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On the mod-2 cohomology of $\mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$

Hans-Werner Henn

Let $\Gamma = \mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$, let X be any mod-2 acyclic Γ-CW complex on which Γ acts with finite stabilizers and let X_s be the 2-singular locus of X. We calculate the mod-2 cohomology of the Borel construction of X_s with respect to the action of Γ. This cohomology coincides with the mod-2 cohomology of Γ in cohomological degrees bigger than 8 and the result is compatible with a conjecture of Quillen which predicts the structure of the cohomology ring $H^*(\Gamma; \mathbb{F}_2)$.

1. Introduction

The main motivation for this paper comes from a conjecture of Quillen [1971, Conjecture 14.7] which concerns the structure of the mod-p cohomology ring of the group $\mathrm{GL}_n(\Lambda)$ of invertible matrices of rank n with coefficients in a ring Λ of S-integers in a number field; the assumption on Λ is that p is invertible in Λ and Λ contains a primitive p-th root of unity. The conjecture stipulates that under these assumptions $H^*(\mathrm{GL}_n(\Lambda); \mathbb{Z}/p)$ is a free module over the polynomial algebra $\mathbb{Z}/p[c_1, \dots, c_n]$ where the c_i are the mod-p Chern classes associated to an embedding of Λ into the complex numbers. In the sequel we will denote this conjecture by $C(n, \Lambda, p)$.

For $p = 2$ the simplest ring for which the assumptions of Quillen's conjecture hold is the ring $\mathbb{Z}\big[\frac{1}{2}\big]$. Let $\mathbb{Z}\big[\frac{1}{2}, i\big]$ be the ring obtained from the Gaussian integers $\mathbb{Z}[i]$ by inverting 2.

Conjecture $C\big(n, \mathbb{Z}\big[\frac{1}{2}\big], 2\big)$ is trivially true for $n = 1$ and known to be true for $n = 2$ by [Mitchell 1992] and $n = 3$ by [Henn 1999]; it is known to be false for $n = 32$ by [Dwyer 1998] and even for $n \geq 14$ (Henn and Lannes, unpublished). The positive results have been established by direct calculation and while a direct calculation is perhaps still doable for $n = 4$, it would be extremely involved. For larger n a complete calculation does not look realistic. In fact, the negative results have been obtained by very indirect methods which depend on étale approximations for the homotopy type of the 2-completion of $\mathrm{BGL}_n\big(\mathbb{Z}\big[\frac{1}{2}\big]\big)$. These étale approximations can also be used to show that if $C\big(2n, \mathbb{Z}\big[\frac{1}{2}\big], 2\big)$ holds then $C\big(n, \mathbb{Z}\big[\frac{1}{2}, i\big], 2\big)$ holds

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as well (Henn and Lannes, unpublished). This gives particular motivation to study conjecture $C\big(n, \mathbb{Z}\big[\tfrac{1}{2}, i\big], 2\big)$.

We will show in Theorem 5.1 that $C\big(n, \mathbb{Z}\big[\tfrac{1}{2}, i\big], 2\big)$ holds if and only if there is an isomorphism

$$H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big) \cong \mathbb{F}_2[c_1, \dots, c_n] \otimes E(e_1, e_1', \dots, e_{2n-1}, e_{2n-1}')$$

where the classes c_i are the Chern classes of the tautological n-dimensional complex representation of $\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$, E denotes an exterior algebra and the classes e_{2i-1}, e_{2i-1}' are of cohomological degree $2i-1$ for $i = 1, \dots, n$. These exterior classes are closely related to Quillen's exterior classes in the mod-2 cohomology of $\mathrm{GL}_n(\mathbb{F}_p)$ if p is a prime such that $p \equiv 1 \bmod 4$ (see (5-1) for more details).

Conjecture $C\big(n, \mathbb{Z}\big[\tfrac{1}{2}, i\big], 2\big)$ is again trivially true for $n = 1$ and has been verified by direct calculation for $n = 2$ in [Weiss 2006]. Dwyer's method [1998] using étale approximations X_n for the homotopy type of the 2-completion of $\mathrm{BGL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big)$ and comparing the set of homotopy classes of $[\mathrm{BP}, X_n]$ with that of $\big[\mathrm{BP}, \mathrm{BGL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big)\big]$ for suitable cyclic groups P of order 2^n can be adapted to disprove $C\big(16, \mathbb{Z}\big[\tfrac{1}{2}, i\big], 2\big)$. However, we will not dwell on this in this paper.

This paper embarks on a study of conjecture $C\big(3, \mathbb{Z}\big[\tfrac{1}{2}, i\big], 2\big)$ which is more accessible than conjecture $C\big(4, \mathbb{Z}[\tfrac{1}{2}], 2\big)$. In order to calculate $H^*\big(\mathrm{GL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ we first try to calculate $H^*\big(\mathrm{SL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big)$. For this we propose the same strategy as the one which was used in the case of $\mathrm{SL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big)$ and which finally led to a verification of conjecture $C\big(3, \mathbb{Z}\big[\tfrac{1}{2}\big], 2\big)$. In a first step one uses a centralizer spectral sequence introduced in [Henn 1997] in order to calculate the mod-2 Borel cohomology $H^*_G(X_s; \mathbb{F}_2)$ where X is any mod-2 acyclic G-CW complex on which a given discrete group G acts with finite stabilizers and X_s is the 2-singular locus of X, i.e., the subcomplex consisting of all points for which the isotropy group of the action of G is of even order. For $G = \mathrm{SL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big)$ this step was carried out in [Henn 1997] and for $G = \mathrm{SL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$ it is carried out in this paper. The precise form of X does not really matter in this step.

The second step involves a very laborious analysis of the relative mod-2 Borel cohomology $H^*_G(X, X_s; \mathbb{F}_2)$ and of the connecting homomorphism for the Borel cohomology of the pair (X, X_s). In the case of $G = \mathrm{SL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big)$ this was carried out by hand in [Henn 1999]. A by hand calculation looks forbidding in the case of $G = \mathrm{SL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$ and this paper makes no attempt on such a calculation. However, we do make some comments on what is likely to be involved in such an attempt.

Here are the main results of this paper. In these results the elements b_2 and b_3 are of degree 4 and 6, respectively. They are given as Chern classes of the tautological 3-dimensional complex representation of $\mathrm{SL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$. The indices of the other elements give their cohomological degrees. These elements come from Quillen's exterior cohomology classes in the cohomology of $\mathrm{GL}_3(\mathbb{F}_p)$ for suitable primes p, for example $p = 5$ (see Section 3.2 for more details). Furthermore Σ^n denotes n-fold suspension so that $\Sigma^4\mathbb{F}_2$ is a one dimensional $\mathbb{F}_2$-vector space concentrated in degree 4.

Theorem 1.1. *Let $\Gamma = \mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$ and let X be any mod-2 acyclic Γ-CW complex such that the isotropy group of each cell is finite. Then the centralizer spectral sequence of [Henn 1997] collapses at E_2 and gives a short exact sequence*

$$0 \to \Sigma^4\mathbb{F}_2 \oplus \Sigma^4\mathbb{F}_2 \oplus \Sigma^7\mathbb{F}_2 \to H^*_\Gamma(X_s; \mathbb{F}_2) \to \mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5') \to 0$$

in which the epimorphism is a map of graded commutative algebras with unit.

Next let

$$\psi : H^*(\Gamma; \mathbb{F}_2) = H^*_\Gamma(X; \mathbb{F}_2) \to H^*_\Gamma(X_s; \mathbb{F}_2) \to \mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5')$$

be the composition of the map induced by the inclusion $X_s \subset X$ and the epimorphism of Theorem 1.1.

Theorem 1.2. *Let $\Gamma = \mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$ and X be as in the previous theorem.*

(a) *If $\mathrm{SD}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$ denotes the subgroup of diagonal matrices of Γ then the target of ψ can be identified with a subalgebra of $H^*\big(\mathrm{SD}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ in such a way that ψ is induced by the restriction homomorphism*

$$H^*(\Gamma; \mathbb{F}_2) \to H^*\big(\mathrm{SD}_3\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big).$$

(b) *The homomorphism ψ admits a multiplicative section*

$$\varphi : \mathbb{F}_2[c_2, c_3] \otimes E(e_3, e_3', e_5, e_5') \to H^*(\Gamma; \mathbb{F}_2)$$

that sends c_i to b_i for $i = 2, 3$ and sends e_i and e_i' respectively to d_i and d_i' for $i = 3, 5$.

(c) *The homomorphism ψ is surjective in all degrees, an isomorphism in degrees $* > 8$ and its kernel is finite-dimensional in degrees $* \leq 8$.*

Remark 1.3. Conjecture $C(3, \mathbb{Z}\big[\frac{1}{2}, i\big], 2)$ would hold if the maps ψ and φ of part (b) of Theorem 1.2 turned out to be isomorphisms (see Proposition 5.5).

The following result is an immediate consequence of Theorem 1.2.

