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Grothendieck–Messing deformation theory for varieties of K3 type

Andreas Langer and Thomas Zink

Let R be an artinian local ring with perfect residue class field k. We associate to certain 2-displays over the small ring of Witt vectors $\widehat{W}(R)$ a crystal on $\operatorname{Spec} R$.

Let X be a scheme of K3 type over $\operatorname{Spec} R$. We define a perfect bilinear form on the second crystalline cohomology group X which generalizes the Beauville–Bogomolov form for hyper-Kähler varieties over $\mathbb{C}$. We use this form to prove a lifting criterion of Grothendieck–Messing type for schemes of K3 type. The crystalline cohomology $H^2_{\text{crys}}(X/\widehat{W}(R))$ is endowed with the structure of a 2-display such that the Beauville–Bogomolov form becomes a bilinear form in the sense of displays. If X is ordinary, the infinitesimal deformations of X correspond bijectively to infinitesimal deformations of the 2-display of X with its Beauville–Bogomolov form. For ordinary K3 surfaces X/R we prove that the slope spectral sequence of the de Rham–Witt complex degenerates and that $H^2_{\text{crys}}(X/W(R))$ has a canonical Hodge–Witt decomposition.

Introduction

Displays were introduced in [Zink 2002] to classify formal p-divisible groups over a ring R where p is nilpotent. They form a subcategory of the exact tensor category of higher displays constructed in [Langer and Zink 2007]. Such displays arise naturally for a certain class of projective smooth schemes over R (abelian

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schemes, K3 surfaces, complete intersections) and equip the crystalline cohomology with an additional structure, in particular the existence of divided Frobenius homomorphisms which satisfy a relative version of Fontaine's strong divisibility condition.

Let p be a prime number such that $p \geq 3$. Let R be an artinian local ring with perfect residue field k of characteristic p. We denote by $\widehat{W}(R)$ the small Witt ring [Zink 2001a]. Displays over the small Witt ring are called Dieudonné displays. They classify all p-divisible groups over R [Zink 2001a; Lau 2014]. In particular a Dieudonné display defines a crystal of locally free modules on the site $(\operatorname{Spec} R/W(k))_{\mathrm{crys}}$. This crystal has an elementary description in terms of linear algebra. Moreover, there is a Grothendieck–Messing criterion for lifting Dieudonné displays.

In [Langer and Zink 2007] we associated to a projective variety X/R whose cohomology has good base change properties a display of higher degree over $W(R)$. We define in this paper under more restrictive conditions on X listed at the beginning of Section 2 a Dieudonné 2-display associated to X. This can be regarded as an additional structure on the crystalline cohomology $H^2_{\mathrm{crys}}(X/\widehat{W}(R))$ (Proposition 19). Let $R' \to R$ be a pd-thickening in the category of local artinian rings with residue field k. This means that $R' \to R$ is a surjective ring homomorphism and that its kernel is endowed with divided powers which are compatible with the canonical divided powers on the ideal pR'. We define the notion of a relative Dieudonné 2-display with respect to such a pd-thickening. We obtain a crystal of relative Dieudonné 2-displays which may be regarded as an additional structure on the crystal

$$R' \mapsto H^2_{\mathrm{crys}}(X/\widehat{W}(R')). \tag{1}$$

In Section 3 we define schemes of K3 type. The main examples are the Hilbert schemes of zero-dimensional subschemes of K3 surfaces denoted by K3n in the literature. We introduce for a scheme of K3 type $X \to T$ a Beauville–Bogomolov form (Definition 23) on the de Rham cohomology $H^2_{\mathrm{DR}}(X/T)$. It coincides with the usual Beauville–Bogomolov form if $T = \operatorname{Spec} \mathbb{C}$. We prove under mild conditions that this form is horizontal for the Gauss–Manin connection (Proposition 26). In the notation above we obtain for a scheme X of K3 type over the artinian ring R a perfect pairing on the crystalline cohomology

$$H^2_{\mathrm{crys}}(X/\widehat{W}(R')).$$

In analogy to the Grothendieck–Messing lifting theory we have Theorem 31:

Theorem. *The liftings of X to R' correspond to selfdual liftings of the Hodge filtration.*

This is proved in the case $R = k$ and $R' = W_n(k)$ for K3 surfaces in [Deligne 1981b]. In this case the Beauville–Bogomolov form coincides with the cup product. The Beauville–Bogomolov form makes the crystal of Dieudonné 2-displays (1) selfdual.

Let X_0/k be a scheme of K3 type such that the Frobenius induces a Frobenius linear bijection on the k-vector space $H^2(X_0, \mathcal{O}_{X_0})$. We say that X_0 is F-ordinary. Let $f : X \to \operatorname{Spec} R$ be a deformation of X_0. We prove that there is a unique functorial extension of the Dieudonné 2-display $H^2_{\text{crys}}(X/\widehat{W}(R))$ to a crystal of relative Dieudonné 2-displays (1). In particular the crystal $R^2 f_{\text{crys},*}\mathcal{O}_X^{\text{crys}}$ in $(X/W(k))_{\text{crys}}$ [Berthelot 1974, Chapitre 5, Proposition 3.6.4] can be constructed from this Dieudonné 2-display. Then we obtain from the Grothendieck–Messing criterion Theorem 36:

Theorem. *Assume that X_0 is F-ordinary and lifts to a smooth projective scheme over $W(k)$. The functor which associates to a deformation X/R of X_0 the Dieudonné 2-display $H^2_{\text{crys}}(X/\widehat{W}(R))$ with its Beauville–Bogomolov form is an equivalence to the category of selfdual deformations of the Dieudonné 2-display $H^2_{\text{crys}}(X_0/W(k))$ endowed with the Beauville–Bogomolov form.*

In Section 5 we exhibit the second crystalline cohomology of an ordinary K3 surface X over the usual Witt ring $W(R)$ and its associated display and prove (Theorem 40) a Hodge–Witt decomposition which induces a decomposition of the display into a direct sum of displays attached to the formal Brauer group $\widehat{\mathrm{Br}}_X$, the étale part of the extended Brauer group and the Cartier dual of $\widehat{\mathrm{Br}}_X$, shifted by -1. The proof uses the relative de Rham–Witt complex of [Langer and Zink 2007]. We show that the hypercohomology spectral sequence of the relative de Rham–Witt complex degenerates.

1. Displays

We fix a prime number p.

Definition 1. A frame $\mathcal{F}$ consists of the following data $\mathcal{F} = (W, J, R, \sigma, \dot{\sigma})$. Here W is a commutative ring, $J \subset W$ is an ideal, and $R = W/J$ is the factor ring. The map $\sigma : W \to W$ is a ring homomorphism and $\dot{\sigma} : J \to W$ is a σ-linear homomorphism of W-modules. We assume that the following conditions are satisfied:

(i) The ideal J and the prime number p are contained in the Jacobson radical of W.

(ii) For each $s \in W$ we have

$$\sigma(s) \equiv s^p \mod pW.$$

(iii) The set $\dot{\sigma}(J)$ generates W as an W-module.

There is a unique element $\theta \in W$ such that $\sigma(\eta) = \theta\dot{\sigma}(\eta)$ for all $\eta \in J$. We will assume that $\theta = p$. In the following the ring W is local.

Suppose $f : M \to N$ is a σ-linear map of W-modules. Then we define a new σ-linear map

$$\tilde{f} : J \otimes_S M \to N, \quad \tilde{f}(\eta \otimes m) = \dot{\sigma}(\eta) f(m) \quad \text{for } \eta \in J.$$

Definition 2. An $\mathcal{F}$-predisplay $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ consists of the following data:

(1) A sequence of W-modules P_i for $i \geq 0$.

(2) Two sequences of W-module homomorphisms

$$\iota_i : P_{i+1} \to P_i, \quad \alpha_i : J \otimes_S P_i \to P_{i+1}, \quad \text{for } i \geq 0.$$

(3) A sequence of σ-linear maps for $i \geq 0$

$$F_i : P_i \to P_0.$$

These data satisfy the following properties:

(i) Consider the following morphisms:

$$\begin{array}{ccc} J \otimes P_i & \xrightarrow{\alpha_i} & P_{i+1} \\ {\scriptstyle \mathrm{id}_J \otimes \iota_{i-1}} \big\downarrow & & \big\downarrow {\scriptstyle \iota_i} \\ J \otimes P_{i-1} & \xrightarrow[\alpha_{i-1}]{} & P_i \end{array}$$

The composites $\iota_i \circ \alpha_i$ and $\alpha_{i-1} \circ (\mathrm{id}_J \otimes \iota_{i-1})$ are the multiplication $J \otimes P_i \to P_i$ for each i where the composites make sense.

(ii) $F_{i+1} \circ \alpha_i = \widetilde{F_i}$.

If we have only the data $\mathcal{P} = (P_i, \iota_i, \alpha_i)$ such that property (i) holds we say that $\mathcal{P}$ is an $\mathcal{F}$-module.

We will denote the morphisms in the category of $\mathcal{F}$-predisplays and in the category of $\mathcal{F}$-modules by

$$\mathrm{Hom}_{\mathcal{F}}(\mathcal{P}, \mathcal{P}') \quad \text{and} \quad \mathrm{Hom}_{\mathcal{F}\text{-mod}}(\mathcal{P}, \mathcal{P}')$$

respectively.

This is a generalization of Definition 2.1 of [Langer and Zink 2007]. The arguments there imply

$$F_i(\iota_i(x)) = pF_{i+1}(x) \quad \text{for } x \in P_{i+1}.$$

If $i, k \geq 0$ we will denote the map

$$\iota_{i+k-1} \circ \cdots \circ \iota_i : P_{i+k} \to P_i$$

simply by ι^{iter}.

We are going to associate a frame to the following situation. Let R and S be p-adic rings. Let

$$S \to R \tag{2}$$

be a surjective ring homomorphism such that the kernel $\mathfrak{a}$ is endowed with divided powers. We will assume that $\mathfrak{a}$ becomes nilpotent in the ring S/pS.

Let $W(S) \xrightarrow{\boldsymbol{w}_0} S \to R$ be the composite with the Witt polynomial $\boldsymbol{w}_0$. Let $\mathcal{J}$ be the kernel of this composite. We set $I_S = VW(S)$ and we denote by $\tilde{\mathfrak{a}} \subset W(S)$ the logarithmic Teichmüller representatives of elements of $\mathfrak{a}$. These are the elements $\tilde{a} := \log^{-1}[a, 0, 0, \ldots]$ in the notation of [Zink 2002, (48)]. Then we have a direct decomposition of $\mathcal{J}$ as a sum of two ideals of $W(S)$:

$$\mathcal{J} = \tilde{\mathfrak{a}} \oplus I_S.$$

We will denote the Frobenius endomorphism F of the ring of Witt vectors $W(S)$ also by σ. We have

$$\sigma(\tilde{\mathfrak{a}}) = 0, \quad I_S \cdot \tilde{\mathfrak{a}} = 0.$$

We define a map

$$\dot{\sigma} : \mathcal{J} \to W(S), \quad \dot{\sigma}(a + {}^V\xi) = \xi, \quad a \in \tilde{\mathfrak{a}},\ \xi \in W(S).$$

This map is σ-linear. We note that the ideal $\mathcal{J}$ inherits from $\mathfrak{a}$ divided powers which extend the natural divided powers on $I_S \subset W(S)$ [Zink 2002, (89)].

Let us assume that the divided powers on $\mathfrak{a}$ are compatible with the canonical divided powers on pS. This implies the canonical divided powers on $pW(S) + VW(S)$ are compatible with the divided powers on $\tilde{\mathfrak{a}}$ given by the isomorphism with $\mathfrak{a}$. In this sense $W(S) \to W(R/pR)$ is then a pd-thickening.

We call $\mathcal{W}_{S/R} = (W(S), \mathcal{J}, R, \sigma, \dot{\sigma})$ the relative Witt frame. If $S = R$ we call it the Witt frame and write $\mathcal{W}_R$. If $S \to R$ is fixed as above we call a $\mathcal{W}_R$-predisplay simply a predisplay and a $\mathcal{W}_{S/R}$-predisplay a relative predisplay.

Suppose S and R are artinian local rings with perfect residue field of characteristic $p \geq 3$. Let $S \to R$ be a surjective homomorphism with kernel $\mathfrak{a}$. We assume that $\mathfrak{a}$ is endowed with nilpotent divided powers. We call this a nilpotent pd-thickening. In this situation we can also use the small rings of Witt vectors $\widehat{W}(R)$ and $\widehat{W}(S)$ defined in [Zink 2001a] to define a version of the relative Witt frame. For this we use that the divided Witt polynomial defines an isomorphism

$$\widehat{W}(\mathfrak{a}) \to \bigoplus_{i \geq 0} \mathfrak{a}$$

by [Zink 2002, Remark after Corollary 82]. By this isomorphism the logarithmic Teichmüller elements are defined. Then we obtain the small relative Witt frame $\widehat{\mathcal{W}}_{S/R} = (\widehat{W}(S), \widehat{\mathcal{J}}, R, \sigma, \dot{\sigma})$, where $\widehat{\mathcal{J}} = \tilde{\mathfrak{a}} \oplus V\widehat{W}(S)$ is the kernel of the homomorphism $\widehat{W}(S) \to R$.

These frames are endowed with a Verjüngung: Let $\mathcal{F} = (W, J, R, \sigma, \dot{\sigma})$ be a frame. The structure of a Verjüngung on $\mathcal{F}$ consists of two W-module homomorphisms

$$\nu : J \otimes_W J \to J, \quad \pi : J \to J$$

such that ν is associative. We will also write

$$\nu(y_1 \otimes y_2) = y_1 * y_2, \quad y_1, y_2 \in J.$$

The iteration of ν is well-defined:

$$\nu^{(k)} : J \otimes_W \cdots \otimes_W J \to J,$$

where the tensor product on the left-hand side has k factors. We have $\nu^{(2)} = \nu$ and $\nu^{(1)} = \mathrm{id}_J$. The image of $\nu^{(k)}$ is an ideal $J_k \subset W$. By associativity, $\nu^{(k+1)}$ factors through a map

$$J \otimes_W J_k \to J_{k+1}. \tag{3}$$

We also require, that the following properties hold:

$$\begin{gathered}\pi(y_1 * y_2) = y_1 y_2, \quad \text{where } y_1, y_2 \in J,\\ \dot{\sigma}(y_1 * y_2) = \dot{\sigma}(y_1)\dot{\sigma}(y_2),\\ \dot{\sigma}(\pi(y_1)) = \sigma(y_1),\\ (\operatorname{Ker} \dot{\sigma}) \cap (\operatorname{Ker} \pi) = 0.\end{gathered} \tag{4}$$

These properties imply

$$y_1 * y_2 = y_2 * y_1, \quad \pi(y_1) * y_2 = y_1 y_2.$$

Indeed, for each of these equations the difference between the two sides lies in $(\operatorname{Ker} \dot{\sigma}) \cap (\operatorname{Ker} \pi)$.

In the case of the frame $\mathcal{W}_{S/R}$ we define the Verjüngung as

$$(a_1 + {}^V\xi_1) * (a_2 + {}^V\xi_2) = a_1 a_2 + {}^V(\xi_1 \xi_2), \quad \pi(a + {}^V\xi) = a + p\,{}^V\xi. \tag{5}$$

Then we have

$$J_i = \tilde{\mathfrak{a}}^i + VW(S).$$

The map (3) is given by the first formula of (5).

In the same way we obtain a Verjüngung for the frames $\widehat{\mathcal{W}}_{S/R}$. These are the only examples we are interested in.

We define the notion of a standard display over a frame $\mathcal{F}$ with Verjüngung ν, π. In the case of $\mathcal{W}_R$ it coincides with the notion given in [Langer and Zink 2007].

A standard datum consists of a sequence of finitely generated projective W-modules $L_0, \ldots, L_d$ and σ-linear homomorphisms

$$\Phi_i : L_i \to L_0 \oplus \cdots \oplus L_d.$$

We assume that

$$\Phi_0 \oplus \cdots \oplus \Phi_d : L_0 \oplus \cdots \oplus L_d \to L_0 \oplus \cdots \oplus L_d$$

is a σ-linear isomorphism.

We set

$$P_i = J_i L_0 \oplus J_{i-1} \cdots \oplus J L_{i-1} \oplus L_i \oplus \cdots \oplus L_d.$$

The map ι is defined by the following diagram:

$$\begin{array}{ccccccccccc}
J_{i+1}L_0 & \oplus & J_i L_1 & \oplus\cdots\oplus & JL_i & \oplus & L_{i+1} & \oplus\cdots\oplus & L_d \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & \cdots & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\mathrm{id}} & \cdots & \downarrow{\scriptstyle\mathrm{id}} \\
J_i L_0 & \oplus & J_{i-1}L_1 & \oplus\cdots\oplus & L_i & \oplus & L_{i+1} & \oplus\cdots\oplus & L_d
\end{array}$$

We remark that $\pi(J_{i+1}) \subset J_i$ because of the formula

$$\pi(y_1 * y_2 * \cdots * y_{i+1}) = y_1(y_2 * \cdots * y_{i+1}).$$

The homomorphism $\alpha_i : J \otimes P_i \to P_{i+1}$ is defined as follows:

$$\begin{array}{ccccccccccc}
J\otimes J_i L_0 & \oplus & J\otimes J_{i-1}L_1 & \oplus\cdots\oplus & J\otimes L_i & \oplus & J\otimes L_{i+1} & \oplus\cdots\oplus & J\otimes L_d \\
\downarrow{\scriptstyle\nu} & & \downarrow{\scriptstyle\nu} & \cdots & \downarrow{\scriptstyle\mathrm{mult}} & & \downarrow{\scriptstyle\mathrm{mult}} & \cdots & \downarrow{\scriptstyle\mathrm{mult}} \\
J_{i+1}L_0 & \oplus & J_i L_1 & \oplus\cdots\oplus & JL_i & \oplus & L_{i+1} & \oplus\cdots\oplus & L_d.
\end{array}$$

Here the arrows denoted by ν are induced by the maps (3), and mult denotes the multiplication.

Finally we define σ-linear maps $F_i : P_i \to P_0$:

$$\begin{array}{ccccccccccc}
J_i L_0 & \oplus\cdots\oplus & JL_{i-1} & \oplus & L_i & \oplus & L_{i+1} & \oplus & L_{i+2} & \cdots \\
\downarrow{\scriptstyle\widetilde{\Phi}_0} & \cdots & \downarrow{\scriptstyle\widetilde{\Phi}_{i-1}} & & \downarrow{\scriptstyle\Phi_i} & & \downarrow{\scriptstyle p\Phi_{i+1}} & & \downarrow{\scriptstyle p^2\Phi} & \cdots \\
L_0 & \oplus\cdots\oplus & L_{i-1} & \oplus & L_i & \oplus & L_{i+1} & \oplus & L_{i+2} & \cdots
\end{array}$$

The maps $\widetilde{\Phi}_j$ are by definition

$$\widetilde{\Phi}_j(\eta\ell_j) = \dot{\sigma}(\eta)\Phi_j(\ell_j) \quad \text{for } \eta \in J_j,\ \ell_j \in L_j,\ j < i.$$

The data $(P_i, \iota_i, \alpha_i, F_i)$ meet the requirements for a predisplay. This predisplay is called the $\mathcal{F}$-display of a standard datum.

Definition 3. Let $\mathcal{F} = (W, J, R, \sigma, \dot{\sigma}, \nu, \pi)$ be a frame with Verjüngung. An $\mathcal{F}$-display $\mathcal{P}$ is an $\mathcal{F}$-predisplay which is isomorphic to the display of a standard datum. The choice of such an isomorphism is called a normal decomposition of $\mathcal{P}$.

We call an $\mathcal{F}$-predisplay $\mathcal{P}$ separated if the commutative diagram

$$\begin{array}{ccc} P_i & \xrightarrow{F_i} & P_0 \\ {\scriptstyle\iota_i}\uparrow & & \uparrow{\scriptstyle p} \\ P_{i+1} & \xrightarrow[F_{i+1}]{} & P_0 \end{array}$$

induces an injective map from P_{i+1} to the fibre product $P_i \times_{F_i, P_0, p} P_0$.

One checks easily that an $\mathcal{F}$-display is separated and one proves immediately:

Proposition 4. *Let $\mathcal{F}$ be a frame with Verjüngung. Let $\mathcal{P}$ be a separated $\mathcal{F}$-predisplay. Let $\mathcal{P}'$ be an $\mathcal{F}$-predisplay. Then the natural map*

$$\mathrm{Hom}_{\mathcal{F}\text{-dsp}}(\mathcal{P}', \mathcal{P}) \to \mathrm{Hom}_{W\text{-mod}}(P_0', P_0)$$

from the Hom-*group of homomorphisms of predisplays to the* Hom-*group of homomorphisms of W-modules is injective.*

An $\mathcal{F}$-display $\mathcal{P}$ is a separated $\mathcal{F}$-predisplay.

Let $\mathcal{P}$ be an $\mathcal{F}$-predisplay. Iterating the homomorphisms α_i in Definition 2 we obtain W-module homomorphisms for $i, k \geq 0$:

$$\alpha_i^{(k)} : J \otimes_W J \cdots \otimes_W J \otimes_W P_i \to P_{i+k}. \tag{6}$$

By definition we have $\alpha_i^{(0)} = \mathrm{id}_{P_i}$ and $\alpha_i^{(1)} = \alpha_i$. We say that $\mathcal{P}$ satisfies the condition **alpha** if the map (6) factors through a homomorphism $\bar{\alpha}_i^{(k)}$:

$$\textbf{alpha}: \quad J \otimes_W \cdots \otimes_W J \otimes_W P_i \xrightarrow{\nu^{(k)} \otimes \mathrm{id}} J_k \otimes_W P_i \xrightarrow{\bar{\alpha}_i^{(k)}} P_{i+k}. \tag{7}$$

Obviously $\bar{\alpha}_i^{(k)}$ is uniquely determined. A display satisfies the condition **alpha**.

Proposition 5. *Let $\mathcal{Q}$ be the display associated to a standard datum (L_i, Φ_i), $i = 0, \ldots, d$. Let $\mathcal{P}$ be a predisplay which satisfies the condition* **alpha**.

Let $\rho : \mathcal{Q} \to \mathcal{P}$ be a morphism of $\mathcal{F}$-modules. We denote for $i \leq d$ the restriction of $\rho_i : Q_i \to P_i$ to $L_i \subset Q_i$ by $\rho_{|i} : L_i \to P_i$.

With this notation let $\rho_{|i} : L_i \to P_i$ for $i = 0, \ldots, d$ be arbitrary W-module homomorphisms $\rho_{|i} : L_i \to P_i$ for $i = 0, \ldots, d$. Then there exists a unique $\mathcal{F}$-module homomorphism $\rho : \mathcal{Q} \to \mathcal{P}$ which induces the given $\rho_{|i}$.

Moreover the morphism of $\mathcal{F}$-modules ρ defined by a sequence of homomorphisms $\rho_{|i} : L_i \to P_i$ is a morphism of predisplays if and only if the following diagrams are commutative:

$$\begin{array}{ccc} L_i & \xrightarrow{\Phi_i} & Q_0 \\ {\scriptstyle\rho_{|i}}\downarrow & & \downarrow{\scriptstyle\rho_0} \\ P_i & \xrightarrow[F_i]{} & P_0 \end{array} \tag{8}$$

We remark that the morphism $\rho_0 : Q_0 = \bigoplus_{i=1}^{d} L_i \to P_0$ is given on the summand L_i as the composite $L_i \subset Q_i \to P_i \xrightarrow{\iota^{\text{iter}}} P_0$, where the composition of the first two arrows is $\rho_{|i}$ and the last arrow is the composition $\iota^{\text{iter}} = \iota_0 \circ \cdots \circ \iota_{i-1}$.

Proof. We have

$$Q_i = J_i L_0 \oplus \cdots \oplus J_{i-k} L_k \oplus \cdots \oplus J L_{i-1} \oplus L_i \oplus \cdots .$$

We will define $\rho_i : Q_i \to P_i$. We do this for each of the summands above separately. For $k < i$ we obtain by tensoring $\rho_{|k}$ with J_{i-k} a homomorphism

$$J_{i-k} L_k \to J_{i-k} \otimes_W P_k.$$

Composing the last arrow with $\bar{\alpha}_k^{i-k}$ from the condition **alpha** we obtain ρ_i on the summand $J_{i-k} L_k$.

For $j \geq i$ the map $\iota^{\text{iter}} : Q_j \to Q_i$ induces the identity on L_j. Therefore we define the restriction of ρ_i to the summand L_j as the composite

$$L_j \xrightarrow{\rho_{|j}} P_j \xrightarrow{\iota^{\text{iter}}} P_i.$$

One checks that the ρ_i define a morphism of $\mathcal{F}$-modules and, if the diagrams (8) commute, a morphism of $\mathcal{F}$-predisplays. □

We will now define the base change of displays. We consider a morphism of frames with Verjüngung $u : \mathcal{F} \to \mathcal{F}'$. Let $\mathcal{P}'$ be an $\mathcal{F}'$-predisplay. This may be regarded as an $\mathcal{F}$-predisplay with the same P_i' but regarded as W-modules. Only the maps α_i need a definition:

$$\alpha_i : J \otimes_W P_i' \to J' \otimes_{W'} P_i' \xrightarrow{\alpha_i'} P_{i+1}'.$$

We denote the $\mathcal{F}$-predisplay obtained in this way by $u^{\bullet}\mathcal{P}'$. Let $\mathcal{P}$ be an $\mathcal{F}$-display. We say that an $\mathcal{F}'$-display $u_{\bullet}\mathcal{P}$ is a base change of $\mathcal{P}$ if there exists for each $\mathcal{F}'$-display $\mathcal{P}'$ a bijection

$$\mathrm{Hom}_{\mathcal{F}'}(u_{\bullet}\mathcal{P}, \mathcal{P}') \cong \mathrm{Hom}_{\mathcal{F}}(\mathcal{P}, u^{\bullet}\mathcal{P}')$$

which is functorial in $\mathcal{P}'$.

Proposition 6 (base change). *Let $u : \mathcal{F} \to \mathcal{F}'$ be a morphism of frames with Verjüngung. Then the base change of an $\mathcal{F}$-display $\mathcal{P}$ exists. Moreover for $\mathcal{F}'$-predisplays $\mathcal{P}'$ which satisfy the condition* **alpha** *we have a functorial bijection*

$$\mathrm{Hom}_{\mathcal{F}'}(u_{\bullet}\mathcal{P}, \mathcal{P}') \cong \mathrm{Hom}_{\mathcal{F}}(\mathcal{P}, u^{\bullet}\mathcal{P}').$$

Proof. We choose a normal decomposition (L_i, Φ_i) of $\mathcal{P}$. Then a morphism $\rho : \mathcal{P} \to u^{\bullet}\mathcal{P}'$ is given by a set of W-module homomorphisms

$$\rho_{|i} : L_i \to P_i'$$

such that the analogues of (8) are commutative. From the standard datum (L_i, Φ_i) we obtain a standard datum $(L'_i = W' \otimes_W L_i, \Phi'_i = \sigma' \otimes \Phi_i)$ for the frame $\mathcal{F}'$ which defines an $\mathcal{F}'$-display $\mathcal{Q}$. From $\rho_{|i}$ we obtain W'-module homomorphisms

$$\rho'_{|i} : W' \otimes_W L_i \to P'_i.$$

By Proposition 5, these homomorphisms define a morphism of $\mathcal{F}'$-predisplays $\mathcal{Q} \to \mathcal{P}'$. This shows that $u_\bullet \mathcal{P} := \mathcal{Q}$ is a base change and has the claimed property. □

We apply the base change to the following obvious morphisms of frames with Verjüngung:

$$\mathcal{W}_S \to \mathcal{W}_{S/R} \to \mathcal{W}_R.$$

More generally let

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow & & \downarrow \\ R & \longrightarrow & R' \end{array}$$

be a morphism of pd-extension of the type (2). We obtain a morphism of frames with Verjüngung $\mathcal{W}_{S/R} \to \mathcal{W}_{S'/R'}$. We have this for small Witt frames too.

We will give now an intrinsic characterization of a display which doesn't use a normal decomposition. Let $\mathcal{F}$ be a frame with Verjüngung. Let $\mathcal{P}$ be an $\mathcal{F}$-predisplay. Then we denote the image of the homomorphism

$$P_i \xrightarrow{\iota^{\text{iter}}} P_0 \to P_0/J P_0 \tag{9}$$

by E^i or more precisely by $\text{Fil}^i_{\mathcal{P}}$. This is called the Hodge filtration on the R-module $P_0/J P_0$:

$$\cdots E^{i+1} \to E^i \to \cdots \to E^0 = P_0/J P_0. \tag{10}$$

If $\mathcal{P}$ is a display, this is a filtration by direct summands.

Proposition 7. *Let $\mathcal{F}$ be a frame with Verjüngung. We assume that each finitely generated projective R-module may be lifted to a finitely generated projective W-module. Let $\mathcal{P}$ be an $\mathcal{F}$-predisplay with Hodge filtration E^i such that the following properties hold*:

(1) *$\mathcal{P}$ is separated and satisfies the condition* **alpha**.

(2) *P_0 is a finitely generated projective W-module.*

(3) *The Hodge filtration consists of direct summands $E^i \subset P_0/\mathcal{J} P_0$.*

(4) *There is an exact sequence*

$$\mathcal{J} \otimes P_i \xrightarrow{\alpha_i} P_{i+1} \to E^{i+1} \to 0.$$

(5) *The subgroups $F_i P_i$ for $i \geq 0$ generate the W-module P_0.*

Then $\mathcal{P}$ is an $\mathcal{F}$-display.

We omit the proof. We note that the assumption that liftings of finitely generated projective modules exist is trivial if R is a local ring.

Let $\mathcal{F} = \mathcal{W}_{S/R}$ or $\mathcal{F} = \widehat{\mathcal{W}}_{S/R}$. Let $\mathcal{P}$ be an $\mathcal{F}$-display. A lifting of the Hodge filtration of $\mathcal{P}$ is a sequence of split injections of projective finitely generated S-modules

$$\cdots \widetilde{E}^{i+1} \to \widetilde{E}^{i} \to \cdots \to \widetilde{E}^{0} = P_0/I_S P_0, \tag{11}$$

which coincides with (10) when tensored with R.

We will now discuss the notion of an extended display. Let $\mathcal{F}$ be a frame with Verjüngung. Let (L_i, Φ_i) be a standard datum. If we replace in the definition of the display associated to this datum all J_i simply by J we obtain an $\mathcal{F}$-predisplay $\widetilde{\mathcal{P}}$. We consider this notion only for the frames $\mathcal{W}_{S/R}$ and $\widehat{\mathcal{W}}_{S/R}$. Let $\widetilde{\mathcal{P}}$ be an extended display and let E^i be its Hodge filtration. Then $\widetilde{\mathcal{P}}$ satisfies all conditions of Proposition 7 except for the condition (4).

We note that there is no difference between displays and extended displays in the case $S = R$ because then $J = J_i$.

Let $\mathcal{Q}$ be a $\mathcal{W}_{S/R}$-predisplay or a $\widehat{\mathcal{W}}_{S/R}$-predisplay. For this discussion we denote by $\overline{Q}_i \subset Q_i$ the intersection of all images of maps

$$Q_{i+k} \xrightarrow{\iota^{\text{iter}}} Q_i.$$

If $\mathcal{Q}$ is a display then $\overline{Q}_i = 0$ for all i because $W(S)$ is a p-adic ring. If $\mathcal{Q}$ is an extended display we have that the map $\iota^{\text{iter}} : Q_i \to Q_0$ induces an isomorphism

$$\iota^{\text{iter}} : \overline{Q}_i \to \tilde{\mathfrak{a}} Q_0. \tag{12}$$

Note that for $k > i$ we have that $\tilde{\mathfrak{a}} L_k \subset (\tilde{\mathfrak{a}} \oplus I_S) L_k \subset Q_i$ is a direct summand of Q_i which is mapped isomorphically to a direct summand of $\overline{Q}_i$ and further by (12) isomorphically to $\tilde{\mathfrak{a}} L_k \subset \tilde{\mathfrak{a}} Q_0$.

We note that an extended display satisfies the condition **alpha**. We have the following version of Proposition 5 which is proved by the same argument.

Proposition 8. *We consider a predisplay for the frame $\mathcal{F} = \mathcal{W}_{S/R}$ or $\mathcal{F} = \widehat{\mathcal{W}}_{S/R}$. Let $\widetilde{\mathcal{Q}}$ be the extended display associated to a standard datum (L_i, Φ_i), $i = 0, \ldots, d$. Let $\mathcal{P}$ be a predisplay which satisfies the condition* **alpha** *and* (12).

Let $\rho : \widetilde{\mathcal{Q}} \to \mathcal{P}$ be a morphism of $\mathcal{F}$-modules. We denote for $i \leq d$ the restriction of $\rho_i : Q_i \to P_i$ to $L_i \subset Q_i$ by $\rho_{|i} : L_i \to P_i$.

Conversely arbitrary $W(S)$-module homomorphisms $\rho_{|i} : L_i \to P_i$ for $i = 0, \ldots, d$ define uniquely a morphism of $\mathcal{F}$-modules $\rho : \widetilde{\mathcal{Q}} \to \mathcal{P}$.

Moreover the morphism of $\mathcal{F}$-modules ρ defined by a sequence of homomorphisms $\rho_{|i} : L_i \to P_i$ is a morphism of predisplays if and only if the following diagrams are commutative:

$$\begin{array}{ccc} L_i & \xrightarrow{\Phi_i} & \widetilde{\mathcal{Q}}_0 \\ {\scriptstyle \rho_{|i}}\downarrow & & \downarrow{\scriptstyle \rho_0} \\ P_i & \xrightarrow[F_i]{} & P_0 \end{array} \tag{13}$$

Corollary 9. *Let $\mathcal{Q}$ be the display associated to the standard datum (L_i, Φ_i). Then we have a canonical bijection*

$$\mathrm{Hom}_{\mathcal{F}}(\widetilde{\mathcal{Q}}, \mathcal{P}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{F}}(\mathcal{Q}, \mathcal{P}).$$

We conclude that we have a functor $\mathcal{Q} \mapsto \widetilde{\mathcal{Q}}$ from the category of displays to the category of extended displays, because the construction of $\widetilde{\mathcal{Q}}$ does not depend on the normal decomposition.

Let (L_i, Φ_i) be a standard datum for the frame $\mathcal{W}_{S/R}$; let $\mathcal{Q}$ be the associated display and $\widetilde{\mathcal{Q}}$ the extended display.

Let $\widehat{\mathcal{Q}}$ be the $\mathcal{W}_R$-display associated to $(W(R) \otimes_{W(S)} L_i, \sigma \otimes \Phi_i)$. Then $\widehat{\mathcal{Q}}$ is the base change of $\mathcal{Q}$ via $\mathcal{W}_{S/R} \to \mathcal{W}_R$. Since a $\mathcal{W}_R$-display $\mathcal{P}$ regarded as a $\mathcal{W}_{S/R}$-predisplay satisfies the condition **alpha** and (12) we obtain

$$\mathrm{Hom}_{\mathcal{W}_R}(\widehat{\mathcal{Q}}, \mathcal{P}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{W}_{S/R}}(\widetilde{\mathcal{Q}}, \mathcal{P}).$$

This shows that we have also a functor $\widetilde{\mathcal{Q}} \mapsto \widehat{\mathcal{Q}}$. Therefore we have functors

$$(\mathcal{W}_{S/R}\text{-displays}) \to (\mathcal{W}_{S/R}\text{-extended displays}) \to (\mathcal{W}_R\text{-displays})$$

such that the composition of these functors is base change. The same functors exist if the small Witt frame $\widehat{\mathcal{W}}_{S/R}$ is defined.

We have defined what is a lifting of the Hodge filtration for a $\mathcal{W}_{S/R}$-display $\mathcal{P}$. We will now construct the functor

$$\begin{pmatrix} \text{extended } \mathcal{W}_{S/R}\text{-displays} \\ \text{and a lift of the Hodge filtration} \end{pmatrix} \to (\mathcal{W}_S\text{-displays}). \tag{14}$$

Again the construction will be the same for small Witt frames.

Let $\widetilde{\mathcal{P}}$ be an extended $\mathcal{W}_{S/R}$-display. Let

$$\cdots \widetilde{E}^{i+1} \to \widetilde{E}^i \to \cdots \to \widetilde{E}^0 = \widetilde{P}_0 / I_S \widetilde{P}_0$$

be a lift of the Hodge filtration. We construct in a functorial way a $\mathcal{W}_S$-display $\mathcal{P}$. We denote by $\widehat{E}^i \subset \widetilde{P}_0 / I_S \widetilde{P}_0$ the preimage of the Hodge filtration $E^i \subset \widetilde{P}_0 / J \widetilde{P}_0$. By choosing an arbitrary normal decomposition of $\widetilde{\mathcal{P}}$ we find that the map

$$\widetilde{P}_i \to \widetilde{P}_0 / I_S \widetilde{P}_0$$

has image $\widehat{E}^i$.

We choose a splitting of the lifted Hodge filtration and obtain a decomposition into S-submodules of $\widetilde{P}_0/I_S\widetilde{P}_0$:

$$\widetilde{E}^i = \bar{L}_i \oplus \bar{L}_{i+1} \oplus \cdots \oplus \bar{L}_d.$$

We choose a finitely generated projective $W(S)$-module L_i which lifts the S-module $\bar{L}_i$ and we choose a commutative diagram:

$$\begin{array}{ccc} L_i & & \\ \downarrow & \searrow & \\ \widetilde{P}_i & \longrightarrow & \widehat{E}_i \end{array}$$

A composite of the ι maps yields $L_i \to \widetilde{P}_i \to \widetilde{P}_0 = P_0$. We obtain a homomorphism

$$L_0 \oplus L_1 \oplus \cdots \oplus L_m \to P_0.$$

We see that this map is an isomorphism by taking it modulo J.

The maps $F_i : \widetilde{P}_i \to \widetilde{P}_0$ give by restriction maps $\Phi_i : L_i \to P_i$. We will show that the map

$$L_0 \oplus L_1 \oplus \cdots \oplus L_m \to P_0 \tag{15}$$

is a Frobenius linear isomorphism. Then we obtain standard data (L_i, Φ_i) for $\mathcal{P}$.

To show that (15) is an isomorphism we consider the $\mathcal{W}_R$ display $\bar{\mathcal{P}}$ obtained by base change from $\mathcal{P}$. We have natural maps $P_i \to \widetilde{P}_i \to \bar{P}_i$. The images of the L_i in $\bar{P}_i$ give a normal decomposition of those displays. Therefore the map (15) becomes a Frobenius linear isomorphism when tensored with $W(R)$. Then the map itself is a Frobenius linear isomorphism.

Then we define the desired $\mathcal{W}_S$-display $\mathcal{P}$ by the standard datum (L_i, Φ_i). Our construction gives that $P_i \subset \widetilde{P}_i$ is the preimage of $\widetilde{E}^i$ under the map

$$\widetilde{P}_i \to P_0/I_S P_0.$$

This shows that the assignment $\widetilde{\mathcal{P}} \mapsto \mathcal{P}$ is functorial and does not depend on the construction of the normal decomposition chosen above.

Proposition 10. *The functor* (14) *defines an equivalence of the category of extended $\mathcal{W}_{S/R}$-displays together with a Hodge filtration and the category of $\mathcal{W}_S$-displays. The same holds for the small Witt frames.*

Proof. Indeed there is an obvious inverse functor. We denote by

$$u_\bullet : (\mathcal{W}_S\text{-displays}) \to (\text{extended } \mathcal{W}_{S/R}\text{-displays})$$

the functor induced by base change. An extended $\mathcal{W}_{S/R}$-display $\mathcal{P}$ may be regarded as a $\mathcal{W}_S$-predisplay. Then we denote it by $u^\bullet\mathcal{P}$. By Propositions 5 and 8 we have for a $\mathcal{W}_S$-display $\mathcal{Q}$ a functorial bijection,

$$\mathrm{Hom}_{\mathcal{W}_{S/R}}(u_\bullet \mathcal{Q}, \mathcal{P}) \cong \mathrm{Hom}_{\mathcal{W}_S}(\mathcal{Q}, u^\bullet \mathcal{P}).$$

We set

$$\widehat{\mathcal{Q}} = u_\bullet \mathcal{Q}.$$

The canonical map $\mathcal{Q} \to u^\bullet \widehat{\mathcal{Q}}$ induces natural injections $Q_i \to \widehat{Q}_i$. This provides a lifting of the Hodge filtration of the extended display $\widehat{\mathcal{Q}}$. Clearly this functor is inverse to the functor (14). □

Let $\mathcal{P}$ be a $\mathcal{W}_{S/R}$-display. We say that a lifting of the Hodge filtration $\widetilde{E}^i \subset P_0/I_S P_0$ for $i \geq 0$ is admissible if $\widetilde{E}^i$ is in the image of $P_i \xrightarrow{\iota^{\text{iter}}} P_0/I_S P_0$. If $\mathcal{Q}$ is a $\mathcal{W}_S$-display and $\widetilde{\mathcal{Q}}$ is the $\mathcal{W}_{S/R}$-display by base change then we have a natural inclusion $Q_i \to \widetilde{Q}_i$. This shows that the induced Hodge filtration on $\widetilde{\mathcal{Q}}$ is admissible. From the proof of the last proposition we obtain:

Corollary 11. *The functor*

$$(\mathcal{W}_S\textit{-displays}) \to \begin{pmatrix} \mathcal{W}_{S/R}\textit{-displays with an admissible} \\ \textit{lift of the Hodge filtration} \end{pmatrix}$$

is an equivalence of categories.

We consider a frame with Verjüngung $\mathcal{F} = (W, J, R, \sigma, \dot{\sigma}, \nu, \pi)$. We consider 2-displays, i.e., displays which are defined by standard data (L_i, Φ_i) with $L_i = 0$ for $i > 2$ [Langer and Zink 2007, Definitions 2.4 and 2.5]. Let $\mathcal{P}$ and $\mathcal{P}'$ given by standard data $(L_0, L_1, L_2, \Phi_0, \Phi_1, \Phi_2)$, $(L'_0, L'_1, L'_2, \Phi'_0, \Phi'_1, \Phi'_2)$. We assume that the W-modules L_i and L'_i are free. We choose a W-basis of each of these modules. A morphism of displays $\rho : \mathcal{P} \to \mathcal{P}'$ is given by three maps

$$\rho_{|i} : L_i \to P'_i = J_i L'_0 \oplus \cdots \oplus J L'_{i-1} \oplus L'_i \cdots$$

for $i = 0, 1, 2$. We represent each of these maps by a column vector. These are the columns of the matrix

$$\begin{pmatrix} X_{00} & Y_{01} & Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix}. \tag{16}$$

The X_{ij} are matrices with coefficients in W. They represent in the chosen basis the homomorphisms $L_j \to L'_i$ obtained from $\rho_{|j}$. The matrices Y_{01} and Y_{12} have coefficients in J and Y_{02} has coefficients in J_2. Since a morphism of 2-displays commutes with ι, one can see that the map $P_0 \to P'_0$ is given by the matrix

$$\begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix}.$$

By Proposition 5 the matrix (16) defines a morphism of displays if and only if the following diagram is commutative for $i = 0, 1, 2$:

$$\begin{array}{ccc} L_i & \xrightarrow{\Phi_i} & P_0 \\ {\scriptstyle \rho_{|i}}\downarrow & & \downarrow{\scriptstyle \rho_0} \\ P'_i & \xrightarrow[F'_i]{} & P'_0 \end{array} \tag{17}$$

The σ-linear maps Φ_i and Φ'_i are given by the row vectors of matrices with coefficients in W:

$$\begin{pmatrix} A_{0i} \\ A_{1i} \\ A_{2i} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A'_{0i} \\ A'_{1i} \\ A'_{2i} \end{pmatrix}.$$

We write these vectors in a matrix. For example, the standard data $(L_0, L_1, L_2, \Phi_1, \Phi_2, \Phi_3)$ for $\mathcal{P}$ are equivalent to the block matrix:

$$\boldsymbol{A} = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix}. \tag{18}$$

We will call this a structure matrix for the display $\mathcal{P}$. It is by definition a matrix in $\mathrm{GL}_h(W)$, where $h = \mathrm{rank}_W P_0$.

From the definition of F'_i in terms of standard data, these σ-linear maps have the following matrix representations:

$$F'_0\left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} A'_{00} & pA'_{01} & p^2A'_{02} \\ A'_{10} & pA'_{11} & p^2A'_{12} \\ A'_{20} & pA'_{21} & p^2A'_{22} \end{pmatrix} \begin{pmatrix} \sigma(x_0) \\ \sigma(x_1) \\ \sigma(x_2) \end{pmatrix},$$

$$F'_1\left(\begin{pmatrix} y_0 \\ x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} A'_{00} & A'_{01} & pA'_{02} \\ A'_{10} & A'_{11} & pA'_{12} \\ A'_{20} & A'_{21} & pA'_{22} \end{pmatrix} \begin{pmatrix} \dot{\sigma}(y_0) \\ \sigma(x_1) \\ \sigma(x_2) \end{pmatrix},$$

$$F'_2\left(\begin{pmatrix} y_0 \\ y_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} A'_{00} & A'_{01} & A'_{02} \\ A'_{10} & A'_{11} & A'_{12} \\ A'_{20} & A'_{21} & A'_{22} \end{pmatrix} \begin{pmatrix} \dot{\sigma}(y_0) \\ \dot{\sigma}(y_1) \\ \sigma(x_2) \end{pmatrix}.$$

The vectors x_i have coefficients in W, and the y_i have coefficients in J, but in the equation for F'_2 the vector y_0 has even coefficients in J_2.

Then the commutativity of the diagram (17) for $i = 0$ amounts to

$$\begin{pmatrix} A'_{00} & pA'_{01} & p^2A'_{02} \\ A'_{10} & pA'_{11} & p^2A'_{12} \\ A'_{20} & pA'_{21} & p^2A'_{22} \end{pmatrix} \begin{pmatrix} \sigma(X_{00}) \\ \sigma(X_{10}) \\ \sigma(X_{20}) \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} A_{00} \\ A_{10} \\ A_{20} \end{pmatrix}.$$

For $i = 1$ it amounts to

$$\begin{pmatrix} A'_{00} & A'_{01} & pA'_{02} \\ A'_{10} & A'_{11} & pA'_{12} \\ A'_{20} & A'_{21} & pA'_{22} \end{pmatrix} \begin{pmatrix} \dot{\sigma}(Y_{01}) \\ \sigma(X_{11}) \\ \sigma(X_{21}) \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} A_{01} \\ A_{11} \\ A_{21} \end{pmatrix}.$$

Finally for $i = 2$ it amounts to

$$\begin{pmatrix} A'_{00} & A'_{01} & A'_{02} \\ A'_{10} & A'_{11} & A'_{12} \\ A'_{20} & A'_{21} & A'_{22} \end{pmatrix} \begin{pmatrix} \dot{\sigma}(Y_{02}) \\ \dot{\sigma}(Y_{12}) \\ \sigma(X_{22}) \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} A_{02} \\ A_{12} \\ A_{22} \end{pmatrix}.$$

We may write the last three equations as a single matrix equation,

$$\boldsymbol{A}' \begin{pmatrix} \sigma(X_{00}) & \dot{\sigma}(Y_{01}) & \dot{\sigma}(Y_{02}) \\ p\sigma(X_{10}) & \sigma(X_{11}) & \dot{\sigma}(Y_{12}) \\ p^2\sigma(X_{20}) & p\sigma(X_{21}) & \sigma(X_{22}) \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \boldsymbol{A}, \tag{19}$$

where $\boldsymbol{A}$ and $\boldsymbol{A}'$ are the structure matrices (18).

Let $\mathcal{F}$ be a frame with Verjüngung such that any finitely generated projective R-module is free. Then the category of $\mathcal{F}$-2-displays is equivalent to the following category $\mathcal{M}_{\mathcal{F}}$: The objects are invertible matrices $\boldsymbol{A}$ with coefficients in W with a 3×3-partition into blocks such that the blocks on the diagonal are quadratic matrices. The morphisms $\boldsymbol{A} \to \boldsymbol{A}'$ are block matrices (16) such that (19) is satisfied. Of course we have to say what is the composite of two matrices, but we omit this. In this direction we make only the following remark: the maps $\rho_i : P_i \to P'_i$ are explicitly given by the matrix equations

$$\rho_0\left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix},$$

$$\rho_1\left(\begin{pmatrix} y_0 \\ x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} y_0 \\ x_1 \\ x_2 \end{pmatrix},$$

$$\rho_2\left(\begin{pmatrix} y_0 \\ y_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} X_{00} & \check{Y}_{01} & Y_{02} \\ \pi X_{10} & X_{11} & Y_{12} \\ \pi X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ x_2 \end{pmatrix}.$$

We need to explain the last equation. Here y_0 is a vector with entries in J_2 and y_1 a vector with entries in J. The entries πX_{10}, πX_{20} and $\check{Y}_{01}$ are only symbols. But the matrix multiplication becomes meaningful with the definitions

$$\check{Y}_{01} y_1 = Y_{01} * y_1, \quad \pi X_{10} y_0 = X_{10}\pi(y_0), \quad \pi X_{20} y_0 = X_{10}\pi(y_0). \tag{20}$$

Note that the vectors of (20) have entries in J_2.

Using these expressions for ρ_i we see that (19) amounts to the commutativity of the following diagram:

$$\begin{array}{ccc} P_2 & \xrightarrow{\rho_2} & P_2' \\ {\scriptstyle F_2}\downarrow & & \downarrow{\scriptstyle F_2'} \\ P_0 & \xrightarrow[\rho_0]{} & P_0' \end{array}$$

Finally we give the description of the dual display in terms of standard data. Let $\mathcal{F}$ be a frame with Verjüngung as before. Assume $\mathcal{P}$ is the display associated to the standard data:

$$\Phi : L_0 \oplus L_1 \oplus L_2 \to L_0 \oplus L_1 \oplus L_2. \tag{21}$$

We write Φ in matrix form:

$$\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \sigma(x) \\ \sigma(y) \\ \sigma(z) \end{pmatrix}.$$

Then the dual display $\widehat{\mathcal{P}}$ is formed from the following standard data. We take the dual modules $L_i^* = \operatorname{Hom}_W(L_i, W)$ but in the order L_2^*, L_1^*, L_0^*. Changing the order in (21) and taking the dual of this σ-linear map we obtain a linear map

$$\Phi^* : L_2^* \oplus L_1^* \oplus L_0^* \to W \otimes_{\sigma, W} (L_2^* \oplus L_1^* \oplus L_0^*). \tag{22}$$

We set $\widehat{\Phi} = (\Phi^*)^{-1}$. We regard this as a σ-linear map. We obtain a standard datum,

$$(L_2^*, L_1^*, L_0^*, \widehat{\Phi}),$$

which is by definition the standard datum of $\widehat{\mathcal{P}}$. In particular

$$\widehat{P_0} = L_2^* \oplus L_1^* \oplus L_0^*.$$

In matrix form $\widehat{\Phi}$ takes the form

$$\widehat{\Phi} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} {}^tA_{22} & {}^tA_{12} & {}^tA_{02} \\ {}^tA_{21} & {}^tA_{11} & {}^tA_{01} \\ {}^tA_{20} & {}^tA_{10} & {}^tA_{00} \end{pmatrix}^{-1} \begin{pmatrix} \sigma(x') \\ \sigma(y') \\ \sigma(z') \end{pmatrix}.$$

Let us denote by d_0, d_1, d_2 the ranks of the modules L_0, L_1, L_2 respectively. Consider the block matrix

$$\boldsymbol{B} := \begin{pmatrix} \mathbf{0} & \mathbf{0} & E_{d_0} \\ \mathbf{0} & E_{d_1} & \mathbf{0} \\ E_{d_2} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where E denotes a unit matrix. This matrix defines a bilinear form:

$$\langle\ ,\ \rangle : P_0 \times \widehat{P}_0 \to W, \quad \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \right\rangle = (x \ \ y \ \ z)\boldsymbol{B} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \tag{23}$$

In this notation the definition of $\widehat{\Phi}$ reads

$$\langle \Phi(u), \widehat{\Phi}(\hat{u}) \rangle = \sigma \langle u, \hat{u} \rangle, \quad u \in P_0,\ \hat{u} \in \widehat{P}_0.$$

One deduces the formula

$$\langle F_0(u), \widehat{F}_0(\hat{u}) \rangle = p^2 \sigma \langle u, \hat{u} \rangle.$$

If we denote by $\mathcal{U}(2)$ the $\mathcal{F}$-2-display associated to the standard datum $(0, 0, W; \sigma)$ we obtain from $\langle\ ,\ \rangle$ a bilinear pairing of $\mathcal{F}$-displays,

$$\langle\ ,\ \rangle : \mathcal{P} \times \widehat{\mathcal{P}} \to \mathcal{U}(2).$$

The complete definition of a bilinear form is given in [Langer and Zink 2007] after equation (15), pp.160–163. In the case $\mathcal{U}(2)$ we have for each $i, j \geq 0$ a W-bilinear pairing

$$P_i \times P_j \to W.$$

The most important formulas of this definition are

$$\begin{aligned} \langle F_i x_i, \widehat{F}_j x_j \rangle &= p^{2-i-j} \sigma \langle x_i, \hat{x}_j \rangle \quad &&\text{if } i + j \leq 2, \text{ for } x_i \in P_i, \\ \langle F_i x_i, \widehat{F}_j x_j \rangle &= \dot{\sigma} \langle x_i, \hat{x}_j \rangle &&\text{if } i + j > 2 \text{ and } \hat{x}_j \in \widehat{P}_j. \end{aligned}$$

One should also keep in mind that the bilinear form of displays is already uniquely determined by the induced W-bilinear form

$$P_0 \times \widehat{P}_0 \to W.$$

We note that the Hodge filtrations

$$\{0\} \subset L_2/JL_2 \subset L_1/JL_1 \oplus L_2/JL_2 \subset P_0/JP_0,$$

and

$$\{0\} \subset \widehat{L}_2/J\widehat{L}_2 \subset \widehat{L}_1/J\widehat{L}_1 \oplus \widehat{L}_2/J\widehat{L}_2 \subset \widehat{P}_0/J\widehat{P}_0$$

are dual with respect to $\langle\ ,\ \rangle$.

In particular an isomorphism of 2-displays $\mathcal{P} \to \widehat{\mathcal{P}}$ defines a bilinear form of displays

$$\mathcal{P} \times \mathcal{P} \to \mathcal{U}(2)$$

such that the Hodge filtration of $\mathcal{P}$ is selfdual with respect to this pairing.

Let R, S be an artinian local ring with perfect residue field. Let $S \to R$ be a nilpotent pd-thickening and let $\mathfrak{a}$ be the kernel. Then have defined the small relative Witt frame $\widehat{\mathcal{W}}_{S/R}$ with Verjüngung.

For a $\widehat{\mathcal{W}}_{S/R}$-display $\mathcal{P}$ the Frobenius $F_0 : P_0 \to P_0$ induces a map

$$\bar{F}_0 : P_0/(JP_0 + pP_0 + \iota_0(P_1)) \to P_0/(JP_0 + pP_0 + \iota_0(P_1)). \tag{24}$$

Definition 12. We say that a 2-display $\mathcal{P}$ is $\bar{F}_0$-étale if the map (24) is an isomorphism.

If a 2-display $\mathcal{P}$ and the dual display $\widehat{\mathcal{P}}$ are $\bar{F}_0$-étale, we call the display $\mathcal{P}$ F-ordinary.

This makes sense for other frames but then we can do nothing with the definition.

Let $S' \to R$ be a second nilpotent pd-thickening of the same type. We denote the kernel by $\mathfrak{a}'$. Let $S' \to S$ a morphism of pd-thickenings of R. Then the kernel $\mathfrak{b}$ of this morphism is a sub-pd-ideal of $\mathfrak{a}'$. We obtain a morphism of frames

$$\widehat{\mathcal{W}}_{S'/R} \to \widehat{\mathcal{W}}_{S/R}. \tag{25}$$

Proposition 13. *Let R, S, S' be artinian local rings with perfect residue field of characteristic $p > 0$. Let $S' \to S$ be a surjective morphism of nilpotent pd-thickenings of R. Let $\mathfrak{a}$ and $\mathfrak{a}'$ be the kernels of the pd-thickenings.*

Let $\mathcal{P}$ and $\mathcal{Q}$ be two $\widehat{\mathcal{W}}_{S/R}$-2-displays which are F-ordinary. Let $\mathcal{P}'$ and $\mathcal{Q}'$ be liftings to $\widehat{\mathcal{W}}_{S'/R}$-2-displays.

Then each homomorphism $\rho : \mathcal{P} \to \mathcal{Q}$ lifts to a homomorphism of $\widehat{\mathcal{W}}_{S'/R}$-displays $\rho' : \mathcal{P}' \to \mathcal{Q}'$.

If we assume moreover that $(\mathfrak{a}')^2 = 0$, the homomorphism ρ' is uniquely determined by ρ.

Proof. By the usual argument, compare [Zink 2002, proof of Theorem 46], we may assume that $\mathcal{P} = \mathcal{Q}$ and that ρ is the identity. We choose such a normal decomposition and a basis in each module of this decomposition. We lift this to a normal decomposition of $\mathcal{P}'$ or $\mathcal{Q}'$ respectively and we also lift the given basis.

Then we may represent the 2-display $\mathcal{P}'$ by the structure matrix

$$\boldsymbol{A} = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix} \in \mathrm{GL}(\widehat{W}(S')),$$

and similarly $\mathcal{Q}'$ by the structure matrix $\boldsymbol{A}' = (A'_{ij})$. We will write $\boldsymbol{A}^{-1} = (\breve{A}_{ij})$ and $(\boldsymbol{A}')^{-1} = (\breve{A}'_{ij})$. Then our assumption says that the following matrices are invertible:

$$A_{00}, \quad A'_{00}, \quad \breve{A}_{22}, \quad \breve{A}'_{22}.$$

Let $\mathfrak{c}$ be the kernel of $S' \to S$. Decomposing $S' \to S$ in a series of pd-morphisms, we may assume that $\mathfrak{c}^2 = 0$ and $p\mathfrak{c} = 0$. A morphism $\rho' : \mathcal{P}' \to \mathcal{Q}'$ which lifts the identity may be represented by a matrix of the form

$$\boldsymbol{E} + \begin{pmatrix} X_{00} & Y_{01} & Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix}.$$

The entries of the matrices X_{ij} and Y_{ij} are in $\widehat{W}(\mathfrak{c})$ and the entries of $\boldsymbol{w}_0(Y_{02})$ are moreover in $(\mathfrak{a}')^2$.

