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Ramification groups of coverings and valuations

Takeshi Saito

Dedicated to the memory of Michel Raynaud

We give a purely scheme theoretic construction of the filtration by ramification groups of the Galois group of a covering. The valuation need not be discrete but the normalizations are required to be locally of complete intersection.

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For a Galois extension of a complete discrete valuation field with not necessarily perfect residue field, the filtration by ramification groups on the Galois group is defined in a joint article [Abbes and Saito 2002] with Ahmed Abbes. Although the definition there is based on rigid geometry, it was later observed that the use of rigid geometry can be avoided and the conventional language of schemes suffices [Saito 2009]. In this article, we reformulate the construction in [Abbes and Saito 2002] in the language of schemes. As a byproduct, we give a generalization for ramified finite Galois coverings of normal and universally Japanese noetherian schemes and valuations not necessarily discrete.

All the ideas are present in the 2002 article, possibly in different formulation. As in that article, the main ingredients in the definition of ramification groups are the following: First, we interpret a subgroup as a quotient of the fiber functor with a cocartesian property, Proposition 1.4.2. Thus, the definition of ramification groups is a consequence of a construction of quotients of the fiber functor, indexed by elements of the rational value group of valuation.

The required quotients of the fiber functor are constructed as the sets of connected components of geometric fibers of dilatations [Abbes and Saito 2011; Saito 2009] defined by an immersion of the covering to a smooth scheme over the base scheme. Here a crucial ingredient is the reduced fiber theorem of Bosch,

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Lütkebohmert and Raynaud [Bosch et al. 1995] recalled in Theorem 1.2.5. This specializes to the finiteness theorem of Grauert and Remmert in the classical case where the base is a discrete valuation ring. A variant of the filtration is defined using the underlying sets of geometric fibers of quasifinite schemes without using the sets of connected components.

To prove the basic properties of ramification groups stated in Theorem 3.3.1 including the rationality of breaks, semicontinuity etc., a key ingredient is a generalization due to Temkin [2011] of the semistable reduction theorem of curves recalled in Theorem 1.3.5.

Let X be a normal noetherian scheme and let $U \subset X$ be a dense open subscheme. The Zariski–Riemann space $\widetilde{X}$ is defined as the inverse limit of proper schemes X' over X such that $U' = U \times_X X' \to U$ is an isomorphism. Points of $\widetilde{X}$ on the boundary $\widetilde{X} - U$ correspond bijectively to the inverse limits of the images of the closed points by the liftings of the morphisms $T = \operatorname{Spec} A \to X$ for valuation rings $A \subsetneqq K = k(t)$ for points $t \in U$ such that $T \times_X U$ consists of the single point t.

Let $W \to U$ be a finite étale connected Galois covering of the Galois group G. We will construct in Theorem 3.3.1 filtrations (G_T^γ) and $(G_T^{\gamma+})$ on G by ramification groups for a morphism $T \to X$ as above indexed by the positive part

$$(0, \infty)_{\Gamma_{\mathbb{Q}}} \subset \Gamma_{\mathbb{Q}} = \Gamma \otimes \mathbb{Q}$$

for the value group $\Gamma = K^\times / A^\times$. To complete the definition, we need to assume that for every intermediate covering $V \to U$, the normalization Y of X in V is locally of complete intersection over X to assure the cocartesian property in Proposition 1.4.2. The required cocartesian property Proposition 3.1.2 is then a consequence of a lifting property in commutative algebra recalled in Proposition 1.1.5.

The definition depends on X, not only on $W \to U$. In other words, for a normal noetherian scheme X' over X as above, the filtrations (G_T^γ) and $(G_T^{\gamma+})$ defined for X and those for X' may be different. This arises from the fact that the formation of the normalization Y need not commute with base change $X' \to X$. To obtain a definition depending only on $W \to U$, one would need to take the inverse limit with respect to X'. This requires that the normalizations over T to be locally of complete intersection.

By Proposition 1.4.2, the definition of the filtrations (G_T^γ) and $(G_T^{\gamma+})$ are reduced to the construction of surjections $F_T^\infty \to F_T^\gamma$ and $F_T^\infty \to F_T^{\gamma+}$ for a fiber functor F_T^∞. To define them, for each intermediate covering $V \to U$, we take an embedding $Y \to Q$ of the normalization to a smooth scheme over X. Further taking a ramified covering and a blow-up X', we find an effective Cartier divisor $R' \subset X'$ and a lifting $T' \to X'$ of $T \to X$ such that the valuation $v'(R')$ of R' is γ for each $\gamma \in \Gamma_{\mathbb{Q}}$. Then, we define a dilatation $Q'^{(R')}$ over X' to be the normalization of an open subscheme $Q'^{[R']}$ of the blow-up of the base change $Q' = Q \times_X X'$ at the closed subscheme $Y \times_X R' \subset Q \times_X X'$. To obtain a construction independent of the choice of X', we apply the reduced fiber theorem of Bosch–Lütkebohmert–Raynaud for $Q'^{(R')} \to X'$ to be flat and to have reduced geometric fibers.

Now the desired functor $F_T^{\gamma}(Y/X)$ is defined as the set of connected components of the geometric fiber of $Q'^{(R')} \to X'$ at the image of the closed point by $T' \to X'$. We recover the construction in [Abbes and Saito 2002] in the classical case where $X = T$ is the spectrum of a complete discrete valuation ring as we show in Lemma 3.3.2 using Example 2.1.1(1) and Remark 1.1.2. Its variant $F_T^{\gamma+}(Y/X)$ is defined more simply as the geometric fiber of the inverse image $Y' \times_{Q'^{[R']}} Q'^{(R')}$ with respect to the morphism $Y' = Y \times_X X' \to Q'^{[R']}$ lifting the original immersion $Y \to Q$. The fact that the construction is independent of the choice of immersion $Y \to Q$ is based on a homotopy invariance of dilatations proved in Proposition 2.1.5.

To study the behavior of the functors F_T^{γ} and $F_T^{\gamma+}$ thus defined for the variable γ, we use a semistable curve C over X defined by $st = f$ for a nonzero divisor f on X defining an effective Cartier divisor $D \subset X$ such that $D \cap U = \varnothing$ as a parameter space for γ. Let $\widetilde{D} \subset C$ denote the effective Cartier divisor defined by t. Then, for $\gamma \in [0, v(D)]_{\Gamma_{\mathbb{Q}}}$, there is a lifting $T' \to C$ of $T \to X$ such that $v'(\widetilde{D}) = \gamma$. Using this together with a local description (Proposition 1.3.3) of Cartier divisors on a semistable curve over a normal noetherian scheme and a combination of the reduced fiber theorem and the semistable reduction theorem over a general base scheme, we derive basic properties of F_T^{γ} and $F_T^{\gamma+}$ in Proposition 3.1.8 to prove Theorems 3.2.6 and 3.3.1.

Convention. In this article, we assume that for a noetherian scheme X, the normalization of the reduced part of a scheme of finite type over X remains to be of finite type over X. This property is satisfied if X is of finite type over a field, $\mathbb{Z}$, or a complete discrete valuation ring, for example.

1. Preliminaries

1.1. *Connected components.*

Definition 1.1.1 [EGA IV$_2$ 1965, définition (6.8.1)]. Let $f : X \to S$ be a flat morphism locally of finite presentation of schemes. We say that f is *reduced* if for every geometric point s of S, the geometric fiber X_s is *reduced*.

In [SGA 1 1971, exposé X, définition 1.1], reduced morphism is called separable morphism. A morphism f of finite presentation is étale if and only if f is quasifinite, flat and reduced.

We study the sets of connected components of geometric fibers of a flat and reduced morphism of finite type. Let S be a scheme and let s and t be geometric points of S. Let $S_{(s)}$ denote the strict localization. A specialization $s \leftarrow t$ of geometric points means a morphism $S_{(s)} \leftarrow t$ over S.

Assume that S is noetherian. Let $X \to S$ be a flat and reduced morphism of finite type and let $s \leftarrow t$ be a specialization of geometric points of S. We define the cospecialization mapping

$$\pi_0(X_s) \to \pi_0(X_t) \tag{1-1}$$

as follows. By replacing S by the closure of the image of t, we may assume that S is integral and that t is above the generic point η of S. By replacing S further by a quasifinite scheme over S such that the function field is a finite extension of $\kappa(\eta)$ in $\kappa(t)$, we may assume that the canonical mapping $\pi_0(X_t) \to \pi_0(X_\eta)$ is a bijection. Let $U \subset S$ be a dense open subset such that the canonical mapping $\pi_0(X_\eta) \to \pi_0(X_U)$ is a bijection. Then, by [EGA IV$_4$ 1967, corollaire (18.9.11)], the canonical mapping $\pi_0(X_U) \to \pi_0(X)$ is also a bijection. Thus, we define the cospecialization mapping (1-1) to be the composition

$$\pi_0(X_s) \to \pi_0(X) \xleftarrow{\simeq} \pi_0(X_\eta) \xleftarrow{\simeq} \pi_0(X_t).$$

We say that the sets of connected components of geometric fibers of $X \to S$ are locally constant if for every specialization $s \leftarrow t$ of geometric points of S, the cospecialization mapping $\pi_0(X_s) \to \pi_0(X_t)$ is a bijection. By [EGA IV$_3$ 1966, théorème (9.7.7)] and by noetherian induction, there exists a finite stratification $S = \coprod_i S_i$ by locally closed subschemes such that the sets of connected components of geometric fibers of the base change $X \times_S S_i \to S_i$ are locally constant for every i. We call this fact that the sets of connected components of geometric fibers of $X \to S$ are constructible.

Remark 1.1.2. Let $S = \operatorname{Spec} \mathcal{O}_K$ for a discrete valuation ring $\mathcal{O}_K$ and let $X = \operatorname{Spec} A$ be an affine scheme of finite type over S. Let $\bar{s} \to S$ be a geometric closed point. Let $\mathfrak{X} = \operatorname{Spf} \hat{A}$ be the formal completion along the closed fiber and let $\mathfrak{X}_{\overline{K}} = \operatorname{Sp} \hat{A} \otimes_{\mathcal{O}_K} \overline{K}$ be the associated affinoid variety over an algebraic closure $\overline{K}$ of the fraction field K of $\mathcal{O}_K$. If X is flat and reduced over S, the cospecialization mapping $\pi_0(X_{\bar{s}}) \to \pi_0(\mathfrak{X}_{\overline{K}})$ is a bijection.

Let $Y \to S$ be another flat and reduced morphism of finite type and let $f : X \to Y$ be a morphism over S. The cospecialization mappings (1-1) form a commutative diagram

$$\begin{array}{ccc} \pi_0(X_s) & \longrightarrow & \pi_0(X_t) \\ \downarrow & & \downarrow \\ \pi_0(Y_s) & \longrightarrow & \pi_0(Y_t) \end{array} \tag{1-2}$$

Lemma 1.1.3. *Let $f : X \to Y$ be a morphism of schemes of finite type over a noetherian scheme S. Assume that X is étale over S and that Y is flat and reduced over S. Let A denote the subset of X consisting of the images of geometric points x of X satisfying the following condition*:

> *Let s be the geometric point of S defined as the image of x and let $C \subset Y_s$ be the connected component of the fiber containing the image of x. Then, $f_s^{-1}(C) \subset X_s$ consists of a single point x.*

Then A is closed.

Proof. By the constructibility of connected components of geometric fibers of Y, the subset $A \subset X$ is constructible. For a specialization $s \leftarrow t$ of geometric points of S, the upper horizontal arrow in the commutative diagram

$$\begin{array}{ccc} X_s & \longrightarrow & X_t \\ \downarrow & & \downarrow \\ \pi_0(Y_s) & \longrightarrow & \pi_0(Y_t) \end{array}$$

is an injection since $X \to S$ is étale. Hence A is closed under specialization and is closed. □

We have specialization mappings going the other way for proper morphisms. Let X be a proper scheme over S. Let $s \leftarrow t$ be a specialization of geometric points of S. Then, the inclusion $X_s \to X \times_S S_{(s)}$ induces a bijection $\pi_0(X_s) \to \pi_0(X \times_S S_{(s)})$ by [SGA 4½ 1977, IV proposition (2.1)]. Its composition with the mapping $\pi_0(X_t) \to \pi_0(X \times_S S_{(s)})$ induced by the morphism $X_t \to X \times_S S_{(s)}$ defines the specialization mapping

$$\pi_0(X_s) \leftarrow \pi_0(X_t). \tag{1-3}$$

For a morphism $X \to Y$ of proper schemes over S, the specialization mappings make a commutative diagram

$$\begin{array}{ccc} \pi_0(X_s) & \longleftarrow & \pi_0(X_t) \\ \downarrow & & \downarrow \\ \pi_0(Y_s) & \longleftarrow & \pi_0(Y_t). \end{array} \tag{1-4}$$

Lemma 1.1.4. *Let $f : X \to Y$ be a finite unramified morphism of schemes. Let B denote the subset of X consisting of the images of geometric points x of X satisfying the following condition*:

> *For the geometric point y of Y defined as the image of x, the fiber $X \times_Y y$ consists of a single point x.*

Then, B is open.

Proof. The complement $X - B$ equals the image of the complement $X \times_Y X - X$ of the diagonal by a projection. Since $X \to Y$ is unramified, the complement $X \times_Y X - X \subset X \times_Y X$ is closed. Since the projection $X \times_Y X \to X$ is finite, the image $X - B$ is closed. $\square$

Proposition 1.1.5. *Let*

$$\begin{array}{ccc} Z' & \longrightarrow & X' \\ \downarrow & \square & \downarrow f \\ Z & \longrightarrow & X \end{array} \tag{1-5}$$

be a cartesian diagram of noetherian schemes. Assume that X is normal, the horizontal arrows are closed immersion, the right vertical arrow is quasifinite and the left vertical arrow is finite. Assume further that there exists a dense open subscheme $U \subset X$ such that $U' = U \times_X X' \to U$ is faithfully flat and that $U' \subset X'$ is also dense.

1. *Let $C \subset Z$ be an irreducible closed subset and let $C' \subset f^{-1}(C)$ be an irreducible component. Then, $C' \to C$ is surjective.*
2. *Let $C \subset Z$ be a connected closed subset and let $C' \subset f^{-1}(C)$ be a connected component. Then, $C' \to C$ is surjective.*

Proof. (1) By replacing U by a dense open subscheme if necessary, we may assume that $U' \to U$ is finite. By Zariski's main theorem, there exists a scheme $\bar{X}'$ finite over X containing X' as an open subscheme. By replacing $\bar{X}'$ by the closure of U', we may assume that U' is dense in $\bar{X}'$. Since U' is closed in $\bar{X}' \times_X U$, we have $\bar{X}' \times_X U = U'$. Since $Z' = (\bar{X}' \times_X Z) \cap X'$ is closed and open in $\bar{X}' \times_X Z$, by replacing X' by $\bar{X}'$, we may assume that f is finite.

Since f is a closed mapping, it suffices to show that the generic point z of C is the image of the generic point z' of C'. Let x' be a point of C'. Replacing X by an affine neighborhood of $x = f(x') \in C$, we may assume $X = \operatorname{Spec} A$ and $X' = \operatorname{Spec} B$ are affine. Then, the assumption implies that $A \to B$ is an injection and B is finite over A. Since x is a point of the closure $C = \overline{\{z\}}$, the assertion follows from [Bourbaki 1985, Chapter V, Section 2.4, Theorem 3].

(2) Let $C_1 \subset C$ be an irreducible component such that $C_1 \cap f(C')$ is not empty. Then, there exists an irreducible component C_1' of $f^{-1}(C_1) \subset f^{-1}(C)$ such that $C_1' \cap C'$ is not empty. By (1), we have $C_1 = f(C_1')$. Since C' is a connected component of $f^{-1}(C)$ and $C_1' \cap C' \neq \varnothing$, we have $C_1' \subset C'$ and hence $C_1 = f(C_1') \subset f(C')$. Thus, the complement $C - f(C')$ is the union of irreducible components of C not meeting $f(C')$ and is closed. Since $f(C') \subset C$ is also closed and is nonempty, we have $C = f(C')$. $\square$

Corollary 1.1.6. *Let*

$$\begin{array}{ccccccc} Z' & \longrightarrow & X' & \longleftarrow & Y'_1 & \xleftarrow{g'} & Y' \\ \downarrow & \square & \downarrow f & \square & \downarrow f_1 & & \downarrow f' \\ Z & \longrightarrow & X & \longleftarrow & Y_1 & \xleftarrow{g} & Y \end{array} \tag{1-6}$$

be a commutative diagram of noetherian schemes such that the left square is cartesian and satisfies the conditions in Proposition 1.1.5. Assume that $Y_1 \subset X$ is a closed subscheme, that the middle square is cartesian and that the four arrows in the right square are finite. Assume that there exists a dense open subscheme $V_1 \subset Y_1$ such that $V = V_1 \times_{Y_1} Y \subset Y$ is also dense and that $g|_V : V \to V_1$ and $g'|_{V'} : V' = V \times_Y Y' \to V'_1 = V_1 \times_{Y_1} Y'_1$ are isomorphisms.

(1) *For any irreducible (resp. connected) component C of Y, we have $f^{-1}(g(C)) = g'(f'^{-1}(C))$. Consequently, we have $f^{-1}(g(Y)) = g'(Y')$.*

(2) *Suppose that the mapping $Z \times_X Y \to \pi_0(Y)$ is a bijection. Then, the diagram*

$$\begin{array}{ccc} Z' \cap Y'_1 & \longleftarrow & Z' \times_{X'} Y' \\ \downarrow & & \downarrow \\ Z \cap Y_1 & \longleftarrow & Z \times_X Y \end{array} \tag{1-7}$$

of underlying sets induces a surjection $Z' \times_{X'} Y' \to (Z' \cap Y'_1) \times_{Z \cap Y_1} (Z \times_X Y)$ of sets. If $Z \times_X Y \to Z \cap Y_1$ is surjective, then $Z' \times_{X'} Y' \to Z' \cap Y'_1$ is also surjective. Further, if $Y' \to Y$ is surjective, then $Z' \cap Y'_1 \to Z \cap Y_1$ is also surjective and the diagram (1-7) is a cocartesian diagram of underlying sets.

(3) *The diagram*

$$\begin{array}{ccc} \pi_0(Z') & \longleftarrow & Z' \cap Y'_1 \\ \downarrow & & \downarrow \\ \pi_0(Z) & \longleftarrow & Z \cap Y_1 \end{array} \tag{1-8}$$

of sets induces a surjection $Z' \cap Y'_1 \to \pi_0(Z') \times_{\pi_0(Z)} (Z \cap Y_1)$ of sets. If $Z \cap Y_1 \to \pi_0(Z)$ is surjective, then $Z' \cap Y'_1 \to \pi_0(Z')$ is also surjective. Further if $Z' \cap Y'_1 \to Z \cap Y_1$ is surjective, the diagram (1-8) is a cocartesian diagram of sets.

Proof. (1) Let $C \subset Y$ be an irreducible component. The inclusion $f^{-1}(g(C)) \supset g'(f'^{-1}(C))$ is clear. We show the other inclusion. Since V is dense in Y, the intersection $C \cap V$ and hence its image $g(C) \cap V_1$ are not empty. Let C' be an irreducible component of $f^{-1}(g(C)) \subset Y'_1$. Since $Y'_1 \to Y_1$ is finite and $g(C) \subset Y_1$ is an irreducible closed subset, we have $g(C) = f(C')$ by Proposition 1.1.5(1). Since $f(C' \cap V'_1) = f(C') \cap V_1 = g(C) \cap V_1$ is not empty, $C' \cap V'_1 = g'(g'^{-1}(C' \cap V'_1))$ is also nonempty and hence is dense in C'. Since $g'^{-1}(C'\cap V_1') = g'^{-1}(C')\cap V' \subset f'^{-1}(C)$ and since $g' : Y' \to X'$ is proper, we have $C' \subset g'(f'^{-1}(C))$.

Since a connected component of Y and Y itself are unions of irreducible components of Y, the remaining assertions follow from the assertion for irreducible components.

(2) Let $z' \in Z'\cap Y_1'$ and $y \in Z\times_X Y$ be points satisfying $f(z') = g(y)$ in $Z\cap Y_1$. Let $C \subset Y$ be the unique connected component containing y. Since $z' \in f^{-1}(g(C)) = g'(f'^{-1}(C))$ by (1), there exists a point $y' \in Z' \times_{X'} f'^{-1}(C) \subset Z' \times_{X'} Y'$ such that $z' = g'(y')$. Since $f'(y') \in Z \times_X Y$ is a unique point contained in $C \in \pi_0(Y)$, we have $y = f'(y')$. Thus, $(z', y) \in (Z' \cap Y_1') \times_{Z\cap Y_1} (Z \times_X Y)$ is the image of $y' \in Z' \times_{X'} Y'$.

If $Z\times_X Y \to Z\cap Y_1$ is surjective, then $(Z'\cap Y_1')\times_{Z\cap Y_1}(Z\times_X Y) \to Z'\cap Y_1'$ is surjective and hence the first assertion implies the surjectivity of $Z' \times_{X'} Y' \to Z' \cap Y_1'$.

If both $Z\times_X Y \to Z\cap Y_1$ and $Y' \to Y$ are surjective, then $Z'\times_{X'} Y' = Z\times_X Y' \to Z\times_X Y$ is also surjective and hence by the commutative diagram (1-7), the mapping $Z'\cap Y_1' \to Z\cap Y_1$ is a surjection. This implies that the diagram (1-7) with $Z'\times_{X'} Y'$ replaced by $(Z' \cap Y_1') \times_{Z\cap Y_1} (Z \times_X Y)$ is a cocartesian diagram of underlying sets. Hence the surjectivity of $Z' \times_{X'} Y' \to Z' \cap Y_1' \times_{Z\cap Y_1} (Z \times_X Y)$ implies that the diagram (1-7) is a cocartesian diagram of underlying sets.

