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# Looijenga line bundles in complex analytic elliptic cohomology

Charles Rezk

We present a calculation that shows how the moduli of complex analytic elliptic curves arises naturally from the Borel cohomology of an extended moduli space of $U(1)$-bundles on a torus. Furthermore, we show how the analogous calculation, applied to a moduli space of principal bundles for a $K(\mathbb{Z}, 2)$ central extension of $U(1)^d$, gives rise to Looijenga line bundles. We then speculate on the relation of these calculations to the construction of complex analytic equivariant elliptic cohomology.

## 1. Introduction

In this note, we describe some aspects of how complex analytic elliptic curves arise naturally from the cohomology of certain spaces which parametrize principal bundles on orientable genus-1 surfaces. This suggests how elliptic cohomology emerges from certain derived complex analytic spaces associated to dimensional reduction applied to 2-dimensional field theories.

**1.1.** ***Complex analytic elliptic cohomology.*** Complex analytic equivariant elliptic cohomology was first defined by Grojnowski [2007].[1] In its most basic formulation, given

- a compact connected abelian Lie group $G$ (i.e., $G \approx U(1)^d$), with cocharacter lattice $B = \mathrm{Hom}(U(1), G)$, and
- an elliptic curve $C_\tau = \mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$ for $\mathrm{Im}\,\tau > 0$,

he obtains an equivariant cohomology theory

$$\mathrm{Ell}_G^* : h\mathrm{Top}_G^{\mathrm{fin}} \to \mathrm{Coh}(C_\tau \otimes B)$$

on $G$-spaces homotopy equivalent to finite $G$-CW-complexes, taking values in coherent sheaves of $\mathcal{O}_{C_\tau \otimes B}$-modules on the complex analytic abelian variety $C_\tau \otimes B \approx C_\tau^d$.

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[1] Originally circulated as a preprint in 1994; see [Ando and Miller 2007].

Grojnowski describes his construction as "delocalized". That is, $\mathrm{Ell}^*_G(X)$ is produced by gluing together certain localizations of the values of Borel equivariant cohomology rings $H^*(X^H \times_G EG; \mathbb{C})$ for various subgroups $H$ of $G$. Conceptually, one can regard this as a "reverse engineered" version of a character sheaf, by analogy with the interpretation of $\mathbb{C} \otimes K_G(X)$ as a sheaf over the multiplicative group $\mathbb{G}_m$, whose localizations at various points of $\mathbb{G}_m$ are computed, in terms of standard localization theorems, in terms of Borel cohomology (e.g., as in [Baum et al. 1985]).

Grojnowski's theory has been extended and used to explain aspects of elliptic genera, notably the rigidity of the Ochanine genus [Rosu 2001], and the modularity of the Witten genus [Ando and Basterra 2002; Ando 2003]. A significant feature of this theory is the ability to twist by a *level*, which in the above formulation is described in terms of tensoring sheaves with the *Looijenga line bundle* associated to a quadratic form on the cocharacter lattice $B$ [Grojnowski 2007, §3.3]. Looijenga's theta functions appear explicitly in the Kac character formula, which can be identified with the calculation of a Gysin map in elliptic cohomology [Ando 2000; Ganter 2014].

This construction of analytic elliptic cohomology, though productive, is somewhat ad hoc, and technically rather intricate. Furthermore, we should expect more from the theory. In particular,

(1) it should take values not (merely) in sheaves on a scheme or complex analytic space, but rather in sheaves on a *derived* scheme or complex analytic space, and

(2) it should in some sense classify, or at least give invariants of, *two-dimensional reductions* of certain kinds 2-dimensional field theories.

These should nowadays be much more approachable goals than was the case when Grojnowski originally defined the theory. For point (1), there is well-developed machinery for constructing cohomology theories from *derived* geometric objects [Lurie 2009]. Furthermore, there is a direct construction of a derived algebraic scheme realizing rational equivariant elliptic cohomology for $G = U(1)$ following Grojnowski's delocalized approach [Greenlees 2005]. Point (2) is more difficult; however, there has been partial success in relating elliptic cohomology to field theory (following the program of Segal [1988]), and many features of the relationship are understood (see, e.g., [Stolz and Teichner 2011]). We note the recent work of Berwick-Evans and Tripathy [2018] in this direction.

**1.2.** ***The purpose and results of this paper.*** We are motivated by the observation that elliptic cohomology at the Tate curve should be associated to a *one-dimensional reduction* of 1-dimensional field theories. Very roughly, Tate elliptic cohomology should arise as some kind of equivariant $K$-theory for the "rotationally extended loop group" $\mathcal{L}^{\mathrm{rot}}(G)$ of $G$. By the latter[2] we really mean the topological groupoid whose objects are certain principal $G$-bundles $P \to \mathbb{T}$, over a circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and whose morphisms are maps $(P \to \mathbb{T}) \to (P' \to \mathbb{T})$ of $G$-bundles covering a rotation of the circle; when $G$ is connected, the groupoid $\mathcal{L}^{\mathrm{rot}}(G)$ is equivalent to the group $\mathrm{Map}(\mathbb{T}, G) \rtimes \mathbb{T}$.

Our point of view is inspired by that of [Ganter 2007; Ganter 2013], which considers the special case of finite groups $G$, in which case $\mathcal{L}^{\mathrm{rot}}G$ is a Lie groupoid, and thus comes with a well-defined equivariant $K$-theory. Furthermore, Kitchloo [2009] has defined a version of equivariant $K$-theory for certain Kac–Moody groups. Using this, he constructs [Kitchloo 2014], for loop groups on simple and simply connected $G$, a version of $G$-equivariant elliptic cohomology on an arbitrary complex analytic elliptic curve. It turns out that Looijenga line bundles arise naturally in this framework. Kitchloo also gives a partial modularity result [Kitchloo 2014, §6], so that the theory he constructs takes values in sheaves on a moduli stack of curves; his modularity result only applies to a restricted class of $G$-spaces.

The purpose of this note is to describe calculations inspired by the idea of *two-dimensional reduction*. Thus, (i) the circle $\mathbb{T}$ is replaced with an orientable genus-1 surface $\Sigma$ (e.g., $\mathbb{T}^2$), and (ii) equivariant $K$-theory is replaced with Borel cohomology with complex coefficients. We restrict attention to a limited class of equivariance groups $G$, namely (i) tori $G = U(1)^d$, or (ii) "central extensions" $\widetilde{G} = U(1)^d \times_\phi K(\mathbb{Z}, 2)$ of a torus $G$ by $K(\mathbb{Z}, 2)$, according to a class $\phi \in H^4(BG; \mathbb{Z})$.

We summarize our calculations as follows; precise statements are given in Sections 2 and 3. Fix

$$\Sigma = \text{orientable genus-1 surface}, \qquad G = \text{topological group},$$

and consider the "wreath product" group

$$\mathcal{W}(G) = \mathcal{W}^\Sigma(G) := \mathrm{Map}(\Sigma, G) \rtimes \mathrm{Diff}(\Sigma),$$

where $\mathrm{Diff}(\Sigma)$ is the group of diffeomorphisms[3] (not necessarily orientation preserving). Note that $\mathrm{Map}(\Sigma, G)$ is the gauge group of $\Sigma \times G \to \Sigma$, the trivial $G$-bundle over $\Sigma$, and thus $\mathcal{W}(G)$ is an "extended gauge group". Its classifying space $B\mathcal{W}(G)$ is thus a homotopy theoretic moduli space for the data (smooth genus-1 surface, trivializable principal $G$-bundle).

[2] The "rotational extension" is not to be confused with a central $U(1)$-extension of a loop group.

[3] That we use the diffeomorphism group here is not essential, since we will only use homotopy invariant features of this action. Thus, in its place we could use the homeomorphism group of $\Sigma$, or even the monoid of self-homotopy equivalences of $\Sigma$, each of which have the same homotopy type as $\mathrm{Diff}(\Sigma)$.

Let $\mathcal{W}_0(G) \subseteq \mathcal{W}(G)$ denote the identity component, with discrete quotient $\overline{\mathcal{W}}(G) = \mathcal{W}(G)/\mathcal{W}_0(G)$. Thus there is a natural action $\overline{\mathcal{W}}(G) \curvearrowright B\mathcal{W}_0(G)$ on the classifying space of the connected subgroup. For the $G$ we will consider, the cohomology ring $H^*(B\mathcal{W}_0(G); \mathbb{C})$ is concentrated in even degree, whence we obtain an action

$$(\overline{\mathcal{W}}(G) \times \mathbb{C}^\times)^{\mathrm{op}} \curvearrowright \operatorname{Spec} H^*(B\mathcal{W}_0(G); \mathbb{C})$$

on an affine complex variety, where $\mathbb{C}^\times$ acts linearly on $H^2$. Let

$$\mathcal{X}_G := [\operatorname{Spec} H^*(B\mathcal{W}_0(G); \mathbb{C})]_{\mathrm{an}} \smallsetminus \{\mathrm{bad}\},$$

which is a complex analytic space obtained as the "analytification" of the complex variety, with a certain closed subset (described in Section 2.10) removed. In our examples $\mathcal{X}_G$ is always smooth. The object we are interested in is

$$\mathcal{M}_G := (\overline{\mathcal{W}}(G) \times \mathbb{C}^\times) \backslash\backslash \mathcal{X}_G,$$

the stacky quotient in complex manifolds. We compute that

$$\begin{aligned}
\mathcal{M}_e &\approx \mathcal{M} = \text{the moduli stack of (complex analytic) elliptic curves},\\
\mathcal{M}_{U(1)} &\approx \mathcal{E} = \text{the universal elliptic curve over } \mathcal{M},\\
\mathcal{M}_{U(1)^d} &\approx \mathcal{E}^d = \mathcal{E} \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{E} = \text{the } d\text{-fold product of } \mathcal{E},\\
\mathcal{M}_{K(\mathbb{Z},2)} &\approx \mathbb{G}_m \times \mathcal{M} = \text{the multiplicative group}\\
&\qquad\qquad \text{as a trivial bundle of groups over } \mathcal{M},\\
\mathcal{M}_{U(1)^d \times_\phi K(\mathbb{Z},2)} &\approx \mathcal{P}_\phi = \text{principal } \mathbb{G}_m\text{-bundle associated to } \mathcal{L}_\phi,
\end{aligned}$$

where $\mathcal{L}_\phi \to \mathcal{E}^d$ is the "Looijenga line bundle" associated to $\phi \in H^4(BU(1)^d, \mathbb{Z})$, regarded as a quadratic function $\phi : H_2BU(1)^d = \mathbb{Z}^d \to \mathbb{Z}$. The first three cases of the computation are easy observations, and are described in Section 2. The main purpose of this paper is prove the last two cases, which are stated in Section 3.

**1.3.** ***Organization of this paper.*** In Section 2, we present the basic observation mentioned above: that the universal complex analytic elliptic curve arises *naturally* from the cohomology of certain spaces. I have not seen this observation stated in this way before; however, it is closely related to an observation by Etingof and Frenkel about coadjoint actions in double loop groups, a relationship we describe briefly in Section 2.12.

In Section 3, we replace $G = U(1)^d$ with $\widetilde{G} = U(1)^d \times_\phi K(\mathbb{Z}, 2)$, the extension associated to a class $\phi \in H^4(BG; \mathbb{Z})$, and observe that our formulation naturally gives Looijenga-type line bundles. This is stated as Theorem 3.7, which is our main result.

In Section 4 we observe how isogenies of complex analytic elliptic curves fit naturally into this story, via finite covering maps of genus-1 surfaces.

In Section 5 we speculate as to how these constructions might give rise to *derived* elliptic curves (in an analytic setting) following the pattern described in [Lurie 2009], and to elliptic cohomology theories of Grojnowski type. We only sketch a picture here; setting this up formally would involve confronting a definition of derived complex analytic space, which is beyond the scope of this note. We also describe the "stacky" dependence of our constructions on the group $G$, and note what happens in the simpler 1-dimensional case (where $\Sigma$ is a circle).

The remainder of the paper (Sections 6–9) is taken up with the proof of the main result, Theorem 3.7, which is itself obtained from a more general and coordinate invariant formulation, Theorem 7.6.

**1.4.** ***Conventions.*** At various points we need to consider the action of a group on another group (always from the left). We will sometimes use the notation $g \propto h$ for such an action, so as to typographically distinguish it from $gh$ a product of group elements. We will also occasionally use this notation for the action of a group on a space.

When $G$ acts on $H$ from the left, a semidirect product $K$ is always a group with subgroups $G$ and $H$ that $GH = HG = K$ and $G \cap H = \{1\}$, and such that $ghg^{-1} = g \propto h$. There are two distinct but canonically isomorphic constructions of such: $G \ltimes H$ and $H \rtimes G$ with group laws $(g, h) \cdot (g', h') = (gg', (g'^{-1} \propto h)h')$ and $(h, g) \cdot (h', g') = (h(g \propto h'), gg')$ respectively. In Section 10, we describe the homotopy theoretic conventions we use, primarily in order to establish the sign conventions we need in Sections 6–9.

## 2. Analytic moduli of elliptic curves vs. homotopic moduli of genus-1 surfaces

**2.1.** ***Moduli of elliptic curves over*** $\mathbb{C}$***.*** The classical uniformization theory of Weierstrass says that

(1) every elliptic curve is isomorphic, as a complex manifold, to $\mathbb{C}/\Lambda$ for some lattice $\Lambda$, (i.e., a subgroup $\Lambda = \mathbb{Z}t_1 + \mathbb{Z}t_2$ such that $\mathbb{R} \otimes \Lambda = \mathbb{C}$), with neutral element at the origin, and

(2) every map $\mathbb{C}/\Lambda \to \mathbb{C}/\Lambda'$ between such complex manifolds fixing the neutral element is given by multiplication by a complex scalar.

That is, such curves correspond to lattices in $\mathbb{C}$ up to scaling by a nonzero complex number.

This can be enriched to a description of the moduli stack of such curves. Let

$$\mathcal{X} := \{ (t_1, t_2) \mid \mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C} \} \subset \mathbb{C}^2.$$

We have a group action

$$\mathrm{GL}_2(\mathbb{Z}) \times \mathbb{C}^\times \curvearrowright \mathcal{X}$$

by

$$A \propto (t_1, t_2) = (at_1 + bt_2, ct_1 + dt_2), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}), \tag{2.2}$$

and

$$\lambda \propto (t_1, t_2) = (\lambda t_1, \lambda t_2), \quad \lambda \in \mathbb{C}^\times. \tag{2.3}$$

Points of the quotient space $(\mathrm{GL}_2(\mathbb{Z}) \times \mathbb{C}^\times)\backslash\mathcal{X}$ are in bijective correspondence to homothety-equivalence classes ($\Lambda \sim \lambda\Lambda$) of lattices, i.e., to isomorphism classes of elliptic curves. It turns out that the moduli stack is in fact the stack quotient

$$\mathcal{M} := (\mathrm{GL}_2(\mathbb{Z}) \times \mathbb{C}^\times) \mathbin{\backslash\!\backslash} \mathcal{X}.$$

For our purposes, we do not need to worry about the general notion of stacks. It is sufficient to remember that information defining $\mathcal{M}$ is precisely contained in the group action, so that (for instance), sheaves on the stack $\mathcal{M}$ are precisely equivariant sheaves on $\mathcal{X}$.

**2.4. Remark.** The stack $\mathcal{M}$ is an orbifold, though the above does not present it as such. The continuous group $\mathbb{C}^\times$ acts freely on $\mathcal{X}$, so that $\mathbb{C}^\times\backslash\mathcal{X} \approx \mathbb{C} \smallsetminus \mathbb{R}$ defined by $(t_1, t_2) \mapsto \tau = t_1/t_2$ gives an identification with the double-half plane. The residual $\mathrm{GL}_2(\mathbb{Z})$-action descends to an action on $\mathbb{C} \smallsetminus \mathbb{R}$ with finite isotropy, whence $\mathcal{M} \approx \mathrm{GL}_2(\mathbb{Z}) \mathbin{\backslash\!\backslash} (\mathbb{C} \smallsetminus \mathbb{R})$.

**2.5. Remark.** Instead of $\mathcal{X}$ we could use $\mathcal{X}^+ = \{\, (t_1, t_2) \in \mathcal{X} \mid \mathrm{Im}(t_1/t_2) > 0 \,\}$, so $\mathcal{M} \approx (\mathrm{SL}_2(\mathbb{Z}) \times \mathbb{C}^\times) \mathbin{\backslash\!\backslash} \mathcal{X}^+ \approx \mathrm{SL}_2(\mathbb{Z}) \mathbin{\backslash\!\backslash} \mathcal{H}$ where $\mathcal{H} = \{\, \tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0 \,\}$.

**2.6. *The universal elliptic curve.*** The universal elliptic curve $\mathcal{E} \to \mathcal{M}$ can be modeled by a map $C \to \mathcal{X}$, with fiber $C_{(t_1,t_2)} = \mathbb{C}/(\mathbb{Z}t_1 + \mathbb{Z}t_2)$ over $(t_1, t_2) \in \mathcal{X}$, together with a lift of the group action on $\mathcal{X}$. Since the fibers are themselves quotients by a free action, we can describe the universal curve as a stack quotient, via the action

$$(\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \times \mathbb{C}^\times \curvearrowright \mathcal{X} \times \mathbb{C} = \left\{ (t_1, t_2, y) \in \mathbb{C}^2 \times \mathbb{C} \mid \mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C} \right\}$$

defined by

$$\begin{aligned} A \propto (t_1, t_2, y) &= (at_1 + bt_2, ct_1 + dt_2, y), & A &\in \mathrm{GL}_2(\mathbb{Z}), \\ (m_1, m_2) \propto (t_1, t_2, y) &= (t_1, t_2, y + m_1 t_1 + m_2 t_2), & (m_1, m_2) &\in \mathbb{Z}^2, \\ \lambda \propto (t_1, t_2, y) &= (\lambda t_1, \lambda t_2, \lambda y), & \lambda &\in \mathbb{C}^\times. \end{aligned} \tag{2.7}$$

Thus, the stack quotient

$$\mathcal{E} := \big((\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \times \mathbb{C}^\times\big) \mathbin{\backslash\!\backslash} \mathcal{X} \times \mathbb{C}$$

presents the universal curve over $\mathcal{M}$.

**2.8.** ***Moduli of genus-1 surfaces.*** Fix a smooth surface $\Sigma$, closed and orientable of genus-1. We write

$$\mathrm{Diff}(\Sigma) \supset \mathrm{Diff}_0(\Sigma),$$

for the group of diffeomorphisms and its identity component. The classifying space $B\,\mathrm{Diff}(\Sigma)$ can be viewed as a homotopy-theoretic moduli space of orientable (but not oriented) genus-1 surfaces.

For convenience in describing calculations, we use the model $\Sigma := \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. In this case $\mathrm{Diff}(\Sigma)$ is weakly equivalent, as a topological group, to the subgroup $\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$ (acting on $\mathbb{T}^2$ in the evident way from the left) [Earle and Eells 1967].

Therefore $B\,\mathrm{Diff}_0(\Sigma) \approx B\mathbb{T}^2$, which carries an evident action by $\mathrm{GL}_2(\mathbb{Z}) = \mathrm{Diff}_+(\Sigma)/\,\mathrm{Diff}_0(\Sigma)$. We may thus consider the induced action

$$\mathrm{GL}_2(\mathbb{Z})^{\mathrm{op}} \curvearrowright H^*(B\,\mathrm{Diff}_0(\Sigma); \mathbb{C}).$$

It is immediate that

$$H^*(B\,\mathrm{Diff}_0(\Sigma); \mathbb{C}) = H^*(B\mathbb{T}^2; \mathbb{C}) \approx \mathbb{C}[t_1, t_2], \quad t_1, t_2 \in H^2,$$

with $\mathrm{GL}_2(\mathbb{Z})$ action given by the precisely the formula (2.2). The cohomology also carries a natural $\mathbb{C}^\times$ action, determined by the grading, which coincides with (2.3).

**2.9.** ***Universal degree-0 line bundle on a genus-1 surface.*** Now consider the group

$$\mathcal{W}(U(1)) := \mathrm{Map}(\Sigma, U(1)) \rtimes \mathrm{Diff}(\Sigma),$$

which has identity component $\mathcal{W}_0(U(1)) := \mathrm{Map}_0(\Sigma, U(1)) \rtimes \mathrm{Diff}_0(\Sigma)$, and set $\overline{\mathcal{W}}(U(1)) := \pi_0\mathcal{W}(U(1)) = \mathcal{W}(U(1))/\mathcal{W}_0(U(1))$. The classifying space $B\mathcal{W}(U(1))$ carries the universal example of a smooth orientable genus-one surface together with a degree-0 complex line bundle over it.

Using $\Sigma = \mathbb{T}^2$, we obtain an explicit finite dimensional model for $\mathcal{W}(U(1))$ (up to homotopy equivalence), namely

$$\big(\mathrm{Hom}(\mathbb{T}^2, U(1)) \times U(1)\big) \rtimes \big(\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2\big).$$

That is, the homomorphism $\mathrm{Hom}(\mathbb{T}^2, U(1)) \times U(1) \to \mathrm{Map}(\Sigma, U(1))$ defined by[4] $(m, y) \mapsto ((s_1, s_2) \mapsto y + m_1 s_1 + m_2 s_2)$ is a homotopy equivalence, and is invariant under the evident action of $\mathrm{GL}_2(\mathbb{Z}) \rtimes \mathbb{T}^2 \subset \mathrm{Diff}(\Sigma)$. We can rebracket this as

$$\big(\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2\big) \ltimes \big(\mathbb{T}^2 \times U(1)\big),$$

using the left action $\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \curvearrowright \mathbb{T}^2 \times U(1)$ given by

$$(A, m) \propto (t, y) = (At, y + m_1 t_1 + m_2 t_2).$$

[4]We write the group laws on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $U(1) \approx \mathbb{R}/\mathbb{Z}$ additively, and use the evident isomorphism $\mathbb{Z}^2 = \mathbb{Z}^{1\times 2} \approx \mathrm{Hom}(\mathbb{T}^2, U(1))$.

The induced action $\overline{\mathcal{W}}(U(1))^{\mathrm{op}} \curvearrowright H^*(B\mathcal{W}_0(U(1)); \mathbb{C})$ thus has the form

$$(\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2)^{\mathrm{op}} \curvearrowright H^*(B(\mathbb{T}^2 \times U(1)); \mathbb{C}).$$

We easily read off that

$$H^*\big(B(\mathbb{T}^2 \times U(1)); \mathbb{C}\big) \approx \mathbb{C}[t_1, t_2, y], \quad t_1, t_2, y \in H^2,$$

with $\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ action given precisely by the first two formulas from (2.7).[5] The grading of cohomology corresponds to the $\mathbb{C}^\times$-action from (2.7).

**2.10.** ***The geometric picture.*** As in the introduction (Section 1.2), we write

$$\mathcal{W}(G) = \mathcal{W}^{\Sigma}(G) := \mathrm{Map}(\Sigma, G) \rtimes \mathrm{Diff}(\Sigma),$$

with group law $(\psi, \phi) \cdot (\psi', \phi') = (\psi \cdot (\psi' \circ \phi^{-1}), \phi \circ \phi')$, for the extended gauge group of a trivial principal $G$-bundle over $\Sigma$; hence the classifying space $B\mathcal{W}(G)$ carries the universal example of a trivializable principal $G$-bundle over a genus-1 surface. We let $\mathcal{W}_0(G) \subseteq \mathcal{W}(G)$ denote the identity component, and set $\overline{\mathcal{W}}(G) = \mathcal{W}(G)/\mathcal{W}_0(G) = \pi_0\mathcal{W}(G)$.

Now assume that we restrict to groups $G$ for which $H^*(B\mathcal{W}_0(G); \mathbb{C})$ is concentrated in even degrees; for connected compact Lie groups, this means $G$ must be a abelian. We obtain an action

$$(\overline{\mathcal{W}}(G) \times \mathbb{C}^\times)^{\mathrm{op}} \curvearrowright H^*(B\mathcal{W}_0(G); \mathbb{C}),$$

where $\mathbb{C}^\times$ acts by scalar multiplication on $H^2$, and note that this action is *functorial* with respect to the group $G$ and homomorphisms, i.e., $\phi\colon G \to G'$ induces a map of cohomology rings which is compatible with the group actions in the evident way. In particular, the tautological homomorphism $G \to e$ induces a map $\pi\colon B\mathcal{W}_0(G) \to B\,\mathrm{Diff}_0(\Sigma)$ which is invariant under the action of $\overline{W}(G)$.

We can now take the analytification of the resulting affine scheme over $\mathbb{C}$. Define

$$\mathcal{X}_G := [\mathrm{Spec}\, H^*(B\mathcal{W}_0(G); \mathbb{C})]_{\mathrm{an}} \smallsetminus B_G,$$

where $B_G$ is the closed (in the analytic topology) subset consisting of $\mathbb{C}$-points $p$ such that the composite

$$H^2(B\,\mathrm{Diff}_0(\Sigma); \mathbb{R}) \to H^2(B\,\mathrm{Diff}_0(\Sigma); \mathbb{C}) \xrightarrow{\pi^*} H^2(B\mathcal{W}_0(G); \mathbb{C}) \xrightarrow{p} \mathbb{C}$$

is *not* a bijection. Note in particular that if $G = e$ is the trivial group, then (in terms of a choice of basis of $H^2 B\mathcal{W}_0(e) = H^2 B\,\mathrm{Diff}_0(\Sigma)$),

$$\mathcal{X}_e \approx \{\mathbb{R}\text{-linear bijections } \mathbb{R}^2 \to \mathbb{C}\} \approx \mathcal{X} = \{\, (t_1, t_2) \mid \mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C} \,\} \subset \mathbb{C}^2.$$

[5] We can regard cohomology classes "$t_1$", "$t_2$" and "$y$" as coordinate functions on the space $\mathcal{X} \times \mathbb{C} = \{(t_1, t_2, y)\}$, so the formulas of (2.7) also describe how to pull back such functions.

Therefore $\mathcal{X}_G$ is the preimage of $\mathcal{X}_e \subset [\operatorname{Spec} H^*(B\,\mathrm{Diff}_0(\Sigma);\mathbb{C})]_{\mathrm{an}} \approx \mathbb{C}^2$ with respect to the map induced by $\pi$, which is therefore invariant under the action of $\overline{\mathcal{W}}(G) \times \mathbb{C}^\times$. Hence we obtain

$$\mathcal{M}_G := \overline{\mathcal{W}}(G) \times \mathbb{C}^\times \backslash\backslash \mathcal{X}_G.$$

**2.11.** ***Products of elliptic curves vs. degree-0 torus-bundles.*** Consider $G = U(1)^d$, with $d \geq 1$. As in the case of $d = 1$, we have a finite dimensional model

$$\big(\mathrm{Hom}(\mathbb{T}^2, U(1)^d) \times U(1)^d\big) \rtimes \big(\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2\big) \xrightarrow{\sim} \mathcal{W}(U(1)^d)$$

which can be rebracketed as

$$\big(\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^{d\times 2}\big) \ltimes \big(\mathbb{T}^2 \times U(1)^d\big).$$

Thus

$$H^*(B\mathcal{W}_0(U(1)^d);\mathbb{C}) \approx H^*(B(\mathbb{T}^2 \times U(1)^d);\mathbb{C}) \approx \mathbb{C}[t_1, t_2, y_1, \dots, y_d],$$

with induced action by $(\overline{\mathcal{W}}(G) \times \mathbb{C}^\times)^{\mathrm{op}}$ described much as in (2.7), except that we have

$$m \propto (t_1, t_2, y) = (t_1, t_2, y + m_1 t_1 + m_2 t_2), \quad m = (m_1, m_2) \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}^d) \approx (\mathbb{Z}^d)^2,$$

where $y = (y_1, \dots, y_d)$. Geometrically, this gives

$$\begin{aligned}\mathrm{GL}_2(\mathbb{Z}) \rtimes \mathbb{Z}^{d\times 2} \times \mathbb{C}^\times \curvearrowright \mathcal{X}_{U(1)^d}& \\ &= \mathcal{X} \times \mathbb{C}^d \\ &= \big\{ (t_1, t_2, y_1, \dots, y_d) \in \mathbb{C}^2 \times \mathbb{C}^d \mid \mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C} \big\},\end{aligned}$$

whence $\mathcal{M}_{U(1)^d} \to \mathcal{M}_e$ describes the $d$-fold fiber product of $\mathcal{E}$ over $\mathcal{M}$.

**2.12.** ***Remarks on the relation to double loop groups.*** The construction we just described seems to be a variant of one described in [Etingof and Frenkel 1994]. Here we will briefly describe how to relate the two.

Fix a compact and simply connected Lie group $G$, with maximal torus $T$ and Weyl group $W$. By analogy with loop groups, one has the *double loop group*

$$LLG := \mathrm{Map}(\mathbb{T}^2, G)$$

(where now we consider smooth maps), and also the *rotationally extended double loop group*

$$LL^{\mathrm{rot}}G := \mathrm{Map}(\mathbb{T}^2, G) \rtimes \mathbb{T}^2,$$

where the $\mathbb{T}^2$ acts by rotations. The group $LL^{\mathrm{rot}}G$ itself has an action of $\mathrm{Aut}(\mathbb{T}^2) = \mathrm{GL}_2(\mathbb{Z})$. This describes a subgroup $LL^{\mathrm{rot}}G \rtimes \mathrm{GL}_2(\mathbb{Z})$ of $\mathcal{W}(G)$.

The rotationally extended double loop group contains a finite dimensional torus

$$T^{\mathrm{rot}} = T^{\mathrm{rot}}_G := T \times \mathbb{T}^2,$$

where the $T$ corresponds to constant maps $\mathbb{T}^2 \to \{*\} \to T \subseteq G$. The Weyl group of $T^{\mathrm{rot}} \subset LL^{\mathrm{rot}}G$ is the *elliptic Weyl group*

$$W_{\mathrm{Ell}} = W \ltimes \mathrm{Hom}(\mathbb{Z}^2, \check{T})$$

of $G$, where $\check{T}$ = the cocharacter lattice of $T$. The Lie algebra $\mathrm{Lie}(T^{\mathrm{rot}}) = \mathrm{Lie}(T \times \mathbb{T}^2)$ inherits an action by $W_{\mathrm{Ell}}$, as well as an action by $\mathrm{GL}_2(\mathbb{Z})$.

For the trivial group $e$ we have $T^{\mathrm{rot}}_e = \mathbb{T}^2$, and $\mathrm{Lie}(\mathbb{T}^2) \otimes \mathbb{C} \approx \mathbb{C}^2$. Let $\mathcal{X}_e \subseteq \mathrm{Lie}(\mathbb{T}^2) \otimes \mathbb{C}$ denote the subset consisting of pairs of elements in $\mathbb{C}$ which generate a lattice, and define $\mathcal{X}_G$ as the preimage with respect to the evident projection $\pi$:

$$\begin{array}{ccc} \mathcal{X}_G & \rightarrowtail & \mathrm{Lie}(T \times \mathbb{T}^2) \otimes \mathbb{C} \\ \downarrow & & \downarrow \pi \\ \mathcal{X}_e & \rightarrowtail & \mathrm{Lie}(\mathbb{T}^2) \otimes \mathbb{C} \end{array}$$

The action by $\mathrm{GL}_2(\mathbb{Z}) \ltimes W_{\mathrm{Ell}}$ restricts to one $\mathcal{X}_G$, and acts fiberwise with respect to $\pi$, so that for each $t \in \mathcal{X}_e$ we obtain $W_{\mathrm{Ell}} \curvearrowright \pi^{-1}(t)$. When $G = T$ is itself a torus, this is evidently the same action as the one we described in the previous section, related via the Chern–Weil isomorphism

$$\mathrm{Sym}(\mathrm{Lie}(T \times \mathbb{T}^2)^* \otimes \mathbb{C}) \xrightarrow{\sim} H^*(B(T \times \mathbb{T}^2); \mathbb{C}) = H^*(B\mathcal{W}_0(G); \mathbb{C}).$$

Etingof and Frenkel [1994] describe the following construction. Given a *simply connected* $G$ with complexification $G_{\mathbb{C}}$, together with a choice of holomorphic structure (=complex structure + invariant holomorphic 1-form) on $\Sigma$, they describe a "coadjoint action" of $\mathrm{Map}(\Sigma, G_{\mathbb{C}})$ on $\mathrm{Lie}(\mathrm{Map}(\Sigma, G_{\mathbb{C}}))$ (actually a twisted version of the usual coadjoint action which depends on the chosen holomorphic structure on $\Sigma$). They show that orbits for this action correspond to isomorphism classes of holomorphic principal $G$-bundles on $\Sigma$. A generic class of orbits are given by the restriction to the maximal torus: the orbits of $W_{\mathrm{Ell}}$ acting on $\mathrm{Lie}(T_{\mathbb{C}})$ correspond to the "flat and unitary" holomorphic $G$-bundles on $\Sigma$.

Examining the formulas in Etingof and Frenkel, one sees that the holomorphic data for $\Sigma$ corresponds to a choice of point $t \in \mathcal{X}_e \subset \mathrm{Lie}(\mathbb{T}^2) \otimes \mathbb{C}$, and that their action $W_{\mathrm{Ell}} \curvearrowright \mathrm{Lie}(T_{\mathbb{C}})$ coincides with the action of $W_{\mathrm{Ell}}$ on the fiber $\pi^{-1}(t) \subset \mathrm{Lie}(T \times \mathbb{T}^2) \otimes \mathbb{C}$ that we described above. (Note: in the formulation of Etingof and Frenkel, they do not identify holomorphic structures on $\Sigma$ with points in $\mathrm{Lie}(\mathbb{T}^2) \otimes \mathbb{C}$; rather, they use the holomorphic structure to construct a central $\mathbb{C}^\times$-extension of $\mathrm{Map}(\Sigma, \mathbb{C}^\times)$, so that their coadjoint action is the natural one on a slice of the Lie algebra of their central extension [Etingof and Frenkel 1994, §3].)

## 3. Looijenga line bundles

We now describe the main result of this paper: if in our construction we replace $G = U(1)^d$ with $\widetilde{G} = U(1)^d \times_\phi K(\mathbb{Z}, 2) =$ a "central" extension of $U(1)^d$ by $K(\mathbb{Z}, 2)$, we get the total space of the principal bundle of a Looijenga line bundle. We start with the special case of $d = 0$, i.e., $\widetilde{G} = K(\mathbb{Z}, 2)$.

**3.1.** $\widetilde{G} = K(\mathbb{Z}, 2)$ ***gives the multiplicative group.*** We describe our results in the case that $\widetilde{G} = K(\mathbb{Z}, 2)$. We have that

$$\overline{\mathcal{W}}(K(\mathbb{Z}, 2)) \approx \mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z},$$

where $\mathrm{GL}_2(\mathbb{Z})$ acts on $\mathbb{Z} = H^2(\Sigma; \mathbb{Z})$ via the determinant. We have

$$H^*(B\mathcal{W}_0(K(\mathbb{Z}, 2)); \mathbb{C}) \approx \mathbb{C}[t_1, t_2, x_1, x_2]/(t_1x_1 + t_2x_2), \quad t_i, x_k \in H^2.$$

The resulting action

$$(\overline{\mathcal{W}}(K(\mathbb{Z}, 2)) \times \mathbb{C}^\times)^{\mathrm{op}} \curvearrowright H^*(B\mathcal{W}_0(K(\mathbb{Z}, 2)); \mathbb{C})$$

is described by

$$\begin{aligned} n \propto (t_1, t_2, x_1, x_2) &= (t_1, t_2, x_1 - nt_2, x_2 + nt_1) & n \in \mathbb{Z}, \\ A \propto (t_1, t_2, x_1, x_2) &= \left(at_1 + bt_2, ct_1 + dt_2, \frac{dx_1 - cx_2}{\det A}, \frac{-bx_1 + ax_2}{\det A}\right), \\ & & A \in \mathrm{GL}_2(\mathbb{Z}), \\ \lambda \propto (t_1, t_2, x_1, x_2) &= (\lambda t_1, \lambda t_2, \lambda x_1, \lambda x_2), & \lambda \in \mathbb{C}^\times. \end{aligned} \tag{3.2}$$

The associated geometric object is

$$\mathcal{X}_{K(\mathbb{Z},2)} = \left\{ (t, x) \in \mathbb{C}^2 \mid t_1x_2 + t_2x_2 = 0,\ \mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C} \right\} \subset \mathcal{X} \times \mathbb{C}^2.$$

The projection $\mathcal{X}_{K(\mathbb{Z},2)} \to \mathcal{X}$ is a trivial line bundle over $\mathcal{X}$, via the nowhere vanishing section $(t_1, t_2) \mapsto (t_1, t_2, -t_2, t_1)$. The action $\mathbb{Z} \curvearrowright \mathcal{X}_{K(\mathbb{Z},2)}$ is fiber-by-fiber, via translation along this section, and so acts freely. Thus

$$\mathbb{Z}\backslash\backslash\mathcal{X}_{K(\mathbb{Z},2)} \approx \mathbb{Z}\backslash\mathcal{X}_{K(\mathbb{Z},2)} \approx \mathcal{X} \times \mathbb{C}^\times.$$

Explicitly, $\mathbb{Z}\backslash\mathcal{X}_{K(\mathbb{Z},2)} \xrightarrow{\sim} \mathcal{X} \times \mathbb{C}^\times$ is given by

$$(t_1, t_2, x_1, x_2) \mapsto (t_1, t_2, e^{2\pi i(x_1/t_2)}).$$

The $\mathrm{GL}_2(\mathbb{Z}) \times \mathbb{C}^\times$ action descends to an action on $\mathcal{X} \times \mathbb{C}^\times$ of the form

$$(A, \lambda) \propto (t_1, t_2, u) = \big(\lambda(at_1 + bt_2), \lambda(ct_1 + dt_2), u^{1/\det A}\big).$$

Since elements $A \in \mathrm{GL}_2(\mathbb{Z})$ with $\det A = -1$ switch the two components of $\mathcal{X}$, we see that

$$\mathcal{M}_{K(\mathbb{Z},2)} \approx \mathcal{M} \times \mathbb{C}^\times.$$

This is naturally a group object over $\mathcal{M}$, via the group structure on $K(\mathbb{Z}, 2)$.

**3.3. Remark.** This will follow from the general theorem Theorem 7.6. To see how it arises, consider the Serre spectral sequence for

$$B\,\mathrm{Map}_0(\Sigma, K(\mathbb{Z}, 2)) \to B\mathcal{W}_0(K(\mathbb{Z}, 2)) \to B\,\mathrm{Diff}_0(\Sigma),$$

which has $E_2^{p,q} = \mathbb{C}[t_1, t_2] \otimes \mathbb{C}[x_1, x_2, \epsilon]$, with $|\epsilon| = (0, 3)$. The only differential is $d_2(\epsilon) = \pm(t_1x_1 + t_2x_2)$. The terms $(\dots, \dots - nt_2, \dots + nt_1)$ in the first line of (3.2) ultimately derive from the nondegenerate pairing $H_1\mathbb{T}^2 \xrightarrow{\sim} H^1\mathbb{T}^2$ adjoint to the Pontryagin product on $H_*\mathbb{T}^2$.

**3.4. *Quadratic functions.*** Let $B$ and $C$ be finitely generated free abelian groups. A *quadratic function* $\phi : B \to C$ is a function such that

- $\beta(b, b') := \phi(b + b') - \phi(b) - \phi(b')$ is bilinear, and
- $\phi(nb) = n^2\phi(b)$ for $n \in \mathbb{Z}$.

The symmetric bilinear form $\beta : B \otimes B \to C$ is called the *Hessian form* of $\phi$. Note that $\phi(b) = \frac{1}{2}\beta(b, b)$, so $\beta$ determines $\phi$.

Let $\Gamma_2 B$ be the second degree part of the divided power algebra on $B$; since $B$ is 2-torsion free, $\Gamma_2 B \approx (B \otimes B)^{\Sigma_2}$. The function $\gamma_2 : B \to \Gamma_2 B$ given by $b \mapsto b \otimes b$ is the universal quadratic function out of $B$, so that

$$\mathrm{Hom}(\Gamma_2 B, C) \xrightarrow[\sim]{\widetilde{\phi} \mapsto \widetilde{\phi} \circ \gamma_2} \{\text{quadratic } B \xrightarrow{\phi} C\}$$

is a bijection. We will use the notation $\widetilde{\phi}$ for the homomorphism associated to a quadratic function $\phi$.

A *bilinear extension* of $\phi$ is any bilinear (but not necessarily symmetric) map $\omega : B \times B \to C$ such that $\phi(b) = \omega(b, b)$. Such extensions always exist (because the exact sequence $0 \to \Gamma_2 B \to B \otimes B \to \Lambda^2 B \to 0$ splits), and any two such extensions differ by an alternating form.

In terms of a choice of coordinates $B \approx \mathbb{Z}^d$, we have

$$\phi(y) = \tfrac{1}{2} \sum_{i,j} c_{ij} y_i y_j, \quad \beta(y, y') = \sum_{i,j} c_{ij} y_i y'_j, \quad \omega(y, y') = \sum_{i,j} d_{ij} y_i y'_j, \quad (3.5)$$

where $(c_{ij})$ is a symmetric integer matrix with $c_{ii} \in 2\mathbb{Z}$, and $(d_{ij})$ any integer matrix such that $c_{ij} = d_{ij} + d_{ji}$.

**3.6.** ***Case of*** $\widetilde{G}$ ***= extension of*** $U(1)^d$ ***by*** $K(\mathbb{Z},2)$**.** Given a topological group $G$ and a map $\widetilde{\phi}\colon BG \to K(\mathbb{Z},4)$, we have a fibration sequence of the form

$$B\widetilde{G} \to BG \xrightarrow{\widetilde{\phi}} K(\mathbb{Z},4).$$

We define $\widetilde{G}$ to be the (based) loop space of the fiber $B\widetilde{G}$, modeled as a topological group. We call this $\widetilde{G}$ the $K(\mathbb{Z},2)$*-central extension of* $G$ corresponding to $\phi$ (though as realized above the extension might not be central).

Given $G = U(1)^d$, set $B := \pi_1 G = H_2(BG,\mathbb{Z}) = \mathbb{Z}^d$, so that up to homotopy maps $\widetilde{\phi}$ correspond to elements

$$\widetilde{\phi} \in H^4(BU(1)^d;\mathbb{Z}) \approx \operatorname{Sym}^2 H^2(BU(1)^d;\mathbb{Z}) \approx \operatorname{Hom}(\Gamma_2 B,\mathbb{Z}),$$

and thus to quadratic functions $\phi\colon B \to \mathbb{Z}$.

**3.7. Theorem.** *Let* $\widetilde{G}$ *be a* $K(\mathbb{Z},2)$*-central extension of* $G = U(1)^d$ *associated to a quadratic function* $\phi$ *with Hessian form* $\beta$*, and choose a bilinear extension* $\omega$ *of* $\phi$*.*

(1) *We have*

$$\overline{\mathcal{W}}(\widetilde{G}) \approx \mathrm{GL}_2(\mathbb{Z}) \ltimes E,$$

*where* $E$ *is a central extension*

$$0 \to \mathbb{Z} \to E \to \operatorname{Hom}(\mathbb{Z}^2,\mathbb{Z}^d) \to 0, \tag{3.8}$$

*defined so that the group law on* $E = \operatorname{Hom}(\mathbb{Z}^2,\mathbb{Z}^d)\times\mathbb{Z}$ *takes the form*

$$(m_1,m_2,n)\cdot(m_1',m_2',n') = \big(m_1+m_1',\ m_2+m_2',\ n+n'+(\omega(m_1,m_2')-\omega(m_2,m_1'))\big),$$

*where* $n,n' \in \mathbb{Z}$ *and* $m,m' \in \operatorname{Hom}(\mathbb{Z}^2,\mathbb{Z}^d) \approx (\mathbb{Z}^d)^2$.

*The group* $\mathrm{GL}_2(\mathbb{Z})$ *acts on* $E$ (*from the left*) *by*

$$A \propto (m_1,m_2,n) = \left(\frac{dm_1 - cm_2}{\det A},\ \frac{-bm_1+am_2}{\det A},\ n\right).$$

(2) *We have*

$$H^*(B\mathcal{W}_0(\widetilde{G});\mathbb{C}) \approx \mathbb{C}[t_1,t_2,y_1,\dots,y_d,x_1,x_2]/\big(\phi(y)+(t_1x_1+t_2x_2)\big),\qquad t_i,y_j,x_k \in H^2.$$

(3) *The action* $\overline{\mathcal{W}}(\widetilde{G})^{\mathrm{op}} \curvearrowright H^*(B\mathcal{W}_0(\widetilde{G});\mathbb{C})$ *is given* (*in terms of the description in* (1)) *by*

$$n \propto (t,y,x) = (t_1,t_2,\ y,\ x_1 - nt_2,\ x_2 + nt_1), \qquad n \in \mathbb{Z},$$

$$m \propto (t,y,x) = (t,\ y+mt,\ x-\beta(y,m)-\omega(mt,m)), \quad m \in \operatorname{Hom}(\mathbb{Z}^2,\mathbb{Z}^d),$$

$$A \propto (t,y,x) = \left(at_1+bt_2, ct_1+dt_2,\ y,\ \frac{dx_1-cx_2}{\det A}, \frac{-bx_1+ax_2}{\det A}\right), \qquad A \in \mathrm{GL}_2(\mathbb{Z}), \tag{3.9}$$

*where* $t=(t_1,t_2)$, $y=(y_1,\dots,y_d)$, $x=(x_1,x_2)$.

**3.10. Remark.** The second line of (3.9) is in compressed form. In full it means

$$\begin{aligned}(m_1,m_2)&\propto(t,y,x)\\ &=\bigl(t_1,t_2,\ y+m_1t_1+m_2t_2,\ x_1-\beta(y,m_1)-\omega(m_1t_1+m_2t_2,m_1),\\ &\qquad\qquad x_2-\beta(y,m_2)-\omega(m_1t_1+m_2t_2,m_2)\bigr),\end{aligned}\tag{3.11}$$

where $m=(m_1,m_2)\in\operatorname{Hom}(\mathbb{Z}^2,\mathbb{Z}^d)\approx(\mathbb{Z}^d)^2$.

**3.12. Remark.** Up to isomorphism, the central extension (3.8) depends only on the antisymmetrization of the 2-cocycle $\gamma(m,m')=\omega(m_1,m_2')-\omega(m_2,m_1')$, which is $\gamma_{\text{antisym}}(m,m')=\gamma(m,m')-\gamma(m',m)=\beta(m_1,m_2')-\beta(m_2,m_1')$, and thus depends only on $\phi$, not on $\omega$.

The corresponding geometric object $\mathcal{X}_{\widetilde{G}}\subseteq\mathcal{X}\times\mathbb{C}^d\times\mathbb{C}^2$ is the locus of $t_1x_1+x_2t_2=-\phi(y)$, subject to $\mathbb{R}t_1+\mathbb{R}t_2=\mathbb{C}$. The free quotient $\mathbb{Z}\backslash\mathcal{X}_{\widetilde{G}}$ is a principal $\mathbb{C}^\times$-bundle over $\mathcal{X}\times\mathbb{C}^d$. Thus, $\mathcal{M}_{\widetilde{G}}$ is the total space of a principal $\mathbb{C}^\times$-bundle over $\mathcal{E}^d$.

In fact, let us consider the quotient of $\mathcal{X}_{\widetilde{G}}$ under the free action by $\mathbb{Z}\times\mathbb{C}^\times$. Explicitly,

$$(\mathbb{Z}\times\mathbb{C}^\times)\backslash\mathcal{X}_{\widetilde{G}}\xrightarrow{\sim}(\mathbb{C}\smallsetminus\mathbb{R})\times\mathbb{C}^d\times\mathbb{C}^\times$$

is given by

$$(t_1,t_2,\ y_1,\dots,y_d,\ x_1,x_2)\mapsto\bigl(\tfrac{t_1}{t_2},\ \tfrac{y_1}{t_2},\dots,\tfrac{y_d}{t_2},\ e^{2\pi i(x_1/t_2)}\bigr)=(\tau,\ z_1,\dots,z_d,\ u).$$

The $\overline{\mathcal{W}}(G)$-action descends to an action by $\operatorname{Hom}(\mathbb{Z}^2,\mathbb{Z}^d)\rtimes\mathrm{GL}_2(\mathbb{Z})$ on the quotient, given by

$$\begin{aligned}m\propto(\tau,z,u)&=\bigl(\tau,\ z+m_1\tau+m_2,\ ue^{2\pi i[-\beta(z,m_1)-\phi(m_1)\tau]}\bigr),\quad m\in\operatorname{Hom}(\mathbb{Z}^2,\mathbb{Z}^d),\\ A\propto(\tau,z,u)&=\bigl(A\tau,\ (c\tau+d)^{-1}z,\ u^{1/\det A}e^{2\pi i(1/\det A)[c(c\tau+d)^{-1}\phi(z)]}\bigr),\\ &\qquad A\in\mathrm{GL}_2(\mathbb{Z}),\end{aligned}\tag{3.13}$$

where $A\tau=(a\tau+b)/(c\tau+d)$ and $z=(z_1,\dots,z_d)$. This describes the principal $\mathbb{C}^\times$-bundle over $\mathcal{E}^{\times d}$ whose associated line bundle has as sections $\theta(\tau,z)$ such that (for $\operatorname{Im}(\tau)>0$ and $A\in\mathrm{SL}_2(\mathbb{Z})$)

$$\begin{aligned}\theta(\tau,\ z+m_1\tau+m_2)&=\theta(\tau,z)\,e^{2\pi i[-\beta(z,m_1)-\phi(m_1)\tau]},\\ \theta(A\tau,\ (c\tau+d)^{-1}z)&=\theta(\tau,z)\,e^{2\pi i[c(c\tau+d)^{-1}\phi(z)]}.\end{aligned}$$

In other words, we obtain the *Looijenga line bundle* associated to the quadratic form $\phi$ [Looijenga 1976/77].

**3.14. Remark.** Suppose $\phi:B=\mathbb{Z}^d\to\mathbb{Z}$ is a nondegenerate quadratic function. Then with our conventions, the line bundle $L_\phi$ associated to $\phi$, admits a nontrivial *holomorphic* section over $C_\tau^d$ (for any chosen $C_\tau:=\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})$ with $\operatorname{Im}(\tau)>0$)

if and only if $\phi$ is *positive definite*. The main example of interest is the positive definite quadratic function $\phi$ associated to the Killing form on the coroot lattice of a simply connected compact Lie group; in this case $\phi$ is invariant under the action of the Weyl group, so the bundle $L_\phi$ is equivariant for the Weyl group.

To see the existence of sections in this case, we use [Mumford 1970, I.2 and I.3]. In the notation of [Mumford 1970, I.2, p. 15–16; I.3, p. 24–25], the line bundle $L_\phi|C^d_\tau$ is described by a 1-cocycle $e$ on $U = \mathbb{Z}^d\tau + \mathbb{Z}^d \subseteq \mathbb{C}^d$ with coefficients in holomorphic functions $\mathbb{C}^d \to \mathbb{C}^\times$, given by

$$e_u(z) = e^{2\pi i f_u(z)}, \qquad f_{m_1\tau+m_2}(z) = -\beta(z, m_1) - \tfrac{1}{2}\beta(m_1, m_1)\tau, \quad m_1, m_2 \in \mathbb{Z}^d.$$

By [Mumford 1970, I.2, p. 18, Proposition],

$$E(u, u') := f_{u'}(z+u) + f_u(z) - f_u(z+u') - f_{u'}(z), \quad \text{any } z \in \mathbb{C}^d,$$

defines an alternating 2-form $E : U \times U \to \mathbb{Z}$ which represents the Chern class of $L_\phi|C^d_\tau$. We calculate that in our case,

$$E(m_1\tau + m_2, m_1'\tau + m_2') = \beta(m_1, m_2') - \beta(m_2, m_1').$$

Extend $E$ to an $\mathbb{R}$-linear form $\mathbb{C}^d \times \mathbb{C}^d \to \mathbb{R}$ and set $H(x, y) := E(ix, y) + iE(x, y)$. Then $H$ is a Hermitian form with $\operatorname{Im} H = E$. By the proposition and preceding discussion in I.3 of [Mumford 1970, p. 26], if $H$ is nondegenerate, then $L_\phi|C^d_\tau$ admits nonzero holomorphic sections if and only if $H$ is positive definite, in which case $\dim H^0(C^d_\tau, L_\phi|C^d_\tau) = \sqrt{\det E}$ (express $E$ as a matrix using a $\mathbb{Z}$-basis of $U$). We calculate that in our case,

$$H(x, x) = (\operatorname{Im}\tau)^{-1}\beta(x, \bar{x}), \quad x \in \mathbb{C}^d.$$

As $\operatorname{Im}\tau > 0$ and $\beta(x, y) = \sum c_{ij}x_i y_j$ is a symmetric form on $\mathbb{C}^d$ with $c_{ij} \in \mathbb{Z} \subseteq \mathbb{R}$, we see that $H$ is nondegenerate/positive definite on $\mathbb{C}^d$ if and only if $\beta$ is nondegenerate/positive definite on $\mathbb{R}^d$, and if so we have $\sqrt{\det E} = \det(c_{ij})$.

**3.15. Remark.** For $\phi : B \approx \mathbb{Z}^d \to \mathbb{Z}$ positive definite, sections $\theta_u$ of $L_\phi|C^d_\tau$ are given by $\theta_u(\tau, z) = \sum_{v\in B} e^{2\pi i[-\beta(z,u+v)+\phi(u+v)\tau]}$ for $u \in B \otimes \mathbb{R}$ such that $\beta(u, B) \subseteq \mathbb{Z}$, [Looijenga 1976/77, §4].

**3.16.** ***Proof of the theorem.*** We will derive Theorem 3.7 from a more general (and coordinate invariant) statement Theorem 7.6, whose setup and proof takes up Sections 6–10. It is entirely calculational, and amounts to completely describing the homotopy type of the spaces $B\mathcal{W}(\widetilde{G})$. In particular, the key is to compute all Whitehead products in the homotopy groups of this space.

We note that one can instead regard $\widetilde{G}$ as arising from a *Lie 2-group*, specifically as a 2-group extension as considered in [Ganter 2018]. It seems likely that 2-group methods should lead to a more informative proof of the results shown here.

## 4. Isogenies

We describe how, according to the picture of the previous sections, finite coverings of genus-1 surfaces correspond to isogenies of elliptic curves.

Fix a finite covering map $f : \Sigma' \to \Sigma$ between two surfaces. Let $\mathrm{Diff}(f) \subset \mathrm{Diff}(\Sigma) \times \mathrm{Diff}(\Sigma')$ denote the group of pairs of diffeomorphisms compatible with $f$. We note that the projection map $\mathrm{Diff}(f) \xrightarrow{t} \mathrm{Diff}(\Sigma)$ is a finite covering map, while the projection map $\mathrm{Diff}(f) \xrightarrow{s} \mathrm{Diff}(\Sigma')$ is injective and induces a homotopy equivalence between $\mathrm{Diff}(f)$ and a union of path components of $\mathrm{Diff}(\Sigma')$, corresponding to a finite index subgroup of $\pi_0 \mathrm{Diff}(\Sigma')$.

Given any group $G$, we can form a diagram as follows:

$$\begin{array}{ccccccc}
B\mathcal{W}^{\Sigma}(G) & \longleftarrow & (Bt)^* B\mathcal{W}^{\Sigma}(G) & \xrightarrow{\quad f^* \quad} & (Bs)^* B\mathcal{W}^{\Sigma'}(G) & \longrightarrow & B\mathcal{W}^{\Sigma'}(G) \\
\downarrow & & \searrow & & \swarrow & & \downarrow \\
B\,\mathrm{Diff}(\Sigma) & \xleftarrow[\quad Bt \quad]{} & & B\,\mathrm{Diff}(f) & & \xrightarrow[\quad Bs \quad]{} & B\,\mathrm{Diff}(\Sigma')
\end{array} \tag{4.1}$$

where the trapezoids are homotopy pullbacks. That is,

$$(Bt)^* B\mathcal{W}^{\Sigma}(G) \approx B\big(\mathrm{Map}(\Sigma, G) \rtimes \mathrm{Diff}(f)\big),$$
$$(Bs)^* B\mathcal{W}^{\Sigma'}(G) \approx B\big(\mathrm{Map}(\Sigma', G) \rtimes \mathrm{Diff}_+(f)\big),$$

while the map labeled $f^*$ is obtained from the map $\mathrm{Map}(\Sigma, G) \to \mathrm{Map}(\Sigma', G)$ given by restriction along $f$.

The observation is that, after applying the construction of Section 2.10, the map $f^*$ presents an isogeny of curves, of degree equal to the degree of $f$. To see this, we consider an explicit example.

**4.2. Example.** Fix $\Sigma = \Sigma' = \mathbb{T}^2$, and let $f : \Sigma' \to \Sigma$ be the map induced by left multiplication by some rationally invertible integer matrix $B$. Set $\Gamma_B := \mathrm{GL}_2(\mathbb{Z}) \cap B^{-1}\,\mathrm{GL}_2(\mathbb{Z})\,B$. Note that, using a suitable choice of bases of $H_1\Sigma$ and $H_1\Sigma'$, the matrix $B$ can be given the form $B = \left(\begin{smallmatrix} M & 0 \\ 0 & MN \end{smallmatrix}\right)$ for some $M, N \geq 1$, in which case $\Gamma_B = \Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\}$.

Then there is a weak equivalence of topological groups

$$\Gamma_B \ltimes \mathbb{T}^2 \xrightarrow{\sim} \mathrm{Diff}(f),$$

so that the projections $\mathrm{Diff}(\Sigma) \leftarrow \mathrm{Diff}(f) \to \mathrm{Diff}(\Sigma')$ correspond to

$$\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \xleftarrow{(BAB^{-1}, Bt) \leftarrow\!\shortmid (A,t)} \Gamma_B \ltimes \mathbb{T}^2 \xrightarrow{(A,t) \mapsto (A,t)} \mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2.$$

Let $G = U(1)$, form $\mathrm{Spec}_{\mathrm{an}}$ of the cohomology of universal covers of objects in (4.1), and restrict to the subset $\mathcal{X} = \{\, (t_1, t_2) \mid \mathbb{R}t_1 + \mathbb{R}t_2 = \mathbb{C} \,\}$. Together with actions of fundamental groups and the grading action by $\mathbb{C}^\times$, the middle triangle of (4.1) is seen to have the form

$$\begin{array}{ccc} (\Gamma_B \ltimes' \mathbb{Z}^2) \times \mathbb{C}^\times \backslash\!\backslash \mathcal{X} \times \mathbb{C} & \xrightarrow{\quad f^* \quad} & (\Gamma_B \ltimes \mathbb{Z}^2) \times \mathbb{C}^\times \backslash\!\backslash \mathcal{X} \times \mathbb{C} \\ & \searrow \quad \swarrow & \\ & \Gamma_B \times \mathbb{C}^\times \backslash\!\backslash \mathcal{X} & \end{array}$$

where $f^*$ is induced by the identity on $\mathcal{X} \times \mathbb{C}$, and both maps $\mathcal{X} \times \mathbb{C} \to \mathcal{X}$ are the evident projection. The semidirect product $\Gamma_B \ltimes \mathbb{Z}^2$ is induced by the tautological action $\Gamma_B \subset \mathrm{GL}_2(\mathbb{Z})$, while the semidirect product $\Gamma_B \ltimes' \mathbb{Z}^2$ is induced by the homomorphism $A \mapsto BAB^{-1} : \Gamma_B \to \mathrm{GL}_2(\mathbb{Z})$. The action in the upper-right corner is

$$A \propto (t, y) = (At, y), \quad m \propto (t, y) = (t, y + mt), \quad \lambda \propto (t, y) = (\lambda t, \lambda y),$$

while the action in the upper-left corner is

$$A \propto (t, y) = (At, y), \quad m \propto (t, y) = (t, y + mBt), \quad \lambda \propto (t, y) = (\lambda t, \lambda y),$$

where $A \in \Gamma_B$, $m \in \mathbb{Z}^2$ (treated as a row vector), $t \in \mathcal{X}$ (treated as a column vector), $y \in \mathbb{C}$, and $\lambda \in \mathbb{C}^\times$.

Thus, in the "fibers" over $(t_1, t_2) \in \mathcal{X}$ we obtain (after taking quotients by $\mathbb{Z}^2$-actions) the projection $\mathbb{C}/((Bt)_1\mathbb{Z} + (Bt)_2\mathbb{Z}) \to \mathbb{C}/(t_1\mathbb{Z} + t_2\mathbb{Z})$, an isogeny of degree $\det B$. E.g., for $B = \left(\begin{smallmatrix} M & 0 \\ 0 & MN \end{smallmatrix}\right)$ we get $\mathbb{C}/(Mt_1\mathbb{Z} + MNt_2\mathbb{Z}) \to \mathbb{C}/(t_1\mathbb{Z} + t_2\mathbb{Z})$.

## 5. Remarks on the formalism

**5.1.** ***Remarks on the construction of an equivariant cohomology theory.*** We can easily produce for each group $G$ that we consider an equivariant cohomology theory of the form

$$E_G^* : \big(G\text{-CW-complexes}\big)^{\mathrm{op}} \to \big(\overline{\mathcal{W}}(G)\text{-equivariant } H^*(B\mathcal{W}_0(G); \mathbb{C})\text{-algebras}\big).$$

Given a $G$-space $X$ let

$$\mathrm{Map}_G^{\mathrm{gh}}(\Sigma \times G, X) \subseteq \mathrm{Map}_G(\Sigma \times G, X)$$

be the subspace consisting of *ghost maps*, i.e., $G$-equivariant maps $f : \Sigma \times G \to X$ such that $f(\Sigma \times G)$ is contained in a single $G$-orbit. The ghost maps are invariant under the evident action of $\mathcal{W}(G)$ on $\mathrm{Map}_G(G \times \Sigma, X)$, so we can define

$$E_G^*(X) := \big(\overline{\mathcal{W}}(G)^{\mathrm{op}} \curvearrowright H^*\big(\mathrm{Map}_G^{\mathrm{gh}}(\Sigma \times G, X)_{h\mathcal{W}_0(G)}; \mathbb{C}\big)\big).$$

That this is a cohomology theory amounts to the observations that

(i) $X \mapsto \mathrm{Map}_G^{\mathrm{gh}}(\Sigma \times G, X)$ preserves pushouts along cofibrations, and

(ii) $\mathrm{Map}_G^{\mathrm{gh}}(\Sigma \times G, T \times X) \approx T \times \mathrm{Map}_G^{\mathrm{gh}}(\Sigma \times G, X)$ when $T$ has trivial $G$-action.

**5.2. Example.** Let $G = U(1)$ and $X = U(1)/\mu_N$. Then

$$E^*_{U(1)}(U(1)/\mu_N) \approx \prod_{(n_1,n_2)\in\mathbb{Z}^2} \mathbb{C}[t_1, t_2, y]/\big(y - (n_1/N)t_1 - (n_2/N)t_2\big),$$

which is an algebra over $H^*(B\mathcal{W}_0(U(1)); \mathbb{C}) \approx \mathbb{C}[t_1, t_2, y]$ in the obvious way and which carries an evident compatible action by $\overline{\mathcal{W}}(U(1)) = \mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.

Ideally one would like to "analytify" the equivariant module $E^*_G(X)$, to obtain a sheaf of $\mathcal{O}_{\mathcal{M}_G}$-algebras on $\mathcal{M}_G$, to be coherent at least if $X$ is a finite $G$-CW-complex; we would then hope to take it as a model for Grojnowski's equivariant elliptic cohomology. Unfortunately, the most obvious way to do this (e.g., by tensoring up from algebraic to holomorphic functions), though exact, behaves poorly on most $E^*_G(X)$ (which are often non-Noetherian, even when $X$ is a $G$-orbit).

This is perhaps a bit disappointing. I suspect that an additional ingredient is needed here in order to supply a rigorous "analytification" of this construction, which will probably come from mathematical physics and supergeometry. For instance see the constructions in [Berwick-Evans and Tripathy 2018], where complex analytic structure arises from "cs manifolds", a kind of complexified supermanifold.

**5.3. *Remarks on derived constructions.*** In this paper we have been content to produce examples of "classical" geometric objects, e.g., complex analytic spaces. However, we know that elliptic cohomology wants to take values in sheaves on a *derived* geometric object, along the lines of [Lurie 2009]. I don't know how to make such a derived construction; however, I'll give some speculation here.

Fix a commutative dga[6] $\mathbb{C}[u^{\pm}]$, where $|u| = 2$ with $du = 0$. This admits an evident grading coaction by the Hopf algebra $\mathbb{C}[\lambda^{\pm}]$ with $|\lambda| = 0$ and $d(\lambda) = 0$, by $u \mapsto \lambda \otimes u$. Thus $\mathbb{G}_m := \mathrm{Spec}^{\mathrm{der}}\, \mathbb{C}[\lambda^{\pm}]$ acts on $\mathrm{Spec}^{\mathrm{der}}\, \mathbb{C}[u^{\pm}]$.

For a space $X$, let $C^*X$ denote a functorial commutative dga model for the cochains on $X$ with $\mathbb{C}$ coefficients; e.g., we could take $C^*X$ to be the PL-de Rham forms on $X$. Thus $\mathrm{Spec}^{\mathrm{der}}\, C^*X \otimes_{\mathbb{C}} \mathbb{C}[u^{\pm}]$ inherits an action by $\mathbb{G}_m$. We can then plug in $\overline{W}(G) \curvearrowright B\mathcal{W}_0(G)$ as above to obtain

$$\overline{W}(G) \times \mathbb{G}_m \curvearrowright \mathrm{Spec}^{\mathrm{der}}\, C^*B\mathcal{W}_0(G) \otimes_{\mathbb{C}} \mathbb{C}[u^{\pm}],$$

a derived scheme equipped with an action by a group scheme.

At this point we posit the existence of a *derived analytification* functor $\mathrm{Spec}^{\mathrm{der}}_{\mathrm{an}}$, which takes as input a commutative dga $A^*$ over $\mathbb{C}$, as gives as output a derived complex analytic space $\mathrm{Spec}^{\mathrm{der}}_{\mathrm{an}}\, A^* = (X, \mathcal{O})$, in some suitable $\infty$-category $\mathrm{An}^{\mathrm{der}}$

[6] We use homological grading here, so $x \in C^q$ has $|x| = -q$.

of derived analytic spaces. It should have the property that (at least in the examples we care about), the underlying complex analytic space $(X, H^0(\mathcal{O}))$ is equivalent to $[\operatorname{Spec} H^0 A^*]_{\mathrm{an}}$. Given this, we would could then proceed to construct derived versions of $\mathcal{X}_G$ and $\mathcal{M}_G$ as desired.

An $\infty$-category $\mathrm{An}^{\mathrm{der}}$ has been constructed in work of Lurie [2011] and Porto [2015], and in fact comes equipped with an analytification functor. A significant issue in carrying out this program (as pointed out to me by Mauro Porto) is that the "rings" which appear in this model are fundamentally $(-1)$-connected objects, whereas the rings we want to consider are naturally nonconnected, and in fact are generally 2-periodic.

**5.4.** ***Remarks on functoriality.*** As we have described it, our construction $G \mapsto \mathcal{M}_G$ is functorial with respect to homomorphisms of groups. For a derived version of this construction it is highly desirable to have an enhanced "stacky" version of this functoriality, where homomorphisms are enriched to maps between classifying spaces (not necessarily basepoint preserving), i.e., we should have

$$\mathrm{Map}_{\mathrm{Top}}(BG, BG') \to \mathrm{Map}_{\mathrm{An}^{\mathrm{der}}}(\mathcal{M}_G^{\mathrm{der}}, \mathcal{M}_{G'}^{\mathrm{der}}).$$

This extended functoriality should apply not just to tori but to $K(\mathbb{Z}, 2)$-extensions of them, and thus should be consistent with Lurie's notion [2009, §5] of *2-equivariance*. I'll briefly indicate how to achieve this; it may be enlightening even in the nonderived case.

Consider $\mathrm{Map}(\Sigma, BG)$, the space parametrizing principal $G$-bundles over $\Sigma$. There is a distinguished path component $\mathrm{Map}_0(\Sigma, BG) \subseteq \mathrm{Map}(\Sigma, BG)$ corresponding to trivializable bundles, which is equivalent to $B\,\mathrm{Map}(\Sigma, G)$. There is a corresponding path component

$$\mathcal{P}(BG) \subseteq \mathrm{Map}(\Sigma, BG)_{h\,\mathrm{Diff}(\Sigma)}$$

equivalent to $B\mathcal{W}(G)$. Note that a map $BG \to BG'$ sends $\mathcal{P}(BG)$ to $\mathcal{P}(BG')$.

Thus, "enhanced functoriality" follows once we describe how to functorially obtain a derived stack from a suitable connected space $X$ (such as $X = \mathcal{P}(BG)$).

For each path connected space $X$, make an arbitrary choice of universal cover $p : \widetilde{X} \to X$, and write $G$ for its group of deck transformations. Note that $p$ is a principal $G$-bundle. We get a topological quotient stack $G \mathbin{\backslash\backslash} \widetilde{X}$ which is equivalent to $X$. After taking cohomology we obtain a total quotient stack $G \mathbin{\backslash\backslash} \operatorname{Spec} H^*\widetilde{X}$; replacing cohomology with cochains gives the corresponding derived object. When $X = \mathcal{P}(BG)$ this recovers the construction $\overline{W}(G) \mathbin{\backslash\backslash} H^* B\mathcal{W}_0(G)$.

Consider a map $f : X \to Y$ to another path connected space, we and write $(\widetilde{Y}, q, H)$ for the analogous choices for $Y$. Then $f$ induces a map $G \mathbin{\backslash\backslash} \operatorname{Spec} H^*\widetilde{X} \to$

$H \mathbin{\backslash\!\backslash} \operatorname{Spec} H^*\widetilde{Y}$ of stacks which is represented by a *bibundle*, as follows. Consider

$$\widetilde{X} \xleftarrow{\pi} \operatorname{Lift}(f) \times \widetilde{X} \xrightarrow{\epsilon} \widetilde{Y}$$

where $\operatorname{Lift}(f) = \{\, \widetilde{f} : \widetilde{X} \to \widetilde{Y} \mid q\widetilde{f} = fp \,\}$ is the set of lifts of $f$ to the universal covers, $\pi$ is the projection map, and $\epsilon$ is the evaluation map. We have:

- $G$ acts on $\operatorname{Lift}(f) \times \widetilde{X}$ from the left by $g \propto (\widetilde{f}, \widetilde{x}) = (\widetilde{f}g^{-1}, g\widetilde{x})$,
- $H$ acts on $\operatorname{Lift}(f) \times \widetilde{X}$ from the left by $h \propto (\widetilde{f}, \widetilde{x}) = (h\widetilde{f}, \widetilde{x})$,
- the group actions on $\operatorname{Lift}(f) \times \widetilde{X}$ commute,
- $\pi$ is equivariant with respect to $G$ and $H$ (where $H$ acts trivially on $\widetilde{X}$),
- $\epsilon$ is equivariant with respect to $G$ and $H$ (where $G$ acts trivially on $\widetilde{Y}$), and
- $\pi$ describes a *$G$-equivariant principal $H$-bundle* over $\widetilde{X}$.

That is, the diagram describes a bibundle from the topological groupoid $G \mathbin{\backslash\!\backslash} \widetilde{X}$ to $H \mathbin{\backslash\!\backslash} \widetilde{Y}$; i.e, it is a "stacky" presentation of $f$ in terms of the chosen covers.

Taking cohomology (or in the derived context, cochains) gives

$$\operatorname{Spec} H^*\widetilde{X} \xleftarrow{\pi} \operatorname{Lift}(f) \times \operatorname{Spec} H^*\widetilde{X} \xrightarrow{\epsilon} \operatorname{Spec} H^*\widetilde{Y},$$

exhibiting a bibundle between groupoid schemes, i.e., representing a map

$$G \mathbin{\backslash\!\backslash} \operatorname{Spec} H^*\widetilde{X} \to H \mathbin{\backslash\!\backslash} \operatorname{Spec} H^*\widetilde{Y}$$

of stacks.

**5.5. Example.** Applying this to our set-up in the case of a map $f : * \approx Be \to BU(1)$ gives

$$\mathcal{X} \xleftarrow{\pi} (\mathrm{GL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \times \mathcal{X} \xrightarrow{\epsilon} \mathcal{X} \times \mathbb{C}$$

with $\epsilon((B,n),t) = (Bt, nBt)$. The groups $G = \mathrm{GL}_2(\mathbb{Z})$ and $H = \mathrm{GL}_2(\mathbb{Z}) \rtimes \mathbb{Z}^2$ act on $\mathbb{Z}^2 \times \mathcal{X}$ by $A \cdot ((B,n),t) = ((BA^{-1}, n), At)$ and $((B,n),t) \cdot (A', m) = ((A'B, (n+m)A'^{-1}), t)$.

**5.6. *Remarks on the 1-dimensional case.*** We can carry out the analogue of our constructions in the case that $\Sigma$ is a circle rather than a torus. The relevant calculations can be read off from Theorem 7.6. The main differences are that in this case we take

$$\mathcal{X} = \mathcal{X}_e = \{\, t \in \mathbb{C} \mid \mathbb{R}t = \mathbb{C} \,\} = \mathbb{C} \smallsetminus \{0\}.$$

Then we easily discover that $\mathcal{M}_{U(1)}$ is the "universal multiplicative group" living over $\mathcal{M}_e \approx (\operatorname{Aut}(\mathbb{G}_m) \mathbin{\backslash\!\backslash} *) \approx (\{\pm 1\} \mathbin{\backslash\!\backslash} *)$. The central extension groups turn out to be invisible from this point of view, since $\mathcal{M}_{K(\mathbb{Z},2)} \approx \mathcal{M}_e$.

## 6. Some spaces and groups

The spaces $X = B\mathcal{W}_0(\widetilde{G})$ that we need to deal with are simply connected 3-types such that $\pi_2$ and $\pi_3$ are finitely generated and free. We first discuss some general facts and conventions about such spaces, concluding with the calculation of $H^*(X; \mathbb{Q})$ in terms of the Whitehead product in $\pi_* X$ in good cases; all of this material is surely standard. We next describe an explicit topological group model for the central extensions $\widetilde{G} = U(1)^d \times_\phi K(2, \mathbb{Z})$ that we need to consider.

**6.1.** ***Simply connected 3-types with all homotopy groups finitely generated and free.*** Let $\mathcal{C}$ denote the full subcategory of spaces $X$ which are

(i) simply connected,

(ii) have $\pi_k X \approx 0$ for $k \geq 4$, and

(iii) have $\pi_2 X$ and $\pi_3 X$ which are finitely generated free abelian groups.

Write $h\mathcal{C}$ for the associated homotopy category.

For $X \in \mathcal{C}$ with $\pi_2 X = B$ and $\pi_3 X = C$, the Whitehead product (Section 10.7) $[-, -] : \pi_2 X \times \pi_2 X \to \pi_3 X$ defines a bilinear symmetric form

$$\beta : B \otimes B \to C.$$

Precomposition with the Hopf map $\eta : \pi_2 X \to \pi_3 X,\ \eta \in \pi_3 S^2$ is a function

$$\phi : B \to C,$$

quadratic in the sense of Section 3.6, which satisfies

$$\phi(y + y') = \phi(y) + \beta(y, y') + \phi(y').$$

(This identity fixes our preferred choice of generator $\eta$ of $\pi_3 S^2$.) We call $\phi$ the *quadratic invariant* of $X$, and $\beta$ the associated *Hessian form*.

It is classical that the data of $(B, C, \phi)$ is a complete invariant for the homotopy type of $X \in \mathcal{C}$. In fact, $X \mapsto \phi$ defines an equivalence between the homotopy category $h\mathcal{C}$ of such spaces, and the category of quadratic functions between finitely generated free groups.

Let

$$i : K(C, 3) \to X \quad \text{and} \quad j : X \to K(B, 2)$$

be maps, unique up to homotopy, which induce identity on the relevant homotopy groups. We can furthermore extend to a fibration sequence

$$X \xrightarrow{j} K(B, 2) \xrightarrow{\psi} K(C, 4),$$

i.e., so that $j$ is identified with the tautological map from the homotopy fiber of $\psi$.

**6.2. Proposition.** *Let $X \in \mathcal{C}$ with quadratic invariant $\phi$ and Hessian form $\beta$, and consider $b, b' \in B = \pi_2 X$. There exists a homotopy commutative diagram*

$$\begin{array}{ccccccc}
S^3 & \xrightarrow{w} & S^2 \vee S^2 & \rightarrowtail & S^2 \times S^2 & \twoheadrightarrow & S^4 \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (b,b')} & & {\scriptstyle g}\downarrow & & {\scriptstyle \widetilde{f}}\downarrow \\
K(C,3) & \xrightarrow[i]{} & X & \xrightarrow[j]{} & K(B,2) & \xrightarrow[\psi]{} & K(C,4)
\end{array}$$

*where $w$ is the universal Whitehead product, and $\widetilde{f} : S^4 \approx S^3 \wedge S^1 \to K(C,4)$ is adjoint to $f : S^3 \to \Omega K(C,4) \approx K(C,3)$. Furthermore,*

(1) *$g_* : H_4(S^2 \times S^2) \to H_4 K(B,2) \approx \Gamma_2 B \subseteq (B \otimes B)^{\Sigma_2}$ sends $[S^2] \times [S^2]$ to $b \otimes b' + b' \otimes b$,*

(2) *$\widetilde{f}_* : H_4 S^4 \to H_4 K(C,4) \approx C$ sends $[S^4]$ to $\beta(b,b')$, and*

(3) *$\psi_* : H_4 K(B,2) \approx \Gamma_2 B \to H_4 K(C,4) \approx C$ coincides with $\widetilde{\phi} : \Gamma_2 B \to C$, the homomorphism associated to $\phi$ as defined in Section 3.4.*

*Proof.* We are using the tautological identification $H_4 K(B,2) \approx (B \otimes B)^{\Sigma_2}$ dual to $H^4 K(B,2) \approx (B \otimes B)_{\Sigma_2}$ defined by the cup product. With respect to this identification, the H-space structure on $K(B,2)$ induces a Pontryagin product $H_2 K(B,2) \otimes H_2 K(B,2) \to H_4 K(B,2)$ given by $b \otimes b' \mapsto b \otimes b' + b' \otimes b : B \otimes B \to \Gamma_2(B)$.

The construction of the diagram is straightforward. In particular, we can use the H-space structure on $K(B,2)$ to define $g$ as the composite

$$S^2 \times S^2 \xrightarrow{b \times b'} K(B,2) \times K(B,2) \to K(B,2),$$

from which statement (1) follows immediately. Statement (2) is immediate from the fact that $f$ and $\widetilde{f}$ are adjoint, and that $if = [b,b'] : S^3 \to X$. Statement (3) then follows from the commutativity of the diagram. □

Thus, any $X \in \mathcal{C}$ is the homotopy fiber of the characteristic class in $H^4(K(B,2),C)$ corresponding to its quadratic invariant.

**6.3.** ***Rational cohomology ring of*** **$X \in \mathcal{C}$.** Say that a quadratic function $\phi : B \to C$ is *regular* if the function

$$\widetilde{\phi}^* : C^* \otimes \mathbb{Q} \to \operatorname{Hom}(\Gamma_2 B, \mathbb{Q}) \approx \operatorname{Sym}^2(B^* \otimes \mathbb{Q})$$

dual to $\widetilde{\phi} : \Gamma_2 B \to C$ sends some basis of $C^* \otimes \mathbb{Q}$ to a regular sequence in the ring $\operatorname{Sym}(B^* \otimes \mathbb{Q})$.

**6.4. Remark.** If $B = \mathbb{Z}^d$, $C = \mathbb{Z}^e$, and $\phi(y) = (\phi_1(y), \ldots, \phi_e(y))$ with $\phi_k(y) = \frac{1}{2}\sum_{i,j} c^k_{ij} y_i y_j$, then $\phi$ is regular if and only if the sequence $\phi_1(y), \ldots, \phi_e(y)$ forms a regular sequence in $\mathbb{Q}[y_1, \ldots, y_d]$.

In particular, if $e = 1$, then $\phi$ is regular if and only if $\phi \neq 0$.

**6.5. Proposition.** *Let $X \in \mathcal{C}$ with quadratic invariant $\phi$. Then the map $j^* : H^*(K(B,2);\mathbb{Q}) \to H^*(X;\mathbb{Q})$ factors through*

$$\mathrm{Sym}(B^* \otimes \mathbb{Q})/(\widetilde{\phi}^*(C^* \otimes \mathbb{Q})) \to H^*(X;\mathbb{Q}),$$

*where $\widetilde{\phi}^* : C^* \to (\Gamma_2 B)^*$ is the $\mathbb{Z}$-dual to $\widetilde{\phi}$. Furthermore, the above map is an isomorphism of rings when $\phi$ is regular.*

*Proof.* The Serre spectral sequence for the fibration sequence $K(C,3) \xrightarrow{i} F \xrightarrow{j} K(B,2)$ has

$$E_2 = E_4 = H^*\big(K(B,2); H^*(K(C,3);\mathbb{Q})\big) \approx \mathrm{Sym}(B^* \otimes \mathbb{Q}) \otimes \Lambda(C^* \otimes \mathbb{Q}).$$

The first nontrivial differential is $d_4 : E_4^{0,3} \to E_4^{4,0}$, which is $\pm\widetilde{\phi}^*$, by Proposition 6.2. The regularity condition is what is needed for $E_5^{p,q}$ with $q > 0$ to vanish, so that the spectral sequence collapses to $E_\infty^{*,*} = E_5^{*,0}$. □

**6.6. *An explicit group model for central extensions.*** Every space $X \in \mathcal{C}$ is equivalent to the classifying space of a topological group. We give an explicit construction of such a group as a central extension. In particular, given a bilinear map $\omega : B \otimes B \to C$ between finitely generated free groups, we construct a topological group $G_\omega$ so that $X = BG_\omega \in \mathcal{C}$ has quadratic invariant $\phi$ with $\omega$ as its bilinear extension, and thus sits in a fiber sequence $K(C,3) \to X \to K(B,2) \xrightarrow{\widetilde{\phi}} K(C,4)$. In particular, this produces an explicit model for our extension groups $U(1)^d \times_\phi K(\mathbb{Z},2)$.

Let $K(B,1)_\bullet$ and $K(C,2)_\bullet$ be simplicial abelian groups, degreewise free, together with identifications $B \approx \pi_1 K(B,1)_\bullet$ and $C \approx \pi_2 K(C,2)_\bullet$, and all other homotopy groups trivial. There exists a map

$$\kappa : K(B,1)_\bullet \otimes K(B,1)_\bullet \to K(C,2)_\bullet$$

of simplicial abelian groups inducing $\omega$ on $\pi_2$, which is unique up to homotopy. We fix such a choice of $\kappa$.