We set $C_{ij} = A'_{ij} - A_{ij}$. These are matrices with entries in $\widehat{W}(\mathfrak{c})$. Since $\sigma(X_{ij}) = 0$, (19) may be rewritten as

$$\boldsymbol{C} + \boldsymbol{A}' \begin{pmatrix} 0 & \dot{\sigma}(Y_{01}) & \dot{\sigma}(Y_{02}) \\ 0 & 0 & \dot{\sigma}(Y_{12}) \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \boldsymbol{A}. \tag{26}$$

We used the notation $\boldsymbol{C} := (C_{ij})$. We have to show that there are matrices X_{ij} and Y_{ij} which satisfy this equation. We write $Y_{02} = \mathfrak{y}_{02} + {}^V Z_{02}$, where $\mathfrak{y}_{02} \in \tilde{\mathfrak{c}} \cap \widetilde{(\mathfrak{a}')^2}$. We note that $\pi Y_{02} = \mathfrak{y}_{02}$. In particular we need $\pi Y_{02} = 0$ if we want to prove the second assertion of the proposition, that the solutions X_{ij}, Y_{ij} are unique.

We set $\boldsymbol{D} = \boldsymbol{C}\boldsymbol{A}^{-1}$. Then (26) becomes

$$\boldsymbol{D} + \boldsymbol{A}' \begin{pmatrix} 0 & \dot{\sigma}(Y_{01}) & \dot{\sigma}(Y_{02}) \\ 0 & 0 & \dot{\sigma}(Y_{12}) \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{A}^{-1} = \begin{pmatrix} X_{00} & Y_{01} & \mathfrak{y}_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix}. \tag{27}$$

We have

$$\boldsymbol{A}' \begin{pmatrix} 0 & \dot{\sigma}(Y_{01}) & \dot{\sigma}(Y_{02}) \\ 0 & 0 & \dot{\sigma}(Y_{12}) \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{A}^{-1}$$
$$= \begin{pmatrix} * & A'_{00}\dot{\sigma}(Y_{01})\breve{A}_{11} + A'_{00}\dot{\sigma}(Y_{02})\breve{A}_{21} + A'_{01}\dot{\sigma}(Y_{12})\breve{A}_{21} & ?_1 \\ * & * & ?_2 \\ * & * & ?_3 \end{pmatrix},$$

where

$$?_1 = A'_{00}\dot{\sigma}(Y_{01})\breve{A}_{12} + A'_{00}\dot{\sigma}(Y_{02})\breve{A}_{22} + A'_{01}\dot{\sigma}(Y_{12})\breve{A}_{22},$$
$$?_2 = A'_{10}\dot{\sigma}(Y_{01})\breve{A}_{12} + A'_{10}\dot{\sigma}(Y_{02})\breve{A}_{22} + A'_{11}\dot{\sigma}(Y_{12})\breve{A}_{22},$$
$$?_3 = *.$$

The entries $*$ are irrelevant for the following and therefore not specified. Since the X_{ij} don't appear on the left-hand side of (27) we see that it is enough to satisfy the following equations if we want to solve (27):

$$\begin{aligned} D_{01} + A'_{00}\dot\sigma(Y_{01})\breve A_{11} + A'_{00}\dot\sigma(Y_{02})\breve A_{21} + A'_{01}\dot\sigma(Y_{12})\breve A_{21} &= Y_{01}, \\ D_{12} + A'_{10}\dot\sigma(Y_{01})\breve A_{12} + A'_{10}\dot\sigma(Y_{02})\breve A_{22} + A'_{11}\dot\sigma(Y_{12})\breve A_{22} &= Y_{12}, \\ D_{02} - \mathfrak{y}_{02} + A'_{00}\dot\sigma(Y_{01})\breve A_{12} + A'_{01}\dot\sigma(Y_{12})\breve A_{22} &= -A'_{00}\dot\sigma(Y_{02})\breve A_{22}. \end{aligned} \tag{28}$$

In this equation the D_{ij} are matrices with entries in $\widehat W(\mathfrak{c})$. We note that for any given matrices $\mathfrak{y}_{02}$ and $\dot\sigma(Y_{02})$ there is a unique Y_{02}.

Therefore the proposition follows if we show that for any given $\mathfrak{y}_{02}$ with entries in $\mathfrak{c}\cap(\mathfrak{a}')^2$ the equation above has a unique solution for the unknowns

$$Z_0 = Y_{01}, \quad Z_1 = Y_{12}, \quad Z_2 = -A'_{00}\dot\sigma(Y_{02})\breve A_{22},$$

with entries in $\widehat W(\mathfrak{c})$. This is because the matrices A'_{00} and $\breve A_{22}$ are invertible.

We denote by $\mathfrak{c}_{[n]}$ the $\widehat W(S')$-module obtained from the S'-module $\mathfrak{c}$ via restriction of scalars by the homomorphism $\boldsymbol{w}_n : \widehat W(S') \to S'$. The divided powers on $\mathfrak{c}$ allow us to divide the Witt polynomial $\boldsymbol{w}_n$ by p^n. The divided Witt polynomials $\boldsymbol{w}'_n$ define an isomorphism

$$\widehat W(\mathfrak{c}) \to \bigoplus_{n=0}^{\infty} \mathfrak{c}_{[n]} \tag{29}$$

of $\widehat W(S')$-modules.

For a matrix M with entries in $\widehat W(\mathfrak{c})$ of suitable size we define the operator

$$\mathcal{L}_{00}(M) = A'_{00} M \breve A_{11}.$$

If M has entries in the ideal $\bigoplus_{i=0}^{n} \mathfrak{c}_{[i]}$ in the sense of (29) then $\mathcal{L}_{00}(M)$ has entries in the same ideal. In this case we write length $M \le n$. It follows that length $\dot\sigma(M) \le n-1$.

With obvious definitions of the operators $\mathcal{L}_{ij}$ we may write the system of equations (28) as

$$\begin{aligned} D_{01} + \mathcal{L}_{00}(\dot\sigma(Z_0)) + \mathcal{L}_{01}(\dot\sigma(Z_1)) + \mathcal{L}_{02}(Z_2) &= Z_0, \\ D_{12} + \mathcal{L}_{10}(\dot\sigma(Z_0)) + \mathcal{L}_{11}(\dot\sigma(Z_1)) + \mathcal{L}_{12}(Z_2) &= Z_1, \\ D'_{02} + \mathcal{L}_{20}(\dot\sigma(Z_0)) + \mathcal{L}_{21}(\dot\sigma(Z_1)) &= Z_2. \end{aligned}$$

Here we write $D'_{02} := D_{02} - \mathfrak{y}_{02}$. We look for solutions in the space of matrices (Z_0, Z_1, Z_2) with entries in $\widehat W(\mathfrak{c})$. On this space we consider the operator U given by

$$U\left(\begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix}\right) = \begin{pmatrix} \mathcal{L}_{00}(\dot\sigma(Z_0)) + \mathcal{L}_{01}(\dot\sigma(Z_1)) + \mathcal{L}_{02}(Z_2) \\ \mathcal{L}_{10}(\dot\sigma(Z_0)) + \mathcal{L}_{11}(\dot\sigma(Z_1)) + \mathcal{L}_{12}(Z_2) \\ \mathcal{L}_{20}(\dot\sigma(Z_0)) + \mathcal{L}_{21}(\dot\sigma(Z_1)) \end{pmatrix}.$$

Clearly it suffices to prove that the operator U is pointwise nilpotent. Assume we are given Z_0, Z_1, Z_2. We set

$$U\left(\begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix}\right) = \begin{pmatrix} Z_0' \\ Z_1' \\ Z_2' \end{pmatrix}, \quad U\left(\begin{pmatrix} Z_0' \\ Z_1' \\ Z_2' \end{pmatrix}\right) = \begin{pmatrix} Z_0'' \\ Z_1'' \\ Z_2'' \end{pmatrix}.$$

Let m be a natural number such that length $Z_0 \leq m$, length $Z_1 \leq m$ and length $Z_2 \leq m-1$. Since $\dot{\sigma}$ decreases the length by 1, we obtain

$$\text{length } Z_0' \leq m-1, \quad \text{length } Z_1' \leq m-1, \quad \text{length } Z_2' \leq m-1.$$

And in the next step we find

$$\text{length } Z_0'' \leq m-1, \quad \text{length } Z_1'' \leq m-1, \quad \text{length } Z_2'' \leq m-2.$$

The nilpotence of U is now clear. This proves the uniqueness of the solutions. □

Remark. With the assumptions of the last proposition we assume that the kernel $\mathfrak{c}$ of the pd-morphism satisfies $\mathfrak{c}^2 = 0$ and $p\mathfrak{c} = 0$. Then the group of isomorphisms $\mathcal{P}' \to \mathcal{P}'$ which lift the identity $\mathrm{id}_{\mathcal{P}}$ is isomorphic to the additive group of $\mathfrak{c} \cap (\mathfrak{a}')^2$. (The assumptions ensure that such a lift is the same as a solution for (27).) This is because $\mathfrak{y}_{02}$ determines the lifting uniquely and because one can check that the composite of two endomorphisms of $\mathcal{P}'$ which lift zero is zero.

Corollary 14. *Let $S \to R$ be a surjective morphism of artinian local rings with perfect residue class field. Let $\mathcal{P}$ and $\mathcal{P}'$ be two F-ordinary $\widehat{\mathcal{W}}_S$-2-displays. Let $\rho, \tau : \mathcal{P} \to \mathcal{P}'$ be two homomorphisms such that their base changes ρ_R and τ_R are equal.*

Then $\rho = \tau$.

Corollary 15. *Let R, S' be an artinian local ring with perfect residue field as above. Let $S' \to R$ be a nilpotent pd-thickening with kernel $\mathfrak{a}'$ such that $(\mathfrak{a}')^2 = 0$. Let $\mathcal{Q}$ be an F-ordinary $\widehat{\mathcal{W}}_R$-2-display over R. By Proposition 13 there is up to canonical isomorphism a unique $\widehat{\mathcal{W}}_{S'/R}$-2-display $\widehat{\mathcal{Q}}$ which lifts $\mathcal{Q}$.*

The category of F-ordinary $\widehat{\mathcal{W}}_{S'}$-2-displays is equivalent to the category of pairs $(\mathcal{Q}, \mathrm{Fil})$ *where $\mathcal{Q}$ is an F-ordinary $\widehat{\mathcal{W}}_R$-2-displays and where* Fil *is an admissible lifting of the Hodge filtration of $\widehat{\mathcal{Q}}$.*

Let k be a perfect field of characteristic $p > 2$. Let $\mathcal{A}rt_k$ be the category of artinian local rings with residue class field k. Let $\mathcal{S}$ be an ordinary $\mathcal{W}_k$-2-display. Let $\mathcal{D}$ be the functor that associates to $R \in \mathcal{A}rt_k$ the isomorphism classes of pairs $(\mathcal{P}, \iota)$, where $\mathcal{P}$ is a $\widehat{\mathcal{W}}_R$-2-display and $\iota : \mathcal{S} \to \mathcal{P}_k$ is an isomorphism. If we have a diagram $R_1 \to R \leftarrow R_2$ then the canonical map

$$\mathcal{D}(R_1 \times_R R_2) \to \mathcal{D}(R_1) \times_{\mathcal{D}(R)} \mathcal{D}(R_2) \tag{30}$$

is surjective. To see this, one can for example use the interpretation of a display as a block matrix. It is injective because of Corollary 14.

By Corollary 15 the tangent space of the functor $\mathcal{D}$ is finite-dimensional.

Corollary 16. *The functor $\mathcal{D}$ is prorepresentable by a power series ring over $W(k)$ in finitely many variables.*

Proof. By what we have said the prorepresentability is standard. It remains to check that the functor is smooth. But this follows again by representing a display by a matrix. □

We will use the following version of the deformation functor. We take a $\mathcal{W}_k$-2-display $\mathcal{S}$ as above and we assume moreover that $\mathcal{S}$ is endowed with an isomorphism

$$\lambda_0 : \mathcal{S} \to \widehat{\mathcal{S}}. \tag{31}$$

We can also regard λ_0 as a pairing $\mathcal{S} \times \mathcal{S} \to \mathcal{U}(2)$. Then we define the deformation functor $\widehat{\mathcal{D}} : \mathcal{A}rt_k \to$ (sets). For $R \in \mathcal{A}rt_k$ we define $\widehat{\mathcal{D}}(R)$ as the set of isomorphism classes of $\widehat{\mathcal{W}}_R$-2-displays $\mathcal{P}$ together with an isomorphism $\lambda : \mathcal{P} \to \widehat{\mathcal{P}}$ and an isomorphism $\iota : \mathcal{S} \to \mathcal{P}_k$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\iota} & \mathcal{P}_k \\ {\scriptstyle\lambda_0}\downarrow & & \downarrow{\scriptstyle\lambda_k} \\ \widehat{\mathcal{S}} & \xleftarrow[\hat{\iota}]{} & \widehat{\mathcal{P}}_k \end{array}$$

We note that by the diagram and Corollary 14 the morphism λ is uniquely determined if it exists. Therefore we have an inclusion $\widehat{\mathcal{D}}(R) \subset \mathcal{D}(R)$. The map (30) for the functor $\widehat{\mathcal{D}}$ is also bijective. We will now find the tangent space of $\widehat{\mathcal{D}}$. More generally we consider a surjective homomorphism $S' \to R$ in $\mathcal{A}rt_k$ with kernel $\mathfrak{a}'$ such that $(\mathfrak{a}')^2 = 0$. We endow this with the trivial divided powers. Assume we are given $(\mathcal{P}, \iota, \lambda) \in \widehat{\mathcal{D}}(R)$. Then $\mathcal{P}$ lifts uniquely to a $\widehat{\mathcal{W}}_{S'/R}$-display $\mathcal{P}_{\mathrm{rel}}$ and λ lifts to an isomorphism $\lambda_{\mathrm{rel}} : \mathcal{P}_{\mathrm{rel}} \to \widehat{\mathcal{P}}_{\mathrm{rel}}$. Let $\mathcal{Q}$ be a $\widehat{\mathcal{W}}_{S'}$-2-display which lifts $\mathcal{P}$. Giving $\mathcal{Q}$ is the same as giving an admissible lifting of the Hodge filtration of $\mathcal{P}_{\mathrm{rel}}$. The dual display $\widehat{\mathcal{Q}}$ corresponds to the dual filtration of $\widehat{\mathcal{P}}_{\mathrm{rel}}$. But then $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$ are isomorphic if and only if λ_{rel} takes the filtration $\mathrm{Fil}_{\mathcal{Q}}$ given by $\mathcal{Q}$ to the dual filtration, i.e., $\mathrm{Fil}_{\mathcal{Q}}$ is selfdual with respect to the bilinear form

$$(\ ,\ ) : P_{\mathrm{rel},0}/I_{S'}P_{\mathrm{rel},0} \times P_{\mathrm{rel},0}/I_{S'}P_{\mathrm{rel},0} \to S' \tag{32}$$

induced by λ_{rel}.

Proposition 17. *Let $S' \to R$ be a pd-thickening with kernel $\mathfrak{a}'$ such that $(\mathfrak{a}')^2 = 0$. Let $(\mathcal{P}, \lambda)$ be an ordinary $\widehat{\mathcal{W}}_R$-2-display which is endowed with an isomorphism*

$\lambda : \mathcal{P} \to \widehat{\mathcal{P}}$. *We assume that* λ *is symmetric* (*i.e.*, $\lambda = \hat{\lambda}$) *and such that for the Hodge filtration* $\operatorname{rank}_R \operatorname{Fil}^2_{\mathcal{P}} = 1$. *We denote by* $\mathcal{P}_{\text{rel}}$ *the unique* $\widehat{\mathcal{W}}_{S'/R}$ *which lifts* $\mathcal{P}$.

The liftings of $(\mathcal{P}, \lambda)$ *to a* $\widehat{\mathcal{W}}_{S'}$*-2-display* $\mathcal{Q}$ *together with a lift of* λ *to an isomorphism* $\mu : \mathcal{Q} \to \widehat{\mathcal{Q}}$ *are in bijection with the liftings of* $\operatorname{Fil}^2_{\mathcal{P}}$ *to a isotropic direct summand of* $P_{\text{rel},0}/I_{S'}P_{\text{rel},0}$.

Proof. We know that λ lifts to an isomorphism $\lambda_{\text{rel}} : \mathcal{P}_{\text{rel}} \to \widehat{\mathcal{P}}_{\text{rel}}$. It follows from Corollary 15 that the liftings $(\mathcal{Q}, \mu)$ of $(\mathcal{P}, \lambda)$ are in bijection with selfdual admissible liftings of the Hodge filtration of $\mathcal{P}$.

The isomorphism λ_{rel} induces a perfect pairing (32) of S'-modules. We claim that the image of

$$P_{\text{rel},2} \to P_{\text{rel},0}/I_{S'}P_{\text{rel},0}$$

is isotropic under this pairing (32).

To verify this we take a normal decomposition of $\mathcal{P}_{\text{rel}}$,

$$P_{\text{rel},0} = L_0 \oplus L_1 \oplus L_2.$$

This induces the dual normal decomposition of $\widehat{\mathcal{P}}_{\text{rel}}$ (compare (22))

$$\widehat{P}_{\text{rel},0} = \widehat{L}_0 \oplus \widehat{L}_1 \oplus \widehat{L}_2,$$

where $\widehat{L}_0 = L_2^*,\ \widehat{L}_1 = L_1^*,\ \widehat{L}_2 = L_0^*$.

We set $L'_i = L_i/I_{S'}L_i \subset P_{\text{rel},0}/I_{S'}P_{\text{rel},0}$ and $\widehat{L}'_i = \widehat{L}_i/I_{S'}\widehat{L}_i \subset \widehat{P}_{\text{rel},0}/I_{S'}\widehat{P}_{\text{rel},0}$. Then the images of the two maps

$$P_{\text{rel},2} \to P_{\text{rel},0}/I_{S'}P_{\text{rel},0} \quad \text{and} \quad \widehat{P}_{\text{rel},2} \to \widehat{P}_{\text{rel},0}/I_{S'}\widehat{P}_{\text{rel},0}$$

are

$$L'_2 \oplus \mathfrak{a}'L_1 \quad \text{and} \quad \widehat{L}'_2 \oplus \mathfrak{a}'\widehat{L}_1$$

respectively. Since $(\mathfrak{a}')^2 = 0$ the last two modules are orthogonal with respect to the perfect pairing,

$$P_{\text{rel},0}/I_{S'}P_{\text{rel},0} \times \widehat{P}_{\text{rel},0}/I_{S'}\widehat{P}_{\text{rel},0} \to S'$$

induced by (23). Composing this with the isomorphism λ_{rel} we obtain the claim.

Next we show that any lift of $\operatorname{Fil}^2_{\mathcal{P}}$ to an isotropic direct summand $U \subset P_{\text{rel},0}/I_{S'}P_{\text{rel},0}$ is contained in the image of $P_{\text{rel},2}$. We take a splitting of the selfdual Hodge filtration of $\mathcal{P}$:

$$\operatorname{Fil}^2_{\mathcal{P}} = N'_2 \subset N'_2 \oplus N'_1 \subset N'_2 \oplus N'_1 \oplus N_0 = P_0/I_R P_0.$$

Then the perfect pairing induced by λ induces a perfect pairing $N'_2 \times N'_0 \to R$ and $(N'_2)^{\perp} = N'_2 \oplus N'_0$. It is easy to see that this lifts to a selfdual filtration

$$L'_2 \subset L'_2 \oplus L'_1 \subset L'_2 \oplus L'_1 \subset L'_2 \oplus L'_1 \oplus L'_0 = P_{\text{rel},0}/I_{S'}P_{\text{rel},0}$$

with respect to (32). Let l_2 be a basis of L'_2. Then U has a basis

$$u = l_2 + l_1 + l_0, \quad l_1 \in \mathfrak{a}' L'_1,\ l_2 \in \mathfrak{a}' L'_2.$$

We find

$$(u, u) = (l_2, l_0) + (l_0, l_2) = 2(l_2, l_0).$$

Because $L'_2 \times L'_0 \to S'$ is perfect, this implies $l_0 = 0$. Therefore u is in the image of $P_{\mathrm{rel},2}$.

From an isotropic lift $U \subset P_{\mathrm{rel},0}/I_{S'} P_{\mathrm{rel},0}$ we obtain a selfdual admissible lift of the Hodge filtration of $\mathcal{P}$, if we define $U = \mathrm{Fil}^2$ and Fil^1 to be the orthogonal complement of U. By Corollary 15 this gives a lifting $(\mathcal{Q}, \mu)$. □

In particular we see that lifts of $\mathcal{P}$ always exist. Therefore we obtain:

Corollary 18. *Let $\mathcal{S}, \lambda_0$ be a $\mathcal{W}_k$-2-display with a symmetric isomorphism* (31) *and such that for the Hodge filtration* $\dim_k \mathrm{Fil}^2_{\mathcal{S}} = 1$.

Then the functor $\widehat{\mathcal{D}}$ is prorepresentable by a power series ring over $W(k)$ in finitely many variables.

2. 2-displays of schemes

Let X_0 be a projective and smooth scheme over a perfect field k of characteristic $p > 2$. We make the following assumptions:

(1) Let $T_{X_0/k}$ be the tangent bundle of X_0. Then

$$H^0(X_0, T_{X_0/k}) = H^2(X_0, T_{X_0/k}) = 0. \tag{33}$$

(2) Let R be a local artinian $W(k)$-algebra with residue class field k and let $f : X \to \operatorname{Spec} R$ be an arbitrary deformation of X_0. Then the R-modules

$$R^j f_* \Omega^i_{X/R} \tag{34}$$

are free for $i + j \leq 2$ and commute with base change for morphisms $\operatorname{Spec} R' \to \operatorname{Spec} R$, where R' is a local artinian $W(k)$-algebra with residue class field k.

(3) The spectral sequence

$$E_1^{ij} = R^j f_* \Omega^i_{X/R} \Rightarrow \mathbb{R}^{j+i} f_* \Omega^{\cdot}_{X/R} \tag{35}$$

degenerates for $i + j \leq 2$, i.e., all differentials starting or ending at E_r^{ij} for $i + j \leq 2$, $r \geq 1$ are zero.

We remark that the last two requirements are fulfilled if

$$H^j(X_0, \Omega^i_{X_0/k}) = 0 \quad \text{for } i + j = 1 \text{ or } 3.$$

Assume that X_0 satisfies the three conditions above. Let R be a local $W(k)$-algebra whose maximal ideal $\mathfrak{m}$ is nilpotent and such that $R/\mathfrak{m} = k$. Let $g : Y \to R$

be a deformation of X_0. Then the last two conditions are also satisfied for g. Indeed R is the filtered union of local $W(k)$-algebras of finite type and X is automatically defined over a $W(k)$-algebra of finite type.

Assume again that the three conditions are fulfilled for X_0. Then there is a universal deformation, i.e., a morphism of formal schemes,

$$\mathfrak{X} \to \operatorname{Spf} A. \tag{36}$$

The adic ring A is the ring $W(k)[[T_1, \dots, T_r]]$ with $(p, T_1, \dots, T_r)$ as the ideal of definition. We have $r = \dim H^1(X_0, T_{X_0/k})$. We denote by σ the endomorphism of A such that $\sigma(T_i) = T_i^p$ and such that σ is the Frobenius on $W(k)$.

We are going to define a display structure on the de Rham–Witt cohomology of (36). For this we use the frames introduced in [Zink 2001b]. We call them w-frames in order to distinguish them from the frames introduced above. With respect to a w-frame we have the category of windows [Langer and Zink 2007, §5]. We use here the base change which associates to a window a display in our sense [loc. cit., Remark after Definition 5.1, pp. 181–182].

Let $n \geq 1$ be an integer. We set $C_n = W(k)[[T_1, \dots, T_r]]/(T_1^n, \dots, T_r^n)$ and $R_n = C_n/p^nC_n$. Then σ induces an endomorphism on C_n denoted by the same letter. We obtain that $\mathcal{C}_n = (C_n, p^nC_n, R_n, \sigma)$ is a w-frame. An obvious modification of [loc. cit., Corollary 5.6] shows that we have the structure of a $\widehat{\mathcal{W}}_{R_n}$-display on

$$H^2_{\text{crys}}(\mathcal{X}_{R_n}/\widehat{W}(R_n)).$$

This is obtained from the Lazard morphism $C_n \to W(R_n)$, which factors through

$$C_n \to \widehat{W}(R_n) \to W(R_n).$$

By Theorem 5.5 of [loc. cit.] we have a $\mathcal{C}_n$-window structure on $H^2_{\text{crys}}(\mathcal{X}_{R_n}/C_n)$. We can apply the base change of [loc. cit., Remark, p. 181] to obtain from a $\mathcal{C}_n$-window a $\widehat{\mathcal{W}}_{R_n}$-display.

This is functorial in R_n. If $f : X \to R$ is a deformation as in (34) we obtain for n big enough a unique $W(k)$-algebra homomorphism $R_n \to R$. Therefore we obtain by base change:

Proposition 19. *Let $f : X \to R$ be as above. Then the crystalline cohomology $H^2_{\text{crys}}(X/\widehat{W}(R))$ has the structure of a $\widehat{\mathcal{W}}_R$-display which is functorial in R.*

The uniqueness follows from the functoriality and the fact that $\widehat{W}(A)$ has no p-torsion.

We now show that X/R defines a crystal of displays in the following sense:

Corollary 20. *With the assumptions of the proposition let $S \to R$ be a pd-thickening where S is an artinian $W(k)$-algebra. Then we have the natural structure of a*

$\widehat{\mathcal{W}}_{S/R}$-display on $H^2_{\mathrm{crys}}(X/S)$. More precisely this structure is functorial with respect to morphisms of pd-thickenings and uniquely determined by this property.

Proof. We obtain a $\widehat{\mathcal{W}}_{S/R}$-2-display structure by lifting X to a smooth scheme X' over S and then making the base change with respect to $\widehat{\mathcal{W}}_S \to \widehat{\mathcal{W}}_{S/R}$. We show that the result is independent of the chosen lifting X'. Assume we have two liftings X' and X'' which are induced from the universal family (36) by two morphisms $A \to S$. We consider the following commutative diagram:

$$\begin{array}{ccc} A \,\widehat{\otimes}_{W(k)}\, A & \longrightarrow & S \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & R \end{array} \tag{37}$$

The left vertical arrow is the multiplication. Let J be the kernel. We denote the divided power hull of $(B := A \,\widehat{\otimes}_{W(k)}\, A, J)$ by P. It is obtained as follows: Let $A_0 = W(k)[T_1, \dots, T_r]$ and J_0 be the kernel of the multiplication $B_0 := A_0 \otimes_{W(k)} A_0 \to A_0$. We denote by P_0 the divided power hull of (B_0, J_0). Then P_0 is isomorphic to the divided power algebra of the free A_0-module with r generators. In particular P_0 is a free A_0-module for the two natural A_0-module structures. We have $P = P_0 \otimes_{B_0} B$. Then P is flat as a P_0-module and therefore without p-torsion. Then the diagram (37) extends to the following diagram:

$$\begin{array}{ccc} P & \longrightarrow & S \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & R \end{array} \tag{38}$$

Let $\widehat{P}$ be the p-adic completion of P. Then $\widehat{P} \to A$ is a frame $\mathcal{D}$. By [Langer and Zink 2007, Theorem 5.5] the universal family $\mathfrak{X}$ defines a $\mathcal{D}$-display $\mathcal{U}$. We consider also the trivial w-frame $\mathcal{D}_0 = (A, 0, A, \sigma)$. Again $\mathfrak{X}$ defines a $\mathcal{D}_0$-window $\mathcal{U}_0$. The two natural sections $A \to \widehat{P}$ define two morphisms of w-frames $\mathcal{D}_0 \to \mathcal{D}$. Since the construction of [loc. cit., Theorem 5.5] is compatible with base change, we obtain $\mathcal{U}$ from $\mathcal{U}_0$ by base change with respect to both of these two morphisms.

We consider the morphism of frames

$$\mathcal{D}_0 \rightrightarrows \mathcal{D} \to \widehat{\mathcal{W}}_{S/R}.$$

The two $\widehat{\mathcal{W}}_{S/R}$-displays associated with X' and X'' are obtained by base change from $\mathcal{U}_0$ by the two morphisms

$$\mathcal{D}_0 \rightrightarrows \widehat{\mathcal{W}}_{S/R}.$$

We see that these two $\widehat{\mathcal{W}}_{S/R}$-displays are both obtained by the base change of $\mathcal{U}$ with respect to $\mathcal{D} \to \widehat{\mathcal{W}}_{S/R}$. This shows that the $\widehat{\mathcal{W}}_{S/R}$ display does not, up to canonical isomorphism, depend on the lifting X' of X. □

Let $\mathfrak{X} \to \operatorname{Spf} A$ be the universal family (36). We set $A_0 = A/pA$ and we consider the formal scheme $\mathfrak{X}_0 = \mathfrak{X} \otimes_A A_0$. We write $H^2_{\mathrm{DR}}(\mathfrak{X}/A) = \mathbb{H}^2(\mathfrak{X}, \Omega^{\cdot}_{\mathfrak{X}/A})$ for the de Rham cohomology. By the isomorphism

$$H^2_{\mathrm{DR}}(\mathfrak{X}/A) = H^2_{\mathrm{crys}}(\mathfrak{X}_0/A),$$

we obtain a σ-linear endomorphism $F : H^2_{\mathrm{DR}}(\mathfrak{X}/A) \to H^2_{\mathrm{DR}}(\mathfrak{X}/A)$.

Lemma 21. *We consider the following subcomplex of* $\Omega^{\cdot}_{\mathfrak{X}/A}$:

$$\mathcal{F}^1\Omega^{\cdot}_{\mathfrak{X}/A} : p\Omega^0_{\mathfrak{X}/A} \to \Omega^1_{\mathfrak{X}/A} \to \Omega^2_{\mathfrak{X}/A} \to \cdots .$$

The natural map

$$\mathbb{H}^2(\mathfrak{X}, \mathcal{F}^1\Omega^{\cdot}_{\mathfrak{X}/A}) \to \mathbb{H}^2(\mathfrak{X}, \Omega^{\cdot}_{\mathfrak{X}/A})$$

is injective and the image is the set

$$\{x \in H^2_{\mathrm{DR}}(\mathfrak{X}/A) \mid Fx \in pH^2_{\mathrm{DR}}(\mathfrak{X}/A)\}.$$

The image of the natural map

$$\mathbb{H}^2(\mathfrak{X}, \mathcal{F}^1\Omega^{\cdot}_{\mathfrak{X}/A}) \to \mathbb{H}^2(\mathfrak{X}_0, \Omega^{\cdot}_{\mathfrak{X}_0/A_0})$$

is the Hodge filtration $\mathrm{Fil}^1 \subset H^2_{\mathrm{DR}}(\mathfrak{X}_0/A_0)$.

Proof. We use the notation before Proposition 19. If we take the projective limit of the $\mathcal{C}_n$-displays

$$H^2_{\mathrm{crys}}(\mathfrak{X}_{R_n}/C_n) = H^2_{\mathrm{crys}}(\mathfrak{X}_{C_n/pC_n}/C_n),$$

we obtain a display structure on $H^2_{\mathrm{DR}}(\mathfrak{X}, A)$ with respect to the frame $\mathcal{A} = (A, pA, A_0, \sigma, \sigma/p)$. We denote this display by $\mathcal{P} = (P_i, F_i, \iota_i, \alpha_i)$. It follows from [Langer and Zink 2007, Theorem 5.5] that P_1 is the hypercohomology $\mathbb{H}^2(\mathfrak{X}, \mathcal{F}^1\Omega_{\mathfrak{X}/A})$.

The first assertion of the lemma follows from the fact that for an $\mathcal{A}$-display

$$P_1 = \{x \in P_0 \mid F_0 x \in pP_0\}.$$

Indeed, we take a normal decomposition of $\mathcal{P}$, which in our case is a 2-display:

$$P_0 = L_0 \oplus L_1 \oplus L_2, \quad P_1 = pL_0 \oplus L_1 \oplus L_2.$$

Then the Frobenius F_0 is given by an invertible block matrix $\boldsymbol{D}$ with coefficients in A:

$$F_0\left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}\right) = \boldsymbol{D}\begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p^2 \end{pmatrix}\begin{pmatrix} \sigma(x_0) \\ \sigma(x_1) \\ \sigma(x_2) \end{pmatrix}.$$

Multiplying this equation by $\boldsymbol{D}^{-1}$ we see that the right-hand side is in pP_0 if and only if $\sigma(x_0) \in pL_0$ or equivalently $x_0 \in pL_0$. This proves the first assertion. The second assertion follows because by the definition of $\mathcal{P}$, the image of P_1 in $\mathbb{H}^2(\mathfrak{X}_0, \Omega_{\mathfrak{X}_0/A_0})$ is the Hodge filtration. □

3. The Beauville–Bogomolov form

Definition 22. A scheme of K3 type is a smooth and proper morphism $f : X \to S$ of relative dimension $2n$ with the following properties.

For each geometric point $\eta \to S$ we have

$$\begin{aligned} H^q(X_\eta, \Omega^p_{X_\eta/\eta}) &= 0 \quad \text{for } p+q=1,\ p+q=3, \\ \dim_{\kappa(\eta)} H^q(X_\eta, \mathcal{O}_{X_\eta}) &= 1 \quad \text{for } q=0,2, \\ \dim_{\kappa(\eta)} H^0(X_\eta, \Omega^2_{X_\eta/\eta}) &= 1. \end{aligned} \tag{39}$$

We assume that for each η there is a nowhere degenerate $\sigma \in H^0(X_\eta, \Omega^2_{X_\eta/\eta})$; i.e., $\sigma^n \in H^0(X_\eta, \Omega^{2n}_{X_\eta/\eta})$ defines an isomorphism,

$$\mathcal{O}_{X_\eta} \to \Omega^{2n}_{X_\eta/\eta}.$$

We assume that for each η there is a class $\rho \in H^2(X_\eta, \mathcal{O}_{X_\eta})$ such that ρ^n generates $H^{2n}(X_\eta, \mathcal{O}_{X_\eta})$. (We note that this is a 1-dimensional vector space by Serre duality.)

Finally we require that for each η the pairing

$$H^1(X_\eta, \Omega^1_{X_\eta/\eta}) \times H^1(X_\eta, \Omega^1_{X_\eta/\eta}) \to R, \quad \omega_1 \times \omega_2 \mapsto \int \omega_1\omega_2\sigma^{n-1}\rho^{n-1}, \tag{40}$$

is perfect.

We note that (40) is equivalent to saying that the cup product

$$\sigma^{n-1}\rho^{n-1} : H^1(X_\eta, \Omega^1_{X_\eta/\eta}) \to H^{2n-1}(X_\eta, \Omega^{2n-1}_{X_\eta/\eta})$$

is an isomorphism.

We denote by $\mathcal{T}_{X/S}$ the dual $\mathcal{O}_X$-module of $\Omega^1_{X/S}$. By definition σ induces a perfect pairing

$$\mathcal{T}_{X/S} \times \mathcal{T}_{X/S} \to \mathcal{O}_X, \tag{41}$$

or equivalently an isomorphism $\mathcal{T}_{X/S} \cong \Omega^1_{X/S}$.

Remarks. If X is a hyper-Kähler variety over $\mathbb{C}$ such that $H^3(X, \mathbb{C}) = 0$, then X is of K3 type. [Salamon 1996, Introduction; Huybrechts 1999, 1.1–1.3 and 1.7].

Over any field, a K3 surface is of K3 type. Over a field of characteristic 0, the Hilbert scheme $K^{[n]}$ of zero-dimensional subschemes of a K3 surface K is of K3 type [Beauville 1983, Theorem 3; Salamon 1996, Remark between 5.6 and 5.7 and Example, p. 149]. In fact all odd Betti numbers of $K^{[n]}$ vanish. This follows from a general formula proven by Göttsche [1990, Theorem 0.1].

By [Mumford 1970; Hartshorne 1977, Chapter III, §12] it follows that for $f : X \to S$ of K3 type the direct images $R^q f_* \Omega^p_{X/S}$ for $p + q = 2$ are locally free and commute with arbitrary base change and $f_* \mathcal{O}_X = \mathcal{O}_S$. Therefore locally on S we have sections $\sigma \in H^0(S, f_* \Omega^2_{X/S})$ and $\rho \in H^2(X, \mathcal{O}_X)$ which induce in each geometric fibre the classes required in the definition.

It follows from [Hartshorne 1977] that the set of points of S where a smooth and proper morphism $f : X \to S$ is of K3 type is open.

We therefore obtain varieties of K3 type as follows. Let S be a scheme of finite type and flat over $\mathbb{Z}$. We consider a smooth and proper morphism $f : X \to S$ such that $f_{\mathbb{Q}}$ is of K3 type. Then over an open subset of S, the morphism is of K3 type. In particular the schemes K3n are for almost all prime numbers p of K3 type over a field of characteristic p. The set of such primes p contains the set of primes for which Charles [2013, Corollary 5] recently proved the Tate conjecture for varieties of K3 type in characteristic p.

In the following we will assume without loss of generality that $S = \operatorname{Spec} R$ and that σ and ρ are globally defined. We note that σ is closed because $H^0(X, \Omega^3_{X/S}) = 0$.

Let X be a scheme of K3 type of dimension $2n$ over a ring R. It follows from our assumptions that $\epsilon := \int (\sigma\rho)^n \in R$ is a unit.

We regard $\sigma \in H^2_{\mathrm{DR}}(X/R)$ and we choose an arbitrary lifting $\tau \in H^2_{\mathrm{DR}}(X/R)$ of ρ. We have

$$\epsilon = \int \sigma^n \tau^n.$$

Definition 23. We assume that n, $n+1$ are units in R. We define the quadratic form $\mathbb{B}_{\sigma,\tau}(\alpha)$ on $H^2_{\mathrm{DR}}(X/R)$ by

$$\mathbb{B}_{\sigma,\tau}(\alpha) = \frac{n}{2} \int (\sigma\tau)^{n-1} \alpha^2 + \frac{1-n}{\epsilon} \Big(\int \sigma^{n-1} \tau^n \alpha \Big) \Big(\int \sigma^n \tau^{n-1} \alpha \Big) + \frac{1}{\epsilon^2} \frac{n(n-1)}{2(n+1)} \Big(\int \sigma^n \tau^{n-1} \alpha \Big)^2 \Big(\int \sigma^{n-1} \tau^{n+1} \Big). \tag{42}$$

Later we will consider the case $\epsilon = 1$.

Lemma 24. *We assume that R is an integral domain whose fraction field has characteristic 0. Let $\tau, \tau' \in H^2_{\mathrm{DR}}(X/R)$ be liftings of ρ. Then we have*

$$\mathbb{B}_{\sigma,\tau} = \mathbb{B}_{\sigma,\tau'}. \tag{43}$$

Up to a factor in R^* the form $\mathbb{B}_{\sigma,\tau}$ doesn't depend on the choice of σ, ρ, and τ.

If $R = \mathbb{C}$ it coincides with the usual Beauville–Bogomolov form [Beauville 1983, p. 772] up to a constant.

Proof. Let us assume the assertion (43). Then we show the uniqueness up to a constant. Let $\sigma_1 = u\sigma$ and $\tau_1 = v\tau$, where $u, v \in R^*$. We set $\mathbb{B} = \mathbb{B}_{\sigma,\tau}$ and

$\mathbb{B}_1 = \mathbb{B}_{\sigma_1,\tau_1}$. We have $\epsilon_1 = (\sigma_1\tau_1)^n = (uv)^n\epsilon$. We obtain

$$\mathbb{B}_1(\alpha) = \frac{n}{2}(uv)^{n-1}\int(\sigma\tau)^{n-1}\alpha^2 + \frac{1-n}{(uv)^n\epsilon}(uv)^{2n-1}\Big(\int\sigma^{n-1}\tau^n\alpha\Big)\Big(\int\sigma^n\tau^{n-1}\alpha\Big)$$
$$+\frac{1}{(uv)^{2n}\epsilon^2}\frac{n(n-1)}{2(n+1)}\Big(\int\sigma^n\tau^{n-1}\alpha\Big)^2(u^nv^{n-1})^2\Big(\int\sigma^{n-1}\tau^{n+1}\Big)u^{n-1}v^{n+1}.$$

The (uv)-factors in the last summand together yield the factor

$$\frac{1}{(uv)^{2n}}u^{3n-1}v^{3n-1} = (uv)^{n-1}.$$

Hence we get

$$\mathbb{B}_1(\alpha) = (uv)^{n-1}\mathbb{B}(\alpha). \tag{44}$$

So the form $\mathbb{B}(\alpha)$ changes by a unit in R.

We will now consider the case $R = \mathbb{C}$. We can use the Hodge decomposition. We take for ρ and τ the complex conjugate of σ; i.e., we set $\tau := \bar{\sigma}$. As $\bar{\sigma}^{n+1} = 0$ we obtain for the Beauville–Bogomolov form

$$\mathbb{B}_{\sigma,\bar{\sigma}}(\alpha) = \frac{n}{2}\int(\sigma\bar{\sigma})^{n-1}\alpha^2 + \frac{1-n}{\epsilon}\Big(\int\sigma^{n-1}\bar{\sigma}^n\alpha\Big)\Big(\int\sigma^n\bar{\sigma}^{n-1}\alpha\Big). \tag{45}$$

This is the usual Beauville–Bogomolov form, if we change σ by a constant such that $\epsilon = \int(\sigma\bar{\sigma})^n = 1$; see (42), [Beauville 1983, p. 772; Huybrechts 1999, 1.9].

Now let $\tau = a\sigma + \gamma + \bar{\sigma}$, where γ is a closed 1-1-form and $a \in \mathbb{C}$, so $\tau \in H^2_{\mathrm{DR}}(X/\mathbb{C})$ is a lifting of $\bar{\sigma} \in H^2(X, \mathcal{O}_X)$. We evaluate the forms $\mathbb{B}_{\sigma,\bar{\sigma}}$ and $\mathbb{B}_{\sigma,\tau}$ (the form in the definition) in an arbitrary form $\alpha \in H^2_{\mathrm{DR}}(X/\mathbb{C})$ and show that $\mathbb{B}_{\sigma,\bar{\sigma}}(\alpha) = \mathbb{B}_{\sigma,\tau}(\alpha)$. Without loss of generality let $\alpha = c\sigma + \beta + c'\bar{\sigma}$ with $c' \neq 0$ and after multiplication with a constant we can assume that $\alpha = c\sigma + \beta + \bar{\sigma}$, with β a closed 1-1-form, also without loss of generality $\epsilon = \int(\sigma\bar{\sigma})^n = 1$. Therefore $\alpha^2 = c^2\sigma^2 + 2c\sigma\beta + 2c\sigma\bar{\sigma} + 2\beta\bar{\sigma} + \bar{\sigma}^2 + \beta^2$.

Now we compute $\frac{n}{2}\int(\sigma\tau)^{n-1}\alpha^2$ for each summand:

$\underline{c^2\sigma^2}$:

$$\frac{n}{2}\int(\sigma\tau)^{n-1}c^2\sigma^2 = \frac{n}{2}\int\sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}c^2\sigma^2.$$

The p-degree (with respect to the p,q-Hodge decomposition) is $\geq 2n+2$. So this integral is zero.

$\underline{2c\sigma\beta}$:

$$n\int\sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}\sigma\beta = 0$$

because $\sigma^{n-1}\sigma\beta$ has already p-degree $2n+1$.

$\underline{2c\sigma\bar{\sigma}}$:

$$nc\int\sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}\sigma\bar{\sigma} = nc\int\sigma\bar{\sigma}^n.$$

So the third summand is independent from the choice of τ.

$\underline{2\beta\bar{\sigma}}$:
$$n\int\sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}\beta\bar{\sigma}=n\int\sum_{i+j+k=n-1}\frac{(n-1)!}{i!\,j!\,k!}\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\beta\bar{\sigma}.$$

The form $\sigma^{n-1+i}\bar{\sigma}^{k+1}\gamma^j\beta$ has p-degree $2n-2+2i+j+1$.

Hence the integral can be nonzero only for $i=0$ and $j=1$ and $k=n-2$. So the above integral is
$$n\int\frac{(n-1)!}{(n-2)!}\sigma^{n-1}\bar{\sigma}^{n-1}\beta\gamma.$$

This term depends on the choice of τ.

$\underline{\bar{\sigma}^2}$:
$$\frac{n}{2}\int\sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}\cdot\bar{\sigma}^2=\frac{n}{2}\sum_{i+j+k=n-1}\int\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^{k+2}\cdot\frac{(n-1)!}{i!j!k!}.$$

The p-degree of $\sigma^{n-1}\sigma^i\gamma^j\bar{\sigma}^{k+2}$ is $2(n-1)+2i+j$, which is $2n$ only for $i=1$, $j=0$ and $k=n-2$ or $i=0$, $j=2$, $k=n-3$.

In the first case ($i=1$, $j=0$ and $k=n-2$) the integral depends on the choice of τ. One gets the summand $\frac{1}{2}an(n-1)$.

In the second case ($i=0$, $j=2$, $k=n-3$) one gets the summand
$$\frac{(n-1)!}{2!\,(n-3)!}\frac{n}{2}\int\sigma^{n-1}\bar{\sigma}^{n-1}\gamma^2,$$
which depends on the choice of τ.

$\underline{\beta^2}$:
$$\frac{n}{2}\int\sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}\beta^2=\frac{n}{2}\sum_{i+j+k=n-1}\frac{(n-1)!}{i!\,j!\,k!}\int\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\beta^2.$$

The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\beta^2$ is $2n-2+2i+j+2$, which is $2n$ only if $i=0$ and $j=0$ and $k=n-1$.

Hence this integral does not depend on the choice of τ.

Adding up all cases we obtain (one may assume $\int(\sigma\bar{\sigma})^n=1$)
$$\frac{n}{2}\int(\sigma\bar{\sigma})^{n-1}\alpha^2=nc+n(n-1)\int(\sigma\bar{\sigma})^{n-1}\beta\gamma+(n-1)\frac{n}{2}a \\ +\frac{n(n-1)!}{4(n-3)!}\int(\sigma\bar{\sigma})^{n-1}\gamma^2+\frac{n}{2}\int\sigma^{n-1}\bar{\sigma}^{n-1}\beta^2. \quad (46)$$

Now we compute the other summands in $\mathbb{B}_{\sigma,\tau}(\alpha)$.

We have
$$\int\sigma^n\tau^{n-1}\alpha=\int\sigma^n(a\sigma+\gamma+\bar{\sigma})^{n-1}(c\sigma+\beta+\bar{\sigma})=\int\sigma^n\bar{\sigma}^n=1.$$

Then

$$\int \sigma^{n-1}\tau^n\alpha = \int \sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^n(c\sigma+\beta+\bar{\sigma})$$
$$= \sum_{i+j+k=n} \frac{n!}{i!\,j!\,k!}\int \sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k(c\sigma+\beta+\bar{\sigma}).$$

The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k c\sigma$ is $2n+2i+j$, which is $2n$ only for $i=j=0$ and $k=n$.

The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\beta$ is $2n-2+2i+j+1$, which is $2n$ only for $i=0,\ j=1$ and $k=n-1$.

The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\bar{\sigma}$ is $2n-2+2i+j$, which is $2n$ only for $i=1$, $j=0,\ k=n-1$ or for $i=0,\ j=2,\ k=n-2$.

Using this we get the formula

$$\int \sigma^{n-1}\tau^n\alpha$$
$$= c\int(\sigma\bar{\sigma})^n + n\int \sigma^{n-1}\gamma\bar{\sigma}^{n-1}\beta + n\int a\sigma^n\bar{\sigma}^n + \frac{n!}{2(n-2)!}\int \sigma^{n-1}\gamma^2\bar{\sigma}^{n-1}$$
$$= c+na+n\int(\sigma\bar{\sigma})^{n-1}\beta\gamma + \frac{n}{2}(n-1)\int(\sigma\bar{\sigma})^{n-1}\gamma^2. \tag{47}$$

Equations (46) and (47) then give us the following formula for the first two summands in $\mathbb{B}_{\sigma,\tau}(\alpha)$:

$$\frac{n}{2}\int(\sigma\tau)^{n-1}\alpha^2 + (1-n)\Big(\int \sigma^{n-1}\tau^n\alpha\Big)\Big(\int \sigma^n\tau^{n-1}\alpha\Big)$$
$$= nc + \frac{n}{2}(n-1)a + \frac{n(n-1)(n-2)}{4}\int(\sigma\bar{\sigma})^{n-1}\gamma^2$$
$$+ n(n-1)\int(\sigma\bar{\sigma})^{n-1}\beta\gamma + \frac{n}{2}\int(\sigma\bar{\sigma})^{n-1}\beta^2 + (1-n)c + (1-n)na$$
$$+ n(1-n)\int(\sigma\bar{\sigma})^{1-n}\beta\gamma + (1-n)\frac{n}{2}(n-1)\int(\sigma\bar{\sigma})^{n-1}\gamma^2. \tag{48}$$

A small calculation yields

$$\frac{n(n-1)(n-2)}{4} + \frac{(1-n)n(n-1)}{2} = -\frac{n^2}{4}(n-1).$$

Hence we can simplify (48) and get

$$\frac{n}{2}\int(\sigma\tau)^{n-1}\alpha^2 + (1-n)\Big(\int \sigma^{n-1}\tau^n\alpha\Big)\Big(\int \sigma^n\tau^{n-1}\alpha\Big)$$
$$= c + \frac{n}{2}\int(\sigma\bar{\sigma})^{n-1}\beta^2 - \frac{n}{2}(n-1)a - \frac{n^2}{4}(n-1)\int(\sigma\bar{\sigma})^{n-1}\gamma^2. \tag{49}$$

Now we compute the last summand in $\mathbb{B}_{\sigma,\tau}(\alpha)$:

$$\int \sigma^{n-1}\tau^{n+1} = \sum_{i+j+k=n+1} \frac{(n+1)!}{i!j!k!} \int \sigma^{n-1} a^i \sigma^i \gamma^j \bar{\sigma}^k.$$

The p-degree of $\sigma^{n-1} a^i \sigma^i \gamma^j \bar{\sigma}^k$ is $2(n-1)+2i+j$, which is $2n$ only for $i=1$, $j=0,\ k=n$ or for $i=0,\ j=2,\ k=n-1$.

So we get for the last summand in $\mathbb{B}_{\sigma,\tau}(\alpha)$

$$\frac{n(n-1)}{2(n+1)} \int \sigma^{n-1}\tau^{n+1} = \frac{n}{2}(n-1)a + \frac{n^2}{4}(n-1) \int (\sigma\bar{\sigma})^{n-1}\gamma^2. \tag{50}$$

(We have that $\int \sigma^n \tau^{n-1}\alpha = 1$ because of $\int (\sigma\bar{\sigma})^n = 1$.)

Then (49) and (50) yield

$$\mathbb{B}_{\sigma,\tau}(\alpha) = c + \frac{n}{2} \int (\sigma\bar{\sigma})^{n-1}\beta^2 = \mathbb{B}_{\sigma,\bar{\sigma}}(\alpha).$$

The last equality follows from [Huybrechts 1999, 1.9]. It proves (43) in the case $R = \mathbb{C}$.

We now prove (43) for arbitrary R. By base change we can replace R by it field of fractions K. Finally we may assume $K = \mathbb{C}$ by the Lefschetz principle. □

With the notation of Definition 23, let R be arbitrary. Let S be an integral domain whose field of fractions has characteristic 0 and let Y/S be a scheme of K3 type. We assume that there is a surjective ring homomorphism $S \to R$ such that X is obtained from Y by base change. Then Lemma 24 is true for R. This follows because the second cohomology groups commute with base change by assumption.

Remark. In the following proofs we will assume that there is a form $\rho \in H^2(X, \mathcal{O}_X)$ such that $1 = \epsilon = \int (\sigma\rho)^n$. If R is a strict henselian local ring whose residue characteristic is relatively prime to n, such a form ρ always exists. For $\epsilon = 1$ the form (42) doesn't up to a root of unity depend on the choices of σ and τ. For $R = \mathbb{C}$ it coincides then with the usual Beauville–Bogomolov form up to a root of unity of order n.

Lemma 25. *Let X be a scheme of K3 type over S. Then the form*

$$\mathbb{B}_{\sigma,\tau} : H^2_{\mathrm{DR}}(X/R) \times H^2_{\mathrm{DR}}(X/R) \to R$$

is perfect.

Proof. We can reduce to the case where R is a complete local ring with separably closed residue field. Then we may assume that $\int (\sigma\tau)^n = 1$. We do so to simplify the computation. We consider the Hodge filtration

$$H^0(X, \Omega^2_{X/R}) = \mathrm{Fil}^2 \subset \mathrm{Fil}^1 \subset H^2_{\mathrm{DR}}(X/R).$$

We claim that with respect to $\mathbb{B}_{\sigma,\tau}$

$$\mathrm{Fil}^1 \subset (\mathrm{Fil}^2)^{\perp}.$$

Let $\alpha \in \mathrm{Fil}^1$; we have to show that

$$\mathbb{B}_{\sigma,\tau}(\sigma, \alpha) = \tfrac{1}{2}(\mathbb{B}_{\sigma,\tau}(\sigma+\alpha) - \mathbb{B}_{\sigma,\tau}(\sigma) - \mathbb{B}_{\sigma,\tau}(\alpha)) = 0. \tag{51}$$

The second summand on the right-hand side is clearly 0. We note that $\sigma^n\alpha = 0$ because $\mathrm{Fil}^{2n} \cup \mathrm{Fil}^1 \subset \mathrm{Fil}^{2n+1} = 0$. We compute

$$\mathbb{B}_{\sigma,\tau}(\sigma+\alpha) = \frac{n}{2}\Big(\int (\sigma\tau)^{n-1}\sigma^2 + 2\int (\sigma\tau)^{n-1}\sigma\alpha + \int (\sigma\tau)^{n-1}(\alpha^2)\Big). \tag{52}$$

The other terms on the right-hand side of (42) vanish because $\sigma+\alpha \in \mathrm{Fil}^1$. We see that the first two terms on the right-hand side of (52) vanish. This shows that (51) vanishes too.

Therefore $\mathbb{B}_{\sigma,\tau}$ induces a bilinear form $\overline{\mathbb{B}}_{\sigma,\tau}$ on $\mathrm{Fil}^1/\mathrm{Fil}^2 = H^1(X, \Omega_{X/R})$. By the verification above we obtain

$$\overline{\mathbb{B}}_{\sigma,\tau}(\alpha) = \frac{n}{2}\int (\sigma\tau)^{n-1}\alpha^2 = \frac{n}{2}\int (\sigma\rho)^{n-1}\alpha^2.$$

By the requirement (40) this is perfect and

$$(\mathrm{Fil}^2)^{\perp} = \mathrm{Fil}^1. \tag{53}$$

Finally one has

$$\mathbb{B}_{\sigma,\tau}(\sigma, \tau) = \tfrac{1}{2},$$

which is a unit. We omit the easy verification. Together with the perfectness of $\overline{\mathbb{B}}_{\sigma,\tau}$ this implies perfectness. □

We prove the following proposition for the universal deformation of a variety of K3 type. The general case of a scheme of K3 type over an artinian local ring will follow from this (compare (73)).

Proposition 26. *Let X_0 be a scheme of K3 type over an algebraically closed field k. We assume that X_0 lifts to a smooth projective scheme over a discrete valuation ring O of characteristic zero with residue class field k. We consider the universal deformation of X_0*

$$X \to S = \mathrm{Spf}\, W(k)[[T_1, \dots, T_r]].$$

Assume that σ and τ are chosen such that $\epsilon = 1$. Then the form

$$\mathbb{B}_{\sigma,\tau} : H^2_{\mathrm{DR}}(X/S) \times H^2_{\mathrm{DR}}(X/S) \to W(k)[[T_1, \dots, T_r]]$$

is horizontal with respect to the Gauss–Manin connection (see [Deligne 1981b, Corollaire 2.3] for the definition).

Remark. Using the arguments of [Deligne 1981b] we will show in Proposition 29 that X_k lifts over some O if $H^1(X_0, \mathcal{T}_{X_0/k}) \neq 0$.