(3) Let $C' \subset Z'$ be a connected component and let $z \in Z \cap Y_1$ be a point such that the connected component $C \subset Z$ satisfying $f(C') \subset C$ contains z. Since $f(C') = C$ by Proposition 1.1.5(2), the intersection $C' \cap f^{-1}(z) \subset Z' \cap Y_1'$ is not empty. Hence $(C', z) \in \pi_0(Z') \times_{\pi_0(Z)} (Z \cap Y_1)$ is in the image of $C' \cap f^{-1}(z) \subset Z' \cap Y_1'$.

The remaining assertions are proved similarly as in (2). □

1.2. ***Flat and reduced morphisms.*** Let $k \geqq 0$ be an integer. Recall that a noetherian scheme X satisfies the condition (R_k) if for every point $x \in X$ of $\dim \mathcal{O}_{X,x} \leqq k$, the local ring $\mathcal{O}_{X,x}$ is regular [EGA IV$_2$ 1965, définition (5.8.2)]. Recall also that a noetherian scheme X satisfies the condition (S_k) if for every point $x \in X$, we have $\mathrm{prof}\, \mathcal{O}_{X,x} \geqq \inf(k, \dim \mathcal{O}_{X,x})$ [EGA IV$_2$ 1965, définition (5.7.2)].

Proposition 1.2.1. *Let $f : X \to S$ be a flat morphism of finite type of noetherian schemes and let $k \geqq 0$ be an integer. We define a function $k : S \to \mathbb{N}$ by $k(s) = \max(k - \dim \mathcal{O}_{S,s}, 0)$.*

(1) *If S satisfies the condition (R_k) and if the fiber $X_s = X \times_S s$ satisfies $(\mathrm{R}_{k(s)})$ for every $s \in S$, then X satisfies the condition (R_k).*

(2) *If X satisfies the condition (R_k) and if $f : X \to S$ is faithfully flat, then S satisfies the condition (R_k).*

Proof. (1) Assume $\dim \mathcal{O}_{X,x} \leqq k$ and set $s = f(x)$. Then, we have $\dim \mathcal{O}_{S,s} \leqq \dim \mathcal{O}_{X,x} \leqq k$ and $\dim \mathcal{O}_{X_s,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{S,s} \leqq k(s)$ by [EGA IV$_2$ 1965, proposition (6.1.1)]. Hence $\mathcal{O}_{S,s}$ and $\mathcal{O}_{X_s,x}$ are regular by the assumption. Thus $\mathcal{O}_{X,x}$ is regular by [EGA IV$_1$ 1964, chapitre 0_{IV} proposition (17.3.3)(ii)].

(2) This follows from [EGA IV$_2$ 1965, proposition (6.5.3)(i)]. □

Proposition 1.2.2. *Let $f : X \to S$ be a flat morphism of finite type of noetherian schemes and let $k \geqq 0$ be an integer. Let the function $k : S \to \mathbb{N}$ be as in Proposition 1.2.1.*

(1) *If S satisfies the condition* (S_k) *and if the fiber X_s satisfies* ($\mathrm{S}_{k(s)}$) *for every $s \in S$, then X satisfies the condition* (S_k).

(2) *If X satisfies the condition* (S_k) *and if $f : X \to S$ is faithfully flat, then S satisfies the condition* (S_k).

(3) *If X satisfies the condition* (S_k) *and if S is of Cohen–Macaulay, then the fiber X_s satisfies* ($\mathrm{S}_{k(s)}$) *for every $s \in S$.*

Proof. (1) Let $x \in X$ and $s = f(x)$. Then, we have $\operatorname{prof} \mathcal{O}_{S,s} \geqq \inf(k, \dim \mathcal{O}_{S,s})$ and $\operatorname{prof} \mathcal{O}_{X_s,x} \geqq \inf(k(s), \dim \mathcal{O}_{X_s,x})$ by the assumption. By $\dim \mathcal{O}_{X_s,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{S,s}$ [EGA IV$_2$ 1965, proposition (6.1.1)], we have

$$\inf(k, \dim \mathcal{O}_{S,s}) + \inf(k(s), \dim \mathcal{O}_{X_s,x}) = \inf(k, \dim \mathcal{O}_{X,x}).$$

Hence the claim follows from $\operatorname{prof} \mathcal{O}_{X,x} = \operatorname{prof} \mathcal{O}_{S,s} + \operatorname{prof} \mathcal{O}_{X_s,x}$ [EGA IV$_2$ 1965, proposition (6.3.1)].

(2) This follows from [EGA IV$_2$ 1965, proposition (6.4.1)(i)].

(3) Let $x \in X$ and $s = f(x)$. Then by the assumption, we have

$$\operatorname{prof} \mathcal{O}_{X,x} \geqq \inf(k, \dim \mathcal{O}_{X,x}) \quad \text{and} \quad \operatorname{prof} \mathcal{O}_{S,s} = \dim \mathcal{O}_{S,s}.$$

By $\operatorname{prof} \mathcal{O}_{X_s,x} = \operatorname{prof} \mathcal{O}_{X,x} - \operatorname{prof} \mathcal{O}_{S,s} \geqq 0$ [EGA IV$_2$ 1965, proposition (6.3.1)] and $\dim \mathcal{O}_{X_s,x} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{S,s}$ [EGA IV$_2$ 1965, proposition (6.1.1)] we have $\operatorname{prof} \mathcal{O}_{X_s,x} \geqq \inf(k - \dim \mathcal{O}_{S,s}, \dim \mathcal{O}_{X_s,x}) \geqq k(s)$ and the assertion follows. □

Corollary 1.2.3. *Let $f : X \to S$ be a flat morphism of finite type of noetherian schemes and let $U \subset X$ be the largest open subset smooth over S.*

(1) *Assume that the fiber X_s is reduced for every $s \in S$. Assume further that S is normal and that for the generic point s of each irreducible component, X_s is normal. Then X is normal.*

(2) *For $s \in S$ and a geometric point $\bar{s}$ above s, we consider the following conditions:*

   (i) *The geometric fiber $X_{\bar{s}}$ is reduced.*
   (ii) *U_s is dense in X_s.*

Then, we have (i) $\Rightarrow$ (ii). *Conversely, if X is normal and S is regular of dimension* $\leqq 1$, *then we have* (ii) $\Rightarrow$ (i).

Proof. (1) By Serre's criterion [EGA IV$_2$ 1965, théorème (5.8.6)], S satisfies (R_2) and (S_1). By [EGA IV$_2$ 1965, proposition (5.8.5)], every fiber X_s satisfies (R_1) and (S_0). Further if s is the generic point of an irreducible component, the fiber X_s satisfies (R_2) and (S_1). Since the function $k(s)$ for $k = 2$ satisfies $k(s) \leqq 1$ unless s is the generic point an irreducible component and $k(s) = 2$ for such point, the scheme X satisfies the conditions (R_2) and (S_1) by Propositions 1.2.1(1) and 1.2.2(1). Thus the assertion follows by [EGA IV$_2$ 1965, théorème (5.8.6)].

(2) (i) $\Rightarrow$ (ii): Since $X_{\bar{s}}$ is reduced, there exists a dense open subset $V \subset X_{\bar{s}}$ smooth over $\bar{s}$. Since f is flat, the image of V in X_s is a subset of U_s.

(ii) $\Rightarrow$ (i): Since X satisfies (S_2) and S is Cohen–Macaulay of dimension $\leqq 1$, the fiber X_s satisfies (S_1) by Proposition 1.2.2(3). Hence the geometric fiber $X_{\bar{s}}$ also satisfies (S_1) by [EGA IV$_2$ 1965, proposition (6.7.7)]. By (ii), $X_{\bar{s}}$ satisfies (R_0). Hence the assertion follow from [EGA IV$_2$ 1965, proposition (5.8.5)]. □

Lemma 1.2.4. *Let S be a noetherian scheme and let $f : Y \to X$ be a quasifinite morphism of schemes of finite type over S. Assume that X is smooth over S and that Y is flat and reduced over S. Assume that there exist dense open subschemes $U \subset S$ and $U \times_S X \subset W \subset X$ such that $Y \times_X W \to W$ is étale and that for every point $s \in S$, the inverse image $f_s^{-1}(W_s) \subset Y_s = Y \times_S s$ of $W_s = W \times_S s \subset X_s = X \times_S s$ by $f_s : Y_s \to X_s$ is dense. Then, $Y \to X$ is étale.*

Proof. If S is regular, the assumption that $Y \times_X W \to W$ is étale and Corollary 1.2.3(1) implies that the quasifinite morphism $Y \to X$ of normal noetherian schemes is étale in codimension $\leqq 1$. Since X is regular, the assertion follows from the purity theorem of Zariski–Nagata.

Since X and Y are flat over S, it suffices to show that for every point $s \in S$, the morphism $Y_s = Y \times_S s \to X_s$ is étale. Let $S' \to S$ be the normalization of the blow-up at the closure of $s \in S$. Then, there exists a point $s' \in S'$ above $s \in S$ such that the local ring $\mathcal{O}_{S',s'}$ is a discrete valuation ring. Since the assumption is preserved by the base change $\operatorname{Spec} \mathcal{O}_{S',s'} \to S$, the morphism $Y_{s'} = Y \times_S s' \to X_{s'} = X \times_S s'$ is étale. Hence $Y_s \to X_s$ is also étale as required. □

The following statement is a combination of the reduced fiber theorem and the flattening theorem.

Theorem 1.2.5 [Bosch et al. 1995, Theorem 2.1′; Raynaud and Gruson 1971, théorème (5.2.2)]. *Let S be a noetherian scheme and let $U \subset S$ be a schematically dense open subscheme. Let X be a scheme of finite type over S such that $X_U = X \times_S U$ is schematically dense in X and that $X_U \to U$ is flat and reduced.*

Then there exists a commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array} \tag{1-9}$$

of schemes satisfying the following conditions:

(i) *The morphism $S' \to S$ is the composition of a blow-up $S^* \to S$ with center supported in $S - U$ and a faithfully flat morphism $S' \to S^*$ of finite type such that $U' = S' \times_S U \to U$ is étale.*

(ii) *The morphism $X' \to S'$ is* ***flat*** *and* ***reduced***. *The induced morphism $X' \to X \times_S S'$ is finite and its restriction $X' \times_{S'} U' \to X \times_S U'$ is an isomorphism.*

If $X_U \to U$ is smooth and if S' is normal, then X' is the normalization of $X \times_S S'$ by Corollary 1.2.3(1). If $X_U \to U$ is étale, the first condition in (ii) implies that $X' \to S'$ is étale.

For the morphism $S' \to S$ satisfying the condition (i) in Theorem 1.2.5, we have the following variant of the valuative criterion.

Lemma 1.2.6. *Let S be a scheme and let U be a dense open subscheme. Let $S_1 \to S$ be a proper morphism such that $U_1 = U \times_S S_1 \to U$ is an isomorphism and let $S' \to S_1$ be a quasifinite faithfully flat morphism. Let $t \in U$, let $A \subset K = k(t)$ be a valuation ring and let $T = \operatorname{Spec} A \to S$ be a morphism extending $t \to U$. Then, there exist $t' \in U' = U \times_S S'$ above t, a valuation ring $A' \subset K' = k(t')$ such that $A = A' \cap K$ and a commutative diagram*

$$\begin{array}{ccc} T' & \longrightarrow & S' \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array} \tag{1-10}$$

for $T' = \operatorname{Spec} A'$. Further, if $t = T \times_S U$, then we have $t' = T' \times_{S'} U'$.

Proof. Since $S_1 \to S$ is proper and $U_1 \to U$ is an isomorphism, the morphism $T \to S$ is uniquely lifted to $T \to S_1$ by the valuative criterion of properness. Let $x_1 \in T \times_{S_1} S'$ be a closed point and let $t' \in t \times_{S_1} S'$ be a point above t such that x_1 is contained in the closure $T_1 = \overline{\{t'\}} \subset T \times_{S_1} S'$ with the reduced scheme structure. Let $A' \subset k(t')$ be a valuation ring dominating the local ring $\mathcal{O}_{T_1,x_1}$. Then, we have the commutative diagram (1-10) for $T' = \operatorname{Spec} A'$.

Since t' is the unique point of $t \times_T T'$, the equality $t = T \times_S U$ implies $t' = T' \times_{S'} U'$. $\square$

1.3. ***Semistable curves.*** Let S be a scheme. Recall that a flat separated scheme X of finite presentation over S is a semistable curve, if every geometric fiber is purely of dimension 1 and has at most nodes as singularities.

Example 1.3.1. Let S be a scheme and let $D \subset S$ be an effective Cartier divisor. Let $C' \to \mathbb{A}^1_S$ be the blow-up at $D \subset S \subset \mathbb{A}^1_S$ regarded as a closed subscheme by the 0-section. Then, the complement $C_D \subset C'$ of the proper transform of the 0-section is a semistable curve over S and is smooth over the complement $U = S - D$. The exceptional divisor $\widetilde{D} \subset C_D$ is an effective Cartier divisor satisfying $0 \leqq \widetilde{D} \leqq D \times_S C_D$. The difference $D \times_S C_D - \widetilde{D}$ equals the proper transform of $\mathbb{A}^1_D$.

If $S = \operatorname{Spec} A$ is affine, $\mathbb{A}^1_S = \operatorname{Spec} A[t]$ and if D is defined by a nonzero divisor $f \in A$, we have $C_D = \operatorname{Spec} A[s,t]/(st - f)$ and $\widetilde{D} \subset C_D$ is defined by t.

Lemma 1.3.2. *Let S be a scheme and let $U \subset S$ be a schematically dense open subscheme. Let C be a separated flat scheme of finite presentation over S such that the base change $C_U = C \times_S U$ is a smooth curve over U. Then, the following conditions are equivalent*:

(1) *C is a semistable curve over S.*

(2) *Étale locally on C and on S, there exist an effective Cartier divisor $D \subset S$ such that $D \cap U$ is empty and an étale morphism $C \to C_D$ over S to the semistable curve C_D defined in Example 1.3.1.*

Proof. This is a special case of [SGA 7_{II} 1973, corollaire 1.3.2]. □

Let S be a normal noetherian scheme and let $j : U = S - D \to S$ be the open immersion of the complement of an effective Cartier divisor D. Let $i : D \to S$ be the closed immersion and let $\pi_D : \bar{D} \to D$ denote the normalization. Then, the valuations at the generic points of irreducible components of D define an exact sequence $0 \to \mathbb{G}_{m,S} \to j_*\mathbb{G}_{m,U} \to i_*\pi_{D*}\mathbb{Z}_{\bar{D}}$ of étale sheaves on S.

Let $f : C = C_D \to S$ be the semistable curve over S defined in Example 1.3.1. Let $\tilde{j} : U_C = C \times_S U \to C$ denote the open immersion and let $\tilde{i} : D_C = C \times_S D \to C$ denote the closed immersion. Let $A \subset C$ be the exceptional divisor and let $B = D_C - A \subset C$ be the effective Cartier divisor defined as the proper transform of $\mathbb{A}^1_D$. Let $a : A \to C$ and $b : B \to C$ and $e : E = A \cap B \to C$ denote the closed immersions. Then, the Cartier divisors $A, B, D_C \subset C$ defines a commutative diagram

$$\begin{array}{ccc} f^*i_*\mathbb{Z} & \longrightarrow & a_*\mathbb{Z} \oplus b_*\mathbb{Z} \\ \downarrow & & \downarrow \\ f^*(j_*\mathbb{G}_{m,U}/\mathbb{G}_{m,S}) & \longrightarrow & \tilde{j}_*\mathbb{G}_{m,U_C}/\mathbb{G}_{m,C} \end{array} \tag{1-11}$$

of étale sheaves on C.

Proposition 1.3.3. *Let S be a normal noetherian scheme and let $D \subset S$ be an effective Cartier divisor. Let $f : C = C_D \to S$ be the semistable curve defined in Example 1.3.1. Then, the diagram (1-11) induces an exact sequence*

$$0 \to f^* i_* \mathbb{Z} \to f^*(j_* \mathbb{G}_{m,U}/\mathbb{G}_{m,S}) \oplus (a_* \mathbb{Z} \oplus b_* \mathbb{Z}) \to \tilde{j}_* \mathbb{G}_{m,U_C}/\mathbb{G}_{m,C} \to 0 \quad (1\text{-}12)$$

of étale sheaves on D_C.

Proof. Let z be a geometric point of C; we will show the exactness of the stalks of (1-12) at z. Replacing S by the strict localization at the image x of z, we may assume that S is strict local and that x is the closed point. For $t \in S = S_{(x)}$, the Milnor fiber $C_{(z)} \times_S t$ at t of the strict localization $C_{(z)}$ at z is geometrically connected by [EGA IV$_4$ 1967, théorème (18.9.7)]. Further, if $z \in E$ and if $t \in D$, the fiber at t of $C_{(z)} - E_{(z)}$ has 2 geometrically connected components.

First, we consider the case where C is smooth over S at z. Then, since the Milnor fiber $C_{(z),t}$ is connected, the canonical morphism $f^* i_* \mathbb{Z}_{\bar{D}} \to i_{C*} \mathbb{Z}_{\bar{D}_C}$ is an isomorphism. Hence, the stalk of the lower horizontal arrow (1-11) at z is an injection. Further, this is a surjection by flat descent.

We assume that $C \to S$ is not smooth at z. Let $\widetilde{D}$ be a Cartier divisor of $C_{(z)}$ supported on $D_{C_{(z)}} = C_{(z)} \times_S D$. Then similarly as above, there exists a Cartier divisor D_1 on S supported on D such that $D_0 = \widetilde{D} - f^* D_1$ is supported on the inverse image of A. Define a $\mathbb{Z}$-valued function n on $y \in E_{(z)} = D$ as the intersection number of D_0 with the fiber $B \times_S y$. We show that the function n is constant. By adding some multiple of A to $\widetilde{D}$ if necessary, we may assume that D_0 is an effective Cartier divisor of C supported on A. Since B is flat over D, the pull-back $D_0 \times_C B$ is an effective Cartier divisor of B finite flat over D by [EGA IV$_1$ 1964, 0_{IV} proposition (15.1.16) c)$\Rightarrow$b)]. Hence the function n is constant. Thus we have $\widetilde{D} = f^* D_1 + n \cdot A$ and the exactness of the stalks of (1-12) at z follows. □

Corollary 1.3.4. *Let S be a normal noetherian scheme and let $C \to S$ be a semistable curve. Let $x \in S$ be a point and let $z \in C \times_S x$ be a singular point of the fiber. Assume that z is contained in the intersection of two irreducible components C_1 and C_2 of $C \times_S x$. Let $s_1 : S \to C$ and $s_2 : S \to C$ be sections meeting with the smooth parts of C_1 and C_2 respectively.*

Let $U \subset S$ be a dense open subscheme such that $C_U = C \times_S U$ is smooth over U and let $\widetilde{D} \subset C$ be an effective Cartier divisor such that $\widetilde{D} \cap C_U$ is empty. Define effective Cartier divisors $D_1 = s_1^* \widetilde{D}$ and $D_2 = s_2^* \widetilde{D}$ of S as the pull-back of $\widetilde{D}$.

Then, on a neighborhood of x, we have either $D_1 \leqq D_2$ or $D_2 \leqq D_1$. Suppose we have $D_1 \leqq D_2$ on a neighborhood of x. Then, we have $D_1 \times_S C \leqq \widetilde{D} \leqq D_2 \times_S C$ on a neighborhood of z.

Proof. In the notation of the proof of Proposition 1.3.3, we have $\widetilde{D} = f^* D_1 + nA$ for an integer n on an étale neighborhood of z. Hence the assertion follows. □

We recall a combination the flattening theorem and a strong version of the semistable reduction theorem for curves over a general base scheme.

Theorem 1.3.5 [Raynaud and Gruson 1971, théorème (5.2.2); Temkin 2011, Theorem 2.3.3]. *Let S be a noetherian scheme and let $U \subset S$ be a schematically dense open subscheme. Let $C \to S$ be a separated morphism of finite type such that $C \times_S U \to U$ is a smooth relative curve and that $C \times_S U \subset C$ is schematically dense. Then, there exists a commutative diagram*

$$\begin{array}{ccc} C & \longleftarrow & C' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

of schemes satisfying the following conditions:

(i) *The morphism $S' \to S$ is the composition of a proper modification $S_1 \to S$ such that $U_1 = U \times_S S_1 \to U$ is an isomorphism and a faithfully flat morphism $S' \to S_1$ such that $U' = U \times_S S' \to U_1$ is étale and $U' \subset S'$ is schematically dense.*

(ii) *The morphism $C' \to S'$ is a* semistable *curve and the morphism $C' \to C \times_S S'$ is a proper modification such that $C' \times_{S'} U' \to C \times_S U'$ is an isomorphism.*

Corollary 1.3.6. *Let S be a noetherian scheme and let $U \subset S$ be a schematically dense open subscheme. Let $C \to S$ be a separated morphism of finite type such that $C_U = C \times_S U \to U$ is a smooth relative curve and that $C_U \subset C$ is schematically dense. Let $X \to C$ be a separated morphism of finite type such that $X_U = X \times_S U \subset X$ is schematically dense and that $X_U \to C_U$ is flat and reduced. Then, there exists a commutative diagram*

$$\begin{array}{ccc} X & \longleftarrow & X' \\ \downarrow & & \downarrow \\ C & \longleftarrow & C' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

of schemes satisfying the following conditions:

(i) *The morphism $S' \to S$ is the composition of a proper modification $S_1 \to S$ such that $U_1 = U \times_S S_1 \to U$ is an isomorphism and a faithfully flat morphism $S' \to S_1$ such that $U' = U \times_S S' \to U_1$ is étale and $U' \subset S'$ is schematically dense.*

(ii) *The morphism $C' \to S'$ is a semistable curve and the morphism $C' \to C \times_S S'$ is the composition of a proper modification $C'_0 \to C \times_S S'$ such that $C'_0 \times_{S'} U' \to$*

$C \times_S U'$ is an isomorphism, a faithfully flat morphism $C'_1 \to C'_0$ such that $C'_1 \times_{S'} U' \to C'_0 \times_{S'} U'$ is étale and of a proper modification $C' \to C'_1$ such that $C' \times_{S'} U' \to C'_1 \times_{S'} U'$ is an isomorphism.

