Tunisian Journal of Mathematics
an international publication organized by the Tunisian Mathematical Society

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2019 vol. 1 no. 4

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TUNISIAN JOURNAL OF MATHEMATICS
Vol. 1, No. 4, 2019
dx.doi.org/10.2140/tunis.2019.1.455

Grothendieck–Messing deformation theory for varieties of K3 type

Andreas Langer and Thomas Zink

Let R be an artinian local ring with perfect residue class field k. We associate to certain 2-displays over the small ring of Witt vectors $\widehat{W}(R)$ a crystal on $\operatorname{Spec} R$.

Let X be a scheme of K3 type over $\operatorname{Spec} R$. We define a perfect bilinear form on the second crystalline cohomology group X which generalizes the Beauville–Bogomolov form for hyper-Kähler varieties over $\mathbb{C}$. We use this form to prove a lifting criterion of Grothendieck–Messing type for schemes of K3 type. The crystalline cohomology $H^2_{\text{crys}}(X/\widehat{W}(R))$ is endowed with the structure of a 2-display such that the Beauville–Bogomolov form becomes a bilinear form in the sense of displays. If X is ordinary, the infinitesimal deformations of X correspond bijectively to infinitesimal deformations of the 2-display of X with its Beauville–Bogomolov form. For ordinary K3 surfaces X/R we prove that the slope spectral sequence of the de Rham–Witt complex degenerates and that $H^2_{\text{crys}}(X/W(R))$ has a canonical Hodge–Witt decomposition.

Introduction

Displays were introduced in [Zink 2002] to classify formal p-divisible groups over a ring R where p is nilpotent. They form a subcategory of the exact tensor category of higher displays constructed in [Langer and Zink 2007]. Such displays arise naturally for a certain class of projective smooth schemes over R (abelian

MSC2010: 14F30, 14F40.
Keywords: crystalline cohomology, displays, Dieudonné 2-displays, F-ordinary schemes.

schemes, K3 surfaces, complete intersections) and equip the crystalline cohomology with an additional structure, in particular the existence of divided Frobenius homomorphisms which satisfy a relative version of Fontaine's strong divisibility condition.

Let p be a prime number such that $p \geq 3$. Let R be an artinian local ring with perfect residue field k of characteristic p. We denote by $\widehat{W}(R)$ the small Witt ring [Zink 2001a]. Displays over the small Witt ring are called Dieudonné displays. They classify all p-divisible groups over R [Zink 2001a; Lau 2014]. In particular a Dieudonné display defines a crystal of locally free modules on the site $(\operatorname{Spec} R/W(k))_{\text{crys}}$. This crystal has an elementary description in terms of linear algebra. Moreover, there is a Grothendieck–Messing criterion for lifting Dieudonné displays.

In [Langer and Zink 2007] we associated to a projective variety X/R whose cohomology has good base change properties a display of higher degree over $W(R)$. We define in this paper under more restrictive conditions on X listed at the beginning of Section 2 a Dieudonné 2-display associated to X. This can be regarded as an additional structure on the crystalline cohomology $H^2_{\text{crys}}(X/\widehat{W}(R))$ (Proposition 19). Let $R' \to R$ be a pd-thickening in the category of local artinian rings with residue field k. This means that $R' \to R$ is a surjective ring homomorphism and that its kernel is endowed with divided powers which are compatible with the canonical divided powers on the ideal pR'. We define the notion of a relative Dieudonné 2-display with respect to such a pd-thickening. We obtain a crystal of relative Dieudonné 2-displays which may be regarded as an additional structure on the crystal

$$R' \mapsto H^2_{\text{crys}}(X/\widehat{W}(R')). \tag{1}$$

In Section 3 we define schemes of K3 type. The main examples are the Hilbert schemes of zero-dimensional subschemes of K3 surfaces denoted by K3n in the literature. We introduce for a scheme of K3 type $X \to T$ a Beauville–Bogomolov form (Definition 23) on the de Rham cohomology $H^2_{\text{DR}}(X/T)$. It coincides with the usual Beauville–Bogomolov form if $T = \operatorname{Spec} \mathbb{C}$. We prove under mild conditions that this form is horizontal for the Gauss–Manin connection (Proposition 26). In the notation above we obtain for a scheme X of K3 type over the artinian ring R a perfect pairing on the crystalline cohomology

$$H^2_{\text{crys}}(X/\widehat{W}(R')).$$

In analogy to the Grothendieck–Messing lifting theory we have Theorem 31:

Theorem. *The liftings of X to R' correspond to selfdual liftings of the Hodge filtration.*

This is proved in the case $R = k$ and $R' = W_n(k)$ for K3 surfaces in [Deligne 1981b]. In this case the Beauville–Bogomolov form coincides with the cup product. The Beauville–Bogomolov form makes the crystal of Dieudonné 2-displays (1) selfdual.

Let X_0/k be a scheme of K3 type such that the Frobenius induces a Frobenius linear bijection on the k-vector space $H^2(X_0, \mathcal{O}_{X_0})$. We say that X_0 is F-ordinary. Let $f : X \to \operatorname{Spec} R$ be a deformation of X_0. We prove that there is a unique functorial extension of the Dieudonné 2-display $H^2_{\mathrm{crys}}(X/\widehat{W}(R))$ to a crystal of relative Dieudonné 2-displays (1). In particular the crystal $R^2 f_{\mathrm{crys},*}\mathcal{O}_X^{\mathrm{crys}}$ in $(X/W(k))_{\mathrm{crys}}$ [Berthelot 1974, Chapitre 5, Proposition 3.6.4] can be constructed from this Dieudonné 2-display. Then we obtain from the Grothendieck–Messing criterion Theorem 36:

Theorem. *Assume that X_0 is F-ordinary and lifts to a smooth projective scheme over $W(k)$. The functor which associates to a deformation X/R of X_0 the Dieudonné 2-display $H^2_{\mathrm{crys}}(X/\widehat{W}(R))$ with its Beauville–Bogomolov form is an equivalence to the category of selfdual deformations of the Dieudonné 2-display $H^2_{\mathrm{crys}}(X_0/W(k))$ endowed with the Beauville–Bogomolov form.*

In Section 5 we exhibit the second crystalline cohomology of an ordinary K3 surface X over the usual Witt ring $W(R)$ and its associated display and prove (Theorem 40) a Hodge–Witt decomposition which induces a decomposition of the display into a direct sum of displays attached to the formal Brauer group $\widehat{\mathrm{Br}}_X$, the étale part of the extended Brauer group and the Cartier dual of $\widehat{\mathrm{Br}}_X$, shifted by -1. The proof uses the relative de Rham–Witt complex of [Langer and Zink 2007]. We show that the hypercohomology spectral sequence of the relative de Rham–Witt complex degenerates.

1. Displays

We fix a prime number p.

Definition 1. A frame $\mathcal{F}$ consists of the following data $\mathcal{F} = (W, J, R, \sigma, \dot{\sigma})$. Here W is a commutative ring, $J \subset W$ is an ideal, and $R = W/J$ is the factor ring. The map $\sigma : W \to W$ is a ring homomorphism and $\dot{\sigma} : J \to W$ is a σ-linear homomorphism of W-modules. We assume that the following conditions are satisfied:

(i) The ideal J and the prime number p are contained in the Jacobson radical of W.

(ii) For each $s \in W$ we have

$$\sigma(s) \equiv s^p \mod pW.$$

(iii) The set $\dot{\sigma}(J)$ generates W as an W-module.

There is a unique element $\theta \in W$ such that $\sigma(\eta) = \theta\dot{\sigma}(\eta)$ for all $\eta \in J$. We will assume that $\theta = p$. In the following the ring W is local.

Suppose $f : M \to N$ is a σ-linear map of W-modules. Then we define a new σ-linear map

$$\tilde{f} : J \otimes_S M \to N, \quad \tilde{f}(\eta \otimes m) = \dot{\sigma}(\eta) f(m) \quad \text{for } \eta \in J.$$

Definition 2. An $\mathcal{F}$-predisplay $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ consists of the following data:

(1) A sequence of W-modules P_i for $i \geq 0$.

(2) Two sequences of W-module homomorphisms

$$\iota_i : P_{i+1} \to P_i, \quad \alpha_i : J \otimes_S P_i \to P_{i+1}, \quad \text{for } i \geq 0.$$

(3) A sequence of σ-linear maps for $i \geq 0$

$$F_i : P_i \to P_0.$$

These data satisfy the following properties:

(i) Consider the following morphisms:

$$\begin{array}{ccc} J \otimes P_i & \xrightarrow{\alpha_i} & P_{i+1} \\ {\scriptstyle \mathrm{id}_J \otimes \iota_{i-1}} \downarrow & & \downarrow {\scriptstyle \iota_i} \\ J \otimes P_{i-1} & \xrightarrow[\alpha_{i-1}]{} & P_i \end{array}$$

The composites $\iota_i \circ \alpha_i$ and $\alpha_{i-1} \circ (\mathrm{id}_J \otimes \iota_{i-1})$ are the multiplication $J \otimes P_i \to P_i$ for each i where the composites make sense.

(ii) $F_{i+1} \circ \alpha_i = \widetilde{F}_i$.

If we have only the data $\mathcal{P} = (P_i, \iota_i, \alpha_i)$ such that property (i) holds we say that $\mathcal{P}$ is an $\mathcal{F}$-module.

We will denote the morphisms in the category of $\mathcal{F}$-predisplays and in the category of $\mathcal{F}$-modules by

$$\mathrm{Hom}_{\mathcal{F}}(\mathcal{P}, \mathcal{P}') \quad \text{and} \quad \mathrm{Hom}_{\mathcal{F}\text{-mod}}(\mathcal{P}, \mathcal{P}')$$

respectively.

This is a generalization of Definition 2.1 of [Langer and Zink 2007]. The arguments there imply

$$F_i(\iota_i(x)) = pF_{i+1}(x) \quad \text{for } x \in P_{i+1}.$$

If $i, k \geq 0$ we will denote the map

$$\iota_{i+k-1} \circ \cdots \circ \iota_i : P_{i+k} \to P_i$$

simply by ι^{iter}.

We are going to associate a frame to the following situation. Let R and S be p-adic rings. Let

$$S \to R \tag{2}$$

be a surjective ring homomorphism such that the kernel $\mathfrak{a}$ is endowed with divided powers. We will assume that $\mathfrak{a}$ becomes nilpotent in the ring S/pS.

Let $W(S) \xrightarrow{\boldsymbol{w}_0} S \to R$ be the composite with the Witt polynomial $\boldsymbol{w}_0$. Let $\mathcal{J}$ be the kernel of this composite. We set $I_S = VW(S)$ and we denote by $\tilde{\mathfrak{a}} \subset W(S)$ the logarithmic Teichmüller representatives of elements of $\mathfrak{a}$. These are the elements $\tilde{a} := \log^{-1}[a, 0, 0, \ldots]$ in the notation of [Zink 2002, (48)]. Then we have a direct decomposition of $\mathcal{J}$ as a sum of two ideals of $W(S)$:

$$\mathcal{J} = \tilde{\mathfrak{a}} \oplus I_S.$$

We will denote the Frobenius endomorphism F of the ring of Witt vectors $W(S)$ also by σ. We have

$$\sigma(\tilde{\mathfrak{a}}) = 0, \quad I_S \cdot \tilde{\mathfrak{a}} = 0.$$

We define a map

$$\dot{\sigma} : \mathcal{J} \to W(S), \quad \dot{\sigma}(a + {}^V\xi) = \xi, \quad a \in \tilde{\mathfrak{a}},\ \xi \in W(S).$$

This map is σ-linear. We note that the ideal $\mathcal{J}$ inherits from $\mathfrak{a}$ divided powers which extend the natural divided powers on $I_S \subset W(S)$ [Zink 2002, (89)].

Let us assume that the divided powers on $\mathfrak{a}$ are compatible with the canonical divided powers on pS. This implies the canonical divided powers on $pW(S) + VW(S)$ are compatible with the divided powers on $\tilde{\mathfrak{a}}$ given by the isomorphism with $\mathfrak{a}$. In this sense $W(S) \to W(R/pR)$ is then a pd-thickening.

We call $\mathcal{W}_{S/R} = (W(S), \mathcal{J}, R, \sigma, \dot{\sigma})$ the relative Witt frame. If $S = R$ we call it the Witt frame and write $\mathcal{W}_R$. If $S \to R$ is fixed as above we call a $\mathcal{W}_R$-predisplay simply a predisplay and a $\mathcal{W}_{S/R}$-predisplay a relative predisplay.

Suppose S and R are artinian local rings with perfect residue field of characteristic $p \geq 3$. Let $S \to R$ be a surjective homomorphism with kernel $\mathfrak{a}$. We assume that $\mathfrak{a}$ is endowed with nilpotent divided powers. We call this a nilpotent pd-thickening. In this situation we can also use the small rings of Witt vectors $\widehat{W}(R)$ and $\widehat{W}(S)$ defined in [Zink 2001a] to define a version of the relative Witt frame. For this we use that the divided Witt polynomial defines an isomorphism

$$\widehat{W}(\mathfrak{a}) \to \bigoplus_{i \geq 0} \mathfrak{a}$$

by [Zink 2002, Remark after Corollary 82]. By this isomorphism the logarithmic Teichmüller elements are defined. Then we obtain the small relative Witt frame $\widehat{\mathcal{W}}_{S/R} = (\widehat{W}(S), \widehat{\mathcal{J}}, R, \sigma, \dot{\sigma})$, where $\widehat{\mathcal{J}} = \tilde{\mathfrak{a}} \oplus V\widehat{W}(S)$ is the kernel of the homomorphism $\widehat{W}(S) \to R$.

These frames are endowed with a Verjüngung: Let $\mathcal{F} = (W, J, R, \sigma, \dot{\sigma})$ be a frame. The structure of a Verjüngung on $\mathcal{F}$ consists of two W-module homomorphisms

$$\nu : J \otimes_W J \to J, \quad \pi : J \to J$$

such that ν is associative. We will also write

$$\nu(y_1 \otimes y_2) = y_1 * y_2, \quad y_1, y_2 \in J.$$

The iteration of ν is well-defined:

$$\nu^{(k)} : J \otimes_W \cdots \otimes_W J \to J,$$

where the tensor product on the left-hand side has k factors. We have $\nu^{(2)} = \nu$ and $\nu^{(1)} = \mathrm{id}_J$. The image of $\nu^{(k)}$ is an ideal $J_k \subset W$. By associativity, $\nu^{(k+1)}$ factors through a map

$$J \otimes_W J_k \to J_{k+1}. \tag{3}$$

We also require, that the following properties hold:

$$\begin{aligned} \pi(y_1 * y_2) &= y_1 y_2, \quad \text{where } y_1, y_2 \in J, \\ \dot{\sigma}(y_1 * y_2) &= \dot{\sigma}(y_1)\dot{\sigma}(y_2), \\ \dot{\sigma}(\pi(y_1)) &= \sigma(y_1), \\ (\operatorname{Ker} \dot{\sigma}) \cap (\operatorname{Ker} \pi) &= 0. \end{aligned} \tag{4}$$

These properties imply

$$y_1 * y_2 = y_2 * y_1, \quad \pi(y_1) * y_2 = y_1 y_2.$$

Indeed, for each of these equations the difference between the two sides lies in $(\operatorname{Ker} \dot{\sigma}) \cap (\operatorname{Ker} \pi)$.

In the case of the frame $\mathcal{W}_{S/R}$ we define the Verjüngung as

$$(a_1 + {}^V\xi_1) * (a_2 + {}^V\xi_2) = a_1 a_2 + {}^V(\xi_1 \xi_2), \quad \pi(a + {}^V\xi) = a + p\,{}^V\xi. \tag{5}$$

Then we have

$$J_i = \tilde{\mathfrak{a}}^i + VW(S).$$

The map (3) is given by the first formula of (5).

In the same way we obtain a Verjüngung for the frames $\widehat{\mathcal{W}}_{S/R}$. These are the only examples we are interested in.

We define the notion of a standard display over a frame $\mathcal{F}$ with Verjüngung ν, π. In the case of $\mathcal{W}_R$ it coincides with the notion given in [Langer and Zink 2007].

A standard datum consists of a sequence of finitely generated projective W-modules $L_0, \ldots, L_d$ and σ-linear homomorphisms

$$\Phi_i : L_i \to L_0 \oplus \cdots \oplus L_d.$$

We assume that

$$\Phi_0 \oplus \cdots \oplus \Phi_d : L_0 \oplus \cdots \oplus L_d \to L_0 \oplus \cdots \oplus L_d$$

is a σ-linear isomorphism.

We set

$$P_i = J_i L_0 \oplus J_{i-1} \cdots \oplus J L_{i-1} \oplus L_i \oplus \cdots \oplus L_d.$$

The map ι is defined by the following diagram:

$$\begin{array}{ccccccccccc}
J_{i+1}L_0 & \oplus & J_i L_1 & \oplus\cdots\oplus & J L_i & \oplus & L_{i+1} & \oplus\cdots\oplus & L_d \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & \cdots & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & \cdots & \downarrow{\scriptstyle \mathrm{id}} \\
J_i L_0 & \oplus & J_{i-1}L_1 & \oplus\cdots\oplus & L_i & \oplus & L_{i+1} & \oplus\cdots\oplus & L_d
\end{array}$$

We remark that $\pi(J_{i+1}) \subset J_i$ because of the formula

$$\pi(y_1 * y_2 * \cdots * y_{i+1}) = y_1(y_2 * \cdots * y_{i+1}).$$

The homomorphism $\alpha_i : J \otimes P_i \to P_{i+1}$ is defined as follows:

$$\begin{array}{ccccccccccc}
J\otimes J_i L_0 & \oplus & J\otimes J_{i-1}L_1 & \oplus\cdots\oplus & J\otimes L_i & \oplus & J\otimes L_{i+1} & \oplus\cdots\oplus & J\otimes L_d \\
\downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} & \cdots & \downarrow{\scriptstyle \mathrm{mult}} & & \downarrow{\scriptstyle \mathrm{mult}} & \cdots & \downarrow{\scriptstyle \mathrm{mult}} \\
J_{i+1}L_0 & \oplus & J_i L_1 & \oplus\cdots\oplus & J L_i & \oplus & L_{i+1} & \oplus\cdots\oplus & L_d.
\end{array}$$

Here the arrows denoted by ν are induced by the maps (3), and mult denotes the multiplication.

Finally we define σ-linear maps $F_i : P_i \to P_0$:

$$\begin{array}{cccccccccc}
J_i L_0 & \oplus\cdots\oplus & J L_{i-1} & \oplus & L_i & \oplus & L_{i+1} & \oplus & L_{i+2} & \cdots \\
\downarrow{\scriptstyle \widetilde{\Phi}_0} & \cdots & \downarrow{\scriptstyle \widetilde{\Phi}_{i-1}} & & \downarrow{\scriptstyle \Phi_i} & & \downarrow{\scriptstyle p\Phi_{i+1}} & & \downarrow{\scriptstyle p^2\Phi} & \cdots \\
L_0 & \oplus\cdots\oplus & L_{i-1} & \oplus & L_i & \oplus & L_{i+1} & \oplus & L_{i+2} & \cdots
\end{array}$$

The maps $\widetilde{\Phi}_j$ are by definition

$$\widetilde{\Phi}_j(\eta\ell_j) = \dot{\sigma}(\eta)\Phi_j(\ell_j) \quad \text{for } \eta \in J_j,\ \ell_j \in L_j,\ j < i.$$

The data $(P_i, \iota_i, \alpha_i, F_i)$ meet the requirements for a predisplay. This predisplay is called the $\mathcal{F}$-display of a standard datum.

Definition 3. Let $\mathcal{F} = (W, J, R, \sigma, \dot{\sigma}, \nu, \pi)$ be a frame with Verjüngung. An $\mathcal{F}$-display $\mathcal{P}$ is an $\mathcal{F}$-predisplay which is isomorphic to the display of a standard datum. The choice of such an isomorphism is called a normal decomposition of $\mathcal{P}$.

We call an $\mathcal{F}$-predisplay $\mathcal{P}$ separated if the commutative diagram

$$\begin{tikzcd} P_i \arrow[r, "F_i"] & P_0 \\ P_{i+1} \arrow[u, "\iota_i"] \arrow[r, "F_{i+1}"'] & P_0 \arrow[u, "p"'] \end{tikzcd}$$

induces an injective map from P_{i+1} to the fibre product $P_i \times_{F_i, P_0, p} P_0$.

One checks easily that an $\mathcal{F}$-display is separated and one proves immediately:

Proposition 4. *Let $\mathcal{F}$ be a frame with Verjüngung. Let $\mathcal{P}$ be a separated $\mathcal{F}$-predisplay. Let $\mathcal{P}'$ be an $\mathcal{F}$-predisplay. Then the natural map*

$$\mathrm{Hom}_{\mathcal{F}\text{-dsp}}(\mathcal{P}', \mathcal{P}) \to \mathrm{Hom}_{W\text{-mod}}(P_0', P_0)$$

from the Hom-*group of homomorphisms of predisplays to the* Hom-*group of homomorphisms of W-modules is injective.*

An $\mathcal{F}$-display $\mathcal{P}$ is a separated $\mathcal{F}$-predisplay.

Let $\mathcal{P}$ be an $\mathcal{F}$-predisplay. Iterating the homomorphisms α_i in Definition 2 we obtain W-module homomorphisms for $i, k \geq 0$:

$$\alpha_i^{(k)} : J \otimes_W J \cdots \otimes_W J \otimes_W P_i \to P_{i+k}. \tag{6}$$

By definition we have $\alpha_i^{(0)} = \mathrm{id}_{P_i}$ and $\alpha_i^{(1)} = \alpha_i$. We say that $\mathcal{P}$ satisfies the condition **alpha** if the map (6) factors through a homomorphism $\bar{\alpha}_i^{(k)}$:

$$\textbf{alpha}: \quad J \otimes_W \cdots \otimes_W J \otimes_W P_i \xrightarrow{\nu^{(k)} \otimes \mathrm{id}} J_k \otimes_W P_i \xrightarrow{\bar{\alpha}_i^{(k)}} P_{i+k}. \tag{7}$$

Obviously $\bar{\alpha}_i^{(k)}$ is uniquely determined. A display satisfies the condition **alpha**.

Proposition 5. *Let $\mathcal{Q}$ be the display associated to a standard datum (L_i, Φ_i), $i = 0, \ldots, d$. Let $\mathcal{P}$ be a predisplay which satisfies the condition* **alpha**.

Let $\rho : \mathcal{Q} \to \mathcal{P}$ be a morphism of $\mathcal{F}$-modules. We denote for $i \leq d$ the restriction of $\rho_i : Q_i \to P_i$ to $L_i \subset Q_i$ by $\rho_{|i} : L_i \to P_i$.

With this notation let $\rho_{|i} : L_i \to P_i$ for $i = 0, \ldots, d$ be arbitrary W-module homomorphisms $\rho_{|i} : L_i \to P_i$ for $i = 0, \ldots, d$. Then there exists a unique $\mathcal{F}$-module homomorphism $\rho : \mathcal{Q} \to \mathcal{P}$ which induces the given $\rho_{|i}$.

Moreover the morphism of $\mathcal{F}$-modules ρ defined by a sequence of homomorphisms $\rho_{|i} : L_i \to P_i$ is a morphism of predisplays if and only if the following diagrams are commutative:

$$\begin{tikzcd} L_i \arrow[r, "\Phi_i"] \arrow[d, "\rho_{|i}"'] & Q_0 \arrow[d, "\rho_0"] \\ P_i \arrow[r, "F_i"'] & P_0 \end{tikzcd} \tag{8}$$

We remark that the morphism $\rho_0 : Q_0 = \bigoplus_{i=1}^{d} L_i \to P_0$ is given on the summand L_i as the composite $L_i \subset Q_i \to P_i \xrightarrow{\iota^{\mathrm{iter}}} P_0$, where the composition of the first two arrows is $\rho_{|i}$ and the last arrow is the composition $\iota^{\mathrm{iter}} = \iota_0 \circ \cdots \circ \iota_{i-1}$.

Proof. We have

$$Q_i = J_i L_0 \oplus \cdots \oplus J_{i-k} L_k \oplus \cdots \oplus J L_{i-1} \oplus L_i \oplus \cdots .$$

We will define $\rho_i : Q_i \to P_i$. We do this for each of the summands above separately. For $k < i$ we obtain by tensoring $\rho_{|k}$ with J_{i-k} a homomorphism

$$J_{i-k} L_k \to J_{i-k} \otimes_W P_k.$$

Composing the last arrow with $\bar{\alpha}_k^{i-k}$ from the condition **alpha** we obtain ρ_i on the summand $J_{i-k} L_k$.

For $j \geq i$ the map $\iota^{\mathrm{iter}} : Q_j \to Q_i$ induces the identity on L_j. Therefore we define the restriction of ρ_i to the summand L_j as the composite

$$L_j \xrightarrow{\rho_{|j}} P_j \xrightarrow{\iota^{\mathrm{iter}}} P_i.$$

One checks that the ρ_i define a morphism of $\mathcal{F}$-modules and, if the diagrams (8) commute, a morphism of $\mathcal{F}$-predisplays. □

We will now define the base change of displays. We consider a morphism of frames with Verjüngung $u : \mathcal{F} \to \mathcal{F}'$. Let $\mathcal{P}'$ be an $\mathcal{F}'$-predisplay. This may be regarded as an $\mathcal{F}$-predisplay with the same P_i' but regarded as W-modules. Only the maps α_i need a definition:

$$\alpha_i : J \otimes_W P_i' \to J' \otimes_{W'} P_i' \xrightarrow{\alpha_i'} P_{i+1}'.$$

We denote the $\mathcal{F}$-predisplay obtained in this way by $u^\bullet \mathcal{P}'$. Let $\mathcal{P}$ be an $\mathcal{F}$-display. We say that an $\mathcal{F}'$-display $u_\bullet \mathcal{P}$ is a base change of $\mathcal{P}$ if there exists for each $\mathcal{F}'$-display $\mathcal{P}'$ a bijection

$$\mathrm{Hom}_{\mathcal{F}'}(u_\bullet \mathcal{P}, \mathcal{P}') \cong \mathrm{Hom}_{\mathcal{F}}(\mathcal{P}, u^\bullet \mathcal{P}')$$

which is functorial in $\mathcal{P}'$.

