{\infty} \frac{\gamma^k}{1 - \gamma^{2k+1} e^{i\varphi}} + \sum_{\ell=0}^{\infty} \frac{\gamma^{-\ell-1}}{1 - \gamma^{-2\ell-1} e^{i\varphi}} \\
&= \sum_{k=0}^{\infty} \frac{\gamma^k}{1 - \gamma^{2k+1} e^{i\varphi}} + \sum_{\ell=0}^{\infty} \frac{\gamma^\ell}{\gamma^{2\ell+1} - e^{i\varphi}} \\
&= \sum_{\ell=0}^{\infty} \frac{\gamma^\ell (1 + \gamma^{2\ell+1})(1 - e^{-i\varphi})}{1 + \gamma^{4\ell+2} - 2\gamma^{2\ell+1} \cos\varphi},
\end{aligned}
$$

so that

$$|F_\gamma(\zeta)| = 2|\sin(\varphi/2)| \sum_{\ell=0}^{\infty} \frac{\gamma^\ell(1+\gamma^{2\ell+1})}{1+\gamma^{4\ell+2}-2\gamma^{2\ell+1}\cos\varphi}$$

$$= |\sin(\varphi/2)| \sum_{\ell\in\mathbb{Z}} \frac{\gamma^\ell(1+\gamma^{2\ell+1})}{1+\gamma^{4\ell+2}-2\gamma^{2\ell+1}\cos\varphi}.$$

Consequently,

$$\min_{|\zeta|=1} |F_\gamma(\zeta)| - \gamma^{\frac{1}{2}} \max_{|\zeta|=\gamma} |F_\gamma(\zeta)| = \min_{\theta\in\mathbb{T}}|\sin(\theta/2)| \sum_{\ell\in\mathbb{Z}} \frac{\gamma^\ell(1+\gamma^{2\ell})}{1+\gamma^{4\ell}-2\gamma^{2\ell}\cos\theta}$$

$$- \max_{\varphi\in\mathbb{T}}|\sin(\varphi/2)| \sum_{\ell\in\mathbb{Z}} \frac{\gamma^{\ell+1/2}(1+\gamma^{2\ell+1})}{1+\gamma^{4\ell+2}-2\gamma^{2\ell+1}\cos\varphi}$$

Set, for $x\in\mathbb{R}$, $\theta\in\mathbb{T}\setminus\{0\}$,

$$f_{\gamma,\theta}(x) = |\sin(\theta/2)| \frac{\gamma^x(1+\gamma^{2x})}{1+\gamma^{4x}-2\gamma^{2x}\cos\theta}.$$

Then we are reduced to proving that

$$\inf_{\theta\in\mathbb{T}\setminus\{0\}} \sum_{k\in\mathbb{Z}} f_{\gamma,\theta}(k) - \sup_{\varphi\in\mathbb{T}\setminus\{0\}} \sum_{k\in\mathbb{Z}} f_{\gamma,\varphi}\left(k+\tfrac{1}{2}\right) > 0.$$

Applying the Poisson summation formula, we have

$$\sum_{k\in\mathbb{Z}} f_{\gamma,\theta}(k) = \sum_{n\in\mathbb{Z}} \hat{f}_{\gamma,\theta}(2\pi n), \quad \sum_{k\in\mathbb{Z}} f_{\gamma,\varphi}\left(k+\tfrac{1}{2}\right) = \sum_{n\in\mathbb{Z}}(-1)^n \hat{f}_{\gamma,\varphi}(2\pi n),$$

where

$$\hat{f}_{\gamma,\theta}(\xi) = |\sin(\theta/2)| \int_\mathbb{R} \frac{\gamma^x(1+\gamma^{2x})}{1+\gamma^{4x}-2\gamma^{2x}\cos\theta} e^{-ix\xi}\, dx$$

$$= \frac{|\sin(\theta/2)|}{|\log\gamma|} \int_0^\infty \frac{(1+t^2)t^{-i\xi/\log\gamma}}{1+t^4-2t^2\cos\theta}\, dt$$

$$= \frac{|\sin(\theta/2)|}{2|\log\gamma|} \int_0^\infty \frac{(1+y)y^{-i\xi/2\log\gamma-\frac{1}{2}}}{1+y^2-2y\cos\theta}\, dy,$$

