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Purity of crystalline strata

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Let p be a prime. Let $n \in \mathbb{N}^*$. Let $\mathcal{C}$ be an F^n-crystal over a locally noetherian $\mathbb{F}_p$-scheme S. Let $(a, b) \in \mathbb{N}^2$. We show that the reduced locally closed subscheme of S whose points are exactly those $x \in S$ such that (a, b) is a break point of the Newton polygon of the fiber $\mathcal{C}_x$ of $\mathcal{C}$ at x is pure in S, i.e., it is an affine S-scheme. This result refines and reobtains previous results of de Jong and Oort, of Vasiu, and of Yang. As an application, we show that for all $m \in \mathbb{N}$ the reduced locally closed subscheme of S whose points are exactly those $x \in S$ for which the p-rank of $\mathcal{C}_x$ is m is pure in S; the case $n = 1$ was previously obtained by Deligne (unpublished) and the general case $n \geq 1$ refines and reobtains a result of Zink.

1. Introduction

For a reduced locally closed subscheme Z of a locally noetherian scheme Y, let $\bar{Z}$ be the schematic closure of Z in Y. We recall from [Nicole et al. 2010, Definition 1.1] that Z is called *pure* in Y if it is an affine Y-scheme. The paper [Nicole et al. 2010] also uses a weaker variant of this purity which in [Li 2015] is called *weakly pure*: we say Z is weakly pure in Y if each nonempty irreducible component of the complement $\bar{Z} - Z$ is of pure codimension 1 in $\bar{Z}$. It is well known that if Z is pure in Y, then Z is also weakly pure in Y (for instance, see Proposition 13 of Section 4.4).

Let n and r be natural numbers. Let p be a prime. Let S be a locally noetherian $\mathbb{F}_p$-scheme. Let $\Phi_S : S \to S$ be the Frobenius endomorphism of S. Let $\mathcal{M}$ be a *crystal* of the gross absolute crystalline site $CRIS(S/\operatorname{Spec}(\mathbb{Z}_p))$ introduced in [Berthelot 1974, Chapter III, Example 1.1.3 and Definition 4.1.1] in locally free $\mathcal{O}_{S/\operatorname{Spec}(\mathbb{Z}_p)}$-modules of rank r. We assume that we have an *isogeny* $\phi_{\mathcal{M}} : (\Phi_S^n)^*(\mathcal{M}) \to \mathcal{M}$; thus the pair $\mathcal{C} = (\mathcal{M}, \phi_{\mathcal{M}})$ is an F^n-crystal of $CRIS(S/\operatorname{Spec}(\mathbb{Z}_p))$. If the $\mathbb{F}_p$-scheme $S = \operatorname{Spec} A$ is affine, then the pair $\mathcal{C} = (\mathcal{M}, \phi_{\mathcal{M}})$ is canonically identified with a $\sigma^n\!-\!F$-crystal on A in the sense of [Katz 1979, Subsection (2.1)].

Let $\nu : [0, r] \to [0, \infty)$ be a *Newton polygon*, i.e., a nondecreasing piecewise linear continuous function such that $\nu(0) = 0$ and the coordinates of all its *break*

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points are natural numbers. For $x \in S$, let ν_x be the *Newton polygon* of the fiber $\mathcal{C}_x$ of $\mathcal{C}$ at x. Let S_ν be the reduced locally closed subscheme of S whose points are exactly those $x \in S$ such that we have $\nu_x = \nu$; see the Grothendieck–Katz theorem [Katz 1979, Corollary 2.3.2]. If nonempty, S_ν is a *stratum* of the Newton polygon *stratification* of S defined by $\mathcal{C}$.

Let $a, b \in \mathbb{N}$ be such that $0 \leq a \leq r$. Let $T = T_{(a,b)}(\mathcal{C})$ be the reduced locally closed subscheme of S whose points are those $x \in S$ such that (a, b) is a break point of ν_x. The end break point $(r, \nu_x(r))$ remains constant under specializations of $x \in S$. Thus locally in the Zariski topology of S, we can assume that there exists $d \in \mathbb{N}$ such that for all $x \in S$ we have $\nu_x(r) = d$ and this implies that T is the reduced locally closed subscheme of S which is a finite union $\bigcup_{\nu \in N_{r,d,a,b}} S_\nu$ of Newton polygon strata S_ν indexed by the set $N_{r,d,a,b}$ of all Newton polygons $\nu : [0, r] \to [0, \infty)$ with the two properties that $\nu(r) = d$ and (a, b) is a break point of ν.

It is known that T is weakly pure in S; see [Yang 2011, Theorem 1.1.] It is also known that S_ν is pure in S; see [Vasiu 2006, Main Theorem B]. This last result implies the celebrated result of de Jong and Oort [2000, Theorem 4.1] which asserts that S_ν is weakly pure in S. Strictly speaking, the references of this paragraph work with $n = 1$ but their proofs apply to all $n \in \mathbb{N}^*$.

In general, a finite union of locally closed subschemes of S which are pure in S is not pure in S. Therefore the following purity result which refines and reobtains the mentioned results of de Jong and Oort, of Vasiu, and of Yang, comes as a surprise.

Theorem 1. *With the above notation*, T *is pure in* S.

In Section 2 we gather the few preliminary steps that are required to prove Theorem 1 in Section 3. The following two corollaries are direct consequences of Theorem 1. The first one for $n = 1$ just reobtains [Vasiu 2006, Main Theorem B] in the locally noetherian case.

Corollary 2. *Each Newton polygon stratum* S_ν *is pure in* S.

The p-rank $\chi(x)$ of $\mathcal{C}_x$ is the multiplicity of the Newton polygon slope 0 of ν_x. Equivalently, $\chi(x)$ is the unique natural number such that $(0, 0)$ and $(\chi(x), 0)$ are the only break points of ν_x on the horizontal axis (i.e., which have the second coordinate 0).

Corollary 3. *Let* $m \in \mathbb{N}$. *We consider the reduced locally closed subscheme* S_m *of* S *whose points are exactly those* $x \in S$ *such that the* p*-rank* $\chi(x)$ *of* $\mathcal{C}_x$ *is* m. *Then* S_m *is pure in* S.

If $m > 0$, then we have $S_m = T_{(m,0)}(\mathcal{C})$ and if $m = 0$, then we have $S_0 = T_{(1,0)}(\mathcal{C} \oplus \mathcal{E}_0)$ where $\mathcal{E}_0$ is the pullback to S of the F^n-crystal over $\mathrm{Spec}(\mathbb{F}_p)$ of rank 1 and Newton polygon slope 0 which has a Frobenius invariant global section; therefore, regardless of what m is, Corollary 3 follows from Theorem 1.

For $n = 1$ Corollary 3 was first obtained by Deligne [2011] and more recently by Vasiu [2014] and Li [2015]. Corollary 3 also refines and reobtains a prior result of Zink which asserts that S_m is weakly pure in S (see [Zink 2001], Proposition 5).

In Section 4 we first follow [Li 2015] to show that Corollary 2 follows directly from Theorem 1 and then we follow [Vasiu 2014] to include a second proof of Corollary 3 in the more general context provided by a functorial version of the *Artin–Schreier stratifications* introduced in [Vasiu 2013, Definition 2.4.2] which is simpler, does not rely on Theorem 1, and is based on Theorem 12 of Section 4.2.

Theorem 1 is due to Li [2015]. While the proof of [Yang 2011, Theorem 1.1] follows the proof of [de Jong and Oort 2000, Theorem 4.1], the proof of Theorem 1 presented follows [Li 2015] and thus the proofs of [Vasiu 2006, Main Theorem B and Theorem 6.1]. It is known (see [Nicole et al. 2010, Example 7.1]) that in general S_m is not strongly pure in S in the sense of [Nicole et al. 2010, Definition 7.1], and therefore Theorem 1 and Corollary 3 cannot be improved in general (i.e., are optimal).

We refer to $T_{(a,b)}(\mathcal{C})$, S_ν, and S_m as crystalline strata of S associated to $\mathcal{C}$ and certain (basic) discrete invariants of F^n-crystals. Cases of nondiscrete invariants stemming from isomorphism classes are also studied in the literature (for instance, see [Vasiu 2006, Section 5.3] and [Nicole et al. 2010, Theorem 1.2 and Corollary 1.5]). Crystalline strata have applications to the study in positive characteristic of different moduli spaces and schemes such as special fibers of Shimura varieties of Hodge type (for instance, see [Vasiu 2006] and [Nicole et al. 2010]).

2. Standard reduction steps

The above notation $p,\ S,\ \Phi_S,\ \bar{Z},\ n,\ r,\ \mathcal{C} = (\mathcal{M}, \phi_{\mathcal{M}}),\ \mathcal{C}_x,\ \nu_x,\ (a,b) \in \mathbb{N}^2$, $T = T_{(a,b)}(\mathcal{C}),\ S_\nu,\ m,\ S_m,\ \chi(x)$, and $\mathcal{E}_0$ will be used throughout the paper. For a fixed Newton polygon ν, let $S_{\geq\nu}$ be the reduced closed subscheme of S whose points are exactly those $x \in S$ such that the Newton polygon ν_x is above ν, see [Katz 1979, Corollary 2.3.2].