Proposition 6 (base change). *Let $u : \mathcal{F} \to \mathcal{F}'$ be a morphism of frames with Verjüngung. Then the base change of an $\mathcal{F}$-display $\mathcal{P}$ exists. Moreover for $\mathcal{F}'$-predisplays $\mathcal{P}'$ which satisfy the condition* **alpha** *we have a functorial bijection*

$$\mathrm{Hom}_{\mathcal{F}'}(u_\bullet \mathcal{P}, \mathcal{P}') \cong \mathrm{Hom}_{\mathcal{F}}(\mathcal{P}, u^\bullet \mathcal{P}').$$

Proof. We choose a normal decomposition (L_i, Φ_i) of $\mathcal{P}$. Then a morphism $\rho : \mathcal{P} \to u^\bullet \mathcal{P}'$ is given by a set of W-module homomorphisms

$$\rho_{|i} : L_i \to P_i'$$

such that the analogues of (8) are commutative. From the standard datum (L_i, Φ_i) we obtain a standard datum $(L'_i = W' \otimes_W L_i, \Phi'_i = \sigma' \otimes \Phi_i)$ for the frame $\mathcal{F}'$ which defines an $\mathcal{F}'$-display $\mathcal{Q}$. From $\rho_{|i}$ we obtain W'-module homomorphisms

$$\rho'_{|i} : W' \otimes_W L_i \to P'_i.$$

By Proposition 5, these homomorphisms define a morphism of $\mathcal{F}'$-predisplays $\mathcal{Q} \to \mathcal{P}'$. This shows that $u_\bullet \mathcal{P} := \mathcal{Q}$ is a base change and has the claimed property. $\square$

We apply the base change to the following obvious morphisms of frames with Verjüngung:

$$\mathcal{W}_S \to \mathcal{W}_{S/R} \to \mathcal{W}_R.$$

More generally let

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow & & \downarrow \\ R & \longrightarrow & R' \end{array}$$

be a morphism of pd-extension of the type (2). We obtain a morphism of frames with Verjüngung $\mathcal{W}_{S/R} \to \mathcal{W}_{S'/R'}$. We have this for small Witt frames too.

We will give now an intrinsic characterization of a display which doesn't use a normal decomposition. Let $\mathcal{F}$ be a frame with Verjüngung. Let $\mathcal{P}$ be an $\mathcal{F}$-predisplay. Then we denote the image of the homomorphism

$$P_i \xrightarrow{\iota^{\text{iter}}} P_0 \to P_0/JP_0 \tag{9}$$

by E^i or more precisely by $\mathrm{Fil}^i_{\mathcal{P}}$. This is called the Hodge filtration on the R-module P_0/JP_0:

$$\cdots E^{i+1} \to E^i \to \cdots \to E^0 = P_0/JP_0. \tag{10}$$

If $\mathcal{P}$ is a display, this is a filtration by direct summands.

Proposition 7. *Let $\mathcal{F}$ be a frame with Verjüngung. We assume that each finitely generated projective R-module may be lifted to a finitely generated projective W-module. Let $\mathcal{P}$ be an $\mathcal{F}$-predisplay with Hodge filtration E^i such that the following properties hold*:

(1) *$\mathcal{P}$ is separated and satisfies the condition* **alpha**.

(2) *P_0 is a finitely generated projective W-module.*

(3) *The Hodge filtration consists of direct summands $E^i \subset P_0/\mathcal{J}P_0$.*

(4) *There is an exact sequence*

$$\mathcal{J} \otimes P_i \xrightarrow{\alpha_i} P_{i+1} \to E^{i+1} \to 0.$$

(5) *The subgroups F_iP_i for $i \geq 0$ generate the W-module P_0.*

Then $\mathcal{P}$ is an $\mathcal{F}$-display.

We omit the proof. We note that the assumption that liftings of finitely generated projective modules exist is trivial if R is a local ring.

Let $\mathcal{F} = \mathcal{W}_{S/R}$ or $\mathcal{F} = \widehat{\mathcal{W}}_{S/R}$. Let $\mathcal{P}$ be an $\mathcal{F}$-display. A lifting of the Hodge filtration of $\mathcal{P}$ is a sequence of split injections of projective finitely generated S-modules

$$\cdots \widetilde{E}^{i+1} \to \widetilde{E}^{i} \to \cdots \to \widetilde{E}^{0} = P_0/I_S P_0, \tag{11}$$

which coincides with (10) when tensored with R.

We will now discuss the notion of an extended display. Let $\mathcal{F}$ be a frame with Verjüngung. Let (L_i, Φ_i) be a standard datum. If we replace in the definition of the display associated to this datum all J_i simply by J we obtain an $\mathcal{F}$-predisplay $\widetilde{\mathcal{P}}$. We consider this notion only for the frames $\mathcal{W}_{S/R}$ and $\widehat{\mathcal{W}}_{S/R}$. Let $\widetilde{\mathcal{P}}$ be an extended display and let E^i be its Hodge filtration. Then $\widetilde{\mathcal{P}}$ satisfies all conditions of Proposition 7 except for the condition (4).

We note that there is no difference between displays and extended displays in the case $S = R$ because then $J = J_i$.

Let $\mathcal{Q}$ be a $\mathcal{W}_{S/R}$-predisplay or a $\widehat{\mathcal{W}}_{S/R}$-predisplay. For this discussion we denote by $\bar{Q}_i \subset Q_i$ the intersection of all images of maps

$$Q_{i+k} \xrightarrow{\iota^{\text{iter}}} Q_i.$$

If $\mathcal{Q}$ is a display then $\bar{Q}_i = 0$ for all i because $W(S)$ is a p-adic ring. If $\mathcal{Q}$ is an extended display we have that the map $\iota^{\text{iter}} : Q_i \to Q_0$ induces an isomorphism

$$\iota^{\text{iter}} : \bar{Q}_i \to \tilde{\mathfrak{a}} Q_0. \tag{12}$$

Note that for $k > i$ we have that $\tilde{\mathfrak{a}} L_k \subset (\tilde{\mathfrak{a}} \oplus I_S) L_k \subset Q_i$ is a direct summand of Q_i which is mapped isomorphically to a direct summand of $\bar{Q}_i$ and further by (12) isomorphically to $\tilde{\mathfrak{a}} L_k \subset \tilde{\mathfrak{a}} Q_0$.

We note that an extended display satisfies the condition **alpha**. We have the following version of Proposition 5 which is proved by the same argument.

Proposition 8. *We consider a predisplay for the frame $\mathcal{F} = \mathcal{W}_{S/R}$ or $\mathcal{F} = \widehat{\mathcal{W}}_{S/R}$. Let $\widetilde{\mathcal{Q}}$ be the extended display associated to a standard datum (L_i, Φ_i), $i = 0, \ldots, d$. Let $\mathcal{P}$ be a predisplay which satisfies the condition* **alpha** *and (12).*

Let $\rho : \widetilde{\mathcal{Q}} \to \mathcal{P}$ be a morphism of $\mathcal{F}$-modules. We denote for $i \leq d$ the restriction of $\rho_i : Q_i \to P_i$ to $L_i \subset Q_i$ by $\rho_{|i} : L_i \to P_i$.

Conversely arbitrary $W(S)$-module homomorphisms $\rho_{|i} : L_i \to P_i$ for $i = 0, \ldots, d$ define uniquely a morphism of $\mathcal{F}$-modules $\rho : \widetilde{\mathcal{Q}} \to \mathcal{P}$.

Moreover the morphism of $\mathcal{F}$-modules ρ defined by a sequence of homomorphisms $\rho_{|i} : L_i \to P_i$ is a morphism of predisplays if and only if the following diagrams are commutative:

$$\begin{tikzcd} L_i \arrow[r, "\Phi_i"] \arrow[d, "\rho_{|i}"'] & \widetilde{Q}_0 \arrow[d, "\rho_0"] \\ P_i \arrow[r, "F_i"'] & P_0 \end{tikzcd} \tag{13}$$

Corollary 9. *Let $\mathcal{Q}$ be the display associated to the standard datum (L_i, Φ_i). Then we have a canonical bijection*

$$\mathrm{Hom}_{\mathcal{F}}(\widetilde{\mathcal{Q}}, \mathcal{P}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{F}}(\mathcal{Q}, \mathcal{P}).$$

We conclude that we have a functor $\mathcal{Q} \mapsto \widetilde{\mathcal{Q}}$ from the category of displays to the category of extended displays, because the construction of $\widetilde{\mathcal{Q}}$ does not depend on the normal decomposition.

Let (L_i, Φ_i) be a standard datum for the frame $\mathcal{W}_{S/R}$; let $\mathcal{Q}$ be the associated display and $\widetilde{\mathcal{Q}}$ the extended display.

Let $\widehat{\mathcal{Q}}$ be the $\mathcal{W}_R$-display associated to $(W(R) \otimes_{W(S)} L_i, \sigma \otimes \Phi_i)$. Then $\widehat{\mathcal{Q}}$ is the base change of $\mathcal{Q}$ via $\mathcal{W}_{S/R} \to \mathcal{W}_R$. Since a $\mathcal{W}_R$-display $\mathcal{P}$ regarded as a $\mathcal{W}_{S/R}$-predisplay satisfies the condition **alpha** and (12) we obtain

$$\mathrm{Hom}_{\mathcal{W}_R}(\widehat{\mathcal{Q}}, \mathcal{P}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{W}_{S/R}}(\widetilde{\mathcal{Q}}, \mathcal{P}).$$

This shows that we have also a functor $\widetilde{\mathcal{Q}} \mapsto \widehat{\mathcal{Q}}$. Therefore we have functors

$$(\mathcal{W}_{S/R}\text{-displays}) \to (\mathcal{W}_{S/R}\text{-extended displays}) \to (\mathcal{W}_R\text{-displays})$$

such that the composition of these functors is base change. The same functors exist if the small Witt frame $\widehat{\mathcal{W}}_{S/R}$ is defined.

We have defined what is a lifting of the Hodge filtration for a $\mathcal{W}_{S/R}$-display $\mathcal{P}$. We will now construct the functor

$$\left(\begin{array}{c} \text{extended } \mathcal{W}_{S/R}\text{-displays} \\ \text{and a lift of the Hodge filtration} \end{array} \right) \to (\mathcal{W}_S\text{-displays}). \tag{14}$$

Again the construction will be the same for small Witt frames.

Let $\widetilde{\mathcal{P}}$ be an extended $\mathcal{W}_{S/R}$-display. Let

$$\cdots \widetilde{E}^{i+1} \to \widetilde{E}^i \to \cdots \to \widetilde{E}^0 = \widetilde{P}_0 / I_S \widetilde{P}_0$$

be a lift of the Hodge filtration. We construct in a functorial way a $\mathcal{W}_S$-display $\mathcal{P}$. We denote by $\widehat{E}^i \subset \widetilde{P}_0 / I_S \widetilde{P}_0$ the preimage of the Hodge filtration $E^i \subset \widetilde{P}_0 / J \widetilde{P}_0$. By choosing an arbitrary normal decomposition of $\widetilde{\mathcal{P}}$ we find that the map

$$\widetilde{P}_i \to \widetilde{P}_0 / I_S \widetilde{P}_0$$

has image $\widehat{E}^i$.

We choose a splitting of the lifted Hodge filtration and obtain a decomposition into S-submodules of $\widetilde{P}_0/I_S\widetilde{P}_0$:

$$\widetilde{E}^i = \bar{L}_i \oplus \bar{L}_{i+1} \oplus \cdots \oplus \bar{L}_d.$$

We choose a finitely generated projective $W(S)$-module L_i which lifts the S-module $\bar{L}_i$ and we choose a commutative diagram:

$$\begin{tikzcd} L_i \arrow[d] \arrow[dr] & \\ \widetilde{P}_i \arrow[r] & \widehat{E}_i \end{tikzcd}$$

A composite of the ι maps yields $L_i \to \widetilde{P}_i \to \widetilde{P}_0 = P_0$. We obtain a homomorphism

$$L_0 \oplus L_1 \oplus \cdots \oplus L_m \to P_0.$$

We see that this map is an isomorphism by taking it modulo J.

The maps $F_i : \widetilde{P}_i \to \widetilde{P}_0$ give by restriction maps $\Phi_i : L_i \to P_i$. We will show that the map

$$L_0 \oplus L_1 \oplus \cdots \oplus L_m \to P_0 \tag{15}$$

is a Frobenius linear isomorphism. Then we obtain standard data (L_i, Φ_i) for $\mathcal{P}$.

To show that (15) is an isomorphism we consider the $\mathcal{W}_R$ display $\bar{\mathcal{P}}$ obtained by base change from $\mathcal{P}$. We have natural maps $P_i \to \widetilde{P}_i \to \bar{P}_i$. The images of the L_i in $\bar{P}_i$ give a normal decomposition of those displays. Therefore the map (15) becomes a Frobenius linear isomorphism when tensored with $W(R)$. Then the map itself is a Frobenius linear isomorphism.

Then we define the desired $\mathcal{W}_S$-display $\mathcal{P}$ by the standard datum (L_i, Φ_i). Our construction gives that $P_i \subset \widetilde{P}_i$ is the preimage of $\widetilde{E}^i$ under the map

$$\widetilde{P}_i \to P_0/I_S P_0.$$

This shows that the assignment $\widetilde{\mathcal{P}} \mapsto \mathcal{P}$ is functorial and does not depend on the construction of the normal decomposition chosen above.

Proposition 10. *The functor (14) defines an equivalence of the category of extended $\mathcal{W}_{S/R}$-displays together with a Hodge filtration and the category of $\mathcal{W}_S$-displays. The same holds for the small Witt frames.*

Proof. Indeed there is an obvious inverse functor. We denote by

$$u_\bullet : (\mathcal{W}_S\text{-displays}) \to (\text{extended } \mathcal{W}_{S/R}\text{-displays})$$

the functor induced by base change. An extended $\mathcal{W}_{S/R}$-display $\mathcal{P}$ may be regarded as a $\mathcal{W}_S$-predisplay. Then we denote it by $u^\bullet\mathcal{P}$. By Propositions 5 and 8 we have for a $\mathcal{W}_S$-display $\mathcal{Q}$ a functorial bijection,

$$\operatorname{Hom}_{\mathcal{W}_{S/R}}(u_\bullet \mathcal{Q}, \mathcal{P}) \cong \operatorname{Hom}_{\mathcal{W}_S}(\mathcal{Q}, u^\bullet \mathcal{P}).$$

We set

$$\widehat{\mathcal{Q}} = u_\bullet \mathcal{Q}.$$

The canonical map $\mathcal{Q} \to u^\bullet \widehat{\mathcal{Q}}$ induces natural injections $Q_i \to \widehat{Q}_i$. This provides a lifting of the Hodge filtration of the extended display $\widehat{\mathcal{Q}}$. Clearly this functor is inverse to the functor (14). □

Let $\mathcal{P}$ be a $\mathcal{W}_{S/R}$-display. We say that a lifting of the Hodge filtration $\widetilde{E}^i \subset P_0/I_S P_0$ for $i \geq 0$ is admissible if $\widetilde{E}^i$ is in the image of $P_i \xrightarrow{\iota^{\text{iter}}} P_0/I_S P_0$. If $\mathcal{Q}$ is a $\mathcal{W}_S$-display and $\widetilde{\mathcal{Q}}$ is the $\mathcal{W}_{S/R}$-display by base change then we have a natural inclusion $Q_i \to \widetilde{Q}_i$. This shows that the induced Hodge filtration on $\widetilde{\mathcal{Q}}$ is admissible. From the proof of the last proposition we obtain:

Corollary 11. *The functor*

$$(\mathcal{W}_S\text{-displays}) \to \begin{pmatrix} \mathcal{W}_{S/R}\text{-displays with an admissible} \\ \text{lift of the Hodge filtration} \end{pmatrix}$$

is an equivalence of categories.

We consider a frame with Verjüngung $\mathcal{F} = (W, J, R, \sigma, \dot{\sigma}, \nu, \pi)$. We consider 2-displays, i.e., displays which are defined by standard data (L_i, Φ_i) with $L_i = 0$ for $i > 2$ [Langer and Zink 2007, Definitions 2.4 and 2.5]. Let $\mathcal{P}$ and $\mathcal{P}'$ given by standard data $(L_0, L_1, L_2, \Phi_0, \Phi_1, \Phi_2)$, $(L'_0, L'_1, L'_2, \Phi'_0, \Phi'_1, \Phi'_2)$. We assume that the W-modules L_i and L'_i are free. We choose a W-basis of each of these modules. A morphism of displays $\rho : \mathcal{P} \to \mathcal{P}'$ is given by three maps

$$\rho_{|i} : L_i \to P'_i = J_i L'_0 \oplus \cdots \oplus J L'_{i-1} \oplus L'_i \cdots$$

for $i = 0, 1, 2$. We represent each of these maps by a column vector. These are the columns of the matrix

$$\begin{pmatrix} X_{00} & Y_{01} & Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix}. \tag{16}$$

The X_{ij} are matrices with coefficients in W. They represent in the chosen basis the homomorphisms $L_j \to L'_i$ obtained from $\rho_{|j}$. The matrices Y_{01} and Y_{12} have coefficients in J and Y_{02} has coefficients in J_2. Since a morphism of 2-displays commutes with ι, one can see that the map $P_0 \to P'_0$ is given by the matrix

$$\begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix}.$$

By Proposition 5 the matrix (16) defines a morphism of displays if and only if the following diagram is commutative for $i = 0, 1, 2$:

$$\begin{tikzcd} L_i \arrow[r, "\Phi_i"] \arrow[d, "\rho_{|i}"'] & P_0 \arrow[d, "\rho_0"] \\ P'_i \arrow[r, "F'_i"'] & P'_0 \end{tikzcd} \tag{17}$$

The σ-linear maps Φ_i and Φ'_i are given by the row vectors of matrices with coefficients in W:

$$\begin{pmatrix} A_{0i} \\ A_{1i} \\ A_{2i} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A'_{0i} \\ A'_{1i} \\ A'_{2i} \end{pmatrix}.$$

We write these vectors in a matrix. For example, the standard data $(L_0, L_1, L_2, \Phi_1, \Phi_2, \Phi_3)$ for $\mathcal{P}$ are equivalent to the block matrix:

$$\boldsymbol{A} = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix}. \tag{18}$$

We will call this a structure matrix for the display $\mathcal{P}$. It is by definition a matrix in $\mathrm{GL}_h(W)$, where $h = \operatorname{rank}_W P_0$.

From the definition of F'_i in terms of standard data, these σ-linear maps have the following matrix representations:

$$F'_0\left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} A'_{00} & pA'_{01} & p^2A'_{02} \\ A'_{10} & pA'_{11} & p^2A'_{12} \\ A'_{20} & pA'_{21} & p^2A'_{22} \end{pmatrix} \begin{pmatrix} \sigma(x_0) \\ \sigma(x_1) \\ \sigma(x_2) \end{pmatrix},$$

$$F'_1\left(\begin{pmatrix} y_0 \\ x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} A'_{00} & A'_{01} & pA'_{02} \\ A'_{10} & A'_{11} & pA'_{12} \\ A'_{20} & A'_{21} & pA'_{22} \end{pmatrix} \begin{pmatrix} \dot\sigma(y_0) \\ \sigma(x_1) \\ \sigma(x_2) \end{pmatrix},$$

$$F'_2\left(\begin{pmatrix} y_0 \\ y_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} A'_{00} & A'_{01} & A'_{02} \\ A'_{10} & A'_{11} & A'_{12} \\ A'_{20} & A'_{21} & A'_{22} \end{pmatrix} \begin{pmatrix} \dot\sigma(y_0) \\ \dot\sigma(y_1) \\ \sigma(x_2) \end{pmatrix}.$$

The vectors x_i have coefficients in W, and the y_i have coefficients in J, but in the equation for F'_2 the vector y_0 has even coefficients in J_2.

Then the commutativity of the diagram (17) for $i = 0$ amounts to

$$\begin{pmatrix} A'_{00} & pA'_{01} & p^2A'_{02} \\ A'_{10} & pA'_{11} & p^2A'_{12} \\ A'_{20} & pA'_{21} & p^2A'_{22} \end{pmatrix} \begin{pmatrix} \sigma(X_{00}) \\ \sigma(X_{10}) \\ \sigma(X_{20}) \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} A_{00} \\ A_{10} \\ A_{20} \end{pmatrix}.$$

For $i = 1$ it amounts to

$$\begin{pmatrix} A'_{00} & A'_{01} & pA'_{02} \\ A'_{10} & A'_{11} & pA'_{12} \\ A'_{20} & A'_{21} & pA'_{22} \end{pmatrix} \begin{pmatrix} \dot{\sigma}(Y_{01}) \\ \sigma(X_{11}) \\ \sigma(X_{21}) \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} A_{01} \\ A_{11} \\ A_{21} \end{pmatrix}.$$

Finally for $i = 2$ it amounts to

$$\begin{pmatrix} A'_{00} & A'_{01} & A'_{02} \\ A'_{10} & A'_{11} & A'_{12} \\ A'_{20} & A'_{21} & A'_{22} \end{pmatrix} \begin{pmatrix} \dot{\sigma}(Y_{02}) \\ \dot{\sigma}(Y_{12}) \\ \sigma(X_{22}) \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} A_{02} \\ A_{12} \\ A_{22} \end{pmatrix}.$$

We may write the last three equations as a single matrix equation,

$$\boldsymbol{A}' \begin{pmatrix} \sigma(X_{00}) & \dot{\sigma}(Y_{01}) & \dot{\sigma}(Y_{02}) \\ p\sigma(X_{10}) & \sigma(X_{11}) & \dot{\sigma}(Y_{12}) \\ p^2\sigma(X_{20}) & p\sigma(X_{21}) & \sigma(X_{22}) \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \boldsymbol{A}, \tag{19}$$

where $\boldsymbol{A}$ and $\boldsymbol{A}'$ are the structure matrices (18).

Let $\mathcal{F}$ be a frame with Verjüngung such that any finitely generated projective R-module is free. Then the category of $\mathcal{F}$-2-displays is equivalent to the following category $\mathcal{M}_{\mathcal{F}}$: The objects are invertible matrices $\boldsymbol{A}$ with coefficients in W with a 3×3-partition into blocks such that the blocks on the diagonal are quadratic matrices. The morphisms $\boldsymbol{A} \to \boldsymbol{A}'$ are block matrices (16) such that (19) is satisfied. Of course we have to say what is the composite of two matrices, but we omit this. In this direction we make only the following remark: the maps $\rho_i : P_i \to P'_i$ are explicitly given by the matrix equations

$$\rho_0\left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix},$$

$$\rho_1\left(\begin{pmatrix} y_0 \\ x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} y_0 \\ x_1 \\ x_2 \end{pmatrix},$$

$$\rho_2\left(\begin{pmatrix} y_0 \\ y_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} X_{00} & \check{Y}_{01} & Y_{02} \\ \pi X_{10} & X_{11} & Y_{12} \\ \pi X_{20} & X_{21} & X_{22} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ x_2 \end{pmatrix}.$$

We need to explain the last equation. Here y_0 is a vector with entries in J_2 and y_1 a vector with entries in J. The entries πX_{10}, πX_{20} and $\check{Y}_{01}$ are only symbols. But the matrix multiplication becomes meaningful with the definitions

$$\check{Y}_{01} y_1 = Y_{01} * y_1, \quad \pi X_{10} y_0 = X_{10}\pi(y_0), \quad \pi X_{20} y_0 = X_{10}\pi(y_0). \tag{20}$$

Note that the vectors of (20) have entries in J_2.