Consider the composite map of simplicial sets

$$K(B,1)_\bullet \times K(B,1)_\bullet \xrightarrow{(x,y)\mapsto x\otimes y} K(B,1)_\bullet \otimes K(B,1)_\bullet \xrightarrow{\kappa} K(C,2)_\bullet.$$

Taking geometric realization produces a map of spaces which we also denote

$$\kappa : K(B,1) \times K(B,1) \to K(C,2),$$

which is a bilinear map between topological abelian groups (and so factors through $K(B,1) \wedge K(B,1)$). Let $G_\omega$ be the space $K(B,1) \times K(C,2)$ with group law[7]

$$(y,x)\cdot(y',x') := (y+y', -\kappa(y,y')+x+x'), \quad (y,x),(y',x') \in G_\omega.$$

Note that inversion in $G_\omega$ is given by

$$(y,x)^{-1} = (-y, -\kappa(y,y)-x),$$

while the commutator is given by

$$(y,x)\cdot(y',x')\cdot(y,x)^{-1}\cdot(y',x')^{-1} = (0, -\kappa(y,y')+\kappa(y',y)). \tag{6.7}$$

Thus $G_\omega$ is a central extension of $K(B,1)$ by $K(C,2)$, and we have evident isomorphisms $\pi_1 G_\omega \approx B$ and $\pi_2 G_\omega \approx C$.

The commutator $G_\omega \wedge G_\omega \to G_\omega$ defines the Samelson product

$$\langle -,- \rangle : \pi_p G_\omega \times \pi_q G_\omega \to \pi_{p+q} G_\omega.$$

**6.8. Proposition.** *The Samelson product $\pi_1 G_\omega \times \pi_1 G_\omega \to \pi_2 G_\omega$ is given by*

$$\langle b, b' \rangle = -\omega_{\text{sym}}(b,b') := -\omega(b,b') - \omega(b',b).$$

*Proof.* The map $\kappa : K(B,1) \wedge K(B,1) \to K(C,2)$ induces $\omega$ on homotopy groups by construction, and therefore $(y,y') \mapsto \kappa(y',y)$ induces $(b,b') \mapsto -\omega(b',b)$ on homotopy groups, with sign introduced by switching the order of the two classes in $\pi_1 G_\omega$. The result follows from (6.7). □

**6.9. Proposition.** *Let $X = BG_\omega$. Then the Whitehead product $\pi_2 X \times \pi_2 X \to \pi_3 X$ is given by*

$$[b,b'] = \omega_{\text{sym}}(b,b') = \omega(b,b') + \omega(b',b).$$

*Proof.* This is a special case of the relation between the Whitehead and Samelson products (10.9); in this dimension, the two differ by a sign. □

Thus, the space $X = BG_\omega$ has quadratic invariant $\phi : B \to C$, with associated Hessian form $\beta = \omega_{\text{sym}} : B \otimes B \to C$.

## 7. The main theorem

In this section we restate our main theorem, Theorem 3.7, but in terms of our explicit models for $G$, and in somewhat more generality, in that we allow $\Sigma$ to be a torus of rank other than 2.

[7] The group $G_\omega$ really depends on the choice of $\kappa$, but all our computations about it will only depend on $\omega$.

**7.1.** ***The group*** $W^T(G)$**.** Fix a finitely generated free abelian group $L$, and let $T := L \otimes \mathbb{T}$ be the associated torus. We consider the semidirect product group $D(T) := \mathrm{Aut}(T) \ltimes T$ with group law

$$(A, t) \cdot (A', t') := (AA', (A')^{-1}t + t').$$

Note that $D(T)$ acts on the space $T$ by

$$(A, t) \cdot s = A(s + t).$$

Given a topological group $G$, we define a group

$$W^T(G) := \mathrm{Map}(T, G) \rtimes D(T)$$

with group law given by

$$(g, f) \cdot (g', f') = \big(\big(s \mapsto g(s) \cdot g'(f^{-1}(s))\big), ff'\big).$$

We write $W_0^T(G)$ for the identity component of $W^T(G)$, and write $\overline{W}^T(G)$ for the quotient $W^T(G)/W_0^T(G)$.

Given $\omega : B \otimes B \to C$, we will compute the homotopy type of the classifying space $BW_0^T(G_\omega)$, together with the evident action of $\overline{W}^T(G_\omega)$ on its homotopy groups.

**7.2.** ***Homology and cohomology of*** $T$**.** Because $T$ is an abelian group, $H_*T$ is naturally a graded commutative Hopf algebra. The iterated coproduct $\psi : H_pT \to H_1T \otimes \cdots \otimes H_1T$ gives an identification of $H_pT$ with the antisymmetric invariants $\Lambda_p L \subseteq L^{\otimes p}$. In terms of this identification the Pontryagin product $H_1T \otimes H_1T \to H_2T$ is given by $t \otimes t' \mapsto t \wedge t' := t \otimes t' - t' \otimes t \in \Lambda_2 L$. This is a direct consequence of the fact that $H_*T$ is a *graded* Hopf algebra:

$$\psi(tt') = \psi(t)\psi(t') = (t\otimes 1 + 1\otimes t)(t'\otimes 1 + 1\otimes t') = tt'\otimes 1 + (t\otimes t' - t'\otimes t) + 1\otimes tt'.$$

The Kronecker pairing $(-,-) : H^*(T; B) \otimes H_*T \to B$ then gives an identification $H^p(T; B) \xrightarrow{\sim} \mathrm{Hom}(H_pT, B) = \mathrm{Hom}(\Lambda_p L, B)$. We note the following formula for the cup product in these terms, which involves a tricky sign.

**7.3. Proposition.** *Let* $f \in H^1(T; B)$ *and* $f' \in H^1(T; B')$ *be cohomology classes corresponding to* $m \in \mathrm{Hom}(L, B)$ *and* $m' \in \mathrm{Hom}(L, B')$ *via the Kronecker pairing. Then with respect to the Kronecker pairing, the cup product* $f \smile f' \in H^2(T; B \otimes B')$ *corresponds to*

$$(\Lambda_2 L \hookrightarrow L \otimes L \xrightarrow{-m\otimes m'} B \otimes B') \in \mathrm{Hom}(\Lambda_2 L, B \otimes B').$$

*Thus,* $f \smile f'$ *corresponds to the function*

$$t \wedge t' \mapsto (-m \otimes m')(t \wedge t') = -m(t) \otimes m'(t') + m(t') \otimes m'(t).$$

*Proof.* Using the *graded* Kronecker pairing

$$(-,-) : (H^*(T;\,B) \otimes H^*(T;\,B')) \otimes (H_*T \otimes H_*T) \to B \otimes B'$$

we have

$$(f \smile f', u) = \left(f \otimes f', \sum v \otimes v'\right) = -\sum (f, v)(f', v'),$$

for $u \in H_2T$ where $\psi(u) = \sum v \otimes v' \in H_1T \otimes H_1T$ (the component of the coproduct in degree (1, 1)). In terms of our identifications, $\psi : H_2T \to H_1T \otimes H_1T$ is the inclusion $\Lambda_2 L \to L \otimes L$, and the formula follows.

(The additional sign here comes from a conflict of two sign conventions: the graded Kronecker pairing $H^1(T;\,B) \otimes H^1(T;\,B') \otimes H_1T \otimes H_1T \to B \otimes B'$, which is what is used to identify the coproduct $\psi$ as dual to cup product, and which introduces a sign, vs. the evaluation pairing $\operatorname{Hom}(\Lambda_2 L, B) \otimes \operatorname{Hom}(\Lambda_2 L, B') \otimes L \otimes L \to B \otimes B'$, which does not introduce a sign.) □

The Kronecker pairing generalizes to the “slant product”

$$f, v \mapsto f \nabla v : H^{p+q}(T;\,M) \times H_qT \to H^p(T;\,M)$$

by

$$f \nabla v = \sum f'(f'', v),$$

where $f \in H^{p+q}(T;\,M)$, $v \in H^qT$, and $\sum f' \otimes f'' = \operatorname{Mult}_* f$, the image of $f$ under the map $H^*(T;\,M) \to H^*(T \times T;\,M) \approx H^*(T;\,M) \otimes H^*T$ induced by multiplication in $T$. Thus if $|f| = |v|$ then $f \nabla v = (f, v)$.

Given $n \in \operatorname{Hom}(\Lambda_k L, M)$ and $t \in L$, we define the contraction operation $n \nabla t \in \operatorname{Hom}(\Lambda_{k-1} L, M)$ by

$$(n \nabla t)(\tau) := n(t \wedge \tau).$$

**7.4. Proposition.** *With respect to the usual identifications $H_1T = L$ and $H^k(T;M) = \operatorname{Hom}(\Lambda_k L, M)$, the slant product coincides with the contraction pairing.*

*Proof.* Let $f \in H^k(T;\,M)$ so $n = (f, -) : H_kT = \Lambda_k L \to M$. For $t \in H_1T = L$ and $u \in H_{k-1}T = \Lambda_{k-1}L$ we have

$$\begin{aligned}
(f \nabla t, u) &= \left(\sum f'(f'', t), u\right) = \sum (f', u)(f'', t) \\
&= (-1)^{k-1} \sum (f' \otimes f'', u \otimes t) \\
&= (-1)^{k-1} (f, \operatorname{Mult}_*(u \otimes t)) = (-1)^{k-1}(f, u \wedge t) = (f, t \wedge u) \\
&= n(t \wedge u) = (n \nabla t)(u),
\end{aligned}$$

so $f \nabla v$ corresponds to $n \nabla v$. □

For instance, if $k = 2$, then $H^2(T; M) \otimes H_1 T \to H^1(T; M)$ is described by $(n\nabla t)(t') = n(t \wedge t') = n(t \otimes t' - t' \otimes t)$.

Finally, given a bilinear map $\gamma : L \otimes L \to C$, we will use the same symbol $\gamma$ for its restriction $\Lambda_2 L \to C$ (e.g., $\gamma = \omega(m \otimes m')$ in the statement of the Theorem 7.6 below).

**7.5. *The homotopy groups of* $BW_0^T(G_\omega)$.** We now describe $\pi_* BW_0^T(G_\omega)$, its quadratic invariant, the evident action of $\overline{W}(G_\omega)$ on homotopy groups, and its cohomology ring. After this we briefly explain how Theorem 3.7 is read off from this calculation.

**7.6. Theorem.** *Let $\omega : B \otimes B \to C$ be a bilinear function with associated quadratic function $\phi$ and Hessian form $\beta$, and let $G_\omega$ be a topological group associated with $\omega$ as in Section 6.6. The space $X = BW_0^T(G_\omega)$ is an object of $\mathcal{C}$, with*

$$\pi_3 X \approx C, \quad \pi_2 X \approx L \times B \times \mathrm{Hom}(L, C),$$

*and with quadratic invariant $\phi^\sharp : \pi_2 X \to \pi_3 X$ given by*

$$\phi^\sharp(t, y, x) = \phi(y) + xt, \qquad t \in L, \quad y \in B, \quad x \in \mathrm{Hom}(L, C).$$

*Furthermore we have*

$$\overline{W}^T(G_\omega) \approx \mathrm{Aut}(L) \ltimes E,$$

*where $E$ is a group with underlying set*

$$E = \mathrm{Hom}(L, B) \times \mathrm{Hom}(\Lambda_2 L, C)$$

*and group law*

$$(m, n) \cdot (m', n') = (m + m', \omega(m \otimes m') + n + n'),$$
$$m, m' \in \mathrm{Hom}(L, B), \quad n, n' \in \mathrm{Hom}(\Lambda_2 L, C),$$

*while the semidirect product is defined via the action of* $\mathrm{Aut}(L)$ *on $E$ given by*

$$A \propto (m, n) = (mA^{-1}, n(\Lambda_2 A^{-1})).$$

*The action of $E \subseteq \overline{W}^T(G_\omega)$ on $\pi_* X$ is given by*

$$(m, n) \propto c = c, \qquad c \in \pi_3 X$$
$$(m, n) \propto (t, y, x) = (t, y + mt, x - \beta(y, m) - \omega(mt, m) + n\nabla t), \quad (t, y, x) \in \pi_2 X,$$

*while the action of* $\mathrm{Aut}(L) \subseteq \overline{W}^T(G_\omega)$ *on $\pi_* X$ is given by*

$$A \propto c = c, \qquad c \in \pi_3 X,$$
$$A \propto (t, y, x) = (At, y, xA^{-1}), \quad (t, y, x) \in \pi_2 X.$$

**7.7. Remark.** The central extension $E$ corresponds to the cocycle

$$\gamma : \mathrm{Hom}(L, B) \times \mathrm{Hom}(L, B) \to \mathrm{Hom}(\Lambda^2 L, C)$$

defined by $(m, m') \mapsto \omega(m \otimes m')$. Up to isomorphism, this central extension depends only on the antisymmetrization of $\gamma$, which satisfies $\gamma_{\mathrm{antisym}}(m, m') = \beta(m \otimes m') - \beta(m' \otimes m)$ and so depends only on the quadratic function $\phi$.

The proof of Theorem 7.6 is given in the next several sections, culminating in Section 9.5.

It is straightforward to check that $\phi^\sharp$ is regular. For instance, if we choose coordinates and write $t = (t_1, \dots, t_r) \in L = \mathbb{Z}^r$, $(y_1, \dots, y_d) \in B = \mathbb{Z}^d$, $C = \mathbb{Z}^e$, and $(x_{11}, \dots, x_{re}) \in \mathrm{Hom}(L, C) = \mathbb{Z}^{r\times e}$, and write $\phi = (\phi_1, \dots, \phi_e)$, then $\phi^\sharp = (\phi_1^\sharp, \dots, \phi_e^\sharp)$, where $\phi_k^\sharp(t, y, x) = \phi_k(y) + t_1 x_{1k} + \cdots + t_r x_{rk}$. This sequence of polynomials is easily seen to be regular (when $r \geq 1$); in fact, it remains regular after passing to the quotient ring in which $y_1 = \cdots = y_d = 0$. From Proposition 6.5 we obtain the following.

**7.8. Corollary.** *If the rank of $T$ is positive, then*

$$\begin{aligned} H^*(BW_0^T(G_\omega); \mathbb{Q}) &\approx \mathrm{Sym}\big((L \times B \times \mathrm{Hom}(L, C))^* \otimes \mathbb{Q}\big)/(\textit{image of } (\widetilde{\phi}^\sharp)^*) \\ &\approx \mathbb{Q}[t_1, \dots, t_r, y_1, \dots, y_d, x_{11}, \dots, x_{re}]/(\phi_1^\sharp, \dots, \phi_e^\sharp), \end{aligned}$$

*with the evident action by $\overline{W}^T(G_\omega)^{\mathrm{op}}$.*

We obtain the statement of Theorem 3.7 by:

- setting $B = \mathbb{Z}^d$, so $K(B, 1) \approx U(1)^d$,
- setting $C = \mathbb{Z}$,
- taking $\omega : B \otimes B \to C$ to be any quadratic refinement of $\phi : B \to C$,
- setting $T = \mathbb{T}^2$ so $L = H_1\mathbb{T}^2 = \mathbb{Z}^2$, and
- identifying $\mathbb{Z} \approx \Lambda_2 L$ via the generator $e_1 \wedge e_2 \in \Lambda_2 L$, where $e_1 = (1, 0)$, $e_2 = (0, 1) \in \mathbb{Z}^2 = L$.

This last identification implies that for $n \in \mathbb{Z} \approx \mathrm{Hom}(\Lambda^2 L, \mathbb{Z})$ and $t = (t_1, t_2) \in L$, we have that $(n\nabla t)(e_1) = -t_2$ and $(n\nabla t)(e_2) = t_1$. With these choices, $W^T(G_\omega)$ is a model for $\mathcal{W}^\Sigma(U(1)^d \times_\phi K(\mathbb{Z}, 2))$, and the results of Theorem 3.7 follow.

## 8. Computation of $\pi_* \mathrm{Map}(\Sigma, G_\omega)$

We fix a topological group $G = G_\omega$ associated to a homomorphism $\omega : B \otimes B \to C$. In this section we compute invariants of the space $\mathrm{Map}(\Sigma, G_\omega)$ for an arbitrary space $\Sigma$.

**8.1.** ***Bilinear cohomology operations.*** Any bilinear map $\alpha : B \otimes B' \to C$ induces a bilinear cohomology operation

$$\widetilde{\alpha} : H^p(-; B) \times H^q(-; B') \to H^{p+q}(-; C)$$

by

$$\widetilde{\alpha}(x, y) := \alpha_*(x \smile y),$$

where the cup product is $\smile: H^p(-; B) \times H^q(-; B') \to H^{p+q}(-; B \otimes B')$, and $\alpha_* : H^*(-; B \otimes B') \to H^*(-; C)$ is the natural map induced by the homomorphism on coefficients.

**8.2. Remark.** We note that

$$\widetilde{\alpha}(x, y) = (-1)^{|x||y|}(\widetilde{\alpha\tau})(y, x),$$

where $\tau : B' \otimes B \xrightarrow{\sim} B \otimes B'$ is the evident transposition map. In particular, if $\alpha : B \otimes B \to C$ is symmetric, then $\widetilde{\alpha}(x, y) = (-1)^{|x||y|}\widetilde{\alpha}(y, x)$.

**8.3.** ***The functors*** $\pi_*$ **Map**$(-, G_\omega)$**.** We are going to describe $\pi_k \operatorname{Map}(-, G_\omega)$ (basepoint at the identity element) as functors on the homotopy category of spaces, together with their Samelson products.

**8.4. Proposition.** *Let $\Sigma$ be an arbitrary space.*

(1) *We have natural bijections of sets*

$$\pi_k \operatorname{Map}(\Sigma, G_\omega) \approx H^{1-k}(\Sigma; B) \times H^{2-k}(\Sigma; C)$$

*for all $k$.*

(2) *The group law on $\pi_k \operatorname{Map}(\Sigma, G_\omega)$ for $k \geq 1$ is the evident additive one. The group law on $\pi_0$ is given by*

$$(u, v) \cdot (u', v') = (u + u', -\widetilde{\omega}(u, u') + v + v').$$

(3) *Samelson products $\pi_p \times \pi_q \to \pi_{p+q}$ on $\operatorname{Map}(\Sigma, G_\omega)$ are given, in terms of the above bijection, by*

$$\langle (y, x), (y', x') \rangle = \big(0, -(-1)^{p(1-q)}\widetilde{\omega_{\text{sym}}}(y, y')\big).$$

Of course, $\pi_k \operatorname{Map}(\Sigma, G_\omega) \approx 0$ for $k \geq 3$, and the first factor is always 0 in $\pi_2 \operatorname{Map}(\Sigma, G_\omega)$. In low dimensions, the Samelson products take the form

$$\begin{aligned}
\langle (u, v), (y, x) \rangle &= (0, -\widetilde{\omega_{\text{sym}}}(u, y)) = (0, -\widetilde{\omega_{\text{sym}}}(y, u)), && \pi_0 \times \pi_1 \to \pi_0, \\
\langle (y, x), (u, v) \rangle &= (0, \widetilde{\omega_{\text{sym}}}(y, u)) = (0, \widetilde{\omega_{\text{sym}}}(u, y)), && \pi_1 \times \pi_0 \to \pi_1, \\
\langle (y, x), (y', x') \rangle &= -\widetilde{\omega_{\text{sym}}}(y, y') = \widetilde{\omega_{\text{sym}}}(y', y), && \pi_1 \times \pi_1 \to \pi_2.
\end{aligned}$$

The Samelson product $\pi_0 \times \pi_2 \to \pi_2$ is trivial.

**8.5. Remark.** We record here the inversion formula for $\pi_0 \operatorname{Map}(\Sigma, G_\gamma)$:

$$(u, v)^{-1} = (-u, -\widetilde{\omega}(u, u) - v).$$

**8.6. Remark.** Suppose $\Sigma = T$ is a torus so that as in Section 7.2 we have identifications $L = H_1 T$, and

$$H^1(\Sigma; B) \xrightarrow{\sim} \operatorname{Hom}(L, B), \quad H^2(\Sigma; C) \xrightarrow{\sim} \operatorname{Hom}(\Lambda_2 L, C).$$

Then

$$\pi_0 \operatorname{Map}(T, G_\omega) \approx \operatorname{Hom}(L, B) \times \operatorname{Hom}(\Lambda_2 L, C)$$

with group law given by

$$(m, n) \cdot (m', n') = (m + m', \omega(m \otimes m') + n + n'),$$
$$m, m' \in \operatorname{Hom}(L, B), \quad n, n' \in \operatorname{Hom}(\Lambda_2 L, C),$$

where $\omega(m \otimes m')$ represents the composite $\Lambda_2 L \hookrightarrow L \otimes L \xrightarrow{m \otimes m'} B \otimes B \xrightarrow{\omega} C$. The change sign in the formula is a consequence of Proposition 7.3.

**8.7. Remark.** Let $X = B \operatorname{Map}(T, G_\omega)$. Then we have $\pi_k X = \pi_{k-1} \operatorname{Map}(T, G_\omega)$. The identifications of Proposition 8.4 give

$$\pi_1 X \approx \operatorname{Hom}(L, B) \times \operatorname{Hom}(\Lambda_2 L, C), \quad \pi_2 X \approx B \times \operatorname{Hom}(L, C), \quad \pi_3 X \approx C,$$

with group law on $\pi_1 X = \pi_0 \operatorname{Map}(T, G_\omega)$ given as above.

Tracing through: with the relation between Samelson products and Whitehead products (Section 10.8), the expression of $\pi_1 \curvearrowright \pi_n$ in terms of Whitehead products (Section 10.7), and the relation between Whitehead products and the quadratic invariant (Section 6.1), together with the identifications and formulas of Proposition 8.4, we find that:

- The action of $\pi_1 X$ on $\pi_2 X$ is given by
$$(m, n) \propto (y, x) = (y, x) + [(m, n), (y, x)] = (y, x) + \langle (m, n), (y, x) \rangle$$
$$= (b, x - \beta(y, m)).$$
- The action of $\pi_1 X$ on $\pi_3 X$ is trivial.
- The quadratic invariant $\phi^\sharp : \pi_2 X \to \pi_3 X$ is given by $\phi^\sharp(y, x) = \phi(y)$.

*Proof of Proposition 8.4, parts* (1) *and* (2). The bijections of (1) are immediate from the description of $G_\omega$. The formula for the group law in (2) amounts to the fact that the topological cocycle $-\kappa : K(B, 1) \times K(B, 1) \to K(C, 2)$ represents the cohomology operation $-\widetilde{\omega}$. That the group structure for $\pi_{*\geq 1}$ is the additive one is straightforward; i.e., $\Omega G \approx K(B, 0) \times K(C, 1)$ as an $H$-space. □

It remains to prove part (3) of Proposition 8.4, for which we need a suspension map.

**8.8.** ***The suspension map.*** There is a natural *suspension map*

$$S_k : \pi_p \operatorname{Map}(\Sigma, X) \to \pi_{p-k} \operatorname{Map}(\Sigma \times S^k, X)$$

induced by the evident inclusion

$$\operatorname{Map}_*(S^k, \operatorname{Map}(\Sigma, X)) \subseteq \operatorname{Map}(S^k, \operatorname{Map}(\Sigma, X)) \approx \operatorname{Map}(\Sigma \times S^k, X).$$

We compute the effect of suspension for $X = G_\omega$ in terms of the isomorphisms of Proposition 8.4(1).

**8.9. Proposition.** *When $X = G_\omega$, the suspension map $S_k$ is given by*

$$S_k(x, y) = (x \times \epsilon_k, y \times \epsilon_k), \qquad x \in H^{1-p}(\Sigma; B), \quad y \in H^{2-p}(\Sigma; C),$$

*where $\epsilon_k \in H^k S^k$ is the canonical generator. In particular, the suspension map is injective.*

*Proof.* Using $G_\omega = K(B, 1) \times K(C, 2)$, we see that $S_k$ is the same as the suspension map in cohomology. (See Section 10.5.) □

**8.10.** ***Computation of Samelson products, and proof of Proposition 8.4*(3).** When $p = q = 0$, the Samelson product is just the commutator on $\pi_0 \operatorname{Map}(\Sigma, G_\omega)$. Thus, in terms of Proposition 8.4(1),

$$\langle (y, x), (y', x') \rangle = (0, -\widetilde{\omega}(y, y') + \widetilde{\omega}(y', y)) = (0, -\widetilde{\omega_{\mathrm{sym}}}(y, y')).$$

The last equality is because $\widetilde{\gamma_{\mathrm{sym}}} : H^1(\Sigma; B) \times H^1(\Sigma; B) \to H^2(\Sigma; C)$ is the *antisymmetrization* of $\widetilde{\gamma}$, for degree reasons (Remark 8.2).

**8.11. Proposition.** *Let $G$ be a topological group. There is a commutative diagram of the form*

$$\begin{array}{ccc}
\pi_p \operatorname{Map}(\Sigma, G) \times \pi_q \operatorname{Map}(\Sigma, G) & \xrightarrow{\langle -,- \rangle} & \pi_{p+q} \operatorname{Map}(\Sigma, G) \\
\Big\downarrow{\scriptstyle S_p \times S_q} & & \Big\downarrow{\scriptstyle S_{p+q}} \\
\pi_0 \operatorname{Map}(\Sigma \times S^p, G) \times \pi_0 \operatorname{Map}(\Sigma \times S^q, G) & & \pi_0 \operatorname{Map}(\Sigma \times S^{p+q}, G) \\
\Big\downarrow{\scriptstyle \pi_p^* \times \pi_q^*} & & \Big\downarrow{\scriptstyle \pi_{p,q}^*} \\
\pi_0 \operatorname{Map}(\Sigma \times S^p \times S^q, G) \times \pi_0 \operatorname{Map}(\Sigma \times S^p \times S^q, G) & \xrightarrow{\langle -,- \rangle} & \pi_0 \operatorname{Map}(\Sigma \times S^p \times S^q, G)
\end{array}$$

*where $\pi_p : S^p \times S^q \to S^p$, $\pi_q : S^p \times S^q \to S^q$, and $\pi_{p,q} : S^p \times S^q \to S^p \wedge S^q \approx S^{p+q}$ are the evident projections.*

*Proof.* The underlying point-set diagram commutes. □

*Proof of Proposition 8.4*(3). We have already proved the commutator formula for $\pi_0$ above. Inserting $(y, x) \in \pi_p \operatorname{Map}(\Sigma, G)$ and $(y', x') \in \pi_q \operatorname{Map}(\Sigma, G)$, going around the upper/right side of the square of Proposition 8.11 gives

$$\langle (y, x), (y', x') \rangle \times \epsilon_p \times \epsilon_q,$$

while going around the left/lower side gives

$$\big(0, -\widetilde{\omega_{\mathrm{sym}}}\big(y \times \epsilon_p \times 1, y' \times 1 \times \epsilon_q\big)\big) = \big(0, -(-1)^{p(1-q)}\widetilde{\omega_{\mathrm{sym}}}(y, y') \times \epsilon_p \times \epsilon_q\big),$$

since $y' \in H^{1-q}(\Sigma; B)$, using the formula for $\pi_0$. Therefore we arrive at the formula

$$\langle (y, x), (y', x') \rangle = (0, -(-1)^{p(1-q)}\widetilde{\omega_{\mathrm{sym}}}(y, y')),$$
$$(y, x) \in \pi_p \operatorname{Map}(\Sigma, G), \quad (y', x') \in \pi_q \operatorname{Map}(\Sigma, G). \quad \square$$

**8.12.** ***A desuspension map.*** Consider the basepoint preserving map

$$D : \operatorname{Map}(\Sigma \times S^1, G) \to \Omega \operatorname{Map}(\Sigma, G)$$

defined by

$$(Df)(t)(s) := f(s, t) \cdot f(s, *)^{-1}, \qquad t \in S^1, \quad s \in \Sigma,$$

using the group law of $G$; here $* \in S^1$ represents the basepoint. This induces a map on homotopy groups

$$D_* : \pi_k \operatorname{Map}(\Sigma \times S^1, G) \xrightarrow{\pi_k D} \pi_k \Omega \operatorname{Map}(\Sigma, G) \underset{\sim}{\xrightarrow{\nu}} \pi_{k+1} \operatorname{Map}(\Sigma, G).$$

**8.13. Proposition.** *For $k = 0, 1$ the map*

$$D_* : \pi_k \operatorname{Map}(\Sigma \times S^1, G_\omega) \to \pi_{k+1} \operatorname{Map}(\Sigma, G_\omega)$$

*is given by*

$$D_*(y \times 1 + y' \times \epsilon,\ x \times 1 + x' \times \epsilon) = (y',\ -\widetilde{\omega}(y', y) + x'),$$

*where $y \times 1 + y' \times \epsilon \in H^{1-k}(\Sigma \times S^1; B)$, $x \times 1 + x' \times \epsilon \in H^{2-k}(\Sigma \times S^1; C)$, and where $\epsilon \in H^1 S^1$ is such that $(\epsilon, [S^1]) = 1$.*

*In this case $k = 1$ we must have $y' = 0$ and this simplifies to*

$$D_*(y \times 1,\ x \times 1 + x' \times \epsilon) = x'.$$

*Proof.* Consider the composite $S_1 \circ D$ with the suspension map, which is the self-map of $\operatorname{Map}(\Sigma \times S^1, G)$ which sends $f$ to $(s, t) \mapsto f(s, t) \cdot f(s, *)^{-1}$.

For $k = 0$, the effect of $S_1 \circ D$ on $(y + y' \times \epsilon, x + x' \times \epsilon)$ (we omit "$\times 1$" from the notation), using the formula for the group law on $\pi_0$, including the formula of

Remark 8.5 for inversion, is

$$\begin{aligned}(y+y'\times\epsilon, x+x'\times\epsilon)\cdot(y,x)^{-1} &= (y+y'\times\epsilon, x+x'\times\epsilon)\cdot(-y, -\widetilde{\omega}(y,y)-x)\\ &= \big(y'\times\epsilon, \widetilde{\gamma}(y,y)-\widetilde{\omega}(y+y'\times\epsilon, -y)+x'\times\epsilon\big)\\ &= (y'\times\epsilon, (-\widetilde{\omega}(y',y)+x')\times\epsilon).\end{aligned}$$

Note that $\widetilde{\omega}(y'\times\epsilon, y) = \omega_*((y'\times\epsilon)\smile y) = -\omega_*(y'\smile y)\times\epsilon = -\widetilde{\omega}(y',y)\times\epsilon$, since $y\in H^1$. We read off the formula we need using Proposition 8.9.

For $k=1$, the calculation of $S_1\circ D$ on $(y, x+x'\times\epsilon)$ is as expected:

$$(y, x+x'\times\epsilon)\cdot(y,x)^{-1} = (y, x+x'\times\epsilon)-(y,x) = (0, x'\times\epsilon),$$

whence the desired formula. □

## 9. Computation of $\pi_* W^T(G_\omega)$ and proof of Theorem 7.6

**9.1.** ***Certain Samelson products in*** $\pi_* W^T(G_\omega)$**.** For a based map $v : S^1\to T$, define

$$C_v : T\times S^1\to T, \quad C_v(t,s) = v(s)^{-1}\cdot t = t\cdot v(s)^{-1}.$$

That is, $C_v$ is the composite

$$T\times S^1 \xrightarrow{\text{id}\times v} T\times T \xrightarrow{\text{id}\times\text{Inv}} T\times T \xrightarrow{\text{Mult}} T.$$

**9.2. Lemma.** *The induced map* $C_v^* : H^k(T;M)\to H^k(T\times S^1;M)$ *is given by*

$$C_v^*(f) = f\times 1 - (f\nabla v)\times\epsilon,$$

*where we also write* $v$ *for* $v_*[S^1]\in H_1T$.

*Proof.* If $\text{Mult}^* f = \sum f'\otimes f''$, then

$$(\text{id}\times\text{Inv})^*\,\text{Mult}^* f = \sum(-1)^{|f''|} f'\otimes f''.$$

Since $v^* f = (f, v_*[*])\times 1 + (f, v_*[S^1])\times\epsilon$, we have

$$\begin{aligned}(\text{id}\times v\,\text{Inv})^*\,\text{Mult}^* f &= \sum(-1)^{|f''|} f'(f'', [*])\times 1 + (-1)^{|f''|} f'(f'', v)\times\epsilon\\ &= f\nabla[*]\times 1 - f\nabla v\times\epsilon.\end{aligned}$$

□

Given elements $(1,f), (g,1)\in \text{Map}(T,G)\rtimes T\subset W^T(G)$, their commutator is given by

$$(1,f)\cdot(g,1)\cdot(1,f^{-1})\cdot(g^{-1},1) = \big(1, t\mapsto g(f^{-1}\cdot t)\cdot g^{-1}(t)\big).$$

**9.3. Lemma.** *For $v \in \pi_1 T$ and $w \in \pi_k \operatorname{Map}(T, G)$ we have the Samelson product*

$$\langle (0, v), (w, 0)\rangle = \big((D \circ C_v^*)_*(w), 0\big) \in \pi_{k+1} \operatorname{Map}(T, G) \rtimes T = \pi_{k+1} \operatorname{Map}(T, G) \times \pi_{k+1} T,$$

*where $D : \operatorname{Map}(T \times S^1, G) \to \Omega \operatorname{Map}(T, G)$ is the desuspension map, and*

$$C_v^* = \operatorname{Map}(C_v, \mathrm{id}) : \operatorname{Map}(T, G) \to \operatorname{Map}(T \times S^1, G).$$

*Proof.* Evidently $D \circ C_v^*$ sends $g \in \operatorname{Map}(T, G)$ to $h \in \Omega \operatorname{Map}(T, G)$ defined by $h(s) := \big(t \mapsto g(v(s)^{-1} \cdot t) \cdot g(t)^{-1}\big)$. That is, $h(s) = v(s) \cdot g \cdot v(s)^{-1} \cdot g^{-1}$, the commutator in $\operatorname{Map}(T, G) \rtimes T$ of $v(s) \in T$ and $g \in \operatorname{Map}(T, G)$. Thus for a based map $w : S^k \to \operatorname{Map}(T, G)$, the composite $D \circ C_v^* \circ w$ is adjoint to the desired Samelson product. □

Now fix $G = G_\omega$. We compute the Samelson product

$$\langle -, -\rangle : \pi_1 W^T(G_\omega) \times \pi_k W^T(G_\omega) \to \pi_{k+1} W^T(G_\omega)$$

on elements which come from the subgroups $T$ and $\operatorname{Map}(T, G)$.

**9.4. Proposition.** *For $t \in L = \pi_1 T \subset \pi_1 D(T)$ and*

$$(y, x) \in H^{1-k}(T; B) \times H^{2-k}(T; C) = \pi_k \operatorname{Map}(T, G_\omega) \subset \pi_1 W^T(G_\omega),$$

*we have, for $k = 0, 1$,*

$$\langle (0, t), ((y, x), 0)\rangle = \big(-y \nabla t,\ \widetilde{\omega}(y \nabla t, y) - x \nabla t\big) \in \pi_{k+1} \operatorname{Map}(T, G_\omega) \subset \pi_{k+1} W^T(G_\omega).$$

*For $k = 1$ this simplifies to*

$$\langle (0, t), ((y, x), 0)\rangle = (0, -x \nabla t).$$

*Proof.* Combine Lemma 9.3, the formula in Proposition 8.13 for the desuspension map, and the effect of $C_t^*$ on $\pi_* \operatorname{Map}(T, G_\omega)$ (for $t \in L = \pi_1 T$), which is computed by Lemma 9.2. Explicitly, in terms of the isomorphism $\pi_k \operatorname{Map}(\Sigma, G_\omega) \approx H^{1-k}(\Sigma; B) \times H^{2-k}(\Sigma; C)$ of Proposition 8.4, the formula of Lemma 9.2 gives

$$C_t^*(y, x) = \big(y \times 1 - (y \nabla t) \times \epsilon, x \times 1 - (x \nabla t) \times \epsilon\big),$$

whence Proposition 8.13 gives

$$D(C_t^*(y, x)) = \big(-y \nabla t,\ -\widetilde{\omega}(-y \nabla t, y) - x \nabla t\big) = \big(-y \nabla t,\ \widetilde{\omega}(y \nabla t, y) - x \nabla t\big). \ \square$$

**9.5.** ***Structure of*** $\pi_* W^T(G_\omega)$**.** We now put this together to compute the homotopy groups of $\mathrm{Map}(T, G_\omega) \rtimes T \subset W^T(G_\omega)$ together with its Samelson products.

**9.6. Theorem.** *Fix* $\omega : B \otimes B \to C$, *and set* $L = H_1 T = \pi_1 T$.

(1) *We have set bijections*

$$\begin{aligned}
\pi_2(\mathrm{Map}(T, G_\omega) \rtimes T) &\approx H^0(T; C),\\
\pi_1(\mathrm{Map}(T, G_\omega) \rtimes T) &\approx H_1 T \times H^0(T; B) \times H^1(T; C),\\
\pi_0(\mathrm{Map}(T, G_\omega) \rtimes T) &\approx H^1(T; B) \times H^2(T; C).
\end{aligned}$$

(2) *The group structure on* $\pi_1$ *and* $\pi_2$ *is the evident additive one, while the group law in* $\pi_0$ *is given by*

$$(m, n) \cdot (m', n') = \big(m + m', -\widetilde{\omega}(m, m') + n + n'\big).$$

(3) *Samelson products on* $\pi_*$ *are given in terms of the above bijections by*

$$\langle (m, n), (t, y, x)\rangle = \big(0,\ m\nabla t,\ -\widetilde{\beta}(m, y) - \widetilde{\omega}(m\nabla t, m) + n\nabla t\big), \qquad \pi_0 \times \pi_1 \to \pi_1, \tag{9.7}$$

$$\langle (t, y, x), (m, n)\rangle = \big(0,\ -m\nabla t,\ \widetilde{\beta}(m, y) + \widetilde{\omega}(m\nabla t, m) - n\nabla t\big), \qquad \pi_1 \times \pi_0 \to \pi_1, \tag{9.8}$$

$$\langle (t, y, x), (t', y', x')\rangle = -\widetilde{\beta}(y, y') - x\Delta t' - x'\Delta t, \qquad \pi_1 \times \pi_1 \to \pi_2, \tag{9.9}$$

*while* $\pi_0 \times \pi_2 \to \pi_2$ *is trivial.*

*Proof.* This is a combination of what we have proved up to now. The set bijections are from the evident product decomposition of $\mathrm{Map}(T, G_\omega) \rtimes T$ and Proposition 8.4(1). The group structure in $\pi_0$ is by Proposition 8.4(2), while in higher degrees it is follows easily from Proposition 8.4(2) since the groups must be abelian.

Part (3) is a consequence of the bilinearity of the Samelson product, combined with Proposition 8.4(3), Proposition 9.4, and the fact that Samelson products in $\pi_* T$ vanish since $T$ is abelian. □

We can add in the action of $\mathrm{Aut}(L)$.

**9.10. Proposition.** *With respect to the identification of the homotopy groups of* $\mathrm{Map}(T, G_\omega) \rtimes T$ *described in Theorem 9.6, the (left) action of* $\mathrm{Aut}(L)$ *on* $\mathrm{Map}(T, G_\omega)$ *induces the following action on homotopy groups, written in terms of the evident left actions of* $\mathrm{Aut}(L) = \mathrm{Aut}(T)$ *on* $H^*(T; M)$:

$$\begin{aligned}
A \propto (m, n) &= ((A^{-1})^* m, (A^{-1})^* n), && (m, n) \in H^1(T; B) \times H^2(T; C),\\
A \propto (t, y, x) &= (At, y, (A^{-1})^* x), && (t, y, x) \in L \times H^0(T; B) \times H^1(T; C),\\
A \propto c &= c, && c \in H^0(T; C).
\end{aligned}$$

*Proof.* This is immediate by functoriality. □

If we introduce the isomorphisms

$$H^0(T; M) \approx M, \quad H^1(T; M) \approx \mathrm{Hom}(L, M), \quad H^2(T; M) \approx \mathrm{Hom}(\Lambda_2 L, M),$$

then we have

$$\begin{aligned}
\pi_2(\mathrm{Map}(T, G_\omega) \rtimes T) &\approx C, \\
\pi_1(\mathrm{Map}(T, G_\omega) \rtimes T) &\approx L \times B \times \mathrm{Hom}(L, C), \\
\pi_0(\mathrm{Map}(T, G_\omega) \rtimes T) &\approx \mathrm{Hom}(L, B) \times \mathrm{Hom}(\Lambda_2 L, C),
\end{aligned}$$

and the formulas take the form

$$\begin{aligned}
(m, n) \cdot (m', n') &= \big(m + m',\ \omega(m \otimes m') + n + n'\big), \\
\langle (m, n), (t, y, x)\rangle &= \big(0,\ mt,\ -\beta(m, y) - \omega(mt, m) + n\nabla t\big), \\
\langle (t, y, x), (m, n)\rangle &= \big(0,\ -mt,\ \beta(m, y) + \omega(mt, m) - n\nabla t\big), \\
\langle (t, y, x), (t', y', x')\rangle &= -\beta(y, y') - x\nabla t' - x'\nabla t, \\
A \propto (m, n) &= (mA^{-1}, n(\Lambda_2 A^{-1})) \\
A \propto (t, y, x) &= (At, y, xA^{-1}),
\end{aligned}$$

where:

- The pairings $m, t \mapsto m\nabla t$ and $x, t \mapsto x\nabla t$ are examples of Kronecker pairings $H^1(T; M) \times H_1 T \to H^0(T; M)$, and so become evaluation maps
$$\mathrm{Hom}(L, M) \otimes L \to M.$$
- The pairing $n, t \mapsto n\nabla t : H^2(T; C) \times H_1 T \to H^1(T; C)$ becomes the contraction pairing $\nabla : \mathrm{Hom}(\Lambda_2 L, C) \times L \to \mathrm{Hom}(L, C)$ (Proposition 7.4).
- The expression $-\widetilde{\omega}(m, m')$ becomes $\omega(m \otimes m')$ as explained in Remark 8.6.
- The action of $\mathrm{Aut}(L) = \mathrm{Aut}(T)$ on $H^k(T; M)$ becomes the evident action on $\mathrm{Hom}(\Lambda_k L, M)$ defined by functoriality of $\Lambda_k$ and composition.

This is read off from the calculations of Theorem 9.6 and Proposition 9.10, together with the relation between Whitehead and Samelson products (10.9), and the fact that Whitehead products $\pi_1 \times \pi_k \to \pi_k$ describe the action $\pi_1 \curvearrowright \pi_k$ (Section 10.7).

We can now give the proof of the general result.

*Proof of Theorem 7.6.* We read off the results using the isomorphism $\pi_* BW^T(\Sigma) \approx \pi_{*-1} W^T(\Sigma)$. The Samelson products become Whitehead products, up to a sign as described in (10.9). □

## 10. Conventions

**10.1.** ***Orientations.*** Let $I=[0,1]$. We use $S^n = I^n/\partial I^n$. We orient $I^n$, and thus $S^n$, using the Künneth map, e.g., using $I^n/\partial I^n \approx (I/\partial I)^{\wedge n}$. The boundary $\partial I^n$ is homeomorphic to $S^{n-1}$, and its orientation is fixed by the boundary map in singular homology. When necessary, we take $(0,0)$ as the base point of $\partial I^n$.

Examining the Eilenberg–Zilber map shows that the face $\{\,(t_1,\dots,t_n) \mid t_i = 1\,\} \subset I^n$ receives a positive orientation when $i$ is odd, and a negative orientation when $i$ is even. In particular, we see that an orientation preserving equivalence $I/\partial I \to \partial I^2$ goes counterclockwise around the square; e.g., $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,1)$ to $(0,0)$.

**10.2.** ***Loops.*** We compose paths in temporal order: thus $\gamma \cdot \delta$ is defined when $\gamma(1) = \delta(0)$.

The standard action of the fundamental group $\pi_1 X \curvearrowright \pi_p X$ is from the *left*. We write it as $\gamma \propto \alpha$, $\gamma \in \pi_1 X$ and $\alpha \in \pi_p X$. It is necessarily defined, in terms of the "balloon on a string" picture, by putting the *end* of the string $\gamma(1)$ at the basepoint of the $p$-sphere: i.e., via $S^p \to I \cup_{\{1\}} S^p \xrightarrow{(\gamma,\alpha)} X$.

With this convention, we may extend this action to a functor $\pi_p : \Pi_1 X \to \mathrm{Ab}$ from the fundamental groupoid, for $p \geq 2$.

We have for $k \geq 1$ a standard isomorphism

$$\nu : \pi_k X \xrightarrow{\sim} \pi_{k-1}\Omega X,$$

defined so that $f : S^k \to X$ corresponds to $\nu f : S^{k-1} \to X$ given by

$$(\nu f)(x_1,\dots,x_{k-1})(t) = f(x_1,\dots,x_{k-1},t).$$

This convention matches the most syntactically convenient convention for the tensor-hom adjunction, i.e., $\mathrm{Map}_*(X \wedge Y, Z) \approx \mathrm{Map}_*(X, \mathrm{Map}_*(Y,Z))$. In particular, the standard isomorphism $\mathrm{Map}_*(X \wedge S^p, Z) \approx \mathrm{Map}_*(X, \Omega^p Z)$ is determined in this way.

**10.3. Remark.** If instead we consider $\nu' : \pi_k X \xrightarrow{\sim} \pi_{k-1}\Omega X$, so that

$$(\nu' f)(x_1,\dots,x_{k-1})(t) = f(t, x_1,\dots,x_{k-1}),$$

then we have $(\nu' f) \sim (-1)^{k-1}(\nu f)$, with the sign coming from the evident transposition $S^{k-1} \wedge S^1 \to S^1 \wedge S^{k-1}$. (The map $\nu'$ is the standard identification used in [Whitehead 1978], for instance.)

**10.4.** ***Eilenberg–MacLane spaces.*** For us, an Eilenberg–MacLane space consists of a based space $K$ *equipped* with a choice of isomorphism $A \xrightarrow{\sim} \pi_n K$ to its only nontrivial homotopy group. Such an object is unique up to canonical homotopy equivalence (and in fact, up to contractible choice), and so can be unambiguously denoted $K(A,n)$.

We thus obtain a canonical weak equivalence $\Omega K(A,n) \approx K(A,n-1)$, using the standard isomorphism $\nu : \pi_n \xrightarrow{\sim} \pi_{n-1}\Omega$ to fix the identification of the homotopy group.

The identification $\widetilde{H}^n(X, A) \approx \pi_0 \operatorname{Map}_*(X, K(A,n))$ is fixed in the usual way, via a choice of tautological class

$$\iota \in \widetilde{H}^n(K(A,n); A) \approx \operatorname{Hom}(H_n(K(A,n); A), A) \approx \operatorname{Hom}(\pi_n K(A,n), A)$$

corresponding to the identity map of $A$. (Ultimately, this depends on the choice of the fundamental class in $H_n S^n$, which defines the Hurewicz map.)

Using the standard identification $\Omega^p K(A,n) \approx K(A,n-p)$, we obtain a standard natural isomorphism

$$\pi_p \operatorname{Map}(X, K(A,n)) \approx \pi_0 \operatorname{Map}(X, \Omega^p K(A,n)) \approx H^{n-p}(X; A).$$

Similarly, we obtain a natural suspension isomorphism

$$\widetilde{H}^{n-p}(X; A) \approx [X, \Omega^p K(A,n)]_* \approx [X \wedge S^p, K(A,n)]_* \approx \widetilde{H}^n(X \wedge S^p; A).$$

**10.5.** ***Cup product.*** The cup product

$$\smile : H^m(X; A) \otimes H^n(X; B) \to H^{m+n}(X; A \otimes B)$$

is represented by a map $\smile : K(A,m) \wedge K(B,n) \to K(A \otimes B, m+n)$, characterized by the property that $\pi_m K(A,m) \otimes \pi_n K(B,n) \to \pi_{m+n} K(A \otimes B, m+n)$ induces the identity map of $A \otimes B$.

Let $\epsilon \in H^1 S^1$ be the tautological class, corresponding to the map $S^1 \to K(\mathbb{Z}, 1)$ sending the tautological class $\iota_1 \in \pi_1 S^1$ to $1 \in \mathbb{Z} = \pi_1 K(\mathbb{Z}, 1)$. The map

$$- \times \epsilon : \widetilde{H}^{n-1}(X; A) \to \widetilde{H}^n(X \wedge S^1; A)$$

coincides with the standard isomorphisms

$$[X, K(A, n-1)]_* \approx [X, \Omega K(A,n)]_* \approx [X \wedge S^1, K(A,n)]_*,$$

and thus with the suspension isomorphism described above.

More precisely: using our standard identifications $K(A, n-1) \approx \Omega K(A,n)$ and $H^k(X, B) \approx [X, K(B,k)]$, the composite

$$[X, K(A, n-1)] \approx [X, \Omega K(A,n)] \to [X \times S^1, K(A,n)]$$

coincides with $- \times \epsilon : H^{n-1}(X; A) \to H^n(X \times S^1; A)$, whereas the composite

$$[X, K(A, n-1)] \approx [X, \Omega K(A,n)] \to [S^1 \times X, K(A,n)]$$

coincides with $(-1)^{n-1}(\epsilon\times -) : H^{n-1}(X;A) \to H^n(S^1\times X;A)$ (and not with $\epsilon\times -$ as one might naively think).

***Cohomology operations.*** A based map $\psi : K(A,m)\to K(B,n)$ induces a cohomology operation $\psi : H^m(X;A)\to H^n(X;B)$. Taking loops gives a map

$$\Omega\psi : K(A,m-1)\to K(B,n-1)$$

and a corresponding operation $\Omega\psi : H^{m-1}(X;A)\to H^{n-1}(X;B)$. The diagram

$$\begin{array}{ccc} H^{m-1}(X;A) & \xrightarrow{\Omega\psi} & H^{n-1}(X;B) \\ {\scriptstyle\times\epsilon}\downarrow & & \downarrow{\scriptstyle\times\epsilon} \\ H^m(X\wedge S^1;A) & \xrightarrow[\psi]{} & H^n(X\wedge S^1;B) \end{array}$$

commutes, i.e.,

$$\Omega\psi(x)\times\epsilon = \psi(x\times\epsilon).$$

Note that the analogous diagram involving $\epsilon\times$ only commutes *up to sign* depending on $m-n$, i.e., $(-1)^{n-1}\epsilon\times\Omega\psi(x) = \psi((-1)^{m-1}\epsilon\times x)$.

A based map $\psi : K(A,p)\wedge K(B,q)\to K(C,n)$ determines a cohomology operation $\psi : H^p(X;A)\times H^q(X;B)\to H^n(X;C)$ in two variables. Using the evident maps $\Omega X\wedge Y\to\Omega(X\wedge Y)$ and $X\wedge\Omega Y\to\Omega(X\wedge Y)$, we can loop $\psi$ in either of its two inputs, obtaining

$$\Omega_1\psi : K(A,p-1)\wedge K(B,q)\to K(C,n-1),$$
$$\Omega_2\psi : K(A,p)\wedge K(B,q-1)\to K(C,n-1).$$

The relation between these are given by

$$\Omega_1\psi(x,y)\times\epsilon = (-1)^{|y|}\psi(x\times\epsilon,y),\quad \Omega_2\psi(x,y)\times\epsilon = \psi(x,y\times\epsilon),$$

**10.6.** ***Groups and loop spaces.*** Let $G$ be a topological group, and let $EG\to BG$ be the universal principal bundle. We may regard $EG$ as having a *left* action by $G$.

For a based space $(X,x_0)$, the path fibration $PX\to X$ may be defined by

$$PX = \{\,\gamma\in\mathrm{Map}([0,1],X)\mid\gamma(0)=x_0\,\}.$$

With this definition (with the free end of the path at $t=1$), the composition law on $\Omega X$ extends naturally to an "action" $*:\Omega X\times PX\to PX$.

Taking $X=BG$, any lift $EG\to P(BG)$ of the two projections gives rise to a weak equivalence $G\to\Omega G$, which furthermore is an $H$-space map.

**10.7.** ***Whitehead products.*** The Whitehead product $[-,-]\colon \pi_p \times \pi_q \to \pi_{p+q-1}$ is defined via precomposition with

$$\big(\partial I^{p+q} \to \partial I^{p+q}/\sim\big) = \big(I^p \times \partial I^q \cup \partial I^p \times I^q \to I^p/\partial I^p \vee I^q/\partial I^q\big),$$

using the orientation convention of Section 10.1.

- If $p=q=1$, the Whitehead product is the loop commutator $[\gamma,\delta]=\gamma\delta\gamma^{-1}\delta^{-1}$.
- If $\gamma \in \pi_1$ and $\alpha \in \pi_{q\geq 2}$, then the Whitehead product is $[\gamma,\alpha] = (\gamma \propto \alpha) - \alpha$.
- On inputs in dimensions $\geq 2$, $[-,-]$ is bilinear.
- In general, we have the commutation relation $[\beta,\alpha]=(-1)^{pq}[\alpha,\beta]$ for $\alpha \in \pi_p$ and $\beta \in \pi_q$.
- We only need to care about the cases when $p, q \leq 2$:

$$\begin{aligned} &[\delta,\gamma] = [\gamma,\delta]^{-1}, && \gamma,\delta \in \pi_1,\\ &[\alpha,\gamma] = [\gamma,\alpha], && \gamma \in \pi_1,\ \alpha \in \pi_2,\\ &[\beta,\alpha] = [\alpha,\beta], && \alpha,\beta \in \pi_1. \end{aligned}$$

  These conventions agree with those of [Whitehead 1978, §7.4].[8]

**10.8.** ***Samelson products.*** For a topological group $G$, the Samelson product $\langle -,-\rangle\colon \pi_p \times \pi_q \to \pi_{p+q}$ is defined in the evident way using the commutator map $G \wedge G \to G$ sending $(x,y) \mapsto xyx^{-1}y^{-1}$.

The definition admits an extension to loop spaces. For such spaces, the Samelson product agrees with the Whitehead product up to a sign. In fact, for $\alpha \in \pi_p X$ and $\beta \in \pi_q X$ we have that

$$\nu[\alpha,\beta] = (-1)^{p-1}\langle \nu(\alpha), \nu(\beta)\rangle, \tag{10.9}$$

according to [Whitehead 1978, 7.10].[9] In particular, we have

$$\begin{aligned} &\langle \nu(\gamma), \nu(\delta)\rangle = \nu[\gamma,\delta], && \gamma,\delta \in \pi_1,\\ &\langle \nu(\gamma), \nu(\alpha)\rangle = \nu[\gamma,\alpha], && \gamma \in \pi_1,\ \alpha \in \pi_2,\\ &\langle \nu(\alpha), \nu(\gamma)\rangle = -\nu[\alpha,\gamma] = -\nu[\gamma,\alpha], && \gamma \in \pi_1,\ \alpha \in \pi_2,\\ &\langle \nu(\alpha), \nu(\beta)\rangle = -\nu[\alpha,\beta], && \alpha,\beta \in \pi_2. \end{aligned}$$

We note the commutation relation

$$\langle \alpha,\beta\rangle = -(-1)^{pq}\langle \beta,\alpha\rangle, \qquad \alpha \in \pi_p G, \quad \beta \in \pi_q G.$$

[8] Although he defines the products using disks, not cubes. It is also unclear to me where he puts his basepoint; when $p, q \geq 2$, the choice of basepoint "shouldn't matter".

[9] In fact, his formula uses $\nu'$ instead of $\nu$. But this makes no difference, as can be shown using Remark 10.3.

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# Fronts d'onde des représentations tempérées et de réduction unipotente pour SO(2$n$ + 1)

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Soit $G$ le groupe spécial orthogonal SO$(2n+1)$ défini sur un corps $p$-adique $F$. Soit $\pi$ une représentation admissible et irreductible de $G(F)$ qui est tempérée et de réduction unipotente. On démontre que $\pi$ admet un front d'onde et l'on en donne une méthode de calcul dans certains cas particuliers.

Let $G$ be a special orthogonal group SO$(2n+1)$ defined over a $p$-adic field $F$. Let $\pi$ be an admissible irreducible representation of $G(F)$ which is tempered and of unipotent reduction. We prove that $\pi$ has a wave front set. In some particular cases, we give a method to compute this wave front set.

## Introduction

Soit $F$ un corps local non archimédien et de caractéristique nulle et soit $n \geq 1$ un entier. On suppose $p > 6n+4$, où $p$ est la caractéristique résiduelle de $F$. Le groupe spécial orthogonal SO$(2n+1)$ a deux formes possibles définies sur $F$. Une forme déployée que nous notons $G_{\text{iso}}$ et une forme non quasi-déployée, qui est une forme intérieure de la précédente et que nous notons $G_{\text{an}}$. Soit $\sharp = \text{iso}$ ou an et soit $\pi$ une représentation admissible irréductible de $G_\sharp(F)$ dans un espace complexe $E$. Pour tout sous-groupe parahorique $K$ de $G_\sharp(F)$, notons $K^u$ son radical pro-$p$-unipotent et $E^{K^u}$ le sous-espace des éléments de $E$ fixés par $K^u$. De $\pi$ se déduit une représentation de $K/K^u$ dans $E^{K^u}$. Le groupe $K/K^u$ s'identifie au groupe des points sur le corps résiduel $\mathbb{F}_q$ de $F$ d'un groupe algébrique connexe défini sur $\mathbb{F}_q$. Lusztig a défini la notion de représentation unipotente d'un tel groupe. On dit que $\pi$ est de réduction unipotente si et seulement s'il existe $K$ comme ci-dessus de sorte que $E^{K^u}$ soit non nul et que la représentation de $K/K^u$ dans $E^{K^u}$ soit unipotente.

Soit $\pi$ une représentation admissible irréductible de $G_\sharp(F)$. Notons $\mathfrak{g}_\sharp$ l'algèbre de Lie de $G_\sharp$. D'après Harish-Chandra, dans un voisinage de l'origine dans $\mathfrak{g}_\sharp(F)$, le caractère de $\pi$, descendu par l'exponentielle à $\mathfrak{g}_\sharp(F)$, est combinaison linéaire de transformées de Fourier d'intégrales orbitales nilpotentes. Fixons une clôture algébrique $\bar{F}$ de $F$ et notons $\bar{\mathcal{N}}(\pi)$ l'ensemble des orbites nilpotentes $\mathcal{O}$ dans $\mathfrak{g}_\sharp(\bar{F})$ qui

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vérifient la condition suivante : il existe une orbite nilpotente $\mathcal{O}$ dans $\mathfrak{g}_\sharp(F)$, qui est incluse dans $\boldsymbol{\mathcal{O}}$ et qui intervient avec un coefficient non nul dans le développement du caractère de $\pi$. On dit que $\pi$ admet un front d'onde si $\overline{\mathcal{N}}(\pi)$ admet un plus grand élément (pour l'ordre usuel sur les orbites nilpotentes). Si c'est le cas, on appelle ce plus grand élément le front d'onde de $\pi$. Le théorème principal de l'article est le suivant.

**Théorème.** *Soit* $\sharp =$ iso *ou* an. *Alors toute représentation admissible irréductible de* $G_\sharp(F)$, *qui est tempérée et de réduction unipotente*, *admet un front d'onde.*

Pour tout entier $N \in \mathbb{N}$, notons $\mathcal{P}^{\text{symp}}(2N)$ l'ensemble des partitions symplectiques de $2N$ (une partition est dite symplectique si tout entier impair y intervient avec une multiplicité paire). Pour une telle partition $\lambda$, notons $\text{Jord}^{\text{bp}}(\lambda)$ l'ensemble (sans multiplicités) des entiers pairs strictement positifs qui interviennent dans $\lambda$. Notons $\boldsymbol{\mathcal{P}}^{\text{symp}}(2N)$ l'ensemble des couples $(\lambda, \epsilon)$ où $\lambda \in \mathcal{P}^{\text{symp}}(2N)$ et $\epsilon \in \{\pm 1\}^{\text{Jord}^{\text{bp}}(\lambda)}$. Notons $\mathfrak{Irr}_{\text{quad}}(2n)$ l'ensemble des quadruplets $(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$ pour lesquels il existe deux entiers $n^+$ et $n^-$ de sorte que $n^+ + n^- = n$, $(\lambda^+, \epsilon^+) \in \boldsymbol{\mathcal{P}}^{\text{symp}}(2n^+)$ et $(\lambda^-, \epsilon^-) \in \boldsymbol{\mathcal{P}}^{\text{symp}}(2n^-)$. À un tel quadruplet $(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$, on peut associer un indice $\sharp =$ iso ou an et une représentation admissible irréductible $\pi(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$ de $G_\sharp(F)$, qui est tempérée et de réduction unipotente. L'indice $\sharp$ est déterminé par une formule simple rappelée en 1.5 Indiquons brièvement quel est le paramètre de Langlands de cette représentation. Notons $W_F$ le groupe de Weil de $F$ et $W_{\text{DF}} = W_F \times \text{SL}(2, \mathbb{C})$ le groupe de Weil–Deligne. Un paramètre de Langlands est un couple $(\rho, \chi)$, où $\rho$ est un homomorphisme de $W_{\text{DF}}$ dans $\text{Sp}(2n; \mathbb{C})$ et $\chi$ est un caractère du groupe des composantes connexes du commutant dans $\text{Sp}(2n; \mathbb{C})$ de l'image de $\rho$. Dans le cas d'une représentation $\pi(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$, la restriction de $\rho$ à $W_F$ est la somme directe de $2n^+$ fois le caractère trivial et de $2n^-$ fois l'unique caractère non ramifié d'ordre 2. Le commutant de l'image de cette restriction est un groupe $\text{Sp}(2n^+; \mathbb{C}) \times \text{Sp}(2n^-; \mathbb{C})$. Les classes de conjugaison d'éléments unipotents dans ce groupe sont paramétrées par $\mathcal{P}^{\text{symp}}(2n^+) \times \mathcal{P}^{\text{symp}}(2n^-)$. La restriction de $\rho$ à $\text{SL}(2; \mathbb{C})$ prend ses valeurs dans ce groupe et l'image d'un unipotent non trivial de $\text{SL}(2; \mathbb{C})$ est paramétré par $(\lambda^+, \lambda^-)$. On voit que le groupe des composantes connexes du commutant dans $\text{Sp}(2n; \mathbb{C})$ de l'image de $\rho$ est isomorphe à $(\mathbb{Z}/2\mathbb{Z})^{\text{Jord}^{\text{bp}}(\lambda^+)} \times (\mathbb{Z}/2\mathbb{Z})^{\text{Jord}^{\text{bp}}(\lambda^-)}$. Le couple $(\epsilon^+, \epsilon^-)$ s'identifie à un caractère de ce groupe, qui n'est autre que le caractère $\chi$ du couple $(\rho, \chi)$.

On note $\mathfrak{Irr}^{\text{bp}}_{\text{quad}}(2n)$ le sous ensemble des $(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-) \in \mathfrak{Irr}_{\text{quad}}(2n)$ tels que tous les termes de $\lambda^+$ et $\lambda^-$ soient pairs. Selon [Waldspurger 2018b, 3.4], pour démontrer le théorème, il suffit de prouver que, pour tout $(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$ élément de $\mathfrak{Irr}^{\text{bp}}_{\text{quad}}(2n)$, la représentation $\pi(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$ admet un front d'onde (cela résulte d'un argument trivial d'induction).

Pour une représentation $\pi(\lambda^+,\epsilon^+,\lambda^-,\epsilon^-)$, où $(\lambda^+,\epsilon^+,\lambda^-,\epsilon^-)\in\mathfrak{Irr}^{bp}_{quad}(2n)$, on a un résultat un peu plus précis. Dans [Waldspurger 2017], on a étudié une certaine représentation d'un groupe de Weyl définie par Lusztig. En supposant, comme c'est ici le cas, que tous les termes de $\lambda^+$ sont pairs, on a associé à $(\lambda^+,\epsilon^+)\in\boldsymbol{\mathcal{P}}^{symp}(2n^+)$ un autre couple $(\lambda^{+,min},\epsilon^{+,min})\in\boldsymbol{\mathcal{P}}^{symp}(2n^+)$ (voir ci-dessous). De même, à $(\lambda^-,\epsilon^-)\in\boldsymbol{\mathcal{P}}^{symp}(2n^-)$, on associe un autre couple $(\lambda^{-,min},\epsilon^{-,min})\in\boldsymbol{\mathcal{P}}^{symp}(2n^-)$. Introduisons la réunion usuelle de $\lambda^{+,min}$ et $\lambda^{-,min}$, que l'on note $\lambda^{+,min}\cup\lambda^{-,min}$. C'est une partition symplectique de $2n$. Notons $\mathcal{P}^{orth}(2n+1)$ l'ensemble des partitions orthogonales de $2n+1$ (une partition est orthogonale si et seulement si tout entier pair strictement positif y intervient avec multiplicité paire). On sait bien que l'ensemble $\mathcal{P}^{orth}(2n+1)$ paramètre les orbites nilpotentes dans $\mathfrak{g}_\sharp(\bar{F})$. Un front d'onde est donc paramétré par un élément de cet ensemble. D'autre part, à la suite de Spaltenstein, on définit une dualité $d:\mathcal{P}^{symp}(2n)\to\mathcal{P}^{orth}(2n+1)$, cf. 2.6 (elle n'est ni injective, ni surjective, son image est le sous-ensemble des partitions spéciales dans $\mathcal{P}^{orth}(2n+1)$).

**Théorème.** *Soit*
$$(\lambda^+,\epsilon^+,\lambda^-,\epsilon^-)\in\mathfrak{Irr}^{bp}_{quad}(2n).$$
*Alors la représentation $\pi(\lambda^+,\epsilon^+,\lambda^-,\epsilon^-)$ admet un front d'onde. Celui-ci est paramétré par la partition $d(\lambda^{+,min}\cup\lambda^{-,min})$.*

La preuve de ce théorème reprend celle de [Waldspurger 2018b]. Posons $\pi=\pi(\lambda^+,\epsilon^+,\lambda^-,\epsilon^-)$. L'existence d'un front d'onde pour $\pi$ se lit sur le caractère de cette représentation. Celui-ci se calcule en fonction des représentations des différents groupes finis $K/K^u$ dans $E^{K^u}$, avec les notations du premier paragraphe ci-dessus (en vérité, le groupe fini est $K^\dagger/K^u$, où $K^\dagger$ est le normalisateur de $K$ dans $G_\sharp(F)$). La construction de la représentation $\pi$ (qui est due à Lusztig) permet d'expliciter ces représentations de groupes finis. On les décrit à l'aide de représentations de groupes de Weyl $W_m$ de type $B_m$ ou $C_m$. Une vieille combinatoire tirée de [Waldspurger 2001] permet alors de traduire l'existence d'un front d'onde et son calcul en un problème concernant exclusivement des représentations de tels groupes $W_m$, cf. 1.4. Les objets cruciaux qui interviennent ici sont les représentations $\boldsymbol{\rho}_{\lambda^+,\epsilon^+}$ et $\boldsymbol{\rho}_{\lambda^-,\epsilon^-}$ définies par Lusztig (ce ne sont pas ses notations) auxquelles on a fait allusion ci-dessus. Elles ne sont pas irréductibles en général et on connaît peu de choses sur leur décomposition en représentations irréductibles. On sait toutefois que, disons dans la décomposition de $\boldsymbol{\rho}_{\lambda^+,\epsilon^+}$, il y a un élément minimal qui est la représentation $\rho_{\lambda^+,\epsilon^+}$ associée à $(\lambda^+,\epsilon^+)$ par la correspondance de Springer généralisée. Dans [Waldspurger 2018b], cela nous a suffi pour traiter non pas la représentation $\pi$, mais son image par l'involution d'Aubert–Zelevinsky. Le point nouveau est le résultat de [Waldspurger 2017] qui affirme (sous l'hypothèse que tous les termes de $\lambda^+$ sont pairs) que la décomposition de $\boldsymbol{\rho}_{\lambda^+,\epsilon^+}$ admet aussi un élément maximal pour un ordre convenable (cf. 4.1 pour un énoncé précis). C'est $\rho_{\lambda^{+,\min},\epsilon^{+,\min}} \otimes \mathrm{sgn}$, où sgn est le caractère signature. Cette propriété nous permet de conclure.

Les paragraphes 1 à 3 sont surtout consacrés à des rappels de résultats antérieurs. On a amélioré certains d'entre eux quand c'était nécessaire. Le théorème ci-dessus est démontré au paragraphe 4. Dans le paragraphe 5, nous indiquons comment se calculent les partitions $\lambda^{+,\min}$ et $\lambda^{-,\min}$ (en fait leurs transposées) et nous donnons quelques exemples de fronts d'onde.

## 1. Rappel pas très bref des résultats de [Waldspurger 2018b]

**1.1.** ***Partitions, notations.*** Soit $\lambda = (\lambda_1, \dots, \lambda_r)$ une suite finie de nombres réels. Notons $t(\lambda)$ le plus grand entier $j \in \{1, \dots, r\}$ tel que $\lambda_j \neq 0$. On identifie deux suites $\lambda$ et $\lambda'$ si $t(\lambda) = t(\lambda')$ et $\lambda_j = \lambda'_j$ pour tout $j \leq t(\lambda)$. Soit $\lambda$ une telle suite et soit $k \in \mathbb{N}$. Quitte à adjoindre à $\lambda$ des termes nuls, on peut écrire $\lambda = (\lambda_1, \dots, \lambda_r)$ avec $r \geq k$. On pose $S_k(\lambda) = \lambda_1 + \cdots + \lambda_k$. Évidemment, $S_k(\lambda)$ ne dépend plus de $k$ dès que $k \geq t(\lambda)$. On pose $S(\lambda) = S_{t(\lambda)}(\lambda)$. On définit la somme $\lambda + \lambda'$ de deux suites $\lambda$ et $\lambda'$ : $(\lambda + \lambda')_j = \lambda_j + \lambda'_j$ pour tout $j \geq 1$.

Une partition est une suite finie décroissante d'entiers positifs ou nuls. On identifie comme ci-dessus deux partitions qui ne diffèrent que par des termes nuls. Pour une partition $\lambda = (\lambda_1, \dots, \lambda_r)$ et pour un entier $i \geq 1$, on note $\mathrm{mult}_\lambda(i)$ le nombre d'indices $j$ tels que $\lambda_j = i$. On note $\mathrm{Jord}(\lambda)$ l'ensemble des $i \geq 1$ tels que $\mathrm{mult}_\lambda(i) > 0$. Pour tout $N \in \mathbb{N}$, on note $\mathcal{P}(N)$ l'ensemble des partitions $\lambda$ telles que $S(\lambda) = N$ et on note $\mathcal{P}_2(N)$ l'ensemble des couples $(\alpha, \beta)$ de partitions telles que $S(\alpha) + S(\beta) = N$. On ordonne les éléments de $\mathcal{P}(N)$ de la façon usuelle : $\lambda \leq \lambda'$ si et seulement si $S_k(\lambda) \leq S_k(\lambda')$ pour tout $k \in \mathbb{N}$. On définit la réunion $\lambda \cup \lambda'$ de deux partitions $\lambda$ et $\lambda'$ : pour tout entier $i \geq 1$, $\mathrm{mult}_{\lambda\cup\lambda'}(i) = \mathrm{mult}_\lambda(i) + \mathrm{mult}_{\lambda'}(i)$.

Soit $\lambda$ une partition. Pour tout $i \in \mathbb{N}$, on note $J(i)$ l'ensemble des $j \geq 1$ tels que $\lambda_j = i$. Si $i = 0$, on considère que $J(0)$ est l'intervalle infini $\{t(\lambda) + 1, \dots\}$. Pour $i \in \mathrm{Jord}(\lambda)$, $J(i)$ est non vide. On note $j_{\min}(i)$, resp. $j_{\max}(i)$, le plus petit, resp. grand, élément de $J(i)$. On pose $j_{\min}(0) = t(\lambda) + 1$.

On note $W_N$ le groupe de Weyl d'un système de racines de type $B_N$ ou $C_N$, avec la convention $W_0 = \{1\}$. On note sgn le caractère signature usuel de $W_N$ et $\mathrm{sgn}_{\mathrm{CD}}$ le caractère dont le noyau est le sous-groupe $W_N^D$ d'un système de racines de type $D_N$. Les représentations irréductibles de $W_N$ sont paramétrées par $\mathcal{P}_2(N)$. Pour $(\alpha, \beta) \in \mathcal{P}_2(N)$, on note $\rho(\alpha, \beta)$ la représentation paramétrée par $(\alpha, \beta)$. Les représentations irréductibles de $W_N^D$ sont presque paramétrées par le quotient de $\mathcal{P}_2(N)$ par la relation d'équivalence $(\alpha, \beta) \equiv (\beta, \alpha)$. Presque, parce qu'un couple de la forme $(\alpha, \alpha)$ paramètre deux représentations irréductibles.

Pour tout ensemble $E$, on note $\mathbb{C}[E]$ le $\mathbb{C}$-espace vectoriel de base $E$. Pour tout groupe fini $W$, on note $\widehat{W}$ l'ensemble des classes d'équivalence de représentations irréductibles de $W$. En identifiant une représentation à son caractère, $\mathbb{C}[\widehat{W}]$ est aussi l'espace des fonctions de $W$ dans $\mathbb{C}$ qui sont invariantes par conjugaison.

**1.2.** ***L'espace $\mathcal{R}$.*** *On fixe pour tout l'article un entier $n \geq 1$.* On note $\Gamma$ l'ensemble des quadruplets $\gamma = (r', r'', N^+, N^-)$ tels que

$$r' \in \mathbb{N}, \quad r'' \in \mathbb{Z}, \quad N^+ \in \mathbb{N}, \quad N^- \in \mathbb{N}, \quad r'^2 + r' + N^+ + r''^2 + N^- = n.$$

Pour un tel $\gamma$, on pose $\mathcal{R}(\gamma) = \mathbb{C}[\widehat{W}_{N^+}] \otimes \mathbb{C}[\widehat{W}_{N^-}]$. On pose

$$\mathcal{R} = \bigoplus_{\gamma \in \Gamma} \mathcal{R}(\gamma).$$

On définit un endomorphisme $\varphi \mapsto \mathrm{sgn} \otimes \varphi$ de $\mathcal{R}$ de la façon suivante. Il respecte chaque sous-espace $\mathcal{R}(\gamma)$. Pour $\gamma$ comme ci-dessus, pour $\rho^+ \in \widehat{W}_{N^+}$ et $\rho^- \in \widehat{W}_{N^-}$, on pose $\mathrm{sgn} \otimes (\rho^+ \otimes \rho^-) = (\rho^+ \otimes \mathrm{sgn}) \otimes (\rho^- \otimes \mathrm{sgn})$.

On a défini en [Waldspurger 2004, 1.10] un endomorphisme $\rho\iota$. Puisqu'il est essentiel à nos constructions, rappelons sa définition. Soit $\gamma = (r', r'', N^+, N^-) \in \Gamma$ et $\varphi \in \mathcal{R}(\gamma)$. Posons $N = N^+ + N^-$. L'élément $\rho\iota(\varphi)$ appartient à

$$\bigoplus_{\substack{N_1, N_2 \in \mathbb{N} \\ N_1 + N_2 = N}} \mathcal{R}(r', (-1)^{r'} r'', N_1, N_2).$$

Soit $\delta = (r', (-1)^{r'} r'', N_1, N_2) \in \Gamma$. Décrivons la composante $\rho\iota(\varphi)_\delta$ de $\rho\iota(\varphi)$ dans $\mathcal{R}(\delta)$.

On définit un quadruplet d'entiers $\boldsymbol{a} = (a_1^+, a_1^-, a_2^+, a_2^-)$ par les formules suivantes :

$$\begin{aligned}
\boldsymbol{a} &= (0, 0, 0, 1) \quad \text{si } 0 < r'' \leq r' \text{ ou si } r'' = 0 \text{ et } r' \text{ est pair;} \\
\boldsymbol{a} &= (0, 0, 1, 0) \quad \text{si } -r' \leq r'' < 0 \text{ ou si } r'' = 0 \text{ et } r' \text{ est impair;} \\
\boldsymbol{a} &= (0, 1, 0, 0) \quad \text{si } r' < r''; \\
\boldsymbol{a} &= (1, 0, 0, 0) \quad \text{si } r'' < -r'.
\end{aligned}$$

Notons $\mathcal{N}$ l'ensemble des quadruplets $\boldsymbol{N} = (N_1^+, N_1^-, N_2^+, N_2^-)$ d'entiers positifs ou nuls tels que

$$N^+ = N_1^+ + N_2^+, \quad N^- = N_1^- + N_2^-, \quad N_1 = N_1^+ + N_1^-, \quad N_2 = N_2^+ + N_2^-.$$

Pour un tel quadruplet, posons $W_{\boldsymbol{N}} = W_{N_1^+} \times W_{N_1^-} \times W_{N_2^+} \times W_{N_2^-}$. Ce groupe se plonge de façon évidente dans $W_{N_1} \times W_{N_2}$, resp. $W_{N^+} \times W_{N^-}$, et ces plongements sont bien définis à conjugaison près. On a donc des foncteurs de restriction $\mathrm{res}_{W_{\boldsymbol{N}}}^{W_{N^+} \times W_{N^-}}$ et d'induction $\mathrm{ind}_{W_{\boldsymbol{N}}}^{W_{N_1} \times W_{N_2}}$. On note $\mathrm{sgn}_{\mathrm{CD}}^{\boldsymbol{a}}$ le caractère de $W_{\boldsymbol{N}}$ qui est le produit tensoriel des caractères $\mathrm{sgn}_{\mathrm{CD}}^{a_1^+}$, $\mathrm{sgn}_{\mathrm{CD}}^{a_1^-}$, $\mathrm{sgn}_{\mathrm{CD}}^{a_2^+}$, $\mathrm{sgn}_{\mathrm{CD}}^{a_2^-}$ sur chacun des facteurs de $W_N$. Alors

$$\rho\iota(\varphi)_\delta = \sum_{N\in\mathcal{N}} \mathrm{ind}_{W_N}^{W_{N_1}\times W_{N_2}}\big(\mathrm{sgn}_{\mathrm{CD}}^{\boldsymbol{a}} \otimes \mathrm{res}_{W_N}^{W_{N^+}\times W_{N^-}}(\varphi)\big).$$

**1.3.** ***Correspondance de Springer généralisée.*** Soit $N\in\mathbb{N}$. On a défini l'ensemble $\mathcal{P}^{\mathrm{symp}}(2N)$ dans l'introduction. La correspondance de Springer généralisée dans le cas symplectique est une bijection de $\mathcal{P}^{\mathrm{symp}}(2N)$ sur l'ensemble des couples $(k,\rho)$ où

$$k\in\mathbb{N} \quad \text{et} \quad k(k+1)\leq 2N; \qquad \rho\in \widehat{W}_{N-k(k+1)/2}.$$

Pour $(\lambda,\epsilon)\in\mathcal{P}^{\mathrm{symp}}(2N)$, on note $(k_{\lambda,\epsilon},\rho_{\lambda,\epsilon})$ le couple qui lui correspond et on pose $N_{\lambda,\epsilon}=N-k_{\lambda,\epsilon}(k_{\lambda,\epsilon}+1)/2$. Rappelons comment on calcule $k_{\lambda,\epsilon}$. On note $i_1>\cdots>i_t$ les entiers $i\in\mathrm{Jord}^{\mathrm{bp}}(\lambda)$ tels que $\mathrm{mult}_\lambda(i)$ soit impair. On pose

$$M=\big|\{h=1,\dots,t;\ h \text{ est pair et } \epsilon_{i_h}=-1\}\big| \\ -\big|\{h=1,\dots,t;\ h \text{ est impair et } \epsilon_{i_h}=-1\}\big|.$$

Alors, d'après [Waldspurger 2001] XI.3, on a

$$k_{\lambda,\epsilon}=2M \quad \text{si } M\geq 0, \qquad k_{\lambda,\epsilon}=-2M-1 \quad \text{si } M<0. \tag{1}$$

On définit une autre représentation $\boldsymbol{\rho}_{\lambda,\epsilon}$ du même groupe $W_{N_{\lambda,\epsilon}}$, cf. [Waldspurger 2004, 5.1]. En gros, $\rho_{\lambda,\epsilon}$ est l'action de $W_{N_{\lambda,\epsilon}}$ sur un sous-espace déterminé par $\epsilon$ de l'espace de cohomologie de plus haut degré d'une certaine variété algébrique, tandis que $\boldsymbol{\rho}_{\lambda,\epsilon}$ est l'action de $W_{N_{\lambda,\epsilon}}$ sur un sous-espace analogue de la somme de tous les espaces de cohomologie de cette variété.

Soit $(\lambda^+,\epsilon^+,\lambda^-,\epsilon^-)\in\mathfrak{Irr}_{\mathrm{quad}}(2n)$. Pour $\zeta=\pm$, posons $2n^\zeta=S(\lambda^\zeta)$, $k^\zeta=k_{\lambda^\zeta,\epsilon^\zeta}$, $N^\zeta=n^\zeta-k^\zeta(k^\zeta+1)/2$. On définit des entiers $r'\in\mathbb{N}$, $r''\in\mathbb{Z}$ par les formules suivantes :

$$\begin{aligned} &\text{si } k^+\equiv k^- \bmod 2\mathbb{Z}, && r'=\frac{k^++k^-}{2}, && r''=\frac{k^+-k^-}{2};\\ &\text{si } k^+\not\equiv k^- \bmod 2\mathbb{Z} \quad \text{et } k^+>k^-, && r'=\frac{k^+-k^--1}{2}, && r''=\frac{k^++k^-+1}{2};\\ &\text{si } k^+\not\equiv k^- \bmod 2\mathbb{Z} \quad \text{et } k^+<k^-, && r'=\frac{k^--k^+-1}{2}, && r''=-\frac{k^++k^-+1}{2}.\end{aligned}$$

Le quadruplet $\gamma=(r',r'',N^+,N^-)$ appartient à $\Gamma$. Puisque

$$\mathcal{R}(\gamma)=\mathbb{C}[\widehat{W}_{N^+}]\otimes\mathbb{C}[\widehat{W}_{N^-}],$$

on peut identifier $\boldsymbol{\rho}_{\lambda^+,\epsilon^+}\otimes\boldsymbol{\rho}_{\lambda^-,\epsilon^-}$ à un élément de $\mathcal{R}(\gamma)$, a fortiori à un élément de $\mathcal{R}$. Dans la suite $\boldsymbol{\rho}_{\lambda^+,\epsilon^+}\otimes\boldsymbol{\rho}_{\lambda^-,\epsilon^-}$ désignera cet élément.

Pour $M\in\mathbb{N}$, on note $\mathcal{P}^{\mathrm{orth}}(M)$ l'ensemble des partitions orthogonales de $M$. Pour une telle partition $\lambda$, on note $\mathrm{Jord}^{\mathrm{bp}}(\lambda)$ l'ensemble des entiers impairs $i\geq 1$ tels que $\text{mult}_\lambda(i) > 0$. On note $\boldsymbol{\mathcal{P}}^{\text{orth}}(M)$ l'ensemble des couples $(\lambda, \epsilon)$ où $\lambda \in \mathcal{P}^{\text{orth}}(M)$ et $\epsilon \in \{\pm 1\}^{\text{Jord}^{\text{bp}}(\lambda)}/\{\pm 1\}$, le groupe $\{\pm 1\}$ s'envoyant diagonalement dans $\{\pm 1\}^{\text{Jord}^{\text{bp}}(\lambda)}$.

