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# Horn's problem and Fourier analysis

Jacques Faraut

Let $A$ and $B$ be two $n \times n$ Hermitian matrices. Assume that the eigenvalues $\alpha_1, \dots, \alpha_n$ of $A$ are known, as well as the eigenvalues $\beta_1, \dots, \beta_n$ of $B$. What can be said about the eigenvalues of the sum $C = A + B$? This is Horn's problem. We revisit this question from a probabilistic viewpoint. The set of Hermitian matrices with spectrum $\{\alpha_1, \dots, \alpha_n\}$ is an orbit $\mathcal{O}_\alpha$ for the natural action of the unitary group $U(n)$ on the space of $n \times n$ Hermitian matrices. Assume that the random Hermitian matrix $X$ is uniformly distributed on the orbit $\mathcal{O}_\alpha$ and, independently, the random Hermitian matrix $Y$ is uniformly distributed on $\mathcal{O}_\beta$. We establish a formula for the joint distribution of the eigenvalues of the sum $Z = X + Y$. The proof involves orbital measures with their Fourier transforms, and Heckman's measures.

## Introduction

Consider two Hermitian matrices $A$ and $B$, and their sum $C = A + B$. Assume that the eigenvalues $\alpha_1, \dots, \alpha_n$ of $A$ and the eigenvalues $\beta_1, \dots, \beta_n$ of $B$ are known. Here is Horn's problem: what can be said about the eigenvalues $\gamma_1, \dots, \gamma_n$ of $C$? Horn's conjecture [1962] says that the set of possible eigenvalues $\gamma_1, \dots, \gamma_n$ for $C$ is determined by a family of inequalities of the form

$$\sum_{k \in K} \gamma_k \le \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j,$$

for certain "admissible" triples $(I, J, K)$ of subsets of $\{1, 2, \dots, n\}$. Weyl inequalities [1912] are of this type. Klyachko [1998] describes these admissible triplets in terms of Schubert calculus. To a subset $I \subset \{1, \dots, n\}$ one associates a Schubert variety. The admissible triplets are those for which the associated Schubert varieties have a nonempty intersection. We will not go further in this direction. See for instance the survey paper [Bhatia 2001].

It is possible to consider Horn's problem from a probabilistic point of view (see [Frumkin and Goldberger 2006; Zuber 2018]). The set of $n \times n$ Hermitian matrices $X$ with eigenvalues $\alpha_1, \dots, \alpha_n$ is an orbit $\mathcal{O}_\alpha$ for the action of the unitary

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group $U(n)$. Assume the random Hermitian matrix $X$ to be uniformly distributed on $\mathcal{O}_\alpha$ and, independently, the matrix $Y$ uniformly distributed on $\mathcal{O}_\beta$. The question is now: what is the distribution of the eigenvalues $\gamma_1, \dots, \gamma_n$ of the sum $Z = X + Y$? We follow this approach to determine explicitly the distribution $\nu_{\alpha,\beta}$.

The proof uses the celebrated Harish-Chandra–Itzykson–Zuber integral and Heckman's measures. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ the orbit

$$\mathcal{O}_\alpha = \{U \operatorname{diag}(\alpha_1, \dots, \alpha_n) U^* \mid U \in U(n)\}$$

carries a natural probability, the orbital measure $\mu_\alpha$. The Fourier–Laplace transform of $\mu_\alpha$ is given by the Harish-Chandra–Itzykson–Zuber formula. Heckman's measure $M_\alpha$ is the projection of the orbital measure $\mu_\alpha$ on the space of diagonal matrices. Heckman [1982] studied this measure in a more general setting and gave an explicit formula for it. Our main result is an explicit formula for the distribution $\nu_{\alpha,\beta}$ (Theorem 4.1):

$$\nu_{\alpha,\beta} = C_n V_n(x) \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \delta_{\sigma(\alpha)} * M_\beta,$$

where $V_n$ denotes the Vandermonde polynomial in $n$ variables,

$$V_n(x) = \prod_{i<j} (x_i - x_j),$$

and $\mathfrak{S}_n$ is the symmetric group which acts on $\mathbb{R}^n$ as follows:

$$\sigma((x_1, \dots, x_n)) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

The support $S(\alpha, \beta)$ of the measure $\nu_{\alpha,\beta}$ is the set of possible systems of eigenvalues for the matrix $C = A + B$, if $\alpha_1, \dots, \alpha_n$ are the eigenvalues of $A$ and $\beta_1, \dots, \beta_n$ the eigenvalues of $B$.

Horn's problem is related to representation theory. If $\alpha$ and $\beta$ are highest weights of two irreducible representations $\pi_\alpha$ and $\pi_\beta$ of the unitary group $U(n)$, the spectrum of the tensor product $\pi_\alpha \otimes \pi_\beta$ is contained in the support of $\nu_{\alpha,\beta}$. But we will not consider this aspect of Horn's problem. See [Fulton 1998; 2000; Knutson and Tao 1999; Knutson et al. 2004].

We introduce in Section 1 the orbital measures on the space of Hermitian matrices and the radial part of a measure which is invariant under the action of the unitary group. In Section 2 we recall the Harish-Chandra–Itzykson–Zuber integral and, in Section 3, some properties of Heckman's measures. We state and prove our main result in Section 4. The case of a rank-one matrix $B$ is considered in Section 5, and our result is compared to results of Frumkin and Goldberger [2006]. In Section 6 we give some formulas related to the case of $2 \times 2$ real symmetric matrices. We conclude with a few remarks.

## 1. Orbital measures

Let $\mathcal{H}_n(\mathbb{R}) = \mathrm{Sym}(n, \mathbb{R})$, the space of $n \times n$ real symmetric matrices, and $\mathcal{H}_n(\mathbb{C}) = \mathrm{Herm}(n, \mathbb{C})$, the space of $n \times n$ Hermitian matrices. For a matrix $X \in \mathcal{H}_n(\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) the classical spectral theorem says that the eigenvalues are real and the corresponding eigenspaces are orthogonal. We will denote by $D_n$ the space of real diagonal matrices, $D_n \simeq \mathbb{R}^n$, and define the chamber

$$C_n = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid t_1 \geq t_2 \geq \cdots \geq t_n\}.$$

Let $U_n(\mathbb{R}) = O(n)$, the orthogonal group, and $U_n(\mathbb{C}) = U(n)$, the unitary group. The group $U_n(\mathbb{F})$ acts on the space $\mathcal{H}_n(\mathbb{F})$ by the transformations $X \mapsto UXU^*$ ($U \in U_n(\mathbb{F})$). Let $\mathcal{O}_\alpha$ denote the orbit of the diagonal matrix $A = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$ with $(\alpha_1, \dots, \alpha_n) \in C_n$:

$$\mathcal{O}_\alpha = \{UAU^* \mid U \in U_n(\mathbb{F})\}.$$

From the spectral theorem it follows that

$$\mathcal{O}_\alpha = \{X \in \mathcal{H}_n(\mathbb{F}) \mid \mathrm{spectrum}(X) = \{\alpha_1, \dots, \alpha_n\}\}.$$

The orbit $\mathcal{O}_\alpha$ carries a natural probability measure: the orbital measure $\mu_\alpha$, which is the image of the normalized Haar measure $\omega$ of the compact group $U_n(\mathbb{F})$ under the map

$$U_n(\mathbb{F}) \to \mathcal{H}_n(\mathbb{F}), \quad U \mapsto UAU^*.$$

For a continuous function $f$ on $\mathcal{O}_\alpha$,

$$\int_{\mathcal{O}_\alpha} f(X)\mu_\alpha(dX) = \int_{U_n(\mathbb{F})} f(UAU^*)\omega(dU).$$