(iii) *The morphism $X' \to C'$ is flat and reduced, the morphism $X' \to X \times_C C'$ is finite and $X' \times_{S'} U' \to X \times_C C' \times_{S'} U'$ is an isomorphism.*

Proof. By the reduced fiber theorem (Theorem 1.2.5) applied to $X \to C$, there exists a commutative diagram

$$\begin{array}{ccc} X & \longleftarrow & X_1 \\ \downarrow & & \downarrow \\ C & \longleftarrow & C_1 \end{array}$$

satisfying the conditions (i) and (ii) of Theorem 1.2.5. Since $C_1 \times_S U \to C \times_S U$ is étale and $C_1 \times_S U \subset C_1$ is schematically dense, by the combination of the stable reduction theorem and the flattening theorem (Theorem 1.3.5), there exists a commutative diagram

$$\begin{array}{ccc} C_1 & \longleftarrow & C' \\ \downarrow & & \downarrow \\ S & \longleftarrow & S' \end{array}$$

satisfying the conditions (i) and (ii) of Theorem 1.3.5.

We show that $X' = X_1 \times_{C_1} C' \to C' \to S'$ satisfy the required conditions. By the construction, $S' \to S$ satisfies the condition (i) and $C' \to S'$ is a semistable curve. Since $C_1 \to C$ is obtained by applying Theorem 1.2.5 and $C' \to S'$ is obtained by applying Theorem 1.3.5, the composition $C' \to C'_1 = C_1 \times_S S' \to C \times_S S'$ satisfies the condition in (ii). Finally, the base change $X' \to C'$ of a flat and reduced morphism $X_1 \to C_1$ is flat and reduced. Since $X' \to C'$ is obtained by applying Theorem 1.2.5, the morphism $X' \to X \times_C C'$ satisfies the condition (iii). □

1.4. ***Subgroups and fiber functor.*** For a finite group G, let (Finite G-sets) denote the category of finite sets with left G-actions.

Definition 1.4.1. We say that a category C is a *finite Galois category* if there exist a finite group G and an equivalence of categories $F : C \to$ (Finite G-sets). If $F : C \to$ (Finite G-sets) is an equivalence of categories, we say that G is the *Galois group* of the finite Galois category C and call the functor F itself or the composition $C \to$ (Finite-sets) with the forgetful functor also denoted by F a *fiber functor* of C.

We say that a morphism $F \to F'$ of functors $F, F' : C \to$ (Finite-sets) is a surjection if $F(X) \to F'(X)$ is a surjection for every object X of C. For a subgroup $H \subset G$ and for a fiber functor $F : C \to$ (Finite G-sets), let F_H denote the functor $C \to$ (Finite-sets) defined by $F_H(X) = H\backslash F(X)$. The canonical morphism $F \to F_H$ is a surjection.

Surjections $F \to F_H$ are characterized as follows.

Proposition 1.4.2 (cf. [Abbes and Saito 2002, Proposition 2.1]). *Let C be a finite Galois category of the Galois group G and let $F : C \to$ (Finite-sets) be a fiber functor. Let $F' : C \to$ (Finite-sets) be another functor and let $F \to F'$ be a surjection of functors. Then, the following conditions are equivalent*:

(1) *For every surjection $X \to Y$ in C, the diagram*

$$\begin{array}{ccc} F(X) & \longrightarrow & F'(X) \\ \downarrow & & \downarrow \\ F(Y) & \longrightarrow & F'(Y) \end{array} \tag{1-13}$$

is a cocartesian diagram of finite sets. For every pair of objects X and Y of C, the morphism $F'(X) \amalg F'(Y) \to F'(X \amalg Y)$ is a bijection.

(2) *There exists a subgroup $H \subset G$ such that $F \to F'$ induces an isomorphism $F_H \to F'$.*

Proof. (1) $\Rightarrow$ (2): We may assume $C =$ (Finite G-sets) and F is the forgetful functor. For $X = G$, the mapping $F(G) = G \to F'(G)$ is a surjection of finite sets. Define an equivalence relation $\sim$ on G by requiring that $G/{\sim} \to F'(G)$ be a bijection and set $H = \{x \in G \mid x \sim e\}$. Then, since the group G acts on the object G of C by the right action, the relation $x \sim y$ is equivalent to $xy^{-1} \in H$. Since $\sim$ is an equivalence relation, the transitivity implies that H is stable under the multiplication, the reflexivity implies $e \in H$ and the symmetry implies that H is stable under the inverse. Hence H is a subgroup and the surjection $F(G) = G \to F'(G)$ induces a bijection $H\backslash G \to F'(G)$.

Let X be an object of $C =$ (Finite G-sets) and regard $G \times X$ as a G-set by the left action on G. Then, since the functor F' preserves the disjoint union, we have a canonical isomorphism $F'(G \times X) \to F'(G) \times X \to (H\backslash G) \times X$. Further, the cocartesian diagram (1-13) for the surjection $G \times X \to X$ in C defined by the action of G is given by

$$\begin{array}{ccc} G \times X & \longrightarrow & (H\backslash G) \times X \\ \downarrow & & \downarrow \\ X & \longrightarrow & F'(X) \end{array} \tag{1-14}$$

Thus we obtain a bijection $H\backslash X \to F'(X)$.

(2) $\Rightarrow$ (1): This is clear. □

Corollary 1.4.3. *Let the notation be as in Proposition 1.4.2 and let G' be a quotient group. Let $C' \subset C$ be the full subcategory consisting of objects X such that $F(X)$ are G'-sets. Then the subgroup $H' \subset G'$ defined by the surjection $F|_{C'} \to F'|_{C'}$ of the restrictions of the functors equals the image of $H \subset G$ in G'.*

Proof. If a G-set X is a G'-set, the quotient $H\backslash X$ is $H'\backslash X$. □

Corollary 1.4.4. *Let C be a finite Galois category of Galois group G and let $F : C \to$* (Finite G-sets) *be a fiber functor. Let $G' \to G$ be a morphism of groups and let F also denote the functor $C \to$* (Finite G'-sets) *defined as the composition defined by $G' \to G$. Let $F' : C \to$* (Finite G'-sets) *be another functor and let $F \to F'$ be a surjection of functors such that the composition with the forgetful functor satisfies the condition* (1) *in Proposition 1.4.2.*

Let $H \subset G$ be the subgroup satisfying the condition (2) *in Proposition 1.4.2 and let $G'_1 \subset G$ be the image of $G' \to G$. Then, the functor F' induces a functor $C \to$* (Finite G'_1-sets) *and $G'_1 \subset G$ is a subgroup of the normalizer $N_G(H)$ of H.*

Proof. For an object X of C, $F(X)$ regarded as a G'-set is a G'_1-set. Since $F(X) \to F'(X)$ is a surjection of G'-sets, $F'(X)$ is also a G'_1-set. Since the left action of $G'_1 \subset G$ on the G-set $F(G) = G$ induces an action on $F'(G) = H\backslash G$, the subgroup H is normalized by G'_1. □

2. Dilatations

2.1. *Functoriality of dilatations.* Let X be a noetherian scheme and we consider morphisms

$$D \to X \leftarrow Q \leftarrow Y \tag{2-1}$$

of separated schemes of finite type over X satisfying the following condition:

(i) $D \subset X$, $D_Y = D \times_X Y \subset Y$ and $D_Q = D \times_X Q \subset Q$ are effective Cartier divisors and $Y \to Q$ is a closed immersion.

In later subsections, we will further assume the following condition:

(ii) X is normal and Q is smooth over X.

We give examples of constructions of Q for a given Y over X.

Example 2.1.1. Assume that X and Y are separated schemes of finite type over a noetherian scheme S.

(1) Assume $S = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine. Then, taking a surjection $A[T_1, \dots, T_n] \to B$, we obtain a closed immersion $Y \to Q = \mathbb{A}^n_S \times_S X$.

(2) Assume that Y is smooth over S. Then, $Q = Y \times_S X \to X$ is smooth and the canonical morphism $Y \to Q = Y \times_S X$ is a closed immersion.

(3) Assume that $\pi : Y \to X$ is finite flat and define a vector bundle Q over X by the symmetric $\mathcal{O}_X$-algebra $S^\bullet \pi_* \mathcal{O}_Y$. Then the canonical surjection $S^\bullet \pi_* \mathcal{O}_Y \to \pi_* \mathcal{O}_Y$ defines a closed immersion $Y \to Q$.

For morphisms (2-1) satisfying the condition (i) above, we construct a commutative diagram

$$\begin{array}{ccccc} \bar{Y} & \longrightarrow & Q^{(D)} & & \\ \downarrow & & \downarrow & \searrow & \\ Y & \longrightarrow & Q^{[D]} & \longrightarrow & Q \end{array} \tag{2-2}$$

of schemes over X as follows. Let $\mathcal{I}_D \subset \mathcal{O}_X$ and $\mathcal{I}_Y \subset \mathcal{O}_Q$ be the ideal sheaves defining the closed subschemes $D \subset X$ and $Y \subset Q$. Let $Q' \to Q$ be the blow-up at $D_Y = D \times_X Y \subset Q$ and define the dilatation $Q^{[D]}$ at $Y \to Q$ and D to be the largest open subset of Q' where $\mathcal{I}_D \mathcal{O}_{Q'} \supset \mathcal{I}_Y \mathcal{O}_{Q'}$. Since D_Y is a divisor of Y, by the functoriality of blow-up, the immersion $Y \to Q$ is uniquely lifted to a closed immersion $Y \to Q^{[D]}$. Let $\bar{Y}$ and $Q^{(D)}$ be the normalizations of Y and $Q^{[D]}$ and let $\bar{Y} \to Q^{(D)}$ be the morphism induced by the morphism $Y \to Q^{[D]}$. If there is a risk of confusion, we also write $Q^{[D]}$ and $Q^{(D)}$ as $Q^{[D \cdot Y]}$ and $Q^{(D \cdot Y)}$ in order to make Y explicit.

Locally, if $Q = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A/I$ are affine and if $D \subset X$ is defined by a nonzero divisor f, we have

$$Q^{[D]} = \operatorname{Spec} A[I/f] \tag{2-3}$$

for the subring $A[I/f] \subset A[1/f]$ and the immersion $Y \to Q^{[D]}$ is defined by the isomorphism $A[I/f]/(I/f)A[I/f] \to A/I$.

Example 2.1.2. Let X be a noetherian scheme and let $D \subset X$ be an effective Cartier divisor.

(1) Let Q be a smooth separated scheme over X and let $s : X \to Q$ be a section. Let $Y = s(X) \subset Q$ be the closed subscheme. Then, $Q^{[D]}$ is smooth over X. If X is normal, the canonical morphism $Q^{(D)} \to Q^{[D]}$ is an isomorphism.

(2) Assume that X is normal. Let Q be a smooth curve over X and let $s_1, \dots, s_n : X \to Q$ be sections. Define a closed subscheme $Y \subset Q$ as the sum $\sum_{i=1}^n s_i(X)$ of the sections regarded as effective Cartier divisors of Q. Assume that $D \subset s_n^*(s_i(X))$ for $i = 1, \dots, n-1$. Then $Q^{(nD)} \to X$ is smooth and $Y \times_{Q^{[nD]}} Q^{(nD)} \subset Q^{(nD)}$ is the sum $\sum_{i=1}^n \tilde{s}_i(X)$ of the sections $\tilde{s}_i : X \to Q^{(nD)}$ lifting $s_i : X \to Q$.

In fact, we may assume that $X = \operatorname{Spec} A$ is affine and, locally on Q, take an étale morphism $Q \to \mathbb{A}^1_X$. Then, we may assume that $Q = \mathbb{A}^1_X = \operatorname{Spec} A[T]$ and Y is defined by $P = \prod_{i=1}^n (T - a_i)$ for $a_i \in A$. We may further assume that D is defined by a nonzero divisor $a \in A$ dividing $a_1, \dots, a_n$. Then, we have

$Q^{[nD]} = \operatorname{Spec} A[T][P/a^n]$ and $T' = T/a$ satisfies $\prod_{i=1}^{n}(T' - a_i/a) = P/a^n$ in $A[T][1/a]$. Hence we have $Q^{(nD)} = \operatorname{Spec} A[T']$ and this equals $Q^{[D.s_n(X)]}$ and is smooth over X. The section $Y \to Q^{[nD]}$ is defined by $P/a^n = 0$ and hence $Y \times_{Q^{[nD]}} Q^{(nD)} \subset Q^{(nD)}$ is defined by $A[T']/\prod_{i=1}^{n}(T' - a_i/a)$.

We study the base change $Q^{[D]} \times_X D$.

Lemma 2.1.3. (1) *The canonical morphism* $Q^{[D]} \to Q$ *induces*

$$Q^{[D]} \times_X D = Q^{[D]} \times_Q D_Y \to D_Y. \tag{2-4}$$

(2) *If* $Y \to Q$ *is a regular immersion and if* $T_Y Q$ *and* $T_D X$ *denote the normal bundles, we have a canonical isomorphism*

$$T_Y Q(-D_Y) \times_Y D_Y = (T_Y Q \times_Y D_Y) \otimes (T_D X \times_D D_Y)^{\otimes -1} \to Q^{[D]} \times_X D. \tag{2-5}$$

The isomorphism (2-5) *depends only on the restriction* $D_Y \to Q$ *and not on* $Y \to Q$ *itself.*

(3) *Assume that* Q *is smooth over* X *and* $X = Y \to Q$ *is a section. Let* $T(Q/X)$ *denote the relative tangent bundle defined by the symmetric* $\mathcal{O}_Q$*-algebra* $S^{\bullet}_{\mathcal{O}_Q}\Omega^1_{Q/X}$. *Then, we have a canonical isomorphism*

$$T(Q/X)(-D) \times_Q D = (T(Q/X) \times_Q D) \otimes T_D X^{\otimes -1} \to Q^{[D]} \times_X D. \tag{2-6}$$

The isomorphism (2-6) *depends only on the restriction* $D \to Q$ *and not on the section* $X \to Q$ *itself.*

Proof. (1) Since $\mathcal{I}_D \mathcal{O}_{Q^{[D]}} \supset \mathcal{I}_Y \mathcal{O}_{Q^{[D]}}$ on $Q^{[D]}$ by the definition of $Q^{[D]}$, we have $Q^{[D]} \times_X D = Q^{[D]} \times_Q D_Y$. Hence, we obtain a morphism $Q^{[D]} \times_X D \to D_Y$.

(2) Assume that $Y \to Q$ is a regular immersion. Then, $D_Y \to Q$ is also a regular immersion and the normal bundle $T_{D_Y} Q$ fits in an exact sequence

$$0 \to T_{D_Y} D_Q \to T_{D_Y} Q \to T_D X \times_D D_Y \to 0$$

depending only on $D \to X$ and $D_Y \to Q$ and not on $Y \to Q$. Let $Q' \to Q$ be the blow-up at $D_Y \subset Q$. Then, the exceptional divisor $Q' \times_Q D_Y$ is canonically identified with the projective space bundle $\mathbb{P}(T_{D_Y} Q)$ over D_Y. Its open subset $Q^{[D]} \times_Q D_Y$ is identified as in (2-5) since $T_{D_Y} D_Q = T_Y Q \times_Y D_Y$.

(3) Since the normal bundle $T_X Q$ is canonically identified with the restriction $T(Q/X) \times_Q X$ of the relative tangent bundle, the assertion follows from (2). □

We give a sufficient condition for the morphism $\overline{Y} \to Q^{(D)}$ to be an immersion.

Lemma 2.1.4. *Assume that* X *and* $Y - D_Y$ *are normal and let* $\pi : \overline{Y} \to Y$ *be the normalization. Assume that* $\overline{Y} \to X$ *is étale and that* $\pi_* \mathcal{O}_{\overline{Y}}/\mathcal{O}_Y$ *is an* $\mathcal{O}_{D_Y}$*-module. Then, the finite morphism* $\overline{Y} \to Q^{(2D)}$ *is a closed immersion.*

Proof. Since the assertion is étale local on Q and X, we may assume that $Y \to X$ is finite and that the étale covering $\overline{Y} \to X$ is split. We may further assume that X, Y and Q are affine and that D is defined by a nonzero divisor f on X. Let $Y = \operatorname{Spec} A$, $\overline{Y} = \operatorname{Spec} \bar{A}$, $Q = \operatorname{Spec} B$, $Q^{[2D]} = \operatorname{Spec} B^{[2D]}$, $Q^{(2D)} = \operatorname{Spec} B^{(2D)}$ for $A = B/I$, $B^{[2D]} = B[I/f^2] \subset B[1/f]$ and the normalization $B^{(2D)}$ of $B^{[2D]}$. Since $\overline{Y} \to X$ is a split étale covering, it suffices to show that for every idempotent $e \in \bar{A}$, there exists a lifting $\tilde{e} \in B^{(2D)}$.

Since $\bar{A}/A$ is annihilated by f, the product $fe = g$ is an element of A. Let $\tilde{g} \in B$ be a lifting of g. Since $e^2 = e$, the element $h = \tilde{g}^2 - f\tilde{g} \in B$ is contained in I and hence $h/f^2 \in B[1/f]$ is an element of $B^{[2D]}$. Thus $\tilde{e} = \tilde{g}/f \in B[1/f]$ is a root of the polynomial $T^2 - T - h/f^2 \in B^{[2D]}[T]$ and is an element of $B^{(2D)}$. Since $\tilde{e}$ is a lifting of e, the assertion follows. $\square$

We study the functoriality of the construction. We consider a commutative diagram

$$\begin{array}{ccccccc} D \times_X X' \hookrightarrow & D' \hookrightarrow & X' & \longleftarrow & Q' & \longleftarrow & Y' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D & \hookrightarrow & X & \longleftarrow & Q & \longleftarrow & Y \end{array} \tag{2-7}$$

of schemes such that the both lines satisfy the condition (i) on the diagram (2-1). Then, by the functoriality of dilatations and normalizations, we obtain a commutative diagram

$$\begin{array}{ccccccc} Y' & \longrightarrow & Q'^{[D']} & \longleftarrow & Q'^{(D')} & \longleftarrow & \overline{Y}' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Q^{[D]} & \longleftarrow & Q^{(D)} & \longleftarrow & \overline{Y}. \end{array} \tag{2-8}$$

The diagram (2-8) induces a morphism

$$Q'^{(D')} \times_{Q'^{[D']}} Y' \to Q^{(D)} \times_{Q^{[D]}} Y. \tag{2-9}$$

Let $\bar{x}$ be a geometric point of D and let $\bar{x}'$ be a geometric point of $D \times_X X'$ above $\bar{x}$. Then the diagram (2-8) also induces a mapping

$$\pi_0(Q'^{(D')}_{\bar{x}'}) \to \pi_0(Q^{(D)}_{\bar{x}}) \tag{2-10}$$

of the sets of connected components of the geometric fibers.

First we study the dependence on Q.

Proposition 2.1.5. *Suppose $X = X'$, $Y = Y'$ and $D = D'$ and let $\bar{x}$ be a geometric point of D.*

(1) *Assume that $Q' \to Q$ is smooth and let $T = T(Q'/Q)$ denote the relative tangent bundle of Q' over Q. Then $Q'^{[D]} \to Q^{[D]}$ is also smooth and there exists a cartesian diagram*

$$\begin{array}{ccc} T(-D) \times_{Q'} D_Y & \longleftarrow & Q'^{[D]} \times_X D \\ \downarrow & \square & \downarrow \\ D_Y & \xleftarrow{\text{(2-4)}} & Q^{[D]} \times_X D \end{array} \tag{2-11}$$

(2) *Assume that Q and Q' are smooth over X. Then, the square*

$$\begin{array}{ccc} Q'^{[D]} & \longleftarrow & Q'^{(D)} \\ \downarrow & & \downarrow \\ Q^{[D]} & \longleftarrow & Q^{(D)} \end{array} \tag{2-12}$$

is cartesian. The induced morphism $Q'^{(D)} \times_{Q'^{[D]}} Y \to Q^{(D)} \times_{Q^{[D]}} Y$ (2-9) is an isomorphism over Y and the induced mapping $\pi_0(Q'^{(D)}_{\bar{x}}) \to \pi_0(Q^{(D)}_{\bar{x}})$ (2-10) is a bijection.

Proof. (1) First, we show the case where $Q' \to Q$ admits a section $Q \to Q'$ extending $Y \to Q'$. The section $Q \to Q'$ defines a section $Q^{[D]} \to Q' \times_Q Q^{[D]}$. Define $(Q' \times_Q Q^{[D]})^{[D_{Q^{[D]}} \cdot Q^{[D]}]}$ to be the dilatation of $Q' \times_Q Q^{[D]}$ for the section $Q^{[D]} \to Q' \times_Q Q^{[D]}$ and a divisor $D_{Q^{[D]}} = D \times_X Q^{[D]}$ over $Q^{[D]}$. We show that the canonical morphism $Q'^{[D]} \to Q' \times_Q Q^{[D]}$ induces an isomorphism

$$Q'^{[D]} \to (Q' \times_Q Q^{[D]})^{[D_{Q^{[D]}} \cdot Q^{[D]}]}. \tag{2-13}$$

Since the question is étale local on Q', we may assume that $Q' = \mathbb{A}^n_Q$ and the section $Q \to Q'$ is the 0-section. Further, we may assume that $Q = \operatorname{Spec} A$ and $Y = \operatorname{Spec} A/I$ are affine and that $D \subset X$ is defined by a nonzero divisor f on X. We set $A' = A[T_1, \dots, T_n]$ and $Q' = \operatorname{Spec} A'$. The 0-section $Q \to Q'$ is defined by the ideal $J = (T_1, \dots, T_n) \subset A'$. We have $Q^{[D]} = \operatorname{Spec} A[I/f]$ and $Q'^{[D]} = \operatorname{Spec} A'[I'/f]$ for $I' = IA' + J$. Since $A'[I'/f] = A[I/f][T_1/f, \dots, T_n/f]$ as a subring of $A'[1/f]$, we obtain an isomorphism (2-13).