In what follows, by an étale cover we mean a surjective finite étale morphism of schemes. For basic properties of excellent rings we refer to [Matsumura 1980, Chapter 13]. If $V \to Y$ is a morphism of $\mathbb{F}_p$-schemes and if $\mathcal{F}$ (or $\mathcal{F}_Y$) is an F^n-crystal over Y, let $\mathcal{F}_V$ be the pullback of $\mathcal{F}$ (or $\mathcal{F}_Y$) to an F^n-crystal over V, i.e., of $\mathit{CRIS}(V/\operatorname{Spec}(\mathbb{Z}_p))$. Let $k(y)$ be the residue field of a point $y \in Y$. If $V = \operatorname{Spec}(k(y)) \to Y$ is the natural morphism, then we denote $\mathcal{F}_V = \mathcal{F}_{\operatorname{Spec}(k(y))}$ simply by $\mathcal{F}_y$ (the fiber of $\mathcal{F}$ at y).

For an $\mathbb{F}_p$-algebra R, let $W(R)$ be the ring of p-typical Witt vectors with coefficients in R. Let $\mathbb{W}(R) = (\operatorname{Spec} R, \operatorname{Spec}(W(R)), \text{can})$ be the thickening in which "can" stands for the canonical divided power structure of the kernel of the epimorphism $W(R) \to W_1(R) = R$. For $s \in \mathbb{N}^*$, let $W_s(R)$ be the ring of p-typical Witt vectors of length s with coefficients in R. Let $\mathbb{W}_s(R) = (\operatorname{Spec} R, \operatorname{Spec}(W_s(R)), \text{can})$ be the thickening defined naturally by $\mathbb{W}(R)$. Let Φ_R be the Frobenius endomorphism of either $W(R)$ or $W_s(R)$.

The property of a reduced locally closed subscheme being pure in S is local for the faithfully flat topology of S, and thus until the end we will also assume that $S = \operatorname{Spec} A$ is an affine $\mathbb{F}_p$-scheme and that there exists $d \in \mathbb{N}$ such that for all $x \in S$ we have $\nu_x(r) = d$. As the scheme S is locally noetherian and affine, it is noetherian. To prove Theorem 1, we have to prove that T is an affine scheme.

2.1. ***Some abelian categories.*** Let $\mathcal{M}(W_s(R))$ be the abelian category whose objects are pairs (O, ϕ_O), comprised of a $W_s(R)$-module O and a Φ_R^n-linear endomorphism $\phi_O : O \to O$ (i.e., ϕ_O is additive and for all $z \in O$ and $\sigma \in W_s(R)$ we have $\phi_O(\sigma z) = \Phi_R^n(\sigma)\phi_O(z)$) and whose morphisms $f : (O_1, \phi_{O_1}) \to (O_2, \phi_{O_2})$ are $W_s(R)$-linear maps $f : O_1 \to O_2$ satisfying $f \circ \phi_{O_1} = \phi_{O_2} \circ f$. If $t \in \{0, \dots, s-1\}$, then by a *quasi-isogeny* of $\mathcal{M}(W_s(R))$ whose cokernel is annihilated by p^t we mean a morphism $f : (O_1, \phi_{O_1}) \to (O_2, \phi_{O_2})$ of $\mathcal{M}(W_s(R))$ which has the following two properties: (i) both O_1 and O_2 are projective $W_s(R)$-modules which have the same positive rank locally in the Zariski topology of $\operatorname{Spec}(W_s(R))$, and (ii) the cokernel $O_2/f(O_1)$ is annihilated by p^t. An object (O, ϕ_O) of $\mathcal{M}(W_s(R))$ is called *divisible* by $t \in \{1, \dots, s-1\}$ if O is a projective $W_s(R)$-module such that $\operatorname{Im}(\phi_O) \subseteq p^t O$.

For $l \in \mathbb{N}^*$ we have a natural functor

$$\mathcal{M}(W_{s+l}(R)) \to \mathcal{M}(W_s(R))$$

to be referred to, by abuse of language, as the reduction modulo p^s functor.

If Y is a $\operatorname{Spec}(\mathbb{F}_p)$-scheme, in a similar way we define the scheme $W_s(Y)$, its Frobenius endomorphism Φ_Y, and the abelian category $\mathcal{M}(W_s(Y))$, and speak about quasi-isogenies of $\mathcal{M}(W_s(Y))$ whose cokernels are annihilated by p^t with $t \in \{0, \dots, s-1\}$, about objects of $\mathcal{M}(W_s(Y))$ divisible by $t \in \{1, \dots, s-1\}$, and about reduction modulo p^s functors $\mathcal{M}(W_{s+l}(Y)) \to \mathcal{M}(W_s(Y))$. We have canonical identifications

$$\mathcal{M}(W_s(R)) = \mathcal{M}(W_s(\operatorname{Spec} R)).$$

For homomorphisms $R \to R_1$ and morphisms $Y_1 \to Y$, we have natural pullback functors $\mathcal{M}(W_s(R)) \to \mathcal{M}(W_s(R_1))$ and $\mathcal{M}(W_s(Y)) \to \mathcal{M}(W_s(Y_1))$.

To prove that T is an affine scheme, we can also assume that the *evaluation* M of $\mathcal{M}$ at the thickening $\mathbb{W}_1(A)$ is a free A-module of rank r. The evaluation of $\phi_{\mathcal{M}}$ at this thickening is a Φ_A^n-linear endomorphism $\phi_M : M \to M$.

In what follows we will apply twice the following elementary general fact which can be also deduced easily from the elementary divisor theorem.

Fact 4. Let D be a discrete valuation ring and let $\pi \in D$ be a uniformizer of it. Let $s, t \in \mathbb{N}$ be such that $s > t$. Let $D_s = D/(\pi^s)$. Let $g_s : D_s^r \to D_s^r$ be a D_s-linear endomorphism such that its cokernel is annihilated by π^t. Then for each $x \in D_s^r - \pi D_s^r$, we have $g_s(x) \in D_s^r - \pi^{t+1} D_s^r$.

Proof. Let $g : D^r \to D^r$ be a D-linear endomorphism which lifts g_s. Let $E = \mathrm{Im}(g) + \pi^s D^r$ (one can easily check that $E = \mathrm{Im}(g)$ but we will not stop to argue this). It is a free D-module of rank r which (as $\pi^t\,\mathrm{Coker}(g_s) = 0$) contains $\pi^t D^r$. Thus $\pi^s D^r \subseteq pE$ and therefore $\mathrm{Im}(g)$ surjects onto the D_1-vector space $E/\pi E$ of rank r. Hence a D_s-basis of D_s^r maps via g_s to a D_1-basis of $E/\pi E$. From this and the fact that $\pi^{t+1} D^r \subseteq \pi E$ we get that no element of a D_s-basis of D_s^r is mapped by g_s to $\pi^{t+1} D_s^r$. Thus the fact holds. □

2.2. ***On (a, b).*** If (a, b) is $(0, 0)$ or (r, d), then $T = S$. If $a = 0$ and $b > 0$ or if $a = r$ and $b \neq d$, then $T = \varnothing$. Thus, to prove that T is an affine scheme we can assume that $1 \leq a \leq r - 1$.

Lemma 5. *Let k be a field of characteristic p. Let $\nu : [0, r] \to [0, \infty)$ be the Newton polygon of an F^n-crystal $\mathcal{F}$ over k of rank r. Let $a, b \in \mathbb{N}$ be such that $1 \leq a \leq r - 1$. Then (a, b) is a break point of ν if and only if $(1, b)$ is a break point of the Newton polygon $\bigwedge^a(\nu)$ of the F^n-crystal over k of rank $\binom{r}{a}$ which is the exterior power $\bigwedge^a(\mathcal{F})$ of $\mathcal{F}$.*

Proof. Let $\alpha_1 \leq \cdots \leq \alpha_r$ be the Newton polygon slopes of ν. Let $\beta_1 \leq \cdots \leq \beta_{\binom{r}{a}}$ be the Newton polygon slopes of $\bigwedge^a(\nu)$. We have

$$\beta_1 = \sum_{i=1}^{a} \alpha_i \quad \text{and} \quad \beta_2 = \left(\sum_{i=1}^{a-1} \alpha_i\right) + \alpha_{a+1} = \beta_1 + \alpha_{a+1} - \alpha_a.$$

Thus $\beta_1 < \beta_2$ if and only if $\alpha_a < \alpha_{a+1}$. Moreover, (a, b) is a break point of ν if and only if we have $\alpha_a < \alpha_{a+1}$, and $(1, b)$ is a break point of the Newton polygon $\bigwedge^a(\nu)$ if and only if we have $\beta_1 < \beta_2$. The lemma follows from the last two sentences. □

Based on Lemma 5, to prove that T is an affine scheme, by replacing $\mathcal{C}$ with its exterior power $\bigwedge^a(\mathcal{C})$, we can assume that $a = 1$.