where we have set $t=\gamma^x$, $y=t^2$. We calculate the above integral by introducing the holomorphic function

$$g(z) = \frac{|\sin(\theta/2)|}{2|\log\gamma|} \frac{(1+z)z^{-i\xi/2\log\gamma-\frac{1}{2}}}{1+z^2-2z\cos\theta},$$

on the domain $\mathbb{C}\setminus\mathbb{R}_+$, where the argument of z belongs to $(0,2\pi)$. Integrating on the contour of Figure 1 and making $R\to\infty$, $\varepsilon\to 0$, we obtain, by the residue
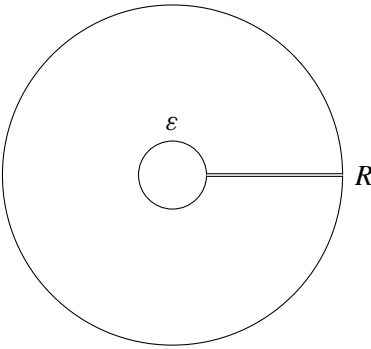
Figure 1. Contour for the proof of Lemma 17.

theorem, assuming $\theta \in (0, 2\pi)$ with no loss of generality,

$$
\hat{f}_{\gamma,\theta}(\xi)(1 + e^{\pi\xi/\log\gamma})
$$
$$
= 2i\pi\left[\operatorname{Res}(g(z), z = e^{i\theta}) + \operatorname{Res}(g(z), z = e^{-i\theta})\right]
$$
$$
= \frac{i\pi\,\sin(\theta/2)}{|\log\gamma|}\left(\frac{2\cos(\theta/2)}{2i\sin\theta}e^{\theta\xi/2\log\gamma} + \frac{2\cos(\theta/2)}{2i\sin\theta}e^{(2\pi-\theta)\xi/2\log\gamma}\right)
$$
$$
= \frac{\pi}{2|\log\gamma|}\left(e^{\theta\xi/2\log\gamma} + e^{(2\pi-\theta)\xi/2\log\gamma}\right).
$$

We infer

$$
\hat{f}_{\gamma,\theta}(\xi) = \frac{\pi}{2|\log\gamma|}\frac{\cosh\big((\pi-\theta)\xi/(2\log\gamma)\big)}{\cosh\big(\pi\xi/(2\log\gamma)\big)}, \quad \theta \in (0, 2\pi).
$$

Finally, for $\theta, \varphi \in (0, 2\pi)$,

$$
\sum_{k\in\mathbb{Z}} f_{\gamma,\theta}(k) - \sum_{k\in\mathbb{Z}} f_{\gamma,\varphi}\big(k+\tfrac{1}{2}\big)
$$
$$
= \frac{\pi}{2|\log\gamma|}\left(\sum_{n\in\mathbb{Z}}\frac{\cosh\big((\pi-\theta)\pi n/(\log\gamma)\big)}{\cosh\big(\pi^2 n/(\log\gamma)\big)}\right.
$$
$$
\left. - \sum_{n\in\mathbb{Z}}(-1)^n\frac{\cosh\big((\pi-\varphi)\pi n/(\log\gamma)\big)}{\cosh\big(\pi^2 n/(\log\gamma)\big)}\right)
$$
$$
= \frac{\pi}{|\log\gamma|}\left(\sum_{n=1}^{\infty}\frac{\cosh\big((\pi-\theta)\pi n/(\log\gamma)\big)}{\cosh\big(\pi^2 n/(\log\gamma)\big)}\right.
$$
$$
\left. + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cosh\big((\pi-\varphi)\pi n/(\log\gamma)\big)}{\cosh\big(\pi^2 n/(\log\gamma)\big)}\right)
$$
$$
\geq \frac{\pi}{|\log\gamma|}\sum_{n=1}^{\infty}\frac{1}{\cosh\big(\pi^2 n/(\log\gamma)\big)},
$$

since the second series is an alternating series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n,$$

with a_n decaying to 0 as $n \to \infty$. Therefore

$$\min_{|\zeta|=1} |F_\gamma(\zeta)| - \gamma^{\frac{1}{2}} \max_{|\zeta|=\gamma} |F_\gamma(\zeta)| \geq \frac{\pi}{|\log \gamma|} \sum_{n=1}^{\infty} \frac{1}{\cosh\left(\pi^2 n/(\log \gamma)\right)} > 0. \qquad \square$$

Lemma 17 implies that Φ has no zeroes for $|\zeta| = 1$ and $|z| \leq 1$. By continuity, it has no zeroes in a neighbourhood of this set. Hence Lemma 16 is proved.