By the isomorphism (2-13) and Example 2.1.2(1), the morphism $Q'^{[D]} \to Q^{[D]}$ is smooth. Further, by Lemma 2.1.3(3), we obtain a cartesian diagram (2-11), depending only on $D \to X$, $D_Y \to Q$ and $D_Y \to Q'$ but not on the choice of section $Q \to Q'$ extending $Y \to Q'$.

We prove the general case. Since $Q' \to Q$ has a section on $Y \subset Q$, locally on Q, there exist a closed subscheme $Q_1 \subset Q'$ étale over Q such that $Y \to Q'$ is induced by $Y \to Q_1$. For the smoothness of $Q'^{[D]} \to Q^{[D]}$, since the assertion is étale local, we may assume that $Q_1 = Q$ is a section. Hence the smoothness

$Q'^{[D]} \to Q^{[D]}$ follows. Further since the cartesian diagram (2-11) defined étale locally is independent of the choice of section, we obtain (2-11) for Q' by patching.

(2) First, we show the case where $Q' \to Q$ is smooth. Then by (1), $Q'^{[D]} \to Q^{[D]}$ is also smooth and the fibered product $Q^{(D)} \times_{Q^{[D]}} Q'^{[D]}$ is normal. Hence the square (2-12) is cartesian and the morphism (2-9) is an isomorphism. By the cartesian squares (2-12) and (2-11), $Q'^{(D)}_{\bar{x}}$ is a vector bundle over $Q^{(D)}_{\bar{x}}$. Hence (2-10) is a bijection.

We show the general case. A morphism $f : Q' \to Q$ is decomposed as the composition of the projection $\mathrm{pr}_2 : Q' \times_X Q \to Q$ and a section of the projection $\mathrm{pr}_1 : Q' \times_X Q \to Q'$. Hence, the cartesian squares (2-12) and the bijections (2-10) for the projections imply those for f respectively. The cartesian square (2-12) for f implies an isomorphism (2-9) for f. □

Corollary 2.1.6. *Assume that Q and Q' are smooth over X. Then, the morphism $Q'^{(D')} \times_{Q'^{[D']}} Y' \to Q^{(D)} \times_{Q^{[D]}} Y$ (2-9) is independent of $Q' \to Q$. Let $\bar{x}$ be a geometric point of D and let $\bar{x}'$ be a geometric point of D' above $\bar{x}$. Then the mapping $\pi_0(Q'^{(D')}_{\bar{x}'}) \to \pi_0(Q^{(D)}_{\bar{x}})$ (2-10) is independent of morphism $Q' \to Q$.*

Proof. Decompose a morphism $Q' \to Q$ as $Q' \to Q' \times_X Q \to Q$. Then the isomorphism (2-9) and the bijection (2-10) for $Q' \to Q' \times_X Q$ are the inverses of those for the projection $Q' \times_X Q \to Q'$. Hence the assertion follows. □

By the canonical isomorphism (2-9), the finite scheme $Y \times_{Q^{[D]}} Q^{(D)}$ over Y is independent of Q. We write it as $Y^{(D)}$.

Lemma 2.1.7. *Suppose that the squares*

$$\begin{array}{ccc} D' & \longrightarrow & X' \\ \downarrow & \square & \downarrow \\ D & \longrightarrow & X \end{array} \qquad \begin{array}{ccc} Q' & \longrightarrow & Y' \\ \downarrow & \square & \downarrow \\ Q & \longrightarrow & Y \end{array}$$

are cartesian.

(1) *The morphism $Q'^{[D']} \to Q^{[D]} \times_Q Q'$ is a closed immersion and $Q'^{(D')} \to Q^{(D)} \times_Q Q'$ is finite. Consequently, the morphism $Q'^{(D')} \times_{Q'} Y' \to Q^{(D)} \times_Q Y$ is finite if $Y' \to Y$ is finite. Further, if Q and Q' are normal, then $Q'^{(D')}$ equals the normalization of $Q^{(D)} \times_Q Q'$ in $Q' - D' \times_{X'} Q'$.*

(2) *If $Q' \to Q$ is flat, the square*

$$\begin{array}{ccc} Q'^{[D']} & \longrightarrow & Q' \\ \downarrow & \square & \downarrow \\ Q^{[D]} & \longrightarrow & Q \end{array}$$

is cartesian.

Proof. Since the assertion is local on a neighborhood of $Y' \subset Q'$, we may assume that $Q = \operatorname{Spec} A$, $Y = \operatorname{Spec} A/I$, $Q' = \operatorname{Spec} A'$ and $Y' = \operatorname{Spec} A'/IA'$ are affine and that D is defined by a nonzero divisor f on X. Then, we have $Q^{[D]} = \operatorname{Spec} A[I/f]$ and $Q'^{[D']} = \operatorname{Spec} A'[IA'/f]$.

(1) Since $A[I/f] \otimes_A A' \to A'[IA'/f]$ is a surjection, the morphism $Q'^{[D']} \to Q^{[D]} \times_Q Q'$ is a closed immersion. The remaining assertions follow from this immediately.

(2) If $A \to A'$ is flat, the injection $A[I/f] \to A[1/f]$ induces an injection

$$A' \otimes_A A[I/f] \to A' \otimes_A A[1/f] = A'[1/f].$$

Hence the surjection $A' \otimes_A A[I/f] \to A'[IA'/f]$ is an isomorphism. □

The construction of $Q^{(D)}$ commutes with base change if $Q^{(D)} \to X$ is flat and reduced.

Lemma 2.1.8. *Suppose that the diagram (2-7) is cartesian and $D' = D \times_X X'$. Assume that one of the following conditions is satisfied*:

(i) *X' is normal, $Q \to X$ is smooth and $Q^{(D)} \to X$ is flat and reduced.*

(ii) *$X' \to X$ is smooth.*

Then the square

$$\begin{array}{ccc} Q^{(D)} & \longleftarrow & Q'^{(D')} \\ \downarrow & & \downarrow \\ X & \longleftarrow & X' \end{array} \tag{2-14}$$

is cartesian.

Proof. By Lemma 2.1.7(1), $Q'^{(D')}$ is the normalization of $Q^{(D)} \times_X X'$. If the condition (i) is satisfied, then $Q^{(D)} \times_X X'$ is normal by Corollary 1.2.3(1). If $X' \to X$ is smooth, then $Q^{(D)} \times_X X'$ is smooth over $Q^{(D)}$ and is normal. Hence the square (2-14) is cartesian in both cases. □

We study the dependence on D and show that the canonical morphism contracts the closed fiber.

Lemma 2.1.9. *Suppose $X = X'$, $Y = Y'$ and $Q = Q'$, and that $D_1 = D' - D$ is an effective Cartier divisor of X. Then, the morphism $Q^{[D']} \to Q^{[D]}$ (resp. $Q^{(D')} \to Q^{(D)}$) induces a morphism $Q^{[D']} \times_Q D_{1,Y} \to D_{1,Y} \subset Y \subset Q^{[D]}$ (resp. $Q^{(D')} \times_Q D_{1,Y} \to Q^{(D)} \times_{Q^{[D]}} D_{1,Y} \subset Q^{(D)}$).*

Proof. We consider the immersion $Y \to Q^{[D]}$ lifting $Y \to Q$. Then, the morphism $Q^{[D']} \to Q^{[D]}$ induces an isomorphism $Q^{[D']} \to (Q^{[D]})^{[D_1]}$ to the dilatation $(Q^{[D]})^{[D_1]}$ of $Q^{[D]}$ for $Y \to Q^{[D]}$ and $D_1 \subset X$. Hence the morphism (2-4) defines a morphism $Q^{[D']} \times_Q D_{1,Y} \to D_{1,Y}$. The assertion for $Q^{(D')}$ follows from this. □

2.2. ***Dilatations and complete intersection.*** We give a condition for the right square in (2-7) to be cartesian.

Lemma 2.2.1. *Let S be a noetherian scheme and let $Q \to P$ be a quasifinite morphism of smooth schemes of finite type over S. If $Q \to P$ is flat on dense open subschemes, then $Q \to P$ is flat and locally of complete intersection of relative virtual dimension* 0.

Proof. Let $U \subset P$ and $V \subset Q$ be dense open subschemes such that $V \to U$ is flat. Then the relative dimension of $V \to S$ is the same as that of $U \to S$. Hence, we may assume that the relative dimensions of $P \to S$ and $Q \to S$ are the same integer n.

The morphism $Q \to P$ is the composition of the graph $Q \to Q \times_S P$ and the projection $Q \times_S P \to P$. For every point $x \in P$, the fiber $Q \times_P x \to Q \times_S x$ is a regular immersion of codimension n. Hence by [EGA IV$_3$ 1966, proposition (15.1.16) c)$\Rightarrow$b)] applied to the immersion $Q \to Q \times_S P$ over P, the immersion $Q \to Q \times_S P$ is also a regular immersion of codimension n and $Q \to P$ is flat. □

Lemma 2.2.2. *Let S be a noetherian scheme and let $Y \to X$ be a morphism of schemes of finite type over S.*

(1) *Suppose that there exists a cartesian diagram*

$$\begin{array}{ccc} Q & \longleftarrow & Y \\ \downarrow & \square & \downarrow \\ P & \longleftarrow & X \end{array} \tag{2-15}$$

of schemes of finite type over S satisfying the following conditions:

> *P and Q are smooth over S and $Q \to P$ is quasifinite and is flat on dense open subschemes. The horizontal arrows are closed immersions.*

Then $Y \to X$ is quasifinite, flat and locally of complete intersection of relative virtual dimension 0.

(2) *Conversely, suppose that $Y \to X$ is finite* (*resp. quasifinite*) *and locally of complete intersection of relative virtual dimension* 0. *Then $Y \to X$ is flat and, locally on X* (*resp. locally on X and on Y*), *there exists a cartesian diagram* (2-15) *satisfying the following conditions*:

> *P and Q are smooth of the same relative dimension over S and $Q \to P$ is quasifinite and flat. The horizontal arrows are closed immersions.*

Proof. (1) By Lemma 2.2.1, the quasifinite morphism $Q \to P$ is flat and locally of complete intersection. Hence $Y \to X$ is also quasifinite, flat and locally of complete intersection of relative virtual dimension 0.

(2) Since the assertion is local, we may assume that S, X and Y are affine. Take a closed immersion

$$Q_1 = \mathbb{A}^m_X \leftarrow Y.$$

Since the immersion $Y \to Q_1$ is a regular immersion of codimension m and since $Y \to X$ is finite (resp. quasifinite), after shrinking X (resp. Q_1 and Y), we may assume that the ideal defining $Y \subset Q_1$ is generated by m sections $f_1, \dots, f_m$ of $\mathcal{O}_{Q_1}$. Also take a closed immersion $P_1 = \mathbb{A}^n_S \leftarrow X$ and an open subscheme $Q \subset \mathbb{A}^m_{P_1}$ to obtain a cartesian diagram

$$\begin{array}{ccccc} Q & \longleftarrow & Q_1 & \longleftarrow & Y \\ \downarrow & \square & \downarrow & & \\ P_1 & \longleftarrow & X & & \end{array} \tag{2-16}$$

Taking sections $\tilde{f}_1, \dots, \tilde{f}_m$ of $\mathcal{O}_Q$ lifting $f_1, \dots, f_m$ after shrinking Q if necessary, define a morphism $Q \to P = \mathbb{A}^m_{P_1}$. Then, we obtain a cartesian diagram

$$\begin{array}{ccccc} Q & \longleftarrow & Q_1 & \longleftarrow & Y \\ \downarrow & \square & \downarrow & \square & \downarrow \\ P & \longleftarrow & \mathbb{A}^m_X & \longleftarrow & X \end{array} \tag{2-17}$$

where the lower right horizontal arrow $\mathbb{A}^m_X \to X$ is the 0-section.

The schemes $P = \mathbb{A}^{n+m}_S$ and $Q \subset \mathbb{A}^{n+m}_S$ are smooth over S. Since $Y \to X$ is quasifinite, after replacing Q by a neighborhood of Y if necessary, the morphism $Q \to P$ is quasifinite. Since Q and P are smooth of the same relative dimension over S, the morphism $Q \to P$ is flat on dense open subschemes. By Lemma 2.2.1, the quasifinite morphism $Q \to P$ is flat and hence $Y \to X$ is also flat. □

We give examples of construction of the diagram (2-15).

Example 2.2.3. Assume that X and Y are schemes of finite type over a noetherian scheme S.

(1) Assume $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine. Let

$$A[T_1, \dots, T_n]/(f_1, \dots, f_n) \to B$$

be an isomorphism and define a morphism

$$Q = \mathbb{A}^n_X = \operatorname{Spec} A[T_1, \dots, T_n] \to P = \mathbb{A}^n_X$$

by $f_1, \dots, f_n$. Then, we obtain a cartesian diagram (2-15) by defining the section $X \to P = \mathbb{A}^n_X$ to be the 0-section.

(2) Assume that X and Y are smooth over a noetherian scheme S. Then, we obtain a cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & Q = Y \times_S X \\ \big\downarrow & \square & \big\downarrow \\ X & \longrightarrow & P = X \times_S X \end{array}$$

Consider a cartesian diagram (2-15) satisfying the conditions of Lemma 2.2.2(1) and let $D \subset X$ be an effective Cartier divisor. Assume that $Q^{(D)} \to P^{(D)}$ is étale on a neighborhood of $Q^{(D)} \times_X D$. Let $\bar{x}$ be a geometric point of D and let $0_{\bar{x}}$ denote the geometric point above the origin of the vector space $P^{(D)}_{\bar{x}}$ over $\bar{x}$. Then, since $Q^{(D)}_{\bar{x}} \to P^{(D)}_{\bar{x}}$ is finite étale, we have an action of the fundamental group $\pi_1(P^{(D)}_{\bar{x}}, 0_{\bar{x}})$ on

$$Y^{(D)}_{\bar{x}} = Q^{(D)}_{\bar{x}} \times_{P^{(D)}_{\bar{x}}} 0_{\bar{x}}.$$

The action on $Y^{(D)}_{\bar{x}}$ is compatible with the canonical mapping $Y^{(D)}_{\bar{x}} \to \pi_0(Q^{(D)}_{\bar{x}})$ with respect to the trivial action on $\pi_0(Q^{(D)}_{\bar{x}})$ and is transitive on the inverse image of each element of $\pi_0(Q^{(D)}_{\bar{x}})$.

Since $Q^{[D]} \to P^{[D]} \times_P Q$ is an isomorphism by Lemma 2.1.7(2), for a geometric point $\bar{y}$ of $Y_{\bar{x}}$ and for the geometric point $0_{\bar{y}}$ of $Q^{[D]}_{\bar{y}}$ above $P^{(D)}_{\bar{x}}$, we have canonical isomorphisms $Q^{[D]}_{\bar{y}} = Q^{[D]} \times_Q \bar{y} \to P^{[D]}_{\bar{x}} = P^{(D)}_{\bar{x}}$ and $\pi_1(Q^{[D]}_{\bar{y}}, 0_{\bar{y}}) \to \pi_1(P^{(D)}_{\bar{x}}, 0_{\bar{x}})$. The action of $\pi_1(P^{(D)}_{\bar{x}}, 0_{\bar{x}})$ on $Y^{(D)}_{\bar{x}}$ is compatible with the action of $\pi_1(Q^{[D]}_{\bar{y}}, 0_{\bar{y}})$ on $Y^{(D)}_{\bar{x}} \times_{Y_{\bar{x}}} \bar{y}$. For a morphism $Q' \to Q$, the canonical morphism $\pi_1(Q'^{[D]}_{\bar{y}}, 0_{\bar{y}}) \to \pi_1(Q^{[D]}_{\bar{y}}, 0_{\bar{y}})$ is compatible with the actions on $Y^{(D)}_{\bar{x}} \times_{Y_{\bar{x}}} \bar{y}$.

We study the relation between the étaleness of $Q^{(D)} \to P^{(D)}$ and the annihilator of $\mathcal{O}_{Y^{(D)}} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/X}$.

Lemma 2.2.4. *Let*

$$\begin{array}{ccc} Q & \longleftarrow & Y \\ \big\downarrow & \square & \big\downarrow \\ P & \longleftarrow & X \end{array}$$

be a cartesian diagram of separated schemes of finite type over X. Assume that P and Q are smooth over X and that the vertical arrows are quasifinite and flat.

Assume that there exists an effective Cartier divisor $D_1 \subset D = D_1 + D_0$ of X such that $\mathcal{O}_{Y^{(D)}} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/X}$ is annihilated by $\mathcal{I}_{D_1} \subset \mathcal{O}_X$ and that we have an equality $D_0 = D$ of underlying sets. Then, there exists an open neighborhood $W \subset Q^{[D]}$ of $Q^{[D]} \times_X D$ such that $Q^{[D]} \to P^{[D]}$ is étale on $W - (Q^{[D]} \times_X D)$.

Proof. It suffices to show that each irreducible component $Z \subset Q^{[D]}$ of the inverse image of the support of $\Omega^1_{Q/P}$ is either a subset of $Q^{[D]} \times_X D$ or does not meet $Q^{[D]} \times_X D$, since $Q^{[D]} \to Q$ is an isomorphism on the complement of the inverse images of D. Assume that Z is not a subset of $Q^{[D]} \times_X D$ but does meet $Q^{[D]} \times_X D$

and regard Z as an integral closed subscheme of $Q^{[D]}$. Then, $D \times_X Z \subset Z$ is a nonempty effective Cartier divisor.

Since the assertion is étale local on Q and X, we may assume that $Y \to X$ is faithfully flat and finite. Let $T_0 \subset Z \times_X Y^{(D)}$ be the closure of the complement $Z \times_X Y^{(D)} - D \times_X (Z \times_X Y^{(D)})$ and let T be its normalization. Then, since $Y \to X$ is finite surjective, $T \to Z$ is also finite surjective. Hence $D_T = D \times_X T \subset T$ is a nonempty effective Cartier divisor.

By the assumption that $\mathcal{O}_{Y^{(D)}} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/X}$ is annihilated by $\mathcal{I}_{D_1} \subset \mathcal{O}_X$, the $\mathcal{O}_T$-module $\mathcal{O}_T \otimes_{\mathcal{O}_Y} \Omega^1_{Y/X}$ is annihilated by $\mathcal{I}_{D_1} \cdot \mathcal{O}_T$. Since D_T is a scheme over $Q^{[D]} \times_X D$, we have an isomorphism $\mathcal{O}_{D_T} \otimes_{\mathcal{O}_Q} \Omega^1_{Q/P} \to \mathcal{O}_{D_T} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/X}$ by Lemma 2.1.3(1). Thus $\mathcal{O}_{D_T} \otimes_{\mathcal{O}_Q} \Omega^1_{Q/P}$ is also annihilated by $\mathcal{I}_{D_1} \cdot \mathcal{O}_{D_T}$. Since $D = D_1 + D_0$, this means an inclusion $\mathcal{I}_{D_1} \cdot \mathcal{O}_T \otimes_{\mathcal{O}_Q} \Omega^1_{Q/P} \subset \mathcal{I}_{D_0} \cdot \mathcal{I}_{D_1} \cdot \mathcal{O}_T \otimes_{\mathcal{O}_Q} \Omega^1_{Q/P}$. By Nakayama's lemma, we have $\mathcal{I}_{D_1} \cdot \mathcal{O}_T \otimes_{\mathcal{O}_Q} \Omega^1_{Q/P} = 0$ on a neighborhood of $D_0 \times_X T$.

Since Z is a subset of the inverse image of support of $\Omega^1_{Q/P}$, the annihilator ideal of $\mathcal{O}_T \otimes_{\mathcal{O}_Q} \Omega^1_{Q/P}$ is 0. This contradicts to that $D_0 \times_X T = D_T$ is nonempty. $\square$

Lemma 2.2.5. *Assume X is normal and let*

$$\begin{array}{ccc} Q & \longleftarrow & Y \\ \downarrow & \square & \downarrow \\ P & \longleftarrow & X \end{array}$$

be a cartesian diagram of separated schemes of finite type over X. Assume that P and Q are smooth over X and that the vertical arrows are quasifinite and flat.

Let Y_0 be a closed subscheme of Y étale over X satisfying an equality $D_{Y_0} = D_Y$ of underlying sets and let $\mathcal{J}_0 \subset \mathcal{O}_{D_Y}$ be the nilpotent ideal defining $D_{Y_0} \subset D_Y$. Let $n \geqq 1$ be an integer satisfying $\mathcal{J}_0^n = 0$ and let $D_0 \subset D$ be an effective Cartier divisor on X satisfying $nD_0 \leqq D$.

Assume that $Y^{(D)} = Y \times_{Q^{[D]}} Q^{(D)}$ is étale over X. Then $\mathcal{O}_{Y^{(D)}} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/X}$ is annihilated by the ideal $\mathcal{I}_{D-D_0} \subset \mathcal{O}_X$ defining $D - D_0 \subset X$.

Proof. Let $\mathcal{I} \subset \mathcal{I}_0 \subset \mathcal{O}_Q$ and $\mathcal{I}_D \subset \mathcal{I}_{D_0} \subset \mathcal{O}_X$ be the ideals defining the closed subschemes $Y_0 \subset Y \subset Q$ and $D_0 \subset D \subset X$. Let $Y_0^{(n)} \subset Q$ denote the closed scheme defined by the ideal $\mathcal{I}_0^n \subset \mathcal{O}_Q$. Let $Q^{[D_0.Y_0]} \to Q$ denote the dilatation for $Y_0 \to Q$ and D_0. We also define a dilatation $Q^{[nD_0.Y_0^{(n)}]} \to Q$ for $Y_0^{(n)} \to Q$ and nD_0.

Since Y_0 is étale over X, the scheme $Q^{[D_0.Y_0]}$ is smooth over X by Example 2.1.2(1) and equals its normalization $Q^{(D_0.Y_0)}$. The canonical morphism $Q^{[D_0.Y_0]} \to Q^{[nD_0.Y_0^{(n)}]}$ is finite and induces an isomorphism $Q^{(D_0.Y_0)} \to Q^{(nD_0.Y_0^{(n)})}$ on the normalizations.