Corollary 1.4. *Let $\Gamma = \mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$ and X be as in Theorem 1.1. Then the following conditions are equivalent*:

(a) *The restriction homomorphism $H^*(\Gamma; \mathbb{F}_2) \to H^*\big(\mathrm{SD}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ is injective and $H^*(\Gamma; \mathbb{F}_2)$ is isomorphic as a graded $\mathbb{F}_2$-algebra to $\mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5')$.*

(b) *There is an isomorphism*

$$H^*_\Gamma(X, X_s; \mathbb{F}_2) \cong \Sigma^5\mathbb{F}_2 \oplus \Sigma^5\mathbb{F}_2 \oplus \Sigma^8\mathbb{F}_2$$

and the connecting homomorphism $H^*_\Gamma(X_s; \mathbb{F}_2) \to H^{*+1}_\Gamma(X, X_s; \mathbb{F}_2)$ *is surjective.* □

The paper is organized as follows. In Section 2 we recall the centralizer spectral sequence and in Section 3 we prove Theorems 1.1 and 1.2 In Section 4 we make some comments on step 2 of the program of a complete calculation of $H^*(\Gamma; \mathbb{F}_2)$. Finally in Section 5 we establish Theorem 5.1 and discuss the relation between Theorem 1.2 and conjecture $C\big(3, \mathbb{Z}\big[\frac{1}{2}, i\big], 2\big)$.

2. The centralizer spectral sequence

We recall the centralizer spectral sequence introduced in [Henn 1997].

Let G be a discrete group and let p be a fixed prime. Let $\mathcal{A}(G)$ be the category whose objects are the elementary abelian p-subgroups E of G, i.e., subgroups which are isomorphic to $(\mathbb{Z}/p)^k$ for some integer k; if E_1 and E_2 are elementary abelian p-subgroups of G, then the set of morphisms from E_1 to E_2 in $\mathcal{A}(G)$ consists precisely of those group homomorphisms $\alpha : E_1 \to E_2$ for which there exists an element $g \in G$ with $\alpha(e) = geg^{-1}$ for all $e \in E_1$. Let $\mathcal{A}_*(G)$ be the full subcategory of $\mathcal{A}(G)$ whose objects are the nontrivial elementary abelian p-subgroups.

For an elementary abelian p-subgroup E we denote its centralizer in G by $C_G(E)$. Then the assignment $E \mapsto H^*(C_G(E); \mathbb{F}_p)$ determines a functor from $\mathcal{A}_*(G)$ to the category $\mathcal{E}$ of graded $\mathbb{F}_p$-vector spaces. The inverse limit functor is a left exact functor from the functor category $\mathcal{E}^{\mathcal{A}_*(G)}$ to $\mathcal{E}$. Its right derived functors are denoted by $\lim^s$. The p-rank $r_p(G)$ of a group G is defined as the supremum of all k such that G contains a subgroup isomorphic to $(\mathbb{Z}/p)^k$.

For a G-space X and a fixed prime p we denote by X_s the p-singular locus, i.e., the subspace of X consisting of points whose isotropy group contains an element of order p. Let EG be the total space of the universal principal G-bundle. The mod-p cohomology of the Borel construction $EG \times_G X$ of a G space X will be denoted $H^*_G(X; \mathbb{F}_p)$. The following result is a special case of part (a) of Corollary 0.4 of [Henn 1997].

Theorem 2.1. *Let G be a discrete group and assume there exists a finite-dimensional mod-p acyclic G-CW complex X such that the isotropy group of each cell is finite. Then there exists a cohomological second quadrant spectral sequence*

$$E_2^{s,t} = \lim^s_{\mathcal{A}_*(G)} H^t(C_G(E); \mathbb{F}_p) \Rightarrow H^{s+t}_G(X_s; \mathbb{F}_p)$$

with $E_2^{s,t} = 0$ if $s \geq r_p(G)$ and $t \geq 0$.

Remark 2.2. The edge homomorphism in this spectral sequence is a map of algebras

$$H_G^*(X_s; \mathbb{F}_p) \to \lim_{\mathcal{A}_*(G)} H^*(C_G(E); \mathbb{F}_p),$$

which is given as follows.

Let X^E be the fixed points for the action of E on X. The G-action on X restricts to an action of the centralizer $C_G(E)$ on X^E and the G-equivariant maps

$$G \times_{C_G(E)} X^E \to X_s, \quad (g, x) \mapsto gx.$$

for $E \in \mathcal{A}_*(G)$ induce compatible maps in Borel cohomology

$$H_G^*(X_s; \mathbb{F}_2) \to H_G^*(G \times_{C_G(E)} X^E; \mathbb{F}_2) \cong H_{C_G(E)}^*(X^E; \mathbb{F}_2) \cong H^*(C_G(E); \mathbb{F}_2)$$

which assemble to give the map to the inverse limit. Here we have used that by classical Smith theory X^E is mod p-acyclic if X is mod-p acyclic and hence we get canonical isomorphisms $H_{C_G(E)}^*(X^E; \mathbb{F}_2) \cong H_{C_G(E)}^*(*; \mathbb{F}_2) \cong H^*(C_G(E); \mathbb{F}_2)$.

Furthermore the composition

$$H^*(G; \mathbb{F}_p) = H_G^*(X; \mathbb{F}_p) \to H_G^*(X_s; \mathbb{F}_p) \to H^*(C_G(E); \mathbb{F}_p) \tag{2-1}$$

is induced by the inclusions $C_G(E) \to G$ as E varies through $\mathcal{A}_*(G)$.

In [Henn 1997] we have used this spectral sequence in the case $p = 2$ and $G = \mathrm{SL}_3(\mathbb{Z})$. Here we will use it in the case $p = 2$ and $G = \mathrm{SL}\big(3, \mathbb{Z}\big[\frac{1}{2}, i\big]\big)$. In both cases we have $r_2(G) = 2$ and hence the spectral sequence collapses at E_2 and degenerates into a short exact sequence

$$0 \to \lim\nolimits^1_{\mathcal{A}_*(G)} H^*(C_G(E); \mathbb{F}_2) \to H_G^{*+1}(X_s; \mathbb{F}_2) \to \lim_{\mathcal{A}_*(G)} H^{*+1}(C_G(E); \mathbb{F}_2) \to 0. \tag{2-2}$$

3. The centralizer spectral sequence for $\mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$

3.1. ***The Quillen category.*** Let K be any number field, let $\mathcal{O}_K$ be its ring of integers and consider the ring of S-integers $\mathcal{O}_K\big[\frac{1}{2}\big]$. Then, up to equivalence, the Quillen category of $G := \mathrm{SL}_3\big(\mathcal{O}_K\big[\frac{1}{2}\big]\big)$ for the prime 2 is independent of K. In fact, because 2 is invertible every elementary abelian 2-subgroup is conjugate to a diagonal subgroup, and hence $\mathcal{A}_*(G)$ has a skeleton, say $\mathcal{A}$, with exactly two objects, say E_1 and E_2 of rank 1 and 2, respectively. We take E_1 to be the subgroup generated by the diagonal matrix whose first two diagonal entries are -1 and whose third diagonal entry is 1, and E_2 to be the subgroup of all diagonal matrices with diagonal entries 1 or -1 and determinant 1.

The automorphism group of E_1 is trivial, of course, while $\mathrm{Aut}_{\mathcal{A}}(E_2)$ is isomorphic to the group of all abstract automorphisms of E_2 which we can identify with $\mathfrak{S}_3$, the symmetric group on three elements. There are three morphisms from E_1 to E_2 and $\mathrm{Aut}_{\mathcal{A}}(E_2)$ acts transitively on them.

3.2. ***The centralizers and their cohomology.*** For centralizers in $H := \mathrm{GL}_3\big(\mathcal{O}_K\big[\frac{1}{2}\big]\big)$ we find $C_H(E_1) = \mathrm{GL}_2\big(\mathcal{O}_K\big[\frac{1}{2}\big]\big) \times \mathrm{GL}_1\big(\mathcal{O}_K\big[\frac{1}{2}\big]\big)$ and $C_H(E_2) = D_3\big(\mathcal{O}_K\big[\frac{1}{2}\big]\big)$ if $D_n\big(\mathcal{O}_K\big[\frac{1}{2}\big]\big)$ denotes the subgroup of diagonal matrices in $\mathrm{GL}_n\big(\mathcal{O}_K\big[\frac{1}{2}\big]\big)$. This implies

$$\begin{aligned} C_G(E_1) &\cong \mathrm{GL}_2\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big), \\ C_G(E_2) &= \mathrm{SD}_3\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big) \cong D_2\big(\mathcal{O}_K\big[\tfrac{1}{2}\big]\big) \cong \mathcal{O}_K\big[\tfrac{1}{2}\big]^\times \times \mathcal{O}_K\big[\tfrac{1}{2}\big]^\times, \end{aligned}$$

where as before $\mathrm{SD}_3\big(\mathcal{O}_K\big[\frac{1}{2}\big]\big)$ denotes special diagonal matrices with coefficients in $\mathcal{O}_K\big[\frac{1}{2}\big]$.

From now on we specialize to the case $K = \mathbb{Q}[i]$ where we have $\mathcal{O}_K\big[\frac{1}{2}\big] = \mathbb{Z}\big[\frac{1}{2}, i\big]$. In this case the cohomology of the centralizers is explicitly known. In the sequel we abbreviate $\mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$ by Γ.