4.4. *Studying the index.* Let us first recall the definition of the index. For $0 < R < \infty$, we denote by $\mathcal{C}_R$ the circle

$$\{z \in \mathbb{C} : |z| = R\}.$$

Let f be a holomorphic function near $\mathcal{C}_R$, with no zero on $\mathcal{C}_R$. The index on $\mathcal{C}_R$ around 0 of f is given by

$$\mathrm{Ind}_{f(\mathcal{C}_R)}(0) := \frac{1}{2i\pi} \int_{\mathcal{C}_R} \frac{f'(\zeta)}{f(\zeta)} \, d\zeta.$$

In this section, we prove the following lemma.

Lemma 18. *For any $r < 1$ sufficiently close to 1, the function*

$$\zeta \mapsto F_\gamma(r\zeta)$$

has index zero on the unit circle.

Notice that $\Phi(0, r\zeta) = F_\gamma(r\zeta)$. As the index is valued in $\mathbb{Z}$ and the map $z \mapsto \Phi(z, \zeta)$ is smooth, Lemma 18 implies that the index of $\zeta \mapsto \Phi(z, r\zeta)$ is zero for any z with $|z| \leq 1 + \rho$ as long as r is sufficiently close to 1.

Corollary 19. *For any $r < 1$ sufficiently close to 1, the function*

$$\zeta \mapsto \Phi(z, r\zeta)$$

has index zero for any z.

This corollary will complete the proof of Proposition 13.

Proof of Lemma 18. We could use Remark 15 in order to reduce to properties of the Weierstrass $\wp$ function. However, for the convenience of the reader, we prefer

to give a self-contained proof. Let us assume that R is chosen so that $R \neq \gamma^{2\ell}$, $\ell \in \mathbb{Z}$ and $F_\gamma \neq 0$ on $\mathcal{C}_R$. We consider the index of F_γ on $\mathcal{C}_R$ around 0:

$$I(R) := \mathrm{Ind}_{F_\gamma(\mathcal{C}_R)}(0) := \frac{1}{2i\pi} \int_{\mathcal{C}_R} \frac{F_\gamma'(\zeta)}{F_\gamma(\zeta)}\, d\zeta.$$

The statement of Lemma 18 is equivalent to

$$I(1^-) := \lim_{R \to 1^-} I(R) = 0.$$

By definition, I is valued in $\mathbb{Z}$ and is continuous on the intervals corresponding to the circles avoiding the zeroes and the poles of F_γ. From properties (11), one has

$$I(R) + I\left(\frac{1}{R}\right) = -1, \quad I(R\gamma^2) = I(R). \tag{12}$$

In particular,

$$I(R) + I\left(\frac{\gamma^2}{R}\right) = -1 \tag{13}$$

and

$$I(1^+) = I((\gamma^2)^+), \tag{14}$$

where $I(r^\pm) = \lim_{t \to r^\pm} I(t)$. We are going to compute $I((\gamma^2)^+)$ in another way, using the zeroes and the poles of F_γ.

Let us first collect some basic relations. Let n be the number of zeroes in the annulus

$$\{z \in \mathbb{C} : \gamma < |z| < 1\}.$$

Since there are no poles inside this annulus, one has

$$n = I(1^-) - I(\gamma^+). \tag{15}$$

From Equation (13) with $R = 1^-$ and $R = \gamma^+$,

$$I(1^-) + I((\gamma^2)^+) = -1 \quad \text{and} \quad I(\gamma^+) + I(\gamma^-) = -1.$$

Subtracting these equalities gives $I(1^-) - I(\gamma^+) = I(\gamma^-) - I((\gamma^2)^+)$, hence

$$n = I(\gamma^-) - I((\gamma^2)^+). \tag{16}$$

Denote by m the number of zeroes on $\mathcal{C}_\gamma$. As γ is a zero of F_γ, $m \geq 1$, and

$$m = I(\gamma^+) - I(\gamma^-) \tag{17}$$

since there is no pole on $\mathcal{C}_\gamma$. Denote by N the number of zeroes on $\mathcal{C}_1$. Then

$$I(1^+) - I(1^-) = N - 1, \tag{18}$$

since 1 is the only pole on $\mathcal{C}_1$.