Let $\mu$ be a measure on $\mathcal{H}_n(\mathbb{F})$ which is invariant under $U_n(\mathbb{F})$. The integral of a function $f$ can be decomposed as follows

$$\int_{\mathcal{H}_n(\mathbb{F})} f(X)\mu(dX) = \int_{\mathbb{R}^n} \left( \int_{U_n(\mathbb{F})} f(U \,\mathrm{diag}(t_1, \dots, t_n)U^*)\omega(dU) \right) \nu(dt),$$

where $\nu$ is a measure on $\mathbb{R}^n$ which is invariant under the symmetric group $\mathfrak{S}_n$. For a function $F$ on $\mathbb{R}^n$ and $\sigma \in \mathfrak{S}_n$

$$\int_{\mathbb{R}^n} F(t_{\sigma(1)}, \dots, t_{\sigma(n)})\nu(dt) = \int_{\mathbb{R}^n} F(t_1, \dots, t_n)\nu(dt).$$

The measure $\nu$ is called the *radial part* of the measure $\mu$. If $\mu$ is a probability measure on $\mathcal{H}_n(\mathbb{F})$ which is $U_n(\mathbb{F})$-invariant, its radial part $\nu$ is the joint distribution of the eigenvalues of a random matrix $X$ whose distribution is the measure $\mu$. For instance, the radial part $\nu_\alpha$ of the orbital measure $\mu_\alpha$ is

$$\nu_\alpha = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \delta_{\sigma(\alpha)},$$

where $\sigma(\alpha) = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)})$. If the measure $\mu$ has a density $h$ with respect to the Lebesgue measure $m$ on the real vector space $\mathcal{H}_n(\mathbb{F})$, $\mu(dX) = h(X)m(dX)$, then, by the Weyl integration formula,

$$\nu(dt) = Ch(t)|V_n(t)|^d dt_1 \cdots dt_n,$$

where $V_n$ is the Vandermonde polynomial,

$$V_n(t) = \prod_{1 \le i < j \le n} (t_i - t_j),$$

$d = 1$ if $\mathbb{F} = \mathbb{R}$, $d = 2$ if $\mathbb{F} = \mathbb{C}$, and $C$ is a constant which depends on $d$ and $n$. In this paper the radial part $\nu$ is defined as a $\mathfrak{S}_n$-invariant measure on $\mathbb{R}^n$. It is more usual to define the radial part as a measure on the chamber $C_n$. This is a slight difference, but responsible, in some explicit formulas, for the appearance of a factor $n!$ which does not show up in some other papers.

Assume that the random Hermitian matrix $X$ is uniformly distributed on the orbit $\mathcal{O}_\alpha$, i.e., according to the orbital measure $\mu_\alpha$, and, independently the random Hermitian matrix $Y$ is uniformly distributed on $\mathcal{O}_\beta$, i.e., according to $\mu_\beta$. Then the sum $Z = X + Y$ is distributed according to the convolution product $\mu_\alpha * \mu_\beta$ and the joint distribution of the eigenvalues of $Z$ is equal to the radial part $\nu_{\alpha,\beta}$ of $\mu_\alpha * \mu_\beta$. In case of $\mathbb{F} = \mathbb{C}$ we will determine explicitly the measure $\nu_{\alpha,\beta}$ by using Fourier analysis (Theorem 4.1).

## 2. Fourier–Laplace transform

The Fourier–Laplace transform of a bounded measure $\mu$ on $\mathcal{H}_n(\mathbb{F})$ is given by

$$\mathcal{F}\mu(Z) = \int_{\mathcal{H}_n(\mathbb{F})} e^{\mathrm{tr}(ZX)} \mu(dX).$$

The function $\mathcal{F}\mu$ is defined on $i\mathcal{H}_n(\mathbb{F})$. If the support of $\mu$ is compact, then $\mathcal{F}\mu$ is defined on $\mathrm{Sym}(n, \mathbb{C})$ if $\mathbb{F} = \mathbb{R}$, on $M_n(n, \mathbb{C})$ if $\mathbb{F} = \mathbb{C}$. If the measure $\mu$ is $U_n(\mathbb{F})$-invariant, its Fourier–Laplace transform $\mathcal{F}\mu$ is $U_n(\mathbb{F})$-invariant as well and determined by its restriction to the space $D_n$ of diagonal matrices. For

$$Z = \mathrm{diag}(z_1, \ldots, z_n), \qquad z = (z_1, \ldots, z_n) \in \mathbb{C}^n,$$
$$T = \mathrm{diag}(t_1, \ldots, t_n), \qquad t = (t_1, \ldots, t_n) \in \mathbb{R}^n,$$

define the function

$$\mathcal{E}_n(z,t) = \int_{U_n(\mathbb{F})} e^{\mathrm{tr}(ZUTU^*)}\omega(dU).$$

The Fourier–Laplace transform of a $U_n(\mathbb{F})$-invariant bounded measure $\mu$ can be written, for $Z = \mathrm{diag}(z_1, \dots, z_n)$,

$$\mathcal{F}\mu(Z) = \int_{\mathbb{R}^n} \mathcal{E}_n(z,t)\nu(dt),$$

where $\nu$ is the radial part of $\mu$. Observe that the Fourier–Laplace transform of the orbital measure $\mu_\alpha$ is given by

$$\mathcal{F}\mu_\alpha(Z) = \mathcal{E}_n(z,\alpha).$$

Since $\mathcal{F}(\mu_\alpha * \mu_\beta) = \mathcal{F}\mu_\alpha \mathcal{F}\mu_\beta$, we obtain the following key relation for determining the measure $\nu_{\alpha,\beta}$.

**Proposition 2.1.** *The measure $\nu_{\alpha,\beta}$ is determined by the relation, for $z \in \mathbb{C}^n$,*

$$\int_{\mathbb{R}^n} \mathcal{E}_n(z,t)\nu_{\alpha,\beta}(dt) = \mathcal{E}_n(z,\alpha)\mathcal{E}_n(z,\beta).$$

This relation is nothing but the product formula for the spherical functions of the following Gelfand pair $(G, K)$:

$$G = U_n(\mathbb{F}) \ltimes \mathcal{H}_n(\mathbb{F}), \qquad K = U_n(\mathbb{F}).$$

The group $G$ acts on $\mathcal{H}_n(\mathbb{F})$ by the transformations

$$g \cdot X = UXU^* + A \quad (g = (U, A)).$$

A function $f$ on $G$ which is $K$-biinvariant can be seen as a $U_n(\mathbb{F})$-invariant function on $\mathcal{H}_n(\mathbb{F})$ and such a function only depends on the eigenvalues. Hence we can identify a $K$-biinvariant function $f$ on $G$ to a $\mathfrak{S}_n$-invariant function $F$ on $\mathbb{R}^n$:

$$f(g) = F(t_1, \dots, t_n),$$

if $t_1, \dots, t_n$ are the eigenvalues of $g \cdot 0$. The spherical functions of the Gelfand pair $(G, K)$ are given by

$$\varphi_z(g) = \mathcal{E}_n(z,t) \quad (t = (t_1, \dots, t_n),\ z \in \mathbb{C}^n).$$

They satisfy the functional equation:

$$\int_K \varphi_z(g_1 U g_2)\omega(dU) = \varphi_z(g_1)\varphi_z(g_2) \quad (g_1, g_2 \in G).$$

With the identifications

$$\varphi_z(g_1) = \mathcal{E}_n(z,\alpha), \qquad \varphi_z(g_2) = \mathcal{E}_n(z,\beta)$$

the functional equation can be written as

$$\int_{\mathbb{R}^n} \mathcal{E}_n(z,t)\nu_{\alpha,\beta}(dt) = \mathcal{E}_n(z,\alpha)\mathcal{E}_n(z,\beta).$$

For this viewpoint see the inspiring paper [Berezin and Gelfand 1962]. See also the recent paper [Kuijlaars and Román 2016]. Closely related are the papers [Dooley et al. 1993; Graczyk and Sawyer 2002], and Section 7 in [Rösler 2003].

In the case $\mathbb{F} = \mathbb{C}$, there is an explicit formula for $\mathcal{E}_n(z,t)$, the Harish-Chandra–Itzykson–Zuber formula [Itzykson and Zuber 1980]. In fact it is a special case of a formula established by Harish-Chandra [1957] for the adjoint action of a compact Lie group on its Lie algebra.