Using these expressions for ρ_i we see that (19) amounts to the commutativity of the following diagram:

$$\begin{tikzcd} P_2 \arrow[r, "\rho_2"] \arrow[d, "F_2"'] & P_2' \arrow[d, "F_2'"] \\ P_0 \arrow[r, "\rho_0"'] & P_0' \end{tikzcd}$$

Finally we give the description of the dual display in terms of standard data. Let $\mathcal{F}$ be a frame with Verjüngung as before. Assume $\mathcal{P}$ is the display associated to the standard data:

$$\Phi : L_0 \oplus L_1 \oplus L_2 \to L_0 \oplus L_1 \oplus L_2. \tag{21}$$

We write Φ in matrix form:

$$\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \sigma(x) \\ \sigma(y) \\ \sigma(z) \end{pmatrix}.$$

Then the dual display $\widehat{\mathcal{P}}$ is formed from the following standard data. We take the dual modules $L_i^* = \mathrm{Hom}_W(L_i, W)$ but in the order L_2^*, L_1^*, L_0^*. Changing the order in (21) and taking the dual of this σ-linear map we obtain a linear map

$$\Phi^* : L_2^* \oplus L_1^* \oplus L_0^* \to W \otimes_{\sigma, W} (L_2^* \oplus L_1^* \oplus L_0^*). \tag{22}$$

We set $\widehat{\Phi} = (\Phi^*)^{-1}$. We regard this as a σ-linear map. We obtain a standard datum,

$$(L_2^*, L_1^*, L_0^*, \widehat{\Phi}),$$

which is by definition the standard datum of $\widehat{\mathcal{P}}$. In particular

$$\widehat{P}_0 = L_2^* \oplus L_1^* \oplus L_0^*.$$

In matrix form $\widehat{\Phi}$ takes the form

$$\widehat{\Phi} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} {}^tA_{22} & {}^tA_{12} & {}^tA_{02} \\ {}^tA_{21} & {}^tA_{11} & {}^tA_{01} \\ {}^tA_{20} & {}^tA_{10} & {}^tA_{00} \end{pmatrix}^{-1} \begin{pmatrix} \sigma(x') \\ \sigma(y') \\ \sigma(z') \end{pmatrix}.$$

Let us denote by d_0, d_1, d_2 the ranks of the modules L_0, L_1, L_2 respectively. Consider the block matrix

$$\boldsymbol{B} := \begin{pmatrix} \mathbf{0} & \mathbf{0} & E_{d_0} \\ \mathbf{0} & E_{d_1} & \mathbf{0} \\ E_{d_2} & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where E denotes a unit matrix. This matrix defines a bilinear form:

$$\langle\ ,\ \rangle : P_0 \times \widehat{P}_0 \to W, \quad \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \right\rangle = (x \;\; y \;\; z)\boldsymbol{B} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \tag{23}$$

In this notation the definition of $\widehat{\Phi}$ reads

$$\langle \Phi(u), \widehat{\Phi}(\hat{u}) \rangle = \sigma\langle u, \hat{u}\rangle, \quad u \in P_0,\ \hat{u} \in \widehat{P}_0.$$

One deduces the formula

$$\langle F_0(u), \widehat{F}_0(\hat{u})\rangle = p^2\sigma\langle u, \hat{u}\rangle.$$

If we denote by $\mathcal{U}(2)$ the $\mathcal{F}$-2-display associated to the standard datum $(0, 0, W; \sigma)$ we obtain from $\langle\ ,\ \rangle$ a bilinear pairing of $\mathcal{F}$-displays,

$$\langle\ ,\ \rangle : \mathcal{P} \times \widehat{\mathcal{P}} \to \mathcal{U}(2).$$

The complete definition of a bilinear form is given in [Langer and Zink 2007] after equation (15), pp.160–163. In the case $\mathcal{U}(2)$ we have for each $i, j \geq 0$ a W-bilinear pairing

$$P_i \times P_j \to W.$$

The most important formulas of this definition are

$$\begin{aligned} \langle F_i x_i, \widehat{F}_j x_j\rangle &= p^{2-i-j}\sigma\langle x_i, \hat{x}_j\rangle && \text{if } i + j \leq 2, \text{ for } x_i \in P_i, \\ \langle F_i x_i, \widehat{F}_j x_j\rangle &= \dot{\sigma}\langle x_i, \hat{x}_j\rangle && \text{if } i + j > 2 \text{ and } \hat{x}_j \in \widehat{P}_j. \end{aligned}$$

One should also keep in mind that the bilinear form of displays is already uniquely determined by the induced W-bilinear form

$$P_0 \times \widehat{P}_0 \to W.$$

We note that the Hodge filtrations

$$\{0\} \subset L_2/JL_2 \subset L_1/JL_1 \oplus L_2/JL_2 \subset P_0/JP_0,$$

and

$$\{0\} \subset \widehat{L}_2/J\widehat{L}_2 \subset \widehat{L}_1/J\widehat{L}_1 \oplus \widehat{L}_2/J\widehat{L}_2 \subset \widehat{P}_0/J\widehat{P}_0$$

are dual with respect to $\langle\ ,\ \rangle$.

In particular an isomorphism of 2-displays $\mathcal{P} \to \widehat{\mathcal{P}}$ defines a bilinear form of displays

$$\mathcal{P} \times \mathcal{P} \to \mathcal{U}(2)$$

such that the Hodge filtration of $\mathcal{P}$ is selfdual with respect to this pairing.

Let R, S be an artinian local ring with perfect residue field. Let $S \to R$ be a nilpotent pd-thickening and let $\mathfrak{a}$ be the kernel. Then have defined the small relative Witt frame $\widehat{\mathcal{W}}_{S/R}$ with Verjüngung.

For a $\widehat{\mathcal{W}}_{S/R}$-display $\mathcal{P}$ the Frobenius $F_0 : P_0 \to P_0$ induces a map

$$\bar{F}_0 : P_0/(JP_0 + pP_0 + \iota_0(P_1)) \to P_0/(JP_0 + pP_0 + \iota_0(P_1)). \tag{24}$$

Definition 12. We say that a 2-display $\mathcal{P}$ is $\bar{F}_0$-étale if the map (24) is an isomorphism.

If a 2-display $\mathcal{P}$ and the dual display $\widehat{\mathcal{P}}$ are $\bar{F}_0$-étale, we call the display $\mathcal{P}$ F-ordinary.

This makes sense for other frames but then we can do nothing with the definition.

Let $S' \to R$ be a second nilpotent pd-thickening of the same type. We denote the kernel by $\mathfrak{a}'$. Let $S' \to S$ a morphism of pd-thickenings of R. Then the kernel $\mathfrak{b}$ of this morphism is a sub-pd-ideal of $\mathfrak{a}'$. We obtain a morphism of frames

$$\widehat{\mathcal{W}}_{S'/R} \to \widehat{\mathcal{W}}_{S/R}. \tag{25}$$

Proposition 13. *Let R, S, S' be artinian local rings with perfect residue field of characteristic $p > 0$. Let $S' \to S$ be a surjective morphism of nilpotent pd-thickenings of R. Let $\mathfrak{a}$ and $\mathfrak{a}'$ be the kernels of the pd-thickenings.*

Let $\mathcal{P}$ and $\mathcal{Q}$ be two $\widehat{\mathcal{W}}_{S/R}$-2-displays which are F-ordinary. Let $\mathcal{P}'$ and $\mathcal{Q}'$ be liftings to $\widehat{\mathcal{W}}_{S'/R}$-2-displays.

Then each homomorphism $\rho : \mathcal{P} \to \mathcal{Q}$ lifts to a homomorphism of $\widehat{\mathcal{W}}_{S'/R}$-displays $\rho' : \mathcal{P}' \to \mathcal{Q}'$.

If we assume moreover that $(\mathfrak{a}')^2 = 0$, the homomorphism ρ' is uniquely determined by ρ.

Proof. By the usual argument, compare [Zink 2002, proof of Theorem 46], we may assume that $\mathcal{P} = \mathcal{Q}$ and that ρ is the identity. We choose such a normal decomposition and a basis in each module of this decomposition. We lift this to a normal decomposition of $\mathcal{P}'$ or $\mathcal{Q}'$ respectively and we also lift the given basis.

Then we may represent the 2-display $\mathcal{P}'$ by the structure matrix

$$\boldsymbol{A} = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix} \in \mathrm{GL}(\widehat{W}(S')),$$

and similarly $\mathcal{Q}'$ by the structure matrix $\boldsymbol{A}' = (A'_{ij})$. We will write $\boldsymbol{A}^{-1} = (\breve{A}_{ij})$ and $(\boldsymbol{A}')^{-1} = (\breve{A}'_{ij})$. Then our assumption says that the following matrices are invertible:

$$A_{00}, \quad A'_{00}, \quad \breve{A}_{22}, \quad \breve{A}'_{22}.$$

Let $\mathfrak{c}$ be the kernel of $S' \to S$. Decomposing $S' \to S$ in a series of pd-morphisms, we may assume that $\mathfrak{c}^2 = 0$ and $p\mathfrak{c} = 0$. A morphism $\rho' : \mathcal{P}' \to \mathcal{Q}'$ which lifts the identity may be represented by a matrix of the form

$$\boldsymbol{E} + \begin{pmatrix} X_{00} & Y_{01} & Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix}.$$

The entries of the matrices X_{ij} and Y_{ij} are in $\widehat{W}(\mathfrak{c})$ and the entries of $\boldsymbol{w}_0(Y_{02})$ are moreover in $(\mathfrak{a}')^2$.

We set $C_{ij} = A'_{ij} - A_{ij}$. These are matrices with entries in $\widehat{W}(\mathfrak{c})$. Since $\sigma(X_{ij}) = 0$, (19) may be rewritten as

$$\boldsymbol{C} + \boldsymbol{A}' \begin{pmatrix} 0 & \dot{\sigma}(Y_{01}) & \dot{\sigma}(Y_{02}) \\ 0 & 0 & \dot{\sigma}(Y_{12}) \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} X_{00} & Y_{01} & \pi Y_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix} \boldsymbol{A}. \tag{26}$$

We used the notation $\boldsymbol{C} := (C_{ij})$. We have to show that there are matrices X_{ij} and Y_{ij} which satisfy this equation. We write $Y_{02} = \mathfrak{y}_{02} + {}^{V}Z_{02}$, where $\mathfrak{y}_{02} \in \tilde{\mathfrak{c}} \cap (\tilde{\mathfrak{a}'})^2$. We note that $\pi Y_{02} = \mathfrak{y}_{02}$. In particular we need $\pi Y_{02} = 0$ if we want to prove the second assertion of the proposition, that the solutions X_{ij}, Y_{ij} are unique.

We set $\boldsymbol{D} = \boldsymbol{C}\boldsymbol{A}^{-1}$. Then (26) becomes

$$\boldsymbol{D} + \boldsymbol{A}' \begin{pmatrix} 0 & \dot{\sigma}(Y_{01}) & \dot{\sigma}(Y_{02}) \\ 0 & 0 & \dot{\sigma}(Y_{12}) \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{A}^{-1} = \begin{pmatrix} X_{00} & Y_{01} & \mathfrak{y}_{02} \\ X_{10} & X_{11} & Y_{12} \\ X_{20} & X_{21} & X_{22} \end{pmatrix}. \tag{27}$$

We have

$$\begin{aligned} &\boldsymbol{A}' \begin{pmatrix} 0 & \dot{\sigma}(Y_{01}) & \dot{\sigma}(Y_{02}) \\ 0 & 0 & \dot{\sigma}(Y_{12}) \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{A}^{-1} \\ &\qquad = \begin{pmatrix} * & A'_{00}\dot{\sigma}(Y_{01})\breve{A}_{11} + A'_{00}\dot{\sigma}(Y_{02})\breve{A}_{21} + A'_{01}\dot{\sigma}(Y_{12})\breve{A}_{21} & ?_1 \\ * & * & ?_2 \\ * & * & ?_3 \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} ?_1 &= A'_{00}\dot{\sigma}(Y_{01})\breve{A}_{12} + A'_{00}\dot{\sigma}(Y_{02})\breve{A}_{22} + A'_{01}\dot{\sigma}(Y_{12})\breve{A}_{22}, \\ ?_2 &= A'_{10}\dot{\sigma}(Y_{01})\breve{A}_{12} + A'_{10}\dot{\sigma}(Y_{02})\breve{A}_{22} + A'_{11}\dot{\sigma}(Y_{12})\breve{A}_{22}, \\ ?_3 &= *. \end{aligned}$$

The entries $*$ are irrelevant for the following and therefore not specified. Since the X_{ij} don't appear on the left-hand side of (27) we see that it is enough to satisfy the following equations if we want to solve (27):

$$\begin{aligned} D_{01}+A'_{00}\dot{\sigma}(Y_{01})\breve{A}_{11}+A'_{00}\dot{\sigma}(Y_{02})\breve{A}_{21}+A'_{01}\dot{\sigma}(Y_{12})\breve{A}_{21}&=Y_{01},\\ D_{12}+A'_{10}\dot{\sigma}(Y_{01})\breve{A}_{12}+A'_{10}\dot{\sigma}(Y_{02})\breve{A}_{22}+A'_{11}\dot{\sigma}(Y_{12})\breve{A}_{22}&=Y_{12},\\ D_{02}-\mathfrak{y}_{02}+A'_{00}\dot{\sigma}(Y_{01})\breve{A}_{12}+A'_{01}\dot{\sigma}(Y_{12})\breve{A}_{22}&=-A'_{00}\dot{\sigma}(Y_{02})\breve{A}_{22}. \end{aligned} \tag{28}$$

In this equation the D_{ij} are matrices with entries in $\widehat{W}(\mathfrak{c})$. We note that for any given matrices $\mathfrak{y}_{02}$ and $\dot{\sigma}(Y_{02})$ there is a unique Y_{02}.

Therefore the proposition follows if we show that for any given $\mathfrak{y}_{02}$ with entries in $\mathfrak{c}\cap(\mathfrak{a}')^2$ the equation above has a unique solution for the unknowns

$$Z_0=Y_{01},\quad Z_1=Y_{12},\quad Z_2=-A'_{00}\dot{\sigma}(Y_{02})\breve{A}_{22},$$

with entries in $\widehat{W}(\mathfrak{c})$. This is because the matrices A'_{00} and $\breve{A}_{22}$ are invertible.

We denote by $\mathfrak{c}_{[n]}$ the $\widehat{W}(S')$-module obtained from the S'-module $\mathfrak{c}$ via restriction of scalars by the homomorphism $\boldsymbol{w}_n:\widehat{W}(S')\to S'$. The divided powers on $\mathfrak{c}$ allow us to divide the Witt polynomial $\boldsymbol{w}_n$ by p^n. The divided Witt polynomials $\boldsymbol{w}'_n$ define an isomorphism

$$\widehat{W}(\mathfrak{c})\to\bigoplus_{n=0}^{\infty}\mathfrak{c}_{[n]} \tag{29}$$

of $\widehat{W}(S')$-modules.

For a matrix M with entries in $\widehat{W}(\mathfrak{c})$ of suitable size we define the operator

$$\mathcal{L}_{00}(M)=A'_{00}M\breve{A}_{11}.$$

If M has entries in the ideal $\bigoplus_{i=0}^{n}\mathfrak{c}_{[i]}$ in the sense of (29) then $\mathcal{L}_{00}(M)$ has entries in the same ideal. In this case we write length $M\le n$. It follows that length $\dot{\sigma}(M)\le n-1$.

With obvious definitions of the operators $\mathcal{L}_{ij}$ we may write the system of equations (28) as

$$\begin{aligned} D_{01}+\mathcal{L}_{00}(\dot{\sigma}(Z_0))+\mathcal{L}_{01}(\dot{\sigma}(Z_1))+\mathcal{L}_{02}(Z_2)&=Z_0,\\ D_{12}+\mathcal{L}_{10}(\dot{\sigma}(Z_0))+\mathcal{L}_{11}(\dot{\sigma}(Z_1))+\mathcal{L}_{12}(Z_2)&=Z_1,\\ D'_{02}+\mathcal{L}_{20}(\dot{\sigma}(Z_0))+\mathcal{L}_{21}(\dot{\sigma}(Z_1))&=Z_2. \end{aligned}$$

Here we write $D'_{02}:=D_{02}-\mathfrak{y}_{02}$. We look for solutions in the space of matrices (Z_0,Z_1,Z_2) with entries in $\widehat{W}(\mathfrak{c})$. On this space we consider the operator U given by

$$U\left(\begin{pmatrix}Z_0\\Z_1\\Z_2\end{pmatrix}\right)=\begin{pmatrix}\mathcal{L}_{00}(\dot{\sigma}(Z_0))+\mathcal{L}_{01}(\dot{\sigma}(Z_1))+\mathcal{L}_{02}(Z_2)\\ \mathcal{L}_{10}(\dot{\sigma}(Z_0))+\mathcal{L}_{11}(\dot{\sigma}(Z_1))+\mathcal{L}_{12}(Z_2)\\ \mathcal{L}_{20}(\dot{\sigma}(Z_0))+\mathcal{L}_{21}(\dot{\sigma}(Z_1))\end{pmatrix}.$$

Clearly it suffices to prove that the operator U is pointwise nilpotent. Assume we are given Z_0, Z_1, Z_2. We set

$$U\left(\begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \end{pmatrix}\right) = \begin{pmatrix} Z_0' \\ Z_1' \\ Z_2' \end{pmatrix}, \quad U\left(\begin{pmatrix} Z_0' \\ Z_1' \\ Z_2' \end{pmatrix}\right) = \begin{pmatrix} Z_0'' \\ Z_1'' \\ Z_2'' \end{pmatrix}.$$

Let m be a natural number such that length $Z_0 \leq m$, length $Z_1 \leq m$ and length $Z_2 \leq m-1$. Since $\dot{\sigma}$ decreases the length by 1, we obtain

$$\text{length } Z_0' \leq m-1, \quad \text{length } Z_1' \leq m-1, \quad \text{length } Z_2' \leq m-1.$$

And in the next step we find

$$\text{length } Z_0'' \leq m-1, \quad \text{length } Z_1'' \leq m-1, \quad \text{length } Z_2'' \leq m-2.$$

The nilpotence of U is now clear. This proves the uniqueness of the solutions. $\square$

Remark. With the assumptions of the last proposition we assume that the kernel $\mathfrak{c}$ of the pd-morphism satisfies $\mathfrak{c}^2 = 0$ and $p\mathfrak{c} = 0$. Then the group of isomorphisms $\mathcal{P}' \to \mathcal{P}'$ which lift the identity $\mathrm{id}_{\mathcal{P}}$ is isomorphic to the additive group of $\mathfrak{c} \cap (\mathfrak{a}')^2$. (The assumptions ensure that such a lift is the same as a solution for (27).) This is because $\mathfrak{y}_{02}$ determines the lifting uniquely and because one can check that the composite of two endomorphisms of $\mathcal{P}'$ which lift zero is zero.

Corollary 14. *Let $S \to R$ be a surjective morphism of artinian local rings with perfect residue class field. Let $\mathcal{P}$ and $\mathcal{P}'$ be two F-ordinary $\widehat{\mathcal{W}}_S$-2-displays. Let $\rho, \tau : \mathcal{P} \to \mathcal{P}'$ be two homomorphisms such that their base changes ρ_R and τ_R are equal.*

Then $\rho = \tau$.

Corollary 15. *Let R, S' be an artinian local ring with perfect residue field as above. Let $S' \to R$ be a nilpotent pd-thickening with kernel $\mathfrak{a}'$ such that $(\mathfrak{a}')^2 = 0$. Let $\mathcal{Q}$ be an F-ordinary $\widehat{\mathcal{W}}_R$-2-display over R. By Proposition 13 there is up to canonical isomorphism a unique $\widehat{\mathcal{W}}_{S'/R}$-2-display $\widehat{\mathcal{Q}}$ which lifts $\mathcal{Q}$.*

The category of F-ordinary $\widehat{\mathcal{W}}_{S'}$-2-displays is equivalent to the category of pairs $(\mathcal{Q}, \mathrm{Fil})$ *where $\mathcal{Q}$ is an F-ordinary $\widehat{\mathcal{W}}_R$-2-displays and where* Fil *is an admissible lifting of the Hodge filtration of $\widehat{\mathcal{Q}}$.*

Let k be a perfect field of characteristic $p > 2$. Let $\mathcal{A}rt_k$ be the category of artinian local rings with residue class field k. Let $\mathcal{S}$ be an ordinary $\mathcal{W}_k$-2-display. Let $\mathcal{D}$ be the functor that associates to $R \in \mathcal{A}rt_k$ the isomorphism classes of pairs $(\mathcal{P}, \iota)$, where $\mathcal{P}$ is a $\widehat{\mathcal{W}}_R$-2-display and $\iota : \mathcal{S} \to \mathcal{P}_k$ is an isomorphism. If we have a diagram $R_1 \to R \leftarrow R_2$ then the canonical map

$$\mathcal{D}(R_1 \times_R R_2) \to \mathcal{D}(R_1) \times_{\mathcal{D}(R)} \mathcal{D}(R_2) \tag{30}$$

is surjective. To see this, one can for example use the interpretation of a display as a block matrix. It is injective because of Corollary 14.

By Corollary 15 the tangent space of the functor $\mathcal{D}$ is finite-dimensional.

Corollary 16. *The functor $\mathcal{D}$ is prorepresentable by a power series ring over $W(k)$ in finitely many variables.*

Proof. By what we have said the prorepresentability is standard. It remains to check that the functor is smooth. But this follows again by representing a display by a matrix. □

We will use the following version of the deformation functor. We take a $\mathcal{W}_k$-2-display $\mathcal{S}$ as above and we assume moreover that $\mathcal{S}$ is endowed with an isomorphism

$$\lambda_0 : \mathcal{S} \to \widehat{\mathcal{S}}. \tag{31}$$

We can also regard λ_0 as a pairing $\mathcal{S} \times \mathcal{S} \to \mathcal{U}(2)$. Then we define the deformation functor $\widehat{\mathcal{D}} : \mathcal{A}rt_k \to$ (sets). For $R \in \mathcal{A}rt_k$ we define $\widehat{\mathcal{D}}(R)$ as the set of isomorphism classes of $\widehat{\mathcal{W}}_R$-2-displays $\mathcal{P}$ together with an isomorphism $\lambda : \mathcal{P} \to \widehat{\mathcal{P}}$ and an isomorphism $\iota : \mathcal{S} \to \mathcal{P}_k$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{S} \arrow[r, "\iota"] \arrow[d, "\lambda_0"'] & \mathcal{P}_k \arrow[d, "\lambda_k"] \\
\widehat{\mathcal{S}} & \widehat{\mathcal{P}}_k \arrow[l, "\hat{\iota}"]
\end{tikzcd}$$

We note that by the diagram and Corollary 14 the morphism λ is uniquely determined if it exists. Therefore we have an inclusion $\widehat{\mathcal{D}}(R) \subset \mathcal{D}(R)$. The map (30) for the functor $\widehat{\mathcal{D}}$ is also bijective. We will now find the tangent space of $\widehat{\mathcal{D}}$. More generally we consider a surjective homomorphism $S' \to R$ in $\mathcal{A}rt_k$ with kernel $\mathfrak{a}'$ such that $(\mathfrak{a}')^2 = 0$. We endow this with the trivial divided powers. Assume we are given $(\mathcal{P}, \iota, \lambda) \in \widehat{\mathcal{D}}(R)$. Then $\mathcal{P}$ lifts uniquely to a $\widehat{\mathcal{W}}_{S'/R}$-display $\mathcal{P}_{\text{rel}}$ and λ lifts to an isomorphism $\lambda_{\text{rel}} : \mathcal{P}_{\text{rel}} \to \widehat{\mathcal{P}}_{\text{rel}}$. Let $\mathcal{Q}$ be a $\widehat{\mathcal{W}}_{S'}$-2-display which lifts $\mathcal{P}$. Giving $\mathcal{Q}$ is the same as giving an admissible lifting of the Hodge filtration of $\mathcal{P}_{\text{rel}}$. The dual display $\widehat{\mathcal{Q}}$ corresponds to the dual filtration of $\widehat{\mathcal{P}}_{\text{rel}}$. But then $\mathcal{Q}$ and $\widehat{\mathcal{Q}}$ are isomorphic if and only if λ_{rel} takes the filtration $\operatorname{Fil}_{\mathcal{Q}}$ given by $\mathcal{Q}$ to the dual filtration, i.e., $\operatorname{Fil}_{\mathcal{Q}}$ is selfdual with respect to the bilinear form

$$(\ ,\ ) : P_{\text{rel},0}/I_{S'}P_{\text{rel},0} \times P_{\text{rel},0}/I_{S'}P_{\text{rel},0} \to S' \tag{32}$$

induced by λ_{rel}.

Proposition 17. *Let $S' \to R$ be a pd-thickening with kernel $\mathfrak{a}'$ such that $(\mathfrak{a}')^2 = 0$. Let $(\mathcal{P}, \lambda)$ be an ordinary $\widehat{\mathcal{W}}_R$-2-display which is endowed with an isomorphism*

$\lambda : \mathcal{P} \to \widehat{\mathcal{P}}$. We assume that λ is symmetric (i.e., $\lambda = \hat{\lambda}$) and such that for the Hodge filtration $\operatorname{rank}_R \operatorname{Fil}^2_{\mathcal{P}} = 1$. *We denote by $\mathcal{P}_{\mathrm{rel}}$ the unique $\widehat{\mathcal{W}}_{S'/R}$ which lifts $\mathcal{P}$.*

The liftings of $(\mathcal{P}, \lambda)$ to a $\widehat{\mathcal{W}}_{S'}$-2-display $\mathcal{Q}$ together with a lift of λ to an isomorphism $\mu : \mathcal{Q} \to \widehat{\mathcal{Q}}$ are in bijection with the liftings of $\operatorname{Fil}^2_{\mathcal{P}}$ *to a isotropic direct summand of $P_{\mathrm{rel},0}/I_{S'}P_{\mathrm{rel},0}$.*

Proof. We know that λ lifts to an isomorphism $\lambda_{\mathrm{rel}} : \mathcal{P}_{\mathrm{rel}} \to \widehat{\mathcal{P}}_{\mathrm{rel}}$. It follows from Corollary 15 that the liftings $(\mathcal{Q}, \mu)$ of $(\mathcal{P}, \lambda)$ are in bijection with selfdual admissible liftings of the Hodge filtration of $\mathcal{P}$.

The isomorphism λ_{rel} induces a perfect pairing (32) of S'-modules. We claim that the image of

$$P_{\mathrm{rel},2} \to P_{\mathrm{rel},0}/I_{S'}P_{\mathrm{rel},0}$$

is isotropic under this pairing (32).

To verify this we take a normal decomposition of $\mathcal{P}_{\mathrm{rel}}$,

$$P_{\mathrm{rel},0} = L_0 \oplus L_1 \oplus L_2.$$

This induces the dual normal decomposition of $\widehat{\mathcal{P}}_{\mathrm{rel}}$ (compare (22))

$$\widehat{P}_{\mathrm{rel},0} = \widehat{L}_0 \oplus \widehat{L}_1 \oplus \widehat{L}_2,$$

where $\widehat{L}_0 = L_2^*$, $\widehat{L}_1 = L_1^*$, $\widehat{L}_2 = L_0^*$.