Proof. We begin by proving a complex-analytic version of this proposition. Let $X \to S$ be a proper and smooth morphism of complex analytic manifolds. Let $\Lambda \in H^0(X, R^2 f_* \mathbb{Q})$. Then we have a pairing

$$q_\Lambda : R^2 f_* \mathbb{Q} \times R^2 f_* \mathbb{Q} \to R^{2n} f_* \mathbb{Q} \xrightarrow{\int} \mathbb{Q}_S, \tag{54}$$

defined by

$$q_\Lambda(\alpha, \beta) = \int \Lambda^{2n-2} \alpha\beta, \quad \alpha, \beta \in R^2 f_* \mathbb{Q}.$$

If $\Lambda \in H^0(X, R^2 f_* \mathbb{Q})$ is the class of a relative ample line bundle on X then the pairing (54) is nondegenerate and we have

$$\nu := \nu(\Lambda) := \int \Lambda^{2n} \neq 0.$$

If we tensor (54) with $\mathcal{O}_S$ we obtain a horizontal pairing with respect to the Gauss–Manin connection:

$$q_\Lambda : H^2_{\mathrm{DR}}(X/S) \times H^2_{\mathrm{DR}}(X/S) \to \mathcal{O}_S. \tag{55}$$

Assume that Λ is a cohomology class such that q_Λ is nondegenerate and $\nu(\Lambda) \neq 0$. Then we denote by $(R^2 f_* \mathbb{Q})_0 \subset R^2 f_* \mathbb{Q}$ the local system which is the orthogonal complement of Λ. The vector bundle $H^2_{\mathrm{DR}}(X/S)$ decomposes as a vector bundle with connection

$$H^2_{\mathrm{DR}}(X/S) = (H^2_{\mathrm{DR}}(X/S))_0 \oplus \mathcal{O}_S \Lambda.$$

Lemma 27. *Let $\Lambda \in H^2_{\mathrm{DR}}(X/S)$ and $\nu(\Lambda) = \int \Lambda^{2n} \in \mathcal{O}_S$. Then we have the following formula for all $\alpha \in H^2_{\mathrm{DR}}(X/S)$:*

$$\nu(\Lambda)^2 \mathbb{B}_{\sigma,\tau}(\alpha) = \mathbb{B}_{\sigma,\tau}(\Lambda)\left[(2n-1)\nu(\Lambda)\int \Lambda^{2n-2}\alpha^2 - (2n-2)\left(\int \Lambda^{2n-1}\alpha\right)^2\right].$$

Proof. Both terms on each side are functions in $\mathcal{O}_S$. We consider them as functions on the complex manifold S. For each $s \in S(\mathbb{C})$ we evaluate the functions at s. The analogous equality in s, namely for $\Lambda_s, \alpha_s, \sigma_s, \tau_s \in H^2_{\mathrm{DR}}(X_s/\mathbb{C}) = H^2_{\mathrm{DR}}(X/S) \otimes_{\mathcal{O}_S} k(s)$,

$$\begin{aligned} &\nu(\Lambda_s)^2 \mathbb{B}_{\sigma_s,\tau_s}(\alpha_s) \\ &\quad = \mathbb{B}_{\sigma_s,\tau_s}(\Lambda_s)\left[(2n-1)\nu(\Lambda_s)\int \Lambda_s^{2n-2}\alpha_s^2 - (2n-2)\left(\int \Lambda_s^{2n-1}\alpha_s\right)^2\right] \end{aligned} \tag{56}$$

was shown in [Beauville 1983, Théorème 5(c)] for the Beauville–Bogomolov form over $\mathbb{C}$. Since this form differs from $\mathbb{B}_{\sigma_s,\tau_s}$ by a constant in $\mathbb{C}$ we obtain (56). Hence both functions coincide on $S(\mathbb{C})$. But then the algebraic functions in $\mathcal{O}_S$ coincide as well. $\square$

The formula (56) shows that for $\alpha \in (H^2_{\mathrm{DR}}(X/S))_0$

$$\nu^2 \mathbb{B}_{\sigma,\tau}(\alpha) = \mathbb{B}_{\sigma,\tau}(\Lambda)(2n-1)\nu q_\Lambda(\alpha). \tag{57}$$

Let $s \in S$. Since $\mathbb{B}_{\sigma_s,\tau_s}$ is up to a root of unity of order n the Beauville–Bogomolov form, we know that $\mathbb{B}_{\sigma_s,\tau_s}(\Lambda_s)$ is a real number times an n-th root of unity by [Beauville 1983, Théorème 5(a)]. From this it follows that the analytic function $\mathbb{B}_{\sigma,\tau}(\Lambda)$ on S is constant. Therefore $\mathbb{B}_{\sigma,\tau}$ is by (57) a horizontal form with respect to the Gauss–Manin connection on the bundle $(H^2_{\mathrm{DR}}(X/S))_0$.

We show that $(H^2_{\mathrm{DR}}(X/S))_0$ and $\mathcal{O}_S$ are orthogonal for the form $\mathbb{B}_{\sigma,\tau}$ too. We have to show that

$$\nu^2(\mathbb{B}_{\sigma,\tau}(\alpha+\Lambda) - \mathbb{B}_{\sigma,\tau}(\alpha) - \mathbb{B}_{\sigma,\tau}(\Lambda)) = 0 \tag{58}$$

for all $\alpha \in (H^2_{\mathrm{DR}}(X/S))_0$. From Lemma 27 we obtain

$$\nu^2 \mathbb{B}_{\sigma,\tau}(\alpha+\Lambda) = \mathbb{B}_{\sigma,\tau}(\Lambda)\left[(2n-1)\nu \int \Lambda^{2n-2}\alpha^2 + (2n-1)\nu^2 - (2n-2)\nu^2\right].$$

From this one obtains (58). Therefore it suffices to show that $\mathbb{B}_{\sigma,\tau}$ is horizontal on the subbundle $\mathcal{O}_S\Lambda \subset H^2_{\mathrm{DR}}(X/S)$. This is equivalent to saying that $\mathbb{B}_{\sigma,\tau}(\Lambda) \in \mathcal{O}_S$ is a constant function. This we have seen above.

Now we prove Proposition 26. The formal scheme $\operatorname{Spf} W(k)[[T_1, \dots, T_r]]$ has $(p, T_1, \dots, T_r)$ as an ideal of definition and this is also an ideal of definition for X.

On the other hand we can consider the universal deformation of a lifting $\widetilde{X}/O$ of X_0, which exists by assumption. Then we obtain a formal scheme Y over $\operatorname{Spf} O[[T_1, \dots, T_d]]$, where the ideal of definition in the last ring is now $(T_1, \dots, T_d)$. We may assume that O is complete. Then we a have a natural map

$$W(k)[[T_1, \dots, T_r]] \to O[[U_1, \dots, U_d]], \tag{59}$$

which corresponds on the tangent spaces to the natural homomorphism

$$H^1(\widetilde{X}, \Omega_{\widetilde{X}/O}) \to H^1(X_0, \Omega_{X_0/k}). \tag{60}$$

Therefore we have $r = d$ and we may arrange after a coordinate transformation that $T_i \mapsto U_i + a_i$ by the map (59), where the a_i are in the maximal ideal of O. We see that the regular parameter system $(p, T_1, \dots, T_d)$ of the local ring on the left-hand side of (59) is mapped to a parameter system on the right-hand side. Therefore the morphism (59) is injective. By definition, the push-forward of X by (59) is the completion of Y in the adic topology defined by the maximal ideal. Because the map induced by (59) on the de Rham cohomology is also injective, it suffices to show that the Beauville–Bogomolov form of the family Y is horizontal. We take an embedding $O \to \mathbb{C}$. Then we obtain the universal deformation of $\widetilde{X}_{\mathbb{C}}$. It suffices to show that $\mathbb{B}_{\sigma,\tau}$ is horizontal for the Gauss–Manin connection of this family. Since we obtain this by completion of the Kuranishi family $f : \mathfrak{X} \to S$ of $\widetilde{X}_{\mathbb{C}}$, we are reduced to the case above. We have to ensure that there is a cohomology class $\Lambda \in H^0(S, R^2 f_*\mathbb{Q})$ such that

$$q_\Lambda \text{ is nondegenerate} \quad \text{and} \quad \nu(\Lambda) \neq 0. \tag{61}$$

Let $s_0 \in S$ be the point such that $f^{-1}(s_0) = \widetilde{X}_{\mathbb{C}}$. Let

$$\Lambda_0 \in (R^2 f_*\mathbb{Q})_{s_0} = H^2(\widetilde{X}_{\mathbb{C}}, \mathbb{Q})$$

be the cohomology class of an ample line bundle on $\widetilde{X}_{\mathbb{C}}$. By shrinking S we may assume that $R^2 f_*\mathbb{Q}$ is a constant local system on S. But then Λ_0 extends to a global section Λ of $R^2 f_*\mathbb{Q}$. Then Λ meets the requirements (61). This proves Proposition 26. □

4. Deformations of varieties of K3 type

Let X_0/k be a projective and smooth scheme of K3 type over a perfect field k of characteristic $p \geq 3$. We consider the universal deformation

$$\mathfrak{X} \to S = \operatorname{Spf} A,$$

where

$$A = W[\![T_1, \dots, T_r]\!], \quad r = \dim_k H^1(X_0, \mathcal{T}_{X_0/k}).$$

We consider the Gauss–Manin connection

$$\nabla : H^2_{\mathrm{DR}}(\mathfrak{X}/S) \to H^2_{\mathrm{DR}}(\mathfrak{X}/S) \otimes_A \Omega^1_{S/W}.$$

If we compose this with the natural maps

$$\partial/\partial t_i : \Omega^1_{S/W} \to A, \quad i = 1, \dots, r,$$

we obtain the maps

$$\nabla_i : H^2_{\mathrm{DR}}(\mathfrak{X}/S) \to H^2_{\mathrm{DR}}(\mathfrak{X}/S).$$

The de Rham cohomology is endowed with the Hodge filtration

$$0 \subset \operatorname{Fil}^2 H^2_{\mathrm{DR}}(\mathfrak{X}/S) \subset \operatorname{Fil}^1 H^2_{\mathrm{DR}}(\mathfrak{X}/S) \subset \operatorname{Fil}^0 H^2_{\mathrm{DR}}(\mathfrak{X}/S) = H^2_{\mathrm{DR}}(\mathfrak{X}/S).$$

We have $\operatorname{Fil}^2 H^2_{\mathrm{DR}}(\mathfrak{X}/S) = H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S})$. We denote by $\operatorname{gr}^t H^2_{\mathrm{DR}}(\mathfrak{X}/S)$ the subquotients of this filtration. By Griffiths transversality, ∇ induces a map

$$\operatorname{gr}^t \nabla : \operatorname{gr}^t H^2_{\mathrm{DR}}(\mathfrak{X}/S) \to \operatorname{gr}^{t-1} H^2_{\mathrm{DR}}(\mathfrak{X}/S) \otimes_A \Omega^1_{S/W}, \tag{62}$$

which is A-linear. We are interested in this map for $t = 2$. By duality we obtain an A-linear map

$$\mathcal{T}_{S/W} \to \operatorname{Hom}_A(H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S}), H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S})). \tag{63}$$

It is proved for K3 surfaces in [Deligne 1981b] that this is an isomorphism. The same argument works for varieties of K3 type. Indeed, the map (63) factors as

$$\mathcal{T}_{S/W} \to H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/S}) \to \operatorname{Hom}_A(H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S}), H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S})).$$

The first arrow is the Kodaira–Spencer map, which is an isomorphism, and the second map is the cup product. To see that the second map is an isomorphism we choose a generator $\omega \in H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S})$. The multiplication with ω induces an isomorphism $\mathcal{T}_{\mathfrak{X}/S} \cong \Omega^1_{\mathfrak{X}/S}$. Therefore the cup product with ω is an isomorphism

$$H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/S}) \cong H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S}).$$

This proves that (63) is an isomorphism. The isomorphism (63) signifies that $\nabla_1(\omega), \dots, \nabla_r(\omega)$ is a basis of $H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S})$.

Lemma 28. *The maps* $\operatorname{gr}^t \nabla$ (62) *are for* $t = 1, 2$ *split injections of* A*-modules.*

Proof. Clearly it is enough to show that the maps

$$\mathcal{T}_{S/W} \to \operatorname{Hom}_A(\operatorname{gr}^t H^2_{\mathrm{DR}}(\mathfrak{X}/S), \operatorname{gr}^{t-1} H^2_{\mathrm{DR}}(\mathfrak{X}/S))$$

induced by $\operatorname{gr}^t \nabla$, where $t = 1, 2$, are isomorphisms. We have already seen this for $t = 2$. For $t = 1$ we have the A-module homomorphism

$$\mathcal{T}_{S/W} \to \operatorname{Hom}_A(H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S}), H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})). \tag{64}$$

The Kodaira–Spencer map gives an isomorphism $\mathcal{T}_{S/W} \otimes_A k \cong H^1(X_0, \mathcal{T}_{X_0/k})$. Using the Nakayama's lemma we see that (64) is an isomorphism if and only if the cup product induces a perfect pairing of k-vector spaces:

$$H^1(X_0, \mathcal{T}_{X_0/k}) \times H^1(X_0, \Omega^1_{X_0/k}) \to H^2(X_0, \mathcal{O}_{X_0}). \tag{65}$$

This follows from the definition of a variety of K3 type and the following commutative diagram in the notation of Definition 22:

$$\begin{array}{ccccc}
H^1(X_0, \mathcal{T}_{X_0/k}) \times & H^1(X_0, \Omega^1_{X_0/k}) & \longrightarrow & H^2(X_0, \mathcal{O}_{X_0}) \\
\big\downarrow \cup\sigma & \big\downarrow \mathrm{id} & & \big\downarrow \cup\sigma \\
H^1(X_0, \Omega^1_{X_0/k}) \times & H^1(X_0, \Omega^1_{X_0/k}) & \longrightarrow & H^2(X_0, \Omega^2_{X_0/k}) \\
 & & & \big\downarrow \cup\sigma^{n-1}\rho^{n-1} \\
 & & & H^{2n}(X_0, \Omega^{2n}_{X_0/k})
\end{array}$$

The composition of the two vertical arrows on the right-hand side is an isomorphism because $\sigma^n \rho^n$ is a generator of $H^{2n}(X_0, \Omega^{2n}_{X_0/k})$. The perfectness follows therefore from the perfectness of the pairing (40). □

Proposition 29. *Let X_0/k be a scheme of K3 type over a perfect field k and let L_0 be a very ample line bundle on X_0. We assume that* $\dim H^1(X_0, \mathcal{T}_{X_0/k}) \geq 1$.

Then there is a projective scheme X over a discrete valuation ring O of mixed characteristic with residue class field k and a very ample line bundle L on X such that

$$(X, L) \otimes_O k \cong (X_0, L_0).$$

If we assume moreover that the first Chern class $c_1(L_0) \in H^1(X_0, \Omega^1_{X_0/k})$ is not zero, a projective scheme X exists over $W = O$.

Proof. We follow the proof of [Deligne 1981b] for K3 surfaces but we indicate in detail the necessary changes for varieties of K3 type. We consider the functor $\mathrm{Def}(X_0, L_0)$ of infinitesimal deformations of the pair (X_0, L_0). This functor is represented by a closed subscheme of $S = \mathrm{Spf}\, A$ given by a single equation $f \in A$. For this the arguments of [Deligne 1981b] apply with no changes.

We will show that $f \notin pA$. We assume $f \in pA$ and deduce a contradiction. In this case L_0 lifts to a line bundle $\mathcal{L}_0$ on $\mathfrak{X}_0 = \mathfrak{X} \otimes_A A_0$, where $A_0 = A/pA$. We consider the Chern class

$$x = c_1(\mathcal{L}_0) \in H^2_{\mathrm{crys}}(\mathfrak{X}_0/A) = H^2_{\mathrm{DR}}(\mathfrak{X}/A).$$

It is not zero because $c_1(L_0)$ is not zero by our assumption that L_0 is very ample. By general facts on Chern classes we have [Deligne 1981b, Proposition 2.9]

$$Fx = px, \quad \nabla x = 0.$$

We write $x = p^m y$, where $y \in H^2_{\mathrm{DR}}(\mathfrak{X}/A)$ is not divisible by p. Then $y_0 = y \mod p \in H^2_{\mathrm{DR}}(\mathfrak{X}_0/A_0)$ is not trivial. Since p is not a zero divisor in $H^2_{\mathrm{DR}}(\mathfrak{X}/A)$ we have

$$Fy = py, \quad \nabla y = 0.$$

It follows from Lemma 21 that $y_0 \in \mathrm{Fil}^1 H^2_{\mathrm{DR}}(\mathfrak{X}_0/A_0)$. Lemma 28 shows that the following maps are for $t = 1, 2$ injective:

$$\mathrm{gr}^t \nabla : \mathrm{gr}^t H^2_{\mathrm{DR}}(\mathfrak{X}_0/A_0) \to \mathrm{gr}^{t-1} H^2_{\mathrm{DR}}(\mathfrak{X}_0/A_0) \otimes_A \Omega^1_{A_0/W}.$$

The injectivity of these maps and $\nabla y_0 = 0$ implies that $y_0 = 0$. This contradiction shows that $f \notin pA$.

We set $B = A/fA$. The universal line bundle $\mathcal{L}$ on $\mathfrak{X}_B$ is very ample. Since A is factorial, the prime p is not a zero divisor in B and in particular $B[1/p] \neq 0$.

Let $\tilde{\mathfrak{q}}$ a maximal ideal in $B[1/p]$. We set $\mathfrak{q} = \tilde{\mathfrak{q}} \cap B$. The normalization O of $B/\mathfrak{q}$ is the desired ring.

It remains to prove the last statement. It follows from [Ogus 1979, Corollary 1.14] and the perfectness of the pairing (65) that (X_0, L_0) doesn't lift to $A_0/\mathfrak{m}_0^2$, where $\mathfrak{m}_0$ denotes the maximal ideal of A_0. But this implies that $B = A/fA$ is a power series ring over W. In particular we find an augmentation $B \to W$. We obtain the desired scheme by base change of $\mathfrak{X}_B$. □

Let $\alpha : R' \to R$ be a surjective homomorphism of local artinian W-algebras with residue class field k. We set $\mathfrak{a} = \operatorname{Ker} \alpha$. We assume that $\mathfrak{a}\mathfrak{m}_{R'} = 0$, where $\mathfrak{m}_{R'}$ denotes the maximal ideal of R'. Let X/R be a deformation of X_0 and let X' be a deformation of X over R'. We have a natural isomorphism $H^2_{\text{crys}}(X/R') \cong H^2_{\text{DR}}(X'/R')$.

Let Y/R' be another deformation of X. Then we obtain a natural isomorphism

$$H^2_{\text{DR}}(X'/R') \to H^2_{\text{DR}}(Y/R'). \tag{66}$$

There is an explicit formula for this isomorphism in terms of the Gauss–Manin connection on the universal deformation S; see (69) below.

We denote by $F_Y \in H^2_{\text{DR}}(X'/R')$ the preimage of

$$H^0(Y, \Omega^2_{Y/R'}) = \operatorname{Fil}^2 H^2_{\text{DR}}(Y/R') \subset H^2_{\text{DR}}(Y/R')$$

by the isomorphism (66).

Proposition 30. *We assume that* $\mathfrak{a}\mathfrak{m}_{R'} = 0$. *The direct summand* $F_Y \subset H^2_{\text{DR}}(X'/R')$ *is contained in* $\operatorname{Fil}^1 H^2_{\text{DR}}(X'/R')$. *The map* $Y \mapsto F_Y$ *is a bijection between isomorphism classes of liftings* Y/R' *of* X/R *and direct summands* $F \subset \operatorname{Fil}^1 H^2_{\text{DR}}(X'/R')$ *which lift the direct summand* $\operatorname{Fil}^2 H^2_{\text{DR}}(X/R) \subset \operatorname{Fil}^1 H^2_{\text{DR}}(X/R)$.

Proof. We set $F' = \operatorname{Fil}^2 H^2_{\text{DR}}(X'/R')$. Let $F \subset H^2_{\text{DR}}(X'/R')$ be an arbitrary direct summand which lifts $\operatorname{Fil}^2 H^2_{\text{DR}}(X/R)$. We call this a lift of the Hodge filtration.

We consider the canonical map

$$F \to H^2_{\text{DR}}(X'/R')/F'. \tag{67}$$

Its image is in $\mathfrak{a}(H^2_{\text{DR}}(X'/R')/F') \cong \mathfrak{a} \otimes_k (H^2_{\text{DR}}(X_0/k)/\operatorname{Fil}^2 H^2_{\text{DR}}(X_0/k))$. The map (67) factors through $F \to \operatorname{Fil}^2 H^2_{\text{DR}}(X_0/k)$. Therefore liftings of the Hodge filtration are classified by homomorphisms of k-vector spaces

$$\varkappa(F) : H^0(X_0, \Omega^2_{X_0/k}) \to \mathfrak{a} \otimes_k (H^2_{\text{DR}}(X_0/k)/\operatorname{Fil}^2 H^2_{\text{DR}}(X_0/k)). \tag{68}$$

The assertion that $F_Y \subset \operatorname{Fil}^1 H^2_{\text{DR}}(X'/R')$ is equivalent to saying that

$$\varkappa(F_Y)(H^0(X_0, \Omega^2_{X_0/k})) \subset \mathfrak{a} \otimes_k (\operatorname{Fil}^1 H^2_{\text{DR}}(X_0/k)/\operatorname{Fil}^2 H^2_{\text{DR}}(X_0/k)).$$

The deformation X'/R' of X_0 is given by a uniquely determined W-algebra homomorphism $f : A \to R'$ and the deformation Y is given by $g : A \to R'$. We obtain a diagram such that the two compositions are equal:

$$A \overset{f}{\underset{g}{\rightrightarrows}} R' \longrightarrow R.$$

The isomorphism (66) is obtained as follows. Let $u \in H^2_{\mathrm{DR}}(X'/R')$. We find $\tilde{u} \in H^2_{\mathrm{DR}}(\mathfrak{X}/S)$ such that $u = f_*(\tilde{u})$. We set $v = g_*(\tilde{u})$. Then (66) is given as follows [Deligne 1981a, Lemma 1.1.2]:

$$H^2_{\mathrm{DR}}(X'/R') \to H^2_{\mathrm{DR}}(Y/R'), \quad u \mapsto v + \sum_{i=1}^{r} (f(t_i) - g(t_i)) \breve{\nabla}_i(\tilde{u}). \tag{69}$$

We denote here by $\breve{\nabla}_i(\tilde{u})$ the image of $\nabla_i(\tilde{u})$ in $H^2_{\mathrm{DR}}(X_0/k)$. The formula (69) makes sense because $f(t_i) - g(t_i) \in \mathfrak{a}$.

Now we take for $\tilde{u}$ a generator of the free A-module $\mathrm{Fil}^2\, H^2_{\mathrm{DR}}(\mathfrak{X}/S)$. We deduce from (66) that

$$u - \sum_{i=1}^{r} (f(t_i) - g(t_i)) \breve{\nabla}_i(\tilde{u})$$

is a generator of F_Y. Let $u_0 \in \mathrm{Fil}^2\, H^2_{\mathrm{DR}}(X_0/k)$ be the image of $\tilde{u}$. Then the map $\varkappa(F_Y)$ is given by

$$\varkappa(F_Y)(u_0) = -\sum_{i=1}^{r} (f(t_i) - g(t_i)) \otimes \breve{\nabla}_i(\tilde{u}) \in \mathfrak{a} \otimes_k \mathrm{gr}^1\, H^2_{\mathrm{DR}}(X_0/k).$$

This formula shows that $F_Y \subset \mathrm{Fil}^1\, H^2_{\mathrm{DR}}(X/R)$. As we remarked, (63) implies that $\breve{\nabla}_i(\tilde{u})$ form a basis of $\mathrm{gr}^1\, H^2_{\mathrm{DR}}(X_0/k)$. It follows that F_Y determines the elements $a_i := f(t_i) - g(t_i) \in \mathfrak{a}$ for $i = 1, \dots, r$. Given such elements a_i we define $g(t_i) = f(t_i) - a_i$. The homomorphism $g : A \to R'$ thus defined gives the desired variety of K3 type. □

We will now extend the proposition to the case where $R' \to R$ is an arbitrary pd-thickening with nilpotent divided powers on $\mathfrak{a}$.

We assume now that k is algebraically closed. We assume that $2n = \dim X_0$ is prime to the characteristic p of k. We also assume that X_0 lifts to a smooth projective scheme over some discrete valuation ring O with residue class field k. We fix generators σ and ρ of the 1-dimensional k-modules $H^0(X_0, \Omega^2_{X_0/k})$ and $H^2(X_0, \mathcal{O}_{X_0})$ respectively such that $\int (\sigma\rho)^n = 1$. We can lift them to generators $\tilde{\sigma}$ and $\tilde{\rho}$ of the cohomology groups $H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S})$ and $H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ respectively. Then we obtain by Proposition 26 and Lemma 25 a horizontal perfect symmetric pairing

$$(\ ,\ ) : H^2_{\mathrm{DR}}(\mathfrak{X}/S) \times H^2_{\mathrm{DR}}(\mathfrak{X}/S) \to S, \tag{70}$$

which depends only on σ and ρ. With respect to this pairing the Hodge filtration is self dual:

$$(\mathrm{Fil}^1)^{\perp} = \mathrm{Fil}^2, \quad (\mathrm{Fil}^2)^{\perp} = \mathrm{Fil}^1 .$$

In the situation of the proposition it is equivalent to say that the lift of the Hodge filtration $F \subset H^2_{\mathrm{DR}}(X'/R')$ is in $\mathrm{Fil}^1 H^2_{\mathrm{DR}}(X'/R')$ or that $F \subset H^2_{\mathrm{DR}}(X'/R')$ is isotropic. Indeed, we take $\tilde{c} \in H^2_{\mathrm{DR}}(\mathfrak{X}/S)$, which induces a generator of the A-module $H^2_{\mathrm{DR}}(\mathfrak{X}/S)/\mathrm{Fil}^1$. Then $(\tilde{u}, \tilde{c})$ is a unit in A. The image c of $\tilde{c}$ in $H^2_{\mathrm{DR}}(X'/R')$ induces a basis of $H^2_{\mathrm{DR}}(X'/R')/\mathrm{Fil}^1 H^2_{\mathrm{DR}}(X'/R')$.

Any lifting of the Hodge filtration has a generator of the form

$$u + \beta c + \sum_{i=1}^{r} \alpha_i \check{\nabla}_i(\tilde{u}), \quad \alpha, \beta \in \mathfrak{a}. \tag{71}$$

Assume F is isotropic. Since u is orthogonal to $\check{\nabla}_i(\tilde{u})$ we obtain $2\beta(u, c) = 0$ which implies $\beta = 0$. This implies $F \subset \mathrm{Fil}^1 H^2_{\mathrm{DR}}(X'/R')$. On the other hand the vector (71) is isotropic if $\beta = 0$.

Theorem 31. *Let X_0 be a projective scheme of K3 type over an algebraically closed field k of characteristic $p > 0$.*

Let $\alpha : R' \to R$ be a surjective morphism of artinian local W-algebras with residue class field k. We assume that the kernel $\mathfrak{a}$ of α is endowed with nilpotent divided powers which are compatible with the canonical divided powers on pW.

Let X/R be a deformation of X_0 and X'/R' a lifting of X.

If Y/R' is an arbitrary lifting of X, the Gauss–Manin connection provides an isomorphism

$$H^2_{\mathrm{DR}}(X'/R') \to H^2_{\mathrm{DR}}(Y/R') \tag{72}$$

which respects the symmetric bilinear forms on both sides. We denote by F_Y the preimage of $\mathrm{Fil}^2 H^2_{\mathrm{DR}}(Y/R')$ *by this isomorphism.*

The map $Y \mapsto F_Y$ is a bijection between liftings Y/R' of X and liftings of the Hodge filtration $\mathrm{Fil}^2 H^2_{\mathrm{DR}}(X/R) \subset H^2_{\mathrm{DR}}(X/R)$ *to isotropic direct summands $F \subset H^2_{\mathrm{DR}}(X'/R')$.*

Proof. The assertion that (72) respects the pairing $(\ ,\ )$ follows because the pairing is horizontal. Therefore F_Y is isotropic.

We consider the divided powers of the ideal $\mathfrak{a}$:

$$\mathfrak{a} \supset \mathfrak{a}^{[2]} \supset \cdots \supset \mathfrak{a}^{[t-1]} \supset \mathfrak{a}^{[t]} = 0.$$

If $t = 2$, the theorem follows from the proposition. We consider the nilpotent pd-thickenings

$$R' \to R'/\mathfrak{a}^{[t-1]} \to R.$$

By induction we may assume that the theorem holds for the second thickening.

We start with an isotropic lifting $F \subset H^2_{\mathrm{DR}}(X'/R')$ of the Hodge filtration. Let $R_1 = R/\mathfrak{a}^{[t-1]}$. Then F induces a filtration $F_1 \subset H^2_{\mathrm{DR}}(X'_{R_1}/R_1)$. By induction there is a lifting Z/R_1 of X which corresponds to F_1. We choose an arbitrary lifting Z'/R' of Z. Since Z' is also a lifting of X, we have an isomorphism

$$H^2_{\mathrm{DR}}(X'/R') \to H^2_{\mathrm{DR}}(Z'/R').$$

Let G be the image of F under this isomorphism. Then G is a lifting of the Hodge filtration $\mathrm{Fil}^2\, H^2_{\mathrm{DR}}(Z/R_1) \subset H^2_{\mathrm{DR}}(Z/R_1)$. If the proposition is applicable to $R' \to R_1$ we find a lifting Y/R' of Z/R_1 which corresponds to $G \subset H^2_{\mathrm{DR}}(Z'/R')$ and therefore to $F \subset H^2_{\mathrm{DR}}(X'/R')$. Thus our map is surjective.

Therefore it suffices to show our theorem for $R' \to R_1$. The kernel $\mathfrak{b} = \mathfrak{a}^{[t-1]}$ is endowed with the trivial divided powers and we have $\mathfrak{b}^2 = 0$. Decomposing $R' \to R_1$ into a series of small surjections (as in the proposition) $R' \to R_m \to \cdots \to R_1$, we may argue as above.

The injectivity follows easily in the same manner. □

We may reformulate this in the language of crystals. Let X be the deformation of X_0 over an artinian local ring R with residue class field k (or equivalently a continuous homomorphism $A \to R$).

Suppose $R' \to R$ is a nilpotent pd-thickening where $R' \to R$ is a homomorphism of local artinian rings with residue field k. We consider the crystalline cohomology

$$H^2_{\mathrm{crys}}(X/R').$$

This is a crystal in R' which is induced from the Gauss–Manin connection on $H^2_{\mathrm{DR}}(\mathfrak{X}/S)$. Therefore (70) induces a bilinear form of crystals

$$H^2_{\mathrm{crys}}(X/R') \times H^2_{\mathrm{crys}}(X/R') \to R'. \tag{73}$$

The Hodge filtration on $H^2_{\mathrm{DR}}(X/R) = H^2_{\mathrm{crys}}(X/R)$ is selfdual with respect to this bilinear form.

We may reformulate the last theorem.

Corollary 32. *Let $R' \to R$ be a surjective homomorphism of artinian local rings with algebraically closed residue class field k whose kernel is endowed with nilpotent divided powers compatible with p.*

Let X/R be a deformation of X_0. Then the liftings of X to X' correspond bijectively to liftings of the Hodge filtration to selfdual filtrations of $H^2_{\mathrm{crys}}(X/R')$.

Corollary 33. *Let $R' \to R$ be a nilpotent pd-thickening and let X/R be as in Corollary 32. Let X'/R' be a lifting of X. Let $\alpha : X \to X$ be an automorphism of the R-scheme X (but not necessarily of the deformation).*

Then α lifts to an automorphism $\alpha' : X' \to X'$ if and only if $\alpha^* : H^2_{\mathrm{crys}}(X/R') \to H^2_{\mathrm{crys}}(X/R')$ respects the Hodge filtration given by X'.

Proof. The universal deformation space S classifies pairs (X, ρ) where X is a scheme of K3 type over R and $\rho : X_0 \to X_k$ is an isomorphism.

Since X is a deformation of X_0, the map ρ is given. Let $\alpha_0 : X_0 \to X_0$ be the automorphism induced by α. The data α is equivalent to saying that the two pairs (X, ρ) and $(X, \rho\alpha_0)$ are isomorphic as deformations.

The existence of a lifting α' is equivalent to saying that the pairs (X', ρ) and $(X', \rho\alpha_0)$ are isomorphic as deformations. Thus we conclude by Corollary 32. □

We will now prove the compatibility of the Beauville–Bogomolov form with the Frobenius endomorphism. Let X_0/k be a projective and smooth scheme of K3 type over an algebraically closed field k of characteristic p. We will write $W = W(k)$ for the ring of Witt vectors. We consider the universal deformation

$$\mathfrak{X} \to S = \operatorname{Spf} A,$$

as in Section 4. By Proposition 26 we have a perfect and horizontal Beauville–Bogomolov form,

$$\mathbb{B} : H^2_{\mathrm{DR}}(\mathfrak{X}/A) \times H^2_{\mathrm{DR}}(\mathfrak{X}/A) \to A.$$

We set $\mathfrak{X}_0 = \mathfrak{X} \otimes_A A_0$. Then we have the canonical isomorphisms

$$H^2_{\mathrm{DR}}(\mathfrak{X}/A) \cong H^2_{\mathrm{crys}}(\mathfrak{X}_0/A).$$

We will denote by σ a ring endomorphism of $A = W[[T_1, \dots, T_r]]$ which extends the Frobenius on W and induces the Frobenius endomorphism modulo p. We will denote by $\rho : \operatorname{Spf} A \to \operatorname{Spf} A$ the morphism $\operatorname{Spf} \sigma$. The relative Frobenius Fr is given by the diagram

$$\begin{array}{ccccc} \mathfrak{X}_0 & \xrightarrow{\mathrm{Fr}} & \mathfrak{X}_0^{(p)} & \longrightarrow & \mathfrak{X}_0 \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spf} A & \longrightarrow & \operatorname{Spf} A & \xrightarrow[\rho]{} & \operatorname{Spf} A \end{array}$$

where the second square is cartesian and the composition of the upper horizontal arrows is the absolute Frobenius. Taking the crystalline cohomology we obtain the morphisms

$$A \otimes_{\sigma,A} H^2_{\mathrm{crys}}(\mathfrak{X}_0/A) \cong H^2_{\mathrm{crys}}(\mathfrak{X}_0^{(p)}/A) \xrightarrow{F} H^2_{\mathrm{crys}}(\mathfrak{X}_0/A).$$

We may view this as morphisms of crystals on $\operatorname{Spf} A_0$. We will set

$$\mathcal{H} = H^2_{\mathrm{DR}}(\mathfrak{X}/A) \quad \text{and} \quad \mathcal{H}^{(\sigma)} = A \otimes_{\sigma,A} \mathcal{H}.$$

Then we may write

$$F : \mathcal{H}^{(\sigma)} \to \mathcal{H}.$$

Since this A-module homomorphism is induced by a morphism of crystals we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H}^{(\sigma)} & \xrightarrow{\quad F \quad} & \mathcal{H} \\ {\scriptstyle \nabla^{(\sigma)}}\downarrow & & \downarrow{\scriptstyle \nabla} \\ \Omega^1_{A/W} \otimes_A \mathcal{H}^{(\sigma)} & \xrightarrow[\mathrm{id}\otimes F]{} & \Omega^1_{A/W} \otimes_A \mathcal{H} \end{array}$$

We denote here by $\Omega^1_{A/W}$ the continuous differentials, by ∇ the Gauss–Manin connection and by $\nabla^{(\sigma)}$ the inverse image of the Gauss–Manin connection by ρ.

Proposition 34. *We assume that X_0 lifts to a projective smooth scheme X over W* (*compare Proposition 29*). *We have on $\mathcal{H}^{(\sigma)}$ two horizontal forms*

$$\mathbb{B}^{(\sigma)} := A \otimes_{\sigma,A} \mathbb{B}, \quad \mathbb{B}(F\tilde{\alpha}, F\tilde{\beta}), \quad \text{where } \tilde{\alpha}, \tilde{\beta} \in \mathcal{H}^{(\sigma)}. \tag{74}$$

For a suitable choice of the Frobenius lift $\sigma : A \to A$, we have after multiplying $\mathbb{B}$ by a unit in W the relation

$$\mathbb{B}(F\tilde{\alpha}, F\tilde{\beta}) = p^2 \mathbb{B}^{(\sigma)}(\tilde{\alpha}, \tilde{\beta}). \tag{75}$$

If we regard F as a σ-linear homomorphism $\mathcal{H} \to \mathcal{H}$ and if we take $\tilde{\alpha} = 1 \otimes \alpha$ and $\tilde{\beta} = 1 \otimes \beta$ for $\alpha, \beta \in \mathcal{H}$ with $\alpha, \beta \in \mathcal{H}$, we may rewrite the last relation as

$$\mathbb{B}(F\alpha, F\beta) = p^2 \sigma(\mathbb{B}(\alpha, \beta)). \tag{76}$$

Proof. We consider the relative Frobenius

$$W \otimes_{\sigma,W} H^2_{\mathrm{crys}}(X_0/W) \cong H^2_{\mathrm{crys}}(X_0^{(p)}/W) \xrightarrow{F} H^2_{\mathrm{crys}}(X_0/W). \tag{77}$$

We take a very ample line bundle L on X_0. It defines a cohomology class,

$$\Lambda \in H^2_{\mathrm{crys}}(X_0/W) = H^2_{\mathrm{DR}}(X/W). \tag{78}$$

By Lemma 27 the form

$$\mathbb{B}_\Lambda(\alpha) := (2n-1)\nu(\Lambda) \int \Lambda^{2n-2}\alpha^2 - (2n-2)\Big(\int \Lambda^{2n-1}\alpha\Big)^2$$

coincides with the Beauville–Bogomolov form induced by $\mathbb{B}$ on the right-hand side of (78) up to a factor in $W \otimes \mathbb{Q}$. Indeed, to reduce this to the case of analytic manifolds we take an embedding of W into a field. Finally we may assume that the field is $\mathbb{C}$.

We denote by $L^{(p)}$ the inverse image of L by the map $X_0^{(p)} \to X_0$ and by $\Lambda^{(\sigma)}$ its cohomology class. It is the inverse image of Λ by the map $H^2_{\mathrm{crys}}(X_0/W) \to H^2_{\mathrm{crys}}(X_0^{(p)}/W)$. Since we may interpret $\nu(\Lambda)$ as an intersection product we have

$$\nu(\Lambda) = \nu(\Lambda^{(\sigma)}) \in \mathbb{Z}$$

is nonzero.

We claim that

$$\mathbb{B}_\Lambda(F\tilde{\alpha}) = p^2\mathbb{B}_{\Lambda^{(\sigma)}}(\tilde{\alpha}), \quad \tilde{\alpha} \in H^2_{\text{crys}}(X_0^{(p)}/W). \tag{79}$$

For this we use that $F(\Lambda^{(\sigma)}) = p\Lambda$ and that we have a commutative diagram:

$$\begin{array}{ccc} H^{4n}_{\text{crys}}(X_0^{(p)}/W) & \xrightarrow{F} & H^{4n}_{\text{crys}}(X_0/W) \\ {\scriptstyle\int}\downarrow & & \downarrow{\scriptstyle\int} \\ W & \xrightarrow{p^{2n}} & W \end{array} \tag{80}$$

Now we can compute

$$\begin{aligned} \mathbb{B}_\Lambda(F\tilde{\alpha}) &= (2n-1)\nu(\Lambda)\int \Lambda^{2n-2}(F\tilde{\alpha})^2 - (2n-2)\left(\int \Lambda^{2n-1}F\tilde{\alpha}\right)^2 \\ &= (2n-1)\nu(\Lambda)(1/p^{2n-2})\int (F\Lambda^{(\sigma)})^{2n-2}(F\tilde{\alpha})^2 \\ &\qquad\qquad - (2n-2)(1/p^{2(2n-1)})\left(\int (F\Lambda^{(\sigma)})^{2n-1}F\tilde{\alpha}\right)^2 \\ &= (2n-1)\nu(\Lambda)(1/p^{2n-2})p^{2n}\int (\Lambda^{(\sigma)})^{2n-2}(\tilde{\alpha})^2 \\ &\qquad\qquad - (2n-2)(1/p^{2(2n-1)})p^{4n}\left(\int (\Lambda^{(\sigma)})^{2n-1}\tilde{\alpha}\right)^2 \\ &= p^2\mathbb{B}_{\Lambda^{(\sigma)}}(\tilde{\alpha}). \end{aligned}$$

The last equation holds by (80). This shows (79).

We may multiply the Beauville–Bogomolov form $\mathbb{B}$ by a constant in $W \otimes \mathbb{Q}$ such that it induces on $H^2_{\text{crys}}(X_0/W) = H^2_{\text{DR}}(X/W)$ the form $\mathbb{B}_\Lambda$ if we make base change by the natural map $\rho : A \to W$ induced by X. There is a lift σ of the Frobenius to A such that ρ commutes with Frobenii. Indeed, after a coordinate change we may write $A = W[\![T_1, \dots, T_r]\!]$ in such a way that $\rho(T_i) = 0$. We take for σ the Frobenius such that $\sigma(T_i) = T_i^p$.

The two horizontal forms (74) may be regarded as two horizontal sections of the bundle $\mathcal{H}^{(\sigma)} \otimes (\mathcal{H}^{(\sigma)})^{\text{dual}}$ endowed with its natural integrable connection. We have shown that these two sections differ by the factor p^2 if we make the base change $A \to W$. Hence the sections themselves differ by the factor p^2. This shows (75). □

We can now prove a refinement of Proposition 19.

Proposition 35. *Let k be an algebraically closed field and let X_0 be a projective scheme of K3 type which lifts to a projective smooth scheme over $W(k)$.*

Let $f : X \to \operatorname{Spec} R$ be a deformation of X_0 over an artinian local ring R with residue class field k. Then the crystalline cohomology $H^2_{\text{crys}}(X/\widehat{W}(R))$ has the unique structure of a selfdual $\widehat{\mathcal{W}}_R$-display which is functorial in R.

Proof. We can use the w-frames $\mathcal{C}_n$ introduced before Proposition 19. We consider the $\mathcal{C}_n$-window $\mathcal{P}$ we introduced on $H^2_{\text{crys}}(\mathfrak{X}_{R_n}/C_n)$. We have to show that the Beauville–Bogomolov form induces a bilinear form of $\mathcal{C}_n$-displays

$$\mathcal{P} \times \mathcal{P} \to \mathcal{U}(2). \tag{81}$$

Here we choose the Beauville–Bogomolov form in such a way that $\epsilon = 1$, which is possible by the remark preceding Lemma 25. Because we are in the torsion-free case it suffices to show that this pairing is compatible with F_0, which follows from Proposition 34. We already know that the Beauville–Bogomolov form induces a selfdual pairing on

$$H^2_{\text{crys}}(\mathfrak{X}_{R_n}/C_n) = H^2_{\text{DR}}(\mathfrak{X}_{R_n}/C_n)$$

with respect to the Hodge filtration on the right-hand side. This shows that (81) is perfect. The assertion of the proposition is obtained by base change. □

Remark. In the same way, one can generalize the corollary of Proposition 19 and obtain duality on the displays there.

We denote by $(\mathcal{P}_0, \mathbb{B}_0)$ the selfdual $\widehat{\mathcal{W}}_k$-2-display associated to X_0. We assume that this 2-display is F_0-étale (Definition 12). By Corollary 18 the deformation functor of $(\mathcal{P}_0, \mathbb{B}_0)$ is prorepresentable by

$$S_{\text{disp}} = \operatorname{Spf} A_{\text{disp}},$$

where A_{disp} is a power series ring over $W(k)$. The universal object is a $\widehat{\mathcal{W}}_{A_{\text{disp}}}$-display. Let $\mathfrak{X} \to S$ be the universal deformation of X_0. By Proposition 35 we have a general morphism

$$S \to S_{\text{disp}}. \tag{82}$$

Let $f : X \to \operatorname{Spec} R$ be as in Proposition 35 and let $(\mathcal{P}, \lambda)$ be the corresponding $\widehat{\mathcal{W}}_R$ display. Let $R' \to R$ be a surjection of artinian local rings with residue class field k and kernel $\mathfrak{a}'$. We assume that $(\mathfrak{a}')^2 = 0$. Then the liftings of X to R' and of $(\mathcal{P}, \lambda)$ to R' are by Proposition 17 and Corollary 32 in natural bijection. In particular (82) is an isomorphism. We obtain:

Theorem 36. *Let k be an algebraically closed field. Let X_0 be a scheme of K3 type over k which lifts to a projective scheme over $W(k)$. We assume that the associated selfdual $\widehat{\mathcal{W}}_k$-2-display $(\mathcal{P}_0, \lambda_0)$ is F_0-étale.*

Let R be a local artinian ring with residue class field k. The map which associates to a deformation of X/R of X_0 its selfdual $\widehat{\mathcal{W}}_R$-2-display $(\mathcal{P}, \lambda)$ is a bijection to the deformations of $(\mathcal{P}_0, \lambda_0)$ to R.

Moreover an automorphism of X_0 lifts to an automorphism of X (necessarily unique) if and only if the induced automorphism of $(\mathcal{P}_0, \lambda_0)$ lifts to $(\mathcal{P}, \lambda)$.

Proof. The last statement is a consequence of Corollary 33. □

5. The relative de Rham–Witt complex of an ordinary K3 surface

We now relate our results to the results of [Nygaard 1983] and prove the degeneration of the integral de Rham–Witt spectral sequence for ordinary K3 surfaces.

Let R be a ring such that p is nilpotent on R, and let $X/\operatorname{Spec} R$ be a smooth projective scheme.

We assume that there exists a formal lifting $\mathfrak{X}$ of X over $\operatorname{Spf} W(R)$ and let $\Omega^{\cdot}_{\mathfrak{X}/W(R)}$ be its de Rham complex. We recall the following complex from [Langer and Zink 2007, Section 4] denoted by $\mathcal{F}^m\Omega^{\cdot}_{\mathfrak{X}/W(R)}$:

$$I_R \otimes_{W(R)} \Omega^0_{\mathfrak{X}/W(R)} \xrightarrow{pd} \cdots \xrightarrow{pd} I_R \otimes_{W(R)} \Omega^{m-1}_{\mathfrak{X}/W(R)} \xrightarrow{d} \Omega^m_{W(R)} \xrightarrow{d} \cdots,$$

where $I_R = VW(R)$.

Let $W\Omega_{X/R}$ denote the relative de Rham–Witt complex and $N^m W\Omega^{\cdot}_{X/R}$ the Nygaard complex (compare [Langer and Zink 2007, Introduction]):

$$(W\mathcal{O}_{X/R})_{[F]} \xrightarrow{d} \cdots \xrightarrow{d} (W\Omega^{m-1}_{X/R})_{[F]} \xrightarrow{dV} W\Omega^m_{X/R} \xrightarrow{d} W\Omega^{m+1}_{X/R} \xrightarrow{d} \cdots.$$

Here F means the restriction of scalars via $F : W(R) \to W(R)$.

Then we recall the following.

Conjecture 37. *There exists a canonical isomorphism in the derived category $D^+(X_{\mathrm{zar}}, W(R))$ between the Nygaard complex and the complex $\mathcal{F}^m\Omega_{\mathfrak{X}/W(R)}$:*

$$N^m W\Omega^{\cdot}_{X/R} \cong \mathcal{F}^m\Omega^{\cdot}_{\mathfrak{X}/W(R)}.$$

This is proved in [Langer 2018, Theorem 0.2] for $m < p$.

Remark 38. Let us assume that the de Rham spectral sequences associated to $\Omega^{\cdot}_{X/R}$ and $\Omega^{\cdot}_{\mathfrak{X}/W(R)}$ degenerate and commute with base change. Then it is proved in [Gregory and Langer 2017] under some additional assumptions that the hypercohomology groups $\mathbb{H}^n(X, N^m W\Omega_{X/R})$ define for varying m a display structure on the crystalline cohomology $H^n_{\mathrm{crys}}(X/W(R))$.

In [Langer and Zink 2007, Conjecture 5.8], this was predicted in general.

Before we state the main result we give a very general fact.

Lemma 39. *Let $X \to \operatorname{Spec} R$ be a proper scheme over the spectrum of a complete local ring.*

Then for all integers $r, s \geq 0$ the cohomology group $H^s(X, W\Omega^r_{X/R})$ is V-separated; *i.e.*,

$$\bigcap_{n>0} V^n H^s(X, W\Omega^r_{X/R}) = 0. \tag{83}$$

Proof. The composition of the following maps is zero:

$$W\Omega^r_{X/R} \xrightarrow{V^n} W\Omega^r_{X/R} \to W_n\Omega^r_{X/R}.$$

Indeed, V^n maps a differential of the type $\xi d\eta_1 \cdots d\eta_r$ to ${}^{V^n}\xi\, d\,{}^{V^n}\eta_1 \cdots d\,{}^{V^n}\eta_r$. Therefore the composition of the following arrows is zero:

$$H^s(X, W\Omega^r_{X/R}) \xrightarrow{V^n} H^s(X, W\Omega^r_{X/R}) \to H^s(X, W_n\Omega^r_{X/R}). \tag{84}$$

Let us denote by $M \subset H^s(X, W\Omega^r_{X/R})$ the left-hand side of (83). We conclude from (84) that for each n the group M is mapped to zero by

$$H^s(X, W\Omega^r_{X/R}) \to H^s(X, W_n\Omega^r_{X/R}).$$

On the other hand we have

$$H^s(X, W\Omega^r_{X/R}) = \varprojlim_n H^s(X, W_n\Omega^r_{X/R})$$

by [Langer and Zink 2004, Corollary 1.14]. Since the map from M to each group in the projective system is zero we conclude that $M = 0$. □

Theorem 40. *Let X/R be a smooth projective scheme such that R is artinian with perfect residue field k of characteristic $p > 2$ and such that the closed fibre X_k is an ordinary $K3$ surface.*

Then the de Rham–Witt spectral sequence associated to the relative de Rham–Witt complex

$$E_1^{i,j} = H^j(X, W\Omega^i_{X/R}) \to \mathbb{H}^{i+j}(W\Omega^{\cdot}_{X/R})$$

degenerates. Moreover, one has the following properties:

- $H^0_{\text{crys}}(X/W(R)) = H^0(X, W\mathcal{O}_{X/R}) = W(R)$.
- $H^1_{\text{crys}}(X/W(R)) = H^3_{\text{crys}}(X/W(R)) = 0$.
- $H^i(X, W\Omega^j_{X/R}) = 0$ *for* $i + j$ *odd, or* $i + j > 4$, *or* $i + j = 4$, $i \neq j$.
- $H^2(X, W\Omega^2_{X/R}) = H^4_{\text{crys}}(X/W(R)) = W(R)$.
- $H^2_{\text{crys}}(X/W(R)) \cong H^0(X, W\Omega^2_{X/R}) \oplus H^1(X, W\Omega^1_{X/R}) \oplus H^2(X, W\mathcal{O}_X)$, *which is a Hodge-de Rham–Witt decomposition (slope decomposition) in degree* 2, *lifting the slope decomposition over* $W(k)$.

$H^2(X, W\mathcal{O}_X)$ inherits from $W\mathcal{O}_X$ the operators F and V and it is with this structure the Cartier module of $\widehat{\mathrm{Br}}_{X/R} \cong \widehat{\mathbb{G}}_m/R$, the formal Brauer group of X. The Frobenius $F : W\Omega^1_{X/R} \to W\Omega^1_{X/R}$ induces an endomorphism of $H^1(X, W\Omega^1_{X/R})$, which we denote by F_1. Let $\mathcal{P} = (P, Q, F, F_1)$ be the display defined by $P = Q = H^1(X, W\Omega^1_{X/R})$, $F_1 := F_1$ and $F := pF_1$. Then $\mathcal{P}$ is the display of the étale part $\Psi^{\text{et}}_{X/R}$ of the extended Brauer group $\Psi_{X/R}$.

Remark 41. This is the first nontrivial example where the spectral sequence of the relative de Rham–Witt complex degenerates. Note that for the absolute de Rham–

Witt complex in the case $R = k$, it is known that the de Rham–Witt spectral sequence degenerates modulo torsion [Illusie 1979, II, Théorème 3.2] and degenerates at E_1 in the following two cases:

(a) All $H^i(X, W\Omega^j_X)$ are $W(k)$-modules of finite type [Illusie 1979, II, Théoréme 3.7].

(b) All $H^i(X, W\Omega^j_X)$ are p-torsion free [Illusie 1979, II, Corollaire 4.9].

Proof. First we prove Theorem 40 using Conjecture 37 (proven in [Langer 2018, Theorem 0.2]) and then give an alternative proof using the universal deformation of X_k over the universal deformation ring and applying [Langer and Zink 2007, Corollary 4.7].

It is well known that the crystalline cohomology $H^i_{\text{crys}}(X/W(R))$ is isomorphic to the de Rham cohomology $H^i_{\text{DR}}(\mathfrak{X})$ of a smooth formal lifting $\mathfrak{X}$ over $\operatorname{Spf} W(R)$, commutes with base change and is locally free of rank $1, 0, 22, 0, 1$ for $i = 0, 1, 2, 3, 4$ respectively; see [Langer and Zink 2007, p. 151] and [Illusie 1979, II, Section 7.2].

It is known that, as we are in the ordinary case, $\widehat{\text{Br}}_{X/R} \cong \widehat{\mathbb{G}}_{m,R}$ by [Artin and Mazur 1977, IV, Proposition 1.8; Nygaard 1983, Introduction] and $H^2(X, W\mathcal{O}_X)$ is the Cartier module of $\widehat{\text{Br}}_{X/R}$; hence $H^2(X, W\mathcal{O}_X) = W(R)$ by [Artin and Mazur 1977, II, Proposition 2.13].

Let G be an arbitrary p-divisible group over R. We will denote by $\mathbb{D}(G)$ the Grothendieck–Messing crystal of G. Its evaluation at the pd-thickening $W(R) \to R$ will be denoted by

$$D(G) = \mathbb{D}(G)_{W(R)}.$$

Note that in [Nygaard and Ogus 1985] this $W(R)$-module is denoted by $D(G^*)_{W(R)}$. It is endowed with a display structure.

By [Nygaard and Ogus 1985, Theorem 3.16] we have a Frobenius equivariant map,

$$D(\widehat{\text{Br}}_{X/R}) \to D(\Psi_{X/R}) \to H^2_{\text{crys}}(X/W(R)) = \mathbb{H}^2(X, W\Omega^{\cdot}_{X/R}).$$

Using the natural Frobenius equivariant map of complexes $W\Omega^{\cdot}_{X/R} \to W\mathcal{O}_X$ we obtain a Frobenius equivariant map,

$$D(\widehat{\text{Br}}_{X/R}) \to H^2_{\text{crys}}(X/W(R)) \to H^2(X, W\mathcal{O}_X). \tag{85}$$

The first and the last $W(R)$-module in this sequence are free of rank 1. Therefore we conclude by reduction to the case $R = k$ (compare [Nygaard and Ogus 1985, p. 490]) that the composite of the arrows in (85) is an isomorphism. Therefore we obtain an F-equivariant section ρ of the last map.

Since $H^1(X, \mathcal{O}_X) = 0$ we conclude that V is surjective on $H^1(X, W\mathcal{O}_X)$. We conclude by Lemma 39 that $H^1(X, W\mathcal{O}_X) = 0$. We consider the exact sequence of complexes

$$0 \to W\Omega^{\geq 1}_{X/R} \to W\Omega^{\cdot}_{X/R} \to W\mathcal{O}_X \to 0.$$

If we take hypercohomology and use the section ρ above, we obtain an F-equivariant decomposition,

$$H^2_{\mathrm{crys}}(X/W(R)) = H^2(X, W\mathcal{O}_X) \oplus \mathbb{H}^2(W\Omega^{\geq 1}_{X/R}). \tag{86}$$

Let $\mathfrak{X}$ be a formal lifting of X over $\operatorname{Spf} W(R)$. It is known that the Hodge–de Rham spectral sequence of $\mathfrak{X}$ degenerates; moreover the Hodge–de Rham spectral sequence associated to the complex $\mathcal{F}^m\Omega^{\cdot}_{\mathfrak{X}/W(R)}$ degenerates too; see [Langer and Zink 2007, Propositions 3.1 and 3.2].

Remark 42. Using the isomorphism $N^2W\Omega^{\bullet}_{X/R} \simeq \mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}$ we compute the cohomology of the Nygaard complex:

$$\mathbb{H}^0(N^2W\Omega^{\bullet}_{X/R}) \cong \mathbb{H}^0(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}) \cong I_R H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}),$$

$$\mathbb{H}^1(N^2W\Omega^{\bullet}_{X/R}) \cong \mathbb{H}^1(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}) \cong I_R H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \oplus I_R H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}}) = 0,$$

$$\mathbb{H}^2(N^2W\Omega^{\bullet}_{X/R}) \cong \mathbb{H}^2(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}) \cong I_R H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \oplus I_R H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}}) \oplus H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}}),$$

$$\mathbb{H}^3(N^2W\Omega^{\bullet}_{X/R}) \cong \mathbb{H}^3(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}) \cong I_R H^3(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \oplus I_R H^2(\mathfrak{X}, \Omega^1_{\mathfrak{X}}) \oplus H^1(\mathfrak{X}, \Omega^2_{\mathfrak{X}}) = 0,$$

$$\mathbb{H}^4(N^2W\Omega^{\bullet}_{X/R}) \cong \mathbb{H}^4(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}) \cong H^2(\mathfrak{X}, \Omega^2_{\mathfrak{X}}) \cong H^4_{\mathrm{DR}}(\mathfrak{X}) \cong W(R).$$

We will consider the following map from the Nygaard complex to the usual de Rham–Witt complex:

$$\begin{array}{ccccccl} W\mathcal{O}_X & \xrightarrow{d} & W\Omega^1_{X/R} & \xrightarrow{dV} & W\Omega^2_{X/R} & : & N^2W\Omega^{\cdot}_{X/R} \\ \downarrow{\scriptstyle pV} & & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle =} & & \\ W\mathcal{O}_X & \xrightarrow{d,} & W\Omega^1_{X/R} & \xrightarrow{d} & W\Omega^2_{X/R} & : & W\Omega^{\cdot}_{X/R} \end{array} \tag{87}$$

Lemma 43. *We consider the complexes* $W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}$ *and* $W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}$ *in degree* 0 *and* 1. *Then we have*

$$\mathbb{H}^0(X, W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) = 0, \quad \mathbb{H}^0(X, W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}) = 0.$$

Proof. Since $H^1_{\mathrm{crys}}(X/W(R)) = 0 = H^1(X, W\mathcal{O}_X)$ we have

$$\mathbb{H}^0(W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) = 0.$$

From Remark 42 we get an exact sequence

$$\begin{array}{llll} 0 \to \mathbb{H}^0(N^2 W\Omega^\bullet_{X/R}) \to & H^0(X, W\mathcal{O}_X) \xrightarrow{\partial} & \mathbb{H}^0(W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}) \to 0, \\ \quad \cong I_R H^0(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) & \cong W(R) & \\ \quad \cong I_R W(R) & & \end{array} \tag{88}$$

where ∂ is induced by the differential d and therefore is the zero map since d vanishes on $W(R) \cong H^0(X, W\mathcal{O}_X)$. □

Now consider the following commutative diagram of de Rham complexes:

$$\begin{array}{ccccccc} \mathcal{O}_\mathfrak{X} & \xrightarrow{\ d\ } & \Omega^1_\mathfrak{X} & \xrightarrow{\ d\ } & \Omega^2_\mathfrak{X} & : & \Omega^\bullet_{\mathfrak{X}/W(R)} \\ \uparrow{\scriptstyle \cdot p} & & \uparrow & & \uparrow{\scriptstyle =} & & \\ I_R \otimes \mathcal{O}_\mathfrak{X} & \xrightarrow[pd]{} & I_R \otimes \Omega^1_\mathfrak{X} & \xrightarrow[d]{} & \Omega^2_\mathfrak{X} & : & \mathcal{F}^2\Omega^\bullet_{\mathfrak{X}/W(R)} \end{array} \tag{89}$$

The diagram (89) and the degeneracy of the Hodge–de Rham spectral sequences associated to $\Omega^\bullet_{\mathfrak{X}/W(R)}$ and $\mathcal{F}^2\Omega^\bullet_{\mathfrak{X}/W(R)}$ yield a commutative diagram of exact rows:

$$\begin{array}{ccccc} \mathbb{H}^2(\Omega^{\geq 1}_{\mathfrak{X}/W(R)}[-1]) & \hookrightarrow & H^2_{dR}(\mathfrak{X}/W(R)) & \twoheadrightarrow & H^2(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle \cdot p} \\ \mathbb{H}^2(0 \to I_R \otimes \Omega^1 \to \Omega^2 \to 0) & \hookrightarrow & \mathbb{H}^2(\mathcal{F}^2\Omega_{\mathfrak{X}/W(R)}) & \twoheadrightarrow & I_R H^2(\mathfrak{X}, \mathcal{O}_\mathfrak{X}) \end{array} \tag{90}$$

The vertical map on the left-hand side may be identified with

$$I_R H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/W(R)}) \oplus H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/W(R)}) \to H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/W(R)}) \oplus H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/W(R)}).$$

Since the cohomology $H^j(\mathfrak{X}, \Omega^i_{\mathfrak{X}/W(R)})$ commutes with arbitrary base change we obtain that the cokernel of the left vertical map coincides with $H^1(X, \Omega^1_{X/R})$. The vertical map in the middle is the map described in [Langer and Zink 2007, Definitions 2.1 and 2.5], in terms of the predisplay structure on $H^2_{dR}(\mathfrak{X}/W(R))$. Then we have:

Lemma 44. *The map* (87) *induces the following diagram with exact rows*:

$$\begin{array}{ccccc} \mathbb{H}^1(W\Omega^1 \xrightarrow{d} W\Omega^2) & \hookrightarrow & H^2_{\text{crys}}(X/W(R)) & \twoheadrightarrow & H^2(X, W\mathcal{O}_X) \\ {\scriptstyle (V,\text{id})}\uparrow & & \uparrow & & \uparrow{\scriptstyle pV} \\ \mathbb{H}^1(W\Omega^1 \xrightarrow{dV} W\Omega^2) & \hookrightarrow & \mathbb{H}^2(N^2 W\Omega^\cdot_{X/R}) & \twoheadrightarrow & H^2(X, W\mathcal{O}_X) \end{array} \tag{91}$$

(*We wrote here* $W\Omega^? = W\Omega^?_{X/R}$).