**Theorem 2.2.** *Let $A, B, \in \mathcal{H}_n(\mathbb{C})$ with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$. Then*

$$\int_{U_n(\mathbb{C})} e^{\operatorname{tr}(AUBU^*)}\omega(dU) = \delta_n! \frac{1}{V_n(\alpha)V_n(\beta)} \det(e^{\alpha_i\beta_j})_{1\le i,j\le n},$$

*where $\delta_n = (n-1, n-2, \ldots, 1, 0)$, $\delta_n! = (n-1)!(n-2)!\cdots 2!$.*

Then we get

$$\mathcal{E}_n(z,t) = \delta_n! \frac{1}{V_n(z)V_n(t)} \det(e^{z_it_j})_{1\le i,j\le n}.$$

The formula can be seen as the Fourier–Laplace transform of an orbital measure:

$$\mathcal{F}\mu_\alpha(Z) = \delta_n! \frac{1}{V_n(z)V_n(\alpha)} \det(e^{z_i\alpha_j})_{1\le i,j\le n},$$

for $Z = \operatorname{diag}(z_1, \ldots, z_n)$.

## 3. Heckman's measure

Let us consider the projection $q$ of the space $\mathcal{H}_n(\mathbb{F})$ onto the subspace $D_n \simeq \mathbb{R}^n$ of real diagonal matrices,

$$q : \mathcal{H}_n(\mathbb{F}) \to \mathbb{R}^n, \quad X \mapsto (x_1, \ldots, x_n), \quad x_i = X_{ii}.$$

Recall Horn's convexity theorem [1954]: the image $q(\mathcal{O}_\alpha)$ of the orbit $\mathcal{O}_\alpha$ is equal to the convex hull $C(\alpha)$ of the points $\sigma(\alpha)$,

$$q(\mathcal{O}_\alpha) = C(\alpha) := \operatorname{Conv}(\{\sigma(\alpha) \mid \sigma \in \mathfrak{S}_n\}).$$

From now on, in this section, we assume $\mathbb{F} = \mathbb{C}$. The image $M_\alpha = q(\mu_\alpha)$ of the orbital measure $\mu_\alpha$ is called Heckman's measure. In fact this measure has been described by Heckman [1982] in a more general setting (see also [Duflo et al. 1984]). The measure $M_\alpha$ has support $q(\mathcal{O}_\alpha)$ which is contained in the hyperplane $x_1 + \cdots + x_n = \alpha_1 + \cdots + \alpha_n$. It is symmetric, i.e., invariant under the group $\mathfrak{S}_n$, acting by permuting the coordinates. If the eigenvalues $\alpha_1, \dots, \alpha_n$ are distinct, Heckman's measure $M_\alpha$ is absolutely continuous with respect to the Lebesgue measure of this hyperplane and its density is piecewise polynomial. These facts have been established by Heckman. Let us recall their proof in the present special case. For a bounded measure $M$ on $\mathbb{R}^n$ we will denote by $\widehat{M}$ its Fourier–Laplace transform:

$$\widehat{M}(z) = \int_{\mathbb{R}^n} e^{(z|x)} M(dx).$$

For $\alpha \in \mathbb{R}^n$ with the $\alpha_i$ all distinct, define the skew-symmetric measure

$$\eta_\alpha = \frac{\delta_n!}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \delta_{\sigma(\alpha)}.$$

The Fourier–Laplace transform of $\eta_\alpha$ is given by

$$\widehat{\eta_\alpha}(z) = \frac{\delta_n!}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) e^{(z|\sigma(\alpha))} = \frac{\delta_n!}{V_n(\alpha)} \det(e^{z_i \alpha_j})_{1 \le i,j \le n}.$$

The map $\alpha \mapsto \eta_\alpha$ extends as a continuous map $\mathbb{R}^n \to \mathcal{E}'(\mathbb{R}^n)$, the space of distributions on $\mathbb{R}^n$ with compact support. In particular

$$\eta_0 = V_n\left(\frac{\partial}{\partial x}\right)\delta_0.$$

**Proposition 3.1.** *Heckman's measure $M_\alpha$ satisfies the following equation*

$$V_n\left(-\frac{\partial}{\partial x}\right) M_\alpha = \eta_\alpha.$$

*Proof.* For a bounded measure $\mu$ on $\mathcal{H}_n(\mathbb{C})$, the Fourier–Laplace transform of the projection $M = q(\mu)$ of $\mu$ on $D_n$ is equal to the restriction to $D_n$ of the Fourier–Laplace transform of $\mu$: $\widehat{M}(z) = \mathcal{F}\mu(Z)$, for $Z = \operatorname{diag}(z_1, \dots, z_n)$. Hence

$$\widehat{M_\alpha}(z) = \mathcal{F}\mu_\alpha(Z) = \mathcal{E}_n(z, \alpha).$$

Therefore, by the Harish-Chandra–Itzykson–Zuber formula (Theorem 2.2),

$$\widehat{M_\alpha}(z) = \delta_n! \frac{1}{V_n(\alpha) V_n(z)} \det(e^{z_i \alpha_j})_{1 \le i,j \le n} = \frac{1}{V_n(z)} \widehat{\eta_\alpha}(z).$$

This equality, which can be written $V_n(z)\widehat{M_\alpha}(z) = \widehat{\eta_\alpha}(z)$, means an equality between two Fourier–Laplace transforms of compactly supported distributions, and implies the following differential equation

$$V_n\left(-\frac{\partial}{\partial x}\right) M_\alpha = \eta_\alpha. \qquad \square$$

For solving this equation we will use an elementary solution of the differential operator $V_n\big(\frac{\partial}{\partial x}\big)$. Let us define the distribution $E_n$ on $\mathbb{R}^n$:

$$\langle E_n, \varphi\rangle = \int_{\mathbb{R}_+^{n(n-1)/2}} \varphi\Big(\sum_{i<j} t_{ij}\varepsilon_{ij}\Big)\, dt_{ij},$$

where $\varepsilon_{ij} = e_i - e_j$ ($\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$).

**Proposition 3.2.** *The distribution $E_n$ is an elementary solution of the differential operator $V_n\big(\frac{\partial}{\partial x}\big)$:*

$$V_n\Big(\frac{\partial}{\partial x}\Big) E_n = \delta_0.$$

*The support of $E_n$ is the convex cone in the hyperplane $x_1 + \cdots + x_n = 0$ generated by the vectors $\varepsilon_{ij}$, with $i < j$. The distribution $E_n$ is absolutely continuous with respect to the Lebesgue measure of the hyperplane $x_1 + \cdots + x_n = 0$. The cone* $\mathrm{supp}(E_n)$ *decomposes into a finite union of cones, and the restriction of the density to each of these cones is a polynomial, homogeneous of degree $\frac{1}{2}(n-1)(n-2)$.*

*Proof.* The differential operator $V_n\big(\frac{\partial}{\partial x}\big)$ is a product of degree one differential operators:

$$V_n\Big(\frac{\partial}{\partial x}\Big) = \prod_{i<j}\Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Big).$$

An elementary solution of $\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}$ is the Heaviside distribution $Y_{ij}$ defined by

$$\langle Y_{ij}, \varphi\rangle = \int_0^\infty \varphi(t\varepsilon_{ij})\, dt.$$

Hence the convolution product

$$E_n = \prod_{i<j}^{*} Y_{ij}$$

is an elementary solution of $V_n\big(\frac{\partial}{\partial x}\big)$. □

For a function $\varphi$ define $\check{\varphi}(x) = \varphi(-x)$, and for a distribution $T$, $\langle \check{T}, \varphi\rangle = \langle T, \check{\varphi}\rangle$.