By the assumptions $\mathcal{J}_0^n = 0$ and $nD_0 \leqq D$, we have $\mathcal{I}_0^n \subset \mathcal{I} + \mathcal{I}_D \subset \mathcal{I} + \mathcal{I}_{nD_0}$. Hence we have a morphism $Q^{[nD_0]} \to Q^{[nD_0.Y_0^{(n)}]}$. Further, by $nD_0 \leqq D$, we obtain a morphism $Q^{(D)} \to Q^{(D_0.Y_0)}$ of normalizations.

The dilatation $P^{[D]}$ of P for the section $X \to P$ and D is smooth over X by Example 2.1.2(1) and hence is equal to the normalization $P^{(D)}$. Since $Y^{(D)} \to X$ is étale and since each square of the diagram

$$\begin{array}{ccccc} Y^{(D)} & \longrightarrow & Q^{(D)} & & \\ \downarrow & & \downarrow & & \\ Y & \longrightarrow & Q^{[D]} & \longrightarrow & Q \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & P^{[D]} & \longrightarrow & P \end{array}$$

is cartesian by Lemma 2.1.7(2), the quasifinite morphism $Q^{(D)} \to P^{(D)}$ of normal schemes is étale on a neighborhood $W \subset Q^{(D)}$ of $Y^{(D)}$ by [EGA IV$_4$ 1967, théorème (18.10.16)].

The commutative diagram

$$\begin{array}{ccccc} Q^{(D)} & \longrightarrow & Q^{(D_0 \cdot Y_0)} & \longrightarrow & Q \\ \downarrow & & & & \downarrow \\ P^{(D)} & & \longrightarrow & & P \end{array}$$

of schemes defines a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_W \otimes \Omega^1_{Q^{(D)}/X} & \longleftarrow & \mathcal{O}_W \otimes \Omega^1_{Q^{(D_0 \cdot Y_0)}/X} & \longleftarrow & \mathcal{O}_W \otimes \Omega^1_{Q/X} \\ \uparrow & & & & \uparrow \\ \mathcal{O}_W \otimes \Omega^1_{P^{(D)}/X} & & \longleftarrow & & \mathcal{O}_W \otimes \Omega^1_{P/X} \end{array}$$

of locally free $\mathcal{O}_W$-modules. Since $Q^{(D)} \to P^{(D)}$ is étale on W, the left vertical arrow is an isomorphism.

Since $X \to X$ and $Y_0 \to X$ are étale, the lower horizontal arrow (resp. the upper right horizontal arrow) induces an isomorphism $\mathcal{O}_W \otimes \Omega^1_{P/X} \to \mathcal{I}_D \cdot \mathcal{O}_W \otimes \Omega^1_{P^{(D)}/X}$ (resp. $\mathcal{O}_W \otimes \Omega^1_{Q/X} \to \mathcal{I}_{D_0} \cdot \mathcal{O}_W \otimes \Omega^1_{Q^{(D_0 \cdot Y_0)}/X}$). Hence $\mathcal{I}_{D-D_0} \cdot \mathcal{O}_W \otimes \Omega^1_{Q/X} = \mathcal{I}_D \cdot \mathcal{O}_W \otimes \Omega^1_{Q^{(D_0 \cdot Y_0)}/X}$ is contained in the image of $\mathcal{O}_W \otimes \Omega^1_{P/X}$. Or equivalently, $\mathcal{O}_W \otimes_{\mathcal{O}_Q} \Omega^1_{Q/P}$ is annihilated by $\mathcal{I}_{D-D_0}$. Hence its pull-back $\mathcal{O}_{Y^{(D)}} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/X}$ is also annihilated by $\mathcal{I}_{D-D_0}$. □

3. Ramification

3.1. ***Ramification of quasifinite schemes.*** Let X be a normal noetherian scheme and let D be an effective Cartier divisor of X. Let Y be a quasifinite scheme over X such that $D_Y = D \times_X Y \subset Y$ is a Cartier divisor.

Locally on X, there exists a smooth scheme Q over X and a closed immersion $Y \to Q$ over X. Then, by Proposition 2.1.5 and Corollary 2.1.6, the scheme $Y^{(D)}$ over Y defined as $Y \times_{Q^{[D]}} Q^{(D)}$ is canonically independent of Q. Hence a finite scheme $Y^{(D)}$ over Y is defined by patching. Similarly, for a geometric point $\bar{x}$ above $x \in D$, the set $\pi_0(Q^{(D)}_{\bar{x}})$ of connected components of the geometric fiber is canonically independent of Q.

Definition 3.1.1. Let X be a normal noetherian scheme and let D be an effective Cartier divisor of X. Let Y be a quasifinite scheme over X such that $D_Y = D \times_X Y \subset Y$ is a Cartier divisor and let $\overline{Y}$ be the normalization of Y. Let $\bar{x}$ be a geometric point above a point $x \in D$.

By taking a closed immersion $Y \to Q$ to a smooth scheme Q over X defined on a neighborhood of x, we define finite sets $F^D_{\bar{x}}(Y/X)$ and $F^{D+}_{\bar{x}}(Y/X)$ by

$$F^D_{\bar{x}}(Y/X) = \pi_0(Q^{(D)}_{\bar{x}}), \qquad F^{D+}_{\bar{x}}(Y/X) = Y^{(D)}_{\bar{x}} \tag{3-1}$$

equipped with canonical mappings

$$\begin{array}{ccc} \overline{Y}_{\bar{x}} & \xrightarrow{\varphi^{D+}_{\bar{x}}} & F^{D+}_{\bar{x}}(Y/X) \\ {\scriptstyle \varphi^D_{\bar{x}}}\downarrow & \swarrow & \downarrow \\ F^D_{\bar{x}}(Y/X) & \longrightarrow & Y_{\bar{x}} \end{array} \tag{3-2}$$

induced by the morphisms in (2-2):

$$\begin{array}{ccc} \overline{Y} & \longrightarrow & Y^{(D)} \\ \downarrow & \swarrow & \downarrow \\ Q^{(D)} & \longrightarrow & Q \end{array}$$

We consider a commutative diagram

$$\begin{array}{ccccccccc} Y' & \longrightarrow & X' & \overset{\supset}{\hookleftarrow} & D' & \overset{\supset}{\hookleftarrow} & D \times_X X' & \longleftarrow & \bar{x}' \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ Y & \longrightarrow & X & & \overset{\supset}{\hookleftarrow} & & D & \longleftarrow & \bar{x} \end{array} \tag{3-3}$$

of noetherian schemes. We assume that X' is normal, $D' \subset X'$ is an effective Cartier divisor, Y' is quasifinite over X' and that $D'_{Y'} \subset Y'$ is an effective Cartier divisor.

Then, the commutative diagram (2-8) induces a commutative diagram

$$\begin{array}{ccccccc} \overline{Y}'_{\bar{x}'} & \xrightarrow{\varphi^{D'+}_{\bar{x}'}} & F^{D'+}_{\bar{x}'}(Y'/X') & \longrightarrow & F^{D'}_{\bar{x}'}(Y'/X') & \longrightarrow & Y'_{\bar{x}'} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \overline{Y}_{\bar{x}} & \xrightarrow{\varphi^{D+}_{\bar{x}}} & F^{D+}_{\bar{x}}(Y/X) & \longrightarrow & F^{D}_{\bar{x}}(Y/X) & \longrightarrow & Y_{\bar{x}}. \end{array} \tag{3-4}$$

For effective Cartier divisors D and D' of a scheme X defined by the ideal sheaves $\mathcal{I}_D, \mathcal{I}_{D'} \subset \mathcal{O}_X$ and for $x \in D$, we write $D < D'$ at x if we have a strict inclusion $\mathcal{I}_{D,x} \supsetneqq \mathcal{I}_{D',x}$. If $X = X'$, $Y = Y'$, $\bar{x} = \bar{x}'$ and if $D < D'$ at the image x of $\bar{x}$ as Cartier divisors, further we have an arrow $F^{D'}_{\bar{x}}(Y/X) \to F^{D+}_{\bar{x}}(Y/X)$ making the two triangles obtained by dividing the middle square commutative by Lemma 2.1.9.

Proposition 3.1.2. *Assume that $Y \to X$ is quasifinite, flat and locally of complete intersection and that the normalization $\overline{Y}$ of Y is étale over X.*

(1) *The arrows in diagram (3-2)*

$$\begin{array}{ccc} \overline{Y}_{\bar{x}} & \xrightarrow{\varphi^{D+}_{\bar{x}}} & F^{D+}_{\bar{x}}(Y/X) \\ {\scriptstyle \varphi^{D}_{\bar{x}}}\downarrow & \swarrow & \downarrow \\ F^{D}_{\bar{x}}(Y/X) & \longrightarrow & Y_{\bar{x}} \end{array}$$

are surjections.

(2) *Let $Y' \to Y$ be a surjective morphism locally of complete intersection of quasifinite and flat schemes over X. Assume that the normalization $\overline{Y}'$ of Y' is étale over X. Then, the diagram*

$$\begin{array}{ccccccc} \overline{Y}'_{\bar{x}} & \xrightarrow{\varphi^{D+}_{\bar{x}}} & F^{D+}_{\bar{x}}(Y'/X) & \longrightarrow & F^{D}_{\bar{x}}(Y'/X) & \longrightarrow & Y'_{\bar{x}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \overline{Y}_{\bar{x}} & \xrightarrow{\varphi^{D+}_{\bar{x}}} & F^{D+}_{\bar{x}}(Y/X) & \longrightarrow & F^{D}_{\bar{x}}(Y/X) & \longrightarrow & Y_{\bar{x}} \end{array} \tag{3-5}$$

is a cocartesian diagram of surjections.

Proof. By replacing X by the strict localization $X_{(\bar{x})}$, we may assume that $\bar{x} \to X$ is a closed immersion and that $Y \to X$ is finite.

(1) By Lemma 2.2.2(2), we may assume that there exist smooth schemes P and Q over X and a cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & Q \\ \downarrow & \square & \downarrow \\ X & \longrightarrow & P \end{array}$$

of schemes over X such that the horizontal arrows are closed immersions and that the vertical arrows are quasifinite and flat. We verify that the diagram

$$\begin{array}{ccccccc} Q^{(D)}_{\bar{x}} & \longrightarrow & Q^{(D)} & \longleftarrow & Y^{(D)} & \longleftarrow & \overline{Y} \\ \downarrow & \square & \downarrow & & \downarrow & & \downarrow \\ P^{(D)}_{\bar{x}} & \longrightarrow & P^{(D)} & \longleftarrow & X & = & X \end{array} \tag{3-6}$$

satisfies the assumptions in Corollary 1.1.6. Since $P^{[D]} \to X$ is smooth, we have $P^{(D)} = P^{[D]}$. By Lemma 2.1.7.2, the diagram

$$\begin{array}{ccccc} Q & \longleftarrow & Q^{[D]} & \longleftarrow & Y \\ \downarrow & \square & \downarrow & \square & \downarrow \\ P & \longleftarrow & P^{[D]} & \longleftarrow & X \end{array}$$

is cartesian. Hence the middle square in (3-6) is also cartesian.

The diagram (3-6) satisfies the finiteness assumption in Corollary 1.1.6, by Lemma 2.1.7(1). Since $X = X_{(\bar{x})}$ is strictly local, the assumption that the canonical mapping $\bar{x} \to \pi_0(\overline{X})$ is a bijection is satisfied. Since $P^{(D)}_{\bar{x}}$ is a vector space over $\bar{x}$ and is connected, the mapping $\bar{x} \to P^{(D)}_{\bar{x}} \cap X \to \pi_0(P^{(D)}_{\bar{x}})$ are bijections of sets consisting of single elements. We may assume that the finite étale morphism $\overline{Y} \to X$ is surjective since if otherwise the assertion is trivial. Hence by Corollary 1.1.6(2) (resp. (3)), the mapping $\overline{Y}_{\bar{x}} \to Y^{(D)}_{\bar{x}} = F^{D+}_{\bar{x}}(Y/X)$ $\big($resp. $F^{D+}_{\bar{x}}(Y/X) = Y^{(D)}_{\bar{x}} \to \pi_0(Q^{(D)}_{\bar{x}}) = F^{D}_{\bar{x}}(Y/X)\big)$ is surjective.

Similarly, applying Corollary 1.1.6(2) to the diagram

$$\begin{array}{ccccccc} Y_{\bar{x}} & \longrightarrow & Y & \longleftarrow & Y & \longleftarrow & \overline{Y} \\ \downarrow & \square & \downarrow & \square & \downarrow & & \downarrow \\ \bar{x} & \longrightarrow & X & \longleftarrow & X & = & X \end{array}$$

we see that $\overline{Y}_{\bar{x}} \to Y_{\bar{x}}$ is a surjection.

(2) By Lemma 2.2.2(2), we may assume that there exists smooth schemes Q and Q' over X and a cartesian diagram

$$\begin{array}{ccc} Y' & \longrightarrow & Q' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Q \end{array}$$

of schemes over X such that the horizontal arrows are closed immersions and that the vertical arrows are quasifinite and flat.

We verify that the diagram

$$\begin{array}{ccccccc}
Q'^{(D)}_{\bar{x}} & \longrightarrow & Q'^{(D)} & \longleftarrow & Y'^{(D)} & \longleftarrow & \overline{Y}' \\
\downarrow & \square & \downarrow & & \downarrow & & \downarrow \\
Q^{(D)}_{\bar{x}} & \longrightarrow & Q^{(D)} & \longleftarrow & Y^{(D)} & \longleftarrow & \overline{Y}
\end{array}$$

satisfies the assumptions in Corollary 1.1.6. The middle square is cartesian by Lemma 2.1.7(2). The finiteness assumption in Corollary 1.1.6 is satisfied by Lemma 2.1.7(1). Since the finite étale covering $\overline{Y} \to X$ is split and X is connected, the assumption that the canonical mapping $\overline{Y}_{\bar{x}} \to \pi_0(\overline{Y})$ is a bijection is satisfied. By (1), $\overline{Y}_{\bar{x}} \to Y^{(D)}_{\bar{x}} \to \pi_0(Q^{(D)}_{\bar{x}})$ are surjective. We may assume that Y and Y' are finite over X. Since $Y' \to Y$ is surjective, the morphism $\overline{Y}' \to \overline{Y}$ of finite étale schemes over X is also surjective. Hence by Corollary 1.1.6(2) (resp. (3)), the right square (resp. the middle square) of (3-5) is a cocartesian diagram of surjections.

Similarly, applying Corollary 1.1.6(2) to the diagram

$$\begin{array}{ccccccc}
Y'_{\bar{x}} & \longrightarrow & Y' & \longleftarrow & Y' & \longleftarrow & \overline{Y}' \\
\downarrow & \square & \downarrow & \square & \downarrow & & \downarrow \\
Y_{\bar{x}} & \longrightarrow & Y & \longleftarrow & Y & \longleftarrow & \overline{Y}
\end{array}$$

we see that the big rectangle in (3-5) is a cocartesian diagram of surjections. □

Corollary 3.1.3. *Assume that $Y \to X$ is locally of complete intersection and that the normalization $\overline{Y}$ is étale over X. Let P and Q be smooth schemes over X and let*

$$\begin{array}{ccc}
Y & \longrightarrow & Q \\
\downarrow & \square & \downarrow \\
X & \longrightarrow & P
\end{array}$$

be a cartesian diagram of schemes over X such that the horizontal arrows are closed immersions and that the vertical arrows are quasifinite and flat. Then, the mapping $\overline{Y}_{\bar{x}} \to F^{D+}_{\bar{x}}(Y/X)$ is an injection on the inverse image of $y \in Y$ if and only if $Q^{(D)} \to P^{(D)}$ is étale on the inverse image of y by $Y^{(D)} \to Y$.

Proof. Since the assertion is étale local, we may assume that $Y \to X$ and $Q \to P$ are finite and that y is the unique point of the inverse image of x. Then, by Proposition 3.1.2(1), $\overline{Y}_{\bar{x}} \to Y^{(D)}_{\bar{x}} = F^{D+}_{\bar{x}}(Y/X) \subset Q^{(D)}_{\bar{x}}$ is a bijection of finite sets. Hence $Q^{(D)} \to P^{(D)}$ is étale at x by [EGA IV$_4$ 1967, théorème (18.10.16)]. □

Definition 3.1.4. Let X be a normal noetherian scheme and let $U \subset X$ be a dense open subscheme. Let Y be a quasifinite scheme over X such that $V = U \times_X Y \to U$ is étale. Let D be an effective Cartier divisor of X such that $U \cap D$ is empty and that $D_Y = D \times_X Y$ is an effective Cartier divisor.

(1) For $x \in D$, we consider the following condition on X, Y and D:

(RF) There exist an open neighborhood W of $x \in X$, a smooth scheme Q over W and a closed immersion $Y \times_X W \to Q$ such that the normalization $\overline{Y}$ of Y is étale over W and that the normalization $Q^{(D)}$ of the dilatation $Q^{[D]}$ is flat and reduced over W.

If the condition (RF) is satisfied at every $x \in D$, we say that Y over X satisfies the condition (RF) for D.

(2) Let $x \in D$ and assume that Y over X satisfies the condition (RF) for D at x. Let y be a point of $\overline{Y} \times_X x \subset \overline{Y} \times_X D$. We say that the ramification of $Y \to X$ is bounded by D (resp. by $D+$) at y, if the mapping $\varphi^D_{\bar{x}} : \overline{Y}_{\bar{x}} \to F^D_{\bar{x}}(Y/X)$ $\big($resp. $\varphi^{D+}_{\bar{x}} : \overline{Y}_{\bar{x}} \to F^{D+}_{\bar{x}}(Y/X)\big)$ is an injection on the inverse image of y.

We say that the ramification of $Y \to X$ is bounded by D (resp. by $D+$) at x, if the mapping $\varphi^D_{\bar{x}} : \overline{Y}_{\bar{x}} \to F^D_{\bar{x}}(Y/X)$ (resp. $\varphi^{D+}_{\bar{x}} : \overline{Y}_{\bar{x}} \to F^{D+}_{\bar{x}}(Y/X)$) is an injection.

If ramification is bounded by D, it is bounded by $D+$. We show that the condition (RF) is independent of the choice of Q.

Lemma 3.1.5. *Let X be a normal noetherian scheme and let $U \subset X$ be a dense open subscheme. Let Y be a quasifinite scheme over X such that $V = U \times_X Y \to U$ is étale. Let D be an effective Cartier divisor of X such that $U \cap D$ is empty and that $D_Y \subset Y$ is an effective Cartier divisor. Let $x \in D$.*

(1) *Assume that Y over X satisfies (RF) for D at x. Let $W \subset X$ be an open neighborhood of x, let Q be a smooth scheme over W and let $Y \times_X W \to Q$ be a closed immersion. Then, there exists an open neighborhood $W' \subset W$ of x, such that $(Q \times_W W')^{(D \times_X W')} \to W'$ is flat and reduced.*

(2) *Let $X' \to X$ be a morphism of normal noetherian scheme such that $U' = U \times_X X'$ is a dense open subscheme and that $D'_{Y'} = D_Y \times_X X' \subset Y' = Y \times_X X'$ is an effective Cartier divisor. Let x' be a point of $D' = D \times_X X'$ above x. We consider the following conditions*:

(i) *Y over X satisfies (RF) for D at x.*

(ii) *Y' over X' satisfies (RF) for D' at x'.*

We have (i) $\Rightarrow$ (ii). *Conversely, if $X' \to X$ is smooth at x', we have* (ii) $\Rightarrow$ (i).

Proof. (1) Set $D_W = D \times_X W$. After shrinking W if necessary, we may assume that there exist a smooth scheme Q_0 over W and a closed immersion $Y \times_X W \to Q_0$ such that $Q_0^{(D_W)} \to W$ is flat and reduced. Since $Q^{(D_W)} \leftarrow (Q \times_W Q_0)^{(D_W)} \to Q_0^{(D_W)}$ are smooth by Proposition 2.1.5, the assertion follows.

(2) (i)⇒(ii): This follows from Lemma 2.1.8.

(ii)⇒(i): After shrinking X' if necessary, we may assume that $X' \to X$ is smooth. Let W be an open neighborhood of x, let $Y \times_X W \to Q$ be a closed immersion to a smooth scheme Q over W and let $W' = W \times_X X'$. Then the morphism

$$(Q \times_W W')^{(D' \times_{X'} W')} \to Q^{(D \times_X W)} \times_W W'$$

is an isomorphism by Lemma 2.1.8. Hence the assertion follows. □

Lemma 3.1.6. *Let X be a normal noetherian scheme and let $U \subset X$ be a dense open subscheme. Let Y be a quasifinite scheme over X such that $V = U \times_X Y \to U$ is étale. Let D and D', with $D \subset D'$, be effective Cartier divisors of X such that $U \cap D'$ is empty and that $D'_Y \subset Y$ is an effective Cartier divisor. Let $x \in D$ and assume that Y over X satisfies (RF) for D and D' at x.*

Let $y \in Y$ be a point above x. If the ramification of Y over X is bounded by $D+$ at y and if $D < D'$ at x, then the ramification of Y over X is bounded by D' at y.

Proof. It follows from Lemma 2.1.9. □

Lemma 3.1.7. *Let X be a normal noetherian scheme and let $U \subset X$ be a dense open subscheme. Let Y be a quasifinite scheme over X such that $V = U \times_X Y \to U$ is étale. Let D be an effective Cartier divisor of X such that $U \cap D$ is empty and that $D_Y \subset Y$ is an effective Cartier divisor. Assume that Y over X satisfies the condition (RF) for D.*

Let $S \subset D_Y$ (resp. $S^+ \subset D_Y$) denote the subset consisting of points $y \in D_Y$ where the ramification of $Y \to X$ is bounded by D (resp. by $D+$).

(1) *We have $S \subset S^+$.*

(2) *The subset $S \subset D_Y$ is closed and the subset $S^+ \subset D_Y$ is open.*

Proof. (1) It follows from the commutative diagram (3-2).