2.3. ***A description of T.*** Let $q \in \mathbb{N}^*$ be such that for each $x \in S$ the Newton polygon slopes of the F^{nq}-crystal over $\mathrm{Spec}(k(x))$ which is the q-th iterate of $\mathcal{C}_x$, are all integers. For instance, as each Newton polygon slope of $\mathcal{C}_x$ is a rational number whose denominator is a natural number at most equal to r, we can take $q = r!$. Thus by replacing n by nq and $\mathcal{C}$ by its q-th iterate, we can assume that, for each $x \in S$, the Newton polygon slopes of $\mathcal{C}_x$ are natural numbers.

We consider the Newton polygon $\nu_1 : [0, r] \to [0, \infty)$ whose graph is Figure 1.

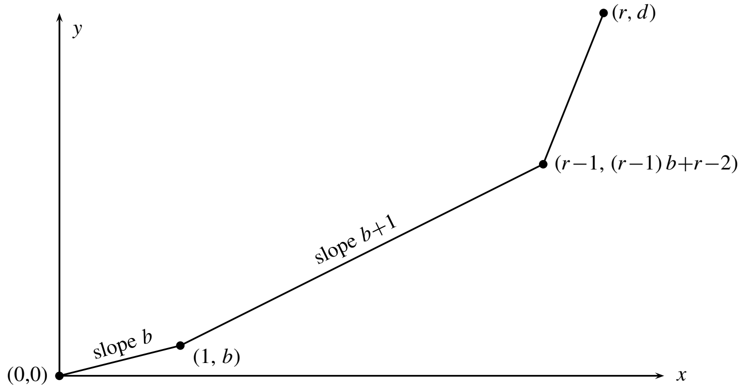

Figure 1. The Newton polygon $\nu_1 : [0, r] \to [0, \infty)$.

If $x \in T$, then because all Newton polygon slopes of $\mathcal{C}_x$ are natural numbers, these Newton polygon slopes are $\alpha_1 = b, \ \alpha_2 \geq b+1, \ \alpha_{r-1} \geq b+1$, and $\alpha_r = d - \sum_{i=1}^{r-1} \alpha_i \geq b+1$. Therefore, if $x \in T$ then we have $x \in S_{\geq \nu_1}$. This implies that T is a subscheme of the closed subscheme $S_{\geq \nu_1}$ of S. By replacing S with $S_{\geq \nu_1}$ we can assume that $S = S_{\geq \nu_1}$. Thus S is reduced.

If $r(b+1) > d$, then $S = S_{\geq \nu_1} = S_{\nu_1} = T$ and thus T is affine. Thus we can assume that $r(b+1) \leq d$ and therefore there exists a Newton polygon $\nu_2 : [0, r] \to [0, \infty)$ whose graph is Figure 2.

If $x \in S - T = S_{\geq \nu_1} - T$, then all Newton polygon slopes of $\mathcal{C}_x$ are natural numbers $\alpha_1 \geq b+1, \ \alpha_2 \geq b+1, \ \alpha_{r-1} \geq b+1$, and $\alpha_r = d - \sum_{i=1}^{r-1} \alpha_i \geq b+1$ and thus ν_x is above ν_2. If ν_x is not above ν_2, then as ν_x is above ν_1 (as $S = S_{\geq \nu_1}$) we have $\alpha_1 = b$ and $\alpha_i \geq b+1$ for $i \in \{2, \ldots, r\}$.

With the last two sentences, we have the identities

$$T = T_{(1,b)} = S - S_{\geq \nu_2} = S_{\geq \nu_1} - S_{\geq \nu_2}.$$

Thus, under all the above reduction steps, T is an open subscheme of S.

2.4. ***On S.*** The statement that T is an affine scheme is local in the faithfully flat topology of S and therefore until the end of Section 3 we will assume that A is a complete local reduced noetherian ring. Thus A is also excellent and therefore its normalization in its ring of fractions is a finite product of normal complete local noetherian integral domains. Based on [Vasiu 2006, Lemma 2.9.2], which is a standard application of Chevalley's theorem of [Grothendieck 1961, Chapter II, (6.7.1)], to prove that T is an affine scheme we can replace A by one of the factors of the mentioned finite product. Thus we can assume that A is a normal complete local noetherian integral domain. We can also assume that T is nonempty and therefore it is an open dense subscheme of S. Let K be the field of factions of A and let $\overline{K}$ be an algebraic closure of it.

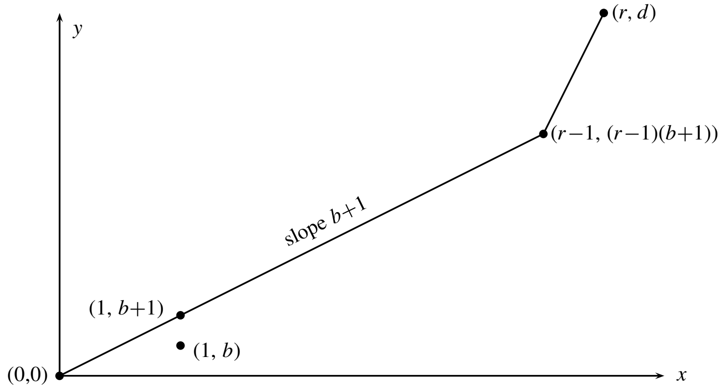

Figure 2. The Newton polygon $\nu_2 : [0, r] \to [0, \infty)$.

3. Proof of Theorem 1

In this section we complete the proof of Theorem 1, i.e., we prove that T is an affine scheme when $a = 1 < r$, when for each $x \in S$ all Newton polygon slopes of $\mathcal{C}_x$ are natural numbers, when we have $S = S_{\geq \nu_1} = \operatorname{Spec} A$ with A a normal complete local noetherian integral domain, and when $T = T_{(1,b)} = S - S_{\geq \nu_2}$ is open dense in S. Let $\mathcal{E}_b = (\mathcal{M}_b, \phi_{\mathcal{M}_b})$ be the pullback to S of the F^n-crystal over $\operatorname{Spec}(\mathbb{F}_p)$ of rank 1 and Newton polygon slope b defined by the pair $(\mathbb{Z}_p, p^b 1_{\mathbb{Z}_p})$. Let η be the generic point $\operatorname{Spec} K \to S$ of S. Let $s, l \in \mathbb{N}^*$.

In Section 3.1 we consider commutative affine group schemes $\mathbb{H}_s$ over S of morphisms between certain evaluations of $\mathcal{E}_b$ and $\mathcal{C}$. In Section 3.2 we glue morphisms between different such evaluations in order to introduce good sections above T of the morphisms $\mathbb{H}_s \to S$ in Section 3.3. In Section 3.4 we complete the proof of Theorem 1. The key idea (the plan) can be summarized as follows: under suitable reductions, for $s \gg 0$ via such good sections above T we can identify T with a closed subscheme of $\mathbb{H}_s$ and therefore we can conclude that T is an affine scheme.

If R is a reduced perfect ring of characteristic p, following [Katz 1979] we say that an F^n-crystal $\mathcal{F}$ over $\operatorname{Spec} R$ is divisible by b if its evaluation at the endomorphism Φ_R^n of the thickening $\mathbb{W}(R)$ is defined by a Φ_R^n-linear endomorphism whose q-th iterate for all $q \in \mathbb{N}^*$ is congruent to 0 modulo p^{bq}. Thus if $y \in \operatorname{Spec} R$, then the Hodge polygon slopes of $\mathcal{F}_y$ are all greater than or equal to b.

3.1. ***Moduli group schemes of morphisms.*** For an A-algebra B and an F^n-crystal $\mathcal{F}$ over B, let $\mathbb{E}_s(\mathcal{F})$ be the evaluation of $\mathcal{F}$ at the thickening $\mathbb{W}_s(B)$; it is an object of the category $\mathcal{M}(W_s(B))$. In particular, we write $\mathbb{E}_s(\mathcal{C}_B) = (M_{s,B}, \phi_{M_{s,B}})$ and let $\mathbb{E}_s(\mathcal{E}_{b,B}) = (N_{s,B}, \phi_{N_{s,B}})$. Thus we have $M = M_{1,A}$, $\phi_M = \phi_{M_{1,A}}$, and $N_{s,B} = W_s(B)$. Moreover $\phi_{N_{s,B}} : N_{s,B} \to N_{s,B}$ is the Φ_B^n-linear endomorphism which maps 1 to p^b and $\phi_{M_{s,B}} : M_{s,B} \to M_{s,B}$ is a Φ_B^n-linear endomorphism and we have

$M_{s,B} = W_s(B) \otimes_{W_s(A)} M_{s,A}$. The kernel of the epimorphism $W_s(B) \to W_1(B) = B$ is a nilpotent ideal. Based on this and the fact that M is a free A-module of rank r, we get that each $M_{s,B}$ is a free $W_s(B)$-module of rank r.