Moreover, *the diagram is isomorphic to the diagram* (90); *hence the left vertical arrow is injective and its cokernel is* $H^1(X, \Omega^1_{X/R})$.

Proof. The last vertical arrow of the diagram factors through

$$H^2(X, W\mathcal{O}_X) \xrightarrow{V} H^2(X, VW\mathcal{O}_X) \xrightarrow{\cdot p} H^2(X, W\mathcal{O}_X).$$

Indeed, this follows from the exact sequence

$$0 \to W\mathcal{O}_X \xrightarrow{V} W\mathcal{O}_X \to \mathcal{O}_X \to 0.$$

Since $H^1(X, \mathcal{O}_X) = 0$, the map $H^2(X, W\mathcal{O}_X) \xrightarrow{V} H^2(X, W\mathcal{O}_X)$ is injective and we may identify its image with $H^2(X, VW\mathcal{O}_X)$. Moreover $H^1(X, W\mathcal{O}_X) = 0$ by Lemma 39, since $VH^1(X, W\mathcal{O}_X) = H^1(X, W\mathcal{O}_X)$. Therefore the horizontal left-hand maps in (91) are injective. Since $\mathbb{D}(\widehat{\mathrm{Br}}_{X/R})$ is a crystal and $\widehat{\mathrm{Br}}_{\mathfrak{X}/W(R)} = \mathbb{G}_{m/\mathfrak{X}}$ by rigidity, we have

$$\mathbb{D}(\widehat{\mathrm{Br}}_{X/R})_{W(R)} = \mathbb{D}(\widehat{\mathrm{Br}}_{\mathfrak{X}/W(R)})_{W(R)} = H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = \mathrm{Lie}\, \widehat{\mathrm{Br}}_{\mathfrak{X}/W(R)}$$

(compare the bottom lines in [Nygaard and Ogus 1985, p. 492]). Under the identification $H^2(X, W\mathcal{O}_X) = H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, the top exact sequences in (90) and (91) are isomorphic under the isomorphism $H^2_{dR}(\mathfrak{X}/W(R)) \cong H^2_{\mathrm{crys}}(X/W(R))$. The exact sequence

$$0 \to I_R \to W(R) \to R \to 0$$

yields

$$0 \to I_R H^2(X, W\mathcal{O}_X) \to H^2(X, W\mathcal{O}_X) \to H^2(X, \mathcal{O}_X) \to 0$$

and

$$0 \to I_R H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to H^2(X, \mathcal{O}_X) \to 0;$$

hence

$$I_R H^2(X, W\mathcal{O}_X) \cong H^2(X, VW\mathcal{O}_X) \cong I_R H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}).$$

Using the isomorphism $N^2 W\Omega^\bullet_{X/R} \cong \mathcal{F}^2\Omega^\bullet_{\mathfrak{X}/W(R)}$ we can identify the middle vertical arrows in diagrams (90) and (91). Moreover, since $H^2(X, W\mathcal{O}_X)$ is isomorphic to $H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \cong W(R)$, we see that the whole diagram (91) is isomorphic to the diagram (90). By the remark after (90) this implies that the left vertical map in (91) is injective and has cokernel $H^1(X, \Omega^1_{X/R})$. □

Lemma 45. *We have* $H^0(X, W\Omega^1_{X/R}) = 0$.

Proof. Lemma 43 implies that the rows in the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0(X, W\Omega^1) & \xrightarrow{d} & H^0(X, W\Omega^2) & \longrightarrow & \mathbb{H}^1(W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) \\
& \uparrow V & & \uparrow = & & \uparrow (V, \mathrm{id}) \\
0 \longrightarrow & H^0(X, W\Omega^1) & \xrightarrow{dV} & H^0(X, W\Omega^2) & \longrightarrow & \mathbb{H}^1(W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R})
\end{array}$$

are exact. Since (V, id) is injective by Lemma 44, the map

$$V : H^0(X, W\Omega^1) \to H^0(X, W\Omega^1)$$

is an isomorphism. We conclude by Lemma 39. □

Now consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} H^0(X,W\Omega^2_{X/R}) & \hookrightarrow & \mathbb{H}^1(X,[W\Omega^1_{X/R}\xrightarrow{d} W\Omega^2_{X/R}]) & \to & H^1(X,W\Omega^1_{X/R}) & \to & H^1(X,W\Omega^2_{X/R}) \\ \uparrow{=} & & \uparrow{\hat\alpha} & & \uparrow{V} & & \uparrow{=} \\ H^0(X,W\Omega^2_{X/R}) & \hookrightarrow & \mathbb{H}^1(X,[W\Omega^1_{X/R}\xrightarrow{dV} W\Omega^2_{X/R}]) & \to & H^1(X,W\Omega^1_{X/R}) & \to & H^1(X,W\Omega^2_{X/R}) \end{array} \tag{92}$$

$\hat\alpha$ denotes the left vertical arrow of (91). By Lemma 44, $\hat\alpha$ is injective and has cokernel $H^1(X, \Omega^1_{X/R})$.

Lemma 46. *The sequence*

$$0 \to H^1(X, W\Omega^1_{X/R}) \xrightarrow{V} H^1(X, W\Omega^1_{X/R}) \to H^1(X, \Omega^1_{X/R}) \to 0 \tag{93}$$

is exact and $H^i(X, W\Omega^1_{X/R}) = 0$ *for* $i \geq 2$.

Proof. Let us begin with the short exact sequence. We have to show that the kernels (and cokernels) of $\hat\alpha$ and V in the diagram (92) are the same. This follows formally if we prove that the last two horizontal arrows in this diagram are surjective. The continuation of the diagram (91) gives a commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{H}^2(W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) & \longrightarrow & H^3_{\mathrm{crys}}(X/W(R)) \\ & & \uparrow{\hat\alpha} & & \uparrow \\ 0 & \longrightarrow & \mathbb{H}^2(W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}) & \longrightarrow & \mathbb{H}^3(N^2 W\Omega^{\cdot}_{X/R}) \end{array} \tag{94}$$

By Remark 42 we see that all terms in the diagram (94) vanish. This shows that in the diagram (92) the last horizontal maps are surjective. The exactness of (93) follows.

For the last assertion we need only to consider the case $i = 2$ because the cohomological dimension of X is 2 by Grothendieck's theorem.

To show the vanishing we continue the exact cohomology sequences which lead to (92):

$$\begin{array}{ccccc} H^2(X, W\Omega^1_{X/R}) & \longrightarrow & H^2(X, W\Omega^2_{X/R}) & \longrightarrow & \mathbb{H}^3(W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) \\ \uparrow{V} & & \uparrow{=} & & \uparrow{\hat\alpha} \\ H^2(X, W\Omega^1_{X/R}) & \longrightarrow & H^2(X, W\Omega^2_{X/R}) & \longrightarrow & \mathbb{H}^3(W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}) \end{array} \tag{95}$$

The horizontal arrows on the left are injective by the vanishing of (94). By Lemma 39 it suffices to show that the $\hat{\alpha}$ on the right-hand side is an isomorphism because then V on the left-hand side is bijective. To see this we continue the diagram (91) in higher degrees and obtain:

$$\begin{array}{ccc} \mathbb{H}^3(W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) & \xrightarrow{\simeq} & H^4_{\mathrm{crys}}(X/W(R)) \\ \uparrow{\scriptstyle \hat{\alpha}} & & \uparrow \\ \mathbb{H}^3(W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}) & \xrightarrow{\simeq} & \mathbb{H}^4(N^2 W\Omega^\bullet_{X/R}) \end{array}$$

The horizontal arrows are isomorphisms because X is a noetherian space of Zariski dimension 2 and therefore $H^3(X, W\mathcal{O}_X) = H^4(X, W\mathcal{O}_X) = 0$.

Under the isomorphism $H^4_{\mathrm{crys}}(X/W(R)) \cong H^4_{\mathrm{DR}}(\mathfrak{X}) \cong H^2(\mathfrak{X}, \Omega^2_{\mathfrak{X}})$, Remark 42 implies that the right vertical arrow is an isomorphism; hence $\hat{\alpha}$ is an isomorphism as well. □

From the F-equivariant decomposition (86) we obtain by projection an F-equivariant morphism

$$D(\Psi_{X/R}) \to \mathbb{H}^2(X, W\Omega^{\geq 1}_{X/R}). \tag{96}$$

The Frobenius on the right-hand side is inherited from $H^2_{\mathrm{crys}}(X/W(R))$. It is in a natural way divisible by p. By the definition of the decomposition (86), the submodule $D(\widehat{\mathrm{Br}}_{X/R}) \subset D(\Psi_{X/R})$ is mapped to zero by the map (96). Therefore we obtain a map

$$D(\Psi^{\mathrm{et}}_{X/R}) \to \mathbb{H}^2(X, W\Omega^{\geq 1}_{X/R}). \tag{97}$$

We will write $(P, Q, F, F_1) = D(\Psi^{\mathrm{et}}_{X/R})$. Then $P = Q$ because this is a display of an étale group. The Frobenius F and the Verschiebung V on $W\Omega^1_{X/R}$ induce maps on the cohomology $H^1(X, W\Omega^1_{X/R})$ which we denote by the same letter. Composing the map (97) with $\mathbb{H}^2(X, W\Omega^{\geq 1}_{X/R}) \to H^1(X, W\Omega^1_{X/R})$ We obtain a map

$$\varsigma : P \to H^1(X, W\Omega^1_{X/R}) \tag{98}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} P & \longrightarrow & H^1(X, W\Omega^1_{X/R}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle pF} \\ P & \longrightarrow & H^1(X, W\Omega^1_{X/R}) \end{array}$$

Lemma 47. *The $W(R)$-module homomorphism*

$$\varsigma : P = D(\Psi^{\mathrm{et}}_{X/R}) \to H^1(X, W\Omega^1_{X/R})$$

is an isomorphism. We have a commutative diagram:

$$\begin{array}{ccc} P & \longrightarrow & H^1(X, W\Omega^1_{X/R}) \\ {\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F} \\ P & \longrightarrow & H^1(X, W\Omega^1_{X/R}) \end{array}$$

Proof. We already know that the last diagram is commutative if we multiply the vertical arrows by p. Indeed, on the left-hand side we have $pF_1 = F$ by the definition of a display. To prove the commutativity we may replace X by the universal deformation

$$\mathfrak{X} \to \operatorname{Spf} W(k)[[t_1, \dots, t_{22}]].$$

Then the groups of the diagram have no p-torsion and the result follows. We will write $I_R = VW(R)$ as before. We define $F_2 : I_R P \to P$ by

$$F_2({}^V\xi x) = \xi F_1 x, \quad \xi \in W(R),\ x \in P.$$

Because F_1 is an F-linear isomorphism, F_2 is bijective. Then we have a commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\varsigma} & H^1(X, W\Omega^1_{X/R}) \\ \uparrow{\scriptstyle F_2} & & \downarrow{\scriptstyle V} \\ I_R P & \xrightarrow{\varsigma} & H^1(X, W\Omega^1_{X/R}) \end{array}$$

Indeed,

$$V\varsigma(F_2({}^V\xi x)) = V(\varsigma(\xi F_1 x)) = V\xi F\varsigma(x) = {}^V\xi\varsigma(x) = \varsigma({}^V\xi x).$$

The next-to-last equation holds because the corresponding equation holds for $W\Omega^1_{X/R}$ by the last equation of [Langer and Zink 2004, Definition 1.4].

We set $I_n = V^n W(R)$. From the preceding remark we obtain

$$\varsigma(I_n P) \subset V^n H^1(X, \Omega^1_{X/R}).$$

Then we also have the following diagram:

$$\begin{array}{ccc} I_{n-1} P & \xrightarrow{\varsigma} & V^{n-1} H^1(X, W\Omega^1_{X/R}) \\ \uparrow{\scriptstyle F_2} & & \downarrow{\scriptstyle V} \\ I_n P & \xrightarrow{\varsigma} & V^n H^1(X, W\Omega^1_{X/R}) \end{array} \tag{99}$$

Here again the map F_2 is bijective.

The map ς induces an R-module homomorphism

$$\bar{\varsigma} : P/I_R P \to H^1(X, \Omega^1_{X/R}). \tag{100}$$

We claim that this is an isomorphism. To see this we note that both sides are free R-modules of the same rank and that both sides commute with arbitrary base change $R \to R'$. Therefore it suffices to prove that (100) is surjective in the case $R = k$. In this case the F-crystal $H^2_{\text{crys}}(X_k/W(k))$ decomposes into a direct sum of isoclinic crystals. This has the consequence that [Illusie 1979, II, Section 7.2]

$$H^2_{\text{crys}}(X_k/W(k)) = H^2(X_k, W\mathcal{O}_{X_k}) \oplus H^1(X_k, W\Omega^1_{X_k/k}) \oplus H^0(X_k, W\Omega^2_{X_k/k}) \quad (101)$$

is this decomposition as a sum of isoclinic crystals, i.e., the isoclinic components are canonically isomorphic to the cohomology groups on the right-hand side. In particular these groups are free $W(k)$-modules of ranks 1, 20, 1 respectively. The F-crystal of the extended Brauer group is the part of the F-crystal $H^2_{\text{crys}}(X_k/W(k))$ with slopes in [0, 1]; i.e., it corresponds to the first two direct summands of (101). Therefore the map (100) is induced by

$$H^1(X_k, W\Omega^1_{X_k/k}) \to H^1(X_k, \Omega^1_{X_k/k}).$$

But we know by Lemma 46 that this map is surjective. Then (100) is also surjective in the case $R = k$ and this proves our claim that (100) is an isomorphism.

Now we can check easily by induction that the maps induced by ς,

$$I_n P/I_{n+1}P \to V^n H^1(X, W\Omega^1_{X/R})/V^{n+1}H^1(X, W\Omega^1_{X/R}), \quad (102)$$

are surjective. In the case $n = 0$ this follows from the isomorphism (100) and Lemma 46.

Let $m \in H^1(X, W\Omega^1_{X/R})$. Then we find by induction elements $x \in I_{n-1}P$ and $m_1 \in H^1(X, W\Omega^1_{X/R})$ such that

$$V^{n-1}m = \varsigma(x) + V^n m_1.$$

We write $x = F_2(y)$ for $y \in I_n P$. Then we obtain

$$V^n m = V\varsigma(F_2 y) + V^{n+1}m_1 = \varsigma(y) + V^{n+1}m_1.$$

This ends the induction.

Recall $H^1(X, W\Omega^1_{X/R})$ is V-separated. By [Bourbaki 1961, §2.8, Théorème 1] it follows from (102) that $\varsigma : P \to H^1(X, W\Omega^1_{X/R})$ is surjective and $H^1(X, W\Omega^1_{X/R})$ is complete in the V-adic topology. Since V is also injective by Lemma 46, $H^1(X, W\Omega^1_{X/R})$ is a reduced Cartier module.

We consider (102) as a homomorphism of $W_{n+1}(R)$-modules.

Assertion. Both sides of (102) are isomorphic as $W_{n+1}(R)$-modules and are noetherian.

Because a surjective endomorphism of a noetherian module is an isomorphism, the assertion implies that (102) is an isomorphism. But then ς is an isomorphism.

To finish the proof it remains to show the assertion. From the bijection $F_2 : I_n P \to I_{n-1}P$, $F_2({}^{V^n}\xi x) \mapsto {}^{V^{n-1}}\xi F_1 x$ we deduce the bijection

$$F_2 : I_n P/I_{n+1}P \to I_{n-1}P/I_n P.$$

We denote by $P/I_1 P_{[F^n]}$ the $W_{n+1}(R)$-module obtained by restriction of scalars with respect to $F^n : W_{n+1}(R) \to R$. Iterating F_2 we obtain an isomorphism of $W_{n+1}(R)$-modules

$$F_2^n : I_n P/I_{n+1}P \to (P/I_1 P)_{[F^n]}.$$

Because R is F-finite, the last module is a noetherian $W_{n+1}(R)$-module [Langer and Zink 2004, Proposition A.2]. We note that $W_{n+1}(R)$ is a noetherian ring because it is a $W(k)$-module of finite length.

For the reduced Cartier module $M = H^1(X, W\Omega^1_{X/R})$ we obtain in the same way the isomorphism

$$V^n : M/VM_{[F^n]} \to V^n M/V^{n+1}M.$$

Therefore the isomorphism (100) together with Lemma 46 shows the assertion above. □

The isomorphism ς of Lemma 47 factors by definition through (97):

$$D(\Psi^{\mathrm{et}}_{X/R}) \to \mathbb{H}^2(X, W\Omega^{\geq 1}_{X/R}) \to H^1(X, W\Omega^1_{X/R}). \tag{103}$$

Therefore the last arrow is a split surjection. Since $H^0(X, W\Omega^1_{X/R}) = 0$ we obtain a split exact sequence

$$0 \to H^0(X, W\Omega^2_{X/k}) \to \mathbb{H}^2(W\Omega^{\geq 1}_{X/R}) \to H^1(X, W\Omega^1_{X/R}) \to 0.$$

Together with (86) this gives the Hodge–Witt decomposition

$$H^2_{\mathrm{crys}}(X/W(R)) = H^2(X, W\mathcal{O}_X) \oplus H^1(X, W\Omega^1_{X/R}) \oplus H^0(X, W\Omega^2_{X/R}).$$

We see that the free $W(R)$-modules on the right-hand side have ranks 1, 20, 1 since we know the height of the formal Brauer group and the extended formal Brauer group.

It follows from the above that

$$D(\Psi_X)_{W(R)} \cong H^2(X, W\mathcal{O}_X) \oplus H^1(X, W\Omega^1_{X/R})$$

and since

$$D(\Psi_X)_{W(R)} = \ker(H^2_{\mathrm{crys}}(X/W(R)) \xrightarrow{\pi} D(\widehat{\mathrm{Br}}^*_{X/R})(-1)_{W(R)})$$

(this surjective map is defined in [Nygaard and Ogus 1985, (3.20.1)]), π factors through an isomorphism

$$H^0(X, W\Omega^2_{X/R}) \xrightarrow{\sim} D(\widehat{\mathrm{Br}}^*_{X/R})(-1)_{W(R)}$$

of rank-1-$W(R)$-modules. This identifies all direct summands of $H^2_{\text{crys}}(X/W(R))$ as Cartier–Dieudonné modules as in Theorem 40.

The Hodge–Witt decomposition for $H^2_{\text{crys}}(X/W(R))$ implies a surjection,

$$H^2_{\text{crys}}(X/W(R)) \twoheadrightarrow \mathbb{H}^2(X, W\Omega^{\leq 1}_{X/R}) = H^2(X, W\mathcal{O}_X) \oplus H^1(X, W\Omega^1_{X/R})$$

Then the map $H^1(X, W\Omega^2_{X/R}) \to H^3_{\text{crys}}(X/W(R))$ is injective and therefore $H^1(X, W\Omega^2_{X/R})$ vanishes too, because $H^3_{\text{crys}}(X/W(R)) = 0$.

We get the exact sequence

$$0 \to H^2(X, W\Omega^2_{X/R}) \to \underset{= W(R)}{H^4_{\text{crys}}(X/W(R))} \to \mathbb{H}^4(W\mathcal{O}_X \xrightarrow{d} W\Omega^1_{X/R}).$$

We have seen that $H^3(X, W\mathcal{O}_X) = H^4(X, W\mathcal{O}_X) = 0$.

By the same arguments one shows that $V : H^3(X, W\Omega^1_{X/R}) \to H^3(X, W\Omega^1_{X/R})$ is injective with vanishing cokernel $= H^3(\Omega^1_{X/R})$.

So $H^3(W\Omega^1_{X/R}) = 0$; this means

$$H^2(X, W\Omega^2_{X/R}) \cong H^4_{\text{crys}}(X/W(R))$$

and this finishes the proof of the theorem. □

Proposition 48. *Under the assumptions of Theorem 40, the Hodge–de Rham–Witt decomposition of $H^2_{\text{crys}}(X/W(R))$ extends to a direct sum decomposition of displays (over the usual Witt ring $W(R)$) associated to the formal Brauer group, the étale part of the extended Brauer group and its Cartier dual, twisted by -1 and where $H^2_{\text{crys}}(X/W(R))$ is equipped with the display structure arising from the Nygaard complex* (see [Langer and Zink 2007]).

Proof. This is clear. □

Alternatively we can derive a Hodge–Witt decomposition for H^2_{crys} using the universal deformation ring.

Let as before B be the universal deformation ring of X_k, X_B be the universal family of X_k over Spf B, and define $X_n = X_B \times_{\text{Spf}\, B} \text{Spec}\, B/m^n$. Let $\widetilde{\mathcal{Y}}$ be a formal p-adic lifting to $W(B)$ with induced liftings $\mathcal{Y}^k_n$ over Spec $W_k(B/m^n)$, compatible with the liftings X_n. We assume that n is big enough so that $B \to R$ factors through $B/m^n \to R$.

By [Langer and Zink 2007, Theorem 4.6] we have for $r < p$ a quasiisomorphism

$$Ru_{n*}J^{[r]}_{X_n/W_k(B/m^n)} \to I^r W_k\Omega^{\cdot}_{X_n/(B/m^n)},$$

where $u_n : \text{Crys}(X_n/W_k(B/m^n)) \to X_n$ is the canonical morphism of sites and $I^r W_k\Omega^{\cdot}_{X_n/(B/m^n)}$ denotes the complex

$$p^{r-1}VW_{k-1}(\mathcal{O}_{X_n}) \to \cdots \to VW_{k-1}\Omega^{r-1}_{X_n/(B/m^n)} \to W_k\Omega^r_{X_n/(B/m^n)} \cdots .$$

By [Berthelot and Ogus 1978, Theorem 7.2], $Ru_{n*}J^{[r]}_{X_n/W_k(B/m^n)}$ is represented by the complex ($I_n^k := VW_{k-1}(B/m^n)$)

$$p^{r-1}I_n^k\Omega^0_{\mathcal{Y}_n^k/W_k(B/m^n)} \to \cdots \to I_n^k\Omega^{r-1}_{\mathcal{Y}_n^k/W_k(B/m^n)} \to \Omega^r_{\mathcal{Y}_n^k/W_k(B/m^n)} \to \cdots .$$

As we pass to the projective limit with respect to k, n and note that all inverse systems of sheaves in the above complexes are Mittag–Leffler systems, we get an isomorphism of complexes in the derived category of $W(B)$-modules between

$$p^{r-1}VW(B)\Omega^0_{\widetilde{\mathcal{Y}}/W(B)} \to \cdots \to VW(B)\Omega^{r-1}_{\widetilde{\mathcal{Y}}/W(B)} \to \Omega^r_{\widetilde{\mathcal{Y}}/W(B)} \to \cdots$$

and

$$p^{r-1}VW\mathcal{O}_{X_B} \to \cdots \to VW\Omega^{r-1}_{X_B/B} \to W\Omega^r_{X_B/B} \to \cdots$$

which is the inverse limit of the complexes $I^rW_k\Omega^{\cdot}_{X_n/(B/m^n)}$ with respect to k, n.

As multiplication by p is injective on $\Omega^{\cdot}_{\widetilde{\mathcal{Y}}/W(B)}$ and $W\Omega^{\cdot}_{X_B/B}$ (this can be reduced to a local argument, and can be made explicit for polynomial algebras), the first complex is isomorphic to $\mathcal{F}^r\Omega_{\widetilde{\mathcal{Y}}/W(B)}$ (notation as in Conjecture 37) and the second complex is isomorphic to the Nygaard complex $N^rW\Omega^{\cdot}_{X_B/B}$. The above considerations hold for any smooth proper X/R with a smooth deformation ring B and $r < p$. For K3 surfaces we take $r = 2$ and see that Conjecture 37 holds for the universal family X_B over B. Thus the statement of Theorem 40 holds for X_B over Spf B. In particular the de Rham–Witt spectral sequence

$$H^j(X_B, W\Omega^i_{X_B/B}) \to \mathbb{H}^{i+j}(X_B, W\Omega^{\cdot}_{X_B/B})$$

degenerates and we have a Hodge–Witt decomposition

$$H^2_{\text{crys}}(X_B/W(B)) = \bigoplus_{i+j=2} H^i(X_B, W\Omega^j_{X_B/B}). \tag{104}$$

By base change we get a decomposition over $W(R)$ as follows:

$$H^2_{\text{crys}}(X/W(R)) = \bigoplus_{i+j=2} H^i(X_B, W\Omega^j_{X_B/B}) \otimes_{W(B)} W(R). \tag{105}$$

Moreover we have the following evident properties of the direct summands:

- $H^2(X_B, W\mathcal{O}_{X_B}) \otimes_{W(B)} W(R) = H^2(X, W\mathcal{O}_X)$ is the Cartier–Dieudonné module of $\widehat{\mathrm{Br}}_{X_R} = \widehat{\mathbb{G}}_m/R$.
- $H^1(X_B, W\Omega^1_{X_B/B}) \otimes_{W(B)} W(R) = H^1(X, W\Omega^1_{X/R})$ is the Dieudonné module of $\Psi^{\text{et}}_{X/R}$.
- $H^0(X_B, W\Omega^2_{X_B/B}) \otimes_{W(B)} W(R) = H^0(X, W\Omega^2_{X/R})$ is the (shifted by -1) Dieudonné module of the Cartier dual $\widehat{\mathrm{Br}}^*_{X/R}$.

As in Proposition 48, the decomposition (105), which is a direct sum decomposition of Dieudonné modules of p-divisible groups, extends to a direct sum decomposition of the corresponding displays, where $H^2_{\text{crys}}(X/W(R))$ carries the display structure obtained by base change via $B \to R$ from the display structure on $H^2_{\text{crys}}(X_B/W(B))$.

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Purity of crystalline strata

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Let p be a prime. Let $n \in \mathbb{N}^*$. Let $\mathcal{C}$ be an F^n-crystal over a locally noetherian $\mathbb{F}_p$-scheme S. Let $(a, b) \in \mathbb{N}^2$. We show that the reduced locally closed subscheme of S whose points are exactly those $x \in S$ such that (a, b) is a break point of the Newton polygon of the fiber $\mathcal{C}_x$ of $\mathcal{C}$ at x is pure in S, i.e., it is an affine S-scheme. This result refines and reobtains previous results of de Jong and Oort, of Vasiu, and of Yang. As an application, we show that for all $m \in \mathbb{N}$ the reduced locally closed subscheme of S whose points are exactly those $x \in S$ for which the p-rank of $\mathcal{C}_x$ is m is pure in S; the case $n = 1$ was previously obtained by Deligne (unpublished) and the general case $n \geq 1$ refines and reobtains a result of Zink.

1. Introduction

For a reduced locally closed subscheme Z of a locally noetherian scheme Y, let $\bar{Z}$ be the schematic closure of Z in Y. We recall from [Nicole et al. 2010, Definition 1.1] that Z is called *pure* in Y if it is an affine Y-scheme. The paper [Nicole et al. 2010] also uses a weaker variant of this purity which in [Li 2015] is called *weakly pure*: we say Z is weakly pure in Y if each nonempty irreducible component of the complement $\bar{Z} - Z$ is of pure codimension 1 in $\bar{Z}$. It is well known that if Z is pure in Y, then Z is also weakly pure in Y (for instance, see Proposition 13 of Section 4.4).

Let n and r be natural numbers. Let p be a prime. Let S be a locally noetherian $\mathbb{F}_p$-scheme. Let $\Phi_S : S \to S$ be the Frobenius endomorphism of S. Let $\mathcal{M}$ be a *crystal* of the gross absolute crystalline site $CRIS(S/\operatorname{Spec}(\mathbb{Z}_p))$ introduced in [Berthelot 1974, Chapter III, Example 1.1.3 and Definition 4.1.1] in locally free $\mathcal{O}_{S/\operatorname{Spec}(\mathbb{Z}_p)}$-modules of rank r. We assume that we have an *isogeny* $\phi_{\mathcal{M}}$: $(\Phi_S^n)^*(\mathcal{M}) \to \mathcal{M}$; thus the pair $\mathcal{C} = (\mathcal{M}, \phi_{\mathcal{M}})$ is an F^n-crystal of $CRIS(S/\operatorname{Spec}(\mathbb{Z}_p))$. If the $\mathbb{F}_p$-scheme $S = \operatorname{Spec} A$ is affine, then the pair $\mathcal{C} = (\mathcal{M}, \phi_{\mathcal{M}})$ is canonically identified with a $\sigma^n - F$-crystal on A in the sense of [Katz 1979, Subsection (2.1)].

Let $\nu : [0, r] \to [0, \infty)$ be a *Newton polygon*, i.e., a nondecreasing piecewise linear continuous function such that $\nu(0) = 0$ and the coordinates of all its *break*

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points are natural numbers. For $x \in S$, let ν_x be the *Newton polygon* of the fiber $\mathcal{C}_x$ of $\mathcal{C}$ at x. Let S_ν be the reduced locally closed subscheme of S whose points are exactly those $x \in S$ such that we have $\nu_x = \nu$; see the Grothendieck–Katz theorem [Katz 1979, Corollary 2.3.2]. If nonempty, S_ν is a *stratum* of the Newton polygon *stratification* of S defined by $\mathcal{C}$.

Let $a, b \in \mathbb{N}$ be such that $0 \leq a \leq r$. Let $T = T_{(a,b)}(\mathcal{C})$ be the reduced locally closed subscheme of S whose points are those $x \in S$ such that (a, b) is a break point of ν_x. The end break point $(r, \nu_x(r))$ remains constant under specializations of $x \in S$. Thus locally in the Zariski topology of S, we can assume that there exists $d \in \mathbb{N}$ such that for all $x \in S$ we have $\nu_x(r) = d$ and this implies that T is the reduced locally closed subscheme of S which is a finite union $\bigcup_{\nu \in N_{r,d,a,b}} S_\nu$ of Newton polygon strata S_ν indexed by the set $N_{r,d,a,b}$ of all Newton polygons $\nu : [0, r] \to [0, \infty)$ with the two properties that $\nu(r) = d$ and (a, b) is a break point of ν.

It is known that T is weakly pure in S; see [Yang 2011, Theorem 1.1.] It is also known that S_ν is pure in S; see [Vasiu 2006, Main Theorem B]. This last result implies the celebrated result of de Jong and Oort [2000, Theorem 4.1] which asserts that S_ν is weakly pure in S. Strictly speaking, the references of this paragraph work with $n = 1$ but their proofs apply to all $n \in \mathbb{N}^*$.

In general, a finite union of locally closed subschemes of S which are pure in S is not pure in S. Therefore the following purity result which refines and reobtains the mentioned results of de Jong and Oort, of Vasiu, and of Yang, comes as a surprise.

Theorem 1. *With the above notation*, T *is pure in* S.

In Section 2 we gather the few preliminary steps that are required to prove Theorem 1 in Section 3. The following two corollaries are direct consequences of Theorem 1. The first one for $n = 1$ just reobtains [Vasiu 2006, Main Theorem B] in the locally noetherian case.

Corollary 2. *Each Newton polygon stratum* S_ν *is pure in* S.

The p-rank $\chi(x)$ of $\mathcal{C}_x$ is the multiplicity of the Newton polygon slope 0 of ν_x. Equivalently, $\chi(x)$ is the unique natural number such that $(0, 0)$ and $(\chi(x), 0)$ are the only break points of ν_x on the horizontal axis (i.e., which have the second coordinate 0).

Corollary 3. *Let* $m \in \mathbb{N}$. *We consider the reduced locally closed subscheme* S_m *of* S *whose points are exactly those* $x \in S$ *such that the* p*-rank* $\chi(x)$ *of* $\mathcal{C}_x$ *is* m. *Then* S_m *is pure in* S.

If $m > 0$, then we have $S_m = T_{(m,0)}(\mathcal{C})$ and if $m = 0$, then we have $S_0 = T_{(1,0)}(\mathcal{C} \oplus \mathcal{E}_0)$ where $\mathcal{E}_0$ is the pullback to S of the F^n-crystal over $\mathrm{Spec}(\mathbb{F}_p)$ of rank 1 and Newton polygon slope 0 which has a Frobenius invariant global section; therefore, regardless of what m is, Corollary 3 follows from Theorem 1.

For $n = 1$ Corollary 3 was first obtained by Deligne [2011] and more recently by Vasiu [2014] and Li [2015]. Corollary 3 also refines and reobtains a prior result of Zink which asserts that S_m is weakly pure in S (see [Zink 2001], Proposition 5).

In Section 4 we first follow [Li 2015] to show that Corollary 2 follows directly from Theorem 1 and then we follow [Vasiu 2014] to include a second proof of Corollary 3 in the more general context provided by a functorial version of the *Artin–Schreier stratifications* introduced in [Vasiu 2013, Definition 2.4.2] which is simpler, does not rely on Theorem 1, and is based on Theorem 12 of Section 4.2.

Theorem 1 is due to Li [2015]. While the proof of [Yang 2011, Theorem 1.1] follows the proof of [de Jong and Oort 2000, Theorem 4.1], the proof of Theorem 1 presented follows [Li 2015] and thus the proofs of [Vasiu 2006, Main Theorem B and Theorem 6.1]. It is known (see [Nicole et al. 2010, Example 7.1]) that in general S_m is not strongly pure in S in the sense of [Nicole et al. 2010, Definition 7.1], and therefore Theorem 1 and Corollary 3 cannot be improved in general (i.e., are optimal).

We refer to $T_{(a,b)}(\mathcal{C})$, S_ν, and S_m as crystalline strata of S associated to $\mathcal{C}$ and certain (basic) discrete invariants of F^n-crystals. Cases of nondiscrete invariants stemming from isomorphism classes are also studied in the literature (for instance, see [Vasiu 2006, Section 5.3] and [Nicole et al. 2010, Theorem 1.2 and Corollary 1.5]). Crystalline strata have applications to the study in positive characteristic of different moduli spaces and schemes such as special fibers of Shimura varieties of Hodge type (for instance, see [Vasiu 2006] and [Nicole et al. 2010]).

2. Standard reduction steps

The above notation p, S, Φ_S, $\bar{Z}$, n, r, $\mathcal{C} = (\mathcal{M}, \phi_{\mathcal{M}})$, $\mathcal{C}_x$, ν_x, $(a, b) \in \mathbb{N}^2$, $T = T_{(a,b)}(\mathcal{C})$, S_ν, m, S_m, $\chi(x)$, and $\mathcal{E}_0$ will be used throughout the paper. For a fixed Newton polygon ν, let $S_{\geq\nu}$ be the reduced closed subscheme of S whose points are exactly those $x \in S$ such that the Newton polygon ν_x is above ν, see [Katz 1979, Corollary 2.3.2].

In what follows, by an étale cover we mean a surjective finite étale morphism of schemes. For basic properties of excellent rings we refer to [Matsumura 1980, Chapter 13]. If $V \to Y$ is a morphism of $\mathbb{F}_p$-schemes and if $\mathcal{F}$ (or $\mathcal{F}_Y$) is an F^n-crystal over Y, let $\mathcal{F}_V$ be the pullback of $\mathcal{F}$ (or $\mathcal{F}_Y$) to an F^n-crystal over V, i.e., of $\mathit{CRIS}(V/\operatorname{Spec}(\mathbb{Z}_p))$. Let $k(y)$ be the residue field of a point $y \in Y$. If $V = \operatorname{Spec}(k(y)) \to Y$ is the natural morphism, then we denote $\mathcal{F}_V = \mathcal{F}_{\operatorname{Spec}(k(y))}$ simply by $\mathcal{F}_y$ (the fiber of $\mathcal{F}$ at y).

For an $\mathbb{F}_p$-algebra R, let $W(R)$ be the ring of p-typical Witt vectors with coefficients in R. Let $\mathbb{W}(R) = (\operatorname{Spec} R, \operatorname{Spec}(W(R)), \text{can})$ be the thickening in which “can” stands for the canonical divided power structure of the kernel of the epimorphism $W(R) \to W_1(R) = R$. For $s \in \mathbb{N}^*$, let $W_s(R)$ be the ring of p-typical Witt vectors of length s with coefficients in R. Let $\mathbb{W}_s(R) = (\operatorname{Spec} R, \operatorname{Spec}(W_s(R)), \text{can})$ be the thickening defined naturally by $\mathbb{W}(R)$. Let Φ_R be the Frobenius endomorphism of either $W(R)$ or $W_s(R)$.

The property of a reduced locally closed subscheme being pure in S is local for the faithfully flat topology of S, and thus until the end we will also assume that $S = \operatorname{Spec} A$ is an affine $\mathbb{F}_p$-scheme and that there exists $d \in \mathbb{N}$ such that for all $x \in S$ we have $\nu_x(r) = d$. As the scheme S is locally noetherian and affine, it is noetherian. To prove Theorem 1, we have to prove that T is an affine scheme.

2.1. ***Some abelian categories.*** Let $\mathcal{M}(W_s(R))$ be the abelian category whose objects are pairs (O, ϕ_O), comprised of a $W_s(R)$-module O and a Φ_R^n-linear endomorphism $\phi_O : O \to O$ (i.e., ϕ_O is additive and for all $z \in O$ and $\sigma \in W_s(R)$ we have $\phi_O(\sigma z) = \Phi_R^n(\sigma)\phi_O(z)$) and whose morphisms $f : (O_1, \phi_{O_1}) \to (O_2, \phi_{O_2})$ are $W_s(R)$-linear maps $f : O_1 \to O_2$ satisfying $f \circ \phi_{O_1} = \phi_{O_2} \circ f$. If $t \in \{0, \dots, s-1\}$, then by a *quasi-isogeny* of $\mathcal{M}(W_s(R))$ whose cokernel is annihilated by p^t we mean a morphism $f : (O_1, \phi_{O_1}) \to (O_2, \phi_{O_2})$ of $\mathcal{M}(W_s(R))$ which has the following two properties: (i) both O_1 and O_2 are projective $W_s(R)$-modules which have the same positive rank locally in the Zariski topology of $\operatorname{Spec}(W_s(R))$, and (ii) the cokernel $O_2/f(O_1)$ is annihilated by p^t. An object (O, ϕ_O) of $\mathcal{M}(W_s(R))$ is called *divisible* by $t \in \{1, \dots, s-1\}$ if O is a projective $W_s(R)$-module such that $\operatorname{Im}(\phi_O) \subseteq p^t O$.

For $l \in \mathbb{N}^*$ we have a natural functor

$$\mathcal{M}(W_{s+l}(R)) \to \mathcal{M}(W_s(R))$$

to be referred to, by abuse of language, as the reduction modulo p^s functor.

If Y is a $\operatorname{Spec}(\mathbb{F}_p)$-scheme, in a similar way we define the scheme $W_s(Y)$, its Frobenius endomorphism Φ_Y, and the abelian category $\mathcal{M}(W_s(Y))$, and speak about quasi-isogenies of $\mathcal{M}(W_s(Y))$ whose cokernels are annihilated by p^t with $t \in \{0, \dots, s-1\}$, about objects of $\mathcal{M}(W_s(Y))$ divisible by $t \in \{1, \dots, s-1\}$, and about reduction modulo p^s functors $\mathcal{M}(W_{s+l}(Y)) \to \mathcal{M}(W_s(Y))$. We have canonical identifications

$$\mathcal{M}(W_s(R)) = \mathcal{M}(W_s(\operatorname{Spec} R)).$$

For homomorphisms $R \to R_1$ and morphisms $Y_1 \to Y$, we have natural pullback functors $\mathcal{M}(W_s(R)) \to \mathcal{M}(W_s(R_1))$ and $\mathcal{M}(W_s(Y)) \to \mathcal{M}(W_s(Y_1))$.

To prove that T is an affine scheme, we can also assume that the *evaluation* M of $\mathcal{M}$ at the thickening $\mathbb{W}_1(A)$ is a free A-module of rank r. The evaluation of $\phi_{\mathcal{M}}$ at this thickening is a Φ_A^n-linear endomorphism $\phi_M : M \to M$.

In what follows we will apply twice the following elementary general fact which can be also deduced easily from the elementary divisor theorem.

Fact 4. Let D be a discrete valuation ring and let $\pi \in D$ be a uniformizer of it. Let $s, t \in \mathbb{N}$ be such that $s > t$. Let $D_s = D/(\pi^s)$. Let $g_s : D_s^r \to D_s^r$ be a D_s-linear endomorphism such that its cokernel is annihilated by π^t. Then for each $x \in D_s^r - \pi D_s^r$, we have $g_s(x) \in D_s^r - \pi^{t+1} D_s^r$.

Proof. Let $g : D^r \to D^r$ be a D-linear endomorphism which lifts g_s. Let $E = \mathrm{Im}(g) + \pi^s D^r$ (one can easily check that $E = \mathrm{Im}(g)$ but we will not stop to argue this). It is a free D-module of rank r which (as $\pi^t \,\mathrm{Coker}(g_s) = 0$) contains $\pi^t D^r$. Thus $\pi^s D^r \subseteq pE$ and therefore $\mathrm{Im}(g)$ surjects onto the D_1-vector space $E/\pi E$ of rank r. Hence a D_s-basis of D_s^r maps via g_s to a D_1-basis of $E/\pi E$. From this and the fact that $\pi^{t+1} D^r \subseteq \pi E$ we get that no element of a D_s-basis of D_s^r is mapped by g_s to $\pi^{t+1} D_s^r$. Thus the fact holds. □

2.2. ***On*** (a, b)**.** If (a, b) is $(0, 0)$ or (r, d), then $T = S$. If $a = 0$ and $b > 0$ or if $a = r$ and $b \neq d$, then $T = \varnothing$. Thus, to prove that T is an affine scheme we can assume that $1 \leq a \leq r - 1$.

Lemma 5. *Let k be a field of characteristic p. Let $\nu : [0, r] \to [0, \infty)$ be the Newton polygon of an F^n-crystal $\mathcal{F}$ over k of rank r. Let $a, b \in \mathbb{N}$ be such that $1 \leq a \leq r - 1$. Then (a, b) is a break point of ν if and only if $(1, b)$ is a break point of the Newton polygon $\bigwedge^a(\nu)$ of the F^n-crystal over k of rank $\binom{r}{a}$ which is the exterior power $\bigwedge^a(\mathcal{F})$ of $\mathcal{F}$.*

Proof. Let $\alpha_1 \leq \cdots \leq \alpha_r$ be the Newton polygon slopes of ν. Let $\beta_1 \leq \cdots \leq \beta_{\binom{r}{a}}$ be the Newton polygon slopes of $\bigwedge^a(\nu)$. We have

$$\beta_1 = \sum_{i=1}^{a} \alpha_i \quad \text{and} \quad \beta_2 = \left(\sum_{i=1}^{a-1} \alpha_i\right) + \alpha_{a+1} = \beta_1 + \alpha_{a+1} - \alpha_a.$$

Thus $\beta_1 < \beta_2$ if and only if $\alpha_a < \alpha_{a+1}$. Moreover, (a, b) is a break point of ν if and only if we have $\alpha_a < \alpha_{a+1}$, and $(1, b)$ is a break point of the Newton polygon $\bigwedge^a(\nu)$ if and only if we have $\beta_1 < \beta_2$. The lemma follows from the last two sentences. □

Based on Lemma 5, to prove that T is an affine scheme, by replacing $\mathcal{C}$ with its exterior power $\bigwedge^a(\mathcal{C})$, we can assume that $a = 1$.

2.3. ***A description of*** T**.** Let $q \in \mathbb{N}^*$ be such that for each $x \in S$ the Newton polygon slopes of the F^{nq}-crystal over $\mathrm{Spec}(k(x))$ which is the q-th iterate of $\mathcal{C}_x$, are all integers. For instance, as each Newton polygon slope of $\mathcal{C}_x$ is a rational number whose denominator is a natural number at most equal to r, we can take $q = r!$. Thus by replacing n by nq and $\mathcal{C}$ by its q-th iterate, we can assume that, for each $x \in S$, the Newton polygon slopes of $\mathcal{C}_x$ are natural numbers.

We consider the Newton polygon $\nu_1 : [0, r] \to [0, \infty)$ whose graph is Figure 1.

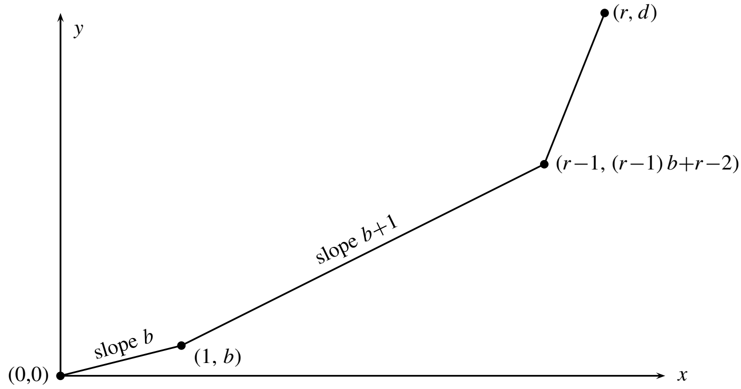


Figure 1. The Newton polygon $\nu_1 : [0, r] \to [0, \infty)$.

If $x \in T$, then because all Newton polygon slopes of $\mathcal{C}_x$ are natural numbers, these Newton polygon slopes are $\alpha_1 = b$, $\alpha_2 \geq b+1$, $\alpha_{r-1} \geq b+1$, and $\alpha_r = d - \sum_{i=1}^{r-1} \alpha_i \geq b+1$. Therefore, if $x \in T$ then we have $x \in S_{\geq \nu_1}$. This implies that T is a subscheme of the closed subscheme $S_{\geq \nu_1}$ of S. By replacing S with $S_{\geq \nu_1}$ we can assume that $S = S_{\geq \nu_1}$. Thus S is reduced.

If $r(b+1) > d$, then $S = S_{\geq \nu_1} = S_{\nu_1} = T$ and thus T is affine. Thus we can assume that $r(b+1) \leq d$ and therefore there exists a Newton polygon $\nu_2 : [0, r] \to [0, \infty)$ whose graph is Figure 2.

If $x \in S - T = S_{\geq \nu_1} - T$, then all Newton polygon slopes of $\mathcal{C}_x$ are natural numbers $\alpha_1 \geq b+1$, $\alpha_2 \geq b+1$, $\alpha_{r-1} \geq b+1$, and $\alpha_r = d - \sum_{i=1}^{r-1} \alpha_i \geq b+1$ and thus ν_x is above ν_2. If ν_x is not above ν_2, then as ν_x is above ν_1 (as $S = S_{\geq \nu_1}$) we have $\alpha_1 = b$ and $\alpha_i \geq b+1$ for $i \in \{2, \ldots, r\}$.

With the last two sentences, we have the identities

$$T = T_{(1,b)} = S - S_{\geq \nu_2} = S_{\geq \nu_1} - S_{\geq \nu_2}.$$

Thus, under all the above reduction steps, T is an open subscheme of S.

2.4. ***On S.*** The statement that T is an affine scheme is local in the faithfully flat topology of S and therefore until the end of Section 3 we will assume that A is a complete local reduced noetherian ring. Thus A is also excellent and therefore its normalization in its ring of fractions is a finite product of normal complete local noetherian integral domains. Based on [Vasiu 2006, Lemma 2.9.2], which is a standard application of Chevalley's theorem of [Grothendieck 1961, Chapter II, (6.7.1)], to prove that T is an affine scheme we can replace A by one of the factors of the mentioned finite product. Thus we can assume that A is a normal complete local noetherian integral domain. We can also assume that T is nonempty and therefore it is an open dense subscheme of S. Let K be the field of factions of A and let $\overline{K}$ be an algebraic closure of it.

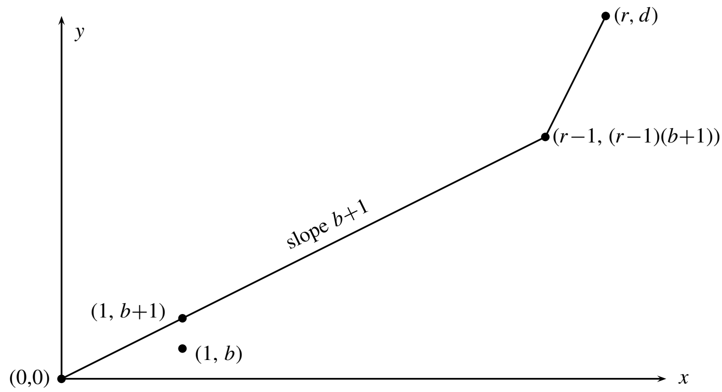


Figure 2. The Newton polygon $\nu_2 : [0, r] \to [0, \infty)$.

3. Proof of Theorem 1

In this section we complete the proof of Theorem 1, i.e., we prove that T is an affine scheme when $a = 1 < r$, when for each $x \in S$ all Newton polygon slopes of $\mathcal{C}_x$ are natural numbers, when we have $S = S_{\geq \nu_1} = \operatorname{Spec} A$ with A a normal complete local noetherian integral domain, and when $T = T_{(1,b)} = S - S_{\geq \nu_2}$ is open dense in S. Let $\mathcal{E}_b = (\mathcal{M}_b, \phi_{\mathcal{M}_b})$ be the pullback to S of the F^n-crystal over $\operatorname{Spec}(\mathbb{F}_p)$ of rank 1 and Newton polygon slope b defined by the pair $(\mathbb{Z}_p, p^b 1_{\mathbb{Z}_p})$. Let η be the generic point $\operatorname{Spec} K \to S$ of S. Let $s, l \in \mathbb{N}^*$.

In Section 3.1 we consider commutative affine group schemes $\mathbb{H}_s$ over S of morphisms between certain evaluations of $\mathcal{E}_b$ and $\mathcal{C}$. In Section 3.2 we glue morphisms between different such evaluations in order to introduce good sections above T of the morphisms $\mathbb{H}_s \to S$ in Section 3.3. In Section 3.4 we complete the proof of Theorem 1. The key idea (the plan) can be summarized as follows: under suitable reductions, for $s \gg 0$ via such good sections above T we can identify T with a closed subscheme of $\mathbb{H}_s$ and therefore we can conclude that T is an affine scheme.

If R is a reduced perfect ring of characteristic p, following [Katz 1979] we say that an F^n-crystal $\mathcal{F}$ over $\operatorname{Spec} R$ is divisible by b if its evaluation at the endomorphism Φ_R^n of the thickening $\mathbb{W}(R)$ is defined by a Φ_R^n-linear endomorphism whose q-th iterate for all $q \in \mathbb{N}^*$ is congruent to 0 modulo p^{bq}. Thus if $y \in \operatorname{Spec} R$, then the Hodge polygon slopes of $\mathcal{F}_y$ are all greater than or equal to b.

3.1. ***Moduli group schemes of morphisms.*** For an A-algebra B and an F^n-crystal $\mathcal{F}$ over B, let $\mathbb{E}_s(\mathcal{F})$ be the evaluation of $\mathcal{F}$ at the thickening $\mathbb{W}_s(B)$; it is an object of the category $\mathcal{M}(W_s(B))$. In particular, we write $\mathbb{E}_s(\mathcal{C}_B) = (M_{s,B}, \phi_{M_{s,B}})$ and let $\mathbb{E}_s(\mathcal{E}_{b,B}) = (N_{s,B}, \phi_{N_{s,B}})$. Thus we have $M = M_{1,A}$, $\phi_M = \phi_{M_{1,A}}$, and $N_{s,B} = W_s(B)$. Moreover $\phi_{N_{s,B}} : N_{s,B} \to N_{s,B}$ is the Φ_B^n-linear endomorphism which maps 1 to p^b and $\phi_{M_{s,B}} : M_{s,B} \to M_{s,B}$ is a Φ_B^n-linear endomorphism and we have

$M_{s,B} = W_s(B) \otimes_{W_s(A)} M_{s,A}$. The kernel of the epimorphism $W_s(B) \to W_1(B) = B$ is a nilpotent ideal. Based on this and the fact that M is a free A-module of rank r, we get that each $M_{s,B}$ is a free $W_s(B)$-module of rank r.

We consider the commutative affine group scheme $\mathbb{H}_s$ over S which represents the following functor: for an A-algebra B, the abelian group

$$\mathbb{H}_s(B) = \mathrm{Hom}_{\mathcal{M}(W_s(B))}(\mathbb{E}_s(\mathcal{E}_{b,B}), \mathbb{E}_s(\mathcal{C}_B))$$

is the group of all $W_s(B)$-linear maps $f : N_{s,B} \to M_{s,B}$ which satisfy the identity $f \circ \phi_{N_{s,B}} = \phi_{M_{s,B}} \circ f$. The S-scheme $\mathbb{H}_s$ is of finite presentation (for $n = 1$, see [Vasiu 2006, Lemma 2.8.4.1], the proof of which applies to all $n \in \mathbb{N}^*$).

Let $x \in S$ be a point of codimension 1. Thus the local ring $D_x := \mathcal{O}_{S,x}$ of S at x is a discrete valuation ring. Let E_x be a complete discrete valuation ring which dominates D_x and has a residue field which is algebraically closed. Let P_x be the perfection of E_x. We recall that $\mathcal{C}_{P_x}$ is the pullback of $\mathcal{C}$ via the natural morphism Spec $P_x \to S$. As $S = S_{\geq \nu_1}$, the Newton polygon slopes of the two fibers of $\mathcal{C}_{P_x}$ are greater than or equal to b. Thus from [Katz 1979, Theorem 2.6.1], we get the existence of an F^n-crystal $\mathcal{D}$ over Spec P_x which is divisible by b and which is equipped with an isogeny

$$\psi_x : \mathcal{D} \to \mathcal{C}_{P_x}$$

whose cokernel is annihilated by p^t for some $t \in \mathbb{N}$. Based on the proof of [loc. cit.], we can assume that

$$t = (r-1)b$$

depends only on r and b.

Proposition 6. *We assume that the point $x \in S$ of codimension 1 belongs to T. Then there exists a unique F^n-subcrystal $\mathcal{D}_b$ of $\mathcal{D}$ which is isomorphic to the pullback $\mathcal{E}_{b,P_x}$ of $\mathcal{E}_b$. Moreover, $\mathcal{D}_b$ has a unique direct supplement in $\mathcal{D}$.*

Proof. We know that for $y \in$ Spec P_x, all Hodge polygon slopes of $\mathcal{D}_y$ are at least b. If all Hodge polygon slopes of $\mathcal{D}_y$ are at least $b+1$, then all Newton polygon slopes of $\mathcal{D}_y$ are at least $b+1$. As under the morphism Spec $P_x \to S$, the point y maps to either $x \in T$ or $\eta \in T$ and as ψ_x is an isogeny, $(1, b)$ is a break point of the Newton polygon of $\mathcal{D}_y$. From the last three sentences we get that $(1, b)$ is a point of the Hodge polygon of $\mathcal{D}_y$.

Thus for each point $y \in$ Spec P_x, $(1, b)$ is a break point of the Newton polygon of $\mathcal{D}_y$ and is a point of the Hodge polygon of $\mathcal{D}_y$. Due to this, from [Katz 1979, Theorem 2.4.2] we get that there exists a unique direct sum decomposition,

$$\mathcal{D} = \mathcal{D}_b \oplus \mathcal{D}_{>b},$$

into F^n-crystals over Spec P_x, where $\mathcal{D}_b$ is of rank 1 and each fiber of it at a point $y \in$ Spec P_x has all Hodge and Newton polygon slopes equal to b and where $\mathcal{D}_{>b}$

is of rank $r-1$ and each fiber of it at a point $y \in \operatorname{Spec} P_x$ has all Newton polygon slopes greater than b (and has all Hodge polygon slopes greater than or equal to b).

As $\mathcal{D}$ is divisible by b, $\mathcal{D}_b$ and $\mathcal{D}_{>b}$ are also divisible by b.

As P_x is perfect, for each $l \in \mathbb{N}^*$ we have $W(P_x)/(p^l) = W_l(P_x)$ and the module of differentials $\Omega^1_{W_l(P_x)}$ is 0. Thus, from [Berthelot and Messing 1990, Proposition 1.3.3] we get that an F^n-crystal over $\operatorname{Spec} P_x$ is uniquely determined by its evaluation at the thickening $\mathbb{W}(P_x)$. The evaluation of $\mathcal{E}_{b,P_x}$ at the thickening $\mathbb{W}(P_x)$ is canonically identified with $(W(P_x), p^b\Phi^n_{P_x})$ and the evaluation of $\mathcal{D}_b$ at the thickening $\mathbb{W}(P_x)$ can be identified with $(W(P_x), p^b\Phi_b)$, where $\Phi_b : W(P_x) \to W(P_x)$ is a $\Phi^n_{P_x}$-linear endomorphism such that $\Phi_b(1)$ generates $W(P_x)$.