Soit $N \in \mathbb{N}$. La correspondance de Springer généralisée dans le cas orthogonal impair est une bijection de $\boldsymbol{\mathcal{P}}^{\text{orth}}(2N+1)$ sur l'ensemble des couples $(k, \rho)$ tels que

$$k \in \mathbb{N}, \quad k \text{ est impair et } k^2 \leq 2N+1; \qquad \rho \in \widehat{W}_{N-(k^2-1)/2}.$$

Soit $(k_{\lambda,\epsilon}, \rho_{\lambda,\epsilon})$ le couple associé à $(\lambda, \epsilon) \in \boldsymbol{\mathcal{P}}^{\text{orth}}(2N+1)$. Soit $\boldsymbol{\mathcal{P}}^{\text{orth}}(2N+1)_{k=1}$ le sous-ensemble des $(\lambda, \epsilon) \in \boldsymbol{\mathcal{P}}^{\text{orth}}(2N+1)$ tels que $k_{\lambda,\epsilon} = 1$.

La correspondance de Springer généralisée dans le cas orthogonal pair est une bijection entre $\boldsymbol{\mathcal{P}}^{\text{orth}}(2N)$ et l'ensemble des couples $(k, \rho)$ tels que

$k \in \mathbb{N}$, $k$ est pair et $k^2 \leq 2N$;

si $k > 0$, $\rho \in \widehat{W}_{N-k^2/2}$;

si $k = 0$, $\rho$ est une classe d'équivalence dans $\widehat{W}_{N-k^2/2}$, deux représentations irréductibles $\rho'$ et $\rho''$ étant ici équivalentes si et seulement si $\rho' = \rho''$ ou $\rho' = \rho'' \otimes \text{sgn}_{\text{CD}}$.

On note $(k_{\lambda,\epsilon}, \rho_{\lambda,\epsilon})$ le couple associé à $(\lambda, \epsilon) \in \boldsymbol{\mathcal{P}}^{\text{orth}}(2N)$. On note $\boldsymbol{\mathcal{P}}^{\text{orth}}(2N)_{k=0}$ le sous-ensemble des $(\lambda, \epsilon) \in \boldsymbol{\mathcal{P}}^{\text{orth}}(2N)$ tels que $k_{\lambda,\epsilon} = 0$. Quand $k_{\lambda,\epsilon} = 0$, $\rho_{\lambda,\epsilon}$ n'est qu'une classe d'équivalence comme on vient de le dire. Autrement dit, $\rho_{\lambda,\epsilon}$ est paramétrée par un couple $(\alpha, \beta) \in \mathcal{P}_2(N)$ à l'ordre près. Si $\alpha = \beta$, on pose $\rho^+_{\lambda,\epsilon} = \rho^-_{\lambda,\epsilon} = \rho(\alpha, \beta)$. Si $\alpha \neq \beta$, on choisit $\alpha$ et $\beta$ de sorte que $\alpha > \beta$ pour l'ordre lexicographique. On pose $\rho^+_{\lambda,\epsilon} = \rho(\alpha, \beta)$ et $\rho^-_{\lambda,\epsilon} = \rho(\beta, \alpha)$.

**1.4.** ***Caractérisation du front d'onde.*** On a introduit les groupes $G_{\text{iso}}$ et $G_{\text{an}}$. Pour $\sharp = \text{iso}$ ou an, on note $\text{Irr}_{\text{tunip},\sharp}$ l'ensemble des classes d'isomorphismes de représentations admissibles irréductibles de $G_\sharp(F)$ qui sont tempérées et de réduction unipotente. On note $\text{Irr}_{\text{tunip}}$ la réunion disjointe de $\text{Irr}_{\text{tunip,iso}}$ et $\text{Irr}_{\text{tunip,an}}$. On a défini en [Waldspurger 2018a, 1.5] un espace $\mathcal{R}^{\text{par}}$ et une application linéaire $\text{Rep} : \mathbb{C}[\text{Irr}_{\text{tunip}}] \to \mathcal{R}^{\text{par}}$. A la suite de Lusztig, on a défini en [Mœglin et Waldspurger 2003, 3.16] deux isomorphismes $\text{Rep} : \mathcal{R} \to \mathcal{R}^{\text{par}}$ et $k : \mathcal{R} \to \mathcal{R}^{\text{par}}$. On note $\mathcal{F}$ l'automorphisme de $\mathcal{R}$ tel que $\text{Rep} \circ \mathcal{F} = k$. C'est une involution sur le calcul de laquelle nous reviendrons en 2.5. Pour $\pi \in \text{Irr}_{\text{tunip}}$, on note $\kappa_\pi$ l'élément de $\mathcal{R}$ tel que $k(\kappa_\pi) = \text{Res}(\pi)$. Soient $n_1, n_2 \in \mathbb{N}$ et $\rho_1 \in \widehat{W}_{n_1}$, $\rho_2 \in \widehat{W}_{n_2}$. Le quadruplet $\gamma = (0, 0, n_1, n_2)$ appartient à $\Gamma$ et on a $\mathcal{R}(\gamma) = \mathbb{C}[\widehat{W}_{n_1}] \otimes \mathbb{C}[\widehat{W}_{n_2}]$. Notons $\kappa_\pi(\gamma)$ la composante de $\kappa_\pi$ dans $\mathcal{R}(\gamma)$. C'est une combinaison linéaire de représentations irréductibles avec des coefficients complexes. On note $m_\pi(\rho_1, \rho_2)$ le coefficient de $\rho_1 \otimes \rho_2$ dans cette combinaison linéaire.

On pose $\text{sgn}_{\text{iso}} = 1$, $\text{sgn}_{\text{an}} = -1$. Soit $\sharp = \text{iso}$ ou an, soit $\pi \in \text{Irr}_{\text{tunip},\sharp}$, soient $n_1, n_2 \in \mathbb{N}$ tels que $n_1 + n_2 = n$ et soient $(\mu_1, \eta_1) \in \boldsymbol{\mathcal{P}}^{\text{orth}}(2n_1+1)_{k=1}$ et $(\mu_2, \eta_2) \in$

$\boldsymbol{\mathcal{P}}^{\text{orth}}(2n_2)_{k=0}$. Comme cas particulier de la définition ci-dessus, pour $\zeta = \pm$, on définit les multiplicités $m_\pi(\rho_{\mu_1,\eta_1} \otimes \text{sgn}, \rho^{\zeta}_{\mu_2,\eta_2} \otimes \text{sgn})$. On pose

$$M_\pi(\mu_1, \eta_1; \mu_2, \eta_2) = m_\pi(\rho_{\mu_1,\eta_1} \otimes \text{sgn}, \rho^{+}_{\mu_2,\eta_2} \otimes \text{sgn}) + \text{sgn}_\sharp m_\pi(\rho_{\mu_1,\eta_1} \otimes \text{sgn}, \rho^{-}_{\mu_2,\eta_2} \otimes \text{sgn}).$$

**Proposition.** *Soient* $\sharp = \text{iso}$ *ou* an, $\pi \in \text{Irr}_{\text{tunip},\sharp}$ *et* $\mu \in \mathcal{P}^{\text{orth}}(2n+1)$. *Alors* $\pi$ *admet un front d'onde paramétré par* $\mu$ *si et seulement si les deux conditions suivantes sont vérifiées.*

(i) *Soient* $n_1, n_2 \in \mathbb{N}$ *tels que* $n_1 + n_2 = n$ *et* $n_2 \geq 1$ *si* $\sharp =$ an. *Soient*

$$(\mu_1, \eta_1) \in \boldsymbol{\mathcal{P}}^{\text{orth}}(2n_1+1)_{k=1} \quad \textit{et} \quad (\mu_2, \eta_2) \in \boldsymbol{\mathcal{P}}^{\text{orth}}(2n_2)_{k=0}.$$

*Supposons* $M_\pi(\mu_1, \eta_1; \mu_2, \eta_2) \neq 0$. *Alors* $\mu_1 \cup \mu_2 \leq \mu$.

(ii) *Il existe* $n_1, n_2 \in \mathbb{N}$ *tels que* $n_1 + n_2 = n$ *et* $n_2 \geq 1$ *si* $\sharp =$ an *et il existe*

$$(\mu_1, \eta_1) \in \boldsymbol{\mathcal{P}}^{\text{orth}}(2n_1+1)_{k=1} \quad \textit{et} \quad (\mu_2, \eta_2) \in \boldsymbol{\mathcal{P}}^{\text{orth}}(2n_2)_{k=0}$$

*tels que* $M_\pi(\mu_1, \eta_1; \mu_2, \eta_2) \neq 0$ *et* $\mu_1 \cup \mu_2 = \mu$.

Cf. [Waldspurger 2018b, 3.7] . Les notations de cette référence étaient légèrement différente, les multiplicités étaient dans certains cas divisées par 2 mais cela ne change évidemment pas l'énoncé. D'autre part, dans [Waldspurger 2018b], la représentation $\pi$ était d'une forme particulière, mais cela n'était utilisé que pour décrire explicitement la fonction $\kappa_\pi$ dans [loc. cit., 3.8], cela n'intervient pas à ce point.

**1.5.** ***Les représentations*** $\boldsymbol{\pi(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)}$**.** Soit $(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-) \in \mathfrak{Irr}_{\text{quad}}(2n)$. En utilisant une construction de Lusztig, on a défini en [Waldspurger 2018a, 1.3] la représentation $\pi(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$. Sa paramétrisation de Langlands a été rappelée rapidement dans l'introduction. C'est une représentation admissible, irréductible et tempérée de $G_\sharp(F)$, où l'indice $\sharp$ est déterminé par la formule

$$\text{sgn}_\sharp = \left( \prod_{i \in \text{Jord}^{\text{bp}}(\lambda^+)} \epsilon^+(i)^{\text{mult}_{\lambda^+}(i)} \right) \left( \prod_{i \in \text{Jord}^{\text{bp}}(\lambda^-)} \epsilon^-(i)^{\text{mult}_{\lambda^-}(i)} \right), \tag{1}$$

cf. 1.3 pour la définition de $\text{sgn}_\sharp$. Notons $D$ l'involution de Aubert–Zelevinski. Elle permute les représentations admissibles irréductibles de $G_\sharp(F)$. On a l'égalité

$$\text{Res} \circ D(\pi(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)) = \text{Rep} \circ \rho\iota(\boldsymbol{\rho}_{\lambda^+,\epsilon^+} \otimes \boldsymbol{\rho}_{\lambda^-,\epsilon^-}), \tag{2}$$

cf. [Waldspurger 2018a, proposition 1.11].

L'espace $\mathcal{R}^{\text{par}}$ est somme directe finie d'espaces vectoriels ayant pour base les classes d'équivalence de représentations irréductibles et unipotentes de groupes finis de la forme $\text{SO}(2n'+1; \mathbb{F}_q) \times O(2n''; \mathbb{F}_q)$, avec des notations compréhensibles.

Chacun de ces espaces est muni d'involutions du même type que $D$. L'espace $\mathcal{R}^{\text{par}}$ est ainsi muni d'une involution $D^{\text{par}}$ et on a prouvé en [Waldspurger 2018a, 1.7] l'égalité $\text{Res} \circ D = D^{\text{par}} \circ \text{Res}$. Montrons que l'on a aussi

$$D^{\text{par}} \circ \text{Rep}(\varphi) = \text{Rep}(\text{sgn} \otimes \varphi) \quad \text{pour tout } \varphi \in \mathcal{R}. \tag{3}$$

*Preuve.* Fixons $\gamma = (r', r'', N_1, N_2) \in \Gamma$, $\rho_1 \in \widehat{W}_{N_1}$, $\rho_2 \in \widehat{W}_{N_2}$ et considérons l'élément $\varphi = \rho_1 \otimes \rho_2 \in \mathcal{R}(\gamma)$. D'après Lusztig, le couple $(r', \rho_1)$ paramètre une représentation irréductible $\pi_1$ d'un groupe fini $\text{SO}(2n_1+1, \mathbb{F}_q)$, où $n_1 = N_1 + r'^2 + r'$ et $\mathbb{F}_q$ est le corps résiduel de $F$. De même, le couple $(r'', \rho_2)$ paramètre une représentation irréductible $\pi_2$ d'un groupe fini $\text{SO}(2n_2, \mathbb{F}_q)$, où $n_2 = N_2 + r''^2$ (c'est la forme déployée du groupe si $r''$ est pair, non déployée si $r''$ est impair). Le terme $\text{Rep}(\varphi)$ n'est autre que $\pi_1 \otimes \pi_2$. On définit usuellement une involution du groupe de Grothendieck des représentations de longueur finie de tels groupes finis (cf. [Carter 1985, 8.2] dans le cas d'un groupe connexe et [Digne et Michel 1994, 3.10] dans le cas non connexe). C'est une somme alternée de composés de foncteurs de restriction et d'induction. D'après notre définition de [Waldspurger 2018a, 1.7], $D^{\text{par}}(\pi_1 \otimes \pi_2)$ est le produit tensoriel des images de $\pi_1$ et $\pi_2$ par ces involutions multipliées par des signes de sorte que ces images soient des représentations irréductibles. D'autre part, pour tout $m \in \mathbb{N}$, on définit une involution $D_{W_m}$ de $\mathbb{C}[\widehat{W}_m]$ par une formule analogue : c'est une somme alternée de composés de foncteurs de restriction et d'induction, cf. [Howlett et Lehrer 1982, corollaire 1]. Les paramétrages $(r', \rho_1) \mapsto \pi_1$ et $(r'', \rho_2) \mapsto \pi_2$ étant compatibles en un sens plus ou moins évident aux foncteurs de restriction et d'induction, $D^{\text{par}}(\pi_1 \otimes \pi_2)$ est égal à l'image par Rep de $\pm D_{W_{N_1}}(\rho_1) \otimes D_{W_{N_2}}(\rho_2)$, le signe étant choisi de sorte que ce terme soit le produit tensoriel de deux représentations irréductibles. D'après [Howlett et Lehrer 1982, corollaire 1], on a $D_{W_m}(\rho) = \pm \rho \otimes \text{sgn}$ pour tout $m \in \mathbb{N}$ et tout $\rho \in \widehat{W}_m$. Donc $D^{\text{par}}(\pi_1 \otimes \pi_2)$ est égal à l'image par Rep de $(\rho_1 \otimes \text{sgn}) \otimes (\rho_2 \otimes \text{sgn})$, c'est-à-dire de $\text{sgn} \otimes \varphi$. $\square$

Il est clair d'après sa définition que l'endomorphisme $\rho\iota$ commute à la tensorisation $\varphi \mapsto \text{sgn} \otimes \varphi$. Alors la formule (2) se transforme en

$$\text{Res} \circ \pi(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-) = \text{Rep} \circ \rho\iota\big((\boldsymbol{\rho}_{\lambda^+,\epsilon^+} \otimes \text{sgn}) \otimes (\boldsymbol{\rho}_{\lambda^-,\epsilon^-} \otimes \text{sgn})\big).$$

En utilisant l'égalité $\text{Rep} = k \circ \mathcal{F}$, on obtient finalement l'égalité

$$\kappa_{\pi(\lambda^+,\epsilon^+,\lambda^-,\epsilon^-)} = \mathcal{F} \circ \rho\iota\big((\boldsymbol{\rho}_{\lambda^+,\epsilon^+} \otimes \text{sgn}) \otimes (\boldsymbol{\rho}_{\lambda^-,\epsilon^-} \otimes \text{sgn})\big). \tag{4}$$

## 2. Symboles, partitions spéciales, dualité

**2.1.** ***Symboles.*** Pour un couple $\Lambda = (X, Y)$ de sous-ensembles finis de $\mathbb{N}$, on définit le rang $\mathrm{rg}(\Lambda)$ et le défaut $\mathrm{def}(\Lambda)$ par

$$\mathrm{rg}(\Lambda) = S(X) + S(Y) - [(|X| + |Y| - 1)^2/4],$$

où $[\cdot]$ désigne la partie entière,

$$\mathrm{def}(\Lambda) = \sup(|X| - |Y|, |Y| - |X|).$$

On définit une relation d'équivalence entre couples de sous-ensembles finis de $\mathbb{N}$, engendrée par $(X, Y) \sim (X', Y')$, où

$$X' = \{x + 1;\ x \in X\} \cup \{0\}, \quad Y' = \{y + 1;\ y \in Y\} \cup \{0\}.$$

Le rang et le défaut sont constants sur toute classe d'équivalence. On appelle symbole de défaut impair une classe d'équivalence de couples $\Lambda = (X, Y)$ tels que $|X| > |Y|$ et $\mathrm{def}(\Lambda)$ est impair. On appelle symbole de défaut pair une classe d'équivalence de couples $\Lambda = (X, Y)$ tels que $\mathrm{def}(\Lambda)$ est pair (dans le cas pair, on n'impose pas $|X| \geq |Y|$).

Soit $m \in \mathbb{N}$. On note $S_{m,\mathrm{imp}}$ l'ensemble des classes d'équivalence de symboles de défaut impair et de rang $m$. Pour $\Lambda \in S_{m,\mathrm{imp}}$, on pose $r(\Lambda) = (\mathrm{def}(\Lambda) - 1)/2$. On note $S_{m,\mathrm{pair}}$ l'ensemble des classes d'équivalence de symbole de défaut pair et de rang $m$. Pour $\Lambda = (X, Y) \in S_{m,\mathrm{pair}}$, on pose $r(\Lambda) = (|X| - |Y|)/2$. On a $\mathrm{def}(\Lambda) = 2|r(\Lambda)|$.

**Remarque.** La définition que l'on utilise ici des symboles de défaut pair est différente de celle de [Waldspurger 2018b, 1.2] où l'on avait identifié les couples $(X, Y)$ et $(Y, X)$.

Notons $\Sigma_{m,\mathrm{imp}}$ l'ensemble des triplets $(r, \alpha, \beta)$ où $r \in \mathbb{N}$, $\alpha$ et $\beta$ sont des partitions et $r^2 + r + S(\alpha) + S(\beta) = m$. Remarquons que, puisque les couples de partitions $(\alpha, \beta)$ vérifiant la relation précédente paramètrent les représentations irréductibles de $W_{m-r^2-r}$, on peut identifier $\Sigma_{m,\mathrm{imp}}$ à l'ensemble des couples $(r, \rho)$, où $r \in \mathbb{N}$ vérifie $r^2 + r \leq m$ et $\rho \in \widehat{W}_{m-r^2-r}$. On définit une application $\mathrm{symb} : \Sigma_{m,\mathrm{imp}} \to S_{m,\mathrm{imp}}$ de la façon suivante. Soit $(r, \alpha, \beta) \in \Sigma_{m,\mathrm{imp}}$. On suppose que $\beta$ a $a$ termes pour un entier $a \geq 0$ et que $\alpha$ en a $a + 2r + 1$. On pose $X = \alpha + \{a + 2r, a + 2r - 1, \ldots, 0\}$, $Y = \beta + \{a - 1, a - 2, \ldots, 0\}$, $\Lambda = (X, Y)$. Alors, $\mathrm{symb}(r, \alpha, \beta) = \Lambda$. Remarquons que $r = r(\Lambda)$. L'application symb ainsi définie est une bijection de $\Sigma_{m,\mathrm{imp}}$ sur $S_{m,\mathrm{imp}}$.

Notons $\Sigma_{m,\mathrm{pair}}$ l'ensemble des triplets $(r, \alpha, \beta)$ où $r \in \mathbb{Z}$, $\alpha$ et $\beta$ sont des partitions et $r^2 + S(\alpha) + S(\beta) = m$. On peut identifier $\Sigma_{m,\mathrm{pair}}$ à l'ensemble des couples $(r, \rho)$, où $r \in \mathbb{Z}$ vérifie $r^2 \leq m$ et $\rho \in \widehat{W}_{m-r^2}$. On définit une application symb : $\Sigma_{m,\mathrm{pair}} \to S_{m,\mathrm{pair}}$ de la façon suivante. Soit $(r, \alpha, \beta) \in \Sigma_{m,\mathrm{pair}}$. On suppose que $\beta$ a $a$

termes et que $\alpha$ en a $a+2|r|$. Si $r \geq 0$, on pose $X = \alpha + \{a+2r-1, a+2r-2, \ldots, 0\}$, $Y = \beta + \{a-1, a-2, \ldots, 0\}$. Si $r < 0$, on pose $X = \beta + \{a-1, a-2, \ldots, 0\}$, $Y = \alpha + \{a+2|r|-1, a-2|r|-2, \ldots, 0\}$. On pose $\Lambda = (X, Y)$. Alors $\mathrm{symb}(r, \alpha, \beta) = \Lambda$. Remarquons que $r = r(\Lambda)$. L'application symb ainsi définie est une bijection de $\Sigma_{m,\mathrm{pair}}$ sur $S_{m,\mathrm{pair}}$.

Posons

$$\mathcal{S} = \bigoplus_{n'+n''=n} \mathbb{C}[S_{n',\mathrm{imp}}] \otimes \mathbb{C}[S_{n'',\mathrm{pair}}].$$

D'après la construction de 1.2, l'espace $\mathcal{R}$ s'identifie à

$$\bigoplus_{n'+n''=n} \mathbb{C}[\Sigma_{n',\mathrm{imp}}] \otimes \mathbb{C}[\Sigma_{n'',\mathrm{pair}}].$$

Des bijections symb ci-dessus se déduisent donc un isomorphisme encore noté $\mathrm{symb} : \mathcal{R} \to \mathcal{S}$.

**2.2.** ***Partitions spéciales, cas symplectique.*** Soit $m \in \mathbb{N}$. Une partition symplectique $\lambda \in \mathcal{P}^{\mathrm{symp}}(2m)$ est dite spéciale si $\lambda_{2j-1}$ et $\lambda_{2j}$ sont de même parité pour tout $j \geq 1$. On note $\mathcal{P}^{\mathrm{symp,sp}}(2m)$ le sous-ensemble des partitions spéciales. Soit $\lambda$ une telle partition spéciale. Considérons l'ensemble des éléments $i \in \mathrm{Jord}^{\mathrm{bp}}(\lambda)$ tels que $\mathrm{mult}_\lambda(i)$ soit impair. S'il a un nombre pair d'éléments, on les note $i_1 > i_2 > \cdots > i_t$. S'il a un nombre impair d'éléments, on les note $i_1 > i_2 > \cdots > i_{t-1}$ et on pose $i_t = 0$. Ainsi, $t$ est toujours pair. On appelle intervalle de $\lambda$ un sous-ensemble de $\mathrm{Jord}(\lambda) \cup \{0\}$ de l'une des formes suivantes :

$$\big\{i \in \mathrm{Jord}(\lambda) \cup \{0\};\, i_{2h-1} \geq i \geq i_{2h}\big\} \quad \text{pour } h = 1, \ldots, t/2;$$

$\{i\}$ pour $i \in \mathrm{Jord}^{\mathrm{bp}}(\lambda) \cup \{0\}$ tel qu'il n'existe pas de $h = 1, \ldots, t/2$ de sorte que $i_{2h-1} \geq i \geq i_{2h}$.

Parce que $\lambda$ est spéciale, on voit que les intervalles sont formés d'entiers pairs. On note $\widetilde{\mathrm{Int}}(\lambda)$ l'ensemble de ces intervalles. Il est ordonné de façon naturelle : $\Delta > \Delta'$ si et seulement si $i > i'$ pour tous $i \in \Delta$ et $i' \in \Delta'$. L'élément minimal est celui qui contient 0, on le note $\Delta_{\min}$ et on pose $\mathrm{Int}(\lambda) = \widetilde{\mathrm{Int}}(\lambda) - \{\Delta_{\min}\}$. Pour $\Delta \in \widetilde{\mathrm{Int}}(\lambda)$, on note $J(\Delta)$ l'ensemble des $j \geq 1$ tels que $\lambda_j \in \Delta$. C'est un intervalle de $\mathbb{N}$, qui est infini dans le cas $\Delta = \Delta_{\min}$. On note $j_{\min}(\Delta)$ le plus petit élément de $J(\Delta)$ et, si $\Delta \neq \Delta_{\min}$, on note $j_{\max}(\Delta)$ le plus grand élément de $J(\Delta)$. On vérifie que

$\{j_{\min}(\Delta);\, \Delta \in \widetilde{\mathrm{Int}}(\lambda)\}$ est l'ensemble des $j \geq 1$ tels que $j$ soit impair, $\lambda_j$ soit pair et $\lambda_{j-1} > \lambda_j$, avec la convention $\lambda_0 = \infty$;

$\{j_{\max}(\Delta);\, \Delta \in \mathrm{Int}(\lambda)\}$ est l'ensemble des $j \geq 1$ tels que $j$ soit pair, $\lambda_j$ soit pair et $\lambda_j > \lambda_{j+1}$.

Par la correspondance de Springer, on associe à $(\lambda, 1)$ une représentation irréductible de $W_m$. Elle est paramétrée par un couple $(\alpha(\lambda), \beta(\lambda))$. On note $(X(\lambda), Y(\lambda)) \in S_{m,\text{imp}}$ l'image de $(0, \alpha(\lambda), \beta(\lambda))$ par l'application symb. C'est un symbole spécial, c'est-à-dire que $|X(\lambda)| = |Y(\lambda)| + 1$ et, si on note $X(\lambda) = (x_1 > \cdots > x_{a+1})$, $Y(\lambda) = (y_1 > \cdots > y_a)$, on a $x_1 \geq y_1 \geq x_2 \geq y_2 \geq \cdots \geq x_a \geq y_a \geq x_{a+1}$. On appelle famille de $\lambda$ l'ensemble des symboles $(X, Y) \in S_{m,\text{imp}}$ tels que, quitte à remplacer $(X, Y)$ et $(X(\lambda), Y(\lambda))$ par des symboles équivalents, on ait

$$X \cup Y = X(\lambda) \cup Y(\lambda), \quad X \cap Y = X(\lambda) \cap Y(\lambda). \tag{1}$$

On note $\text{Fam}(\lambda)$ la famille de $\lambda$. On montre que $S_{m,\text{imp}}$ est la réunion disjointe des $\text{Fam}(\lambda)$ quand $\lambda$ décrit l'ensemble $\mathcal{P}^{\text{symp,sp}}(2m)$.

Soit $\lambda \in \mathcal{P}^{\text{symp,sp}}(2m)$. On montre qu'il y a une unique bijection croissante $\Delta \mapsto x_\Delta$ de $\widetilde{\text{Int}}(\lambda)$ sur $X(\lambda) - (X(\lambda) \cap Y(\lambda))$ et une unique bijection croissante $\Delta \mapsto y_\Delta$ de $\text{Int}(\lambda)$ sur $Y(\lambda) - (X(\lambda) \cap Y(\lambda))$. A un symbole $\Lambda = (X, Y) \in \text{Fam}(\lambda)$, on associe deux éléments $\tau \in (\mathbb{Z}/2\mathbb{Z})^{\widetilde{\text{Int}}(\lambda)}$ et $\delta \in (\mathbb{Z}/2\mathbb{Z})^{\text{Int}(\lambda)}$ par les formules suivantes. On suppose les symboles choisis de sorte que (1) soit vérifié. Alors, pour $\Delta \in \widetilde{\text{Int}}(\lambda)$, resp. $\Delta \in \text{Int}(\lambda)$, on pose

$$\tau(\Delta) = \left|\{\Delta' \in \widetilde{\text{Int}}(\lambda);\ \Delta' \geq \Delta,\ x_{\Delta'} \in Y\}\right| + \left|\{\Delta' \in \text{Int}(\lambda);\ \Delta' > \Delta,\ y_{\Delta'} \in X\}\right| + r(\Lambda) \mod 2\mathbb{Z};$$

resp.

$$\delta(\Delta) = \left|\{\Delta' \in \text{Int}(\lambda);\ \Delta' \geq \Delta,\ x_{\Delta'} \in Y\}\right| + \left|\{\Delta' \in \text{Int}(\lambda);\ \Delta' \geq \Delta,\ y_{\Delta'} \in X\}\right| \mod 2\mathbb{Z}.$$

Par cette construction, la famille $\text{Fam}(\lambda)$ s'identifie à l'ensemble des couples $(\tau, \delta) \in (\mathbb{Z}/2\mathbb{Z})^{\widetilde{\text{Int}}(\lambda)} \times (\mathbb{Z}/2\mathbb{Z})^{\text{Int}(\lambda)}$ tels que $\tau(\Delta_{\min}) = 0$. On note $\mathcal{F}am(\lambda)$ cet ensemble. Pour $(\tau, \delta)$ dans cet ensemble, provenant du symbole $\Lambda$, on pose $r(\tau, \delta) = r(\Lambda)$.

**2.3.** ***Partitions spéciales, cas orthogonal impair.*** Soit $m \in \mathbb{N}$. Une partition orthogonale $\lambda \in \mathcal{P}^{\text{orth}}(2m+1)$ est dite spéciale si $\lambda_{2j}$ et $\lambda_{2j+1}$ sont de même parité pour tout $j \geq 1$. Il en résulte que $\lambda_1$ est impair. On note $\mathcal{P}^{\text{orth,sp}}(2m+1)$ le sous-ensemble des partitions spéciales. Soit $\lambda$ une telle partition spéciale. Les constructions du paragraphe précédent s'appliquent. Par la correspondance de Springer, on associe à $(\lambda, 1)$ une représentation irréductible de $W_m$, puis un symbole appartenant à $S_{m,\text{imp}}$. Il est spécial. On définit la famille de $\lambda$, que l'on note $\text{Fam}(\lambda)$. On montre que $S_{m,\text{imp}}$ est la réunion disjointe des $\text{Fam}(\lambda)$ quand $\lambda$ décrit l'ensemble $\mathcal{P}^{\text{orth,sp}}(2m+1)$.

Remarquons que la conjonction des propriétés énoncées ici et dans le paragraphe précédent entraîne qu'il y a une bijection entre $\mathcal{P}^{\text{symp,sp}}(2m)$ et $\mathcal{P}^{\text{orth,sp}}(2m+1)$ : $\lambda \in \mathcal{P}^{\text{symp,sp}}(2m)$ correspond à $\mu \in \mathcal{P}^{\text{orth,sp}}(2m+1)$ si et seulement si $\text{Fam}(\lambda) = \text{Fam}(\mu)$. En fait, nous utiliserons une autre bijection, la dualité, cf. 2.6.

**2.4.** ***Partitions spéciales, cas orthogonal pair.*** Soit $m \in \mathbb{N}$. Une partition orthogonale $\lambda \in \mathcal{P}^{\text{orth}}(2m)$ est dite spéciale si $\lambda_{2j-1}$ et $\lambda_{2j}$ sont de même parité pour tout $j \geq 1$. On note $\mathcal{P}^{\text{orth,sp}}(2m)$ le sous-ensemble des partitions spéciales. Soit $\lambda$ une telle partition spéciale. Considérons l'ensemble des éléments $i \in \text{Jord}^{\text{bp}}(\lambda)$ tels que $\text{mult}_\lambda(i)$ soit impair. On les note $i_1 > i_2 > \cdots > i_t$. L'entier $t$ est forcément pair. On appelle intervalle de $\lambda$ un sous-ensemble de $\text{Jord}(\lambda)$ de l'une des formes suivantes :

$$\left\{i \in \text{Jord}(\lambda);\ i_{2h-1} \geq i \geq i_{2h}\right\} \quad \text{pour } h = 1, \ldots, t/2;$$

$$\{i\} \quad \text{pour } i \in \text{Jord}^{\text{bp}}(\lambda) \text{ tel qu'il n'existe pas de } h = 1, \ldots, t/2 \text{ de sorte que } i_{2h-1} \geq i \geq i_{2h}.$$

Parce que $\lambda$ est spéciale, on voit que les intervalles sont formés d'entiers impairs. On note $\text{Int}(\lambda)$ l'ensemble de ces intervalles. Comme dans le cas symplectique, il est ordonné de façon naturelle. Pour $\Delta \in \text{Int}(\lambda)$, on définit $J(\Delta)$, $j_{\min}(\Delta)$ et $j_{\max}(\Delta)$ comme dans le cas symplectique. On vérifie que

$\{j_{\min}(\Delta);\ \Delta \in \text{Int}(\lambda)\}$ est l'ensemble des $j \geq 1$ tels que $j$ soit impair, $\lambda_j$ soit impair et $\lambda_{j-1} > \lambda_j$, avec la convention $\lambda_0 = \infty$;

$\{j_{\max}(\Delta);\ \Delta \in \text{Int}(\lambda)\}$ est l'ensemble des $j \geq 1$ tels que $j$ soit pair, $\lambda_j$ soit impair et $\lambda_j > \lambda_{j+1}$.

Par la correspondance de Springer, on associe à $(\lambda, 1)$ une représentation irréductible de $W_m^D$. Elle est paramétrée par un couple $(\alpha(\lambda), \beta(\lambda))$, qui n'est déterminé qu'à l'ordre près. On impose que $\alpha(\lambda) \geq \beta(\lambda)$ pour l'ordre lexicographique (s'il existe $j$ tel que $\alpha(\lambda)_j \neq \beta(\lambda)_j$, on a $\alpha(\lambda)_j > \beta(\lambda)_j$ pour le plus petit de ces entiers $j$). On note $(X(\lambda), Y(\lambda)) \in S_{m,\text{pair}}$ l'image de $(0, \alpha(\lambda), \beta(\lambda))$ par l'application symb. C'est un symbole spécial, c'est-à-dire que $|X(\lambda)| = |Y(\lambda)|$ et, si on note $X(\lambda) = (x_1 > \cdots > x_a)$, $Y(\lambda) = (y_1 > \cdots > y_a)$, on a $x_1 \geq y_1 \geq x_2 \geq y_2 \geq \cdots \geq x_a \geq y_a$. On appelle famille de $\lambda$ l'ensemble des symboles $(X, Y) \in S_{m,\text{pair}}$ tels que, quitte à remplacer $(X, Y)$ et $(X(\lambda), Y(\lambda))$ par des symboles équivalents, on ait

$$X \cup Y = X(\lambda) \cup Y(\lambda), \quad X \cap Y = X(\lambda) \cap Y(\lambda). \tag{1}$$

On note $\text{Fam}(\lambda)$ la famille de $\lambda$. On montre que $S_{m,\text{pair}}$ est la réunion disjointe des familles $\text{Fam}(\lambda)$ quand $\lambda$ décrit l'ensemble $\mathcal{P}^{\text{orth,sp}}(2m)$.

Soit $\lambda \in \mathcal{P}^{\text{orth,sp}}(\lambda)$. On montre qu'il y a une unique bijection croissante $\Delta \mapsto x_\Delta$ de $\text{Int}(\lambda)$ sur $X(\lambda) - (X(\lambda) \cap Y(\lambda))$ et une unique bijection croissante $\Delta \mapsto y_\Delta$ de $\text{Int}(\lambda)$ sur $Y(\lambda) - (X(\lambda) \cap Y(\lambda))$. À un symbole $\Lambda = (X, Y) \in \text{Fam}(\lambda)$, on associe deux éléments $\tau, \delta \in (\mathbb{Z}/2\mathbb{Z})^{\text{Int}(\lambda)}$ par les mêmes formules qu'en 2.2 (à ceci près qu'un $\widetilde{\text{Int}}(\lambda)$ figurant dans ces dernières est remplacé par $\text{Int}(\lambda)$). Par cette construction, la famille $\text{Fam}(\lambda)$ s'identifie à l'ensemble des couples $(\tau, \delta) \in (\mathbb{Z}/2\mathbb{Z})^{\text{Int}(\lambda)} \times (\mathbb{Z}/2\mathbb{Z})^{\text{Int}(\lambda)}$. On note $\mathcal{F}am(\lambda)$ cet ensemble. Pour $(\tau, \delta)$ dans cet ensemble, provenant du symbole $\Lambda$, on pose $r(\tau,\delta)=r(\Lambda)$. Si $\mathrm{Int}(\lambda)\neq\emptyset$, on note $\Delta_{\min}$ son plus petit élément et on vérifie que

$$\delta(\Delta_{\min})\equiv r(\tau,\delta) \mod 2\mathbb{Z}; \tag{2}$$

si $\mathrm{Int}(\lambda)=\emptyset$, $\mathcal{F}am(\lambda)$ a un unique élément $(\emptyset,\emptyset)$ et on a $r(\emptyset,\emptyset)=0$.

**2.5.** ***L'involution de Lusztig.*** Soient $m\in\mathbb{N}$ et $\lambda\in\mathcal{P}^{\mathrm{symp,sp}}(2m)$. On note $\Lambda(\lambda)=(X(\lambda),Y(\lambda))$ le symbole spécial associé à $\lambda$. On représente tout élément de la famille de $\lambda$ par un symbole $\Lambda=(X,Y)$ vérifiant la condition 2.2(1). Soient $\Lambda=(X,Y)$, $\Lambda'=(X',Y')$ deux éléments de $\mathrm{Fam}(\lambda)$. On pose

$$\langle\Lambda,\Lambda'\rangle=r(\Lambda)+r(\Lambda')+|X\cap X'\cap Y(\lambda)|+|Y\cap Y'\cap X(\lambda)| \mod 2\mathbb{Z}.$$

Cela définit une application :

$$\langle\cdot,\cdot\rangle:\mathrm{Fam}(\lambda)\times\mathrm{Fam}(\lambda)\to\mathbb{Z}/2\mathbb{Z}.$$

On définit un automorphisme $\mathcal{F}$ de l'espace $\mathbb{C}[\mathrm{Fam}(\lambda)]$ par la formule

$$\mathcal{F}(\Lambda)=|\,\mathrm{Fam}(\lambda)|^{-\frac{1}{2}}\sum_{\Lambda'\in\mathrm{Fam}(\lambda)}(-1)^{\langle\Lambda,\Lambda'\rangle}\Lambda',$$

les symboles étant ici identifiés aux éléments de base de $\mathbb{C}[\mathrm{Fam}(\lambda)]$. On vérifie qu'il est involutif. D'après ce que l'on a dit en 2.2, l'espace $\mathbb{C}[S_{m,\mathrm{imp}}]$ est somme directe des sous-espaces $\mathbb{C}[\mathrm{Fam}(\lambda)]$ quand $\lambda$ décrit $\mathcal{P}^{\mathrm{symp,sp}}(2m)$. On note $\mathcal{F}$ l'automorphisme de $\mathbb{C}[S_{m,\mathrm{imp}}]$ qui est la somme directe des automorphismes de ces sous-espaces que l'on vient de construire.

Pour $\lambda\in\mathcal{P}^{\mathrm{orth,sp}}(2m)$, on définit exactement de la même façon un automorphisme $\mathcal{F}$ de $\mathbb{C}[\mathrm{Fam}(\lambda)]$. Puis, par somme directe, on en déduit un automorphisme de $\mathbb{C}[S_{m,\mathrm{pair}}]$.

Dans le cas orthogonal pair, on dispose d'une involution $\sigma$ de $\mathrm{Fam}(\lambda)$ : si $\Lambda=(X,Y)$, $\sigma(\Lambda)=(Y,X)$. Pour $\Lambda,\Lambda'\in\mathrm{Fam}(\lambda)$, on vérifie la formule

$$\langle\sigma(\Lambda),\Lambda'\rangle\equiv r(\Lambda')+\langle\Lambda,\Lambda'\rangle \mod 2\mathbb{Z}. \tag{1}$$

Rappelons que

$$\mathcal{S}=\bigoplus_{\substack{n',n''\in\mathbb{N}\\ n'+n''=n}}\mathbb{C}[S_{n',\mathrm{imp}}]\otimes\mathbb{C}[S_{n'',\mathrm{pair}}].$$

On a défini des automorphismes $\mathcal{F}$ de chacun des espaces qui interviennent ici. Par produit tensoriel et sommation, on en déduit un automorphisme $\mathcal{F}$ de $\mathcal{S}$. On a défini en 2.1 un isomorphisme $\mathrm{symb}:\mathcal{R}\to\mathcal{S}$. Par celui-ci, on transporte l'automorphisme $\mathcal{F}$ de $\mathcal{S}$ en un automorphisme $\mathcal{F}$ de $\mathcal{R}$. C'est l'automorphisme de Lusztig introduit en 1.4.

**2.6.** ***Dualité.*** Soit $m \in \mathbb{N}$. On a introduit en 2.1 l'ensemble $\Sigma_{m,\mathrm{imp}}$, que l'on voit ici comme un ensemble de couples $(r, \rho)$, où $\rho \in \widehat{W}_{m-r^2-r}$. On définit une involution de cet ensemble par $(r, \rho) \mapsto (r, \rho \otimes \mathrm{sgn})$. Transportons-la en une involution de $S_{m,\mathrm{imp}}$ par la bijection symb. On note $d$ l'involution obtenue. Elle se calcule ainsi. Soit $\Lambda = (X, Y) \in S_{m,\mathrm{imp}}$. Fixons un entier $a$ plus grand que tous les termes de $X$ et $Y$. Posons

$$X' = \{a, \dots, 0\} - \{a - y;\ y \in Y\}, \quad Y' = \{a, \dots, 0\} - \{a - x;\ x \in X\}.$$

Alors $d(\Lambda) = (X', Y')$. Cette formule montre que $d$ conserve la décomposition en familles, c'est-à-dire que si $\Lambda$ et $\Lambda'$ sont dans une même famille, alors $d(\Lambda)$ et $d(\Lambda')$ sont aussi dans une même famille. On définit une application appelée dualité $d : \mathcal{P}^{\mathrm{symp,sp}}(2m) \to \mathcal{P}^{\mathrm{orth,sp}}(2m+1)$ ou $d : \mathcal{P}^{\mathrm{orth,sp}}(2m+1) \to \mathcal{P}^{\mathrm{symp,sp}}(2m)$ par la condition $\mathrm{Fam}(d(\lambda)) = d(\mathrm{Fam}(\lambda))$. Les deux applications sont inverses l'une de l'autre.

Ces dualités s'étendent en des applications $d : \mathcal{P}^{\mathrm{symp}}(2m) \to \mathcal{P}^{\mathrm{orth,sp}}(2m+1)$ ou $d : \mathcal{P}^{\mathrm{orth}}(2m+1) \to \mathcal{P}^{\mathrm{symp,sp}}(2m)$. Rappelons la définition de la première, celle de la seconde étant similaire. Soit $\lambda \in \mathcal{P}^{\mathrm{symp}}(2m)$. La correspondance de Springer associe au couple $(\lambda, 1) \in \boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2m)$ une représentation $\rho_{\lambda,1}$ de $W_m$. Le couple $(0, \rho_{\lambda,1})$ appartient à $\Sigma_{m,\mathrm{imp}}$. Il existe une unique partition symplectique spéciale, que l'on note $\mathrm{sp}(\lambda)$, dont la famille contient le symbole $\mathrm{symb}(0, \rho_{\lambda,1})$. En fait, on montre que $\mathrm{sp}(\lambda)$ est la plus petite partition symplectique spéciale $\lambda'$ telle que $\lambda \leq \lambda'$. On pose $d(\lambda) = d(\mathrm{sp}(\lambda))$. Cette dualité est décroissante : $\lambda \leq \lambda'$ entraîne $d(\lambda') \leq d(\lambda)$.

On peut remplacer $\Sigma_{m,\mathrm{imp}}$ par $\Sigma_{m,\mathrm{pair}}$ dans la construction ci-dessus. On obtient une dualité $d$ qui est une involution de $\mathcal{P}^{\mathrm{orth,sp}}(2m)$. Celle-ci s'étend en une application $d : \mathcal{P}^{\mathrm{orth}}(2m) \to \mathcal{P}^{\mathrm{orth,sp}}(2m)$, qui est décroissante.

**2.7.** ***Calcul de $d(\lambda)$.*** Soient $m \in \mathbb{N}$ et $\lambda \in \mathcal{P}^{\mathrm{symp}}(2m)$. Pour $i \in \mathrm{Jord}(\lambda) \cup \{0\}$, notons $J(i)$ l'ensemble des indices $j \geq 1$ tels que $\lambda_j = i$. C'est un intervalle de $\mathbb{N} - \{0\}$, infini si $i = 0$. On note $j_{\min}(i)$ son plus petit élément et, si $i \neq 0$, $j_{\max}(i)$ son plus grand élément. Considérons l'ensemble des éléments $i$ de $\mathrm{Jord}^{\mathrm{bp}}(\lambda)$ tels que $\mathrm{mult}_\lambda(i)$ soit impaire. Comme en 2.2, si cet ensemble a un nombre pair d'éléments, on les note $i_1 > \dots > i_t$. S'il a un nombre impair d'éléments, on les note $i_1 > \dots > i_{t-1}$ et on pose $i_t = 0$. Pour $h = 1, \dots, t$, on vérifie que

$$j_{\min}(i_h) \equiv h \mod 2\mathbb{Z} \quad \text{et,} \quad \text{si } i_h \neq 0,\ j_{\max}(i_h) \equiv h \mod 2\mathbb{Z}.$$

Considérons les éléments de $\mathrm{Jord}(\lambda) \cup \{0\}$ qui n'interviennent pas dans la suite $i_1, \dots, i_t$, c'est-à-dire les $i \in \mathrm{Jord}(\lambda)$ tels que $\mathrm{mult}_\lambda(i)$ soit pair et aussi 0 dans le cas où $i_t \neq 0$. Notons $\mathcal{J}(\lambda)$ cet ensemble. On décompose $\mathcal{J}(\lambda)$ en union disjointe $\mathcal{J}'(\lambda) \sqcup \mathcal{J}''(\lambda)$ : $\mathcal{J}''(\lambda)$ est l'ensemble des $i \in \mathcal{J}(\lambda)$ tels qu'il existe $h = 1, \dots, t/2$

de sorte que $i_{2h-1} > i > i_{2h}$ ; $\mathcal{J}'(\lambda)$ est son complémentaire. On vérifie que

pour $i \in \mathcal{J}'(\lambda)$, $j_{\min}(i)$ est impair et, si $i \neq 0$, $j_{\max}(i)$ est pair;

pour $i \in \mathcal{J}''(\lambda)$, $j_{\min}(i)$ est pair et, si $i \neq 0$, $j_{\max}(i)$ est impair.

Notons

$P^+(\lambda)$ l'ensemble des entiers impairs $j \geq 1$ tels que $\lambda_j$ est pair et $\lambda_{j-1} > \lambda_j$, avec la convention $\lambda_0 = \infty$;

$P^-(\lambda)$ l'ensemble des entiers pairs $j \geq 2$ tels que $\lambda_j$ est pair et $\lambda_j > \lambda_{j+1}$;

$Q^+(\lambda)$ l'ensemble des entiers pairs $j \geq 2$ tels que $\lambda_j$ est impair et $\lambda_{j-1} > \lambda_j$;

$Q^-(\lambda)$ l'ensemble des entiers impairs $j \geq 1$ tels que $\lambda_j$ est impair et $\lambda_j > \lambda_{j+1}$.

Ces ensembles sont disjoints. A l'aide des propriétés précédentes, on voit que

$P^+(\lambda)$ est l'ensemble des $j_{\min}(i)$ pour $i = i_h$ avec $h$ impair, ou pour un élément pair $i \in \mathcal{J}'(\lambda)$;

$P^-(\lambda)$ est l'ensemble des $j_{\max}(i)$ pour $i = i_h$ avec $h$ pair et $i_h \neq 0$ ou pour un élément pair non nul $i \in \mathcal{J}'(\lambda)$;

$Q^+(\lambda)$ est l'ensemble des $j_{\min}(i)$ pour un élément impair $i \in \mathcal{J}''(\lambda)$;

$Q^-(\lambda)$ est l'ensemble des $j_{\max}(i)$ pour un élément impair $i \in \mathcal{J}''(\lambda)$.

Ces ensembles sont disjoints. Les éléments de $P^+(\lambda) \cup P^-(\lambda)$ apparaissent presque tous par paires. Un élément de $P^+(\lambda)$ de la forme $j_{\min}(i)$ pour $i = i_h$ avec $h$ impair est suivi de l'élément $j_{\max}(i_{h+1}) \in P^-(\lambda)$ sauf si $i_{h+1} = 0$. Un élément de $P^+(\lambda)$ de la forme $j_{\min}(i)$ pour un élément pair $i \in \mathcal{J}'(\lambda)$ est suivi de l'élément $j_{\max}(i) \in P^-(\lambda)$ sauf si $i = 0$. Le plus petit élément de $P^+(\lambda)$ est $j_{\min}(i_{t-1})$ si $i_t = 0$ ou $j_{\min}(0)$ si $i_t \neq 0$. Il n'est suivi d'aucun élément de $P^-(\lambda)$. Il en résulte que $|P^+\lambda)| = |P^-(\lambda)| + 1$ et que, si on note ces ensembles

$$P^+(\lambda) = \{p_1^+ < \cdots < p_{a+1}^+\}, \quad P^-(\lambda) = \{p_1^- < \cdots < p_a^-\},$$

on a les relations

$$p_1^+ < p_1^- < p_2^+ < p_2^- < \cdots < p_a^+ < p_a^- < p_{a+1}^+.$$

On voit de même que $|Q^+(\lambda)| = |Q^-(\lambda)|$ et que, si on note ces ensembles

$$Q^+(\lambda) = \{q_1^+ < \cdots < q_b^+\}, \quad Q^-(\lambda) = \{q_1^- < \cdots < q_b^-\},$$

on a les relations

$$q_1^+ < q_1^- < q_2^+ < q_2^- < \cdots < q_b^+ < q_b^-.$$

Remarquons que, pour $j \in Q^+(\lambda)$, on a $\lambda_j = \lambda_{j+1}$. En effet, $\lambda_j$ est impair, donc $\mathrm{mult}_\lambda(\lambda_j)$ est pair. Mais on a aussi $\lambda_{j-1} > \lambda_j$, d'où l'égalité cherchée. De même, pour $j \in Q^-(\lambda)$, on a $j \geq 2$ et $\lambda_{j-1} = \lambda_j$.

Définissons deux suites de nombres

$$\zeta(\lambda) = (\zeta(\lambda)_1, \zeta(\lambda)_2, \dots) \quad \text{et} \quad s(\lambda) = (s(\lambda)_1, s(\lambda)_2, \dots)$$

par les égalités

$$\zeta(\lambda)_j = \begin{cases} 1 & \text{si } j \in P^+(\lambda), \\ -1 & \text{si } j \in P^-(\lambda), \\ 0 & \text{sinon}\,; \end{cases} \qquad s(\lambda)_j = \begin{cases} 1 & \text{si } j \in Q^+(\lambda), \\ -1 & \text{si } j \in Q^-(\lambda), \\ 0 & \text{sinon}\,; \end{cases}$$

**Lemme.** *On a les égalités* (i) $\mathrm{sp}(\lambda) = \lambda + s(\lambda)$ ; (ii) ${}^t d(\lambda) = \lambda + \zeta(\lambda)$.

*Preuve.* Posons $\nu = \lambda + s(\lambda)$. Montrons que $\nu$ est une partition, c'est-à-dire que $\nu_j \geq \nu_{j+1}$ pour tout $j \geq 1$. Puisque le couple $(\nu_j, \nu_{j+1})$ s'obtient en ajoutant à $(\lambda_j, \lambda_{j+1})$ un couple qui appartient à $\{-1, 0, 1\} \times \{-1, 0, 1\}$ et puisque $\lambda_j \geq \lambda_{j+1}$, la conclusion est claire sauf si le couple ajouté est $(-1, 0)$, $(0, 1)$ ou $(-1, 1)$. Le premier cas se produit seulement si $j \in Q^-(\lambda)$. Dans ce cas on a $\lambda_j > \lambda_{j+1}$ par définition de $Q^-(\lambda)$ et alors $\lambda_j - 1 \geq \lambda_{j+1}$. Le deuxième cas se produit seulement si $j+1 \in Q^+(\lambda)$. Dans ce cas, on a encore $\lambda_j > \lambda_{j+1}$ par définition de $Q^+(\lambda)$ et alors $\lambda_j \geq \lambda_{j+1} + 1$. Le dernier cas se produit quand $j \in Q^-(\lambda)$ et $j+1 \in Q^+(\lambda)$. On a encore $\lambda_j > \lambda_{j+1}$ De plus, $\lambda_j$ et $\lambda_{j+1}$ sont tous deux impairs. Donc $\lambda_j \geq \lambda_{j+1} + 2$. Alors $\lambda_j - 1 \geq \lambda_{j+1} + 1$.

L'égalité des nombres d'éléments de $Q^+(\lambda)$ et de $Q^-(\lambda)$ et la définition de $s(\lambda)$ entraînent que $S(\nu) = 2n$. Une partition $\mu$ de $2n$ est symplectique et spéciale si et seulement si, pour tout entier $j \geq 1$ impair, $\mu_j$ et $\mu_{j+1}$ sont de même parité et si, lorsque ces nombres sont impairs, ils sont égaux. Cela équivaut à : pour tout $j \geq 1$ impair, si $\mu_j$ ou $\mu_{j+1}$ est impair, alors $\mu_j = \mu_{j+1}$. Montrons que $\nu$ vérifie cette condition. Soit un entier $j \geq 1$ impair, supposons $\nu_j$ impair. L'entier $j$ n'appartient pas à $Q^+(\lambda)$ car il est impair. Il n'appartient pas à $Q^-(\lambda)$ : sinon $\lambda_j$ serait impair et $\nu_j = \lambda_j - 1$ serait pair. Donc $s(\lambda)_j = 0$ et $\nu_j = \lambda_j$. Puisque $j$ est impair, que $\lambda_j = \nu_j$ est impair et que $j \notin Q^-(\lambda)$, on a $\lambda_j = \lambda_{j+1}$. Cette égalité entraîne que $j+1$ n'appartient pas à $Q^+(\lambda)$. Il n'appartient pas non plus à $Q^-(\lambda)$ car $j+1$ est pair. Donc $s(\lambda)_{j+1} = 0$, $\nu_{j+1} = \lambda_{j+1}$ et on conclut $\nu_j = \nu_{j+1}$. Une preuve analogue montre que, si $\nu_{j+1}$ est impair, on a $\nu_j = \nu_{j+1}$. Donc $\nu$ est symplectique et spéciale.

Soit $j \geq 1$. Par construction et d'après la description des ensembles $Q^+(\lambda)$ et $Q^-(\lambda)$, $S_j(\nu) = S_j(\lambda)$ sauf s'il existe un élément impair $i \in \mathcal{J}''(\lambda)$ tel que $j_{\min}(i) \leq j < j_{\max}(i)$. S'il existe un tel $i$, on a $S_j(\nu) = S_j(\lambda) + 1$. Cela montre que $\lambda \leq \nu$. Soit $\mu \in \mathcal{P}^{\mathrm{symp,sp}}(2n)$ telle que $\lambda \leq \mu$. On a $S_j(\lambda) \leq S_j(\mu)$. Cela entraîne

$$S_j(\nu) \leq S_j(\mu), \tag{1}$$

sauf s'il existe $i$ comme ci-dessus. Supposons qu'il existe un tel $i$ et notons simplement $j^+ = j_{\min}(i)$, $j^- = j_{\max}(i)$. On a $j^+ \in Q^+(\lambda)$ et $j^- \in Q^-(\lambda)$. Par définition de $j_{\min}(i)$, on a $\lambda_{j^+-1} > \lambda_{j^+}$. Les entiers $\lambda_1, \dots, \lambda_{j^+-1}$ sont tous les entiers strictement supérieurs à $\lambda_{j^+} = i$ qui interviennent dans $\lambda$, comptés avec leurs multiplicités. Puisque $\lambda$ est symplectique, l'entier $S_{j^+-1}(\lambda)$ est pair. Puisque $j^+ \in Q^+(\lambda)$, $j^+$ est pair et on sait que $i$ est impair. Donc $S_{j^+}(\lambda) = S_{j^+-1}(\lambda) + i$ est impair et aussi $S_{j^+}(\lambda) + j^+$. Puisque $\lambda_{j'} = i$ est impair pour $j' \in \{j^+, \dots, j\}$, on voit que $S_j(\lambda) + j$ est aussi impair. Supposons $j$ pair. Alors $S_j(\lambda)$ est impair. Or le fait que $\mu$ soit spéciale entraîne que $S_j(\mu)$ est pair. L'inégalité $S_j(\lambda) \leq S_j(\mu)$ est alors stricte et on conclut $S_j(\nu) = S_j(\lambda) + 1 \leq S_j(\mu)$. Supposons $j$ impair. On sait que $j^+$ est pair et que $j^-$ est impair par définition des ensembles $Q^+(\lambda)$ et $Q^-(\lambda)$. Les hypothèses $j \in \{j^+, \dots, j^- - 1\}$ et $j$ impair entraînent alors que $j - 1 \in \{j^+, \dots, j^- - 1\}$ et $j, j + 1 \in \{j^+ + 1 \dots, j^- - 1\}$. L'égalité (1) est démontrée pour $j - 1$ et pour $j + 1$ puisque ces entiers sont pairs. D'où

$$S_{j-1}(\nu) \leq S_{j-1}(\mu) \quad \text{et} \quad S_{j+1}(\nu) \leq S_{j+1}(\mu).$$

De plus, puisque $j$ et $j + 1$ appartiennent tous deux à $\{j^+ + 1, \dots, j^- - 1\}$, on a $\nu_j = i = \nu_{j+1}$. La seconde inégalité ci-dessus se récrit

$$S_{j-1}(\nu) + 2i \leq S_{j-1}(\mu) + \mu_j + \mu_{j+1}.$$

On additionne cette inégalité avec la première inégalité ci-dessus et on obtient

$$S_{j-1}(\nu) + i \leq S_{j-1}(\mu) + (\mu_j + \mu_{j+1})/2.$$

Évidemment, $(\mu_j + \mu_{j+1})/2 \leq \mu_j$, d'où

$$S_{j-1}(\nu) + i \leq S_{j-1}(\mu) + \mu_j.$$

Le membre de gauche est $S_j(\nu)$, celui de droite $S_j(\mu)$. Cela achève de démontrer (1).

L'inégalité (1) signifie que $\nu \leq \mu$. On a ainsi démontré que $\nu$ était la plus petite partition symplectique spéciale $\mu$ telle que $\lambda \leq \mu$. Cette propriété caractérise $\mathrm{sp}(\lambda)$, ce qui démontre le (i) de l'énoncé.

Prouvons maintenant que

$$\zeta(\lambda) = \zeta(\nu) + s(\lambda). \tag{2}$$

Par définition de ces suites, cela équivaut aux égalités

$$P^+(\nu) = P^+(\lambda) \cup Q^-(\lambda), \quad P^-(\nu) = P^-(\lambda) \cup Q^+(\lambda). \tag{3}$$

La première égalité concerne des indices $j \geq 1$ impairs. Soit un tel $j$. Supposons d'abord $j \in P^+(\lambda)$. On a $\nu_j = \lambda_j$ et ce terme est pair. On a de plus $\lambda_{j-1} > \lambda_j$. Si $j = 1$, on a trivialement $\nu_{j-1} > \nu_j$ et on conclut $j \in P^+(\nu)$. Supposons $j \geq 2$.

Certainement, $j-1 \notin Q^-(\lambda)$ puisque $j-1$ est pair. Donc $\nu_{j-1} \geq \lambda_{j-1}$, d'où $\nu_{j-1} > \nu_j$. Alors $j$ appartient à $P^+(\nu)$. Supposons maintenant $j \in Q^-(\lambda)$. Alors $\nu_j = \lambda_j - 1$ et $\lambda_j$ est impair, donc $\nu_j$ est pair. Comme on l'a vu, l'hypothèse $j \in Q^-(\lambda)$ entraîne $\lambda_{j-1} = \lambda_j$. Comme ci-dessus, $j-1$ n'appartient pas à $Q^-(\lambda)$ donc $\nu_{j-1} \geq \lambda_{j-1} = \lambda_j > \nu_j$. D'où $j \in P^+(\nu)$. Supposons enfin que $j \in P^+(\nu)$. En particulier $\nu_j$ est pair. Si $\lambda_j$ est impair, on a nécessairement $s(\lambda)_j \neq 0$ et, puisque $j$ est impair, $j$ appartient à $Q^-(\lambda)$. Supposons $\lambda_j$ pair. Alors $s(\lambda)_j$ est pair donc nul. Si $j=1$, on a trivialement $\lambda_{j-1} > \lambda_j$ et $j$ appartient à $P^+(\lambda)$. Supposons $j \geq 2$. Puisque $j \in P^+(\nu)$, on a $\nu_{j-1} > \nu_j$, autrement dit $\lambda_{j-1} + s(\lambda)_{j-1} > \lambda_j$. On n'a pas $j-1 \in Q^+(\lambda)$ car cette relation entraîne que $\lambda_{j-1} = \lambda_j$ est impair contrairement à l'hypothèse. Donc $s(\lambda)_{j-1} \leq 0$. L'inégalité $\lambda_{j-1} + s(\lambda)_{j-1} > \lambda_j$ entraîne alors $\lambda_{j-1} > \lambda_j$, donc $j \in P^+(\lambda)$. Cela démontre la première égalité de (3). La seconde se démontre de façon analogue. Cela prouve (3), d'où (2).

Dans le cas où $\lambda$ est spéciale, on a défini l'ensemble d'intervalles $\widetilde{\mathrm{Int}}(\lambda)$. On voit que $P^+(\lambda)$ est l'ensemble des $j_{\min}(\Delta)$ quand $\Delta$ décrit $\widetilde{\mathrm{Int}}(\lambda)$ et que $P^-(\lambda)$ est l'ensemble des $j_{\max}(\Delta)$ pour $\Delta \in \mathrm{Int}(\lambda)$. Alors $\zeta(\lambda)$ est la suite que l'on a définie en [Waldspurger 2018b, 1.6]. On a démontré dans cette référence l'égalité ${}^t d(\lambda) = \lambda + \zeta(\lambda)$. Supprimons l'hypothèse que $\lambda$ est spéciale. Par définition, $d(\lambda) = d(\mathrm{sp}(\lambda))$. D'où

$$ {}^t d(\lambda) = {}^t d(\mathrm{sp}(\lambda)) = \mathrm{sp}(\lambda) + \zeta(\mathrm{sp}(\lambda)). $$

Puisque $\mathrm{sp}(\lambda) = \nu = \lambda + s(\lambda)$, l'égalité (2) entraîne la deuxième assertion de l'énoncé. □

Soit maintenant $\lambda \in \mathcal{P}^{\mathrm{orth}}(2m)$. On définit $P^+(\lambda)$ et $P^-(\lambda)$ en échangeant les conditions de parité sur les $\lambda_j$. C'est-à-dire

- $P^+(\lambda)$ l'ensemble des entiers impairs $j \geq 1$ tels que $\lambda_j$ est impair et $\lambda_{j-1} > \lambda_j$, avec la convention $\lambda_0 = \infty$;
- $P^-(\lambda)$ l'ensemble des entiers pairs $j \geq 2$ tels que $\lambda_j$ est impair et $\lambda_j > \lambda_{j+1}$.

Dans ce cas, on a $|P^+(\lambda)| = |P^-(\lambda)|$. On définit la suite $\zeta$ comme plus haut. Nous aurons besoin de l'analogue du (ii) du lemme ci-dessus, mais seulement dans le cas où $\lambda$ est spéciale. C'est-à-dire

$$ \text{si } \lambda \in \mathcal{P}^{\mathrm{orth,sp}}(2m), \quad \text{on a} \quad {}^t d(\lambda) = \lambda + \zeta(\lambda). \tag{4} $$

Cf. [Waldspurger 2018b, 1.7].

## 3. Induction endoscopique

**3.1.** ***L'induite endoscopique de deux partitions spéciales.*** Soient $n_1, n_2 \in \mathbb{N}$ tels que $n_1 + n_2 = n$ et soient $\lambda_1 \in \mathcal{P}^{\mathrm{symp,sp}}(2n_1)$ et $\lambda_2 \in \mathcal{P}^{\mathrm{orth,sp}}(2n_2)$. Pour un indice

$j \geq 1$, on dit que $\lambda_{1,j}$, resp. $\lambda_{2,j}$, est de bonne parité si $\lambda_{1,j}$ est pair, resp. $\lambda_{2,j}$ est impair. Notons

$J^+$ l'ensemble des $j \geq 1$ tels que $j$ soit impair, $\lambda_{1,j}$ et $\lambda_{2,j}$ soient de bonne parité et il existe $d = 1, 2$ de sorte que $\lambda_{d,j-1} > \lambda_{d,j}$ (avec toujours la convention $\lambda_{d,0} = \infty$);

$J^-$ l'ensemble des $j \geq 1$ tels que $j$ soit pair, $\lambda_{1,j}$ et $\lambda_{2,j}$ soient de bonne parité et il existe $d = 1, 2$ de sorte que $\lambda_{d,j} > \lambda_{d,j+1}$.

On vérifie que $|J^+| = |J^-|$ et que, si on note $j_1^+ < \cdots < j_a^+$ les éléments de $J^+$ et $j_1^- < \cdots < j_a^-$ ceux de $J^-$, on a $j_1^+ < j_1^- < j_2^+ < \cdots < j_a^+ < j_a^-$. On définit une suite $\xi = (\xi_1, \xi_2, \dots)$ de nombres entiers par $\xi_j = 1$ si $j \in J^+$, $\xi_j = -1$ si $j \in J^-$ et $\xi_j = 0$ si $j \notin J^+ \cup J^-$. On pose

$$\lambda = \lambda_1 + \lambda_2 + \xi.$$

C'est une partition symplectique de $2n$, appelée l'induite endoscopique de $\lambda_1$ et $\lambda_2$.

Pour unifier les notations, on pose $\widetilde{\text{Int}}(\lambda_2) = \text{Int}(\lambda_2)$. Pour $d = 1, 2$, posons $J_{d,\min} = \{j_{\min}(\Delta);\ \Delta \in \widetilde{\text{Int}}(\lambda_d)\}$, $J_{d,\max} = \{j_{\max}(\Delta);\ \Delta \in \text{Int}(\lambda_d)\}$. On note $\mathcal{J}^+ = J_{1,\min} \cap J_{2,\min}$, $\mathcal{J}^- = J_{1,\max} \cap J_{2,\max}$,

$$\mathcal{J} = J_{1,\min} \cup J_{2,\min} \cup J_{1,\max} \cup J_{2,\max} \cup \{\infty\}.$$

Appelons intervalle relatif d'indices un sous-ensemble de $\mathbb{N} - \{0\}$ de l'une des formes suivantes :

(1) $\{j\}$ pour $j \in \mathcal{J}^+ \cup \mathcal{J}^-$ ;

(2) $\{j, \dots, j'\}$ où $j$ et $j'$ sont deux éléments consécutifs de $\mathcal{J}$ tels qu'il existe un unique $d = 1, 2$ de sorte que $\{j, \dots, j'\} \subset J(\Delta)$ pour un $\Delta \in \widetilde{\text{Int}}(\lambda_d)$.

Pour un intervalle relatif d'indices $J$, on pose $D(J) = \{\lambda_j;\ j \in J\}$. On appelle intervalle de $\lambda$ relatif à $(\lambda_1, \lambda_2)$ un sous-ensemble de $\text{Jord}(\lambda) \cup \{0\}$ de la forme $D(J)$. On note $\widetilde{\text{Int}}_{\lambda_1,\lambda_2}(\lambda)$ l'ensemble de ces intervalles relatifs. On montre qu'ils sont disjoints, formés de nombres pairs et que $\widetilde{\text{Int}}_{\lambda_1,\lambda_2}(\lambda)$ est une partition de $\text{Jord}^{\text{bp}}(\lambda) \cup \{0\}$. Pour un intervalle relatif $D$, on note $J(D)$ l'intervalle relatif d'indices $J$ tel que $D = D(J)$. Les intervalles relatifs sont ordonnés de façon naturelle : $D > D'$ si et seulement si $i > i'$ pour tous $i \in D$, $i' \in D'$. L'intervalle minimal est celui qui contient 0, on le note $D_{\min}$ et on pose $\text{Int}_{\lambda_1,\lambda_2}(\lambda) = \widetilde{\text{Int}}_{\lambda_1,\lambda_2}(\lambda) - \{D_{\min}\}$. Pour $D \in \widetilde{\text{Int}}_{\lambda_1,\lambda_2}(\lambda)$, on note $j_{\min}(D)$, resp. $j_{\max}(D)$, le plus petit, resp. grand, élément de $J(D)$ (on considère que $j_{\max}(D_{\min}) = \infty$).

Montrons que

pour tout $j \in \mathcal{J}$,

il existe un unique intervalle relatif $D$ tel que $j \in \{j_{\min}(D), j_{\max}(D)\}$. (3)

*Preuve.* L'unicité est claire puisque, quand $D$ parcourt $\widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$, les $J(D)$ sont disjoints. Pour $j=\infty$, on a $j=j_{\max}(D_{\min})$. Soit $j\in\mathcal{J}$ différent de $\infty$. Supposons par exemple $j$ pair, le cas $j$ impair étant similaire. La définition de $\mathcal{J}$ et cette hypothèse de parité imposent qu'il existe $d=1,2$ et $\Delta_d\in\mathrm{Int}(\lambda_d)$ de sorte que $j=j_{\max}(\Delta_d)$. Pour fixer la notation, on suppose qu'il en est ainsi pour $d=1$. L'ensemble des $j'\in\mathcal{J}$ tels que $j'<j$ n'est pas vide : il contient $j_{\min}(\Delta_1)$. Notons $j^-$ le plus grand de ces éléments. On a donc $j_{\min}(\Delta_1)\leq j^-$ et $\{j^-,\ldots,j\}$ est contenu dans $J(\Delta_1)$. Si $\{j^-,\ldots,j\}$ n'est contenu dans $J(\Delta_2)$ pour aucun $\Delta_2\in\widetilde{\mathrm{Int}}(\lambda_2)$, il existe par définition des intervalles relatifs un tel intervalle $D$ tel que $J(D)=\{j^-,\ldots,j\}$ et on a $j=j_{\max}(D)$. Supposons qu'il existe un $\Delta_2\in\widetilde{\mathrm{Int}}(\lambda_2)$ de sorte que $\{j^-,\ldots,j\}\subset J(\Delta_2)$. Si $j=j_{\max}(\Delta_2)$, alors, par définition des intervalles relatifs, il existe un tel intervalle $D$ tel que $\{j\}=J(D)$ et on conclut. Supposons $j<j_{\max}(\Delta_2)$. On note $j^+$ le plus petit élément de $\mathcal{J}$ qui soit strictement supérieur à $j$. Comme précédemment, on a $j^+\leq j_{\max}(\Delta_2)$, d'où $\{j,\ldots,j^+\}\subset J(\Delta_2)$. S'il existait $\Delta_1'\in\widetilde{\mathrm{Int}}(\lambda_1)$ vérifiant $\{j,\ldots,j^+\}\subset J(\Delta_1')$, on aurait $\Delta_1'=\Delta_1$ puisque $j\in J(\Delta_1)$ et aussi $j_{\max}(\Delta_1')\geq j^+>j$. Cela contredit l'hypothèse $j=j_{\max}(\Delta_1)$. Un tel $\Delta_1'$ n'existe donc pas et, par définition des intervalles relatifs, il existe un tel intervalle $D$ tel que $J(D)=\{j,\ldots,j^+\}$. Alors $j=j_{\min}(D)$. □

On définit une fonction $\chi_{\lambda_1,\lambda_2}:\widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)\to\mathbb{Z}/2\mathbb{Z}$ de la façon suivante. Soit $D\in\widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$. Si $|J(D)|=1$, $\chi_{\lambda_1,\lambda_2}(D)=0$. Si $|J(D)|\geq 2$, $J(D)$ est de la forme (2) ci-dessus et cette relation nous fournit un indice $d\in\{1,2\}$. On note $\chi_{\lambda_1,\lambda_2}(D)$ l'image de $d$ dans $\mathbb{Z}/2\mathbb{Z}$. Remarquons que l'on a $\chi_{\lambda_1,\lambda_2}(D_{\min})=0$.

On définit l'ensemble $P^+_{\lambda_1,\lambda_2}(\lambda)$ formé des $j_{\min}(D)$ qui sont impairs, pour $D\in\widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}$ et l'ensemble $P^-_{\lambda_1,\lambda_2}(\lambda)$ formé des $j_{\max}(D)$ qui sont pairs, pour $D\in\mathrm{Int}_{\lambda_1,\lambda_2}(\lambda)$. On définit une suite $\zeta_{\lambda_1,\lambda_2}(\lambda)=(\zeta_{\lambda_1,\lambda_2}(\lambda)_1,\zeta_{\lambda_1,\lambda_2}(\lambda)_2,\ldots)$ par $\zeta_{\lambda_1,\lambda_2}(\lambda)_j=1$ si $j\in P^+_{\lambda_1,\lambda_2}(\lambda)$, $\zeta_{\lambda_1,\lambda_2}(\lambda)_j=-1$ si $j\in P^-_{\lambda_1,\lambda_2}(\lambda)$, $\zeta_{\lambda_1,\lambda_2}(\lambda)_j=0$ si $j\notin P^+_{\lambda_1,\lambda_2}(\lambda)\cup P^-_{\lambda_1,\lambda_2}(\lambda)$.

**Lemme.** $$\zeta(\lambda_1)+\zeta(\lambda_2)=\zeta_{\lambda_1,\lambda_2}(\lambda)+\xi.$$

*Preuve.* Restreignons-nous d'abord à l'ensemble des $j\geq 1$ impairs. Alors les fonctions ci-dessus sont les fonctions caractéristiques des ensembles $P^+(\lambda_1)$, $P^+(\lambda_2)$, $P_{\lambda_1,\lambda_2}(\lambda)$ et $J^+$. Il s'agit donc de prouver les égalités

$$P^+(\lambda_1)\cup P^+(\lambda_2)=P^+_{\lambda_1,\lambda_2}(\lambda)\cup J^+; \tag{4}$$

$$P^+(\lambda_1)\cap P^+(\lambda_2)=P^+_{\lambda_1,\lambda_2}(\lambda)\cap J^+. \tag{5}$$

Rappelons que, puisque $\lambda_d$ est spéciale pour $d=1,2$, $P^+(\lambda_d)$ est l'ensemble des $j_{\min}(\Delta_d)$ pour $\Delta_d\in\mathrm{Int}(\lambda_d)$. Considérons un $j$ appartenant à l'ensemble de gauche de (4). Pour fixer la notation, supposons $j\in P^+(\lambda_1)$. Alors $j=j_{\min}(\Delta_1)$ pour un $\Delta_1\in\mathrm{Int}(\lambda_1)$, en particulier $j$ appartient à l'ensemble $\mathcal{J}$. Si $\lambda_{2,j}$ est impair, $j$ appartient à $J^+$ par définition de cet ensemble. Supposons $\lambda_{2,j}$ pair. Soit $j^+$

le plus petit élément du sous-ensemble des éléments de l'ensemble $\mathcal{J}$ qui sont strictement supérieurs à $j$. Ce sous-ensemble contenant $j_{\max}(\Delta_1)$ (où il convient ici de considérer que $j_{\max}(\Delta_{1,\min}) = \infty$), $j^+$ existe et on a $j^+ \leq j_{\max}(\Delta_1)$. L'ensemble $\{j, \dots, j^+\}$ est contenu dans $J(\Delta_1)$ mais, puisque $\lambda_{2,j}$ est de mauvaise parité, il n'existe pas de $\Delta_2 \in \mathrm{Int}(\lambda_2)$ tel que $\{j, \dots, j^+\}$ soit contenu dans $J(\Delta_2)$. Par définition $\{j, \dots, j^+\}$ est alors égal à $J(D)$ pour un intervalle relatif $D$ et on a $j = j_{\min}(D)$. Donc $j \in P^+_{\lambda_1,\lambda_2}(\lambda)$. Inversement, considérons un $j$ qui appartient à l'ensemble de droite de (4). Si $j \in J^+$, il est par définition de la forme $j_{\min}(\Delta_d)$ pour un $d = 1, 2$ et un $\Delta_d \in \mathrm{Int}(\lambda_d)$. C'est-à-dire $j \in P^+(\lambda_d)$. Supposons $j \in P^+_{\lambda_1,\lambda_2}(\lambda)$. Alors $j = j_{\min}(D)$ pour un $D \in \widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$. Par définition des intervalles relatifs, $j$ appartient à $\mathcal{J}$. Puisque $j$ est impair, $j$ est forcément de la forme $j_{\min}(\Delta_d)$ pour un $d = 1, 2$ et un $\Delta_d \in \widetilde{\mathrm{Int}}(\lambda_d)$. C'est-à-dire $j \in P^+(\lambda_d)$. Cela prouve (4).

Soit $j \in P^+(\lambda_1) \cap P^+(\lambda_2)$. Alors, pour $d = 1, 2$, $j$ est de la forme $j_{\min}(\Delta_d)$ pour un $\Delta_d \in \mathrm{Int}(\lambda_d)$. En particulier, $\lambda_{d,j}$ est de la bonne parité. Par définition de $J^+$, on a $j \in J^+$. Cela implique que $\lambda_j$ est pair. Donc il existe un intervalle relatif $D \in \widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$ tel que $j \in J(D)$. Si $j = 1$, on a forcément $j = j_{\min}(D)$ et $j \in P^+_{\lambda_1,\lambda_2}(\lambda)$. Supposons $j \geq 2$. Pour $d = 1, 2$, l'hypothèse $j = j_{\min}(\Delta_d)$ implique que $\{j-1, j\}$ n'est contenu dans $J(\Delta'_d)$ pour aucun $\Delta'_d \in \mathrm{Int}(\lambda_d)$. Par définition des intervalles relatifs, $\{j-1, j\}$ n'est donc contenu dans $J(D')$ pour aucun $D' \in \widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$. En particulier $j - 1 \notin J(D)$, d'où $j = j_{\min}(D)$ et $j \in P^+_{\lambda_1,\lambda_2}(\lambda)$. Inversement, soit $j \in P^+_{\lambda_1,\lambda_2}(\lambda) \cap J^+$. Par définition de $J^+$, $\lambda_{1,j}$ et $\lambda_{2,j}$ sont de bonne parité et il existe $d = 1, 2$ et $\Delta_d \in \mathrm{Int}(\lambda_d)$ de sorte que $j = j_{\min}(\Delta_d)$. Pour fixer la notation, on suppose que ce $d$ est égal à 1. Donc $j \in P^+(\lambda_1)$. L'hypothèse que $\lambda_{2,j}$ est de bonne parité implique qu'il existe $\Delta_2 \in \mathrm{Int}(\lambda_2)$ de sorte que $j \in J(\Delta_2)$. Supposons d'abord que tous les éléments de $\mathcal{J}$ soient supérieurs ou égaux à $j$. Dans ce cas, $j = j_{\min}(\Delta_2)$ et $j \in P^+(\lambda_2)$. Supposons maintenant qu'il existe des éléments de $\mathcal{J}$ strictement inférieurs à $j$, notons $j^-$ le plus grand d'entre eux. L'hypothèse $j \in P^+_{\lambda_1,\lambda_2}(\lambda)$ signifie que $j = j_{\min}(D)$ pour un intervalle relatif $D$. Donc $\{j^-, \dots, j\}$ n'est de la forme $J(D')$ pour aucun $D' \in \widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$. Les entiers $j^-$ et $j$ sont deux éléments consécutifs de $\mathcal{J}$. Ces deux propriétés et la définition des intervalles relatifs entraînent que le nombre de $d$ pour lesquels il existe $\Delta'_d \in \mathrm{Int}(\lambda_d)$ tel que $\{j^-, \dots, j\} \subset J(\Delta'_d)$ est pair. Pour $d = 1$, il n'existe pas de tel $\Delta'_1$ car $j = j_{\min}(\Delta_1)$. Donc il n'existe pas non plus de tel $\Delta'_2$. En particulier $\{j^-, \dots, j\} \not\subset J(\Delta_2)$. Puisque $\{j_{\min}(\Delta_2), \dots, j\} \subset J(\Delta_2)$, cela entraîne $j^- < j_{\min}(\Delta_2)$, et, puisque $j_{\min}(\Delta_2) \in \mathcal{J}$, la définition de $j^-$ entraîne $j \leq j_{\min}(\Delta_2)$, d'où forcément $j = j_{\min}(\Delta_2)$. Donc $j \in P^+(\lambda_2)$. Cela prouve (5).

Un raisonnement analogue vaut en se restreignant à l'ensemble des entiers pairs $j \geq 2$. □

On dit que $\lambda_1$ et $\lambda_2$ induisent régulièrement $\lambda$ si et seulement si $\widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$ est la partition la plus fine de $\mathrm{Jord}^{\mathrm{bp}}(\lambda) \cup \{0\}$, c'est-à-dire si et seulement si tout intervalle relatif est réduit à un seul élément. Dans ce cas, $\chi_{\lambda_1,\lambda_2}$ est définie sur $\mathrm{Jord}^{\mathrm{bp}}(\lambda)\cup\{0\}$ et on a $\chi_{\lambda_1,\lambda_2}(0)=0$.

**3.2. *Une proposition d'existence.*** Soient $n\in\mathbb{N}$ et $\lambda\in\mathcal{P}^{\mathrm{symp}}(2n)$. Fixons une fonction $\chi:\mathrm{Jord}^{\mathrm{bp}}(\lambda)\cup\{0\}\to\mathbb{Z}/2\mathbb{Z}$ telle que $\chi(i)=0$ pour tout $i\in\mathrm{Jord}^{\mathrm{bp}}(\lambda)$ tel que $\mathrm{mult}_\lambda(i)=1$ et telle que $\chi(0)=0$.

**Proposition.** *Il existe $n_1,n_2\in\mathbb{N}$ tels que $n_1+n_2=n$ et il existe $\lambda_1\in\mathcal{P}^{\mathrm{symp,sp}}(2n_1)$ et $\lambda_2\in\mathcal{P}^{\mathrm{orth,sp}}(2n_2)$ tels que*

(i) $\lambda_1$ *et* $\lambda_2$ *induisent régulièrement* $\lambda$ ;

(ii) $d(\lambda_1)\cup d(\lambda_2)=d(\lambda)$ ;

(iii) $\chi_{\lambda_1,\lambda_2}=\chi$.

La preuve est identique à celle de [Waldspurger 2018b, 1.11]. On la refait car, dans cette référence, on avait bêtement supposé que tous les termes de $\lambda$ étaient pairs. On utilise les notations de 2.7.

*Preuve.* Notons $\mathfrak{J}^+$ l'ensemble des $j\geq 1$ tels que $j$ soit impair, $\lambda_j$ soit pair et $\lambda_j>\lambda_{j+1}$. Notons $\mathfrak{J}^-$ l'ensemble des $j\geq 2$ tels que $j$ et $\lambda_j$ soient pairs et $\lambda_{j-1}>\lambda_j$. On voit que $\mathfrak{J}^+$ est l'ensemble des $j_{\max}(i)$ pour $i=i_h$ avec $h$ impair ou pour $i\in\mathcal{J}''(\lambda)\cap\mathrm{Jord}^{\mathrm{bp}}(\lambda)$. De même, $\mathfrak{J}^-$ est l'ensemble des $j_{\min}(i)$ pour $i=i_h$ avec $h$ pair ou pour $i\in\mathcal{J}''(\lambda)\cap\mathrm{Jord}^{\mathrm{bp}}(\lambda)$. On en déduit que $\mathfrak{J}^+$ et $\mathfrak{J}^-$ ont le même nombre d'éléments et que, si on note $\mathfrak{J}^+=\{j_1^+<\cdots<j_c^+\}$ et $\mathfrak{J}^-=\{j_1^-<\cdots<j_c^-\}$, on a

$$j_1^+<j_1^-<j_2^+<j_2^-<\ldots.<j_c^+<j_c^-.$$

On note $\mathfrak{r}=(\mathfrak{r}_1,\mathfrak{r}_2,\ldots)$ la suite de nombres définie par $\mathfrak{r}_j=1$ si $j\in\mathfrak{J}^+$, $\mathfrak{r}_j=-1$ si $j\in\mathfrak{J}^-$ et $\mathfrak{r}_j=0$ si $j\notin\mathfrak{J}^+\cup\mathfrak{J}^-$.