We consider the commutative affine group scheme $\mathbb{H}_s$ over S which represents the following functor: for an A-algebra B, the abelian group

$$\mathbb{H}_s(B) = \mathrm{Hom}_{\mathcal{M}(W_s(B))}(\mathbb{E}_s(\mathcal{E}_{b,B}), \mathbb{E}_s(\mathcal{C}_B))$$

is the group of all $W_s(B)$-linear maps $f : N_{s,B} \to M_{s,B}$ which satisfy the identity $f \circ \phi_{N_{s,B}} = \phi_{M_{s,B}} \circ f$. The S-scheme $\mathbb{H}_s$ is of finite presentation (for $n = 1$, see [Vasiu 2006, Lemma 2.8.4.1], the proof of which applies to all $n \in \mathbb{N}^*$).

Let $x \in S$ be a point of codimension 1. Thus the local ring $D_x := \mathcal{O}_{S,x}$ of S at x is a discrete valuation ring. Let E_x be a complete discrete valuation ring which dominates D_x and has a residue field which is algebraically closed. Let P_x be the perfection of E_x. We recall that $\mathcal{C}_{P_x}$ is the pullback of $\mathcal{C}$ via the natural morphism $\mathrm{Spec}\, P_x \to S$. As $S = S_{\geq \nu_1}$, the Newton polygon slopes of the two fibers of $\mathcal{C}_{P_x}$ are greater than or equal to b. Thus from [Katz 1979, Theorem 2.6.1], we get the existence of an F^n-crystal $\mathcal{D}$ over $\mathrm{Spec}\, P_x$ which is divisible by b and which is equipped with an isogeny

$$\psi_x : \mathcal{D} \to \mathcal{C}_{P_x}$$

whose cokernel is annihilated by p^t for some $t \in \mathbb{N}$. Based on the proof of [loc. cit.], we can assume that

$$t = (r-1)b$$

depends only on r and b.

Proposition 6. *We assume that the point $x \in S$ of codimension* 1 *belongs to T. Then there exists a unique F^n-subcrystal $\mathcal{D}_b$ of $\mathcal{D}$ which is isomorphic to the pullback $\mathcal{E}_{b,P_x}$ of $\mathcal{E}_b$. Moreover, $\mathcal{D}_b$ has a unique direct supplement in $\mathcal{D}$.*

Proof. We know that for $y \in \mathrm{Spec}\, P_x$, all Hodge polygon slopes of $\mathcal{D}_y$ are at least b. If all Hodge polygon slopes of $\mathcal{D}_y$ are at least $b+1$, then all Newton polygon slopes of $\mathcal{D}_y$ are at least $b+1$. As under the morphism $\mathrm{Spec}\, P_x \to S$, the point y maps to either $x \in T$ or $\eta \in T$ and as ψ_x is an isogeny, $(1, b)$ is a break point of the Newton polygon of $\mathcal{D}_y$. From the last three sentences we get that $(1, b)$ is a point of the Hodge polygon of $\mathcal{D}_y$.

Thus for each point $y \in \mathrm{Spec}\, P_x$, $(1, b)$ is a break point of the Newton polygon of $\mathcal{D}_y$ and is a point of the Hodge polygon of $\mathcal{D}_y$. Due to this, from [Katz 1979, Theorem 2.4.2] we get that there exists a unique direct sum decomposition,

$$\mathcal{D} = \mathcal{D}_b \oplus \mathcal{D}_{>b},$$

into F^n-crystals over $\mathrm{Spec}\, P_x$, where $\mathcal{D}_b$ is of rank 1 and each fiber of it at a point $y \in \mathrm{Spec}\, P_x$ has all Hodge and Newton polygon slopes equal to b and where $\mathcal{D}_{>b}$

is of rank $r-1$ and each fiber of it at a point $y \in \operatorname{Spec} P_x$ has all Newton polygon slopes greater than b (and has all Hodge polygon slopes greater than or equal to b).

As $\mathcal{D}$ is divisible by b, $\mathcal{D}_b$ and $\mathcal{D}_{>b}$ are also divisible by b.

As P_x is perfect, for each $l \in \mathbb{N}^*$ we have $W(P_x)/(p^l) = W_l(P_x)$ and the module of differentials $\Omega^1_{W_l(P_x)}$ is 0. Thus, from [Berthelot and Messing 1990, Proposition 1.3.3] we get that an F^n-crystal over $\operatorname{Spec} P_x$ is uniquely determined by its evaluation at the thickening $\mathbb{W}(P_x)$. The evaluation of $\mathcal{E}_{b,P_x}$ at the thickening $\mathbb{W}(P_x)$ is canonically identified with $(W(P_x), p^b\Phi^n_{P_x})$ and the evaluation of $\mathcal{D}_b$ at the thickening $\mathbb{W}(P_x)$ can be identified with $(W(P_x), p^b\Phi_b)$, where $\Phi_b : W(P_x) \to W(P_x)$ is a $\Phi^n_{P_x}$-linear endomorphism such that $\Phi_b(1)$ generates $W(P_x)$.

As P_x is the perfection of E_x and as E_x is complete and has an algebraically closed residue field, the rings $W(P_x)$ and $W_l(P_x)$ are strictly henselian and p-adically complete. We check that these properties imply that there exists a unit υ of $W(P_x)$ such that we have

$$\Phi_b(\upsilon) = \Phi^n_{P_x}(\upsilon)\Phi_b(1) = \upsilon.$$

If $n = 1$, then from [Berthelot and Messing 1990, Proposition 2.4.9] we get that for each $l \in \mathbb{N}^*$ there exists a unit $\upsilon_l \in W(P_x)$ such that $\Phi_b(\upsilon_l) - \upsilon_l \in p^l W(P_x)$, and the proof of [loc. cit.] confirms that we can assume that $\upsilon_{l+1} - \upsilon_l \in p^l W(P_x)$. Thus for $n = 1$ we can take υ to be the p-adic limit of the sequence $(\upsilon_l)_{l\geq 1}$. This argument applies entirely for $n > 1$.

Multiplication by υ defines an isomorphism

$$(W(P_x), p^b\Phi^n_{P_x}) \to (W(P_x), p^b\Phi_b)$$

which defines an isomorphism $\mathcal{E}_{b,P_x} \to \mathcal{D}_b$. □

From now we will assume that $x \in T$. We consider a composite morphism

$$j_x[s] : \mathbb{E}_s(\mathcal{E}_{b,P_x}) \to \mathbb{E}_s(\mathcal{D}_b) \to \mathbb{E}_s(\mathcal{D}) = \mathbb{E}_s(\mathcal{D}_b) \oplus \mathbb{E}_s(\mathcal{D}_{>b})$$

in which the first arrow is an isomorphism and the second arrow is the split monomorphism associated to the direct sum decomposition.

Let

$$i_x(s) : \mathbb{E}_s(\mathcal{E}_{b,P_x}) \to \mathbb{E}_s(\mathcal{C}_{P_x})$$

be the composite of $j_x[s]$ with the morphism $\psi_x[s] : \mathbb{E}_s(\mathcal{D}) \to \mathbb{E}_s(\mathcal{C}_{P_x})$ which is the evaluation of the isogeny ψ_x at the thickening $\mathbb{W}_s(P_x)$ (i.e., which is the reduction modulo p^s of ψ_x). From now on, we will take $s > t = (r-1)b$. We note that $\psi_x[s]$ is a quasi-isogeny whose cokernel is annihilated by p^t and whose domain is divisible by b.

3.2. ***Gluing morphisms.*** For each point $x \in T$ of codimension 1 (i.e., whose local ring D_x is a discrete valuation ring), we follow [Vasiu 2006, Section 2.8.3] to show the existence of a finite field extension K_x of K and of an open subset T_x of the normalization of T in Spec K_x such that T_x has a local ring which is a discrete valuation ring D_x^+ that dominates D_x and moreover we have a morphism

$$i_{T_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,T_x}) \to \mathbb{E}_s(\mathcal{C}_{T_x})$$

of the category $\mathcal{M}(W_s(T_x))$ which is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^t and whose domain is divisible by b.