**Theorem 3.3.** *The Heckman measure $M_\alpha$ is given by*

$$M_\alpha = \check{E}_n * \eta_\alpha.$$

*If the $\alpha_i$ are all distinct, the measure $M_\alpha$ is absolutely continuous with respect to the Lebesgue measure of the hyperplane $x_1+\cdots+x_n = \alpha_1+\cdots+\alpha_n$ and the density is piecewise polynomial. This density is continuous for $n \geq 3$. The map $\alpha \mapsto M_\alpha$ extends as a continuous map $\mathbb{R}^n \to \mathcal{M}_c^1(\mathbb{R}^n)$, the set of probability measures on $\mathbb{R}^n$ with compact support.*

*Proof.* Let $F$ and $G$ be distributions on $\mathbb{R}^n$. Assume the support of $F$ to be compact. Let $D = P\big(\frac{\partial}{\partial x}\big)$ be a differential operator with constant coefficients. Then

$$DF * G = F * DG = D(F * G).$$

Therefore

$$\check{E}_n * V_n\left(-\frac{\partial}{\partial x}\right)M_\alpha = V_n\left(-\frac{\partial}{\partial x}\right)\check{E}_n * M_\alpha = M_\alpha.$$

By Proposition 3.1,

$$V_n\left(-\frac{\partial}{\partial x}\right)M_\alpha = \eta_\alpha.$$

Hence

$$M_\alpha = \check{E}_n * \eta_\alpha.$$

□

**Example 3.4.** $n = 2$ The elementary solution $E_2$ is given by

$$\langle E_2, \varphi\rangle = \int_0^\infty \varphi(t\varepsilon_{1,2})\, dt.$$

In the present case

$$\mathfrak{S}_2 = \{\mathrm{Id}, \tau\}, \qquad \tau : (x_1, x_2) \mapsto (x_2, x_1).$$

By Theorem 3.3,

$$\begin{aligned}\langle M_\alpha, \varphi\rangle &= \frac{1}{\alpha_1 - \alpha_2}\left(\int_0^\infty \varphi(\alpha - t_1\varepsilon_{1,2})\, dt_1 - \int_0^\infty \varphi(\tau(\alpha) - t_2\varepsilon_{1,2})\, dt_2\right).\\ &= \int_0^1 \varphi((1-t)\alpha + t\tau(\alpha))\, dt.\end{aligned}$$

The support of $M_\alpha$ is the segment $[\alpha, \tau(\alpha)]$.

**Example 3.5.** $n = 3$ The elementary solution $E_3$ is given by

$$\begin{aligned}\langle E_3, \varphi\rangle &= \int_{(\mathbb{R}_+)^3} \varphi(u\varepsilon_{1,2} + v\varepsilon_{2,3} + w\varepsilon_{1,3})\, du\, dv\, dw\\ &= \int_{(\mathbb{R}_+)^3} \varphi((u+w)\varepsilon_{1,2} + (v+w)\varepsilon_{2,3})\, du\, dv\, dw.\\ &= \int_{\{0\le w\le s, 0\le w\le t\}} \varphi(s\varepsilon_{1,2} + t\varepsilon_{2,3})\, ds\, dt\, dw\\ &= \int_{(\mathbb{R}+)^2} \inf(s,t)\varphi(s\varepsilon_{1,2} + t\varepsilon_{2,3})\, ds\, dt.\end{aligned}$$

Hence the support of $E_3$ is the angle defined by the rays generated by $\varepsilon_{1,2}$ and $\varepsilon_{2,3}$ with density, if $x = s\varepsilon_{1,2} + t\varepsilon_{2,3}$, $f(x) = \inf(s,t)$.

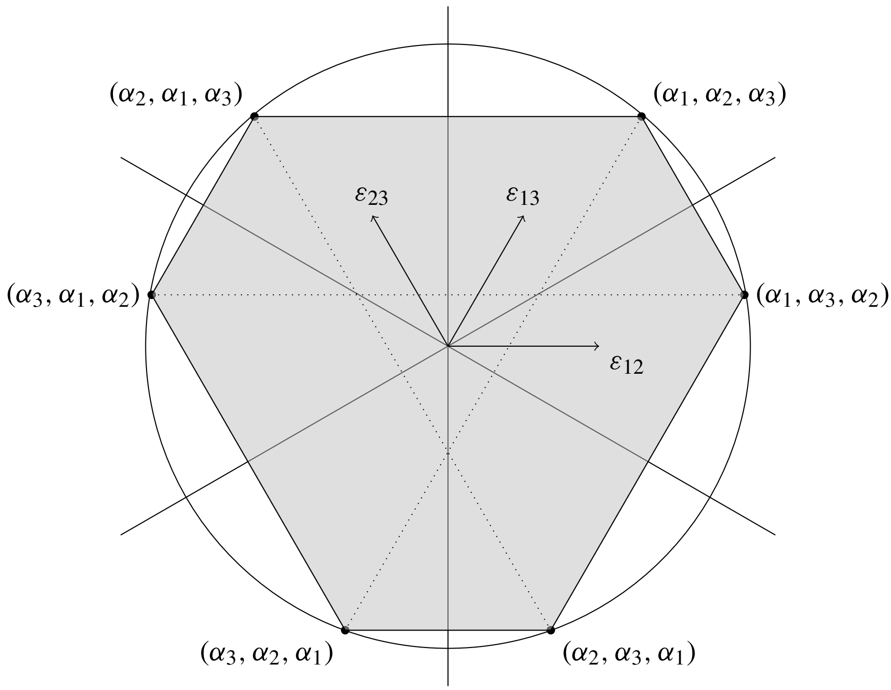


**Figure 1.** *Heckman's measure*, $n = 3$. For $\alpha_1 > \alpha_2 > \alpha_3$, the support of the measure $M_\alpha$ is the convex hull of the six points $\sigma(\alpha)$ ($\sigma \in \mathfrak{S}_3$). The density of $M_\alpha$ is affine linear in the three trapezia (trapezoids) around the rim and in the intervening triangles, and constant in the central triangle.

## 4. The radial part of the convolution product of two orbital measures

Recall that $\nu_{\alpha,\beta}$ denotes the radial part of the convolution product $\mu_\alpha * \mu_\beta$. (The convolution is with respect to $\mathcal{H}_n(\mathbb{F})$.) By Proposition 2.1, the measure $\nu_{\alpha,\beta}$ is determined by the relation

$$\int_{\mathbb{R}^n} \mathcal{E}_n(z,t)\nu_{\alpha,\beta}(dt) = \mathcal{E}_n(z,\alpha)\mathcal{E}_n(z,\beta).$$

**Theorem 4.1.** *Assume that $\mathbb{F} = \mathbb{C}$, the eigenvalues $\alpha_1, \dots, \alpha_n$ are distinct, and the eigenvalues $\beta_1, \dots, \beta_n$ are distinct as well. The radial part $\nu_{\alpha,\beta}$ is given by*

$$\nu_{\alpha,\beta} = \frac{1}{n!}\frac{1}{\delta_n!}V_n(x)\; M_\alpha * \eta_\beta = \frac{1}{n!}\frac{1}{\delta_n!}V_n(x)\; \eta_\alpha * M_\beta,$$

*or*

$$\nu_{\alpha,\beta} = \frac{1}{n!}\frac{V_n(x)}{V_n(\alpha)}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\delta_{\sigma(\alpha)} * M_\beta.$$

*The map* $(\alpha, \beta) \mapsto \nu_{\alpha,\beta}$ *extends continuously as a map* $\mathbb{R}^n \times \mathbb{R}^n \to \mathcal{M}_c^1(\mathbb{R}^n)$.

Here the convolutions are with respect to $\mathbb{R}^n$. The measure $\nu_{\alpha,\beta}$ is a $\mathfrak{S}_n$-invariant probability measure on $\mathbb{R}^n$. Observe that

$$\nu_{\alpha,0} = \nu_\alpha = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \delta_{\sigma(\alpha)}.$$

Theorem 4.1 is related to Theorem 3.4 in [Dooley et al. 1993] and to Theorem 2.1 in [Graczyk and Sawyer 2002]. A similar result, but slightly different, is given in [Rösler 2003, p.2436].