We set $L'_i = L_i/I_{S'}L_i \subset P_{\mathrm{rel},0}/I_{S'}P_{\mathrm{rel},0}$ and $\widehat{L}'_i = \widehat{L}_i/I_{S'}\widehat{L}_i \subset \widehat{P}_{\mathrm{rel},0}/I_{S'}\widehat{P}_{\mathrm{rel},0}$. Then the images of the two maps

$$P_{\mathrm{rel},2} \to P_{\mathrm{rel},0}/I_{S'}P_{\mathrm{rel},0} \quad \text{and} \quad \widehat{P}_{\mathrm{rel},2} \to \widehat{P}_{\mathrm{rel},0}/I_{S'}\widehat{P}_{\mathrm{rel},0}$$

are

$$L'_2 \oplus \mathfrak{a}' L_1 \quad \text{and} \quad \widehat{L}'_2 \oplus \mathfrak{a}' \widehat{L}_1$$

respectively. Since $(\mathfrak{a}')^2 = 0$ the last two modules are orthogonal with respect to the perfect pairing,

$$P_{\mathrm{rel},0}/I_{S'}P_{\mathrm{rel},0} \times \widehat{P}_{\mathrm{rel},0}/I_{S'}\widehat{P}_{\mathrm{rel},0} \to S'$$

induced by (23). Composing this with the isomorphism λ_{rel} we obtain the claim.

Next we show that any lift of $\operatorname{Fil}^2_{\mathcal{P}}$ to an isotropic direct summand $U \subset P_{\mathrm{rel},0}/I_{S'}P_{\mathrm{rel},0}$ is contained in the image of $P_{\mathrm{rel},2}$. We take a splitting of the selfdual Hodge filtration of $\mathcal{P}$:

$$\operatorname{Fil}^2_{\mathcal{P}} = N'_2 \subset N'_2 \oplus N'_1 \subset N'_2 \oplus N'_1 \oplus N_0 = P_0/I_R P_0.$$

Then the perfect pairing induced by λ induces a perfect pairing $N'_2 \times N'_0 \to R$ and $(N'_2)^{\perp} = N'_2 \oplus N'_0$. It is easy to see that this lifts to a selfdual filtration

$$L'_2 \subset L'_2 \oplus L'_1 \subset L'_2 \oplus L'_1 \subset L'_2 \oplus L'_1 \oplus L'_0 = P_{\mathrm{rel},0}/I_{S'}P_{\mathrm{rel},0}$$

with respect to (32). Let l_2 be a basis of L'_2. Then U has a basis

$$u = l_2 + l_1 + l_0, \quad l_1 \in \mathfrak{a}' L'_1,\ l_2 \in \mathfrak{a}' L'_2.$$

We find

$$(u, u) = (l_2, l_0) + (l_0, l_2) = 2(l_2, l_0).$$

Because $L'_2 \times L'_0 \to S'$ is perfect, this implies $l_0 = 0$. Therefore u is in the image of $P_{\mathrm{rel},2}$.

From an isotropic lift $U \subset P_{\mathrm{rel},0}/I_{S'} P_{\mathrm{rel},0}$ we obtain a selfdual admissible lift of the Hodge filtration of $\mathcal{P}$, if we define $U = \mathrm{Fil}^2$ and Fil^1 to be the orthogonal complement of U. By Corollary 15 this gives a lifting $(\mathcal{Q}, \mu)$. □

In particular we see that lifts of $\mathcal{P}$ always exist. Therefore we obtain:

Corollary 18. *Let $\mathcal{S}, \lambda_0$ be a $\mathcal{W}_k$-2-display with a symmetric isomorphism (31) and such that for the Hodge filtration* $\dim_k \mathrm{Fil}^2_{\mathcal{S}} = 1$.

Then the functor $\widehat{\mathcal{D}}$ is prorepresentable by a power series ring over $W(k)$ in finitely many variables.

2. 2-displays of schemes

Let X_0 be a projective and smooth scheme over a perfect field k of characteristic $p > 2$. We make the following assumptions:

(1) Let $T_{X_0/k}$ be the tangent bundle of X_0. Then

$$H^0(X_0, T_{X_0/k}) = H^2(X_0, T_{X_0/k}) = 0. \tag{33}$$

(2) Let R be a local artinian $W(k)$-algebra with residue class field k and let $f : X \to \operatorname{Spec} R$ be an arbitrary deformation of X_0. Then the R-modules

$$R^j f_* \Omega^i_{X/R} \tag{34}$$

are free for $i + j \leq 2$ and commute with base change for morphisms $\operatorname{Spec} R' \to \operatorname{Spec} R$, where R' is a local artinian $W(k)$-algebra with residue class field k.

(3) The spectral sequence

$$E_1^{ij} = R^j f_* \Omega^i_{X/R} \Rightarrow \mathbb{R}^{j+i} f_* \Omega^{\cdot}_{X/R} \tag{35}$$

degenerates for $i + j \leq 2$, i.e., all differentials starting or ending at E_r^{ij} for $i + j \leq 2$, $r \geq 1$ are zero.

We remark that the last two requirements are fulfilled if

$$H^j(X_0, \Omega^i_{X_0/k}) = 0 \quad \text{for } i + j = 1 \text{ or } 3.$$

Assume that X_0 satisfies the three conditions above. Let R be a local $W(k)$-algebra whose maximal ideal $\mathfrak{m}$ is nilpotent and such that $R/\mathfrak{m} = k$. Let $g : Y \to R$

be a deformation of X_0. Then the last two conditions are also satisfied for g. Indeed R is the filtered union of local $W(k)$-algebras of finite type and X is automatically defined over a $W(k)$-algebra of finite type.

Assume again that the three conditions are fulfilled for X_0. Then there is a universal deformation, i.e., a morphism of formal schemes,

$$\mathfrak{X} \to \operatorname{Spf} A. \tag{36}$$

The adic ring A is the ring $W(k)[[T_1, \dots, T_r]]$ with $(p, T_1, \dots, T_r)$ as the ideal of definition. We have $r = \dim H^1(X_0, T_{X_0/k})$. We denote by σ the endomorphism of A such that $\sigma(T_i) = T_i^p$ and such that σ is the Frobenius on $W(k)$.

We are going to define a display structure on the de Rham–Witt cohomology of (36). For this we use the frames introduced in [Zink 2001b]. We call them w-frames in order to distinguish them from the frames introduced above. With respect to a w-frame we have the category of windows [Langer and Zink 2007, §5]. We use here the base change which associates to a window a display in our sense [loc. cit., Remark after Definition 5.1, pp. 181–182].

Let $n \geq 1$ be an integer. We set $C_n = W(k)[[T_1, \dots, T_r]]/(T_1^n, \dots, T_r^n)$ and $R_n = C_n/p^n C_n$. Then σ induces an endomorphism on C_n denoted by the same letter. We obtain that $\mathcal{C}_n = (C_n, p^n C_n, R_n, \sigma)$ is a w-frame. An obvious modification of [loc. cit., Corollary 5.6] shows that we have the structure of a $\widehat{\mathcal{W}}_{R_n}$-display on

$$H^2_{\text{crys}}(\mathcal{X}_{R_n}/\widehat{W}(R_n)).$$

This is obtained from the Lazard morphism $C_n \to W(R_n)$, which factors through

$$C_n \to \widehat{W}(R_n) \to W(R_n).$$

By Theorem 5.5 of [loc. cit.] we have a $\mathcal{C}_n$-window structure on $H^2_{\text{crys}}(\mathcal{X}_{R_n}/C_n)$. We can apply the base change of [loc. cit., Remark, p. 181] to obtain from a $\mathcal{C}_n$-window a $\widehat{\mathcal{W}}_{R_n}$-display.

This is functorial in R_n. If $f : X \to R$ is a deformation as in (34) we obtain for n big enough a unique $W(k)$-algebra homomorphism $R_n \to R$. Therefore we obtain by base change:

Proposition 19. *Let $f : X \to R$ be as above. Then the crystalline cohomology $H^2_{\text{crys}}(X/\widehat{W}(R))$ has the structure of a $\widehat{\mathcal{W}}_R$-display which is functorial in R.*

The uniqueness follows from the functoriality and the fact that $\widehat{W}(A)$ has no p-torsion.

We now show that X/R defines a crystal of displays in the following sense:

Corollary 20. *With the assumptions of the proposition let $S \to R$ be a pd-thickening where S is an artinian $W(k)$-algebra. Then we have the natural structure of a*

$\widehat{\mathcal{W}}_{S/R}$-display on $H^2_{\mathrm{crys}}(X/S)$. More precisely this structure is functorial with respect to morphisms of pd-thickenings and uniquely determined by this property.

Proof. We obtain a $\widehat{\mathcal{W}}_{S/R}$-2-display structure by lifting X to a smooth scheme X' over S and then making the base change with respect to $\widehat{\mathcal{W}}_S \to \widehat{\mathcal{W}}_{S/R}$. We show that the result is independent of the chosen lifting X'. Assume we have two liftings X' and X'' which are induced from the universal family (36) by two morphisms $A \to S$. We consider the following commutative diagram:

$$\begin{array}{ccc} A \mathbin{\widehat{\otimes}}_{W(k)} A & \longrightarrow & S \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & R \end{array} \tag{37}$$

The left vertical arrow is the multiplication. Let J be the kernel. We denote the divided power hull of $(B := A \mathbin{\widehat{\otimes}}_{W(k)} A, J)$ by P. It is obtained as follows: Let $A_0 = W(k)[T_1, \ldots, T_r]$ and J_0 be the kernel of the multiplication $B_0 := A_0 \otimes_{W(k)} A_0 \to A_0$. We denote by P_0 the divided power hull of (B_0, J_0). Then P_0 is isomorphic to the divided power algebra of the free A_0-module with r generators. In particular P_0 is a free A_0-module for the two natural A_0-module structures. We have $P = P_0 \otimes_{B_0} B$. Then P is flat as a P_0-module and therefore without p-torsion. Then the diagram (37) extends to the following diagram:

$$\begin{array}{ccc} P & \longrightarrow & S \\ \big\downarrow & & \big\downarrow \\ A & \longrightarrow & R \end{array} \tag{38}$$

Let $\widehat{P}$ be the p-adic completion of P. Then $\widehat{P} \to A$ is a frame $\mathcal{D}$. By [Langer and Zink 2007, Theorem 5.5] the universal family $\mathfrak{X}$ defines a $\mathcal{D}$-display $\mathcal{U}$. We consider also the trivial w-frame $\mathcal{D}_0 = (A, 0, A, \sigma)$. Again $\mathfrak{X}$ defines a $\mathcal{D}_0$-window $\mathcal{U}_0$. The two natural sections $A \to \widehat{P}$ define two morphisms of w-frames $\mathcal{D}_0 \to \mathcal{D}$. Since the construction of [loc. cit., Theorem 5.5] is compatible with base change, we obtain $\mathcal{U}$ from $\mathcal{U}_0$ by base change with respect to both of these two morphisms.

We consider the morphism of frames

$$\mathcal{D}_0 \rightrightarrows \mathcal{D} \to \widehat{\mathcal{W}}_{S/R}.$$

The two $\widehat{\mathcal{W}}_{S/R}$-displays associated with X' and X'' are obtained by base change from $\mathcal{U}_0$ by the two morphisms

$$\mathcal{D}_0 \rightrightarrows \widehat{\mathcal{W}}_{S/R}.$$

We see that these two $\widehat{\mathcal{W}}_{S/R}$-displays are both obtained by the base change of $\mathcal{U}$ with respect to $\mathcal{D} \to \widehat{\mathcal{W}}_{S/R}$. This shows that the $\widehat{\mathcal{W}}_{S/R}$ display does not, up to canonical isomorphism, depend on the lifting X' of X. $\square$

Let $\mathfrak{X} \to \operatorname{Spf} A$ be the universal family (36). We set $A_0 = A/pA$ and we consider the formal scheme $\mathfrak{X}_0 = \mathfrak{X} \otimes_A A_0$. We write $H^2_{\mathrm{DR}}(\mathfrak{X}/A) = \mathbb{H}^2(\mathfrak{X}, \Omega^{\cdot}_{\mathfrak{X}/A})$ for the de Rham cohomology. By the isomorphism

$$H^2_{\mathrm{DR}}(\mathfrak{X}/A) = H^2_{\mathrm{crys}}(\mathfrak{X}_0/A),$$

we obtain a σ-linear endomorphism $F : H^2_{\mathrm{DR}}(\mathfrak{X}/A) \to H^2_{\mathrm{DR}}(\mathfrak{X}/A)$.

Lemma 21. *We consider the following subcomplex of* $\Omega^{\cdot}_{\mathfrak{X}/A}$:

$$\mathcal{F}^1\Omega^{\cdot}_{\mathfrak{X}/A} : p\Omega^0_{\mathfrak{X}/A} \to \Omega^1_{\mathfrak{X}/A} \to \Omega^2_{\mathfrak{X}/A} \to \cdots .$$

The natural map

$$\mathbb{H}^2(\mathfrak{X}, \mathcal{F}^1\Omega^{\cdot}_{\mathfrak{X}/A}) \to \mathbb{H}^2(\mathfrak{X}, \Omega^{\cdot}_{\mathfrak{X}/A})$$

is injective and the image is the set

$$\{x \in H^2_{\mathrm{DR}}(\mathfrak{X}/A) \mid Fx \in pH^2_{\mathrm{DR}}(\mathfrak{X}/A)\}.$$

The image of the natural map

$$\mathbb{H}^2(\mathfrak{X}, \mathcal{F}^1\Omega^{\cdot}_{\mathfrak{X}/A}) \to \mathbb{H}^2(\mathfrak{X}_0, \Omega^{\cdot}_{\mathfrak{X}_0/A_0})$$

is the Hodge filtration $\mathrm{Fil}^1 \subset H^2_{\mathrm{DR}}(\mathfrak{X}_0/A_0)$.

Proof. We use the notation before Proposition 19. If we take the projective limit of the $\mathcal{C}_n$-displays

$$H^2_{\mathrm{crys}}(\mathfrak{X}_{R_n}/C_n) = H^2_{\mathrm{crys}}(\mathfrak{X}_{C_n/pC_n}/C_n),$$

we obtain a display structure on $H^2_{\mathrm{DR}}(\mathfrak{X}, A)$ with respect to the frame $\mathcal{A} = (A, pA, A_0, \sigma, \sigma/p)$. We denote this display by $\mathcal{P} = (P_i, F_i, \iota_i, \alpha_i)$. It follows from [Langer and Zink 2007, Theorem 5.5] that P_1 is the hypercohomology $\mathbb{H}^2(\mathfrak{X}, \mathcal{F}^1\Omega_{\mathfrak{X}/A})$.

The first assertion of the lemma follows from the fact that for an $\mathcal{A}$-display

$$P_1 = \{x \in P_0 \mid F_0 x \in pP_0\}.$$

Indeed, we take a normal decomposition of $\mathcal{P}$, which in our case is a 2-display:

$$P_0 = L_0 \oplus L_1 \oplus L_2, \quad P_1 = pL_0 \oplus L_1 \oplus L_2.$$

Then the Frobenius F_0 is given by an invertible block matrix $\boldsymbol{D}$ with coefficients in A:

$$F_0\left(\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}\right) = \boldsymbol{D}\begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p^2 \end{pmatrix}\begin{pmatrix} \sigma(x_0) \\ \sigma(x_1) \\ \sigma(x_2) \end{pmatrix}.$$

Multiplying this equation by $\boldsymbol{D}^{-1}$ we see that the right-hand side is in pP_0 if and only if $\sigma(x_0) \in pL_0$ or equivalently $x_0 \in pL_0$. This proves the first assertion. The second assertion follows because by the definition of $\mathcal{P}$, the image of P_1 in $\mathbb{H}^2(\mathfrak{X}_0, \Omega_{\mathfrak{X}_0/A_0})$ is the Hodge filtration. □

3. The Beauville–Bogomolov form

Definition 22. A scheme of K3 type is a smooth and proper morphism $f : X \to S$ of relative dimension $2n$ with the following properties.

For each geometric point $\eta \to S$ we have

$$\begin{aligned} H^q(X_\eta, \Omega^p_{X_\eta/\eta}) &= 0 \quad \text{for } p+q=1,\ p+q=3, \\ \dim_{\kappa(\eta)} H^q(X_\eta, \mathcal{O}_{X_\eta}) &= 1 \quad \text{for } q=0,2, \\ \dim_{\kappa(\eta)} H^0(X_\eta, \Omega^2_{X_\eta/\eta}) &= 1. \end{aligned} \tag{39}$$

We assume that for each η there is a nowhere degenerate $\sigma \in H^0(X_\eta, \Omega^2_{X_\eta/\eta})$; i.e., $\sigma^n \in H^0(X_\eta, \Omega^{2n}_{X_\eta/\eta})$ defines an isomorphism,

$$\mathcal{O}_{X_\eta} \to \Omega^{2n}_{X_\eta/\eta}.$$

We assume that for each η there is a class $\rho \in H^2(X_\eta, \mathcal{O}_{X_\eta})$ such that ρ^n generates $H^{2n}(X_\eta, \mathcal{O}_{X_\eta})$. (We note that this is a 1-dimensional vector space by Serre duality.)

Finally we require that for each η the pairing

$$H^1(X_\eta, \Omega^1_{X_\eta/\eta}) \times H^1(X_\eta, \Omega^1_{X_\eta/\eta}) \to R, \quad \omega_1 \times \omega_2 \mapsto \int \omega_1\omega_2\sigma^{n-1}\rho^{n-1}, \tag{40}$$

is perfect.

We note that (40) is equivalent to saying that the cup product

$$\sigma^{n-1}\rho^{n-1} : H^1(X_\eta, \Omega^1_{X_\eta/\eta}) \to H^{2n-1}(X_\eta, \Omega^{2n-1}_{X_\eta/\eta})$$

is an isomorphism.

We denote by $\mathcal{T}_{X/S}$ the dual $\mathcal{O}_X$-module of $\Omega^1_{X/S}$. By definition σ induces a perfect pairing

$$\mathcal{T}_{X/S} \times \mathcal{T}_{X/S} \to \mathcal{O}_X, \tag{41}$$

or equivalently an isomorphism $\mathcal{T}_{X/S} \cong \Omega^1_{X/S}$.

Remarks. If X is a hyper-Kähler variety over $\mathbb{C}$ such that $H^3(X, \mathbb{C}) = 0$, then X is of K3 type. [Salamon 1996, Introduction; Huybrechts 1999, 1.1–1.3 and 1.7].

Over any field, a K3 surface is of K3 type. Over a field of characteristic 0, the Hilbert scheme $K^{[n]}$ of zero-dimensional subschemes of a K3 surface K is of K3 type [Beauville 1983, Theorem 3; Salamon 1996, Remark between 5.6 and 5.7 and Example, p. 149]. In fact all odd Betti numbers of $K^{[n]}$ vanish. This follows from a general formula proven by Göttsche [1990, Theorem 0.1].

By [Mumford 1970; Hartshorne 1977, Chapter III, §12] it follows that for $f : X \to S$ of K3 type the direct images $R^q f_* \Omega^p_{X/S}$ for $p+q=2$ are locally free and commute with arbitrary base change and $f_*\mathcal{O}_X = \mathcal{O}_S$. Therefore locally on S we have sections $\sigma \in H^0(S, f_*\Omega^2_{X/S})$ and $\rho \in H^2(X, \mathcal{O}_X)$ which induce in each geometric fibre the classes required in the definition.

It follows from [Hartshorne 1977] that the set of points of S where a smooth and proper morphism $f : X \to S$ is of K3 type is open.

We therefore obtain varieties of K3 type as follows. Let S be a scheme of finite type and flat over $\mathbb{Z}$. We consider a smooth and proper morphism $f : X \to S$ such that $f_{\mathbb{Q}}$ is of K3 type. Then over an open subset of S, the morphism is of K3 type. In particular the schemes K3n are for almost all prime numbers p of K3 type over a field of characteristic p. The set of such primes p contains the set of primes for which Charles [2013, Corollary 5] recently proved the Tate conjecture for varieties of K3 type in characteristic p.

In the following we will assume without loss of generality that $S = \operatorname{Spec} R$ and that σ and ρ are globally defined. We note that σ is closed because $H^0(X, \Omega^3_{X/S}) = 0$.

Let X be a scheme of K3 type of dimension $2n$ over a ring R. It follows from our assumptions that $\epsilon := \int (\sigma\rho)^n \in R$ is a unit.

We regard $\sigma \in H^2_{\mathrm{DR}}(X/R)$ and we choose an arbitrary lifting $\tau \in H^2_{\mathrm{DR}}(X/R)$ of ρ. We have

$$\epsilon = \int \sigma^n \tau^n.$$

Definition 23. We assume that n, $n+1$ are units in R. We define the quadratic form $\mathbb{B}_{\sigma,\tau}(\alpha)$ on $H^2_{\mathrm{DR}}(X/R)$ by

$$\mathbb{B}_{\sigma,\tau}(\alpha) = \frac{n}{2} \int (\sigma\tau)^{n-1}\alpha^2 + \frac{1-n}{\epsilon}\Big(\int \sigma^{n-1}\tau^n\alpha\Big)\Big(\int \sigma^n\tau^{n-1}\alpha\Big) + \frac{1}{\epsilon^2}\frac{n(n-1)}{2(n+1)}\Big(\int \sigma^n\tau^{n-1}\alpha\Big)^2\Big(\int \sigma^{n-1}\tau^{n+1}\Big). \tag{42}$$

Later we will consider the case $\epsilon = 1$.

Lemma 24. *We assume that R is an integral domain whose fraction field has characteristic 0. Let $\tau, \tau' \in H^2_{\mathrm{DR}}(X/R)$ be liftings of ρ. Then we have*

$$\mathbb{B}_{\sigma,\tau} = \mathbb{B}_{\sigma,\tau'}. \tag{43}$$

Up to a factor in R^* the form $\mathbb{B}_{\sigma,\tau}$ doesn't depend on the choice of σ, ρ, and τ.

If $R = \mathbb{C}$ it coincides with the usual Beauville–Bogomolov form [Beauville 1983, p. 772] up to a constant.

Proof. Let us assume the assertion (43). Then we show the uniqueness up to a constant. Let $\sigma_1 = u\sigma$ and $\tau_1 = v\tau$, where $u, v \in R^*$. We set $\mathbb{B} = \mathbb{B}_{\sigma,\tau}$ and

$\mathbb{B}_1 = \mathbb{B}_{\sigma_1,\tau_1}$. We have $\epsilon_1 = (\sigma_1\tau_1)^n = (uv)^n\epsilon$. We obtain

$$\mathbb{B}_1(\alpha) = \frac{n}{2}(uv)^{n-1}\int(\sigma\tau)^{n-1}\alpha^2 + \frac{1-n}{(uv)^n\epsilon}(uv)^{2n-1}\Big(\int\sigma^{n-1}\tau^n\alpha\Big)\Big(\int\sigma^n\tau^{n-1}\alpha\Big) + \frac{1}{(uv)^{2n}\epsilon^2}\frac{n(n-1)}{2(n+1)}\Big(\int\sigma^n\tau^{n-1}\alpha\Big)^2(u^nv^{n-1})^2\Big(\int\sigma^{n-1}\tau^{n+1}\Big)u^{n-1}v^{n+1}.$$

The (uv)-factors in the last summand together yield the factor

$$\frac{1}{(uv)^{2n}}u^{3n-1}v^{3n-1} = (uv)^{n-1}.$$

Hence we get

$$\mathbb{B}_1(\alpha) = (uv)^{n-1}\mathbb{B}(\alpha). \tag{44}$$

So the form $\mathbb{B}(\alpha)$ changes by a unit in R.

We will now consider the case $R = \mathbb{C}$. We can use the Hodge decomposition. We take for ρ and τ the complex conjugate of σ; i.e., we set $\tau := \bar\sigma$. As $\bar\sigma^{n+1} = 0$ we obtain for the Beauville–Bogomolov form

$$\mathbb{B}_{\sigma,\bar\sigma}(\alpha) = \frac{n}{2}\int(\sigma\bar\sigma)^{n-1}\alpha^2 + \frac{1-n}{\epsilon}\Big(\int\sigma^{n-1}\bar\sigma^n\alpha\Big)\Big(\int\sigma^n\bar\sigma^{n-1}\alpha\Big). \tag{45}$$

This is the usual Beauville–Bogomolov form, if we change σ by a constant such that $\epsilon = \int(\sigma\bar\sigma)^n = 1$; see (42), [Beauville 1983, p. 772; Huybrechts 1999, 1.9].

Now let $\tau = a\sigma + \gamma + \bar\sigma$, where γ is a closed 1-1-form and $a \in \mathbb{C}$, so $\tau \in H^2_{\mathrm{DR}}(X/\mathbb{C})$ is a lifting of $\bar\sigma \in H^2(X, \mathcal{O}_X)$. We evaluate the forms $\mathbb{B}_{\sigma,\bar\sigma}$ and $\mathbb{B}_{\sigma,\tau}$ (the form in the definition) in an arbitrary form $\alpha \in H^2_{\mathrm{DR}}(X/\mathbb{C})$ and show that $\mathbb{B}_{\sigma,\bar\sigma}(\alpha) = \mathbb{B}_{\sigma,\tau}(\alpha)$. Without loss of generality let $\alpha = c\sigma + \beta + c'\bar\sigma$ with $c' \neq 0$ and after multiplication with a constant we can assume that $\alpha = c\sigma + \beta + \bar\sigma$, with β a closed 1-1-form, also without loss of generality $\epsilon = \int(\sigma\bar\sigma)^n = 1$. Therefore $\alpha^2 = c^2\sigma^2 + 2c\sigma\beta + 2c\sigma\bar\sigma + 2\beta\bar\sigma + \bar\sigma^2 + \beta^2$.

Now we compute $\frac{n}{2}\int(\sigma\tau)^{n-1}\alpha^2$ for each summand:

$\underline{c^2\sigma^2}$:

$$\frac{n}{2}\int(\sigma\tau)^{n-1}c^2\sigma^2 = \frac{n}{2}\int\sigma^{n-1}(a\sigma+\gamma+\bar\sigma)^{n-1}c^2\sigma^2.$$

The p-degree (with respect to the p,q-Hodge decomposition) is $\geq 2n+2$. So this integral is zero.