To check this, with the notations of Section 3.1 we consider four identifications,

$$E_s(\mathcal{C}_{D_x}) = (W_s(D_x)^r, \phi_{s,x}), \qquad \mathbb{E}_s(\mathcal{E}_{b,D_x}) = (W_s(D_x),\, p^b\Phi^n_{D_x}),$$
$$\mathbb{E}_s(\mathcal{D}_b) = (W_s(P_x),\, p^b\Phi^n_{P_x}), \qquad \mathbb{E}_s(\mathcal{D}_{>b}) = (W_s(P_x)^{r-1},\, p^b\phi_{s,>b,x}).$$

Now, the $W_s(P_x)$-linear map $\psi_{s,P_x} : W_s(P_x)^r \to W_s(P_x)^r$ defining $\psi_x[s]$ and the $\Phi^n_{P_x}$-linear map $\phi_{s,>b,x} : W_s(P_x)^{r-1} \to W_s(P_x)^{r-1}$ involve a finite number of coordinates of Witt vectors of length s and therefore are defined over $W_s(B_x)$, where B_x is a finitely generated D_x-subalgebra of P_x. We can choose B_x such that the resulting $W_s(B_x)$-linear map $\psi_{s,B_x} : W_s(B_x)^r \to W_s(B_x)^r$ has a cokernel annihilated by p^t. The faithfully flat morphism Spec $B_x \to$ Spec D_x has quasisections (see [Grothendieck 1967, Corollary 17.16.2]) and therefore there exists a finite field extension K_x of K and a discrete valuation ring D_x^+ of the normalization T in K_x which dominates D_x and for which we have a D_x-homomorphism $B_x \to D_x^+$. The $W_s(D_x^+)$-linear map $\psi_{s,D_x^+} : W_s(D_x^+)^r \to W_s(D_x^+)^r$ which is the natural tensorization of ψ_{s,B_x} induces (via restriction to the first factor $W_s(D_x^+)$ of $W_s(D_x^+)^r$) a morphism $i_{D_x^+}(s) : \mathbb{E}_s(\mathcal{E}_{b,D_x^+}) \to \mathbb{E}_s(\mathcal{C}_{D_x^+})$ of the category $\mathcal{M}(W_s(D_x^+))$ which is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^t and whose domain is divisible by b. It is easy to see that there exists an open subset T_x of the normalization of T in K_x which has D_x^+ as a local ring and for which there exists a morphism $i_{T_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,T_x}) \to \mathbb{E}_s(\mathcal{C}_{T_x})$ of the category $\mathcal{M}(W_s(T_x))$ that has all the desired properties and that extends the morphism $i_{D_x^+}(s)$ of the category $\mathcal{M}(W_s(D_x^+))$.

By working with $s+l$ instead of s, we can assume that there exists $l \in \mathbb{N},\ l \gg 0$ such that $i_{T_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,T_x}) \to \mathbb{E}_s(\mathcal{C}_{T_x})$ is the reduction modulo p^s of a morphism

$$i_{T_x}(s+l) : \mathbb{E}_{s+l}(\mathcal{E}_{b,T_x}) \to \mathbb{E}_{s+l}(\mathcal{C}_{T_x})$$

of the category $\mathcal{M}(W_{s+l}(T_x))$.

Let I_s be the set of morphisms $\mathbb{E}_s(\mathcal{E}_{b,\overline{K}}) \to \mathbb{E}_s(\mathcal{C}_{\overline{K}})$ which lift to morphisms $\mathbb{E}_{s+l}(\mathcal{E}_{b,\overline{K}}) \to \mathbb{E}_{s+l}(\mathcal{C}_{\overline{K}})$ for some $l \gg 0$. From [Vasiu 2006, Theorem 5.1.1(a)] (applied for $l \gg 0$ which depends only on b and r) we get that each element of I_s is the evaluation at the thickening $\mathbb{W}_s(\overline{K})$ of a morphism of F^n-crystals $\mathcal{E}_{b,\overline{K}} \to \mathcal{C}_{\overline{K}}$ (strictly speaking [loc. cit.] is stated for $n = 1$ but its proof works for all $n \in \mathbb{N}^*$). This implies that I_s is a finite set whose elements are all pullbacks of morphisms of $\mathcal{M}(W_s(L))$, where L is a suitable finite field extension of K contained in $\overline{K}$. By replacing S with its normalization in L, we can assume that $L = K$. As inside K_x we have an identity $D_x^+ \cap K = D_x$, inside $W_s(K_x)$ we have an identity $W_s(D_x^+) \cap W_s(K) = W_s(D_x)$. From the last three sentences we get that the pullback $i_{D_x^+}(s)$ of $i_{T_x}(s)$ to a morphism of $\mathcal{M}(W_s(D_x^+))$ is the pullback of a morphism of $\mathcal{M}(W_s(D_x))$. Based on this we can assume that there exists an open subscheme U_x of T which contains x and which has the property that there exists a morphism

$$i_{U_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,U_x}) \to \mathbb{E}_s(\mathcal{C}_{U_x})$$

of the category $\mathcal{M}(W_s(U_x))$ such that $i_{T_x}(s)$ is the pullback of it.

We consider an identification $\mathcal{C}_{\overline{K}} = (Q, \phi_Q)$, where $Q = W(\overline{K})^r$ and $\phi_Q : Q \to Q$ is a $\Phi^n_{\overline{K}}$-linear endomorphism. The Newton polygon ν_η of $\mathcal{C}_{\overline{K}}$ has the Newton polygon slope b with multiplicity 1 and therefore there exists a unique nonzero direct summand Q_b of Q such that we have $\phi_Q(Q_b) = p^b Q_b$. The rank of the $W(\overline{K})$-module Q_b is 1. Let $z_b \in Q_b$ be such that $Q_b = W(\overline{K}) z_b$ and $\phi_Q(z_b) = p^b z_b$; it is unique up to multiplication by units of $W(\mathbb{F}_{p^n})$.

We have a canonical identification $\mathcal{E}_{b,\overline{K}} = (W(\overline{K}), p^b \Phi^n_{\overline{K}})$. The morphism $\mathbb{E}_s(\mathcal{E}_{b,\overline{K}}) \to \mathbb{E}_s(\mathcal{C}_{\overline{K}})$ defined by $i_{T_x}(s)$ is an element of I_s and therefore it is the reduction modulo p^s of a morphism $\lambda_x : (W(\overline{K}), p^b \Phi^n_{\overline{K}}) \to (Q, \phi_Q)$ of F^n-crystals over $\overline{K}$. Clearly $\lambda_x(1) \in Q_b$ and thus there exists a unique element $\tau_x \in W(\mathbb{F}_{p^n})$ such that we have

$$\lambda_x(1) = \tau_x z_b.$$

As $i_{T_x}(s)$ is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^t from Fact 4 applied with $D = W(\overline{K})$ we get that τ_x modulo p^{t+1} is a nonzero element of $W_{t+1}(\mathbb{F}_{p^n})$. Therefore we can write $\tau_x = p^{t_x} u_x$, where $u_x \in W(\mathbb{F}_{p^n})$ is a unit and where $t_x \in \{0, \ldots, t\}$.

From now on, we will take $s > 2t$. We consider the morphism

$$\theta_x := p^{t-t_x} u_x^{-1} i_{U_x}(s) : \mathbb{E}_s(\mathcal{E}_{b,U_x}) \to \mathbb{E}_s(\mathcal{C}_{U_x})$$

of the category $\mathcal{M}(W_s(U_x))$; its pullback to a morphism of $\mathcal{M}(W_s(T_x))$ is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^{t+t_x} and thus also by p^{2t} and whose domain is divisible by b. The pullback of θ_x to a morphism of $\mathcal{M}(W_s(\overline{K}))$ is the reduction modulo p^s of the morphism $p^{t-t_x} u_x^{-1} \lambda_x : (W(\overline{K}), p^b \Phi^n_{\overline{K}}) \to (Q, \phi_Q)$ which maps 1 to $p^t z_b$ and which does not depend on the point $x \in T$ of codimension 1.

Let U be the open subscheme of T which is the union of all U_x's. From the previous paragraph we get that the θ_x's glue together to define a morphism

$$\theta : \mathbb{E}_s(\mathcal{E}_{b,U}) \to \mathbb{E}_s(\mathcal{C}_U)$$

of the category $\mathcal{M}(W_s(U))$.

By replacing S with its normalization in any of the finite field extensions K_x of K, we can assume that there exists an open dense subscheme U_0 of U such that the pullback $\theta_{U_0} : \mathbb{E}_s(\mathcal{E}_{b,U_0}) \to \mathbb{E}_s(\mathcal{C}_{U_0})$ of θ to a morphism of $\mathcal{M}(W_s(U_0))$ is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^{2t} and whose domain is divisible by b: under such a replacement, we can take U_0 to be T_x itself.

3.3. *Good section of* $\mathbb{H}_s$. We have $\operatorname{codim}_T(T-U) \geq 2$ and the morphism θ is defined by a section $\theta : U \to \mathbb{H}_s$ denoted in the same way.