*Proof.* Define $\nu = V_n(x)\ M_\alpha * \eta_\beta$ and let us compute

$$I(z) = \int_{\mathbb{R}^n} \mathcal{E}_n(z, x)\nu\,(dx).$$

The measure $M_\alpha$ is symmetric and $\eta_\beta$ is skew symmetric, therefore $M = M_\alpha * \eta_\beta$ is skew symmetric as is its Fourier–Laplace transform $\widehat{M}$. We obtain

$$\begin{aligned} I(z) &= \frac{\delta_n!}{V_n(z)} \int_{\mathbb{R}^n} \det(e^{z_i x_j})_{1\le i,j\le n} M\,(dx) \\ &= \frac{\delta_n!}{V_n(z)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \int_{\mathbb{R}^n} e^{(\sigma(z)|x)} M\,(dx) \\ &= \frac{\delta_n!}{V_n(z)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \widehat{M}(\sigma(z)) = \frac{\delta_n!}{V_n(z)} n! \widehat{M}(z). \end{aligned}$$

Since

$$\widehat{M}(z) = \widehat{M_\alpha}(z)\widehat{\eta_\beta}(z) = \mathcal{E}_n(z, \alpha) \frac{\delta_n!}{V_n(\beta)} \det(e^{z_i \beta_j})_{1\le i,j\le n},$$

we obtain

$$I(z) = n!\delta_n!\mathcal{E}(z, \alpha)\mathcal{E}(z, \beta).$$

By Proposition 2.1 this proves the formula of Theorem 4.1. □

Recall that $S(\alpha, \beta)$ denotes the support of the measure $\nu_{\alpha,\beta}$. The $\mathfrak{S}_n$-invariant compact set $S(\alpha, \beta) \subset \mathbb{R}^n$ is the set of possible systems of eigenvalues for $C = A + B$, if $\alpha_1, \dots, \alpha_n$ are the eigenvalues of $A$ and $\beta_1, \dots, \beta_n$ the eigenvalues of $B$.

**Corollary 4.2.** (i) *We have the following inclusion*:

$$S(\alpha, \beta) \subset \bigcup_{\sigma \in \mathfrak{S}_n} (\sigma(\alpha) + C(\beta)).$$

(ii) *If*

$$\min_{i<j}(\alpha_i - \alpha_j) \ge \max_{k,\ell} |\beta_k - \beta_\ell|,$$

*then*:

$$S(\alpha,\beta)\cap C_n=\alpha+C(\beta).$$

Recall that $C(\beta)$ is the convex hull of the points $\sigma(\beta)$ ($\sigma\in\mathfrak{S}_n$), and $C_n$ is the chamber:

$$C_n=\{t=(t_1,\dots,t_n)\in\mathbb{R}^n\mid t_1\geq\cdots\geq t_n\}.$$

Part (i) is related to Lidskii's theorem [1950] and can be equivalently written as a system of inequalities

$$\sum_{k\in K}x_k\leq\sum_{i\in I}\alpha_i+\sum_{j\in J}\beta_j,$$

with suitable triples $\{I,J,K\}$. See [Bhatia 2001, p.295; 1997, Theorem II.1.10].

*Proof.*

(a) The support of the measure $\eta_\alpha$ is the orbit of $\alpha$ under the action of $\mathfrak{S}_n$,

$$\operatorname{supp}(\eta_\alpha)=\{\sigma(\alpha)\mid\sigma\in\mathfrak{S}_n\},$$

and, by Horn's Theorem, the support of Heckman's measure $M_\beta$ is

$$\operatorname{supp}(M_\beta)=q(\mathcal{O}_\beta)=C(\beta).$$

Statement (i) follows since

$$\operatorname{supp}(\eta_\alpha*M_\beta)\subset\operatorname{supp}(\eta_\alpha)+\operatorname{supp}(M_\beta).$$

In general this is an inclusion and not an equality, because the measure $\eta_\alpha$ has positive and negative parts, and cancellations are possible.

(b) Under the condition

$$\min_{i<j}(\alpha_i-\alpha_j)>\max_{k,\ell}|\beta_k-\beta_\ell|,$$

the sets $\sigma(\alpha)+C(\beta)$ are disjoint and there is one of them in each chamber $\sigma(C_n)$ ($\sigma\in\mathfrak{S}_n$). Hence no cancellation is possible. □

Theorem 4.1 can be extended as follows. For $\alpha,\beta,\gamma\in\mathbb{R}^n$, the radial part of $\mu_\alpha*\mu_\beta*\mu_\gamma$ is given by

$$\nu_{\alpha,\beta,\gamma}=\frac{1}{n!}\frac{1}{\delta_n!}V_n(x)\eta_\alpha*M_\beta*M_\gamma.$$

This generalizes to any finite convolution product. For $\alpha^{(1)},\dots,\alpha^{(k)}\in\mathbb{R}^n$, the radial part of $\mu_{\alpha^{(1)}}*\cdots*\mu_{\alpha^{(k)}}$ is given by

$$\nu_{\alpha^{(1)},\dots,\alpha^{(k)}}=\frac{1}{n!}\frac{1}{\delta_n!}V_n(x)\eta_{\alpha^{(1)}}*M_{\alpha^{(2)}}*\cdots*M_{\alpha^{(k)}}.$$

**Example 4.3.** $n = 2$ We use the same notation as in Example 3.4. We saw that

$$\langle M_\alpha, \varphi\rangle = \int_0^1 \varphi((1-t)\alpha + t\tau(\alpha))\,dt.$$

In this special case, with $a := V_2(\alpha) = \alpha_1 - \alpha_2$, the measure $\eta_\alpha$ is

$$\eta_\alpha = \tfrac{1}{a}(\delta_\alpha - \delta_{\tau(\alpha)}).$$

One can check the following formula for the Fourier–Laplace transform of $\eta_\alpha$:

$$\widehat{\eta_\alpha}(z) = e^{z_1+z_2)(\alpha_1+\alpha_2)/2}\,\tfrac{1}{a}(e^{a(z_1-z_2)/2} - e^{-a(z_1-z_2)/2}).$$

By Theorem 4.1,

$$\nu_{\alpha,\beta} = \tfrac{1}{2}V_2(x)\; M_\alpha * \eta_\beta = \tfrac{1}{2}V_2(x)\; \eta_\alpha * M_\beta.$$

Let us explicit the measure $\nu_{\alpha,\beta}$ by using the second expression:

$$\begin{aligned}\langle \nu_{\alpha,\beta}, \varphi\rangle = \frac{1}{2a}&\int_0^1 (a + (1-2t)b)\varphi((1-t)(\alpha+\beta) + t(\alpha+\tau(\beta)))\,dt\\ &+ \frac{1}{2a}\int_0^1 (a - (1-2t)b)\varphi((1-t)(\tau(\alpha)+\beta) + t(\tau(\alpha)+\tau(\beta)))\,dt,\end{aligned}$$

where $b = V_2(\beta) = \beta_1 - \beta_2$. The support $S(\alpha, \beta)$ of $\nu_{\alpha,\beta}$ is the union of two segments. If $a < b$, then

$$S(\alpha, \beta) = [\alpha+\beta, \alpha+\tau(\beta)] \cup [\tau(\alpha)+\beta, \tau(\alpha)+\tau(\beta)].$$

If $a < b$, there are some cancellations and one obtains

$$S(\alpha, \beta) = [\alpha+\beta, \tau(\alpha)+\beta] \cup [\alpha+\tau(\beta), \tau(\alpha)+\tau(\beta)],$$

and one checks the symmetry $\nu_{\beta,\alpha} = \nu_{\alpha,\beta}$.