Now, we compute $I((\gamma^2)^+)$:

$$\begin{aligned}
I((\gamma^2)^+) &= I(\gamma^-) - n && \text{from (16)}\\
&= I(\gamma^+) - m - n && \text{from (17)}\\
&= I(1^-) - n - m - n && \text{from (15)}\\
&= I(1^+) - (N-1) - m - 2n && \text{from (18)}.
\end{aligned}$$

Recalling (14), we conclude that $N + 2n + m = 1$, so $n = 0$ and $N + m = 1$. Since $m \geq 1$, this implies $N = 0$ and $m = 1$. From (18) and the equality $I(1^+) + I(1^-) = -1$ (Equation (13) with $R = 1^+$), one concludes $I(1^+) = -1$ and $I(1^-) = 0$ as required. $\qquad\square$

Appendix: A formula for the C^1 norm

Let $u \in L^2_+(\mathbb{T})$ be a rational function corresponding to the finite list of singular values $\rho_1 > \sigma_1 > \cdots > \rho_N > \sigma_N$ and angles $\psi_r = 0$ for $r = 1, \ldots, 2N$. Then we checked in [Gérard and Grellier 2012; 2014] that this cancellation of the angles precisely corresponds to the positivity of the operators $\Gamma_{\hat{u}}$ and $\tilde{\Gamma}_{\hat{u}}$ on $\ell^2(\mathbb{Z}_+)$. The representation formula (2), (3) then reduces to

$$u(z) = \langle \mathscr{C}_N(z)^{-1}(\mathbb{1}_N), \mathbb{1}_N \rangle,$$

with

$$\mathscr{C}_N(z) := \left(\frac{s_{2j-1} - s_{2k}z}{s_{2j-1}^2 - s_{2k}^2} \right)_{1 \leq j,k \leq N}, \tag{19}$$

Furthermore, the positivity of the Hankel matrices $\Gamma_{\hat{u}}$ and $\tilde{\Gamma}_{\hat{u}}$ implies the positivity of the Fourier coefficients of u, since, denoting by $(e_n)_{n \geq 0}$ the canonical basis of $\ell^2(\mathbb{Z}_+)$,

$$\langle \Gamma_{\hat{u}} e_n, e_n \rangle = \hat{u}(2n), \quad \langle \tilde{\Gamma}_{\hat{u}} e_n, e_n \rangle = \hat{u}(2n+1).$$

Therefore the C^1 norm of u on $\mathbb{T}$ is given by

$$S(u) := \sum_{n=1}^{\infty} n\hat{u}(n).$$

The lemma below explicitly computes $S(u)$.

Lemma 20.
$$S(u) = \sum_{k=1}^{N} \sigma_k \left(\prod_{j=1}^{N} \frac{\rho_j + \sigma_k}{\rho_j - \sigma_k} \right) \left(\prod_{\ell \neq k} \frac{\sigma_k + \sigma_\ell}{\sigma_\ell - \sigma_k} \right),$$

where every term in the above sum is positive.

Proof. We have

$$S(u) = u'(1) = \langle \dot{\mathscr{C}}\mathscr{C}(1)^{-1}(\mathbb{1}), \, {}^t\mathscr{C}(1)^{-1}(\mathbb{1}) \rangle,$$

with

$$\mathscr{C}(1) := \left(\frac{1}{\rho_j + \sigma_k}\right)_{1 \le j,k \le N}, \quad \dot{\mathscr{C}} := \left(\frac{\sigma_k}{\rho_j^2 - \sigma_k^2}\right)_{1 \le j,k \le N}.$$

Notice that $\mathscr{C}(1)$ is a Cauchy matrix, so that the expression of $\mathscr{C}(1)^{-1}(\mathbb{1})$ is explicit. We have

$$\mathscr{C}(1)^{-1}(\mathbb{1}) = \left(\frac{\prod_{j=1}^{N}(\rho_j + \sigma_k)}{\prod_{\ell \ne k}(\sigma_k - \sigma_\ell)}\right)_{1 \le k \le N}. \tag{20}$$