3.2.1. *The cohomology of* $C_\Gamma(E_2)$. There is an isomorphism of groups

$$\mathbb{Z}/4 \times \mathbb{Z} \cong \mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times, \quad (n, m) \mapsto i^n(1+i)^m$$

and therefore we get an isomorphism

$$H^*(C_\Gamma(E_2); \mathbb{F}_2) \cong H^*\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times \times \mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times; \mathbb{F}_2\big) \cong \mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_1', x_2, x_2') \tag{3-1}$$

with y_1 and y_2 in degree 2 and the other generators in degree 1. We agree to choose the generators so that y_1, x_1 and x_1' come from the first factor with x_1 and x_1' being the dual basis to the basis of

$$H_1\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times; \mathbb{F}_2\big) \cong \mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times / \big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]^\times\big)^2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$$

given by the image of i and $(1+i)$ in the mod-2 reduction of the abelian group $\mathbb{Z}\big[\frac{1}{2}, i\big]^\times$ and y_1 coming from $H^2(\mathbb{Z}/4; \mathbb{F}_2)$; likewise with y_2, x_2 and x_2' coming from the second factor.

3.2.2. *The cohomology of* $C_\Gamma(E_1)$. This cohomology has been calculated in [Weiss 2006]. In fact, from Theorem 1 of [Weiss 2006] we know

$$H^*(C_\Gamma(E_1); \mathbb{F}_2) \cong H^*\big(\mathrm{GL}_2\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big) \cong \mathbb{F}_2[c_1, c_2] \otimes E(e_1, e_1', e_3, e_3'). \tag{3-2}$$

Here we give a short summary of this calculation. The classes e_1, e_1', e_3 and e_3' are pulled back from Quillen's exterior classes q_1 and q_3 [1972] in

$$H^*(\mathrm{GL}_2(\mathbb{F}_5); \mathbb{F}_2) \cong \mathbb{F}_2[c_1, c_2] \otimes E(q_1, q_3) \tag{3-3}$$

via two ring homomorphisms

$$\pi : \mathbb{Z}[\tfrac{1}{2}, i] \to \mathbb{F}_5, \quad \pi' : \mathbb{Z}[\tfrac{1}{2}, i] \to \mathbb{F}_5. \tag{3-4}$$

We choose π such that i is sent to 3 and π' such that i is sent to 2.

Now consider the two commutative diagrams (with horizontal arrows induced by inclusion and vertical arrows induced by π and, respectively, π')

$$\begin{tikzcd} D_2\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big) \arrow[r] \arrow[d] & \mathrm{GL}_2\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big) \arrow[d] \\ D_2(\mathbb{F}_5) \arrow[r] & \mathrm{GL}_2(\mathbb{F}_5). \end{tikzcd} \tag{3-5}$$

By abuse of notation we can write

$$H^*(D_2(\mathbb{F}_5); \mathbb{F}_2) \cong H^*(\mathbb{F}_5^\times \times \mathbb{F}_5^\times; \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_2) \tag{3-6}$$

with $y_1 \in H^2(\mathbb{F}_5^\times; \mathbb{F}_2)$ and $x_1 \in H^2(\mathbb{F}_5^\times; \mathbb{F}_2)$ coming from the first factor and likewise with y_2 and x_2 coming from the second factor. Then π and π' induce two homomorphisms

$$\pi^*, \pi'^* : H^*(D_2(\mathbb{F}_5); \mathbb{F}_2) \to H^*\big(D_2\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big); \mathbb{F}_2\big)$$

which in terms of the isomorphisms (3-6) and (3-1) are explicitly given by

$$\pi^*(y_i) = y_i = \pi'^*(y_i), \quad \pi^*(x_i) = x_i, \quad \pi'^*(x_i) = x_i + x_i' \quad \text{for } i = 1, 2. \tag{3-7}$$

The cohomology of $\mathrm{GL}_2(\mathbb{F}_5)$ is detected by restriction to the cohomology of diagonal matrices and restriction is given explicitly as follows:

$$c_1 \mapsto y_1 + y_2, \quad c_2 \mapsto y_1 y_2, \quad q_1 \mapsto x_1 + x_2, \quad q_3 \mapsto y_1 x_2 + y_2 x_1. \tag{3-8}$$

Then e_1, e_1', e_3, e_3' are defined via

$$e_1 = \pi^*(q_1), \quad e_3 = \pi^*(q_3), \quad e_1' = \pi'^*(q_1), \quad e_3' = \pi'^*(q_3). \tag{3-9}$$

If c_1 and c_2 are the Chern classes of the tautological 2-dimensional complex representation of $\mathrm{GL}_2\big(\mathbb{Z}\big[\frac{1}{2}\big], i\big)$, then the restriction homomorphism which sends $H^*\big(\mathrm{GL}_2\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ to the cohomology of the subgroup of diagonal matrices is injective and by using (3-5) and (3-8) we see that it is explicitly given by

$$\begin{aligned} c_1 &\mapsto y_1 + y_2, & c_2 &\mapsto y_1 y_2, \\ e_1 &\mapsto x_1 + x_2, & e_3 &\mapsto y_1 x_2 + y_2 x_1, \\ e_1' &\mapsto x_1 + x_1' + x_2 + x_2', & e_3' &\mapsto y_1(x_2 + x_2') + y_2(x_1 + x_1'). \end{aligned} \tag{3-10}$$

3.2.3. *Functoriality.* We note that together with the isomorphisms (3-1) and (3-2) the restriction (3-10) also describes the map

$$\alpha_* : H^*(C_\Gamma(E_1); \mathbb{F}_2) \to H^*(C_\Gamma(E_2); \mathbb{F}_2)$$

induced from the standard inclusion of E_1 into E_2.

To finish the description of $H^*(C_\Gamma(-);\mathbb{F}_2)$ as a functor on $\mathcal{A}$ it remains to describe the action of the symmetric group $\mathrm{Aut}_{\mathcal{A}}(E_2)\cong\mathfrak{S}_3$ of rank 3 on

$$H^*(C_\Gamma(E_2);\mathbb{F}_2)\cong\mathbb{F}_2[y_1,y_2]\otimes\Lambda(x_1,x_1',x_2,x_2').$$

Because of the multiplicative structure we need it only on the generators.

If $\tau\in\mathrm{Aut}_{\mathcal{A}}(E_2)$ corresponds to permuting the factors in $C_\Gamma(E_2)\cong\mathrm{GL}_1\bigl(\mathbb{Z}\bigl[\frac{1}{2},i\bigr]\bigr)\times\mathrm{GL}_1\bigl(\mathbb{Z}\bigl[\frac{1}{2},i\bigr]\bigr)$ then

$$\begin{aligned}\tau_*(y_1)&=y_2, & \tau_*(x_1)&=x_2, & \tau_*(x_1')&=x_2',\\ \tau_*(y_2)&=y_1, & \tau_*(x_2)&=x_1, & \tau_*(x_2')&=x_1',\end{aligned}\tag{3-11}$$

and if $\sigma\in\mathrm{Aut}_{\mathcal{A}}(E_2)$ corresponds to the cyclic permutation of the diagonal entries (in suitable order) then

$$\begin{aligned}\sigma_*(y_1)&=y_2, & \sigma_*(x_1)&=x_2, & \sigma_*(x_1')&=x_2',\\ \sigma_*(y_2)&=y_1+y_2, & \sigma_*(x_2)&=x_1+x_2, & \sigma_*(x_2')&=x_1'+x_2'.\end{aligned}\tag{3-12}$$

3.3. ***Calculating the limit and its derived functors.*** In Proposition 4.3 of [Henn 1997] we showed that for any functor F from $\mathcal{A}$ to $\mathbb{Z}_{(2)}$-modules there is an exact sequence

$$0\to\lim_{\mathcal{A}}F\to F(E_1)\xrightarrow{\varphi}\mathrm{Hom}_{\mathbb{Z}[\mathfrak{S}_3]}(\mathrm{St}_{\mathbb{Z}},F(E_2))\to\lim\nolimits^1_{\mathcal{A}}F\to 0\tag{3-13}$$

where $\mathrm{St}_{\mathbb{Z}}$ is the $\mathbb{Z}[\mathfrak{S}_3]$-module given by the kernel of the augmentation map $\mathbb{Z}[\mathfrak{S}_3/\mathfrak{S}_2]\to\mathbb{Z}$, and if a and b are chosen to give an integral basis of $\mathrm{St}_{\mathbb{Z}}$ on which τ and σ act via

$$\begin{aligned}\tau_*(a)&=b, & \tau_*(b)&=a,\\ \sigma_*(a)&=-b, & \sigma_*(b)&=a-b,\end{aligned}\tag{3-14}$$

then $\varphi(x)(a)=\alpha_*(x)-(\sigma_*)^2\alpha_*(x)$ and $\varphi(x)(b)=\alpha_*(x)-\sigma_*\alpha_*(x)$ if $x\in F(E_1)$.

Because in our case the functor takes values in $\mathbb{F}_2$-vector spaces we can replace $\mathrm{Hom}_{\mathbb{Z}[\mathfrak{S}_3]}$ by $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}$ and $\mathrm{St}_{\mathbb{Z}}$ by its mod-2 reduction. The following elementary lemma is needed in the analysis of the third term in the exact sequence (3-13).