## 5. The case of a rank-one matrix $B$

In this section we consider the special case of a rank-one matrix $B$. In such a case $\beta = (b, 0, \dots, 0)$ with $b > 0$ or $\beta = (0, \dots, 0, b)$ with $b < 0$. We assume first that $\beta = (1, 0, \dots, 0)$. The orbit $\mathcal{O}_\beta$ is the set of Hermitian matrices $Y = (u_i\bar{u}_j)$, where $u = (u_1, \dots, u_n)$ is a unit vector, $u \in S(\mathbb{F}^n)$. In case of $\mathbb{F} = \mathbb{R}$, the orbit $\mathcal{O}_\beta$ can be identified with $S(\mathbb{R}^n)/\{+1, -1\}^n$ and, in case of $\mathbb{F} = \mathbb{C}$, with $S(\mathbb{C}^n)/\mathbb{T}^n$.

Recall that $q$ denotes the projection $q : \mathcal{H}_n(\mathbb{F}) \to D_n \simeq \mathbb{R}^n$. Then

$$q(\mathcal{O}_\beta) = \{(|u_1|^2, \dots, |u_n|^2) \mid u \in S(\mathbb{F}^n)\}$$

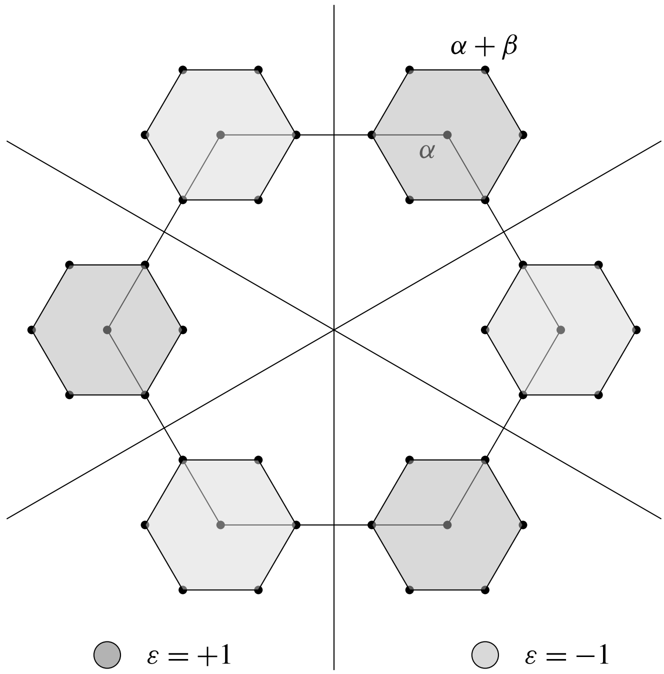


**Figure 2.** *Support* $S(\alpha, \beta)$ *of* $\nu_{\alpha,\beta}$, $\alpha = (3, 0, -3)$, $\beta = (1, 0, -1)$. The support is the union of the six hexagons.

is the simplex $\Sigma_n = \text{Conv}(e_1, \ldots, e_n)$ contained in the hyperplane $x_1 + \cdots + x_n = 1$. The orbital measure $\mu_\beta$ is the image of the normalized uniform measure on the sphere $S(\mathbb{F}^n)$.

We assume that $\mathbb{F} = \mathbb{C}$ for the rest of this section.

**Proposition 5.1.** *Heckman's measure* $M_\beta = q(\mu_\beta)$ *is the normalized uniform measure on the simplex* $\text{Conv}(e_1, \ldots, e_n)$, *i.e., the normalized restriction to the simplex* $\Sigma_n$ *of the Lebesgue measure of the hyperplane* $x_1 + \cdots + x_n = 1$.

*Proof.* The image of the normalized uniform measure on the sphere $S(\mathbb{C}^n)$ under the map

$$S(\mathbb{C}^n) \to \Sigma_n, \quad u \mapsto (|u_1|^2, \ldots, |u_n|^n),$$

is the normalized restriction to $\Sigma_n$ of the Lebesgue measure on the hyperplane $x_1 + \cdots + x_n = 1$. □

Consider on the hyperplane $x_1 + \cdots + x_n = 1$ the differential form

$$w = dx_1 \wedge \cdots \wedge dx_{n-1}.$$

Then

$$\int_{\Sigma_n} w = \frac{1}{(n-1)!},$$

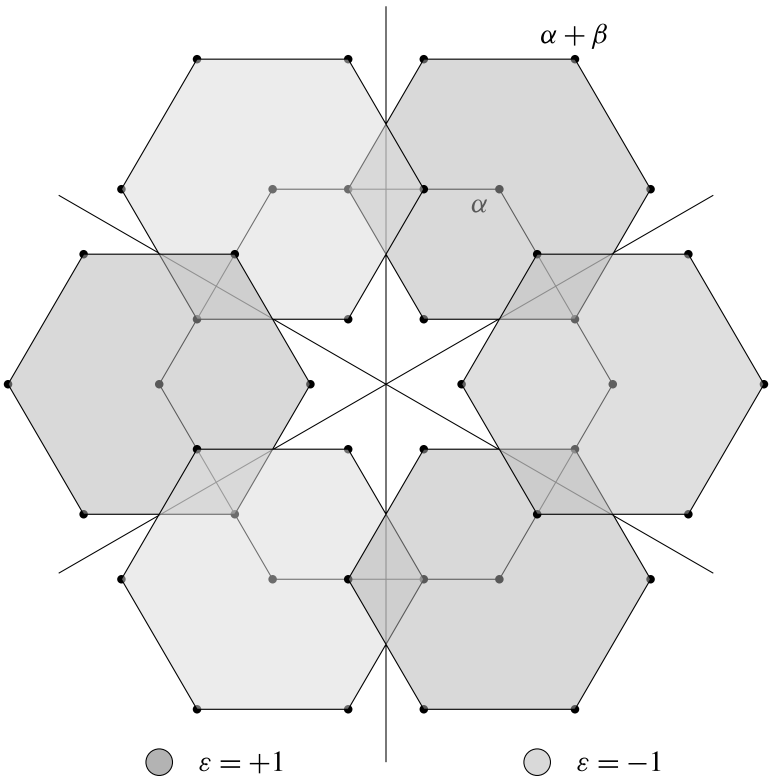


**Figure 3.** *Support* $S(\alpha, \beta)$ *of* $\nu_{\alpha,\beta}$, $\alpha = (3, 0, -3)$, $\beta = (2, 0, -2)$.
The support is the union of the six hexagons.

and Heckman's measure $M_\beta$ can be given by

$$\langle M_\beta, \varphi\rangle = (n-1)! \int_{\Sigma_n} \varphi(x) w.$$

Whereas it will not be used in the sequel we give a formula for the Fourier–Laplace transform of Heckman's measure $M_\beta$ in this special case:

$$\widehat{M_\beta}(z) = \int_{\mathbb{R}^n} e^{(z|x)} M_\beta\,(dx) = (n-1)! \frac{1}{V_n(z)} \begin{vmatrix} e^{z_1} & \cdots & e^{z_n} \\ z_1^{n-2} & \cdots & z_n^{n-2} \\ \vdots & & \vdots \\ z_1 & \cdots & z_n \\ 1 & \cdots & 1 \end{vmatrix}.$$

(This formula can be obtained by using Theorem 4.1 in [Faraut 2015].)

Recall that, for $\alpha = (\alpha_1, \dots, \alpha_n) \in C_n$, $\nu_{\alpha,\beta}$ denotes the radial part of the measure $\mu_\alpha * \mu_\beta$. The following result has been obtained by Frumkin and Goldberger [2006, Theorem 6.1 and Theorem 6.7].