(2) By Lemma 1.1.3 applied to $\overline{Y} \to Q^{(D)}$, we see that S is closed. Similarly, by Lemma 1.1.4 applied to $\overline{Y} \to Y^{(D)}$ we see that S^+ is open. □

Proposition 3.1.8. *Let X be a normal noetherian scheme and let $U \subset X$ be a dense open subscheme. Let Y be a quasifinite scheme over X such that $V = U \times_X Y \to U$ is étale. Let D be an effective Cartier divisor of X such that $U \cap D$ is empty and that $D_Y \subset Y$ is an effective Cartier divisor.*

Let C be a semistable curve over X such that $C_U = C \times_X U \to U$ is smooth. Let $x \in X$ be a point of D and let $z \in C$ be a singular point of the fiber C_x. Assume that there exist two irreducible components C_1 and C_2 of the fiber C_x meeting at z and let ζ_1 and ζ_2 be their generic points. Let $D_1 \subset D_2$ be effective Cartier divisors *on X and let $\widetilde{D} \subset C$ be an effective Cartier divisor such that $D_1 < D_2$ at x and that $\widetilde{D} = D_i \times_X C = D_{i,C}$ on a neighborhood of ζ_i for $i = 1, 2$.*

Assume that $Y_C = Y \times_X C$ over C satisfies the condition (RF) for $\widetilde{D}$ at z.

(1) *Y over X satisfies the condition (RF) for D_1 and D_2 at x.*

(2) *We have a commutative diagram*

$$\begin{array}{ccccc} F_{\bar{x}}^{D_2+}(Y/X) & \longrightarrow & F_{\bar{z}}^{\widetilde{D}+}(Y_C/C) & \longrightarrow & F_{\bar{x}}^{D_1+}(Y/X) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ F_{\bar{x}}^{D_2}(Y/X) & \longrightarrow & F_{\bar{z}}^{\widetilde{D}}(Y_C/C) & \longrightarrow & F_{\bar{x}}^{D_1}(Y/X) \end{array} \tag{3-7}$$

(3) *The lower left horizontal arrow $F_{\bar{x}}^{D_2}(Y/X) \to F_{\bar{z}}^{\widetilde{D}}(Y_C/C)$ in (3-7) is an injection. The upper right horizontal arrow $F_{\bar{z}}^{\widetilde{D}+}(Y_C/C) \to F_{\bar{x}}^{D_1+}(Y/X)$ in (3-7) is an injection on the image of $\overline{Y}_{\bar{x}}$.*

Proof. (1) Since ζ_1 and ζ_2 are contained in any open neighborhood of z, the scheme Y_C over C satisfies (RF) for $\widetilde{D}$ at ζ_1 and ζ_2. Since $C \to X$ is smooth at ζ_1 and ζ_2, the scheme Y over X satisfies (RF) for D_1 and D_2 at x by Lemma 3.1.5(2).

(2) Let $D_{1,C}$ and $D_{2,C}$ be the pull-backs of D_1 and D_2 to C. Then, we have $D_{1,C} < \widetilde{D} < D_{2,C}$ at z. Hence by (3-4) with the slant arrow added, we obtain a commutative diagram

$$\begin{array}{ccccc} F_{\bar{z}}^{D_{2,C}+}(Y_C/C) & \longrightarrow & F_{\bar{z}}^{\widetilde{D}+}(Y_C/C) & \longrightarrow & F_{\bar{z}}^{D_{1,C}+}(Y_C/C) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ F_{\bar{z}}^{D_{2,C}}(Y_C/C) & \longrightarrow & F_{\bar{z}}^{\widetilde{D}}(Y_C/C) & \longrightarrow & F_{\bar{z}}^{D_{1,C}}(Y_C/C) \end{array} \tag{3-8}$$

Since Y over X satisfies (RF) for D_1 and D_2 at x by (1), the pull-back defines canonical isomorphisms from the left and right columns of (3-7) to those of (3-8) by Lemma 2.1.8. Thus we obtain (3-7).

(3) By functoriality of cospecialization mappings, we obtain a commutative diagram

$$\begin{array}{ccccc} F_{\bar{\zeta}_2}^{D_{2,C}}(Y_C/C) & \xleftarrow{\text{cosp.}} & F_{\bar{z}}^{D_{2,C}}(Y_C/C) & \longleftarrow & F_{\bar{x}}^{D_2}(Y/X) \\ \downarrow & & \downarrow & \swarrow & \\ F_{\bar{\zeta}_2}^{\widetilde{D}}(Y_C/C) & \xleftarrow{\text{cosp.}} & F_{\bar{z}}^{\widetilde{D}}(Y_C/C). & & \end{array} \tag{3-9}$$

By Lemma 2.1.8 and by $\widetilde{D} = D_{2,C}$ at ζ_2, the composition $F_{\bar{x}}^{D_2}(Y/X) \to F_{\bar{\zeta}_2}^{\widetilde{D}}(Y_C/C)$ is a bijection. Hence $F_{\bar{x}}^{D_2}(Y/X) \to F_{\bar{z}}^{\widetilde{D}}(Y_C/C)$ is injective.

Since the second assertion is étale local on X, we may assume that $Y \to X$ is finite. By functoriality of specialization mappings, we obtain a commutative diagram

$$\begin{array}{ccccccc}
 & & F_{\bar z}^{\widetilde{D}+}(Y_C/C) & \xleftarrow{\text{sp.}} & F_{\bar\zeta_1}^{\widetilde{D}+}(Y_C/C) & \longleftarrow & \overline{Y}_{\bar x} \\
 & \swarrow & \downarrow & & \downarrow & \swarrow & \\
F_{\bar x}^{D_1+}(Y/X) & \longrightarrow & F_{\bar z}^{D_{1,C}+}(Y_C/C) & \xleftarrow{\text{sp.}} & F_{\bar\zeta_1}^{D_{1,C}+}(Y_C/C). & &
\end{array}$$

The vertical arrow $F_{\bar z}^{\widetilde{D}+}(Y_C/C) \to F_{\bar z}^{D_{1,C}+}(Y_C/C)$ is an injection on the image of $\overline{Y}_{\bar x}$, since the composition $F_{\bar\zeta_1}^{\widetilde{D}+}(Y_C/C) \to F_{\bar z}^{D_{1,C}+}(Y_C/C)$ is a bijection. Hence the assertion follows. □

3.2. ***Ramification and valuations.*** In the rest of the article, A denotes a valuation ring and K denotes its fraction field. Let $v : K^\times \to \Gamma = K^\times/A^\times$ denote the valuation.

Definition 3.2.1. Let X be a normal separated noetherian scheme, let $U \subset X$ be a dense open subscheme and let A be a valuation ring. We say that a morphism $T = \operatorname{Spec} A \to X$ is *U-external* if $T \times_X U$ consists of a single point t.

For a morphism $T = \operatorname{Spec} A \to X$ and an effective Cartier divisor $D \subset X$, let $v(D) \in \Gamma$ denote the valuation $v(f)$ of a nonzero divisor f defining $D \subset X$ on a neighborhood of the image of T.

Let $\widetilde{X} = \varprojlim X'$ be the inverse limit of proper schemes $X' \to X$ such that $U' = U \times_X X' \to U$ is an isomorphism. Then, points of $\widetilde{X} - U$ correspond bijectively to the inverse limits of the images of the closed points by the liftings of U-external morphisms $T \to X$ defined by valuation rings of the residue fields of points of U by [Fujiwara and Kato 2018, Theorem E.2.11].

Lemma 3.2.2. *Let X be a normal noetherian scheme, let $U \subset X$ be a dense open subscheme, let $t \in U$ be a point, let $A \subsetneqq K = k(t)$ be a valuation ring and let $T = \operatorname{Spec} A \to X$ be a U-external morphism.*

(1) *Let $g \in \Gamma(U', \mathcal{O}_{U'}^\times)$ be an invertible function defined on an open neighborhood $U' \subset U$ of $t \in U$ such that $v(g) = \gamma \geqq 0$. Then, there exists a normal scheme X' of finite type over X such that $U \times_X X' = U'$, g is extended to a nonzero divisor on X' defining an effective Cartier divisor $R' \subset X'$, and $U' = X' - D'$ is the complement of an effective Cartier divisor $D' \subset X'$ and a U'-external morphism $T \to X'$ lifting $T \to X$ and $v(R') = \gamma$.*

(2) *Let K' be a finite separable extension of $K = k(t)$ and let $A' \subsetneqq K'$ be a valuation ring such that $A' \cap K = A$. Set $T' = \operatorname{Spec} A'$ and let $\gamma > 0$ be a*

positive element of the value group Γ' of A'. Then, there exist a commutative diagram

$$\begin{array}{ccccc} U' & \longrightarrow & X' & \longleftarrow & T' \\ \downarrow & \square & \downarrow & & \downarrow \\ U & \longrightarrow & X & \longleftarrow & T \end{array}$$

of schemes, a point $t' \in U'$ above t, an isomorphism $K' \to k(t')$ over K and an effective Cartier divisor R' of X' satisfying the following conditions (i)–(iv):

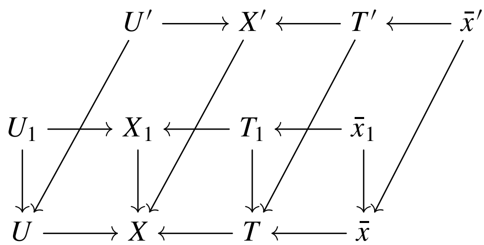
(i) *X' is a normal scheme of finite type over X.*

(ii) *The left square is cartesian and U' is a dense open subscheme of X' étale over U.*

(iii) *$T' \to X'$ is a U'-external morphism extending $t' \to U'$.*

(iv) *$R' \cap U' = \varnothing$ and $v'(R') = \gamma$.*

(3) *Let*



be a commutative diagram, let $t_1 \in U_1$ and $t' \in U'$ be points above $t \in U$ and let $R_1 \subset X_1$ and $R' \subset X'$ be effective Cartier divisors satisfying the following conditions (a)–(d):

(a) *X_1 and X' are normal noetherian schemes and $X_1 \to X$ is of finite type.*

(b) *The left square and the left parallelogram are cartesian and $U_1 \to U$ is étale. The open subschemes $U_1 \subset X_1$ and $U' \subset X'$ are dense.*

(c) *$T_1 = \operatorname{Spec} A_1$ and $T' = \operatorname{Spec} A'$ for valuation rings $A_1 \subsetneqq K_1 = k(t_1)$ and $A' \subsetneqq K' = k(t')$ satisfying $A_1 \cap K = A' \cap K = A$. The morphism $T_1 \to X_1$ is U_1-external and $T' \to X'$ is U'-external.*

(d) *$R_1 \cap U_1$ and $R' \cap U'$ are empty and we have $v_1(R_1) \leqq v'(R')$ in $\Gamma'_{\mathbb{Q}}$.*

Then, there exist a commutative diagram

$$\begin{array}{ccccccc} U'_1 & \longrightarrow & X'_1 & \longleftarrow & T'_1 & \longleftarrow & \bar{x}'_1 \\ \| & & \downarrow & & \downarrow & & \downarrow \\ U' \times_U U_1 & \longrightarrow & X' \times_X X_1 & \longleftarrow & T' \times_T T_1 & \longleftarrow & \bar{x}' \times_{\bar{x}} \bar{x}_1 \end{array}$$

and $t'_1 \in U'_1$ above t satisfying the following conditions (i)–(iv):

(i) *X'_1 is a normal scheme of finite type over X'.*

(ii) *U'_1 is $U' \times_{X'} X'_1$ and is a dense open subscheme of X'_1.*
(iii) *$T'_1 = \operatorname{Spec} A'_1$ for a valuation ring $A'_1 \subset K'_1 = k(t'_1)$.*
(iv) *For the pull-backs $R'_1 = R_1 \times_{X_1} X'_1$ and $R'_2 = R' \times_{X'} X'_1$, we have $R'_1 \leqq R'_2$.*

Proof. (1) Let Z and Z' be closed subschemes such that $U = X - Z$ and $U' = X - Z'$. By replacing X by the normalization of the blow-up at Z and at Z' and by the valuative criterion of properness, we may assume that $U = X - D$ and $U' = X - D'$ are the complements of effective Cartier divisors $D, D' \subset X$.

Let $x \in X$ be the image of the closed point of T and let $W = \operatorname{Spec} B \subset X$ be an open neighborhood of x such that $D \cap W$, $D' \cap W$ are principal divisors defined by $f, f' \in B$. Then we have $U' \cap W = W - D' \cap W = \operatorname{Spec} B[1/f']$. Set $g = h/f'^n \in B[1/f']$. The function g and hence also $h \in B$ are also invertible on $U' \cap W$. Set $\alpha = v(f), \alpha' = v(f') \in \Gamma$. Since $T \to X$ is U-external, we have $\Gamma^+[1/\alpha] = \Gamma$. Hence after replacing f by its power, we may assume that $\alpha' \leqq \alpha$.

Let $W' \to W$ be the normalization of the blow-up at the ideals (f'^n, h) and (f, f'). Since f, f' and h are invertible on $U' \cap W$, the morphism $W' \to W$ induces an isomorphism $U' \times_{X'} W' \to U' \cap W$. Since $W' \to W$ is proper, the morphism $T \to W$ is uniquely lifted to $T \to W'$. Since the generic point $t \in T$ is the unique point of $U \times_X T \supset (U' \times_{X'} W') \times_{W'} T$, the morphism $T \to W'$ is U'-external.

Let $x' \in W'$ be the image of the closed point of T. Since the ideals (f'^n, h) and (f, f') of $\mathcal{O}_{W,x'}$ are principal ideals and since $v(h) \geqq v(f'^n)$ and $v(f') \geqq v(f)$, there exists an open neighborhood X' of $x' \in W$ such that $U' \subset X'$, where we have inclusions $(f'^n) \supset (h)$ and $(f) \supset (f')$. Then, $g = h/f'^n$ defines a Cartier divisor R' on X' satisfying $R' \cap U' = \varnothing$ and $v(R') = \gamma$. We also have an inclusion $U \times_X X' = X' - D \times_X X' \subset X' - D' \times_X X' = U' \times_X X' = U'$. Since the other inclusion is obvious, we have $U' = U \times_X X'$.

(2) We may take an étale scheme $U_1 \to U$ such that $t' = \operatorname{Spec} K' = t \times_U U_1$ and a finite scheme $X_1 \to X$ containing U_1 as a dense open scheme. After shrinking U_1 if necessary, we may take an invertible function $g \in \Gamma(U_1, \mathcal{O}_{U_1}^\times)$ such that $\gamma = v'(g)$. Since T' is a localization of the normalization of $T \times_X X_1$, the morphism $t' \to U_1 \subset X_1$ is uniquely extended to $T' \to X_1$.

Then, by (1) applied to the open subschemes $U_1 \subset U \times_X X_1 \subset X_1$, to the morphism $T' \to X_1$ and to the invertible function $g \in \Gamma(U_1, \mathcal{O}_{U_1}^\times)$, the assertion follows.

(3) Let $T_{(\bar{x})}$, $T_{1,(\bar{x}_1)}$ and $T'_{(\bar{x}')}$ denote the strict localizations. We take a point $\tilde{t}'_1 \in T'_{(\bar{x}')} \times_{T_{(\bar{x})}} T_{1,(\bar{x}_1)}$ above the generic point of $T'_{(\bar{x}')}$. Then the normalization $\tilde{T}'_1$ of $T'_{(\bar{x}')}$ in $\tilde{t}'_1$ is $\tilde{T}'_1 = \operatorname{Spec} A'^{sh}_1$ for a strictly local valuation ring A'^{sh}_1. Let $t'_1 \in t' \times_t t_1 \subset T' \times_T T_1$ be the image of $\tilde{t}'_1$ and set $K'_1 = k(t'_1)$ and $A'_1 = A'^{sh}_1 \cap K'_1$. Let $T'_1 = \operatorname{Spec} A'_1$ and $\bar{x}'_1$ be the geometric point of T'_1 defined by a geometric closed point of $\tilde{T}'_1$.

Let X'_0 be the normalization of $X' \times_X X_1$ in $U'_1 = U' \times_U U_1$. Define effective Cartier divisors of X'_0 by $R'_{0,1} = R_1 \times_{X_1} X'_0$ and $R'_{0,2} = R' \times_{X'} X'_0$. Let $\bar{X}'_1 \to X'_0$ be the normalization of the blow-up at $R'_0 \cap R'_1 = R'_0 \times_{X'_0} R'_1$ and define effective Cartier divisors of $\bar{X}'_1$ by $\bar{R}'_1 = R_1 \times_{X_1} \bar{X}'_1$ and $\bar{R}'_2 = R' \times_{X'} \bar{X}'_1$. Since $\bar{X}'_1 \to X' \times_X X_1$ is proper, the morphism $t'_1 \to t' \times_t t_1 \subset U' \times_U U_1$ is uniquely lifted to $T'_1 \to \bar{X}'_1$ by the valuative criterion of properness.

Let $x'_1 \in \bar{X}'_1$ be the image of the closed point of T'_1. The intersection $\bar{R}'_1 \cap \bar{R}'_2 \subset \bar{X}'_1$ is the exceptional divisor and hence is an effective Cartier divisor. Since $v'_1(\bar{R}'_1) \leqq v'_1(\bar{R}'_2)$, on an open neighborhood $X'_1 \subset \bar{X}'_1$ of x'_1, we have $\bar{R}'_1 \cap \bar{R}'_2 = \bar{R}'_1 \leqq \bar{R}'_2$ by Nakayama's lemma. □

Let X be a normal noetherian scheme and let $U \subset X$ be a dense open subscheme. Let $t \in U$ and let $T = \operatorname{Spec} A \to X$ be a U-external morphism defined by a valuation ring $A \subsetneqq K = k(t)$ of the residue field at a point $t \in U$. Let $\bar{x}$ and $\bar{t}$ be geometric points of T supported on the closed point and on the generic point respectively. Recall that $T_{(\bar{x})}$ denotes the strict localization and that a specialization $\bar{x} \leftarrow \bar{t}$ is a morphism $T_{(\bar{x})} \leftarrow \bar{t}$ of schemes.

Let A' be a valuation ring and let $T' = \operatorname{Spec} A' \to T$ be a faithfully flat morphism. We identify Γ as a subgroup of the value group Γ' of A' by the canonical injection $\Gamma \to \Gamma'$. Let $\bar{x}'$ and $\bar{t}'$ be geometric points of T' above $\bar{x}$ and $\bar{t}$ respectively. We say that a specialization $\bar{x}' \leftarrow \bar{t}'$ is a lifting of $\bar{x} \leftarrow \bar{t}$ if the diagram

$$\begin{array}{ccccccc} \bar{x}' & \longrightarrow & T' & \longleftarrow & T'_{(\bar{x}')} & \longleftarrow & \bar{t}' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bar{x} & \longrightarrow & T & \longleftarrow & T_{(\bar{x})} & \longleftarrow & \bar{t} \end{array}$$

is commutative.

We consider a commutative diagram

$$\begin{array}{ccc} X' & \longleftarrow & T' \\ \downarrow & & \downarrow \\ X & \longleftarrow & T \end{array} \tag{3-10}$$

of schemes equipped with an effective Cartier divisor $R' \subset X'$ and a lifting $\bar{x}' \leftarrow \bar{t}'$ to T' of the specialization $\bar{x} \leftarrow \bar{t}$ satisfying the following conditions (i)–(iii):

(i) X' is a normal noetherian scheme of finite type over X such that $U' = U \times_X X' \subset X'$ is a dense open subscheme étale over U.

(ii) $T' = \operatorname{Spec} A' \to X'$ is a U'-external morphism defined by a valuation ring $A' \subsetneqq K' = k(t')$ of the residue field at a point $t' \in U'$ above t such that $A' \cap K = A$.

(iii) $R' \cap U' = \varnothing$ and $v'(R') = \gamma$ in the value group Γ' of A'.

For elements $\alpha \leqq \beta$ of a totally ordered group Γ, let $(\alpha, \beta)_\Gamma \subset \Gamma$ denote the subset $\{\gamma \in \Gamma \mid \alpha < \gamma < \beta\}$. Similarly, we define $(\alpha, \beta]_\Gamma$, $(\alpha, \infty)_\Gamma \subset \Gamma$ etc.

Definition 3.2.3. Let X be a normal noetherian scheme and let $U \subset X$ be a dense open subscheme. Let $t \in U$, let $A \subsetneqq k(t)$ be a valuation ring of the residue field at t and let $T = \operatorname{Spec} A \to X$ be a U-external morphism. Let $\gamma \in (0, \infty)_{\Gamma_{\mathbb{Q}}}$ for $\Gamma_{\mathbb{Q}} = \Gamma \otimes \mathbb{Q}$. Let Y be a quasifinite flat scheme over X such that $V = Y \times_X U \to U$ is étale.

We define a commutative diagram

$$\begin{array}{ccc} F_T^{\infty}(Y/X) & \xrightarrow{\varphi_T^{\gamma+}} & F_T^{\gamma+}(Y/X) \\ {\scriptstyle \varphi_T^{\gamma}}\downarrow & \swarrow & \downarrow \\ F_T^{\gamma}(Y/X) & \longrightarrow & F_T^{0+}(Y/X) \end{array} \tag{3-11}$$

as the inverse limit of

$$\begin{array}{ccc} \overline{Y}'_{\bar{x}'} & \xrightarrow{\varphi_{\bar{x}'}^{R'+}} & F_{\bar{x}'}^{R'+}(Y'/X') \\ {\scriptstyle \varphi_{\bar{x}'}^{R'}}\downarrow & \swarrow & \downarrow \\ F_{\bar{x}'}^{R'}(Y'/X') & \longrightarrow & Y_{\bar{x}} \end{array} \tag{3-12}$$

for commutative diagrams (3-10) satisfying the conditions (i)–(iii) and for $Y' = Y \times_X X'$.

We say that the ramification of Y over X at T is bounded by γ (resp. by $\gamma+$) if $F_T^{\infty}(Y/X) \to F_T^{\gamma}(Y/X)$ (resp. $F_T^{\infty}(Y/X) \to F_T^{\gamma+}(Y/X)$) is an injection.

By Lemma 3.2.2, the limit is a filtered limit.