As P_x is the perfection of E_x and as E_x is complete and has an algebraically closed residue field, the rings $W(P_x)$ and $W_l(P_x)$ are strictly henselian and p-adically complete. We check that these properties imply that there exists a unit υ of $W(P_x)$ such that we have

$$\Phi_b(\upsilon) = \Phi^n_{P_x}(\upsilon)\Phi_b(1) = \upsilon.$$

If $n = 1$, then from [Berthelot and Messing 1990, Proposition 2.4.9] we get that for each $l \in \mathbb{N}^*$ there exists a unit $\upsilon_l \in W(P_x)$ such that $\Phi_b(\upsilon_l) - \upsilon_l \in p^l W(P_x)$, and the proof of [loc. cit.] confirms that we can assume that $\upsilon_{l+1} - \upsilon_l \in p^l W(P_x)$. Thus for $n = 1$ we can take υ to be the p-adic limit of the sequence $(\upsilon_l)_{l \geq 1}$. This argument applies entirely for $n > 1$.

Multiplication by υ defines an isomorphism

$$(W(P_x), p^b\Phi^n_{P_x}) \to (W(P_x), p^b\Phi_b)$$

which defines an isomorphism $\mathcal{E}_{b,P_x} \to \mathcal{D}_b$. □

From now we will assume that $x \in T$. We consider a composite morphism

$$j_x[s] : \mathbb{E}_s(\mathcal{E}_{b,P_x}) \to \mathbb{E}_s(\mathcal{D}_b) \to \mathbb{E}_s(\mathcal{D}) = \mathbb{E}_s(\mathcal{D}_b) \oplus \mathbb{E}_s(\mathcal{D}_{>b})$$

in which the first arrow is an isomorphism and the second arrow is the split monomorphism associated to the direct sum decomposition.

Let

$$i_x(s) : \mathbb{E}_s(\mathcal{E}_{b,P_x}) \to \mathbb{E}_s(\mathcal{C}_{P_x})$$

be the composite of $j_x[s]$ with the morphism $\psi_x[s] : \mathbb{E}_s(\mathcal{D}) \to \mathbb{E}_s(\mathcal{C}_{P_x})$ which is the evaluation of the isogeny ψ_x at the thickening $\mathbb{W}_s(P_x)$ (i.e., which is the reduction modulo p^s of ψ_x). From now on, we will take $s > t = (r-1)b$. We note that $\psi_x[s]$ is a quasi-isogeny whose cokernel is annihilated by p^t and whose domain is divisible by b.

3.2. ***Gluing morphisms.*** For each point $x \in T$ of codimension 1 (i.e., whose local ring D_x is a discrete valuation ring), we follow [Vasiu 2006, Section 2.8.3] to show the existence of a finite field extension K_x of K and of an open subset T_x of the normalization of T in $\operatorname{Spec} K_x$ such that T_x has a local ring which is a discrete valuation ring D_x^+ that dominates D_x and moreover we have a morphism

$$i_{T_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,T_x}) \to \mathbb{E}_s(\mathcal{C}_{T_x})$$

of the category $\mathcal{M}(W_s(T_x))$ which is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^t and whose domain is divisible by b.

To check this, with the notations of Section 3.1 we consider four identifications,

$$E_s(\mathcal{C}_{D_x}) = (W_s(D_x)^r, \phi_{s,x}), \qquad \mathbb{E}_s(\mathcal{E}_{b,D_x}) = (W_s(D_x), p^b\Phi^n_{D_x}),$$
$$\mathbb{E}_s(\mathcal{D}_b) = (W_s(P_x), p^b\Phi^n_{P_x}), \qquad \mathbb{E}_s(\mathcal{D}_{>b}) = (W_s(P_x)^{r-1}, p^b\phi_{s,>b,x}).$$

Now, the $W_s(P_x)$-linear map $\psi_{s,P_x} : W_s(P_x)^r \to W_s(P_x)^r$ defining $\psi_x[s]$ and the $\Phi^n_{P_x}$-linear map $\phi_{s,>b,x} : W_s(P_x)^{r-1} \to W_s(P_x)^{r-1}$ involve a finite number of coordinates of Witt vectors of length s and therefore are defined over $W_s(B_x)$, where B_x is a finitely generated D_x-subalgebra of P_x. We can choose B_x such that the resulting $W_s(B_x)$-linear map $\psi_{s,B_x} : W_s(B_x)^r \to W_s(B_x)^r$ has a cokernel annihilated by p^t. The faithfully flat morphism $\operatorname{Spec} B_x \to \operatorname{Spec} D_x$ has quasisections (see [Grothendieck 1967, Corollary 17.16.2]) and therefore there exists a finite field extension K_x of K and a discrete valuation ring D_x^+ of the normalization T in K_x which dominates D_x and for which we have a D_x-homomorphism $B_x \to D_x^+$. The $W_s(D_x^+)$-linear map $\psi_{s,D_x^+} : W_s(D_x^+)^r \to W_s(D_x^+)^r$ which is the natural tensorization of ψ_{s,B_x} induces (via restriction to the first factor $W_s(D_x^+)$ of $W_s(D_x^+)^r$) a morphism $i_{D_x^+}(s) : \mathbb{E}_s(\mathcal{E}_{b,D_x^+}) \to \mathbb{E}_s(\mathcal{C}_{D_x^+})$ of the category $\mathcal{M}(W_s(D_x^+))$ which is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^t and whose domain is divisible by b. It is easy to see that there exists an open subset T_x of the normalization of T in K_x which has D_x^+ as a local ring and for which there exists a morphism $i_{T_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,T_x}) \to \mathbb{E}_s(\mathcal{C}_{T_x})$ of the category $\mathcal{M}(W_s(T_x))$ that has all the desired properties and that extends the morphism $i_{D_x^+}(s)$ of the category $\mathcal{M}(W_s(D_x^+))$.

By working with $s+l$ instead of s, we can assume that there exists $l \in \mathbb{N}$, $l \gg 0$ such that $i_{T_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,T_x}) \to \mathbb{E}_s(\mathcal{C}_{T_x})$ is the reduction modulo p^s of a morphism

$$i_{T_x}(s+l) : \mathbb{E}_{s+l}(\mathcal{E}_{b,T_x}) \to \mathbb{E}_{s+l}(\mathcal{C}_{T_x})$$

of the category $\mathcal{M}(W_{s+l}(T_x))$.

Let I_s be the set of morphisms $\mathbb{E}_s(\mathcal{E}_{b,\overline{K}}) \to \mathbb{E}_s(\mathcal{C}_{\overline{K}})$ which lift to morphisms $\mathbb{E}_{s+l}(\mathcal{E}_{b,\overline{K}}) \to \mathbb{E}_{s+l}(\mathcal{C}_{\overline{K}})$ for some $l \gg 0$. From [Vasiu 2006, Theorem 5.1.1(a)] (applied for $l \gg 0$ which depends only on b and r) we get that each element of I_s is the evaluation at the thickening $\mathbb{W}_s(\overline{K})$ of a morphism of F^n-crystals $\mathcal{E}_{b,\overline{K}} \to \mathcal{C}_{\overline{K}}$ (strictly speaking [loc. cit.] is stated for $n = 1$ but its proof works for all $n \in \mathbb{N}^*$). This implies that I_s is a finite set whose elements are all pullbacks of morphisms of $\mathcal{M}(W_s(L))$, where L is a suitable finite field extension of K contained in $\overline{K}$. By replacing S with its normalization in L, we can assume that $L = K$. As inside K_x we have an identity $D_x^+ \cap K = D_x$, inside $W_s(K_x)$ we have an identity $W_s(D_x^+) \cap W_s(K) = W_s(D_x)$. From the last three sentences we get that the pullback $i_{D_x^+}(s)$ of $i_{T_x}(s)$ to a morphism of $\mathcal{M}(W_s(D_x^+))$ is the pullback of a morphism of $\mathcal{M}(W_s(D_x))$. Based on this we can assume that there exists an open subscheme U_x of T which contains x and which has the property that there exists a morphism

$$i_{U_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,U_x}) \to \mathbb{E}_s(\mathcal{C}_{U_x})$$

of the category $\mathcal{M}(W_s(U_x))$ such that $i_{T_x}(s)$ is the pullback of it.

We consider an identification $\mathcal{C}_{\overline{K}} = (Q, \phi_Q)$, where $Q = W(\overline{K})^r$ and $\phi_Q : Q \to Q$ is a $\Phi_{\overline{K}}^n$-linear endomorphism. The Newton polygon ν_η of $\mathcal{C}_{\overline{K}}$ has the Newton polygon slope b with multiplicity 1 and therefore there exists a unique nonzero direct summand Q_b of Q such that we have $\phi_Q(Q_b) = p^b Q_b$. The rank of the $W(\overline{K})$-module Q_b is 1. Let $z_b \in Q_b$ be such that $Q_b = W(\overline{K}) z_b$ and $\phi_Q(z_b) = p^b z_b$; it is unique up to multiplication by units of $W(\mathbb{F}_{p^n})$.

We have a canonical identification $\mathcal{E}_{b,\overline{K}} = (W(\overline{K}), p^b \Phi_{\overline{K}}^n)$. The morphism $\mathbb{E}_s(\mathcal{E}_{b,\overline{K}}) \to \mathbb{E}_s(\mathcal{C}_{\overline{K}})$ defined by $i_{T_x}(s)$ is an element of I_s and therefore it is the reduction modulo p^s of a morphism $\lambda_x : (W(\overline{K}), p^b \Phi_{\overline{K}}^n) \to (Q, \phi_Q)$ of F^n-crystals over $\overline{K}$. Clearly $\lambda_x(1) \in Q_b$ and thus there exists a unique element $\tau_x \in W(\mathbb{F}_{p^n})$ such that we have

$$\lambda_x(1) = \tau_x z_b.$$

As $i_{T_x}(s)$ is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^t from Fact 4 applied with $D = W(\overline{K})$ we get that τ_x modulo p^{t+1} is a nonzero element of $W_{t+1}(\mathbb{F}_{p^n})$. Therefore we can write $\tau_x = p^{t_x} u_x$, where $u_x \in W(\mathbb{F}_{p^n})$ is a unit and where $t_x \in \{0, \ldots, t\}$.

From now on, we will take $s > 2t$. We consider the morphism

$$\theta_x := p^{t-t_x} u_x^{-1} i_{U_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,U_x}) \to \mathbb{E}_s(\mathcal{C}_{U_x})$$

of the category $\mathcal{M}(W_s(U_x))$; its pullback to a morphism of $\mathcal{M}(W_s(T_x))$ is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^{t+t_x} and thus also by p^{2t} and whose domain is divisible by b. The pullback of θ_x to a morphism of $\mathcal{M}(W_s(\overline{K}))$ is the reduction modulo p^s of the morphism $p^{t-t_x} u_x^{-1} \lambda_x : (W(\overline{K}), p^b \Phi_{\overline{K}}^n) \to (Q, \phi_Q)$ which maps 1 to $p^t z_b$ and which does not depend on the point $x \in T$ of codimension 1.

Let U be the open subscheme of T which is the union of all U_x's. From the previous paragraph we get that the θ_x's glue together to define a morphism

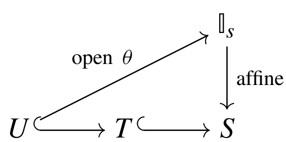
$$\theta : \mathbb{E}_s(\mathcal{E}_{b,U}) \to \mathbb{E}_s(\mathcal{C}_U)$$

of the category $\mathcal{M}(W_s(U))$.

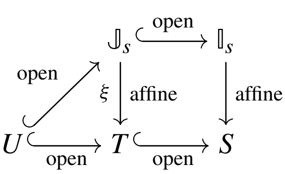
By replacing S with its normalization in any of the finite field extensions K_x of K, we can assume that there exists an open dense subscheme U_0 of U such that the pullback $\theta_{U_0} : \mathbb{E}_s(\mathcal{E}_{b,U_0}) \to \mathbb{E}_s(\mathcal{C}_{U_0})$ of θ to a morphism of $\mathcal{M}(W_s(U_0))$ is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^{2t} and whose domain is divisible by b: under such a replacement, we can take U_0 to be T_x itself.

3.3. *Good section of* $\mathbb{H}_s$. We have $\operatorname{codim}_T(T - U) \geq 2$ and the morphism θ is defined by a section $\theta : U \to \mathbb{H}_s$ denoted in the same way.

Let $\mathbb{I}_s$ be the schematic closure $\overline{\theta(U)}$ of $\theta(U)$ in $\mathbb{H}_s$. As the scheme $\mathbb{H}_s$ is affine and noetherian and as U is an integral scheme, the scheme $\mathbb{I}_s$ is also affine, noetherian, and integral. We have a commutative diagram:



We consider the pullback $\mathbb{J}_s$ of $\mathbb{I}_s$ to T:



Lemma 7. *The affine morphism $\xi : \mathbb{J}_s \to T$ is an isomorphism.*

Proof. To prove that ξ is an isomorphism, we can assume that $T = S = \operatorname{Spec} A$ is an affine scheme. As ξ is an affine morphism, $\mathbb{J}_s = \operatorname{Spec} B$ is also an affine scheme. Since U is open dense in both T and $\mathbb{I}_s$, T and $\mathbb{J}_s$ have the same field of fractions K. As $\operatorname{codim}_T(T - U) \geq 2$ and as U is an open subscheme of both T and $\mathbb{J}_s$, we have $A_{\mathfrak{p}} = B_{\mathfrak{p}}$ for each prime $\mathfrak{p} \in S = T$ of height 1. As A is a noetherian normal domain, inside K we have

$$A \subseteq B \subseteq \bigcap_{\mathfrak{q} \in \operatorname{Spec} B \text{ of height } 1} B_{\mathfrak{q}} \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Spec} A \text{ of height } 1} A_{\mathfrak{p}} = A$$

(see [Matsumura 1980, (17.H), Theorem 38] for the equality part; the first inclusion is defined by ξ). Therefore $A = B$. □

Lemma 7 allows us in what follows to identify T itself with an open dense subscheme of $\mathbb{I}_s$ (i.e., with $\mathbb{J}_s$).

3.4. ***End of the proof.*** In this subsection we will show that for $s \gg 0$, we have $T = \mathbb{I}_s$. This will complete the proof of Theorem 1 as $\mathbb{I}_s$ is an affine scheme.

It remains to show that the assumption that for $s \gg 0$ we have $T \neq \mathbb{I}_s$ leads to a contradiction. This assumption implies that there exists an algebraically closed field k of characteristic p and a morphism $\zeta_0 : \mathrm{Spec}(k[\![X]\!]) \to \mathbb{I}_s$ with the properties that under it the generic point of $\mathrm{Spec}(k[\![X]\!])$ maps to U_0 and its special point maps to $\mathbb{I}_s - T$.

Let $P = k[\![X]\!]^{\mathrm{perf}}$ be the perfection of $k[\![X]\!]$, let κ be the perfect field which is the field of fractions of P, and let $\zeta : \mathrm{Spec}\, P \to \mathbb{I}_s$ be the morphism defined naturally by ζ_0. To the composite of ζ with the closed embedding $\mathbb{I}_s \to \mathbb{H}_s$ corresponds a morphism

$$\omega : \mathbb{E}_s(\mathcal{E}_{b,P}) \to \mathbb{E}_s(\mathcal{C}_P)$$

of the category $\mathcal{M}(W_s(P))$ whose pullback ω_κ to a morphism of $\mathcal{M}(W_s(\kappa))$ is equal to the pullback $\theta_\kappa : \mathbb{E}_s(\mathcal{E}_{b,\kappa}) \to \mathbb{E}_s(\mathcal{C}_\kappa)$ of θ.

We have a natural identification $\mathbb{E}_s(\mathcal{E}_{b,P}) = (W_s(P), p^b\Phi_P^n)$ and we consider an identification $\mathbb{E}_s(\mathcal{C}_P) = (W_s(P)^r, \phi)$. Thus we have a $W_s(P)$-linear map

$$\omega : W_s(P) \to W_s(P)^r$$

such that $\omega \circ p^b\Phi_P^n = \phi \circ \omega$. We consider an isogeny $\mathcal{D} \to \mathcal{C}_P$ whose cokernel is annihilated by p^t and with $\mathcal{D}$ divisible by b, again see [Katz 1979, Theorem 2.6.1] (here $t = (r-1)b$ as stated before Proposition 6). Thus we also have an isogeny $\iota : \mathcal{C}_P \to \mathcal{D}$ whose cokernel is annihilated by p^t. We consider its evaluation

$$\iota[s] : \mathbb{E}_s(\mathcal{C}_P) \to \mathbb{E}_s(\mathcal{D})$$

at the thickening $\mathbb{W}_s(P)$. Under an identification $\mathbb{E}_s(\mathcal{D}) = (W_s(P)^r, p^b\varphi)$ with $\varphi : W_s(P)^r \to W_s(P)^r$ as a Φ_P^n-linear endomorphism, we get a $W_s(P)$-linear endomorphism $\iota[s] : W_s(P)^r \to W_s(P)^r$ such that we have $\iota[s] \circ \phi = p^b\varphi \circ \iota[s]$. We consider the composite morphism

$$\rho = \iota[s] \circ \omega : \mathbb{E}_s(\mathcal{E}_{b,P}) \to \mathbb{E}_s(\mathcal{D})$$

identified with a $W_s(P)$-linear map $\rho : W_s(P) \to W_s(P)^r$ such that $\rho \circ p^b\Phi_P^n = p^b\varphi \circ \rho$. Let

$$\gamma = \rho(1) = (\gamma_1, \dots, \gamma_r) \in W_s(P)^r.$$

From the identity $\rho \circ p^b\Phi_P = p^b\varphi \circ \rho$ we get that the image of $\varphi(\gamma) - \gamma$ in $W_{s-b}(P)^r$ is 0. Writing $\gamma = p^u\delta$, where $u \in \mathbb{N}$ and $\delta \in W_s(P)^r - pW_s(P)^r$, we get that the image of $\varphi(\delta)-\delta$ in $W_{s-b-u}(P)^r$ is 0. Let $\bar{\delta}\in P^r-0$ be the image in $P^r=W_1(P)$ of δ (i.e., the reduction modulo p of δ).

Lemma 8. *If $s\geq 3t+1$, then we have $u\leq 3t$. Therefore, if moreover we have $s\geq 3t+b+1$, then the image of $\varphi(\delta)-\delta$ in $W_{s-b-3t}(P)^r$ is* 0.

Proof. To check this we can work over $W_s(\kappa)$. As the generic point of Spec P maps to U_0, $\omega_\kappa=\theta_\kappa:\mathbb{E}_s(\mathcal{E}_{b,\kappa})\to\mathbb{E}_s(\mathcal{C}_\kappa)$ is the pullback of the morphism θ_{U_0}. The pullback ρ_κ of ρ to $\mathcal{M}(W_s(\kappa))$ is a composite morphism

$$\rho_\kappa=\iota[s]_\kappa\circ\theta_\kappa:\mathbb{E}_s(\mathcal{E}_{b,\kappa})\to\mathbb{E}_s(\mathcal{D}_\kappa)$$

and therefore it is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^{2t} (as θ_{U_0} has this property) and with a quasi-isogeny whose cokernel is annihilated by p^t (as ι is an isogeny whose cokernel is annihilated by p^t). Therefore, ρ_κ is also the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^{3t}. This implies that the image of γ in $W_{3t+1}(\kappa)$ is nonzero (see Fact 4 applied with $D=W(\kappa)$) and therefore we have $u\leq 3t$. □

Lemma 9. *If $s\geq 3t+b+1$, then the image of $\bar{\delta}$ in $k^r=W_1(k)^r$ is nonzero.*

Proof. We show that the assumption that the image of $\bar{\delta}\in P^r-0$ in $k^r=W_1(k)^r$ is 0 leads to a contradiction. This assumption implies that there exists a largest positive rational number c of denominator a power of p such that we have

$$\bar{\delta}\in X^cP^r\subset P^r=(k[\![X]\!]^{\mathrm{perf}})^r.$$

Let $\bar{\varphi}:P^r\to P^r$ be the P-linear endomorphism which is the reduction modulo p of φ. From Lemma 8 we get that $\bar{\delta}=\bar{\varphi}(\bar{\delta})$. Thus $\bar{\delta}\in\bar{\varphi}(X^cP^r)\subseteq X^{p^nc}P^r$ and this implies that $p^nc\leq c$ which is a contradiction. □

From the inequality $u\leq 3t$ (see Lemma 8) and from Lemma 9 we get that for $s\geq 3t+b+1$ the pullback ω_k of ω to a morphism of $\mathcal{M}(W_s(k))$ is such that its reduction modulo p^{3t+1} is nonzero. For $s>3t+b+1+l$ with $l\in\mathbb{N}^*$ large enough but depending only on b and r, the reduction of ω_k modulo p^{s-l} lifts to a morphism $\mathcal{E}_{0,k}\to\mathcal{D}_k$ (see [Vasiu 2006, Theorem 5.1.1(a)], which, again, stated for $n=1$, applies to all $n\in\mathbb{N}^*$) which is nonzero. Thus $\mathcal{D}_k$ has Newton polygon slope b with multiplicity at least 1. From this and the existence of the isogeny ι we get that $\mathcal{C}_k$ has Newton polygon slope b with multiplicity at least 1. This implies that the special point of Spec$(k[\![X]\!])$ under the composite of $\zeta_0:\mathrm{Spec}(k[\![X]\!])\to\mathbb{I}_s$ with the morphism $\mathbb{I}_s\to S$ does not map to a point of $S_{\nu_2}=S-T$ and so it maps to a point of T. This is a contradiction, and ends the proof of Theorem 1. □

4. Applications of Theorem 1

In Section 4.1 we prove Corollary 2. In Section 4.2 we follow [Vasiu 2013] to introduce generalized Artin–Schreier systems of equations and Artin–Schreier stratifications. In Section 4.3 we refine and reobtain Corollary 3 in the context of these stratifications. Section 4.4 contains some complements, including Proposition 13, which prove that "pure in" implies "weakly pure in". Until the end let A be an arbitrary $\mathbb{F}_p$-algebra.

4.1. ***Proof of Corollary 2.*** To prove Corollary 3, in this subsection we can assume that $S = \operatorname{Spec} A$ and $d \in \mathbb{N}$ are as in the paragraph before Section 2.1. We can also assume that $\nu(r) = d$ as otherwise $S_\nu = \varnothing$ is pure in S. Let $l \in \mathbb{N}$ be such that the Newton polygon ν has exactly $l+1$ breaking points denoted as $(a_0, b_0) = (0, 0), \ldots, (a_l, b_l) = (r, d)$.

We have obvious identities

$$S_\nu = \Big[S_{\geq\nu} \bigcap_{i=0}^{l} T_{(a_l,b_l)}(\mathcal{C})\Big]_{\text{red}} = [S_{\geq\nu} \times_S (T_{(a_0,b_0)}(\mathcal{C}))_S \times \cdots \times_S T_{(a_l,b_l)}(\mathcal{C})]_{\text{red}}.$$

From Theorem 1 we get that each $T_{(a_l,b_l)}(\mathcal{C})$ is an affine scheme. We recall that $S_{\geq\nu}$ is a reduced closed subscheme of S. From the last three sentences we get that S_ν is an affine scheme, i.e., is pure in S. □

4.2. ***Artin–Schreier stratifications.*** Let $x_0, x_1, \ldots, x_r$ be free variables. For $i, j \in \{1, \ldots, r\}$ let $P_{i,j}(x_0) \in A[x_0]$ be a polynomial which is a linear combination with coefficients in A of the monomials x_0^q with $q \in \mathbb{N}$ either 0 or a power of p. By a generalized Artin–Schreier system of equations in r variables over A we mean a system of equations of the form

$$x_i = \sum_{j=1}^{r} P_{i,j}(x_j^p) \quad i \in \{1, \ldots, r\}$$

to which we associate the A-algebra

$$B = A[x_1, \ldots, x_r]/\bigg(x_1 - \sum_{j=1}^{r} P_{1,j}(x_j^p), x_2 - \sum_{j=1}^{r} P_{2,j}(x_j^p), \ldots, x_r - \sum_{j=1}^{r} P_{r,j}(x_j^p)\bigg).$$

Each equation of the form $x_i = \sum_{j=1}^{r} P_{i,j}(x_j^p)$ will be called as a generalized Artin–Schreier equation, and its degree $e_i \in \mathbb{N}$ is defined as follows. We have $e_i = 0$ if and only if for all $j \in \{1, \ldots, r\}$ the polynomial $P_{i,j}(x_0)$ is a constant, and if $e_i > 0$ then e_i is the largest integer such that there exists a $j \in \{1, \ldots, r\}$ with the property that the degree of $P_{i,j}(x_j^p)$ is p^{e_i}.

Let $e = \max\{e_1, \dots, e_r\}$; we call it the degree of the generalized Artin–Schreier system of equations in r variables over A. Following [Vasiu 2013], when $e \leq 1$ we drop the word "generalized".

Proposition 10. *The morphism $\epsilon : \operatorname{Spec} B \to \operatorname{Spec} A$ is étale and surjective and its geometric fibers have a number of points equal to a power of p.*

Proof. If $e_i > 1$, then by adding, for each $j \in \{1, \dots, r\}$ such that the degree of $P_{i,j}(x_j^p)$ is p^{e_i}, an extra variable $y_{i,j}$ and an equation of the form $y_{i,j} = x_j^p$, the generalized Artin–Schreier equation $x_i = \sum_{j=1}^r P_{i,j}(x_j^p)$ gets replaced by several generalized Artin–Schreier equations of degrees less than e_i. By repeating this process of adding extra variables and equations which (up to isomorphisms between $\operatorname{Spec} A$-schemes) do not change the morphism $\epsilon : \operatorname{Spec} B \to \operatorname{Spec} A$, we can assume that $e \leq 1$. Thus the proposition follows from [Vasiu 2013, Theorem 2.4.1(a) and (b)]. □

Definition 11 is a natural extrapolation of [Vasiu 2013, Definition 2.4.2] which applies to étale morphisms $\epsilon : \operatorname{Spec} B \to \operatorname{Spec} A$ as in Proposition 10.

Definition 11. Let $\varepsilon : \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} A$ be an étale morphism between affine $\mathbb{F}_p$-schemes.

(a) We assume that A is noetherian. Then by the Artin–Schreier stratification of $\operatorname{Spec} A$ associated to $\varepsilon : \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} A$ in reduced locally closed subschemes $V_1, \dots, V_q$ we mean the stratification defined inductively by the following property: for each $l \in \{1, \dots, q\}$ the scheme V_l is the maximal open subscheme of the reduced scheme of $(\operatorname{Spec} A) - \left(\bigcup_{q=1}^{l-1} V_q\right)$ which has the property that the morphism $\epsilon_{V_l} : (\operatorname{Spec} B) \times_{\operatorname{Spec} A} V_l \to V_l$ is an étale cover.

(b) Let $\mu_1 > \mu_2 > \cdots > \mu_v$ be the shortest sequence of strictly decreasing natural numbers such that each fiber of the morphism $\epsilon : \operatorname{Spec} B \to \operatorname{Spec} A$ has a number of geometric points equal to μ_l for some $l \in \{1, \dots, v\}$. Then by the functorial Artin–Schreier stratification of $\operatorname{Spec} A$ associated to $\varepsilon : \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} A$ we mean the stratification of $\operatorname{Spec} A$ in reduced locally closed subschemes $U_1, \dots, U_v$ defined inductively by the following property: for each $l \in \{1, \dots, v\}$ the scheme U_l is the maximal open subscheme of the reduced scheme of $(\operatorname{Spec} A) - \left(\bigcup_{q=1}^{l-1} U_q\right)$ which has the property that the morphism $\epsilon_{U_l} : (\operatorname{Spec} B) \times_{\operatorname{Spec} A} U_l \to U_l$ is an étale cover whose fibers all have a number of geometric points equal to μ_l.

The existence of the stratification $V_1, \dots, V_q$ of $\operatorname{Spec} A$ is a standard piece of algebraic geometry. The existence of the sequence $\mu_1 > \mu_2 > \cdots > \mu_v$ follows from the facts that each étale morphism is locally quasifinite and that $\operatorname{Spec} \mathcal{B}$ is quasicompact. The existence of the stratification $U_1, \dots, U_v$ of $\operatorname{Spec} A$ is implied by [Grothendieck 1967, Proposition 18.2.8 and Corollary 18.2.9], which show that one can define U_l directly and functorially as follows: each U_l is the set of all points $x \in \operatorname{Spec} A$ such that the fiber of ε at x has exactly μ_l geometric points.

Theorem 12. *Let $\varepsilon : \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} A$ be an étale morphism between affine $\mathbb{F}_p$-schemes. Then the functorial Artin–Schreier stratification of* Spec A *associated to* $\varepsilon : \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} A$ *in reduced locally closed subschemes $U_1, \ldots, U_v$ is pure, i.e., for each $l \in \{1, \ldots, v\}$, the stratum U_l is pure in* Spec A.

Proof. As the étale morphism $\varepsilon : \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} A$ is of finite presentation and due to the functorial part, we can assume that A is a finitely generated $\mathbb{F}_p$-algebra and thus an excellent ring. We follow [Vasiu 2014]. By replacing Spec A by its closed subscheme $(\operatorname{Spec} A) - \left(\bigcup_{q=1}^{l-1} U_q\right)$ endowed with the reduced structure, we can assume that $l = 1$ and that A is reduced. Thus U_1 is an open dense subscheme of Spec A. Based again on [Vasiu 2006, Lemma 2.9.2], to prove that U_1 is an affine scheme, we can replace A by its normalization in its ring of fractions. Thus by passing to connected components of Spec A, we can assume that A is an excellent normal domain. Thus $B = \prod_{l=1}^{w} B_l$ is a finite product of excellent normal domains which are étale A-algebras. Let K_l be the field of fractions of B_l. Let L be the finite Galois extension of the field of fractions K of A generated by the finite separable extensions K_l's of K. By replacing A by its normalization in L (again based on [Vasiu 2006, Lemma 2.9.2]), we can assume $K = K_1 = \cdots = K_w$. This implies that each $\operatorname{Spec}(B_l)$ is an open subscheme of Spec A and thus

$$\begin{aligned} U_1 &= \bigcap_{l=1}^{w} \operatorname{Spec}(B_l) \\ &= (\operatorname{Spec}(B_1)) \times_{\operatorname{Spec} A} (\operatorname{Spec}(B_2)) \times_{\operatorname{Spec} A} \cdots \times_{\operatorname{Spec} A} (\operatorname{Spec}(B_w)) \end{aligned}$$

is the affine scheme $\operatorname{Spec}(B_1 \otimes_A \otimes \cdots \otimes_A B_w)$. □

4.3. ***A second proof of Corollary 3.*** We will use Theorem 12 to obtain a second proof of Corollary 3 which is simpler and independent of Theorem 1. We can assume that $S = \operatorname{Spec} A$ is affine and let $\phi_M : M \to M$ be as in Section 2.1.

The identities $S_m = T_{(m,0)}(\mathcal{C})$ if $m > 0$ and $S_0 = T_{(1,0)}(\mathcal{C} \oplus \mathcal{E}_0)$ show that S_m is a reduced locally closed subscheme of S. Thus by replacing S by $\bar{S}_m$, we can assume that S_m is an open dense subscheme of $S = \bar{S}_m$.

We consider the equation

$$\phi_M(z) = z \tag{1}$$

in $z \in M$. For $x \in S$ we have $\chi(x) = \dim_{\mathbb{F}_{p^n}}(\vartheta_x)$, where ϑ_x is the $\mathbb{F}_{p^n}$-vector space of solutions of the tensorization of (1) over A with an algebraic closure of the residue field $k(x)$ of S at x.

From now on we will forget about $\mathcal{C}$ and just work with the free A-module M of rank r and its Φ_A^n-linear endomorphism $\phi_M : M \to M$ and we only assume that we have an open dense subset S_m of $S = \operatorname{Spec} A$ defined by the following property: for $x \in S$, we have $x \in S_m$ if and only if $\dim_{\mathbb{F}_{p^n}}(\vartheta_x) = m$.

With respect to a fixed A-basis $\{v_1, \ldots, v_r\}$ of M, by writing $z = \sum_{i=1}^r x_i v_i$, (1) defines a generalized Artin–Schreier system of equations in the r variables $x_1, \ldots, x_r$ of the form

$$x_i = L_i(x_1^{p^n}, \ldots, x_r^{p^n}), \quad i \in \{1, \ldots, r\},$$

where each L_i is a homogeneous polynomial of total degree at most 1. Let

$$B = A[x_1, \ldots, x_r]/(x_1 - L_1(x_1^{p^n}, \ldots, x_r^{p^n}), \ldots, x_r - L_r(x_1^{p^n}, \ldots, x_r^{p^n})),$$

let $\epsilon : \operatorname{Spec} B \to S$ and let $U_1, \ldots, U_v$ be the functorial Artin–Schreier stratification of S associated to $\epsilon : \operatorname{Spec} B \to S$. Let $p^{\mu_1} > p^{\mu_2} > \cdots > p^{\mu_v}$ be the shortest sequence of strictly decreasing of powers of p by natural numbers such that for each $l \in \{1, \ldots, v\}$, every geometric fiber of the morphism $\epsilon_{U_l} : \operatorname{Spec} B \times_S U_l \to U_l$ has a number of geometric points equal to p^{μ_l}, see Proposition 10 and Definition 11(b).

The fact that the morphism $\epsilon : \operatorname{Spec} B \to S$ is étale (see Proposition 10) is equivalent to [Zink 2001, Proposition 3]. We consider the lower semicontinuous function (see [Grothendieck 1967, Proposition 18.2.8])

$$\mu : S \to \mathbb{N}$$

defined by the rule: $\mu(x) = p^{n \dim_{\mathbb{F}_{p^n}}(\vartheta_x)}$ is the number of geometric points of $\epsilon : \operatorname{Spec} B \to S$ above x (i.e., is the number of elements of ϑ_x). We get that μ_l is divisible by n for all $l \in \{1, \ldots, v\}$ and (as S_m is dense in S) we have $\mu_1 = mn$. Moreover, for $x \in S$ and $q \in \mathbb{N}$ we have $\mu(x) = p^{nq}$ if and only if $x \in S_q$. We conclude that $S_m = U_1$ and therefore (see Theorem 12) S_m is an affine scheme. □

4.4. *Complements.* For the sake of completeness, we include a proof of the following well-known result (to be compared with [Vasiu 2006, Remark 6.3(a)]).

Proposition 13. *Let Z be a reduced locally closed subscheme of a locally noetherian scheme Y. If Z is pure in Y, then Z is weakly pure in Y.*

Proof. We can assume that $Z \subsetneq \overline{Z} = Y$. By localizing Y at the generic point of an irreducible component of $\overline{Z} - Z$, we can assume that $Y = \overline{Z} = \operatorname{Spec} C$ is a local affine scheme of dimension at least 1 and Z is the complement in Y of the closed point of Y and we have to prove that C has dimension 1. By passing to a connected component of the normalization of the reduced completion $\hat{C}_{\text{red}}$ of C in the ring of fractions of $\hat{C}_{\text{red}}$, we can assume that C is in fact an integral normal local ring which is not a field.

We show that the assumption that $\dim(C) \geq 2$ leads to a contradiction. As the open dense subscheme Z of Y is pure in Y, Z is the spectrum of a C-subalgebra of the field of fractions of C which contains C and which is contained in the intersection of all the localizations of C at points of Y of codimension 1 in Y (as these points belong to Z). As $\dim(C) \geq 2$, from [Matsumura 1980, (17H), Theorem 38] we get that this intersection is C and thus we have $Z = \operatorname{Spec} C = Y$. This is a contradiction. Thus $\dim(C) = 1$. □

Remark 14. Suppose A is a local noetherian $\mathbb{F}_p$-algebra of dimension at least 2. Let $\mathfrak{m}$ be the maximal ideal of A. Suppose $M = A^r$ is equipped with a Φ_A^n-linear endomorphism $\phi_M : M \to M$ such that for each nonclosed point x of $S = \operatorname{Spec} A$, with the notation of Section 4.3 we have $\dim_{\mathbb{F}_{p^n}}(\vartheta_x) = m$. Then $S_m = U_1$ being pure in S, it is also weakly pure in S (see Proposition 13) and thus $S - S_m$ cannot be $\mathfrak{m}$ as $\operatorname{codim}_S(\mathfrak{m}) \geq 2$. Therefore we have $S_m = S$ and in this way we reobtain [Zink 2001, Proposition 5] . One can view Theorem 12 as a generalization and a refinement of [Zink 2001, Proposition 5].

Remark 15. For $q \in \mathbb{N}^*$ we define recursively an A-linear map

$$\phi_M^{(q)} : A \otimes_{F_A^{nq}, A} M \to M$$

as follows: let $\phi_M^{(1)} : A \otimes_{F_A^n, A} M \to M$ be the A-linear map defined by ϕ_M, and we have the recursive formula $\phi_M^{(q)} = \phi_M^{(1)} \circ (1_A \otimes_{F_A^n, A} \phi_M^{(q-1)})$. Deligne [2011] proved the case $n = 1$ of Theorem 12 using ranks of images of $\phi_M^{(q)}$ for $q \gg 0$ at points $x \in S = \operatorname{Spec} A$ and properties of Grassmannians.

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On the mod-2 cohomology of $\mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$

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Let $\Gamma = \mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$, let X be any mod-2 acyclic Γ-CW complex on which Γ acts with finite stabilizers and let X_s be the 2-singular locus of X. We calculate the mod-2 cohomology of the Borel construction of X_s with respect to the action of Γ. This cohomology coincides with the mod-2 cohomology of Γ in cohomological degrees bigger than 8 and the result is compatible with a conjecture of Quillen which predicts the structure of the cohomology ring $H^*(\Gamma; \mathbb{F}_2)$.

1. Introduction

The main motivation for this paper comes from a conjecture of Quillen [1971, Conjecture 14.7] which concerns the structure of the mod-p cohomology ring of the group $\mathrm{GL}_n(\Lambda)$ of invertible matrices of rank n with coefficients in a ring Λ of S-integers in a number field; the assumption on Λ is that p is invertible in Λ and Λ contains a primitive p-th root of unity. The conjecture stipulates that under these assumptions $H^*(\mathrm{GL}_n(\Lambda); \mathbb{Z}/p)$ is a free module over the polynomial algebra $\mathbb{Z}/p[c_1, \dots, c_n]$ where the c_i are the mod-p Chern classes associated to an embedding of Λ into the complex numbers. In the sequel we will denote this conjecture by $C(n, \Lambda, p)$.

For $p = 2$ the simplest ring for which the assumptions of Quillen's conjecture hold is the ring $\mathbb{Z}\big[\frac{1}{2}\big]$. Let $\mathbb{Z}\big[\frac{1}{2}, i\big]$ be the ring obtained from the Gaussian integers $\mathbb{Z}[i]$ by inverting 2.

Conjecture $C\big(n, \mathbb{Z}\big[\frac{1}{2}\big], 2\big)$ is trivially true for $n = 1$ and known to be true for $n = 2$ by [Mitchell 1992] and $n = 3$ by [Henn 1999]; it is known to be false for $n = 32$ by [Dwyer 1998] and even for $n \geq 14$ (Henn and Lannes, unpublished). The positive results have been established by direct calculation and while a direct calculation is perhaps still doable for $n = 4$, it would be extremely involved. For larger n a complete calculation does not look realistic. In fact, the negative results have been obtained by very indirect methods which depend on étale approximations for the homotopy type of the 2-completion of $\mathrm{BGL}_n\big(\mathbb{Z}\big[\frac{1}{2}\big]\big)$. These étale approximations can also be used to show that if $C\big(2n, \mathbb{Z}\big[\frac{1}{2}\big], 2\big)$ holds then $C\big(n, \mathbb{Z}\big[\frac{1}{2}, i\big], 2\big)$ holds

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as well (Henn and Lannes, unpublished). This gives particular motivation to study conjecture $C\big(n, \mathbb{Z}\big[\frac{1}{2}, i\big], 2\big)$.

We will show in Theorem 5.1 that $C\big(n, \mathbb{Z}\big[\frac{1}{2}, i\big], 2\big)$ holds if and only if there is an isomorphism

$$H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big) \cong \mathbb{F}_2[c_1, \dots, c_n] \otimes E(e_1, e_1', \dots, e_{2n-1}, e_{2n-1}')$$

where the classes c_i are the Chern classes of the tautological n-dimensional complex representation of $\mathrm{GL}_n\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$, E denotes an exterior algebra and the classes e_{2i-1}, e_{2i-1}' are of cohomological degree $2i-1$ for $i = 1, \dots, n$. These exterior classes are closely related to Quillen's exterior classes in the mod-2 cohomology of $\mathrm{GL}_n(\mathbb{F}_p)$ if p is a prime such that $p \equiv 1 \bmod 4$ (see (5-1) for more details).

Conjecture $C\big(n, \mathbb{Z}\big[\frac{1}{2}, i\big], 2\big)$ is again trivially true for $n = 1$ and has been verified by direct calculation for $n = 2$ in [Weiss 2006]. Dwyer's method [1998] using étale approximations X_n for the homotopy type of the 2-completion of $\mathrm{BGL}_n\big(\mathbb{Z}\big[\frac{1}{2}\big]\big)$ and comparing the set of homotopy classes of $[\mathrm{BP}, X_n]$ with that of $\big[\mathrm{BP}, \mathrm{BGL}_n\big(\mathbb{Z}\big[\frac{1}{2}\big]\big)\big]$ for suitable cyclic groups P of order 2^n can be adapted to disprove $C\big(16, \mathbb{Z}\big[\frac{1}{2}, i\big], 2\big)$. However, we will not dwell on this in this paper.

This paper embarks on a study of conjecture $C\big(3, \mathbb{Z}\big[\frac{1}{2}, i\big], 2\big)$ which is more accessible than conjecture $C\big(4, \mathbb{Z}[\frac{1}{2}], 2\big)$. In order to calculate $H^*\big(\mathrm{GL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ we first try to calculate $H^*\big(\mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big); \mathbb{F}_2\big)$. For this we propose the same strategy as the one which was used in the case of $\mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}\big]\big)$ and which finally led to a verification of conjecture $C\big(3, \mathbb{Z}\big[\frac{1}{2}\big], 2\big)$. In a first step one uses a centralizer spectral sequence introduced in [Henn 1997] in order to calculate the mod-2 Borel cohomology $H_G^*(X_s; \mathbb{F}_2)$ where X is any mod-2 acyclic G-CW complex on which a given discrete group G acts with finite stabilizers and X_s is the 2-singular locus of X, i.e., the subcomplex consisting of all points for which the isotropy group of the action of G is of even order. For $G = \mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}\big]\big)$ this step was carried out in [Henn 1997] and for $G = \mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$ it is carried out in this paper. The precise form of X does not really matter in this step.

The second step involves a very laborious analysis of the relative mod-2 Borel cohomology $H_G^*(X, X_s; \mathbb{F}_2)$ and of the connecting homomorphism for the Borel cohomology of the pair (X, X_s). In the case of $G = \mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}\big]\big)$ this was carried out by hand in [Henn 1999]. A by hand calculation looks forbidding in the case of $G = \mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$ and this paper makes no attempt on such a calculation. However, we do make some comments on what is likely to be involved in such an attempt.

Here are the main results of this paper. In these results the elements b_2 and b_3 are of degree 4 and 6, respectively. They are given as Chern classes of the tautological 3-dimensional complex representation of $\mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$. The indices of the other elements give their cohomological degrees. These elements come from Quillen's exterior cohomology classes in the cohomology of $\mathrm{GL}_3(\mathbb{F}_p)$ for suitable primes p, for example $p = 5$ (see Section 3.2 for more details). Furthermore Σ^n denotes n-fold suspension so that $\Sigma^4\mathbb{F}_2$ is a one dimensional $\mathbb{F}_2$-vector space concentrated in degree 4.

Theorem 1.1. *Let $\Gamma = \mathrm{SL}_3(\mathbb{Z}[\frac{1}{2}, i])$ and let X be any mod-2 acyclic Γ-CW complex such that the isotropy group of each cell is finite. Then the centralizer spectral sequence of [Henn 1997] collapses at E_2 and gives a short exact sequence*

$$0 \to \Sigma^4\mathbb{F}_2 \oplus \Sigma^4\mathbb{F}_2 \oplus \Sigma^7\mathbb{F}_2 \to H^*_\Gamma(X_s; \mathbb{F}_2) \to \mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5') \to 0$$

in which the epimorphism is a map of graded commutative algebras with unit.

Next let

$$\psi : H^*(\Gamma; \mathbb{F}_2) = H^*_\Gamma(X; \mathbb{F}_2) \to H^*_\Gamma(X_s; \mathbb{F}_2) \to \mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5')$$

be the composition of the map induced by the inclusion $X_s \subset X$ and the epimorphism of Theorem 1.1.

Theorem 1.2. *Let $\Gamma = \mathrm{SL}_3(\mathbb{Z}[\frac{1}{2}, i])$ and X be as in the previous theorem.*

(a) *If $\mathrm{SD}_3(\mathbb{Z}[\frac{1}{2}, i])$ denotes the subgroup of diagonal matrices of Γ then the target of ψ can be identified with a subalgebra of $H^*(\mathrm{SD}_3(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ in such a way that ψ is induced by the restriction homomorphism*

$$H^*(\Gamma; \mathbb{F}_2) \to H^*(\mathrm{SD}_3(\mathbb{Z}[\tfrac{1}{2}, i]); \mathbb{F}_2).$$

(b) *The homomorphism ψ admits a multiplicative section*

$$\varphi : \mathbb{F}_2[c_2, c_3] \otimes E(e_3, e_3', e_5, e_5') \to H^*(\Gamma; \mathbb{F}_2)$$

that sends c_i to b_i for $i = 2, 3$ and sends e_i and e_i' respectively to d_i and d_i' for $i = 3, 5$.

(c) *The homomorphism ψ is surjective in all degrees, an isomorphism in degrees $* > 8$ and its kernel is finite-dimensional in degrees $* \leq 8$.*

Remark 1.3. Conjecture $C(3, \mathbb{Z}[\frac{1}{2}, i], 2)$ would hold if the maps ψ and φ of part (b) of Theorem 1.2 turned out to be isomorphisms (see Proposition 5.5).

The following result is an immediate consequence of Theorem 1.2.

Corollary 1.4. *Let $\Gamma = \mathrm{SL}_3(\mathbb{Z}[\frac{1}{2}, i])$ and X be as in Theorem 1.1. Then the following conditions are equivalent*:

(a) *The restriction homomorphism $H^*(\Gamma; \mathbb{F}_2) \to H^*(\mathrm{SD}_3(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ is injective and $H^*(\Gamma; \mathbb{F}_2)$ is isomorphic as a graded $\mathbb{F}_2$-algebra to $\mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5')$.*

(b) *There is an isomorphism*

$$H^*_\Gamma(X, X_s; \mathbb{F}_2) \cong \Sigma^5\mathbb{F}_2 \oplus \Sigma^5\mathbb{F}_2 \oplus \Sigma^8\mathbb{F}_2$$

and the connecting homomorphism $H^*_\Gamma(X_s; \mathbb{F}_2) \to H^{*+1}_\Gamma(X, X_s; \mathbb{F}_2)$ *is surjective.* □

The paper is organized as follows. In Section 2 we recall the centralizer spectral sequence and in Section 3 we prove Theorems 1.1 and 1.2 In Section 4 we make some comments on step 2 of the program of a complete calculation of $H^*(\Gamma; \mathbb{F}_2)$. Finally in Section 5 we establish Theorem 5.1 and discuss the relation between Theorem 1.2 and conjecture $C\big(3, \mathbb{Z}\big[\frac{1}{2}, i\big], 2\big)$.

2. The centralizer spectral sequence

We recall the centralizer spectral sequence introduced in [Henn 1997].

Let G be a discrete group and let p be a fixed prime. Let $\mathcal{A}(G)$ be the category whose objects are the elementary abelian p-subgroups E of G, i.e., subgroups which are isomorphic to $(\mathbb{Z}/p)^k$ for some integer k; if E_1 and E_2 are elementary abelian p-subgroups of G, then the set of morphisms from E_1 to E_2 in $\mathcal{A}(G)$ consists precisely of those group homomorphisms $\alpha : E_1 \to E_2$ for which there exists an element $g \in G$ with $\alpha(e) = geg^{-1}$ for all $e \in E_1$. Let $\mathcal{A}_*(G)$ be the full subcategory of $\mathcal{A}(G)$ whose objects are the nontrivial elementary abelian p-subgroups.

For an elementary abelian p-subgroup E we denote its centralizer in G by $C_G(E)$. Then the assignment $E \mapsto H^*(C_G(E); \mathbb{F}_p)$ determines a functor from $\mathcal{A}_*(G)$ to the category $\mathcal{E}$ of graded $\mathbb{F}_p$-vector spaces. The inverse limit functor is a left exact functor from the functor category $\mathcal{E}^{\mathcal{A}_*(G)}$ to $\mathcal{E}$. Its right derived functors are denoted by $\lim^s$. The p-rank $r_p(G)$ of a group G is defined as the supremum of all k such that G contains a subgroup isomorphic to $(\mathbb{Z}/p)^k$.

For a G-space X and a fixed prime p we denote by X_s the p-singular locus, i.e., the subspace of X consisting of points whose isotropy group contains an element of order p. Let EG be the total space of the universal principal G-bundle. The mod-p cohomology of the Borel construction $EG \times_G X$ of a G space X will be denoted $H^*_G(X; \mathbb{F}_p)$. The following result is a special case of part (a) of Corollary 0.4 of [Henn 1997].

Theorem 2.1. *Let G be a discrete group and assume there exists a finite-dimensional mod-p acyclic G-CW complex X such that the isotropy group of each cell is finite. Then there exists a cohomological second quadrant spectral sequence*

$$E_2^{s,t} = \lim\nolimits^s_{\mathcal{A}_*(G)} H^t(C_G(E); \mathbb{F}_p) \Rightarrow H^{s+t}_G(X_s; \mathbb{F}_p)$$

with $E_2^{s,t} = 0$ *if* $s \geq r_p(G)$ *and* $t \geq 0$.

Remark 2.2. The edge homomorphism in this spectral sequence is a map of algebras

$$H_G^*(X_s; \mathbb{F}_p) \to \lim_{\mathcal{A}_*(G)} H^*(C_G(E); \mathbb{F}_p),$$

which is given as follows.

Let X^E be the fixed points for the action of E on X. The G-action on X restricts to an action of the centralizer $C_G(E)$ on X^E and the G-equivariant maps

$$G \times_{C_G(E)} X^E \to X_s, \quad (g, x) \mapsto gx.$$

for $E \in \mathcal{A}_*(G)$ induce compatible maps in Borel cohomology

$$H_G^*(X_s; \mathbb{F}_2) \to H_G^*(G \times_{C_G(E)} X^E; \mathbb{F}_2) \cong H_{C_G(E)}^*(X^E; \mathbb{F}_2) \cong H^*(C_G(E); \mathbb{F}_2)$$

which assemble to give the map to the inverse limit. Here we have used that by classical Smith theory X^E is mod p-acyclic if X is mod-p acyclic and hence we get canonical isomorphisms $H_{C_G(E)}^*(X^E; \mathbb{F}_2) \cong H_{C_G(E)}^*(*; \mathbb{F}_2) \cong H^*(C_G(E); \mathbb{F}_2)$.

Furthermore the composition

$$H^*(G; \mathbb{F}_p) = H_G^*(X; \mathbb{F}_p) \to H_G^*(X_s; \mathbb{F}_p) \to H^*(C_G(E); \mathbb{F}_p) \qquad (2\text{-}1)$$

is induced by the inclusions $C_G(E) \to G$ as E varies through $\mathcal{A}_*(G)$.

In [Henn 1997] we have used this spectral sequence in the case $p = 2$ and $G = \mathrm{SL}_3(\mathbb{Z})$. Here we will use it in the case $p = 2$ and $G = \mathrm{SL}\big(3, \mathbb{Z}\big[\frac{1}{2}, i\big]\big)$. In both cases we have $r_2(G) = 2$ and hence the spectral sequence collapses at E_2 and degenerates into a short exact sequence

$$0 \to \lim\nolimits^1_{\mathcal{A}_*(G)} H^*(C_G(E); \mathbb{F}_2) \to H_G^{*+1}(X_s; \mathbb{F}_2) \to \lim_{\mathcal{A}_*(G)} H^{*+1}(C_G(E); \mathbb{F}_2) \to 0. \qquad (2\text{-}2)$$

3. The centralizer spectral sequence for $\mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$

3.1. ***The Quillen category.*** Let K be any number field, let $\mathcal{O}_K$ be its ring of integers and consider the ring of S-integers $\mathcal{O}_K\big[\frac{1}{2}\big]$. Then, up to equivalence, the Quillen category of $G := \mathrm{SL}_3\big(\mathcal{O}_K\big[\frac{1}{2}\big]\big)$ for the prime 2 is independent of K. In fact, because 2 is invertible every elementary abelian 2-subgroup is conjugate to a diagonal subgroup, and hence $\mathcal{A}_*(G)$ has a skeleton, say $\mathcal{A}$, with exactly two objects, say E_1 and E_2 of rank 1 and 2, respectively. We take E_1 to be the subgroup generated by the diagonal matrix whose first two diagonal entries are -1 and whose third diagonal entry is 1, and E_2 to be the subgroup of all diagonal matrices with diagonal entries 1 or -1 and determinant 1.

The automorphism group of E_1 is trivial, of course, while $\mathrm{Aut}_{\mathcal{A}}(E_2)$ is isomorphic to the group of all abstract automorphisms of E_2 which we can identify with $\mathfrak{S}_3$, the symmetric group on three elements. There are three morphisms from E_1 to E_2 and $\mathrm{Aut}_{\mathcal{A}}(E_2)$ acts transitively on them.

3.2. ***The centralizers and their cohomology.*** For centralizers in $H:=\mathrm{GL}_3\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big)$ we find $C_H(E_1) = \mathrm{GL}_2\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big) \times \mathrm{GL}_1\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big)$ and $C_H(E_2) = D_3\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big)$ if $D_n\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big)$ denotes the subgroup of diagonal matrices in $\mathrm{GL}_n\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big)$. This implies

$$
\begin{aligned}
C_G(E_1) &\cong \mathrm{GL}_2\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big),\\
C_G(E_2) &= \mathrm{SD}_3\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big) \cong D_2\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big) \cong \mathcal{O}_K\big[\tfrac{1}{2}\big]^\times \times \mathcal{O}_K\big[\tfrac{1}{2}\big]^\times,
\end{aligned}
$$

where as before $\mathrm{SD}_3\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big)$ denotes special diagonal matrices with coefficients in $\mathcal{O}_K\big[\tfrac{1}{2}\big]$.

From now on we specialize to the case $K = \mathbb{Q}[i]$ where we have $\mathcal{O}_K\big[\tfrac{1}{2}\big] = \mathbb{Z}\big[\tfrac{1}{2}, i\big]$. In this case the cohomology of the centralizers is explicitly known. In the sequel we abbreviate $\mathrm{SL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$ by Γ.

3.2.1. *The cohomology of* $C_\Gamma(E_2)$. There is an isomorphism of groups

$$\mathbb{Z}/4 \times \mathbb{Z} \cong \mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times, \quad (n, m) \mapsto i^n(1+i)^m$$

and therefore we get an isomorphism

$$H^*(C_\Gamma(E_2); \mathbb{F}_2) \cong H^*\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times \times \mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times; \mathbb{F}_2\big) \cong \mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_1', x_2, x_2') \tag{3-1}$$

with y_1 and y_2 in degree 2 and the other generators in degree 1. We agree to choose the generators so that y_1, x_1 and x_1' come from the first factor with x_1 and x_1' being the dual basis to the basis of

$$H_1\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times; \mathbb{F}_2\big) \cong \mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times / \big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times\big)^2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

given by the image of i and $(1+i)$ in the mod-2 reduction of the abelian group $\mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times$ and y_1 coming from $H^2(\mathbb{Z}/4; \mathbb{F}_2)$; likewise with y_2, x_2 and x_2' coming from the second factor.

3.2.2. *The cohomology of* $C_\Gamma(E_1)$. This cohomology has been calculated in [Weiss 2006]. In fact, from Theorem 1 of [Weiss 2006] we know

$$H^*(C_\Gamma(E_1); \mathbb{F}_2) \cong H^*\big(\mathrm{GL}_2\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big) \cong \mathbb{F}_2[c_1, c_2] \otimes E(e_1, e_1', e_3, e_3'). \tag{3-2}$$

Here we give a short summary of this calculation. The classes e_1, e_1', e_3 and e_3' are pulled back from Quillen's exterior classes q_1 and q_3 [1972] in

$$H^*(\mathrm{GL}_2(\mathbb{F}_5); \mathbb{F}_2) \cong \mathbb{F}_2[c_1, c_2] \otimes E(q_1, q_3) \tag{3-3}$$

via two ring homomorphisms

$$\pi : \mathbb{Z}[\tfrac{1}{2}, i] \to \mathbb{F}_5, \quad \pi' : \mathbb{Z}[\tfrac{1}{2}, i] \to \mathbb{F}_5. \tag{3-4}$$

We choose π such that i is sent to 3 and π' such that i is sent to 2.