$\underline{2c\sigma\beta}$:

$$n\int\sigma^{n-1}(a\sigma+\gamma+\bar\sigma)^{n-1}\sigma\beta = 0$$

because $\sigma^{n-1}\sigma\beta$ has already p-degree $2n+1$.

$\underline{2c\sigma\bar\sigma}$:

$$nc\int\sigma^{n-1}(a\sigma+\gamma+\bar\sigma)^{n-1}\sigma\bar\sigma = nc\int\sigma\bar\sigma^n.$$

So the third summand is independent from the choice of τ.

$\underline{2\beta\bar{\sigma}}$:
$$n\int \sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}\beta\bar{\sigma} = n\int \sum_{i+j+k=n-1} \frac{(n-1)!}{i!\,j!\,k!}\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\beta\bar{\sigma}.$$

The form $\sigma^{n-1+i}\bar{\sigma}^{k+1}\gamma^j\beta$ has p-degree $2n-2+2i+j+1$.

Hence the integral can be nonzero only for $i=0$ and $j=1$ and $k=n-2$. So the above integral is
$$n\int \frac{(n-1)!}{(n-2)!}\sigma^{n-1}\bar{\sigma}^{n-1}\beta\gamma.$$

This term depends on the choice of τ.

$\underline{\bar{\sigma}^2}$:
$$\frac{n}{2}\int \sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}\cdot\bar{\sigma}^2 = \frac{n}{2}\sum_{i+j+k=n-1}\int \sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^{k+2}\cdot\frac{(n-1)!}{i!j!k!}.$$

The p-degree of $\sigma^{n-1}\sigma^i\gamma^j\bar{\sigma}^{k+2}$ is $2(n-1)+2i+j$, which is $2n$ only for $i=1$, $j=0$ and $k=n-2$ or $i=0$, $j=2$, $k=n-3$.

In the first case ($i=1$, $j=0$ and $k=n-2$) the integral depends on the choice of τ. One gets the summand $\frac{1}{2}an(n-1)$.

In the second case ($i=0$, $j=2$, $k=n-3$) one gets the summand
$$\frac{(n-1)!}{2!\,(n-3)!}\frac{n}{2}\int \sigma^{n-1}\bar{\sigma}^{n-1}\gamma^2,$$
which depends on the choice of τ.

$\underline{\beta^2}$:
$$\frac{n}{2}\int \sigma^{n-1}(a\sigma+\gamma+\bar{\sigma})^{n-1}\beta^2 = \frac{n}{2}\sum_{i+j+k=n-1}\frac{(n-1)!}{i!\,j!\,k!}\int \sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\beta^2.$$

The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k\beta^2$ is $2n-2+2i+j+2$, which is $2n$ only if $i=0$ and $j=0$ and $k=n-1$.

Hence this integral does not depend on the choice of τ.

Adding up all cases we obtain (one may assume $\int(\sigma\bar{\sigma})^n=1$)
$$\frac{n}{2}\int(\sigma\bar{\sigma})^{n-1}\alpha^2 = nc+n(n-1)\int(\sigma\bar{\sigma})^{n-1}\beta\gamma+(n-1)\frac{n}{2}a + \frac{n(n-1)!}{4(n-3)!}\int(\sigma\bar{\sigma})^{n-1}\gamma^2+\frac{n}{2}\int\sigma^{n-1}\bar{\sigma}^{n-1}\beta^2. \quad (46)$$

Now we compute the other summands in $\mathbb{B}_{\sigma,\tau}(\alpha)$.

We have
$$\int \sigma^n\tau^{n-1}\alpha = \int \sigma^n(a\sigma+\gamma+\bar{\sigma})^{n-1}(c\sigma+\beta+\bar{\sigma}) = \int \sigma^n\bar{\sigma}^n = 1.$$

Then

$$\int \sigma^{n-1}\tau^n\alpha = \int \sigma^{n-1}(a\sigma+\gamma+\bar\sigma)^n(c\sigma+\beta+\bar\sigma)$$
$$= \sum_{i+j+k=n} \frac{n!}{i!\,j!\,k!}\int \sigma^{n-1}a^i\sigma^i\gamma^j\bar\sigma^k(c\sigma+\beta+\bar\sigma).$$

The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar\sigma^k c\sigma$ is $2n+2i+j$, which is $2n$ only for $i=j=0$ and $k=n$.

The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar\sigma^k\beta$ is $2n-2+2i+j+1$, which is $2n$ only for $i=0,\ j=1$ and $k=n-1$.

The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar\sigma^k\bar\sigma$ is $2n-2+2i+j$, which is $2n$ only for $i=1$, $j=0,\ k=n-1$ or for $i=0,\ j=2,\ k=n-2$.

Using this we get the formula

$$\int \sigma^{n-1}\tau^n\alpha$$
$$= c\int(\sigma\bar\sigma)^n + n\int \sigma^{n-1}\gamma\bar\sigma^{n-1}\beta + n\int a\sigma^n\bar\sigma^n + \frac{n!}{2(n-2)!}\int \sigma^{n-1}\gamma^2\bar\sigma^{n-1}$$
$$= c+na+n\int(\sigma\bar\sigma)^{n-1}\beta\gamma + \frac{n}{2}(n-1)\int(\sigma\bar\sigma)^{n-1}\gamma^2. \tag{47}$$

Equations (46) and (47) then give us the following formula for the first two summands in $\mathbb{B}_{\sigma,\tau}(\alpha)$:

$$\frac{n}{2}\int(\sigma\tau)^{n-1}\alpha^2 + (1-n)\Big(\int\sigma^{n-1}\tau^n\alpha\Big)\Big(\int\sigma^n\tau^{n-1}\alpha\Big)$$
$$= nc + \frac{n}{2}(n-1)a + \frac{n(n-1)(n-2)}{4}\int(\sigma\bar\sigma)^{n-1}\gamma^2$$
$$+ n(n-1)\int(\sigma\bar\sigma)^{n-1}\beta\gamma + \frac{n}{2}\int(\sigma\bar\sigma)^{n-1}\beta^2 + (1-n)c + (1-n)na$$
$$+ n(1-n)\int(\sigma\bar\sigma)^{1-n}\beta\gamma + (1-n)\frac{n}{2}(n-1)\int(\sigma\bar\sigma)^{n-1}\gamma^2. \tag{48}$$

A small calculation yields

$$\frac{n(n-1)(n-2)}{4} + \frac{(1-n)n(n-1)}{2} = -\frac{n^2}{4}(n-1).$$

Hence we can simplify (48) and get

$$\frac{n}{2}\int(\sigma\tau)^{n-1}\alpha^2 + (1-n)\Big(\int\sigma^{n-1}\tau^n\alpha\Big)\Big(\int\sigma^n\tau^{n-1}\alpha\Big)$$
$$= c + \frac{n}{2}\int(\sigma\bar\sigma)^{n-1}\beta^2 - \frac{n}{2}(n-1)a - \frac{n^2}{4}(n-1)\int(\sigma\bar\sigma)^{n-1}\gamma^2. \tag{49}$$

Now we compute the last summand in $\mathbb{B}_{\sigma,\tau}(\alpha)$:

$$\int \sigma^{n-1}\tau^{n+1} = \sum_{i+j+k=n+1} \frac{(n+1)!}{i!j!k!} \int \sigma^{n-1} a^i \sigma^i \gamma^j \bar{\sigma}^k.$$

The p-degree of $\sigma^{n-1}a^i\sigma^i\gamma^j\bar{\sigma}^k$ is $2(n-1)+2i+j$, which is $2n$ only for $i=1$, $j=0,\ k=n$ or for $i=0,\ j=2,\ k=n-1$.

So we get for the last summand in $\mathbb{B}_{\sigma,\tau}(\alpha)$

$$\frac{n(n-1)}{2(n+1)} \int \sigma^{n-1}\tau^{n+1} = \frac{n}{2}(n-1)a + \frac{n^2}{4}(n-1)\int (\sigma\bar{\sigma})^{n-1}\gamma^2. \tag{50}$$

(We have that $\int \sigma^n\tau^{n-1}\alpha = 1$ because of $\int(\sigma\bar{\sigma})^n = 1$.)

Then (49) and (50) yield

$$\mathbb{B}_{\sigma,\tau}(\alpha) = c + \frac{n}{2}\int (\sigma\bar{\sigma})^{n-1}\beta^2 = \mathbb{B}_{\sigma,\bar{\sigma}}(\alpha).$$

The last equality follows from [Huybrechts 1999, 1.9]. It proves (43) in the case $R=\mathbb{C}$.

We now prove (43) for arbitrary R. By base change we can replace R by it field of fractions K. Finally we may assume $K=\mathbb{C}$ by the Lefschetz principle. □

With the notation of Definition 23, let R be arbitrary. Let S be an integral domain whose field of fractions has characteristic 0 and let Y/S be a scheme of K3 type. We assume that there is a surjective ring homomorphism $S \to R$ such that X is obtained from Y by base change. Then Lemma 24 is true for R. This follows because the second cohomology groups commute with base change by assumption.

Remark. In the following proofs we will assume that there is a form $\rho \in H^2(X, \mathcal{O}_X)$ such that $1=\epsilon=\int(\sigma\rho)^n$. If R is a strict henselian local ring whose residue characteristic is relatively prime to n, such a form ρ always exists. For $\epsilon=1$ the form (42) doesn't up to a root of unity depend on the choices of σ and τ. For $R=\mathbb{C}$ it coincides then with the usual Beauville–Bogomolov form up to a root of unity of order n.

Lemma 25. *Let X be a scheme of K3 type over S. Then the form*

$$\mathbb{B}_{\sigma,\tau} : H^2_{\mathrm{DR}}(X/R) \times H^2_{\mathrm{DR}}(X/R) \to R$$

is perfect.

Proof. We can reduce to the case where R is a complete local ring with separably closed residue field. Then we may assume that $\int(\sigma\tau)^n = 1$. We do so to simplify the computation. We consider the Hodge filtration

$$H^0(X, \Omega^2_{X/R}) = \mathrm{Fil}^2 \subset \mathrm{Fil}^1 \subset H^2_{\mathrm{DR}}(X/R).$$

We claim that with respect to $\mathbb{B}_{\sigma,\tau}$

$$\mathrm{Fil}^1 \subset (\mathrm{Fil}^2)^{\perp}.$$

Let $\alpha \in \mathrm{Fil}^1$; we have to show that

$$\mathbb{B}_{\sigma,\tau}(\sigma,\alpha) = \tfrac{1}{2}(\mathbb{B}_{\sigma,\tau}(\sigma+\alpha) - \mathbb{B}_{\sigma,\tau}(\sigma) - \mathbb{B}_{\sigma,\tau}(\alpha)) = 0. \tag{51}$$

The second summand on the right-hand side is clearly 0. We note that $\sigma^n\alpha = 0$ because $\mathrm{Fil}^{2n} \cup \mathrm{Fil}^1 \subset \mathrm{Fil}^{2n+1} = 0$. We compute

$$\mathbb{B}_{\sigma,\tau}(\sigma+\alpha) = \frac{n}{2}\biggl(\int (\sigma\tau)^{n-1}\sigma^2 + 2\int (\sigma\tau)^{n-1}\sigma\alpha + \int (\sigma\tau)^{n-1}(\alpha^2)\biggr). \tag{52}$$

The other terms on the right-hand side of (42) vanish because $\sigma+\alpha \in \mathrm{Fil}^1$. We see that the first two terms on the right-hand side of (52) vanish. This shows that (51) vanishes too.

Therefore $\mathbb{B}_{\sigma,\tau}$ induces a bilinear form $\overline{\mathbb{B}}_{\sigma,\tau}$ on $\mathrm{Fil}^1 / \mathrm{Fil}^2 = H^1(X, \Omega_{X/R})$. By the verification above we obtain

$$\overline{\mathbb{B}}_{\sigma,\tau}(\alpha) = \frac{n}{2}\int (\sigma\tau)^{n-1}\alpha^2 = \frac{n}{2}\int (\sigma\rho)^{n-1}\alpha^2.$$

By the requirement (40) this is perfect and

$$(\mathrm{Fil}^2)^{\perp} = \mathrm{Fil}^1. \tag{53}$$

Finally one has

$$\mathbb{B}_{\sigma,\tau}(\sigma,\tau) = \tfrac{1}{2},$$

which is a unit. We omit the easy verification. Together with the perfectness of $\overline{\mathbb{B}}_{\sigma,\tau}$ this implies perfectness. □

We prove the following proposition for the universal deformation of a variety of K3 type. The general case of a scheme of K3 type over an artinian local ring will follow from this (compare (73)).

Proposition 26. *Let X_0 be a scheme of K3 type over an algebraically closed field k. We assume that X_0 lifts to a smooth projective scheme over a discrete valuation ring O of characteristic zero with residue class field k. We consider the universal deformation of X_0*

$$X \to S = \operatorname{Spf} W(k)[[T_1, \dots, T_r]].$$

Assume that σ and τ are chosen such that $\epsilon = 1$. Then the form

$$\mathbb{B}_{\sigma,\tau} : H^2_{\mathrm{DR}}(X/S) \times H^2_{\mathrm{DR}}(X/S) \to W(k)[[T_1, \dots, T_r]]$$

is horizontal with respect to the Gauss–Manin connection (see [Deligne 1981b, Corollaire 2.3] for the definition).

Remark. Using the arguments of [Deligne 1981b] we will show in Proposition 29 that X_k lifts over some O if $H^1(X_0, \mathcal{T}_{X_0/k}) \neq 0$.

Proof. We begin by proving a complex-analytic version of this proposition. Let $X \to S$ be a proper and smooth morphism of complex analytic manifolds. Let $\Lambda \in H^0(X, R^2 f_* \mathbb{Q})$. Then we have a pairing

$$q_\Lambda : R^2 f_* \mathbb{Q} \times R^2 f_* \mathbb{Q} \to R^{2n} f_* \mathbb{Q} \xrightarrow{\int} \mathbb{Q}_S, \tag{54}$$

defined by

$$q_\Lambda(\alpha, \beta) = \int \Lambda^{2n-2} \alpha\beta, \quad \alpha, \beta \in R^2 f_* \mathbb{Q}.$$

If $\Lambda \in H^0(X, R^2 f_* \mathbb{Q})$ is the class of a relative ample line bundle on X then the pairing (54) is nondegenerate and we have

$$\nu := \nu(\Lambda) := \int \Lambda^{2n} \neq 0.$$

If we tensor (54) with $\mathcal{O}_S$ we obtain a horizontal pairing with respect to the Gauss–Manin connection:

$$q_\Lambda : H^2_{\mathrm{DR}}(X/S) \times H^2_{\mathrm{DR}}(X/S) \to \mathcal{O}_S. \tag{55}$$

Assume that Λ is a cohomology class such that q_Λ is nondegenerate and $\nu(\Lambda) \neq 0$. Then we denote by $(R^2 f_* \mathbb{Q})_0 \subset R^2 f_* \mathbb{Q}$ the local system which is the orthogonal complement of Λ. The vector bundle $H^2_{\mathrm{DR}}(X/S)$ decomposes as a vector bundle with connection

$$H^2_{\mathrm{DR}}(X/S) = (H^2_{\mathrm{DR}}(X/S))_0 \oplus \mathcal{O}_S \Lambda.$$

Lemma 27. *Let $\Lambda \in H^2_{\mathrm{DR}}(X/S)$ and $\nu(\Lambda) = \int \Lambda^{2n} \in \mathcal{O}_S$. Then we have the following formula for all $\alpha \in H^2_{\mathrm{DR}}(X/S)$:*

$$\nu(\Lambda)^2 \mathbb{B}_{\sigma,\tau}(\alpha) = \mathbb{B}_{\sigma,\tau}(\Lambda) \left[(2n-1)\nu(\Lambda) \int \Lambda^{2n-2} \alpha^2 - (2n-2) \left(\int \Lambda^{2n-1} \alpha \right)^2 \right].$$

Proof. Both terms on each side are functions in $\mathcal{O}_S$. We consider them as functions on the complex manifold S. For each $s \in S(\mathbb{C})$ we evaluate the functions at s. The analogous equality in s, namely for $\Lambda_s, \alpha_s, \sigma_s, \tau_s \in H^2_{\mathrm{DR}}(X_s/\mathbb{C}) = H^2_{\mathrm{DR}}(X/S) \otimes_{\mathcal{O}_S} k(s)$,

$$\begin{aligned} &\nu(\Lambda_s)^2 \mathbb{B}_{\sigma_s,\tau_s}(\alpha_s) \\ &\quad = \mathbb{B}_{\sigma_s,\tau_s}(\Lambda_s) \left[(2n-1)\nu(\Lambda_s) \int \Lambda_s^{2n-2} \alpha_s^2 - (2n-2) \left(\int \Lambda_s^{2n-1} \alpha_s \right)^2 \right] \end{aligned} \tag{56}$$

was shown in [Beauville 1983, Théorème 5(c)] for the Beauville–Bogomolov form over $\mathbb{C}$. Since this form differs from $\mathbb{B}_{\sigma_s,\tau_s}$ by a constant in $\mathbb{C}$ we obtain (56). Hence both functions coincide on $S(\mathbb{C})$. But then the algebraic functions in $\mathcal{O}_S$ coincide as well. $\square$

The formula (56) shows that for $\alpha \in (H^2_{\mathrm{DR}}(X/S))_0$

$$\nu^2 \mathbb{B}_{\sigma,\tau}(\alpha) = \mathbb{B}_{\sigma,\tau}(\Lambda)(2n-1)\nu q_\Lambda(\alpha). \tag{57}$$

Let $s \in S$. Since $\mathbb{B}_{\sigma_s,\tau_s}$ is up to a root of unity of order n the Beauville–Bogomolov form, we know that $\mathbb{B}_{\sigma_s,\tau_s}(\Lambda_s)$ is a real number times an n-th root of unity by [Beauville 1983, Théorème 5(a)]. From this it follows that the analytic function $\mathbb{B}_{\sigma,\tau}(\Lambda)$ on S is constant. Therefore $\mathbb{B}_{\sigma,\tau}$ is by (57) a horizontal form with respect to the Gauss–Manin connection on the bundle $(H^2_{\mathrm{DR}}(X/S))_0$.

We show that $(H^2_{\mathrm{DR}}(X/S))_0$ and $\mathcal{O}_S$ are orthogonal for the form $\mathbb{B}_{\sigma,\tau}$ too. We have to show that

$$\nu^2(\mathbb{B}_{\sigma,\tau}(\alpha+\Lambda) - \mathbb{B}_{\sigma,\tau}(\alpha) - \mathbb{B}_{\sigma,\tau}(\Lambda)) = 0 \tag{58}$$

for all $\alpha \in (H^2_{\mathrm{DR}}(X/S))_0$. From Lemma 27 we obtain

$$\nu^2 \mathbb{B}_{\sigma,\tau}(\alpha+\Lambda) = \mathbb{B}_{\sigma,\tau}(\Lambda)\left[(2n-1)\nu \int \Lambda^{2n-2}\alpha^2 + (2n-1)\nu^2 - (2n-2)\nu^2\right].$$

From this one obtains (58). Therefore it suffices to show that $\mathbb{B}_{\sigma,\tau}$ is horizontal on the subbundle $\mathcal{O}_S\Lambda \subset H^2_{\mathrm{DR}}(X/S)$. This is equivalent to saying that $\mathbb{B}_{\sigma,\tau}(\Lambda) \in \mathcal{O}_S$ is a constant function. This we have seen above.

Now we prove Proposition 26. The formal scheme $\operatorname{Spf} W(k)[[T_1, \dots, T_r]]$ has $(p, T_1, \dots, T_r)$ as an ideal of definition and this is also an ideal of definition for X.

On the other hand we can consider the universal deformation of a lifting $\widetilde{X}/O$ of X_0, which exists by assumption. Then we obtain a formal scheme Y over $\operatorname{Spf} O[[T_1, \dots, T_d]]$, where the ideal of definition in the last ring is now $(T_1, \dots, T_d)$. We may assume that O is complete. Then we a have a natural map

$$W(k)[[T_1, \dots, T_r]] \to O[[U_1, \dots, U_d]], \tag{59}$$

which corresponds on the tangent spaces to the natural homomorphism

$$H^1(\widetilde{X}, \Omega_{\widetilde{X}/O}) \to H^1(X_0, \Omega_{X_0/k}). \tag{60}$$

Therefore we have $r = d$ and we may arrange after a coordinate transformation that $T_i \mapsto U_i + a_i$ by the map (59), where the a_i are in the maximal ideal of O. We see that the regular parameter system $(p, T_1, \dots, T_d)$ of the local ring on the left-hand side of (59) is mapped to a parameter system on the right-hand side. Therefore the morphism (59) is injective. By definition, the push-forward of X by (59) is the completion of Y in the adic topology defined by the maximal ideal. Because the map induced by (59) on the de Rham cohomology is also injective, it suffices to show that the Beauville–Bogomolov form of the family Y is horizontal. We take an embedding $O \to \mathbb{C}$. Then we obtain the universal deformation of $\widetilde{X}_{\mathbb{C}}$. It suffices to show that $\mathbb{B}_{\sigma,\tau}$ is horizontal for the Gauss–Manin connection of this family. Since we obtain this by completion of the Kuranishi family $f : \mathfrak{X} \to S$ of $\widetilde{X}_{\mathbb{C}}$, we are reduced to the case above. We have to ensure that there is a cohomology class $\Lambda \in H^0(S, R^2 f_*\mathbb{Q})$ such that

$$q_\Lambda \text{ is nondegenerate} \quad \text{and} \quad \nu(\Lambda) \neq 0. \tag{61}$$

Let $s_0 \in S$ be the point such that $f^{-1}(s_0) = \widetilde{X}_{\mathbb{C}}$. Let

$$\Lambda_0 \in (R^2 f_*\mathbb{Q})_{s_0} = H^2(\widetilde{X}_{\mathbb{C}}, \mathbb{Q})$$

be the cohomology class of an ample line bundle on $\widetilde{X}_{\mathbb{C}}$. By shrinking S we may assume that $R^2 f_*\mathbb{Q}$ is a constant local system on S. But then Λ_0 extends to a global section Λ of $R^2 f_*\mathbb{Q}$. Then Λ meets the requirements (61). This proves Proposition 26. □

4. Deformations of varieties of K3 type

Let X_0/k be a projective and smooth scheme of K3 type over a perfect field k of characteristic $p \geq 3$. We consider the universal deformation

$$\mathfrak{X} \to S = \operatorname{Spf} A,$$

where

$$A = W[\![T_1, \ldots, T_r]\!], \quad r = \dim_k H^1(X_0, \mathcal{T}_{X_0/k}).$$

We consider the Gauss–Manin connection

$$\nabla : H^2_{\mathrm{DR}}(\mathfrak{X}/S) \to H^2_{\mathrm{DR}}(\mathfrak{X}/S) \otimes_A \Omega^1_{S/W}.$$

If we compose this with the natural maps

$$\partial/\partial t_i : \Omega^1_{S/W} \to A, \quad i = 1, \ldots, r,$$

we obtain the maps

$$\nabla_i : H^2_{\mathrm{DR}}(\mathfrak{X}/S) \to H^2_{\mathrm{DR}}(\mathfrak{X}/S).$$

The de Rham cohomology is endowed with the Hodge filtration

$$0 \subset \operatorname{Fil}^2 H^2_{\mathrm{DR}}(\mathfrak{X}/S) \subset \operatorname{Fil}^1 H^2_{\mathrm{DR}}(\mathfrak{X}/S) \subset \operatorname{Fil}^0 H^2_{\mathrm{DR}}(\mathfrak{X}/S) = H^2_{\mathrm{DR}}(\mathfrak{X}/S).$$

We have $\operatorname{Fil}^2 H^2_{\mathrm{DR}}(\mathfrak{X}/S) = H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S})$. We denote by $\operatorname{gr}^t H^2_{\mathrm{DR}}(\mathfrak{X}/S)$ the subquotients of this filtration. By Griffiths transversality, ∇ induces a map

$$\operatorname{gr}^t \nabla : \operatorname{gr}^t H^2_{\mathrm{DR}}(\mathfrak{X}/S) \to \operatorname{gr}^{t-1} H^2_{\mathrm{DR}}(\mathfrak{X}/S) \otimes_A \Omega^1_{S/W}, \tag{62}$$

which is A-linear. We are interested in this map for $t = 2$. By duality we obtain an A-linear map

$$\mathcal{T}_{S/W} \to \mathrm{Hom}_A(H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S}), H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S})). \tag{63}$$

It is proved for K3 surfaces in [Deligne 1981b] that this is an isomorphism. The same argument works for varieties of K3 type. Indeed, the map (63) factors as

$$\mathcal{T}_{S/W} \to H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/S}) \to \mathrm{Hom}_A(H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S}), H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S})).$$

The first arrow is the Kodaira–Spencer map, which is an isomorphism, and the second map is the cup product. To see that the second map is an isomorphism we choose a generator $\omega \in H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S})$. The multiplication with ω induces an isomorphism $\mathcal{T}_{\mathfrak{X}/S} \cong \Omega^1_{\mathfrak{X}/S}$. Therefore the cup product with ω is an isomorphism

$$H^1(\mathfrak{X}, \mathcal{T}_{\mathfrak{X}/S}) \cong H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S}).$$

This proves that (63) is an isomorphism. The isomorphism (63) signifies that $\nabla_1(\omega), \dots, \nabla_r(\omega)$ is a basis of $H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S})$.