Soit $d\in\{1,2\}$. Pour $j\geq 1$, disons que $j$ et $j+1$ sont $d$-liés si et seulement si l'une des conditions suivantes est vérifiée :

$$\lambda_j=\lambda_{j+1}\text{ est pair et }\chi(\lambda_j)=d+1(\text{c'est-à-dire }\chi(\lambda_j)\equiv d+1\bmod 2\mathbb{Z});\tag{1a}$$

$$j\in\mathfrak{J}^+;\tag{1b}$$

$$j+1\in\mathfrak{J}^-;\tag{1c}$$

$$\lambda_j\text{ et }\lambda_{j+1}\text{ sont impairs et }\lambda_j\in\mathcal{J}''(\lambda).\tag{1d}$$

Remarquons que cette dernière condition équivaut à

$$\lambda_j\text{ et }\lambda_{j+1}\text{ sont impairs et }\lambda_{j+1}\in\mathcal{J}''(\lambda).\tag{1d$'$}$$

En effet, si (1d) est vérifiée, on a $i_h>\lambda_j>i_{h+1}$ pour un $h$ impair. Alors $i_h>\lambda_{j+1}\geq i_{h+1}$. Mais $\lambda_{j+1}\neq i_{h+1}$ puisque $\lambda_{j+1}$ est impair et $i_{h+1}$ est pair. Donc $i_h>\lambda_{j+1}>i_{h+1}$ et $\lambda_{j+1}\in\mathcal{J}''(\lambda)$. La réciproque est similaire.

Pour deux entiers $1 \leq j \leq j'$, disons qu'ils sont $d$-liés si et seulement si $k$ et $k+1$ sont $d$-liés pour tout $k = j, \dots, j'-1$. C'est une relation d'équivalence et les classes sont des intervalles de $\mathbb{N} - \{0\}$, éventuellement infinis. On note $\widetilde{\mathfrak{Int}}_d$ l'ensemble des classes d'équivalence ayant au moins deux éléments. Pour $\mathfrak{I} \in \widetilde{\mathfrak{Int}}_d$, on note $j_{\min}(\mathfrak{I})$, resp. $j_{\max}(\mathfrak{I})$, le plus petit, resp. grand, élément de $\mathfrak{I}$ (avec $j_{\max}(\mathfrak{I}) = \infty$ si $\mathfrak{I}$ est infini). Pour $d = 1, 2$ définissons une fonction $p_d : \mathbb{N} - \{0\} \to \mathbb{Z}/2\mathbb{Z}$ par $p_d(j) = 1$ s'il existe $\mathfrak{I} \in \widetilde{\mathfrak{Int}}_d$ tel que $j \in \mathfrak{I}$, $p_d(j) = 0$ sinon. Montrons que

(2) l'ensemble $\widetilde{\mathfrak{Int}}_d$ est fini ; il contient un élément infini si et seulement si $d = 1$ ; on note $\mathfrak{Int}_1$ l'ensemble $\widetilde{\mathfrak{Int}}_1$ privé de cet élément infini et on pose $\mathfrak{Int}_2 = \widetilde{\mathfrak{Int}}_2$ ;

(3) pour $\mathfrak{I} \in \widetilde{\mathfrak{Int}}_d$, $j_{\min}(\mathfrak{I})$ est impair et $j_{\max}(\mathfrak{I})$ est pair ou infini ;

(4) pour $j \geq 1$, on a

$$p_1(j) + p_2(j) = \begin{cases} 2 & \text{si } j \in \mathfrak{J}^+ \cup \mathfrak{J}^-; \\ 1 & \text{si } \lambda_j \text{ est pair, et } j \notin \mathfrak{J}^+ \cup \mathfrak{J}^-; \\ 0 & \text{si } \lambda_j \text{ est impair et } \lambda_j \in \mathcal{J}'(\lambda); \\ 2 & \text{si } \lambda_j \text{ est impair et } \lambda_j \in \mathcal{J}''(\lambda); \end{cases}$$

(5) $\mathfrak{J}^+$ est égal à l'ensemble des $j \geq 1$ tels que $p_1(j) = p_2(j) = 1$ et qu'il existe $d = 1, 2$ et un élément de $\mathfrak{I} \in \mathfrak{Int}_d$ de sorte que $j = j_{\min}(\mathfrak{I})$ ;

(6) $\mathfrak{J}^-$ est égal à l'ensemble des $j \geq 1$ tels que $p_1(j) = p_2(j) = 1$ et qu'il existe $d = 1, 2$ et un élément de $\mathfrak{I} \in \mathfrak{Int}_d$ de sorte que $j = j_{\max}(\mathfrak{I})$.

Soit $t(\lambda)$ le plus grand entier $l$ tel que $\lambda_l > 0$. Parce que $\chi(0) = 0$, on voit que, pour $j > t(\lambda)$, $j$ et $j+1$ sont 1-liés mais pas 2-liés. Donc $\{t(\lambda)+1, \dots\}$ est contenu dans un intervalle infini $\mathfrak{I}_{1,\min} \in \widetilde{\mathfrak{Int}}_1$ tandis que, pour $j \geq t(\lambda)+2$, $\{j\}$ est une classe d'équivalence pour la 2-liaison et $j$ n'est pas contenu dans un élément de $\mathfrak{Int}_2$. Cela prouve (2).

Soit $\mathfrak{I} \in \widetilde{\mathfrak{Int}}_d$. On pose simplement $j = j_{\min}(\mathfrak{I})$. Montrons que $j$ est impair. C'est évident si $j = 1$. On suppose $j \geq 2$. Par définition, $j$ et $j+1$ sont $d$-liés tandis que $j-1$ et $j$ ne le sont pas. Si (1b) ou (1c) est vérifiée, $j$ est trivialement impair. Supposons vérifiée (1a). On n'a pas $\lambda_{j-1} = \lambda_j$ : sinon ces entiers seraient pairs, on aurait $\chi(\lambda_{j-1}) = \chi(\lambda_j) = d+1$ et $j-1$ et $j$ vérifieraient l'analogue de (1a) et seraient $d$-liés. Donc $\lambda_{j-1} > \lambda_j$. Alors $j$ est impair ou appartient à $\mathfrak{J}^-$. Or cette dernière relation est exclue car elle entraîne que $j-1$ et $j$ vérifient l'analogue de (1c) et sont $d$-liés. Donc $j$ est impair. Supposons maintenant que (1d) soit vérifiée. Supposons d'abord que $\lambda_{j-1}$ est impair. Alors $j-1$ et $j$ vérifient l'analogue de (1d′) et sont $d$-liés, ce qui n'est pas le cas. Donc $\lambda_{j-1}$ est pair et $\lambda_{j-1} > \lambda_j$. Alors $j-1$ est pair ou $j-1 \in \mathfrak{J}^+$. Or cette dernière relation est exclue car elle entraîne que $j-1$ et $j$ vérifient l'analogue de (1b) et sont $d$-liés. Donc $j-1$ est pair et $j$ est impair. Cela montre que $j_{\min}(\mathfrak{I})$ est impair. Une preuve analogue montre que $j_{\max}(\mathfrak{I})$ est pair s'il n'est pas infini. Cela prouve (3).

Soit $j \in \mathfrak{J}^+$. Alors (1b) est vérifié et $j$ et $j+1$ sont $d$-liés pour $d = 1, 2$. Donc $p_1(j) = p_2(j) = 1$. Soit maintenant $j \in \mathfrak{J}^-$. Alors $j$ est pair donc différent de 1. L'analogue de (1c) pour le couple $(j-1, j)$ est vérifiée et $j-1$ et $j$ sont $d$-liés pour $d = 1, 2$. Donc $p_1(j) = p_2(j) = 1$. Supposons maintenant $\lambda_j$ pair mais $j \notin \mathfrak{J}^+ \cup \mathfrak{J}^-$. Supposons par exemple $j$ impair, le cas où $j$ est pair se traitant de façon analogue. Puisque $j \notin \mathfrak{J}^+$, on a $\lambda_j = \lambda_{j+1}$. Les entiers $j$ et $j+1$ sont $d$-liés pour l'unique $d$ tel que $\chi(\lambda_j) = d+1$. Pour ce $d$, $p_d(j) = 1$. Soit $d'$ l'autre élément de $\{1, 2\}$. On doit prouver que $j$ n'appartient à aucun élément de $\widetilde{\mathfrak{Int}}_{d'}$. On vient de voir que $j$ et $j+1$ ne sont pas $d'$-liés. Si $j$ appartenait à un élément $\mathfrak{I}' \in \widetilde{\mathfrak{Int}}_{d'}$, cet intervalle serait fini et $j$ serait égal à $j_{\max}(\mathfrak{I}')$. Mais alors $j$ serait pair d'après (3), contrairement à l'hypothèse. Supposons maintenant $\lambda_j$ impair, $j$ impair et $\lambda_j \in \mathcal{J}'(\lambda)$. Cette dernière condition implique d'après 2.7 que $j_{\max}(\lambda_j)$ est pair, donc $j < j_{\max}(\lambda_j)$, donc $\lambda_{j+1} = \lambda_j$. Pour $d = 1, 2$, les conditions (1a), (1b) et (1c) ne sont pas vérifiées : elles imposent que $\lambda_j$ ou $\lambda_{j+1}$ est pair. La condition (1d) ne l'est pas puisque $\lambda_j \in \mathcal{J}'(\lambda)$. Donc $j$ et $j+1$ ne sont pas $d$-liés. Si $j = 1$, $j$ n'appartient donc à aucun élément de $\widetilde{\mathfrak{Int}}_d$. Si $j > 1$, les analogues des conditions (1a) et (1c) pour le couple $(j-1, j)$ ne sont pas vérifiées : elles imposent que $\lambda_j$ est pair. L'analogue de (1c) n'est pas vérifiée : elle impose $j-1$ impair donc $j$ pair. L'analogue de (1d′) n'est pas vérifiée puisque $\lambda_j \in \mathcal{J}'(\lambda)$. Donc $j-1$ et $j$ ne sont pas $d$-liés. Donc $p_d(j) = 0$. Supposons maintenant $\lambda_j$ impair, $j$ pair et $j \in \mathcal{J}'(\lambda)$. Cette dernière condition implique d'après 2.7 que $j_{\min}(\lambda_j)$ est impair, donc $j_{\min}(\lambda_j) < j$, donc $\lambda_{j-1} = \lambda_j$. Des arguments analogues à ceux ci-dessus montrent que, pour $d = 1, 2$, $p_d(j) = 0$. Supposons enfin que $\lambda_j$ est impair et que $j \in \mathcal{J}''(\lambda)$. Puisque $\mathrm{mult}_\lambda(\lambda_j)$ est paire, on a $\lambda_{j-1} = \lambda_j$ ou $\lambda_{j+1} = \lambda_j$. Dans le premier cas, $j-1$ et $j$ vérifient l'analogue de (1d′) et sont $d$-liés pour tout $d$. Dans le deuxième cas, $j$ et $j+1$ vérifient (1d) et sont $d$-liés pour tout $d$. Donc $p_d(j) = 1$ pour tout $d$. Cela démontre (4).

Soit $j \in \mathfrak{J}^+$. D'après (4), on a $p_1(j) = p_2(j) = 1$, c'est-à-dire que, pour tout $d$, il existe $\mathfrak{I}_d \in \widetilde{\mathfrak{Int}}_d$ tel que $j \in \mathfrak{I}_d$. Si $j = 1$, on a forcément $j = j_{\min}(\mathfrak{I}_d)$ pour tout $d$. Supposons $j > 1$. On veut montrer que $j = j_{\min}(\mathfrak{I}_d)$ pour au moins un $d$, autrement dit que $j-1$ et $j$ ne sont pas $d$-liés pour au moins un $d$. Les analogues pour le couple $(j-1, j)$ des conditions (1b) et (1c) ne sont pas vérifiées : elles impliquent que $j$ est pair, alors que $j$ est impair puisque $j \in \mathfrak{J}^+$. L'analogue de (1d) n'est pas vérifiée, puisque $\lambda_j$ est pair. Donc $j-1$ et $j$ ne sont $d$-liés que si l'analogue de (1a) est vérifiée. Mais cette analogue ne peut être vérifiée que pour un unique $d$. Cela démontre que $\mathfrak{J}^+$ est contenu dans l'ensemble décrit en (5). Inversement, soit $j \geq 1$, supposons que $p_1(j) = p_2(j) = 1$ et qu'il existe $d = 1, 2$ et un élément de $\mathfrak{I} \in \widetilde{\mathfrak{Int}}_d$ de sorte que $j = j_{\min}(\mathfrak{I})$. Autrement dit, ou bien $j = 1$, ou bien il existe $d$ tel que $j-1$ et $j$ ne sont pas $d$-liés. D'après (3), $j$ est impair. D'après (4), on a soit $j \in \mathfrak{J}^+ \cup \mathfrak{J}^-$, soit $\lambda_j$ est impair et $\lambda_j \in \mathcal{J}''(\lambda)$. Dans le premier cas, l'imparité de $j$ entraîne $j \in \mathfrak{J}^+$, ce que l'on veut prouver. Supposons donc que $\lambda_j$ est impair et $\lambda_j \in \mathcal{J}''(\lambda)$. D'après 2.7, cette condition entraîne que $j_{\min}(\lambda_j)$ est pair, donc $j_{\min}(\lambda_j) < j$ et $\lambda_{j-1} = \lambda_j$. Alors $j-1$ et $j$ vérifient l'analogue de (1d′) et sont $d$-liés. Cela contredit l'hypothèse. On a ainsi prouvé (5). La preuve de (6) est similaire.

La relation (3) entraîne

$$p_d(j) = p_d(j+1) \quad \text{si } j \text{ est impair.} \tag{7}$$

La définition de $\mathfrak{r}$ et l'assertion (4) entraînent

$$\mathfrak{r}_j \equiv p_1(j) + p_2(j) + 1 + \lambda_j \mod 2\mathbb{Z}. \tag{8}$$

On va montrer qu'il existe des suites d'entiers positifs ou nuls $\lambda_1$ et $\lambda_2$ vérifiant les conditions suivantes, pour tout $j \geq 1$ :

(9) $\lambda_{1,j} + \lambda_{2,j} + \mathfrak{r}_j = \lambda_j$ ;

(10) pour $d = 1, 2$, $\quad \lambda_{d,j} \equiv d + p_d(j) \mod 2\mathbb{Z}$ ;

(11) pour $d = 1, 2$, on a

(a) $\lambda_{d,j} = \lambda_{d,j+1}$ si $j$ est pair, $p_d(j) = 1$ et il n'existe pas de $\mathfrak{I} \in \mathfrak{Int}_d$ tel que $j = j_{\max}(\mathfrak{I})$ ou si $j$ est impair et $p_d(j) = 0$ ;

(b) $\lambda_{d,j} > \lambda_{d,j+1}$ si $j$ est pair et il existe $\mathfrak{I} \in \mathfrak{Int}_d$ tel que $j = j_{\max}(\mathfrak{I})$ (la condition que $j$ est pair est redondante d'après (3)) ;

(c) $\lambda_{d,j} \geq \lambda_{d,j+1}$ si $j$ est impair et $p_d(j) = 1$ ou si $j$ est pair et $p_d(j) = 0$.

On raisonne par récurrence descendante sur $j$. Pour $j \geq t(\lambda) + 2$, on pose $\lambda_{1,j} = \lambda_{2,j} = 0$. On a vu dans la preuve de (2) que $j$ était contenu dans $\mathfrak{I}_{1,\min}$ mais dans aucun élément de $\mathfrak{Int}_2$. Donc $p_1(j) = 1$ et $p_2(j) = 0$. De plus, $j$ n'appartient pas à $\mathfrak{J}^+ \cup \mathfrak{J}^-$ donc $\mathfrak{r}_j = 0$. On voit alors que toutes les conditions ci-dessus sont vérifiées.

On fixe $j$ et on suppose que l'on a fixé des termes $\lambda_{1,j'}$, $\lambda_{2,j'}$ pour $j' > j$ de sorte que les conditions ci-dessus soient vérifiées pour ces $j'$. Pour $d = 1, 2$, on pose $\lambda_{d,j} = \lambda_{d,j+1} + e_d$, avec $e_d \in \mathbb{Z}$. Les conditions ci-dessus se traduisent en termes de ces entiers $e_d$. L'analogue de (9) étant vérifiée pour $j+1$, cette condition (9) se traduit par

$$e_1 + e_2 = \lambda_j - \lambda_{j+1} + \mathfrak{r}_{j+1} - \mathfrak{r}_j. \tag{12}$$

De même, la condition (10) se traduit par

$$e_d \equiv p_d(j) + p_d(j+1) \mod 2\mathbb{Z}. \tag{13}$$

Remarquons que, si (12) est vérifiée, la relation (8) entraîne

$$e_1 + e_2 \equiv p_1(j) + p_1(j+1) + p_2(j) + p_2(j+1) \mod 2\mathbb{Z}.$$

Donc (13) est vérifiée pour un $d$ si et seulement si elle l'est pour les deux $d$.

La condition (11) se traduit par $e_d = 0$ dans le cas (a), $e_d > 0$ dans le cas (b) et $e_d \geq 0$ dans le cas (c). Remarquons que, dans le cas (a), la condition $e_d = 0$ est compatible avec (13), autrement dit $p_d(j) = p_d(j+1)$. En effet, si $j$ est impair, cette relation est toujours vraie d'après (5). Si $j$ est pair, la condition (11)(a) impose que $j$ et $j+1$ sont $d$-liés donc $p_d(j) = p_d(j+1) = 1$.

Supposons la condition (11)(a) vérifiée pour un $d$, disons pour $d = 1$. On n'a pas le choix pour $e_1$ : on pose $e_1 = 0$. La condition (12) impose $e_2 = \lambda_j - \lambda_{j+1} + \mathfrak{r}_{j+1} - \mathfrak{r}_j$. Comme on vient de le dire, la condition (13) est vérifiée pour $d = 1$. Elle l'est donc aussi pour $d = 2$. Il reste à vérifier les conditions provenant de (11) pour $d = 2$.

Supposons $j$ impair. Supposons d'abord que la condition (11)(a) soit vérifiée pour $d = 2$, auquel cas on doit vérifier que $e_2 = 0$. La condition (11)(a) pour $j$ impair est que $p_d(j) = 0$. Cette condition est vérifiée pour $d = 1, 2$. D'après (4), $\lambda_j$ est impair et $\lambda_j \in \mathcal{J}'(\lambda)$. D'après 2.7, $j_{\max}(\lambda_j)$ est pair, donc $j < j_{\max}(\lambda_j)$ et $\lambda_j = \lambda_{j+1}$. Évidemment, $j, j+1 \notin \mathfrak{J}^+ \cup \mathfrak{J}^-$, donc $\mathfrak{r}_j = \mathfrak{r}_{j+1} = 0$. Alors $e_2 = \lambda_j - \lambda_{j+1} + \mathfrak{r}_{j+1} - \mathfrak{r}_j = 0$. La condition (11)(b) n'est pas vérifiée pour $d = 2$ puisque $j$ est impair. Supposons la condition (11)(c) vérifiée pour $d = 2$. On doit alors prouver que $e_2 \geq 0$. Puisque $j$ est impair, cette condition est que $p_2(j) = 1$. On a aussi $p_1(j) = 0$ puisque (11)(a) est vérifiée pour $d = 1$. D'après (7), on a aussi $p_1(j+1) = 0$ et $p_2(j+1) = 1$. Alors, d'après (4), ni $j$, ni $j+1$ n'appartiennent à $\mathfrak{J}^+ \cup \mathfrak{J}^-$. Donc $\mathfrak{r}_j = \mathfrak{r}_{j+1} = 0$. Donc $e_2 = \lambda_j - \lambda_{j+1} \geq 0$.

Supposons plutôt $j$ pair. Supposons d'abord que la condition (11)(a) soit vérifiée pour $d = 2$, auquel cas on doit vérifier que $e_2 = 0$. Pour $j$ pair, la condition (11)(a) pour $d$ est que $p_d(j) = 1$ et qu'il n'existe pas de $\mathfrak{I} \in \mathfrak{Int}_d$ tel que $j = j_{\max}(\mathfrak{I})$. Cette condition est vérifiée pour $d = 1, 2$. D'après (4), on a soit $j \in \mathfrak{J}^+ \cup \mathfrak{J}^-$, soit $\lambda_j$ est impair et $\lambda_j \in \mathcal{J}''(\lambda)$. Dans le premier cas, la parité de $j$ impose $j \in \mathfrak{J}^-$. Mais alors la relation (6) implique l'existence de $d$ et de $\mathfrak{I} \in \mathfrak{Int}_d$ tels que $j = j_{\max}(\mathfrak{I})$, contrairement aux hypothèses. Supposons donc que $\lambda_j$ soit impair et que $\lambda_j \in \mathcal{J}''(\lambda)$. D'après 2.7, $j_{\max}(\lambda_j)$ est impair, donc $j < j_{\max}(\lambda_j)$ et $\lambda_j = \lambda_{j+1}$. Évidemment, $j, j+1 \notin \mathfrak{J}^+ \cup \mathfrak{J}^-$, donc $\mathfrak{r}_j = \mathfrak{r}_{j+1} = 0$. Alors

$$e_2 = \lambda_j - \lambda_{j+1} + \mathfrak{r}_{j+1} - \mathfrak{r}_j = 0.$$

Supposons maintenant vérifiée la condition (11)(b) pour $d = 2$. On doit prouver que $e_2 > 0$. La condition est que $p_2(j) = 1$ et qu'il existe $\mathfrak{I} \in \mathfrak{Int}_2$ tel que $j = j_{\max}(\mathfrak{I})$. On a aussi $p_1(j) = 1$ puisque (11)(a) est vérifiée pour $d = 1$. D'après (6), on a $j \in \mathfrak{J}^-$. Cela entraîne $\mathfrak{r}_j = -1$. Puisque $j+1$ est impair, on a $j+1 \notin \mathfrak{J}^-$ donc $\mathfrak{r}_{j+1} \leq 0$. Alors $e_2 = \lambda_j - \lambda_{j+1} + \mathfrak{r}_{j+1} - \mathfrak{r}_j \geq \lambda_j - \lambda_{j+1} + 1 > 0$. Supposons enfin vérifiée la condition (11)(c) pour $d = 2$, autrement dit $p_2(j) = 0$. On doit vérifier que $e_2 \geq 0$. Puisque $p_1(j) = 1$, on a $\lambda_j$ pair et $j \notin \mathfrak{J}^+ \cup \mathfrak{J}^-$ d'après (4). Le même raisonnement que dans le cas $j$ impair s'applique et on conclut $e_2 \geq 0$.

On peut maintenant supposer que la condition (11)(a) n'est vérifiée pour aucun $d$. Supposons la condition (11)(b) vérifiée pour $d = 1$. On choisit pour $e_1$ le plus petit entier strictement positif vérifiant la relation (13). On a $e_1 = 1$ ou 2. La condition résultant de (11)(b) pour $d = 1$ est $e_1 > 0$, elle est vérifiée. On pose

$$e_2 = \lambda_j - \lambda_{j+1} + \mathfrak{r}_{j+1} - \mathfrak{r}_j - e_1.$$

Comme précédemment, il reste seulement à prouver que $e_2$ vérifie les conditions résultant de (11) pour $d = 2$. On a exclu la condition (11)(a). Supposons que la condition (11)(b) soit vérifiée pour $d = 2$. On doit montrer que $e_2 > 0$. Les conditions (11)(b) sont vérifiées pour $d = 1, 2$, c'est-à-dire que $j$ est pair et qu'il existe $\mathfrak{I}_d \in \mathfrak{Int}_d$ de sorte que $j = j_{\max}(\mathfrak{I}_d)$. Autrement dit, $p_d(j) = 1$ mais $j$ et $j + 1$ ne sont pas $d$-liés. D'après (6), on a $j \in \mathfrak{J}^-$, donc $\lambda_j$ est pair. Si $\lambda_{j+1} = \lambda_j$, $j$ et $j + 1$ vérifient (1a) pour un $d$ et sont $d$-liés contrairement à l'hypothèse. Donc $\lambda_j > \lambda_{j+1}$. Puisque $j \in \mathfrak{J}^-$, on a aussi $\mathfrak{r}_j = -1$. Le nombre $j + 1$ est impair donc n'appartient pas à $\mathfrak{J}^-$, d'où $\mathfrak{r}_{j+1} \geq 0$. On voit alors que $e_2 = \lambda_j - \lambda_{j+1} + \mathfrak{r}_{j+1} - \mathfrak{r}_j - e_1$ est strictement positif sauf si les trois conditions suivantes sont vérifiées : $\lambda_j = \lambda_{j+1} + 1$, $\mathfrak{r}_{j+1} = 0$ et $e_1 = 2$. Supposons ces conditions vérifiées. Puisque $p_1(j) = 1$ et $e_1 = 2$, la condition (13) pour $d = 1$, qui est vérifiée par définition de $e_1$, implique $p_1(j+1) = 1$. Puisque $\lambda_j = \lambda_{j+1} + 1$, $\lambda_{j+1}$ est impair. Puisque $\mathfrak{r}_{j+1} = 0$, la relation (8) implique que $p_2(j + 1) = 1$. Alors, pour $d = 1, 2$, $j + 1$ appartient à un élément $\mathfrak{I}'_d \in \widetilde{\mathfrak{Int}}_d$. Puisque $j$ et $j + 1$ ne sont pas $d$-liés, on a forcément $j + 1 = j_{\min}(\mathfrak{I}'_d)$. D'après (5), cela entraîne $j + 1 \in \mathfrak{J}^+$. Donc $\mathfrak{r}_{j+1} = 1$ contrairement à l'hypothèse. Cette contradiction conclut. Supposons maintenant que la condition (11)(c) soit vérifiée pour $d = 2$. On doit montrer que $e_2 \geq 0$. On a toujours la condition (11)(b) pour $d = 1$, c'est-à-dire que $j$ est pair, que $p_1(j) = 1$ mais que $j$ et $j + 1$ ne sont pas 1-liés. La condition (11)(c) pour $d = 2$ dit que $p_2(j) = 0$. Alors $j$ et $j + 1$ ne sont pas non plus 2-liés. D'autre part, la relation (4) entraîne que $\lambda_j$ est pair et que $j \notin \mathfrak{J}^+ \cup \mathfrak{J}^-$. D'où $\mathfrak{r}_j = 0$. On ne peut pas avoir $\lambda_j = \lambda_{j+1}$ sinon la relation (1a) serait vérifiée pour un $d$ et $j$ et $j + 1$ seraient $d$-liés, ce qui n'est pas le cas. On n'a pas $j + 1 \in \mathfrak{J}^-$ puisque $j + 1$ est impair. Donc $\mathfrak{r}_{j+1} \geq 0$. On voit alors que $e_2 = \lambda_j - \lambda_{j+1} + \mathfrak{r}_{j+1} - \mathfrak{r}_j - e_1$ est positif ou nul sauf si les mêmes conditions que ci-dessus sont vérifiées : $\lambda_j = \lambda_{j+1} + 1$, $\mathfrak{r}_{j+1} = 0$ et $e_1 = 2$. Ces conditions sont exclues par le même raisonnement que ci-dessus. D'où $e_2 \geq 0$.

Il nous reste à traiter le cas où (11)(c) est vérifiée pour $d = 1, 2$. On choisit pour $e_1$ le plus petit entier positif ou nul vérifiant la relation (13). On a $e_1 = 0$ ou 1. La condition résultant de (11)(c) pour $d = 1$ est $e_1 \geq 0$, elle est vérifiée. On pose $e_2 = \lambda_j - \lambda_{j+1} + \mathfrak{r}_{j+1} - \mathfrak{r}_j - e_1$. Comme précédemment, il reste seulement à prouver que $e_2$ vérifie la condition résultant de (11)(c) pour $d = 2$, c'est-à-dire $e_2 \geq 0$.

Supposons d'abord $j$ impair. Les conditions (11)(c) pour $d = 1, 2$ disent que $p_1(j) = p_2(j) = 1$. D'après (7), on a aussi $p_1(j + 1) = p_2(j + 1) = 1$. La relation

(13) pour $d=1$ implique $e_1=0$. Si ni $j$, ni $j+1$ n'appartiennent à $\mathfrak{J}^+\cup\mathfrak{J}^-$, on a $\mathfrak{r}_j=\mathfrak{r}_{j+1}=0$ et $e_2=\lambda_j-\lambda_{j+1}\geq 0$. Si un seul des éléments $j$ et $j+1$ appartiennent à $\mathfrak{J}^+\cup\mathfrak{J}^-$, on a par parité $j\in\mathfrak{J}^+$ et $j+1\notin\mathfrak{J}^+\cup\mathfrak{J}^-$, ou $j+1\in\mathfrak{J}^-$ et $j\notin\mathfrak{J}^+\cup\mathfrak{J}^-$. Alors $\mathfrak{r}_{j+1}-\mathfrak{r}_j=-1$. Mais l'hypothèse $j\in\mathfrak{J}^+$ ou $j+1\in\mathfrak{J}^-$ implique $\lambda_j>\lambda_{j+1}$. Alors $e_2=\lambda_j-\lambda_{j+1}-1\geq 0$. Enfin supposons que $j$ et $j+1$ appartiennent tous deux à $\mathfrak{J}^+\cup\mathfrak{J}^-$. La parité impose $j\in\mathfrak{J}^+$ et $j+1\in\mathfrak{J}^-$. Alors $\mathfrak{r}_{j+1}-\mathfrak{r}_j=-2$. Mais les hypothèses $j\in\mathfrak{J}^+$ et $j+1\in\mathfrak{J}^-$ imposent non seulement $\lambda_j>\lambda_{j+1}$ mais aussi que $\lambda_j$ et $\lambda_{j+1}$ sont pairs. Donc $\lambda_j\geq\lambda_{j+1}+2$. Alors $e_2=\lambda_j-\lambda_{j+1}-2\geq 0$.

Supposons maintenant $j$ pair. Les conditions (11)(c) pour $d=1,2$ disent que $p_1(j)=p_2(j)=0$. D'après (4), $\lambda_j$ est impair donc $\mathfrak{r}_j=0$. On n'a pas $j+1\in\mathfrak{J}^-$ puisque $j+1$ est impair. Donc $\mathfrak{r}_{j+1}\geq 0$. On voit alors que $e_2=\lambda_j-\lambda_{j+1}+\mathfrak{r}_{j+1}-e_1$ est positif ou nul sauf si les trois conditions suivantes sont vérifiées : $\lambda_j=\lambda_{j+1}$, $\mathfrak{r}_{j+1}=0$ et $e_1=1$. Supposons ces conditions vérifiées. D'après (13) pour $d=1$, on a $p_1(j+1)=1$. Puisque $\lambda_j=\lambda_{j+1}$, $\lambda_{j+1}$ est impair. L'égalité $\mathfrak{r}_{j+1}=0$ et la relation (8) entraînent alors $p_2(j+1)=1$. Pour $d=1,2$, $j+1$ appartient donc à un élément $\mathfrak{J}_d\in\widetilde{\mathfrak{Int}}_d$. Puisque $p_d(j)=0$, $j$ et $j+1$ ne sont pas $d$-liés, donc $j+1=j_{\min}(\mathfrak{J}_d)$. Mais alors, (5) nous dit que $j+1$ appartient à $\mathfrak{J}^+$, donc $\mathfrak{r}_{j+1}=1$ contrairement à l'hypothèse. Cette contradiction conclut. Cela achève la preuve de l'existence de nos suites $\lambda_1$ et $\lambda_2$.

Fixons donc de telles suites $\lambda_1$ et $\lambda_2$. La condition (11) entraîne que ce sont des partitions, c'est-à-dire qu'elles sont décroissantes. Montrons que

(14) il existe des entiers positifs ou nuls $n_1$ et $n_2$ tels que $n_1+n_2=n$, que $\lambda_1$ appartienne à $\mathcal{P}^{\mathrm{symp,sp}}(2n_1)$ et que $\lambda_2$ appartienne à $\mathcal{P}^{\mathrm{orth,sp}}(2n_2)$.

Si les deux dernières conditions sont vérifiées, on a forcément $n_1+n_2=n$. En effet, la relation (9) implique que $S(\lambda_1)+S(\lambda_2)+S(\mathfrak{r})=S(\lambda)$ et on a $S(\mathfrak{r})=0$. Pour prouver les deux dernières conditions, on doit prouver que, pour $d=1,2$ et $k\geq 1$, les termes $\lambda_{d,2k-1}$ et $\lambda_{d,2k}$ sont de même parité et que, quand cette parité est celle de $d$, on a $\lambda_{d,2k-1}=\lambda_{d,2k}$. La première condition résulte de (10) et (7). Si $\lambda_{d,2k-1}\equiv d \bmod 2\mathbb{Z}$, la condition (10) impose $p_d(2k-1)=0$. Alors les conditions de (11)(a) sont vérifiées pour $j=2k-1$, d'où $\lambda_{d,2k-1}=\lambda_{d,2k}$. Cela prouve (14).

Grâce à (14), on définit comme en 3.1 les ensembles d'intervalles $\widetilde{\mathrm{Int}}(\lambda_1)$, $\widetilde{\mathrm{Int}}(\lambda_2)$, les ensembles $J^+$ et $J^-$ et la fonction $\xi$. Montrons que

(15) on a $\{J(\Delta);\ \Delta\in\widetilde{\mathrm{Int}}(\lambda_d)\}=\widetilde{\mathfrak{Int}}_d$ pour $d=1,2$ ; on a $J^+=\mathfrak{J}^+$, $J^-=\mathfrak{J}^-$ et $\xi=\mathfrak{r}$.

Soit $d=1,2$. La réunion des $J(\Delta)$ quand $\Delta$ décrit $\widetilde{\mathrm{Int}}(\lambda_d)$ est l'ensemble des $j\geq 1$ tels que $\lambda_{d,j}$ soit de bonne parité. D'après (10), c'est l'ensemble des $j\geq 1$ tels que $p_d(j)=1$. Cet ensemble d'indices est donc découpé de deux façons en intervalles : les $J(\Delta)$ pour $\Delta\in\widetilde{\mathrm{Int}}(\lambda_d)$ et les $\mathfrak{J}\in\widetilde{\mathfrak{Int}}_d$. Pour prouver que ces découpages coïncident, il suffit de prouver que les ensembles d'éléments maximaux de ces intervalles coïncident (en admettant ici que l'élément maximal d'un intervalle infini est $\infty$). C'est-à-dire qu'il suffit de prouver l'égalité

$$\{j_{\max}(\Delta);\ \ \Delta \in \widetilde{\mathrm{Int}}(\lambda_d)\} = \{j_{\max}(\mathfrak{I});\ \ \mathfrak{I} \in \widetilde{\mathfrak{Int}}_d\}.$$

L'infini intervient dans les deux ensembles pour $d = 1$ et n'intervient dans aucun d'eux pour $d = 2$ (d'après (2) pour l'ensemble de droite). On élimine ces termes. Pour $j \geq 1$, $j$ n'intervient dans ces ensembles que si $j$ est pair (d'après (3) pour celui de droite) et $\lambda_{d,j} \equiv d + 1 \bmod 2\mathbb{Z}$ autrement dit $p_d(j) = 1$. Supposons ces conditions vérifiées. Alors $j$ intervient dans l'ensemble de gauche si et seulement si $\lambda_{d,j} > \lambda_{d,j+1}$. Si $j$ intervient dans l'ensemble de droite, la condition (11)(b) est vérifiée et l'inégalité précédente l'est aussi. Si $j$ n'intervient pas dans l'ensemble de droite, la condition (11)(a) est vérifiée et l'inégalité précédente ne l'est pas. Cela démontre l'égalité de ces ensembles, d'où la première assertion de (15).

Par définition, $J^+$ est l'ensemble des $j \geq 1$ pour lesquels $\lambda_{1,j}$ et $\lambda_{2,j}$ sont de bonne parité et il existe $\Delta \in \widetilde{\mathrm{Int}}(\lambda_1) \cup \widetilde{\mathrm{Int}}(\lambda_2)$ tel que $j = j_{\min}(\Delta)$. En utilisant ce que l'on vient de démontrer, il suffit d'appliquer (5) pour conclure $J^+ = \mathfrak{J}^+$. On prouve de même que $J^- = \mathfrak{J}^-$. Alors $\xi = \mathfrak{x}$ par définition de ces fonctions. Cela prouve (15).

On a $\mathrm{Ind}(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2 + \xi$ par définition, d'où $\mathrm{Ind}(\lambda_1, \lambda_2) = \lambda$ d'après (15) et (9). Montrons que

(16) $\lambda_1$ et $\lambda_2$ induisent régulièrement $\lambda$.

Il s'agit de prouver que tout intervalle relatif est réduit à un seul élément. Soit $D$ un intervalle relatif. Si $J(D)$ est réduit à un seul élément, $D$ aussi. Supposons que $J(D)$ a au moins deux éléments. Par définition, il existe un unique $d = 1, 2$ pour lequel il existe $\Delta_d \in \mathrm{Int}(\lambda_d)$ de sorte que $J(D) \subset J(\Delta_d)$. Pour fixer la notation, on suppose $d = 1$. Cela entraîne : pour $j, j+1 \in J(D)$, il n'existe pas de $\Delta_2 \in \mathrm{Int}(\lambda_2)$ tel que $\{j, j+1\} \subset J(\Delta_2)$. En effet, les extrémités $j_{\min}(D)$ et $j_{\max}(D)$ sont par définition des éléments consécutifs de l'ensemble $\mathcal{J}$ de 3.1. Un $\Delta_2$ comme ci-dessus vérifierait donc $j_{\min}(\Delta_2) \leq j_{\min}(D)$ et $j_{\max}(D) \leq j_{\max}(\Delta_2)$, donc $J(D) \subset J(\Delta_2)$, ce qui est exclu. On traduit d'après (15) : il existe $\mathfrak{I}_1 \in \widetilde{\mathfrak{Int}}_1$ tel que $J(D) \subset \mathfrak{I}_1$ et, pour $j, j+1 \in J(D)$, $j$ et $j+1$ ne sont pas 2-liés. Soient $j, j+1 \in J(D)$. Les indices $j, j+1$ n'étant pas 2-liés, ils ne vérifient pas les conditions (1b), (1c) et (1d) (cette dernière étant de toute façon exclue puisque $\lambda_j$ et $\lambda_{j+1}$ sont pairs par définition des intervalles relatifs). Puisque $j$ et $j+1$ sont 1-liés, ils vérifient forcément la condition (1a) pour $d = 1$. Donc $\lambda_j = \lambda_{j+1}$. Cela étant vrai pour tout couple $\{j, j+1\} \subset J(D)$, $\lambda_j$ est constant pour $j \in J(D)$. Autrement dit, $D$ est réduit à un seul élément.

Montrons que

$$\chi_{\lambda_1,\lambda_2} = \chi. \tag{17}$$

On a $\chi_{\lambda_1,\lambda_2}(0)=0$ par définition et $\chi(0)=0$ par hypothèse. Soit $i\in \mathrm{Jord}^{bp}(\lambda)$. Si $\mathrm{mult}_\lambda(i)=1$, $\chi_{\lambda_1;\lambda_2}(i)=0$ par définition et $\chi(i)=0$ par hypothèse. Supposons $\mathrm{mult}_\lambda(i)\geq 2$. Comme dans la preuve de (16), il existe un unique $d=1,2$ de sorte qu'il existe $\Delta_d\in\widetilde{\mathrm{Int}}(\lambda_d)$ tel que $J(i)\subset J(\Delta_d)$. On a $\chi_{\lambda_1,\lambda_2}(i)=d+1$ par définition. Toujours comme dans la preuve de (16), pour $j$, $j+1\in J(i)$, la condition (1a) est vérifiée pour ce $d$. Alors $\chi(i)=d+1$. D'où (17).

Montrons que

$$\zeta(\lambda_1)+\zeta(\lambda_2)=\zeta(\lambda)+\xi. \tag{18}$$

On a défini en 3.1 les ensembles $P^+_{\lambda_1,\lambda_2}(\lambda)$ et $P^-_{\lambda_1,\lambda_2}(\lambda)$ et la suite $\zeta_{\lambda_1,\lambda_2}(\lambda)$. Puisque $\lambda_1$ et $\lambda_2$ induisent régulièrement $\lambda$, on a les égalités $P^+_{\lambda_1,\lambda_2}(\lambda)=P^+(\lambda)$, $P^-_{\lambda_1,\lambda_2}(\lambda)=P^-(\lambda)$. Donc $\zeta_{\lambda_1,\lambda_2}(\lambda)=\zeta(\lambda)$. Alors le lemme 3.1 implique (18).

L'égalité (18) entraîne

$$\lambda_1+\zeta(\lambda_1)+\lambda_2+\zeta(\lambda_2)=\lambda_1+\lambda_2+\xi+\zeta(\lambda)=\lambda+\zeta(\lambda).$$

Le lemme 2.7 et l'assertion 2.7(4) transforment cette égalité en

$${}^td(\lambda_1)+{}^td(\lambda_2)={}^td(\lambda),$$

d'où $d(\lambda_1)\cup d(\lambda_2)=d(\lambda)$. □

**3.3. *Les fonctions $\tau^\zeta$, $\delta^\zeta$.*** Soient $n_1,n_2\in\mathbb{N}$ tels que $n_1+n_2=n$ et soient $\lambda_1\in\mathcal{P}^{symp,sp}(2n_1)$ et $\lambda_2\in\mathcal{P}^{orth,sp}(2n_2)$. Soit $\lambda$ l'induite endoscopique de $\lambda_1$ et $\lambda_2$. On considère de plus des éléments $\iota_1=(\tau_1,\delta_1)\in\mathcal{F}am(\lambda_1)$ et $\iota_2=(\tau_2,\delta_2)\in\mathcal{F}am(\lambda_2)$. On pose $r_1=r(\tau_1,\delta_1)$, $r_2=r(\tau_2,\delta_2)$.

Pour $d=1,2$ et $\Delta\in\widetilde{\mathrm{Int}}(\lambda_d)$, on note $\Delta^+$ le plus petit $\Delta'\in\mathrm{Int}(\lambda_d)$ tel que $\Delta'>\Delta$, pour peu qu'il existe un tel $\Delta'$ (sinon, $\Delta^+$ n'existe pas). Pour $D\in\widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$, on définit $D^+$ de façon similaire.

Pour $D\in\mathrm{Int}_{\lambda_1,\lambda_2}(\lambda)$ et pour $d=1,2$, considérons l'ensemble des $\Delta\in\widetilde{\mathrm{Int}}(\lambda_d)$ tels que $j_{\max}(D)\leq j_{\max}(\Delta)$ (ici, on pose par convention $j_{\max}(\Delta_{1,\min})=\infty$ où $\Delta_{1,\min}$ est le plus petit élément de $\widetilde{\mathrm{Int}}(\lambda_1)$). Si cet ensemble est non vide (ce qui est le cas si $d=1$ par la convention que l'on vient de poser), on note $\Delta_d(D)$ son plus grand élément. On pose $\Delta_1(D_{\min})=\Delta_{1,\min}$ tandis que $\Delta_2(D_{\min})$ n'existe pas. Si $\Delta_2(D)$ n'existe pas et si $\mathrm{Int}(\lambda_2)$ n'est pas vide, on note $\Delta_2(D)^+$ le plus petit élément de $\mathrm{Int}(\lambda_2)$ (si $\mathrm{Int}(\lambda_2)$ est vide, $\Delta_2(D)$ et $\Delta_2(D)^+$ n'existent pas).

Pour $\zeta=\pm$, on définit une fonction $\delta^\zeta\in(\mathbb{Z}/2\mathbb{Z})^{\mathrm{Int}_{\lambda_1,\lambda_2}(\lambda)}$ par les formules ci-dessous. Soit $D\in\mathrm{Int}_{\lambda_1,\lambda_2}(\lambda)$. On pose $\Delta_d=\Delta_d(D)$ pour $d=1,2$. Ce terme existe toujours dans chaque cas ci-dessous. Par contre, $\Delta_d^+$ n'existe pas toujours. Dans ce cas, on considère que $\delta_d(\Delta_d^+)=0$. On écrit les formules comme des égalités, en fait, il s'agit de congruences modulo $2\mathbb{Z}$. On pose

si $j_{\max}(D) \in J^+$,
$$\delta^+(D) = \tau_1(\Delta_1) + \tau_2(\Delta_2) + r_1 + r_2 + 1, \quad \delta^-(D) = \delta^+(D) + 1;$$
si $j_{\max}(D) \in J^-$, $$\delta^+(D) = \delta^-(D) = \delta_1(\Delta_1) + \delta_2(\Delta_2);$$
si $j_{\max}(D) \notin J^+ \cup J^-$ et $J(D) \subset J(\Delta_1)$,
$$\delta^+(D) = \delta^-(D) = \delta_1(\Delta_1) + \delta_2(\Delta_2^+);$$
si $j_{\max}(D) \notin J^+ \cup J^-$ et $J(D) \subset J(\Delta_2)$,
$$\delta^+(D) = \delta^-(D) = \delta_1(\Delta_1^+) + \delta_2(\Delta_2).$$

Avec les mêmes notations, on définit une fonction $\tau^\zeta \in (\mathbb{Z}/2\mathbb{Z})^{\widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)}$ par

si $|J(D)| \geq 2$ et $J(D) \subset J(\Delta_1)$, $\tau^+(D) = \tau^-(D) = \tau_1(\Delta_1) + \delta_2(\Delta_2^+) + r_2$;
si $|J(D)| \geq 2$ et $J(D) \subset J(\Delta_2)$,
$$\tau^+(D) = \delta_1(\Delta_1^+) + \tau_2(\Delta_2) + r_1, \quad \tau^-(D) = \tau^+(D) + 1;$$
si $|J(D)| = 1$ et $j_{\min}(D) = j_{\max}(D) \in J^+$,
$$\tau^+(D) = \tau^-(D) = \tau_1(\Delta_1) + \delta_2(\Delta_2^+) + r_2;$$
si $|J(D)| = 1$ et $j_{\min}(D) = j_{\max}(D) \in J^-$,
$$\tau^+(D) = \tau^-(D) = \tau_1(\Delta_1) + \delta_2(\Delta_2) + r_2.$$

Tous ces cas sont exclusifs l'un de l'autre. On a évidemment
$$\begin{aligned} &\delta^-(D) = \delta^+(D) + 1 \quad \text{si et seulement si} \quad j_{\max}(D) \in J^+;\\ &\tau^-(D) = \tau^+(D) + 1 \quad \text{si et seulement si} \quad |J(D)| \geq 2 \text{ et } J(D) \subset J(\Delta_2). \end{aligned} \tag{1}$$

On a aussi
$$\tau^+(D_{\min}) = \tau^-(D_{\min}) = 0. \tag{2}$$

En effet, $J(D_{\min})$ est infini. Il ne peut qu'être contenu dans $J(\Delta_{1,\min})$. Donc $\tau^+(D_{\min}) = \tau^-(D_{\min}) = \tau_1(\Delta_1(D_{\min})) + \delta_2(\Delta_2(D_{\min})^+) + r_2$. On a $\Delta_1(D_{\min}) = \Delta_{1,\min}$ et $\Delta_2(D_{\min})$ n'existe pas. On a $\tau_1(\Delta_{1,\min}) = 0$. D'après 2.4(2) et nos conventions, $\delta_2(\Delta_2(D_{\min})^+) = r_2$. D'où (2).

Pour $\zeta = \pm$, posons
$$C^\zeta = \sum_{D \in \mathrm{Int}_{\lambda_1,\lambda_2}(\lambda)} (1 - (-1)^{\tau^\zeta(D)})((-1)^{\delta^\zeta(D)} - (-1)^{\delta^\zeta(D^+)}).$$

Ici encore, on considère que $\delta^\zeta(D^+) = 1$ si $D^+$ n'existe pas. On a
$$C^\zeta = \begin{cases} 2(r_1 + \zeta r_2) & \text{si } r_1 + r_2 \text{ est pair,} \\ -2(r_1 + \zeta r_2 + 1) & \text{si } r_1 + r_2 \text{ est impair.} \end{cases} \tag{3}$$

Cela résulte de [Waldspurger 2001, XI.24], à ceci près que les hypothèses de cette référence étaient plus restrictives que les nôtres. On renvoie pour ce problème aux explications que l'on donnera après la proposition du paragraphe suivant.

**3.4.** ***Le résultat de*** [Waldspurger 2001]. Les données sont les mêmes que dans le paragraphe précédent. Pour $d = 1, 2$, le couple $\iota_d = (\tau_d, \delta_d)$ provient d'un symbole $\Lambda_d$ dans la famille de $\lambda_d$. On note $(r_d, \rho_d)$ l'élément de $\Sigma_{n_1,\mathrm{imp}}$ si $d = 1$, $\Sigma_{n_2,\mathrm{pair}}$ si $d = 2$, tel que $\mathrm{symb}(r_d, \rho_d) = \Lambda_d$. On pose $N_1 = n_1 - r_1^2 - r_1$, $N_2 = n_2 - r_2^2$. On fixe un élément $\zeta \in \{\pm 1\}$, que l'on considérera souvent comme un simple signe $\pm$. Si $\zeta = 1$, on pose $h^+ = r_1 + |r_2|$, $h^- = \sup(r_1 - |r_2|, |r_2| - r_1 - 1)$. Si $\zeta = -1$, on pose $h^+ = \sup(r_1 - |r_2|, |r_2| - r_1 - 1)$, $h^- = r_1 + |r_2|$. On vérifie que $h^+(h^+ + 1)/2 + h^-(h^- + 1)/2 = r_1^2 + r_1 + r_2^2$. On fixe des entiers $n^+, n^- \in \mathbb{N}$ tels que $n^+ + n^- = n$, $n^+ \geq h^+(h^+ + 1)/2$, $n^- \geq h^-(h^- + 1)/2$ et on pose $N^+ = n^+ - h^+(h^+ + 1)/2$, $N^- = n^- - h^-(h^- + 1)/2$. On a $N^+ + N^- = N_1 + N_2$. On définit un quadruplet d'entiers $\boldsymbol{a} = (a_1^+, a_1^-, a_2^+, a_2^-)$ par les formules suivantes :

$$\begin{aligned}
\boldsymbol{a} &= (0, 0, 0, 1) \quad \text{si } \zeta = 1 \quad \text{et} \quad r_1 \geq |r_2|;\\
\boldsymbol{a} &= (0, 0, 1, 0) \quad \text{si } \zeta = -1 \quad \text{et} \quad r_1 \geq |r_2|;\\
\boldsymbol{a} &= (0, 1, 0, 0) \quad \text{si } \zeta = 1 \quad \text{et} \quad r_1 < |r_2|;\\
\boldsymbol{a} &= (1, 0, 0, 0) \quad \text{si } \zeta = -1 \quad \text{et} \quad r_1 < |r_2|.
\end{aligned}$$

Avec les mêmes notations qu'en 1.2, on définit une représentation $\Pi^\zeta(\iota_1, \iota_2)$ de $W_{N^+} \times W_{N^-}$ par la formule

$$\Pi^\zeta(\iota_1, \iota_2) = \bigoplus_{N \in \mathcal{N}} \mathrm{ind}_{W_N}^{W_{N^+} \times W_{N^-}} \big(\mathrm{sgn}_{\mathrm{CD}}^{\boldsymbol{a}} \otimes \mathrm{res}_{W_N}^{W_{N_1} \times W_{N_2}} (\rho_1 \otimes \rho_2)\big).$$

On note $\mathcal{I}^\zeta(\iota_1, \iota_2)$ l'ensemble des quadruplets

$$(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-) \in \boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2n^+) \times \boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2n^-)$$

vérifiant les conditions suivantes :

(1) $k_{\lambda^+,\epsilon^+} = h^+$, $\quad k_{\lambda^-,\epsilon^-} = h^-$ ;

(2) la représentation $\rho_{\lambda^+,\epsilon^+} \otimes \rho_{\lambda^-,\epsilon^-}$ de $W_{N^+} \times W_{N^-}$ intervient dans $\Pi^\zeta(\iota_1, \iota_2)$ avec une multiplicité strictement positive.

Pour poser la définition suivante, on a besoin d'introduire deux notations. Pour $D \in \mathrm{Int}_{\lambda_1,\lambda_2}(\lambda)$, notons $i_{\min}(D)$ le plus petit élément de $D$. On a $i_{\min}(D) \geq 1$ puisque $D \neq D_{\min}$. Pour toute partition $\mu$, on pose $\mathrm{mult}_\mu(\geq D) = \sum_{i \in \mathbb{N}, i \geq i_{\min}(D)} \mathrm{mult}_\mu(i)$. D'autre part, on pose $\nu = 1$ si $r_2 \geq 0$, $\nu = -1$ si $r_2 < 0$.

On note $\mathcal{I}^{\zeta,\max}(\iota_1, \iota_2)$ l'ensemble des quadruplets

$$(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-) \in \boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2n^+) \times \boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2n^-)$$

vérifiant les conditions suivantes :

(3) $\lambda^+ \cup \lambda^- = \lambda$ ;

(4) pour tout $D \in \mathrm{Int}_{\lambda_1,\lambda_2}(\lambda)$, on a

$$\mathrm{mult}_{\lambda^+}(\geq D) \equiv \delta^{\zeta\nu}(D) \bmod 2\mathbb{Z}, \quad \text{et} \quad \mathrm{mult}_{\lambda^-}(\geq D) \equiv \delta^{-\zeta\nu}(D) \bmod 2\mathbb{Z};$$

(5) pour tout $D \in \widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$ et tout $i \in D$ tel que $i \neq 0$ et $\mathrm{mult}_{\lambda^+}(i) > 0$, resp. $\mathrm{mult}_{\lambda^-}(i) > 0$, on a

$$\epsilon_i^+ = (-1)^{\tau^{\zeta\nu}(D)}, \quad \text{resp.} \quad \epsilon_i^- = (-1)^{\tau^{-\zeta\nu}(D)}.$$

Dans ces formules, on a évidemment identifié les signes $\pm$ des définitions de $\tau^+$, $\tau^-$, etc. à des éléments de $\{\pm 1\}$. On a montré en [Waldspurger 2001, XI.29, remarque 4] que, sous l'hypothèse (3), les deux congruences de (4) étaient équivalentes.

**Proposition.** (i) *Soit* $(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-) \in \mathcal{I}^\zeta(\iota_1, \iota_2)$. *Alors* $\lambda^+ \cup \lambda^- \leq \lambda$.

(ii) *L'ensemble* $\mathcal{I}^{\zeta,\max}(\iota_1, \iota_2)$ *est égal au sous-ensemble des* $(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-) \in \mathcal{I}^\zeta(\iota_1, \iota_2)$ *tels que* $\lambda^+ \cup \lambda^- = \lambda$. *Pour* $(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-) \in \mathcal{I}^{\zeta,\max}(\iota_1, \iota_2)$, *la représentation* $\rho_{\lambda^+,\epsilon^+} \otimes \rho_{\lambda^-,\epsilon^-}$ *intervient avec multiplicité* 1 *dans* $\Pi^\zeta(\iota_1, \iota_2)$.

Cela résulte de [Waldspurger 2001, propositions XI.28 et XI.29], ainsi qu'on l'a expliqué dans la preuve de la proposition XII.7 de cette référence (voir aussi [Waldspurger 2018b, propositions 1.12 et 1.13]). A ceci près qu'alors, les hypothèses sur $\iota_2$ étaient restrictives : on supposait que $r_2$ était pair et positif ou nul ; dans le cas $r_2 = 0$, on supposait que le symbole $(X, Y)$ correspondant à $\iota_2$ vérifiait $X \geq Y$ pour l'ordre lexicographique. En fait, cette dernière hypothèse était utilisée dans d'autres passages de [Waldspurger 2001] mais pas dans les démonstrations des propositions utilisées. Pour traiter le cas où $r_2$ est impair et positif, il n'y a pas d'autre méthode que de reprendre la démonstration. C'est ce que l'on a fait mais elle est trop longue pour la récrire. Le cas où $r_2 < 0$ se déduit du cas $r_2 > 0$ de la façon suivante. On suppose donc $r_2 < 0$. On a dit que $\iota_2$ correspondait à un symbole $\Lambda_2 = (X_2, Y_2)$, puis à un couple $(r_2, \rho_2)$. Inversement, on voit que $(-r_2, \rho_2)$ correspond au symbole $\Lambda_2' = (Y_2, X_2)$, puis à un élément $\iota_2' \in \mathcal{F}am(\lambda_2)$. Quand on remplace $\iota_2$ par $\iota_2'$ dans les constructions ci-dessus, la représentation $\Pi^\zeta(\iota_1, \iota_2)$ ne change pas. Donc la proposition ci-dessus étant vérifiée pour $\iota_2'$, elle le restera pourvu que l'on ait les égalités $\mathcal{I}^\zeta(\iota_1, \iota_2) = \mathcal{I}^\zeta(\iota_1, \iota_2')$ et $\mathcal{I}^{\zeta,\max}(\iota_1, \iota_2) = \mathcal{I}^{\zeta,\max}(\iota_1, \iota_2')$. La première égalité est claire d'après (1) et (2). La deuxième ne l'est pas car les fonctions $\tau^\pm$ et $\delta^\pm$ dépendent de $\iota_2$. Mais, puisqu'on passe de $\Lambda_2$ à $\Lambda_2'$ en permutant $X_2$ et $Y_2$, on voit sur les formules de 2.2 que changer $\iota_2$ en $\iota_2'$ ne change pas $\delta_2$ et remplace $\tau_2$ par $\tau_2 + 1$. On voit ensuite sur les formules de 3.3 que cela échange les couples $(\tau^+, \delta^+)$ et $(\tau^-, \delta^-)$. Mais alors, parce qu'il figure dans les conditions (4) et (5) un signe terme $\nu$, qui vaut 1 pour $\iota_2'$ et $-1$ pour $\iota_2$, on voit que ces conditions ne changent pas quand on remplace $\iota_2$ par $\iota_2'$. C'est ce qu'on voulait.

**3.5.** ***Réciproque de la construction des fonctions $\tau^\zeta$ et $\delta^\zeta$.*** Soient $n_1, n_2 \in \mathbb{N}$ tels que $n_1+n_2=n$ et soient $\lambda_1 \in \mathcal{P}^{\text{symp,sp}}(2n_1)$ et $\lambda_2 \in \mathcal{P}^{\text{orth,sp}}(2n_2)$. Notons $\lambda$ l'induite endoscopique de $\lambda_1$ et $\lambda_2$. Pour $\iota_1 = (\tau_1, \delta_1) \in \mathcal{F}am(\lambda_1)$ et $\iota_2 = (\tau_2, \delta_2) \in \mathcal{F}am(\lambda_2)$, on a construit en 3.3 des fonctions $\tau^\zeta$ et $\delta^\zeta$ pour $\zeta = \pm$. Dans ce paragraphe, il convient de les noter plus précisément $\tau^\zeta_{\iota_1,\iota_2}$ et $\delta^\zeta_{\iota_1,\iota_2}$. On note aussi $C^\zeta_{\iota_1,\iota_2}$ la somme définie en 3.3.

Soient $r_1 \in \mathbb{N}$, $r_2 \in \mathbb{Z}$ et, pour $\zeta = \pm$, soient $\tau^\zeta \in (\mathbb{Z}/2\mathbb{Z})^{\widetilde{\text{Int}}_{\lambda_1,\lambda_2}(\lambda)}$ et $\delta^\zeta \in (\mathbb{Z}/2\mathbb{Z})^{\text{Int}_{\lambda_1,\lambda_2}(\lambda)}$. On pose

$$C^\zeta = \sum_{D \in \text{Int}_{\lambda_1,\lambda_2}(\lambda)} (1-(-1)^{\tau^\zeta(D)})\big((-1)^{\delta^\zeta(D)} - (-1)^{\delta^\zeta(D^+)}\big).$$

On suppose que ces données vérifient les conditions

$$\begin{aligned} \delta^-(D) = \delta^+(D)+1 \quad &\text{si et seulement si} \quad j_{\max}(D) \in J^+; \\ \tau^-(\delta) = \tau^+(D)+1 \quad &\text{si et seulement si} \quad |J(D)| \geq 2 \quad \text{et} \quad J(D) \subset J(\Delta_2(D)); \end{aligned} \tag{1}$$

$$\tau^+(D_{\min}) = \tau^-(D_{\min}) = 0; \tag{2}$$

$$C^\zeta = \begin{cases} 2(r_1+\zeta r_2) & \text{si } r_1+r_2 \text{ est pair,} \\ -2(r_1+\zeta r_2+1) & \text{si } r_1+r_2 \text{ est impair.} \end{cases} \tag{3}$$

**Lemme.** *Sous ces hypothèses, il existe d'uniques*

$$\iota_1 = (\tau_1, \delta_1) \in \mathcal{F}am(\lambda_1) \quad \textit{et} \quad \iota_2 = (\tau_2, \delta_2) \in \mathcal{F}am(\lambda_2)$$

*tels que, pour $\zeta = \pm$, on ait les égalités $\tau^\zeta = \tau^\zeta_{\iota_1,\iota_2}$ et $\delta^\zeta = \delta^\zeta_{\iota_1,\iota_2}$. De plus, on a $r_1 = r(\tau_1, \delta_1)$ et $r_2 = r(\tau_2, \delta_2)$.*

*Preuve.* S'il existe $(\tau_1, \delta_1)$ et $(\tau_2, \delta_2)$ vérifiant la première assertion de l'énoncé, les fonctions $\tau^\zeta$ et $\delta^\zeta$ sont données par les formules du paragraphe 3.3, où l'on remplace $r_1$ et $r_2$ par $r_1' = r(\tau_1, \delta_1)$ et $r_2' = r(\tau_2, \delta_2)$. Remarquons que ces formules ne dépendent que des images de $r_1'$ et $r_2'$ dans $\mathbb{Z}/2\mathbb{Z}$. On note symboliquement $(X_{r_1',r_2'})$ ces formules.

Commençons par prouver que, pour deux éléments donnés $r_1', r_2' \in \mathbb{Z}/2\mathbb{Z}$, il existe d'uniques

$$(\tau_1, \delta_1) \in (\mathbb{Z}/2\mathbb{Z})^{\widetilde{\text{Int}}(\lambda_1)} \times (\mathbb{Z}/2\mathbb{Z})^{\text{Int}(\lambda_1)}, \quad (\tau_2, \delta_2) \in (\mathbb{Z}/2\mathbb{Z})^{\text{Int}(\lambda_2)} \times (\mathbb{Z}/2\mathbb{Z})^{\text{Int}(\lambda_2)}$$

telles que les formules $(X_{r_1',r_2'})$ soient vérifiées. Remarquons que l'on peut considérer uniquement les formules exprimant $\tau^+$ et $\delta^+$ : celles concernant $\tau^-$ et $\delta^-$ s'en déduisent d'après l'hypothèse (1).

Pour $D \in \widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$ et pour $d = 1, 2$, notons $\mathfrak{T}_d(\geq D)$ l'ensemble des $\Delta_d \in \widetilde{\mathrm{Int}}(\lambda_d)$ tels que $j_{\min}(\Delta_d) \leq j_{\max}(D)$ (en convenant que $j_{\max}(D_{\min}) = \infty$) et notons $\mathfrak{D}_d(\geq D)$ l'ensemble des $\Delta_d \in \mathrm{Int}(\lambda_d)$ tels que $j_{\max}(\Delta_d) \leq j_{\max}(D)$. Remarquons que $\mathfrak{T}_d(\geq D_{\min}) = \widetilde{\mathrm{Int}}(\lambda_d)$ et $\mathfrak{D}_d(\geq D_{\min}) = \mathrm{Int}(\lambda_d)$. Pour deux intervalles relatifs $D > D'$, il est clair que $\mathfrak{T}_d(\geq D)$ est inclus dans $\mathfrak{T}_d(\geq D')$ et que $\mathfrak{D}_d(\geq D)$ est inclus dans $\mathfrak{D}_d(\geq D')$. On pose

$$\mathfrak{T}_d(D) = \mathfrak{T}_d(\geq D) - \mathfrak{T}_d(\geq D^+), \quad \mathfrak{D}_d(D) = \mathfrak{D}_d(\geq D) - \mathfrak{D}_d(\geq D^+),$$

avec la convention $\mathfrak{T}_d(\geq D^+) = \mathfrak{D}_d(\geq D^+) = \varnothing$ si $D^+$ n'existe pas, c'est-à-dire si $D$ est l'intervalle relatif maximal. Cette définition entraîne :

(4) pour deux intervalles relatifs $D \neq D'$, on a

$$\mathfrak{T}_d(D) \cap \mathfrak{T}_d(D') = \varnothing \quad \text{et} \quad \mathfrak{D}_d(D) \cap \mathfrak{D}_d(D') = \varnothing.$$

Montrons que

(5) $\mathfrak{T}_d(D)$ est l'ensemble des $\Delta_d \in \widetilde{\mathrm{Int}}(\lambda_d)$ tels que
$$j_{\min}(\Delta_d) \in \{j_{\min}(D), j_{\max}(D)\};$$
$\mathfrak{D}_d(D)$ est l'ensemble des $\Delta_d \in \mathrm{Int}(\lambda_d)$ tels que
$$j_{\max}(\Delta_d) \in \{j_{\min}(D), j_{\max}(D)\};$$
ces ensembles ont au plus un élément.

Soit $\Delta_d \in \widetilde{\mathrm{Int}}(\lambda_d)$, supposons $j_{\min}(\Delta_d) \in \{j_{\min}(D), j_{\max}(D)\}$. Alors $j_{\min}(\Delta_d) \leq j_{\max}(D)$ et $\Delta_d$ appartient à $\mathfrak{T}_d(\geq D)$. Si $D$ est l'intervalle relatif maximal, cela entraîne $\Delta_d \in \mathfrak{T}_d(D)$. Sinon, on a $j_{\max}(D^+) < j_{\min}(D) \leq j_{\min}(\Delta_d)$ donc $\Delta_d$ n'appartient pas à $\mathfrak{T}_d(\geq D^+)$. D'où $\Delta_d \in \mathfrak{T}_d(D)$. Réciproquement, supposons $\Delta_d \in \mathfrak{T}_d(D)$. L'entier $j_{\min}(\Delta_d)$ appartient à l'ensemble $\mathcal{J}$ de 3.1. D'après 3.1(3), il existe $D' \in \widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$ tel que $j_{\min}(\Delta_d) \in \{j_{\min}(D'), j_{\max}(D')\}$. D'après ce que l'on vient de prouver, on a $\Delta_d \in \mathfrak{T}_d(D')$. Alors (4) entraîne $D' = D$, donc $j_{\min}(\Delta_d) \in \{j_{\min}(D), j_{\max}(D)\}$. Cela prouve la première assertion de (4). Supposons encore que $\Delta_d \in \mathfrak{T}_d(D)$ et considérons un intervalle $\Delta'_d \in \widetilde{\mathrm{Int}}(\lambda_d)$ distinct de $\Delta_d$. Si $\Delta'_d > \Delta_d$, on a $j_{\max}(\Delta'_d) < j_{\min}(\Delta_d) \leq j_{\max}(D)$. Le nombre $j_{\max}(\Delta'_d)$ appartient à $\mathcal{J}$. Par définition des intervalles relatifs, $j_{\min}(D)$ et $j_{\max}(D)$ sont soit égaux, soit des éléments consécutifs de $\mathcal{J}$. Cela entraîne en tout cas $j_{\max}(\Delta'_d) \leq j_{\min}(D)$. Puisque $j_{\min}(\Delta'_d) < j_{\max}(\Delta'_d)$, on a donc $j_{\min}(\Delta'_d) \notin \{j_{\min}(D), j_{\max}(D)\}$, d'où $\Delta'_d \notin \mathfrak{T}_d(D)$. Si maintenant $\Delta'_d < \Delta_d$, on a $j_{\min}(D) \leq j_{\min}(\Delta_d) < j_{\max}(\Delta_d)$. Comme ci-dessus, on en déduit $j_{\max}(D) \leq j_{\max}(\Delta_d)$, puis $j_{\max}(D) < j_{\min}(\Delta'_d)$ et on conclut $\Delta'_d \notin \mathfrak{T}_d(\geq D)$. Donc $\mathfrak{T}_d(D)$ a au plus un élément. Les assertions concernant $\mathfrak{D}_d(D)$ se démontrent de la même façon. Cela prouve (5).

On va montrer que, pour tout intervalle relatif $D$ les formules $(X_{r'_1,r'_2})$ exprimant $\tau^+(D)$ et $\delta^+(D)$, d'une part ne font intervenir des $\tau_d(\Delta_d)$ que pour des $\Delta_d \in \mathfrak{T}_d(\geq D)$ et des $\delta_d(\Delta_d)$ que pour des $\Delta_d \in \mathfrak{D}_d(\geq D)$, d'autre part que, quand

$\mathfrak{T}_d(D)$, resp. $\mathfrak{D}_d(D)$, est non vide, elles font intervenir $\tau_d(\Delta_d)$, resp. $\delta_d(\Delta_d)$, pour l'unique élément $\Delta_d$ de cet ensemble. On étudie les différents cas possibles, pour $D \in \mathrm{Int}_{\lambda_1,\lambda_2}(\lambda)$. On suppose d'abord $D \neq D_{\min}$. On pose simplement $\Delta_d = \Delta_d(D)$.

(a) Supposons que $|J(D)| = 1$ et que $j_{\min}(D) = j_{\max}(D) \in J^+$. Dans ce cas, on a $j_{\max}(D) = j_{\min}(\Delta_1) = j_{\min}(\Delta_2)$. D'après (5), on a $\mathfrak{T}_1(D) = \{\Delta_1\}$, $\mathfrak{T}_2(D) = \{\Delta_2\}$. Si $\Delta'_1 \in \mathfrak{D}_1(D)$, on a $j_{\max}(D) = j_{\min}(D) \in J(\Delta'_1)$, donc $J(\Delta'_1) \cap J(\Delta_1) \neq \varnothing$, donc $\Delta'_1 = \Delta_1$. Or $j_{\max}(\Delta_1) > j_{\min}(\Delta_1) = j_{\max}(D)$, donc $\Delta_1 \notin \mathfrak{D}_1(D)$. Donc $\mathfrak{D}_1(D) = \varnothing$ et, de même, $\mathfrak{D}_2(D) = \varnothing$. Par ailleurs, si $\Delta_2^+$ existe, on a $j_{\max}(\Delta_2^+) < j_{\min}(\Delta_2) = j_{\max}(D)$, donc $\Delta_2^+ \in \mathfrak{D}_2(\geq D)$. Enfin, les formules dans notre cas sont

$$\delta^+(D) = \tau_1(\Delta_1) + \tau_2(\Delta_2) + r'_1 + r'_2 + 1, \quad \tau^+(D) = \tau_1(\Delta_1) + \delta_2(\Delta_2^+) + r'_2.$$

On voit que les propriétés requises sont vérifiées.

(b) Supposons que $|J(D)| = 1$ et que $j_{\min}(D) = j_{\max}(D) \in J^-$. Ce cas est similaire au précédent. On a cette fois $j_{\max}(D) = j_{\max}(\Delta_1) = j_{\max}(\Delta_2)$. On a $\mathfrak{T}_d(D) = \varnothing$ pour $d = 1, 2$, $\mathfrak{D}_1(D) = \{\Delta_1\}$, $\mathfrak{D}_2(D) = \{\Delta_2\}$ et $\Delta_1 \in \mathfrak{T}_1(\geq D)$. Les formules sont

$$\delta^+(D) = \delta_1(\Delta_1) + \delta_2(\Delta_2), \quad \tau^+(D) = \tau_1(\Delta_1) + \delta_2(\Delta_2) + r'_2.$$

Les propriétés requises sont vérifiées.

(c) Supposons que $|J(D)| \geq 2$, que $J(D) \subset J(\Delta_1)$ et que $j_{\min}(D)$ et $j_{\max}(D)$ soient impairs. Puisque ces termes appartiennent à l'ensemble $\mathcal{J}$, l'imparité impose qu'ils sont de la forme $j_{\min}(D) = j_{\min}(\Delta'_{d'})$ et $j_{\max}(D) = j_{\min}(\Delta''_{d''})$ pour des entiers $d', d'' = 1, 2$ et des intervalles $\Delta'_{d'} \in \widetilde{\mathrm{Int}}(\lambda_{d'})$ et $\Delta''_{d''} \in \widetilde{\mathrm{Int}}(\lambda_{d''})$. Si $d' = 2$, puisque $j_{\min}(D)$ et $j_{\max}(D)$ sont des éléments consécutifs de $\mathcal{J}$, on a $j_{\max}(D) \leq j_{\max}(\Delta'_2)$, d'où $J(D) \subset J(\Delta'_2)$, ce qui est interdit par définition des intervalles et par l'hypothèse $J(D) \subset J(\Delta_1)$. Donc $d' = 1$ et forcément $\Delta'_1 = \Delta_1$, c'est-à-dire $j_{\min}(D) = j_{\min}(\Delta_1)$. Si $d'' = 1$, on a $J(\Delta''_1) \cap J(\Delta_1) \neq \varnothing$ donc $\Delta'_1 = \Delta_1$. Mais $j_{\min}(\Delta_1) \leq j_{\min}(D)$ par hypothèse, donc $j_{\min}(\Delta_1)$ ne peut pas être égal à $j_{\max}(D)$. Donc $d'' = 2$ et forcément $\Delta'_2 = \Delta_2$. C'est-à-dire $j_{\max}(D) = j_{\min}(\Delta_2)$. Alors $\mathfrak{T}_1(D) = \{\Delta_1\}$, $\mathfrak{T}_2(D) = \{\Delta_2\}$. Pour $d = 1, 2$ et $\Delta'_d \in \mathrm{Int}(\lambda_d)$, on a

$$j_{\max}(\Delta'_d) \neq j_{\min}(D), \quad j_{\max}(\Delta'_d) \neq j_{\max}(D)$$

par comparaison des parités. D'après (5), cela entraîne $\Delta'_d \notin \mathfrak{D}_d(D)$. Donc $\mathfrak{D}_1(D) = \mathfrak{D}_2(D) = \varnothing$. Si $\Delta_2^+$ existe, on a

$$j_{\max}(\Delta_2^+) < j_{\min}(\Delta_2) = j_{\max}(D), \quad \text{d'où } \Delta_2^+ \in \mathfrak{D}_2(\geq D).$$

Enfin, l'égalité $j_{\max}(D) = j_{\min}(\Delta_2)$ et la relation $j_{\max}(D) \in J(\Delta_1)$ entraînent $j_{\max}(D) \in J^+$. Alors

$$\delta^+(D) = \tau_1(\Delta_1) + \tau_2(\Delta_2) + r'_1 + r'_2 + 1, \quad \tau^+(D) = \tau_1(\Delta_1) + \delta_2(\Delta_2^+) + r'_2.$$

Les propriétés requises sont vérifiées.

(d) Supposons que $|J(D)| \geq 2$, que $J(D) \subset J(\Delta_1)$, que $j_{\min}(D)$ soit pair et que $j_{\max}(D)$ soit impair. Comme en (c), on a $j_{\max}(D) = j_{\min}(\Delta_2)$. On a $j_{\min}(D) = j_{\max}(\Delta'_d)$ pour un $d = 1, 2$ et un $\Delta'_d \in \mathrm{Int}(\lambda_d)$. Si $d = 1$, on a $J(\Delta'_1) \cap J(\Delta_1) \neq \varnothing$ donc $\Delta'_1 = \Delta_1$. Mais c'est impossible puisque $j_{\max}(\Delta_1) \geq j_{\max}(D) > j_{\min}(D)$. Donc $d = 2$ et forcément $\Delta'_2 = \Delta_2^+$ (ce raisonnement montre que $\Delta_2^+$ existe). D'où $j_{\min}(D) = j_{\max}(\Delta_2^+)$. On voit que $\mathfrak{T}_2(D) = \{\Delta_2\}$ et $\mathfrak{D}_2(D) = \{\Delta_2^+\}$. Pour $\Delta'_1 \in \widetilde{\mathrm{Int}}(\lambda_1)$, on ne peut avoir $j_{\min}(\Delta'_1) \in J(D)$ ou $j_{\max}(\Delta'_1) \in J(D)$ que si $\Delta'_1 = \Delta_1$. On sait que $j_{\min}(\Delta_1) \leq j_{\min}(D)$ et $j_{\max}(D) \leq j_{\max}(\Delta_1)$. Par comparaison des parités, ces inégalités sont strictes. Donc $j_{\min}(\Delta_1)$ et $j_{\max}(\Delta_1)$ n'appartiennent pas à $J(D)$ et, grâce à (5), on conclut $\mathfrak{T}_1(D) = \mathfrak{D}_1(D) = \varnothing$. Enfin, l'inégalité $j_{\min}(\Delta_1) \leq j_{\max}(D)$ montre que $\Delta_1 \in \mathfrak{T}_1(\geq D)$. On a les mêmes formules que dans le cas (c) :

$$\delta^+(D) = \tau_1(\Delta_1) + \tau_2(\Delta_2) + r'_1 + r'_2 + 1, \quad \tau^+(D) = \tau_1(\Delta_1) + \delta_2(\Delta_2^+) + r'_2.$$

Les propriétés requises sont vérifiées.

(e) Supposons que $|J(D)| \geq 2$, que $J(D) \subset J(\Delta_1)$, que $j_{\min}(D)$ soit impair et que $j_{\max}(D)$ soit pair. Comme en (c), on a $j_{\min}(D) = j_{\min}(\Delta_1)$. Un raisonnement similaire à ceux ci-dessus montre que $j_{\max}(D) = j_{\max}(\Delta_1)$. Donc $\mathfrak{T}_1(D) = \mathfrak{D}_1(D) = \{\Delta_1\}$. Si $\Delta_2$, resp. $\Delta_2^+$, existe, on a forcément $j_{\max}(D) \leq j_{\min}(\Delta_2)$ et $j_{\max}(\Delta_2^+) \leq j_{\min}(D)$. Ces inégalités sont strictes par comparaison des parités. Cela entraîne qu'il n'existe pas de $\Delta'_2 \in \mathrm{Int}(\lambda_2)$ tel que $j_{\min}(\Delta'_2)$ ou $j_{\max}(\Delta'_2)$ appartiennent à $J(D)$. Donc $\mathfrak{T}_2(D) = \mathfrak{D}_2(D) = \varnothing$. Par contre, si $\Delta_2^+$ existe, on a $\Delta_2^+ \in \mathfrak{D}_2(\geq D)$. Puisque $j_{\max}(D)$ est pair, on a $j_{\max}(D) \notin J^+$. On a alors

$$\delta^+(D) = \delta_1(\Delta_1) + \delta_2(\Delta_2^+), \quad \tau^+(D) = \tau_1(\Delta_1) + \delta_2(\Delta_2^+) + r'_2.$$

Les propriétés requises sont vérifiées.

(f) Supposons que $|J(D)| \geq 2$, que $J(D) \subset J(\Delta_1)$ et que $j_{\min}(D)$ et $j_{\max}(D)$ soient pairs. En utilisant des résultats extraits de (d) et (e), on a $j_{\min}(D) = j_{\max}(\Delta_2^+)$ et $j_{\max}(D) = j_{\max}(\Delta_1)$. De plus, $j_{\min}(\Delta_1) < j_{\min}(D)$ et $j_{\max}(D) < j_{\min}(\Delta_2)$ si $\Delta_2$ existe. Donc $\mathfrak{T}_1(D) = \mathfrak{T}_2(D) = \varnothing$, $\mathfrak{D}_1(D) = \{\Delta_1\}$ et $\mathfrak{D}_2(D) = \{\Delta_2^+\}$. On a encore $j_{\max}(D) \notin J^+$. Puisque $j_{\min}(\Delta_1) \leq j_{\max}(D)$, on a $\Delta_1 \in \mathfrak{T}_1(\geq D)$. On a les mêmes relations que dans le cas (e) :

$$\delta^+(D) = \delta_1(\Delta_1) + \delta_2(\Delta_2^+), \quad \tau^+(D) = \tau_1(\Delta_1) + \delta_2(\Delta_2^+) + r'_2.$$

On a des cas (g), (h), (i), (j) qui sont les symétriques de (c), (d), (e), (f) : on remplace la condition $J(D) \subset J(\Delta_1)$ par $J(D) \subset J(\Delta_2)$. Les formules que l'on obtient sont les exactes symétriques de celles obtenues dans les cas traités.

Comme on l'a dit, les formules ci-dessus supposaient $D \neq D_{\min}$. Supposons maintenant $D = D_{\min}$. On a $D_1 = D_{1,\min}$, $J(D_{\min}) \subset J(\Delta_{1,\min})$ et $D_2$ n'existe pas.

(k) Supposons que $j_{\min}(D_{\min})$ soit impair. On a alors $j_{\min}(D_{\min}) = j_{\min}(\Delta_{1,\min})$ comme en (c). On en déduit $\mathfrak{T}_1(D_{\min}) = \{\Delta_{1,\min}\}$ mais $\mathfrak{D}_1(D_{\min}) = \varnothing$ (par définition, l'ensemble $\mathfrak{D}_1(D_{\min})$ est un sous-ensemble de $\mathrm{Int}(\lambda_1)$, lequel ne contient pas $\Delta_{1,\min}$). Si $\mathrm{Int}(\lambda_2) \neq \varnothing$, on a $j_{\max}(\Delta_2^+) > j_{\min}(D)$, donc

$$\mathfrak{T}_2(D_{\min}) = \mathfrak{D}_2(D_{\min}) = \varnothing.$$

Par contre, $\Delta_2^+$ appartient à $\mathfrak{D}_2(\geq D_{\min})$. L'unique formule est

$$\tau^+(D_{\min}) = \tau_1(\Delta_{1,\min}) + \delta_2(\Delta_2^+) + r_2'$$

et les propriétés requises sont vérifiées.

(l) Supposons que $j_{\min}(D_{\min})$ soit pair. Alors $j_{\min}(D_{\min}) = j_{\max}(\Delta_2^+)$ comme en (d). On voit que $\mathfrak{T}_1(D_{\min}) = \mathfrak{D}_1(D_{\min}) = \mathfrak{T}_2(D_{\min}) = \varnothing$ et $\mathfrak{D}_2(D_{\min}) = \{\Delta_2^+\}$. On a aussi $\Delta_{1,\min} \in \mathfrak{T}_1(\geq D_{\min})$. La formule est la même que ci-dessus :

$$\tau^+(D_{\min}) = \tau_1(\Delta_{1,\min}) + \delta_2(\Delta_2^+) + r_2'$$

et les propriétés requises sont vérifiées.