**Theorem 5.2.** *Assume that* $\beta = (b, 0, \dots, 0)$ *with* $b > 0$.

(i) *The support* $S(\alpha, \beta)$ *of* $\nu_{\alpha,\beta}$ *is given by*

$$S(\alpha, \beta) \cap C_n = \{x \in \mathbb{R}^n \mid x_1 \geq \alpha_1 \geq \dots \geq x_n \geq \alpha_n,\ x_1 + \dots + x_n = \alpha_1 + \dots + \alpha_n + b\}.$$

(ii) *The measure* $\nu_{\alpha,\beta}$ *is absolutely continuous with respect to the Lebesgue measure of the hyperplane* $x_1 + \dots + x_n = \alpha_1 + \dots + \alpha_n + b$ *with the density*

$$h(x) = \frac{1}{n} \frac{1}{b^{n-1}} \frac{1}{V_n(\alpha)} V_n(x).$$

(It is assumed that the Lebesgue measure on the hyperplane $x_1 + \dots + x_n = \alpha_1 + \dots + \alpha_n + b$ is associated to the differential form $w = dx_1 \wedge \dots \wedge dx_{n-1}$.)

The inclusion

$$S(\alpha, \beta) \cap C_n \subset \{x \in \mathbb{R}^n \mid x_1 \geq \alpha_1 \geq \dots \geq x_n \geq \alpha_n,\ x_1 + \dots + x_n = \alpha_1 + \dots + \alpha_n + b\}.$$

can be found in [Horn and Johnson 1985, Theorem 4.3.4].

By Theorem 4.1, the density is given in the present case by

$$h(x) = \frac{1}{n} \frac{V_n(x)}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \frac{1}{b^{n-1}} \chi\left(\frac{x - \delta_{\sigma(\alpha)}}{b}\right),$$

where $\chi$ is the indicatrix of the simplex $\Sigma_n$.

Let us comment how Theorem 5.2 is related to Theorem 4.1 and Corollary 4.2. The conditions in (i) can be split in two parts:

$$\text{(I)} \quad x_1 \geq \alpha_1, \dots, x_n \geq \alpha_n, \quad x_1 + \dots + x_n = \alpha_1 + \dots + \alpha_n + b.$$

$$\text{(II)} \quad x_2 \leq \alpha_1, \dots, x_n \leq \alpha_{n-1}.$$

Let us introduce barycentric coordinates $s_i$:

$$x_i = \alpha_i + b s_i \quad (i = 1, \dots, n).$$

Conditions (I) gives

$$s_1 \geq 0, \dots, s_n \geq 0,\ s_1 + \dots + s_n = 1,$$

which means that $x \in \alpha + b\Sigma_n$. If

$$b \leq \alpha_{i-1} - \alpha_i \quad (i = 2, \dots, n),$$

then (I) implies (II). Therefore, in this case, $S(\alpha, \beta) \cap C_n = \alpha + b\Sigma_n$.

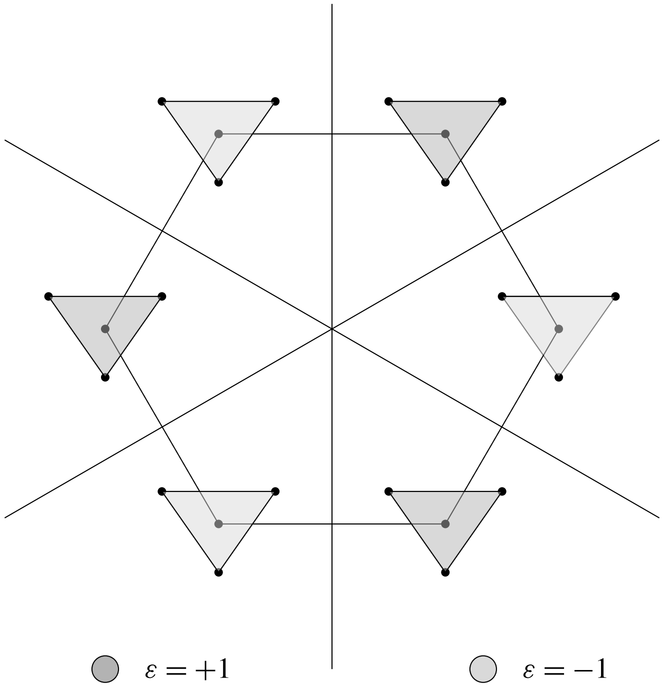


**Figure 4.** $n = 3$. *Support* $S(\alpha, \beta)$ *of* $\nu_{\alpha,\beta}$ *with* $\alpha = (3, 0, -3)$ *and* $\beta = (3, 0, 0) \sim (2, -1, -1)$. The support is the union of the six triangles.

The measure $\nu_{\alpha,\beta}$ does not change essentially if one replaces $\alpha = (\alpha_1, \dots, \alpha_n)$ by $(\alpha_1 + c, \dots, \alpha_n + c)$ and $\beta = (\beta_1, \dots, \beta_n)$ by $(\beta_1 + d, \dots, \beta_n + d)$ $(c, d \in \mathbb{R})$. We will write $(\alpha_1 + c, \dots, \alpha_n + c) \sim (\alpha_1, \dots, \alpha_n)$. Hence in this section we have considered the case where $B$ has an eigenvalue of multiplicity $n - 1$ rather than having rank one.

In general there are cancellations which should correspond to conditions (II).

## 6. Real symmetric matrices, $n = 2$

In the case of real symmetric matrices, we know explicitly Heckman's measure and the measure $\nu_{\alpha,\beta}$ only in case of $n = 2$. For $\alpha = (\alpha_1, \alpha_2)$, the orbit $\mathcal{O}_\alpha$ is the set of the matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \alpha_1 \cos^2\theta + \alpha_2 \sin^2\theta & (\alpha_1 - \alpha_2)\cos\theta\sin\theta \\ (\alpha_1 - \alpha_2)\cos\theta\sin\theta & \alpha_1 \sin^2\theta + \alpha_2 \cos^2\theta \end{pmatrix}.$$

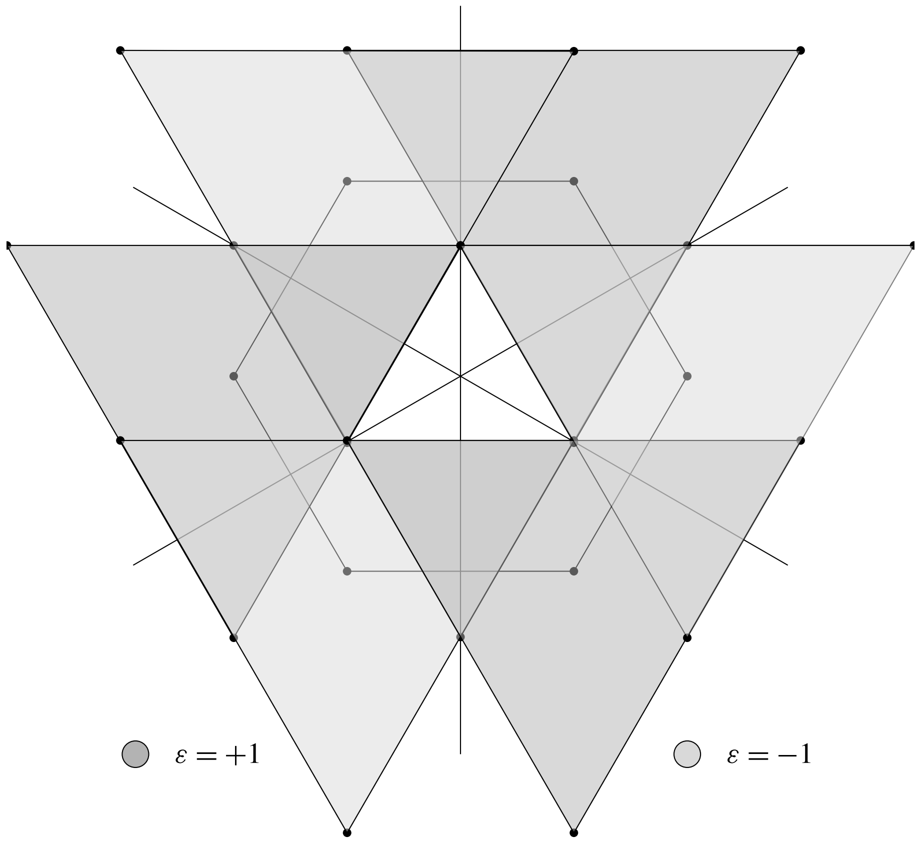


**Figure 5.** $n = 3$. *Support* $S(\alpha, \beta)$ *of* $\nu_{\alpha,\beta}$ *with* $\alpha = (3, 0, -3)$ *and* $\beta = (6, 0, 0) \sim (4, -2, -2)$. The support is the union of the six large gray triangles, minus their six intersections.