Lemma 28. *The maps* $\mathrm{gr}^t \nabla$ (62) *are for* $t = 1, 2$ *split injections of* A*-modules.*

Proof. Clearly it is enough to show that the maps

$$\mathcal{T}_{S/W} \to \mathrm{Hom}_A(\mathrm{gr}^t H^2_{\mathrm{DR}}(\mathfrak{X}/S), \mathrm{gr}^{t-1} H^2_{\mathrm{DR}}(\mathfrak{X}/S))$$

induced by $\mathrm{gr}^t \nabla$, where $t = 1, 2$, are isomorphisms. We have already seen this for $t = 2$. For $t = 1$ we have the A-module homomorphism

$$\mathcal{T}_{S/W} \to \mathrm{Hom}_A(H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/S}), H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})). \tag{64}$$

The Kodaira–Spencer map gives an isomorphism $\mathcal{T}_{S/W} \otimes_A k \cong H^1(X_0, \mathcal{T}_{X_0/k})$. Using the Nakayama's lemma we see that (64) is an isomorphism if and only if the cup product induces a perfect pairing of k-vector spaces:

$$H^1(X_0, \mathcal{T}_{X_0/k}) \times H^1(X_0, \Omega^1_{X_0/k}) \to H^2(X_0, \mathcal{O}_{X_0}). \tag{65}$$

This follows from the definition of a variety of K3 type and the following commutative diagram in the notation of Definition 22:

$$\begin{tikzcd}
H^1(X_0, \mathcal{T}_{X_0/k}) \times H^1(X_0, \Omega^1_{X_0/k}) \arrow[r] \arrow[d, "\cup\sigma", "\mathrm{id}"'] & H^2(X_0, \mathcal{O}_{X_0}) \arrow[d, "\cup\sigma"] \\
H^1(X_0, \Omega^1_{X_0/k}) \times H^1(X_0, \Omega^1_{X_0/k}) \arrow[r] & H^2(X_0, \Omega^2_{X_0/k}) \arrow[d, "\cup\sigma^{n-1}\rho^{n-1}"] \\
 & H^{2n}(X_0, \Omega^{2n}_{X_0/k})
\end{tikzcd}$$

The composition of the two vertical arrows on the right-hand side is an isomorphism because $\sigma^n\rho^n$ is a generator of $H^{2n}(X_0, \Omega^{2n}_{X_0/k})$. The perfectness follows therefore from the perfectness of the pairing (40). □

Proposition 29. *Let X_0/k be a scheme of K3 type over a perfect field k and let L_0 be a very ample line bundle on X_0. We assume that* $\dim H^1(X_0, \mathcal{T}_{X_0/k}) \geq 1$.

Then there is a projective scheme X over a discrete valuation ring O of mixed characteristic with residue class field k and a very ample line bundle L on X such that

$$(X, L) \otimes_O k \cong (X_0, L_0).$$

If we assume moreover that the first Chern class $c_1(L_0) \in H^1(X_0, \Omega^1_{X_0/k})$ is not zero, a projective scheme X exists over $W = O$.

Proof. We follow the proof of [Deligne 1981b] for K3 surfaces but we indicate in detail the necessary changes for varieties of K3 type. We consider the functor $\operatorname{Def}(X_0, L_0)$ of infinitesimal deformations of the pair (X_0, L_0). This functor is represented by a closed subscheme of $S = \operatorname{Spf} A$ given by a single equation $f \in A$. For this the arguments of [Deligne 1981b] apply with no changes.

We will show that $f \notin pA$. We assume $f \in pA$ and deduce a contradiction. In this case L_0 lifts to a line bundle $\mathcal{L}_0$ on $\mathfrak{X}_0 = \mathfrak{X} \otimes_A A_0$, where $A_0 = A/pA$. We consider the Chern class

$$x = c_1(\mathcal{L}_0) \in H^2_{\mathrm{crys}}(\mathfrak{X}_0/A) = H^2_{\mathrm{DR}}(\mathfrak{X}/A).$$

It is not zero because $c_1(L_0)$ is not zero by our assumption that L_0 is very ample. By general facts on Chern classes we have [Deligne 1981b, Proposition 2.9]

$$Fx = px, \quad \nabla x = 0.$$

We write $x = p^m y$, where $y \in H^2_{\mathrm{DR}}(\mathfrak{X}/A)$ is not divisible by p. Then $y_0 = y \bmod p \in H^2_{\mathrm{DR}}(\mathfrak{X}_0/A_0)$ is not trivial. Since p is not a zero divisor in $H^2_{\mathrm{DR}}(\mathfrak{X}/A)$ we have

$$Fy = py, \quad \nabla y = 0.$$

It follows from Lemma 21 that $y_0 \in \operatorname{Fil}^1 H^2_{\mathrm{DR}}(\mathfrak{X}_0/A_0)$. Lemma 28 shows that the following maps are for $t = 1, 2$ injective:

$$\operatorname{gr}^t \nabla : \operatorname{gr}^t H^2_{\mathrm{DR}}(\mathfrak{X}_0/A_0) \to \operatorname{gr}^{t-1} H^2_{\mathrm{DR}}(\mathfrak{X}_0/A_0) \otimes_A \Omega^1_{A_0/W}.$$

The injectivity of these maps and $\nabla y_0 = 0$ implies that $y_0 = 0$. This contradiction shows that $f \notin pA$.

We set $B = A/fA$. The universal line bundle $\mathcal{L}$ on $\mathfrak{X}_B$ is very ample. Since A is factorial, the prime p is not a zero divisor in B and in particular $B[1/p] \neq 0$.

Let $\tilde{\mathfrak{q}}$ a maximal ideal in $B[1/p]$. We set $\mathfrak{q} = \tilde{\mathfrak{q}} \cap B$. The normalization O of $B/\mathfrak{q}$ is the desired ring.

It remains to prove the last statement. It follows from [Ogus 1979, Corollary 1.14] and the perfectness of the pairing (65) that (X_0, L_0) doesn't lift to $A_0/\mathfrak{m}_0^2$, where $\mathfrak{m}_0$ denotes the maximal ideal of A_0. But this implies that $B = A/fA$ is a power series ring over W. In particular we find an augmentation $B \to W$. We obtain the desired scheme by base change of $\mathfrak{X}_B$. □

Let $\alpha : R' \to R$ be a surjective homomorphism of local artinian W-algebras with residue class field k. We set $\mathfrak{a} = \operatorname{Ker}\alpha$. We assume that $\mathfrak{a}\mathfrak{m}_{R'} = 0$, where $\mathfrak{m}_{R'}$ denotes the maximal ideal of R'. Let X/R be a deformation of X_0 and let X' be a deformation of X over R'. We have a natural isomorphism $H^2_{\text{crys}}(X/R') \cong H^2_{\text{DR}}(X'/R')$.

Let Y/R' be another deformation of X. Then we obtain a natural isomorphism

$$H^2_{\text{DR}}(X'/R') \to H^2_{\text{DR}}(Y/R'). \tag{66}$$

There is an explicit formula for this isomorphism in terms of the Gauss–Manin connection on the universal deformation S; see (69) below.

We denote by $F_Y \in H^2_{\text{DR}}(X'/R')$ the preimage of

$$H^0(Y, \Omega^2_{Y/R'}) = \operatorname{Fil}^2 H^2_{\text{DR}}(Y/R') \subset H^2_{\text{DR}}(Y/R')$$

by the isomorphism (66).

Proposition 30. *We assume that $\mathfrak{a}\mathfrak{m}_{R'} = 0$. The direct summand $F_Y \subset H^2_{\text{DR}}(X'/R')$ is contained in $\operatorname{Fil}^1 H^2_{\text{DR}}(X'/R')$. The map $Y \mapsto F_Y$ is a bijection between isomorphism classes of liftings Y/R' of X/R and direct summands $F \subset \operatorname{Fil}^1 H^2_{\text{DR}}(X'/R')$ which lift the direct summand $\operatorname{Fil}^2 H^2_{\text{DR}}(X/R) \subset \operatorname{Fil}^1 H^2_{\text{DR}}(X/R)$.*

Proof. We set $F' = \operatorname{Fil}^2 H^2_{\text{DR}}(X'/R')$. Let $F \subset H^2_{\text{DR}}(X'/R')$ be an arbitrary direct summand which lifts $\operatorname{Fil}^2 H^2_{\text{DR}}(X/R)$. We call this a lift of the Hodge filtration.

We consider the canonical map

$$F \to H^2_{\text{DR}}(X'/R')/F'. \tag{67}$$

Its image is in $\mathfrak{a}(H^2_{\text{DR}}(X'/R')/F') \cong \mathfrak{a} \otimes_k (H^2_{\text{DR}}(X_0/k)/\operatorname{Fil}^2 H^2_{\text{DR}}(X_0/k))$. The map (67) factors through $F \to \operatorname{Fil}^2 H^2_{\text{DR}}(X_0/k)$. Therefore liftings of the Hodge filtration are classified by homomorphisms of k-vector spaces

$$\varkappa(F) : H^0(X_0, \Omega^2_{X_0/k}) \to \mathfrak{a} \otimes_k (H^2_{\text{DR}}(X_0/k)/\operatorname{Fil}^2 H^2_{\text{DR}}(X_0/k)). \tag{68}$$

The assertion that $F_Y \subset \operatorname{Fil}^1 H^2_{\text{DR}}(X'/R')$ is equivalent to saying that

$$\varkappa(F_Y)(H^0(X_0, \Omega^2_{X_0/k})) \subset \mathfrak{a} \otimes_k (\operatorname{Fil}^1 H^2_{\text{DR}}(X_0/k)/\operatorname{Fil}^2 H^2_{\text{DR}}(X_0/k)).$$

The deformation X'/R' of X_0 is given by a uniquely determined W-algebra homomorphism $f : A \to R'$ and the deformation Y is given by $g : A \to R'$. We obtain a diagram such that the two compositions are equal:

$$A \underset{g}{\overset{f}{\rightrightarrows}} R' \longrightarrow R.$$

The isomorphism (66) is obtained as follows. Let $u \in H^2_{\mathrm{DR}}(X'/R')$. We find $\tilde{u} \in H^2_{\mathrm{DR}}(\mathfrak{X}/S)$ such that $u = f_*(\tilde{u})$. We set $v = g_*(\tilde{u})$. Then (66) is given as follows [Deligne 1981a, Lemma 1.1.2]:

$$H^2_{\mathrm{DR}}(X'/R') \to H^2_{\mathrm{DR}}(Y/R'), \quad u \mapsto v + \sum_{i=1}^{r} (f(t_i) - g(t_i)) \check{\nabla}_i(\tilde{u}). \tag{69}$$

We denote here by $\check{\nabla}_i(\tilde{u})$ the image of $\nabla_i(\tilde{u})$ in $H^2_{\mathrm{DR}}(X_0/k)$. The formula (69) makes sense because $f(t_i) - g(t_i) \in \mathfrak{a}$.

Now we take for $\tilde{u}$ a generator of the free A-module $\mathrm{Fil}^2\, H^2_{\mathrm{DR}}(\mathfrak{X}/S)$. We deduce from (66) that

$$u - \sum_{i=1}^{r} (f(t_i) - g(t_i)) \check{\nabla}_i(\tilde{u})$$

is a generator of F_Y. Let $u_0 \in \mathrm{Fil}^2\, H^2_{\mathrm{DR}}(X_0/k)$ be the image of $\tilde{u}$. Then the map $\varkappa(F_Y)$ is given by

$$\varkappa(F_Y)(u_0) = -\sum_{i=1}^{r} (f(t_i) - g(t_i)) \otimes \check{\nabla}_i(\tilde{u}) \in \mathfrak{a} \otimes_k \mathrm{gr}^1\, H^2_{\mathrm{DR}}(X_0/k).$$

This formula shows that $F_Y \subset \mathrm{Fil}^1\, H^2_{\mathrm{DR}}(X/R)$. As we remarked, (63) implies that $\check{\nabla}_i(\tilde{u})$ form a basis of $\mathrm{gr}^1\, H^2_{\mathrm{DR}}(X_0/k)$. It follows that F_Y determines the elements $a_i := f(t_i) - g(t_i) \in \mathfrak{a}$ for $i = 1, \dots, r$. Given such elements a_i we define $g(t_i) = f(t_i) - a_i$. The homomorphism $g : A \to R'$ thus defined gives the desired variety of K3 type. □

We will now extend the proposition to the case where $R' \to R$ is an arbitrary pd-thickening with nilpotent divided powers on $\mathfrak{a}$.

We assume now that k is algebraically closed. We assume that $2n = \dim X_0$ is prime to the characteristic p of k. We also assume that X_0 lifts to a smooth projective scheme over some discrete valuation ring O with residue class field k. We fix generators σ and ρ of the 1-dimensional k-modules $H^0(X_0, \Omega^2_{X_0/k})$ and $H^2(X_0, \mathcal{O}_{X_0})$ respectively such that $\int (\sigma\rho)^n = 1$. We can lift them to generators $\tilde{\sigma}$ and $\tilde{\rho}$ of the cohomology groups $H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/S})$ and $H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ respectively. Then we obtain by Proposition 26 and Lemma 25 a horizontal perfect symmetric pairing

$$(\ ,\ ) : H^2_{\mathrm{DR}}(\mathfrak{X}/S) \times H^2_{\mathrm{DR}}(\mathfrak{X}/S) \to S, \tag{70}$$

which depends only on σ and ρ. With respect to this pairing the Hodge filtration is self dual:

$$(\mathrm{Fil}^1)^{\perp} = \mathrm{Fil}^2, \quad (\mathrm{Fil}^2)^{\perp} = \mathrm{Fil}^1 .$$

In the situation of the proposition it is equivalent to say that the lift of the Hodge filtration $F \subset H^2_{\mathrm{DR}}(X'/R')$ is in $\mathrm{Fil}^1 H^2_{\mathrm{DR}}(X'/R')$ or that $F \subset H^2_{\mathrm{DR}}(X'/R')$ is isotropic. Indeed, we take $\tilde{c} \in H^2_{\mathrm{DR}}(\mathfrak{X}/S)$, which induces a generator of the A-module $H^2_{\mathrm{DR}}(\mathfrak{X}/S)/\mathrm{Fil}^1$. Then $(\tilde{u}, \tilde{c})$ is a unit in A. The image c of $\tilde{c}$ in $H^2_{\mathrm{DR}}(X'/R')$ induces a basis of $H^2_{\mathrm{DR}}(X'/R')/\mathrm{Fil}^1 H^2_{\mathrm{DR}}(X'/R')$.

Any lifting of the Hodge filtration has a generator of the form

$$u + \beta c + \sum_{i=1}^{r} \alpha_i \breve{\nabla}_i(\tilde{u}), \quad \alpha, \beta \in \mathfrak{a}. \tag{71}$$

Assume F is isotropic. Since u is orthogonal to $\breve{\nabla}_i(\tilde{u})$ we obtain $2\beta(u, c) = 0$ which implies $\beta = 0$. This implies $F \subset \mathrm{Fil}^1 H^2_{\mathrm{DR}}(X'/R')$. On the other hand the vector (71) is isotropic if $\beta = 0$.

Theorem 31. *Let X_0 be a projective scheme of K3 type over an algebraically closed field k of characteristic $p > 0$.*

Let $\alpha : R' \to R$ be a surjective morphism of artinian local W-algebras with residue class field k. We assume that the kernel $\mathfrak{a}$ of α is endowed with nilpotent divided powers which are compatible with the canonical divided powers on pW.

Let X/R be a deformation of X_0 and X'/R' a lifting of X.

If Y/R' is an arbitrary lifting of X, the Gauss–Manin connection provides an isomorphism

$$H^2_{\mathrm{DR}}(X'/R') \to H^2_{\mathrm{DR}}(Y/R') \tag{72}$$

which respects the symmetric bilinear forms on both sides. We denote by F_Y the preimage of $\mathrm{Fil}^2 H^2_{\mathrm{DR}}(Y/R')$ by this isomorphism.

The map $Y \mapsto F_Y$ is a bijection between liftings Y/R' of X and liftings of the Hodge filtration $\mathrm{Fil}^2 H^2_{\mathrm{DR}}(X/R) \subset H^2_{\mathrm{DR}}(X/R)$ to isotropic direct summands $F \subset H^2_{\mathrm{DR}}(X'/R')$.

Proof. The assertion that (72) respects the pairing $(\ ,\ )$ follows because the pairing is horizontal. Therefore F_Y is isotropic.

We consider the divided powers of the ideal $\mathfrak{a}$:

$$\mathfrak{a} \supset \mathfrak{a}^{[2]} \supset \cdots \supset \mathfrak{a}^{[t-1]} \supset \mathfrak{a}^{[t]} = 0.$$

If $t = 2$, the theorem follows from the proposition. We consider the nilpotent pd-thickenings

$$R' \to R'/\mathfrak{a}^{[t-1]} \to R.$$

By induction we may assume that the theorem holds for the second thickening.

We start with an isotropic lifting $F \subset H^2_{\mathrm{DR}}(X'/R')$ of the Hodge filtration. Let $R_1 = R/\mathfrak{a}^{[t-1]}$. Then F induces a filtration $F_1 \subset H^2_{\mathrm{DR}}(X'_{R_1}/R_1)$. By induction there is a lifting Z/R_1 of X which corresponds to F_1. We choose an arbitrary lifting Z'/R' of Z. Since Z' is also a lifting of X, we have an isomorphism

$$H^2_{\mathrm{DR}}(X'/R') \to H^2_{\mathrm{DR}}(Z'/R').$$

Let G be the image of F under this isomorphism. Then G is a lifting of the Hodge filtration $\mathrm{Fil}^2\, H^2_{\mathrm{DR}}(Z/R_1) \subset H^2_{\mathrm{DR}}(Z/R_1)$. If the proposition is applicable to $R' \to R_1$ we find a lifting Y/R' of Z/R_1 which corresponds to $G \subset H^2_{\mathrm{DR}}(Z'/R')$ and therefore to $F \subset H^2_{\mathrm{DR}}(X'/R')$. Thus our map is surjective.

Therefore it suffices to show our theorem for $R' \to R_1$. The kernel $\mathfrak{b} = \mathfrak{a}^{[t-1]}$ is endowed with the trivial divided powers and we have $\mathfrak{b}^2 = 0$. Decomposing $R' \to R_1$ into a series of small surjections (as in the proposition) $R' \to R_m \to \cdots \to R_1$, we may argue as above.

The injectivity follows easily in the same manner. $\square$

We may reformulate this in the language of crystals. Let X be the deformation of X_0 over an artinian local ring R with residue class field k (or equivalently a continuous homomorphism $A \to R$).

Suppose $R' \to R$ is a nilpotent pd-thickening where $R' \to R$ is a homomorphism of local artinian rings with residue field k. We consider the crystalline cohomology

$$H^2_{\mathrm{crys}}(X/R').$$

This is a crystal in R' which is induced from the Gauss–Manin connection on $H^2_{\mathrm{DR}}(\mathfrak{X}/S)$. Therefore (70) induces a bilinear form of crystals

$$H^2_{\mathrm{crys}}(X/R') \times H^2_{\mathrm{crys}}(X/R') \to R'. \tag{73}$$

The Hodge filtration on $H^2_{\mathrm{DR}}(X/R) = H^2_{\mathrm{crys}}(X/R)$ is selfdual with respect to this bilinear form.

We may reformulate the last theorem.

Corollary 32. *Let $R' \to R$ be a surjective homomorphism of artinian local rings with algebraically closed residue class field k whose kernel is endowed with nilpotent divided powers compatible with p.*

Let X/R be a deformation of X_0. Then the liftings of X to X' correspond bijectively to liftings of the Hodge filtration to selfdual filtrations of $H^2_{\mathrm{crys}}(X/R')$.

Corollary 33. *Let $R' \to R$ be a nilpotent pd-thickening and let X/R be as in Corollary 32. Let X'/R' be a lifting of X. Let $\alpha : X \to X$ be an automorphism of the R-scheme X (but not necessarily of the deformation).*

Then α lifts to an automorphism $\alpha' : X' \to X'$ if and only if $\alpha^* : H^2_{\mathrm{crys}}(X/R') \to H^2_{\mathrm{crys}}(X/R')$ respects the Hodge filtration given by X'.

Proof. The universal deformation space S classifies pairs (X, ρ) where X is a scheme of K3 type over R and $\rho : X_0 \to X_k$ is an isomorphism.

Since X is a deformation of X_0, the map ρ is given. Let $\alpha_0 : X_0 \to X_0$ be the automorphism induced by α. The data α is equivalent to saying that the two pairs (X, ρ) and $(X, \rho\alpha_0)$ are isomorphic as deformations.

The existence of a lifting α' is equivalent to saying that the pairs (X', ρ) and $(X', \rho\alpha_0)$ are isomorphic as deformations. Thus we conclude by Corollary 32. □

We will now prove the compatibility of the Beauville–Bogomolov form with the Frobenius endomorphism. Let X_0/k be a projective and smooth scheme of K3 type over an algebraically closed field k of characteristic p. We will write $W = W(k)$ for the ring of Witt vectors. We consider the universal deformation

$$\mathfrak{X} \to S = \operatorname{Spf} A,$$

as in Section 4. By Proposition 26 we have a perfect and horizontal Beauville–Bogomolov form,

$$\mathbb{B} : H^2_{\mathrm{DR}}(\mathfrak{X}/A) \times H^2_{\mathrm{DR}}(\mathfrak{X}/A) \to A.$$

We set $\mathfrak{X}_0 = \mathfrak{X} \otimes_A A_0$. Then we have the canonical isomorphisms

$$H^2_{\mathrm{DR}}(\mathfrak{X}/A) \cong H^2_{\mathrm{crys}}(\mathfrak{X}_0/A).$$

We will denote by σ a ring endomorphism of $A = W[\![T_1, \dots, T_r]\!]$ which extends the Frobenius on W and induces the Frobenius endomorphism modulo p. We will denote by $\rho : \operatorname{Spf} A \to \operatorname{Spf} A$ the morphism $\operatorname{Spf} \sigma$. The relative Frobenius Fr is given by the diagram

$$\begin{tikzcd}
\mathfrak{X}_0 \arrow[r, "\mathrm{Fr}"] \arrow[d] & \mathfrak{X}_0^{(p)} \arrow[r] \arrow[d] & \mathfrak{X}_0 \arrow[d] \\
\operatorname{Spf} A \arrow[r] & \operatorname{Spf} A \arrow[r, "\rho"'] & \operatorname{Spf} A
\end{tikzcd}$$

where the second square is cartesian and the composition of the upper horizontal arrows is the absolute Frobenius. Taking the crystalline cohomology we obtain the morphisms

$$A \otimes_{\sigma, A} H^2_{\mathrm{crys}}(\mathfrak{X}_0/A) \cong H^2_{\mathrm{crys}}(\mathfrak{X}_0^{(p)}/A) \xrightarrow{F} H^2_{\mathrm{crys}}(\mathfrak{X}_0/A).$$

We may view this as morphisms of crystals on $\operatorname{Spf} A_0$. We will set

$$\mathcal{H} = H^2_{\mathrm{DR}}(\mathfrak{X}/A) \quad \text{and} \quad \mathcal{H}^{(\sigma)} = A \otimes_{\sigma, A} \mathcal{H}.$$

Then we may write

$$F : \mathcal{H}^{(\sigma)} \to \mathcal{H}.$$

Since this A-module homomorphism is induced by a morphism of crystals we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{H}^{(\sigma)} \arrow[r, "F"] \arrow[d, "\nabla^{(\sigma)}"'] & \mathcal{H} \arrow[d, "\nabla"] \\
\Omega^1_{A/W} \otimes_A \mathcal{H}^{(\sigma)} \arrow[r, "\mathrm{id}\otimes F"'] & \Omega^1_{A/W} \otimes_A \mathcal{H}
\end{tikzcd}$$

We denote here by $\Omega^1_{A/W}$ the continuous differentials, by ∇ the Gauss–Manin connection and by $\nabla^{(\sigma)}$ the inverse image of the Gauss–Manin connection by ρ.

Proposition 34. *We assume that X_0 lifts to a projective smooth scheme X over W (compare Proposition 29). We have on $\mathcal{H}^{(\sigma)}$ two horizontal forms*

$$\mathbb{B}^{(\sigma)} := A \otimes_{\sigma,A} \mathbb{B}, \quad \mathbb{B}(F\tilde{\alpha}, F\tilde{\beta}), \quad \textit{where } \tilde{\alpha}, \tilde{\beta} \in \mathcal{H}^{(\sigma)}. \tag{74}$$

For a suitable choice of the Frobenius lift $\sigma : A \to A$, we have after multiplying $\mathbb{B}$ by a unit in W the relation

$$\mathbb{B}(F\tilde{\alpha}, F\tilde{\beta}) = p^2 \mathbb{B}^{(\sigma)}(\tilde{\alpha}, \tilde{\beta}). \tag{75}$$

If we regard F as a σ-linear homomorphism $\mathcal{H} \to \mathcal{H}$ and if we take $\tilde{\alpha} = 1 \otimes \alpha$ and $\tilde{\beta} = 1 \otimes \beta$ for $\alpha, \beta \in \mathcal{H}$ with $\alpha, \beta \in \mathcal{H}$, we may rewrite the last relation as

$$\mathbb{B}(F\alpha, F\beta) = p^2 \sigma(\mathbb{B}(\alpha, \beta)). \tag{76}$$

Proof. We consider the relative Frobenius

$$W \otimes_{\sigma,W} H^2_{\mathrm{crys}}(X_0/W) \cong H^2_{\mathrm{crys}}(X_0^{(p)}/W) \xrightarrow{F} H^2_{\mathrm{crys}}(X_0/W). \tag{77}$$

We take a very ample line bundle L on X_0. It defines a cohomology class,

$$\Lambda \in H^2_{\mathrm{crys}}(X_0/W) = H^2_{\mathrm{DR}}(X/W). \tag{78}$$

By Lemma 27 the form

$$\mathbb{B}_\Lambda(\alpha) := (2n-1)\nu(\Lambda) \int \Lambda^{2n-2}\alpha^2 - (2n-2)\Big(\int \Lambda^{2n-1}\alpha\Big)^2$$

coincides with the Beauville–Bogomolov form induced by $\mathbb{B}$ on the right-hand side of (78) up to a factor in $W \otimes \mathbb{Q}$. Indeed, to reduce this to the case of analytic manifolds we take an embedding of W into a field. Finally we may assume that the field is $\mathbb{C}$.