On peut alors prouver par récurrence descendante l'assertion suivante : pour $D \in \widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$, il existe pour $d = 1, 2$ d'uniques fonctions $\tau_d$, resp. $\delta_d$, définies sur $\mathfrak{T}_d(\geq D)$, resp. $\mathfrak{D}_d(\geq D)$, de sorte que les formules $(X_{r_1',r_2'})$ soient vérifiées pour tout $D' \geq D$. En effet, soit $D \in \mathrm{Int}_{\lambda_1,\lambda_2}(\lambda)$, supposons que l'assertion ci-dessus soit vérifiée pour $D^+$ (la condition est vide si $D$ est maximal). Les fonctions $\tau_d$ et $\delta_d$ sont donc uniquement définies sur $\mathfrak{T}_d(\geq D^+)$, resp. $\mathfrak{D}_d(\geq D^+)$. Il faut montrer que l'on peut définir d'une seule façon des termes $\tau_d(\Delta_d)$ pour $\Delta_d \in \mathfrak{T}_d(D)$ et $\delta_d(\Delta_d)$ pour $\Delta_d \in \mathfrak{D}_d(D)$ de sorte que les formules soient aussi vérifiées pour l'intervalle $D$. Par exemple, traitons le cas (a). Le terme $\delta_2(\Delta_2^+)$ est déjà défini. On doit définir $\tau_1(\Delta_1)$ et $\tau_2(\Delta_2)$ de sorte que

$$\delta^+(D) = \tau_1(\Delta_1) + \tau_2(\Delta_2) + r_1' + r_2' + 1, \quad \tau^+(D) = \tau_1(\Delta_1) + \delta_2(\Delta_2^+) + r_2'.$$

Il est clair que ces équations ont une solution et que celle-ci est unique. Les autres cas (b) à (l) sont similaires. L'assertion est donc démontrée par récurrence. Pour $D = D_{\min}$, on obtient l'assertion voulue : pour deux éléments donnés $r_1', r_2' \in \mathbb{Z}/2\mathbb{Z}$, il existe d'uniques

$$(\tau_1, \delta_1) \in (\mathbb{Z}/2\mathbb{Z})^{\widetilde{\mathrm{Int}}(\lambda_1)} \times (\mathbb{Z}/2\mathbb{Z})^{\mathrm{Int}(\lambda_1)}, \quad (\tau_2, \delta_2) \in (\mathbb{Z}/2\mathbb{Z})^{\mathrm{Int}(\lambda_2)} \times (\mathbb{Z}/2\mathbb{Z})^{\mathrm{Int}(\lambda_2)}$$

tels que soient vérifiées les formules $(X_{r_1',r_2'})$.

Ces paires $(\tau_1, \delta_1)$ et $(\tau_2, \delta_2)$ ne vérifient pas forcément les conditions imposées au début de la démonstration. Si $(\tau_2, \delta_2)$ est bien un élément de $\mathcal{F}am(\lambda_2)$,

$(\tau_1,\delta_1)$ n'est pas forcément un élément de $\mathcal{F}am(\lambda_1)$ : c'en est un si et seulement si $\tau_1(\Delta_{1,\min})=0$. D'autre part, en admettant que cette condition soit vérifiée, nos paires vérifient les conditions requises si et seulement si $r'_1\equiv r(\tau_1,\delta_1) \bmod 2\mathbb{Z}$ et $r'_2\equiv r(\tau_2,\delta_2) \bmod 2\mathbb{Z}$. Pour démontrer la première assertion du lemme, il suffit de prouver que ces conditions sont vérifiées pour un seul couple $(r'_1,r'_2)$.

Continuons avec un couple quelconque $(r'_1,r'_2)$ et les paires $(\tau_1,\delta_1)$ et $(\tau_2,\delta_2)$ que l'on a construites ci-dessus. Posons $a=\tau_1(\Delta_{1,\min})$. Définissons $\underline{\tau}_1$ par $\underline{\tau}_1(\Delta)=\tau_1(\Delta)+a$. Alors $(\underline{\tau}_1,\delta_1)$ appartient bien à $\mathcal{F}am(\lambda_1)$. On pose $\underline{r}_1=r(\underline{\tau}_1,\delta_1)$, $\underline{r}_2=r(\tau_2,\delta_2)$. Les conditions à vérifier sont

$$a=0,\quad \underline{r}_1\equiv r'_1 \bmod 2\mathbb{Z},\quad \underline{r}_2\equiv r'_2 \bmod 2\mathbb{Z}. \tag{6}$$

Remarquons que la première condition est redondante avec la troisième. En effet, comme on l'a vu dans la preuve de 3.1(2), on a par construction

$$\tau^+(D_{\min})=\tau_1(\Delta_{1,\min})+\delta_2(\Delta_2(D_{\min})^+)+r'_2.$$

On sait que $\delta_2(\Delta_2(D_{\min})^+)=\underline{r}_2$, cf. 2.4(2). On a aussi $\tau^+(D_{\min})=0$ par l'hypothèse (2), d'où $a+r'_2+\underline{r}_2\equiv 0 \bmod 2\mathbb{Z}$.

Construisons les fonctions associées à $\underline{\iota}_1=(\underline{\tau}_1,\delta_1)$ et $\underline{\iota}_2=(\tau_2,\delta_2)$, que l'on note $\underline{\tau}^\zeta=\tau^\zeta_{\underline{\iota}_1,\underline{\iota}_2}$ et $\underline{\delta}^\zeta=\delta^\zeta_{\underline{\iota}_1,\underline{\iota}_2}$. Cela revient, dans la construction des fonctions $\tau^\zeta$ et $\delta^\zeta$ par les formules $(X_{r'_1,r'_2})$, à changer $\tau_1$ en $\underline{\tau}_1$, $r'_1$ en $\underline{r}_1$ et $r'_2$ en $\underline{r}_2$. On remarque que les termes $\tau_1(\Delta_1)$ et $r'_2$ n'interviennent que par leur somme $\tau_1(\Delta_1)+r'_2$. Or, comme on vient de le voir, $\underline{\tau}_1(\Delta_1)+\underline{r}_2=\tau_1(\Delta_1)+a+\underline{r}_2=\tau_1(\Delta_1)+r'_2$. Changer $\tau_1$ en $\underline{\tau}_1$ et $r'_2$ en $\underline{r}_2$ ne change donc pas les fonctions $\tau^\zeta$ et $\delta^\zeta$. On remarque que $r'_1$ intervient exactement dans les expressions $\delta^\zeta(D)$ ou $\tau^\zeta(D)$ telles que $\delta^{-\zeta}(D)=\delta^\zeta(D)+1$ ou $\tau^{-\zeta}(D)=\tau^\zeta(D)+1$. Changer $r'_1$ en $\underline{r}_1$ change donc les fonctions $\tau^\zeta$ et $\delta^\zeta$ en multipliant éventuellement $\zeta$ par $-1$, en identifiant le signe $\zeta$ à un élément de $\{\pm 1\}$. Précisément, posons $u=(-1)^{r'_1+\underline{r}_1}$. On obtient les égalités

$$\underline{\tau}^\zeta=\tau^{u\zeta},\quad \underline{\delta}^\zeta=\delta^{u\zeta}.$$

En posant $\underline{C}^\zeta=C^\zeta_{\underline{\iota}_1,\underline{\iota}_2}$, ces égalités entraînent $\underline{C}^\zeta=C^{u\zeta}$. D'après 3.3(3), on a les égalités

$$\underline{C}^\zeta=\begin{cases} 2(\underline{r}_1+\zeta\underline{r}_2) & \text{si } \underline{r}_1+\underline{r}_2 \text{ est pair,}\\ -2(\underline{r}_1+\zeta\underline{r}_2+1) & \text{si } \underline{r}_1+\underline{r}_2 \text{ est impair.}\end{cases}$$

Par l'hypothèse (3), on a aussi

$$C^{u\zeta}=\begin{cases} 2(r_1+u\zeta r_2) & \text{si } r_1+r_2 \text{ est pair,}\\ -2(r_1+u\zeta r_2+1), & \text{si } r_1+r_2 \text{ est impair.}\end{cases}$$

L'égalité de ces deux expressions est équivalente aux égalités suivantes :

Si $\underline{r}_1+\underline{r}_2$ est pair et

$r_1+r_2$ est pair, $\quad \underline{r}_1+\zeta\underline{r}_2=r_1+u\zeta r_2 \quad$ pour $\zeta=\pm 1$;

$r_1+r_2$ est impair, $\quad \underline{r}_1+\zeta\underline{r}_2=-(r_1+u\zeta r_2+1) \quad$ pour $\zeta=\pm 1$.

Si $\underline{r}_1+\underline{r}_2$ est impair et

$r_1+r_2$ est pair, $\quad -(\underline{r}_1+\zeta\underline{r}_2+1)=r_1+u\zeta r_2 \quad$ pour $\zeta=\pm 1$;

$r_1+r_2$ est impair, $\quad -(\underline{r}_1+\zeta\underline{r}_2+1)=-(r_1+u\zeta r_2+1) \quad$ pour $\zeta=\pm 1$.

En sommant en $\zeta=\pm 1$, le deuxième cas entraîne $\underline{r}_1=-(r_1+1)$. C'est impossible puisque $\underline{r}_1$ et $r_1$ sont positifs ou nuls. Ce cas ne se produit donc pas. Le troisième cas non plus, pour la même raison. Cela montre que $\underline{r}_1+\underline{r}_2$ et $r_1+r_2$ sont de la même parité. Dans ce cas, les égalités ci-dessus entraînent $\underline{r}_1=r_1$ et $\underline{r}_2=ur_2$. Alors les conditions (6) sont vérifiées si et seulement si $r'_1\equiv r_1 \mod 2\mathbb{Z}$ et $r'_2\equiv r_2 \mod 2\mathbb{Z}$. Cela démontre la première assertion du lemme. Pour ce couple $(r'_1,r'_2)$ ainsi déterminé, on vient de voir que $\underline{r}_1=r_1$. On a aussi $u=(-1)^{r'_1+\underline{r}_1}=1$, donc $\underline{r}_2=ur_2=r_2$. Cela démontre la seconde assertion de l'énoncé. □

## 4. Le front d'onde de $\pi(\lambda^+,\epsilon^+,\lambda^-,\epsilon^-)$

**4.1.** ***Le résultat de*** [Waldspurger 2017]. Soit $m\in\mathbb{N}$ et $(\lambda,\epsilon)\in\mathcal{P}^{\mathrm{symp}}(2m)$. On a introduit en 1.3 la représentation $\boldsymbol{\rho}_{\lambda,\epsilon}$ de $W_{N_{\lambda,\epsilon}}$. On sait qu'elle se décompose en

$$\boldsymbol{\rho}_{\lambda,\epsilon}=\bigoplus_{(\lambda',\epsilon')}\mathrm{mult}(\lambda,\epsilon;\lambda',\epsilon')\rho_{\lambda',\epsilon'},$$

où $(\lambda',\epsilon')$ parcourt les éléments de $\mathcal{P}^{\mathrm{symp}}(2m)$ tels que les $\mathrm{mult}(\lambda,\epsilon;\lambda',\epsilon')$ sont des entiers positifs ou nuls et $k_{\lambda',\epsilon'}=k_{\lambda,\epsilon}$. Le couple $(\lambda,\epsilon)$ est minimal dans cette décomposition, c'est-à-dire que l'on a

$$\text{si } \mathrm{mult}(\lambda,\epsilon;\lambda',\epsilon')\neq 0, \text{ alors } \lambda'>\lambda \quad \text{ou } (\lambda',\epsilon')=(\lambda,\epsilon).$$

De plus $\mathrm{mult}(\lambda,\epsilon;\lambda,\epsilon)=1$.

Pour tout couple $(\mu,\nu)\in\mathcal{P}^{\mathrm{symp}}(2m)$, notons $({}^s\mu,{}^s\nu)$ le couple tel que $k_{{}^s\mu,{}^s\nu}=k_{\mu,\nu}$ et $\rho_{{}^s\mu,{}^s\nu}=\rho_{\mu,\nu}\otimes\mathrm{sgn}$.

**Proposition.** *Supposons que tous les termes de $\lambda$ soient pairs. Alors il existe un unique couple $(\lambda^{\min},\epsilon^{\min})\in\mathcal{P}^{\mathrm{symp}}(2m)$ vérifiant les propriétés suivantes*:

(1) $\mathrm{mult}(\lambda,\epsilon;{}^s\lambda^{\min},{}^s\epsilon^{\min})=1$ ;

(2) *pour tout élément $(\lambda',\epsilon')\in\mathcal{P}^{\mathrm{symp}}(2m)$ tel que* $\mathrm{mult}(\lambda,\epsilon;{}^s\lambda',{}^s\epsilon')\neq 0$, *on a* $\lambda^{\min}<\lambda'$ *ou* $(\lambda',\epsilon')=(\lambda^{\min},\epsilon^{\min})$.

Cf. [Waldspurger 2017, théorème 4.7].

**4.2.** ***Calcul de $M_\pi(\mu_1, \eta_1; \mu_2, \eta_2)$.*** On fixe désormais un quadruplet

$$(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-) \in \mathfrak{Irr}^{\mathrm{bp}}_{\mathrm{quad}}(2n).$$

Rappelons que l'exposant bp signifie que tous les termes de $\lambda^+$ et $\lambda^-$ sont pairs. On pose

$$\pi = \pi(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$$

et on note $\sharp$ l'indice iso ou an tel que $\pi \in \mathrm{Irr}_{\mathrm{tunip},\sharp}$.

Soient $n_1, n_2 \in \mathbb{N}$ tels que $n_1 + n_2 = n$. Soient $(\mu_1, \eta_1) \in \boldsymbol{\mathcal{P}}^{\mathrm{orth}}(2n_1+1)_{k=1}$ et $(\mu_2, \eta_2) \in \boldsymbol{\mathcal{P}}^{\mathrm{orth}}(2n_2)_{k=0}$. On a défini le nombre $M_\pi(\mu_1, \eta_1; \mu_2, \eta_2)$ en 1.4. On se propose de le calculer.

Le couple $(0, \rho_{\mu_1,\eta_1})$ appartient à $\Sigma_{n_1,\mathrm{imp}}$ et son symbole à $\mathrm{Fam}(\mathrm{sp}(\mu_1, \eta_1))$ pour une partition spéciale $\mathrm{sp}(\mu_1, \eta_1) \in \mathcal{P}^{\mathrm{orth,sp}}(2n_1+1)$. Posons $\lambda_1 = d(\mathrm{sp}(\mu_1, \eta_1))$. On a $\lambda_1 \in \mathcal{P}^{\mathrm{symp,sp}}(2n_1)$. Il résulte de 2.6 que le symbole $\Lambda_1$ de $(0, \rho_{\mu_1,\eta_1} \otimes \mathrm{sgn})$ appartient à $\mathrm{Fam}(\lambda_1)$.

Pour $\xi = \pm$, le couple $(0, \rho^{\xi}_{\mu_2,\eta_2})$ appartient à $\Sigma_{n_2,\mathrm{pair}}$ et son symbole appartient à $\mathrm{Fam}(\mathrm{sp}(\mu_2, \eta_2))$ pour une partition spéciale $\mathrm{sp}(\mu_2, \eta_2) \in \mathcal{P}^{\mathrm{orth,sp}}(2n_2)$. Celle-ci ne dépend pas du signe $\xi$ : changer de signe revient à échanger les deux termes $X$ et $Y$ du symbole. Posons $\lambda_2 = d(\mathrm{sp}(\mu_2, \eta_2))$. On a $\lambda_2 \in \mathcal{P}^{\mathrm{orth,sp}}(2n_2)$. Le symbole $\Lambda_2^{\xi}$ de $(0, \rho^{\xi}_{\mu_2,\eta_2} \otimes \mathrm{sgn})$ appartient à $\mathrm{Fam}(\lambda_2)$.

Signalons que l'on a les inégalités

$$\mu_1 \le \mathrm{sp}(\mu_1, \eta_1), \quad \mu_2 \le \mathrm{sp}(\mu_2, \eta_2), \tag{1}$$

cf. [Waldspurger 2018b, lemmes 1.4 et 1.5].

Posons $\gamma_0 = (0, 0, n_1, n_2)$. Par définition de la multiplicité

$$m_\pi\big(\rho_{\mu_1,\eta_1} \otimes \mathrm{sgn}, \rho^{\xi}_{\mu_2,\eta_2} \otimes \mathrm{sgn}\big)$$

et d'après 1.5(4), cette multiplicité est celle de $(\rho_{\mu_1,\eta_1} \otimes \mathrm{sgn}) \otimes (\rho^{\xi}_{\mu_2,\eta_2} \otimes \mathrm{sgn})$ dans la composante dans $\mathcal{R}(\gamma_0)$ de

$$\kappa_\pi = \mathcal{F}(\Pi),$$

où on a posé

$$\Pi = \rho\iota\big((\boldsymbol{\rho}_{\lambda^+,\epsilon^+} \otimes \mathrm{sgn}) \otimes (\boldsymbol{\rho}_{\lambda^+,\epsilon^+} \otimes \mathrm{sgn})\big).$$

En 1.3, on a associé à $(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$ un élément $\gamma = (r', r'', N^+, N^-) \in \Gamma$ et identifié $(\boldsymbol{\rho}_{\lambda^+,\epsilon^+} \otimes \mathrm{sgn}) \otimes (\boldsymbol{\rho}_{\lambda^+,\epsilon^+} \otimes \mathrm{sgn})$ à un élément de $\mathcal{R}(\gamma)$. On pose $r_1 = r'$, $r_2 = (-1)^{r'} r''$. Par construction de $\rho\iota$, l'élément $\Pi$ n'a de composante non nulle que dans les composantes $\mathcal{R}(\gamma')$ pour $\gamma'$ de la forme $(r_1, r_2, N_1, N_2)$. Par définition de $\mathcal{F}$, pour un tel $\gamma'$ et pour $\varphi \in \mathcal{R}(\gamma')$, l'élément $\mathcal{F}(\varphi)$ n'a de composante non nulle dans $\mathcal{R}(\gamma_0)$ que si $N_1 + r_1^2 + r_1 = n_1$ et $N_2 + r_2^2 = n_2$. Cela entraîne

$$\text{si } n_1 < r_1^2 + r_1 \text{ ou } n_2 < r_2^2, \quad \text{on a } M_\pi(\mu_1, \eta_1; \mu_2, \eta_2) = 0. \tag{2}$$

Supposons

$$n_1 \geq r_1^2 + r_1 \quad \text{et} \quad n_2 \geq r_2^2. \tag{3}$$

Posons $N_1 = n_1 - r_1^2 - r_1$, $N_2 = n_2 - r_2^2$ et $\underline{\gamma} = (r_1, r_2, N_1, N_2)$. On peut se limiter à considérer la composante $\Pi_{\underline{\gamma}}$ de $\Pi$ dans $\mathcal{R}(\underline{\gamma})$. Plus précisément, pour $d = 1, 2$, notons $\mathcal{F}am_{r_d}(\lambda_d)$ l'ensemble les $\iota_d = (\tau_d, \delta_d) \in \mathcal{F}am(\lambda_d)$ tels que $r(\tau_d, \delta_d) = r_d$. Pour de tels éléments, notons $(r_d, \rho_{\iota_d})$ l'élément de $\Sigma_{n_1,\mathrm{imp}}$ si $d = 1$ et $\Sigma_{n_2,\mathrm{pair}}$ si $d = 2$ associé à $\iota_d$. Posons $\Lambda_{\iota_d} = \mathrm{symb}(r_d, \rho_{\iota_d})$. Notons $m(\Pi_{\underline{\gamma}}, \rho_{\iota_1} \otimes \rho_{\iota_2})$ la multiplicité de $\rho_{\iota_1} \otimes \rho_{\iota_2}$ dans $\Pi_{\underline{\gamma}}$. Alors, par définition de $\mathcal{F}$, on a l'égalité

$$\begin{aligned} & m_\pi\big(\rho_{\mu_1,\eta_1} \otimes \mathrm{sgn}, \rho^{\xi}_{\mu_2,\delta_2} \otimes \mathrm{sgn}\big) \\ & = |\mathrm{Fam}(\lambda_1)|^{-\frac{1}{2}} |\mathrm{Fam}(\lambda_2)|^{-\frac{1}{2}} \sum_{\substack{\iota_1 \in \mathcal{F}am_{r_1}(\lambda_1) \\ \iota_2 \in \mathcal{F}am_{r_2}(\lambda_2)}} (-1)^{\langle \Lambda_1, \Lambda_{\iota_1} \rangle + \langle \Lambda_2^{\xi}, \Lambda_{\iota_2} \rangle} m(\Pi_{\underline{\gamma}}, \rho_{\iota_1} \otimes \rho_{\iota_2}). \end{aligned} \tag{4}$$

Pour $\zeta = \pm$, on pose $n^\zeta = S(\lambda^\zeta)/2$, $k^\zeta = k_{\lambda^\zeta, \epsilon^\zeta}$. Notons $\boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2n^\zeta)_{k^\zeta}$ l'ensemble des $(\lambda', \epsilon') \in \boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2n^\zeta)$ tels que $k_{\lambda',\epsilon'} = k^\zeta$. On peut écrire

$$\begin{aligned} & (\boldsymbol{\rho}_{\lambda^+,\epsilon^+} \otimes \mathrm{sgn}) \otimes (\boldsymbol{\rho}_{\lambda^-,\epsilon^-} \otimes \mathrm{sgn}) \\ & \qquad = \sum_{\substack{(\lambda'^+,\epsilon'^+) \in \boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2n^+)_{k^+} \\ (\lambda'^-,\epsilon'^-) \in \boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2n^-)_{k^-}}} x(\lambda'^+, \epsilon'^+, \lambda'^-, \epsilon'^-) \rho_{\lambda'^+,\epsilon'^+} \otimes \rho_{\lambda'^-,\epsilon'^-}, \end{aligned}$$

où les $x(\lambda'^+, \epsilon'^+, \lambda'^-, \epsilon'^-)$ sont des multiplicités. Précisément, avec les notations de 4.1, on a

$$x(\lambda'^+, \epsilon'^+, \lambda'^-, \epsilon'^-) = \mathrm{mult}(\lambda^+, \epsilon^+; {}^s\lambda'^+, {}^s\epsilon'^+) \mathrm{mult}(\lambda^-, \epsilon^-; {}^s\lambda'^-, {}^s\epsilon'^-). \tag{5}$$

Pour

$$(\lambda'^+, \epsilon'^+) \in \boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2n^+)_{k^+} \quad \text{et} \quad (\lambda'^-, \epsilon'^-) \in \boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2n^-)_{k^-},$$

notons $\Pi_{\underline{\gamma}}(\lambda'^+, \epsilon'^+, \lambda'^-, \epsilon'^-)$ la composante dans $\mathcal{R}(\underline{\gamma})$ de

$$\rho\iota(\rho_{\lambda'^+,\epsilon'^+} \otimes \rho_{\lambda'^-,\epsilon'^-}).$$

Pour $\iota_1 \in \mathcal{F}am_{r_1}(\lambda_1)$ et $\iota_2 \in \mathcal{F}am_{r_2}(\lambda_2)$, notons $m(\Pi_{\underline{\gamma}}(\lambda'^+, \epsilon'^+, \lambda'^-, \epsilon'^-), \rho_{\iota_1} \otimes \rho_{\iota_2})$ la multiplicité de $\rho_{\iota_1} \otimes \rho_{\iota_2}$ dans $\Pi_{\underline{\gamma}}(\lambda'^+, \epsilon'^+, \lambda'^-, \epsilon'^-)$. On a

$$\begin{aligned} & m(\Pi_{\underline{\gamma}}, \rho_{\iota_1} \otimes \rho_{\iota_2}) \\ & = \sum_{\substack{(\lambda'^+,\epsilon'^+) \in \boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2n^+)_{k^+} \\ (\lambda'^-,\epsilon'^-) \in \boldsymbol{\mathcal{P}}^{\mathrm{symp}}(2n^-)_{k^-}}} x(\lambda'^+, \epsilon'^+, \lambda'^-, \epsilon'^-) m\big(\Pi_{\underline{\gamma}}(\lambda'^+, \epsilon'^+, \lambda'^-, \epsilon'^-), \rho_{\iota_1} \otimes \rho_{\iota_2}\big). \end{aligned}$$

En vertu de la définition posée en 1.4, on déduit de (4) la formule finale

$$M_\pi(\mu_1,\eta_1;\mu_2,\eta_2) = |\mathrm{Fam}(\lambda_1)|^{-\frac{1}{2}}|\mathrm{Fam}(\lambda_2)|^{-\frac{1}{2}} \times \sum_{\substack{\iota_1\in\mathcal{F}am_{r_1}(\lambda_1)\\ \iota_2\in\mathcal{F}am_{r_2}(\lambda_2)}} (-1)^{\langle\Lambda_1,\Lambda_{\iota_1}\rangle}\big((-1)^{\langle\Lambda_2^+,\Lambda_{\iota_2}\rangle}+\mathrm{sgn}_\sharp(-1)^{\langle\Lambda_2^-,\Lambda_{\iota_2}\rangle}\big)$$
$$\times \sum_{\substack{(\lambda'^+,\epsilon'^+)\in\mathcal{P}^{\mathrm{symp}}(2n^+)_{k^+}\\ (\lambda'^-,\epsilon'^-)\in\mathcal{P}^{\mathrm{symp}}(2n^-)_{k^-}}} x(\lambda'^+,\epsilon'^+,\lambda'^-,\epsilon'^-)m\big(\Pi_{\underline{\gamma}}(\lambda'^+,\epsilon'^+,\lambda'^-,\epsilon'^-),\rho_{\iota_1}\otimes\rho_{\iota_2}\big). \quad (6)$$

**4.3.** ***Comparaison entre deux constructions.*** On conserve les notations du paragraphe précédent et on impose l'hypothèse (3) de ce paragraphe. Considérons des éléments $\iota_1\in\mathcal{F}am_{r_1}(\lambda_1)$, $\iota_2\in\mathcal{F}am_{r_2}(\lambda_2)$, $(\lambda'^+,\epsilon'^+)\in\mathcal{P}^{\mathrm{symp}}(2n^+)_{k^+}$, $(\lambda'^-,\epsilon'^-)\in\mathcal{P}^{\mathrm{symp}}(2n^-)_{k^-}$. On a défini la multiplicité

$$m\big(\Pi_{\underline{\gamma}}(\lambda'^+,\epsilon'^+,\lambda'^-,\epsilon'^-),\rho_{\iota_1}\otimes\rho_{\iota_2}\big).$$

Un élément $\zeta\in\{\pm1\}$ étant fixé, on a associé en 3.4 à $(r_1,r_2)$ un couple $(h^+,h^-)$. En se reportant à la définition de 1.2 et en se rappelant que $(r_1,r_2)=(r',(-1)^{r'}r'')$, on vérifie cas par cas qu'il est égal à $(k^+,k^-)$ pourvu que $\zeta=1$ si $k^+>k^-$, $\zeta=-1$ si $k^+<k^-$. Notons que $k^+>k^-$ équivaut à $(-1)^{r_1}r_2>0$ et $k^+<k^-$ équivaut à $(-1)^{r_1}r_2<0$. Si $k^+=k^-$, ce qui équivaut à $r_2=0$, $\zeta$ est indifférent, le couple $(h^+,h^-)$ ne dépendant pas de $\zeta$ et étant égal à $(k^+,k^-)$. On suppose que $\zeta$ vérifie ces conditions. On peut donc appliquer la construction de 3.4 aux entiers $n^+$ et $n^-$. On en déduit une représentation $\Pi^\zeta(\iota_1,\iota_2)$ de $W_{N^+}\otimes W_{N^-}$. On note $m\big(\Pi^\zeta(\iota_1,\iota_2),\rho_{\lambda'^+,\epsilon'^+}\otimes\rho_{\lambda'^-,\epsilon'^-}\big)$ la multiplicité de $\rho_{\lambda'^+,\epsilon'^+}\otimes\rho_{\lambda'^-,\epsilon'^-}$ dans $\Pi^\zeta(\iota_1,\iota_2)$. Un jeu habituel avec les restrictions et inductions montre que cette multiplicité est égale à celle de $\rho_1\otimes\rho_2$ dans la représentation

$$\sum_{N\in\mathcal{N}}\mathrm{ind}_{W_N}^{W_{N_1}\times W_{N_2}}\big(\mathrm{sgn}_{\mathrm{CD}}^{\boldsymbol{a}}\otimes\mathrm{res}_{W_N}^{W_{N^+}\times W_{N^-}}(\rho_{\lambda'^+,\epsilon'^+}\otimes\rho_{\lambda'^-,\epsilon'^-})\big),$$

où $\boldsymbol{a}$ est défini comme en 3.4. Un calcul cas par cas montre que ce $\boldsymbol{a}$ est le même qu'en 1.2, pourvu que, dans le cas $r_2=0$, on choisisse $\zeta=1$ si $r_1$ est pair, $\zeta=-1$ si $r_1$ est impair. Le signe $\zeta$ étant ainsi déterminé en tout cas, la représentation ci-dessus n'est autre que la composante dans $\mathcal{R}(\underline{\gamma})$ de

$$\rho\iota(\rho_{\lambda'^+,\epsilon'^+}\otimes\rho_{\lambda'^-,\epsilon'^-}).$$

On conclut

$$m\big(\Pi_{\underline{\gamma}}(\lambda'^+,\epsilon'^+,\lambda'^-,\epsilon'^-),\rho_1\otimes\rho_2\big)=m\big(\Pi^\zeta(\iota_1,\iota_2),\rho_{\lambda'^+,\epsilon'^+}\otimes\rho_{\lambda'^-,\epsilon'^-}\big). \quad (1)$$

Dans les formules 3.4(4) et 3.4(5) intervient le signe $\zeta\nu$, ou $\nu = 1$ si $r_2 \geq 0$, $\nu = -1$ si $r_2 < 0$. Avec la définition de $\zeta$ ci-dessus, on a

$$\zeta\nu = (-1)^{r_1}. \tag{2}$$

**4.4.** ***Démonstration du (i) de la proposition 1.4.*** On considère les données de 4.2 et on suppose $M_\pi(\mu_1, \eta_1; \mu_2, \eta_2) \neq 0$. La relation 4.2(2) entraîne que l'hypothèse 4.2(3) est vérifiée. D'après 4.2(6), on peut fixer des éléments $\iota_1 \in \mathcal{F}am_{r_1}(\lambda_1)$, $\iota_2 \in \mathcal{F}am_{r_2}(\lambda_2)$, $(\lambda'^+, \epsilon'^+) \in \mathcal{P}^{\mathrm{symp}}(2n^+)_{k^+}$, $(\lambda'^-, \epsilon'^-) \in \mathcal{P}^{\mathrm{symp}}(2n^-)_{k^-}$ vérifiant les conditions

$$x(\lambda'^+, \epsilon'^+, \lambda'^-, \epsilon'^-) \neq 0; \tag{1}$$

$$m\big(\Pi_{\underline{\gamma}}(\lambda'^+, \epsilon'^+, \lambda'^-, \epsilon'^-), \rho_{\iota_1} \otimes \rho_{\iota_2}\big) \neq 0. \tag{2}$$

En vertu de la définition 4.2(5) de $x(\lambda'^+, \epsilon'^+, \lambda'^-, \epsilon'^-)$ et de la proposition 4.1, la relation (1) entraîne

$$\lambda^{+,\min} \leq \lambda'^+, \quad \lambda^{-,\min} \leq \lambda'^-. \tag{3}$$

Notons $\lambda$ l'induite endoscopique de $\lambda_1$ et $\lambda_2$. En vertu de 4.3(1) et de la proposition 3.4(i), la relation (2) entraîne

$$\lambda'^+ \cup \lambda'^- \leq \lambda. \tag{4}$$

De ces deux inégalités, on déduit

$$\lambda^{+,\min} \cup \lambda^{-,\min} \leq \lambda.$$

Posons

$$\mu = d(\lambda^{+,\min} \cup \lambda^{-,\min}).$$

La dualité est une application décroissante. L'inégalité précédente entraîne $d(\lambda) \leq \mu$. D'après [Waldspurger 2018b, proposition 1.9], on a aussi $d(\lambda_1) \cup d(\lambda_2) \leq d(\lambda)$, d'où $d(\lambda_1) \cup d(\lambda_2) \leq \mu$. Par construction, $d(\lambda_1) = \mathrm{sp}(\mu_1, \eta_1)$, $d(\lambda_2) = \mathrm{sp}(\mu_2, \eta_2)$. D'où $\mathrm{sp}(\mu_1, \eta_1) \cup \mathrm{sp}(\mu_2, \eta_2) \leq \mu$. En appliquant 4.2(1), on obtient

$$\mu_1 \cup \mu_2 \leq \mu.$$

C'est l'assertion (i) de la proposition 1.4.

**4.5.** ***Démonstration du (ii) de la proposition 1.4.*** La seule donnée est ici le quadruplet $(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-) \in \mathfrak{Irr}^{\mathrm{bp}}_{\mathrm{quad}}(2n)$. On pose $\lambda = \lambda^{+,\min} \cup \lambda^{-,\min}$. Fixons une fonction $\chi$ : $\mathrm{Jord}^{\mathrm{bp}}(\lambda)\cup\{0\}\to\mathbb{Z}/2\mathbb{Z}$ vérifiant les conditions suivantes :

$\chi(i)=0$ pour tout $i\in\mathrm{Jord}^{\mathrm{bp}}(\lambda)$ tel que $\mathrm{mult}_\lambda(i)=1$;

$\chi(i)=0$ pour tout $i\in\mathrm{Jord}^{\mathrm{bp}}(\lambda)$ tel que $\mathrm{mult}_{\lambda^{+,\min}}(i)\geq 1$ et $\mathrm{mult}_{\lambda^{-,\min}}(i)\geq 1$ (ce qui implique $\mathrm{mult}_\lambda(i)\geq 2$);

$\chi(i)=0$ si $\epsilon_i^{+,\min}\neq\epsilon_i^{-,\min}$;

$\chi(i)=1$ si $\epsilon_i^{+,\min}=\epsilon_i^{-,\min}$;

$\chi(i)=1$ pour tout $i\in\mathrm{Jord}^{\mathrm{bp}}(\lambda)$ tel que $\mathrm{mult}_\lambda(i)\geq 2$ et que $\mathrm{mult}_{\lambda^{+,\min}}(i)=0$ ou $\mathrm{mult}_{\lambda^{-,\min}}(i)=0$.

On choisit $n_1,n_2$ et $\lambda_1,\lambda_2$ vérifiant les conditions de la proposition 3.2, pour ce choix de la fonction $\chi$. C'est-à-dire que $\lambda_1\in\mathcal{P}^{\mathrm{symp,sp}}(2n_1)$, $\lambda_2\in\mathcal{P}^{\mathrm{orth,sp}}(2n_2)$, $\lambda_1$ et $\lambda_2$ induisent régulièrement $\lambda$, $d(\lambda_1)\cup d(\lambda_2)=d(\lambda)=\mu$ et $\chi_{\lambda_1,\lambda_2}=\chi$. On pose $\mu_1=d(\lambda_1)$, $\mu_2=d(\lambda_2)$. On a $\mu_1\in\mathcal{P}^{\mathrm{orth,sp}}(2n_1+1)$, $\mu_2\in\mathcal{P}^{\mathrm{orth,sp}}(2n_2)$, et

$$\mu_1\cup\mu_2=\mu. \tag{1}$$

On définit $r_1$ et $r_2$ comme en 4.2 : $r_1=r'$, $r_2=(-1)^{r'}r''$, où $r'$ et $r''$ sont définis en 1.3. Pour une partition $\nu$ et pour $i\in\mathbb{N}-\{0\}$, posons $\mathrm{mult}_\nu(\geq i)=\sum_{i'\geq i}\mathrm{mult}_\nu(i')$. Posons $\eta=(-1)^{r'}$. Pour $\zeta=\pm$, on définit une fonction $\delta^\zeta$ : $\mathrm{Jord}^{\mathrm{bp}}(\lambda)\to\mathbb{Z}/2\mathbb{Z}$ par

$$\delta^\zeta(i)\equiv\mathrm{mult}_{\lambda^{\zeta\eta,\min}}(\geq i) \mod 2\mathbb{Z}.$$

On définit une fonction $\tau^\zeta$ : $\mathrm{Jord}^{\mathrm{bp}}(\lambda)\cup\{0\}\to\mathbb{Z}/2\mathbb{Z}$ par

si $i\neq 0$ et $\mathrm{mult}_{\lambda^{\zeta\eta,\min}}(i)>0$, $\quad\epsilon_i^{\zeta\eta,\min}=(-1)^{\tau^\zeta(i)}$;

si $i\neq 0$ et $\mathrm{mult}_{\lambda^{\zeta\eta,\min}}(i)=0$ (auquel cas $\mathrm{mult}_{\lambda^{-\zeta\eta,\min}}(i)>0$), $\epsilon_i^{-\zeta\eta,\min}=(-1)^{\tau^\zeta(i)}$;

$\tau^\zeta(0)=0$.

On peut considérer que ces fonctions sont définies sur $\mathrm{Int}_{\lambda_1,\lambda_2}(\lambda)$, resp. $\widetilde{\mathrm{Int}}_{\lambda_1,\lambda_2}(\lambda)$, puisque $\lambda_1$ et $\lambda_2$ induisent régulièrement $\lambda$. Montrons que

$$\text{ces fonctions vérifient les conditions de 3.5.} \tag{2}$$

*Preuve.* Soit $i\in\mathrm{Jord}^{\mathrm{bp}}(\lambda)$. D'après la définition ci-dessus, $\delta^-(i)=\delta^+(i)+1$ si et seulement si $\mathrm{mult}_\lambda(\geq i)$ est impair. Remarquons que l'on a l'égalité

$$\mathrm{mult}_\lambda(\geq i)=j_{\max}(i).$$

Si $j_{\max}(i)\in J^+$, $j_{\max}(i)$ est impair. Inversement, supposons $j_{\max}(i)$ impair. Si $\mathrm{mult}_\lambda(i)=1$, $j_{\max}(i)$ appartient à l'ensemble $\mathcal{J}^+\cup\mathcal{J}^-$ de 3.1 par définition des intervalles relatifs. L'imparité impose $j_{\max}(i)\in\mathcal{J}^+$. Or $\mathcal{J}^+\subset J^+$ par définition, donc $j_{\max}(i)\in J^+$. Supposons $\mathrm{mult}_\lambda(i)\geq 2$. Par définition des intervalles relatifs, il existe $d=1,2$ et $\Delta_d \in \widetilde{\mathrm{Int}}(\lambda_d)$ de sorte que $J(i)=\{j_{\min}(i),\dots,j_{\max}(i)\}\subset J(\Delta_d)$. Pour fixer la notation, supposons que $d=1$, donc $J(i)\subset J(\Delta_1)$. Par définition des intervalles relatifs, $j_{\max}(i)$ appartient à l'ensemble $\mathcal{J}$ de 3.1. L'imparité impose alors qu'il existe $d=1,2$ et $\Delta'_d \in \widetilde{\mathrm{Int}}(\lambda_d)$ de sorte que $j_{\max}(i)=j_{\min}(\Delta'_d)$. Si $d=1$, on a $j_{\max}(i)\in J(\Delta_1)\cap J(\Delta'_1)$ donc $\Delta'_1=\Delta_1$. Mais $j_{\min}(\Delta_1)\le j_{\min}(i)<j_{\max}(i)$, ce qui est contradictoire. Donc $d=2$. Alors $j_{\max}(i)=j_{\min}(\Delta'_2)$. Puisque $j_{\max}(i)\in J(\Delta_1)$, $\lambda_{1,j}$ est pair. Alors, par définition de $J^+$, on a $j_{\max}(i)\in J^+$. Cela prouve que les fonctions $\delta^\zeta$ vérifient la première condition de la relation 3.5(1).

Soit $i\in \mathrm{Jord}^{\mathrm{bp}}(\lambda)$. D'après la définition ci-dessus, $\tau^-(i)=\tau^+(i)+1$ si et seulement si $\mathrm{mult}_{\lambda^{+,\min}}(i)>0$, $\mathrm{mult}_{\lambda^{-,\min}}(i)>0$ et $\epsilon_i^{+,\min}\neq\epsilon_i^{-,\min}$. D'après la définition de $\chi$, ces conditions sont équivalentes à $\mathrm{mult}_\lambda(i)\ge 2$ et $\chi(i)=0$. La première condition équivaut à $|J(i)|\ge 2$. Sous cette condition, puisque $\chi=\chi_{\lambda_1,\lambda_2}$, la seconde condition équivaut à $J(i)\subset J(\Delta_2(i))$ avec la notation de 3.5(1). Cela achève de prouver cette condition 3.5(1).

La condition 3.5(2) est claire.

Notons $i_1>\dots>i_t$ les entiers pairs $i\ge 2$ tels que $\mathrm{mult}_{\lambda^{+,\min}}(i)$ soit impair. Pour $i\in\mathrm{Jord}^{\mathrm{bp}}(\lambda)$, on a $(-1)^{\delta^\eta(i)}-(-1)^{\delta^\eta(i^+)}\neq 0$ si et seulement si $\delta^\eta(i)\neq\delta^\eta(i^+)$. Par définition de $\delta^\eta$, cela équivaut à ce que $\mathrm{mult}_{\lambda^{+,\min}}(i)$ soit impair, autrement dit à ce que $i=i_h$ pour un $h=1,\dots,t$. Pour un tel $i_h$, on a

$$(-1)^{\delta^\eta(i_h)}-(-1)^{\delta^\eta(i_h^+)}=2(-1)^{\delta^\eta(i_h)}=2(-1)^{\mathrm{mult}_{\lambda^{+,\min}}(\ge i)}=2(-1)^h.$$

On a aussi

$$1-(-1)^{\tau^\eta(i_h)}=1-\epsilon_{i_h}^{+,\min}=\begin{cases}0 & \text{si } \epsilon_{i_h}^{+,\min}=1,\\ 2 & \text{si } \epsilon_{i_h}^{+,\min}=-1.\end{cases}$$

On en déduit

$$C^\eta=4\left|\{h=1,\dots,t;\ h \text{ pair et } \epsilon_{i_h}^{+,\min}=-1\}\right| \\ -4\left|\{h=1,\dots,t;\ h \text{ impair et } \epsilon_{i_h}^{+,\min}=-1\}\right|.$$

En utilisant 1.3(1), on obtient

$$C^\eta=\begin{cases}2k^+ & \text{si } k^+ \text{ est pair},\\ -2(k^++1) & \text{si } k^+ \text{ est impair}.\end{cases}\tag{3}$$

On a une formule analogue pour $C^{-\eta}$, où $k^+$ est remplacé par $k^-$. En reprenant les définitions de $r'$ et $r''$ donnée en 1.3, un calcul cas par cas montre que (3) équivaut à

$$C^\eta=\begin{cases}2(r'+r'') & \text{si } r'+r'' \text{ est pair},\\ -2(r'+r''+1), & \text{si } r'+r'' \text{ est impair}.\end{cases}$$

De même, l'égalité analogue de (3) pour $C^{-\eta}$ équivaut à

$$C^{-\eta}=\begin{cases} 2(r'-r'') & \text{si } r'+r'' \text{ est pair,}\\ -2(r'-r''+1) & \text{si } r'+r'' \text{ est impair.}\end{cases}$$

Par définition, $r'=r_1$ et $r''=\eta r_2$. Alors les formules ci-dessus sont la condition 3.5(3). Cela prouve (2). □

On peut appliquer le lemme 3.5. On note $\underline{\iota}_1$ et $\underline{\iota}_2$ les termes dont ce lemme affirme l'existence. Avec les notations de 4.2, ils appartiennent à $\mathcal{F}am_{r_1}(\lambda_1)$, resp. $\mathcal{F}am_{r_2}(\lambda_2)$. En conséquence, ces ensembles sont non vides. A fortiori, on a

$$r_1^2+r_1\leq n_1,\quad r_2^2\leq n_2. \tag{4}$$

Appliquons maintenant le calcul de 4.2 aux couples $(\mu_1,1)\in\boldsymbol{\mathcal{P}}^{\mathrm{orth}}(2n_1+1)_{k=1}$ et $(\mu_2,1)\in\boldsymbol{\mathcal{P}}^{\mathrm{orth}}(2n_2)_{k=0}$ (les partitions $\mu_1$ et $\mu_2$ ont été définies ci-dessus avant (1)). On a évidemment $\mathrm{sp}(\mu_1,1)=\mu_1$ et $\mathrm{sp}(\mu_2,1)=\mu_2$. La condition 4.2(3) est vérifiée : c'est (4) ci-dessus. Dans la formule 4.2(6), on peut limiter les sommations aux quadruplets $(\lambda'^+,\epsilon'^+,\lambda'^-,\epsilon'^-)$ et aux couples $(\iota_1,\iota_2)$ tels que $x(\lambda'^+,\epsilon'^+,\lambda'^-,\epsilon'^-)\neq 0$ et

$$m\big(\Pi_{\underline{\gamma}}(\lambda'^+,\epsilon'^+,\lambda'^-,\epsilon'^-),\,\rho_{\iota_1}\otimes\rho_{\iota_2}\big)\neq 0.$$

Comme en 4.4, on déduit de ces conditions les relations 4.4(3) et 4.4(4) :

$$\lambda^{+,\min}\leq\lambda'^+,\quad \lambda^{-,\min}\leq\lambda'^-,\quad \lambda'^+\cup\lambda'^-\leq\lambda.$$

Mais ici $\lambda=\lambda^{+,\min}\cup\lambda^{-,\min}$ par définition. Les inégalités ci-dessus sont donc des égalités. D'après 4.1 et 4.2(5), les conditions $\lambda^{+,\min}=\lambda'^+$, $\lambda^{-,\min}=\lambda'^-$ et $x(\lambda'^+,\epsilon'^+,\lambda'^-,\epsilon'^-)\neq 0$ impliquent

$$(\lambda'^+,\epsilon'^+)=(\lambda^{+,\min},\epsilon^{+,\min})\quad\text{et}\quad(\lambda'^-,\epsilon'^-)=(\lambda^{-,\min},\epsilon^{-,\min}).$$

Dans la somme 4.2(6), il ne reste que le quadruplet $(\lambda^{+,\min},\epsilon^{+,\min},\lambda^{-,\min},\epsilon^{-,\min})$ et on sait d'après 4.1 que, pour celui-là, on a $x(\lambda^{+,\min},\epsilon^{+,\min},\lambda^{-,\min},\epsilon^{-,\min})=1$.

Il ne reste aussi que les couples $(\iota_1,\iota_2)$ tels que

$$m\big(\Pi_{\underline{\gamma}}(\lambda^{+,\min},\epsilon^{+,\min},\lambda^{-,\min},\epsilon^{-,\min}),\,\rho_{\iota_1}\otimes\rho_{\iota_2}\big)\neq 0.$$

Ou encore, d'après 4.3(1), tels que $m\big(\Pi^{\zeta}(\iota_1,\iota_2),\,\rho_{\lambda^{+,\min},\epsilon^{+,\min}}\otimes\rho_{\lambda^{-,\min},\epsilon^{-,\min}}\big)\neq 0$, le signe $\zeta$ étant déterminé comme en 4.3. Cette condition équivaut à ce que

$$(\lambda^{+,\min},\epsilon^{+,\min},\lambda^{-,\min},\epsilon^{-,\min})\in\mathcal{I}^{\zeta}(\iota_1,\iota_2)\quad\text{(l'ensemble défini en 3.4).}$$

Puisque $\lambda^{+,\min}\cup\lambda^{-,\min}=\lambda$, la proposition 3.4(ii) nous dit qu'elle équivaut aussi à ce que $(\lambda^{+,\min},\epsilon^{+,\min},\lambda^{-,\min},\epsilon^{-,\min})$ appartienne à $\mathcal{I}^{\zeta,\max}(\iota_1,\iota_2)$. En outre, on a dans ce cas

$$m\big(\Pi_{\underline{\gamma}}(\lambda^{+,\min},\epsilon^{+,\min},\lambda^{-,\min},\epsilon^{-,\min}),\,\rho_{\iota_1}\otimes\rho_{\iota_2}\big)=1.$$

La condition $(\lambda^{+,\min}, \epsilon^{+,\min}, \lambda^{-,\min}, \epsilon^{-,\min}) \in \mathcal{I}^{\zeta,\max}(\iota_1, \iota_2)$ équivaut à ce que les formules 3.4(4) et 3.4(5) soient vérifiées, avec les modifications suivantes : les couples $(\lambda^+, \epsilon^+)$ et $(\lambda^-, \epsilon^-)$ de ce paragraphe sont remplacés par $(\lambda^{+,\min}, \epsilon^{+,\min})$ et $(\lambda^{-,\min}, \epsilon^{-,\min})$ ; les fonctions $\delta^+$, $\delta^-$, $\tau^+$ et $\tau^-$ sont remplacées par $\delta^+_{\iota_1,\iota_2}$, etc. La condition 3.4(4) détermine entièrement les fonctions $\delta^+_{\iota_1,\iota_2}$ et $\delta^-_{\iota_1,\iota_2}$. En se rappelant que le signe $\zeta\nu$ qui intervient vaut précisément $\eta$ (cf. 4.3(2)), on voit que ces fonctions coïncident avec les fonctions $\delta^+$ et $\delta^-$ construites ci-dessus. Les fonctions $\tau^+_{\iota_1,\iota_2}$ et $\tau^-_{\iota_1,\iota_2}$ ne sont pas à première vue entièrement déterminées par la relation 3.4(5). Toutefois, pour tout $i \in \mathrm{Jord}^{\mathrm{bp}}(\lambda)$, l'une au moins des valeurs $\tau^+_{\iota_1,\iota_2}(i)$ ou $\tau^-_{\iota_1,\iota_2}(i)$ est déterminée et coïncide avec la valeur de $\tau^+(i)$ ou $\tau^-(i)$. Puisque les couples $(\tau^+, \tau^-)$ et $(\tau^+_{\iota_1,\iota_2}, \tau^-_{\iota_1,\iota_2})$ vérifient tous deux la condition 3.5(1), cela suffit à conclure que ces deux couples sont égaux. Alors le lemme 3.5 nous dit que $(\iota_1, \iota_2)$ est égal au couple $(\underline{\iota}_1, \underline{\iota}_2)$ introduit ci-dessus. Inversement, pour ce dernier couple, les conditions 3.4(4) et 3.4(5) sont bien vérifiées. Autrement dit, dans la somme 4.2(6), il ne reste plus que le couple $(\underline{\iota}_1, \underline{\iota}_2)$ et on a

$$m\big(\Pi_{\underline{\gamma}}(\lambda^{+,\min}, \epsilon^{+,\min}, \lambda^{-,\min}, \epsilon^{-,\min}), \rho_{\underline{\iota}_1} \otimes \rho_{\underline{\iota}_2}\big) = 1.$$

Cette formule 4.2(6), devient

$$\begin{aligned} &M_\pi(\mu_1, 1; \mu_2, 1) \\ &= |\mathrm{Fam}(\lambda_1)|^{-\frac{1}{2}} |\mathrm{Fam}(\lambda_2)|^{-\frac{1}{2}} (-1)^{\langle \Lambda_1, \Lambda_{\underline{\iota}_1}\rangle}\big((-1)^{\langle \Lambda_2^+, \Lambda_{\underline{\iota}_2}\rangle} + \mathrm{sgn}_\sharp (-1)^{\langle \Lambda_2^-, \Lambda_{\underline{\iota}_2}\rangle}\big). \end{aligned} \quad (5)$$

Rappelons que $\Lambda_2^+$ et $\Lambda_2^-$ sont les symboles des couples $(0, \rho^+_{\mu_2,1} \otimes \mathrm{sgn})$ et $(0, \rho^-_{\mu_2,1} \otimes \mathrm{sgn})$. Ils se déduisent l'un de l'autre par permutation des deux termes $X$ et $Y$ de chaque symbole. D'après 2.5(1), on a donc

$$(-1)^{\langle \Lambda_2^-, \Lambda_{\underline{\iota}_2}\rangle} = (-1)^{r_2} (-1)^{\langle \Lambda_2^+, \Lambda_{\underline{\iota}_2}\rangle}. \quad (6)$$

Considérons la formule 1.5(1). Notons $i_1 > \cdots > i_t$ les entiers pairs $i \geq 2$ tels que $\mathrm{mult}_{\lambda^+}(i)$ soit impair. Le premier produit de la formule vaut $(-1)^{X^+}$, où

$$X^+ = \big|\{h = 1, \ldots, t;\ \epsilon^+_{i_h} = -1\}\big|.$$

On a

$$X^+ \equiv \big|\{h = 1, \ldots, t;\ h \text{ pair et } \epsilon^+_{i_h} = -1\}\big| - \big|\{h = 1, \ldots, t;\ h \text{ impair et } \epsilon^+_{i_h} = -1\}\big| \mod 2\mathbb{Z}.$$

D'après 1.3(1), le membre de droite vaut $k^+/2$ si $k^+$ est pair, $-(k^+ + 1)/2$ si $k^+$ est impair. D'après le même calcul cas par cas qui a calculé $C^\eta$ ci-dessus, c'est aussi $(r' + r'')/2$ si $r' + r''$ est pair, $-(r' + r'' + 1)/2$ si $r' + r''$ est impair. On obtient

$$(-1)^{X^+} = \begin{cases} (-1)^{(r'+r'')/2} & \text{si } r' + r'' \text{ est pair,} \\ (-1)^{(r'+r''+1)/2} & \text{si } r' + r'' \text{ est impair.} \end{cases}$$

Le deuxième facteur de 1.5(1) se calcule de même, $r''$ étant remplacé par $-r''$. Le produit de ces termes vaut $(-1)^{r''}$, ou encore $(-1)^{r_2}$. La formule 1.5(1) nous dit donc que

$$\mathrm{sgn}_\sharp = (-1)^{r_2}. \tag{7}$$

Grâce à (6) et (7), (5) se simplifie en

$$M_\pi(\mu_1, 1; \mu_2, 1) = 2|\mathrm{Fam}(\lambda_1)|^{-\frac{1}{2}}|\mathrm{Fam}(\lambda_2)|^{-\frac{1}{2}}(-1)^{\langle \Lambda_1, \Lambda_{\iota_1}\rangle + \langle \Lambda_2^+, \Lambda_{\iota_2}\rangle}.$$

Donc $M_\pi(\mu_1, 1; \mu_2, 1) \neq 0$. Alors, en vertu de (1), les couples $(\mu_1, 1)$ et $(\mu_2, 1)$ vérifient le (ii) de la proposition 1.4, à ceci près que l'on doit de plus prouver que $n_2 \geq 1$ si $\sharp = \mathrm{an}$. Mais, si $\sharp = \mathrm{an}$, (7) implique que $r_2$ est impair et (4) implique alors que $n_2 \geq 1$.

**4.6.** ***Conclusion.*** On a prouvé que $\mu$ vérifiait les conditions de la proposition 1.4. Celle-ci implique que $\mu$ est le front d'onde de $\pi(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$. Cela démontre le deuxième théorème de l'introduction. Comme on l'a dit dans celle-ci, le premier théorème s'en déduit grâce à [Waldspurger 2018b, 3.4].

## 5. Sur le calcul effectif du front d'onde

**5.1.** ***Le couple*** $(\boldsymbol{\lambda}^{\max}, \boldsymbol{\epsilon}^{\max})$**.** Soit $(\lambda, \epsilon) \in \mathcal{P}^{\mathrm{symp}}(2n)$, supposons que tous les termes de $\lambda$ sont pairs. On lui associe un couple $(\lambda^{\max}, \epsilon^{\max}) \in \mathcal{P}^{\mathrm{symp}}(2n)$ par récurrence sur $n$, selon la construction qui suit et qui est extraite de [Waldspurger 2017, 5.1]. On représente $\lambda$ sous la forme d'une suite infinie $\lambda = (\lambda_1, \lambda_2, \dots)$. On associe à $\epsilon$ une fonction encore notée $\epsilon$ sur l'ensemble d'indices $\mathbb{N}_{>0} = \mathbb{N} - \{0\}$ par $\epsilon(j) = \epsilon_{\lambda_j}$, avec la convention $\epsilon_0 = 1$. On note $\mathfrak{S}$ la réunion de $\{1\}$ et de l'ensemble des $j \geq 2$ tels que $\epsilon(j)(-1)^j \neq \epsilon(j-1)(-1)^{j-1}$. On note $s_1 = 1 < s_2 < \cdots$ les éléments de $\mathfrak{S}$. Pour $\zeta \in \{\pm 1\}$, notons $J^\zeta = \{j \in \mathbb{N}_{>0};\ (-1)^{j+1}\epsilon(j) = \zeta\}$ et $\tilde{J}^\zeta = J^\zeta - (J^\zeta \cap \mathfrak{S})$. On pose

$$\lambda_1^{\max} = \Big(\sum_{s \in \mathfrak{S}} \lambda_s\Big) - 2|\tilde{J}^{-\epsilon(1)}|.$$

On pose $n' = n - \lambda_1^{\max}/2$. On note $\lambda'$ la réunion des $\lambda_j$ pour $j \in \tilde{J}^{\epsilon(1)}$ et des $\lambda_j + 2$ pour $j \in \tilde{J}^{-\epsilon(1)}$. Pour $i \in \mathrm{Jord}^{\mathrm{bp}}(\lambda')$, on a $i = \lambda_j$ ou $i = \lambda_j + 2$ pour un $j$ comme ci-dessus. On note $h[j]$ le plus grand entier $h \geq 1$ tel que $s_h < j$ et on pose $\epsilon'_i = (-1)^{h[j]+1}\epsilon(j)$ ($j$ n'est pas uniquement déterminé par $i$ mais on montre que cette définition ne dépend pas du choix de $j$). On montre que $n' < n$ (si $n \neq 0$), que le couple $(\lambda', \epsilon')$ appartient à $\mathcal{P}^{\mathrm{symp}}(2n')$ et que tous les termes de $\lambda'$ sont pairs. Par récurrence, on dispose d'un couple $(\lambda'^{\max}, \epsilon'^{\max})$. On pose $\lambda^{\max} = \{\lambda_1^{\max}\} \cup \lambda'^{\max}$. On définit $\epsilon^{\max}$ par $\epsilon^{\max}_{\lambda_1^{\max}} = \epsilon_{\lambda_1}$ et $\epsilon_i^{\max} = \epsilon'^{\max}_i$ pour $i \in \mathrm{Jord}^{\mathrm{bp}}(\lambda'^{\max})$ (c'est possible, c'est-à-dire que, si $\lambda_1^{\max}$ appartient à $\mathrm{Jord}^{\mathrm{bp}}(\lambda'^{\max})$, on a l'égalité $\epsilon_{\lambda_1} = \epsilon'^{\max}_{\lambda_1^{\max}}$). Cela définit le couple $(\lambda^{\max}, \epsilon^{\max})$. Les termes de $\lambda^{\max}$ sont pairs et $\lambda_1^{\max}$ est bien le plus grand terme de $\lambda^{\max}$.

**5.2.** ***La partition*** $^t\boldsymbol{\lambda}^{\mathbf{min}}$**.** On conserve les mêmes hypothèses. On pose $k = k_{\lambda,\epsilon}$. On a $k_{\lambda^{\max},\epsilon^{\max}} = k$. On a rappelé en 1.3(1) comment se calculait l'entier $k$. On écrit $\lambda^{\max} = (\lambda_1^{\max}, \dots, \lambda_{2R+1}^{\max})$ avec $\lambda_{2R+1}^{\max} = 0$. On note $j_1^+ < \dots < j_N^+$ les $j = 1, \dots, 2R+1$ tels que $\epsilon^{\max}(j)(-1)^{j+1} = (-1)^k$ (en considérant comme dans le paragraphe précédent que $\epsilon^{\max}$ se définit sur l'ensemble d'indices). On note $j_1^- < \dots < j_N^-$ les $j = 1, \dots, 2R+1$ tels que $\epsilon^{\max}(j)(-1)^j = (-1)^k$. On vérifie que $N = R+[k/2]+1$, $M = R-[k/2]$. Notons $\nu'$ la réunion disjointe des partitions suivantes :

$$\begin{aligned}
&\{2R+3u-k-1+\lambda^{\max}_{j_u^+}-2j_u^+;\ u=1,\dots,N\};\\
&\{2R+3v+k+\lambda^{\max}_{j_v^-}-2j_v^-;\ v=1,\dots,M\};\\
&\{R+[(k-1)/2],\ R+[(k-1)/2]-1,\ \dots,0\};\\
&\{R-[(k+3)/2],\ R-[(k+3)/2]-1,\ \dots,0\}.
\end{aligned}$$

On note $\nu' = (\nu'_1, \dots, \nu'_{4R+1})$. Pour $j = 1, \dots, 4R+1$, on pose $\nu_j = \nu'_j - 2R + [j/2]$. Cela définit une partition $\nu$ et on a l'égalité $^t\lambda^{\min} = \nu$ (cette égalité se déduit de [Waldspurger 2017, 4.6 et 4.7]).

**5.3.** ***Exemples.*** Soit $(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-) \in \mathfrak{Irr}^{\mathrm{bp}}_{\mathrm{quad}}(2n)$. Les formules des deux paragraphes précédents permettent de calculer les transposées des partitions $\lambda^{+,\min}$ et $\lambda^{-,\min}$. Le front d'onde de $\pi(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$ est $d(\lambda^{+,\min} \cup \lambda^{-,\min})$. Cette partition duale se calcule ainsi : on note $\nu$ la partition obtenue en ajoutant 1 au plus grand terme de $^t\lambda^{+,\min} + {}^t\lambda^{-,\min}$ ; alors $d(\lambda^{+,\min} \cup \lambda^{-,\min})$ est la plus grande partition orthogonale $\mu$ de $2n+1$ telle que $\mu \le \nu$. Le moins que l'on puisse dire est que ce calcul n'est pas simple.

Signalons le cas particulier rassurant où $\epsilon^+ = 1$, c'est-à-dire $\epsilon_i^+ = 1$ pour tout $i \in \mathrm{Jord}^{\mathrm{bp}}(\lambda^+)$, et $\epsilon^- = 1$. Dans ce cas, on voit que $\lambda^{+,\max} = (2n^+)$, $\epsilon_{2n^+}^{+,\max} = 1$, $\lambda^{-,\max} = (2n^-)$ et $\epsilon_{2n^-}^{-,\max} = 1$. On a $k^+ = k^- = 0$. On calcule $^t\lambda^{+,\min} = (2n^+)$, $^t\lambda^{-,\min} = (2n^-)$, puis $d(\lambda^{+,\min} \cup \lambda^{-,\min}) = (2n+1)$. Autrement dit, notre représentation $\pi(\lambda^+, 1, \lambda^-, 1)$ admet un modèle de Whittaker usuel, ce qui est bien connu.

Un autre cas particulier est celui où, pour $\zeta = \pm$, $n^\pm$ est de la forme $h^\pm(h^\pm+1)$, $\lambda^\zeta$ est égal à $(2h^\zeta, 2h^\zeta-2, \dots, 2)$ et où $\epsilon^\zeta$ est alterné, c'est-à-dire $\epsilon_{2i}^\zeta = (-1)^i$ pour $i = 1, \dots, h^\zeta$. Dans ce cas, on vérifie que $\lambda^{\zeta,\max} = \lambda^{\zeta,\min} = \lambda^\zeta$. Le front d'onde de $\pi(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$ est alors $d(\lambda^+ \cup \lambda^-)$. On retrouve le résultat de [Mœglin 1996; Waldspurger 2018b] car notre représentation est ici cuspidale donc égale à son image par l'involution d'Aubert–Zelevinsky.

Donnons enfin comme exemple le calcul du front d'onde de $\pi(\lambda^+, \epsilon^+, \lambda^-, \epsilon^-)$ dans le cas où $\lambda^-$ est vide et où $\lambda^+$ a au plus trois termes non nuls. On pose simplement $\lambda=\lambda^+$, $\epsilon=\epsilon^+$, $\mu=d(\lambda^{\min})$. On identifie $\epsilon$ au triplet $(\epsilon(1), \epsilon(2), \epsilon(3))$ que l'on note comme un triplet de signes $\pm$. Évidemment, certains triplets ne sont autorisés que sous certaines hypothèses sur $\lambda$ : si $\epsilon(j) \neq \epsilon(j+1)$, on doit avoir $\lambda_j > \lambda_{j+1}$ ; si $\epsilon(j) = -$, on doit avoir $\lambda_j > 0$. On note de même $\epsilon^{\max}$ comme une famille de signes. Alors les résultats sont les suivants :

| $\epsilon$ | $k$ | $\lambda^{\max}$ | $\epsilon^{\max}$ | ${}^t\lambda^{\min}$ |
|---|---|---|---|---|
| $(+,+,+)$ | 0 | $(\lambda_1+\lambda_2+\lambda_3)$ | $(+)$ | $(\lambda_1+\lambda_2+\lambda_3)$ |
| $(+,+,-)$ | 1 | $(\lambda_1+\lambda_2-4, \lambda_3+2, 2)$ | $(+,+,-)$ | $(\lambda_1+\lambda_2-2, \lambda_3, 1, 1)$ |
| $(+,-,+)$ | 2 | $(\lambda_1, \lambda_2, \lambda_3)$ | $(+,-,+)$ | $(\lambda_1-2, \lambda_2, \lambda_3+1, 1)$ |
| $(+,-,-)$ | 0 | $(\lambda_1+\lambda_3-2, \lambda_2, 2)$ | $(+,-,-)$ | $(\lambda_1+\lambda_3-2, \lambda_2+2)$ |
| $(-,+,+)$ | 1 | $(\lambda_1+\lambda_3, \lambda_2)$ | $(-,+)$ | $(\lambda_1+\lambda_3-1, \lambda_2+1)$ |
| $(-,+,-)$ | 3 | $(\lambda_1, \lambda_2, \lambda_3)$ | $(-,+,-)$ | $(\lambda_1-3, \lambda_2-1, \lambda_3, 2, 1, 1)$ |
| $(-,-,+)$ | 0 | $(\lambda_1+\lambda_2-2, \lambda_3+2)$ | $(-,-)$ | $(\lambda_1+\lambda_2-1, \lambda_3+1)$ |
| $(-,-,-)$ | 1 | $(\lambda_1+\lambda_2+\lambda_3)$ | $(-)$ | $(\lambda_1+\lambda_2+\lambda_3-1, 1)$ |

| $\epsilon$ | $\mu$ |
|---|---|
| $(+,+,+)$ | $(\lambda_1+\lambda_2+\lambda_3+1)$ |
| $(+,+,-)$ | $(\lambda_1+\lambda_2-1, \lambda_3-1, 1, 1, 1)$ |
| $(+,-,+)$ | $(\lambda_1-1, \lambda_2-1, \lambda_3+1, 1, 1)$ |
| $(+,-,-)$ | $(\lambda_1+\lambda_3-1, \lambda_2+1, 1)$ |
| $(-,+,+)$ | $(\lambda_1+\lambda_3-1, \lambda_2+1, 1)$ |
| $(-,+,-)$ | $(\lambda_1-3, \lambda_2-1, \lambda_3+1, 1, 1, 1, 1)$ |
| $(-,-,+)$ | $(\lambda_1+\lambda_2-1, \lambda_3+1, 1)$ |
| $(-,-,-)$ | $(\lambda_1+\lambda_2+\lambda_3-1, 1, 1)$ |

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# Spectral Mackey functors and equivariant algebraic *K*-theory, II

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We study the "higher algebra" of spectral Mackey functors, which the first named author introduced in Part I of this paper. In particular, armed with our new theory of symmetric promonoidal $\infty$-categories and a suitable generalization of the second named author's Day convolution, we endow the $\infty$-category of Mackey functors with a well-behaved symmetric monoidal structure. This makes it possible to speak of *spectral Green functors* for any operad $O$. We also answer a question of Mathew, proving that the algebraic $K$-theory of group actions is lax symmetric monoidal. We also show that the algebraic $K$-theory of derived stacks provides an example. Finally, we give a very short, new proof of the equivariant Barratt–Priddy–Quillen theorem, which states that the algebraic $K$-theory of the category of finite $G$-sets is simply the $G$-equivariant sphere spectrum.



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## Introduction and summary

This paper is part of an effort to give a complete description of the structures available on the algebraic $K$-theory of varieties and schemes (and even of various derived stacks) with all their concomitant functorialities and homotopy coherences.

So suppose $X$ a scheme (quasicompact and quasiseparated). The derived tensor product $\otimes^{\mathbf{L}}$ on perfect complexes on $X$ defines a symmetric monoidal structure on the derived category $D_X^{perf}$ of perfect complexes on $X$. With a little more effort, one can lift this structure to a symmetric monoidal structure on the stable $\infty$-category of perfect complexes on $X$. This suffices to get a product on algebraic $K$-theory

$$\otimes \colon K(X) \wedge K(X) \longrightarrow K(X)$$

that is associative and commutative up to coherent homotopy. Thus, $K(X)$ has not only the structure of a connective spectrum, but also the structure of a *connective $E_\infty$ ring spectrum*. This is an exceedingly rich structure: not only do the homotopy groups $K_*(X)$ form a graded commutative ring, but these homotopy groups also support (in a functorial way) a tremendous amount of structure involving intricate higher homotopy operations called *Toda brackets*. Still more information (in the form of *Dyer–Lashof operations*) can be found on the $\mathbf{F}_p$-cohomology of $K(X)$.

Now for any morphism $f : Y \longrightarrow X$ of schemes, the derived functor

$$\mathbf{L}f^\star \colon D_X^{qcoh} \longrightarrow D_Y^{qcoh}$$

on the category of complexes with quasicoherent cohomology preserves perfect complexes, and the resulting functor $\mathbf{L}f^\star \colon D_X^{perf} \longrightarrow D_Y^{perf}$ induces a morphism

$$f^\star \colon K(X) \longrightarrow K(Y)$$

on the algebraic $K$-theory. The functor $\mathbf{L}f^\star$ is compatible with the derived tensor product, in the sense that for any perfect complexes $E$ and $F$ on $X$, there is a natural equivalence

$$\mathbf{L}f^\star(E \otimes^{\mathbf{L}} F) \simeq (\mathbf{L}f^\star E) \otimes^{\mathbf{L}} (\mathbf{L}f^\star F).$$

Again this can be lifted to the level of stable $\infty$-categories, whence the induced morphism $f^\star$ on $K$-theory turns out to be a morphism of connective $E_\infty$ ring spectra. This implies that the induced homomorphism on homotopy groups

$$f^\star \colon K_*(X) \longrightarrow K_*(Y)$$

is a homomorphism of graded commutative rings, and it must respect all the higher homotopy operations on $K_*(X)$ as well.

One can fit all the functors $\mathbf{L}f^\star$ together to get a presheaf $U \rightsquigarrow D_U^{perf}$ on the big site of all schemes. This can even be viewed as a presheaf of stable $\infty$-categories, which suffices to give us a presheaf of connective spectra $U \rightsquigarrow K(U)$. Since the morphisms $f^\star$ are morphisms of connective $E_\infty$ ring spectra, we can regard this as presheaf of $E_\infty$ ring spectra.

If one wanted, one might "externalize" the product on $K$-theory in the following manner. For any two schemes $X$ and $Y$ over a base scheme $S$, one may define an external tensor product

$$\boxtimes^{\mathbf{L}} \colon D_X^{perf} \times D_Y^{perf} \longrightarrow D_{X \times_S Y}^{perf}$$

by the assignment $(E, F) \rightsquigarrow (\mathbf{L}\,\mathrm{pr}_1^\star\, E) \otimes^{\mathbf{L}} (\mathbf{L}\,\mathrm{pr}_2^\star\, F)$. Note that we have natural equivalences

$$(\mathbf{L}f^\star E) \boxtimes^{\mathbf{L}} (\mathbf{L}g^\star F) \simeq \mathbf{L}(f \times g)^\star (E \boxtimes^{\mathbf{L}} F).$$

If we lift this to the level of stable $\infty$-categories, this gives rise to an external pairing

$$\boxtimes \colon K(X) \wedge K(Y) \longrightarrow K(X \times_S Y),$$

which is natural (contravariantly) in $X$ and $Y$. The $E_\infty$ product on $K(X)$ can now be obtained by pulling back this external pairing along the diagonal map:

$$K(X) \wedge K(X) \longrightarrow K(X \times_S X) \longrightarrow K(X).$$

A morphism of schemes $f \colon Y \longrightarrow X$ may induce morphisms in the *covariant* direction as well. The pushforward $\mathbf{R}f_\star \colon D_Y^{qcoh} \longrightarrow D_X^{qcoh}$ generally will not preserve perfect complexes. If, however, $f$ is flat and proper, then for any perfect complex $E$, the complex $\mathbf{R}f_\star E$ *is* perfect. Thus in this case $\mathbf{R}f_\star$ restricts to a functor $\mathbf{R}f_\star \colon D_Y^{perf} \longrightarrow D_X^{perf}$, and after lifting this to the stable $\infty$-categories, we find an induced morphism

$$f_\star \colon K(Y) \longrightarrow K(X)$$

on the algebraic $K$-theory. One thus obtains a covariant functor $U \rightsquigarrow K(U)$, but only with respect to flat and proper morphisms. Observe, however, that since the functors $\mathbf{R}f_\star$ do not commute with the derived tensor product, this functor is *not* valued in ring spectra.

Nevertheless, if $f \colon Y \longrightarrow X$ is proper and flat, we do have an algebraic structure preserved by $\mathbf{R}f_\star$. Observe that one may regard $K(Y)$ as a module over the $E_\infty$ ring spectrum $K(X)$ via $f^\star$. For any perfect complexes $E$ on $Y$ and $F$ on $X$, one has a canonical equivalence

$$(\mathbf{R}f_\star E) \otimes^{\mathbf{L}} F \simeq \mathbf{R}f_\star(E \otimes^{\mathbf{L}} \mathbf{L}f^\star F)$$

of perfect complexes; this is the usual projection formula [20, Exp. III, Pr. 3.7]. At the level of $K$-theory, this translates to the observation that the morphism

$$f_\star\colon K(Y) \longrightarrow K(X)$$

is a morphism of connective $K(X)$-modules. The induced map on homotopy groups

$$f_\star\colon K_*(Y) \longrightarrow K_*(X)$$

is therefore a homomorphism of $K_*(X)$-modules.

Note that the *external* tensor product $\boxtimes^{\mathbf{L}}$ is actually perfectly compatible with the pushforwards, in the sense that one has natural equivalences

$$(\mathbf{R}f_\star E) \boxtimes^{\mathbf{L}} (\mathbf{R}g_\star F) \simeq \mathbf{R}(f \times g)_\star(E \boxtimes^{\mathbf{L}} F),$$

so on $K$-theory the external product $\boxtimes\colon K(X) \wedge K(Y) \longrightarrow K(X \times_S Y)$ is natural (covariantly) in $X$ and $Y$ for flat and proper morphisms.

Last, but certainly not least, there is a compatibility between the morphisms $f^\star$ and the morphisms $g_\star$, which results from the base change theorem for complexes [20, Exp. IV, Pr. 3.1.0]. Suppose that

$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[g]{} & X \end{array}$$

is a pullback square of schemes in which the horizontal maps $g$ are flat and proper. Then the canonical morphism

$$\mathbf{L}f^\star \mathbf{R}g_\star \longrightarrow \mathbf{R}g_\star \mathbf{L}f^\star$$

is an objectwise equivalence of functors $D^{perf}_{X'} \longrightarrow D^{perf}_{Y}$. This translates to the condition that there is a canonical homotopy

$$f^\star g_\star \simeq g_\star f^\star\colon K(X') \longrightarrow K(Y)$$

of morphisms of $K(X)$-modules. In fact, this compatibility between the pullbacks and the pushforwards, combined with the compatibility between $f_\star$ and the external tensor product, allows us to *deduce* the projection formula.

Let us summarize the structure we've found on the assignment $U \rightsquigarrow K(U)$:

- For every scheme $X$, we have an $E_\infty$ ring spectrum $K(X)$. Moreover, for any two schemes $X$ and $Y$ over a base $S$, one has an external pairing

$$\boxtimes\colon K(X) \wedge K(Y) \longrightarrow K(X \times_S Y).$$

- For every morphism $f: Y \longrightarrow X$, we have a pullback morphism
$$f^\star: K(X) \longrightarrow K(Y),$$
which is compatible with the external pairings and thus also with the $E_\infty$ product.
- For every flat and proper morphism $f: Y \longrightarrow X$, we have a pushforward morphism
$$f_\star: K(Y) \longrightarrow K(X),$$
which is compatible with the external pairings and thus (in light of the next condition) also with the $K(X)$-module structure.
- For any pullback square
$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[g]{} & X \end{array}$$
in which the horizontal maps $g$ are flat and proper, we have a canonical homotopy
$$f^\star g_\star \simeq g_\star f^\star: K(X') \longrightarrow K(Y).$$
of morphisms of $K(X)$-modules.

In this paper, we will demonstrate that these structures, along with all of their homotopy coherences, are neatly packaged in a *spectral Green functor* on the category of schemes.

This structure is the origin of both the $\mathrm{Gal}(E/F)$-equivariant $E_\infty$ ring spectrum structure on the algebraic $K$-theory of a Galois extension $E \supset F$ and the cyclotomic structure on the $p$-typical curves on a smooth $\mathbf{F}_p$-scheme. For the former, see 9.7, and for the latter, see the forthcoming paper [5].

In order to describe all the structure we see here, we study the "higher algebra" (in the sense of Lurie's book [18], for example) of spectral Mackey functors, which we introduced in Part I of this paper [3]. The $\infty$-category of spectral Mackey functors turns out to admit all the same well-behaved structures as the $\infty$-category of spectra itself. In particular, the $\infty$-category of Mackey functors admits a well-behaved symmetric monoidal structure. This, combined with Saul Glasman's convolution for $\infty$-categories [10], makes it possible to speak of $E_1$ algebras, $E_\infty$ algebras, or indeed $O$-algebras for any operad $O$ in this context; these are called *$O$-Green functors*.

We use this framework to provide a very simple answer to a question posed to us by Akhil Mathew, in which we demonstrate that the functor that assigns to any $\infty$-category with an action of a finite group $G$ its equivariant algebraic $K$-theory is lax symmetric monoidal. We also show that the algebraic $K$-theory of derived stacks with its transfer maps as described above offers an example of an $E_\infty$ Green functor. We also use this theory to give a new proof of the equivariant Barratt–Priddy–Quillen theorem, which states that the algebraic $K$-theory of the category of finite $G$-sets is simply the $G$-equivariant sphere spectrum. (In fact, we will generalize this result dramatically.)

***Warning.*** Let us emphasize that $E_\infty$-Green functors for a finite group $G$ are *not* equivalent to algebras in $G$-equivariant spectra structured by the equivariant linear isometries operad on a complete $G$-universe. To describe the latter in line with the discussion here — and to find such structures on algebraic $K$-theory spectra — it is necessary to develop elements of the theory of $G$-$\infty$-categories. This we do in the forthcoming joint paper [6].

## 1. $\infty$-anti-operads and symmetric promonoidal $\infty$-categories

One of the many complications that arises when one combines an $\infty$-category and its opposite in the way we have in our construction of the effective Burnside $\infty$-category [3, Df. 3.6] is that our constructions are extremely intolerant of asymmetries in basic definitions. This complication rears its head the moment we want to contemplate the symmetric monoidal structure on the Burnside $\infty$-category. In effect, the description of a symmetric monoidal $\infty$-categories given in [18, Ch. 4] forces one to specify the data of maps *out of* various tensor products in a suitably compatible fashion. Thus symmetric monoidal categories are there identified as certain $\infty$-*operads*. But since we are also working with opposites of symmetric monoidal $\infty$-categories, we will come face-to-face with circumstances in which we must identify the data of maps *into* various tensor products in a suitably compatible fashion. We will call the resulting opposites of $\infty$-operads $\infty$-*anti-operads*.[1] Awkward as this may seem, it cannot be avoided.

**1.1. Notation.** Let $\Lambda(\mathbf{F})$ denote the following ordinary category. The objects will be finite sets, and a morphism $J \longrightarrow I$ will be a map $J \longrightarrow I_+$; one composes $\psi\colon K \longrightarrow J_+$ with $\phi\colon J \longrightarrow I_+$ by forming the composite

$$K \xrightarrow{\psi} J_+ \xrightarrow{\phi_+} I_{++} \xrightarrow{\mu} I_+,$$

where $\mu\colon I_{++} \longrightarrow I_+$ is the map that simply identifies the two added points. (Of course $\Lambda(\mathbf{F})$ is equivalent to the category $\mathbf{F}_*$ of pointed finite sets, but we prefer to think of the objects of $\Lambda(\mathbf{F})$ as unpointed. This is the natural perspective on this category from the theory of operator categories [4].)

[1] We do not know a standard name for this structure. In a previous verion of this paper, CB called these “cooperads,” but this conflicts with better-known terminology.

**1.2. Definition.** (1.2.1) An ***∞-anti-operad*** is an inner fibration

$$p\colon O_{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})^{op}$$

whose opposite

$$p^{op}\colon (O_{\otimes})^{op} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

is an ∞-operad.

(1.2.2) If $p\colon O_{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})^{op}$ is an ∞-anti-operad, then an edge of $O_{\otimes}$ will be said to be ***inert*** if it is cartesian over an edge of $\mathrm{N}\Lambda(\mathbf{F})^{op}$ that corresponds to an inert map in $\Lambda(\mathbf{F})$, that is, a map $\phi\colon J \longrightarrow I_{+}$ such that the induced map $\phi^{-1}(I) \longrightarrow I$ is a bijection [18, Df. 2.1.1.8], [4, Df. 8.1].

(1.2.3) A cartesian fibration

$$q\colon X_{\otimes} \longrightarrow O_{\otimes}$$

will be said to ***exhibit*** $X_{\otimes}$ ***as an*** $O_{\otimes}$***-monoidal ∞-category*** just in case the cocartesian fibration

$$q^{op}\colon (X_{\otimes})^{op} \longrightarrow (O_{\otimes})^{op}$$

exhibits $(X_{\otimes})^{op}$ as an $(O_{\otimes})^{op}$-monoidal ∞-category in the sense of [18, Df. 2.1.2.13]. When $O_{\otimes} = \mathrm{N}\Lambda(\mathbf{F})^{op}$, we will say that $q$ ***exhibits*** $X_{\otimes}$ ***as a symmetric monoidal ∞-category***.

(1.2.4) A ***morphism*** $f\colon O_{\otimes} \longrightarrow P_{\otimes}$ ***of ∞-anti-operads*** is a morphism over $\mathrm{N}\Lambda(\mathbf{F})^{op}$ that carries inert edges to inert edges. If $O_{\otimes}$ and $P_{\otimes}$ are symmetric monoidal ∞-categories, then $f$ is a ***symmetric monoidal functor*** if it carries all cartesian edges to cartesian edges.