Lemma 3.1. (a) *Let* St *be the* $\mathbb{F}_2[\mathfrak{S}_3]$*-module given as the kernel of the augmentation* $\mathbb{F}_2[\mathfrak{S}_3/\mathfrak{S}_2]\to\mathbb{F}_2$. *The tensor product* $\mathrm{St}\otimes\mathrm{St}$ *decomposes as* $\mathbb{F}_2[\mathfrak{S}_3]$*-module canonically as*

$$\mathrm{St}\otimes\mathrm{St}\cong\mathbb{F}_2[\mathfrak{S}_3/A_3]\oplus\mathrm{St}$$

where A_3 *denotes the alternating group on three letters. In fact, the decomposition is given by*

$$\mathrm{St}\otimes\mathrm{St}\cong\mathrm{Im}(\mathrm{id}+\sigma_*+\sigma_*^2)\oplus\mathrm{Ker}(\mathrm{id}+\sigma_*+\sigma_*^2)$$

and the first summand is isomorphic to $\mathbb{F}_2[\mathfrak{S}_3/A_3]$ *while the second summand is isomorphic to* St.

(b) *The tensor product* $\mathbb{F}_2[\mathfrak{S}_3/A_3] \otimes \mathrm{St}$ *is isomorphic to* $\mathrm{St} \oplus \mathrm{St}$.

Proof.

(a) It is well-known that St is a projective $\mathbb{F}_2[\mathfrak{S}_3]$-module, hence $\mathrm{St} \otimes \mathrm{St}$ is also projective. It is also well-known that every projective indecomposable $\mathbb{F}_2[\mathfrak{S}_3]$-module is isomorphic to either St or $\mathbb{F}_2[\mathfrak{S}_3/A_3]$. The two modules can be distinguished by the fact that $e := id + \sigma_* + \sigma_*$ acts trivially on St and as the identity on $\mathbb{F}_2[\mathfrak{S}_3/A_3]$.

Furthermore e is a central idempotent in $\mathbb{F}_2[\mathfrak{S}_3]$ and hence each $\mathbb{F}_2[\mathfrak{S}_3]$-module M decomposes as direct sum of $\mathbb{F}_2[\mathfrak{S}_3]$-modules

$$M \cong \mathrm{Im}(e : M \to M) \oplus \mathrm{Ker}(e : M \to M).$$

An easy calculation shows that in the case of $\mathrm{St} \otimes \mathrm{St}$ both submodules are nontrivial and this together with the fact these submodules must be projective proves the claim.

(b) Again each of the factors in the tensor product is a projective $\mathbb{F}_2[\mathfrak{S}_3]$-module, hence the tensor product is a projective $\mathbb{F}_2[\mathfrak{S}_3]$-module. Because σ acts as the identity on $\mathbb{F}_2[\mathfrak{S}_3/A_3]$ we see that the idempotent e acts trivially on the tensor product and this forces the tensor product to be isomorphic to $\mathrm{St} \oplus \mathrm{St}$. □

Lemma 3.2. *The Poincaré series* χ_2 *of* $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, \mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_1', x_2, x_2'))$ *is given by*

$$\chi_2 = \frac{2t^2(1+3t^2+3t^4+t^6)+2t(1+2t^2+2t^4+2t^6+t^8)}{(1-t^4)(1-t^6)}.$$

Proof. The isomorphism of (3-1) is an isomorphism of $\mathbb{F}_2[\mathfrak{S}_3]$-modules where the action of $\mathfrak{S}_3$ is given by equations (3-11) and (3-12). In particular we see that $H^1(\mathrm{GL}_1(\mathbb{Z}[\frac{1}{2}, i]) \times \mathrm{GL}_1(\mathbb{Z}[\frac{1}{2}, i]); \mathbb{F}_2)$ is isomorphic to $\mathrm{St} \oplus \mathrm{St}$ generated by x_1, x_1', x_2, x_2'. The exterior powers of H^1 are given as

$$E^k(x_1, x_2, x_1', x_2') \cong E^k(\mathrm{St} \oplus \mathrm{St}) \cong \bigoplus_{j=0}^{k} E^j\,\mathrm{St} \otimes E^{k-j}\,\mathrm{St}$$

and, because $E^k(\mathrm{St})$ is isomorphic to $\Sigma^k \mathbb{F}_2$ if $k = 0, 2$, isomorphic to $\Sigma\,\mathrm{St}$ if $k = 1$, and trivially otherwise, we obtain

$$E^k(x_1, x_2, x_1', x_2') \cong \begin{cases} \Sigma^k \mathbb{F}_2 & \text{if } k = 0, 4, \\ \Sigma^k(\mathrm{St} \oplus \mathrm{St}) & \text{if } k = 1, 3, \\ \Sigma^2 \mathbb{F}_2 \oplus \Sigma^2(\mathrm{St} \otimes \mathrm{St}) \oplus \Sigma^2 \mathbb{F}_2 & \text{if } k = 2, \\ 0 & \text{if } k \neq 0, 1, 2, 3, 4, \end{cases}$$

where $\mathbb{F}_2$ denotes the trivial $\mathbb{F}_2[\mathfrak{S}_3]$-module whose additive structure is that of $\mathbb{F}_2$.

Therefore the Poincaré series χ_2 of $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, H^*(C_\Gamma(E_2); \mathbb{F}_2))$ decomposes according to the decomposition of $\Lambda(x_1, x_2', x_1', x_2')$ as the sum

$$\chi_2 := (1+2t^2+t^4)\chi_{2,0}+t^2\chi_{2,1}+2(t+t^3)\chi_{2,2} \tag{3-15}$$

where here we denote by $\chi_{2,0}$ the Poincaré series of $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, \mathbb{F}_2[y_1, y_2])$, by $\chi_{2,1}$ the Poincaré series of $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, \mathrm{St}\otimes\mathrm{St}\otimes\mathbb{F}_2[y_1, y_2])$ and by $\chi_{2,2}$ that of $\mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, \mathrm{St}\otimes\mathbb{F}_2[y_1, y_2])$.

It is well-known (and elementary to verify) that there is an isomorphism of $\mathbb{F}_2[\mathfrak{S}_3]$-modules $\mathrm{St}\oplus\mathrm{St}\oplus\mathbb{F}_2[\mathfrak{S}_3/A_3]\cong\mathbb{F}_2[\mathfrak{S}_3]$ and therefore an isomorphism

$$\begin{aligned}\mathbb{F}_2[y_1, y_2] &\cong \mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}\oplus\mathrm{St}\oplus\mathbb{F}_2[\mathfrak{S}_3/A_3], \mathbb{F}_2[y_1, y_2])\\ &\cong \mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, \mathbb{F}_2[y_1, y_2])^{\oplus 2}\oplus\mathbb{F}_2[y_1, y_2]^{A_3}.\end{aligned}$$

Together with the elementary fact that the A_3-invariants $\mathbb{F}_2[y_1, y_2]^{A_3}$ form a free module over $\mathbb{F}_2[y_1, y_2]^{\mathfrak{S}_3}\cong\mathbb{F}_2[c_2, c_3]$ on the two generators 1 and $y_1^3+y_1y_2^2+y_2^3$ of degree 0 and 6, respectively, this implies

$$2\chi_{2,0}+\frac{1+t^6}{(1-t^4)(1-t^6)}=\frac{1}{(1-t^2)^2}$$

and hence

$$\chi_{2,0}=\frac{t^2}{(1-t^2)(1-t^6)}. \tag{3-16}$$

It is elementary to check that St and $\mathbb{F}_2[\mathfrak{S}_3/A_3]$ are both self-dual $\mathbb{F}_2[\mathfrak{S}_3]$-modules and hence Lemma 3.1 gives

$$\mathrm{St}\otimes\mathrm{St}^*\cong\mathbb{F}_2[\mathfrak{S}_3/A_3]\oplus\mathrm{St}$$

and

$$\begin{aligned}\mathrm{St}\otimes\mathrm{St}^*\otimes\mathrm{St}^* &\cong (\mathbb{F}_2[\mathfrak{S}_3/A_3]\oplus\mathrm{St})\otimes\mathrm{St}^*\\ &\cong (\mathbb{F}_2[\mathfrak{S}_3/A_3]\otimes\mathrm{St})\oplus(\mathrm{St}\otimes\mathrm{St})\\ &\cong \mathbb{F}_2[\mathfrak{S}_3/A_3]\oplus\mathrm{St}\oplus\mathrm{St}\oplus\mathrm{St}\,.\end{aligned}$$

Therefore, if $\chi_{\mathbb{F}_2[y_1,y_2]^{A_3}}$ denotes the Poincaré series of the A_3-invariants then

$$\begin{aligned}\chi_{2,1} &= \chi_{\mathbb{F}_2[y_1,y_2]^{A_3}}+3\chi_{2,0}\\ &= \frac{1+t^6}{(1-t^4)(1-t^6)}+\frac{3t^2}{(1-t^2)(1-t^6)}=\frac{1+3t^2+3t^4+t^6}{(1-t^4)(1-t^6)},\end{aligned} \tag{3-17}$$

$$\begin{aligned}\chi_{2,2} &= \chi_{2,0}+\chi_{\mathbb{F}_2[y_1,y_2]^{A_3}}\\ &= \frac{t^2}{(1-t^2)(1-t^6)}+\frac{1+t^6}{(1-t^4)(1-t^6)}=\frac{1+t^2+t^4+t^6}{(1-t^4)(1-t^6)}.\end{aligned} \tag{3-18}$$

Finally (3-15), (3-16), (3-17) and (3-18) give

$$\begin{aligned}\chi_2 &= \frac{(1+2t^2+t^4)t^2(1+t^2)+t^2(1+3t^2+3t^4+t^6)+2(t+t^3)(1+t^2+t^4+t^6)}{(1-t^4)(1-t^6)}\\ &= \frac{2t^2(1+3t^2+3t^4+t^6)+2t(1+2t^2+2t^4+2t^6+t^8)}{(1-t^4)(1-t^6)},\end{aligned}$$

and this finishes the proof. □

Theorem 1.1 is now an immediate consequence of Theorem 2.1 and the following result.