We denote by $L^{(p)}$ the inverse image of L by the map $X_0^{(p)} \to X_0$ and by $\Lambda^{(\sigma)}$ its cohomology class. It is the inverse image of Λ by the map $H^2_{\mathrm{crys}}(X_0/W) \to H^2_{\mathrm{crys}}(X_0^{(p)}/W)$. Since we may interpret $\nu(\Lambda)$ as an intersection product we have

$$\nu(\Lambda) = \nu(\Lambda^{(\sigma)}) \in \mathbb{Z}$$

is nonzero.

We claim that

$$\mathbb{B}_\Lambda(F\tilde{\alpha}) = p^2 \mathbb{B}_{\Lambda^{(\sigma)}}(\tilde{\alpha}), \quad \tilde{\alpha} \in H^2_{\mathrm{crys}}(X_0^{(p)}/W). \tag{79}$$

For this we use that $F(\Lambda^{(\sigma)}) = p\Lambda$ and that we have a commutative diagram:

$$\begin{array}{ccc} H^{4n}_{\mathrm{crys}}(X_0^{(p)}/W) & \xrightarrow{F} & H^{4n}_{\mathrm{crys}}(X_0/W) \\ {\scriptstyle\int}\downarrow & & \downarrow{\scriptstyle\int} \\ W & \xrightarrow{p^{2n}} & W \end{array} \tag{80}$$

Now we can compute

$$\begin{aligned} \mathbb{B}_\Lambda(F\tilde{\alpha}) &= (2n-1)\nu(\Lambda)\int \Lambda^{2n-2}(F\tilde{\alpha})^2 - (2n-2)\Big(\int \Lambda^{2n-1}F\tilde{\alpha}\Big)^2 \\ &= (2n-1)\nu(\Lambda)(1/p^{2n-2})\int (F\Lambda^{(\sigma)})^{2n-2}(F\tilde{\alpha})^2 \\ &\qquad\qquad - (2n-2)(1/p^{2(2n-1)})\Big(\int (F\Lambda^{(\sigma)})^{2n-1}F\tilde{\alpha}\Big)^2 \\ &= (2n-1)\nu(\Lambda)(1/p^{2n-2})p^{2n}\int (\Lambda^{(\sigma)})^{2n-2}(\tilde{\alpha})^2 \\ &\qquad\qquad - (2n-2)(1/p^{2(2n-1)})p^{4n}\Big(\int (\Lambda^{(\sigma)})^{2n-1}\tilde{\alpha}\Big)^2 \\ &= p^2\mathbb{B}_{\Lambda^{(\sigma)}}(\tilde{\alpha}). \end{aligned}$$

The last equation holds by (80). This shows (79).

We may multiply the Beauville–Bogomolov form $\mathbb{B}$ by a constant in $W \otimes \mathbb{Q}$ such that it induces on $H^2_{\mathrm{crys}}(X_0/W) = H^2_{\mathrm{DR}}(X/W)$ the form $\mathbb{B}_\Lambda$ if we make base change by the natural map $\rho : A \to W$ induced by X. There is a lift σ of the Frobenius to A such that ρ commutes with Frobenii. Indeed, after a coordinate change we may write $A = W[\![T_1, \dots, T_r]\!]$ in such a way that $\rho(T_i) = 0$. We take for σ the Frobenius such that $\sigma(T_i) = T_i^p$.

The two horizontal forms (74) may be regarded as two horizontal sections of the bundle $\mathcal{H}^{(\sigma)} \otimes (\mathcal{H}^{(\sigma)})^{\mathrm{dual}}$ endowed with its natural integrable connection. We have shown that these two sections differ by the factor p^2 if we make the base change $A \to W$. Hence the sections themselves differ by the factor p^2. This shows (75). □

We can now prove a refinement of Proposition 19.

Proposition 35. *Let k be an algebraically closed field and let X_0 be a projective scheme of K3 type which lifts to a projective smooth scheme over $W(k)$.*

Let $f : X \to \operatorname{Spec} R$ be a deformation of X_0 over an artinian local ring R with residue class field k. Then the crystalline cohomology $H^2_{\mathrm{crys}}(X/\widehat{W}(R))$ has the unique structure of a selfdual $\widehat{\mathcal{W}}_R$-display which is functorial in R.

Proof. We can use the w-frames $\mathcal{C}_n$ introduced before Proposition 19. We consider the $\mathcal{C}_n$-window $\mathcal{P}$ we introduced on $H^2_{\text{crys}}(\mathfrak{X}_{R_n}/C_n)$. We have to show that the Beauville–Bogomolov form induces a bilinear form of $\mathcal{C}_n$-displays

$$\mathcal{P} \times \mathcal{P} \to \mathcal{U}(2). \tag{81}$$

Here we choose the Beauville–Bogomolov form in such a way that $\epsilon = 1$, which is possible by the remark preceding Lemma 25. Because we are in the torsion-free case it suffices to show that this pairing is compatible with F_0, which follows from Proposition 34. We already know that the Beauville–Bogomolov form induces a selfdual pairing on

$$H^2_{\text{crys}}(\mathfrak{X}_{R_n}/C_n) = H^2_{\text{DR}}(\mathfrak{X}_{R_n}/C_n)$$

with respect to the Hodge filtration on the right-hand side. This shows that (81) is perfect. The assertion of the proposition is obtained by base change. □

Remark. In the same way, one can generalize the corollary of Proposition 19 and obtain duality on the displays there.

We denote by $(\mathcal{P}_0, \mathbb{B}_0)$ the selfdual $\widehat{\mathcal{W}}_k$-2-display associated to X_0. We assume that this 2-display is F_0-étale (Definition 12). By Corollary 18 the deformation functor of $(\mathcal{P}_0, \mathbb{B}_0)$ is prorepresentable by

$$S_{\text{disp}} = \operatorname{Spf} A_{\text{disp}},$$

where A_{disp} is a power series ring over $W(k)$. The universal object is a $\widehat{\mathcal{W}}_{A_{\text{disp}}}$-display. Let $\mathfrak{X} \to S$ be the universal deformation of X_0. By Proposition 35 we have a general morphism

$$S \to S_{\text{disp}}. \tag{82}$$

Let $f : X \to \operatorname{Spec} R$ be as in Proposition 35 and let $(\mathcal{P}, \lambda)$ be the corresponding $\widehat{\mathcal{W}}_R$ display. Let $R' \to R$ be a surjection of artinian local rings with residue class field k and kernel $\mathfrak{a}'$. We assume that $(\mathfrak{a}')^2 = 0$. Then the liftings of X to R' and of $(\mathcal{P}, \lambda)$ to R' are by Proposition 17 and Corollary 32 in natural bijection. In particular (82) is an isomorphism. We obtain:

Theorem 36. *Let k be an algebraically closed field. Let X_0 be a scheme of K3 type over k which lifts to a projective scheme over $W(k)$. We assume that the associated selfdual $\widehat{\mathcal{W}}_k$-2-display $(\mathcal{P}_0, \lambda_0)$ is F_0-étale.*

Let R be a local artinian ring with residue class field k. The map which associates to a deformation of X/R of X_0 its selfdual $\widehat{\mathcal{W}}_R$-2-display $(\mathcal{P}, \lambda)$ is a bijection to the deformations of $(\mathcal{P}_0, \lambda_0)$ to R.

Moreover an automorphism of X_0 lifts to an automorphism of X (necessarily unique) if and only if the induced automorphism of $(\mathcal{P}_0, \lambda_0)$ lifts to $(\mathcal{P}, \lambda)$.

Proof. The last statement is a consequence of Corollary 33. □

5. The relative de Rham–Witt complex of an ordinary K3 surface

We now relate our results to the results of [Nygaard 1983] and prove the degeneration of the integral de Rham–Witt spectral sequence for ordinary K3 surfaces.

Let R be a ring such that p is nilpotent on R, and let $X/\operatorname{Spec} R$ be a smooth projective scheme.

We assume that there exists a formal lifting $\mathfrak{X}$ of X over $\operatorname{Spf} W(R)$ and let $\Omega^{\cdot}_{\mathfrak{X}/W(R)}$ be its de Rham complex. We recall the following complex from [Langer and Zink 2007, Section 4] denoted by $\mathcal{F}^m\Omega^{\cdot}_{\mathfrak{X}/W(R)}$:

$$I_R \otimes_{W(R)} \Omega^0_{\mathfrak{X}/W(R)} \xrightarrow{pd} \cdots \xrightarrow{pd} I_R \otimes_{W(R)} \Omega^{m-1}_{\mathfrak{X}/W(R)} \xrightarrow{d} \Omega^m_{W(R)} \xrightarrow{d} \cdots,$$

where $I_R = VW(R)$.

Let $W\Omega_{X/R}$ denote the relative de Rham–Witt complex and $N^m W\Omega^{\cdot}_{X/R}$ the Nygaard complex (compare [Langer and Zink 2007, Introduction]):

$$(W\mathcal{O}_{X/R})_{[F]} \xrightarrow{d} \cdots \xrightarrow{d} (W\Omega^{m-1}_{X/R})_{[F]} \xrightarrow{dV} W\Omega^m_{X/R} \xrightarrow{d} W\Omega^{m+1}_{X/R} \xrightarrow{d} \cdots.$$

Here F means the restriction of scalars via $F : W(R) \to W(R)$.

Then we recall the following.

Conjecture 37. *There exists a canonical isomorphism in the derived category $D^+(X_{\text{zar}}, W(R))$ between the Nygaard complex and the complex $\mathcal{F}^m\Omega_{\mathfrak{X}/W(R)}$:*

$$N^m W\Omega^{\cdot}_{X/R} \cong \mathcal{F}^m\Omega^{\cdot}_{\mathfrak{X}/W(R)}.$$

This is proved in [Langer 2018, Theorem 0.2] for $m < p$.

Remark 38. Let us assume that the de Rham spectral sequences associated to $\Omega^{\cdot}_{X/R}$ and $\Omega^{\cdot}_{\mathfrak{X}/W(R)}$ degenerate and commute with base change. Then it is proved in [Gregory and Langer 2017] under some additional assumptions that the hypercohomology groups $\mathbb{H}^n(X, N^m W\Omega_{X/R})$ define for varying m a display structure on the crystalline cohomology $H^n_{\text{crys}}(X/W(R))$.

In [Langer and Zink 2007, Conjecture 5.8], this was predicted in general.

Before we state the main result we give a very general fact.

Lemma 39. *Let $X \to \operatorname{Spec} R$ be a proper scheme over the spectrum of a complete local ring.*

Then for all integers $r, s \geq 0$ the cohomology group $H^s(X, W\Omega^r_{X/R})$ is V-separated; *i.e.*,

$$\bigcap_{n>0} V^n H^s(X, W\Omega^r_{X/R}) = 0. \tag{83}$$

Proof. The composition of the following maps is zero:

$$W\Omega^r_{X/R} \xrightarrow{V^n} W\Omega^r_{X/R} \to W_n\Omega^r_{X/R}.$$

Indeed, V^n maps a differential of the type $\xi d\eta_1 \cdots d\eta_r$ to ${}^{V^n}\xi \, d\, {}^{V^n}\eta_1 \cdots d\, {}^{V^n}\eta_r$. Therefore the composition of the following arrows is zero:

$$H^s(X, W\Omega^r_{X/R}) \xrightarrow{V^n} H^s(X, W\Omega^r_{X/R}) \to H^s(X, W_n\Omega^r_{X/R}). \tag{84}$$

Let us denote by $M \subset H^s(X, W\Omega^r_{X/R})$ the left-hand side of (83). We conclude from (84) that for each n the group M is mapped to zero by

$$H^s(X, W\Omega^r_{X/R}) \to H^s(X, W_n\Omega^r_{X/R}).$$

On the other hand we have

$$H^s(X, W\Omega^r_{X/R}) = \varprojlim_n H^s(X, W_n\Omega^r_{X/R})$$

by [Langer and Zink 2004, Corollary 1.14]. Since the map from M to each group in the projective system is zero we conclude that $M = 0$. $\square$

Theorem 40. *Let X/R be a smooth projective scheme such that R is artinian with perfect residue field k of characteristic $p > 2$ and such that the closed fibre X_k is an ordinary $K3$ surface.*

Then the de Rham–Witt spectral sequence associated to the relative de Rham–Witt complex

$$E_1^{i,j} = H^j(X, W\Omega^i_{X/R}) \to \mathbb{H}^{i+j}(W\Omega^{\cdot}_{X/R})$$

degenerates. Moreover, one has the following properties:

- $H^0_{\text{crys}}(X/W(R)) = H^0(X, W\mathcal{O}_{X/R}) = W(R)$.
- $H^1_{\text{crys}}(X/W(R)) = H^3_{\text{crys}}(X/W(R)) = 0$.
- $H^i(X, W\Omega^j_{X/R}) = 0$ *for $i + j$ odd, or $i + j > 4$, or $i + j = 4$, $i \neq j$.*
- $H^2(X, W\Omega^2_{X/R}) = H^4_{\text{crys}}(X/W(R)) = W(R)$.
- $H^2_{\text{crys}}(X/W(R)) \cong H^0(X, W\Omega^2_{X/R}) \oplus H^1(X, W\Omega^1_{X/R}) \oplus H^2(X, W\mathcal{O}_X)$, *which is a Hodge-de Rham–Witt decomposition (slope decomposition) in degree 2, lifting the slope decomposition over $W(k)$.*

$H^2(X, W\mathcal{O}_X)$ inherits from $W\mathcal{O}_X$ the operators F and V and it is with this structure the Cartier module of $\widehat{\mathrm{Br}}_{X/R} \cong \widehat{\mathbb{G}}_m/R$, the formal Brauer group of X. The Frobenius $F : W\Omega^1_{X/R} \to W\Omega^1_{X/R}$ induces an endomorphism of $H^1(X, W\Omega^1_{X/R})$, which we denote by F_1. Let $\mathcal{P} = (P, Q, F, F_1)$ be the display defined by $P = Q = H^1(X, W\Omega^1_{X/R})$, $F_1 := F_1$ and $F := pF_1$. Then $\mathcal{P}$ is the display of the étale part $\Psi^{\text{et}}_{X/R}$ of the extended Brauer group $\Psi_{X/R}$.

Remark 41. This is the first nontrivial example where the spectral sequence of the relative de Rham–Witt complex degenerates. Note that for the absolute de Rham–

Witt complex in the case $R = k$, it is known that the de Rham–Witt spectral sequence degenerates modulo torsion [Illusie 1979, II, Théorème 3.2] and degenerates at E_1 in the following two cases:

(a) All $H^i(X, W\Omega^j_X)$ are $W(k)$-modules of finite type [Illusie 1979, II, Théoréme 3.7].

(b) All $H^i(X, W\Omega^j_X)$ are p-torsion free [Illusie 1979, II, Corollaire 4.9].

Proof. First we prove Theorem 40 using Conjecture 37 (proven in [Langer 2018, Theorem 0.2]) and then give an alternative proof using the universal deformation of X_k over the universal deformation ring and applying [Langer and Zink 2007, Corollary 4.7].

It is well known that the crystalline cohomology $H^i_{\mathrm{crys}}(X/W(R))$ is isomorphic to the de Rham cohomology $H^i_{\mathrm{DR}}(\mathfrak{X})$ of a smooth formal lifting $\mathfrak{X}$ over $\operatorname{Spf} W(R)$, commutes with base change and is locally free of rank 1, 0, 22, 0, 1 for $i = 0, 1, 2, 3, 4$ respectively; see [Langer and Zink 2007, p. 151] and [Illusie 1979, II, Section 7.2].

It is known that, as we are in the ordinary case, $\widehat{\mathrm{Br}}_{X/R} \cong \widehat{\mathbb{G}}_{m,R}$ by [Artin and Mazur 1977, IV, Proposition 1.8; Nygaard 1983, Introduction] and $H^2(X, W\mathcal{O}_X)$ is the Cartier module of $\widehat{\mathrm{Br}}_{X/R}$; hence $H^2(X, W\mathcal{O}_X) = W(R)$ by [Artin and Mazur 1977, II, Proposition 2.13].

Let G be an arbitrary p-divisible group over R. We will denote by $\mathbb{D}(G)$ the Grothendieck–Messing crystal of G. Its evaluation at the pd-thickening $W(R) \to R$ will be denoted by

$$D(G) = \mathbb{D}(G)_{W(R)}.$$

Note that in [Nygaard and Ogus 1985] this $W(R)$-module is denoted by $D(G^*)_{W(R)}$. It is endowed with a display structure.

By [Nygaard and Ogus 1985, Theorem 3.16] we have a Frobenius equivariant map,

$$D(\widehat{\mathrm{Br}}_{X/R}) \to D(\Psi_{X/R}) \to H^2_{\mathrm{crys}}(X/W(R)) = \mathbb{H}^2(X, W\Omega^{\cdot}_{X/R}).$$

Using the natural Frobenius equivariant map of complexes $W\Omega^{\cdot}_{X/R} \to W\mathcal{O}_X$ we obtain a Frobenius equivariant map,

$$D(\widehat{\mathrm{Br}}_{X/R}) \to H^2_{\mathrm{crys}}(X/W(R)) \to H^2(X, W\mathcal{O}_X). \tag{85}$$

The first and the last $W(R)$-module in this sequence are free of rank 1. Therefore we conclude by reduction to the case $R = k$ (compare [Nygaard and Ogus 1985, p. 490]) that the composite of the arrows in (85) is an isomorphism. Therefore we obtain an F-equivariant section ρ of the last map.

Since $H^1(X, \mathcal{O}_X) = 0$ we conclude that V is surjective on $H^1(X, W\mathcal{O}_X)$. We conclude by Lemma 39 that $H^1(X, W\mathcal{O}_X) = 0$. We consider the exact sequence of complexes

$$0 \to W\Omega^{\geq 1}_{X/R} \to W\Omega^{\cdot}_{X/R} \to W\mathcal{O}_X \to 0.$$

If we take hypercohomology and use the section ρ above, we obtain an F-equivariant decomposition,

$$H^2_{\text{crys}}(X/W(R)) = H^2(X, W\mathcal{O}_X) \oplus \mathbb{H}^2(W\Omega^{\geq 1}_{X/R}). \tag{86}$$

Let $\mathfrak{X}$ be a formal lifting of X over $\operatorname{Spf} W(R)$. It is known that the Hodge–de Rham spectral sequence of $\mathfrak{X}$ degenerates; moreover the Hodge–de Rham spectral sequence associated to the complex $\mathcal{F}^m\Omega^{\cdot}_{\mathfrak{X}/W(R)}$ degenerates too; see [Langer and Zink 2007, Propositions 3.1 and 3.2].

Remark 42. Using the isomorphism $N^2W\Omega^{\bullet}_{X/R} \simeq \mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}$ we compute the cohomology of the Nygaard complex:

$$\begin{aligned}
\mathbb{H}^0(N^2W\Omega^{\bullet}_{X/R}) &\cong \mathbb{H}^0(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}) \cong I_R H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}),\\
\mathbb{H}^1(N^2W\Omega^{\bullet}_{X/R}) &\cong \mathbb{H}^1(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}) \cong I_R H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \oplus I_R H^0(\mathfrak{X}, \Omega^1_{\mathfrak{X}}) = 0,\\
\mathbb{H}^2(N^2W\Omega^{\bullet}_{X/R}) &\cong \mathbb{H}^2(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}) \cong I_R H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \oplus I_R H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}}) \oplus H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}}),\\
\mathbb{H}^3(N^2W\Omega^{\bullet}_{X/R}) &\cong \mathbb{H}^3(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)})\\
&\cong I_R H^3(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \oplus I_R H^2(\mathfrak{X}, \Omega^1_{\mathfrak{X}}) \oplus H^1(\mathfrak{X}, \Omega^2_{\mathfrak{X}}) = 0,\\
\mathbb{H}^4(N^2W\Omega^{\bullet}_{X/R}) &\cong \mathbb{H}^4(\mathcal{F}^2\Omega^{\bullet}_{\mathfrak{X}/W(R)}) \cong H^2(\mathfrak{X}, \Omega^2_{\mathfrak{X}}) \cong H^4_{\text{DR}}(\mathfrak{X}) \cong W(R).
\end{aligned}$$

We will consider the following map from the Nygaard complex to the usual de Rham–Witt complex:

$$\begin{array}{ccccccl}
W\mathcal{O}_X & \xrightarrow{d} & W\Omega^1_{X/R} & \xrightarrow{dV} & W\Omega^2_{X/R} & : & N^2W\Omega^{\cdot}_{X/R}\\
\downarrow{\scriptstyle pV} & & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle =} & & \\
W\mathcal{O}_X & \xrightarrow{d,} & W\Omega^1_{X/R} & \xrightarrow{d} & W\Omega^2_{X/R} & : & W\Omega^{\cdot}_{X/R}
\end{array} \tag{87}$$

Lemma 43. *We consider the complexes $W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}$ and $W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}$ in degree 0 and 1. Then we have*

$$\mathbb{H}^0(X, W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) = 0, \quad \mathbb{H}^0(X, W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}) = 0.$$

Proof. Since $H^1_{\text{crys}}(X/W(R)) = 0 = H^1(X, W\mathcal{O}_X)$ we have

$$\mathbb{H}^0(W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) = 0.$$

From Remark 42 we get an exact sequence

$$\begin{array}{l} 0 \to \mathbb{H}^0(N^2 W\Omega^\bullet_{X/R}) \to H^0(X, W\mathcal{O}_X) \xrightarrow{\partial} \mathbb{H}^0(W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}) \to 0, \\ \qquad \cong I_R H^0(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \qquad \cong W(R) \\ \qquad \cong I_R W(R) \end{array} \tag{88}$$

where ∂ is induced by the differential d and therefore is the zero map since d vanishes on $W(R) \cong H^0(X, W\mathcal{O}_X)$. □

Now consider the following commutative diagram of de Rham complexes:

$$\begin{tikzcd} \mathcal{O}_{\mathfrak{X}} \arrow[r, "d"] & \Omega^1_{\mathfrak{X}} \arrow[r, "d"] & \Omega^2_{\mathfrak{X}} & : \Omega^\bullet_{\mathfrak{X}/W(R)} \\ I_R \otimes \mathcal{O}_{\mathfrak{X}} \arrow[u, "\cdot p"] \arrow[r, "pd"'] & I_R \otimes \Omega^1_{\mathfrak{X}} \arrow[u, hook] \arrow[r, "d"'] & \Omega^2_{\mathfrak{X}} \arrow[u, "="'] & : \mathcal{F}^2\Omega^\bullet_{\mathfrak{X}/W(R)} \end{tikzcd} \tag{89}$$

The diagram (89) and the degeneracy of the Hodge–de Rham spectral sequences associated to $\Omega^\bullet_{\mathfrak{X}/W(R)}$ and $\mathcal{F}^2\Omega^\bullet_{\mathfrak{X}/W(R)}$ yield a commutative diagram of exact rows:

$$\begin{tikzcd} \mathbb{H}^2(\Omega^{\geq 1}_{\mathfrak{X}/W(R)}[-1]) \arrow[r, hook] & H^2_{dR}(\mathfrak{X}/W(R)) \arrow[r, two heads] & H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \\ \mathbb{H}^2(0 \to I_R \otimes \Omega^1 \to \Omega^2 \to 0) \arrow[u] \arrow[r, hook] & \mathbb{H}^2(\mathcal{F}^2\Omega_{\mathfrak{X}/W(R)}) \arrow[u] \arrow[r, two heads] & I_R H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \arrow[u, "\cdot p"'] \end{tikzcd} \tag{90}$$

The vertical map on the left-hand side may be identified with

$$I_R H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/W(R)}) \oplus H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/W(R)}) \to H^1(\mathfrak{X}, \Omega^1_{\mathfrak{X}/W(R)}) \oplus H^0(\mathfrak{X}, \Omega^2_{\mathfrak{X}/W(R)}).$$

Since the cohomology $H^j(\mathfrak{X}, \Omega^i_{\mathfrak{X}/W(R)})$ commutes with arbitrary base change we obtain that the cokernel of the left vertical map coincides with $H^1(X, \Omega^1_{X/R})$. The vertical map in the middle is the map described in [Langer and Zink 2007, Definitions 2.1 and 2.5], in terms of the predisplay structure on $H^2_{dR}(\mathfrak{X}/W(R))$. Then we have:

Lemma 44. *The map (87) induces the following diagram with exact rows*:

$$\begin{tikzcd} \mathbb{H}^1(W\Omega^1 \xrightarrow{d} W\Omega^2) \arrow[r, hook] & H^2_{\mathrm{crys}}(X/W(R)) \arrow[r, two heads] & H^2(X, W\mathcal{O}_X) \\ \mathbb{H}^1(W\Omega^1 \xrightarrow{dV} W\Omega^2) \arrow[u, "{(V,\mathrm{id})}"] \arrow[r, hook] & \mathbb{H}^2(N^2 W\Omega^\cdot_{X/R}) \arrow[u] \arrow[r, two heads] & H^2(X, W\mathcal{O}_X) \arrow[u, "pV"'] \end{tikzcd} \tag{91}$$

(*We wrote here* $W\Omega^? = W\Omega^?_{X/R}$).

Moreover, *the diagram is isomorphic to the diagram* (90); *hence the left vertical arrow is injective and its cokernel is* $H^1(X, \Omega^1_{X/R})$.