**1.3. Example.** Suppose $C$ an ∞-category. We define the ***cartesian ∞-anti-operad*** as

$$p\colon C_{\times} := ((C^{op})^{\sqcup})^{op} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})^{op},$$

where the notation $(\cdot)^{\sqcup}$ refers to the cocartesian ∞-operad [18, Cnstr. 2.4.3.1]. If $C$ is an ∞-category that admits all products, then the functor $p$ exhibits $C_{\times}$ as a symmetric monoidal ∞-category [18, Rk. 2.4.3.4].

An object $(I, X)$ of $C_{\times}$ consists of a finite set $I$ and a family $\{X_i \mid i \in I\}$; a morphism $(\phi, \omega)\colon (I, X) \longrightarrow (J, Y)$ of $C_{\times}$ consists of a map of finite sets $\phi\colon J \longrightarrow I_{+}$ and a family of morphisms

$$\left\{\omega_j\colon X_{\phi(j)} \longrightarrow Y_j \;\middle|\; j \in \phi^{-1}(I)\right\}$$

of $C$. If $C$ admits finite products, then the morphisms $\omega_j$ determine and are determined by a family of morphisms

$$\left\{\, \omega_{J_i}\colon X_i \longrightarrow \prod_{j\in J_i} Y_j \;\middle|\; i \in I \,\right\};$$

here $J_i$ denotes the fiber $\phi^{-1}(i)$.

Observe that the cartesian $\infty$-anti-operad is significantly simpler to define than the cartesian $\infty$-operad [18, Cnstr. 2.4.1.4]. Note also that $(\Delta^0)_\times = \mathrm{N}\Lambda(\mathbf{F})^{op}$.

It is extremely useful to note that the condition that an $\infty$-operad $C^\otimes$ be a symmetric monoidal $\infty$-category can be broken into two conditions:

(1) The first of these is *corepresentability* [18, Df. 6.2.4.3]; this is the condition that the functors $\mathrm{Map}^{\xi_I}_{C^\otimes}(x_I, -)\colon C \longrightarrow \mathbf{Top}$ be corepresentable, where $\xi_I$ is the unique active map $I \longrightarrow *$ in $\Lambda(\mathbf{F})$. A compact expression of this is simply to say (as Lurie does) that the inner fibration $C^\otimes \longrightarrow N\Lambda(\mathbf{F})$ is locally cocartesian.

(2) The second condition is *symmetric promonoidality*. This can be expressed in a number of ways. One may say that for any active map $\phi\colon J \longrightarrow I$ of $\Lambda(\mathbf{F})$, for any object $x_J \in C^\otimes_J$, and for any object $z \in C$, the natural map

$$\int^{y_I \in C^\otimes_I} \mathrm{Map}^{\xi_I}_{C^\otimes}(y_I, z) \times \mathrm{Map}^{\phi}_{C^\otimes}(x_J, y_I) \longrightarrow \mathrm{Map}^{\xi_J}_{C^\otimes}(x_J, z)$$

is an equivalence; this is an operadic version of the condition expressed in [18, Ex. 6.2.4.9]. Equivalently, $C^\otimes$ is a symmetric promonoidal $\infty$-category if it represents a commutative algebra object in the $\infty$-category of $\infty$-categories and profunctors. In light of [18, B.3.3], we make the following definition.

**1.4. Definition.** We will say that an $\infty$-operad $C^\otimes$ is ***symmetric promonoidal*** if the structure map $C^\otimes \longrightarrow N\Lambda(\mathbf{F})$ is a flat inner fibration [18, Df. B.3.8]. Similarly, we will say that an $\infty$-anti-operad $C_\otimes$ is ***symmetric promonoidal*** if the structure map $C_\otimes \longrightarrow N\Lambda(\mathbf{F})^{op}$ is a flat inner fibration.

Our claim now is that the conjunction of these two conditions are equivalent to the condition that $C^\otimes$ be a symmetric monoidal $\infty$-category. That is, we claim that a symmetric monoidal $\infty$-category is *precisely* a corepresentable symmetric promonoidal $\infty$-category. This follows immediately from this result:

**1.5. Proposition.** *The following are equivalent for an inner fibration $p\colon X \longrightarrow S$.*

(1.5.1) *The inner fibration $p$ is flat and locally cocartesian.*

(1.5.2) *The inner fibration $p$ is cocartesian.*

*Proof.* The second condition implies the first by [18, Ex. B.3.11]. Let us show that the first condition implies the second. By [15, Pr. 2.4.2.8], it suffices to consider the case in which $S = \Delta^2$, and to show that for any section of $p$ given by a commutative triangle

$$\begin{array}{ccc} & y & \\ {}^{f}\nearrow & & \searrow^{g} \\ x & \xrightarrow[h]{} & z \end{array}$$

in which $f$ and $g$ are locally $p$-cocartesian, the edge $h$ is locally $p$-cocartesian as well.

In this case, by [15, Cor. 3.3.1.2], we can find a cocartesian fibration $q\colon Y \longrightarrow \Delta^2$ along with an equivalence

$$\phi\colon X \times_{\Delta^2} \Lambda^2_1 \xrightarrow{\sim} Y \times_{\Delta^2} \Lambda^2_1$$

of cocartesian fibrations over $\Lambda^2_1$. Now since $p$ is flat, the inclusion $X \times_{\Delta^2} \Lambda^2_1 \hookrightarrow X$ is a categorical equivalence over $\Delta^2$. Consequently, we may lift to obtain a map $\psi\colon X \longrightarrow Y$ over $\Delta^2$ extending $\phi$. This map is a categorical equivalence since both $p$ and $q$ are flat.

Now $\psi(f) = \phi(f)$ and $\psi(g) = \phi(g)$ are $q$-cocartesian, whence so is $\psi(h)$. The stability of relative colimits under categorical equivalences [15, Pr. 4.3.1.6], in light of [15, Ex. 4.3.1.4], now implies that $h$ is $p$-cocartesian. □

One reason to treasure symmetric promonoidal structures is the fact that, as we shall now prove, they are precisely the structure needed on an $\infty$-category $C$ in order for $\mathrm{Fun}(C, D)$ to admit a *Day convolution* symmetric monoidal structure.[2] Indeed, in the context of ordinary categories this was the generality in which Day himself constructed the Day convolution.

To explain, suppose first $C^{\otimes}$ a small symmetric monoidal $\infty$-category, and suppose $D^{\otimes}$ a symmetric monoidal $\infty$-category such that $D$ admits all colimits, and the tensor product preserves colimits separately in each variable. In [10], Glasman constructs a symmetric monoidal structure on the functor $\infty$-category $\mathrm{Fun}(C, D)$ which is the natural $\infty$-categorical generalization of Day's convolution product. As in Day's construction, the convolution $F \otimes G$ of two functors $F, G\colon C \longrightarrow D$ in Glasman's symmetric monoidal structure is given by the left Kan extension of the composite

$$C \times C \xrightarrow{(F,G)} D \times D \xrightarrow{\otimes} D$$

along the tensor product $\otimes\colon C \times C \longrightarrow C$.

In particular, for any finite set $I$, and for any $I$-tuple $\{F_i\}_{i\in I}$ of functors $C \longrightarrow D$, the value of the tensor product is given by the coend

$$\left(\bigotimes_{i\in I} F_i\right)(x) \simeq \int^{u_I \in C^{\otimes}_I} \mathrm{Map}^{\xi_I}_{C^{\otimes}}(u_I, x) \otimes \bigotimes_{i\in I} F_i(u_i).$$

Equivalently, the Day convolution on $\mathrm{Fun}(C, D)$ is the essentially unique symmetric monoidal structure that enjoys the following criteria:

[2] We would like to acknowledge that Dylan Wilson has independently made this observation.

- The tensor product

$$-\otimes-\colon \mathrm{Fun}(C, D)\times\mathrm{Fun}(C, D)\longrightarrow\mathrm{Fun}(C, D)$$

preserves colimits separately in each variable.

- The functor given by the composite

$$C^{op}\times D \xrightarrow{j\times\mathrm{id}} \mathrm{Fun}(C, \mathbf{Kan})\times D \xrightarrow{m} \mathrm{Fun}(C, D)$$

is symmetric monoidal, where $j$ denotes the Yoneda embedding, and $m$ is the functor corresponding to the composition

$$\mathrm{Fun}(C, \mathbf{Kan}) \longrightarrow \mathrm{Fun}(D\times C, D\times\mathbf{Kan}) \longrightarrow \mathrm{Fun}(D\times C, D)$$

in which the first functor is the obvious one, and the functor $D\times\mathbf{Kan}\longrightarrow D$ is the tensor functor $(X, K)\rightsquigarrow X\otimes K$ of [15, §4.4.4].

Conveniently, we can extend Glasman's Day convolution to situations in which $C^{\otimes}$ is only symmetric promonoidal.

**1.6. Proposition.** *Let $C^{\otimes}$ be a symmetric promonoidal $\infty$-category and $D^{\otimes}$ a symmetric monoidal $\infty$-category such that $D$ admits all colimits and $\otimes\colon D\times D\longrightarrow D$ preserves colimits separately in each variable. Then* $\mathrm{Fun}(C, D)$ *admits a symmetric monoidal structure such that the $E_\infty$-algebras therein are morphisms of $\infty$-operads $C^{\otimes}\longrightarrow D^{\otimes}$.*

*Proof.* The results of the first two sections of [10] hold when $C^{\otimes}$ is symmetric promonoidal with only one change: in the proof of [10, Lm. 2.3], the reference to [15, Pr. 3.3.1.3] should be replaced with a reference to [18, Pr. B.3.14]. Consequently, our claim follows from [10, Prs. 2.11 and 2.12]. ☐

**1.7.** Once again, for any finite set $I$, and for any $I$-tuple $\{F_i\}_{i\in I}$ of functors $C\longrightarrow D$, the value of the tensor product is given by the coend

$$\Big(\bigotimes_{i\in I}F_i\Big)(x)\simeq\int^{u_I\in C_I^{\otimes}}\mathrm{Map}^{\xi_I}_{C^{\otimes}}(u_I, x)\;\otimes\;\bigotimes_{i\in I}F_i(u_i).$$

## 2. The symmetric promonoidal structure on the effective Burnside $\infty$-category

Suppose $C$ a disjunctive $\infty$-category [3, Df. 4.2]. The product on $C$ does not induce the product on the effective Burnside $\infty$-category $A^{eff}(C)$. (Indeed, recall that the effective Burnside $\infty$-category admits direct sums, and these direct sums are induced by the *coproduct* in $C$.) However, a product on $C$ (if it exists) *does* induce a symmetric monoidal structure on $A^{eff}(C)$. The construction of the previous example is just what we need to describe this structure, and it will work for a broad class of disjunctive triples — which we call *cartesian* – as well.

It turns out to be convenient to consider situations in which $C$ does not actually have products. In this case, the effective Burnside $\infty$-category $A^{eff}(C)$ admits not a symmetric monoidal structure, but only a symmetric promonoidal structure, which suffices to get the Day convolution on $\infty$-categories of Mackey functors.

**2.1. Notation.** Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple [3, Df. 5.2]. We now define a triple structure $(C_\times, (C_\times)_\dagger, (C_\times)^\dagger)$ on $C_\times$ in the following manner. A morphism

$$(\phi, \omega)\colon (I, X) \longrightarrow (J, Y)$$

of $C_\times$ will be ingressive just in case $\phi$ is a bijection, and each morphism

$$\omega_j\colon X_{\phi(j)} \longrightarrow Y_j$$

is ingressive. The morphism $(\phi, \omega)$ will be egressive just in case each morphism

$$\omega_j\colon X_{\phi(j)} \longrightarrow Y_j$$

is egressive (with no condition on $\phi$).

**2.2. Lemma.** *Suppose* $(C, C_\dagger, C^\dagger)$ *a left complete* ([3, Df. 10.2]) *disjunctive triple. Then the triple*

$$(C_\times, (C_\times)_\dagger, (C_\times)^\dagger)$$

*is adequate* (in the sense of [3, Df. 5.2]).

*Proof.* We first check (5.2.1) of [3, Df. 5.2]. Suppose we have a diagram in $C_\times$

$$\begin{array}{ccc} & & (J, Y) \\ & & \downarrow{\scriptstyle (\psi, g)} \\ (I, X) & \xrightarrow{(\phi, f)} & (K, Z). \end{array}$$

Let $H_+$ be the pushout of the corresponding diagram of finite pointed sets

$$\begin{array}{ccc} K_+ & \xrightarrow[\cong]{\phi_+} & I_+ \\ {\scriptstyle \psi_+}\downarrow & & \downarrow{\scriptstyle \psi'_+} \\ J_+ & \xrightarrow[\cong]{\phi'_+} & H_+. \end{array}$$

For every $h \in H$, let $W_h$ be the iterated fiber product of the objects

$$\left\{ Y_{(\phi')^{-1}(h)} \times_{Z_k} X_{\phi(k)} \,\middle|\, k \in \psi^{-1}(\phi')^{-1}(h) \right\}$$

over $Y_{(\phi')^{-1}(h)}$, which exists in view of our assumption that $(C, C_\dagger, C^\dagger)$ is a left complete disjunctive triple. For every $j \in J$, let $f'_j : W_{\phi'(j)} \longrightarrow Y_j$ be the projection morphism, which is ingressive in $C$; these morphisms assemble into an ingressive morphism $(\phi', f') : (H, W) \longrightarrow (J, Y)$ in $C_\times$. For every $i \in I$ such that $\psi'(i) \neq +$, let $g'_i : W_{\psi'(i)} \longrightarrow X_i$ be the projection morphism, which is egressive in $C$; these morphisms assemble into an egressive morphism $(\psi', g') : (H, W) \longrightarrow (I, X)$ in $C_\times$. Then we may complete the diagram in $C_\times$ to a square

$$\begin{array}{ccc} (H, W) & \xrightarrow{(\phi', f')} & (J, Y) \\ {\scriptstyle (\psi', g')}\downarrow & & \downarrow{\scriptstyle (\psi, g)} \\ (I, X) & \xrightarrow{(\phi, f)} & (K, Z). \end{array}$$

We leave the verification that this is indeed a pullback square in $C_\times$ to the reader. This checks (5.2.1), and the other condition (5.2.2) of [3, Df. 5.2] also follows readily from our description of the pullback. □

In particular, for any left complete disjunctive triple $(C, C_\dagger, C^\dagger)$, one may consider the effective Burnside $\infty$-category

$$A^{\mathit{eff}}(C_\times, (C_\times)_\dagger, (C_\times)^\dagger).$$

**2.3. Example.** Note in particular that

$$((\Delta^0)_\times, ((\Delta^0)_\times)_\dagger, ((\Delta^0)_\times)^\dagger) \simeq (\mathrm{N}\Lambda(\mathbf{F})^{op}, \iota\mathrm{N}\Lambda(\mathbf{F})^{op}, \mathrm{N}\Lambda(\mathbf{F})^{op}),$$

whence one proves easily that the inclusion

$$\mathrm{N}\Lambda(\mathbf{F}) \simeq (((\Delta^0)_\times)^\dagger)^{op} \hookrightarrow A^{\mathit{eff}}((\Delta^0)_\times, ((\Delta^0)_\times)_\dagger, ((\Delta^0)_\times)^\dagger)$$

is an equivalence.

We'll use the following pair of results. They follow the same basic pattern as [3, Lms. 11.4 and 11.5]; in particular, they too follow immediately from the first author's "omnibus theorem" [3, Th. 12.2].

**2.4. Lemma.** *Suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple. Then the natural functor*

$$A^{\mathit{eff}}(C_\times, (C_\times)_\dagger, (C_\times)^\dagger) \longrightarrow A^{\mathit{eff}}((\Delta^0)_\times, ((\Delta^0)_\times)_\dagger, ((\Delta^0)_\times)^\dagger)$$

*is an inner fibration.*

**2.5. Lemma.** *Suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple. Then for any object $Y$ of $C_\times$ lying over an object $J \in (\Delta^0)_\times$ and any inert morphism $\phi\colon I \longrightarrow J$ of $N\Lambda(\mathbf{F})$, there exists a cocartesian edge $Y \longrightarrow X$ for the inner fibration*

$$A^{\mathit{eff}}(C_\times, (C_\times)_\dagger, (C_\times)^\dagger) \longrightarrow A^{\mathit{eff}}((\Delta^0)_\times, ((\Delta^0)_\times)_\dagger, ((\Delta^0)_\times)^\dagger)$$

*lying over the image of $\phi$ under the equivalence of Ex. 2.3.*

Now we can go about defining the symmetric promonoidal structure on the effective Burnside $\infty$-category of a disjunctive triple.

**2.6. Notation.** For any left complete disjunctive triple $(C, C_\dagger, C^\dagger)$, we define $A^{eff}(C, C_\dagger, C^\dagger)^\otimes$ as the pullback

$$A^{eff}(C, C_\dagger, C^\dagger)^\otimes := A^{eff}(C_\times, (C_\times)_\dagger, (C_\times)^\dagger) \times_{A^{eff}((\Delta^0)_\times, ((\Delta^0)_\times)_\dagger, ((\Delta^0)_\times)^\dagger)} N\Lambda(\mathbf{F}),$$

equipped with its canonical projection to $N\Lambda(\mathbf{F})$. Note that because the inclusion

$$N\Lambda(\mathbf{F}) \hookrightarrow A^{eff}((\Delta^0)_\times, (\Delta^0)_{\times,\dagger}, (\Delta^0)^\dagger_\times)$$

is an equivalence, it follows that the projection functor

$$A^{eff}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow A^{eff}(C_\times, (C_\times)_\dagger, (C_\times)^\dagger)$$

is actually an equivalence.

**2.7. Remark.** Suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple. The objects of the total $\infty$-category $A^{eff}(C, C_\dagger, C^\dagger)^\otimes$ are pairs $(I, X_I)$ consisting of a finite set $I$ and an $I$-tuple $X_I = (X_i)_{i \in I}$ of objects of $C$. A morphism

$$(J, Y_J) \longrightarrow (I, X_I)$$

of $A^{eff}(C, C_\dagger, C^\dagger)^\otimes$ can be thought of as a morphism $\phi\colon J \longrightarrow I$ of $\Lambda(\mathbf{F})$ and a collection of diagrams

$$\left\{ \begin{array}{ccc} & U_{\phi(j)} & \\ \swarrow & & \searrow \\ Y_j & & X_{\phi(j)} \end{array} \;\middle|\; j \in \phi^{-1}(I) \right\}$$

such that for any $j \in J$, the morphism $U_{\phi(j)} \rightarrowtail X_{\phi(j)}$ is ingressive, and the morphism

$$U_{\phi(j)} \twoheadrightarrow Y_j$$

is egressive.

Composition is then defined by pullback; that is, a 2-simplex

$$(K, Z_K) \longrightarrow (J, Y_J) \longrightarrow (I, X_I)$$

consists of morphisms $\psi\colon K \longrightarrow J$ and $\phi\colon J \longrightarrow I$ of $\Lambda(\mathbf{F})$ along with a collection of diagrams

$$\left\{ \begin{array}{ccccc} & & W_{\phi(\psi(k))} & & \\ & \swarrow & & \searrow & \\ & V_{\psi(k)} & & U_{\phi(\psi(k))} & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ Z_k & & Y_{\psi(k)} & & X_{\phi(\psi(k))} \end{array} \;\middle|\; k \in (\phi\psi)^{-1}(I) \right\}$$

in which the square in the middle exhibits each $W_i$ (for $i \in I$) as the iterated fiber product over $U_i$ of the set of objects $\{V_j \times_{Y_j} U_i \mid j \in J_i\}$.

In particular, $A^{eff}(C, C_\dagger, C^\dagger)^{\otimes}_{\{1\}}$ may be identified with the effective Burnside $\infty$-category $A^{eff}(C, C_\dagger, C^\dagger)$ itself, and for any finite set $I$, the inert morphisms $\rho^i : I \longrightarrow \{i\}$ together induce an equivalence

$$A^{eff}(C, C_\dagger, C^\dagger)^{\otimes}_{I} \xrightarrow{\sim} \prod_{i \in I} A^{eff}(C, C_\dagger, C^\dagger)^{\otimes}_{\{i\}}.$$

For the proofs of the next few results it is convenient to introduce some notation.

**2.8. Notation.** Suppose $(C, C_\dagger, C^\dagger)$ a triple, suppose $A$ and $B$ are two sets, and suppose $S\colon A \sqcup B \longrightarrow C$ a functor. Then let

$$C'_{/\{S_x \ ; \ S_y\}_{x \in A, y \in B}} \subseteq C_{/\{S_z\}_{z \in A \sqcup B}}$$

denote the full subcategory spanned by those objects such that the morphisms to the objects $S_x$ are egressive and the morphisms to the objects $S_y$ are ingressive. In particular, note that

$$\mathrm{Map}_{A^{eff}(C, C_\dagger, C^\dagger)^{\otimes}}((J, Y_J), (*, X)) \simeq \iota C'_{/\{Y_j \ ; \ X\}_{j \in J}}.$$

We have almost proven the following.

**2.9. Proposition.** *For any left complete disjunctive triple $(C, C_\dagger, C^\dagger)$, the inner fibration*

$$A^{eff}(C, C_\dagger, C^\dagger)^{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

*is an $\infty$-operad.*

*Proof.* Following Rk. 2.7, it only remains to show that given an edge $\alpha\colon I \longrightarrow J$ in $\mathrm{N}\Lambda(\mathbf{F})$ and objects $(I, X)$, $(J, Y)$ in $A^{eff}(C, C_\dagger, C^\dagger)^{\otimes}$, the cocartesian edges

$$\begin{array}{ccc} & (*, Y_j) & \\ \swarrow\!\!\!\!= & & \searrow \\ (J, Y) & & (*, Y_j), \end{array}$$

over the inert edges $\rho^j : J \longrightarrow *$ induce an equivalence

$$\mathrm{Map}^{\alpha}_{A^{\mathit{eff}}(C,C_\dagger,C^\dagger)^{\otimes}}((I, X), (J, Y)) \longrightarrow \prod_{j\in J} \mathrm{Map}^{\rho^j\circ\alpha}_{A^{\mathit{eff}}(C,C_\dagger,C^\dagger)^{\otimes}}((I, X), (*, Y_j)).$$

But this is indeed true, since the map identifies the left-hand side as

$$\prod_{j\in J} \iota C'_{/\{X_i\ ;\ Y_j\}_{i\in\alpha^{-1}(j)}}. \qquad \square$$

We now show that the $\infty$-operad $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}$ is symmetric promonoidal.

**2.10. Proposition.** *Suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple. Then the $\infty$-operad*

$$p\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

*is symmetric promonoidal; that is, $p$ is a flat inner fibration.*

In preparation for the proof, we digress briefly to give the following proposition, which is useful for studying the interaction of the over and undercategory functors with homotopy colimit diagrams.

**2.11. Proposition.** *Suppose $C$ an $\infty$-category, and let $s\mathbf{Set}_{/C}$ be endowed with the model structure created by the forgetful functor to $s\mathbf{Set}$ equipped with the Joyal model structure. Then we have a Quillen adjunction*

$$C_{(-)/}\colon s\mathbf{Set}_{/C} \rightleftarrows (s\mathbf{Set}_{/C})^{op} : C_{/(-)}$$

*between the over and undercategory functors.*

*Proof.* The displayed functors are indeed adjoint to each other, since for objects $\phi\colon X \longrightarrow C$ and $\psi : Y \longrightarrow C$ we have natural isomorphisms

$$\mathrm{Hom}_{/C}(X, C_{\psi/}) \cong \mathrm{Hom}_{(X\sqcup Y)/}(X \star Y, C) \cong \mathrm{Hom}_{/C}(Y, C_{/\phi}).$$

To check that this adjunction is a Quillen adjunction, we check that $C_{(-)/}$ preserves cofibrations and trivial cofibrations. Let $\tau : \phi \longrightarrow \phi'$ be a map in $s\mathbf{Set}_{/C}$, and let $f = d_2(\tau)\colon X \longrightarrow X'$. If $f$ is a monomorphism, by [15, 2.1.2.1] we have that $C_{\phi'/} \longrightarrow C_{\phi/}$ is a left fibration, hence by [15, 2.4.6.5] a categorical fibration. If $f$ is a monomorphism and a categorical equivalence, by [15, 4.1.1.9] and [15, 4.1.1.1(4)] $f$ is right anodyne, hence by [15, 2.1.2.5] $C_{\phi'/} \longrightarrow C_{\phi/}$ is a trivial Kan fibration. $\square$

**2.11.1. Corollary.** *Let $C$ be an $\infty$-category and suppose given a morphism $f : x \longrightarrow y$ in $C$ and a diagram*

$$\begin{array}{ccccc} K & \xhookrightarrow{\phi} & L & \xrightarrow{p} & C \\ \downarrow & & \downarrow & \nearrow_{p'} & \\ K \sqcup \Delta^0 & \xhookrightarrow{\phi'} & L \sqcup \Delta^0 & & \end{array}$$

*of simplicial sets where $\phi' = \phi \sqcup \mathrm{id}$ and $p'|_{\Delta^0}$ selects $y$. Then we have a homotopy pullback square of $\infty$-categories*

$$\begin{array}{ccc} \{x\} \times_C C_{/p} & \xrightarrow{F} & C_{/p'} \\ \downarrow & & \downarrow \\ \{x\} \times_C C_{/p\circ\phi} & \xrightarrow{G} & C_{/p'\circ\phi'} \end{array}$$

*where the vertical functors are given by change of diagram and the horizontal functors are to be defined.*

*Proof.* Define the functor $F$ as follows: the datum of an $n$-simplex

$$\Delta^n \longrightarrow \{x\} \times_C C_{/p}$$

consists of a map $\alpha \colon \Delta^n \star L \longrightarrow C$ which restricts to $p$ on $L$ and to the constant map to $x$ on $\Delta^n$, and we use this to define $\Delta^n \star (L \sqcup \Delta^0) \longrightarrow C$ to be the unique map which restricts to $\alpha$ on $\Delta^n \star L$ and to

$$\Delta^n \star \Delta^0 \longrightarrow \Delta^1 \xrightarrow{f} C$$

on $\Delta^n \star \Delta^0$; this gives the $n$-simplex of $C_{/p'}$. The definition of $G$ is analogous. The square in question then fits into a rectangle

$$\begin{array}{ccccc} \{x\} \times_C C_{/p} & \xrightarrow{F} & C_{/p'} & \longrightarrow & C_{/p} \\ \downarrow & & \downarrow & & \downarrow \\ \{x\} \times_C C_{/p\circ\phi} & \xrightarrow{G} & C_{/p'\circ\phi'} & \longrightarrow & C_{/p\circ\phi} \end{array}$$

where the long horizontal functors are given as the inclusion of the fiber over $x$ and the functors in the righthand square are given by change of diagram. By Prp. 2.11 and left properness of the Joyal model structure, the righthand square is a homotopy pullback square. The vertical functor $C_{/p} \longrightarrow C_{/p\circ\phi}$ is a right fibration, so the outermost square is a homotopy pullback square. The conclusion follows. □

*Proof of Prp. 2.10.* Suppose $\sigma\colon \Delta^2 \longrightarrow N\Lambda(\mathbf{F})$ a 2-simplex given by a diagram

$$\begin{array}{ccccc} & & J & & \\ & {\scriptstyle\alpha}\nearrow & & \searrow{\scriptstyle\beta} & \\ I & & \xrightarrow[\gamma]{\phantom{xxxxxx}} & & K. \end{array}$$

Suppose

$$\begin{array}{ccc} & (K, W) & \\ \swarrow & & \searrow \\ (I, X) & & (K, Z), \end{array}$$

an edge $\widetilde{\gamma}$ of

$$\Sigma := A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger})^{\otimes} \times_{N\Lambda(\mathbf{F}),\sigma} \Delta^2$$

lifting $\gamma$, where we display $\widetilde{\gamma}$ as a span in $C_{\times}$. Let

$$E := \Sigma_{(I,X)/\ /(K,Z)} \times_{N\Lambda(\mathbf{F})} \{J\}$$

be the $\infty$-category of factorizations of $\widetilde{\gamma}$ through $\Sigma_J$. Observe that an $n$-simplex of $E$ is a cartesian functor $\widetilde{\mathscr{O}}(\Delta^{n+2})^{op} \longrightarrow (C_{\times}, (C_{\times})_{\dagger}, (C_{\times})^{\dagger})$ satisfying certain conditions. Here, $\widetilde{\mathscr{O}}(-)$ denotes the *twisted arrow* $\infty$-category [3, Ntn. 2.3], and we use the definition of the effective Burnside $\infty$-category as a simplicial set [3, Df. 3.6].[3]

To demonstrate that the inner fibration $p$ is flat, we need to show that $E$ is weakly contractible. To do this, we

(1) identify a full subcategory $E' \subset E$,

(2) show that $E'$ is a *colocalization* of $E$, i.e., that the inclusion functor admits a right adjoint, and

(3) demonstrate that $E'$ contains a terminal object.

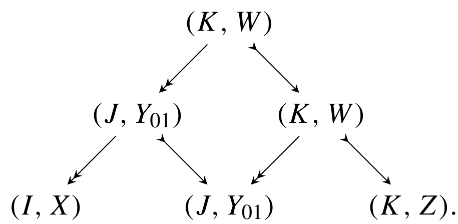
The subcategory $E'$ is the full subcategory spanned by those objects

$$\begin{array}{ccccccccc} & & & & (K, W) & & & & \\ & & & \swarrow & & \searrow & & & \\ & & (J, Y_{01}) & & & & (K, Y_{12}) & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow & \\ (I, X) & & & & (J, Y) & & & & (K, Z) \end{array} \tag{2.11.1}$$

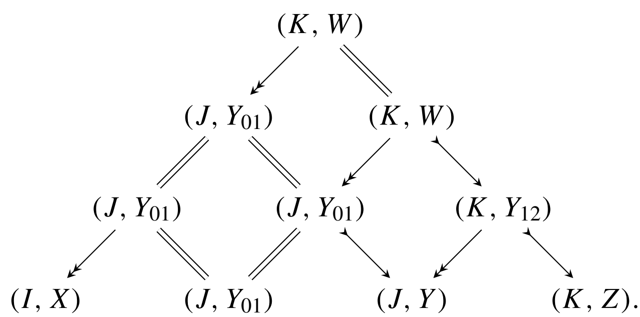
of $E$ such that the morphisms $(J, Y) \longrightarrow (J, Y_{01})$, and thus also $(K, W) \longrightarrow (K, Y_{12})$,

[3]Beware that our convention on the direction of morphisms in $\widetilde{\mathscr{O}}(-)$ is opposite to that of Lurie in [18, §5.2.1]: for us, twisted arrows are covariant in the target and contravariant in the source.

are equivalences; this is point (1). Informally, the right adjoint $R$ to the inclusion $E' \subset E$ is the functor that carries (2.11.1) to the object




To show that this indeed defines a right adjoint, we use the criterion of (the dual of) [15, Pr. 5.2.7.4(3)]. We must therefore construct a functor $\varepsilon\colon E \times \Delta^1 \longrightarrow E$ such that $\epsilon|_{(E\times\{0\})}$ is the functor $R$ we have informally described and $\epsilon|_{(E\times\{1\})}$ is the identity functor. Speaking again informally, $\varepsilon$ carries the object $\tau \in E$ represented by the diagram (2.11.1) to the morphism $\varepsilon_\tau\colon R(\tau) \longrightarrow \tau$ represented by the diagram




Once we've given a precise description of this $\varepsilon$, it will be immediate that for any object $\tau' \in E$, both $R(\varepsilon_{\tau'})$ and $\varepsilon_{R(\tau')}$ are equivalences, so the conditions of [15, Pr. 5.2.7.4(3)] are satisfied, confirming point (2).

The construction of the desired functor $\epsilon\colon E \times \Delta^1 \longrightarrow E$ is as follows: given nonnegative integers $k \leq n$, let $f_{n,k}\colon \widetilde{\mathscr{O}}(\Delta^{n+3}) \longrightarrow \widetilde{\mathscr{O}}(\Delta^{n+2})$ be the unique functor which on objects is given by

$$f_{n,k}(ij) := \begin{cases} 0j & \text{if } i \leq k+1 \text{ and } j \leq k+1; \\ 0(j-1) & \text{if } i \leq k+1 \text{ and } j > k+1; \\ (i-1)(j-1) & \text{if } i > k+1. \end{cases}$$

Then for every $n$-simplex $\nu\colon \Delta^n \longrightarrow E$ corresponding to a functor

$$\overline{\nu}\colon \widetilde{\mathscr{O}}(\Delta^{n+2})^{op} \longrightarrow C_\times,$$

define $\epsilon(\nu)\colon \Delta^n \times \Delta^1 \longrightarrow E$ to be the unique functor which sends the nondegenerate $(n+1)$-simplex

$$(0,0) \longrightarrow \cdots \longrightarrow (0,k) \longrightarrow (1,k) \longrightarrow \cdots \longrightarrow (1,n)$$

to the $(n+1)$-simplex $\Delta^{n+1} \longrightarrow E$ corresponding to the functor

$$\bar{\nu} \circ f_{nk}^{op}\colon \widetilde{\mathscr{O}}(\Delta^{n+3})^{op} \longrightarrow C_\times.$$

It is easy (albeit tedious) to verify that the functors $\epsilon(\nu)$ define the desired functor $\epsilon\colon E \times \Delta^1 \longrightarrow E$. This thus completes the proof of point (2).

We now set about proving point (3) — the existence of a terminal object of $E'$. For this, we will need to use the map of pointed finite sets $\beta\colon J \longrightarrow K$ that came as part of our initial data. We then define $(J, \overline{W}) \in C_\times$ from our given object $(K, W)$ as:

$$\overline{W}_j := \begin{cases} W_{\beta(j)} & \text{if } \beta(j) \neq *; \\ \varnothing & \text{if } \beta(j) = *. \end{cases}$$

There is an obvious factorization of $(K, W) \longrightarrow (I, X)$ through $(J, \overline{W})$, and we define the object $\omega \in E'$ as

$$\begin{array}{ccccccc} & & & (K,W) & & & \\ & & \swarrow & & \diagdown\!\!\diagdown & & \\ & (J,\overline{W}) & & & & (K,W) & \\ \swarrow & & \diagdown\!\!\diagdown & & \swarrow & & \searrow \\ (I,X) & & & (J,\overline{W}) & & & (K,Z) \end{array}$$

To prove point (3), we set about showing that $\omega$ is terminal in $E'$. Let $\tau \in E'$ be any object represented by the diagram

$$\begin{array}{ccccccc} & & & (K,W) & & & \\ & & \swarrow & & \searrow & & \\ & (J,Y_{01}) & & & & (K,Y_{12}) & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ (I,X) & & & (J,Y) & & & (K,Z) \end{array}$$

in which $(J, Y) \longrightarrow (J, Y_{01})$ and therefore also $(K, W) \longrightarrow (K, Y_{12})$ are equivalences. To show that $\mathrm{Map}_{E'}(\tau, \omega)$ is contractible, let us also view $\tau$ and $\omega$ as morphisms in $\Sigma_{(I,X)/}$ in order to write down the following homotopy pullback square

$$\begin{CD} \mathrm{Map}_{E'}(\tau,\omega) @>>> \mathrm{Map}_{\Sigma_{(I,X)/}}(d_2(\tau), d_2(\omega)) \\ @VVV @VV{\omega_*}V \\ \Delta^0 @>>{\tau}> \mathrm{Map}_{\Sigma_{(I,X)/}}(d_2(\tau), \widetilde{\gamma}). \end{CD}$$

The terms on the right-hand side are in turn given as homotopy pullbacks

$$\begin{CD} \mathrm{Map}_{\Sigma_{(I,X)/}}(d_2(\tau), d_2(\omega)) @>>> \mathrm{Map}_{\Sigma}((J,Y),(J,\overline{W})) \\ @VVV @VV{d_2(\tau)^*}V \\ \Delta^0 @>>{d_2(\omega)}> \mathrm{Map}_{\Sigma}((I,X),(J,\overline{W})), \end{CD}$$

and

$$\begin{CD} \mathrm{Map}_{\Sigma_{(I,X)/}}(d_2(\tau), \widetilde{\gamma}) @>>> \mathrm{Map}_{\Sigma}((J,Y),(K,Z)) \\ @VVV @VV{d_2(\tau)^*}V \\ \Delta^0 @>>{\widetilde{\gamma}}> \mathrm{Map}_{\Sigma}((I,X),(K,Z)). \end{CD}$$

In light of the equivalence $(J, Y_{01}) \xrightarrow{\sim} (J, Y)$, we obtain equivalences

$$\mathrm{Map}_{\Sigma}((J,Y),(J,\overline{W})) \simeq \prod_{j\in J} \iota C'_{/\{(Y_{01})_j\ ;\ \overline{W}_j\}};$$

$$\mathrm{Map}_{\Sigma}((I,X),(J,\overline{W})) \simeq \prod_{j\in J} \iota C'_{/\{X_i\ ;\ \overline{W}_j\}_{i\in\alpha^{-1}(j)}}.$$

Under these equivalences the map $d_2(\tau)^*$ is given by $\prod_{j\in J}\phi_j$, where

$$\phi_j : \iota C'_{/\{(Y_{01})_j\ ;\ \overline{W}_j\}} \longrightarrow \iota C'_{/\{X_i\ ;\ \overline{W}_j\}_{i\in\alpha^{-1}(j)}}$$

is defined by postcomposition by the maps $(Y_{01})_j \longrightarrow X_i$ (with $i \in \alpha^{-1}(j)$). Employing Corollary 2.11.1, we may factor the square in question into two homotopy pullback squares:

$$\begin{array}{ccccc} \mathrm{Map}_{\Sigma_{(I,X)/}}(d_2(\tau), d_2(\omega)) & \longrightarrow & \mathrm{Map}_{(C_\times)^\dagger_{\mathrm{id}}}((J,\overline{W}),(J,Y_{01})) & \longrightarrow & \prod_{j\in J}\iota C'_{/\{(Y_{01})_j\,;\,\overline{W}_j\}} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^0 & \longrightarrow & \mathrm{Map}_{(C_\times)^\dagger_{\alpha}}((J,\overline{W}),(I,X)) & \longrightarrow & \prod_{j\in J}\iota C'_{/\{X_i\,;\,\overline{W}_j\}_{i\in\alpha^{-1}(j)}} \end{array}$$

Similarly, we factor the second square into two homotopy pullback squares:

$$\begin{array}{ccccc} \mathrm{Map}_{\Sigma_{(I,X)/}}(d_2(\tau),\widetilde{\gamma}) & \longrightarrow & \mathrm{Map}_{(C_\times)^\dagger_\beta}((K,W),(J,Y_{01})) & \longrightarrow & \prod_{k\in K}\iota C'_{/\{(Y_{01})_j\ ;\ Z_k\}_{j\in\beta^{-1}(k)}} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^0 & \longrightarrow & \mathrm{Map}_{(C_\times)^\dagger_\gamma}((K,W),(I,X)) & \longrightarrow & \prod_{k\in K}\iota C'_{/\{X_i\ ;\ Z_k\}_{i\in\gamma^{-1}(k)}} \end{array}$$

The map $\omega_*$ is then seen to be equivalent to the induced map between the fibers of the horizontal maps in the following commutative square:

$$\begin{array}{ccc} \mathrm{Map}_{(C_\times)^\dagger_{\mathrm{id}}}((J,\overline{W}),(J,Y_{01})) & \longrightarrow & \mathrm{Map}_{(C_\times)^\dagger_\alpha}((J,\overline{W}),(I,X)) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{(C_\times)^\dagger_\beta}((K,W),(J,Y_{01})) & \longrightarrow & \mathrm{Map}_{(C_\times)^\dagger_\gamma}((K,W),(I,X)). \end{array}$$

The left vertical map is the equivalence

$$\prod_{j\in\beta^{-1}(K)}\mathrm{Map}_{C^\dagger}(W_{\beta(j)},(Y_{01})_j)\xrightarrow{\sim}\prod_{k\in K}\prod_{j\in\beta^{-1}(k)}\mathrm{Map}_{C^\dagger}(W_k,(Y_{01})_j),$$

and the right vertical map is the equivalence

$$\prod_{j\in\beta^{-1}(K)}\prod_{i\in\alpha^{-1}(j)}\mathrm{Map}_{C^\dagger}(W_{\beta(j)},X_i)\xrightarrow{\sim}\prod_{k\in K}\prod_{i\in\gamma^{-1}(k)}\mathrm{Map}_{C^\dagger}(W_k,X_i),$$

so the square is in fact a homotopy pullback square and $\omega_*$ is an equivalence. Hence, the mapping space $\mathrm{Map}_{E'}(\tau,\omega)$ is contractible, as desired. This proves point (3), and it completes the proof. □

If we want the symmetric promonoidal $\infty$-category

$$A^{\mathit{eff}}(C,C_\dagger,C^\dagger)^\otimes\longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

to be symmetric monoidal, we need a nontrivial condition on our disjunctive triple.

**2.12. Definition.** A disjunctive triple $(C,C_\dagger,C^\dagger)$ will be said to be ***cartesian*** just in case it enjoys the following properties:

(2.12.1) It is left complete.

(2.12.2) The underlying $\infty$-category $C$ admits finite products.

(2.12.3) For any object $X\in C$, the product functor

$$X\times -\colon C\longrightarrow C$$

preserves finite coproducts; that is, for any finite set $I$ and any collection $\{U_i \mid i \in I\}$ of objects of $C$, the natural map

$$\coprod_{i \in I} (X \times U_i) \longrightarrow X \times \left( \coprod_{i \in I} U_i \right)$$

is an equivalence.

(2.12.4) A morphism $X \longrightarrow \prod_{j \in J} Y_j$ is egressive just in case each of the components $X \longrightarrow Y_j$ is so.

**2.13. Example.** Note that a disjunctive $\infty$-category $C$ that admits a teminal object, when equipped with the maximal triple structure (in which every morphism is both ingressive and egressive) is always cartesian. More generally, any disjunctive triple that contains a terminal object 1 with the property that every morphism $X \longrightarrow 1$ is ingressive and egressive is cartesian.

**2.14. Proposition.** *If* $(C, C_\dagger, C^\dagger)$ *is a cartesian disjunctive triple, then the symmetric promonoidal* $\infty$*-category*

$$p \colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

*is symmetric monoidal; that is,* $p$ *is a cocartesian fibration.*

*Proof.* Since $p$ is flat, by Pr. 1.5 it suffices to verify that $p$ is a locally cocartesian fibration. Since $p$ is an $\infty$-operad, by the dual of [15, Lm. 2.4.2.7] we reduce to checking that for any active edge $\alpha \colon I \longrightarrow J$ and any object $(I, X)$ over $I$, there exists a locally $p$-cocartesian edge $\widetilde{\alpha}$ covering $\alpha$. For each $j \in J$, let $\widetilde{X}_j = \prod_{i \in \alpha^{-1}(j)} X_i$, and define $\widetilde{\alpha}$ to be

$$\begin{array}{ccc} & (J, \widetilde{X}) & \\ \swarrow & & \Vert \\ (I, X) & & (J, \widetilde{X}), \end{array}$$

where the morphism $(J, \widetilde{X}) \to (I, X)$ is defined using the projection maps $\widetilde{X}_{\alpha(i)} \to X_i$. Then $\widetilde{\alpha}$ is a locally $p$-cocartesian edge if for all $(J, Y) \in A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}_J$, the induced map

$$\widetilde{\alpha}^* \colon \mathrm{Map}_{A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}_J}((J, \widetilde{X}), (J, Y)) \longrightarrow \mathrm{Map}_{A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{\otimes}_\alpha}((I, X), (J, Y))$$

is an equivalence. This map is in turn equivalent to the map

$$\prod_{j \in J} \phi_j \colon \prod_{j \in J} \iota C'_{/\left\{\prod_{i \in \alpha^{-1}(j)} X_i \ ; \ Y_j\right\}} \longrightarrow \prod_{j \in J} \iota C'_{/\{X_i \ ; \ Y_j\}_{i \in \alpha^{-1}(j)}}$$

where $\phi_j$ is induced by postcomposition by the projection maps $\prod_{i\in\alpha^{-1}(j)} X_i \longrightarrow X_i$. Since $(C, C_\dagger, C^\dagger)$ is a cartesian disjunctive triple, we have that the functor

$$(C^\dagger)_{/\prod_{i\in\alpha^{-1}(j)} X_i} \longrightarrow (C^\dagger)_{/(X_i, i\in\alpha^{-1}(j))}$$

is an equivalence. Hence in light of Prp. 2.11 we have a homotopy pullback square

$$\begin{array}{ccc} \prod_{j\in J} \iota C'_{/\left\{\prod_{i\in\alpha^{-1}(j)} X_i \ ; \ Y_j\right\}} & \xrightarrow{\phi_j} & \prod_{j\in J} \iota C'_{/\{X_i \ ; \ Y_j\}_{i\in\alpha^{-1}(j)}} \\ \downarrow & & \downarrow \\ (C^\dagger)_{/\prod_{i\in\alpha^{-1}(j)} X_i} & \longrightarrow & (C^\dagger)_{/\{X_i\}_{i\in\alpha^{-1}(j)}} \end{array}$$

where the horizontal maps are equivalences. We deduce that the map $\widetilde{\alpha}^*$ is an equivalence, as desired. □

**2.15.** When $(C, C_\dagger, C^\dagger)$ is a *right* complete disjunctive triple, we may employ duality and write

$$A^{eff}(C, C_\dagger, C^\dagger)_\otimes := (A^{eff}(C, C^\dagger, C_\dagger)^\otimes)^{op}.$$

The functor $A^{eff}(C, C_\dagger, C^\dagger)_\otimes \longrightarrow \mathrm{N}\Lambda(\mathbf{F})^{op}$ is then a symmetric promonoidal structure on the Burnside $\infty$-category $A^{eff}(C, C^\dagger, C_\dagger)^{op} \simeq A^{eff}(C, C_\dagger, C^\dagger)$.

**2.16.** Suppose $(C, C_\dagger, C^\dagger)$ a cartesian disjunctive triple. Note that the formula

$$\coprod_{i\in I}(X \times U_i) \simeq X \times \left(\coprod_{i\in I} U_i\right)$$

implies immediately that the tensor product functor

$$\otimes\colon A^{eff}(C, C_\dagger, C^\dagger) \times A^{eff}(C, C_\dagger, C^\dagger) \longrightarrow A^{eff}(C, C_\dagger, C^\dagger)$$

preserves direct sums separately in each variable.

More generally, suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple, suppose $I$ a finite set, and suppose $\{x_i\}_{i\in I}$ a collection of objects of $C$, which we view, by the standard abuse, as an object of $A^{eff}(C, C_\dagger, C^\dagger)^\otimes_I$. Consider the 1-simplex $\xi_I : \Delta^1 \longrightarrow N\Lambda(\mathbf{F})$, and denote by $h^{\{x_i\}_{i\in I}}$ the restriction of the functor

$$A^{eff}(C, C_\dagger, C^\dagger)^\otimes \times_{N\Lambda(\mathbf{F})} \Delta^1 \longrightarrow \mathbf{Kan}$$

corepresented by $\{x_i\}_{i\in I}$ to $A^{eff}(C, C_\dagger, C^\dagger)$. Informally, this is the functor

$$\mathrm{Map}^{\xi_I}_{C^\otimes}(\{x_i\}_{i\in I}, -).$$

Suppose $j \in I$, and suppose $\{y_k \longrightarrow x_j\}_{k\in K}$ a family of morphisms that together exhibit $x_j$ as the coproduct $\coprod_{k\in K} y_k$. For each $i \in I$ and $k \in K$, write

$$x'_{i,k} := \begin{cases} y_k & \text{if } i = j; \\ x_i & \text{if } i \neq j. \end{cases}$$

Then the natural map

$$h^{\{x_i\}_{i\in I}} \longrightarrow \prod_{k\in K} h^{\{x'_{i,k}\}_{i\in I}}$$

is an equivalence.

**2.17.** For any disjunctive $\infty$-category $C$ that admits a terminal object, the duality functor

$$D\colon A^{\mathit{eff}}(C)^{op} \xrightarrow{\sim} A^{\mathit{eff}}(C)$$

of [3, Nt. 3.10] provides duals for the symmetric monoidal $\infty$-category $A^{\mathit{eff}}(C)^{\otimes}$ [16, Df. 2.3.5]. More precisely, for any object $X$ of $A^{\mathit{eff}}(C)$, there exists an evaluation morphism $X \otimes DX \longrightarrow 1$ given by the diagram

$$\begin{array}{ccc} & X & \\ {}^{\Delta}\swarrow & & \searrow^{!} \\ X \times X & & 1, \end{array}$$

and, dually, there exists a coevaluation morphism $1 \longrightarrow DX \otimes X$ given by the diagram

$$\begin{array}{ccc} & X & \\ {}^{!}\swarrow & & \searrow^{\Delta} \\ 1 & & X \times X. \end{array}$$

Since the square

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta\times \mathrm{id}} \\ X \times X & \xrightarrow[\mathrm{id}\times\Delta]{} & X \times X \times X \end{array}$$

is a pullback, it follows that the composite

$$X \longrightarrow X \otimes DX \otimes X \longrightarrow X$$

in $A^{\mathit{eff}}(C)$ is homotopic to the identity. We conclude that $A^{\mathit{eff}}(C)^{\otimes}$ is a symmetric monoidal $\infty$-category with duals.

**2.18.** If $(C, C_{\dagger}, C^{\dagger})$ is a cartesian disjunctive triple, then in general it is not quite the case that the symmetric monoidal $\infty$-category $A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger})^{\otimes}$ admits duals.

We have an evaluation morphism $X \otimes DX \longrightarrow 1$ in $A^{eff}(C, C_\dagger, C^\dagger)$ just in case the diagonal $\Delta\colon X \longrightarrow X \times X$ of $C$ is egressive, and the morphism $!\colon X \longrightarrow 1$ is ingressive. We have a coevaluation morphism $1 \longrightarrow DX \otimes X$ in $A^{eff}(C, C_\dagger, C^\dagger)$ just in case $\Delta$ is ingressive and ! is egressive.

**2.19.** If $(C, C_\dagger, C^\dagger)$ and $(D, D_\dagger, D^\dagger)$ are left complete disjunctive triples, then it is easy to see that a functor of disjunctive triples

$$f\colon (C, C_\dagger, C^\dagger) \longrightarrow (D, D_\dagger, D^\dagger)$$

induces a functor of adequate triples

$$(C_\times, (C_\times)_\dagger, (C_\times)^\dagger) \longrightarrow (D_\times, (D_\times)_\dagger, (D_\times)^\dagger)$$

and thus a morphism of $\infty$-operads

$$A^{eff}(f)^\otimes\colon A^{eff}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow A^{eff}(D, D_\dagger, D^\dagger)^\otimes.$$

If, furthermore, $(C, C_\dagger, C^\dagger)$ and $(D, D_\dagger, D^\dagger)$ are cartesian and $f$ preserves finite products, then $A^{eff}(f)^\otimes$ is of course a symmetric monoidal functor.

## 3. Green functors

Andreas Dress [9] defined Green functors as Mackey functors equipped with certain pairings. Gaunce Lewis [13] noticed that these pairings made them commutative monoids for the Day convolution tensor product on the category of Mackey functors. By an old observation of Brian Day [8, Ex. 3.2.2], these are precisely the lax symmetric monoidal additive functors on the effective Burnside category. Thanks to recent work of Saul Glasman [10], this characterization of monoids for the Day convolution holds in the $\infty$-categorical context as well.

**3.1. Definition.** We shall say that a symmetric monoidal $\infty$-category $E^\otimes$ is ***additive*** if the underlying $\infty$-category $E$ is additive, and the tensor product functor $\otimes\colon E \times E \longrightarrow E$ preserves direct sums separately in each variable.

**3.2. Definition.** (3.2.1) Suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple and $E^\otimes$ an additive symmetric monoidal $\infty$-category. Then a ***commutative Green functor*** is a morphism of $\infty$-operads

$$A^{eff}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow E^\otimes$$

such that the underlying functor $A^{eff}(C, C_\dagger, C^\dagger) \longrightarrow E$ preserves direct sums.

(3.2.2) More generally, if $O^\otimes$ is an $\infty$-operad, then an ***$O^\otimes$-Green functor*** is a morphism of $\infty$-operads

$$A^{eff}(C, C_\dagger, C^\dagger)^\otimes \times_{N\Lambda(\mathbf{F})} O^\otimes \longrightarrow E^\otimes \times_{N\Lambda(\mathbf{F})} O^\otimes$$

over $O^{\otimes}$ such that for any object $X$ of the underlying $\infty$-category $O$, the functor

$$A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger}) \simeq (A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger})^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} O^{\otimes})_X \longrightarrow (E^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} O^{\otimes})_X \simeq E$$

preserves direct sums.

(3.2.3) Similarly, for any perfect operator category $\Phi$, we may define a ***$\Phi$-Green functor*** as a morphism

$$A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger})^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} \mathrm{N}\Lambda(\Phi) \longrightarrow E^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} \mathrm{N}\Lambda(\Phi)$$

of $\infty$-operads over $\Phi$ such that the underlying functor $A^{\mathit{eff}}(C, C_{\dagger}, C^{\dagger}) \longrightarrow E$ preserves direct sums.

**3.3. Notation.** Suppose $(C, C_{\dagger}, C^{\dagger})$ a left complete disjunctive triple, and suppose $E^{\otimes}$ an additive symmetric monoidal $\infty$-category. For any $\infty$-operad $O^{\otimes}$, let us write, employing the notation of [18, Df. 2.1.3.1]

$$\mathbf{Green}_{O^{\otimes}}(C, C_{\dagger}, C^{\dagger}; E^{\otimes}) \subset \mathbf{Alg}_{A^{\mathit{eff}}(C,C_{\dagger},C^{\dagger})^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} O^{\otimes} \ / O^{\otimes}}(E^{\otimes} \times_{\mathrm{N}\Lambda(\mathbf{F})} O^{\otimes})$$

for the full subcategory spanned by the $O^{\otimes}$-Green functors.

**3.4. Example.** We define *modules* over an *associative Green functor* in this way. Suppose $(C, C_{\dagger}, C^{\dagger})$ a left complete disjunctive triple, and suppose $E^{\otimes}$ an additive symmetric monoidal $\infty$-category. Then we may consider the $\infty$-operad of [18, Df. 4.2.1.7], which we will denote $\mathrm{LM}^{\otimes}$. The inclusion $\mathrm{Ass}^{\otimes} \hookrightarrow \mathrm{LM}^{\otimes}$ induces a functor

$$\mathbf{Green}_{\mathrm{LM}^{\otimes}}(C, C_{\dagger}, C^{\dagger}; E^{\otimes}) \longrightarrow \mathbf{Green}_{\mathrm{Ass}^{\otimes}}(C, C_{\dagger}, C^{\dagger}; E^{\otimes}).$$

An object $A$ of the target may be called an ***associative Green functor***, and an object of the fiber of this functor over $A$ may be called a ***left $A$-module***. We write

$$\mathbf{Mod}^{\ell}_A(C, C_{\dagger}, C^{\dagger}; E^{\otimes}) := \mathbf{Green}_{\mathrm{LM}^{\otimes}}(C, C_{\dagger}, C^{\dagger}; E^{\otimes}) \times_{\mathbf{Green}_{\mathrm{Ass}^{\otimes}}(C,C_{\dagger},C^{\dagger};E^{\otimes})} \{A\}$$

for the $\infty$-category of left $A$-modules. When $A$ is a commutative Green functor, we will drop the superscript $\ell$.

The convolution of two Mackey functors will not in general be a Mackey functor, but it can replaced with one by employing a localization (which we might as well call Mackeyification). To prove that convolution followed by Mackeyification defines a symmetric monoidal structure on the $\infty$-category of Mackey functors, it is necessary to show that Mackeyification is *compatible* with the convolution symmetric monoidal structure in the sense of Lurie [18, Df. 2.2.1.6, Ex. 2.2.1.7].

The following is immediate from [3, Pr. 6.5].

**3.5. Lemma.** *Suppose* $(C, C_\dagger, C^\dagger)$ *a disjunctive triple, and suppose* $E$ *a presentable additive* $\infty$*-category. Then the* $\infty$*-category* $\mathbf{Mack}(C, C_\dagger, C^\dagger; E)$ *is an accessible localization of the* $\infty$*-category* $\mathrm{Fun}(A^{eff}(C, C_\dagger, C^\dagger), E)$.

**3.6. Notation.** Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive $\infty$-category, and suppose $E$ a presentable additive $\infty$-category. Then write $M$ for the left adjoint to the fully faithful inclusion

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; E) \hookrightarrow \mathrm{Fun}(A^{eff}(C, C_\dagger, C^\dagger), E).$$

**3.7. Lemma.** *Let* $(C, C_\dagger, C^\dagger)$ *be a left complete disjunctive* $\infty$*-category and* $E^\otimes$ *a presentable symmetric monoidal additive* $\infty$*-category. Then the left adjoint* $M$ *constructed above is compatible* (in the sense of [18, Df. 2.2.1.6]) *with Glasman's Day convolution symmetric monoidal structure on* $\mathrm{Fun}(A^{eff}(C, C_\dagger, C^\dagger), E)$.

*Proof.* For any collection of objects $\{s_i \mid i \in I\}$ of $C$, let

$$h^{\{s_i\}}\colon A^{eff}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Kan}$$

be as in 2.16, and for any object $x \in E$, let

$$-\otimes x\colon \mathrm{Fun}(A^{eff}(C, C_\dagger, C^\dagger), \mathbf{Kan}) \longrightarrow \mathrm{Fun}(A^{eff}(C, C_\dagger, C^\dagger), E)$$

be the composition with the tensor product $-\otimes x\colon \mathbf{Kan} \longrightarrow E$ with spaces [15, §4.]. Thus objects of the form $h^{\{s_i\}} \otimes x$ generate the $\infty$-category $\mathrm{Fun}(A^{eff}(C, C_\dagger, C^\dagger), E)$ under colimits. It is easy to see that for any functors $f, g\colon A^{eff}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Kan}$ and any object $x \in E$, the map

$$(f \times g) \otimes x \longrightarrow (f \otimes x) \oplus (g \otimes x)$$

is an $M$-equivalence; furthermore, the class of $M$-equivalences is the strongly saturated class generated by the canonical morphisms

$$h^{s\oplus t} \otimes x \longrightarrow (h^s \otimes x) \oplus (h^t \otimes x).$$

This tensor product and the Day convolution are compatible in the sense that there are natural equivalences

$$(h^s \otimes x) \otimes (h^t \otimes y) \simeq h^{\{s,t\}} \otimes (x \otimes y),$$

whence one obtains natural $M$-equivalences

$$\begin{aligned}
((h^s \otimes x) \oplus (h^t \otimes x)) \otimes (h^u \otimes y) &\simeq ((h^s \otimes x) \otimes (h^u \otimes y)) \oplus ((h^t \otimes x) \otimes (h^u \otimes y)) \\
&\simeq (h^{\{s,u\}} \otimes x \otimes y) \oplus (h^{\{t,u\}} \otimes x \otimes y) \\
&\longrightarrow (h^{\{s,u\}} \times h^{\{t,u\}}) \otimes x \otimes y \\
&\simeq h^{\{s\oplus t,u\}} \otimes x \otimes y \\
&\simeq h^{s\oplus t} \otimes x \otimes h^u \otimes y.
\end{aligned}$$

Hence for any $M$-equivalence $X \longrightarrow Y$ and any object $Z \in \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E)$, the morphism

$$X \otimes Z \longrightarrow Y \otimes Z$$

is an $M$-equivalence. □

**3.8.** In particular, if $(C, C_\dagger, C^\dagger)$ is a left complete disjunctive triple, and if $E^\otimes$ a presentable symmetric monoidal additive $\infty$-category, we obtain a symmetric monoidal $\infty$-category $\mathbf{Mack}(C, C_\dagger, C^\dagger; E)^\otimes$, and, in light of [10], for any $\infty$-operad $O^\otimes$, one obtains an equivalence

$$\mathbf{Alg}_{O^\otimes}(\mathbf{Mack}(C, C_\dagger, C^\dagger; E)^\otimes) \simeq \mathbf{Green}_{O^\otimes}(C, C_\dagger, C^\dagger; E).$$

## 4. Green stabilization

Now let us address the issue of multiplicative structures on the Mackey stabilization, as constructed in [3, §7]. In particular, we aim to show that if $E$ is an $\infty$-topos, then the Mackey stabilization of a morphism of operads

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow E^\times$$

naturally admits the structure of a Green functor

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow \mathbf{Sp}(E)^\otimes.$$

**4.1. Definition.** Suppose $(C, C_\dagger, C^\dagger)$ a cartesian disjunctive triple, suppose $E$ an $\infty$-topos, and suppose

$$f\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow E^\times \quad \text{and} \quad F\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow \mathbf{Sp}(E)^\otimes$$

morphisms of $\infty$-operads. Then a morphism of $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes$-algebras

$$\eta\colon f \longrightarrow \Omega^\infty \circ F$$

will be said to exhibit $F$ as the ***Green stabilization*** of $f$ if $F$ is a Green functor, and if, for any Green functor $R\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow \mathbf{Sp}(E)^\otimes$, the map

$$\mathrm{Map}_{\mathbf{Green}_{E_\infty}(C, C_\dagger, C^\dagger; \mathbf{Sp}(E)^\otimes)}(F, R) \longrightarrow \mathrm{Map}_{\mathbf{Alg}_{A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes}(E^\times)}(f, \Omega^\infty \circ R)$$

induced by $\eta$ is an equivalence.

The following result is essentially the same as [1, Pr. 2.1].

**4.2. Proposition.** *Suppose $(C, C_\dagger, C^\dagger)$ a cartesian disjunctive triple. There exists a symmetric monoidal $\infty$-category $\mathrm{D}A(C, C_\dagger, C^\dagger)^\otimes$ and a fully faithful symmetric monoidal functor*

$$j^\otimes\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \hookrightarrow \mathrm{D}A(C, C_\dagger, C^\dagger)^\otimes$$

*with the following properties.*

*(4.2.1) The $\infty$-category $\mathrm{D}A(C, C_\dagger, C^\dagger)$ underlies $\mathrm{D}A(C, C_\dagger, C^\dagger)^\otimes$, and the underlying functor of $j^\otimes$ is the inclusion*

$$j\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \hookrightarrow \mathrm{D}A(C, C_\dagger, C^\dagger)$$

*of* [3, Nt. 7.2].

*(4.2.2) For any symmetric monoidal $\infty$-category $E^\otimes$ whose underlying $\infty$-category admits all sifted colimits such that the tensor product preserves sifted colimits separately in each variable, the induced functor*

$$\mathbf{Alg}_{\mathrm{D}A(C,C_\dagger,C^\dagger)^\otimes}(E^\otimes) \longrightarrow \mathbf{Alg}_{A^{\mathit{eff}}(C,C_\dagger,C^\dagger)^\otimes}(E^\otimes)$$

*exhibits an equivalence from the full subcategory spanned by those morphisms of $\infty$-operads $A$ whose underlying functor $A\colon \mathrm{D}A(C, C_\dagger, C^\dagger) \longrightarrow E$ preserves sifted colimits to the full subcategory spanned by those morphisms of $\infty$-operads $B$ whose underlying functor $B\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow E$ preserves filtered colimits.*

*(4.2.3) The tensor product functor*

$$\otimes\colon \mathrm{D}A(C, C_\dagger, C^\dagger) \times \mathrm{D}A(C, C_\dagger, C^\dagger) \longrightarrow \mathrm{D}A(C, C_\dagger, C^\dagger)$$

*preserves all colimits separately in each variable.*

*Proof.* The only part that is not a consequence of [18, Pr. 4.8.1.10 and Var. 4.8.1.11] is the assertion that the tensor product functor

$$\otimes\colon \mathrm{D}A(C, C_\dagger, C^\dagger) \times \mathrm{D}A(C, C_\dagger, C^\dagger) \longrightarrow \mathrm{D}A(C, C_\dagger, C^\dagger)$$

preserves direct sums separately in each variable. This assertion holds for objects of the effective Burnside category $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ thanks to the universality of coproducts in $C$; the general case follows by exhibiting any object of $\mathrm{D}A(C, C_\dagger, C^\dagger)$ as a colimit of a sifted diagram of objects of $A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$ and using the fact that both the tensor product and the direct sum commute with sifted colimits. □

In light of [1, Pr. 3.5] and [18, Pr. 6.2.4.14 and Th. 6.2.6.2], we now have the following.

**4.3. Proposition.** *Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple, suppose $E$ an $\infty$-topos, and suppose*

$$f\colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow E^\times$$

*a morphism of $\infty$-operads. Then a Green stabilization of $f$ exists. In particular, the functor*

$$\Omega^\infty \circ -\colon \mathbf{Green}(C, C_\dagger, C^\dagger; \mathbf{Sp}(E)^\otimes) \longrightarrow \mathbf{Alg}_{A^{\mathit{eff}}(C,C_\dagger,C^\dagger)^\otimes}(E^\times)$$

*admits a left adjoint that covers the left adjoint of the functor*

$$\Omega^\infty \circ - : \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}(E)) \longrightarrow \mathrm{Fun}(A^{\mathit{eff}}(C, C_\dagger, C^\dagger), E).$$

**4.4. Example.** Suppose $(C, C_\dagger, C^\dagger)$ a cartesian disjunctive triple. Then the functor

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger) \longrightarrow \mathbf{Kan}$$

corepresented by the terminal object 1 of $C$ is the unit for the Day convolution symmetric monoidal structure of Glasman, and hence it is an $E_\infty$ algebra in an essentially unique fashion. Thus we can consider its Green stabilization

$$\mathbf{S}^\otimes = \mathbf{S}^\otimes_{(C,C_\dagger,C^\dagger)} : A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow \mathbf{Sp}^\otimes,$$

whose underlying Mackey functor is the Burnside Mackey functor $\mathbf{S}_{(C,C_\dagger,C^\dagger)}$ of [3]. We call $\mathbf{S}^\otimes$ the ***Burnside Green functor***.

In a similar vein, we immediately have the following:

**4.5. Proposition.** *For any cartesian disjunctive triple* $(C, C_\dagger, C^\dagger)$, *the functor*

$$A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^{op} \longrightarrow \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp})$$

*given by the assignment* $X \rightsquigarrow \mathbf{S}^X$ *is naturally symmetric monoidal. That is, for any two objects* $X, Y \in C$, *one has a canonical equivalence*

$$\mathbf{S}^X \otimes \mathbf{S}^Y \simeq \mathbf{S}^{X \times Y}$$

**4.5.1. Corollary.** *Suppose* $(C, C_\dagger, C^\dagger)$ *a cartesian disjunctive triple. For any spectral Mackey functor* $M$ *thereon, write* $F(M, -)$ *for the right adjoint to the functor*

$$- \otimes M : \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}) \longrightarrow \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}).$$

*Then for any object* $X \in C$, *the Mackey functor* $F(\mathbf{S}^X, M)$ *is given by the assignment*

$$Y \rightsquigarrow M(X \times Y).$$

The following is now immediate.

**4.6. Proposition.** *Suppose* $(C, C_\dagger, C^\dagger)$ *a cartesian disjunctive triple. The Burnside Mackey functor* $\mathbf{S}_{(C,C_\dagger,C^\dagger)}$ *is the unit in the symmetric monoidal* $\infty$*-category* $\mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp})^\otimes$. *Consequently, the Burnside Green functor* $\mathbf{S}^\otimes_{(C,C_\dagger,C^\dagger)}$ *is the initial object in the* $\infty$*-category* $\mathbf{Green}_{\mathrm{N}\Lambda(\mathbf{F})}(C, C_\dagger, C^\dagger; \mathbf{Sp}^\otimes)$, *and the forgetful functor*

$$\mathbf{Mod}_{\mathbf{S}^\otimes}(C, C_\dagger, C^\dagger; \mathbf{Sp}^\otimes) \xrightarrow{\sim} \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp})$$

*is an equivalence.*

## 5. Duality

In this section, suppose $C$ a disjunctive $\infty$-category that admits a terminal object. Since the functor $X \rightsquigarrow \mathbf{S}^X$ is symmetric monoidal, it follows immediately that every representable Mackey functor $\mathbf{S}^X$ is strongly dualizable, and

$$(\mathbf{S}^X)^\vee \simeq \mathbf{S}^{DX}$$

**5.1. Notation.** For any associative spectral Green functor $R$ and for any object $X \in C$, denote by $R^X$ the left $R$-module $R \otimes \mathbf{S}^X$, and denote by ${}^XR$ the right $R$-module $\mathbf{S}^X \otimes R$.

Of course for any left (respectively, right) $R$-module $M$, one has

$$\mathrm{Map}(R^X, M) \simeq \Omega^\infty M(X) \qquad (\text{resp.,} \quad \mathrm{Map}({}^XR, M) \simeq \Omega^\infty M(X) \quad ).$$

**5.2. Definition.** For any associative spectral Green functor $R$ on $C$, denote by $\mathbf{Perf}^\ell_R$ the smallest stable subcategory of the $\infty$-category $\mathbf{Mod}^\ell_R$ that contains the left $R$-modules $R^X$ (for $X \in C$) and is closed under retracts. Similarly, denote by $\mathbf{Perf}^r_R$ the smallest stable subcategory of the $\infty$-category $\mathbf{Mod}^r_R$ that contains the right $R$-modules ${}^XR$ (for $X \in C$) and is closed under retracts.

The objects of $\mathbf{Perf}^\ell_R$ (respectively, $\mathbf{Perf}^r_R$) will be called ***perfect*** left (resp., right) modules over $R$.

Now we obtain the following, which is a straightforward analogue of [18, Pr. 7.2.5.2].

**5.3. Proposition.** *For any associative spectral Green functor $R$, a left $R$-module is compact just in case it is perfect.*

*Proof.* For any $X \in C$, the functor corepresented by $R^X$ is the assignment $M \rightsquigarrow \Omega^\infty M(X)$, which preserves filtered colimits. Hence $R^X$ is compact, and thus any perfect left $R$-module is compact.

Conversely, there is a fully faithful, colimit-preserving functor

$$F\colon \mathrm{Ind}(\mathbf{Perf}^\ell_R) \hookrightarrow \mathbf{Mod}_R$$

induced by the inclusion $\mathbf{Perf}^\ell_R \hookrightarrow \mathbf{Mod}^\ell_R$. If this is not essentially surjective, there exists a nonzero left $R$-module $M$ such that for every $R$-module $N$ in the essential image of $F$, the group $[N, M]$ vanishes. In particular, for any integer $n$ and any object $X \in C$,

$$\pi_n M(X) \cong [R^X[n], M] \cong 0,$$

whence $M \simeq 0$. ☐

The proof of the following is word-for-word identical to that of [18, Pr. 7.2.5.4].

**5.4. Proposition.** *For any associative spectral Green functor $R$ on $C$, a left $R$-module $M$ is perfect just in case there exists a right $R$-module $M^\vee$ that is dual to $M$ in the sense that the functor*

$$\mathrm{Map}(\mathbf{S}, M^\vee \otimes_R -)\colon \mathbf{Mod}^\ell_R \longrightarrow \mathbf{Kan}$$

*is the functor that $M$ corepresents.*

**5.5. Example.** Note that, in particular, for any object $X \in C$, one has

$$(R^X)^\vee \simeq {}^{DX}R.$$

## 6. The Künneth spectral sequence

Let us note that the Künneth spectral sequence works in the Mackey functor context more or less exactly as in the ordinary $\infty$-category of spectra. To this end, let us first discuss $t$-structures on $\infty$-categories of spectral Mackey functors.

**6.1. Proposition.** *Suppose $(C, C_\dagger, C^\dagger)$ a disjunctive triple, and suppose $A$ a stable $\infty$-category equipped with a $t$-structure $(A_{\geq 0}, A_{\leq 0})$. Then the two subcategories*

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; A)_{\geq 0} := \mathbf{Mack}(C, C_\dagger, C^\dagger; A_{\geq 0})$$

*and*

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; A)_{\leq 0} := \mathbf{Mack}(C, C_\dagger, C^\dagger; A_{\leq 0})$$

*define a $t$-structure on* $\mathbf{Mack}(C, C_\dagger, C^\dagger; A)$.

*Proof.* Consider the functor $L\colon \mathbf{Mack}(C, C_\dagger, C^\dagger; A) \longrightarrow \mathbf{Mack}(C, C_\dagger, C^\dagger; A)$ given by composition with $\tau_{\leq -1}$; it is clear that $L$ is a localization functor. Furthermore, the essential image of $L$ is the $\infty$-category $\mathbf{Mack}(C, C_\dagger, C^\dagger; A_{\leq -1})$, which is closed under extensions, since $A_{\leq -1}$ is. Now we apply [18, Pr. 1.2.1.16]. □

**6.2.** Note that if $A$ a stable $\infty$-category equipped with a $t$-structure $(A_{\geq 0}, A_{\leq 0})$, then for any disjunctive triple $(C, C_\dagger, C^\dagger)$, the heart of the induced $t$-structure on $\mathbf{Mack}(C, C_\dagger, C^\dagger; A)$ is given by

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; A)^\heartsuit \simeq \mathbf{Mack}(C, C_\dagger, C^\dagger; A^\heartsuit).$$

Furthermore, it is clear that many properties of the $t$-structure on $A$ are inherited by the induced $t$-structure $\mathbf{Mack}(C, C_\dagger, C^\dagger; A)$: in particular, one verifies easily that the $t$-structure on $\mathbf{Mack}(C, C_\dagger, C^\dagger; A)$ is left bounded, right bounded, left complete, right complete, compatible with sequential colimits, compatible with filtered colimits, or accessible if the $t$-structure on $A$ is so.

**6.3. Example.** For any disjunctive triple $(C, C_\dagger, C^\dagger)$, the $\infty$-category

$$\mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp})$$

admits an accessible $t$-structure that is both left and right complete whose heart is the abelian category $\mathbf{Mack}(C, C_\dagger, C^\dagger; N\mathbf{Ab})$. Observe that the corepresentable functors $\tau_{\leq 0}\mathbf{S}^X$ are projective objects in the heart, and thus the heart has enough projectives.

In particular, if $G$ is a profinite group and if $C$ is the disjunctive $\infty$-category of finite $G$-sets, then the $\infty$-category $\mathbf{Mack}_G$ of spectral Mackey functors for $G$ admits an accessible $t$-structure that is both left and right complete, in which the heart $\mathbf{Mack}_G^\heartsuit$ is the nerve of the usual abelian category of Mackey functors for $G$.

**6.4. Construction.** Suppose $A$ is a stable $\infty$-category equipped with a $t$-structure. Let $(C, C_\dagger, C^\dagger)$ be a disjunctive triple and $X \colon N\mathbf{Z} \longrightarrow \mathbf{Mack}(C, C_\dagger, C^\dagger; A)$ a filtered Mackey functor with colimit $X(+\infty)$. Then we have the spectral sequence

$$E_r^{p,q} := \operatorname{im}\left[\pi_{p+q}\left(\frac{X(p)}{X(p-r)}\right) \longrightarrow \pi_{p+q}\left(\frac{X(p+r-1)}{X(p-1)}\right)\right]$$

associated with $X$ [18, Df. 1.2.2.9].

Note that this is a spectral sequence of $A^\heartsuit$-valued Mackey functors. Since limits and colimits of Mackey functors are defined objectwise, it follows that for any object $U \in A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$, the value $E_r^{p,q}(U)$ is the spectral sequence (in $A^\heartsuit$) associated with the filtered object $X(U) \colon N\mathbf{Z} \longrightarrow A$.

**6.5.** In the setting of Cnstr. 6.4, assume that $A$ admits all sequential colimits and that the $t$-structure is compatible with these colimits. If $X(n) \simeq 0$ for $n \ll 0$, then the associated spectral sequence converges to a filtration on $\pi_{p+q}(X(+\infty))$ [18, 1.2.2.14]. That is:

- For any $p$ and $q$, there is $r \gg 0$ such that the differential $d_r \colon E_r^{p,q} \longrightarrow E_r^{p-r,q+r-1}$ vanishes.
- For any $p$ and $q$, there exist a discrete, exhaustive filtration

$$\cdots \subset F_{p+q}^{-1} \subset F_{p+q}^{0} \subset F_{p+q}^{1} \subset \cdots \subset \pi_{p+q}X(+\infty)$$

  and an isomorphism $E_\infty^{p,q} \cong F_{p+q}^{p}/F_{p+q}^{p-1}$.

In more general circumstances, one can obtain a kind of "local convergence." Suppose again that $A$ admits all sequential colimits, and that the $t$-structure is compatible with these colimits. Now suppose that for every object $U \in A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$, there exists $n \ll 0$ such that $X(n)(U) \simeq 0$. Then for every object $U \in A^{\mathit{eff}}(C, C_\dagger, C^\dagger)$, the spectral sequence $E_r^{p,q}(U)$ converges to $\pi_{p+q}(X(+\infty)(U))$. In finitary cases (e.g., when $C$ is the disjunctive $\infty$-category of finite $G$-sets for a finite group $G$), there is no difference between the local convergence and the global convergence.

Better convergence results can be obtained when the filtered Mackey functor is the skeletal filtration of a simplicial connective object $Y_*$ [18, Pr. 1.2.4.5]. In this case, we do not need to assume that the $t$-structure on $A$ is compatible with sequential colimits, the associated spectral sequence is a first-quadrant spectral sequence, and it converges to a length $p+q$ filtration on $\pi_{p+q}|Y_*|$.

Now, to construct the Künneth spectral sequence for Mackey functors, we can follow very closely the arguments of Lurie [18, §7.2.1].

**6.6. Lemma.** *Suppose* $(C, C_\dagger, C^\dagger)$ *a disjunctive triple. Then the collection of corepresentable Mackey functors* $\{\mathbf{S}^X \mid X \in A^{eff}(C, C_\dagger, C^\dagger)\}$ *is a set of compact projective generators for* $\mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}_{\geq 0})$ *in the sense of* [15, Dfn. 5.5.2.3].

*Proof.* The corepresentable functors provide a set of compact projective generators for the $\infty$-category $\mathrm{Fun}^\times(A^{eff}(C, C_\dagger, C^\dagger), \mathbf{Kan})$ because this category is precisely $P_\Sigma(A^{eff}(C, C_\dagger, C^\dagger)^{op})$. The functor

$$\Omega^\infty \circ -\colon \mathbf{Mack}(C, C_\dagger, C^\dagger; \mathbf{Sp}_{\geq 0}) \longrightarrow \mathrm{Fun}^\times(A^{eff}(C, C_\dagger, C^\dagger), \mathbf{Kan})$$

preserves sifted colimits and is conservative, since $\Omega^\infty\colon \mathbf{Sp}_{\geq 0} \longrightarrow \mathbf{Kan}$ preserves sifted colimits by [18, 1.4.3.9] and is conservative, and the inclusion of both sides into all functors preserves sifted colimits (we use that **Kan** is cartesian closed). We conclude by applying [18, 4.7.4.18]. ☐

To set up the spectral sequence we need to impose the hypotheses of strong dualizability on the $\mathbf{S}^X$. Because of this, we now work in the generality of $C$ a disjunctive $\infty$-category which admits a terminal object.

Suppose

$$R\colon A^{eff}(C)^\otimes \times_{N\Lambda(\mathbf{F})} \mathrm{Ass}^\otimes \longrightarrow \mathbf{Sp}^\wedge \times_{N\Lambda(\mathbf{F})} \mathrm{Ass}^\otimes$$

an associative Green functor, suppose $M$ a right $R$-module, and suppose $N$ a left $R$-module. There is a comparison map

$$\mathrm{Tor}_0^{\pi_* R}(\pi_* M, \pi_* N) \longrightarrow \pi_*(M \otimes_R N)$$

constructed as follows: given $x \in \pi_m M(U)$ and $y \in \pi_n N(V)$, choose representatives $\Sigma^m({}^U R) \longrightarrow M$ and $\Sigma^n(R^V) \longrightarrow N$ and take their smash product to obtain a map

$$\Sigma^{m+n}(\mathbf{S}^{U\times V}) \longrightarrow \Sigma^{m+n}(\mathbf{S}^{U\times V}) \otimes R \simeq \Sigma^m({}^U R) \otimes_R \Sigma^n(R^V) \longrightarrow M \otimes_R N$$

and thus an element $x \otimes y \in \pi_{m+n}(M \otimes_R N)(U \times V)$; this is suitably natural so that it descends to a map out of the Day convolution tensor product $\pi_* M \otimes_{\pi_* R} \pi_* N$ to $\pi_*(M \otimes_R N)$. This map is not usually an isomorphism. Instead, we construct a spectral sequence that converges to $\pi_*(M \otimes_R N)$, in which this map appears as an edge homomorphism.

Let $S$ denote the class of left $R$-modules of the form $\Sigma^n R^X$ for $n \in \mathbf{Z}$ and $X \in C$. By [18, Pr. 7.2.1.4], there exists an $S$-free $S$-hypercovering $P_\bullet \longrightarrow N$ in the (presentable) stable $\infty$-category $\mathbf{Mod}_R^\ell$.

**6.7. Lemma.** *For any $S$-hypercovering $P_\bullet \longrightarrow N$, we have that $|P_\bullet| \simeq N$.*

*Proof.* Let $S_{\geq n}$ be the subset of $S$ on $\Sigma^m \circ R^X$ for $m \geq n$. From our $S$-hypercovering $P_\bullet \longrightarrow N$, we obtain $S_{\geq n}$-hypercoverings $\tau_{\geq n} P_\bullet \longrightarrow \tau_{\geq n} N$ for every $n \in \mathbf{Z}$. Since the $\Sigma^n S^X$, $X \in C$ constitute a set of projective generators for $\mathbf{Mack}(C; \mathbf{Sp}_{\geq n})$ by Lm. 6.6, we have that $\left|\tau_{\geq n} P_\bullet\right| \simeq \tau_{\geq n} N$ by the hypercompleteness of **Kan**. By the right completeness of the $t$-structure, we deduce that $|P_\bullet| \simeq N$. □

By passing to the skeletal filtration of $M \otimes_R |P_\bullet|$, we obtain a spectral sequence $\{E_r^{p,q}, d_r\}_{r \geq 1}$ that converges to $\pi_{p+q}(M \otimes_R N)$. The complex $(E_1^{*,q}, d_1)$ is the normalized chain complex $N_*(\pi_q(M \otimes_R P_\bullet))$.

To proceed, we need to prove the following analogue of [18, Pr. 7.2.1.17].