Let $\mathbb{I}_s$ be the schematic closure $\overline{\theta(U)}$ of $\theta(U)$ in $\mathbb{H}_s$. As the scheme $\mathbb{H}_s$ is affine and noetherian and as U is an integral scheme, the scheme $\mathbb{I}_s$ is also affine, noetherian, and integral. We have a commutative diagram:

$$\begin{array}{ccccc} & & & & \mathbb{I}_s \\ & \nearrow^{\text{open }\theta} & & & \downarrow{\scriptstyle \text{affine}} \\ U & \hookrightarrow & T & \hookrightarrow & S \end{array}$$

We consider the pullback $\mathbb{J}_s$ of $\mathbb{I}_s$ to T:

$$\begin{array}{ccccc} & & \mathbb{J}_s & \xrightarrow[\hookrightarrow]{\text{open}} & \mathbb{I}_s \\ & \nearrow^{\text{open}} & {\scriptstyle \xi}\downarrow{\scriptstyle \text{affine}} & & \downarrow{\scriptstyle \text{affine}} \\ U & \underset{\text{open}}{\hookrightarrow} & T & \underset{\text{open}}{\hookrightarrow} & S \end{array}$$

Lemma 7. *The affine morphism $\xi : \mathbb{J}_s \to T$ is an isomorphism.*

Proof. To prove that ξ is an isomorphism, we can assume that $T = S = \operatorname{Spec} A$ is an affine scheme. As ξ is an affine morphism, $\mathbb{J}_s = \operatorname{Spec} B$ is also an affine scheme. Since U is open dense in both T and $\mathbb{I}_s$, T and $\mathbb{J}_s$ have the same field of fractions K. As $\operatorname{codim}_T(T-U) \geq 2$ and as U is an open subscheme of both T and $\mathbb{J}_s$, we have $A_{\mathfrak{p}} = B_{\mathfrak{p}}$ for each prime $\mathfrak{p} \in S = T$ of height 1. As A is a noetherian normal domain, inside K we have

$$A \subseteq B \subseteq \bigcap_{\mathfrak{q} \in \operatorname{Spec} B \text{ of height } 1} B_{\mathfrak{q}} \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Spec} A \text{ of height } 1} A_{\mathfrak{p}} = A$$

(see [Matsumura 1980, (17.H), Theorem 38] for the equality part; the first inclusion is defined by ξ). Therefore $A = B$. □

Lemma 7 allows us in what follows to identify T itself with an open dense subscheme of $\mathbb{I}_s$ (i.e., with $\mathbb{J}_s$).

3.4. ***End of the proof.*** In this subsection we will show that for $s \gg 0$, we have $T = \mathbb{I}_s$. This will complete the proof of Theorem 1 as $\mathbb{I}_s$ is an affine scheme.

It remains to show that the assumption that for $s \gg 0$ we have $T \neq \mathbb{I}_s$ leads to a contradiction. This assumption implies that there exists an algebraically closed field k of characteristic p and a morphism $\zeta_0 : \mathrm{Spec}(k[\![X]\!]) \to \mathbb{I}_s$ with the properties that under it the generic point of $\mathrm{Spec}(k[\![X]\!])$ maps to U_0 and its special point maps to $\mathbb{I}_s - T$.

Let $P = k[\![X]\!]^{\mathrm{perf}}$ be the perfection of $k[\![X]\!]$, let κ be the perfect field which is the field of fractions of P, and let $\zeta : \mathrm{Spec}\, P \to \mathbb{I}_s$ be the morphism defined naturally by ζ_0. To the composite of ζ with the closed embedding $\mathbb{I}_s \to \mathbb{H}_s$ corresponds a morphism

$$\omega : \mathbb{E}_s(\mathcal{E}_{b,P}) \to \mathbb{E}_s(\mathcal{C}_P)$$

of the category $\mathcal{M}(W_s(P))$ whose pullback ω_κ to a morphism of $\mathcal{M}(W_s(\kappa))$ is equal to the pullback $\theta_\kappa : \mathbb{E}_s(\mathcal{E}_{b,\kappa}) \to \mathbb{E}_s(\mathcal{C}_\kappa)$ of θ.

We have a natural identification $\mathbb{E}_s(\mathcal{E}_{b,P}) = (W_s(P), p^b\Phi_P^n)$ and we consider an identification $\mathbb{E}_s(\mathcal{C}_P) = (W_s(P)^r, \phi)$. Thus we have a $W_s(P)$-linear map

$$\omega : W_s(P) \to W_s(P)^r$$

such that $\omega \circ p^b\Phi_P^n = \phi \circ \omega$. We consider an isogeny $\mathcal{D} \to \mathcal{C}_P$ whose cokernel is annihilated by p^t and with $\mathcal{D}$ divisible by b, again see [Katz 1979, Theorem 2.6.1] (here $t = (r-1)b$ as stated before Proposition 6). Thus we also have an isogeny $\iota : \mathcal{C}_P \to \mathcal{D}$ whose cokernel is annihilated by p^t. We consider its evaluation

$$\iota[s] : \mathbb{E}_s(\mathcal{C}_P) \to \mathbb{E}_s(\mathcal{D})$$

at the thickening $\mathbb{W}_s(P)$. Under an identification $\mathbb{E}_s(\mathcal{D}) = (W_s(P)^r, p^b\varphi)$ with $\varphi : W_s(P)^r \to W_s(P)^r$ as a Φ_P^n-linear endomorphism, we get a $W_s(P)$-linear endomorphism $\iota[s] : W_s(P)^r \to W_s(P)^r$ such that we have $\iota[s] \circ \phi = p^b\varphi \circ \iota[s]$. We consider the composite morphism

$$\rho = \iota[s] \circ \omega : \mathbb{E}_s(\mathcal{E}_{b,P}) \to \mathbb{E}_s(\mathcal{D})$$

identified with a $W_s(P)$-linear map $\rho : W_s(P) \to W_s(P)^r$ such that $\rho \circ p^b\Phi_P^n = p^b\varphi \circ \rho$. Let

$$\gamma = \rho(1) = (\gamma_1, \dots, \gamma_r) \in W_s(P)^r.$$

From the identity $\rho \circ p^b\Phi_P = p^b\varphi \circ \rho$ we get that the image of $\varphi(\gamma) - \gamma$ in $W_{s-b}(P)^r$ is 0. Writing $\gamma = p^u\delta$, where $u \in \mathbb{N}$ and $\delta \in W_s(P)^r - pW_s(P)^r$, we get that the image of $\varphi(\delta)-\delta$ in $W_{s-b-u}(P)^r$ is 0. Let $\bar{\delta} \in P^r - 0$ be the image in $P^r = W_1(P)$ of δ (i.e., the reduction modulo p of δ).

Lemma 8. *If $s \geq 3t+1$, then we have $u \leq 3t$. Therefore, if moreover we have $s \geq 3t+b+1$, then the image of $\varphi(\delta)-\delta$ in $W_{s-b-3t}(P)^r$ is* 0.

Proof. To check this we can work over $W_s(\kappa)$. As the generic point of Spec P maps to U_0, $\omega_\kappa = \theta_\kappa : \mathbb{E}_s(\mathcal{E}_{b,\kappa}) \to \mathbb{E}_s(\mathcal{C}_\kappa)$ is the pullback of the morphism θ_{U_0}. The pullback ρ_κ of ρ to $\mathcal{M}(W_s(\kappa))$ is a composite morphism

$$\rho_\kappa = \iota[s]_\kappa \circ \theta_\kappa : \mathbb{E}_s(\mathcal{E}_{b,\kappa}) \to \mathbb{E}_s(\mathcal{D}_\kappa)$$

and therefore it is the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^{2t} (as θ_{U_0} has this property) and with a quasi-isogeny whose cokernel is annihilated by p^t (as ι is an isogeny whose cokernel is annihilated by p^t). Therefore, ρ_κ is also the composite of a split monomorphism with a quasi-isogeny whose cokernel is annihilated by p^{3t}. This implies that the image of γ in $W_{3t+1}(\kappa)$ is nonzero (see Fact 4 applied with $D = W(\kappa)$) and therefore we have $u \leq 3t$. □

Lemma 9. *If $s \geq 3t+b+1$, then the image of $\bar{\delta}$ in $k^r = W_1(k)^r$ is nonzero.*

Proof. We show that the assumption that the image of $\bar{\delta} \in P^r - 0$ in $k^r = W_1(k)^r$ is 0 leads to a contradiction. This assumption implies that there exists a largest positive rational number c of denominator a power of p such that we have

$$\bar{\delta} \in X^c P^r \subset P^r = (k[\![X]\!]^{\mathrm{perf}})^r.$$

Let $\bar{\varphi} : P^r \to P^r$ be the P-linear endomorphism which is the reduction modulo p of φ. From Lemma 8 we get that $\bar{\delta} = \bar{\varphi}(\bar{\delta})$. Thus $\bar{\delta} \in \bar{\varphi}(X^c P^r) \subseteq X^{p^n c} P^r$ and this implies that $p^n c \leq c$ which is a contradiction. □