Now consider the two commutative diagrams (with horizontal arrows induced by inclusion and vertical arrows induced by π and, respectively, π')

$$\begin{array}{ccc} D_2(\mathbb{Z}[\frac{1}{2}]) & \longrightarrow & \mathrm{GL}_2(\mathbb{Z}[\frac{1}{2}]) \\ \downarrow & & \downarrow \\ D_2(\mathbb{F}_5) & \longrightarrow & \mathrm{GL}_2(\mathbb{F}_5). \end{array} \tag{3-5}$$

By abuse of notation we can write

$$H^*(D_2(\mathbb{F}_5); \mathbb{F}_2) \cong H^*(\mathbb{F}_5^\times \times \mathbb{F}_5^\times; \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_2) \tag{3-6}$$

with $y_1 \in H^2(\mathbb{F}_5^\times; \mathbb{F}_2)$ and $x_1 \in H^2(\mathbb{F}_5^\times; \mathbb{F}_2)$ coming from the first factor and likewise with y_2 and x_2 coming from the second factor. Then π and π' induce two homomorphisms

$$\pi^*, \pi'^* : H^*(D_2(\mathbb{F}_5); \mathbb{F}_2) \to H^*(D_2(\mathbb{Z}[\tfrac{1}{2}]); \mathbb{F}_2)$$

which in terms of the isomorphisms (3-6) and (3-1) are explicitly given by

$$\pi^*(y_i) = y_i = \pi'^*(y_i), \quad \pi^*(x_i) = x_i, \quad \pi'^*(x_i) = x_i + x_i' \quad \text{for } i = 1, 2. \tag{3-7}$$

The cohomology of $\mathrm{GL}_2(\mathbb{F}_5)$ is detected by restriction to the cohomology of diagonal matrices and restriction is given explicitly as follows:

$$c_1 \mapsto y_1 + y_2, \quad c_2 \mapsto y_1 y_2, \quad q_1 \mapsto x_1 + x_2, \quad q_3 \mapsto y_1 x_2 + y_2 x_1. \tag{3-8}$$

Then e_1, e_1', e_3, e_3' are defined via

$$e_1 = \pi^*(q_1), \quad e_3 = \pi^*(q_3), \quad e_1' = \pi'^*(q_1), \quad e_3' = \pi'^*(q_3). \tag{3-9}$$

If c_1 and c_2 are the Chern classes of the tautological 2-dimensional complex representation of $\mathrm{GL}_2(\mathbb{Z}[\frac{1}{2}], i)$, then the restriction homomorphism which sends $H^*(\mathrm{GL}_2(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ to the cohomology of the subgroup of diagonal matrices is injective and by using (3-5) and (3-8) we see that it is explicitly given by

$$\begin{aligned} c_1 &\mapsto y_1 + y_2, & c_2 &\mapsto y_1 y_2, \\ e_1 &\mapsto x_1 + x_2, & e_3 &\mapsto y_1 x_2 + y_2 x_1, \\ e_1' &\mapsto x_1 + x_1' + x_2 + x_2', & e_3' &\mapsto y_1(x_2 + x_2') + y_2(x_1 + x_1'). \end{aligned} \tag{3-10}$$

3.2.3. *Functoriality.* We note that together with the isomorphisms (3-1) and (3-2) the restriction (3-10) also describes the map

$$\alpha_* : H^*(C_\Gamma(E_1); \mathbb{F}_2) \to H^*(C_\Gamma(E_2); \mathbb{F}_2)$$

induced from the standard inclusion of E_1 into E_2.

To finish the description of $H^*(C_\Gamma(-);\mathbb{F}_2)$ as a functor on $\mathcal{A}$ it remains to describe the action of the symmetric group $\mathrm{Aut}_{\mathcal{A}}(E_2)\cong\mathfrak{S}_3$ of rank 3 on

$$H^*(C_\Gamma(E_2);\mathbb{F}_2)\cong\mathbb{F}_2[y_1,y_2]\otimes\Lambda(x_1,x_1',x_2,x_2').$$

Because of the multiplicative structure we need it only on the generators.

If $\tau\in\mathrm{Aut}_{\mathcal{A}}(E_2)$ corresponds to permuting the factors in $C_\Gamma(E_2)\cong\mathrm{GL}_1\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)\times\mathrm{GL}_1\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)$ then

$$\begin{aligned}\tau_*(y_1)&=y_2, &\quad \tau_*(x_1)&=x_2, &\quad \tau_*(x_1')&=x_2',\\ \tau_*(y_2)&=y_1, & \tau_*(x_2)&=x_1, & \tau_*(x_2')&=x_1',\end{aligned}\tag{3-11}$$

and if $\sigma\in\mathrm{Aut}_{\mathcal{A}}(E_2)$ corresponds to the cyclic permutation of the diagonal entries (in suitable order) then

$$\begin{aligned}\sigma_*(y_1)&=y_2, &\quad \sigma_*(x_1)&=x_2, &\quad \sigma_*(x_1')&=x_2',\\ \sigma_*(y_2)&=y_1+y_2, & \sigma_*(x_2)&=x_1+x_2, & \sigma_*(x_2')&=x_1'+x_2'.\end{aligned}\tag{3-12}$$

3.3. ***Calculating the limit and its derived functors.*** In Proposition 4.3 of [Henn 1997] we showed that for any functor F from $\mathcal{A}$ to $\mathbb{Z}_{(2)}$-modules there is an exact sequence

$$0\to\lim_{\mathcal{A}}F\to F(E_1)\xrightarrow{\varphi}\mathrm{Hom}_{\mathbb{Z}[\mathfrak{S}_3]}(\mathrm{St}_{\mathbb{Z}},F(E_2))\to\lim^1_{\mathcal{A}}F\to 0\tag{3-13}$$

where $\mathrm{St}_{\mathbb{Z}}$ is the $\mathbb{Z}[\mathfrak{S}_3]$-module given by the kernel of the augmentation map $\mathbb{Z}[\mathfrak{S}_3/\mathfrak{S}_2]\to\mathbb{Z}$, and if a and b are chosen to give an integral basis of $\mathrm{St}_{\mathbb{Z}}$ on which τ and σ act via

$$\begin{aligned}\tau_*(a)&=b, &\quad \tau_*(b)&=a,\\ \sigma_*(a)&=-b, & \sigma_*(b)&=a-b,\end{aligned}\tag{3-14}$$

then $\varphi(x)(a)=\alpha_*(x)-(\sigma_*)^2\alpha_*(x)$ and $\varphi(x)(b)=\alpha_*(x)-\sigma_*\alpha_*(x)$ if $x\in F(E_1)$.

Because in our case the functor takes values in $\mathbb{F}_2$-vector spaces we can replace $\mathrm{Hom}_{\mathbb{Z}[\mathfrak{S}_3]}$ by $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}$ and $\mathrm{St}_{\mathbb{Z}}$ by its mod-2 reduction. The following elementary lemma is needed in the analysis of the third term in the exact sequence (3-13).

Lemma 3.1. (a) *Let* St *be the* $\mathbb{F}_2[\mathfrak{S}_3]$*-module given as the kernel of the augmentation* $\mathbb{F}_2[\mathfrak{S}_3/\mathfrak{S}_2]\to\mathbb{F}_2$*. The tensor product* $\mathrm{St}\otimes\mathrm{St}$ *decomposes as* $\mathbb{F}_2[\mathfrak{S}_3]$*-module canonically as*

$$\mathrm{St}\otimes\mathrm{St}\cong\mathbb{F}_2[\mathfrak{S}_3/A_3]\oplus\mathrm{St}$$

where A_3 *denotes the alternating group on three letters. In fact, the decomposition is given by*

$$\mathrm{St}\otimes\mathrm{St}\cong\mathrm{Im}(\mathrm{id}+\sigma_*+\sigma_*^2)\oplus\mathrm{Ker}(\mathrm{id}+\sigma_*+\sigma_*^2)$$

and the first summand is isomorphic to $\mathbb{F}_2[\mathfrak{S}_3/A_3]$ *while the second summand is isomorphic to* St.

(b) *The tensor product* $\mathbb{F}_2[\mathfrak{S}_3/A_3] \otimes \mathrm{St}$ *is isomorphic to* $\mathrm{St} \oplus \mathrm{St}$.

Proof.

(a) It is well-known that St is a projective $\mathbb{F}_2[\mathfrak{S}_3]$-module, hence $\mathrm{St} \otimes \mathrm{St}$ is also projective. It is also well-known that every projective indecomposable $\mathbb{F}_2[\mathfrak{S}_3]$-module is isomorphic to either St or $\mathbb{F}_2[\mathfrak{S}_3/A_3]$. The two modules can be distinguished by the fact that $e := id + \sigma_* + \sigma_*$ acts trivially on St and as the identity on $\mathbb{F}_2[\mathfrak{S}_3/A_3]$.

Furthermore e is a central idempotent in $\mathbb{F}_2[\mathfrak{S}_3]$ and hence each $\mathbb{F}_2[\mathfrak{S}_3]$-module M decomposes as direct sum of $\mathbb{F}_2[\mathfrak{S}_3]$-modules

$$M \cong \mathrm{Im}(e : M \to M) \oplus \mathrm{Ker}(e : M \to M).$$

An easy calculation shows that in the case of $\mathrm{St} \otimes \mathrm{St}$ both submodules are nontrivial and this together with the fact these submodules must be projective proves the claim.

(b) Again each of the factors in the tensor product is a projective $\mathbb{F}_2[\mathfrak{S}_3]$-module, hence the tensor product is a projective $\mathbb{F}_2[\mathfrak{S}_3]$-module. Because σ acts as the identity on $\mathbb{F}_2[\mathfrak{S}_3/A_3]$ we see that the idempotent e acts trivially on the tensor product and this forces the tensor product to be isomorphic to $\mathrm{St} \oplus \mathrm{St}$. □

Lemma 3.2. *The Poincaré series* χ_2 *of* $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, \mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_1', x_2, x_2'))$ *is given by*

$$\chi_2 = \frac{2t^2(1+3t^2+3t^4+t^6)+2t(1+2t^2+2t^4+2t^6+t^8)}{(1-t^4)(1-t^6)}.$$

Proof. The isomorphism of (3-1) is an isomorphism of $\mathbb{F}_2[\mathfrak{S}_3]$-modules where the action of $\mathfrak{S}_3$ is given by equations (3-11) and (3-12). In particular we see that $H^1(\mathrm{GL}_1(\mathbb{Z}[\frac{1}{2}, i]) \times \mathrm{GL}_1(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ is isomorphic to $\mathrm{St} \oplus \mathrm{St}$ generated by x_1, x_1', x_2, x_2'. The exterior powers of H^1 are given as

$$E^k(x_1, x_2, x_1', x_2') \cong E^k(\mathrm{St} \oplus \mathrm{St}) \cong \bigoplus_{j=0}^{k} E^j\, \mathrm{St} \otimes E^{k-j}\, \mathrm{St}$$

and, because $E^k(\mathrm{St})$ is isomorphic to $\Sigma^k \mathbb{F}_2$ if $k = 0, 2$, isomorphic to $\Sigma\, \mathrm{St}$ if $k = 1$, and trivially otherwise, we obtain

$$E^k(x_1, x_2, x_1', x_2') \cong \begin{cases} \Sigma^k \mathbb{F}_2 & \text{if } k = 0, 4, \\ \Sigma^k(\mathrm{St} \oplus \mathrm{St}) & \text{if } k = 1, 3, \\ \Sigma^2 \mathbb{F}_2 \oplus \Sigma^2(\mathrm{St} \otimes \mathrm{St}) \oplus \Sigma^2 \mathbb{F}_2 & \text{if } k = 2, \\ 0 & \text{if } k \neq 0, 1, 2, 3, 4, \end{cases}$$

where $\mathbb{F}_2$ denotes the trivial $\mathbb{F}_2[\mathfrak{S}_3]$-module whose additive structure is that of $\mathbb{F}_2$.

Therefore the Poincaré series χ_2 of $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, H^*(C_\Gamma(E_2); \mathbb{F}_2))$ decomposes according to the decomposition of $\Lambda(x_1, x_2', x_1', x_2')$ as the sum

$$\chi_2 := (1+2t^2+t^4)\chi_{2,0}+t^2\chi_{2,1}+2(t+t^3)\chi_{2,2} \tag{3-15}$$

where here we denote by $\chi_{2,0}$ the Poincaré series of $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, \mathbb{F}_2[y_1, y_2])$, by $\chi_{2,1}$ the Poincaré series of $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, \mathrm{St}\otimes\mathrm{St}\otimes\mathbb{F}_2[y_1, y_2])$ and by $\chi_{2,2}$ that of $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, \mathrm{St}\otimes\mathbb{F}_2[y_1, y_2])$.

It is well-known (and elementary to verify) that there is an isomorphism of $\mathbb{F}_2[\mathfrak{S}_3]$-modules $\mathrm{St}\oplus\mathrm{St}\oplus\mathbb{F}_2[\mathfrak{S}_3/A_3]\cong\mathbb{F}_2[\mathfrak{S}_3]$ and therefore an isomorphism

$$\begin{aligned}\mathbb{F}_2[y_1, y_2] &\cong \mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}\oplus\mathrm{St}\oplus\mathbb{F}_2[\mathfrak{S}_3/A_3], \mathbb{F}_2[y_1, y_2])\\ &\cong \mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, \mathbb{F}_2[y_1, y_2])^{\oplus 2}\oplus\mathbb{F}_2[y_1, y_2]^{A_3}.\end{aligned}$$

Together with the elementary fact that the A_3-invariants $\mathbb{F}_2[y_1, y_2]^{A_3}$ form a free module over $\mathbb{F}_2[y_1, y_2]^{\mathfrak{S}_3}\cong\mathbb{F}_2[c_2, c_3]$ on the two generators 1 and $y_1^3+y_1y_2^2+y_2^3$ of degree 0 and 6, respectively, this implies

$$2\chi_{2,0}+\frac{1+t^6}{(1-t^4)(1-t^6)}=\frac{1}{(1-t^2)^2}$$

and hence

$$\chi_{2,0}=\frac{t^2}{(1-t^2)(1-t^6)}. \tag{3-16}$$

It is elementary to check that St and $\mathbb{F}_2[\mathfrak{S}_3/A_3]$ are both self-dual $\mathbb{F}_2[\mathfrak{S}_3]$-modules and hence Lemma 3.1 gives

$$\mathrm{St}\otimes\mathrm{St}^*\cong\mathbb{F}_2[\mathfrak{S}_3/A_3]\oplus\mathrm{St}$$

and

$$\begin{aligned}\mathrm{St}\otimes\mathrm{St}^*\otimes\mathrm{St}^* &\cong (\mathbb{F}_2[\mathfrak{S}_3/A_3]\oplus\mathrm{St})\otimes\mathrm{St}^*\\ &\cong (\mathbb{F}_2[\mathfrak{S}_3/A_3]\otimes\mathrm{St})\oplus(\mathrm{St}\otimes\mathrm{St})\\ &\cong \mathbb{F}_2[\mathfrak{S}_3/A_3]\oplus\mathrm{St}\oplus\mathrm{St}\oplus\mathrm{St}.\end{aligned}$$

Therefore, if $\chi_{\mathbb{F}_2[y_1,y_2]^{A_3}}$ denotes the Poincaré series of the A_3-invariants then

$$\begin{aligned}\chi_{2,1} &= \chi_{\mathbb{F}_2[y_1,y_2]^{A_3}}+3\chi_{2,0}\\ &= \frac{1+t^6}{(1-t^4)(1-t^6)}+\frac{3t^2}{(1-t^2)(1-t^6)}=\frac{1+3t^2+3t^4+t^6}{(1-t^4)(1-t^6)},\end{aligned} \tag{3-17}$$

$$\begin{aligned}\chi_{2,2} &= \chi_{2,0}+\chi_{\mathbb{F}_2[y_1,y_2]^{A_3}}\\ &= \frac{t^2}{(1-t^2)(1-t^6)}+\frac{1+t^6}{(1-t^4)(1-t^6)}=\frac{1+t^2+t^4+t^6}{(1-t^4)(1-t^6)}.\end{aligned} \tag{3-18}$$

Finally (3-15), (3-16), (3-17) and (3-18) give

$$\chi_2 = \frac{(1+2t^2+t^4)t^2(1+t^2)+t^2(1+3t^2+3t^4+t^6)+2(t+t^3)(1+t^2+t^4+t^6)}{(1-t^4)(1-t^6)}$$
$$= \frac{2t^2(1+3t^2+3t^4+t^6)+2t(1+2t^2+2t^4+2t^6+t^8)}{(1-t^4)(1-t^6)},$$

and this finishes the proof. □

Theorem 1.1 is now an immediate consequence of Theorem 2.1 and the following result.

Proposition 3.3. *Let* $p = 2$ *and* $\Gamma = SL_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$.

(a) *There is an isomorphism of graded* $\mathbb{F}_2$*-algebras*

$$\lim_{\mathcal{A}} H^*(C_\Gamma(E); \mathbb{F}_2) \cong \mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5').$$

Furthermore, if we identify this limit with a subalgebra of $H^*(C_\Gamma(E_1); \mathbb{F}_2) \cong \mathbb{F}_2[c_1, c_2] \otimes E(e_1, e_1', e_3, e_3')$ *then*

$$b_2 = c_1^2 + c_2, \qquad b_3 = c_1c_2,$$
$$d_3 = e_3, \qquad d_5 = c_1e_3 + c_2e_1,$$
$$d_3' = e_3', \qquad d_5' = c_1e_3' + c_2e_1'.$$

(b) *There is an isomorphism of graded* $\mathbb{F}_2$*-vector spaces*

$$\lim^1_{\mathcal{A}} H^*(C_\Gamma(E); \mathbb{F}_2) \cong \Sigma^3\mathbb{F}_2 \oplus \Sigma^3\mathbb{F}_2 \oplus \Sigma^6\mathbb{F}_2.$$

(c) *For any* $s > 1$

$$\lim^s_{\mathcal{A}} H^*(C_\Gamma(E); \mathbb{F}_2) = 0.$$

Proof. (a) It is easy to check that the subalgebra of $\mathbb{F}_2[c_1, c_2] \otimes E(e_1, e_1', e_3, e_3')$ generated by the elements $c_1^2 + c_2$, c_1c_2, e_3, e_3', $c_1e_3 + c_2e_1$, and $c_1e_3' + c_2e_1'$ is isomorphic to the tensor product of a polynomial algebra on two generators b_2 and b_3 of degrees 4 and 6 and an exterior algebra on 4 generators d_3, d_3', d_5 and d_5' of degrees 3, 3, 5 and 5. In fact, it is clear that $c_1^2 + c_2$ and c_1c_2 are algebraically independent and the elements e_3, e_3', $c_1e_3 + c_2e_1$, and $c_1e_3' + c_2e_1'$ are exterior classes; their product is given as $c_2^2e_3e_3'e_1e_1' \neq 0$, and this implies easily that the exterior monomials in these elements are linearly independent over the polynomial algebra generated by $c_1^2 + c_2$ and c_1c_2. From now on we identify b_2, b_3, d_3, d_3', d_5 and d_5' with $c_1^2 + c_2$, c_1c_2, e_3, e_3', $c_1e_3 + c_2e_1$ and $c_1e_3' + c_2e_1'$.

Now we use the exact sequence (3-13) and the description of φ to determine the inverse limit. Because α_* is injective, we see that if we identify $H^*(C_\Gamma(E_1); \mathbb{F}_2)$ with its image in $H^*(C_\Gamma(E_2); \mathbb{F}_2)$ then the inverse limit can be identified with the intersection of the image of α_* with the invariants in $\mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_1', x_2, x_2')$

with respect to the action of the cyclic group of order 3 of $\mathrm{Aut}_{\mathcal{A}}(E_2) \cong \mathfrak{S}_3$ generated by σ. This action has been described in (3-12) and with these formulas it is straightforward to check that the elements

$$
\begin{aligned}
b_2 &= y_1^2 + y_1 y_2 + y_2^2,\\
b_3 &= y_1 y_2 (y_1 + y_2),\\
d_3 &= y_1 x_2 + y_2 x_1,\\
d_5 &= (y_1 + y_2)(y_1 x_2 + y_2 x_1) + y_1 y_2 (x_1 + x_2) = y_1^2 x_2 + y_2^2 x_1, \qquad (3\text{-}19)\\
d_3' &= y_1 (x_2 + x_2') + y_2 (x_1 + x_1'),\\
d_5' &= (y_1 + y_2)(y_1 (x_2 + x_2') + y_2 (x_1 + x_1')) + y_1 y_2 (x_1 + x_1' + x_2 + x_2')\\
&= y_1^2 (x_2 + x_2') + y_2^2 (x_1 + x_1')
\end{aligned}
$$

all belong to the inverse limit.

Now consider the Poincaré series

$$
\begin{aligned}
\chi_0 &:= \sum_{n\geq 0} \dim_{\mathbb{F}_2}(\mathbb{F}_2[b_2, b_3] \otimes E(e_3, e_3', e_5, e_5')^n) t^n = \frac{(1+t^3)^2(1+t^5)^2}{(1-t^4)(1-t^6)},\\
\chi_1 &:= \sum_{n\geq 0} \dim_{\mathbb{F}_2} H^n(C_\Gamma(E_1); \mathbb{F}_2) t^n = \frac{(1+t)^2(1+t^3)^2}{(1-t^2)(1-t^4)},\\
\chi_2 &:= \frac{2t^2(1+3t^2+3t^4+t^6)+2t(1+2t^2+2t^4+2t^6+t^8)}{(1-t^4)(1-t^6)}.
\end{aligned}
$$

Then we have the identity

$$\chi_0 + \chi_2 - \chi_1 = \frac{p}{(1-t^4)(1-t^6)}$$

with

$$
\begin{aligned}
p &= (1+t^3)^2(1+t^5)^2 + 2t^2(1+3t^2+3t^4+t^6)\\
&\qquad + 2t(1+2t^2+2t^4+2t^6+t^8) - (1+t)^2(1+t^3)^2(1+t^2+t^4)\\
&= 2t^3 + t^6 - 2t^7 - 2t^9 - t^{10} - t^{12} + 2t^{13} + t^{16}\\
&= (2t^3+t^6)(1-t^4)(1-t^6)
\end{aligned}
$$

and therefore

$$\chi_0 + \chi_2 = \chi_1 + (2t^3 + t^6). \qquad (3\text{-}20)$$

This, together with the fact that $\lim_{\mathcal{A}} H^*(C_\Gamma(E); \mathbb{F}_2)$ contains a subalgebra isomorphic to $\mathbb{F}_2[b_2, b_3] \otimes \Lambda(d_3, d_3', d_5, d_5')$, already implies that the sequence

$$
\begin{aligned}
0 \to \mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5') &\to H^*(C_\Gamma(E_1); \mathbb{F}_2)\\
&\xrightarrow{\varphi} \mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, H^*(C_\Gamma(E_1); \mathbb{F}_2)) \to 0
\end{aligned}
$$

in which the left-hand arrow is given by inclusion is exact except possibly in dimensions 3 and 6.

In order to complete the proof of (a) it is now enough to verify that in degrees 3 and 6 the inverse limit is not bigger than $\mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5')$. We leave this straightforward verification to the reader.

Then (b) follows immediately from (a) together with (3-20) and the exact sequence (3-13), and (c) follows from Theorem 2.1 and the fact that $r_2(G) = 2$. □

We can now give the proof of Theorem 1.2.

Proof.

(a) The exact sequence of Theorem 1.1 is obtained from the exact sequence (2-2) via Proposition 3.3. Therefore the epimorphism of Theorem 1.1 is the edge homomorphism of the centralizer spectral sequence. The result then follows from (2-1) by observing that we have identified the target of the edge homomorphism with the subalgebra $\mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5')$ of $H^*(C_\Gamma(E_1); \mathbb{F}_2)$ and by recalling that $C_\Gamma(E_1)$ is equal to the subgroup of special diagonal matrices $\mathrm{SD}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$.

(b) The two ring homomorphisms $\pi, \pi' : \mathbb{Z}\big[\frac{1}{2}, i\big] \to \mathbb{F}_5$ of (3-4) determine homomorphisms $\mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big) \subset \mathrm{GL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big) \to \mathrm{GL}_3(\mathbb{F}_5)$. By [Quillen 1972] we have

$$H^* \mathrm{GL}_3(\mathbb{F}_5); \mathbb{F}_2) \cong \mathbb{F}_3[c_1, c_2, c_3] \otimes E(q_1, q_3, q_5).$$

We get a well-defined homomorphism of $\mathbb{F}_2$-graded algebras

$$\varphi : \mathbb{F}_2[c_2, c_3] \otimes E(e_3, e_3', e_5, e_5') \to H^*(\Gamma; \mathbb{F}_2)$$

by sending c_i to the i-th Chern class of the tautological 3-dimensional representation of Γ and by declaring $\varphi(e_i) = \pi^*(q_i)$ and $\varphi(e_i') = \pi'^*(q_i')$ for $i = 3, 5$. The classes q_1, q_3 and q_5 are the symmetrizations of x_1, y_1x_2 and $y_1y_2x_3$, respectively, with respect to the natural action of $\mathfrak{S}_3$ on

$$H^*(\mathrm{GL}_3(\mathbb{F}_5); \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2, y_3] \otimes E(x_1, x_2, x_3).$$

Compare (5-1) below.

Next we determine the composition $\psi\varphi$. The universal Chern classes c_i are the elementary symmetric polynomials in variables, say y_i, and the inclusion $\mathrm{GL}_2(\mathbb{C}) \subset \mathrm{SL}_3(\mathbb{C}) \subset \mathrm{GL}_3(\mathbb{C})$ imposes the relation $y_1 + y_2 + y_3 = 0$. This implies that the behavior of ψ on Chern classes is given by

$$c_1 \mapsto 0, \quad c_2 \mapsto c_1^2 + c_2 = y_1^2 + y_1y_2 + y_2^2 = b_2, \quad c_3 \mapsto c_1c_2 = y_1y_2(y_1 + y_2) = b_3.$$

In these equations we have identified $H^*\big(\mathrm{GL}_2\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big); \mathbb{F}_2\big)$, as in the proof of Proposition 3.3, via restriction with a subalgebra of $\mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_1', x_3, x_3')$.

In order to determine the composition $\psi\varphi$ on the classes e_3, e_3', e_5 and e_5' we calculate at the level of $\mathbb{F}_5$ and use naturality with respect to the homomorphisms induced by π and π', i.e., we consider the maps induced in cohomology by the following commutative diagram in which the horizontal maps are induced by inclusion and the vertical maps are induced by π and, respectively, π':

$$\begin{array}{ccccc} \mathrm{GL}_2\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big) & \longrightarrow & \mathrm{SL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big) & \longrightarrow & \mathrm{GL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{GL}_2(\mathbb{F}_5) & \longrightarrow & \mathrm{SL}_3(\mathbb{F}_5) & \longrightarrow & \mathrm{GL}_3(\mathbb{F}_5). \end{array}$$

On the level of $\mathbb{F}_5$ the composition induces in cohomology a map

$$\mathbb{F}_3[c_1, c_2, c_3] \otimes E(q_1, q_3, q_5) \to \mathbb{F}_2[c_1, c_2] \otimes E(e_1, e_3) \subset \mathbb{F}_2[y_1, y_2] \otimes E(q_1, q_3)$$

which is easily determined from (5-1) below by imposing the relations $y_1+y_2+y_3 = 0$ and $x_1 + x_2 + x_3 = 0$ on the symmetrization of the classes y_1x_2 and $y_1y_2x_3$ with respect to the natural action of $\mathfrak{S}_3$ on the cohomology of diagonal matrices $H^*(D_3(\mathbb{F}_5); \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2, y_3] \otimes E(x_1, x_2, x_3)$. Explicitly we get

$$\begin{aligned} &c_1 \mapsto 0, \quad c_2 \mapsto y_1^2 + y_1y_2 + y_2^2, \quad c_3 \mapsto y_1y_2(y_1 + y_2),\\ &q_1 \mapsto 0, \quad q_3 \mapsto y_1x_2 + y_2x_1, \qquad q_5 \mapsto y_1^2x_2 + y_2^2x_1 \end{aligned}$$

and by using (3-7) and (3-19) we see that the composition $\psi\phi$ maps the elements e_3, e_5, e_3', and e_5' as follows:

$$\begin{aligned} e_3 &\mapsto \pi^*(y_1x_2 + y_2x_1) = d_3,\\ e_5 &\mapsto \pi^*(y_1^2x_2 + y_2^2x_1) = d_5,\\ e_3' &\mapsto \pi'^*(y_1x_2 + y_2x_1) = d_3',\\ e_5' &\mapsto \pi'^*(y_1^2x_2 + y_2^2x_1) = d_5'. \end{aligned}$$

Here we have identified the target of ψ with a subalgebra of $H^*\big(\mathrm{GL}_2\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ and the latter via restriction with a subalgebra of $\mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_1', x_3, x_3')$.

(c) The space X can be taken to be the product of symmetric space

$$X_\infty := \mathrm{SL}_3(\mathbb{C})/\,\mathrm{SU}(3)$$

and the Bruhat–Tits building X_2 for $\mathrm{SL}_3(\mathbb{Q}_2[i])$. Now $\mathrm{SL}_3(\mathbb{Q}_2[i])\backslash X_2$ is a 2-simplex [Brown 1989] and the projection map $X \to X_2$ induces a map

$$\mathrm{SL}_3(\mathbb{Q}_2[i])\backslash X \to \mathrm{SL}_3(\mathbb{Q}_2[i])\backslash X_2$$

whose fibers have the homotopy type of a 6-dimensional $\mathrm{SL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$-invariant deformation retract (see Section 4). Therefore we get $H^n_G(X, X_s; \mathbb{F}_2) = 0$ if $n > 8$

and the inclusion $X_s \subset X$ induces an isomorphism $H^n_G(X; \mathbb{F}_2) \cong H^n_G(X_s; \mathbb{F}_2)$ if $n > 8$. Then part (c) simply follows from (a) except for the finiteness statement for the kernel for which we refer to (4-1) and (4-2) below. □

4. Comments on step 2

The situation for $p = 2$ and $G = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ is analogous to the situation for $p = 2$ and $G = SL_3(\mathbb{Z}[\frac{1}{2}])$ for which step 2 was carried out in [Henn 1999] via a detailed study of the relative cohomology $H^*_G(X, X_s; \mathbb{F}_2)$ for X equal to the product of the symmetric space $X_\infty := SL_3(\mathbb{R})/SO(3)$ with the Bruhat–Tits building X_2 for $SL_3(\mathbb{Q}_2)$; the spaces involved had a few hundred cells and the calculation was painful. In the case of $SL_3(\mathbb{Z}[\frac{1}{2}, i])$ with X the product of $SL_3(\mathbb{C})/SU(3)$ with the Bruhat–Tits building for $SL_3(\mathbb{Q}_2[i])$ the calculational complexity of the second step is much more involved and an explicit calculation by hand does not look feasible. However, in recent years there have been a lot of machine aided calculations of the cohomology of various arithmetic groups (for example [Dutour Sikirić et al. 2016; Bui et al. 2016]) and a machine aided calculation seems to be within reach.

The natural strategy for undertaking this second step is to follow the same path as in [Henn 1999]. The equivariant cohomology $H^*_\Gamma(X, X_s; \mathbb{F}_2)$ can be studied via the spectral sequence of the projection map

$$p : X = X_\infty \times X_2 \to X_2.$$

This gives a spectral sequence with

$$E_1^{p,q} \cong \bigoplus_{\sigma \in \Lambda_p} H^q_{\Gamma_\sigma}(X_\infty, X_{\infty,s}; \mathbb{F}_2) \Rightarrow H^{p+q}_\Gamma(X, X_s; \mathbb{F}_2). \tag{4-1}$$

Here Λ_p indexes the p-dimensional cells in the orbit space of X_2 with respect to the action of Γ. The orbit space is a 2-simplex, i.e., Λ_0 and Λ_1 contain 3 elements and Λ_2 is a singleton. Furthermore Γ_σ is the isotropy group of a chosen representative in X_2 of the cell σ in the quotient space. For fixed p all p-dimensional cells have isomorphic isotropy groups because the Γ-action on the Bruhat–Tits building is the restriction of a natural action of $GL_3(\mathbb{Z}[\frac{1}{2}, i])$ on X_2 and this action is transitive on the set of p-dimensional cells [Brown 1989].

Therefore all isotropy subgroups for the action on X_2 are, up to isomorphism, subgroups of $SL_3(\mathbb{Z}[i])$ which itself appears as isotropy group of a 0-dimensional cell in X_2. The isotropy groups of 1-dimensional and 2-dimensional cells are isomorphic to well-known congruence subgroups of $SL_3(\mathbb{Z}[i])$. By the Soulé–Lannes method the fiber X_∞ of the projection map p admits a 6-dimensional $SL_3(\mathbb{Z}[i])$-equivariant deformation retract (the space of "well-rounded hermitian forms" modulo arithmetic equivalence) with compact quotient [Ash 1984] and therefore we have

$$E_1^{s,t}=0 \text{ unless } s=0,1,2,\ 0\le t\le 6, \quad \text{and} \quad \dim_{\mathbb{F}_2} E_1^{s,t}<\infty \text{ for all } (s,t). \tag{4-2}$$

The E_1-term of this spectral sequence should be accessible to machine calculation. The spectral sequence will necessarily degenerate at E_3 and the calculation of the differentials is likely to need human intervention, as in the case of $\mathrm{SL}\big(3,\mathbb{Z}\big[\frac{1}{2}\big]\big)$ (compare Section 3.4 of [Henn 1999]). Likewise the calculation of the connecting homomorphism for the mod-2 Borel cohomology of the pair (X, X_s) is likely to require human intervention.

5. On Quillen's conjecture for $\mathrm{GL}_n\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$

The next result gives two reformulations of the conjecture of Quillen briefly discussed in the introduction. The classes e_{2k-1} and e'_{2k-1} in part (c) will be introduced in (5-1) below.

Theorem 5.1. *Suppose $n\ge 2$. The following statements are equivalent*:

(a) *Conjecture $C\big(n,\mathbb{Z}\big[\frac{1}{2},i\big],2\big)$ holds, i.e., $H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big);\mathbb{F}_2\big)$ is a free module over $\mathbb{Z}/2[c_1,\dots,c_n]$ where the c_i are the mod-2 Chern classes of the tautological n-dimensional complex representation of $\mathrm{GL}_n\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)$.*

(b) *The restriction homomorphism*

$$H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2},i\big]\big);\mathbb{F}_2\big)\to H^*\big(D_n\big(\mathbb{Z}\big[\tfrac{1}{2},i\big]\big);\mathbb{F}_2\big)$$

is injective, where $D_n\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)$ is the subgroup of diagonal matrices in $\mathrm{GL}_n\big(\mathbb{Z}\big[\frac{1}{2}\big]\big)$.

(c) *There are isomorphisms*

$$H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2},i\big]\big);\mathbb{F}_2\big)\cong\mathbb{F}_2[c_1,\dots,c_n]\otimes E(e_1,e'_1,\dots,e_{2n-1},e'_{2n-1})$$

where the classes c_k are the Chern classes of the tautological n-dimensional complex representation of $\mathrm{GL}_n\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)$ and the classes e_{2k-1}, e'_{2k-1} are of cohomological degree $2k-1$ for $k=1,\dots,n$.

Proof. It is trivial that (c) implies (a).

In order to show that (a) implies (b) we observe that $D_n\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)$ is the centralizer of the unique, up to conjugacy, maximal elementary abelian 2-subgroup E_n of $\mathrm{GL}_n\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)$ given by the subgroup of diagonal matrices of order 2. Now consider the top Dickson invariant ω in $H^*(B\mathrm{GL}_n(\mathbb{C});\mathbb{F}_2)$, i.e., the class whose restriction to $H^*B\big(\prod_{i=1}^n \mathrm{GL}_1(\mathbb{C})\big);\mathbb{F}_2)$ is the product of all nontrivial classes of degree 2. The image of ω in $H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big);\mathbb{F}_2\big)$ restricts trivially to the cohomology of all elementary abelian 2-subgroups E of $\mathrm{GL}_n\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)$ of rank less than n. If (a) holds then the image of ω is not a zero divisor in $H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big);\mathbb{F}_2\big)$ and hence

Corollary I.5.8 of [Henn et al. 1995] implies that the restriction to the centralizer of E_n is injective.

The implication (b) $\Rightarrow$ (c) follows from Proposition 5.3 below. □

Before we go on we introduce the classes e_{2k-1} and e'_{2k-1}. As in the case of GL_2 they are obtained from Quillen's classes [1972] $q_{2k-1} \in H^{2k-1}(GL_n(\mathbb{F}_5); \mathbb{F}_2)$ which restrict in the cohomology of diagonal matrices in $\mathbb{F}_5$ to the symmetrization of the classes $y_1 \cdots y_{k-1} x_k$ where y_k is of cohomological degree 2 corresponding to the k-th factor in the product $\prod_{k=1}^n \mathbb{F}_5^\times$ and x_k is of cohomological degree 1 of the same factor. We define

$$e_{2k-1} := \pi^*(q_{2k-1}), \quad e'_{2k-1} := \pi'^*(q_{2k-1}) \tag{5-1}$$

where π and π' are the two ring homomorphisms $\mathbb{Z}\left[\frac{1}{2}, i\right] \to \mathbb{F}_5$ with π sending i to 3 and π' sending i to 2 which we considered earlier in Section 3. We identify the mod-2 cohomology $H^*\big(D_n\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ with $\mathbb{F}_2[y_1, \dots y_n] \otimes E(x_1, x'_1 \dots, x_n, x'_n)$ with y_k, $k = 1, \dots, n$ of degree 2 and x_k, x'_k, $k = 1, \dots, n$ of degree 1 where as before we choose x_k and x'_k to be the basis which is dual to the basis of the k-th factor in

$$D_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)/D_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)^2 \cong \big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times/\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times\big)^2\big)^n$$

given by the classes of i and $1 + i$. Then we get the following lemma which generalizes (3-10) and whose straightforward proof we leave to the reader.

Lemma 5.2. *The class e_{2k-1} restricts in the cohomology of the subgroup of diagonal matrices $H^*\big(D_n\big(\mathbb{Z}\big[\frac{1}{2}, i\big]; \mathbb{F}_2\big)\big)$ to the symmetrization of $y_1 \cdots y_{k-1} x_k$ and the class e'_{2k-1} restricts to the symmetrization of $y_1 \cdots y_{k-1}(x_k + x'_k)$.* □

The following result determines the image of the restriction homomorphism and shows that (b) implies (c) in Theorem 5.1. It resembles results of Mitchell [1992] for $GL_n\big(\mathbb{Z}\big[\frac{1}{2}\big]\big)$ for $p = 2$ and of Anton [1999] for $GL_n\big(\mathbb{Z}\big[\frac{1}{3}, \zeta_3\big]\big)$ for $p = 3$. Its proof uses crucially condition (5-3) below, which also plays a central role in [Anton 2003].

Proposition 5.3. *Let $n \geq 1$ be an integer. The image of the restriction map*

$$i^* : H^*\big(GL_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big) \to H^*\big(D_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big) \cong \mathbb{F}_2[y_1, \dots y_n] \otimes E(x_1, x'_1 \dots, x_n, x'_n)$$

is isomorphic to

$$\mathbb{F}_2[c_1, \dots c_n] \otimes E(e_1, e'_1, \dots, e_{2n-1}, e'_{2n-1}).$$

Here we have identified the Chern classes c_i and the classes e_{2i-1} and e'_{2i-1} with their image via i^*. The images of the elements c_i are, of course, the elementary symmetric polynomials in the y_i and the images of the classes e_{2i-1} and e'_{2i-1} have been determined in Lemma 5.2. We remark that even though i^* need not be injective, it is injective on the subalgebra of $H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ generated by the classes c_i, e_{2i-1} and e'_{2i-1}, $1 \le i \le n$.

Proof. In this proof we denote the subalgebra

$$\mathbb{F}_2[c_1, \dots c_n] \otimes E(e_1, e'_1, \dots, e_{2n-1}, e'_{2n-1}).$$

of $H^*\big(D_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ by C_n and the image of the restriction map by B_n. We need to show that $B_n = C_n$. This is trivial if $n = 1$ and for $n = 2$ this follows from Theorem 1 of [Weiss 2006] (compare (3-2), (3-10) and Lemma 5.2).

The classes $c_1, \dots, c_n$ are in B_n as images of the Chern classes with the same name and the classes $e_1, \dots e_{2n-1}, e'_1, \dots e'_{2n-1}$ are in B_n by Lemma 5.2. Therefore we have $C_n \subset B_n$. We will show $B_n \subset C_n$ for $n \ge 2$ by induction on n. This will be done in three steps:

1. From the inclusions

$$\mathrm{GL}_{n-2}\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big) \times \mathrm{GL}_2\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big) \subset \mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$$
$$\mathrm{GL}_{n-1}\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big) \times \mathrm{GL}_1\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big) \subset \mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$$

given by matrix block sum and the identifications of $D_{n-2}\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big) \times D_2\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$ and of $D_{n-1}\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big) \times D_1\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$ with $D_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$ we see that

$$B_n \subset B_{n-1} \otimes B_1 \cap B_{n-2} \otimes B_2$$

and by the induction hypothesis the latter subalgebra is equal to

$$C_{n-1} \otimes C_1 \cap C_{n-2} \otimes C_2,$$

in particular we have

$$B_n \subset C_{n-1} \otimes C_1 \cap C_{n-2} \otimes C_2. \tag{5-2}$$

2. The monomial basis in

$$H^*\big(D_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big) \cong \mathbb{F}_2[y_1, \dots, y_n] \otimes E(x_1, \dots, x_n, x'_1, \dots, x'_n)$$

is in bijection with the set $S(n)$ of sequences

$$I = (a_1, \varepsilon_{1,1}, \varepsilon_{2,1}, \dots, a_n, \varepsilon_{1,n}, \varepsilon_{2,n})$$

where the a_i are integers ≥ 0 and $\varepsilon_{i,j} \in \{0, 1\}$ for $i = 1, 2$ and $1 \leq j \leq n$. More precisely to I we associate the monomial

$$y^I := y_1^{a_1} \cdots y_n^{a_n} x_1^{\varepsilon_{1,1}} \cdots x_n^{\varepsilon_{1,n}} x_1'^{\,\varepsilon_{2,1}} \cdots x_n'^{\,\varepsilon_{2,n}}.$$

We equip $S(n)$ with the lexicographical order and denote it by $<_n$. This order has the property that for each $1 \leq k < n$ it agrees with the lexicographical order on $S(k) \times S(n-k)$ if $S(k)$ and $S(n-k)$ are equipped with the orders $<_k$ and $<_{n-k}$ and $S(n)$ is identified with $S(k) \times S(n-k)$ via concatenation of sequences.

In what follows we replace the symmetrizations of the elements $y_1 \cdots y_{i-1}(x_i + x_i')$, $i = 1, \ldots, n$, by the symmetrization of $y_1 \cdots y_{i-1} x_i'$ and by abuse of notation we continue to denote them by e'_{2i-1}. This does not change the subalgebra C_n. This subalgebra

$$\mathbb{F}_2[c_1, \ldots c_n] \otimes E(e_1, e'_1, \ldots, e_{2n-1}, e'_{2n-1}) \subset \mathbb{F}_2[y_1, \ldots, y_n] \otimes E(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$$

has a monomial basis which is in bijection with the set $T(n)$ of sequences

$$K = (k_1, \ldots, k_n; \phi_{1,1}, \ldots, \phi_{1,n}; \phi_{2,1} \ldots, \phi_{2,n})$$

where the k_i are integers ≥ 0 and $\phi_{i,j} \in \{0, 1\}$ for $i = 1, 2$ and $1 \leq j \leq n$. More precisely to K we associate the monomial

$$c^K := c_1^{k_1} \cdots c_n^{k_n} e_1^{\phi_{1,1}} \cdots e_n^{\phi_{1,n}} e_1'^{\,\phi_{2,1}} \cdots e_n'^{\,\phi_{2,n}}.$$

We define a map

$$\alpha : T(n) \to S(n)$$

by associating to $K \in T(n)$ the largest monomial in $S(n)$ which occurs in the decomposition of c^K as linear combination of elements x^I with $I \in S(n)$. The proof of the following result is elementary and is left to the reader.

Lemma 5.4. *The map α is explicitly given by*

$$\alpha((k_1, \ldots, k_n; \phi_{1,1}, \ldots, \phi_{1,n}; \phi_{2,1} \ldots, \phi_{2,n})) = (a_1, \varepsilon_{1,1}, \varepsilon_{2,1}, \ldots, a_n, \varepsilon_{1,n}, \varepsilon_{2,n})$$

with

$$\begin{aligned} a_j &= k_j + \sum_{i=j+1}^{n} (k_i + \phi_{1,i} + \phi_{2,i}), \quad 1 \leq j < n, \\ a_n &= k_n, \\ \varepsilon_{i,j} &= \phi_{i,j}, \quad 1 \leq j \leq n, \ i = 1, 2. \end{aligned}$$

□

From this lemma it is obvious that α is injective and a sequence

$$I = (a_1, \varepsilon_{1,1}, \varepsilon_{2,1}, \ldots, a_n, \varepsilon_{1,n}, \varepsilon_{2,n}) \in S(n)$$

is in the image of α if and only if we have

$$a_j - a_{j+1} \geq \varepsilon_{1,j+1} + \varepsilon_{2,j+1} \quad \text{for all } 1 \leq j < n. \tag{5-3}$$

In particular, if an element x is in C_n then the maximal sequence which appears in the decomposition of x as a linear combination of the monomials x^I with $I \in S(n)$ satisfies (5-3) for all $1 \leq j < n$. Likewise, if x is in $C_i \otimes C_{n-i}$ then this maximal sequence is equal to the maximal sequence which appears in the decomposition of x as a linear combination of the monomials x^I with $I \in S(k) \times S(n-k)$ and hence it satisfies (5-3) for all $1 \leq j < i$ and $i+1 \leq j < n$.

3. Now let x be a homogeneous element of B_n and let I_0 be the maximal sequence in $S(n)$ appearing in the decomposition of x as a linear combination of the monomials x^I with $I \in S(n)$. By (5-2) we have $x \in C_{n-1} \otimes C_1$ and $x \in C_{n-2} \otimes C_2$, and I_0 remains the maximal sequence in $S(n-1) \times S(1)$ and $S(n-2) \times S(2)$, respectively, appearing in the decomposition of x as a linear combination of the monomials x^I with, respectively, $I \in S(n-1) \times S(1)$ and $I \in S(n-2) \times S(2)$. Hence I_0 satisfies conditions (5-3) for $1 \leq j < n-1$ and, respectively, $1 \leq j < n-2$ and $j = n-1$. In particular condition (5-3) holds for all $1 \leq j < n$ and therefore there exists $K_0 \in T(n)$ such that $\alpha(K_0) = I_0$. Then $x - c^{K_0}$ is still in B_n and the maximal sequence appearing in the decomposition of $x - c^{K_0}$ is smaller than that of x. By iterating this procedure we see that x belongs to C_n. □

Finally we relate $C\big(3, \mathbb{Z}[\tfrac{1}{2}, i], 2\big)$ to the behavior of the restriction homomorphism

$$H^*(\Gamma; \mathbb{F}_2) \to H^*(C_\Gamma(E_2); \mathbb{F}_2).$$

For this we observe that the subgroups $\Gamma = \mathrm{SL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big)$ and the center $Z \cong \mathbb{Z}[\tfrac{1}{2}, i]^\times$ of $\mathrm{GL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big)$ have trivial intersection and their product is the kernel of the homomorphism

$$\mathrm{GL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big) \to \mathbb{Z}[\tfrac{1}{2}, i]^\times \to \mathbb{Z}[\tfrac{1}{2}, i]^\times / \big(\mathbb{Z}[\tfrac{1}{2}, i]^\times\big)^3 \cong \mathbb{Z}/3$$

given as the composition of the determinant with the natural quotient map. Therefore the spectral sequence of the extension

$$1 \to \mathrm{SL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big) \times Z \to \mathrm{GL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big) \to \mathbb{Z}/3 \to 1$$

gives an isomorphism

$$H^*\big(\mathrm{GL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big); \mathbb{F}_2\big) \cong \big(H^*\big(\mathrm{SL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big); \mathbb{F}_2\big) \otimes H^*(Z; \mathbb{F}_2)\big)^{\mathbb{Z}/3}. \tag{5-4}$$

Proposition 5.5. *Conjecture* $C\big(3, \mathbb{Z}[\tfrac{1}{2}, i], 2\big)$ *holds if and only if either*

(a) $H^*\big(\mathrm{SL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big); \mathbb{F}_2\big) \cong \mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5')$ *or*

(b) *the kernel of the map ψ of Theorem 1.2 is a finite-dimensional vector space for which the action of $\mathbb{Z}/3 \cong \mathbb{Z}\left[\frac{1}{2}, i\right]^{\times}/\left(\mathbb{Z}\left[\frac{1}{2}, i\right]^{\times}\right)^3$ has trivial invariants.*

Proof. Clearly $\mathbb{Z}/3 \cong \mathbb{Z}\left[\frac{1}{2}, i\right]^{\times}/\left(\mathbb{Z}\left[\frac{1}{2}, i\right]^{\times}\right)^3$ acts trivially on $H^*(Z; \mathbb{F}_2)$ and on the image of the homomorphism φ of Theorem 1.2. Hence, the corollary follows immediately from (5-4) and Theorem 1.2. □

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Geometric origin and some properties of the arctangential heat equation

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We establish the geometric origin of the nonlinear heat equation with arctangential nonlinearity: $\partial_t D = \Delta(\arctan D)$ by deriving it, together and in duality with the mean curvature flow equation, from the minimal surface equation in Minkowski space-time, through a suitable quadratic change of time. After examining various properties of the arctangential heat equation (in particular through its optimal transport interpretation à la Otto and its relationship with the Born–Infeld theory of electromagnetism), we briefly discuss its possible use for image processing, once written in nonconservative form and properly discretized.

Introduction. The arctangential heat equation

$$\partial_t D = \Delta(\arctan D) \tag{1}$$

belongs to the class of degenerate nonlinear heat equations

$$\partial_t D = \Delta(\phi(D)),$$

(where ϕ is monotonic with derivative valued in $[0, +\infty]$), usually called "porous medium" (as $\phi(D) = D^m,\ m > 1$) or "fast diffusion" ($m < 1$) and sometimes related to geometry (such as $\phi(D) = \log D$, which corresponds to the Ricci flow in two space dimensions) [Brézis and Crandall 1979; Daskalopoulos and Kenig 2007; Topping and Yin 2017; Vázquez 2007]. The analysis of the arctangential heat equation from the usual PDE viewpoint (existence, uniqueness, regularity theory, ...) is not the point of the present paper. We rather show that the arctangential heat equation has a geometric origin and can be formally derived, together (and in duality) with the well known mean curvature flow for graphs, from the minimal surface equation in the Minkowski space of all $(t, x) \in \mathbb{R} \times \mathbb{R}^d$, with metric $-dt^2 + \delta_{ij} dx^i dx^j$. The minimal surface equation reads (see [Lindblad 2004], for example)

$$\partial_t\Big(\frac{\partial_t \phi}{R}\Big) = \partial_k\Big(\frac{\partial^k \phi}{R}\Big), \qquad R = \sqrt{1 - \partial_t \phi^2 + \partial^k \phi\, \partial_k \phi}, \tag{2}$$

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where

$$\phi = \phi(t,x) \in \mathbb{R}, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}, \quad \partial_k = \frac{\partial}{\partial x^k}, \quad \partial^k = \delta^{kj}\partial_j.$$

From this equation, properly expressed, in Section 1, as a "system of conservation laws with convex entropy" (in the sense of [Dafermos 2016]), we generate in Section 2, using the quadratic change of time method recently discussed in [Brenier and Duan 2018], two "dual" nonlinear parabolic equations: one is the well-known mean curvature flow (for graphs)

$$\partial_t \phi = \sqrt{1 + \partial_k \phi \partial^k \phi}\, \partial_i \left(\frac{\partial^i \phi}{\sqrt{1 + \partial_k \phi \partial^k \phi}} \right), \tag{3}$$

while the second one is precisely the arctangential heat equation (1). The arctangential heat equation seems widely ignored in the literature, but has, in our opinion, many interesting properties, discussed in Sections 3 and 4, on top of being "dual" to the mean curvature flow. First of all, we will compare, in Section 3, the arctangential heat equation, properly rescaled as

$$\partial_t D = \lambda \Delta(\arctan(D\lambda^{-1})$$

with a large parameter $\lambda > 0$, to its formal limit as $\lambda \to +\infty$, namely the linear heat equation

$$\partial_t D = \Delta D,$$

both written à la [Otto 2001; Otto and Westdickenberg 2005], in optimal transport style (for which we refer to [Ambrosio et al. 2008; Otto and Westdickenberg 2005; Santambrogio 2015; Villani 2003]):

$$\partial_t D = \partial_i \big(D\, \partial^i (\mathcal{F}'(D)) \big).$$

In the linear case, $\mathcal{F}(D)$ is just the Boltzmann entropy function $D \log D - D$ (so that $\mathcal{F}'(D) = \log D$), in other words, the Legendre–Fenchel transform of the exponential function

$$\mathcal{F}(D) = D \log D - D = \sup_{u \in \mathbb{R}} uD - \exp u,$$

while, in the arctangential case, as will be seen in Section 3,

$$\mathcal{F}(D) = \sup_{u \le \log \lambda} uD - \lambda \arcsin(\lambda^{-1} \exp u),$$

is the Legendre–Fenchel transform of $u \to \lambda \arcsin(\lambda^{-1} \exp(u))$ (extended by $+\infty$ for $u > \log \lambda$), which can be seen as a "catastrophic" correction to the usual exponential function. (By catastrophic, we mean that this monotonic convex function reaches the value $\lambda\pi/2$ with infinite slope at $u = \log\lambda$ and, then, suddenly jumps to $+\infty$.)

Still in Section 3, we will briefly mention the "Chaplygin heat equation"

$$\partial_t D + \Delta(D^{-1}) = 0,$$

formally obtained in the opposite regime $\lambda \downarrow 0$. (This equation has been named "Chaplygin heat equation" because it can been interpreted as a "friction-dominated" version of the "Chaplygin gas", for which we refer to [Serre 2009].) We will also establish a connection between the arctangential heat equation and the nonlinear theory of Electromagnetism proposed by Max Born and Leopold Infeld [1934]. More precisely, in two space dimensions, both the arctangential heat equation and the mean curvature flow just describe *special* solutions, depending only on two space variables, of the *same* vector-valued diffusion equation in three space dimensions,

$$\partial_t D = \nabla \times \left(B\sqrt{1+D^2} - \frac{(D\cdot B)D}{\sqrt{1+D^2}} \right), \quad B = -\nabla \times \left(\frac{D}{\sqrt{1+D^2}} \right)$$

(written in traditional "nabla" notations, B and D being three-dimensional vectors, $\times$ denoting the vector product, $D \cdot B = D_k B^k$, $D^2 = D_k D^k$), which, itself, can be formally derived, again by quadratic change of time, from the Born and Infeld [1934] equations (for which we also refer to [Brenier 2004; Serre 2004]).

Finally, in Section 4, we discuss the nonconservative form of the arctangential heat equation:

$$\partial_t \psi = \cos(\pi\psi)^2 \Delta\psi \tag{4}$$

(where D is written as $\tan(\pi\psi)$). Properly discretized, this equation might be a valuable tool to treat black and white images (ψ denoting the level of gray), by sharply enhancing the level sets $\{\psi = k + \frac{1}{2}\}$ for $k \in \mathbb{Z}$ as t grows, as shown by several numerical computations in Section 4.

1. Reformulation of the minimal surface equation

It is crucial for our analysis to get a formulation of the minimal surface equation (2) in the framework of "systems of conservation laws with convex entropy", for which we refer to Dafermos' book [2016]. More precisely:

Theorem 1. *Let ϕ be a smooth solution of the minimal surface equation* (2) *and define*

$$D = \frac{\partial_t \phi}{\sqrt{1 - \partial_t\phi^2 + \partial^k\phi\,\partial_k\phi}}, \quad B_i = \partial_i\phi, \quad P_i = \frac{-\partial_t\phi\,\partial_i\phi}{\sqrt{1 - \partial_t\phi^2 + \partial^k\phi\,\partial_k\phi}}. \tag{5}$$

Then (D, B, P) is a solution to the system of conservation laws

$$\partial_t B_i + \partial_i\left(\frac{B_j P^j - D}{h}\right) = 0, \quad \partial_t D + \partial_j\left(\frac{D P^j - B^j}{h}\right) = 0, \tag{6}$$

$$\partial_t P^i + \partial_j\left(\frac{P^i P^j + B^i B^j}{h}\right) = \partial^i\left(\frac{1 + B_j B^j}{h}\right), \tag{7}$$

$$h(D, B, P) = \sqrt{1 + D^2 + B_j B^j + P_j P^j}, \tag{8}$$

which admits, for this strictly convex function h, the extra conservation law

$$\partial_t h + \partial_j\left(P^j - \frac{(D B^j + P^j) + B_k(B^k P^j - P^k B^j)}{h^2}\right) = 0. \tag{9}$$

Notice that the local in time existence of smooth solutions to the minimal surface equation (2) is a well known fact, while the global existence of smooth solutions for "small" (in a suitable sense) initial conditions for $d \geq 2$ is a much more refined result, obtained by Lindblad [2004].

Proof of Theorem 1. The proof follows a strategy similar to the one used for nonlinear Maxwell's equations, in particular for the Born–Infeld equations, in [Brenier 2004; Serre 2004]. In the first step, we get the Hamiltonian form of the minimal surface equations (2), which reads as a system of conservation laws for (D, B) (as defined in (5)), with an extra conservation law for $H(D, B) = \sqrt{(1 + B_k B^k)(1 + D^2)}$, which is a locally (but not globally) convex function of (D, B) about $(0, 0)$. The second step is a suitable augmentation of the Hamiltonian system in order to get a larger system of conservation laws, namely (6)–(7) for (D, B) *and* $P = -DB$. This new system enjoys an extra conservation law for the strictly convex function $h(D, B, P) = \sqrt{1 + D^2 + B_k B^k + P_k P^k}$ which is nothing but $H(D, B)$, written as a function of (D, B, P). The comprehensive proof of Theorem 1 can be found in the Appendix.

2. Recovery of the mean curvature flow and the arctangential heat equation from the minimal surface equation by quadratic change of time

Inspired by our recent work with X. Duan [Brenier and Duan 2018], we investigate the augmented system (6)–(8) under the quadratic change of time: $t \to \theta = t^2/2$. We consider two "dual" regimes of initial conditions at $t = 0$, well suited to this quadratic change of time, respectively $D(0, x) = 0$ and $B(0, x) = 0$ and, in both cases, $P(0, x) = 0$.

In the first regime, we assume $D(0, x) = 0$, $P(0, x) = 0$, i.e., in terms of the original field ϕ, the solution to (2), $\partial_t \phi(0, x) = 0$, by definition (5). We make the consistent ansatz

$$B(t,x)=\mathcal{B}(\theta,x),\quad D(t,x)=t\mathcal{D}(\theta,x),\quad P(t,x)=t\mathcal{P}(\theta,x),\quad \theta=t^2/2, \tag{10}$$

for some smooth fields $(\mathcal{B},\mathcal{D},\mathcal{P})$. In other words, we introduce

$$\mathcal{B}(\theta,x)=B(\sqrt{2\theta},x),\quad \mathcal{D}(\theta,x)=\frac{D(\sqrt{2\theta},x)}{\sqrt{2\theta}},\quad \mathcal{P}(\theta,x)=\frac{P(\sqrt{2\theta},x)}{\sqrt{2\theta}}.$$

In the second, "dual" regime, we assume $B(0,x)=0$, $P(0,x)=0$, which means $\partial_i\phi(0,x)=0$, in terms of ϕ, and, accordingly, we introduce the second ansatz

$$D(t,x)=\mathcal{D}(\theta,x),\quad B(t,x)=t\mathcal{B}(\theta,x),\quad P(t,x)=t\mathcal{P}(\theta,x),\quad \theta=t^2/2, \tag{11}$$

or, equivalently,

$$\mathcal{D}(\theta,x)=D(\sqrt{2\theta},x),\quad \mathcal{B}(\theta,x)=\frac{B(\sqrt{2\theta},x)}{\sqrt{2\theta}},\quad \mathcal{P}(\theta,x)=\frac{P(\sqrt{2\theta},x)}{\sqrt{2\theta}}.$$

Let us now state our main result.