As in the case of $2 \times 2$ Hermitian matrices, the image of the orbit $\mathcal{O}_\alpha$ under the projection $q : \mathcal{H}_2(\mathbb{R}) \to D_2 \simeq \mathbb{R}^2$ is the segment $[\alpha, \tau(\alpha)]$. The projection $M_\alpha$ of the orbital measure $\mu_\alpha$ is given by

$$\begin{aligned}\langle M_\alpha, \varphi\rangle &= \frac{1}{2\pi}\int_0^{2\pi} \varphi(\alpha_1 \cos^2\theta + \alpha_2 \sin^2\theta, \alpha_1 \sin^2\theta + \alpha_2 \cos^2\theta)\, d\theta \\ &= \frac{1}{\pi}\int_0^1 \varphi((1-t)\alpha + t\tau(\alpha)) \frac{dt}{\sqrt{t(1-t)}}.\end{aligned}$$

**Proposition 6.1.** *Let* $J_0$ *be the Bessel function of index* 0. *The Fourier–Laplace transform of the orbital measure* $\mu_\alpha$ *is given*, *if* $Z = \operatorname{diag}(z_1, z_2)$, *by*

$$\mathcal{F}\mu_\alpha(iZ) = \widehat{M_\alpha}(iz) = \int_{\mathbb{R}^2} e^{i(z_1x_1+z_2x_2)} M_\alpha\,(dx) = e^{\frac{1}{2}(z_1+z_2)(\alpha_1+\alpha_2)} J_0\big(\tfrac{1}{2}(z_1-z_2)(\alpha_1-\alpha_2)\big).$$

*Proof.* By the previous formula

$$\widehat{M_\alpha}(iz) = \frac{1}{\pi}\int_0^1 e^{i(z|(1-t)\alpha+t\tau(\alpha))}\frac{dt}{\sqrt{t(1-t)}}.$$

Put $t = \frac{1}{2}(1-\cos\theta)$. Then

$$1-t = \tfrac{1}{2}(1+\cos\theta),\ \ dt = \tfrac{1}{2}\sin\theta d\theta,$$

and

$$(z \mid (1-t\alpha+t\tau(\alpha))) = \tfrac{1}{2}(z_1+z_2)(\alpha_1+\alpha_2)+\tfrac{1}{2}(z_1-z_2)(\alpha_1-\alpha_2)\cos\theta.$$

We obtain

$$\widehat{M_\alpha}(iz) = \frac{1}{\pi}e^{i(z_1+z_2)(\alpha_1+\alpha_2)/2}\int_0^\pi e^{i(z_1-z_2)(\alpha_1-\alpha_2)\cos\theta/2}\,d\theta.$$

Recall the following integral formula for the Bessel function $J_0$:

$$J_0(\zeta) = \frac{1}{\pi}\int_0^\pi e^{i\zeta\cos\theta}d\theta. \qquad \square$$

We introduce the following notation: for $\alpha = (\alpha_1, \alpha_2)$, and $\beta = (\beta_1, \beta_2)$,

$$\tau = \alpha_1+\alpha_2+\beta_1+\beta_2, \quad a = \alpha_1-\alpha_2, \quad b = \beta_1-\beta_2.$$

If $a, b, c$ are the three wedges of a triangle, we denote by $\Delta(a, b, c)$ the area of this triangle. Recall the classical formula

$$\Delta(a, b, c)^2 = p(p-a)(p-b)(p-c),$$

where $p$ is half the perimeter of the triangle.

**Theorem 6.2.** *The measure $\nu_{\alpha,\beta}$ is given by*

$$\langle\nu_{\alpha,\beta}, \varphi\rangle = \frac{1}{8\pi}\int_{|a-b|}^{a+b}\varphi\big(\tfrac{1}{2}(\tau+r)e_1+\tfrac{1}{2}(\tau-r)e_2\big)\frac{2r dr}{\Delta(a, b, r)}$$
$$+\frac{1}{8\pi}\int_{|a-b|}^{a+b}\varphi\big(\tfrac{1}{2}(\tau-r)e_1+\tfrac{1}{2}(\tau+r)e_2\big)\frac{2r\,dr}{\Delta(a, b, r)}.$$

*Proof.* Recall the product formula for the Bessel function $J_0$:

$$J_0(\zeta a)J_0(\zeta b) = \frac{1}{\pi}\int_0^\pi J_0(\zeta\sqrt{a^2+b^2+2ab\cos\theta})\,d\theta.$$

This can be written

$$J_0(\zeta a)J_0(\zeta b) = \frac{1}{\pi}\int_{|a-b|}^{a+b} J_0(\zeta r)\frac{2r\,dr}{\sqrt{(2ab)^2-(a^2+b^2-r^2)^2}}.$$

Since

$$(2ab)^2-(a^2+b^2-r^2)^2=(a+b+r)(a+b-r)(r+a-b)(r-a+b)=16\Delta(a,b,r)^2,$$

it can also be written

$$J_0(\zeta a)J_0(\zeta b)=\frac{1}{2\pi}\int_{|a-b|}^{a+b}J_0(\zeta r)\frac{r\,dr}{\Delta(a,b,r)}.$$

It follows that the function $\mathcal{E}_2(z,\alpha)$ satisfies the following product formula

$$\mathcal{E}_2(z,\alpha)\mathcal{E}_2(z,\beta)=\frac{1}{2\pi}\int_{|a-b|}^{a+b}\mathcal{E}_2(z,\rho)\frac{r\,dr}{\Delta(a,b,r)},$$

with $\rho=(\rho_1,\rho_2)$, $r=\rho_1-\rho_2$. By Proposition 2.1, this establishes Theorem 6.2. □

## Remarks

In the case of the space of real symmetric matrices $\mathcal{H}_n(\mathbb{R})$, with the action of the orthogonal group $O(n)$, for $n\geq 3$, we don't know any explicit formula for Heckman's measure, and for the measures $\nu_{\alpha,\beta}$. This setting is natural, however the problem is more difficult than in the case of the space of Hermitian matrices, and one should not expect any explicit formula. See the recent paper [Coquereaux and Zuber 2018]. However the supports should be the same as in the case of $\mathcal{H}_n(\mathbb{C})$ with the action of the unitary group $U(n)$, according to [Fulton 1998, p.265; 2000, Section 10.7].

There should be an analogue of the results presented in this paper in case of pseudo-Hermitian matrices. In this setting an analogue of Horn's conjecture has been established in [Foth 2010]. An analogue of Theorem 4.1 could probably be obtained by using a formula for the Laplace transform of an orbital measure for the action of the pseudounitary group $U(p,q)$ on the space $\mathcal{H}_n(\mathbb{C}^n)$ $(n=p+q)$. This formula is due Ben Saïd and Ørsted [2005]. A related problem has been studied by using this formula in [Faraut 2017].

More generally one could consider Horn's problem for the adjoint action of a compact Lie group on its Lie algebra. The Fourier transform of an orbital measure is explicitly given by the Harish-Chandra integral formula [1957]. Heckman's paper [1982] is written in this framework. One can expect that there is an analogue of Theorem 4.1 in this setting. In particular one can consider the action of the orthogonal group on the space of real skew-symmetric matrices. See [Zuber 2018] and, for a different problem, [Zubov 2016].

One observes some similarity between the results in [Frumkin and Goldberger 2006], stated in Theorem 5.2, and the classical Cauchy interlacing properties together with Baryshnikov's formula. See [Baryshnikov 2001; Olshanski 2013; Faraut 2015]. There should be an explanation.

## Acknowledgments

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