Lemma 3.2.4. (1) *There exist a commutative diagram* (3-10) *satisfying the conditions* (i)–(iii), *an effective Cartier divisor $R' \subset X'$ satisfying $R' \cap U' = \varnothing$ and $x' \in R'$ such that Y' over X' satisfies* (RF) *for R' at the image $x' \in R'$ of the closed point of T'.*

(2) *For $x' \in R' \subset X'$ satisfying the condition in* (1), *the canonical morphism from* (3-11) *to* (3-12) *is an isomorphism. The diagram* (3-11) *is a diagram of finite sets.*

Proof. (1) By Lemma 3.2.2(1), after replacing X by a normal scheme of finite type over X if necessary, we may assume that there exist an effective Cartier divisor $R \subset X$ such that $v(R) = \gamma$ and a closed immersion $Y \to Q$ over X to a smooth scheme Q over X. Applying Theorem 1.2.5 and the remark following it to $Y \to X$ and to $Q^{(R)} \to X$ and taking the normalizations, we obtain a morphism $X' \to X$ of finite type of normal noetherian schemes satisfying the following properties: The morphism $X' \to X$ is the composition of a blow-up $X^* \to X$ with center supported in $X - U$ and a faithfully flat morphism $X' \to X^*$ of finite type such that $U' = X' \times_X U \to U$ is étale. The normalization of $Y \times_X X'$ is étale over X'. The morphism $Q'^{(R')} \to X'$ is flat and reduced. Hence Y' over X' satisfies the condition (RF) for R'. The morphism $T \to X$ is lifted to $T' \to X'$ by Lemma 1.2.6.

(2) By (1) and Lemma 3.2.2, among commutative diagrams (3-10) those such that the base change $Y' = Y \times_X X'$ over X' satisfies the condition (RF) for R' at x' are cofinal. Hence the assertion follows from Lemma 2.1.8. □

We study functoriality of the construction of $F_T^{\gamma}(Y/X)$ and $F_T^{\gamma+}(Y/X)$. We consider a commutative diagram

$$\begin{array}{ccccccccc} Y' & \longrightarrow & X' & \longleftarrow & T' & \longleftarrow & \bar{x}' & \longleftarrow & \bar{t}' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & X & \longleftarrow & T & \longleftarrow & \bar{x} & \longleftarrow & \bar{t} \end{array} \tag{3-13}$$

together with dense open subschemes $U \subset X$ and $U' \subset U \times_X X' \subset X'$ and $\gamma \in (0, \infty)_{\Gamma_{\mathbb{Q}}}$ and $\gamma' \in (0, \infty)_{\Gamma'_{\mathbb{Q}}}$ satisfying the following properties:

(i) $X' \to X$ is a morphism of normal noetherian schemes.

(ii) $T = \operatorname{Spec} A \to X$ and $T' = \operatorname{Spec} A' \to X'$ are U-external and U'-external morphisms for valuation rings $A \subsetneqq K = k(t)$ and $A' \subsetneqq K' = k(t')$ of the residue fields at $t \in U$ and $t' \in U'$. The morphism $T' \to T$ is faithfully flat.

(iii) $Y \to X$ and $Y' \to X'$ are quasifinite and flat morphisms such that $Y \times_X U \to U$ and $Y' \times_{X'} U' \to U'$ are étale.

(iv) $\gamma \leqq \gamma'$.

(v) $\bar{x}' \leftarrow \bar{t}'$ is a lifting of $\bar{x} \leftarrow \bar{t}$.

Lemma 3.2.5. *We keep the notation above.*

(1) *We have a commutative diagram*

$$\begin{array}{ccccccc} F_{T'}^{\infty}(Y'/X') & \longrightarrow & F_{T'}^{\gamma'+}(Y'/X') & \longrightarrow & F_{T'}^{\gamma'}(Y'/X') & \longrightarrow & F_{T'}^{0+}(Y'/X') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F_T^{\infty}(Y/X) & \longrightarrow & F_T^{\gamma+}(Y/X) & \longrightarrow & F_T^{\gamma}(Y/X) & \longrightarrow & F_T^{0+}(Y/X) \end{array} \tag{3-14}$$

of finite sets. Further if $\gamma < \gamma'$, we have an arrow

$$F_T^{\gamma'}(Y'/X') \to F_T^{\gamma+}(Y/X)$$

making the two triangles obtained by dividing the middle square commutative.

(2) *If the left square in (3-13) is cartesian and if $\gamma = \gamma'$, the vertical arrows in (3-14) are bijections.*

Proof. By Lemma 3.2.4(1), we may assume that there exists an effective Cartier divisor $R \subset X$ such that $R \cap U = \varnothing$ and $v(R) = \gamma$ and that Y over X satisfies the condition (RF) for R. Further by Lemma 3.2.4(1) and Lemma 3.2.2(3), we may assume that there exists an effective Cartier divisor $R' \subset X'$ such that $R' \cap U' = \varnothing$, $v'(R') = \gamma$ and $R' \geqq R \times_X X'$ and that Y' over X' satisfies the condition (RF) for R'. Then, by Lemma 3.2.4(2), we may identify $F_T^{\gamma}(Y/X) = F_{\bar{x}}^{R}(Y/X)$, $F_T^{\gamma+}(Y/X) = F_{\bar{x}}^{R+}(Y/X)$ and $F_{T'}^{\gamma'}(Y'/X') = F_{\bar{x}'}^{R'}(Y'/X')$, $F_{T'}^{\gamma'+}(Y'/X') = F_{\bar{x}'}^{R'+}(Y'/X')$.

(1) The assertion now follows from the functoriality of dilatation (3-4).

(2) In the notation above, we may further assume that $R' = R \times_X X'$. Hence the assertion follows from Lemma 2.1.8. □

Let T^h be the henselization at the closed point $x \in T$ and let $t^h \in T^h$ denote the generic point. Then, the absolute Galois group $D_T = \mathrm{Gal}(\bar{t}/t^h)$ acts on the specialization $\bar{x} \leftarrow \bar{t}$ of geometric points of T. Hence the commutative diagram (3-11) admits a canonical action of D_T.

Theorem 3.2.6. *Let the notation be as in Definition 3.2.3. Then, there exist an element $\beta_0 \in (0, \infty)_{\Gamma_{\mathbb{Q}}}$ and finite pairs $(\alpha_i, \beta_i)_{i \in I}$ of elements of $[0, \beta_0]_{\Gamma_{\mathbb{Q}}}$ satisfying the following properties* (i)–(iii):

(i) $[0, \beta_0]_{\Gamma_{\mathbb{Q}}} = \bigcup_{i \in I} [\alpha_i, \beta_i]_{\Gamma_{\mathbb{Q}}}$.

(ii) *For $\gamma > \beta_0$ (resp. $\gamma \geqq \beta_0$), $F_T^{\gamma}(Y/X) \leftarrow F_T^{\infty}(Y/X)$ (resp. $F_T^{\gamma+}(Y/X) \leftarrow F_T^{\infty}(Y/X)$) is an injection.*

(iii) *Let $i \in I$ and $\gamma \in (\alpha_i, \beta_i)_{\Gamma_{\mathbb{Q}}}$. Then, $F_T^{\gamma}(Y/X) \leftarrow F_T^{\beta_i}(Y/X)$ is an injection and $F_T^{\alpha_i+}(Y/X) \leftarrow F_T^{\gamma+}(Y/X)$ is an injection on the image of $F_T^{\infty}(Y/X)$.*

Proof. Since we may take base change, we may assume that $Y \to X$ is finite and that the normalization $\overline{Y} \to X$ is finite étale. Hence by Lemma 2.1.4, we may assume that there exists an effective Cartier divisor $R \subset X$ such that $R \cap U = \varnothing$ and $\overline{Y} \to Y^{(R)}$ is a closed immersion.

Set $\beta_0 = v(R) \in \Gamma$. Then, by Lemma 3.2.4, after replacing X if necessary, we may assume that Y over X satisfies the condition (RF) for R. Since $\overline{Y} \to Y^{(R)}$ is a closed immersion and $F_T^{\infty}(Y/X) \to \overline{Y}_{\bar{x}}$ is a bijection, $\overline{Y}_{\bar{x}} = F_T^{\infty}(Y/X) \to Y_{\bar{x}}^{(R)} = F_T^{\beta_0+}(Y/X)$ is an injection. For $\gamma > \beta_0$, the composition

$$F_T^{\beta_0+}(Y/X) \leftarrow F_T^{\gamma}(Y/X) \leftarrow F_T^{\gamma+}(Y/X) \leftarrow F_T^{\infty}(Y/X)$$

is an injection. Hence the condition (ii) is satisfied.

Let Q be a smooth scheme over X and let $Y \to Q$ be a closed immersion. As in Example 1.3.1, we define a semistable curve $C_R \to X$ by the effective Cartier divisor $R \subset X$. Define an effective Cartier divisor $\widetilde{R} \subset C_R$ to be the exceptional divisor. Applying Corollary 1.3.6 to $(Q \times_X C_R)^{[\widetilde{R}]} \to C_R \to X$ and taking the normalizations, we obtain a commutative diagram

$$\begin{array}{ccc} C_R & \longleftarrow & C' \\ \downarrow & & \downarrow \\ X & \longleftarrow & X' \end{array}$$

where $Y_{C'} = Y \times_X C'$ over C' satisfies the condition (RF) for $R' = \widetilde{R} \times_{C_R} C'$ and $C' \to X'$ is a semistable curve.

By Lemma 1.2.6, there exist a finite extension K' of K and a valuation ring A' such that $A = A' \cap K$ and that $T \to X$ is lifted to $T' = \operatorname{Spec} A' \to X'$. Let $x' \in X'$ denote the image of the closed point of T'. Further, for $\gamma \in [0, \beta_0]_{\Gamma_{\mathbb{Q}}}$, after replacing K' by a finite extension if necessary, we may assume that γ is an element of $[0, \beta_0]_{\Gamma'}$.

Let I_1 be the set of irreducible components of the fiber $C' \times_{X'} x'$. For $i \in I_1$, let $C_i \subset C' \times_{X'} x'$ denote the corresponding connected component. Let I_2 denote the set of singular points of the fiber $C' \times_{X'} x'$. For $i \in I_2$, let $z_i \subset C' \times_{X'} x'$ denote the corresponding singular point. Set $I = I_1 \amalg I_2$.

Since the assertion is étale local on X', we may assume that for each $i \in I_1$, there exists a section $s_i : X' \to C'$. For $i \in I_1$, set $\alpha_i = \beta_i = v'(s_i^* R') \in \Gamma'^{+}$. Since $\alpha_i = v'(\widetilde{R})$ for the composition $T' \to X' \to C' \to C_R$, we have $\alpha_i \in [0, \beta_0]_{\Gamma_{\mathbb{Q}}}$. For $i \in I_2$, if z_i is contained in two irreducible components C_{i_1} and C_{i_2} such that $\alpha_{i_1} \leqq \alpha_{i_2} \in \Gamma'^{+}$, we define $\alpha_i = \alpha_{i_1} \leqq \beta_i = \alpha_{i_2} \in [0, \beta_0]_{\Gamma_{\mathbb{Q}}}$. If z_i is contained in a unique irreducible component C_{i_1}, we define $\alpha_i = \beta_i = \alpha_{i_1} \in [0, \beta_0]_{\Gamma_{\mathbb{Q}}}$.

We show that the condition (i) is satisfied. Since $\alpha_i, \beta_i \in [0, \beta_0]_{\Gamma_{\mathbb{Q}}}$ for $i \in I$, we have the inclusion

$$[0, \beta_0]_{\Gamma_{\mathbb{Q}}} \supset \textstyle\bigcup_{i \in I} [\alpha_i, \beta_i]_{\Gamma_{\mathbb{Q}}}.$$

Let γ be an element of $[0, \beta_0]_{\Gamma_{\mathbb{Q}}}$. Then, we may assume $\gamma \in [0, \beta_0]_{\Gamma'}$. Then, since $T' \to X$ has a lifting to $T' \to C_R$ such that $v'(\widetilde{R}) = \gamma$ and since $C' \to C_R \times_X X'$ is proper and birational, there exists a unique lifting $T' \to C'$ of $T' \to C_R$ by the valuative criterion. If the image of the closed point by $T' \to C'$ is contained in the smooth part $C_i \cap C_x'^{\text{sm}}$ of an irreducible component $C_i \subset C'_x$ for $i \in I_1$, then we have $\gamma = \alpha_i$. If the image of the closed point by $T \to C'$ is the singular point $z_i \in C'_x$ for $i \in I_2$, then we have $\gamma \in [\alpha_i, \beta_i]_{\Gamma_{\mathbb{Q}}}$ by Corollary 1.3.4. Thus, the condition (i) is also satisfied.

We show that the condition (iii) is satisfied. For $i \in I_1$ or $i \in I_2$ such that $\alpha_i = \beta_i$, there is nothing to prove. Assume that $i \in I_2$ and z_i is contained in two irreducible components C_{i_1} and C_{i_2} such that $\alpha_i = \alpha_{i_1} < \beta_i = \alpha_{i_2} \in \Gamma'^{+}$ and let $\gamma \in (\alpha_i, \beta_i)_{\Gamma_{\mathbb{Q}}}$. Then, we may assume $\gamma \in (\alpha_i, \beta_i)_{\Gamma'}$. By Corollary 1.3.4, after replacing T' by an extension if necessary, we may take a morphism $T' \to C'$ such that the image of the closed point $x' \in T'$ is z_i and $v'(R') = \gamma$. Since $F^{\gamma}_{T}(Y/X) = F^{R'}_{\bar{z}_i}(Y_{C'}/C')$ and $F^{\gamma+}_{T}(Y/X) = F^{R'+}_{\bar{z}_i}(Y_{C'}/C')$ by Lemmas 3.2.5(2) and 3.2.4(2), the assertion follows from Proposition 3.1.8. □

We study some variants.

Let X be a normal noetherian scheme, let U be a dense open subscheme and let $V \to U$ be a finite étale morphism. We consider a cartesian diagram

$$\begin{array}{ccc} Y' & \longleftarrow & V \\ \downarrow & \square & \downarrow \\ X' & \longleftarrow & U \end{array} \tag{3-15}$$

of schemes of finite type over X satisfying the following conditions: the horizontal arrows are dense open immersions, X' is normal, $X' \to X$ is a proper birational morphism inducing the identity on U and Y' is finite flat over X'.

Let $A \subset K = k(t)$ be a valuation ring of the residue field at a point $t \in U$ and let $T = \operatorname{Spec} A \to X$ be a U-external morphism. Let $x \in T$ denote the closed point and let $\bar{x}$ be a geometric point above x. For $\gamma \in \Gamma_{\mathbb{Q},>0}$, we define

$$\begin{array}{ccc} F^{\infty}_{T}(V/U) & \longrightarrow & F^{\gamma+}_{T}(V/U) \\ \downarrow & \swarrow & \downarrow \\ F^{\gamma}_{T}(V/U) & \longrightarrow & F^{0+}_{T}(V/U) \end{array} \tag{3-16}$$

to be the inverse limit of

$$\begin{array}{ccc} F^{\infty}_{T}(Y'/X') & \longrightarrow & F^{\gamma+}_{T}(Y'/X') \\ \downarrow & \swarrow & \downarrow \\ F^{\gamma}_{T}(Y'/X') & \longrightarrow & F^{0+}_{T}(Y'/X'). \end{array} \tag{3-17}$$

Let T_V denote the normalization of T in $V \times_X T$. For X' in (3-15), let $X'_T \subset X'$ denote the reduced closed subscheme supported on the closure of $t \in U \subset X'$ and let $x' \in X'_T$ denote the image of the unique morphism $T \to X'$ lifting $T \to X$. Then, since $A = \varinjlim_{X' \to X} \mathcal{O}_{X'_T, x}$, we have $F^{0+}_{T}(V/U) = T_V \times_T \bar{x}$.

Lemma 3.2.7. *Suppose that the normalization T_V of T in $V \times_X T$ is finite and flat over T. Then, there exists a finite and flat $Y' \to X'$ such that $T_V = Y' \times_{X'} T$. For such $Y' \to X'$, the diagram (3-16) is isomorphic to (3-17).*

Proof. Since $A = \varinjlim_{X' \to X} \mathcal{O}_{X'_T, x}$ in the notation above, the existence of finite flat $Y' \to X'$ such that $T_V = Y' \times_{X'} T$ follows. By the flattening theorem [Raynaud and Gruson 1971, théorème (5.2.2)], such $Y' \to X'$ are cofinal among commutative diagrams (3-15). Hence the assertion follows from Lemma 3.2.5(2). □

For a normal noetherian scheme X, a formal $\mathbb{Q}$-linear combination $R = \sum_i r_i D_i$ with positive coefficients $r_i \geqq 0$ of irreducible closed subsets D_i of codimension 1 is called an effective $\mathbb{Q}$-Cartier divisor if a nonzero multiple is an effective Cartier divisor. The union $\bigcup_i D_i$ for $r_i > 0$ is called the support of R. For an open subset $U \subset X$, if U does not meet the support of R, we write $R \cap U = \varnothing$ by abuse of notation. For a U-external morphism $T = \operatorname{Spec} A \to X$, the valuation $v(R)$ is defined as an element of $[0, \infty)_{\Gamma_{\mathbb{Q}}}$.

Definition 3.2.8. Let X be a normal noetherian scheme and let $U \subset X$ be a dense open subscheme. Let Y be a quasifinite flat scheme over X such that $V = Y \times_X U \to U$ is finite étale. Let R be an effective $\mathbb{Q}$-Cartier divisor of X such that $U \cap R$ is empty and let $x \in X$ be a point contained in the support of R.

We say that the ramification of Y over X is bounded by R (resp. by $R+$) at x, if for every U-external morphism $T \to X$, the ramification of $Y \to X$ is bounded by $v(R)$ (resp. by $v(R)+$) in the sense of Definition 3.2.3.

Lemma 3.2.9. *Let the notation be as in Definition 3.2.8. Then, the following conditions* (1), (1′) *and* (2) *are equivalent*:

(1) *The ramification of $Y \to X$ is bounded by R (resp. by $R+$) in the sense of Definition 3.2.8.*

(1′) *The condition in Definition 3.2.8 with T restricted to be a discrete valuation ring is satisfied.*

(2) *For every morphism $f : X' \to X$ of finite type, of normal noetherian schemes such that $U' = U \times_X X' \to U$ is étale, that $R' = f^*R$ is an effective Cartier divisor and that $Y' = Y \times_X X' \to X'$ satisfies the condition (RF) in Definition 3.1.4 for R', the ramification of $Y' \to X'$ is bounded by R' (resp. by $R'+$) at every point of R' in the sense of Definition 3.1.4.*

Proof. (1′)⇒(2): Let $X' \to X$ be as in (2) and let $x' \in R'$ be a point. Let $X'_1 \to X'$ be the normalization of the blow-up at the closure of x'. Then, the local ring $A' = \mathcal{O}_{X'_1, x'_1}$ at the generic point x'_1 of an irreducible component of the inverse image of x' is a discrete valuation ring. The morphism $T' = \operatorname{Spec} A' \to X'_1 \to X'$ is U'-external and the image of the closed point is x'.

For $\gamma' = v'(R')$, by Lemma 3.2.4(2), the commutative diagram (3-12) is canonically identified with

$$\begin{array}{ccc}
F^{\infty}_{T'}(Y'/X') & \xrightarrow{\varphi^{\gamma'+}_{T'}} & F^{\gamma'+}_{T'}(Y'/X') \\
{\scriptstyle \varphi^{\gamma'}_{T'}}\downarrow & \swarrow & \downarrow \\
F^{\gamma'}_{T'}(Y'/X') & \longrightarrow & Y'_{\bar{x}'}
\end{array}$$

Further, this commutative diagram is canonically identified with (3-11) for $\gamma = v(R)$ by Lemma 3.2.5(2). Hence the assertion follows.

(2)⇒(1): Let $T \to X$ be a U-external morphism and let $\gamma = v(R)$. Then by Lemma 3.2.4(2), the commutative diagram (3-11) is canonically identified with (3-12). Hence the assertion follows.

(1) ⇒ (1′): This implication is obvious. □

Proposition 3.2.10. *Let the notation be as in Definition 3.2.8 and assume that the ramification of Y over X is bounded by $R+$. Assume that Y is locally of complete intersection over X and let*

$$\begin{array}{ccc} Q & \longleftarrow & Y \\ \downarrow & \square & \downarrow \\ P & \longleftarrow & X \end{array}$$

be a cartesian diagram of schemes over X such that P and Q are smooth over X, the vertical arrows are quasifinite and flat and the horizontal arrows are closed immersions.

Let X' be a normal noetherian scheme over X such that $R' = R \times_X X'$ is an effective Cartier divisor and $Y' = Y \times_X X'$ over X' satisfies the condition (RF) for R'.

Then, the morphism $Q'^{(R')} \to P'^{(R')}$ is étale on a neighborhood of $Q'^{(R')} \times_{X'} R'$.

This implies [Saito 2009, Lemma 1.13 6)⇒4)] since $Q'^{(R')} \times_{X'} R' \to P'^{(R')} \times_{X'} R'$ is finite by Lemma 2.1.7(1).

Proof. First, we show that we may assume that there exist a closed subscheme $Y'_0 \subset Y'$ étale over X', an integer $n \geqq 1$ and an effective Cartier divisor $D'_0 \subset R'$ satisfying the following conditions: We have an equality $R'_{Y'_0} = R'_{Y'}$ of underlying sets. Let $\mathcal{J}'_0 \subset \mathcal{O}_{R'_{Y'}}$ be the nilpotent ideal defining $R'_{Y'_0} \subset R'_{Y'}$. Then, we have $\mathcal{J}'^n_0 = 0$ and $(n+1)D'_0 = R'$.