Let us give a simple proof of this formula, inspired from calculations in [Gérard and Pushnitski 2018]. Denote by x_k, $k = 1, \ldots, N$, the components of $\mathscr{C}(1)^{-1}(\mathbb{1})$. We have

$$\sum_{k=1}^{N} \frac{x_k}{\rho_j + \sigma_k} = 1, \quad j = 1, \ldots, N.$$

Consider the polynomial functions

$$Q(\rho) := \prod_{k=1}^{N}(\rho + \sigma_k), \quad P(\rho) := Q(\rho) \sum_{k=1}^{N} \frac{x_k}{\rho + \sigma_k}.$$

Then Q has degree N, P has degree at most $N - 1$ and

$$P(\rho_j) = Q(\rho_j), \quad j = 1, \ldots N.$$

Since $Q - P$ is a unitary polynomial of degree N which cancels at ρ_j, $j = 1, \ldots, N$, we have

$$Q(\rho) - P(\rho) = \prod_{j=1}^{N}(\rho - \rho_j).$$

Consequently,

$$P(-\sigma_k) = -\prod_{j=1}^{N}(-\sigma_k - \rho_j) = (-1)^{N-1}\prod_{j=1}^{N}(\sigma_k + \rho_j).$$

Since

$$x_k = \frac{P(-\sigma_k)}{Q'(-\sigma_k)},$$

this yields (20). Similarly, we have

$${}^t\mathscr{C}(1)^{-1}(\mathbb{1}) = \left(\frac{\prod_{\ell=1}^{N}(\rho_j + \sigma_\ell)}{\prod_{i \ne j}(\rho_j - \rho_i)}\right)_{1 \le j \le N}. \tag{21}$$

Coming back to the proof of Lemma 20, we have, in view of (20) and (21),

$$S(u) = \sum_{j,k=1}^{N} \mu_{jk}^{(N)}, \quad \mu_{jk}^{(N)} := \sigma_k \frac{\rho_j + \sigma_k}{\rho_j - \sigma_k} \left(\prod_{i \neq j} \frac{\rho_i + \sigma_k}{\rho_j - \rho_i} \right) \left(\prod_{\ell \neq k} \frac{\rho_j + \sigma_\ell}{\sigma_k - \sigma_\ell} \right).$$

Multiplying and dividing $\mu_{jk}^{(N)}$ by $\prod_{i \neq j}(\rho_i - \sigma_k)$, we have, for every k,

$$\sum_{j=1}^{N} \mu_{jk}^{(N)} = \frac{\sigma_k R(\sigma_k)}{\prod_{\ell \neq k}(\sigma_k - \sigma_\ell)} \prod_{i=1}^{N} \frac{\rho_i + \sigma_k}{\rho_i - \sigma_k},$$

with

$$R(\sigma) = \sum_{j=1}^{N} \prod_{i \neq j} \frac{\rho_i - \sigma}{\rho_j - \rho_i} \prod_{\ell \neq k}(\rho_j + \sigma_\ell).$$

Notice that, for every $j = 1, \ldots, N$,

$$R(\rho_j) = (-1)^{N-1} \prod_{\ell \neq k}(\rho_j + \sigma_\ell).$$

Since R has degree $N - 1$, we infer

$$R(\sigma) = (-1)^{N-1} \prod_{\ell \neq k}(\sigma + \sigma_\ell),$$

so that

$$\sum_{j=1}^{N} \mu_{jk}^{(N)} = \frac{\sigma_k (-1)^{N-1} \prod_{\ell \neq k}(\sigma_k + \sigma_\ell)}{\prod_{\ell \neq k}(\sigma_k - \sigma_\ell)} \prod_{i=1}^{N} \frac{\rho_i + \sigma_k}{\rho_i - \sigma_k},$$

which is the claimed formula. The positivity of each term is an easy consequence of the inequalities $\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots$. $\qquad \square$

As a consequence of Lemma 20, we retain the following inequality, obtained after discarding most of the factors bigger than 1 in each of the products.

Corollary 21. $$\|u\|_{C^1} \geq \sum_{k=1}^{N} \frac{\sigma_k (\rho_k + \sigma_k)}{\rho_k - \sigma_k}.$$

Notice that this implies inequality (4). Unfortunately, at this stage we do not have arguments allowing us to extend this inequality to nonrational functions u, which would imply that $u \notin C^1$ if

$$\sum_{k=1}^{\infty} \frac{\rho_k \sigma_k}{\rho_k - \sigma_k} = \infty.$$

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