Proposition 3.3. *Let $p = 2$ and $\Gamma = \mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$.*

(a) *There is an isomorphism of graded $\mathbb{F}_2$-algebras*

$$\lim_{\mathcal{A}} H^*(C_\Gamma(E); \mathbb{F}_2) \cong \mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5').$$

Furthermore, if we identify this limit with a subalgebra of $H^*(C_\Gamma(E_1); \mathbb{F}_2) \cong \mathbb{F}_2[c_1, c_2] \otimes E(e_1, e_1', e_3, e_3')$ then

$$\begin{aligned} b_2 &= c_1^2 + c_2, & b_3 &= c_1c_2,\\ d_3 &= e_3, & d_5 &= c_1e_3 + c_2e_1,\\ d_3' &= e_3', & d_5' &= c_1e_3' + c_2e_1'. \end{aligned}$$

(b) *There is an isomorphism of graded $\mathbb{F}_2$-vector spaces*

$$\lim^1_{\mathcal{A}} H^*(C_\Gamma(E); \mathbb{F}_2) \cong \Sigma^3\mathbb{F}_2 \oplus \Sigma^3\mathbb{F}_2 \oplus \Sigma^6\mathbb{F}_2.$$

(c) *For any $s > 1$*

$$\lim^s_{\mathcal{A}} H^*(C_\Gamma(E); \mathbb{F}_2) = 0.$$

Proof. (a) It is easy to check that the subalgebra of $\mathbb{F}_2[c_1, c_2] \otimes E(e_1, e_1', e_3, e_3')$ generated by the elements $c_1^2 + c_2$, c_1c_2, e_3, e_3', $c_1e_3 + c_2e_1$, and $c_1e_3' + c_2e_1'$ is isomorphic to the tensor product of a polynomial algebra on two generators b_2 and b_3 of degrees 4 and 6 and an exterior algebra on 4 generators d_3, d_3', d_5 and d_5' of degrees 3, 3, 5 and 5. In fact, it is clear that $c_1^2 + c_2$ and c_1c_2 are algebraically independent and the elements e_3, e_3', $c_1e_3 + c_2e_1$, and $c_1e_3' + c_2e_1'$ are exterior classes; their product is given as $c_2^2e_3e_3'e_1e_1' \neq 0$, and this implies easily that the exterior monomials in these elements are linearly independent over the polynomial algebra generated by $c_1^2 + c_2$ and c_1c_2. From now on we identify b_2, b_3, d_3, d_3', d_5 and d_5' with $c_1^2 + c_2$, c_1c_2, e_3, e_3', $c_1e_3 + c_2e_1$ and $c_1e_3' + c_2e_1'$.

Now we use the exact sequence (3-13) and the description of φ to determine the inverse limit. Because α_* is injective, we see that if we identify $H^*(C_\Gamma(E_1); \mathbb{F}_2)$ with its image in $H^*(C_\Gamma(E_2); \mathbb{F}_2)$ then the inverse limit can be identified with the intersection of the image of α_* with the invariants in $\mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_1', x_2, x_2')$

with respect to the action of the cyclic group of order 3 of $\mathrm{Aut}_{\mathcal{A}}(E_2) \cong \mathfrak{S}_3$ generated by σ. This action has been described in (3-12) and with these formulas it is straightforward to check that the elements

$$\begin{aligned} b_2 &= y_1^2 + y_1 y_2 + y_2^2, \\ b_3 &= y_1 y_2 (y_1 + y_2), \\ d_3 &= y_1 x_2 + y_2 x_1, \\ d_5 &= (y_1 + y_2)(y_1 x_2 + y_2 x_1) + y_1 y_2 (x_1 + x_2) = y_1^2 x_2 + y_2^2 x_1, \\ d_3' &= y_1 (x_2 + x_2') + y_2 (x_1 + x_1'), \\ d_5' &= (y_1 + y_2)(y_1 (x_2 + x_2') + y_2 (x_1 + x_1')) + y_1 y_2 (x_1 + x_1' + x_2 + x_2') \\ &= y_1^2 (x_2 + x_2') + y_2^2 (x_1 + x_1') \end{aligned} \tag{3-19}$$

all belong to the inverse limit.

Now consider the Poincaré series

$$\begin{aligned} \chi_0 &:= \sum_{n \geq 0} \dim_{\mathbb{F}_2} (\mathbb{F}_2[b_2, b_3] \otimes E(e_3, e_3', e_5, e_5')^n) t^n = \frac{(1+t^3)^2 (1+t^5)^2}{(1-t^4)(1-t^6)}, \\ \chi_1 &:= \sum_{n \geq 0} \dim_{\mathbb{F}_2} H^n (C_\Gamma(E_1); \mathbb{F}_2) t^n = \frac{(1+t)^2 (1+t^3)^2}{(1-t^2)(1-t^4)}, \\ \chi_2 &:= \frac{2t^2(1+3t^2+3t^4+t^6) + 2t(1+2t^2+2t^4+2t^6+t^8)}{(1-t^4)(1-t^6)}. \end{aligned}$$

Then we have the identity

$$\chi_0 + \chi_2 - \chi_1 = \frac{p}{(1-t^4)(1-t^6)}$$

with

$$\begin{aligned} p &= (1+t^3)^2(1+t^5)^2 + 2t^2(1+3t^2+3t^4+t^6) \\ &\qquad + 2t(1+2t^2+2t^4+2t^6+t^8) - (1+t)^2(1+t^3)^2(1+t^2+t^4) \\ &= 2t^3 + t^6 - 2t^7 - 2t^9 - t^{10} - t^{12} + 2t^{13} + t^{16} \\ &= (2t^3 + t^6)(1-t^4)(1-t^6) \end{aligned}$$

and therefore

$$\chi_0 + \chi_2 = \chi_1 + (2t^3 + t^6). \tag{3-20}$$

This, together with the fact that $\lim_{\mathcal{A}} H^*(C_\Gamma(E); \mathbb{F}_2)$ contains a subalgebra isomorphic to $\mathbb{F}_2[b_2, b_3] \otimes \Lambda(d_3, d_3', d_5, d_5')$, already implies that the sequence

$$\begin{aligned} 0 \to \mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5') \to H^*(C_\Gamma(E_1); \mathbb{F}_2) \\ \xrightarrow{\varphi} \mathrm{Hom}_{\mathbb{F}_2[\mathfrak{S}_3]}(\mathrm{St}, H^*(C_\Gamma(E_1); \mathbb{F}_2)) \to 0 \end{aligned}$$

in which the left-hand arrow is given by inclusion is exact except possibly in dimensions 3 and 6.

In order to complete the proof of (a) it is now enough to verify that in degrees 3 and 6 the inverse limit is not bigger than $\mathbb{F}_2[b_2,b_3]\otimes E(d_3,d_3',d_5,d_5')$. We leave this straightforward verification to the reader.

Then (b) follows immediately from (a) together with (3-20) and the exact sequence (3-13), and (c) follows from Theorem 2.1 and the fact that $r_2(G)=2$. $\square$

We can now give the proof of Theorem 1.2.

Proof.

(a) The exact sequence of Theorem 1.1 is obtained from the exact sequence (2-2) via Proposition 3.3. Therefore the epimorphism of Theorem 1.1 is the edge homomorphism of the centralizer spectral sequence. The result then follows from (2-1) by observing that we have identified the target of the edge homomorphism with the subalgebra $\mathbb{F}_2[b_2,b_3]\otimes E(d_3,d_3',d_5,d_5')$ of $H^*(C_\Gamma(E_1);\mathbb{F}_2)$ and by recalling that $C_\Gamma(E_1)$ is equal to the subgroup of special diagonal matrices $\mathrm{SD}_3\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)$.

(b) The two ring homomorphisms $\pi,\pi':\mathbb{Z}\big[\frac{1}{2},i\big]\to\mathbb{F}_5$ of (3-4) determine homomorphisms $\mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)\subset\mathrm{GL}_3\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)\to\mathrm{GL}_3(\mathbb{F}_5)$. By [Quillen 1972] we have

$$H^*\,\mathrm{GL}_3(\mathbb{F}_5);\mathbb{F}_2)\cong\mathbb{F}_3[c_1,c_2,c_3]\otimes E(q_1,q_3,q_5).$$

We get a well-defined homomorphism of $\mathbb{F}_2$-graded algebras

$$\varphi:\mathbb{F}_2[c_2,c_3]\otimes E(e_3,e_3',e_5,e_5')\to H^*(\Gamma;\mathbb{F}_2)$$

by sending c_i to the i-th Chern class of the tautological 3-dimensional representation of Γ and by declaring $\varphi(e_i)=\pi^*(q_i)$ and $\varphi(e_i')=\pi'^*(q_i')$ for $i=3,5$. The classes q_1, q_3 and q_5 are the symmetrizations of x_1, y_1x_2 and $y_1y_2x_3$, respectively, with respect to the natural action of $\mathfrak{S}_3$ on

$$H^*(\mathrm{GL}_3(\mathbb{F}_5);\mathbb{F}_2)\cong\mathbb{F}_2[y_1,y_2,y_3]\otimes E(x_1,x_2,x_3).$$

Compare (5-1) below.