From the inequality $u \leq 3t$ (see Lemma 8) and from Lemma 9 we get that for $s \geq 3t+b+1$ the pullback ω_k of ω to a morphism of $\mathcal{M}(W_s(k))$ is such that its reduction modulo p^{3t+1} is nonzero. For $s > 3t+b+1+l$ with $l \in \mathbb{N}^*$ large enough but depending only on b and r, the reduction of ω_k modulo p^{s-l} lifts to a morphism $\mathcal{E}_{0,k} \to \mathcal{D}_k$ (see [Vasiu 2006, Theorem 5.1.1(a)], which, again, stated for $n=1$, applies to all $n \in \mathbb{N}^*$) which is nonzero. Thus $\mathcal{D}_k$ has Newton polygon slope b with multiplicity at least 1. From this and the existence of the isogeny ι we get that $\mathcal{C}_k$ has Newton polygon slope b with multiplicity at least 1. This implies that the special point of Spec$(k[\![X]\!])$ under the composite of $\zeta_0 : \mathrm{Spec}(k[\![X]\!]) \to \mathbb{I}_s$ with the morphism $\mathbb{I}_s \to S$ does not map to a point of $S_{\nu_2} = S - T$ and so it maps to a point of T. This is a contradiction, and ends the proof of Theorem 1. □

4. Applications of Theorem 1

In Section 4.1 we prove Corollary 2. In Section 4.2 we follow [Vasiu 2013] to introduce generalized Artin–Schreier systems of equations and Artin–Schreier stratifications. In Section 4.3 we refine and reobtain Corollary 3 in the context of these stratifications. Section 4.4 contains some complements, including Proposition 13, which prove that "pure in" implies "weakly pure in". Until the end let A be an arbitrary $\mathbb{F}_p$-algebra.

4.1. ***Proof of Corollary 2.*** To prove Corollary 3, in this subsection we can assume that $S = \operatorname{Spec} A$ and $d \in \mathbb{N}$ are as in the paragraph before Section 2.1. We can also assume that $\nu(r) = d$ as otherwise $S_\nu = \varnothing$ is pure in S. Let $l \in \mathbb{N}$ be such that the Newton polygon ν has exactly $l + 1$ breaking points denoted as $(a_0, b_0) = (0, 0), \ldots, (a_l, b_l) = (r, d)$.

We have obvious identities

$$S_\nu = \Big[S_{\geq\nu} \bigcap_{i=0}^{l} T_{(a_l,b_l)}(\mathcal{C})\Big]_{\mathrm{red}} = [S_{\geq\nu} \times_S (T_{(a_0,b_0)}(\mathcal{C}))_S \times \cdots \times_S T_{(a_l,b_l)}(\mathcal{C})]_{\mathrm{red}}.$$

From Theorem 1 we get that each $T_{(a_l,b_l)}(\mathcal{C})$ is an affine scheme. We recall that $S_{\geq\nu}$ is a reduced closed subscheme of S. From the last three sentences we get that S_ν is an affine scheme, i.e., is pure in S. □

4.2. ***Artin–Schreier stratifications.*** Let $x_0,\ x_1, \ldots, x_r$ be free variables. For $i, j \in \{1, \ldots, r\}$ let $P_{i,j}(x_0) \in A[x_0]$ be a polynomial which is a linear combination with coefficients in A of the monomials x_0^q with $q \in \mathbb{N}$ either 0 or a power of p. By a generalized Artin–Schreier system of equations in r variables over A we mean a system of equations of the form

$$x_i = \sum_{j=1}^{r} P_{i,j}(x_j^p) \quad i \in \{1, \ldots, r\}$$

to which we associate the A-algebra

$$B = A[x_1, \ldots, x_r]/\Big(x_1 - \sum_{j=1}^{r} P_{1,j}(x_j^p), x_2 - \sum_{j=1}^{r} P_{2,j}(x_j^p), \ldots, x_r - \sum_{j=1}^{r} P_{r,j}(x_j^p)\Big).$$

Each equation of the form $x_i = \sum_{j=1}^{r} P_{i,j}(x_j^p)$ will be called as a generalized Artin–Schreier equation, and its degree $e_i \in \mathbb{N}$ is defined as follows. We have $e_i = 0$ if and only if for all $j \in \{1, \ldots, r\}$ the polynomial $P_{i,j}(x_0)$ is a constant, and if $e_i > 0$ then e_i is the largest integer such that there exists a $j \in \{1, \ldots, r\}$ with the property that the degree of $P_{i,j}(x_j^p)$ is p^{e_i}.

Let $e = \max\{e_1, \dots, e_r\}$; we call it the degree of the generalized Artin–Schreier system of equations in r variables over A. Following [Vasiu 2013], when $e \leq 1$ we drop the word "generalized".

Proposition 10. *The morphism* $\epsilon : \operatorname{Spec} B \to \operatorname{Spec} A$ *is étale and surjective and its geometric fibers have a number of points equal to a power of* p.

Proof. If $e_i > 1$, then by adding, for each $j \in \{1, \dots, r\}$ such that the degree of $P_{i,j}(x_j^p)$ is p^{e_i}, an extra variable $y_{i,j}$ and an equation of the form $y_{i,j} = x_j^p$, the generalized Artin–Schreier equation $x_i = \sum_{j=1}^{r} P_{i,j}(x_j^p)$ gets replaced by several generalized Artin–Schreier equations of degrees less than e_i. By repeating this process of adding extra variables and equations which (up to isomorphisms between $\operatorname{Spec} A$-schemes) do not change the morphism $\epsilon : \operatorname{Spec} B \to \operatorname{Spec} A$, we can assume that $e \leq 1$. Thus the proposition follows from [Vasiu 2013, Theorem 2.4.1(a) and (b)]. □

Definition 11 is a natural extrapolation of [Vasiu 2013, Definition 2.4.2] which applies to étale morphisms $\epsilon : \operatorname{Spec} B \to \operatorname{Spec} A$ as in Proposition 10.

Definition 11. Let $\varepsilon : \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} A$ be an étale morphism between affine $\mathbb{F}_p$-schemes.

(a) We assume that A is noetherian. Then by the Artin–Schreier stratification of $\operatorname{Spec} A$ associated to $\varepsilon : \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} A$ in reduced locally closed subschemes $V_1, \dots, V_q$ we mean the stratification defined inductively by the following property: for each $l \in \{1, \dots, q\}$ the scheme V_l is the maximal open subscheme of the reduced scheme of $(\operatorname{Spec} A) - \left(\bigcup_{q=1}^{l-1} V_q\right)$ which has the property that the morphism $\epsilon_{V_l} : (\operatorname{Spec} B) \times_{\operatorname{Spec} A} V_l \to V_l$ is an étale cover.

(b) Let $\mu_1 > \mu_2 > \cdots > \mu_v$ be the shortest sequence of strictly decreasing natural numbers such that each fiber of the morphism $\epsilon : \operatorname{Spec} B \to \operatorname{Spec} A$ has a number of geometric points equal to μ_l for some $l \in \{1, \dots, v\}$. Then by the functorial Artin–Schreier stratification of $\operatorname{Spec} A$ associated to $\varepsilon : \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} A$ we mean the stratification of $\operatorname{Spec} A$ in reduced locally closed subschemes $U_1, \dots, U_v$ defined inductively by the following property: for each $l \in \{1, \dots, v\}$ the scheme U_l is the maximal open subscheme of the reduced scheme of $(\operatorname{Spec} A) - \left(\bigcup_{q=1}^{l-1} U_q\right)$ which has the property that the morphism $\epsilon_{U_l} : (\operatorname{Spec} B) \times_{\operatorname{Spec} A} U_l \to U_l$ is an étale cover whose fibers all have a number of geometric points equal to μ_l.

The existence of the stratification $V_1, \dots, V_q$ of $\operatorname{Spec} A$ is a standard piece of algebraic geometry. The existence of the sequence $\mu_1 > \mu_2 > \cdots > \mu_v$ follows from the facts that each étale morphism is locally quasifinite and that $\operatorname{Spec} \mathcal{B}$ is quasicompact. The existence of the stratification $U_1, \dots, U_v$ of $\operatorname{Spec} A$ is implied by [Grothendieck 1967, Proposition 18.2.8 and Corollary 18.2.9], which show that one can define U_l directly and functorially as follows: each U_l is the set of all points $x \in \operatorname{Spec} A$ such that the fiber of ε at x has exactly μ_l geometric points.

Theorem 12. *Let* $\varepsilon : \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} A$ *be an étale morphism between affine* $\mathbb{F}_p$-*schemes. Then the functorial Artin–Schreier stratification of* $\operatorname{Spec} A$ *associated to* $\varepsilon : \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} A$ *in reduced locally closed subschemes* $U_1, \dots, U_v$ *is pure*, *i.e.*, *for each* $l \in \{1, \dots, v\}$, *the stratum* U_l *is pure in* $\operatorname{Spec} A$.