**6.8. Lemma.** *If $P$ is a direct sum of objects in $S$, then the map*

$$\mathrm{Tor}_0^{\pi_* R}(\pi_* M, \pi_* P) \longrightarrow \pi_*(M \otimes_R P)$$

*is an isomorphism.*

*Proof.* Both sides commute with direct sums and shifts, so we reduce to the case of $P = R^X$. We claim first that for any spectral Mackey functor E,

$$\pi_* E \otimes \tau_{\leq 0} \mathbf{S}^X \cong \pi_*(E \otimes \mathbf{S}^X).$$

Since $\tau_{\leq 0}\mathbf{S}^Y$ corepresents evaluation at $Y$ for **Ab**-valued Mackey functors, and $\tau_{\leq 0}\mathbf{S}^X$ has dual $\tau_{\leq 0}\mathbf{S}^{DX}$, we have $(\pi_* E \otimes \tau_{\leq 0}\mathbf{S}^X)(Y) \cong (\pi_* E)(Y \times DX)$. Similarly, corepresentability and strong dualizability on the level of the **Sp**-valued Mackey functors implies that $\pi_*(E \otimes \mathbf{S}^X)(Y) \cong (\pi_* E)(Y \times DX)$, so we conclude. Now we apply this claim both for $M$ and $R$ to see that

$$\begin{aligned} \pi_* M \otimes_{\pi_* R} \pi_*(R^X) &\cong \pi_* M \otimes_{\pi_* R} (\pi_* R \otimes \tau_{\leq 0} \mathbf{S}^X) \\ &\cong \pi_* M \otimes \tau_{\leq 0} \mathbf{S}^X \\ &\cong \pi_*(M \otimes \mathbf{S}^X) \\ &\cong \pi_*(M \otimes_R R^X). \end{aligned}$$

We leave the identification of the specified map with this isomorphism to the reader. □

We thus obtain an isomorphism

$$\mathrm{Tor}_0^{\pi_* R}(\pi_* M, \pi_* P_\bullet) \cong \pi_*(M \otimes_R P_\bullet).$$

As $P_\bullet$ is an $S$-free $S$-hypercovering of $N$, $N_*(\pi_* P_\bullet)$ is a resolution of $\pi_* N$ by projective $\pi_* R$-modules. It follows that the $E_2$ page is given by

$$E_2^{p,q} \cong \mathrm{Tor}_p^{\pi_* R}(\pi_* M, \pi_* N)_q.$$

As in [18, Cor. 7.2.1.23], we have an immediate corollary.

**6.8.1. Corollary.** *Suppose $C$, $R$, $M$, and $N$ as above. Suppose that $R$, $M$, and $N$ are all connective. Then $M \otimes_R N$ is connective, and one has an isomorphism of ordinary Mackey functors*

$$\pi_0(M \otimes_R N) \cong \pi_0 M \otimes_{\pi_0 R} \pi_0 N.$$

**6.9. Example.** If $C$ is the category of finite $G$-sets for $G$ a finite group, then our Künneth spectral sequence recovers that of Lewis and Mandell in [14]. We refer the reader there to a more extensive discussion of this spectral sequence in that particular case.

## 7. Symmetric monoidal Waldhausen bicartesian fibrations

In [2], we define an $O^\otimes$-monoidal Waldhausen $\infty$-category for any $\infty$-operad $O^\otimes$ as an $O^\otimes$-algebra in the symmetric monoidal $\infty$-category $\mathbf{Wald}_\infty^\otimes$. We give two equivalent fibrational formulations of this notion.

**7.1. Definition.** Suppose $O^\otimes$ an $\infty$-operad. An ***$O^\otimes$-monoidal Waldhausen $\infty$-category*** consists of a pair cocartesian fibration [2, Df. 3.8]

$$p^\otimes : \mathbf{X}^\otimes \longrightarrow O^\otimes$$

such that the following conditions obtain.

(7.1.1) The composite

$$\mathbf{X}^\otimes \longrightarrow O^\otimes \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

exhibits $\mathbf{X}^\otimes$ as an $\infty$-operad.

(7.1.2) The fiber $p \colon \mathbf{X} \longrightarrow O$ over $* \in \mathrm{N}\Lambda(\mathbf{F})$ is a Waldhausen cocartesian fibration.

(7.1.3) For any finite set $I$ and any choice of inert morphisms $\{\rho^i : s \longrightarrow s_i\}_{i \in I}$ covering the inert morphisms $I \longrightarrow \{i\}$, an edge $\eta$ of $\mathbf{X}_s^\otimes$ is ingressive if and only if, for every $i \in I$, the edge $(\rho^i)_!(\eta)$ of $\mathbf{X}_{s_i}$ is ingressive.

(7.1.4) For any finite set $I$, any morphism $\mu \colon s \longrightarrow t$ of $O^\otimes$ covering the unique active morphism $I \longrightarrow \{\xi\}$, and any choice of inert morphisms $\{s \longrightarrow s_i \mid i \in I\}$ covering the inert morphisms $I \longrightarrow \{i\}$, the functor of pairs

$$\mu_! \colon \prod_{i \in I} \mathbf{X}_{s_i} \simeq \mathbf{X}_s^\otimes \longrightarrow \mathbf{X}_t$$

is exact separately in each variable [1].

Dually, suppose $O_{\otimes}$ an $\infty$-anti-operad. Then a ***$O_{\otimes}$-monoidal Waldhausen $\infty$-category*** is a pair cartesian fibration

$$p_{\otimes}\colon \mathbf{X}_{\otimes} \longrightarrow O_{\otimes}$$

such that the following conditions obtain.

(7.1.5) The composition

$$\mathbf{X}_{\otimes} \longrightarrow O_{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})^{op}$$

exhibits $\mathbf{X}_{\otimes}$ as an $\infty$-anti-operad.

(7.1.6) The fiber $p\colon \mathbf{X} \longrightarrow O$ over $* \in \mathrm{N}\Lambda(\mathbf{F})^{op}$ is a Waldhausen cartesian fibration.

(7.1.7) For any finite set $I$ and any choice of inert morphisms $\{\pi_i\colon s \longrightarrow s_i\}_{i\in I}$ covering the inert morphisms $I \longrightarrow \{i\}$, an edge $\eta$ of $\mathbf{X}^{\otimes}_s$ is ingressive if and only if, for every $i \in I$, the edge $\pi_i^{\star}(\eta)$ of $\mathbf{X}_{s_i}$ is ingressive.

(7.1.8) For any finite set $I$, any morphism $\mu\colon t \longrightarrow s$ of $O_{\otimes}$ covering the opposite of the unique active morphism $I \longrightarrow \{\xi\}$, and any choice of inert morphisms $\{s_i \longrightarrow s\}_{i\in I}$ covering the inert morphisms $I \longrightarrow \{i\}$, the functor of pairs

$$\mu^{\star}\colon \prod_{i\in I} \mathbf{X}_{s_i} \simeq \mathbf{X}_{\otimes,s} \longrightarrow \mathbf{X}_t$$

is exact separately in each variable.

Employing [18, Ex. 2.4.2.4 and Pr. 2.4.2.5] and [1, Lm 1.4], one deduces the following.

**7.2. Proposition.** *Suppose $O^{\otimes}$ (respectively, $O_{\otimes}$) an $\infty$-operad (resp., an $\infty$-anti-operad). Then the functor*

$$O^{\otimes} \longrightarrow \mathbf{Cat}_{\infty} \qquad (\textit{resp., the functor} \quad (O_{\otimes})^{op} \longrightarrow \mathbf{Cat}_{\infty} \quad )$$

*classifying an $O^{\otimes}$-monoidal Waldhausen $\infty$-category (resp., an $O_{\otimes}$-monoidal Waldhausen $\infty$-category) factors through an essentially unique morphism of $\infty$-operads*

$$O^{\otimes} \longrightarrow \mathbf{Wald}^{\otimes}_{\infty} \qquad (\textit{resp., the functor} \quad (O_{\otimes})^{op} \longrightarrow \mathbf{Wald}^{\otimes}_{\infty} \quad )$$

**7.3. Definition.** Now suppose $(C, C_{\dagger}, C^{\dagger})$ a left complete disjunctive triple. A ***symmetric monoidal Waldhausen bicartesian fibration***

$$p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$$

over $(C, C_{\dagger}, C^{\dagger})$ is a functor of pairs $\mathbf{X}_{\boxtimes} \longrightarrow (C_{\times})^{\flat}$ with the following properties.

(7.3.1) The underlying functor $p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$ is an inner fibration.

(7.3.2) For any egressive morphism $(\phi, \omega) : (I, X) \twoheadrightarrow (J, Y)$ of $C_\times$ (in the sense of Nt. 2.1) and for any object $Q$ of the fiber $(\mathbf{X}_\boxtimes)_{(J,Y)}$, there exists a $p_\boxtimes$-cartesian morphism $P \longrightarrow Q$ covering $(\phi, \omega)$.

(7.3.3) The composition

$$\mathbf{X}_\boxtimes \longrightarrow C_\times \longrightarrow \mathrm{N}\Lambda(\mathbf{F})^{op}$$

exhibits $\mathbf{X}_\boxtimes$ as an $\infty$-anti-operad.

(7.3.4) The fiber $p\colon \mathbf{X} \longrightarrow C$ over $* \in \mathrm{N}\Lambda(\mathbf{F})^{op}$ is a Waldhausen bicartesian fibration $\mathbf{X} \longrightarrow C$ over $(C, C_\dagger, C^\dagger)$.

**7.4.** This is a lot of data, so let's unpack it a bit.

First, a symmetric monoidal Waldhausen bicartesian fibration

$$p_\boxtimes\colon \mathbf{X}_\boxtimes \longrightarrow C_\times$$

over $(C, C_\dagger, C^\dagger)$ admits an underlying Waldhausen bicartesian fibration $p\colon \mathbf{X} \longrightarrow C$ over $(C, C_\dagger, C^\dagger)$. This provides, for any object $S \in C$, a Waldhausen $\infty$-category $\mathbf{X}_S$, and for any morphism $\phi\colon S \longrightarrow T$ of $C$, it provides an exact "pushforward" functor $\phi_!\colon \mathbf{X}_S \longrightarrow \mathbf{X}_T$ whenever $\phi$ is ingressive and an exact "pullback" functor $\phi^\star\colon \mathbf{X}_T \longrightarrow \mathbf{X}_S$ whenever $\phi$ is egressive. These are compatible with composition, and when $\phi$ is ingressive and (therefore) egressive, these two are adjoint.

There's more structure here: for any finite set $I$ and any $I$-tuple $(S_i)_{i\in I}$ of objects of $C$ with product $S$, consider the cartesian edge

$$(\{\xi\}, S) \longrightarrow (I, S_I)$$

of $C_\times$ lying over the morphism $\{\xi\} \longrightarrow I$ of $\Lambda(\mathbf{F})^{op}$ corresponding to the unique active morphism $I \longrightarrow \{\xi\}$ of $\Lambda(\mathbf{F})$; it is of course egressive in $\mathbf{X}_\boxtimes$. Hence there is a functor

$$\boxtimes_{i\in I}\colon \prod_{i\in I} \mathbf{X}_{S_i} \longrightarrow \mathbf{X}_S,$$

exact separately in each variable. If $(\phi_i : S_i \longrightarrow T_i)_{i\in I}$ is an $I$-tuple of morphisms of $C$ with product $\phi\colon S \longrightarrow T$ then the square

$$\begin{array}{ccc} \prod_{i\in I} \mathbf{X}_{T_i} & \xrightarrow{\boxtimes_{i\in I}} & \mathbf{X}_T \\ {\scriptstyle \prod_{i\in I}\phi_i^\star}\downarrow & & \downarrow{\scriptstyle \phi^\star} \\ \prod_{i\in I} \mathbf{X}_{S_i} & \xrightarrow[\boxtimes_{i\in I}]{} & \mathbf{X}_S \end{array}$$

commutes.

When $(C, C_{\dagger}, C^{\dagger})$ is cartesian, this structure endows each fiber $\mathbf{X}_S$ with a symmetric monoidal structure: indeed, for any finite set $I$, we may define

$$\bigotimes_{i\in I} := \Delta^* \circ \bigboxtimes_{i\in I},$$

where $\Delta\colon S \longrightarrow S^I$ is the diagonal. One sees easily that the commutativity of the square above implies that any functor $\phi^\star$ induced by a morphism $\phi\colon S \longrightarrow T$ is symmetric monoidal in a natural way. Furthermore, a simple argument demonstrates that the external product $\boxtimes_{i\in I}$ can be recovered from the symmetric monoidal structures along with the pullback functors; for example, $X \boxtimes Y \simeq \mathrm{pr}_1^\star X \otimes \mathrm{pr}_2^\star Y$.

Now it follows from [18, Cor. 7.3.2.7] that if $\phi\colon S \longrightarrow T$ is both ingressive and egressive in $C$, then $\phi_!$ extends to a lax symmetric monoidal functor $\mathbf{X}_S^{\otimes} \longrightarrow \mathbf{X}_T^{\otimes}$.

**7.5. Lemma.** *Suppose $(C, C_{\dagger}, C^{\dagger})$ a left complete disjunctive triple, and suppose*

$$p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$$

*a symmetric monoidal Waldhausen bicartesian fibration over $(C, C_{\dagger}, C^{\dagger})$. Then the inner fibration*

$$p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$$

*is an adequate inner fibration* [3, Df. 10.3] *for the triple $(C_{\times}, (C_{\times})_{\dagger}, (C_{\times})^{\dagger})$ (Nt. 2.1).*

*Proof.* The only condition of adequate inner fibrations that isn't explicitly part of the definition above is the assertion that for any ingressive morphism $(\phi, \omega)$ : $(I, X) \rightarrowtail (J, Y)$ of $C_{\times}$ and for any object $P$ of the fiber $(\mathbf{X}_{\boxtimes})_{(I,X)}$, there exists a $p_{\boxtimes}$-cocartesian morphism $P \longrightarrow Q$ covering $(\phi, \omega)$.

So suppose that $(\phi, \omega) : (I, X) \rightarrowtail (J, Y)$ is ingressive — i.e., that $\phi\colon J \longrightarrow I$ is a bijection and each morphism $\omega_{\phi^{-1}(i)}\colon X_i \longrightarrow Y_{\phi^{-1}(i)}$ is ingressive — and suppose that $P$ is an object of $\mathbf{X}_{\boxtimes}$ that lies over $(I, X)$. Then under the equivalence

$$(\mathbf{X}_{\boxtimes})_I \simeq \prod_{i\in I} \mathbf{X}_{\{i\}},$$

the object $P$ corresponds to a family $(P_i)_{i\in I}$ of objects such that $P_i$ lies over $X_i$ for any $i \in I$. For each $i \in I$, select a $p$-cocartesian edge $P_i \longrightarrow Q_{\phi^{-1}(i)}$ covering $\omega_{\phi^{-1}(i)}$. Now there is an essentially unique morphism $P \longrightarrow Q$ covering $(\phi, \omega)$ that corresponds under the equivalence above to the edges $P_i \longrightarrow Q_{\phi^{-1}(i)}$, and it is easy to see that it is $p_{\boxtimes}$-cocartesian. □

If $(C, C_{\dagger}, C^{\dagger})$ is a left complete disjunctive triple, and if $p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$ a symmetric monoidal Waldhausen bicartesian fibration for $(C, C_{\dagger}, C^{\dagger})$, then our goal is now to equip the unfurling of $\mathbf{X}$ with the structure of a $A^{\mathit{eff}}(C)^{\otimes}$-monoidal

Waldhausen structure. It will then follow that the corresponding Mackey functor is in fact a commutative Green functor.

**7.6. Construction.** Suppose $(C, C_\dagger, C^\dagger)$ a left complete disjunctive triple, and suppose

$$p_\boxtimes \colon \mathbf{X}_\boxtimes \longrightarrow C_\times$$

a symmetric monoidal Waldhausen bicartesian fibration over $(C, C_\dagger, C^\dagger)$. Then we define $\Upsilon(\mathbf{X}/(C, C_\dagger, C^\dagger))^\otimes$ as the pullback

$$\Upsilon(\mathbf{X}_\boxtimes/(C_\times, (C_\times)_\dagger, (C_\times)^\dagger)) \times_{A^{\mathit{eff}}(C_\times,(C_\times)_\dagger,(C_\times)^\dagger)} A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes.$$

The inner fibration [3, Lm. 11.4]

$$\Upsilon(\mathbf{X}_\boxtimes/(C_\times, (C_\times)_\dagger, (C_\times)^\dagger)) \longrightarrow A^{\mathit{eff}}(C_\times, (C_\times)_\dagger, (C_\times)^\dagger)$$

pulls back to an inner fibration

$$\Upsilon(p)^\otimes \colon \Upsilon(\mathbf{X}/(C, C_\dagger, C^\dagger))^\otimes \longrightarrow A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes.$$

We call this the ***unfurling*** of the symmetric monoidal Waldhausen bicartesian fibration $p_\boxtimes$.

**7.7.** Suppose, for simplicity, that $(C, C_\dagger, C^\dagger)$ is cartesian. Unwinding the definitions, one sees that the objects of $\Upsilon(\mathbf{X}/(C, C_\dagger, C^\dagger))^\otimes$ are precisely the objects of $\mathbf{X}_\boxtimes$. These, in turn, can be thought of as triples $(I, S_I, P_{S_I})$ consiting of a finite set $I$, an $I$-tuple $S_I := (S_i)_{i\in I}$, and an object $P_{S_I}$ of the fiber

$$(\mathbf{X}_\otimes)_{S_I} \simeq \prod_{i\in I} \mathbf{X}_{S_i},$$

which corresponds to an $I$-tuple $(P_{S_i})_{i\in I}$ of objects of the various Waldhausen $\infty$-categories $\mathbf{X}_{S_i}$. Now a morphism $(J, T_J, Q_{T_J}) \longrightarrow (I, S_I, P_{S_I})$ of the unfurling $\Upsilon(\mathbf{X}/(C, C_\dagger, C^\dagger))^\otimes$ can be thought of as the following data:

(7.7.1) a morphism $\phi \colon J \longrightarrow I$ of $\Lambda(\mathbf{F})$;

(7.7.2) a collection of diagrams

$$\left\{ \begin{array}{ccc|c} & U_{\phi(j)} & & \\ {\scriptstyle\tau_j}\swarrow & & \searrow{\scriptstyle\sigma_{\phi(j)}} & j \in \phi^{-1}(I) \\ T_j & & S_{\phi(j)} & \end{array} \right\}$$

of $C$ such that for any $j \in \phi^{-1}(I)$, the morphism $\sigma_j \colon U_{\phi(j)} \rightarrowtail S_{\phi(j)}$ is ingressive, and the morphism $\tau_j \colon U_{\phi(j)} \twoheadrightarrow T_j$ is egressive; and

(7.7.3) a collection of morphisms

$$\left\{ \sigma_{\phi(j),!}\tau_{J_i}^{\star}\left(\boxtimes_{j\in J_i} Q_{T_j}\right) \longrightarrow P_{S_i} \;\middle|\; i \in I \right\}$$

in the various $\infty$-categories $\mathbf{X}_{S_i}$, where $\tau_{J_i}$ is the edge $(\{i\}, U_i) \longrightarrow (J_i, T_{J_i})$ corresponding to the tuple $(\tau_j)_{j\in J_i}$.

**7.8. Theorem.** *Suppose $(C, C_{\dagger}, C^{\dagger})$ a left complete disjunctive triple, and suppose*

$$p_{\boxtimes}\colon \mathbf{X}_{\boxtimes} \longrightarrow C_{\times}$$

*a symmetric monoidal Waldhausen bicartesian fibration over $(C, C_{\dagger}, C^{\dagger})$. The functor $\Upsilon(p)^{\otimes}$ exhibits the $\infty$-category $\Upsilon(\mathbf{X}/(C, C_{\dagger}, C^{\dagger}))^{\otimes}$ as a $A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes}$-monoidal Waldhausen $\infty$-category.*

*Proof.* We first observe that, in light of [3, Pr. 11.6] and Lm. 7.5, the functor $\Upsilon(p)^{\otimes}$ is a cocartesian fibration. Let us check that the composite cocartesian fibration

$$\Upsilon(\mathbf{X}/(C, C_{\dagger}, C^{\dagger}))^{\otimes} \longrightarrow A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes} \longrightarrow \mathrm{N}\Lambda(\mathbf{F})$$

exhibits $\Upsilon(\mathbf{X}/(C, C_{\dagger}, C^{\dagger}))^{\otimes}$ as a symmetric monoidal $\infty$-category.

To this end, it suffices to show that for any finite set $I$ and any $I$-tuple $S_I := (S_i)_{i\in I}$ of objects of $C$, the functor

$$\prod_{i\in I}\chi_{i,!}\colon (\mathbf{X}_{\boxtimes})_{S_I} \simeq \Upsilon(\mathbf{X}/(C, C_{\dagger}, C^{\dagger}))^{\otimes}_{S_I} \longrightarrow \prod_{i\in I}\Upsilon(\mathbf{X}/(C, C_{\dagger}, C^{\dagger}))_{S_i} \simeq \prod_{i\in I}\mathbf{X}_{S_i}$$

induced by the cocartesian edges covering the inert maps $\chi_i\colon I \longrightarrow \{i\}_+$ is an equivalence. But this morphism can be identified with

$$\prod_{i\in I}\left(\mathrm{id}_! \circ \mathrm{id}^{\star} \circ \boxtimes_{i\in\{i\}}\right)\colon \prod_{i\in I}\mathbf{X}_{S_i} \longrightarrow \prod_{i\in I}\mathbf{X}_{S_i},$$

which is homotopic to the identity.

Now for any finite set $J$, a morphism $T \longrightarrow S$ of $A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes}$ covering the unique active morphism $J \longrightarrow \{\xi\}$ is represented by a collection of spans

$$\left\{ T_j \xleftarrow{\phi_j} U \xrightarrow{\psi} S \;\middle|\; j \in J \right\}.$$

The tensor product functor can therefore be written as

$$\psi_! \circ \phi_J^{\star} \circ \boxtimes_{j\in J}\colon \prod_{j\in J}\mathbf{X}_{T_j} \simeq \mathbf{X}_T \longrightarrow \mathbf{X}_S,$$

which is exact separately in each variable. $\square$

In light of Pr. 7.2, we have the following.

**7.8.1. Corollary.** *Suppose* $(C, C_\dagger, C^\dagger)$ *a cartesian disjunctive triple that is either left complete or right complete, and suppose* $p_\boxtimes \colon \mathbf{X}_\boxtimes \longrightarrow C_\times$ *a symmetric monoidal Waldhausen bicartesian fibration over* $(C, C_\dagger, C^\dagger)$*. Then the cocartesian fibration* $\Upsilon(p)^\otimes$ *is classified by a Green functor*

$$\mathbf{M}_p^\otimes \colon A^{\mathit{eff}}(C, C_\dagger, C^\dagger)^\otimes \longrightarrow \mathbf{Wald}_\infty^\otimes.$$

## 8. Equivariant algebraic $K$-theory of group actions

In this section, we answer a question of Akhil Mathew. Namely, for any Waldhausen $\infty$-category $C$ with an action of a finite group $G$, can one form an equivariant algebraic $K$-theory spectrum $K_G(C)$ whose $H$-fixed point spectrum is the algebraic $K$-theory of the homotopy fixed point $\infty$-category $C^{hH}$? Furthermore, can one do this in a lax symmetric monoidal fashion, so that if $C$ is an algebra in Waldhausen $\infty$-categories over an $\infty$-operad $O^\otimes$, then $K_G(C)$ is an algebra over $O^\otimes$ in $\mathbf{Mack}(\mathbf{F}_G; \mathbf{Sp})$? The answer to both of these questions is yes, and our framework makes it an almost trivial matter to see how.

**8.1. Construction.** Suppose $G$ a finite group. Let us denote by $\mathbf{F}_G^{\mathit{free}} \subset \mathbf{F}_G$ the full subcategory spanned by those finite $G$-sets upon which $G$ acts freely. Observe that $\mathbf{F}_G^{\mathit{free}}$ is the finite-coproduct completion of $BG$; that is, it is the free $\infty$-category with finite coproducts generated by $BG$. Consequently, $A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}})$ is the free semiadditive $\infty$-category generated by $BG$; that is, for any semiadditive $\infty$-category $A$, evaluation at $G/e$ defines an equivalence

$$\mathbf{Mack}(\mathbf{F}_G^{\mathit{free}}; A) \xrightarrow{\sim} \mathrm{Fun}(BG, A).$$

At the same time, the subcategory $\mathbf{F}_G^{\mathit{free}} \subset \mathbf{F}_G$ is clearly closed under coproducts, and since $\mathbf{F}_G^{\mathit{free}}$ is a sieve in $\mathbf{F}_G$, it follows that it is stable under pullbacks and binary products as well. Consequently, we obtain a fully faithful inclusion

$$A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}}) \hookrightarrow A^{\mathit{eff}}(\mathbf{F}_G).$$

We thus obtain, for any semiadditive $\infty$-category $A$, a corresponding restriction functor

$$\mathbf{Mack}(\mathbf{F}_G; A) \longrightarrow \mathbf{Mack}(\mathbf{F}_G^{\mathit{free}}; A).$$

If $A$ is in addition presentable, then the restriction functor admits a right adjoint

$$B_G \colon \mathrm{Fun}(BG, A) \longrightarrow \mathbf{Mack}(\mathbf{F}_G; A),$$

given by right Kan extension. We shall call this the *Borel functor*, since it assigns to any "naïve" $G$-object the corresponding *Borel-equivariant* object.

Applying this when $A = \mathbf{Wald}_\infty$ and applying algebraic $K$-theory, we obtain the *algebraic $K$-theory of group actions*:

$$\mathbf{K} \circ B_G \colon \mathrm{Fun}(BG, \mathbf{Wald}_\infty) \longrightarrow \mathbf{Mack}(\mathbf{F}_G; \mathbf{Sp}).$$

**8.2. Proposition.** *The algebraic $K$-theory of group actions extends naturally to a lax symmetric monoidal functor*

$$\mathbf{K}^{\otimes} \circ B_G^{\otimes} \colon \mathrm{Fun}(BG, \mathbf{Wald}_\infty)^{\otimes} \longrightarrow \mathbf{Mack}(\mathbf{F}_G; \mathbf{Sp})^{\otimes}.$$

*for the objectwise symmetric monoidal structure relative to the symmetric monoidal structure on* $\mathbf{Wald}_\infty$ [1] *and the additivized Day convolution on spectral Mackey functors.*

*Proof.* Since $\mathbf{K}^{\otimes}$ is lax symmetric monoidal [1], it suffices to show that for any presentable semiadditive symmetric monoidal $\infty$-category $E^{\otimes}$, the Borel functor $B_G$ extends to a symmetric monoidal functor

$$B_G^{\otimes} \colon \mathrm{Fun}(BG, E)^{\otimes} \simeq \mathbf{Mack}(\mathbf{F}_G^{\mathit{free}}; E)^{\otimes} \longrightarrow \mathbf{Mack}(\mathbf{F}_G; E)^{\otimes}.$$

This will follow directly from [18, 7.3.2.7], once one knows that the restriction functor

$$\mathbf{Mack}(\mathbf{F}_G; E) \longrightarrow \mathrm{Fun}(BG, E)$$

extends to a symmetric monoidal functor

$$\mathbf{Mack}(\mathbf{F}_G; E)^{\otimes} \longrightarrow \mathbf{Mack}(\mathbf{F}_G^{\mathit{free}}; E)^{\otimes} \simeq \mathrm{Fun}(BG, E)^{\otimes}.$$

For this, observe that since $\mathbf{F}_G^{\mathit{free}} \subset \mathbf{F}_G$ is stable under binary products, the inclusion

$$A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}}) \hookrightarrow A^{\mathit{eff}}(\mathbf{F}_G)$$

extends to a symmetric monoidal functor

$$A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}})^{\otimes} \hookrightarrow A^{\mathit{eff}}(\mathbf{F}_G)^{\otimes}.$$

It thus suffices to note that for any free finite $G$-set $V$, the subcategory

$$(A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}}) \times A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}})) \times_{A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}})} A^{\mathit{eff}}(\mathbf{F}_G^{\mathit{free}})_{/V}$$
$$\subset (A^{\mathit{eff}}(\mathbf{F}_G) \times A^{\mathit{eff}}(\mathbf{F}_G)) \times_{A^{\mathit{eff}}(\mathbf{F}_G)} A^{\mathit{eff}}(\mathbf{F}_G)_{/V}$$

is cofinal. □

## 9. Equivariant algebraic $K$-theory of derived stacks

In this section, we construct two symmetric monoidal Waldhausen bicartesian fibrations that extend the following two Waldhausen bicartesian fibrations introduced in [3, §D]:

- the Waldhausen bicartesian fibration
$$\mathbf{Perf}^{op} \times_{\mathbf{Shv}_{\mathit{flat}}} \mathbf{DM} \longrightarrow \mathbf{DM}$$
for the left complete disjunctive triple $(\mathbf{DM}, \mathbf{DM}_{\mathbf{FP}}, \mathbf{DM})$ of spectral Deligne–Mumford stacks, in which the ingressive morphisms are strongly proper morphisms of finite Tor-amplitude, and all morphisms are egressive [3, Pr. D.18], and
- the Waldhausen bicartesian fibration
$$\mathbf{Perf}^{op} \longrightarrow \mathbf{Shv}_{\mathit{flat}}$$
for the left complete disjunctive triple $(\mathbf{Shv}_{\mathit{flat}}, \mathbf{Shv}_{\mathit{flat},\mathbf{QP}}, \mathbf{Shv}_{\mathit{flat}})$ of flat sheaves in which the ingressive morphisms are the quasi-affine representable and perfect morphisms, and all morphisms are egressive [3, Pr. D.21].

These will give algebraic $K$-theory the structure of a commutative Green functor for these two triples.

**9.1.** To begin, we let

$$\begin{array}{ccc}\mathbf{Mod}^{\otimes} & \longrightarrow & \mathbf{QCoh}^{\otimes} \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathbf{CAlg}^{cn} \times N\Lambda(\mathbf{F}) & \hookrightarrow & \mathbf{Shv}_{\mathit{flat}}^{op} \times N\Lambda(\mathbf{F})\end{array}$$

be a pullback square in which $q$ is the cocartesian fibration of [18, Th. 4.5.3.1], and $p$ is a cocartesian fibration classified by the right Kan extension of the functor that classifies $q$. The objects of $\mathbf{QCoh}^{\otimes}$ can be thought of as triples $(X, I, M_I)$ consisting of a sheaf $X\colon \mathbf{CAlg}^{cn} \longrightarrow \mathbf{Kan}(\kappa_1)$ for the flat topology, a finite set $I$, and an $I$-tuple $M_I = \{M_i\}_{i\in I}$ of quasicoherent modules $M$ over $X$.

**9.2.** We may now pass to the cocartesian $\infty$-operads to obtain a cocartesian fibration of $\infty$-operads

$$p^{\sqcup}\colon (\mathbf{QCoh}^{\otimes})^{\sqcup} \longrightarrow (\mathbf{Shv}_{\mathit{flat}}^{op} \times N\Lambda(\mathbf{F}))^{\sqcup} \simeq (\mathbf{Shv}_{\mathit{flat},\times})^{op} \times_{N\Lambda(\mathbf{F})} N\Lambda(\mathbf{F})^{\sqcup}.$$

Now $N\Lambda(\mathbf{F})^{\sqcup} \longrightarrow N\Lambda(\mathbf{F})$ admits a section that carries any finite set $I$ to the pair $(I, *_I)$, where $*_I = \{*\}_{i\in I}$. Let us pull back $p^{\sqcup}$ along this section to obtain a cocartesian fibration of $\infty$-operads

$$p^{\boxtimes}\colon \mathbf{QCoh}^{\boxtimes} := (\mathbf{QCoh}^{\otimes})^{\sqcup} \times_{N\Lambda(\mathbf{F})^{\sqcup}} N\Lambda(\mathbf{F}) \longrightarrow (\mathbf{Shv}_{\mathit{flat},\times})^{op}.$$

**9.3.** Passing to opposites, we obtain a functor

$$(\mathbf{QCoh}^{op})_{\boxtimes} := (\mathbf{QCoh}^{\boxtimes})^{op} \longrightarrow \mathbf{Shv}_{\mathit{flat},\times}$$

which

- restricts to a symmetric monoidal Waldhausen bicartesian fibration

$$(\mathbf{QCoh}^{op})_{\boxtimes} \times_{\mathbf{Shv}_{\mathit{flat},\times}} \mathbf{DM}_{\times} \longrightarrow \mathbf{DM}_{\times}$$

  that extends the Waldhausen bicartesian fibration of [3, Pr. D.10] for the disjunctive triple of spectral Deligne–Mumford stacks, in which the ingressive morphisms are relatively scalloped, and all morphisms are egressive, and
- gives a symmetric monoidal Waldhausen bicartesian fibration

$$(\mathbf{QCoh}^{op})_{\boxtimes} \longrightarrow \mathbf{Shv}_{\mathit{flat},\times}$$

  that extends the Waldhausen bicartesian fibration of [3, Pr. D.13] for the disjunctive triple of flat sheaves, in which the ingressive morphisms are quasi-affine representable, and all morphisms are egressive.

**9.4.** At last, restricting to perfect modules, we obtain the desired symmetric monoidal Waldhausen bicartesian fibrations

$$(\mathbf{Perf}^{op})_{\boxtimes} \times_{(\mathbf{Shv}_{\mathit{flat}})_{\times}} \mathbf{DM}_{\times} \longrightarrow \mathbf{DM}_{\times}$$

for $(\mathbf{DM}, \mathbf{DM}_{\mathbf{FP}}, \mathbf{DM})$ and

$$(\mathbf{Perf}^{op})_{\boxtimes} \longrightarrow (\mathbf{Shv}_{\mathit{flat}})_{\times}$$

for $(\mathbf{Shv}_{\mathit{flat}}, \mathbf{Shv}_{\mathit{flat},\mathbf{QP}}, \mathbf{Shv}_{\mathit{flat}})$.

Now, passing to the unfurling, we obtain the following pair of results.

**9.5. Proposition.** *The Mackey functor*

$$\mathbf{M}_{\mathbf{DM}}\colon A^{\mathit{eff}}(\mathbf{DM}, \mathbf{DM}_{\mathbf{FP}}, \mathbf{DM}) \longrightarrow \mathbf{Wald}_{\infty}$$

*of* [3, Cor. D.18.1] *admits a natural structure of a commutative Green functor* $\mathbf{M}^{\otimes}_{\mathbf{DM}}$. *In particular, the algebraic $K$-theory of spectral Deligne–Mumford stacks is naturally a commutative spectral Green functor for* $(\mathbf{DM}, \mathbf{DM}_{\mathbf{FP}}, \mathbf{DM})$.

**9.6. Proposition.** *The Mackey functor*

$$\mathbf{M}_{\mathbf{Shv}_{\mathit{flat}}}\colon A^{\mathit{eff}}(\mathbf{Shv}_{\mathit{flat}}, \mathbf{Shv}_{\mathit{flat},\mathbf{QP}}, \mathbf{Shv}_{\mathit{flat}}) \longrightarrow \mathbf{Wald}_{\infty}$$

*of* [3, Cor. D.21.1] *admits a natural structure of a commutative Green functor* $\mathbf{M}^{\otimes}_{\mathbf{Shv}_{flat}}$. *In particular, the algebraic $K$-theory of flat sheaves is naturally a commutative spectral Green functor for* $(\mathbf{Shv}_{flat}, \mathbf{Shv}_{flat,\mathbf{QP}}, \mathbf{Shv}_{flat})$.

**9.7. Construction.** Suppose $X$ a spectral Deligne–Mumford stack. As in [3, Nt. D.23], we denote by $\mathbf{F\acute{E}t}(X)$ the subcategory of $\mathbf{DM}_{/X}$ whose objects are finite [17, Df. 3.2.4] and étale morphisms $Y \longrightarrow X$ and whose morphisms are finite and étale morphisms over $X$. Observe that the fiber product $- \times_X -$ endows $\mathbf{F\acute{E}t}(X)$ with the structure of a cartesian disjunctive $\infty$-category. We will abuse notation and write $A^{eff}(X)^{\otimes}$ for the symmetric monoidal effective Burnside $\infty$-category of $\mathbf{F\acute{E}t}(X)$.

Now the inclusion

$$(\mathbf{F\acute{E}t}(X), \mathbf{F\acute{E}t}(X), \mathbf{F\acute{E}t}(X)) \hookrightarrow (\mathbf{DM}, \mathbf{DM}_{\mathbf{FP}}, \mathbf{DM})$$

is clearly a morphism of cartesian disjunctive triples, whence one can restrict the commutative Green functor $\mathbf{M}^{\otimes}_{\mathbf{DM}}$ above along the morphism of $\infty$-operads

$$A^{eff}(X)^{\otimes} \longrightarrow A^{eff}(\mathbf{DM}, \mathbf{DM}_{\mathbf{FP}}, \mathbf{DM})^{\otimes}$$

to a commutative Green functor

$$\mathbf{M}_X \colon A^{eff}(X)^{\otimes} \longrightarrow \mathbf{Wald}^{\otimes}_{\infty}.$$

Now if $X$ is (say) a connected, noetherian scheme, then a choice of geometric point $x$ of $X$ gives rise to an equivalence

$$A^{eff}(\pi_1^{\acute{e}t}(X, x))^{\otimes} \simeq A^{eff}(X)^{\otimes}.$$

Applying algebraic $K$-theory, we obtain a commutative spectral Green functor for the étale fundamental group:

$$\mathbf{K}^{\otimes}_{\pi_1^{\acute{e}t}(X,x)}(X) \colon A^{eff}(\pi_1^{\acute{e}t}(X, x))^{\otimes} \longrightarrow \mathbf{Sp}^{\otimes}.$$

This commutative Green functor deserves the handle *Galois-equivariant algebraic $K$-theory*.

## 10. An equivariant Barratt–Priddy–Quillen theorem

**10.1. Notation.** In this section, suppose $(C, C_{\dagger}, C^{\dagger})$ a cartesian disjunctive triple.

**10.2. Recollection.** Recall [3, Df. 13.5] that $\mathbf{R}(C) \subset \mathrm{Fun}(\Delta^2/\Delta^{\{0,2\}}, C)$ is the full subcategory spanned by those retract diagrams

$$S_0 \longrightarrow S_1 \longrightarrow S_0;$$

such that the morphism $S_0 \longrightarrow S_1$ is a summand inclusion. We endow $\mathbf{R}(C)$ with the structure of a pair in the following manner. A morphism $T \longrightarrow S$ will be declared ingressive just in case $T_0 \longrightarrow S_0$ is an equivalence, and $T_1 \longrightarrow S_1$ is a summand inclusion. Write $p$ for the functor $\mathbf{R}(C) \longrightarrow C$ given by evaluation at the vertex $0 = 2$:

$$[S_0 \longrightarrow S_1 \longrightarrow S_0] \rightsquigarrow S_0.$$

Recall also that $\mathbf{R}(C, C_\dagger, C^\dagger) \subset \mathbf{R}(C)$ is the full subcategory spanned by those objects

$$S\colon \Delta^2/\Delta^{\{0,2\}} \longrightarrow C$$

such that for any complement $S_0' \hookrightarrow S_1$ of the summand inclusion $S_0 \hookrightarrow S_1$,

(10.2.1) the essentially unique morphism $S_0' \longrightarrow 1$ to the terminal object of $C$ is egressive, and

(10.2.2) the composite $S_0' \longrightarrow S_1 \longrightarrow S_0$ is ingressive.

We endow $\mathbf{R}(C, C_\dagger, C^\dagger)$ with the pair structure induced by $\mathbf{R}(C)$. We will abuse notation by denoting the restriction of the functor $p\colon \mathbf{R}(C) \longrightarrow C$ to the subcategory $\mathbf{R}(C, C_\dagger, C^\dagger) \subset \mathbf{R}(C)$ again by $p$.

We proved in [3, Th. 13.11] that $p$ is a Waldhausen bicartesian fibration over $(C, C_\dagger, C^\dagger)$.

**10.3. Construction.** Recall that an object of the $\infty$-category $\mathbf{R}(C, C_\dagger, C^\dagger)_\times$ can be described as pairs $(I, X)$ consisting of a finite set $I$ and a collection $X = \{X_i \mid i \in I\}$ of objects of $\mathbf{R}(C, C_\dagger, C^\dagger)$ indexed by the elements of $I$. Accordingly, a morphism $(I, X) \longrightarrow (J, Y)$ of $\mathbf{R}(C, C_\dagger, C^\dagger)_\times$ can be described as a map $J \longrightarrow I_+$ of finite sets and a collection

$$\left\{ X_i \longrightarrow \prod_{j \in J_i} Y_j \;\middle|\; i \in I \right\}$$

of morphisms of $\mathbf{R}(C, C_\dagger, C^\dagger)$.

We now define a subcategory $\mathbf{R}(C, C_\dagger, C^\dagger)_\boxtimes \subset \mathbf{R}(C, C_\dagger, C^\dagger)_\times$ that contains all the objects. A morphism $(I, X) \longrightarrow (J, Y)$ of $\mathbf{R}(C, C_\dagger, C^\dagger)_\times$ is a morphism of $\mathbf{R}(C, C_\dagger, C^\dagger)_\boxtimes$ if and only if, for every $i \in I$, every nonempty proper subset $K_i \subset J_i$, and every choice of a complement $Y_{j,0}' \hookrightarrow Y_{j,1}$ of the summand inclusion $Y_{j,0} \hookrightarrow Y_{j,1}$, the square

$$\begin{array}{ccc} \varnothing & \longrightarrow & X_{i,1} \\ \downarrow & & \downarrow \\ \prod_{j \in K_i} Y_{j,0} \times \prod_{j \in J_i \setminus K_i} Y_{j,0}' & \longrightarrow & \prod_{j \in J_i} Y_{j,1}, \end{array}$$

in which $\varnothing$ is initial and the bottom morphism is the obvious summand inclusion, is a pullback.

Let us endow this $\infty$-category with a pair structure in the following manner. We declare a morphism $(I, X) \longrightarrow (J, Y)$ of $\mathbf{R}(C, C_{\dagger}, C^{\dagger})_{\boxtimes}$ to be ingressive just in case the map $J \longrightarrow I_{+}$ represents an isomorphism in $\Lambda(\mathbf{F})$, and, for every $i \in I$, the map $X_i \longrightarrow Y_{\phi(i)}$ of $\mathbf{R}(C, C_{\dagger}, C^{\dagger})$ is ingressive.

The following is now immediate.

**10.4. Proposition.** *The functor*

$$p_{\boxtimes} \colon \mathbf{R}(C, C_{\dagger}, C^{\dagger})_{\boxtimes} \longrightarrow C_{\times}$$

*given by evaluation at* $0 = 2$ *in* $\Delta^2/\Delta^{\{0,2\}}$*exhibits* $\mathbf{R}(C, C_{\dagger}, C^{\dagger})$ *as a symmetric monoidal Waldhausen bicartesian fibration over* $(C, C_{\dagger}, C^{\dagger})$.

**10.5. Construction.** Now we are in a position to apply the unfurling construction of [3, §11] to the symmetric monoidal Waldhausen bicartesian fibration $p_{\boxtimes}$ to obtain an $A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes}$-monoidal Waldhausen $\infty$-category (in the sense of [1])

$$\Upsilon(p)^{\otimes} \colon \Upsilon(\mathbf{R}(C, C_{\dagger}, C^{\dagger})/(C, C_{\dagger}, C^{\dagger}))^{\otimes} \longrightarrow A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes}.$$

As we've demonstrated, $\Upsilon(p)^{\otimes}$ is classified by an $E_{\infty}$ Green functor

$$\mathbf{M}_p^{\otimes} \colon A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes} \longrightarrow \mathbf{Wald}_{\infty}^{\otimes}$$

whose underlying functor is the Mackey functor

$$\mathbf{M}_p \colon A^{eff}(C, C_{\dagger}, C^{\dagger}) \longrightarrow \mathbf{Wald}_{\infty}$$

corresponding to the unfurling of the Waldhausen bicartesian fibration

$$\mathbf{R}(C, C_{\dagger}, C^{\dagger}) \longrightarrow C$$

over $(C, C_{\dagger}, C^{\dagger})$.

In [1], we demonstrated that algebraic $K$-theory lifts in a natural fashion to a morphism of $\infty$-operads, whence we may contemplate the commutative Green functor

$$\mathbf{K}^{\otimes} \circ \mathbf{M}_p^{\otimes} \colon A^{eff}(C, C_{\dagger}, C^{\dagger})^{\otimes} \longrightarrow \mathbf{Sp}^{\otimes}.$$

Observe that by [3, Th. 13.12], the underlying Mackey functor

$$\mathbf{S}_{(C, C_{\dagger}, C^{\dagger})} := \mathbf{K} \circ \mathbf{M}_p$$

of $\mathbf{K}^{\otimes} \circ \mathbf{M}_p^{\otimes}$ is the spectral Burnside Mackey functor for $(C, C_{\dagger}, C^{\dagger})$, as defined in [3, Df. 8.1]. In particular, it is unit for the symmetric monoidal $\infty$-category $\mathbf{Mack}(C, C_{\dagger}, C^{\dagger}; \mathbf{Sp})$, which of course admits an essentially unique $E_{\infty}$ structure. Consequently, we deduce the following.

**10.6. Theorem** (Equivariant Barratt–Priddy–Quillen). *The Green functor* $\mathbf{K}^{\otimes} \circ \mathbf{M}_p^{\otimes}$ *is the spectral Burnside Green functor* $\mathbf{S}_{(C, C_{\dagger}, C^{\dagger})}$.

Of course, this result directly implies the original Barratt–Priddy–Quillen Theorem, which states that the algebraic $K$-theory of the ordinary Waldhausen category $\mathbf{F}_*$ of pointed finite sets (in which the cofibrations are the monomorphisms) is the sphere spectrum $\mathbf{S}$. Furthermore, the essentially unique $E_\infty$ structure on $\mathbf{S}$ is induced by the smash product of pointed finite sets.

## 11. A brief epilogue about the theorems of Guillou–May

Suppose $G$ a finite group. Write $\mathbf{OrthSp}_G$ for the underlying $\infty$-category of the relative category of orthogonal $G$-spectra. The equivariant Barratt–Priddy–Quillen Theorem of Guillou–May [11] provides a similar description in $\mathbf{OrthSp}_G$ of certain mapping spectra. Note that this is not *a priori* related to Th. 10.6 when $C = \mathbf{F}_G$. Nevertheless, a suitable comparison theorem (which of course Guillou–May provide in [12]) offers an implication.

On the other hand, the proof of our result here, combined with work from our forthcoming book [7], will allow us to reprove, using entirely different methods, the comparison result of Guillou–May. Indeed, if we can extend the functor $\Sigma^\infty_+ \colon \mathbf{F}_G \longrightarrow \mathbf{OrthSp}_G$ to a suitable functor $A^{\mathit{eff}}(\mathbf{F}_G) \longrightarrow \mathbf{OrthSp}_G$, then the equivariant Barratt–Priddy–Quillen Theorem above and the Schwede–Shipley theorem [19] together will imply the result of Guillou–May [12] providing the equivalence

$$\mathbf{Sp}^G \simeq \mathbf{OrthSp}_G.$$

It is, however, difficult to construct the desired functor $A^{\mathit{eff}}(\mathbf{F}_G) \longrightarrow \mathbf{OrthSp}_G$ directly, as this involves nontrivial homotopy coherence problems. To surmount this obstacle, we supply a universal property for $A^{\mathit{eff}}(\mathbf{F}_G)$ in [7] using techniques of “$G$-equivariant” $\infty$-category theory. This will provide us with the desired functor, and we will easily deduce the desired equivalence as a corollary.

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# Twisted Calabi–Yau ring spectra, string topology, and gauge symmetry

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In this paper we import the theory of "Calabi–Yau" algebras and categories from symplectic topology and topological field theories, to the setting of spectra in stable homotopy theory. Twistings in this theory will be particularly important. There will be two types of Calabi–Yau structures in the setting of ring spectra: one that applies to compact algebras and one that applies to smooth algebras. The main application of twisted compact Calabi–Yau ring spectra that we will study is to describe, prove, and explain a certain duality phenomenon in string topology. This is a duality between the manifold string topology of Chas and Sullivan (1999) and the Lie group string topology of Chataur and Menichi (2012). This will extend and generalize work of Gruher (2007). Then, generalizing work of Cohen and Jones (2017), we show how the gauge group of the principal bundle acts on this compact Calabi–Yau structure, and we compute some explicit examples. We then extend the notion of the Calabi–Yau structure to smooth ring spectra, and prove that Thom ring spectra of (virtual) bundles over the loop space, $\Omega M$, have this structure. In the case when $M$ is a sphere, we will use these twisted smooth Calabi–Yau ring spectra to study Lagrangian immersions of the sphere into its cotangent bundle. We recast the work of Abouzaid and Kragh (2016) to show that the topological Hochschild homology of the Thom ring spectrum induced by the $h$-principle classifying map of the Lagrangian immersion detects whether that immersion can be Lagrangian isotopic to an embedding. We then compute some examples. Finally, we interpret these Calabi–Yau structures directly in terms of topological Hochschild homology and cohomology.



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## Introduction

The theory of Calabi–Yau algebras and categories has proven to be very important in symplectic topology and the study of topological field theories [Costello 2007; Kontsevich and Soibelman 2009; Kontsevich and Vlassopoulos 2013; Lurie 2009; Cohen and Ganatra 2015]. One of the goals of this paper is to adapt this theory to the setting of spectra in stable homotopy theory, and to apply it to prove and explain a duality relationship between the string topology of a manifold and the string topology of a classifying space of a compact Lie group. We also use this notion to study Lagrangian immersions of spheres.

By way of background, recall that "string topology" is a term that was originally coined by Chas and Sullivan [1999] in their influential paper. In that paper, the term referred to certain algebraic properties of the homology of the loop space of a closed, oriented manifold, $H_*(LM)$, that were the result of a type of intersection theory in $LM$. This intersection theory came about by studying the fibration $\Omega M \to LM \xrightarrow{\text{ev}} M$, where ev evaluates a loop at $1 \in S^1$. Even though the loop space is itself infinite dimensional, the intersection theory defining the string topology operations is ultimately possible because of the finite dimensionality and compactness of $M$, as well as the fiberwise multiplicative properties of this fibration.

Since that time, the subject has expanded considerably. An important variation of the string topology intersection theory was described by Chataur and Menichi [2012], where they defined operations on the cohomology of the loop space of the classifying space of a compact Lie group, $LBG$. In this setting the analogous fibration $G \to LBG \xrightarrow{\text{ev}} BG$ is studied, and the intersection theory defining these operations is possible because of the compactness of $G$ as well as the fiberwise multiplicative properties of this fibration. A theory that includes both the string topology of a manifold and that of classifying spaces was developed in the setting of stacks in [Lupercio et al. 2008; Behrend et al. 2012]. In this setting the intersection theory is done in an appropriate algebraic geometric category.

An observation that helped to shed light on this intersection theory was made by Cohen and Klein [2009] when they classified "umkehr maps" that satisfy appropriate naturality and linearity properties. This led to the observation that the ring spectrum $LM^{-TM}$, which was shown to realize the Chas–Sullivan loop product by Cohen and Jones [2002], can be viewed as a twisted generalized cohomology theory evaluated on the manifold $M$. Specifically, if one takes the fiberwise suspension spectrum of the fibration $\Omega M \to LM \xrightarrow{\text{ev}} M$, and denotes the resulting parametrized spectrum by the notation

$$\Sigma^\infty(\Omega M_+) \to \Sigma^\infty_M(LM_+) \to M,$$

then the result is a parametrized ring spectrum which defines a twisted cohomology theory $\mathcal{S}^\bullet_M$ from the category of spaces over $M$, $\mathcal{T}_M$, to an appropriate category of spectra. If $f : X \to M$ is an object in $\mathcal{T}_M$, then $\mathcal{S}^\bullet_M(X, f) = \Gamma_X\big(f^*(\Sigma^\infty_M(LM_+))\big)$, the spectrum of sections over $X$ of the pullback via $f$ of the parametrized spectrum $\Sigma^\infty_M(LM_+) \to M$. See [Cohen and Klein 2009; May and Sigurdsson 2006] for details. Since $\Sigma^\infty_M(LM_+)$ is a parametrized ring spectrum, this spectrum of sections inherits a ring spectrum structure. Moreover it was proved in [Cohen and Klein 2009] that the value of this cohomology on the identity map $\mathrm{id} : M \xrightarrow{=} M \in \mathcal{T}_M$, has the homotopy type

$$\mathcal{S}^\bullet_M(M) = \Gamma_M(\Sigma^\infty_M(LM_+)) \simeq LM^{-TM}$$

as ring spectra. This equivalence is a type of twisted Poincaré or Atiyah duality as explained in [Cohen and Klein 2009]. Moreover, one sees that the string topology intersection pairing (loop product) on $H_*(LM^{-TM}) \cong H_{*+n}(LM)$ corresponds, via this twisted Poincaré duality, to a generalized cup product pairing in the cohomology $\mathcal{S}^\bullet_M(M)$. This is a twisted generalization of a well-known phenomenon: the intersection product in $H_*(M)$ corresponds up to sign, under traditional Poincaré duality, to the cup product in $H^*(M)$.

As observed by Gruher and Salvatore [2008], the string topology product exists in the presence of any fiberwise monoid over a closed manifold, $Q \to E \to M$. Here $Q$ is a monoid, and the bundle $E$ comes equipped with a fiberwise product $E \times_M E \to E$ over $M$, consistent with the monoid structure of the fiber $Q$. In this case the Thom spectrum $E^{-TM}$ is a ring spectrum. It was also observed in [Gruher and Salvatore 2008] that principal bundles $G \to P \to M$ give rise to fiberwise monoids by taking the associated adjoint bundle, $G \to P^{\mathrm{Ad}} \to M$, where $P^{\mathrm{Ad}} = P \times_G G^{\mathrm{Ad}}$. Here $G^{\mathrm{Ad}}$ denotes $G$ with the left $G$-action given by conjugation.

As observed in [Cohen and Jones 2017], the string topology of principal bundles over manifolds can also be represented by twisted cohomology theories. The representing parametrized spectrum is the fiberwise suspension spectrum

$$\Sigma^\infty(G_+) \to \Sigma^\infty_M(P^{\mathrm{Ad}}_+) \to M.$$

Let $\mathcal{S}^\bullet_P$ denote the corresponding twisted cohomology theory. In particular,

$$\mathcal{S}^\bullet_P(M) = \Gamma_M(\Sigma^\infty_M(P^{\mathrm{Ad}}_+)) \simeq (P^{\mathrm{Ad}})^{-TM},$$

and the ring structure comes from a generalized cup product on $\Sigma^\infty_M(P^{\mathrm{Ad}}_+)^\bullet(M)$. We refer to $\mathcal{S}^\bullet_P(-)$ as the *manifold string topology* structure on the principal bundle $P$.

This perspective on the string topology spectrum, $LM^{-TM}$, or more generally $(P^{\mathrm{Ad}})^{-TM}$, in terms of the sections of a parametrized spectrum was particularly useful in [Cohen and Jones 2017], where the units of these ring spectra were studied. In particular it was shown that the gauge group $\mathcal{G}(P)$ of the principal bundle acts naturally on the string topology spectrum, and so there is a homomorphism,

$$\mathcal{G}(P) \to \mathrm{GL}_1((P^{\mathrm{Ad}})^{-TM})$$

which was studied and computed in [Cohen and Jones 2017].

The first goal of this paper is to show that there is a dual construction for the string topology of the classifying space of a compact Lie group, to investigate this duality using a stable homotopy theoretic version of compact Calabi–Yau algebras, and to compute some of its properties, including gauge symmetry.

We now state the results more precisely. Let $G$ be a compact Lie group, and let $G \to P \to X$ be a principal $G$-bundle. In this, $X$ can be any space of the homotopy type of a CW-complex. It need not be finite. In particular, an important example is the universal principal bundle $G \to EG \to BG$. As before, let $G \to P^{\mathrm{Ad}} \to X$ be the corresponding adjoint bundle. Recall that in the case of the universal bundle, $EG^{\mathrm{Ad}} \simeq LBG$.

Consider the fiberwise suspension spectrum,

$$\Sigma^\infty(G_+) \to \Sigma^\infty_M(P^{\mathrm{Ad}}_+) \to X,$$

and let $\mathcal{D}(\Sigma^\infty_M(P^{\mathrm{Ad}}_+))$ be the fiberwise Spanier–Whitehead dual, as in [May and Sigurdsson 2006]. This is a parametrized spectrum over $X$, whose fibers are the Spanier–Whitehead duals of the fibers of $\Sigma^\infty_M(P^{\mathrm{Ad}}_+)$:

$$G^\vee \to \mathcal{D}(\Sigma^\infty_M(P^{\mathrm{Ad}}_+)) \to X,$$

where $G^\vee = \mathrm{Map}(\Sigma^\infty(G_+), \mathbb{S})$. Here $\mathbb{S}$ denotes the sphere spectrum. Notice that $G^\vee$ is a coalgebra spectrum, with coalgebra structure dual to the ring structure on $\Sigma^\infty(G_+)$.

We denote the twisted homology theory associated to this parametrized spectrum by $\mathcal{S}^P_\bullet : \mathcal{T}_X \to$ Spectra. The following will be proved in Section 1.

**Theorem 1.** *The parametrized spectrum $\mathcal{D}(\Sigma^\infty(G_+)) \to \mathcal{D}(\Sigma^\infty_M(P^{\mathrm{Ad}}_+)) \to X$ is a weak fiberwise coalgebra spectrum satisfying the following properties.*

(1) *Let $f : Y \to X$ be an object in $\mathcal{T}_X$. Then the induced twisted homology $\mathcal{S}^P_\bullet(Y, f)$ is a weak coalgebra spectrum.*

(2) *There is an equivalence of spectra,*

$$\alpha : (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \xrightarrow{\simeq} \mathcal{S}^P_\bullet(X)$$

*where* $(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$ *is the Thom spectrum of minus the vertical tangent bundle* $T_{\mathrm{vert}}P^{\mathrm{Ad}} \to P^{\mathrm{Ad}}$. *Furthermore, a Pontryagin–Thom construction gives* $(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$ *a natural coproduct which is taken by* $\alpha$ *to the coproduct in* $\mathcal{S}^P_\bullet(X)$.

(3) *If one takes the cohomology of the coalgebra spectrum,* $H^*(\mathcal{S}^P_\bullet(Y, f); k)$ (*here the coefficients are in a field* $k$), *one obtains a graded algebra,*

$$H^*(\mathcal{S}^P_\bullet(Y); k) \otimes H^*(\mathcal{S}^P_\bullet(Y); k) \to H^*(\mathcal{S}^P_\bullet(Y); k)$$

*which we call the* ***Lie group string topology algebra of*** $f^*(P)$. *Using the equivalence in part* (2), *when the vertical tangent bundle* $T_{\mathrm{vert}} \to P^{\mathrm{Ad}}$ *is orientable, one obtains a graded algebra of degree* $-d$, *where* $d = \dim G$:

$$H^p(P^{\mathrm{Ad}}) \otimes H^q(P^{\mathrm{Ad}}) \to H^{p+q-d}(P^{\mathrm{Ad}}).$$

(4) *In the case of the universal principal bundle* $G \to EG \to BG$, *this algebra is isomorphic to the algebra structure in the string topology of the classifying space* $BG$ (as described by Chataur and Menichi [2012]),

$$H^*(\mathcal{S}^{EG}_\bullet(BG)) \cong H^*(LBG).$$

**Comments.** (1) The notion of a "weak" fiberwise coalgebra spectrum will be defined in Section 1.

(2) We refer to the coalgebra spectrum $\mathcal{S}^P_\bullet(X) \simeq (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$ as the *Lie group string topology spectrum of the principal bundle* $P$.

(3) The equivalence $\alpha : (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \xrightarrow{\simeq} \mathcal{S}^P_\bullet(X) = \mathcal{D}(\Sigma^\infty_M(P^{\mathrm{Ad}}_+))/X$ can be viewed as a fiberwise Atiyah duality, which on the level of fibers is the classical Atiyah [1961] equivalence,

$$\alpha : G^{-TG} \simeq \Sigma^{-\mathfrak{g}}(G_+) \xrightarrow{\simeq} G^\vee,$$

where $G^{-TG}$ is the Thom spectrum of minus the tangent bundle, which is equivariantly equivalent to the desuspension of $\Sigma^\infty(G_+)$ by the adjoint representation of $G$ on the Lie algebra $\mathfrak{g}$.

(4) The fact that the cohomology algebra $H^*(LBG^{-T_{\mathrm{vert}}}) \cong H^{*+d}(LBG)$ is the string topology of classifying spaces was proved by Gruher [2007].

Once this theorem is established we restrict to the situation where we have a principal $G$-bundle over a closed manifold: $G \to P \to M$. In this case we can study both the "manifold string topology structure" of $P$, that is, the twisted cohomology theory

$$\mathcal{S}_P^{\bullet}(M) = \Gamma_M(\Sigma_M^{\infty}(P_+^{\mathrm{Ad}})) \simeq (P^{\mathrm{Ad}})^{-TM}$$

as well as the "Lie group string topology structure" of $P$, which is to say the twisted homology theory

$$\mathcal{S}_{\bullet}^{P}(M) \xrightarrow{\simeq} (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}.$$

The following is a consequence of Theorem 1 as well as Gruher's [2007] work.

**Corollary 2.** *Let $G \to P \to M$ be a principal bundle, where $G$ is a compact Lie group of dimension $d$ and $M$ is a closed manifold of dimension $n$. The string topology spectra $\mathcal{S}_P^{\bullet}(M) \simeq (P^{\mathrm{Ad}})^{-TM}$ and $\mathcal{S}_{\bullet}^{P}(M) \simeq (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$ are Spanier–Whitehead dual to each other, with the algebra structure of the former corresponding to the coalgebra structure of the latter under this duality. When $M$ is oriented and the bundle $T_{\mathrm{vert}}$ is oriented, this gives $H_*(P^{\mathrm{Ad}})$ the structure of a Frobenius algebra of dimension $n-d$. The multiplication in this Frobenius algebra comes from the manifold string topology, and the comultiplication comes from the Lie group string topology.*

In Section 2 we will define the notion of a "twisted compact Calabi–Yau" ring spectrum ("twisted cCY"), which can be viewed as a strengthened, derived version of Frobenius algebra in the category of spectra. This definition is adapted from the notion of a "compact Calabi–Yau algebra" defined by Kontsevich and Soibelman [2009], as a way of studying two dimensional topological field theories. (We note that Kontsevich and Soibelman used different terminology for this concept.) Related notions were defined by Costello [2007] and Lurie [2009]. In these definitions, the algebra (or ring spectrum) involved must satisfy a finiteness condition called "compactness". In the spectrum setting this means that the spectrum is a perfect module over the sphere spectrum. In our definition of this structure in the setting of spectra, a key role is played by a "twisting bimodule" over the compact ring spectrum. The following is the main result of this section.

**Theorem 3.** *Let $G \to P \to M$ be a principal bundle with compact Lie group fiber and closed manifold base. Then the manifold string topology $\mathcal{S}_P^{\bullet}(M)$ naturally admits the structure of a twisted, compact Calabi–Yau ring spectrum of dimension $n-d$. The Lie group string topology spectrum $\mathcal{S}_{\bullet}^{P}(M)$ is the twisting bimodule spectrum in this structure. Moreover if $E_*$ is a generalized homology theory with respect to which both the vertical tangent bundle $T_{\mathrm{vert}} \to P^{\mathrm{Ad}}$ and the tangent bundle $TM \to M$ are oriented, then the Calabi–Yau structure on $\mathcal{S}_P^{\bullet}(M)$ induces a Frobenius algebra structure on the homology of the manifold string topology, $E_*(\mathcal{S}_P^{\bullet}(M))$, whose dual is the homology of the Lie group string topology spectrum, $E_*(\mathcal{S}_{\bullet}^{P}(M))$.*

In [Cohen and Jones 2017] an action of the gauge group $\mathcal{G}(P)$ of the principal bundle $G \to P \to M$ on the manifold string topology spectrum $\mathcal{S}^{\bullet}_{P}(M) = (P^{\mathrm{Ad}})^{-TM}$ was described and computed. In Section 3 we use Theorems 1 and 3 to describe a similar action of $\mathcal{G}(P)$ on the Lie group string topology spectrum $\mathcal{S}^{P}_{\bullet}(M) = (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$. We also show that this gauge symmetry respects the Calabi–Yau structure. See Theorem 15 below for a precise statement. We then compute some explicit examples of this gauge symmetry.

In Section 4 we introduce the related notion of twisted *smooth* Calabi–Yau ring spectra. Smoothness is a form of smallness property different from compactness. A ring spectrum $A$ is *smooth* if it is perfect as a bimodule over itself. That is, it is a perfect as a left $(A \wedge A^{\mathrm{op}})$-module spectrum. The spectrum notion of a "twisted sCY" structure is adapted from the notion of "sCY" algebras and categories, first defined by Kontsevich and Vlassopoulos [2013], and used by Cohen and Ganatra [2015] to compare the string topology topological field theory to the Floer symplectic field theory of cotangent bundles. In the spectral theory a twisting bimodule spectrum plays an important role. We show that this structure occurs in certain Thom spectra of virtual bundles over the based loop space of a manifold, $\Omega M$. That is, we prove the following theorem:

**Theorem 4.** *Let $M$ be a closed manifold, and $f : M \to BBO$ be a map to a delooping of $BO$. Here, by Bott periodicity we may take $BBO$ to be the infinite homogeneous space* SU / SO. *Consider the induced map of loop spaces, $\Omega f : \Omega M \to BO$. Then its Thom spectrum, which we denote by $(\Omega M)^{\Omega f}$, naturally admits the structure of a twisted, smooth Calabi–Yau ring spectrum.*

**Remark.** When $f : M \to BBO$ is the constant map, this theorem implies that the suspension spectrum $\Sigma^{\infty}(\Omega M_{+})$ has the structure of a twisted sCY ring spectrum. This strengthens a result of Cohen and Ganatra [2015] saying that the singular chain complex $C_{*}(\Omega M)$ admits the structure of a smooth Calabi–Yau differential graded algebra.

Also in Section 4, we describe how these ring spectra arise naturally in the study of Lagrangian immersions. In particular, for the case of spheres, we combine the results of Abouzaid and Kragh [2016] with those of Blumberg, Cohen and Schlichtkrull [Blumberg et al. 2010] to prove the following (see Theorem 25 for a more precise statement).

**Theorem 5.** *Associated to a Lagrangian immersion $\phi : S^n \to T^{*}S^n$ there is a loop map $\Omega\alpha_{\phi} : \Omega S^n \to BU$. If the Lagrangian immersion $\phi$ is Lagrangian isotopic to a Lagrangian embedding, then there is an equivalence of topological Hochschild homology spectra,*

$$\mathrm{THH}((\Omega S^n)^{\Omega\alpha_{\phi}}) \simeq \mathrm{THH}(\Sigma^{\infty}(\Omega S^n_{+})).$$

We then use this theorem, together with homotopy theoretic results about the image of the $J$-homomorphism, to recast results in [Abouzaid and Kragh 2016] giving examples of Lagrangian immersions of spheres that are not Lagrangian isotopic to embeddings, but *are* smoothly isotopic to embeddings.

Finally, in Section 5 we describe this structure from the perspective of topological Hochschild (co)homology. More specifically, let $G \to P \to M$ be a smooth principal bundle, where $G$ is a compact Lie group and $M$ is a smooth, closed manifold. Let $h : \Omega M \to G$ be the holonomy of a connection on $P$. This induces a map of ring spectra $h : \Sigma^\infty(\Omega M_+) \to \Sigma^\infty(G_+)$. Thus $h$ defines bimodule structures on $\Sigma^\infty(G_+)$ over $\Sigma^\infty(\Omega M_+)$. The main result of this section is the following:

**Theorem 6.** *We have the following equivalences involving topological Hochschild homology* $\mathrm{THH}_\bullet$ *and topological Hochschild cohomology* $\mathrm{THH}^\bullet$.

(1) $\mathrm{THH}_\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq \Sigma^\infty(P_+^{\mathrm{Ad}})$

(2) $\mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq (P^{\mathrm{Ad}})^{-TM} \simeq \mathcal{S}_P^\bullet(M)$.
*This equivalence is one of ring spectra.*

(3) $\mathrm{THH}_\bullet\big(\Sigma^\infty(\Omega M_+), G^\vee\big) \simeq (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \simeq \mathcal{S}_\bullet^P(M)$.
*This equivalence is one of coalgebra spectra.*

We end by describing the twisted Calabi–Yau structure on the string topology spectrum from the perspective of these topological Hochschild homology spectra. A consequence of the resulting duality properties is the following:

**Corollary 7.** *If $M$ is oriented and the bundle $T_{\mathrm{vert}}$ is oriented*, *there is a nondegenerate bilinear form on Hochschild homology*,

$$HH_*(C_*(\Omega M), C_*(G)) \times HH_*(C_*(\Omega M), C_*(G)) \to k.$$

*That is*, *this Hochschild homology space is self dual.*

## 1. A twisted homology theory representing Lie group string topology

The goal of this section is to describe Lie group string topology as a twisted generalized homology theory, and to prove Theorem 1. The main issue in proving this theorem is to describe a parametrized form of Atiyah duality. We begin by recalling the specific map yielding the Atiyah duality between the Thom spectrum of minus the tangent bundle of a closed manifold $M$, and the Spanier–Whitehead dual of $M$ [Atiyah 1961; Cohen 2004].

Let $M^n$ be a closed $n$-dimensional manifold and $e : M \hookrightarrow \mathbb{R}^k$ be an embedding into Euclidean space with normal bundle $\eta_e \to M$. By the tubular neighborhood theorem, for sufficiently small $\epsilon > 0$, the open set $\nu_\epsilon(e) \subset \mathbb{R}^k$ consisting of points within a distance of $\epsilon$ of $e(M)$ can be identified with the total space $\eta_e$.

Consider the map

$$\alpha : (\mathbb{R}^k - \nu_\epsilon(e)) \times M \to \mathbb{R}^k - B_\epsilon(0) \simeq S^{k-1}, \qquad (v, y) \mapsto v - e(y), \tag{1}$$

where $B_\epsilon(0)$ is the open ball of radius $\epsilon$. This map induces the Alexander duality isomorphism

$$\tilde{H}_q(\mathbb{R}^k - e(M)) \cong \tilde{H}_q(\mathbb{R}^k - \nu_\epsilon(e)) \xrightarrow{\cong} \tilde{H}^{k-q-1}(M).$$

Atiyah duality [1961] is induced by the same map,

$$M^{\eta_e} \wedge M_+ \cong (\mathbb{R}^k \times M)/\big((\mathbb{R}^k - \nu_\epsilon(e)) \times M\big) \to \mathbb{R}^k/(\mathbb{R}^k - B_\epsilon(0)) \cong S^k, \qquad (v, y) \mapsto v - e(y). \tag{2}$$

The adjoint of this map gives a map from the Thom space of $\eta_e$ to the mapping space, $\alpha : M^{\eta_e} \to \mathrm{Map}(M, S^k)$ which defines the Atiyah duality equivalence of spectra,

$$\alpha : M^{-TM} \to \mathrm{Map}(M, \mathbb{S}). \tag{3}$$

Here this notation refers to the mapping spectrum between the suspension spectrum $\Sigma^\infty(M_+)$ to the sphere spectrum $\mathbb{S}$. This is the Spanier–Whitehead dual of $M$, and will be denoted by $M^\vee$. Indeed in [Cohen 2004] the first author constructed a symmetric ring spectrum (without unit), $M^{-TM}$. The $k$-th space of this spectrum is equivalent, through a range of dimensions that increases with $k$, to the Thom space $M^{\eta_e}$ and is constructed by allowing the embeddings and the choices of $\epsilon$ to vary. The $k$-th space of the mapping spectrum $\mathrm{Map}(M, \mathbb{S})$ has the homotopy type of $\mathrm{Map}(M, S^k)$. It was shown in [Cohen 2004] that the map $\alpha$ induces an equivalence of symmetric ring spectra. We refer the reader to [Cohen 2004] for details.

We now pass to the parametrized setting. Our goal is to describe a parametrized form of this Atiyah duality equivalence. Let $G \to P \to X$ be a principal bundle with compact Lie group fiber. By the fiberwise duality theorem of May and Sigurdsson (Theorem 15.1.1 of [May and Sigurdsson 2006]), the parametrized suspension spectrum $\Sigma^\infty(G_+) \to \Sigma^\infty_X(P^{\mathrm{Ad}}_+) \to X$ is (fiberwise) dualizable because each fiber spectrum is dualizable. This in turn is because every fiber spectrum is equivalent to $\Sigma^\infty(G_+)$, which is dualizable since $G$ is a compact manifold. The parametrized Spanier–Whitehead dual is what we called $G^\vee \to \mathcal{D}(\Sigma^\infty_X(P^{\mathrm{Ad}}_+)) \to X$ in the introduction. The construction in [May and Sigurdsson 2006] is quite general. In this particular case, however, we will describe this fiberwise dual explicitly.

The spectra we work with will be orthogonal spectra, and when we describe a group action, we use $\mathrm{RO}(G)$-indexed orthogonal spectra. We refer the reader to [Mandell and May 2002] for details.

Recall that $P^{\mathrm{Ad}} = P \times_G G^{\mathrm{Ad}}$. Let $V$ be a finite dimensional orthogonal representation of $G$, and let $S^V = V \cup \infty$ be the one-point compactification where the $G$-action fixes $\infty$. The conjugation action of $G$ on itself defines an action of $G$ on $\mathrm{Map}(G, S^V)$,

$$g \cdot \phi : G \to S^V, \quad h \mapsto g\phi(g^{-1}hg). \tag{4}$$

This defines an $\mathrm{RO}(G)$-graded $G$-spectrum, which we call $G^\vee$.

We define the parametrized spectrum $\mathcal{D}(\Sigma^\infty_X(P^{\mathrm{Ad}}_+))$ as an $\mathrm{RO}(G)$-graded spectrum. For a representation $W$, the $W$-space is defined to be

$$\mathcal{D}(\Sigma^\infty_X(P^{\mathrm{Ad}}_+))_W = P \times_G \mathrm{Map}(G, S^W) \tag{5}$$

which fibers over $X = P/G$ with fiber $\mathrm{Map}(G, S^W) = \mathrm{Map}(G, S^k)$, where $k = \dim W$. The fiberwise suspension by a representation $U$ is given by

$$\Sigma^U_X\big(\mathcal{D}(\Sigma^\infty_X(P^{\mathrm{Ad}}_+))_W\big) = P \times_G S^U \wedge \mathrm{Map}(G, S^W)$$

and the structure map $\epsilon_U : \Sigma^U_X\big(\mathcal{D}(\Sigma^\infty_X(P^{\mathrm{Ad}}_+))_W\big) \to \mathcal{D}(\Sigma^\infty_X(P^{\mathrm{Ad}}_+))_{W \oplus U}$ is induced by the $G$-equivariant map

$$\epsilon_U : S^U \wedge \mathrm{Map}(G, S^W) \to \mathrm{Map}(G, S^{W \oplus U}), \quad \epsilon_U(t \wedge \phi)(g) = \phi(g) \wedge t.$$

Notice that since the multiplication map $G \times G \to G$ is equivariant with respect to the adjoint action (the action on $G \times G$ is diagonal), the induced map $G^\vee \to (G \times G)^\vee$ is also equivariant. Along with the natural weak equivalence $G^\vee \wedge G^\vee \to (G \times G)^\vee$, this induces a weak fiberwise coalgebra structure on the parametrized spectrum $\mathcal{D}(\Sigma^\infty_X(P^{\mathrm{Ad}}_+))$.

By a "weak fiberwise coalgebra" structure on a parametrized spectrum, we simply mean the following.

**Definition.** A parametrized spectrum $E \to \mathcal{E} \to X$ is a *weak fiberwise coalgebra* if there is a "comultiplication" map $\gamma : \mathcal{E} \to \mathcal{E} \wedge_X \mathcal{E}$ and a "counit" $\eta : \mathcal{E} \to \mathbb{S}_X$ in the category of parametrized spectra over $X$, that satisfy the usual coassociativity and counit properties up to homotopy. No coherence conditions on the homotopies are assumed. Here $\mathbb{S} \to \mathbb{S}_X \to X$ is the parametrized sphere spectrum. Namely the $n$-th space of $\mathbb{S}_X$ is $X \times S^n$.

Notice that given a fiberwise coalgebra spectrum $E \to \mathcal{E} \to X$, then for any object $f : Y \to X$ in $\mathcal{T}_X$, the twisted homology spectrum $\mathcal{E}_\bullet(Y, f) = \mathcal{E}/X$ is an ordinary coalgebra spectrum.

The source of the parametrized Atiyah duality map is a parametrized Thom spectrum. More precisely, let $e : G \subset V$ be an equivariant embedding of $G$ with its conjugation action into a finite dimensional $G$-representation $V$. Let $k = \dim V$. Let $\nu_\epsilon(e)$ be an equivariant tubular neighborhood as above. It is equivariantly diffeomorphic to the normal bundle $\eta_V \to G$. (We are suppressing the embedding $e$ from the notation.) We let $\eta_V^{\text{vert}} \to P^{\text{Ad}}$ be the vector bundle

$$\eta_V^{\text{vert}} = P \times_G \eta_V \to P \times_G G^{\text{Ad}} = P^{\text{Ad}} \tag{6}$$

The fiberwise Thom space of this bundle is homeomorphic to the fiberwise one-point compactification of the tubular neighborhood,

$$P \times_G G^{\eta_V} \cong P \times_G (\nu_\epsilon(e) \cup \infty).$$

Notice also that there is a map from the fiberwise suspension

$$\epsilon_W : \Sigma_X^W(P \times_G G^{\eta_V}) = P \times_G (S^W \wedge G^{\eta_V}) \xrightarrow{\simeq} P \times_G G^{\eta_V \oplus W}$$

This data defines an RO($G$)-graded parametrized spectrum $P \times_G G^{-TG}$ over $X$ whose $W$-th space is $\Omega^V(P \times_G G^{\eta_V \oplus W})$. Here, for a representation $U$, $\Omega^U$ refers to the $U$-fold loop space, $\text{Map}_\bullet(S^U, -)$.

Furthermore, the Atiyah duality map described above defines a map $\alpha : G^{\eta_V} \cong \nu_\epsilon(e) \cup \infty \to \text{Map}(G, S^V)$. This map is equivariant, and so defines Atiyah duality maps $\overline{\alpha_V} : P \times_G G^{\eta_V} \to P \times_G \text{Map}(G, S^V)$. These maps respect the spectrum structure maps and so this proves the following:

**Lemma 8.** *The maps $\overline{\alpha_k}$ define an equivalence of parametrized spectra over $X$,*

$$\bar{\alpha} : P \times_G G^{-TG} \xrightarrow{\simeq} \mathcal{D}(\Sigma_X^\infty(P_+^{\text{Ad}})).$$

The map $\bar{\alpha}$ therefore defines an equivalence of the generalized twisted homology theories these parametrized spectra represent. Given an object $f : Y \to X$ in $\mathcal{T}_X$, the twisted homology theory that the parametrized spectrum $\mathcal{D}(\Sigma_X^\infty(P_+^{\text{Ad}}))$ represents is what we called $\mathcal{S}_\bullet^P(Y, f)$ in the introduction. The twisted homology theory that the parametrized spectrum $P \times_G G^{-TG}$ represents is given by the (ordinary) spectrum $f^*(P)_+ \wedge_G G^{-TG}$, which is the Thom spectrum of the virtual bundle $-f^*(T_{\text{vert}})$ over $f^*(P^{\text{Ad}})$, where $T_{\text{vert}} \to P^{\text{Ad}}$ is the vertical tangent bundle. Applying this to $X$ itself, we have the equivalence of spectra,

$$\bar{\alpha} : (P^{\text{Ad}})^{-T_{\text{vert}}} \xrightarrow{\simeq} \mathcal{S}_\bullet^P(X) \tag{7}$$

Now recall that given any map $\phi : M \to N$ between closed manifolds, the Pontryagin–Thom construction defines a map $\tau_g : N^{-TN} \to M^{-TM}$ making the following diagram of spectra homotopy commute:

$$\begin{array}{ccc} N^{-TN} & \xrightarrow{\tau_g} & M^{-TM} \\ {\scriptstyle \alpha}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \alpha} \\ N^\vee & \xrightarrow{g^\vee} & M^\vee \end{array} \tag{8}$$

Applying this to the multiplication map $\mu : G \times G \to G$, we get a homotopy commutative diagram

$$\begin{array}{ccc} G^{-TG} & \xrightarrow{\tau_\mu} & G^{-TG} \wedge G^{-TG} \\ {\scriptstyle\alpha}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow{\scriptstyle\alpha} \\ G^\vee & \xrightarrow{\mu^\vee} & G^\vee \wedge G^\vee \end{array}$$

Given the adjoint action of $G$ on itself, and the diagonal adjoint action of $G$ on $G \times G$, the multiplication map $\mu : G \times G \to G$ is equivariant. Therefore there is an induced fiberwise coproduct on the parametrized spectrum $P \times_G G^{-TG}$, as there is on $\mathcal{D}(\Sigma^\infty_X(P^{\mathrm{Ad}}_+))$.

We now verify that the induced map $\bar{\alpha} : P \times_G G^{-TG} \xrightarrow{\simeq} \mathcal{D}(\Sigma^\infty_X(P^{\mathrm{Ad}}_+))$ preserves these coproducts. We do this by studying the definition of the maps involved more carefully.

Toward this end let $e : G \subset V$ be an equivariant embedding of $G$ with its conjugation action into a finite dimensional $G$-representation $V$, as above. Let $k = \dim V$. We then have an induced composition of equivariant embeddings,

$$G \times G \xrightarrow{\mu \times e \times e} G \times V \times V \xrightarrow{e \times 1 \times 1} V \times V \times V. \tag{9}$$

Recall that the tangent bundle of $G$ has an equivariant trivialization $TG \cong G \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra with its adjoint action. Differentiating $e : G \hookrightarrow V$ at the identity gives a linear equivariant embedding $\mathfrak{g} \hookrightarrow V$. We let $\mathfrak{g}^\perp$ be the orthogonal complement with its induced action.