Next we determine the composition $\psi\varphi$. The universal Chern classes c_i are the elementary symmetric polynomials in variables, say y_i, and the inclusion $\mathrm{GL}_2(\mathbb{C})\subset\mathrm{SL}_3(\mathbb{C})\subset\mathrm{GL}_3(\mathbb{C})$ imposes the relation $y_1+y_2+y_3=0$. This implies that the behavior of ψ on Chern classes is given by

$$c_1\mapsto 0,\quad c_2\mapsto c_1^2+c_2=y_1^2+y_1y_2+y_2^2=b_2,\quad c_3\mapsto c_1c_2=y_1y_2(y_1+y_2)=b_3.$$

In these equations we have identified $H^*\big(\mathrm{GL}_2\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big);\mathbb{F}_2\big)$, as in the proof of Proposition 3.3, via restriction with a subalgebra of $\mathbb{F}_2[y_1,y_2]\otimes E(x_1,x_1',x_3,x_3')$.

In order to determine the composition $\psi\varphi$ on the classes e_3, e_3', e_5 and e_5' we calculate at the level of $\mathbb{F}_5$ and use naturality with respect to the homomorphisms induced by π and π', i.e., we consider the maps induced in cohomology by the following commutative diagram in which the horizontal maps are induced by inclusion and the vertical maps are induced by π and, respectively, π':

$$\begin{tikzcd} \mathrm{GL}_2\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big) \arrow[r] \arrow[d] & \mathrm{SL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big) \arrow[r] \arrow[d] & \mathrm{GL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big) \arrow[d] \\ \mathrm{GL}_2(\mathbb{F}_5) \arrow[r] & \mathrm{SL}_3(\mathbb{F}_5) \arrow[r] & \mathrm{GL}_3(\mathbb{F}_5). \end{tikzcd}$$

On the level of $\mathbb{F}_5$ the composition induces in cohomology a map

$$\mathbb{F}_3[c_1, c_2, c_3] \otimes E(q_1, q_3, q_5) \to \mathbb{F}_2[c_1, c_2] \otimes E(e_1, e_3) \subset \mathbb{F}_2[y_1, y_2] \otimes E(q_1, q_3)$$

which is easily determined from (5-1) below by imposing the relations $y_1+y_2+y_3 = 0$ and $x_1 + x_2 + x_3 = 0$ on the symmetrization of the classes y_1x_2 and $y_1y_2x_3$ with respect to the natural action of $\mathfrak{S}_3$ on the cohomology of diagonal matrices $H^*(D_3(\mathbb{F}_5); \mathbb{F}_2) \cong \mathbb{F}_2[y_1, y_2, y_3] \otimes E(x_1, x_2, x_3)$. Explicitly we get

$$\begin{aligned} &c_1 \mapsto 0, \quad c_2 \mapsto y_1^2 + y_1y_2 + y_2^2, \quad c_3 \mapsto y_1y_2(y_1 + y_2),\\ &q_1 \mapsto 0, \quad q_3 \mapsto y_1x_2 + y_2x_1, \qquad q_5 \mapsto y_1^2x_2 + y_2^2x_1 \end{aligned}$$

and by using (3-7) and (3-19) we see that the composition $\psi\phi$ maps the elements e_3, e_5, e_3', and e_5' as follows:

$$\begin{aligned} e_3 &\mapsto \pi^*(y_1x_2 + y_2x_1) &&= d_3,\\ e_5 &\mapsto \pi^*(y_1^2x_2 + y_2^2x_1) &&= d_5,\\ e_3' &\mapsto \pi'^*(y_1x_2 + y_2x_1) &&= d_3',\\ e_5' &\mapsto \pi'^*(y_1^2x_2 + y_2^2x_1) &&= d_5'. \end{aligned}$$

Here we have identified the target of ψ with a subalgebra of $H^*\big(\mathrm{GL}_2\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ and the latter via restriction with a subalgebra of $\mathbb{F}_2[y_1, y_2] \otimes E(x_1, x_1', x_3, x_3')$.

(c) The space X can be taken to be the product of symmetric space

$$X_\infty := \mathrm{SL}_3(\mathbb{C})/\,\mathrm{SU}(3)$$

and the Bruhat–Tits building X_2 for $\mathrm{SL}_3(\mathbb{Q}_2[i])$. Now $\mathrm{SL}_3(\mathbb{Q}_2[i])\backslash X_2$ is a 2-simplex [Brown 1989] and the projection map $X \to X_2$ induces a map

$$\mathrm{SL}_3(\mathbb{Q}_2[i])\backslash X \to \mathrm{SL}_3(\mathbb{Q}_2[i])\backslash X_2$$

whose fibers have the homotopy type of a 6-dimensional $\mathrm{SL}_3\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$-invariant deformation retract (see Section 4). Therefore we get $H^n_G(X, X_s; \mathbb{F}_2) = 0$ if $n > 8$

and the inclusion $X_s \subset X$ induces an isomorphism $H^n_G(X; \mathbb{F}_2) \cong H^n_G(X_s; \mathbb{F}_2)$ if $n > 8$. Then part (c) simply follows from (a) except for the finiteness statement for the kernel for which we refer to (4-1) and (4-2) below. □

4. Comments on step 2

The situation for $p = 2$ and $G = \mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$ is analogous to the situation for $p = 2$ and $G = \mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}\big]\big)$ for which step 2 was carried out in [Henn 1999] via a detailed study of the relative cohomology $H^*_G(X, X_s; \mathbb{F}_2)$ for X equal to the product of the symmetric space $X_\infty := \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}(3)$ with the Bruhat–Tits building X_2 for $\mathrm{SL}_3(\mathbb{Q}_2)$; the spaces involved had a few hundred cells and the calculation was painful. In the case of $\mathrm{SL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$ with X the product of $\mathrm{SL}_3(\mathbb{C})/\mathrm{SU}(3)$ with the Bruhat–Tits building for $\mathrm{SL}_3(\mathbb{Q}_2[i])$ the calculational complexity of the second step is much more involved and an explicit calculation by hand does not look feasible. However, in recent years there have been a lot of machine aided calculations of the cohomology of various arithmetic groups (for example [Dutour Sikirić et al. 2016; Bui et al. 2016]) and a machine aided calculation seems to be within reach.

The natural strategy for undertaking this second step is to follow the same path as in [Henn 1999]. The equivariant cohomology $H^*_\Gamma(X, X_s; \mathbb{F}_2)$ can be studied via the spectral sequence of the projection map

$$p : X = X_\infty \times X_2 \to X_2.$$

This gives a spectral sequence with

$$E_1^{p,q} \cong \bigoplus_{\sigma \in \Lambda_p} H^q_{\Gamma_\sigma}(X_\infty, X_{\infty,s}; \mathbb{F}_2) \Rightarrow H^{p+q}_\Gamma(X, X_s; \mathbb{F}_2). \tag{4-1}$$

Here Λ_p indexes the p-dimensional cells in the orbit space of X_2 with respect to the action of Γ. The orbit space is a 2-simplex, i.e., Λ_0 and Λ_1 contain 3 elements and Λ_2 is a singleton. Furthermore Γ_σ is the isotropy group of a chosen representative in X_2 of the cell σ in the quotient space. For fixed p all p-dimensional cells have isomorphic isotropy groups because the Γ-action on the Bruhat–Tits building is the restriction of a natural action of $\mathrm{GL}_3\big(\mathbb{Z}\big[\frac{1}{2}, i\big]\big)$ on X_2 and this action is transitive on the set of p-dimensional cells [Brown 1989].

Therefore all isotropy subgroups for the action on X_2 are, up to isomorphism, subgroups of $\mathrm{SL}_3(\mathbb{Z}[i])$ which itself appears as isotropy group of a 0-dimensional cell in X_2. The isotropy groups of 1-dimensional and 2-dimensional cells are isomorphic to well-known congruence subgroups of $\mathrm{SL}_3(\mathbb{Z}[i])$. By the Soulé–Lannes method the fiber X_∞ of the projection map p admits a 6-dimensional $\mathrm{SL}_3(\mathbb{Z}[i])$-equivariant deformation retract (the space of "well-rounded hermitian forms" modulo arithmetic equivalence) with compact quotient [Ash 1984] and therefore we have

$$E_1^{s,t} = 0 \text{ unless } s = 0, 1, 2,\ 0 \le t \le 6, \quad \text{and} \quad \dim_{\mathbb{F}_2} E_1^{s,t} < \infty \text{ for all } (s, t). \tag{4-2}$$

The E_1-term of this spectral sequence should be accessible to machine calculation. The spectral sequence will necessarily degenerate at E_3 and the calculation of the differentials is likely to need human intervention, as in the case of $\mathrm{SL}\big(3, \mathbb{Z}\big[\tfrac{1}{2}\big]\big)$ (compare Section 3.4 of [Henn 1999]). Likewise the calculation of the connecting homomorphism for the mod-2 Borel cohomology of the pair (X, X_s) is likely to require human intervention.