Proof. The last vertical arrow of the diagram factors through

$$H^2(X, W\mathcal{O}_X) \xrightarrow{V} H^2(X, VW\mathcal{O}_X) \xrightarrow{\cdot p} H^2(X, W\mathcal{O}_X).$$

Indeed, this follows from the exact sequence

$$0 \to W\mathcal{O}_X \xrightarrow{V} W\mathcal{O}_X \to \mathcal{O}_X \to 0.$$

Since $H^1(X, \mathcal{O}_X) = 0$, the map $H^2(X, W\mathcal{O}_X) \xrightarrow{V} H^2(X, W\mathcal{O}_X)$ is injective and we may identify its image with $H^2(X, VW\mathcal{O}_X)$. Moreover $H^1(X, W\mathcal{O}_X) = 0$ by Lemma 39, since $VH^1(X, W\mathcal{O}_X) = H^1(X, W\mathcal{O}_X)$. Therefore the horizontal left-hand maps in (91) are injective. Since $\mathbb{D}(\widehat{\mathrm{Br}}_{X/R})$ is a crystal and $\widehat{\mathrm{Br}}_{\mathfrak{X}/W(R)} = \mathbb{G}_{m/\mathfrak{X}}$ by rigidity, we have

$$\mathbb{D}(\widehat{\mathrm{Br}}_{X/R})_{W(R)} = \mathbb{D}(\widehat{\mathrm{Br}}_{\mathfrak{X}/W(R)})_{W(R)} = H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = \mathrm{Lie}\,\widehat{\mathrm{Br}}_{\mathfrak{X}/W(R)}$$

(compare the bottom lines in [Nygaard and Ogus 1985, p. 492]). Under the identification $H^2(X, W\mathcal{O}_X) = H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, the top exact sequences in (90) and (91) are isomorphic under the isomorphism $H^2_{dR}(\mathfrak{X}/W(R)) \cong H^2_{\mathrm{crys}}(X/W(R))$. The exact sequence

$$0 \to I_R \to W(R) \to R \to 0$$

yields

$$0 \to I_R H^2(X, W\mathcal{O}_X) \to H^2(X, W\mathcal{O}_X) \to H^2(X, \mathcal{O}_X) \to 0$$

and

$$0 \to I_R H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to H^2(X, \mathcal{O}_X) \to 0;$$

hence

$$I_R H^2(X, W\mathcal{O}_X) \cong H^2(X, VW\mathcal{O}_X) \cong I_R H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}).$$

Using the isomorphism $N^2 W\Omega^\bullet_{X/R} \cong \mathcal{F}^2\Omega^\bullet_{\mathfrak{X}/W(R)}$ we can identify the middle vertical arrows in diagrams (90) and (91). Moreover, since $H^2(X, W\mathcal{O}_X)$ is isomorphic to $H^2(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \cong W(R)$, we see that the whole diagram (91) is isomorphic to the diagram (90). By the remark after (90) this implies that the left vertical map in (91) is injective and has cokernel $H^1(X, \Omega^1_{X/R})$. □

Lemma 45. *We have* $H^0(X, W\Omega^1_{X/R}) = 0$.

Proof. Lemma 43 implies that the rows in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, W\Omega^1) \arrow[r, "d"] & H^0(X, W\Omega^2) \arrow[r] & \mathbb{H}^1(W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) \\
0 \arrow[r] & H^0(X, W\Omega^1) \arrow[r, "dV"] \arrow[u, "V"] & H^0(X, W\Omega^2) \arrow[r] \arrow[u, "="] & \mathbb{H}^1(W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}) \arrow[u, "{(V, \mathrm{id})}"]
\end{tikzcd}$$

are exact. Since (V, id) is injective by Lemma 44, the map

$$V : H^0(X, W\Omega^1) \to H^0(X, W\Omega^1)$$

is an isomorphism. We conclude by Lemma 39. □

Now consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
H^0(X,W\Omega^2_{X/R}) \arrow[r, hook] & \mathbb{H}^1(X,[W\Omega^1_{X/R}\xrightarrow{d} W\Omega^2_{X/R}]) \arrow[r] & H^1(X,W\Omega^1_{X/R}) \arrow[r] & H^1(X,W\Omega^2_{X/R}) \\
H^0(X,W\Omega^2_{X/R}) \arrow[r, hook] \arrow[u, "="] & \mathbb{H}^1(X,[W\Omega^1_{X/R}\xrightarrow{dV} W\Omega^2_{X/R}]) \arrow[r] \arrow[u, "\hat{\alpha}"] & H^1(X,W\Omega^1_{X/R}) \arrow[r] \arrow[u, "V"] & H^1(X,W\Omega^2_{X/R}) \arrow[u, "="]
\end{tikzcd} \tag{92}$$

$\hat{\alpha}$ denotes the left vertical arrow of (91). By Lemma 44, $\hat{\alpha}$ is injective and has cokernel $H^1(X, \Omega^1_{X/R})$.

Lemma 46. *The sequence*

$$0 \to H^1(X, W\Omega^1_{X/R}) \xrightarrow{V} H^1(X, W\Omega^1_{X/R}) \to H^1(X, \Omega^1_{X/R}) \to 0 \tag{93}$$

is exact and $H^i(X, W\Omega^1_{X/R}) = 0$ for $i \geq 2$.

Proof. Let us begin with the short exact sequence. We have to show that the kernels (and cokernels) of $\hat{\alpha}$ and V in the diagram (92) are the same. This follows formally if we prove that the last two horizontal arrows in this diagram are surjective. The continuation of the diagram (91) gives a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{H}^2(W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) \arrow[r] & H^3_{\mathrm{crys}}(X/W(R)) \\
0 \arrow[r] & \mathbb{H}^2(W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}) \arrow[r] \arrow[u, "\hat{\alpha}"] & \mathbb{H}^3(N^2 W\Omega^{\cdot}_{X/R}) \arrow[u]
\end{tikzcd} \tag{94}$$

By Remark 42 we see that all terms in the diagram (94) vanish. This shows that in the diagram (92) the last horizontal maps are surjective. The exactness of (93) follows.

For the last assertion we need only to consider the case $i = 2$ because the cohomological dimension of X is 2 by Grothendieck's theorem.

To show the vanishing we continue the exact cohomology sequences which lead to (92):

$$\begin{tikzcd}
H^2(X, W\Omega^1_{X/R}) \arrow[r] & H^2(X, W\Omega^2_{X/R}) \arrow[r] & \mathbb{H}^3(W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) \\
H^2(X, W\Omega^1_{X/R}) \arrow[r] \arrow[u, "V"] & H^2(X, W\Omega^2_{X/R}) \arrow[r] \arrow[u, "="] & \mathbb{H}^3(W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}) \arrow[u, "\hat{\alpha}"]
\end{tikzcd} \tag{95}$$

The horizontal arrows on the left are injective by the vanishing of (94). By Lemma 39 it suffices to show that the $\hat{\alpha}$ on the right-hand side is an isomorphism because then V on the left-hand side is bijective. To see this we continue the diagram (91) in higher degrees and obtain:

$$\begin{array}{ccc}
\mathbb{H}^3(W\Omega^1_{X/R} \xrightarrow{d} W\Omega^2_{X/R}) & \xrightarrow{\simeq} & H^4_{\mathrm{crys}}(X/W(R)) \\
\uparrow \hat{\alpha} & & \uparrow \\
\mathbb{H}^3(W\Omega^1_{X/R} \xrightarrow{dV} W\Omega^2_{X/R}) & \xrightarrow{\simeq} & \mathbb{H}^4(N^2 W\Omega^\bullet_{X/R})
\end{array}$$

The horizontal arrows are isomorphisms because X is a noetherian space of Zariski dimension 2 and therefore $H^3(X, W\mathcal{O}_X) = H^4(X, W\mathcal{O}_X) = 0$.

Under the isomorphism $H^4_{\mathrm{crys}}(X/W(R)) \cong H^4_{\mathrm{DR}}(\mathfrak{X}) \cong H^2(\mathfrak{X}, \Omega^2_{\mathfrak{X}})$, Remark 42 implies that the right vertical arrow is an isomorphism; hence $\hat{\alpha}$ is an isomorphism as well. □

From the F-equivariant decomposition (86) we obtain by projection an F-equivariant morphism

$$D(\Psi_{X/R}) \to \mathbb{H}^2(X, W\Omega^{\geq 1}_{X/R}). \tag{96}$$

The Frobenius on the right-hand side is inherited from $H^2_{\mathrm{crys}}(X/W(R))$. It is in a natural way divisible by p. By the definition of the decomposition (86), the submodule $D(\widehat{\mathrm{Br}}_{X/R}) \subset D(\Psi_{X/R})$ is mapped to zero by the map (96). Therefore we obtain a map

$$D(\Psi^{\mathrm{et}}_{X/R}) \to \mathbb{H}^2(X, W\Omega^{\geq 1}_{X/R}). \tag{97}$$

We will write $(P, Q, F, F_1) = D(\Psi^{\mathrm{et}}_{X/R})$. Then $P = Q$ because this is a display of an étale group. The Frobenius F and the Verschiebung V on $W\Omega^1_{X/R}$ induce maps on the cohomology $H^1(X, W\Omega^1_{X/R})$ which we denote by the same letter. Composing the map (97) with $\mathbb{H}^2(X, W\Omega^{\geq 1}_{X/R}) \to H^1(X, W\Omega^1_{X/R})$ We obtain a map

$$\varsigma : P \to H^1(X, W\Omega^1_{X/R}) \tag{98}$$

such that the following diagram is commutative:

$$\begin{array}{ccc}
P & \longrightarrow & H^1(X, W\Omega^1_{X/R}) \\
F\downarrow & & \downarrow pF \\
P & \longrightarrow & H^1(X, W\Omega^1_{X/R})
\end{array}$$

Lemma 47. *The $W(R)$-module homomorphism*

$$\varsigma : P = D(\Psi^{\mathrm{et}}_{X/R}) \to H^1(X, W\Omega^1_{X/R})$$

is an isomorphism. We have a commutative diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "F_1"'] & H^1(X, W\Omega^1_{X/R}) \arrow[d, "F"] \\ P \arrow[r] & H^1(X, W\Omega^1_{X/R}) \end{tikzcd}$$

Proof. We already know that the last diagram is commutative if we multiply the vertical arrows by p. Indeed, on the left-hand side we have $pF_1 = F$ by the definition of a display. To prove the commutativity we may replace X by the universal deformation

$$\mathfrak{X} \to \operatorname{Spf} W(k)[[t_1, \dots, t_{22}]].$$

Then the groups of the diagram have no p-torsion and the result follows. We will write $I_R = VW(R)$ as before. We define $F_2 : I_R P \to P$ by

$$F_2({}^V\xi x) = \xi F_1 x, \quad \xi \in W(R),\ x \in P.$$

Because F_1 is an F-linear isomorphism, F_2 is bijective. Then we have a commutative diagram:

$$\begin{tikzcd} P \arrow[r, "\varsigma"] & H^1(X, W\Omega^1_{X/R}) \arrow[d, "V"] \\ I_R P \arrow[u, "F_2"'] \arrow[r, "\varsigma"] & H^1(X, W\Omega^1_{X/R}) \end{tikzcd}$$

Indeed,

$$V\varsigma(F_2({}^V\xi x)) = V(\varsigma(\xi F_1 x)) = V\xi F\varsigma(x) = {}^V\xi\varsigma(x) = \varsigma({}^V\xi x).$$

The next-to-last equation holds because the corresponding equation holds for $W\Omega^1_{X/R}$ by the last equation of [Langer and Zink 2004, Definition 1.4].

We set $I_n = V^n W(R)$. From the preceding remark we obtain

$$\varsigma(I_n P) \subset V^n H^1(X, \Omega^1_{X/R}).$$

Then we also have the following diagram:

$$\begin{tikzcd} I_{n-1} P \arrow[r, "\varsigma"] & V^{n-1} H^1(X, W\Omega^1_{X/R}) \arrow[d, "V"] \\ I_n P \arrow[u, "F_2"'] \arrow[r, "\varsigma"] & V^n H^1(X, W\Omega^1_{X/R}) \end{tikzcd} \tag{99}$$

Here again the map F_2 is bijective.

The map ς induces an R-module homomorphism

$$\bar{\varsigma} : P/I_R P \to H^1(X, \Omega^1_{X/R}). \tag{100}$$

We claim that this is an isomorphism. To see this we note that both sides are free R-modules of the same rank and that both sides commute with arbitrary base change $R \to R'$. Therefore it suffices to prove that (100) is surjective in the case $R = k$. In this case the F-crystal $H^2_{\text{crys}}(X_k/W(k))$ decomposes into a direct sum of isoclinic crystals. This has the consequence that [Illusie 1979, II, Section 7.2]

$$H^2_{\text{crys}}(X_k/W(k)) = H^2(X_k, W\mathcal{O}_{X_k}) \oplus H^1(X_k, W\Omega^1_{X_k/k}) \oplus H^0(X_k, W\Omega^2_{X_k/k}) \quad (101)$$

is this decomposition as a sum of isoclinic crystals, i.e., the isoclinic components are canonically isomorphic to the cohomology groups on the right-hand side. In particular these groups are free $W(k)$-modules of ranks 1, 20, 1 respectively. The F-crystal of the extended Brauer group is the part of the F-crystal $H^2_{\text{crys}}(X_k/W(k))$ with slopes in [0, 1]; i.e., it corresponds to the first two direct summands of (101). Therefore the map (100) is induced by

$$H^1(X_k, W\Omega^1_{X_k/k}) \to H^1(X_k, \Omega^1_{X_k/k}).$$

But we know by Lemma 46 that this map is surjective. Then (100) is also surjective in the case $R = k$ and this proves our claim that (100) is an isomorphism.

Now we can check easily by induction that the maps induced by ς,

$$I_n P / I_{n+1} P \to V^n H^1(X, W\Omega^1_{X/R}) / V^{n+1} H^1(X, W\Omega^1_{X/R}), \quad (102)$$

are surjective. In the case $n = 0$ this follows from the isomorphism (100) and Lemma 46.

Let $m \in H^1(X, W\Omega^1_{X/R})$. Then we find by induction elements $x \in I_{n-1} P$ and $m_1 \in H^1(X, W\Omega^1_{X/R})$ such that

$$V^{n-1} m = \varsigma(x) + V^n m_1.$$

We write $x = F_2(y)$ for $y \in I_n P$. Then we obtain

$$V^n m = V\varsigma(F_2 y) + V^{n+1} m_1 = \varsigma(y) + V^{n+1} m_1.$$

This ends the induction.

Recall $H^1(X, W\Omega^1_{X/R})$ is V-separated. By [Bourbaki 1961, §2.8, Théorème 1] it follows from (102) that $\varsigma : P \to H^1(X, W\Omega^1_{X/R})$ is surjective and $H^1(X, W\Omega^1_{X/R})$ is complete in the V-adic topology. Since V is also injective by Lemma 46, $H^1(X, W\Omega^1_{X/R})$ is a reduced Cartier module.

We consider (102) as a homomorphism of $W_{n+1}(R)$-modules.

Assertion. Both sides of (102) are isomorphic as $W_{n+1}(R)$-modules and are noetherian.

Because a surjective endomorphism of a noetherian module is an isomorphism, the assertion implies that (102) is an isomorphism. But then ς is an isomorphism.

To finish the proof it remains to show the assertion. From the bijection $F_2 : I_n P \to I_{n-1}P$, $F_2({}^{V^n}\xi x) \mapsto {}^{V^{n-1}}\xi F_1 x$ we deduce the bijection

$$F_2 : I_n P/I_{n+1}P \to I_{n-1}P/I_n P.$$

We denote by $P/I_1 P_{[F^n]}$ the $W_{n+1}(R)$-module obtained by restriction of scalars with respect to $F^n : W_{n+1}(R) \to R$. Iterating F_2 we obtain an isomorphism of $W_{n+1}(R)$-modules

$$F_2^n : I_n P/I_{n+1}P \to (P/I_1 P)_{[F^n]}.$$

Because R is F-finite, the last module is a noetherian $W_{n+1}(R)$-module [Langer and Zink 2004, Proposition A.2]. We note that $W_{n+1}(R)$ is a noetherian ring because it is a $W(k)$-module of finite length.

For the reduced Cartier module $M = H^1(X, W\Omega^1_{X/R})$ we obtain in the same way the isomorphism

$$V^n : M/VM_{[F^n]} \to V^n M/V^{n+1}M.$$

Therefore the isomorphism (100) together with Lemma 46 shows the assertion above. □

The isomorphism ς of Lemma 47 factors by definition through (97):

$$D(\Psi^{\mathrm{et}}_{X/R}) \to \mathbb{H}^2(X, W\Omega^{\geq 1}_{X/R}) \to H^1(X, W\Omega^1_{X/R}). \tag{103}$$

Therefore the last arrow is a split surjection. Since $H^0(X, W\Omega^1_{X/R}) = 0$ we obtain a split exact sequence

$$0 \to H^0(X, W\Omega^2_{X/k}) \to \mathbb{H}^2(W\Omega^{\geq 1}_{X/R}) \to H^1(X, W\Omega^1_{X/R}) \to 0.$$

Together with (86) this gives the Hodge–Witt decomposition

$$H^2_{\mathrm{crys}}(X/W(R)) = H^2(X, W\mathcal{O}_X) \oplus H^1(X, W\Omega^1_{X/R}) \oplus H^0(X, W\Omega^2_{X/R}).$$

We see that the free $W(R)$-modules on the right-hand side have ranks 1, 20, 1 since we know the height of the formal Brauer group and the extended formal Brauer group.

It follows from the above that

$$D(\Psi_X)_{W(R)} \cong H^2(X, W\mathcal{O}_X) \oplus H^1(X, W\Omega^1_{X/R})$$

and since

$$D(\Psi_X)_{W(R)} = \ker(H^2_{\mathrm{crys}}(X/W(R)) \xrightarrow{\pi} D(\widehat{\mathrm{Br}}^*_{X/R})(-1)_{W(R)})$$

(this surjective map is defined in [Nygaard and Ogus 1985, (3.20.1)]), π factors through an isomorphism

$$H^0(X, W\Omega^2_{X/R}) \xrightarrow{\sim} D(\widehat{\mathrm{Br}}^*_{X/R})(-1)_{W(R)}$$

of rank-1-$W(R)$-modules. This identifies all direct summands of $H^2_{\mathrm{crys}}(X/W(R))$ as Cartier–Dieudonné modules as in Theorem 40.

The Hodge–Witt decomposition for $H^2_{\mathrm{crys}}(X/W(R))$ implies a surjection,

$$H^2_{\mathrm{crys}}(X/W(R)) \twoheadrightarrow \mathbb{H}^2(X, W\Omega^{\leq 1}_{X/R}) = H^2(X, W\mathcal{O}_X) \oplus H^1(X, W\Omega^1_{X/R})$$

Then the map $H^1(X, W\Omega^2_{X/R}) \to H^3_{\mathrm{crys}}(X/W(R))$ is injective and therefore $H^1(X, W\Omega^2_{X/R})$ vanishes too, because $H^3_{\mathrm{crys}}(X/W(R)) = 0$.

We get the exact sequence

$$0 \to H^2(X, W\Omega^2_{X/R}) \to \underset{= W(R)}{H^4_{\mathrm{crys}}(X/W(R))} \to \mathbb{H}^4(W\mathcal{O}_X \xrightarrow{d} W\Omega^1_{X/R}).$$

We have seen that $H^3(X, W\mathcal{O}_X) = H^4(X, W\mathcal{O}_X) = 0$.

By the same arguments one shows that $V : H^3(X, W\Omega^1_{X/R}) \to H^3(X, W\Omega^1_{X/R})$ is injective with vanishing cokernel $= H^3(\Omega^1_{X/R})$.

So $H^3(W\Omega^1_{X/R}) = 0$; this means

$$H^2(X, W\Omega^2_{X/R}) \cong H^4_{\mathrm{crys}}(X/W(R))$$

and this finishes the proof of the theorem. □

Proposition 48. *Under the assumptions of Theorem 40, the Hodge–de Rham–Witt decomposition of $H^2_{\mathrm{crys}}(X/W(R))$ extends to a direct sum decomposition of displays (over the usual Witt ring $W(R)$) associated to the formal Brauer group, the étale part of the extended Brauer group and its Cartier dual, twisted by -1 and where $H^2_{\mathrm{crys}}(X/W(R))$ is equipped with the display structure arising from the Nygaard complex* (see [Langer and Zink 2007]).

Proof. This is clear. □

Alternatively we can derive a Hodge–Witt decomposition for H^2_{crys} using the universal deformation ring.

Let as before B be the universal deformation ring of X_k, X_B be the universal family of X_k over $\operatorname{Spf} B$, and define $X_n = X_B \times_{\operatorname{Spf} B} \operatorname{Spec} B/m^n$. Let $\widetilde{\mathcal{Y}}$ be a formal p-adic lifting to $W(B)$ with induced liftings $\mathcal{Y}^k_n$ over $\operatorname{Spec} W_k(B/m^n)$, compatible with the liftings X_n. We assume that n is big enough so that $B \to R$ factors through $B/m^n \to R$.

By [Langer and Zink 2007, Theorem 4.6] we have for $r < p$ a quasiisomorphism

$$Ru_{n*} J^{[r]}_{X_n/W_k(B/m^n)} \to I^r W_k\Omega^{\cdot}_{X_n/(B/m^n)},$$

where $u_n : \mathrm{Crys}(X_n/W_k(B/m^n)) \to X_n$ is the canonical morphism of sites and $I^r W_k\Omega^{\cdot}_{X_n/(B/m^n)}$ denotes the complex

$$p^{r-1} V W_{k-1}(\mathcal{O}_{X_n}) \to \cdots \to V W_{k-1}\Omega^{r-1}_{X_n/(B/m^n)} \to W_k\Omega^r_{X_n/(B/m^n)} \cdots .$$

By [Berthelot and Ogus 1978, Theorem 7.2], $Ru_{n*}J^{[r]}_{X_n/W_k(B/m^n)}$ is represented by the complex ($I_n^k := VW_{k-1}(B/m^n)$)

$$p^{r-1}I_n^k\Omega^0_{\mathcal{Y}_n^k/W_k(B/m^n)} \to \cdots \to I_n^k\Omega^{r-1}_{\mathcal{Y}_n^k/W_k(B/m^n)} \to \Omega^r_{\mathcal{Y}_n^k/W_k(B/m^n)} \to \cdots .$$

As we pass to the projective limit with respect to k, n and note that all inverse systems of sheaves in the above complexes are Mittag–Leffler systems, we get an isomorphism of complexes in the derived category of $W(B)$-modules between

$$p^{r-1}VW(B)\Omega^0_{\widetilde{\mathcal{Y}}/W(B)} \to \cdots \to VW(B)\Omega^{r-1}_{\widetilde{\mathcal{Y}}/W(B)} \to \Omega^r_{\widetilde{\mathcal{Y}}/W(B)} \to \cdots$$

and

$$p^{r-1}VW\mathcal{O}_{X_B} \to \cdots \to VW\Omega^{r-1}_{X_B/B} \to W\Omega^r_{X_B/B} \to \cdots$$

which is the inverse limit of the complexes $I^rW_k\Omega^{\cdot}_{X_n/(B/m^n)}$ with respect to k, n.

As multiplication by p is injective on $\Omega^{\cdot}_{\widetilde{\mathcal{Y}}/W(B)}$ and $W\Omega^{\cdot}_{X_B/B}$ (this can be reduced to a local argument, and can be made explicit for polynomial algebras), the first complex is isomorphic to $\mathcal{F}^r\Omega_{\widetilde{\mathcal{Y}}/W(B)}$ (notation as in Conjecture 37) and the second complex is isomorphic to the Nygaard complex $N^rW\Omega^{\cdot}_{X_B/B}$. The above considerations hold for any smooth proper X/R with a smooth deformation ring B and $r < p$. For K3 surfaces we take $r = 2$ and see that Conjecture 37 holds for the universal family X_B over B. Thus the statement of Theorem 40 holds for X_B over Spf B. In particular the de Rham–Witt spectral sequence

$$H^j(X_B, W\Omega^i_{X_B/B}) \to \mathbb{H}^{i+j}(X_B, W\Omega^{\cdot}_{X_B/B})$$

degenerates and we have a Hodge–Witt decomposition

$$H^2_{\text{crys}}(X_B/W(B)) = \bigoplus_{i+j=2} H^i(X_B, W\Omega^j_{X_B/B}). \tag{104}$$

By base change we get a decomposition over $W(R)$ as follows:

$$H^2_{\text{crys}}(X/W(R)) = \bigoplus_{i+j=2} H^i(X_B, W\Omega^j_{X_B/B}) \otimes_{W(B)} W(R). \tag{105}$$

Moreover we have the following evident properties of the direct summands:

- $H^2(X_B, W\mathcal{O}_{X_B}) \otimes_{W(B)} W(R) = H^2(X, W\mathcal{O}_X)$ is the Cartier–Dieudonné module of $\widehat{\mathrm{Br}}_{X_R} = \widehat{\mathbb{G}}_m/R$.
- $H^1(X_B, W\Omega^1_{X_B/B}) \otimes_{W(B)} W(R) = H^1(X, W\Omega^1_{X/R})$ is the Dieudonné module of $\Psi^{\text{et}}_{X/R}$.
- $H^0(X_B, W\Omega^2_{X_B/B}) \otimes_{W(B)} W(R) = H^0(X, W\Omega^2_{X/R})$ is the (shifted by -1) Dieudonné module of the Cartier dual $\widehat{\mathrm{Br}}^*_{X/R}$.

As in Proposition 48, the decomposition (105), which is a direct sum decomposition of Dieudonné modules of p-divisible groups, extends to a direct sum decomposition of the corresponding displays, where $H^2_{\mathrm{crys}}(X/W(R))$ carries the display structure obtained by base change via $B \to R$ from the display structure on $H^2_{\mathrm{crys}}(X_B/W(B))$.

Acknowledgements

The authors would like to thank the referee for pointing out inaccuracies and for his enlightening questions which often led to better arguments. Zink thanks the IHES, Bures-sur-Yvette for hospitality during the preparation of this paper.

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Received 5 Sep 2017. Revised 16 May 2018.

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Tunisian Journal of Mathematics

2019 vol. 1 no. 4