Theorem 2. *After the two "dual" quadratic changes of time* (10) *and* (11), *the minimal surface equations*, *written in augmented form* (6)–(8), *respectively lead on one hand to the mean-curvature flow* (3) *and on the other hand to the arctangential heat equation* (1).

Let us emphasize the formal character of this result, where we are just dealing with the equations. We will not discuss in the present paper the analysis of how close the solutions of the minimal surface equations (2), written in augmented form (6)–(8), after quadratic change of time, are from the solutions of (1) and (3), respectively. Let us just indicate our belief that the "relative entropy" (or "modulated energy") method (as in Dafermos' book [2016], see also [Brenier and Duan 2018; Giesselmann et al. 2017; Serfaty 2017] for very recent occurrences), based on the strict convexity of (8) and the dissipation law (21) established below, is the most appropriate tool to treat this question.

Proof of Theorem 2. Let us transform the augmented system (6)–(8) in both regimes (10)–(11). In the first case, we get the nonautonomous system, where θ features explicitly,

$$\partial_\theta\mathcal{B}_i+\partial_i\left(\frac{\mathcal{B}_j\mathcal{P}^j-\mathcal{D}}{\mathcal{H}}\right)=0,\quad \mathcal{D}+2\theta\left(\partial_\theta\mathcal{D}+\partial_j\left(\frac{\mathcal{D}\mathcal{P}^j}{\mathcal{H}}\right)\right)=\partial_j\left(\frac{\mathcal{B}^j}{\mathcal{H}}\right),$$
$$\mathcal{P}^i+2\theta\left(\partial_\theta\mathcal{P}^i+\partial_j\left(\frac{\mathcal{P}^i\mathcal{P}^j}{\mathcal{H}}\right)\right)+\partial_j\left(\frac{\mathcal{B}^i\mathcal{B}^j}{\mathcal{H}}\right)=\partial^i\left(\frac{1+\mathcal{B}_j\mathcal{B}^j}{\mathcal{H}}\right),$$

where

$$\mathcal{H}=\sqrt{1+\mathcal{B}_j\mathcal{B}^j+2\theta(\mathcal{D}^2+\mathcal{P}_j\mathcal{P}^j)}.$$

Formally, this system admits, as $\theta \downarrow 0$, the following asymptotic system

$$\partial_\theta \mathcal{B}_i + \partial_i \left(\frac{\mathcal{B}_j \mathcal{P}^j - \mathcal{D}}{\sqrt{1 + \mathcal{B}_k \mathcal{B}^k}} \right) = 0, \quad \mathcal{D} = \partial_j \left(\frac{\mathcal{B}^j}{\sqrt{1 + \mathcal{B}_k \mathcal{B}^k}} \right), \tag{12}$$

$$\mathcal{P}^i = -\partial_j \left(\frac{\mathcal{B}^i \mathcal{B}^j}{\sqrt{1 + \mathcal{B}_k \mathcal{B}^k}} \right) + \partial^i \left(\frac{1 + \mathcal{B}_j \mathcal{B}^j}{\sqrt{1 + \mathcal{B}_k \mathcal{B}^k}} \right). \tag{13}$$

In the second regime, when we rather assume $B(0, x) = P(0, x) = 0$ and use ansatz (11) instead of (10), we get from (6)–(8) again a nonautonomous system where θ features explicitly:

$$\partial_\theta \mathcal{D} + \partial_j \left(\frac{\mathcal{D} \mathcal{P}^j - \mathcal{B}^j}{\mathcal{H}} \right) = 0, \quad \mathcal{B}_i + 2\theta \left(\partial_\theta \mathcal{B}_i + \partial_i \left(\frac{\mathcal{B}_j \mathcal{P}^j}{\mathcal{H}} \right) \right) = \partial_i \left(\frac{\mathcal{D}}{\mathcal{H}} \right),$$

$$\mathcal{P}^i + 2\theta \left(\partial_\theta \mathcal{P}^i + \partial_j \left(\frac{\mathcal{P}^i \mathcal{P}^j + \mathcal{B}^i \mathcal{B}^j}{\mathcal{H}} \right) \right) = \partial^i \left(\frac{1}{\mathcal{H}} \right),$$

where

$$\mathcal{H} = \sqrt{1 + \mathcal{D}^2 + 2\theta (\mathcal{B}_j \mathcal{B}^j + \mathcal{P}_j \mathcal{P}^j)},$$

which admits as asymptotic system, as $\theta \downarrow 0$,

$$\partial_\theta \mathcal{D} + \partial_j \left(\frac{\mathcal{D} \mathcal{P}^j - \mathcal{B}^j}{\sqrt{1 + \mathcal{D}^2}} \right) = 0, \quad \mathcal{B}_i = \partial_i \left(\frac{\mathcal{D}}{\sqrt{1 + \mathcal{D}^2}} \right), \tag{14}$$

$$\mathcal{P}_i = \partial_i \left(\frac{1}{\sqrt{1 + \mathcal{D}^2}} \right). \tag{15}$$

Restoring notations (t, D, B, P), instead of $(\theta, \mathcal{D}, \mathcal{B}, \mathcal{P})$, we may write both asymptotic systems (12)–(13) and (14)–(15) respectively as

$$\partial_t B_i + \partial_i \left(\frac{B_j P^j - D}{\eta} \right) = 0, \quad \eta = \sqrt{1 + B_k B^k} \tag{16}$$

$$D = \partial_j \left(\frac{B^j}{\eta} \right), \quad P^i = -\partial_j \left(\frac{B^i B^j}{\eta} \right) + \partial^i \left(\frac{1 + B_j B^j}{\eta} \right) \tag{17}$$

and

$$\partial_t D + \partial_j \left(\frac{D P^j - B^j}{\eta} \right) = 0, \quad \eta = \sqrt{1 + D^2}, \tag{18}$$

$$B_i = \partial_i \left(\frac{D}{\eta} \right), \quad P_i = \partial_i \left(\frac{1}{\eta} \right). \tag{19}$$

Let us first derive the arctangential heat equation (1) from (18)–(19). We get

$$\begin{aligned}\partial_t D &= \partial_j\left(-\frac{DP^j}{\eta}+\frac{B^j}{\eta}\right)=\partial_j\left(-\frac{D}{\eta}\partial^j\left(\frac{1}{\eta}\right)+\frac{1}{\eta}\partial^j\left(\frac{D}{\eta}\right)\right)\\ &=\partial_j\left(\frac{\partial_j D}{\eta^2}\right)=\partial_j\left(\frac{\partial_j D}{1+D^2}\right)=\Delta(\arctan D).\end{aligned}$$

Let us now derive the mean curvature flow (3) from (16)–(17). Writing B as a gradient, i.e., $B_i=\partial_i\phi$, we may integrate (16) just as

$$\partial_t\phi=\frac{D-\partial_i\phi P^i}{\eta}. \tag{20}$$

We have

$$\begin{aligned}P^i &= -\partial_j\left(\frac{\partial^i\phi\,\partial^j\phi}{\eta}\right)+\partial^i\left(\frac{1+\partial_j\phi\,\partial^j\phi}{\eta}\right)\\ &=-\partial_j\left(\frac{\partial^j\phi}{\eta}\right)\partial^i\phi-\left(\frac{\partial^j\phi}{\eta}\right)\partial_j\partial^i\phi+\partial^i\left(\frac{1}{\eta}\right)+\partial^i\left(\frac{\partial_j\phi\,\partial^j\phi}{\eta}\right)\\ &=-\partial_j\left(\frac{\partial^j\phi}{\eta}\right)\partial^i\phi+\partial^i\left(\frac{1}{\eta}\right)+\partial_j\phi\,\partial^i\left(\frac{\partial^j\phi}{\eta}\right)\\ &=-\partial_j\left(\frac{\partial^j\phi}{\eta}\right)\partial^i\phi+\eta\,\partial^i\left(\frac{1}{2\eta^2}\right)+\eta\,\partial^i\left(\frac{\partial_j\partial^j\phi}{2\eta^2}\right)\\ &=-\partial_j\left(\frac{\partial^j\phi}{\eta}\right)\partial^i\phi+\eta\,\partial^i\left(\frac{1+\partial_j\partial^j\phi}{2\eta^2}\right)\\ &=-\partial_j\left(\frac{\partial^j\phi}{\eta}\right)\partial^i\phi\end{aligned}$$

(by definition of η). Thus, by (20)

$$\partial_t\phi=\frac{D}{\eta}+\frac{\partial_i\phi\,\partial^i\phi}{\eta}\,\partial_j\left(\frac{\partial^j\phi}{\eta}\right).$$

Since

$$D=\partial_j\left(\frac{\partial^j\phi}{\eta}\right),$$

we get

$$\partial_t\phi=\frac{1+\partial_i\phi\,\partial^i\phi}{\eta}\,\partial_j\left(\frac{\partial^j\phi}{\eta}\right)=\sqrt{1+\partial_i\phi\,\partial^i\phi}\,\partial_j\left(\frac{\partial^j\phi}{\sqrt{1+\partial_k\phi\,\partial^k\phi}}\right)$$

(by definition of η), which exactly is the mean curvature flow (3). This concludes the proof of Theorem 2.

Remark. It is worth noticing that, in the case of the second ansatz (11), leading to the arctangential equation (1) after quadratic change of time, the extra *conservation* law (9) leads to the *dissipation* law

$$\partial_t \eta + \Delta\left(\frac{1}{\eta}\right) = -\frac{B^k B_k + P^k P_k}{\eta}, \quad \eta = \sqrt{1+D^2}, \quad B_i = \partial_i\left(\frac{D}{\eta}\right), \quad P_i = \partial_i\left(\frac{1}{\eta}\right) \tag{21}$$

on top of conservation laws (18)–(19), which, consistently, can be as well obtained directly from (1).

3. A few properties of the arctangential heat equation

3.1. ***Interpretation à la Otto.*** Assuming that D is nonnegative (which is a consistent assumption due to the maximum principle for (1)), the arctangential heat equation also reads

$$\partial_t D = \partial_i\left(D\,\partial^i\left(\log\left(\frac{D}{\sqrt{1+D^2}}\right)\right)\right), \tag{22}$$

which can be written, in the framework of optimal transport theory [Ambrosio et al. 2008; Santambrogio 2015; Villani 2003],

$$\partial_t D = \partial_i\big(D\,\partial^i(\mathcal{F}'(D))\big), \tag{23}$$

à la Otto, as the gradient flow, with respect to the (so-called) "Wasserstein" or "MK2" metric [Otto 2001; Otto and Westdickenberg 2005], of the functional

$$\mathcal{D} \to \int \mathcal{F}(\mathcal{D}(x))\,dx$$

for a suitable function $\mathcal{F}$. Here $\mathcal{F}$ is a "renormalized" version of the classical Boltzmann entropy, namely

$$\mathcal{F}(D) = D\log\left(\frac{D}{\sqrt{1+D^2}}\right) - \arctan D \tag{24}$$

and should be extended by 0 for $D = 0$ and by $+\infty$ for $D < 0$ to define a globally convex function from $\mathbb{R}$ to $]-\infty, +\infty]$. Its Legendre–Fenchel transform can be explicitly (and easily) computed:

$$u \to \sup_D(uD - \mathcal{F}(D)) = (\mathbb{G}\exp)(u) = \arcsin(\exp(u)), \tag{25}$$

which should be extended by $+\infty$ for $u > 0$ and can be seen as a "generalized" exponential function. (Here the symbol $\mathbb{G}$ is used to note a generalization of a classical special function).

As a matter of fact, if we consistently parametrize the arctangential heat equation, in its formulation à la Otto (22), as

$$\partial_t D = \partial_i \left(D\, \partial^i \left(\log \left(\frac{D}{\sqrt{1 + D^2 \lambda^{-2}}} \right) \right) \right), \tag{26}$$

where $\lambda > 0$ should be understood as a large "cutoff" parameter, the corresponding Boltzmann entropy becomes

$$D \log \left(\frac{D}{\sqrt{1 + D^2 \lambda^{-2}}} \right) - \lambda \arctan(D\lambda^{-1}), \tag{27}$$

whose Legendre–Fenchel transform reads

$$(\mathbb{G}_\lambda \exp)(u) = \lambda \arcsin(\lambda^{-1} \exp(u)), \tag{28}$$

(extended by $+\infty$ for $u > \log \lambda$); see Figure 1. The later function is clearly an approximation of the regular exponential function as λ goes to infinity, with the interesting feature that, at $u = \log \lambda$, it reaches a finite value, namely $\lambda\pi/2$, and suddenly jumps to $+\infty$, while its u-derivative blows up. In some sense, $y = (\mathbb{G}_\lambda \exp)(u)$, which solves the super-nonlinear ODE

$$\frac{dy}{du} = \lambda \tan \left(\frac{y}{\lambda} \right),$$

while its derivative $z = \frac{dy}{du}$, which solves

$$\frac{dz}{du} = \left(1 + \frac{z^2}{\lambda^2} \right) z,$$

is a "catastrophic" version of the exponential function, probably suitable for some applications in geophysics, biology, social sciences and many other fields. Also notice that the inverse of this generalized exponential function provides a generalization of the logarithm, namely,

$$(\mathbb{G}_\lambda \log)(v) = \log(\lambda \sin(v\lambda^{-1})).$$

This function monotonically covers $]-\infty, \log \lambda]$ as $v \in]0, \lambda\pi/2[$ and can be symmetrically and periodically extended to $v \in \mathbb{R}$ as

$$(\mathbb{G}_\lambda \log)(v) = \tfrac{1}{2} \log(\lambda^2 \sin^2(v\lambda^{-1}));$$

see Figure 2. This features in several fields of Mathematics, including the recent theory of "unbalanced optimal transportation" [Chizat et al. 2018; Liero et al. 2018].

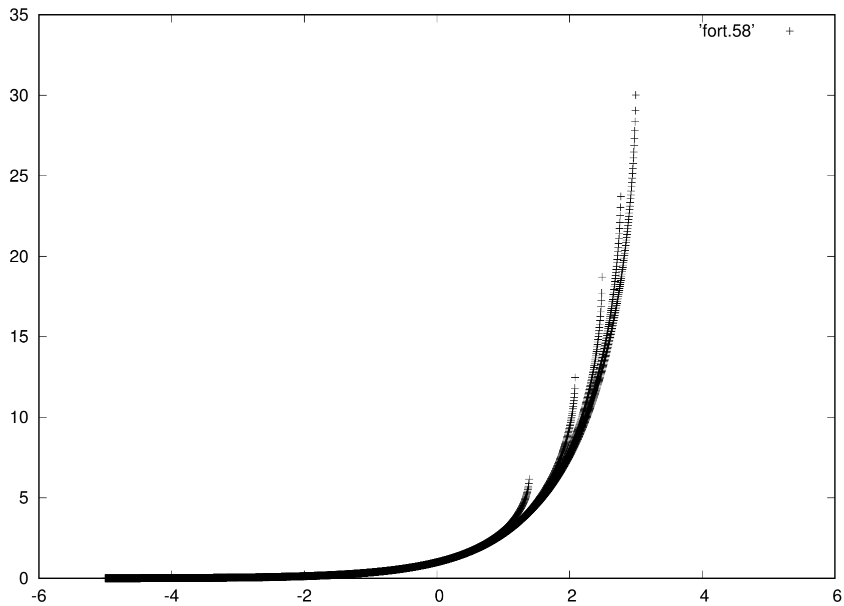

Figure 1. The "catastrophic" exponential $u \to \lambda \arcsin(\lambda^{-1} \exp u)$ (extended by $+\infty$ for $u > \log \lambda$), for different values of λ.

3.2. ***A limit case: the Chaplygin heat equation.*** The arctangential heat equation, in its parametrized form (26), namely

$$\partial_t D = \partial_i \left(D\, \partial^i \left(\log\left(\frac{D}{\sqrt{1 + D^2 \lambda^{-2}}} \right) \right) \right),$$

admits an interesting formal limit as $\lambda \downarrow 0$, Indeed, we have

$$\partial^i \left(\log\left(\frac{D}{\sqrt{1 + D^2 \lambda^{-2}}} \right) \right) = \partial^i \left(\log\left(\frac{\lambda}{\sqrt{1 + \lambda^2 D^{-2}}} \right) \right) = \partial^i \left(\log\left(\frac{1}{\sqrt{1 + \lambda^2 D^{-2}}} \right) \right)$$
$$\sim -\partial^i \left(\frac{\lambda^2 D^{-2}}{2} \right) = -\lambda^2 D^{-1} \partial^i (D^{-1}), \quad \lambda \downarrow 0,$$

so that, as $\lambda \downarrow 0$, after rescaling $t \to \lambda^2 t$, we get from (26) the "Chaplygin heat equation",

$$\partial_t D = -\Delta(D^{-1}). \tag{29}$$

(Notice that this equation can also be directly obtained from the Euler equations of "isentropic fluids",

$$\partial_t D + \partial_k Q^k = 0, \quad \partial_t Q^i + \partial_k \left(\frac{Q^k Q^i}{D} \right) + \partial^i (p(D)) = 0,$$

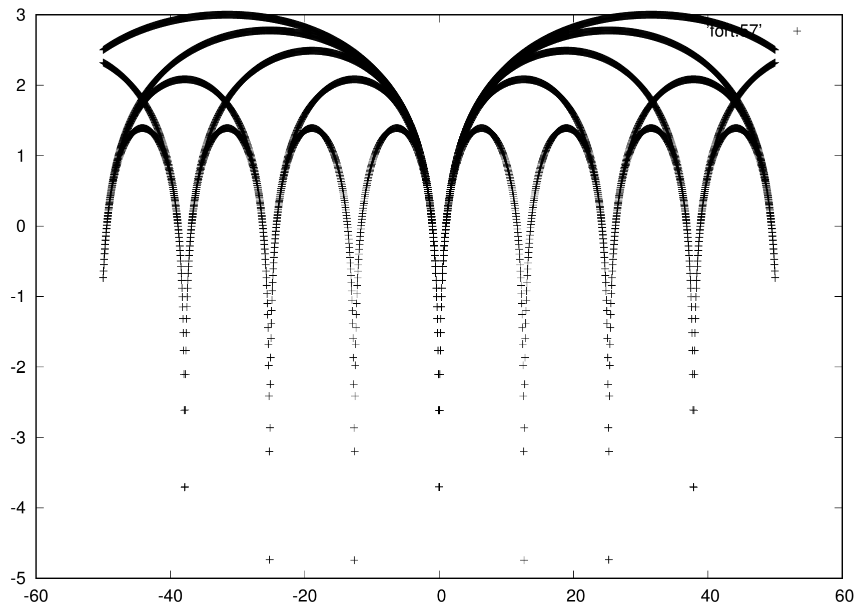

Figure 2. The inverse of the "catastrophic" exponential $v \to \frac{1}{2}\log(\lambda^2 \sin^2(v\lambda^{-1}))$, (after symmetrization and periodization) for different values of λ.

with "Chaplygin pressure" $p(D) = -D^{-1}$ (as in [Serre 2009]), after the same quadratic change of time

$$D(t,x) = \mathcal{D}(\theta,x), \quad Q(t,x) = t\,\mathcal{Q}(\theta,x), \quad \theta = t^2/2,$$

we used above.)

3.3. ***Relationship with nonlinear electromagnetism in two space dimensions.*** Let us now show that, in the case of two space dimensions, $d = 2$, both equations (3) and (1) can be (formally) derived from one of the most famous (and very geometric) models of nonlinear electromagnetism, namely the Born–Infeld equations, again through a suitable quadratic change of time. We first notice that these equations just describe particular solutions, depending only on two space variables, of the *same* three-dimensional vectorial diffusion equation:

Proposition 3. *In two space dimensions, both the mean curvature flow* (3) *and the arctangential heat equation* (1) *correspond to special solutions of the three-dimensional diffusion equation*

$$\partial_t D = \nabla \times \left(B\sqrt{1+D^2} - \frac{(D\cdot B)D}{\sqrt{1+D^2}} \right), \quad B = -\nabla \times \left(\frac{D}{\sqrt{1+D^2}} \right). \tag{30}$$

Proof. Straightforward calculations show that (3) just corresponds to particular solutions D of form

$$D(t,x) = (-\partial_2\phi(t,x^1,x^2), \partial_1\phi(t,x^1,x^2), 0),$$

while (1) rather corresponds to solutions of "dual" form

$$D(t,x) = (0, 0, D(t,x^1,x^2)),$$

which, in both cases, implies $B \cdot D = 0$ and leads to, respectively, (3) and (1). □

The augmented Born–Infeld system. Next, we derive (30) from a suitable quadratic change of time, as a formal asymptotic equation for the nonlinear Maxwell equations

$$\partial_t B + \nabla \times \left(\frac{\partial H}{\partial D}(D,B)\right) = 0, \quad \partial_t D - \nabla \times \left(\frac{\partial H}{\partial B}(D,B)\right) = 0, \tag{31}$$

where the Hamiltonian function H is

$$H(D,B) = \sqrt{(1+B^2)(1+D^2) - (D\cdot B)^2}$$

(while $H(D,B) = (B^2 + D^2)/2$ would correspond to the usual, linear, Maxwell equations). This nonlinear correction to the Maxwell equations was suggested by Born and Infeld [1934]. Let us write (31) more explicitly. Introducing

$$P = D \times B, \quad h = \sqrt{(1+D^2)(1+B^2) - (D\cdot B)^2} = \sqrt{1 + D^2 + B^2 + P^2},$$

we get

$$\partial_t B + \nabla \cdot \left(\frac{B\otimes P - P\otimes B}{h}\right) = -\nabla \times \left(\frac{D}{h}\right), \tag{32}$$

$$\partial_t D + \nabla \cdot \left(\frac{D\otimes P - P\otimes D}{h}\right) = \nabla \times \left(\frac{B}{h}\right). \tag{33}$$

As in Section 1, this system of conservation laws admits an extra conservation law for h. However h, as a function of (D, B), is convex only about $(0, 0)$ and not globally. Following [Brenier 2004; Serre 2004], again as in Section 1, we get a new extra conservation law by considering P as an independent variable and write h as a function of (D, B, P):

$$h = h(D,B,P) = \sqrt{1 + D^2 + B^2 + P^2}. \tag{34}$$

We obtain

$$\partial_t P + \nabla \cdot \left(\frac{P\otimes P - B\otimes B - D\otimes D}{h}\right) = \nabla\left(\frac{1}{h}\right). \tag{35}$$

In [Brenier 2004; Serre 2004], it is proven that the augmented system (32)–(35) enjoys an extra conservation law for h written as a function of (D, B, P).

The diffusive limit of the augmented Born–Infeld system.

Proposition 4. *The diffusion equation* (30) *can be obtained from the augmented Born–Infeld system* (32)–(35) *after the quadratic change of time*

$$D(t,x)=\mathcal{D}(\theta,x),\quad B(t,x)=t\mathcal{B}(\theta,x),\quad P(t,x)=t\mathcal{P}(\theta,x),\quad \theta=t^2/2, \tag{36}$$

Proof. Let us apply ansatz (36) to system (32)–(35). We get the nonautonomous system, where θ features explicitly,

$$\mathcal{B}+2\theta\left(\partial_\theta\mathcal{B}+\nabla\cdot\left(\frac{\mathcal{B}\otimes\mathcal{P}-\mathcal{P}\otimes\mathcal{B}}{\mathcal{H}}\right)\right)=-\nabla\times\left(\frac{\mathcal{D}}{\mathcal{H}}\right),$$
$$\partial_\theta\mathcal{D}+\nabla\cdot\left(\frac{\mathcal{D}\otimes\mathcal{P}-\mathcal{P}\otimes\mathcal{D}}{\mathcal{H}}\right)=\nabla\times\left(\frac{\mathcal{B}}{\mathcal{H}}\right),$$
$$\mathcal{P}+2\theta\left(\partial_\theta\mathcal{P}+\nabla\cdot\left(\frac{\mathcal{P}\otimes\mathcal{P}-\mathcal{B}\otimes\mathcal{B}}{\mathcal{H}}\right)\right)=\nabla\cdot\left(\frac{\mathcal{D}\otimes\mathcal{D}}{\mathcal{H}}\right)+\nabla\left(\frac{1}{\mathcal{H}}\right),$$

where

$$\mathcal{H}=\sqrt{1+\mathcal{D}^2+2\theta(\mathcal{B}^2+\mathcal{P}^2)}.$$

As $\theta\downarrow 0$, we get the asymptotic system

$$\partial_\theta\mathcal{D}+\nabla\cdot\left(\frac{\mathcal{D}\otimes\mathcal{P}-\mathcal{P}\otimes\mathcal{D}}{\mathcal{H}}\right)=\nabla\times\left(\frac{\mathcal{B}}{\mathcal{H}}\right),$$
$$\mathcal{B}=-\nabla\times\left(\frac{\mathcal{D}}{\mathcal{H}}\right),\quad \mathcal{P}=\nabla\cdot\left(\frac{\mathcal{D}\otimes\mathcal{D}}{\mathcal{H}}\right)+\nabla\left(\frac{1}{\mathcal{H}}\right),$$

where, now,

$$\mathcal{H}=\sqrt{1+\mathcal{D}^2}.$$

The equality $\mathcal{P}=\mathcal{D}\times\mathcal{B}$ follows directly from these equations. Thus, since we are in three space dimensions,

$$\nabla\cdot\left(\frac{\mathcal{D}\otimes\mathcal{P}-\mathcal{P}\otimes\mathcal{D}}{\mathcal{H}}\right)=\nabla\times\left(\frac{\mathcal{D}\times\mathcal{P}}{\mathcal{H}}\right)=\nabla\times\left(\frac{\mathcal{D}\times(\mathcal{D}\times\mathcal{B})}{\mathcal{H}}\right)=\nabla\times\left(\frac{(\mathcal{D}\cdot\mathcal{B})\mathcal{D}-\mathcal{D}^2\mathcal{B}}{\mathcal{H}}\right).$$
$$=\nabla\times\left(\frac{(\mathcal{D}\cdot\mathcal{B})\mathcal{D}-(1+\mathcal{D}^2)\mathcal{B})}{\mathcal{H}}\right)+\nabla\times\left(\frac{\mathcal{B}}{\mathcal{H}}\right),$$
$$=\nabla\times\left(\frac{(\mathcal{D}\cdot\mathcal{B})\mathcal{D}}{\mathcal{H}}-\mathcal{H}\mathcal{B}\right)+\nabla\times\left(\frac{\mathcal{B}}{\mathcal{H}}\right).$$

Finally, we have found

$$\partial_\theta\mathcal{D}=\nabla\times\left(\mathcal{B}\sqrt{1+\mathcal{D}^2}-\frac{(\mathcal{D}\cdot\mathcal{B})\mathcal{D}}{\sqrt{1+\mathcal{D}^2}}\right),\quad \mathcal{B}=-\nabla\times\left(\frac{\mathcal{D}}{\sqrt{1+\mathcal{D}^2}}\right)$$

which is nothing but the expected Equation (30), after restoring notations (t, B, D) instead of $(\theta, \mathcal{B}, \mathcal{D})$. ☐

4. The arctangential heat equation in nonconservative form: a tool for image processing?

In nonconservative form, the arctangential heat equation (1) reads (4), namely

$$\partial_t \psi = \left(\frac{\cos(\pi\psi)}{\pi}\right)^2 \Delta\psi.$$

Interestingly enough, in this nonconservative formulation, ψ can take values in the entire real line and not only in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. In sharp contrast with the usual linear heat equation, (4) seems to admit (in a suitable sense) a lot of nontrivial equilibrium solutions, at least in the one dimension case $d = 1$. Such solutions ψ should be continuous piecewise linear functions, with possible change of slope (or plateaus) each time ψ touches the discrete set $\left\{k + \frac{1}{2},\ k \in \mathbb{Z}\right\}$ as $\cos(\pi\psi)$ vanishes. Let us now perform a few numerical experiments based on the very elementary explicit difference scheme (written in two space dimensions with traditional notation of numerical analysis):

$$\psi_{i,j}^{n+1} - \psi_{i,j}^n = \frac{4\tau}{h^2}\cos(\pi\psi_{i,j}^n)^2\left(\frac{\psi_{i+1,j}^n + \psi_{i-1,j}^n + \psi_{i,j+1}^n + \psi_{i,j-1}^n}{4} - \psi_{i,j}^n\right), \quad (37)$$

where τ and h respectively denote the time and space steps. This scheme is stable as long as $4\tau h^{-2} \le 1$ and, in all our numerical experiments, we will choose $4\tau = h^2$.

In the first experiment, we input as initial condition $\psi(0, \cdot)$ a (simulated) Brownian curve on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (made periodic by subtracting a suitable affine function). We draw the initial curve (Figure 3) and the final curve (Figure 4) obtained with 256 grid points after 4096 time steps.

The second experiment is of different nature. We use a 256×256 grid for the periodic square $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ and consider the function

$$\psi(0, x, y) \\ = 4\cos(2\pi(x - 0.25))\cos(2\pi(y - 0.2)) + 3\cos(2\pi(y + x))\cos(2\pi(x - 0.8)).$$

Then we add to $\psi(0, x, y)$, at each grid point, a random number ξ, uniformly independently distributed in $[-0.5, +0.5]$ and run (37). In three successive plots, Figures 5–7, we draw all grid points (i, j) where the resulting function is at a distance less than 0.025 from the set $\left\{k + \frac{1}{2},\ k \in \mathbb{Z}\right\}$ in $\mathbb{R}$, respectively for $n = 0$, first without noise (Figure 5), then with added noise (Figure 6), and, finally, for $n = 8192$ (Figure 7), i.e., after 8192 time steps. We see that the arctangential heat equation, in discretized nonconservative form (37), enjoys some ability at processing black and white images, $\psi_{i,j}^n$ being the (suitably normalized) level of gray at step n and grid point (i, j), by unveiling and enhancing the level sets $\left\{(i, j),\ \psi_{i,j}^n \in \left\{k + \frac{1}{2},\ k \in \mathbb{Z}\right\}\right\}$ as n grows.

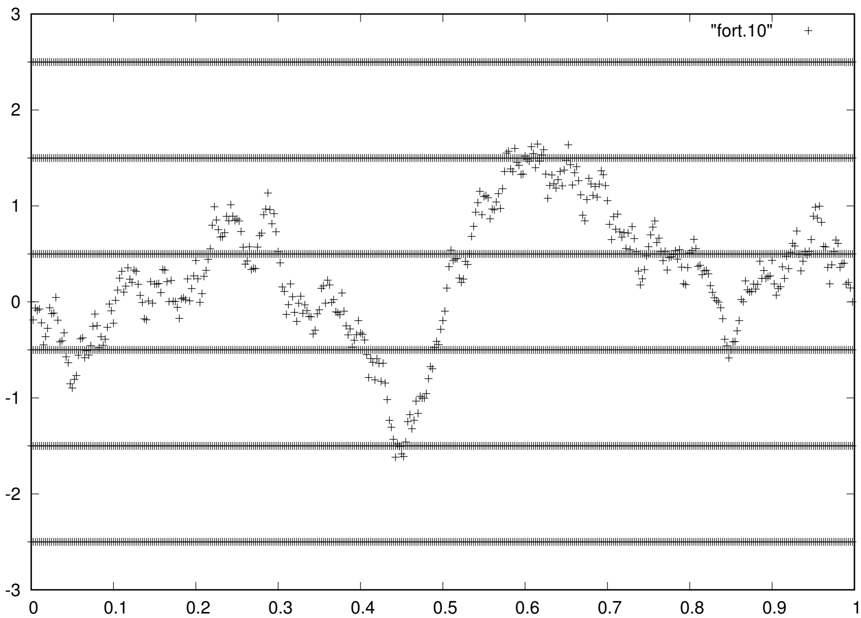

Figure 3. The one-dimensional arctangential heat equation: brownian initial condition, $x \in \mathbb{R}/\mathbb{Z}$.

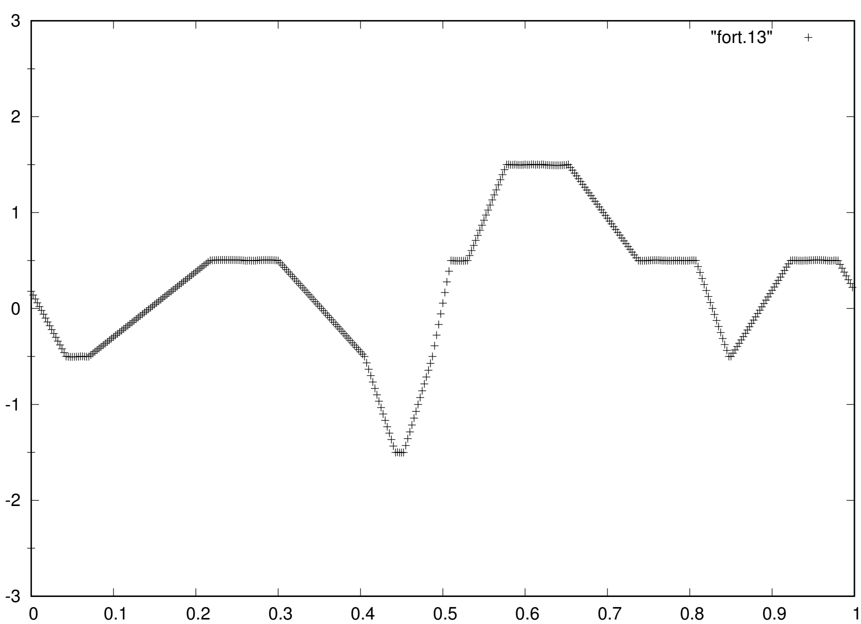

Figure 4. Numerical solution for 256 grid points, 4096 time steps.

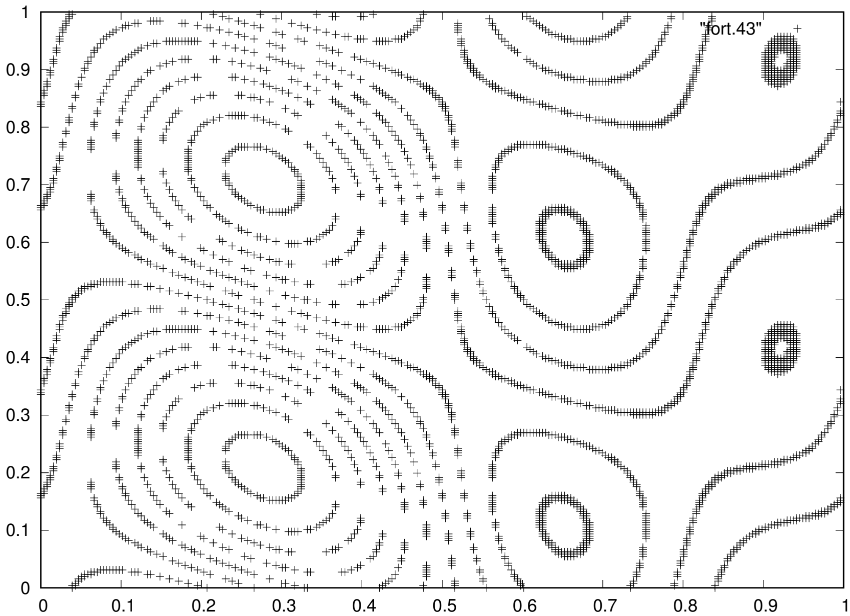

Figure 5. The two-dimensional arctangential heat equation: level sets of the given function with no noise.

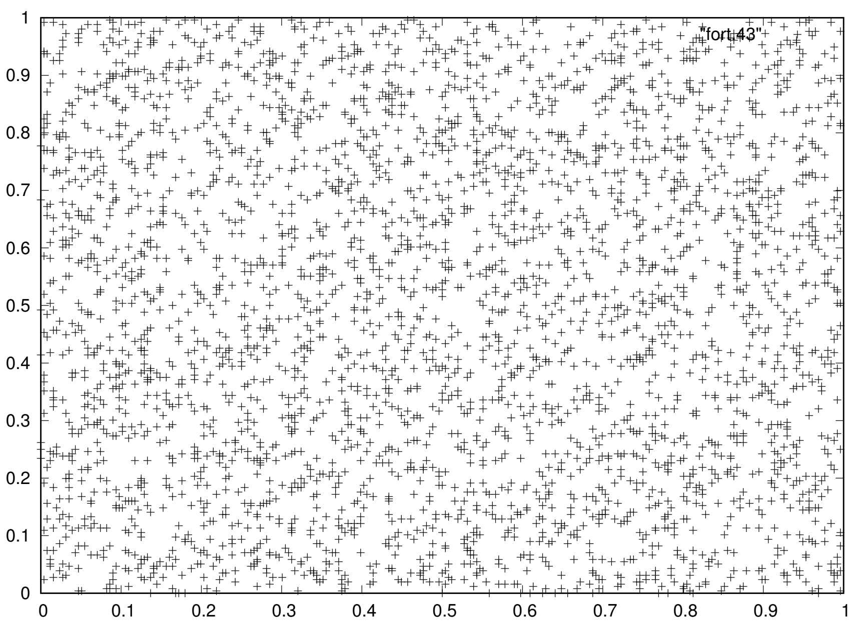

Figure 6. The two-dimensional arctangential heat equation: the initial condition with added noise.

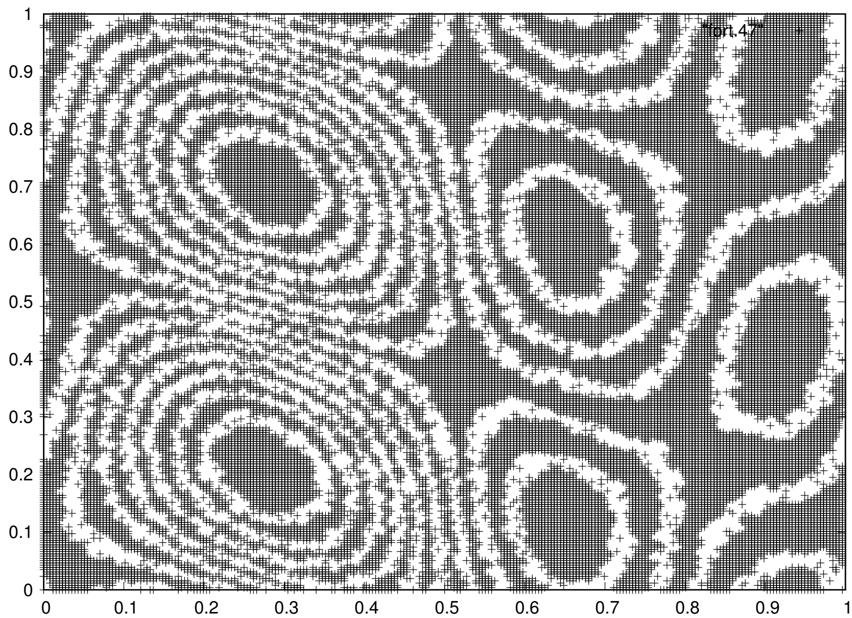

Figure 7. Recovery of the level sets by solving the two-dimensional arctangential heat equation.

Finally, in the third experiment, our initial condition is obtained by adding the same noise ξ as before to the step function with values $\frac{1}{2}$ if

$$4\cos(2\pi(x-0.25))\cos(2\pi(y-0.2))+3\cos(2\pi(y+x))\cos(2\pi(x-0.8))\geq 0.5,$$

and 0 otherwise. We draw the same plots as in the previous experiment, this time for $n=0$ (with and without noise, Figures 8 and 9) and $n=1024$ (Figure 10).

Appendix: Proof of Theorem 1

First step: Hamiltonian form of the minimal surface equations. Equation (2) is easily obtained by finding critical points ϕ of the Minkowski area of the graph $(t,x)\to(t,x,\phi(t,x))$, namely

$$-\iint\sqrt{1-\partial_t\phi^2+\partial^k\phi\,\partial_k\phi}\,dt\,dx, \tag{38}$$

under space-time compactly supported perturbations. For the sequel, it is crucial to use the Hamiltonian form of Equation (2). For that purpose, we introduce the fields

$$E(t,x)=\partial_t\phi(t,x),\quad B_i(t,x)=\partial_i\phi(t,x), \tag{39}$$

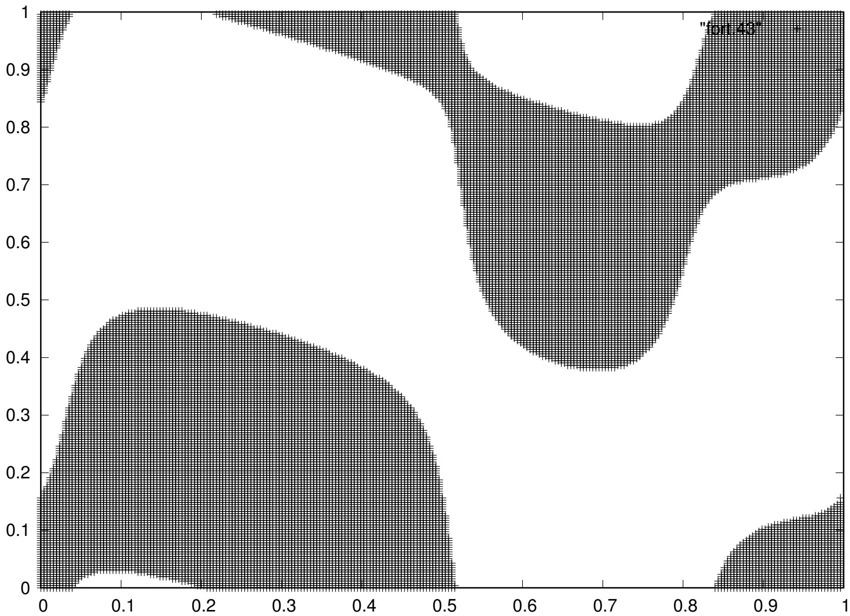

Figure 8. The two-dimensional arctangential heat equation: other choice of data (with binary values).

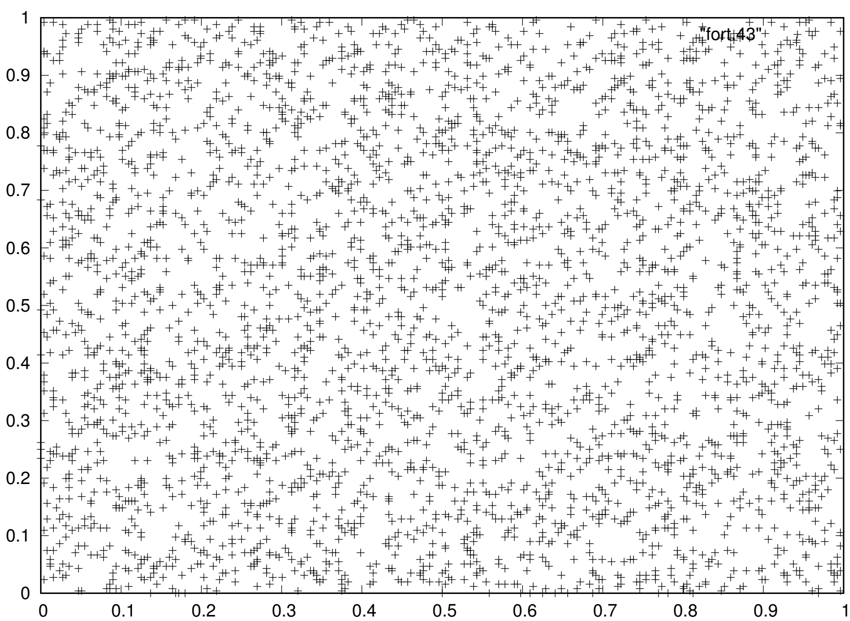

Figure 9. The two-dimensional arctangential heat equation: the initial condition with added noise.

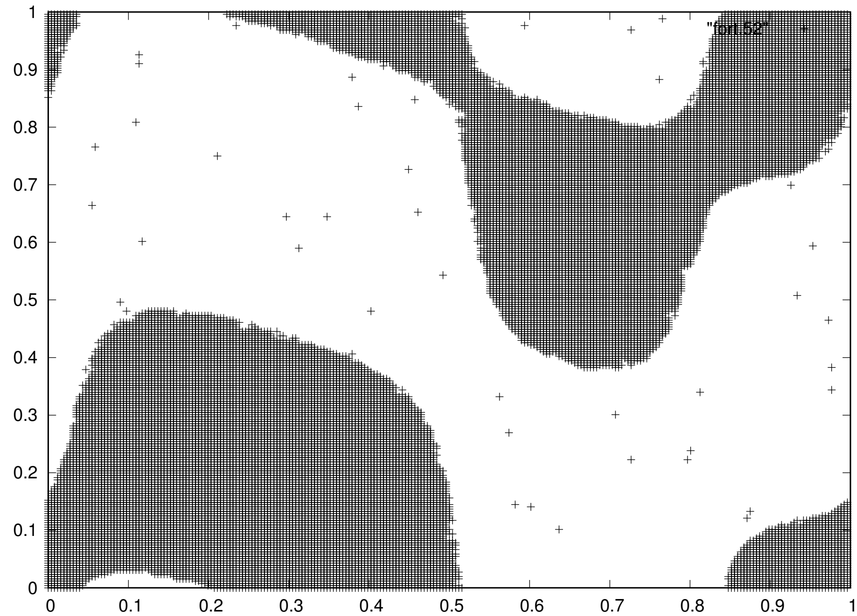

Figure 10. Recovery of the level sets by solving the two-dimensional arctangential heat equation.

which are linked by the differential compatibility condition

$$\partial_t B_i = \partial_i E. \tag{40}$$

Introducing the Lagrangian function

$$L(E, B) = -\sqrt{1 - E^2 + B_k B^k}, \tag{41}$$

we look at critical points (E, B) of

$$\iint L(E(t, x), B(t, x))\, dt\, dx$$

under space-time compactly supported perturbations, subject to constraint (40). In other words, we look for saddle-points (E, B, ψ) of

$$\iint \big(L(E(t, x), B(t, x)) + \partial_t \psi^i B_i(t, x) - \partial_i \psi^i E(t, x)\big)\, dt\, dx,$$

where ψ is a Lagrange multiplier for constraint (40). Independently of the specific definition of L, we may introduce the Hamiltonian H as the partial Legendre–Fenchel transform of the Lagrangian $L(E, B)$ with respect to E,

$$H(D, B) = \sup_{E \in \mathbb{R}} DE - L(E, B) \tag{42}$$

and the corresponding "dual" field

$$D(t,x) = \left(\frac{\partial L}{\partial E}\right)(E(t,x), B(t,x)). \tag{43}$$

Then, we get, by standard differential calculus, the Hamiltonian formulation

$$\partial_t B_i = \partial_i \left(\frac{\partial H}{\partial D}(D,B)\right), \quad \partial_t D = \partial_i \left(\frac{\partial H}{\partial B_i}(D,B)\right), \tag{44}$$

and, as a consequence, an extra conservation law involving H:

$$\partial_t(H(D,B)) + \partial_i(P^i(D,B)) = 0, \quad P^i(D,B) = -\left(\frac{\partial H}{\partial D}\frac{\partial H}{\partial B_i}\right)(D,B). \tag{45}$$

In the case of the minimal surface equations L is given by (41) and we get, explicitly,

$$H(D,B) = \sqrt{(1+B_k B^k)(1+D^2)} \tag{46}$$

and, after elementary calculations, deduce:

Proposition 5. *The minimal surface equations* (2) *can be written in Hamiltonian form*

$$\partial_t B_i = \partial_i \left(\sqrt{\frac{1+B_k B^k}{1+D^2}}D\right), \quad \partial_t D = \partial_i \left(\sqrt{\frac{1+D^2}{1+B_k B^k}}B^i\right), \tag{47}$$

with the extra conservation law

$$\partial_t H + \partial_i P^i = 0, \quad H = \sqrt{(1+B_k B^k)(1+D^2)}, \quad P^i = -DB^i. \tag{48}$$

In addition, (D,B) *are related to the original field* ϕ *involved in* (2) *by*

$$B_i = \partial_i \phi, \quad D = \frac{\partial_t \phi}{\sqrt{1 - \partial_t \phi^2 + \partial^k \phi\, \partial_k \phi}}. \tag{49}$$

Second step: construction of an augmented system with convex entropy. Unfortunately, H, as defined by (46), is not a convex function of (D,B) and, therefore, (47) does not belong to the class of systems of "conservation laws with a convex entropy" which enjoys many interesting properties (as discussed in Dafermos' book [2016]). However, there is also an extra conservation law for $P = -DB$, namely (7). This allows (D,B,P) to be solution of the *augmented* system (6)–(7) of conservation laws which enjoys the extra conservation law (9) for the strictly convex "entropy" $h(D,B,P) = \sqrt{1+D^2+B_k B^k + P_k P^k}$, which is nothing but $H(D,B)$, written as a function of (D,B,P). Let us now provide the detailed calculations.

The first evolution equations (6) are straightforward (just writing (47) with $P = -DB$). The two last ones are much more involved. Let us first prove (7).

Since $P_i = -DB_i$, we get

$$\partial_t P_i = -D\partial_t B_i - B_i \partial_t D = T = T_4 + T_3 + T_1 + T_2,$$

$$T_4 = D\partial_i\left(\frac{B_j P^j}{h}\right), \quad T_3 = -D\partial_i\left(\frac{D}{h}\right), \quad T_1 = B_i \partial_j\left(\frac{DP^j}{h}\right), \quad T_2 = -B_i \partial_j\left(\frac{B^j}{h}\right),$$

using Theorem 1. We have

$$\begin{aligned}
T_4 &= T_4a + T_4b, \quad T_4a = DB_j\partial_i\left(\frac{P^j}{h}\right), \quad T_4b = \frac{DP^j}{h}\partial_i B_j,\\
T_3 &= T_3a + T_3b, \quad T_3a = -\partial_i\left(\frac{D^2}{h}\right), \qquad T_3b = \frac{D}{h}\partial_i D,\\
T_1 &= T_1a + T_1b, \quad T_1a = \partial_j\left(\frac{B_i DP^j}{h}\right), \quad T_1b = -\partial_j B_i \frac{DP^j}{h},\\
T_2 &= T_2a + T_2b, \quad T_2a = -\partial_j\left(\frac{B_i B^j}{h}\right), \quad T_2b = \partial_j B_i \frac{B^j}{h}.
\end{aligned}$$

Since $P_j = -DB_j$, we have

$$T_1a = -\partial_j\left(\frac{P_i P^j}{h}\right),$$

$$T_4a = -P_j\partial_i\left(\frac{P^j}{h}\right) = T_4aa + T_4ab, \quad T_4aa = -\partial_i\left(\frac{P_j P^j}{h}\right), \quad T_4ab = \frac{P^j}{h}\partial_i P_j.$$

Since B is a gradient, we have $\partial_i B_j = \partial_j B_i$ and, therefore,

$$T_4b = -T_1b, \quad T_2b = \partial_i B_j \frac{B^j}{h},$$

so that

$$T_3b + T_2b + T_4ab = \frac{1}{2h}\partial_i(1 + D^2 + B_j B^j + P_j P^j) = \partial_i h = \partial_i\left(\frac{h^2}{h}\right)$$

(by definition (8) of h). Collecting all terms, we find

$$\begin{aligned}
\partial_t P_i = T &= T_4aa + T_4ab + T_4b + T_3a + T_3b + T_1a + T_1b + T_2a + T_2b\\
&= T_4aa + T_3a + T_1a + T_2a + \partial_i h\\
&= -\partial_i\left(\frac{P_j P^j}{h}\right) - \partial_i\left(\frac{D^2}{h}\right) - \partial_j\left(\frac{P_i P^j}{h}\right) - \partial_j\left(\frac{B_i B^j}{h}\right) + \partial_i\left(\frac{h^2}{h}\right)\\
&= \partial_i\left(\frac{1 + B_j B^j}{h}\right) - \partial_j\left(\frac{P_i P^j}{h}\right) - \partial_j\left(\frac{B_i B^j}{h}\right)
\end{aligned}$$

(by definition (8) of h) and we have obtained (7). Let us now prove (9). Notice that, from now on, we no longer can use $P=-DB$. This equation should only follow from the augmented system (6)–(8) and the property that B is a gradient. Using definition (8) of h, we get

$$\begin{aligned}\partial_t h &= \frac{D\partial_t D + B^i\partial_t B_i + P^i\partial_t P_i}{h}\\ &= \frac{D}{h}\partial_j\Big(\frac{-DP^j+B^j}{h}\Big)+\frac{B^i}{h}\partial_i\Big(\frac{-B_jP^j+D}{h}\Big)\\ &\qquad\qquad +\frac{P^i}{h}\Big(\partial_i\Big(\frac{1+B_jB^j}{h}\Big)-\partial_j\Big(\frac{P_iP^j}{h}\Big)-\partial_j\Big(\frac{B_iB^j}{h}\Big)\Big)\\ &= T_1+T_2+T_3+T_4+T_5+T_6+T_7,\end{aligned}$$

where

$$T_1=\frac{D}{h}\partial_j\Big(\frac{-DP^j}{h}\Big),\qquad T_2=\frac{D}{h}\partial_j\Big(\frac{B^j}{h}\Big)$$

$$T_3=\frac{B^i}{h}\partial_i\Big(\frac{-B_jP^j}{h}\Big),\qquad T_4=\frac{B^i}{h}\partial_i\Big(\frac{D}{h}\Big)$$

$$T_5=\frac{P^i}{h}\partial_i\Big(\frac{1+B_jB^j}{h}\Big)$$

$$T_6=-\frac{P^i}{h}\partial_j\Big(\frac{P_iP^j}{h}\Big),\qquad T_7=-\frac{P^i}{h}\partial_j\Big(\frac{B_iB^j}{h}\Big)=-\frac{P^j}{h}\partial_i\Big(\frac{B_jB^i}{h}\Big)$$

We see that

$$T_2+T_4=\partial_i\Big(\frac{DB^i}{h^2}\Big),$$

and

$$\begin{aligned}T_1+T_6 &= -P^j\Big(\frac{D}{h}\partial_j\Big(\frac{D}{h}\Big)+\frac{P^i}{h}\partial_j\Big(\frac{P_i}{h}\Big)\Big)-\partial_jP^j\Big(\frac{D^2+P^2}{h^2}\Big)\\ &= P^j\Big(\frac{1}{h}\partial_j\Big(\frac{1}{h}\Big)+\frac{B^i}{h}\partial_j\Big(\frac{B_i}{h}\Big)\Big)+\partial_jP^j\Big(\frac{1+B_iB^i}{h^2}-1\Big)\qquad\text{(definition of } h)\\ &= \frac{P^j}{h}\partial_j\Big(\frac{1}{h}\Big)+\frac{P^jB^i}{h^2}\partial_jB_i+\frac{P^jB_iB^i}{h}\partial_j\Big(\frac{1}{h}\Big)+\partial_jP^j\Big(\frac{1+B_iB^i}{h^2}-1\Big)\\ &= \frac{P^jB^i}{h^2}\partial_jB_i+\frac{P^j(1+B_iB^i)}{h}\partial_j\Big(\frac{1}{h}\Big)+\partial_jP^j\Big(\frac{1+B_iB^i}{h^2}\Big)-\partial_jP^j\\ &= \frac{P^jB^i}{h^2}\partial_jB_i+\partial_j\Big(\frac{P^j(1+B_iB^i)}{h^2}\Big)-T_5-\partial_jP^j.\end{aligned}$$

We also have

$$T_3 + T_7 = -\partial_i \left(\frac{P^j B_j B^i}{h^2} \right) - \frac{B^i P^j}{h^2} \partial_i B_j,$$

so that (since B is a gradient)

$$T_1 + T_6 + T_3 + T_7 + T_5 = \partial_j \left(\frac{P^j (1 + B_i B^i)}{h^2} \right) - \partial_i \left(\frac{P^j B_j B^i}{h^2} \right) - \partial_j P^j$$

and we have finally obtained

$$\begin{aligned} \partial_t h &= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 \\ &= \partial_j \left(\frac{P^j (1 + B_i B^i)}{h^2} \right) - \partial_i \left(\frac{P^j B_j B^i}{h^2} \right) + \partial_i \left(\frac{D B^i}{h^2} \right) - \partial_j P^j, \end{aligned}$$

in other words,

$$\partial_t h + \partial_j \left(P^j - \frac{(D B^j + P^j) + B_k (B^k P^j - P^k B^j)}{h^2} \right) = 0,$$

which is the desired conservation law (9) and achieves the proof of Theorem 1.

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Horn's problem and Fourier analysis

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Let A and B be two $n \times n$ Hermitian matrices. Assume that the eigenvalues $\alpha_1, \dots, \alpha_n$ of A are known, as well as the eigenvalues $\beta_1, \dots, \beta_n$ of B. What can be said about the eigenvalues of the sum $C = A + B$? This is Horn's problem. We revisit this question from a probabilistic viewpoint. The set of Hermitian matrices with spectrum $\{\alpha_1, \dots, \alpha_n\}$ is an orbit $\mathcal{O}_\alpha$ for the natural action of the unitary group $U(n)$ on the space of $n \times n$ Hermitian matrices. Assume that the random Hermitian matrix X is uniformly distributed on the orbit $\mathcal{O}_\alpha$ and, independently, the random Hermitian matrix Y is uniformly distributed on $\mathcal{O}_\beta$. We establish a formula for the joint distribution of the eigenvalues of the sum $Z = X + Y$. The proof involves orbital measures with their Fourier transforms, and Heckman's measures.

Introduction

Consider two Hermitian matrices A and B, and their sum $C = A + B$. Assume that the eigenvalues $\alpha_1, \dots, \alpha_n$ of A and the eigenvalues $\beta_1, \dots, \beta_n$ of B are known. Here is Horn's problem: what can be said about the eigenvalues $\gamma_1, \dots, \gamma_n$ of C? Horn's conjecture [1962] says that the set of possible eigenvalues $\gamma_1, \dots, \gamma_n$ for C is determined by a family of inequalities of the form

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j,$$

for certain "admissible" triples (I, J, K) of subsets of $\{1, 2, \dots, n\}$. Weyl inequalities [1912] are of this type. Klyachko [1998] describes these admissible triplets in terms of Schubert calculus. To a subset $I \subset \{1, \dots, n\}$ one associates a Schubert variety. The admissible triplets are those for which the associated Schubert varieties have a nonempty intersection. We will not go further in this direction. See for instance the survey paper [Bhatia 2001].