5. On Quillen's conjecture for $\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$

The next result gives two reformulations of the conjecture of Quillen briefly discussed in the introduction. The classes e_{2k-1} and e'_{2k-1} in part (c) will be introduced in (5-1) below.

Theorem 5.1. *Suppose $n \ge 2$. The following statements are equivalent*:

(a) *Conjecture $C\big(n, \mathbb{Z}\big[\tfrac{1}{2}, i\big], 2\big)$ holds, i.e., $H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ is a free module over $\mathbb{Z}/2[c_1, \dots, c_n]$ where the c_i are the mod-2 Chern classes of the tautological n-dimensional complex representation of $\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$.*

(b) *The restriction homomorphism*

$$H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big) \to H^*\big(D_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big)$$

is injective, where $D_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$ is the subgroup of diagonal matrices in $\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big)$.

(c) *There are isomorphisms*

$$H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big) \cong \mathbb{F}_2[c_1, \dots, c_n] \otimes E(e_1, e'_1, \dots, e_{2n-1}, e'_{2n-1})$$

where the classes c_k are the Chern classes of the tautological n-dimensional complex representation of $\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$ and the classes e_{2k-1}, e'_{2k-1} are of cohomological degree $2k-1$ for $k = 1, \dots, n$.

Proof. It is trivial that (c) implies (a).

In order to show that (a) implies (b) we observe that $D_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$ is the centralizer of the unique, up to conjugacy, maximal elementary abelian 2-subgroup E_n of $\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$ given by the subgroup of diagonal matrices of order 2. Now consider the top Dickson invariant ω in $H^*(B\mathrm{GL}_n(\mathbb{C}); \mathbb{F}_2)$, i.e., the class whose restriction to $H^*B\big(\prod_{i=1}^n \mathrm{GL}_1(\mathbb{C})\big); \mathbb{F}_2)$ is the product of all nontrivial classes of degree 2. The image of ω in $H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ restricts trivially to the cohomology of all elementary abelian 2-subgroups E of $\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big)$ of rank less than n. If (a) holds then the image of ω is not a zero divisor in $H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2}, i\big]\big); \mathbb{F}_2\big)$ and hence

Corollary I.5.8 of [Henn et al. 1995] implies that the restriction to the centralizer of E_n is injective.

The implication (b) $\Rightarrow$ (c) follows from Proposition 5.3 below. $\square$

Before we go on we introduce the classes e_{2k-1} and e'_{2k-1}. As in the case of GL_2 they are obtained from Quillen's classes [1972] $q_{2k-1} \in H^{2k-1}(\mathrm{GL}_n(\mathbb{F}_5); \mathbb{F}_2)$ which restrict in the cohomology of diagonal matrices in $\mathbb{F}_5$ to the symmetrization of the classes $y_1 \cdots y_{k-1}x_k$ where y_k is of cohomological degree 2 corresponding to the k-th factor in the product $\prod_{k=1}^n \mathbb{F}_5^\times$ and x_k is of cohomological degree 1 of the same factor. We define

$$e_{2k-1} := \pi^*(q_{2k-1}), \quad e'_{2k-1} := \pi'^*(q_{2k-1}) \tag{5-1}$$

where π and π' are the two ring homomorphisms $\mathbb{Z}\left[\frac{1}{2}, i\right] \to \mathbb{F}_5$ with π sending i to 3 and π' sending i to 2 which we considered earlier in Section 3. We identify the mod-2 cohomology $H^*\left(D_n\left(\mathbb{Z}\left[\frac{1}{2}, i\right]\right); \mathbb{F}_2\right)$ with $\mathbb{F}_2[y_1, \dots y_n] \otimes E(x_1, x'_1 \dots, x_n, x'_n)$ with y_k, $k = 1, \dots, n$ of degree 2 and x_k, x'_k, $k = 1, \dots, n$ of degree 1 where as before we choose x_k and x'_k to be the basis which is dual to the basis of the k-th factor in

$$D_n\left(\mathbb{Z}\left[\tfrac{1}{2}, i\right]\right)/D_n\left(\mathbb{Z}\left[\tfrac{1}{2}, i\right]\right)^2 \cong \left(\mathbb{Z}\left[\tfrac{1}{2}, i\right]^\times / \left(\mathbb{Z}\left[\tfrac{1}{2}, i\right]^\times\right)^2\right)^n$$

given by the classes of i and $1 + i$. Then we get the following lemma which generalizes (3-10) and whose straightforward proof we leave to the reader.

Lemma 5.2. *The class e_{2k-1} restricts in the cohomology of the subgroup of diagonal matrices $H^*\left(D_n\left(\mathbb{Z}\left[\frac{1}{2}, i\right]; \mathbb{F}_2\right)\right)$ to the symmetrization of $y_1 \cdots y_{k-1}x_k$ and the class e'_{2k-1} restricts to the symmetrization of $y_1 \cdots y_{k-1}(x_k + x'_k)$.* $\square$

The following result determines the image of the restriction homomorphism and shows that (b) implies (c) in Theorem 5.1. It resembles results of Mitchell [1992] for $\mathrm{GL}_n\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)$ for $p = 2$ and of Anton [1999] for $\mathrm{GL}_n\left(\mathbb{Z}\left[\frac{1}{3}, \zeta_3\right]\right)$ for $p = 3$. Its proof uses crucially condition (5-3) below, which also plays a central role in [Anton 2003].

Proposition 5.3. *Let $n \geq 1$ be an integer. The image of the restriction map*

$$\begin{aligned} i^* : H^*\left(\mathrm{GL}_n\left(\mathbb{Z}\left[\tfrac{1}{2}, i\right]\right); \mathbb{F}_2\right) \\ \to H^*\left(D_n\left(\mathbb{Z}\left[\tfrac{1}{2}, i\right]\right); \mathbb{F}_2\right) \cong \mathbb{F}_2[y_1, \dots y_n] \otimes E(x_1, x'_1 \dots, x_n, x'_n) \end{aligned}$$

is isomorphic to

$$\mathbb{F}_2[c_1, \dots c_n] \otimes E(e_1, e'_1, \dots, e_{2n-1}, e'_{2n-1}).$$

Here we have identified the Chern classes c_i and the classes e_{2i-1} and e'_{2i-1} with their image via i^*. The images of the elements c_i are, of course, the elementary symmetric polynomials in the y_i and the images of the classes e_{2i-1} and e'_{2i-1} have been determined in Lemma 5.2. We remark that even though i^* need not be injective, it is injective on the subalgebra of $H^*\big(\mathrm{GL}_n\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big);\mathbb{F}_2\big)$ generated by the classes c_i, e_{2i-1} and e'_{2i-1}, $1\le i\le n$.

Proof. In this proof we denote the subalgebra

$$\mathbb{F}_2[c_1,\dots c_n]\otimes E(e_1,e'_1,\dots,e_{2n-1},e'_{2n-1}).$$

of $H^*\big(D_n\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big);\mathbb{F}_2\big)$ by C_n and the image of the restriction map by B_n. We need to show that $B_n=C_n$. This is trivial if $n=1$ and for $n=2$ this follows from Theorem 1 of [Weiss 2006] (compare (3-2), (3-10) and Lemma 5.2).

The classes $c_1,\dots,c_n$ are in B_n as images of the Chern classes with the same name and the classes $e_1,\dots e_{2n-1},e'_1,\dots e'_{2n-1}$ are in B_n by Lemma 5.2. Therefore we have $C_n\subset B_n$. We will show $B_n\subset C_n$ for $n\ge 2$ by induction on n. This will be done in three steps:

1. From the inclusions

$$\mathrm{GL}_{n-2}\big(\mathbb{Z}\big[\tfrac{1}{2},i\big]\big)\times\mathrm{GL}_2\big(\mathbb{Z}\big[\tfrac{1}{2},i\big]\big)\subset\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2},i\big]\big)$$
$$\mathrm{GL}_{n-1}\big(\mathbb{Z}\big[\tfrac{1}{2},i\big]\big)\times\mathrm{GL}_1\big(\mathbb{Z}\big[\tfrac{1}{2},i\big]\big)\subset\mathrm{GL}_n\big(\mathbb{Z}\big[\tfrac{1}{2},i\big]\big)$$

given by matrix block sum and the identifications of $D_{n-2}\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)\times D_2\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)$ and of $D_{n-1}\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)\times D_1\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)$ with $D_n\big(\mathbb{Z}\big[\frac{1}{2},i\big]\big)$ we see that

$$B_n\subset B_{n-1}\otimes B_1\cap B_{n-2}\otimes B_2$$

and by the induction hypothesis the latter subalgebra is equal to

$$C_{n-1}\otimes C_1\cap C_{n-2}\otimes C_2,$$

in particular we have

$$B_n\subset C_{n-1}\otimes C_1\cap C_{n-2}\otimes C_2. \tag{5-2}$$

2. The monomial basis in

$$H^*\big(D_n\big(\mathbb{Z}\big[\tfrac{1}{2},i\big]\big);\mathbb{F}_2\big)\cong\mathbb{F}_2[y_1,\dots,y_n]\otimes E(x_1,\dots,x_n,x'_1,\dots,x'_n)$$

is in bijection with the set $S(n)$ of sequences

$$I=(a_1,\varepsilon_{1,1},\varepsilon_{2,1},\dots,a_n,\varepsilon_{1,n},\varepsilon_{2,n})$$

where the a_i are integers ≥ 0 and $\varepsilon_{i,j} \in \{0, 1\}$ for $i = 1, 2$ and $1 \leq j \leq n$. More precisely to I we associate the monomial

$$y^I := y_1^{a_1} \cdots y_n^{a_n} x_1^{\varepsilon_{1,1}} \cdots x_n^{\varepsilon_{1,n}} x_1'^{\,\varepsilon_{2,1}} \cdots x_n'^{\,\varepsilon_{2,n}}.$$