Proof. As the étale morphism $\varepsilon : \operatorname{Spec} \mathcal{B} \to \operatorname{Spec} A$ is of finite presentation and due to the functorial part, we can assume that A is a finitely generated $\mathbb{F}_p$-algebra and thus an excellent ring. We follow [Vasiu 2014]. By replacing $\operatorname{Spec} A$ by its closed subscheme $(\operatorname{Spec} A) - \left(\bigcup_{q=1}^{l-1} U_q\right)$ endowed with the reduced structure, we can assume that $l = 1$ and that A is reduced. Thus U_1 is an open dense subscheme of $\operatorname{Spec} A$. Based again on [Vasiu 2006, Lemma 2.9.2], to prove that U_1 is an affine scheme, we can replace A by its normalization in its ring of fractions. Thus by passing to connected components of $\operatorname{Spec} A$, we can assume that A is an excellent normal domain. Thus $B = \prod_{l=1}^{w} B_l$ is a finite product of excellent normal domains which are étale A-algebras. Let K_l be the field of fractions of B_l. Let L be the finite Galois extension of the field of fractions K of A generated by the finite separable extensions K_l's of K. By replacing A by its normalization in L (again based on [Vasiu 2006, Lemma 2.9.2]), we can assume $K = K_1 = \cdots = K_w$. This implies that each $\operatorname{Spec}(B_l)$ is an open subscheme of $\operatorname{Spec} A$ and thus

$$\begin{aligned} U_1 &= \bigcap_{l=1}^{w} \operatorname{Spec}(B_l) \\ &= (\operatorname{Spec}(B_1)) \times_{\operatorname{Spec} A} (\operatorname{Spec}(B_2)) \times_{\operatorname{Spec} A} \cdots \times_{\operatorname{Spec} A} (\operatorname{Spec}(B_w)) \end{aligned}$$

is the affine scheme $\operatorname{Spec}(B_1 \otimes_A \otimes \cdots \otimes_A B_w)$. □

4.3. ***A second proof of Corollary 3.*** We will use Theorem 12 to obtain a second proof of Corollary 3 which is simpler and independent of Theorem 1. We can assume that $S = \operatorname{Spec} A$ is affine and let $\phi_M : M \to M$ be as in Section 2.1.

The identities $S_m = T_{(m,0)}(\mathcal{C})$ if $m > 0$ and $S_0 = T_{(1,0)}(\mathcal{C} \oplus \mathcal{E}_0)$ show that S_m is a reduced locally closed subscheme of S. Thus by replacing S by $\bar{S}_m$, we can assume that S_m is an open dense subscheme of $S = \bar{S}_m$.

We consider the equation

$$\phi_M(z) = z \tag{1}$$

in $z \in M$. For $x \in S$ we have $\chi(x) = \dim_{\mathbb{F}_{p^n}}(\vartheta_x)$, where ϑ_x is the $\mathbb{F}_{p^n}$-vector space of solutions of the tensorization of (1) over A with an algebraic closure of the residue field $k(x)$ of S at x.

From now on we will forget about $\mathcal{C}$ and just work with the free A-module M of rank r and its Φ_A^n-linear endomorphism $\phi_M : M \to M$ and we only assume that we have an open dense subset S_m of $S = \operatorname{Spec} A$ defined by the following property: for $x \in S$, we have $x \in S_m$ if and only if $\dim_{\mathbb{F}_{p^n}}(\vartheta_x) = m$.

With respect to a fixed A-basis $\{v_1, \ldots, v_r\}$ of M, by writing $z = \sum_{i=1}^{r} x_i v_i$, (1) defines a generalized Artin–Schreier system of equations in the r variables $x_1, \ldots, x_r$ of the form

$$x_i = L_i(x_1^{p^n}, \ldots, x_r^{p^n}), \quad i \in \{1, \ldots, r\},$$

where each L_i is a homogeneous polynomial of total degree at most 1. Let

$$B = A[x_1, \ldots, x_r]/(x_1 - L_1(x_1^{p^n}, \ldots, x_r^{p^n}), \ldots, x_r - L_r(x_1^{p^n}, \ldots, x_r^{p^n})),$$

let $\epsilon : \operatorname{Spec} B \to S$ and let $U_1, \ldots, U_v$ be the functorial Artin–Schreier stratification of S associated to $\epsilon : \operatorname{Spec} B \to S$. Let $p^{\mu_1} > p^{\mu_2} > \cdots > p^{\mu_v}$ be the shortest sequence of strictly decreasing of powers of p by natural numbers such that for each $l \in \{1, \ldots, v\}$, every geometric fiber of the morphism $\epsilon_{U_l} : \operatorname{Spec} B \times_S U_l \to U_l$ has a number of geometric points equal to p^{μ_l}, see Proposition 10 and Definition 11(b).

The fact that the morphism $\epsilon : \operatorname{Spec} B \to S$ is étale (see Proposition 10) is equivalent to [Zink 2001, Proposition 3]. We consider the lower semicontinuous function (see [Grothendieck 1967, Proposition 18.2.8])

$$\mu : S \to \mathbb{N}$$

defined by the rule: $\mu(x) = p^{n \dim_{\mathbb{F}_{p^n}}(\vartheta_x)}$ is the number of geometric points of $\epsilon : \operatorname{Spec} B \to S$ above x (i.e., is the number of elements of ϑ_x). We get that μ_l is divisible by n for all $l \in \{1, \ldots, v\}$ and (as S_m is dense in S) we have $\mu_1 = mn$. Moreover, for $x \in S$ and $q \in \mathbb{N}$ we have $\mu(x) = p^{nq}$ if and only if $x \in S_q$. We conclude that $S_m = U_1$ and therefore (see Theorem 12) S_m is an affine scheme. □

4.4. *Complements.* For the sake of completeness, we include a proof of the following well-known result (to be compared with [Vasiu 2006, Remark 6.3(a)]).

Proposition 13. *Let Z be a reduced locally closed subscheme of a locally noetherian scheme Y. If Z is pure in Y, then Z is weakly pure in Y.*

Proof. We can assume that $Z \subsetneq \overline{Z} = Y$. By localizing Y at the generic point of an irreducible component of $\overline{Z} - Z$, we can assume that $Y = \overline{Z} = \operatorname{Spec} C$ is a local affine scheme of dimension at least 1 and Z is the complement in Y of the closed point of Y and we have to prove that C has dimension 1. By passing to a connected component of the normalization of the reduced completion $\hat{C}_{\mathrm{red}}$ of C in the ring of fractions of $\hat{C}_{\mathrm{red}}$, we can assume that C is in fact an integral normal local ring which is not a field.

We show that the assumption that $\dim(C) \geq 2$ leads to a contradiction. As the open dense subscheme Z of Y is pure in Y, Z is the spectrum of a C-subalgebra of the field of fractions of C which contains C and which is contained in the intersection of all the localizations of C at points of Y of codimension 1 in Y (as these points belong to Z). As $\dim(C) \geq 2$, from [Matsumura 1980, (17H), Theorem 38] we get that this intersection is C and thus we have $Z = \operatorname{Spec} C = Y$. This is a contradiction. Thus $\dim(C) = 1$. □

Remark 14. Suppose A is a local noetherian $\mathbb{F}_p$-algebra of dimension at least 2. Let $\mathfrak{m}$ be the maximal ideal of A. Suppose $M = A^r$ is equipped with a Φ_A^n-linear endomorphism $\phi_M : M \to M$ such that for each nonclosed point x of $S = \operatorname{Spec} A$, with the notation of Section 4.3 we have $\dim_{\mathbb{F}_{p^n}}(\vartheta_x) = m$. Then $S_m = U_1$ being pure in S, it is also weakly pure in S (see Proposition 13) and thus $S - S_m$ cannot be $\mathfrak{m}$ as $\operatorname{codim}_S(\mathfrak{m}) \geq 2$. Therefore we have $S_m = S$ and in this way we reobtain [Zink 2001, Proposition 5] . One can view Theorem 12 as a generalization and a refinement of [Zink 2001, Proposition 5].

Remark 15. For $q \in \mathbb{N}^*$ we define recursively an A-linear map

$$\phi_M^{(q)} : A \otimes_{F_A^{nq}, A} M \to M$$

as follows: let $\phi_M^{(1)} : A \otimes_{F_A^n, A} M \to M$ be the A-linear map defined by ϕ_M, and we have the recursive formula $\phi_M^{(q)} = \phi_M^{(1)} \circ (1_A \otimes_{F_A^n, A} \phi_M^{(q-1)})$. Deligne [2011] proved the case $n = 1$ of Theorem 12 using ranks of images of $\phi_M^{(q)}$ for $q \gg 0$ at points $x \in S = \operatorname{Spec} A$ and properties of Grassmannians.

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