It is possible to consider Horn's problem from a probabilistic point of view (see [Frumkin and Goldberger 2006; Zuber 2018]). The set of $n \times n$ Hermitian matrices X with eigenvalues $\alpha_1, \dots, \alpha_n$ is an orbit $\mathcal{O}_\alpha$ for the action of the unitary

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group $U(n)$. Assume the random Hermitian matrix X to be uniformly distributed on $\mathcal{O}_\alpha$ and, independently, the matrix Y uniformly distributed on $\mathcal{O}_\beta$. The question is now: what is the distribution of the eigenvalues $\gamma_1, \dots, \gamma_n$ of the sum $Z = X + Y$? We follow this approach to determine explicitly the distribution $\nu_{\alpha,\beta}$.

The proof uses the celebrated Harish-Chandra–Itzykson–Zuber integral and Heckman's measures. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ the orbit

$$\mathcal{O}_\alpha = \{U \operatorname{diag}(\alpha_1, \dots, \alpha_n) U^* \mid U \in U(n)\}$$

carries a natural probability, the orbital measure μ_α. The Fourier–Laplace transform of μ_α is given by the Harish-Chandra–Itzykson–Zuber formula. Heckman's measure M_α is the projection of the orbital measure μ_α on the space of diagonal matrices. Heckman [1982] studied this measure in a more general setting and gave an explicit formula for it. Our main result is an explicit formula for the distribution $\nu_{\alpha,\beta}$ (Theorem 4.1):

$$\nu_{\alpha,\beta} = C_n V_n(x) \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \delta_{\sigma(\alpha)} * M_\beta,$$

where V_n denotes the Vandermonde polynomial in n variables,

$$V_n(x) = \prod_{i<j} (x_i - x_j),$$

and $\mathfrak{S}_n$ is the symmetric group which acts on $\mathbb{R}^n$ as follows:

$$\sigma((x_1, \dots, x_n)) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

The support $S(\alpha, \beta)$ of the measure $\nu_{\alpha,\beta}$ is the set of possible systems of eigenvalues for the matrix $C = A + B$, if $\alpha_1, \dots, \alpha_n$ are the eigenvalues of A and $\beta_1, \dots, \beta_n$ the eigenvalues of B.

Horn's problem is related to representation theory. If α and β are highest weights of two irreducible representations π_α and π_β of the unitary group $U(n)$, the spectrum of the tensor product $\pi_\alpha \otimes \pi_\beta$ is contained in the support of $\nu_{\alpha,\beta}$. But we will not consider this aspect of Horn's problem. See [Fulton 1998; 2000; Knutson and Tao 1999; Knutson et al. 2004].

We introduce in Section 1 the orbital measures on the space of Hermitian matrices and the radial part of a measure which is invariant under the action of the unitary group. In Section 2 we recall the Harish-Chandra–Itzykson–Zuber integral and, in Section 3, some properties of Heckman's measures. We state and prove our main result in Section 4. The case of a rank-one matrix B is considered in Section 5, and our result is compared to results of Frumkin and Goldberger [2006]. In Section 6 we give some formulas related to the case of 2×2 real symmetric matrices. We conclude with a few remarks.

1. Orbital measures

Let $\mathcal{H}_n(\mathbb{R}) = \mathrm{Sym}(n, \mathbb{R})$, the space of $n \times n$ real symmetric matrices, and $\mathcal{H}_n(\mathbb{C}) = \mathrm{Herm}(n, \mathbb{C})$, the space of $n \times n$ Hermitian matrices. For a matrix $X \in \mathcal{H}_n(\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) the classical spectral theorem says that the eigenvalues are real and the corresponding eigenspaces are orthogonal. We will denote by D_n the space of real diagonal matrices, $D_n \simeq \mathbb{R}^n$, and define the chamber

$$C_n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_1 \geq t_2 \geq \cdots \geq t_n\}.$$

Let $U_n(\mathbb{R}) = O(n)$, the orthogonal group, and $U_n(\mathbb{C}) = U(n)$, the unitary group. The group $U_n(\mathbb{F})$ acts on the space $\mathcal{H}_n(\mathbb{F})$ by the transformations $X \mapsto UXU^*$ ($U \in U_n(\mathbb{F})$). Let $\mathcal{O}_\alpha$ denote the orbit of the diagonal matrix $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ with $(\alpha_1, \ldots, \alpha_n) \in C_n$:

$$\mathcal{O}_\alpha = \{UAU^* \mid U \in U_n(\mathbb{F})\}.$$

From the spectral theorem it follows that

$$\mathcal{O}_\alpha = \{X \in \mathcal{H}_n(\mathbb{F}) \mid \mathrm{spectrum}(X) = \{\alpha_1, \ldots, \alpha_n\}\}.$$

The orbit $\mathcal{O}_\alpha$ carries a natural probability measure: the orbital measure μ_α, which is the image of the normalized Haar measure ω of the compact group $U_n(\mathbb{F})$ under the map

$$U_n(\mathbb{F}) \to \mathcal{H}_n(\mathbb{F}), \quad U \mapsto UAU^*.$$

For a continuous function f on $\mathcal{O}_\alpha$,

$$\int_{\mathcal{O}_\alpha} f(X)\mu_\alpha(dX) = \int_{U_n(\mathbb{F})} f(UAU^*)\omega(dU).$$

Let μ be a measure on $\mathcal{H}_n(\mathbb{F})$ which is invariant under $U_n(\mathbb{F})$. The integral of a function f can be decomposed as follows

$$\int_{\mathcal{H}_n(\mathbb{F})} f(X)\mu(dX) = \int_{\mathbb{R}^n} \left(\int_{U_n(\mathbb{F})} f(U\,\mathrm{diag}(t_1, \ldots, t_n)U^*)\omega(dU) \right) \nu(dt),$$

where ν is a measure on $\mathbb{R}^n$ which is invariant under the symmetric group $\mathfrak{S}_n$. For a function F on $\mathbb{R}^n$ and $\sigma \in \mathfrak{S}_n$

$$\int_{\mathbb{R}^n} F(t_{\sigma(1)}, \ldots, t_{\sigma(n)})\nu(dt) = \int_{\mathbb{R}^n} F(t_1, \ldots, t_n)\nu(dt).$$

The measure ν is called the *radial part* of the measure μ. If μ is a probability measure on $\mathcal{H}_n(\mathbb{F})$ which is $U_n(\mathbb{F})$-invariant, its radial part ν is the joint distribution of the eigenvalues of a random matrix X whose distribution is the measure μ. For instance, the radial part ν_α of the orbital measure μ_α is

$$\nu_\alpha = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \delta_{\sigma(\alpha)},$$

where $\sigma(\alpha) = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)})$. If the measure μ has a density h with respect to the Lebesgue measure m on the real vector space $\mathcal{H}_n(\mathbb{F})$, $\mu(dX) = h(X)m(dX)$, then, by the Weyl integration formula,

$$\nu(dt) = Ch(t)|V_n(t)|^d dt_1 \cdots dt_n,$$

where V_n is the Vandermonde polynomial,

$$V_n(t) = \prod_{1 \le i < j \le n} (t_i - t_j),$$

$d = 1$ if $\mathbb{F} = \mathbb{R}$, $d = 2$ if $\mathbb{F} = \mathbb{C}$, and C is a constant which depends on d and n. In this paper the radial part ν is defined as a $\mathfrak{S}_n$-invariant measure on $\mathbb{R}^n$. It is more usual to define the radial part as a measure on the chamber C_n. This is a slight difference, but responsible, in some explicit formulas, for the appearance of a factor $n!$ which does not show up in some other papers.

Assume that the random Hermitian matrix X is uniformly distributed on the orbit $\mathcal{O}_\alpha$, i.e., according to the orbital measure μ_α, and, independently the random Hermitian matrix Y is uniformly distributed on $\mathcal{O}_\beta$, i.e., according to μ_β. Then the sum $Z = X + Y$ is distributed according to the convolution product $\mu_\alpha * \mu_\beta$ and the joint distribution of the eigenvalues of Z is equal to the radial part $\nu_{\alpha,\beta}$ of $\mu_\alpha * \mu_\beta$. In case of $\mathbb{F} = \mathbb{C}$ we will determine explicitly the measure $\nu_{\alpha,\beta}$ by using Fourier analysis (Theorem 4.1).

2. Fourier–Laplace transform

The Fourier–Laplace transform of a bounded measure μ on $\mathcal{H}_n(\mathbb{F})$ is given by

$$\mathcal{F}\mu(Z) = \int_{\mathcal{H}_n(\mathbb{F})} e^{\operatorname{tr}(ZX)} \mu(dX).$$

The function $\mathcal{F}\mu$ is defined on $i\mathcal{H}_n(\mathbb{F})$. If the support of μ is compact, then $\mathcal{F}\mu$ is defined on $\operatorname{Sym}(n, \mathbb{C})$ if $\mathbb{F} = \mathbb{R}$, on $M_n(n, \mathbb{C})$ if $\mathbb{F} = \mathbb{C}$. If the measure μ is $U_n(\mathbb{F})$-invariant, its Fourier–Laplace transform $\mathcal{F}\mu$ is $U_n(\mathbb{F})$-invariant as well and determined by its restriction to the space D_n of diagonal matrices. For

$$Z = \operatorname{diag}(z_1, \dots, z_n), \qquad z = (z_1, \dots, z_n) \in \mathbb{C}^n,$$
$$T = \operatorname{diag}(t_1, \dots, t_n), \qquad t = (t_1, \dots, t_n) \in \mathbb{R}^n,$$

define the function

$$\mathcal{E}_n(z,t) = \int_{U_n(\mathbb{F})} e^{\operatorname{tr}(ZUTU^*)}\omega(dU).$$

The Fourier–Laplace transform of a $U_n(\mathbb{F})$-invariant bounded measure μ can be written, for $Z = \operatorname{diag}(z_1, \ldots, z_n)$,

$$\mathcal{F}\mu(Z) = \int_{\mathbb{R}^n} \mathcal{E}_n(z,t)\nu(dt),$$

where ν is the radial part of μ. Observe that the Fourier–Laplace transform of the orbital measure μ_α is given by

$$\mathcal{F}\mu_\alpha(Z) = \mathcal{E}_n(z,\alpha).$$

Since $\mathcal{F}(\mu_\alpha * \mu_\beta) = \mathcal{F}\mu_\alpha \mathcal{F}\mu_\beta$, we obtain the following key relation for determining the measure $\nu_{\alpha,\beta}$.

Proposition 2.1. *The measure $\nu_{\alpha,\beta}$ is determined by the relation, for $z \in \mathbb{C}^n$,*

$$\int_{\mathbb{R}^n} \mathcal{E}_n(z,t)\nu_{\alpha,\beta}(dt) = \mathcal{E}_n(z,\alpha)\mathcal{E}_n(z,\beta).$$

This relation is nothing but the product formula for the spherical functions of the following Gelfand pair (G, K):

$$G = U_n(\mathbb{F}) \ltimes \mathcal{H}_n(\mathbb{F}), \quad K = U_n(\mathbb{F}).$$

The group G acts on $\mathcal{H}_n(\mathbb{F})$ by the transformations

$$g \cdot X = UXU^* + A \quad (g = (U, A)).$$

A function f on G which is K-biinvariant can be seen as a $U_n(\mathbb{F})$-invariant function on $\mathcal{H}_n(\mathbb{F})$ and such a function only depends on the eigenvalues. Hence we can identify a K-biinvariant function f on G to a $\mathfrak{S}_n$-invariant function F on $\mathbb{R}^n$:

$$f(g) = F(t_1, \ldots, t_n),$$

if $t_1, \ldots, t_n$ are the eigenvalues of $g \cdot 0$. The spherical functions of the Gelfand pair (G, K) are given by

$$\varphi_z(g) = \mathcal{E}_n(z,t) \quad (t = (t_1, \ldots, t_n),\ z \in \mathbb{C}^n).$$

They satisfy the functional equation:

$$\int_K \varphi_z(g_1 U g_2)\omega(dU) = \varphi_z(g_1)\varphi_z(g_2) \quad (g_1, g_2 \in G).$$

With the identifications

$$\varphi_z(g_1) = \mathcal{E}_n(z,\alpha), \quad \varphi_z(g_2) = \mathcal{E}_n(z,\beta)$$

the functional equation can be written as

$$\int_{\mathbb{R}^n} \mathcal{E}_n(z,t)\nu_{\alpha,\beta}(dt) = \mathcal{E}_n(z,\alpha)\mathcal{E}_n(z,\beta).$$

For this viewpoint see the inspiring paper [Berezin and Gelfand 1962]. See also the recent paper [Kuijlaars and Román 2016]. Closely related are the papers [Dooley et al. 1993; Graczyk and Sawyer 2002], and Section 7 in [Rösler 2003].

In the case $\mathbb{F}=\mathbb{C}$, there is an explicit formula for $\mathcal{E}_n(z,t)$, the Harish-Chandra–Itzykson–Zuber formula [Itzykson and Zuber 1980]. In fact it is a special case of a formula established by Harish-Chandra [1957] for the adjoint action of a compact Lie group on its Lie algebra.

Theorem 2.2. *Let $A, B, \in \mathcal{H}_n(\mathbb{C})$ with eigenvalues $\alpha_1,\dots,\alpha_n$ and $\beta_1,\dots,\beta_n$. Then*

$$\int_{U_n(\mathbb{C})} e^{\operatorname{tr}(AUBU^*)}\omega(dU) = \delta_n!\frac{1}{V_n(\alpha)V_n(\beta)}\det(e^{\alpha_i\beta_j})_{1\le i,j\le n},$$

where $\delta_n=(n-1,n-2,\dots,1,0)$, $\delta_n! = (n-1)!(n-2)!\cdots 2!$.

Then we get

$$\mathcal{E}_n(z,t) = \delta_n!\frac{1}{V_n(z)V_n(t)}\det(e^{z_it_j})_{1\le i,j\le n}.$$

The formula can be seen as the Fourier–Laplace transform of an orbital measure:

$$\mathcal{F}\mu_\alpha(Z) = \delta_n!\frac{1}{V_n(z)V_n(\alpha)}\det(e^{z_i\alpha_j})_{1\le i,j\le n},$$

for $Z=\operatorname{diag}(z_1,\dots,z_n)$.

3. Heckman's measure

Let us consider the projection q of the space $\mathcal{H}_n(\mathbb{F})$ onto the subspace $D_n\simeq\mathbb{R}^n$ of real diagonal matrices,

$$q:\mathcal{H}_n(\mathbb{F})\to\mathbb{R}^n,\quad X\mapsto(x_1,\dots,x_n),\quad x_i=X_{ii}.$$

Recall Horn's convexity theorem [1954]: the image $q(\mathcal{O}_\alpha)$ of the orbit $\mathcal{O}_\alpha$ is equal to the convex hull $C(\alpha)$ of the points $\sigma(\alpha)$,

$$q(\mathcal{O}_\alpha) = C(\alpha) := \operatorname{Conv}(\{\sigma(\alpha)\mid\sigma\in\mathfrak{S}_n\}).$$

From now on, in this section, we assume $\mathbb{F}=\mathbb{C}$. The image $M_\alpha=q(\mu_\alpha)$ of the orbital measure μ_α is called Heckman's measure. In fact this measure has been described by Heckman [1982] in a more general setting (see also [Duflo et al. 1984]). The measure M_α has support $q(\mathcal{O}_\alpha)$ which is contained in the hyperplane $x_1+\cdots+x_n=\alpha_1+\cdots+\alpha_n$. It is symmetric, i.e., invariant under the group $\mathfrak{S}_n$, acting by permuting the coordinates. If the eigenvalues $\alpha_1, \dots, \alpha_n$ are distinct, Heckman's measure M_α is absolutely continuous with respect to the Lebesgue measure of this hyperplane and its density is piecewise polynomial. These facts have been established by Heckman. Let us recall their proof in the present special case. For a bounded measure M on $\mathbb{R}^n$ we will denote by $\widehat{M}$ its Fourier–Laplace transform:

$$\widehat{M}(z) = \int_{\mathbb{R}^n} e^{(z|x)} M(dx).$$

For $\alpha \in \mathbb{R}^n$ with the α_i all distinct, define the skew-symmetric measure

$$\eta_\alpha = \frac{\delta_n!}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \delta_{\sigma(\alpha)}.$$

The Fourier–Laplace transform of η_α is given by

$$\widehat{\eta_\alpha}(z) = \frac{\delta_n!}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) e^{(z|\sigma(\alpha))} = \frac{\delta_n!}{V_n(\alpha)} \det(e^{z_i \alpha_j})_{1 \le i, j \le n}.$$

The map $\alpha \mapsto \eta_\alpha$ extends as a continuous map $\mathbb{R}^n \to \mathcal{E}'(\mathbb{R}^n)$, the space of distributions on $\mathbb{R}^n$ with compact support. In particular

$$\eta_0 = V_n\left(\frac{\partial}{\partial x}\right) \delta_0.$$

Proposition 3.1. *Heckman's measure M_α satisfies the following equation*

$$V_n\left(-\frac{\partial}{\partial x}\right) M_\alpha = \eta_\alpha.$$

Proof. For a bounded measure μ on $\mathcal{H}_n(\mathbb{C})$, the Fourier–Laplace transform of the projection $M = q(\mu)$ of μ on D_n is equal to the restriction to D_n of the Fourier–Laplace transform of μ: $\widehat{M}(z) = \mathcal{F}\mu(Z)$, for $Z = \operatorname{diag}(z_1, \dots, z_n)$. Hence

$$\widehat{M_\alpha}(z) = \mathcal{F}\mu_\alpha(Z) = \mathcal{E}_n(z, \alpha).$$

Therefore, by the Harish-Chandra–Itzykson–Zuber formula (Theorem 2.2),

$$\widehat{M_\alpha}(z) = \delta_n! \frac{1}{V_n(\alpha) V_n(z)} \det(e^{z_i \alpha_j})_{1 \le i, j \le n} = \frac{1}{V_n(z)} \widehat{\eta_\alpha}(z).$$

This equality, which can be written $V_n(z)\widehat{M_\alpha}(z) = \widehat{\eta_\alpha}(z)$, means an equality between two Fourier–Laplace transforms of compactly supported distributions, and implies the following differential equation

$$V_n\left(-\frac{\partial}{\partial x}\right) M_\alpha = \eta_\alpha. \qquad \square$$

For solving this equation we will use an elementary solution of the differential operator $V_n\left(\frac{\partial}{\partial x}\right)$. Let us define the distribution E_n on $\mathbb{R}^n$:

$$\langle E_n, \varphi\rangle = \int_{\mathbb{R}_+^{n(n-1)/2}} \varphi\left(\sum_{i<j} t_{ij}\varepsilon_{ij}\right) dt_{ij},$$

where $\varepsilon_{ij} = e_i - e_j$ ($\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$).

Proposition 3.2. *The distribution E_n is an elementary solution of the differential operator $V_n\left(\frac{\partial}{\partial x}\right)$:*

$$V_n\left(\frac{\partial}{\partial x}\right) E_n = \delta_0.$$

The support of E_n is the convex cone in the hyperplane $x_1 + \cdots + x_n = 0$ generated by the vectors ε_{ij}, with $i < j$. The distribution E_n is absolutely continuous with respect to the Lebesgue measure of the hyperplane $x_1 + \cdots + x_n = 0$. The cone $\operatorname{supp}(E_n)$ *decomposes into a finite union of cones, and the restriction of the density to each of these cones is a polynomial, homogeneous of degree $\frac{1}{2}(n-1)(n-2)$.*

Proof. The differential operator $V_n\left(\frac{\partial}{\partial x}\right)$ is a product of degree one differential operators:

$$V_n\left(\frac{\partial}{\partial x}\right) = \prod_{i<j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right).$$

An elementary solution of $\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}$ is the Heaviside distribution Y_{ij} defined by

$$\langle Y_{ij}, \varphi\rangle = \int_0^\infty \varphi(t\varepsilon_{ij})\, dt.$$

Hence the convolution product

$$E_n = \prod_{i<j}^{*} Y_{ij}$$

is an elementary solution of $V_n\left(\frac{\partial}{\partial x}\right)$. □

For a function φ define $\check{\varphi}(x) = \varphi(-x)$, and for a distribution T, $\langle \check{T}, \varphi\rangle = \langle T, \check{\varphi}\rangle$.

Theorem 3.3. *The Heckman measure M_α is given by*

$$M_\alpha = \check{E}_n * \eta_\alpha.$$

If the α_i are all distinct, the measure M_α is absolutely continuous with respect to the Lebesgue measure of the hyperplane $x_1 + \cdots + x_n = \alpha_1 + \cdots + \alpha_n$ and the density is piecewise polynomial. This density is continuous for $n \geq 3$. The map $\alpha \mapsto M_\alpha$ extends as a continuous map $\mathbb{R}^n \to \mathcal{M}_c^1(\mathbb{R}^n)$, the set of probability measures on $\mathbb{R}^n$ with compact support.

Proof. Let F and G be distributions on $\mathbb{R}^n$. Assume the support of F to be compact. Let $D = P\big(\frac{\partial}{\partial x}\big)$ be a differential operator with constant coefficients. Then

$$DF * G = F * DG = D(F * G).$$

Therefore

$$\check{E}_n * V_n\left(-\frac{\partial}{\partial x}\right)M_\alpha = V_n\left(-\frac{\partial}{\partial x}\right)\check{E}_n * M_\alpha = M_\alpha.$$

By Proposition 3.1,

$$V_n\left(-\frac{\partial}{\partial x}\right)M_\alpha = \eta_\alpha.$$

Hence

$$M_\alpha = \check{E}_n * \eta_\alpha.$$

□

Example 3.4. $n = 2$ The elementary solution E_2 is given by

$$\langle E_2, \varphi\rangle = \int_0^\infty \varphi(t\varepsilon_{1,2})\,dt.$$

In the present case

$$\mathfrak{S}_2 = \{\mathrm{Id}, \tau\}, \qquad \tau : (x_1, x_2) \mapsto (x_2, x_1).$$

By Theorem 3.3,

$$\begin{aligned}\langle M_\alpha, \varphi\rangle &= \frac{1}{\alpha_1 - \alpha_2}\left(\int_0^\infty \varphi(\alpha - t_1\varepsilon_{1,2})\,dt_1 - \int_0^\infty \varphi(\tau(\alpha) - t_2\varepsilon_{1,2})\,dt_2\right).\\ &= \int_0^1 \varphi((1-t)\alpha + t\tau(\alpha))\,dt.\end{aligned}$$

The support of M_α is the segment $[\alpha, \tau(\alpha)]$.

Example 3.5. $n = 3$ The elementary solution E_3 is given by

$$\begin{aligned}\langle E_3, \varphi\rangle &= \int_{(\mathbb{R}_+)^3} \varphi(u\varepsilon_{1,2} + v\varepsilon_{2,3} + w\varepsilon_{1,3})\,du\,dv\,dw\\ &= \int_{(\mathbb{R}_+)^3} \varphi((u+w)\varepsilon_{1,2} + (v+w)\varepsilon_{2,3})\,du\,dv\,dw.\\ &= \int_{\{0\le w\le s, 0\le w\le t\}} \varphi(s\varepsilon_{1,2} + t\varepsilon_{2,3})\,ds\,dt\,dw\\ &= \int_{(\mathbb{R}+)^2} \inf(s,t)\varphi(s\varepsilon_{1,2} + t\varepsilon_{2,3})\,ds\,dt.\end{aligned}$$

Hence the support of E_3 is the angle defined by the rays generated by $\varepsilon_{1,2}$ and $\varepsilon_{2,3}$ with density, if $x = s\varepsilon_{1,2} + t\varepsilon_{2,3}$, $f(x) = \inf(s,t)$.

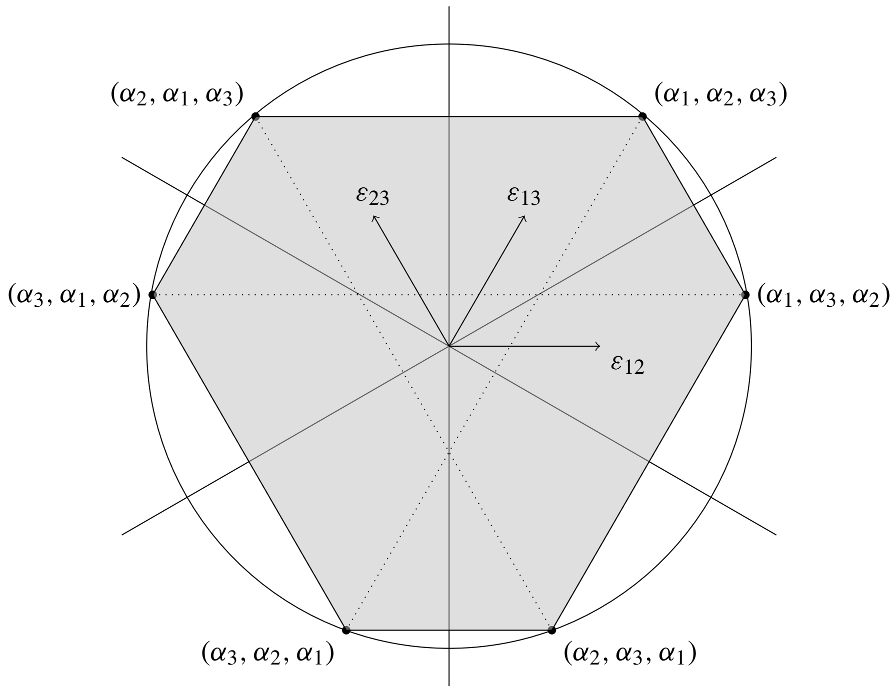

Figure 1. *Heckman's measure*, $n = 3$. For $\alpha_1 > \alpha_2 > \alpha_3$, the support of the measure M_α is the convex hull of the six points $\sigma(\alpha)$ ($\sigma \in \mathfrak{S}_3$). The density of M_α is affine linear in the three trapezia (trapezoids) around the rim and in the intervening triangles, and constant in the central triangle.

4. The radial part of the convolution product of two orbital measures

Recall that $\nu_{\alpha,\beta}$ denotes the radial part of the convolution product $\mu_\alpha * \mu_\beta$. (The convolution is with respect to $\mathcal{H}_n(\mathbb{F})$.) By Proposition 2.1, the measure $\nu_{\alpha,\beta}$ is determined by the relation

$$\int_{\mathbb{R}^n} \mathcal{E}_n(z, t)\nu_{\alpha,\beta}(dt) = \mathcal{E}_n(z, \alpha)\mathcal{E}_n(z, \beta).$$

Theorem 4.1. *Assume that* $\mathbb{F} = \mathbb{C}$, *the eigenvalues* $\alpha_1, \dots, \alpha_n$ *are distinct*, *and the eigenvalues* $\beta_1, \dots, \beta_n$ *are distinct as well. The radial part* $\nu_{\alpha,\beta}$ *is given by*

$$\nu_{\alpha,\beta} = \frac{1}{n!}\frac{1}{\delta_n!} V_n(x)\; M_\alpha * \eta_\beta = \frac{1}{n!}\frac{1}{\delta_n!} V_n(x)\; \eta_\alpha * M_\beta,$$

or

$$\nu_{\alpha,\beta} = \frac{1}{n!}\frac{V_n(x)}{V_n(\alpha)} \sum_{\sigma\in\mathfrak{S}_n} \varepsilon(\sigma)\delta_{\sigma(\alpha)} * M_\beta.$$

The map $(\alpha,\beta)\mapsto\nu_{\alpha,\beta}$ *extends continuously as a map* $\mathbb{R}^n\times\mathbb{R}^n\to\mathcal{M}_c^1(\mathbb{R}^n)$.

Here the convolutions are with respect to $\mathbb{R}^n$. The measure $\nu_{\alpha,\beta}$ is a $\mathfrak{S}_n$-invariant probability measure on $\mathbb{R}^n$. Observe that

$$\nu_{\alpha,0}=\nu_\alpha=\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\delta_{\sigma(\alpha)}.$$

Theorem 4.1 is related to Theorem 3.4 in [Dooley et al. 1993] and to Theorem 2.1 in [Graczyk and Sawyer 2002]. A similar result, but slightly different, is given in [Rösler 2003, p.2436].

Proof. Define $\nu=V_n(x)\ M_\alpha*\eta_\beta$ and let us compute

$$I(z)=\int_{\mathbb{R}^n}\mathcal{E}_n(z,x)\nu\,(dx).$$

The measure M_α is symmetric and η_β is skew symmetric, therefore $M=M_\alpha*\eta_\beta$ is skew symmetric as is its Fourier–Laplace transform $\widehat{M}$. We obtain

$$\begin{aligned}I(z)&=\frac{\delta_n!}{V_n(z)}\int_{\mathbb{R}^n}\det(e^{z_ix_j})_{1\le i,j\le n}M\,(dx)\\&=\frac{\delta_n!}{V_n(z)}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\int_{\mathbb{R}^n}e^{(\sigma(z)|x)}M\,(dx)\\&=\frac{\delta_n!}{V_n(z)}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\widehat{M}(\sigma(z))=\frac{\delta_n!}{V_n(z)}n!\widehat{M}(z).\end{aligned}$$

Since

$$\widehat{M}(z)=\widehat{M_\alpha}(z)\widehat{\eta_\beta}(z)=\mathcal{E}_n(z,\alpha)\frac{\delta_n!}{V_n(\beta)}\det(e^{z_i\beta_j})_{1\le i,j\le n},$$

we obtain

$$I(z)=n!\delta_n!\mathcal{E}(z,\alpha)\mathcal{E}(z,\beta).$$

By Proposition 2.1 this proves the formula of Theorem 4.1. □

Recall that $S(\alpha,\beta)$ denotes the support of the measure $\nu_{\alpha,\beta}$. The $\mathfrak{S}_n$-invariant compact set $S(\alpha,\beta)\subset\mathbb{R}^n$ is the set of possible systems of eigenvalues for $C=A+B$, if $\alpha_1,\dots,\alpha_n$ are the eigenvalues of A and $\beta_1,\dots,\beta_n$ the eigenvalues of B.

Corollary 4.2. (i) *We have the following inclusion*:

$$S(\alpha,\beta)\subset\bigcup_{\sigma\in\mathfrak{S}_n}(\sigma(\alpha)+C(\beta)).$$

(ii) *If*

$$\min_{i<j}(\alpha_i-\alpha_j)\ge\max_{k,\ell}|\beta_k-\beta_\ell|,$$

then:

$$S(\alpha,\beta)\cap C_n=\alpha+C(\beta).$$

Recall that $C(\beta)$ is the convex hull of the points $\sigma(\beta)$ ($\sigma\in\mathfrak{S}_n$), and C_n is the chamber:

$$C_n=\{t=(t_1,\dots,t_n)\in\mathbb{R}^n\mid t_1\ge\cdots\ge t_n\}.$$

Part (i) is related to Lidskii's theorem [1950] and can be equivalently written as a system of inequalities

$$\sum_{k\in K}x_k\le\sum_{i\in I}\alpha_i+\sum_{j\in J}\beta_j,$$

with suitable triples $\{I,J,K\}$. See [Bhatia 2001, p.295; 1997, Theorem II.1.10].

Proof.

(a) The support of the measure η_α is the orbit of α under the action of $\mathfrak{S}_n$,

$$\operatorname{supp}(\eta_\alpha)=\{\sigma(\alpha)\mid\sigma\in\mathfrak{S}_n\},$$

and, by Horn's Theorem, the support of Heckman's measure M_β is

$$\operatorname{supp}(M_\beta)=q(\mathcal{O}_\beta)=C(\beta).$$

Statement (i) follows since

$$\operatorname{supp}(\eta_\alpha*M_\beta)\subset\operatorname{supp}(\eta_\alpha)+\operatorname{supp}(M_\beta).$$

In general this is an inclusion and not an equality, because the measure η_α has positive and negative parts, and cancellations are possible.

(b) Under the condition

$$\min_{i<j}(\alpha_i-\alpha_j)>\max_{k,\ell}|\beta_k-\beta_\ell|,$$

the sets $\sigma(\alpha)+C(\beta)$ are disjoint and there is one of them in each chamber $\sigma(C_n)$ ($\sigma\in\mathfrak{S}_n$). Hence no cancellation is possible. □

Theorem 4.1 can be extended as follows. For $\alpha,\beta,\gamma\in\mathbb{R}^n$, the radial part of $\mu_\alpha*\mu_\beta*\mu_\gamma$ is given by

$$\nu_{\alpha,\beta,\gamma}=\frac{1}{n!}\frac{1}{\delta_n!}V_n(x)\eta_\alpha*M_\beta*M_\gamma.$$

This generalizes to any finite convolution product. For $\alpha^{(1)},\dots,\alpha^{(k)}\in\mathbb{R}^n$, the radial part of $\mu_{\alpha^{(1)}}*\cdots*\mu_{\alpha^{(k)}}$ is given by

$$\nu_{\alpha^{(1)},\dots,\alpha^{(k)}}=\frac{1}{n!}\frac{1}{\delta_n!}V_n(x)\eta_{\alpha^{(1)}}*M_{\alpha^{(2)}}*\cdots*M_{\alpha^{(k)}}.$$

Example 4.3. $n = 2$ We use the same notation as in Example 3.4. We saw that

$$\langle M_\alpha, \varphi\rangle = \int_0^1 \varphi((1-t)\alpha + t\tau(\alpha))\, dt.$$

In this special case, with $a := V_2(\alpha) = \alpha_1 - \alpha_2$, the measure η_α is

$$\eta_\alpha = \tfrac{1}{a}(\delta_\alpha - \delta_{\tau(\alpha)}).$$

One can check the following formula for the Fourier–Laplace transform of η_α:

$$\widehat{\eta_\alpha}(z) = e^{z_1+z_2)(\alpha_1+\alpha_2)/2}\tfrac{1}{a}(e^{a(z_1-z_2)/2} - e^{-a(z_1-z_2)/2}).$$

By Theorem 4.1,

$$\nu_{\alpha,\beta} = \tfrac{1}{2}V_2(x)\ M_\alpha * \eta_\beta = \tfrac{1}{2}V_2(x)\ \eta_\alpha * M_\beta.$$

Let us explicit the measure $\nu_{\alpha,\beta}$ by using the second expression:

$$\begin{aligned}\langle \nu_{\alpha,\beta}, \varphi\rangle = {} & \frac{1}{2a}\int_0^1 (a + (1-2t)b)\varphi((1-t)(\alpha+\beta) + t(\alpha+\tau(\beta)))\, dt \\ & + \frac{1}{2a}\int_0^1 (a - (1-2t)b)\varphi((1-t)(\tau(\alpha)+\beta) + t(\tau(\alpha)+\tau(\beta)))\, dt,\end{aligned}$$

where $b = V_2(\beta) = \beta_1 - \beta_2$. The support $S(\alpha, \beta)$ of $\nu_{\alpha,\beta}$ is the union of two segments. If $a < b$, then

$$S(\alpha, \beta) = [\alpha+\beta, \alpha+\tau(\beta)] \cup [\tau(\alpha)+\beta, \tau(\alpha)+\tau(\beta)].$$

If $a < b$, there are some cancellations and one obtains

$$S(\alpha, \beta) = [\alpha+\beta, \tau(\alpha)+\beta] \cup [\alpha+\tau(\beta), \tau(\alpha)+\tau(\beta)],$$

and one checks the symmetry $\nu_{\beta,\alpha} = \nu_{\alpha,\beta}$.

5. The case of a rank-one matrix B

In this section we consider the special case of a rank-one matrix B. In such a case $\beta = (b, 0, \dots, 0)$ with $b > 0$ or $\beta = (0, \dots, 0, b)$ with $b < 0$. We assume first that $\beta = (1, 0, \dots, 0)$. The orbit $\mathcal{O}_\beta$ is the set of Hermitian matrices $Y = (u_i\bar{u}_j)$, where $u = (u_1, \dots, u_n)$ is a unit vector, $u \in S(\mathbb{F}^n)$. In case of $\mathbb{F} = \mathbb{R}$, the orbit $\mathcal{O}_\beta$ can be identified with $S(\mathbb{R}^n)/\{+1, -1\}^n$ and, in case of $\mathbb{F} = \mathbb{C}$, with $S(\mathbb{C}^n)/\mathbb{T}^n$.

Recall that q denotes the projection $q : \mathcal{H}_n(\mathbb{F}) \to D_n \simeq \mathbb{R}^n$. Then

$$q(\mathcal{O}_\beta) = \{(|u_1|^2, \dots, |u_n|^2) \mid u \in S(\mathbb{F}^n)\}$$

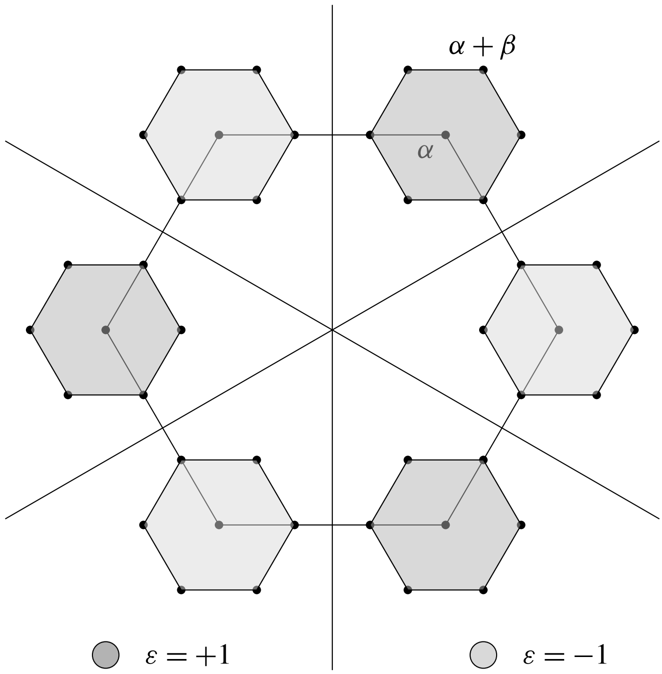

Figure 2. *Support* $S(\alpha,\beta)$ *of* $\nu_{\alpha,\beta}$, $\alpha=(3,0,-3)$, $\beta=(1,0,-1)$. The support is the union of the six hexagons.

is the simplex $\Sigma_n=\mathrm{Conv}(e_1,\ldots,e_n)$ contained in the hyperplane $x_1+\cdots+x_n=1$. The orbital measure μ_β is the image of the normalized uniform measure on the sphere $S(\mathbb{F}^n)$.

We assume that $\mathbb{F}=\mathbb{C}$ for the rest of this section.

Proposition 5.1. *Heckman's measure* $M_\beta=q(\mu_\beta)$ *is the normalized uniform measure on the simplex* $\mathrm{Conv}(e_1,\ldots,e_n)$, *i.e.*, *the normalized restriction to the simplex* Σ_n *of the Lebesgue measure of the hyperplane* $x_1+\cdots+x_n=1$.

Proof. The image of the normalized uniform measure on the sphere $S(\mathbb{C}^n)$ under the map

$$S(\mathbb{C}^n)\to\Sigma_n,\quad u\mapsto(|u_1|^2,\ldots,|u_n|^n),$$

is the normalized restriction to Σ_n of the Lebesgue measure on the hyperplane $x_1+\cdots+x_n=1$. □

Consider on the hyperplane $x_1+\cdots+x_n=1$ the differential form

$$w=dx_1\wedge\cdots\wedge dx_{n-1}.$$

Then

$$\int_{\Sigma_n}w=\frac{1}{(n-1)!},$$

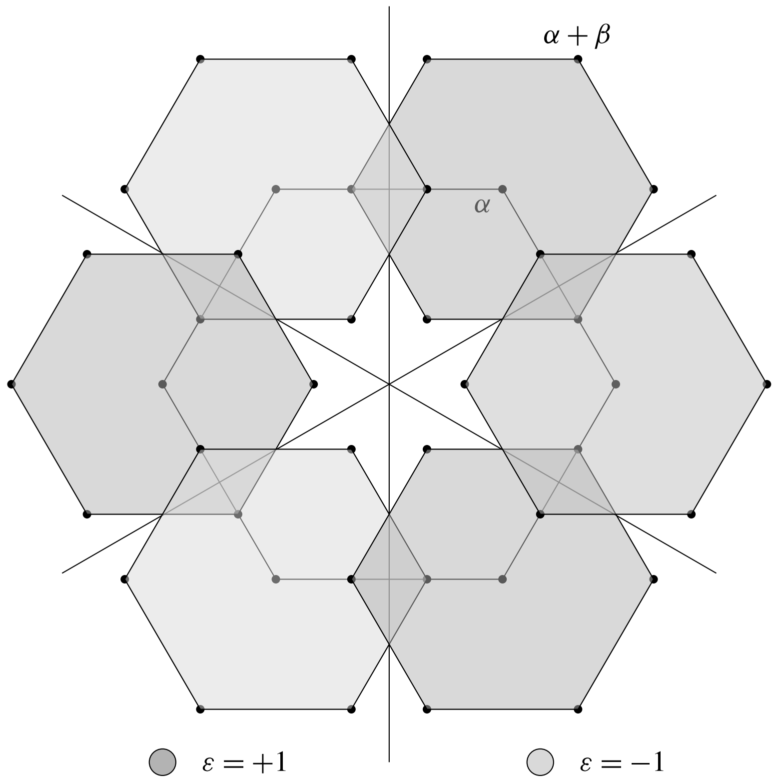

Figure 3. *Support* $S(\alpha, \beta)$ *of* $\nu_{\alpha,\beta}$, $\alpha = (3, 0, -3)$, $\beta = (2, 0, -2)$. The support is the union of the six hexagons.

and Heckman's measure M_β can be given by

$$\langle M_\beta, \varphi\rangle = (n-1)! \int_{\Sigma_n} \varphi(x) w.$$

Whereas it will not be used in the sequel we give a formula for the Fourier–Laplace transform of Heckman's measure M_β in this special case:

$$\widehat{M_\beta}(z) = \int_{\mathbb{R}^n} e^{(z|x)} M_\beta\,(dx) = (n-1)! \frac{1}{V_n(z)} \begin{vmatrix} e^{z_1} & \cdots & e^{z_n} \\ z_1^{n-2} & \cdots & z_n^{n-2} \\ \vdots & & \vdots \\ z_1 & \cdots & z_n \\ 1 & \cdots & 1 \end{vmatrix}.$$

(This formula can be obtained by using Theorem 4.1 in [Faraut 2015].)

Recall that, for $\alpha = (\alpha_1, \dots, \alpha_n) \in C_n$, $\nu_{\alpha,\beta}$ denotes the radial part of the measure $\mu_\alpha * \mu_\beta$. The following result has been obtained by Frumkin and Goldberger [2006, Theorem 6.1 and Theorem 6.7].

Theorem 5.2. *Assume that $\beta = (b, 0, \dots, 0)$ with $b > 0$.*

(i) *The support $S(\alpha, \beta)$ of $\nu_{\alpha,\beta}$ is given by*

$$S(\alpha, \beta) \cap C_n = \{x \in \mathbb{R}^n \mid x_1 \geq \alpha_1 \geq \cdots \geq x_n \geq \alpha_n,\ x_1 + \cdots + x_n = \alpha_1 + \cdots + \alpha_n + b\}.$$

(ii) *The measure $\nu_{\alpha,\beta}$ is absolutely continuous with respect to the Lebesgue measure of the hyperplane $x_1 + \cdots + x_n = \alpha_1 + \cdots + \alpha_n + b$ with the density*

$$h(x) = \frac{1}{n} \frac{1}{b^{n-1}} \frac{1}{V_n(\alpha)} V_n(x).$$

(It is assumed that the Lebesgue measure on the hyperplane $x_1 + \cdots + x_n = \alpha_1 + \cdots + \alpha_n + b$ is associated to the differential form $w = dx_1 \wedge \cdots \wedge dx_{n-1}$.)

The inclusion

$$S(\alpha, \beta) \cap C_n \subset \{x \in \mathbb{R}^n \mid x_1 \geq \alpha_1 \geq \cdots \geq x_n \geq \alpha_n,\ x_1 + \cdots + x_n = \alpha_1 + \cdots + \alpha_n + b\}.$$

can be found in [Horn and Johnson 1985, Theorem 4.3.4].

By Theorem 4.1, the density is given in the present case by

$$h(x) = \frac{1}{n} \frac{V_n(x)}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \frac{1}{b^{n-1}} \chi\left(\frac{x - \delta_{\sigma(\alpha)}}{b}\right),$$

where χ is the indicatrix of the simplex Σ_n.

Let us comment how Theorem 5.2 is related to Theorem 4.1 and Corollary 4.2. The conditions in (i) can be split in two parts:

(I) $x_1 \geq \alpha_1, \dots, x_n \geq \alpha_n, \quad x_1 + \cdots + x_n = \alpha_1 + \cdots + \alpha_n + b.$

(II) $x_2 \leq \alpha_1, \dots, x_n \leq \alpha_{n-1}.$

Let us introduce barycentric coordinates s_i:

$$x_i = \alpha_i + b s_i \quad (i = 1, \dots, n).$$

Conditions (I) gives

$$s_1 \geq 0, \dots, s_n \geq 0,\ s_1 + \cdots + s_n = 1,$$

which means that $x \in \alpha + b\Sigma_n$. If

$$b \leq \alpha_{i-1} - \alpha_i \quad (i = 2, \dots, n),$$

then (I) implies (II). Therefore, in this case, $S(\alpha, \beta) \cap C_n = \alpha + b\Sigma_n$.

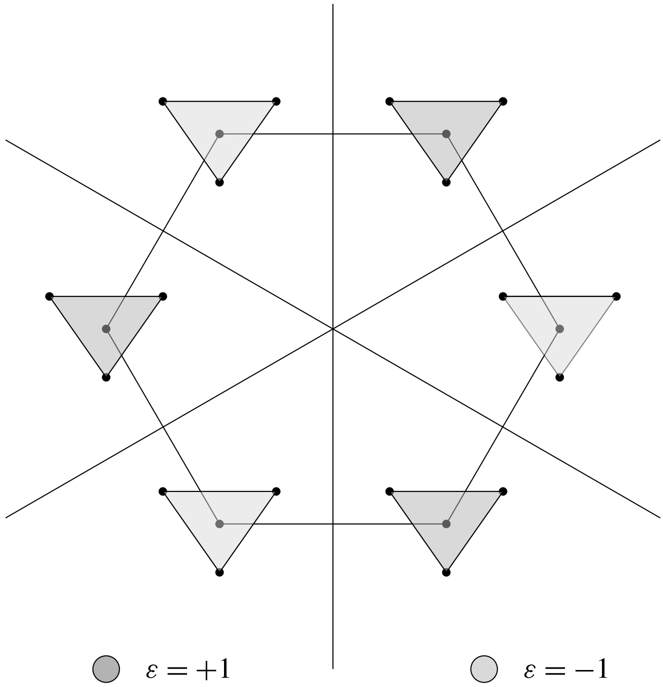

Figure 4. $n = 3$. *Support* $S(\alpha, \beta)$ *of* $\nu_{\alpha,\beta}$ *with* $\alpha = (3, 0, -3)$ *and* $\beta = (3, 0, 0) \sim (2, -1, -1)$. The support is the union of the six triangles.

The measure $\nu_{\alpha,\beta}$ does not change essentially if one replaces $\alpha = (\alpha_1, \dots, \alpha_n)$ by $(\alpha_1 + c, \dots, \alpha_n + c)$ and $\beta = (\beta_1, \dots, \beta_n)$ by $(\beta_1 + d, \dots, \beta_n + d)$ $(c, d \in \mathbb{R})$. We will write $(\alpha_1 + c, \dots, \alpha_n + c) \sim (\alpha_1, \dots, \alpha_n)$. Hence in this section we have considered the case where B has an eigenvalue of multiplicity $n - 1$ rather than having rank one.

In general there are cancellations which should correspond to conditions (II).

6. Real symmetric matrices, $n = 2$

In the case of real symmetric matrices, we know explicitly Heckman's measure and the measure $\nu_{\alpha,\beta}$ only in case of $n = 2$. For $\alpha = (\alpha_1, \alpha_2)$, the orbit $\mathcal{O}_\alpha$ is the set of the matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \alpha_1 \cos^2\theta + \alpha_2 \sin^2\theta & (\alpha_1 - \alpha_2)\cos\theta\sin\theta \\ (\alpha_1 - \alpha_2)\cos\theta\sin\theta & \alpha_1 \sin^2\theta + \alpha_2 \cos^2\theta \end{pmatrix}.$$

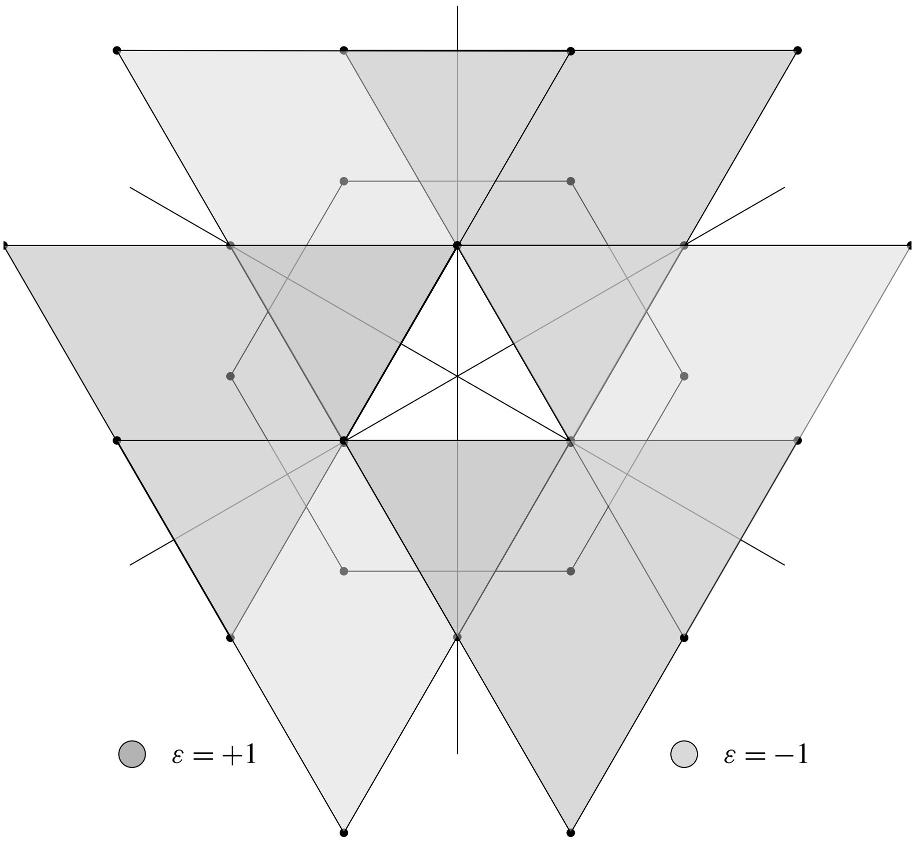

Figure 5. $n = 3$. *Support* $S(\alpha, \beta)$ *of* $\nu_{\alpha,\beta}$ *with* $\alpha = (3, 0, -3)$ *and* $\beta = (6, 0, 0) \sim (4, -2, -2)$. The support is the union of the six large gray triangles, minus their six intersections.

As in the case of 2×2 Hermitian matrices, the image of the orbit $\mathcal{O}_\alpha$ under the projection $q : \mathcal{H}_2(\mathbb{R}) \to D_2 \simeq \mathbb{R}^2$ is the segment $[\alpha, \tau(\alpha)]$. The projection M_α of the orbital measure μ_α is given by

$$\begin{aligned}\langle M_\alpha, \varphi\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \varphi(\alpha_1\cos^2\theta + \alpha_2\sin^2\theta, \alpha_1\sin^2\theta + \alpha_2\cos^2\theta)\,d\theta \\ &= \frac{1}{\pi}\int_0^1 \varphi((1-t)\alpha + t\tau(\alpha))\frac{dt}{\sqrt{t(1-t)}}.\end{aligned}$$

Proposition 6.1. *Let* J_0 *be the Bessel function of index* 0*. The Fourier–Laplace transform of the orbital measure* μ_α *is given, if* $Z = \operatorname{diag}(z_1, z_2)$, *by*

$$\mathcal{F}\mu_\alpha(iZ) = \widehat{M_\alpha}(iz) = \int_{\mathbb{R}^2} e^{i(z_1x_1+z_2x_2)} M_\alpha\,(dx) = e^{\frac{1}{2}(z_1+z_2)(\alpha_1+\alpha_2)} J_0\big(\tfrac{1}{2}(z_1-z_2)(\alpha_1-\alpha_2)\big).$$

Proof. By the previous formula

$$\widehat{M_\alpha}(iz) = \frac{1}{\pi}\int_0^1 e^{i(z|(1-t)\alpha+t\tau(\alpha))}\frac{dt}{\sqrt{t(1-t)}}.$$

Put $t = \frac{1}{2}(1-\cos\theta)$. Then

$$1-t = \tfrac{1}{2}(1+\cos\theta),\ dt = \tfrac{1}{2}\sin\theta d\theta,$$

and

$$(z \mid (1-t\alpha+t\tau(\alpha))) = \tfrac{1}{2}(z_1+z_2)(\alpha_1+\alpha_2) + \tfrac{1}{2}(z_1-z_2)(\alpha_1-\alpha_2)\cos\theta.$$

We obtain

$$\widehat{M_\alpha}(iz) = \frac{1}{\pi}e^{i(z_1+z_2)(\alpha_1+\alpha_2)/2}\int_0^\pi e^{i(z_1-z_2)(\alpha_1-\alpha_2)\cos\theta/2}\,d\theta.$$

Recall the following integral formula for the Bessel function J_0:

$$J_0(\zeta) = \frac{1}{\pi}\int_0^\pi e^{i\zeta\cos\theta}d\theta. \qquad \square$$

We introduce the following notation: for $\alpha = (\alpha_1, \alpha_2)$, and $\beta = (\beta_1, \beta_2)$,

$$\tau = \alpha_1+\alpha_2+\beta_1+\beta_2, \quad a = \alpha_1-\alpha_2, \quad b = \beta_1-\beta_2.$$

If a, b, c are the three wedges of a triangle, we denote by $\Delta(a, b, c)$ the area of this triangle. Recall the classical formula

$$\Delta(a, b, c)^2 = p(p-a)(p-b)(p-c),$$

where p is half the perimeter of the triangle.

Theorem 6.2. *The measure $\nu_{\alpha,\beta}$ is given by*

$$\langle \nu_{\alpha,\beta}, \varphi\rangle = \frac{1}{8\pi}\int_{|a-b|}^{a+b}\varphi\big(\tfrac{1}{2}(\tau+r)e_1+\tfrac{1}{2}(\tau-r)e_2\big)\frac{2r\,dr}{\Delta(a,b,r)} + \frac{1}{8\pi}\int_{|a-b|}^{a+b}\varphi\big(\tfrac{1}{2}(\tau-r)e_1+\tfrac{1}{2}(\tau+r)e_2\big)\frac{2r\,dr}{\Delta(a,b,r)}.$$

Proof. Recall the product formula for the Bessel function J_0:

$$J_0(\zeta a)J_0(\zeta b) = \frac{1}{\pi}\int_0^\pi J_0(\zeta\sqrt{a^2+b^2+2ab\cos\theta})\,d\theta.$$

This can be written

$$J_0(\zeta a)J_0(\zeta b) = \frac{1}{\pi}\int_{|a-b|}^{a+b} J_0(\zeta r)\frac{2r\,dr}{\sqrt{(2ab)^2-(a^2+b^2-r^2)^2}}.$$

Since

$$(2ab)^2-(a^2+b^2-r^2)^2=(a+b+r)(a+b-r)(r+a-b)(r-a+b)=16\Delta(a,b,r)^2,$$

it can also be written

$$J_0(\zeta a)J_0(\zeta b)=\frac{1}{2\pi}\int_{|a-b|}^{a+b}J_0(\zeta r)\frac{r\,dr}{\Delta(a,b,r)}.$$

It follows that the function $\mathcal{E}_2(z,\alpha)$ satisfies the following product formula

$$\mathcal{E}_2(z,\alpha)\mathcal{E}_2(z,\beta)=\frac{1}{2\pi}\int_{|a-b|}^{a+b}\mathcal{E}_2(z,\rho)\frac{r\,dr}{\Delta(a,b,r)},$$

with $\rho=(\rho_1,\rho_2)$, $r=\rho_1-\rho_2$. By Proposition 2.1, this establishes Theorem 6.2. □

Remarks

In the case of the space of real symmetric matrices $\mathcal{H}_n(\mathbb{R})$, with the action of the orthogonal group $O(n)$, for $n\geq 3$, we don't know any explicit formula for Heckman's measure, and for the measures $\nu_{\alpha,\beta}$. This setting is natural, however the problem is more difficult than in the case of the space of Hermitian matrices, and one should not expect any explicit formula. See the recent paper [Coquereaux and Zuber 2018]. However the supports should be the same as in the case of $\mathcal{H}_n(\mathbb{C})$ with the action of the unitary group $U(n)$, according to [Fulton 1998, p.265; 2000, Section 10.7].

There should be an analogue of the results presented in this paper in case of pseudo-Hermitian matrices. In this setting an analogue of Horn's conjecture has been established in [Foth 2010]. An analogue of Theorem 4.1 could probably be obtained by using a formula for the Laplace transform of an orbital measure for the action of the pseudounitary group $U(p,q)$ on the space $\mathcal{H}_n(\mathbb{C}^n)$ $(n=p+q)$. This formula is due Ben Saïd and Ørsted [2005]. A related problem has been studied by using this formula in [Faraut 2017].

More generally one could consider Horn's problem for the adjoint action of a compact Lie group on its Lie algebra. The Fourier transform of an orbital measure is explicitly given by the Harish-Chandra integral formula [1957]. Heckman's paper [1982] is written in this framework. One can expect that there is an analogue of Theorem 4.1 in this setting. In particular one can consider the action of the orthogonal group on the space of real skew-symmetric matrices. See [Zuber 2018] and, for a different problem, [Zubov 2016].

One observes some similarity between the results in [Frumkin and Goldberger 2006], stated in Theorem 5.2, and the classical Cauchy interlacing properties together with Baryshnikov's formula. See [Baryshnikov 2001; Olshanski 2013; Faraut 2015]. There should be an explanation.

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