We equip $S(n)$ with the lexicographical order and denote it by $<_n$. This order has the property that for each $1 \leq k < n$ it agrees with the lexicographical order on $S(k) \times S(n-k)$ if $S(k)$ and $S(n-k)$ are equipped with the orders $<_k$ and $<_{n-k}$ and $S(n)$ is identified with $S(k) \times S(n-k)$ via concatenation of sequences.

In what follows we replace the symmetrizations of the elements $y_1 \cdots y_{i-1}(x_i + x_i')$, $i = 1, \ldots, n$, by the symmetrization of $y_1 \cdots y_{i-1} x_i'$ and by abuse of notation we continue to denote them by e_{2i-1}'. This does not change the subalgebra C_n. This subalgebra

$$\mathbb{F}_2[c_1, \ldots c_n] \otimes E(e_1, e_1', \ldots, e_{2n-1}, e_{2n-1}') \subset \mathbb{F}_2[y_1, \ldots, y_n] \otimes E(x_1, \ldots, x_n, x_1', \ldots, x_n')$$

has a monomial basis which is in bijection with the set $T(n)$ of sequences

$$K = (k_1, \ldots, k_n; \phi_{1,1}, \ldots, \phi_{1,n}; \phi_{2,1} \ldots, \phi_{2,n})$$

where the k_i are integers ≥ 0 and $\phi_{i,j} \in \{0, 1\}$ for $i = 1, 2$ and $1 \leq j \leq n$. More precisely to K we associate the monomial

$$c^K := c_1^{k_1} \cdots c_n^{k_n} e_1^{\phi_{1,1}} \cdots e_n^{\phi_{1,n}} e_1'^{\,\phi_{2,1}} \cdots e_n'^{\,\phi_{2,n}}.$$

We define a map

$$\alpha : T(n) \to S(n)$$

by associating to $K \in T(n)$ the largest monomial in $S(n)$ which occurs in the decomposition of c^K as linear combination of elements x^I with $I \in S(n)$. The proof of the following result is elementary and is left to the reader.

Lemma 5.4. *The map α is explicitly given by*

$$\alpha((k_1, \ldots, k_n; \phi_{1,1}, \ldots, \phi_{1,n}; \phi_{2,1} \ldots, \phi_{2,n})) = (a_1, \varepsilon_{1,1}, \varepsilon_{2,1}, \ldots, a_n, \varepsilon_{1,n}, \varepsilon_{2,n})$$

with

$$\begin{aligned} a_j &= k_j + \sum_{i=j+1}^{n} (k_i + \phi_{1,i} + \phi_{2,i}), \quad 1 \leq j < n, \\ a_n &= k_n, \\ \varepsilon_{i,j} &= \phi_{i,j}, \quad 1 \leq j \leq n,\ i = 1, 2. \end{aligned}$$

□

From this lemma it is obvious that α is injective and a sequence

$$I = (a_1, \varepsilon_{1,1}, \varepsilon_{2,1}, \ldots, a_n, \varepsilon_{1,n}, \varepsilon_{2,n}) \in S(n)$$

is in the image of α if and only if we have

$$a_j - a_{j+1} \geq \varepsilon_{1,j+1} + \varepsilon_{2,j+1} \quad \text{for all } 1 \leq j < n. \tag{5-3}$$

In particular, if an element x is in C_n then the maximal sequence which appears in the decomposition of x as a linear combination of the monomials x^I with $I \in S(n)$ satisfies (5-3) for all $1 \leq j < n$. Likewise, if x is in $C_i \otimes C_{n-i}$ then this maximal sequence is equal to the maximal sequence which appears in the decomposition of x as a linear combination of the monomials x^I with $I \in S(k) \times S(n-k)$ and hence it satisfies (5-3) for all $1 \leq j < i$ and $i+1 \leq j < n$.

3. Now let x be a homogeneous element of B_n and let I_0 be the maximal sequence in $S(n)$ appearing in the decomposition of x as a linear combination of the monomials x^I with $I \in S(n)$. By (5-2) we have $x \in C_{n-1} \otimes C_1$ and $x \in C_{n-2} \otimes C_2$, and I_0 remains the maximal sequence in $S(n-1) \times S(1)$ and $S(n-2) \times S(2)$, respectively, appearing in the decomposition of x as a linear combination of the monomials x^I with, respectively, $I \in S(n-1) \times S(1)$ and $I \in S(n-2) \times S(2)$. Hence I_0 satisfies conditions (5-3) for $1 \leq j < n-1$ and, respectively, $1 \leq j < n-2$ and $j = n-1$. In particular condition (5-3) holds for all $1 \leq j < n$ and therefore there exists $K_0 \in T(n)$ such that $\alpha(K_0) = I_0$. Then $x - c^{K_0}$ is still in B_n and the maximal sequence appearing in the decomposition of $x - c^{K_0}$ is smaller than that of x. By iterating this procedure we see that x belongs to C_n. □

Finally we relate $C\big(3, \mathbb{Z}[\frac{1}{2}, i], 2\big)$ to the behavior of the restriction homomorphism

$$H^*(\Gamma; \mathbb{F}_2) \to H^*(C_\Gamma(E_2); \mathbb{F}_2).$$

For this we observe that the subgroups $\Gamma = \mathrm{SL}_3\big(\mathbb{Z}[\frac{1}{2}, i]\big)$ and the center $Z \cong \mathbb{Z}[\frac{1}{2}, i]^\times$ of $\mathrm{GL}_3\big(\mathbb{Z}[\frac{1}{2}, i]\big)$ have trivial intersection and their product is the kernel of the homomorphism

$$\mathrm{GL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big) \to \mathbb{Z}[\tfrac{1}{2}, i]^\times \to \mathbb{Z}[\tfrac{1}{2}, i]^\times / \big(\mathbb{Z}[\tfrac{1}{2}, i]^\times\big)^3 \cong \mathbb{Z}/3$$

given as the composition of the determinant with the natural quotient map. Therefore the spectral sequence of the extension

$$1 \to \mathrm{SL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big) \times Z \to \mathrm{GL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big) \to \mathbb{Z}/3 \to 1$$

gives an isomorphism

$$H^*\big(\mathrm{GL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big); \mathbb{F}_2\big) \cong \big(H^*\big(\mathrm{SL}_3\big(\mathbb{Z}[\tfrac{1}{2}, i]\big); \mathbb{F}_2\big) \otimes H^*(Z; \mathbb{F}_2)\big)^{\mathbb{Z}/3}. \tag{5-4}$$

Proposition 5.5. *Conjecture $C\big(3, \mathbb{Z}[\frac{1}{2}, i], 2\big)$ holds if and only if either*

(a) $H^*\big(\mathrm{SL}_3\big(\mathbb{Z}[\frac{1}{2}, i]\big); \mathbb{F}_2\big) \cong \mathbb{F}_2[b_2, b_3] \otimes E(d_3, d_3', d_5, d_5')$ *or*

(b) *the kernel of the map ψ of Theorem 1.2 is a finite-dimensional vector space for which the action of $\mathbb{Z}/3 \cong \mathbb{Z}\left[\frac{1}{2}, i\right]^\times/\left(\mathbb{Z}\left[\frac{1}{2}, i\right]^\times\right)^3$ has trivial invariants.*

Proof. Clearly $\mathbb{Z}/3 \cong \mathbb{Z}\left[\frac{1}{2}, i\right]^\times/\left(\mathbb{Z}\left[\frac{1}{2}, i\right]^\times\right)^3$ acts trivially on $H^*(Z; \mathbb{F}_2)$ and on the image of the homomorphism φ of Theorem 1.2. Hence, the corollary follows immediately from (5-4) and Theorem 1.2. □

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Tunisian Journal of Mathematics

2019 vol. 1 no. 4

Grothendieck–Messing deformation theory for varieties for K3 type 455
ANDREAS LANGER and THOMAS ZINK

Purity of crystalline strata 519
JINGHAO LI and ADRIAN VASIU

On the mod-2 cohomology of $SL_3(\mathbb{Z}[\frac{1}{2}, i])$ 539
HANS-WERNER HENN

Geometric origin and some properties of the arctangential heat equation 561
YANN BRENIER

Horn's problem and Fourier analysis 585
JACQUES FARAUT