Under the condition (RF), the formation of $Q'^{(R')} \to P'^{(R')}$ commutes with base change by Lemma 2.1.8 and Example 2.1.2(1). Since $Q'^{(R')}$ and $P'^{(R')}$ are flat over X', the étaleness of $Q'^{(R')} \to P'^{(R')}$ is checked fiberwise. Hence, we may take base change. Let $x' \in R'$ be a point and let $X'' \to X'$ be the normalization of the blow-up at the closure of x'. Then, there exists a point $x'' \in X''$ above x' such that the local ring $\mathcal{O}_{X'',x''}$ is a discrete valuation ring. Hence, by replacing X' by $\operatorname{Spec} \mathcal{O}_{X'',x''}$, we may assume that X' is the spectrum of a discrete valuation ring.

Then, we may assume that $Y' \subset Q'$ is a union of sections $X' \to Q'$. There exists a disjoint union $Y'_0 \subset Y'$ of sections such that we have an equality $R'_{Y'_0} = R'_{Y'}$ of underlying sets. Let $n \geqq 1$ be an integer satisfying $\mathcal{J}'^n_0 = 0$ in the notation above. After replacing X' by a ramified covering if necessary, there exists an effective Cartier divisor D'_0 of X' satisfying $(n+1)D'_0 = R'$.

The finite morphism $Y'^{(R')} \to X'$ is étale by Corollary 3.1.3. Hence by the existence of Y'_0, D'_0 and n and by Lemma 2.2.5, the $\mathcal{O}_{Y'^{(R')}}$-module $\mathcal{O}_{Y'^{(R')}} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/X}$ is annihilated by $\mathcal{I}_{nD'_0}$. Hence by Lemma 2.2.4, there exists an open neighborhood $W_1 \subset Q'^{(R')}$ of $Q'^{(R')} \times_{X'} R'$ such that $Q'^{(R')} \to P'^{(R')}$ is étale on $W_1 - (Q'^{(R')} \times_{X'} R')$.

The morphism $Q'^{(R')} \to P'^{(R')}$ is étale also on a neighborhood W_2 of $Y'^{(R')} \subset Q'^{(R')}$. Since the vector bundle $P'^{(R')} \times_{X'} R' \to R'$ has irreducible fibers, $W_2 \subset Q'^{(R')}$ is dense in the fiber of every point of R' by Proposition 1.1.5(1). Hence the assertion follows from Lemma 1.2.4. □

3.3. *Ramification groups.*

Theorem 3.3.1. *Let X be a connected normal noetherian scheme and let $U \subset X$ be a dense open subscheme. Let G be a finite group, $W \to U$ be a connected G-torsor and let C be the category of finite étale schemes over U trivialized by W. Assume that for every morphism $V_1 \to V_2$ of C, the morphism $Y_1 \to Y_2$ of normalizations of X in V_1 and in V_2 is locally of complete intersection.*

Let $t \in U$ and let $T = \operatorname{Spec} A \to X$ be a U-external morphism for a valuation ring $A \subsetneqq K = k(t)$. Let $\bar{x}$ (resp. $\bar{t}$) be a geometric point above the closed point x (resp. the generic point t) of T and let $\bar{x} \leftarrow \bar{t}$ be a specialization. Fix a lifting of $\bar{x}$ to the normalization T_W of T in $W \times_X T$ and let $I_{\bar{x}} \subset G$ be the inertia group at the image of the lifting of $\bar{x}$ to the normalization Y_W of X in W by $T_W \to Y_W$.

For an object V of C, let Y denote the normalization of X in V and consider the fiber functor sending V to $F^\infty_T(Y/X)$.

(1) *There exist decreasing filtrations $G^\gamma_T \supset G^{\gamma+}_T$ of G indexed by $\gamma \in (0, \infty)_{\Gamma_{\mathbb{Q}}}$ such that, for every object V of C, the canonical surjections $F^\infty_T(Y/X) \to F^{\gamma+}_T(Y/X) \to F^\gamma_T(Y/X)$ induce bijections*

$$G^{\gamma+}_T \backslash F^\infty_T(Y/X) \to F^{\gamma+}_T(Y/X), \quad G^\gamma_T \backslash F^\infty_T(Y/X) \to F^\gamma_T(Y/X). \tag{3-18}$$

For $I_{\bar{x}} = G^{0+}_T$, the mapping

$$G^{0+}_T \backslash F^\infty_T(Y/X) \to F^{0+}_T(Y/X) \tag{3-19}$$

is a bijection.

(2) *There exists a finite increasing sequence $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n$ of elements of $[0, \infty)_{\Gamma_{\mathbb{Q}}}$ such that we have*

$$\begin{aligned} G^{\alpha_{i-1}+} = G^\gamma = G^{\gamma+} = G^{\alpha_i} \quad & \textit{for } \gamma \in (\alpha_{i-1}, \alpha_i)_{\Gamma_{\mathbb{Q}}},\ 1 \leqq i \leqq n, \\ G^{\alpha_n+} = G^\gamma = G^{\gamma+} = 1 \quad & \textit{for } \gamma \in (\alpha_n, \infty)_{\Gamma_{\mathbb{Q}}}. \end{aligned} \tag{3-20}$$

(3) *Let $D_T \subset G$ be the decomposition group of T in $W \times_X T$. Then, D_T normalizes G^γ and $G^{\gamma+}$.*

Proof. (1) Let $V' \to V$ be a morphism in the category C and let $Y' \to Y$ be the morphism of normalizations of X. By Proposition 3.1.2 and Lemma 3.2.4(2), the diagram

$$\begin{array}{ccccccc} F_T^{\infty}(Y'/X) & \longrightarrow & F_T^{\gamma+}(Y'/X) & \longrightarrow & F_T^{\gamma}(Y'/X) & \longrightarrow & F_T^{0+}(Y'/X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F_T^{\infty}(Y/X) & \longrightarrow & F_T^{\gamma+}(Y/X) & \longrightarrow & F_T^{\gamma}(Y/X) & \longrightarrow & F_T^{0+}(Y/X) \end{array} \tag{3-21}$$

is a cocartesian diagram of surjections. Further, the functors F_T^{γ} and $F_T^{\gamma+}$ preserve disjoint unions. Hence by Proposition 1.4.2, we obtain filtrations $(G_T^{\gamma})_{\gamma}$ and $(G_T^{\gamma+})_{\gamma}$ indexed by $\gamma \in (0,\infty)_{\Gamma_{\mathbb{Q}}}$ characterized by the bijections (3-18). For $\gamma = 0$, the bijection (3-19) follows from $F_T^{0+}(Y/X) = Y_{\bar{x}}$.

(2) Since C has only finitely many connected objects and

$$F_T^{\infty}(Y/X) \to F_T^{\gamma+}(Y/X) \to F_T^{\gamma}(Y/X)$$

are surjections, the claim follows from Theorem 3.2.6.

(3) Since the surjections $F_T^{\infty}(Y/X) \to F_T^{\gamma+}(Y/X) \to F_T^{\gamma}(Y/X)$ are compatible with the actions of $D_T \subset G$, the subgroup $D_T \subset D_{\bar{x}}$ normalizes G^{γ} and $G^{\gamma+}$ by Corollary 1.4.4. □

By the definition of the filtrations, the ramification of Y/X at T is bounded by γ (resp. by $\gamma+$) if and only if the action of G_T^{γ} (resp. of $G_T^{\gamma+}$) on $F_T^{\infty}(Y/X)$ is trivial. By Corollary 1.4.3, the filtrations (G^{γ}) and $(G^{\gamma+})$ are compatible with quotients. We have the following functoriality. Let

$$\begin{array}{ccc} X' & \longleftarrow & T' \\ \downarrow & & \downarrow \\ X & \longleftarrow & T \end{array}$$

be a commutative diagram of schemes. Assume that $X' \to X$ is a morphism of normal connected noetherian schemes and let $U' \subset U \times_X X' \subset X'$ be a dense open subscheme. The horizontal arrows $T \to X$ and $T' \to X'$ are U-external and U'-external and the vertical arrow $T' \to T$ is faithfully flat. Let W' be a connected G'-torsor over U' for a finite group G' and let $W' \to W$ be a morphism over $U' \to U$ compatible with a morphism $G' \to G$ of finite groups. Assume that $W' \to U'$ satisfies the complete intersection property as in Theorem 3.3.1 and let $(G'^{\gamma'})$ and $(G'^{\gamma'+})$ be the filtrations of G' indexed by $\gamma' \in (0,\infty)_{\Gamma'_{\mathbb{Q}}}$. Then, for $\gamma \in (0,\infty)_{\Gamma_{\mathbb{Q}}}$, the morphism $G' \to G$ induces

$$G'^{\gamma} \to G^{\gamma}, \quad G'^{\gamma+} \to G^{\gamma+} \tag{3-22}$$

by the functoriality Lemma 3.2.5(1).

We consider a variant. Let $A \subsetneqq K$ be a valuation ring and let L be a finite Galois extension of K of Galois group G. We define a filtration of G by ramification groups under the following assumptions: For every intermediate extension $K \subset M \subset L$, the normalization A_M of A in M is a valuation ring finite flat and of complete intersection over A. There exist an irreducible normal noetherian scheme X such that K is the residue field at the generic point t and a morphism $T = \operatorname{Spec} A \to X$ extending $t \to X$.

Let $T \to X$ be as above and let M be an intermediate extension. Then by Lemma 3.2.2(1), $\operatorname{Spec} A_M$ is the limit of $T \times_{X'} Y'$ for normal schemes X' of finite type over X equipped with a lifting $T \to X'$ of $T \to X$ and finite flat schemes $Y' \to X'$ such that $U' = U \times_X X' \to U$ is an isomorphism for a dense open subscheme $U \subset X$ and $U' \times_{X'} Y' \to U'$ is a finite étale covering corresponding to M. Since A_M is assumed to be finite flat locally of complete intersection over A, there exists a finite flat scheme $Y'_M \to X'$ locally of complete intersection such that $U' \times_{X'} Y'_M \to U'$ is finite étale and $T \times_{X'} Y'_M = \operatorname{Spec} A_M$.

Thus, there exist a dense open subscheme $U \subset X$ and a normal scheme X' of finite type over X satisfying the following conditions: The morphism $U' = U \times_X X' \to U$ is an isomorphism. The morphism $T \to X$ is lifted to $T \to X'$. For every intermediate extension M, there exists a finite flat scheme $Y'_M \to X'$ locally of complete intersection such that $U' \times_{X'} Y'_M \to U'$ is finite étale and $T \times_{X'} Y'_M = \operatorname{Spec} A_M$.

Then applying Theorem 3.3.1, we obtain filtrations (G_T^γ) and $(G_T^{\gamma+})$ by normal subgroups of $G = D_T$ indexed by $(0, \infty)_{\Gamma_\mathbb{Q}}$.

In the rest of the article, we consider the case where $X = T = \operatorname{Spec} \mathcal{O}_K$ for a complete discrete valuation ring $\mathcal{O}_K$. For a finite Galois extension of the fraction field K of the Galois group G, the decreasing filtrations $(G^r)_{r>0}$ and $(G^{r+})_{r \geqq 0}$ by normal subgroups indexed by rational numbers are defined.

Lemma 3.3.2. *Let K be a complete discrete valuation field and let L be a finite Galois extension of the Galois group $G = \operatorname{Gal}(L/K)$. Then, the filtration by ramification groups of G defined in* [Abbes and Saito 2002] *is the same as that defined here.*

Proof. Let M be an intermediate extension and let $Y = \operatorname{Spec} \mathcal{O}_M \to Q = \mathbb{A}^n_{\mathcal{O}_K}$ be a closed immersion defined by taking a system of generators of $\mathcal{O}_M$ over $\mathcal{O}_K$ as in Example 2.1.1(1). Then, the affinoid varieties used in the definition in [Abbes and Saito 2002] are the generic fibers of the formal completions of dilatations of $Q^{(r)}$. Since the geometric connected components of the affinoid varieties are canonically identified with those of the closed fibers as in Remark 1.1.2, the assertion follows. □

Let L be a finite separable extension of degree n of K and let $Y = \operatorname{Spec} \mathcal{O}_L$ for the integer ring $\mathcal{O}_L$. We recall the classical case where $\mathcal{O}_L$ is generated by one element over $\mathcal{O}_K$, using the Herbrand function. Take a closed immersion

$$Y = \operatorname{Spec} \mathcal{O}_L \to Q = \mathbb{A}^1_X = \operatorname{Spec} \mathcal{O}_K[T],$$

and let $P \in \mathcal{O}_K[T]$ be the monic polynomial such that we have an isomorphism $\mathcal{O}_K[T]/(P) \to \mathcal{O}_L$.

Let K' be a finite separable extension containing the Galois closure of L and let $X' = \operatorname{Spec} \mathcal{O}_{K'}$. Let $v' : K' \to \mathbb{Q} \cup \{\infty\}$ be the valuation extending the normalized valuation of K. Let $r > 0$ be a rational number in the image of v' and let $R' \subset X'$ be the effective Cartier divisor such that $v'(R') = r$. Let $Q' \supset Y'$ be the base change of $Q \supset Y$ by $X' \to X$ and let $Q'^{(r)} = Q'^{(R')}$ denote the dilatation. We compute $Q'^{(r)}$ using the Herbrand function, whose definition we briefly recall.

Decompose P as $P = \prod_{i=1}^n (T - a_i)$ in $\mathcal{O}_{K'}[T]$ and set $b_i = a_i - a_n \in \mathcal{O}_{K'}$. Set $P(T_1 + a_n) = \prod_{i=1}^n (T_1 - b_i) = T_1^n + c_1 T_1^{n-1} + \cdots + c_{n-1} T_1$ in $\mathcal{O}_{K'}[T_1]$. Changing the numbering if necessary, we assume that the valuations $s_i = v'(b_i) \in \mathbb{Q}$ are increasing in i. Note that the increasing sequence $s_0 = 0 \leqq s_1 \leqq \cdots \leqq s_{n-1} < s_n = \infty$ is independent of the choice of a_n. The valuation $v'(c_{n-1}) = \sum_{k=1}^{n-1} s_k$ equals the valuation $v'(D_{L/K})$ of the different $D_{L/K}$. It is further equal to the length of the $\mathcal{O}_L$-module $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ divided by the ramification index $e_{L/K}$ by [Serre 1968, Chapter III §6 corollaire 2 à la proposition 11].

The largest piecewise linear convex continuous function

$$p : [0, n-1] \to [0, v'(D_{L/K})]$$

such that the graph is below the points $(0, 0)$ and $(k, v(c_k))$ for $k = 1, \ldots, n-1$ is defined by

$$p(x) = \sum_{i=1}^{k-1} s_i + s_k(x - k + 1) \tag{3-23}$$

on $[k-1, k]$ for $k = 1, \ldots, n-1$. The graph of p is the *Newton polygon* of the polynomial $P(T_1 + a_n)$. The *Herbrand function* $\varphi : [0, \infty) \to [0, \infty)$ is a piecewise linear concave continuous function defined by

$$\varphi(s) = \sum_{i=1}^{n-1} \min(s_i, s) + s. \tag{3-24}$$

We have

$$\varphi(s) = \sum_{i=1}^{k-1} (n - i + 1) \cdot (s_i - s_{i-1}) + (n - k + 1) \cdot (s - s_{k-1}) \tag{3-25}$$

on $[s_{k-1}, s_k)$ for $k = 1, \ldots, n$.

Example 3.3.3. Let $s \in (s_{k-1}, s_k]_{\mathbb{Q}}$, $r = \varphi(s)$ and let t be an element of a finite separable extension K' of K such that $v(t) = s$. By (3-25) and Example 2.1.2, $Q'^{(r)}$ is obtained as an iterated dilatation defined inductively by $Q'_0 = Q'$,

$$Q'_i = Q'^{((n-(i-1))\cdot(s_i - s_{i-1}))}_{i-1} \quad \text{for } 0 < i < k \qquad \text{and} \qquad Q'^{(r)} = Q'^{((n-(k-1))\cdot(s-s_{k-1}))}_{k-1}.$$

Hence $Q'^{(r)} \to X'$ is smooth. Let $C \subset Q'^{(r)} \times_{X'} x'$ be the connected component meeting the section $s'_n : X' \to Q'^{(r)}$ lifting $s_n : X \to Q$ defined by $T = a_n$. Then, $\operatorname{Spec} \mathcal{O}_{K'}[T']$ for $T' = T_1/t$ is a neighborhood of $C \subset Q'^{(r)}$. Further, on $\operatorname{Spec} \mathcal{O}_{K'}[T']$, the closed subscheme $Y'^{(r)} \subset Q'^{(r)}$ is defined by $\prod_{i=k}^{n}(T' - b_i/t)$.

Consequently, the surjections

$$\overline{Y}_{\bar{x}} = \{a_1, \dots, a_n\} \to F^r_X(Y/X)$$

and

$$\overline{Y}_{\bar{x}} \to F^{r+}_X(Y/X)$$

are given by the equivalence relations $v'(a_i - a_j) \geqq s$ and $v'(a_i - a_j) > s$, respectively. In particular, $r_{L/K} = \varphi(s_{n-1}) = v'(D_{L/K}) + s_{n-1}$ is the unique rational number r such that the ramification of Y over X is bounded by $r+$ but not by r.

We give a slightly simplified proof of the proposition below giving characterizations of unramified extensions and tamely ramified extensions.

Lemma 3.3.4 [Serre 1968, chapitre III §7 proposition 13; Abbes and Saito 2002, Proposition A.3]. *Let L be a finite separable extension of a complete discrete valuation field K. Assume that $\mathcal{O}_L$ is generated by one element over $\mathcal{O}_K$ and let $r_{L/K} = \varphi(s_{n-1}) = v'(D_{L/K}) + s_{n-1}$ be as in Example 3.3.3.*

(1) *The following conditions are equivalent*:
  (i) *L is an unramified extension of K.*
  (ii) $r_{L/K} = 0$.
  (iii) $r_{L/K} < 1$.

(2) *The following conditions are equivalent*:
  (i) *L is a tamely ramified extension of K.*
  (ii) $r_{L/K} = 0$ *or* 1.
  (iii) $r_{L/K} \leqq 1$.

Proof. By [Abbes and Saito 2002, Proposition A.3], we have $v'(D_{L/K}) \geqq 1 - 1/e_{L/K}$ and equality holds if and only if L/K is tamely ramified. We have $s_{n-1} \geqq 0$ and equality holds if and only if L is unramified. If L is ramified, we have $s_{n-1} \geqq 1/e_{L/K}$ and equality holds if and only if L is tamely ramified. The assertions follows from these observations. □

Proposition 3.3.5 [Abbes and Saito 2002, Proposition 6.8]. *Let L be a finite separable extension of a complete discrete valuation field K.*

(1) *The following conditions are equivalent*:

  (i) *L is an unramified extension of K.*

  (ii) *The ramification of L over K is bounded by* 1.

(2) *The following conditions are equivalent*:

  (i) *L is a tamely ramified extension of K.*

  (ii) *The ramification of L over K is bounded by* $1+$.

Proof. For both (1) and (2), (i)$\Rightarrow$(ii) follows from Example 3.3.3 and Lemma 3.3.4, since $\mathcal{O}_L$ is generated by one element over $\mathcal{O}_K$.

We show (ii)$\Rightarrow$(i).

(1) Let L be a finite separable extension such that the ramification over K is bounded by 1 and assume that L was ramified over K.

Let G be the Galois group of a Galois closure of L over K and let $1 \subsetneqq I \subset G = \mathrm{Gal}(L/K)$ be the inertia subgroup. By replacing K and L by the subextensions corresponding to I and to a maximal subgroup $H \subsetneqq I$, we may assume that L is a cyclic extension of prime degree since I is solvable.

Then, either the ramification index $e_{L/K}$ is 1 and the residue extension is a purely inseparable extension of degree p or L is totally ramified extension. Hence $\mathcal{O}_L$ is generated by one element and the assertion follows from Example 3.3.3 and Lemma 3.3.4.

(2) If the integer ring $\mathcal{O}_L$ is generated by one element over $\mathcal{O}_K$, the assertion follows from Example 3.3.3 and Lemma 3.3.4. We prove the general case by reducing to this case by contradiction.

Let L be a finite separable extension such that the ramification over K is bounded by $1+$ and assume that L was wildly ramified over K.

Let G be the Galois group of a Galois closure of L over K and let $1 \subsetneqq P \subset I \subset G = \mathrm{Gal}(L/K)$ be the wild inertia subgroup and the inertia subgroup. By replacing K and L by the subextensions corresponding to I and to a maximal subgroup $H \subsetneqq P$, we may assume that $[L : K] = mp$ for an integer m prime to p.

Since an algebraic closure $\widetilde{F}$ of the residue field F of K is a perfect closure of the separable closure, we may construct a henselian separable algebraic extension $\widetilde{K}$ of ramification index 1 of residue field $\widetilde{F}$ as a limit $\varinjlim K_\lambda$ of finite separable extensions of ramification index 1. Since the composition $L\widetilde{K}$ is a totally ramified extension of $\widetilde{K}$, there exists a finite separable extension $K' = K_\lambda$ of ramification index 1 such that $L' = LK_\lambda$ is a totally ramified extension of K'.

By the functoriality (3-22), the ramification of L' over K' is bounded by $1+$. Since L' is totally ramified over K', the integer ring $\mathcal{O}_{L'}$ is generated by one element over $\mathcal{O}_{K'}$. Hence, L' is tamely ramified over K' and we have $[L' : K'] = m$.

By construction, there exists a sequence $K \subset K_0 \subset K_1 \subset \cdots \subset K_n = K'$ such that K_0 is an unramified extension of K and that K_i is an extension of K_{i-1} of degree p of ramification index 1 with inseparable residue field extension for each $i = 1, \dots, n$. Since $[LK_0 : K_0] = mp$, we have $n > 0$. By taking the smallest such n, we may assume $[LK_{n-1} : K_{n-1}] = mp$.

Further, by the functoriality (3-22), we may replace K and L by K_{n-1} and LK_{n-1}. Hence, we may assume that $[K' : K] = p$ and $K' \subset L$. Since $[K' : K] = p$, the integer ring $\mathcal{O}_{K'}$ is generated by one element over $\mathcal{O}_K$. Since $K' \subset L$, the ramification of K' over K is bounded by $1+$. Hence K' is tamely ramified over K. This contradicts the assumption that the residue field extension of K' over K is inseparable. □

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