The total space of the normal bundle of $G \times V \times V \xrightarrow{e \times 1 \times 1} V \times V \times V$ is clearly equivariantly isomorphic to $G \times \mathfrak{g}^\perp \times V \times V$. We perform the Pontryagin–Thom construction on the induced (equivariant) embedding of the restriction of the total space

$$(\mu \times e \times e)^*(G \times \mathfrak{g}^\perp \times V \times V) \hookrightarrow G \times \mathfrak{g}^\perp \times V \times V.$$

This is a codimension $2k - d$ embedding. The Pontryagin–Thom construction gives an equivariant map

$$\tau_\mu^{V \times V \times V} : G_+ \wedge S^{\mathfrak{g}^\perp} \wedge S^V \wedge S^V \to (G \times G)_+ \wedge S^{\mathfrak{g}^\perp} \wedge S^{\mathfrak{g}^\perp} \wedge S^V,$$

or equivalently,

$$\tau_\mu^{V \times V \times V} : G^{\eta_e} \wedge S^V \wedge S^V \to G^{\eta_e} \wedge G^{\eta_e} \wedge S^V.$$

This defines the map

$$\tau_\mu^{V \times V \times V} : P \times_G (G^{\eta_e} \wedge S^V \wedge S^V) \to P \times_G (G^{\eta_e} \wedge G^{\eta_e} \wedge S^V).$$

Similarly, the Atiyah duality map, which as discussed above is defined via a Pontryagin–Thom collapse, is an equivariant map

$$\alpha : G^{\eta_e} \wedge S^V \wedge S^V \to \mathrm{Map}(G, S^V) \wedge S^V \wedge S^V,$$

which induces a map

$$\alpha : P \times_G (G^{\eta_e} \wedge S^V \wedge S^V) \to P \times_G (\mathrm{Map}(G, S^V) \wedge S^V \wedge S^V).$$

The compatibility of these Pontryagin–Thom maps yields that the following diagram commutes:

$$\begin{array}{ccc} P \times_G (G^{\eta_e} \wedge S^V \wedge S^V) & \xrightarrow{\tau_\mu^{V\times V\times V}} & P \times_G (G^{\eta_e} \wedge G^{\eta_e} \wedge S^V) \\ \downarrow \alpha & & \downarrow \alpha \\ P \times_G \big(\mathrm{Map}(G, S^V) \wedge S^V \wedge S^V\big) & \xrightarrow{\mu^\vee} & P \times_G \big(\mathrm{Map}(G \times G, S^V \wedge S^V) \wedge S^V\big) \end{array} \tag{10}$$

Passing to spectra, this says that the following diagram of parametrized spectra over $X$ homotopy commutes:

$$\begin{array}{ccc} P \times_G G^{-TG} & \xrightarrow{\tau_\mu} & P \times_G G^{-TG} \wedge G^{-TG} \\ \alpha \downarrow \simeq & & \simeq \downarrow \alpha \\ P \times_G G^\vee & \xrightarrow{\mu^\vee} & P \times_G (G^\vee \wedge G^\vee) \end{array} \tag{11}$$

Or, written with the notation used above, the following diagram of parametrized spectra over $X$ homotopy commutes:

$$\begin{array}{ccc} P \times_G G^{-TG} & \xrightarrow{\tau_\mu} & P \times_G G^{-TG} \wedge_X P \times_G G^{-TG} \\ \bar{\alpha} \downarrow \simeq & & \simeq \downarrow \bar{\alpha} \\ \mathcal{D}(\Sigma^\infty_X(P^{\mathrm{Ad}}_+)) & \xrightarrow{\mu^\vee} & \mathcal{D}(\Sigma^\infty_X(P^{\mathrm{Ad}}_+)) \wedge_X \mathcal{D}(\Sigma^\infty_X(P^{\mathrm{Ad}}_+)) \end{array} \tag{12}$$

In other words, the induced map

$$\alpha : (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \xrightarrow{\simeq} \mathcal{S}^P_\bullet(X)$$

respects coproducts up to homotopy.

This completes the proof of parts (1) and (2) of Theorem 1. Part (3) of Theorem 1 follows from part (1) and the Thom isomorphism applied to the vertical tangent bundle $T_{\mathrm{vert}} \to P^{\mathrm{Ad}}$. The algebra structure on $H^*(P^{\mathrm{Ad}})$ was discovered first by Gruher [2007]. The main point of part (1) of Theorem 1 is that it realizes the work of Gruher on the level of parametrized spectra and the induced twisted homology theory. In [Gruher 2007] it was shown that in the case of the universal bundle

$$G \to EG \to BG,$$

the algebra structure on $H^*(P^{\mathrm{Ad}} \simeq LBG)$ (or equivalently the coalgebra structure on $H_*(LBG)$) is isomorphic to the Lie group string topology algebra of Chataur and Menichi [2012]. This completes the proof of Theorem 1.

## 2. Twisted, compact Calabi–Yau ring spectra and the duality between manifold and Lie group string topology

The goal of this section is to study duality phenomena in the string topology of a principal bundle $G \to P \to M$, where $G$ is a compact, $d$-dimensional Lie group, and $M$ is a closed, $n$-dimensional manifold. More specifically, our goal is to study the duality between the manifold string topology and the Lie group string topology in this setting. To do this we describe the notion of "twisted, compact Calabi–Yau ring spectra" and show how the string topology of such a principal bundle has this structure. This notion is a lifting to the category of spectra of the notion of "Calabi–Yau" algebras and categories as defined by Costello [2007], Kontsevich and his collaborators [Kontsevich and Soibelman 2009; Kontsevich and Vlassopoulos 2013], Lurie [2009], and Cohen and Ganatra [2015].

Our first result is the following:

**Theorem 9.** *For a principal bundle $G \to P \to M$ where $G$ a compact Lie group and $M$ is a closed manifold, the manifold string topology spectrum $\mathcal{S}^\bullet_P(M)$ and the Lie group string topology spectrum $\mathcal{S}^P_\bullet(M)$ are Spanier–Whitehead dual. Under this duality the ring spectrum structure of $\mathcal{S}^\bullet_P(M)$ corresponds to the coalgebra structure of $\mathcal{S}^P_\bullet(M)$.*

*Proof.* Recall from [Cohen and Jones 2017] (and restated in the introduction above), the manifold string topology $\mathcal{S}^\bullet_P$ is the twisted cohomology theory corresponding to the fiberwise suspension spectrum $\Sigma^\infty(G_+) \to \Sigma^\infty_M(P^{\mathrm{Ad}}_+) \to M$. Using a particular version of Poincaré duality proven by Klein [2001] (called "Atiyah duality" in this paper), Cohen and Klein [2009] showed that

$$\mathcal{S}^\bullet_P(M) = \Gamma_M(\Sigma^\infty_M(P^{\mathrm{Ad}}_+)) \simeq (P^{\mathrm{Ad}})^{-TM},$$

and the ring structure comes from a generalized cup product in this (twisted) cohomology theory arising from the fiberwise ring structure of this parametrized spectrum.

Furthermore, Theorem 1 above states that the Lie group string topology $\mathcal{S}^P_\bullet$ is the twisted homology theory corresponding to the fiberwise Spanier–Whitehead dual spectrum, $G^\vee \to \mathcal{D}(\Sigma^\infty_M(P^{\mathrm{Ad}}_+)) \to M$. It was also shown that this is a fiberwise coalgebra spectrum whose coalgebra structure is (fiberwise) Spanier–Whitehead dual to the ring structure of the parametrized spectrum

$$\Sigma^\infty(G_+) \to \Sigma^\infty_M(P^{\mathrm{Ad}}_+) \to M.$$

Finally it was shown that there is a coproduct-preserving equivalence of spectra,

$$(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \xrightarrow{\simeq} \mathcal{S}^P_\bullet(M).$$

We remark that the fact that the Thom spectra

$$(P^{\mathrm{Ad}})^{-TM} \simeq \mathcal{S}^\bullet_P(M) \quad \text{and} \quad (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \simeq \mathcal{S}^P_\bullet(M)$$

are Spanier–Whitehead dual follows from classical Atiyah [1961] duality.

This completes the proof. □

**Remark.** When $M$ is oriented and the bundle $T_{\mathrm{vert}}$ is oriented, one can apply the two Thom isomorphisms,

$$H_*(P^{\mathrm{Ad}}) \cong H_{*-n}((P^{\mathrm{Ad}})^{-TM}) \quad \text{and} \quad H_*(P^{\mathrm{Ad}}) \cong H_{*-d}((P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}).$$

The Spanier–Whitehead duality above then yields a Frobenius algebra structure on $H_*(P^{\mathrm{Ad}})$ as discovered by Gruher [2007].

We now strengthen this result by proving that in this situation (i.e., a principal bundle $G \to P \to M$, where $G$ is a compact Lie group and $M$ is a closed, smooth manifold), the spectrum $(P^{\mathrm{Ad}})^{-TM}$ is a "twisted, compact Calabi–Yau ring spectrum." The notions of Calabi–Yau differential graded algebras or $A_\infty$ algebras or (higher) categories were introduced in [Costello 2007; Lurie 2009; Kontsevich and Soibelman 2009; Kontsevich and Vlassopoulos 2013] because of their connections with two-dimensional topological field theories. This notion can be viewed as a derived version of a Frobenius algebra. This will be made precise in Proposition 11 below. In this paper we lift these ideas to the category of spectra, where we must deal with "twisted" versions of these notions in order to get many interesting examples. We actually introduce two versions of twisted Calabi–Yau ring spectra: a compact version and a smooth version. This follows the ideas of Kontsevich and his collaborators [Kontsevich and Soibelman 2009; Kontsevich and Vlassopoulos 2013], who worked with $A_\infty$ algebras over a field of characteristic zero, and of Cohen and Ganatra [2015] who worked with $A_\infty$-algebras or categories over arbitrary fields.

We begin with the notion of a "twisted, compact, Calabi–Yau ring spectrum." Recall that a compact $E_1$- ring spectrum $R$ is one that is perfect as an $\mathbb{S}$-module.

**Definition.** A *twisted, compact Calabi–Yau ring spectrum* (or *twisted* cCY) of dimension $n$ is a triple $(R, Q, t)$, where $R$ is a compact ring spectrum, $Q$ is an

$R$-bimodule which is compact as an $\mathbb{S}$-module, and has the same $\mathbb{Z}/2$-homology as $R$:

$$Q \wedge H\mathbb{Z}/2 \simeq R \wedge H\mathbb{Z}/2,$$

as $R$-bimodules.

We refer to $Q$ as the "twisting" bimodule. If $Q = R$ we say that $R$ has *trivial* twisting. The map

$$t : \mathrm{THH}(R;\, Q) \to \Sigma^{-n}\mathbb{S}$$

is a map of spectra we call the $n$-dimensional *trace map* that has the following duality property: The pairing defined by the composition

$$\langle -, -\rangle : R \wedge Q \xrightarrow{\mu} Q \hookrightarrow \mathrm{THH}(R;\, Q) \xrightarrow{t} \Sigma^{-n}\mathbb{S}$$

is nondegenerate in the sense that the adjoint $R \to \Sigma^{-n} Q^{\vee}$ is an equivalence of $R$-bimodule spectra. Here $\mu : R \wedge Q \to Q$ is the module structure, $Q \hookrightarrow \mathrm{THH}(R;\, Q)$ is the inclusion of the spectrum of zero simplices, and $Q^{\vee}$ is the Spanier–Whitehead dual of $Q$, which exists because of the compactness assumption.

The following observation is an immediate consequence of the definition.

**Proposition 10.** *Let $(R, Q, t)$ be a twisted compact Calabi–Yau ring spectrum. Then the duality between R and Q defined by the nondegenerate pairing $\langle -, -\rangle$ defines a coalgebra structure on the twisting bimodule $Q$, whose coproduct is Spanier–Whitehead dual to the product in the ring structure $R$.*

The main applications of compact Calabi–Yau ring spectra occur in the presence of orientations. We now define what we mean by this. In our discussion on orientations, we use the $S^1$-action on topological Hochschild homology given by the cyclic bar construction; a realization of a cyclic spectrum has a circle action. See [Dwyer et al. 1985; Angeltveit et al. 2014].

**Definition.** Let $(R, Q, t)$ be a twisted, cCY ring spectrum of dimension $n$, and let $E$ be a ring spectrum representing a homology theory $E_*$. An *$E_*$-orientation* of $(R, Q, t)$ is a pair $(u, \tilde{t}_E)$, where

$$u : Q \wedge E \xrightarrow{\simeq} R \wedge E$$

is an equivalence of the $E_*$-homology spectra as $R$-bimodules. Here $R$ acts trivially on $E$.

$$\tilde{t}_E : \mathrm{THH}(R;\; R)_{hS^1} \wedge E \to \Sigma^{-n} E$$

is an $E$-module map from the homotopy orbit spectrum of the $S^1$-action induced by the cyclic structure, which factorizes the trace map $t$ in $E$-homology. That is, the induced trace map $t_E = t \wedge 1 : \mathrm{THH}(R;\ Q) \wedge E \to \Sigma^{-n}\mathbb{S} \wedge E$ is homotopic to the composition

$$t_E : \mathrm{THH}(R;\ Q) \wedge E \xrightarrow{u} \mathrm{THH}(R;\ R) \wedge E \xrightarrow{\text{project}} \mathrm{THH}(R;\ R)_{hS^1} \wedge E, \tag{13}$$

$$\xrightarrow{\tilde{t}_E} \Sigma^{-n} E. \tag{14}$$

When $E = Hk$, the Eilenberg–MacLane spectrum for a field $k$, then a twisted, compact Calabi–Yau ring spectrum $(R, Q, t)$ together with an $Hk$-orientation $(u, \tilde{t}_{Hk})$ defines a compact Calabi–Yau algebra structure on the singular chains with $k$-coefficients, $C_*(R; k)$, as defined in [Kontsevich and Vlassopoulos 2013; Cohen and Ganatra 2015].

The following gives a precise relation between twisted cCY ring spectra and Frobenius algebras.

**Proposition 11.** *Let $(R, Q, t)$ be a twisted* cCY *ring spectrum of dimension $n$, and let $E$ be a ring spectrum representing a homology theory $E_*$ with respect to which $(R, Q, t)$ has orientation $(u, \tilde{t}_E)$. Then $R \wedge E$ is a Frobenius algebra over $E$ of dimension $n$. That is, the pairing*

$$\langle -, - \rangle : (R \wedge E) \wedge (R \wedge E) \xrightarrow{\text{multiply}} R \wedge E \xrightarrow{\iota} \mathrm{THH}(R \wedge E; R \wedge E)$$

$$\xrightarrow{\text{project}} \mathrm{THH}(R \wedge E; R \wedge E)_{hS^1} \xrightarrow{\tilde{t}_E} \Sigma^{-n} E \tag{15}$$

*is a nondegenerate pairing of $E$-modules. Here $\iota : R \wedge E \hookrightarrow \mathrm{THH}(R \wedge E; R \wedge E)$ is the inclusion of the spectrum of* 0*-simplices. "Nondegeneracy" means that the adjoint of this pairing,*

$$R \wedge E \to \mathrm{Rhom}_E(R \wedge E, \Sigma^{-n} E)$$

*is an equivalence of $E$-modules.*

*Proof.* It is easily checked from the definition of orientation that the pairing $\langle -, - \rangle$ defined above is homotopic to the composition

$$(R \wedge E) \wedge (R \wedge E) \xrightarrow{1 \wedge u^{-1}} (R \wedge E) \wedge (Q \wedge E) \xrightarrow{\mu} Q \wedge E \hookrightarrow \mathrm{THH}(R; Q) \wedge E$$

$$\xrightarrow{t_E = t \wedge 1} \Sigma^{-n}\mathbb{S} \wedge E.$$

But this pairing is nondegenerate by the definition of the twisted Calabi–Yau structure. □

We now give two important examples of twisted cCY ring spectra.

***Example.*** The first example shows how ordinary Poincaré or Atiyah duality fits the definition of twisted compact Calabi–Yau ring spectra.

**Proposition 12.** *Let $M$ be a closed $n$-dimensional manifold. Then its Spanier–Whitehead dual, $M^\vee$, which, by Atiyah duality is equivalent to $M^{-TM}$, comes naturally equipped with the structure of a twisted* cCY *ring spectrum of dimension $n$.*

*Proof.* The suspension spectrum $\Sigma^\infty(M_+)$ can be viewed as a $M^\vee$ bimodule in the usual way. Notice that since, by Atiyah duality, $M^\vee$ is equivalent to $M^{-TM}$, then the Thom isomorphism gives

$$H_*(\Sigma^\infty(M_+); \mathbb{Z}/2) \cong H_{*-n}(M^\vee; \mathbb{Z}/2).$$

So we let $R = M^\vee$ and let the twisting bimodule $Q = \Sigma^{-n}\Sigma^\infty(M_+)$, which we simply denote $\Sigma^{-n}(M_+)$.

In order to define the $n$-dimensional trace map on $\mathrm{THH}(R; Q)$, we first study its homotopy type. This is a simplicial object in finite type spectra. That is, for each $k$, the spectrum of $k$-simplices is a spectrum of finite type. For such a simplicial spectrum $\mathbb{X}_\bullet$ we define its Spanier–Whitehead dual $\mathbb{X}^\vee$ to be the totalization of the cosimplicial spectrum whose spectrum of $k$-simplices is the Spanier–Whitehead dual $\mathbb{X}_k^\vee = \mathrm{Map}(\mathbb{X}_k, \mathbb{S})$. We then have the following result.

**Lemma 13.** *For $M$ a closed $n$-manifold, $R = M^\vee$ and $Q = \Sigma^{-n}(M_+)$, the Spanier–Whitehead dual of* $\mathrm{THH}(R; Q)$ *is given by*

$$\mathrm{THH}(R; Q)^\vee \simeq \Sigma^n LM^{-TM}.$$

*Proof.* Note that

$$\mathrm{THH}(R; Q)_k = R^{(k)} \wedge Q \simeq (M^k)^\vee \wedge \Sigma^\infty(M_+) \wedge S^{-n}.$$

Therefore in the cosimplicial spectrum $\mathrm{THH}(R; Q)^\vee$, the spectrum of $k$-simplices is equivalent to

$$\mathrm{THH}(R; Q)_k^\vee \simeq \Sigma^\infty(M_+^k) \wedge M^\vee \wedge S^n \simeq \Sigma^\infty(M_+^k) \wedge M^{-TM} \wedge S^n.$$

Under this equivalence, the coface maps are determined by the coalgebra structure of $\Sigma^\infty(M_+)$ defined by the diagonal map of $M$, as well as the bicomodule structure of $M^{-TM}$. This structure is a special case of the bicomodule structure of any Thom spectrum of a bundle (or spherical fibration) $\zeta$ over a space $X$ over the coalgebra $\Sigma^\infty(X_+)$. The bicomodule structure is given by the maps of Thom spectra that are induced by the diagonal maps of the base,

$$X^\zeta \to X_+ \wedge X^\zeta \quad \text{and} \quad X^\zeta \to X^\zeta \wedge X_+.$$

This cosimplicial spectrum is the $n$-fold suspension of the cosimplicial spectrum studied in [Cohen and Jones 2002] where it was shown to have totalization equivalent to $LM^{-TM}$. □

**Remark.** Notice that the inclusion of the spectrum of zero simplices,

$$\Sigma^{-n}(M_+) \hookrightarrow \mathrm{THH}(R;\, Q)$$

is Spanier–Whitehead dual to the map

$$\Sigma^n LM^{-TM} \xrightarrow{\text{eval.}} \Sigma^n M^{-TM} \simeq \Sigma^n M^\vee$$

induced on Thom spectra by the usual evaluation fibration $LM \to M$.

One way of thinking of the $n$-dimensional trace map $t : \mathrm{THH}(R;\, Q) \to \Sigma^{-n}\mathbb{S}$ is that it is Spanier–Whitehead dual to the $n$-fold suspension of the unit map in the ring structure of $LM^{-TM}$:

$$\Sigma^n \mathbb{S} \to \Sigma^n LM^{-TM}.$$

More concretely, notice that the augmentation map of $R$,

$$\epsilon : R = M^\vee \to \mathbb{S}$$

and the map induced by sending all of $M$ to the nonbase point

$$p : \Sigma^{-n}(M_+) \to \Sigma^{-n}\mathbb{S}$$

define a map

$$t : \mathrm{THH}(R;\, Q) = \mathrm{THH}(M^\vee;\, \Sigma^{-n}(M_+)) \xrightarrow{(\epsilon,\, p)} \mathrm{THH}(\mathbb{S};\, \Sigma^{-n}\mathbb{S}) = \Sigma^{-n}\mathbb{S}.$$

The reader can now check that the composition

$$M^\vee \wedge \Sigma^{-n}(M_+) \xrightarrow{\mu} \Sigma^{-n}(M_+) \to \mathrm{THH}(M^\vee;\, \Sigma^{-n}(M_+)) \xrightarrow{t} \Sigma^{-n}\mathbb{S}$$

is simply the $n$-fold desuspension of the duality map, and therefore is nondegenerate. This proves that $(M^\vee, \Sigma^{-n}(M_+), t)$ is a twisted, compact Calabi–Yau ring spectrum of dimension $n$. □

We now consider orientations. Let $E$ be any ring spectrum representing a generalized homology theory with respect to which $M$ is oriented. The Thom isomorphism then defines an equivalence

$$u : \Sigma^{-n}(M_+) \wedge E \xrightarrow{\simeq} M^{-TM} \wedge E \simeq M^\vee \wedge E$$

which is clearly an equivalence of $M^\vee$-bimodules. Again consider the augmentation map $\epsilon : M^\vee \to \mathbb{S}$. Now the orientation induces a Thom class map

$$\tau : M^\vee \simeq M^{-TM} \to \Sigma^{-n}E.$$

These maps define a composition

$$\tilde{t}_E : \mathrm{THH}(M^\vee; M^\vee)_{hS^1} \wedge E \xrightarrow{(\epsilon,\tau)} \mathrm{THH}(\mathbb{S}; \mathbb{S})_{hS^1} \wedge \Sigma^{-n} E \simeq \Sigma^{-n}(BS^1_+) \wedge E \xrightarrow{p\wedge 1} \Sigma^{-n} E,$$

where $p : BS^1_+ \to S^0$ is the projection map.

We leave it to the reader to check that the composition

$$\mathrm{THH}(M^\vee, \Sigma^{-n}(M_+)) \wedge E \xrightarrow{u} \mathrm{THH}(M^\vee, M^\vee) \wedge E \xrightarrow{\text{projection}} \mathrm{THH}(M^\vee, M^\vee)_{hS^1} \wedge E \xrightarrow{\tilde{t}_E} \Sigma^{-n} E$$

is equivalent to $t \wedge 1 : \mathrm{THH}(M^\vee, \Sigma^\infty(M_+)) \wedge E \to \Sigma^{-n} \wedge \mathbb{S} \wedge E$. This proves that the pair $(u, \tilde{t}_E)$ defines an orientation of the twisted cCY structure on $M^\vee$ with respect to $E$.

**Remark.** The above discussion together with Proposition 11 implies that if $M^n$ is an oriented closed manifold, $M^\vee \wedge H\mathbb{Z}$ is a Frobenius algebra over the Eilenberg–MacLane spectrum $H\mathbb{Z}$. Using the Atiyah duality equivalence $M^\vee \simeq M^{-TM}$ we see that $M^{-TM} \wedge H\mathbb{Z} \simeq \Sigma^{-n}(M_+ \wedge H\mathbb{Z})$ is a Frobenius algebra. The multiplication reflects the classical intersection product on the level of chains, $C_{*+n}(M; \mathbb{Z})$. The comultiplication comes from the diagonal, $M \to M \times M$.

***Example.*** The following example supplies the main ingredient for the proof of Theorem 3 as stated in the introduction.

**Proposition 14.** *Let $G \to P \to M$ be a principal bundle where $G$ is a compact Lie group of dimension $d$ and $M$ is a closed manifold of dimension $n$. Then the manifold string topology ring spectrum $R = \mathcal{S}^\bullet_P(M) \simeq (P^{\mathrm{Ad}})^{-TM}$ naturally admits the structure of a twisted, compact Calabi–Yau ring spectrum of dimension $n - d$.*

*Proof.* We need to produce the twisting module $Q$ and a trace map

$$t : \mathrm{THH}(R; Q) \to \Sigma^{d-n}\mathbb{S}.$$

For the twisting module we take the Lie group string topology spectrum $Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \simeq \Sigma^{d-n}(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$. The fact that $R$ and $Q$ have isomorphic mod-2 homology follows from the Thom isomorphism. The fact that $Q$ is indeed an $R$-bimodule follows from the Spanier–Whitehead duality of $R = \mathcal{S}^\bullet_P(M)$ and $\Sigma^{n-d} Q = \mathcal{S}^P_\bullet(M)$ established in Theorem 9, reflecting Gruher's work [2007]. The bimodule structure of $Q$ over $R$ is then the dual of the bimodule structure of $R$ over itself.

Notice also that $R = \mathcal{S}^\bullet_P(M) \simeq (P^{\mathrm{Ad}})^{-TM}$ has an augmentation $\epsilon : R \to \mathbb{S}$. To see this, consider the following diagram, which we view as a map of bundles.

$$\begin{array}{ccc} G & \longrightarrow & \{\mathrm{id}\} \\ \big\downarrow & & \big\downarrow \\ P^{\mathrm{Ad}} & \longrightarrow & M \\ \big\downarrow & & \big\downarrow = \\ M & \xrightarrow{=} & M \end{array}$$

This defines a map of twisted cohomology ring spectra

$$\mathcal{S}^\bullet_P(M) \to \mathcal{S}^\bullet_M(M)$$

or equivalently,

$$\Gamma_M(\Sigma^\infty_M(P^{\mathrm{Ad}}_+)) \to \mathrm{Map}(\Sigma^\infty(M_+), \mathbb{S}) = M^\vee.$$

The augmentation is then given by

$$\epsilon : R = \mathcal{S}^\bullet_P(M) \to M^\vee \to \mathbb{S},$$

where the second map in this composition is the augmentation of $M^\vee \to \mathbb{S}$.

Notice that the above diagram also defines a map of bimodules, induced on the homology spectra,

$$Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \simeq \Sigma^{d-n}(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \to \Sigma^{d-n}\mathcal{S}^M_\bullet(M) = \Sigma^{d-n}(M_+).$$

Composing this map with the projection $p : \Sigma^{d-n}(M_+) \to \Sigma^{d-n}\mathbb{S}$ defines a map $u : Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \to \Sigma^{d-n}\mathbb{S}$. Putting these maps together gives a map of topological Hochschild homologies,

$$t : \mathrm{THH}(R, Q) \xrightarrow{(\epsilon,u)} \mathrm{THH}(\mathbb{S}, \Sigma^{d-n}\mathbb{S}) = \Sigma^{d-n}\mathbb{S}.$$

We leave it to the reader to verify that the pairing defined by the composition

$$\langle -, - \rangle : R \wedge Q = \mathcal{S}^\bullet_P(M) \wedge \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \xrightarrow{\mu} Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \hookrightarrow \mathrm{THH}(R, Q) \xrightarrow{t} \Sigma^{d-n}\mathbb{S} \quad (16)$$

is the duality map given by Theorem 9 above. It is therefore nondegenerate. This proves that the triple $\big(\mathcal{S}^\bullet_P(M), \Sigma^{d-n}\mathcal{S}^P_\bullet(M), t\big)$ is a twisted compact Calabi–Yau ring spectrum of dimension $n - d$. □

Notice that Propositions 14, 10, and 11 imply both Corollary 2 and Theorem 3 as stated in the introduction.

## 3. Gauge symmetry

In this section we continue considering a principal bundle $G \to P \xrightarrow{p} M$, where $G$ is a compact Lie group of dimension $d$, and $M^n$ is a closed manifold of dimension $n$.

Recall that the *gauge group* $\mathcal{G}(P)$ of the bundle $P$ is the group of $G$-equivariant bundle automorphisms of $P$ living over the identity of $M$. Said another way, let $G \to \mathcal{A}ut^G(P) \to M$ be the fibration whose fiber over $x \in M$ is the group of $G$-equivariant automorphisms of the fiber $p^{-1}(x)$. This bundle is a fiberwise group, and the gauge group is the group of sections

$$\mathcal{G}(P) = \Gamma_M(\mathcal{A}ut^G(P)).$$

Now a standard exercise shows that the bundle $\mathcal{A}ut^G(P)$ is isomorphic to the adjoint bundle $G \to P^{\mathrm{Ad}} \to M$. Thus we may identify

$$\mathcal{G}(P) = \Gamma_M(P^{\mathrm{Ad}}).$$

In [Cohen and Jones 2017] a fiberwise stabilization map was defined and studied:

$$\rho : \Sigma^\infty(\mathcal{G}(P)_+) = \Sigma^\infty(\Gamma_M(P^{\mathrm{Ad}})_+) \\ \to \Gamma_M(\Sigma^\infty(P^{\mathrm{Ad}}_+)) \simeq (P^{\mathrm{Ad}})^{-TM} = \mathcal{S}^\bullet_P(M). \quad (17)$$

The map $\rho$ is a map of ring spectra and also defines a map to the group of units of the (manifold) string topology ring spectrum

$$\rho : \mathcal{G}(P) \to \mathrm{GL}_1(\mathcal{S}^\bullet_P(M)). \quad (18)$$

In [Cohen and Jones 2017] this map was studied and computed in several important cases. Now recall from Proposition 14 that in the twisted compact Calabi–Yau structure of $\mathcal{S}^\bullet_P(M)$, the twisting bimodule is given by a suspension of the Lie group string topology spectrum, $Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \simeq \Sigma^{d-n}(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$. In particular, the Lie group string topology spectrum $\mathcal{S}^P_\bullet(M)$ inherits a coalgebra structure. One of the goals of this section is to show that there is a similarly defined and compatible gauge symmetry on this spectrum. We also show how these actions are related, and describe different perspectives on this action. We then compute two examples of this gauge symmetry.

**Theorem 15.** *The twisting bimodule structure on the Lie group string topology spectrum, $Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M)$ has a natural action of the gauge group $\mathcal{G}(P)$. That is, $\Sigma^{d-n}\mathcal{S}^P_\bullet(M)$ is a module spectrum over the ring spectrum $\Sigma^\infty(\mathcal{G}(P)_+)$. Furthermore, this action is compatible with the gauge symmetry on the manifold string topology spectrum $R = \mathcal{S}^\bullet_P(M)$ via its twisted Calabi–Yau duality pairing*

$$\langle -, - \rangle : R \wedge Q = \mathcal{S}^\bullet_P(M) \wedge \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \to \Sigma^{d-n}\mathbb{S}$$

*as defined in the proof of Proposition 14* (*see Equation* (16)). *That is*, *the adjoint equivalence*

$$\mathcal{S}_P^{\bullet}(M) \xrightarrow{\simeq} \mathcal{S}_{\bullet}^P(M)^{\vee}$$

*is equivariant with respect to the gauge symmetry of these spectra.*

*Proof.* Recall that $\mathcal{S}_{\bullet}^P(M)$ is the generalized homology associated to the parametrized spectrum

$$G^{\vee} \to \mathcal{D}(\Sigma_M^{\infty}(P_+^{\mathrm{Ad}})) \to M.$$

We may take $\mathcal{D}(\Sigma_M^{\infty}(P_+^{\mathrm{Ad}}))$ to be the parametrized spectrum whose $k$-th space over $M$ is the fibration

$$\mathrm{Map}(G, S^k) \to P \times_G \mathrm{Map}(G, S^k) \to M$$

where the action of $G$ on $\mathrm{Map}(G, S^k)$ is the dual of the adjoint action as described in (4), with trivial $G$-action on $S^k$. This is because the spaces $\mathrm{Map}(G, S^k)$ with this action form the underlying naive $G$-spectrum of $G^{\vee}$, on which the homotopy theory of $G^{\vee}$ as a $\Sigma^{\infty}(G_+)$-module is determined.

This fibration has a canonical section

$$\sigma : M = P \times_G \mathrm{point} = P \times_G \epsilon \hookrightarrow P \times_G \mathrm{Map}(G, S^k).$$

Here $\epsilon : G \to S^k$ is the constant map at the basepoint $(1, 0, \dots, 0) \in S^k$. Then the $k$-th space of the generalized homology spectrum

$$\mathcal{S}_{\bullet}^P(M) = \mathcal{D}(\Sigma_M^{\infty}(P_+^{\mathrm{Ad}}))/\sigma(M)$$

is given by

$$P_+ \wedge_G \mathrm{Map}(G, S^k).$$

The structure maps are given by

$$\Sigma\big(P_+ \wedge_G \mathrm{Map}(G, S^k)\big) = P_+ \wedge_G \Sigma(\mathrm{Map}(G, S^k)) \xrightarrow{1\wedge s} P_+ \wedge_G \mathrm{Map}(G, S^{k+1}),$$

where $s : \Sigma(\mathrm{Map}(G, S^k)) \to \mathrm{Map}(G, \Sigma S^k)$ is given by $s(t, \phi)(g) = t \wedge \phi(g)$.

Now the bundle $p : P \times_G \mathrm{Map}(G, S^k) \to M$ is $\mathcal{G}(P)$-equivariant with respect to the following action. Let $\phi \in \mathcal{G}(P) = \Gamma_M(P^{\mathrm{Ad}})$, and let $(y, \theta) \in P \times \mathrm{Map}(G, S^k)$ represent an element in $P \times_G \mathrm{Map}(G, S^k)$. Then

$$\phi \cdot (y, \theta) = (y, h \cdot \theta), \tag{19}$$

where $h \in G$ is the unique element such that $\phi(p(y, \theta)) \in P \times_G G^{\mathrm{Ad}}$ is represented by $(y, h) \in P \times G$.

One can check that this action is well-defined, and that the section $\sigma(M = P \times_G \epsilon)$ consists of fixed points of this action. It therefore descends to a $\mathcal{G}(P)$-action on $P_+ \wedge_G \mathrm{Map}(G, S^k)$.

These actions (one for each $k$) clearly respect the structure maps and therefore define an action of $\mathcal{G}(P)$ on the spectrum $\mathcal{D}(\Sigma^\infty_M(P^{\mathrm{Ad}}_+))/\sigma(M) = \mathcal{S}^P_\bullet(M)$.

Now, as seen in Corollary 2, the Lie group string topology spectrum $\mathcal{S}^P_\bullet(M)$ is Spanier–Whitehead dual to the manifold string topology spectrum $\mathcal{S}^\bullet_P(M) = \Gamma_M(\Sigma^\infty_M(P^{\mathrm{Ad}}_+))$. The action of the gauge group on this ring spectrum is given by the stabilization representation (17), (18), and it is immediate that the gauge symmetry defined on the Lie group string topology spectrum $\mathcal{S}^P_\bullet(M)$ in (19) is the dual action. This implies that with respect to the twisted Calabi–Yau duality pairing

$$\langle -, -\rangle : R \wedge Q = \mathcal{S}^\bullet_P(M) \wedge \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \to \Sigma^{d-n}\mathbb{S}$$

the corresponding adjoint equivalence

$$\mathcal{S}^\bullet_P(M) \xrightarrow{\simeq} \mathcal{S}^P_\bullet(M)^\vee$$

is equivariant with respect to the gauge symmetry of these spectra. □

We now study examples of this gauge symmetry and describe this symmetry from different perspectives.

***Example.*** Consider the $U(1)$ Hopf bundle

$$U(1) \to P = S^{2n+1} \to \mathbb{CP}^n.$$

Since $U(1)$ is abelian, the adjoint bundle $P^{\mathrm{Ad}}$ is trivial,

$$U(1) \to \mathbb{CP}^n \times U(1) \to \mathbb{CP}^n.$$

Therefore the gauge group is given by the mapping group,

$$\mathcal{G}(P) = \mathrm{Map}(\mathbb{CP}^n, U(1)).$$

Also, the fiberwise suspension spectrum $\Sigma^\infty(U(1)_+) \to \Sigma^\infty_{\mathbb{CP}^n}(P^{\mathrm{Ad}}_+) \to \mathbb{CP}^n$ is given by the trivially parametrized spectrum

$$\Sigma^\infty(U(1)_+) \to \mathbb{CP}^n_+ \wedge \Sigma^\infty(U(1)_+) \to \mathbb{CP}^n.$$

(By the *trivially parametrized spectrum* $\mathbb{CP}^n_+ \wedge \Sigma^\infty(U(1)_+)$ we mean the parametrized spectrum whose $k$-th space is the trivial fibration $\mathbb{CP}^n \times \Sigma^k(U(1)_+) \to \mathbb{CP}^n$.)

The twisted cohomology theory this parametrized spectrum represents is therefore actually untwisted, and so the defining spectrum of sections is the mapping spectrum,

$$R = \mathcal{S}^\bullet_P(\mathbb{CP}^n) = \mathrm{Map}\big(\Sigma^\infty(\mathbb{CP}^n), \Sigma^\infty(U(1)_+)\big).$$

This is an $E_\infty$-ring spectrum because the source of the mapping spectrum is an $E_\infty$-coalgebra spectrum and the target is an $E_\infty$-ring spectrum.

The action of the gauge group $\mathcal{G}(P) = \mathrm{Map}(\mathbb{CP}^n, U(1))$ is then given by the map of ring spectra

$$\Sigma^\infty(\mathcal{G}(P)_+) = \Sigma^\infty\big(\mathrm{Map}(\mathbb{CP}^n, U(1))_+\big) \xrightarrow{\sigma} \mathrm{Map}\big(\Sigma^\infty(\mathbb{CP}^n), \Sigma^\infty(U(1)_+)\big) = \mathcal{S}^\bullet_P(\mathbb{CP}^n) = R, \quad (20)$$

where $\sigma$ is the obvious stabilization map. The role of stabilization in understanding gauge symmetry on manifold string topology spectra was studied in general in [Cohen and Jones 2017].

Now consider the gauge symmetry on the bimodule $Q = \Sigma^{1-2n}\mathcal{S}^P_\bullet(\mathbb{CP}^n) = \Sigma^{1-2n}(S^{2n+1}_+ \wedge_{U(1)} U(1)^\vee)$ where the action of $U(1)$ on the Spanier–Whitehead dual $U(1)^\vee$ is (the dual of) the conjugation action. Again, since $U(1)$ is abelian this action is trivial, so

$$Q = \Sigma^{1-2n}(\mathbb{CP}^n_+ \wedge U(1)^\vee).$$

Of course $U(1) \cong S^1$, so $U(1)^\vee \simeq \Sigma^\infty(S^{-1} \vee S^0)$. By Spanier–Whitehead duality, the action of the gauge group $\mathcal{G}(P)$ is given by composing the stabilization map described above (20)

$$\sigma : \Sigma^\infty(\mathcal{G}(P)_+) \to R$$

with the $R$-bimodule action on the desuspension of its dual, $Q$, as described in the proof of Proposition 14.

Before we move on to another example, we consider the action of the gauge group on the level of Thom spectra. The point is that the Calabi–Yau ring spectrum in question, $R \simeq (P^{\mathrm{Ad}})^{-TM}$, and the twisting bimodule, $Q \simeq \Sigma^{d-n}(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$, are both Thom spectra. To understand the induced gauge symmetry on these Thom spectra, we first observe that the gauge group actually acts on the space $P^{\mathrm{Ad}}$, and the actions on the Thom spectra are induced from it.

Let $G \to P \xrightarrow{p} M$ be a principal bundle with $M$ a closed $n$-manifold and $G$ a compact Lie group of dimension $d$. By abuse of notation we also call the projection map of the induced bundle $p : P^{\mathrm{Ad}} \to M$. Let $\phi \in \mathcal{G}(P) = \Gamma_M(P^{\mathrm{Ad}})$. Since $\phi$ is a section of $P^{\mathrm{Ad}}$, for $y \in P^{\mathrm{Ad}}$, $\phi(p(y))$ and $y$ live in the same fiber over $M$. That is, $p(\phi(p(y))) = p(y) \in M$. Thus the pair, $(\phi(p(y)), y)$ lies in the fiber product $P^{\mathrm{Ad}} \times_M P^{\mathrm{Ad}}$. Since $P^{\mathrm{Ad}}$ is a fiberwise group, we can compose with the fiberwise multiplication $\mu : P^{\mathrm{Ad}} \times_M P^{\mathrm{Ad}} \to P^{\mathrm{Ad}}$ to produce an element $\phi \cdot y = \mu(\phi(p(y)), y) \in P^{\mathrm{Ad}}$. The map

$$\mathcal{G}(P) \times P^{\mathrm{Ad}} \to P^{\mathrm{Ad}}, \quad (\phi, y) \mapsto \phi \cdot y$$

defines an action of the gauge group on $P^{\mathrm{Ad}}$. This in fact defines a $\mathcal{G}(P)$-equivariance on the fiber bundle, $G \to P^{\mathrm{Ad}} \to M$. That is, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G}(P) \times P^{\mathrm{Ad}} & \xrightarrow{\cdot} & P^{\mathrm{Ad}} \\ \downarrow & & \downarrow p \\ M & \xrightarrow{=} & M \end{array} \tag{21}$$

where the left vertical arrow composes the projection map $\mathcal{G}(P) \times P^{\mathrm{Ad}} \to P^{\mathrm{Ad}}$ with the bundle map $P^{\mathrm{Ad}} \to M$. Therefore this action induces an action on any (virtual) vector bundle over $P^{\mathrm{Ad}}$ that is pulled back from a bundle over $M$. In particular, on the level of Thom spectra, there is an induced action

$$\mathcal{G}(P)_+ \wedge (P^{\mathrm{Ad}})^{-TM} \to (P^{\mathrm{Ad}})^{-TM}.$$

This is easily seen to be equivalent to the action of $\mathcal{G}(P)$ on $\mathcal{S}_P^{\bullet}(M) \simeq (P^{\mathrm{Ad}})^{-TM}$ described above.

We now observe that the gauge symmetry on the Lie group string topology spectrum $\mathcal{S}^P_{\bullet}(M) \simeq (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$ described above can also be viewed in terms of the space level action of $\mathcal{G}(P)$ on $P^{\mathrm{Ad}}$. This is a consequence of the following observation.

**Proposition 16.** *Let* $\mathrm{Act} : \mathcal{G}(P) \times P^{\mathrm{Ad}} \to P^{\mathrm{Ad}}$ *be the action map described above. Then there is an isomorphism of virtual bundles over* $\mathcal{G}(P) \times P^{\mathrm{Ad}}$,

$$\mathcal{G}(P) \times -T_{\mathrm{vert}} P^{\mathrm{Ad}} \xrightarrow{\cong} \mathrm{Act}^*(-T_{\mathrm{vert}} P^{\mathrm{Ad}}).$$

*Proof.* We first observe that the commutativity of diagram (21) says that there is an isomorphism of vector bundles over $\mathcal{G}(P) \times P^{\mathrm{Ad}}$,

$$\mathcal{G}(P) \times p^*(TM) \cong \mathrm{Act}^*(p^*(TM)).$$

Notice also that there is an isomorphism of vector bundles,

$$D : \mathcal{G}(P) \times TP^{\mathrm{Ad}} \xrightarrow{\cong} \mathrm{Act}^*(TP^{\mathrm{Ad}}),$$

where $TP^{\mathrm{Ad}} \to P^{\mathrm{Ad}}$ is the tangent bundle. The isomorphism is given by differentiation of the action. Now notice that there is an induced isomorphism of virtual bundles, which by abuse of notation we call $-D : \mathcal{G}(P) \times -TP^{\mathrm{Ad}} \xrightarrow{\cong} \mathrm{Act}^*(-TP^{\mathrm{Ad}})$. This is defined by the composition of isomorphisms

$$\begin{aligned} \mathrm{Act}^*(-TP^{\mathrm{Ad}}) &= -\,\mathrm{Act}^*(TP^{\mathrm{Ad}}) \\ &\cong -(\mathcal{G}(P) \times TP^{\mathrm{Ad}}) \quad \text{by the above,} \\ &= \mathcal{G}(P) \times -TP^{\mathrm{Ad}}. \end{aligned}$$

Now, using the fact that

$$-T_{\mathrm{vert}} P^{\mathrm{Ad}} \cong -TP^{\mathrm{Ad}} \oplus p^*(TM)$$

we have that

$$\begin{aligned}\mathrm{Act}^*(-T_{\mathrm{vert}} P^{\mathrm{Ad}}) &\cong \mathrm{Act}^*(-TP^{\mathrm{Ad}}) \oplus \mathrm{Act}^*(p^*(TM)) \\ &\cong \mathcal{G}(P) \times \big(-TP^{\mathrm{Ad}} \oplus p^*(TM)\big) \quad \text{by the above} \\ &\cong \mathcal{G}(P) \times -T_{\mathrm{vert}} P^{\mathrm{Ad}}. \end{aligned}$$ □

The last explicit example of this gauge symmetry will be one that was studied initially in [Cohen and Jones 2017].

***Example.*** Consider the principal SU(2)-bundle over an oriented 4-dimensional sphere,

$$\mathrm{SU}(2) \to P_k \to S^4 \tag{22}$$

having second Chern class $c_2(P_k) = k \in H^4(S^4) \cong \mathbb{Z}$.

In this case we restrict our attention to the *based* gauge group $\mathcal{G}^b(P_k)$ which is defined to be the kernel of the homomorphism,

$$\mathcal{G}(P_k) \to \mathrm{SU}(2), \quad \phi \mapsto \phi(\infty).$$

Here we are thinking of $S^4$ as the one point compactification, $\mathbb{R}^4 \cup \infty$.

In the case $k = 1$ the (based) gauge symmetry on the manifold string topology ring spectrum $\mathcal{S}^{\bullet}_{P_k}(S^4)$ was studied in [Cohen and Jones 2017]. We now observe that the argument presented in [Cohen and Jones 2017] quickly extends to $P_k$ for all $k$, and we then show how it gives an understanding of the gauge symmetry of the Lie group string topology spectrum $\mathcal{S}^{P_k}_{\bullet}(S^4)$ as well.

As has become standard notation, given a ring spectrum $R$, let $\mathrm{GL}_1(R)$ denote the "group of units" of $R$. More precisely, $\mathrm{GL}_1(R)$ is defined so that the following diagram of spaces is homotopy cartesian:

$$\begin{array}{ccc} \mathrm{GL}_1(R) & \longrightarrow & \Omega^\infty(R) \\ \downarrow & & \downarrow{\scriptstyle \text{components}} \\ \pi_0(R)^\times & \hookrightarrow & \pi_0(R) \end{array} \tag{23}$$

Here $\pi_0(R)$ is the discrete ring of components and $\pi_0(R)^\times$ is its group of units.

In other words, $\mathrm{GL}_1(R)$ consists of those path components of the zero space $\Omega^\infty(R)$ consisting of homotopy invertible elements. An action of a group $G$ on a ring spectrum $R$ via $R$-module automorphisms is induced by an $A_\infty$-morphism ("representation")

$$\rho : G \to \mathrm{GL}_1(R).$$

(See for example [Lind 2016]). To understand the gauge symmetry on the manifold string topology spectrum $\mathcal{S}^\bullet_{P_k}(S^4)$ we therefore want to describe the representation

$$\mathcal{G}(P_k) \to \mathrm{GL}_1(\mathcal{S}^\bullet_{P_k}(S^4)). \tag{24}$$

Now as observed in [Cohen and Jones 2017], the group-like monoid $\mathrm{GL}_1(\mathcal{S}^\bullet_{P_k}(S^4))$ is equivalent to the grouplike monoid $\mathrm{hAut}(\Sigma^\infty_{S^4}((P_k)_+)$ of homotopy automorphisms of the parametrized spectrum $\Sigma^\infty_{S^4}((P_k)_+)$.

To understand this monoid of homotopy automorphisms, note that given any ring spectrum $R$ and parametrized $R$-line bundle $\mathcal{E}$ over $M$, there is a fibration sequence

$$\mathrm{hAut}^b(\mathcal{E}) \to \mathrm{hAut}(\mathcal{E}) \xrightarrow{\mathrm{ev}} \mathrm{hAut}^R(\mathcal{E}_{x_0}) = \mathrm{GL}_1(R) \tag{25}$$

where the map ev evaluates an automorphism on the fiber over the basepoint $x_0 \in M$. The fiber is $\mathrm{hAut}^b(R)$, which is the $A_\infty$ group-like monoid of based homotopy automorphisms. This is the subgroup of $\mathrm{hAut}(\mathcal{E})$ consisting of those homotopy automorphisms that are equal to the identity on the fiber spectrum at the basepoint $\mathcal{E}_{x_0}$.

Putting these facts together yields a fibration sequence of group-like monoids,

$$\mathrm{hAut}^b(\Sigma^\infty_{S^4}((P_k)_+)) \to \mathrm{GL}_1(\mathcal{S}^\bullet_{P_k}(S^4)) \to \mathrm{GL}_1(\Sigma^\infty(\mathrm{SU}(2)_+)). \tag{26}$$

As was done in [Cohen and Jones 2017], we observe that since $\mathrm{SU}(2) \cong S^3$, the defining diagram (23) becomes, in the case of $R = \Sigma^\infty(\mathrm{SU}(2)_+)$,

$$\begin{array}{ccc} \mathrm{GL}_1(\Sigma^\infty(\mathrm{SU}(2)_+)) & \longrightarrow & Q(S^3_+) \\ \big\downarrow & & \big\downarrow \text{components} \\ \pm 1 & \hookrightarrow & \mathbb{Z} \end{array}$$

That is, $\mathrm{GL}_1(\Sigma^\infty(\mathrm{SU}(2)_+))$ consists of two path components of the infinite loop space $Q(S^3_+)$ corresponding to the units $\pm 1 \in \mathbb{Z} \cong \pi_0(Q(S^3_+))$. We denote this space by $Q_{\pm 1}(S^3_+)$. Therefore fibration (26) has base space $Q_{\pm 1}(S^3_+)$. We now examine the homotopy type of the fiber, $\mathrm{hAut}^b(\Sigma^\infty_{S^4}((P_k)_+))$.

By one of the main results of [Cohen and Jones 2017] (Theorem 3), there is an equivalence

$$\mathrm{hAut}^b(\Sigma^\infty_{S^4}((P_k)_+)) \xrightarrow{\simeq} \Omega\,\mathrm{Map}^b_k\big(S^4, B\mathrm{GL}_1(\Sigma^\infty(\mathrm{SU}(2)_+))\big) = \Omega^4_k\,\mathrm{GL}_1(\Sigma^\infty(\mathrm{SU}(2)_+)),$$

where $\mathrm{Map}^b_k$ denotes the path component of the based mapping space corresponding to

$$k \in \mathbb{Z} = \pi_0\big(\mathrm{Map}^b\big(S^4, B\mathrm{GL}_1(\Sigma^\infty(\mathrm{SU}(2)_+))\big)\big).$$

Similarly $\Omega^4_k$ denotes the corresponding path component in $\Omega^4\,\mathrm{GL}_1(\Sigma^\infty(\mathrm{SU}(2)_+))$. Now, since $\Omega^4\,\mathrm{GL}_1(\Sigma^\infty(\mathrm{SU}(2)_+))$ is a group-like monoid, all of its path components are homotopy equivalent. So we therefore have the following result, which gives a good understanding of the group of units of the manifold string topology spectrum of the principal bundle $\mathrm{SU}(2) \to P_k \to S^4$.

**Lemma 17.** *For any $k$, there is an equivalence of group-like monoids,*

$$\phi_k : \mathrm{hAut}^b\big(\Sigma^\infty_{S^4}((P_k)_+)\big) \xrightarrow{\simeq} \Omega^4 Q(S^3_+).$$

*Furthermore, there are homotopy fibration sequences of group-like monoids*

$$\Omega^4 Q(S^3_+) \xrightarrow{\iota_k} \mathrm{GL}_1(\mathcal{S}^\bullet_{P_k}(S^4)) \xrightarrow{q_k} Q_{\pm 1}(S^3_+).$$

In order to understand the representation $\rho_k : \mathcal{G}^b(P_k) \to \mathrm{GL}_1(\mathcal{S}^\bullet_{P_k}(S^4))$ describing the gauge symmetry of the manifold string topology spectrum, we now consider the homotopy type of the based gauge group $\mathcal{G}^b(P_k)$. Again, for $k = 1$ this was done in [Cohen and Jones 2017], and we simply adapt the argument there to apply to all $k$.

By a basic result on the topology of gauge groups proved by Atiyah and Bott [1983], we have that

$$\mathcal{G}^b(P_k) \simeq \Omega\,\mathrm{Map}^b_k(S^4, B\mathrm{SU}(2)),$$

where, as above, $\mathrm{Map}^b_k$ denotes the path component of degree $k$ based maps. This (based) loop space is equivalent to $\Omega\Omega^3_k\,\mathrm{SU}(2) = \Omega\Omega^3_k S^3$. Since $\Omega^3 S^3$ is a group-like monoid, all of its path components are equivalent, so we have the following.

**Lemma 18.** *For any $k$, there is an equivalence of group-like monoids,*

$$\psi_k : \mathcal{G}^b(P_k) \xrightarrow{\simeq} \Omega^4 S^3.$$

By Proposition 5 of [Cohen and Jones 2017], one knows that given any principal bundle over a manifold, $G \to P \to M$, the action of the gauge group (and therefore the based gauge group) on the manifold string topology spectrum $\mathcal{S}^\bullet_P(M)$ is defined by the representation given by the stabilization map

$$\mathcal{G}^b(P) \xrightarrow{\rho} \mathrm{GL}_1(\mathcal{S}^\bullet_P(M)), \quad \Omega\,\mathrm{Map}^b_P(M, BG) \xrightarrow{\sigma} \Omega\,\mathrm{Map}^b_P(M, B\mathrm{GL}_1(\Sigma^\infty(G_+)),$$

where $\sigma$ is induced by the natural inclusion $G \hookrightarrow \mathrm{GL}_1(\Sigma^\infty(G_+))$. Here $\mathrm{Map}^b_P$ denotes the path component of the based mapping space that classifies the bundle $P$.

In the case of $\mathrm{SU}(2) \to P_k \to S^4$ then Lemma 17 says that the representation $\rho_k : \mathcal{G}^b(P_k) \to \mathrm{GL}_1(\mathcal{S}^\bullet_{P_k}(S^4))$ is given by the stabilization map

$$\Omega^4 S^3 \xrightarrow{\sigma} \Omega^4 Q(S^3_+) \xrightarrow{\iota_k} \mathrm{GL}_1(\mathcal{S}^\bullet_{P_k}(S^4)), \tag{27}$$

where $\sigma$ is induced by the map $u_k : S^3 \to Q(S^3_+) \simeq Q(S^3) \times QS^0$ that sends $S^3$ to a generator of $\pi_3 Q(S^3) \cong \mathbb{Z}$ cross the basepoint of the component $Q_k(S^0)$.

Finally, notice that given any compact ring spectrum $R$, the group of units $\mathrm{GL}_1(R)$ acts on its Spanier–Whitehead dual $R^\vee$ by the dual action of $\mathrm{GL}_1(R)$ on $R$. Given the Spanier–Whitehead duality between the manifold string topology spectrum $\mathcal{S}^\bullet_{P_k}(S^4) \simeq (P_k^{\mathrm{Ad}})^{-TS^4}$ and the Lie group string topology spectrum $\mathcal{S}^{P_k}_\bullet(S^4) \simeq (P_k^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$, (27) describes the action of the based gauge group $\mathcal{G}^b(P_k)$ on the Lie group string topology spectrum as well.

To end this section, we point out that Proposition 11 and the above analysis of gauge symmetry implies the following.

**Theorem 19.** *Let $G \to P \to M$ be a principal bundle over a closed $n$-manifold $M$, with $G$ a $d$-dimensional compact Lie group. Let $E$ be any ring spectrum with respect to which the compact Calabi–Yau structure on $\mathcal{S}^\bullet_P(M)$ given in Proposition 14 is oriented. Then the homology $E_*(\mathcal{S}^\bullet_P(M))$ is a Frobenius algebra over the homology of the gauge group, $E_*(\mathcal{G}(P))$. That is, the following conditions hold*:

- *The homology algebra structure of the manifold string topology ring spectrum $E_*(\mathcal{S}^\bullet_P(M))$ carries the structure of an algebra over $E_*(\mathcal{G}(P))$.*
- *The homology coalgebra structure of the Lie group string topology coalgebra spectrum $E_*(\mathcal{S}^P_\bullet(M))$ is a module over $E_*(\mathcal{G}(P))$.*
- *The duality homomorphism defined by the Frobenius algebra structure induced from the compact Calabi–Yau structure,*

$$E_*(\mathcal{S}^\bullet_P(M)) \xrightarrow{\cong} E_{n-d-*}(\mathcal{S}^P_\bullet(M))^*$$

*is an isomorphism of $E_*(\mathcal{G}(P))$ modules.*

## 4. Twisted, smooth Calabi–Yau ring spectra, Thom ring spectra, and Lagrangian immersions of spheres

We now turn to the notion of twisted Calabi–Yau structures for *smooth* ring spectra.

Recall that a *smooth* ring spectrum $A$ is one that is perfect as an $A$-bimodule. That is, it is perfect as a left $A \wedge A^{\mathrm{op}}$- module. Given a smooth ring spectrum $A$, let $A^!$ be its "bimodule dual". That is,

$$A^! = \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A, A \wedge A^{\mathrm{op}}).$$

A cap product pairing between Hochschild homology and cohomology can then be defined using the fact that $\mathrm{THH}(A, Q) \simeq A \wedge^L_{A\wedge A^{\mathrm{op}}} Q$. Given any $A$-bimodules $P$ and $Q$, this pairing is given by

$$\cap : \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A, P) \wedge (A \wedge^L_{A\wedge A^{\mathrm{op}}} Q) \to P \wedge^L_{A\wedge A^{\mathrm{op}}} Q.$$

When $P = A \wedge A^{\mathrm{op}}$, then one can take a cap product with respect to a map $\rho : \mathbb{S} \to A \wedge_{A \wedge A^{\mathrm{op}}} Q$ to obtain a map

$$\cap\rho : A^! \to Q. \tag{28}$$

**Definition.** A *twisted, smooth Calabi–Yau ring spectrum* (*twisted* sCY) of dimension $n$ is a triple $(A, P, \sigma)$, where $A$ is a smooth ring spectrum and $P$ is a smooth $A$-bimodule that has the same $\mathbb{Z}/2$-homology as $A$:

$$P \wedge H\mathbb{Z}/2 \simeq A \wedge H\mathbb{Z}/2$$

as $A$-bimodules.

We refer to $P$ as the "twisting" bimodule. If $P = A$ we say that $A$ has *trivial* twisting. The map of spectra we call the *$n$-dimensional cotrace map*,

$$\sigma : \Sigma^n \mathbb{S} \to \mathrm{THH}(A,\ P),$$

has the following duality property: the induced cap product pairing

$$\cap\sigma : A^! = \mathrm{Rhom}_{A \wedge A^{\mathrm{op}}}(A,\ A \wedge A^{\mathrm{op}}) \to \Sigma^{-n} P$$

is an equivalence of $A$-bimodule spectra.

**Note.** Given a graded module $P$ over a ring $R$ let $P[-n]$ denote the desuspension $\Sigma^{-n}(P)$.

Like in the compact case, in most applications a twisted, smooth Calabi–Yau spectrum is reduced over a homology theory with respect to which the twisting becomes trivialized, or "oriented." We now make this precise.

**Definition.** Let $(A, P, \sigma)$ be a twisted, sCY ring spectrum of dimension $n$, and let $E$ be a ring spectrum representing a homology theory $E_*$. An *$E_*$-orientation* of $(A, P, \sigma)$ is a pair $(u, \tilde{\sigma}_E)$, where

$$u : P \wedge E \xrightarrow{\simeq} A \wedge E$$

is an equivalence of $E$-module spectra as $A$-bimodules. Here $A$ acts trivially on $E$. $\tilde{\sigma}_E : \Sigma^n E \to \mathrm{THH}(A,\ A)^{hS^1} \wedge E$ is a map to the $E$-homology of the homotopy fixed point spectrum of the $S^1$-action induced by the cyclic structure, which factorizes the cotrace map $\sigma$ in $E_*$-homology. That is, the following diagram homotopy commutes:

$$\begin{array}{ccc} \Sigma^n E & \xrightarrow{\tilde{\sigma}_E} & \mathrm{THH}(A,\ A)^{hS^1} \wedge E \\ {\scriptstyle \sigma\wedge 1}\downarrow & & \downarrow{\scriptstyle j} \\ \mathrm{THH}(A,\ P) \wedge E & \xrightarrow[u_*]{\simeq} & \mathrm{THH}(A, A) \wedge E \end{array}$$

Here $j$ is the natural inclusion of the homotopy fixed points.

Notice that when $E = Hk$, the Eilenberg–MacLane spectrum for a field $k$, then a twisted, smooth Calabi–Yau spectrum $(A, \tau, \sigma)$ together with an $Hk$-orientation $(u, \tilde{\sigma}_{Hk})$ defines a smooth Calabi–Yau algebra structure on the singular chains with $k$-coefficients, $C_*(A; k)$, as defined in [Kontsevich and Vlassopoulos 2013; Cohen and Ganatra 2015].

**4.1.** ***Thom spectra of virtual bundles over the loop space of a manifold.*** We now consider important examples of a twisted, smooth Calabi–Yau spectra. These are Thom ring spectra of virtual bundles over $\Omega M$, for $M$ a closed manifold.

We begin by studying the Thom spectrum of the trivial bundle over $\Omega M$, namely the suspension spectrum $\Sigma^\infty(\Omega M_+)$. The following generalizes the chain complex analogue proven by Cohen and Ganatra [2015].

**Theorem 20.** *Let $M$ be a closed manifold of dimension $n$. Then the suspension spectrum of its based loop space, $\Sigma^\infty(\Omega M_+)$, can be given the structure of a twisted, smooth Calabi–Yau ring spectrum of dimension $n$.*

*Proof.* In order to give $A = \Sigma^\infty(\Omega M_+)$ a twisted sCY structure, we need to define a twisting bimodule and a cotrace map. Consider the virtual bundle $-TM$ over $M$. The associated virtual spherical fibration is classified by a map $B_{-\tau_M} : M \to B\mathrm{GL}_1(\mathbb{S})$. By taking the loop of this map and applying suspension spectra we get a map of ring spectra

$$-\tau_M : A = \Sigma^\infty(\Omega M_+) \to \Sigma^\infty(\mathrm{GL}_1(\mathbb{S})). \tag{29}$$

This defines a $\Sigma^\infty(\Omega M_+)$-bimodule structure on the sphere spectrum. We let $\mathbb{S}_{-\tau_M}$ be the sphere spectrum with this bimodule structure, and we define

$$P = \mathbb{S}_{-\tau_M} \wedge \Sigma^\infty(\Omega M_+) = \mathbb{S}_{-\tau_M} \wedge A$$

to be the induced bimodule. Here $\mathbb{S}_{-\tau_M} \wedge \Sigma^\infty(\Omega M_+)$ is given the diagonal $A$-bimodule structure.

$P$ will be the twisting bimodule. To describe the $n$-dimensional cotrace map, we first need the following observation.

Replace $\Omega M$ by its Kan loop group, which by abuse of notation we still write as $\Omega M$. Consider the $\Omega M$-space $(\Omega M \times \Omega M)^{\mathrm{Ad}}$, which is $\Omega M \times \Omega M$ with the adjoint action of $\Omega M$ defined by

$$g \cdot (h_1, h_2) = (gh_1, h_2 g^{-1}).$$

If $E$ is the homotopy orbit space of this action, $E = \text{point} \times^L_{\Omega M} (\Omega M \times \Omega M)^{\mathrm{Ad}}$, then we have a homotopy fiber sequence (i.e., each successive three terms form a homotopy fibration)

$$\Omega M \xrightarrow{\tilde{\Delta}} \Omega M \times \Omega M \to E \xrightarrow{\pi} M, \tag{30}$$

where $\tilde{\Delta}(g) = (g, g^{-1})$. Note that $(\Omega M \times \Omega M)^{\mathrm{Ad}} \cong \Omega M \times \Omega M$, where the latter has $\Omega M$-action given by $\gamma(a, b) = (\gamma a, b)$. This isomorphism is defined by sending $(a, b)$ to $(a, ba)$. Hence $E \simeq \Omega M$, and $\pi : E \to M$ is nullhomotopic.

The Thom spectrum of the pull-back virtual bundle $\pi^*(-TM)$, which we denote by $E^{-TM}$, is given by

$$\Sigma^{-n}\mathbb{S}_{-\tau_M} \wedge^L_{\Omega M} \Sigma^\infty(\Omega M_+) \wedge \Sigma^\infty(\Omega M_+) = \Sigma^{-n}\mathbb{S}_{-\tau_M} \wedge^L_A (A \wedge A^{\mathrm{op}})^{\mathrm{Ad}}.$$

This is an $A$-bimodule, with action given by right multiplication of $A \wedge A^{\mathrm{op}}$ on itself. The isomorphism $(\Omega M \times \Omega M)^{\mathrm{Ad}} \cong \Omega M \times \Omega M$ above gives an $\Omega M$-equivariant equivalence between the corresponding suspension spectra, and hence

$$h : E^{-TM} = \Sigma^{-n}\, \mathbb{S}_{-\tau_M} \wedge^L_A (A \wedge A^{\mathrm{op}})^{\mathrm{Ad}} \xrightarrow{\simeq} \Sigma^{-n}\mathbb{S}_{-\tau_M} \wedge A = P[-n]. \tag{31}$$

as $A$-bimodules.

We therefore have an equivalence

$$A \wedge^L_{A\wedge A^{\mathrm{op}}} \big(\mathbb{S}_{-\tau_M} \wedge^L_A (A \wedge A^{\mathrm{op}})^{\mathrm{Ad}}\big) \xrightarrow[\simeq]{h} A \wedge^L_{A\wedge A^{\mathrm{op}}} (\mathbb{S}_{-\tau_M} \wedge A) = \mathrm{THH}(A,\ P).$$

Now the map $\tilde{\Delta} : \Omega M \to \Omega M \times \Omega M$ defines a ring map on the level of suspension spectra, which by abuse of notation we still call $\tilde{\Delta}$,

$$\tilde{\Delta} : A \to A \wedge A^{\mathrm{op}}.$$

We then get a change-of-rings equivalence,

$$\phi : A^{\mathrm{Ad}} \wedge^L_A \mathbb{S}_{-\tau_M} \xrightarrow{\simeq} A \wedge^L_{A\wedge A^{\mathrm{op}}} \big((A \wedge A^{\mathrm{op}})^{\mathrm{Ad}} \wedge^L_A \mathbb{S}_{-\tau_M}\big).$$

Consider the unit map $u : \mathbb{S} \to \Sigma^\infty(\Omega M_+) = A$. This defines a map

$$u : \mathbb{S} \wedge^L_A \mathbb{S}_{-\tau_M} \to A^{\mathrm{Ad}} \wedge^L_A \mathbb{S}_{-\tau_M}.$$

Now $\mathbb{S} \wedge^L_A \mathbb{S}_{-\tau_M}$ is the Thom spectrum $\Sigma^n(M^{-TM})$. Thus there is a Pontryagin–Thom map $\gamma : \Sigma^n\mathbb{S} \to \Sigma^n(M^{-TM}) = \mathbb{S} \wedge^L_A \mathbb{S}_{-\tau_M}$. This can be viewed as the $n$-fold suspension of the unit map of the Spanier–Whitehead dual,

$$\mathbb{S} \to M^\vee \simeq M^{-TM}.$$

We can now define the $n$-dimensional cotrace map $\sigma : \Sigma^n\mathbb{S} \to \mathrm{THH}(A,\ P)$ to be the composition

$$\begin{aligned}\sigma : \Sigma^n\mathbb{S} \xrightarrow{\gamma} \mathbb{S} \wedge^L_A \mathbb{S}_{-\tau_M} \xrightarrow{u} A^{\mathrm{Ad}} \wedge^L_A \mathbb{S}_{-\tau_M} \xrightarrow{\phi} A \wedge^L_{A\wedge A^{\mathrm{op}}} \big((A \wedge A^{\mathrm{op}})^{\mathrm{Ad}} \wedge^L_A \mathbb{S}_{-\tau_M}\big)\\ \xrightarrow{h} A \wedge^L_{A\wedge A^{\mathrm{op}}} (\mathbb{S}_{-\tau_M} \wedge A) = \mathrm{THH}(A,\ P).\end{aligned} \tag{32}$$

To show $\sigma$ is a valid cotrace map we need to check that it satisfies the required duality condition. Namely, we need to show that the cap product,

$$\cap\sigma : A^! = \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A,\ A\wedge A^{\mathrm{op}}) \to \Sigma^{-n}\mathbb{S}_{-\tau_M}\wedge A = P[-n] \tag{33}$$

is an equivalence of $A$-bimodules. This map was constructed to be an $A$-bimodule map, so it suffices to check that it is an ordinary weak equivalence.

In order to do this, we study the homotopy type of $\mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A,\ A\wedge A^{\mathrm{op}})$, where $A = \Sigma^\infty(\Omega M_+)$. Notice that since $A$ is a connective Hopf algebra (in the weak sense), we have an equivalence

$$\mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A,\ A\wedge A^{\mathrm{op}}) \simeq \mathrm{Rhom}_A(\mathbb{S},\ (A\wedge A^{\mathrm{op}})^{\mathrm{Ad}}).$$

Consider again the homotopy fibration $\Omega M\times\Omega M\to E\to M$, and its fiberwise suspension spectrum. This is the parametrized spectrum

$$\Sigma^\infty(\Omega M_+)\wedge\Sigma^\infty(\Omega M_+)\to\Sigma^\infty_M(E_+)\to M.$$

The spectrum of sections of this parametrized spectrum is the spectrum whose homotopy type we are trying to compute:

$$\Gamma_M(\Sigma^\infty_M(E_+)) \simeq \mathrm{Rhom}_A(\mathbb{S},\ (A\wedge A^{\mathrm{op}})^{\mathrm{Ad}}) \simeq \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A,\ A\wedge A^{\mathrm{op}}).$$

If we let $\mathcal{E}^\bullet$ be the twisted equivariant cohomology theory represented by the parametrized spectrum $\Sigma^\infty_M(E_+)$, then what we are trying to compute is $\mathcal{E}^\bullet(M)$. But by twisted Poincaré duality theorem of Klein [2001] as described in [Cohen and Klein 2009], this twisted cohomology spectrum is given by the Thom spectrum

$$\mathcal{E}^\bullet(M) = \Gamma_M(\Sigma^\infty_M(E_+)) \simeq E^{-TM}.$$

Now as seen in (31) above, there is an equivalence

$$h : E^{-TM} \xrightarrow{\simeq} \Sigma^{-n}\mathbb{S}_{-\tau_M}\wedge A.$$

Putting these together gives an equivalence,

$$A^! = \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A,\ A\wedge A^{\mathrm{op}}) = \mathcal{E}^\bullet(M) \simeq E^{-TM} \xrightarrow{h} \Sigma^{-n}\mathbb{S}_{-\tau_M}\wedge A = P[-n].$$

We leave it to the reader to check that the cap product map

$$\cap\sigma : A^! \to P[-n]$$

induces such an equivalence.

Given this, the proof that $(\Sigma^\infty(\Omega M_+), -\tau_M, \sigma)$ is a twisted, smooth Calabi–Yau ring spectrum is complete. □

Orientations on this twisted, smooth Calabi–Yau ring spectrum will be addressed in a more general context, in the setting of Thom spectra of virtual bundles over $\Omega M$.

We now generalize the above to the setting of Thom ring spectra. Namely, let $\Omega f : \Omega M \to B\mathrm{GL}_1(\mathbb{S})$ be a loop map. That is, it is obtained by applying the based loop functor to a map $f : M \to B(B\mathrm{GL}_1(\mathbb{S}))$. Let $\Omega M^{\Omega f}$ denote the Thom spectrum of $\Omega f$. By a theorem of Lewis, $\Omega M^{\Omega f}$ is a ring spectrum. Let $E$ be any commutative ($E_\infty$) ring spectrum. As is usual, we say that a virtual bundle $\omega : X \to B\mathrm{GL}_1(\mathbb{S})$ is $E$-orientable if there is a "Thom class" $\tau : X^\omega \to E$ such that the composition

$$\theta_\tau : X^\omega \wedge E \xrightarrow{\Delta\wedge 1} X_+ \wedge X^\omega \wedge E \xrightarrow{1\wedge\tau\wedge 1} X_+ \wedge E \wedge E \xrightarrow{1\wedge \text{multiply}} X_+ \wedge E \tag{34}$$

is an equivalence (the "Thom isomorphism"). Here $\Delta : X^\omega \to X_+ \wedge X^\omega$ is the map of Thom spectra induced by the diagonal map $X \to X \times X$. Again, as is usual, we define an $E$-orientation of a manifold $M$ to be an $E$-orientation of its tangent bundle $\tau_M$ or equivalently of $-\tau_M$. An $E$-orientation of a loop map $\Omega f : \Omega Y \to B\mathrm{GL}_1(\mathbb{S})$ has the additional requirement that the Thom class $\tau : (\Omega Y)^{\Omega f} \to E$ be a map of ring spectra. In this case notice that the orientation equivalence $\theta_\tau$ in (34) is an equivalence of ring spectra.

**Theorem 21.** *$A = \Omega M^{\Omega f}$ naturally has the structure of a twisted* sCY *ring spectrum of dimension $n$. Furthermore, suppose $E$ is a commutative ring spectrum. Then an $E$ orientation $\tau : M^{-TM} \to E$ of $M$ and an $E$-orientation $A = \Omega M^{\Omega f} \to E$ together induce an $E$-orientation on the* sCY *structure on $A$.*

**Remark.** This theorem readily generalizes to the setting of generalized Thom spectra of maps to $B\mathrm{GL}_1(R)$, where $R$ is a commutative ring spectrum.

*Proof.* Let $\mathbb{S}_{-\tau_M}$ denote the sphere spectrum viewed as a $\Sigma^\infty(\Omega M_+)$-module as above. We first observe that by the twisted Poincaré duality theorem of Klein [2001] as described in [Cohen and Klein 2009] (see also [Dwyer et al. 2006; Malm 2011]), there is an equivalence of $\Sigma^\infty(\Omega M_+)$-modules

$$\mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}\big(\mathbb{S}, \Sigma^\infty(\Omega M_+)\big) \simeq \Sigma^{-n}\mathbb{S}_{-\tau_M} \tag{35}$$

Here $\Sigma^\infty(\Omega M_+)$ acts on itself by left multiplication. The reason this equivalence holds is the following. The left-hand side describes the section spectrum of the fiberwise suspension spectrum of the path-loop fibration, $\Omega M \to P(M) \to M$. By Klein's theorem on Poincaré duality for parametrized spectra, the section spectrum (i.e., twisted cohomology spectrum) associated to this parametrized spectrum is equivalent to a twisting by $-TM$ of the homology spectrum,

$$\Sigma^\infty(\Omega M_+) \wedge^L_{\Sigma^\infty(\Omega M_+)} \Sigma^{-n}\mathbb{S}_{-\tau_M} \simeq \Sigma^{-n}\mathbb{S}_{-\tau_M}.$$

This equivalence is given by cap product with the Pontryagin–Thom class

$$t_M : \mathbb{S} \to M^{-TM} = \Sigma^{-n}\big(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}_{-\tau_M}\big).$$

Consider now the map of ring spectra $\Sigma^\infty(\Omega M_+) \to A \wedge A^{\mathrm{op}}$, induced on Thom spectra by the map $\tilde{\Delta} : \Omega M \to \Omega M \times \Omega M$ of (30). The source of this map is indeed $\Sigma^\infty(\Omega M_+)$, as the composite

$$\Omega M \xrightarrow{\tilde{\Delta}} \Omega M \times \Omega M \xrightarrow{\Omega f \times \Omega f} B\mathrm{GL}_1(\mathbb{S}) \times B\mathrm{GL}_1(\mathbb{S}) \xrightarrow{\text{multiply}} B\mathrm{GL}_1(\mathbb{S})$$

is null homotopic. The following is a result of the second author [Klang 2018, Theorem 5.1].

**Theorem 22.** *Under this action of $\Sigma^\infty(\Omega M_+)$ on $A \wedge A^{\mathrm{op}}$, there is an equivalence of $A \wedge A^{\mathrm{op}}$-modules*

$$A \simeq \mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (A \wedge A^{\mathrm{op}}) \tag{36}$$

*Here the module structure on the right hand side is by right action on $A \wedge A^{\mathrm{op}}$.*

Notice that $\mathbb{S}$ is a perfect $\Sigma^\infty(\Omega M_+)$-module since $M$ is assumed to be compact. That is, $\mathbb{S}$ is equivalent to a retract of a finite $\Sigma^\infty(\Omega M_+)$-module. Applying $(-) \wedge^L_{\Sigma^\infty(\Omega M_+)} (A \wedge A^{\mathrm{op}})$, we can conclude from Theorem 22 that $A$ is a retract of a finite $A \wedge A^{\mathrm{op}}$-module, hence a perfect $A \wedge A^{\mathrm{op}}$-module. That is, $A$ is a smooth ring spectrum. Also because $\mathbb{S}$ is perfect as a $\Sigma^\infty(\Omega M_+)$-module, we can apply $(-) \wedge^L_{\Sigma^\infty(\Omega M_+)} (A \wedge A^{\mathrm{op}})$ to both sides of the equivalence (35) to obtain

$$\mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}\big(\mathbb{S}, A \wedge A^{\mathrm{op}}\big) \simeq \Sigma^{-n}\big(\mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} A \wedge A^{\mathrm{op}}\big). \tag{37}$$

We now take our twisting bimodule to be $P = \mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} A \wedge A^{\mathrm{op}}$. Notice that if the original map $f$ is null homotopic, this agrees with the twisting bimodule given in the proof of Theorem 20. Furthermore, by the Thom isomorphism for $-TM$,

$$H\mathbb{Z}/2 \wedge P \simeq H\mathbb{Z}/2 \wedge A.$$

This is an equivalence of $A$-bimodules because the equivalence in Theorem 22 is. Moreover, by the above theorem,

$$\begin{aligned}\mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A, A \wedge A^{\mathrm{op}}) &\simeq \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}\big((A \wedge A^{\mathrm{op}}) \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S},\ A \wedge A^{\mathrm{op}}\big) \\ &\simeq \mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, A \wedge A^{\mathrm{op}}).\end{aligned}$$

So the equivalence (37) becomes

$$A^! = \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A, A \wedge A^{\mathrm{op}}) \simeq \Sigma^{-n} P. \tag{38}$$

Our goal is to show that this equivalence is given by taking the cap product with an appropriate $n$-dimensional cotrace map $\sigma : \Sigma^n\mathbb{S} \to \mathrm{THH}(A, P)$ which we now define.

Since $A$ is smooth,

$$\begin{aligned}\mathrm{THH}(A, P) = \Sigma^n \operatorname{Rhom}_{A\wedge A^{\mathrm{op}}}(A, A \wedge A^{\mathrm{op}}) \wedge^L_{A\wedge A^{\mathrm{op}}} A \simeq \Sigma^n \operatorname{Rhom}_{A\wedge A^{\mathrm{op}}}(A, A)\\ \simeq \Sigma^n \mathrm{THH}^\bullet(A, A),\end{aligned} \tag{39}$$

where the last quantity is the topological Hochschild cohomology.

The inverse equivalence also has a natural description. Consider the $\Sigma^\infty(\Omega M_+)$-module structure on $A = (\Omega M)^{\Omega f}$ given by the generalized conjugation action defined to be the pull-back of the $A\wedge A^{\mathrm{op}}$-action on $A$ to $\Sigma^\infty(\Omega M_+)$ via the ring map $\Sigma^\infty(\Omega M_+) \to A \wedge A^{\mathrm{op}}$ defined above. This action was studied in detail by Klang [2018]. In [Klang 2018] the second author showed that there are equivalences,

$$\mathrm{THH}(A, A) \simeq \mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} A^c, \quad \mathrm{THH}^\bullet(A, A) \simeq \operatorname{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, A^c), \tag{40}$$

where $A^c$ denotes the algebra $A$ with this generalized conjugation action of $\Sigma^\infty(\Omega M_+)$.

Recall the cap product operation

$$\begin{aligned}\big(\operatorname{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, A^c)\big) \wedge \big(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}_{-\tau_M}\big)\\ \xrightarrow{\cap} \mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} A^c \simeq \mathrm{THH}(A, P).\end{aligned} \tag{41}$$

Now $\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}_{-\tau_M} \simeq \Sigma^n M^{-TM}$, so there is a Pontryagin–Thom map $S^n \xrightarrow{t_M} \Sigma^n M^{-TM}$ which corresponds to the unit $\iota \in M^\vee$ under the Atiyah-equivalence of the Spanier–Whitehead dual of a manifold $M^\vee$ and its Thom spectrum $M^{-TM}$. We then get an induced equivalence

$$\begin{aligned}\mathrm{THH}^\bullet(A, A) \wedge S^n \simeq \big(\operatorname{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, A^c)\big) \wedge S^n\\ \xrightarrow{1\wedge t_M} \big(\operatorname{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, A^c)\big) \wedge \big(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}_{-\tau_M}\big)\\ \xrightarrow{\cap} \mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} A^c \simeq \mathrm{THH}(A, P).\end{aligned} \tag{42}$$

This is the inverse to the equivalence (39).

Now, $\mathrm{THH}^\bullet(A, A)$ is an $E_2$-ring spectrum; see, for example, McClure and Smith's [2002] solution of the Deligne conjecture. Let $\iota : \mathbb{S} \to \mathrm{THH}^\bullet(A, A)$ be the unit. Alternatively, recall that $\mathrm{THH}^\bullet(A, A)$ is the spectrum of $A$-bimodule maps $A \to A$; $\iota$ corresponds to the identity map $\mathrm{id} : A \to A$, a characterization which does not rely on the multiplicative structure on $\mathrm{THH}^\bullet(A, A)$. The map $\iota$ allows us to define our $n$-dimensional cotrace map as

$$\sigma : \Sigma^n\mathbb{S} \xrightarrow{\Sigma^n\iota} \Sigma^n \mathrm{THH}^\bullet(A, A) \simeq \mathrm{THH}(A, P). \tag{43}$$

Clearly taking cap product

$$\cap\sigma : A^! = \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A, A\wedge A^{\mathrm{op}}) \to \Sigma^{-n} P$$

defines the equivalence given in (38). This then proves that $(A, P, \sigma)$ is a twisted, smooth Calabi–Yau ring spectrum.

Let $E$ be any commutative ring spectrum satisfying the hypotheses of the theorem. An $E_*$-orientation of $M$ is given by a Thom class $\tau : M^{-TM} \to E$ and induces an equivalence (34) $\theta_\tau : \Sigma^n M^{-TM} \wedge E \xrightarrow{\simeq} M_+ \wedge E$ as in (34). This can be also be written as an equivalence

$$\theta_\tau : (\mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}) \wedge E \xrightarrow{\simeq} (\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}) \wedge E.$$

Given such an orientation $\tau : E \to M$ and an orientation $\nu : A = (\Omega M)^{\Omega f} \to E$ we describe a resulting $E_*$-orientation $(u, \tilde{\sigma}_E)$ of the sCY structure on $A = (\Omega M)^{\Omega f}$.

Notice that the $E_*$ orientation $\tau$ of $M$ induces an equivalence

$$\theta_{\tau,R} : (\mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} R) \wedge E \xrightarrow{\simeq} (\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} R) \wedge E$$

for any left $\Sigma^\infty(\Omega M_+)$-module $R$. Now take $R = A \wedge A^{\mathrm{op}}$ with the $\Sigma^\infty(\Omega M_+)$-action defined by the ring homomorphism $\Sigma^\infty(\Omega M_+) \to A \wedge A^{\mathrm{op}}$ described above. We then define

$$\begin{aligned} u = \theta_{\tau, A\wedge A^{\mathrm{op}}} : P \wedge E = (\mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} A \wedge A^{\mathrm{op}}) \wedge E \\ \xrightarrow{\simeq} (\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} A \wedge A^{\mathrm{op}}) \wedge E = A \wedge E. \end{aligned} \tag{44}$$

We now define the map $\tilde{\sigma}_E : \mathbb{S} \wedge E \to \mathrm{THH}(A, A)^{hS^1} \wedge E$ needed for the orientation of the sCY-structure.

Again, let $\tau : M^{-TM} \to E$ and $\nu : A = (\Omega M)^{\Omega f} \to E$ be orientations of $M$ and $\Omega f$ respectively. Consider the following homotopy commutative diagram.

$$\begin{array}{ccccc}
\Sigma^n \mathbb{S} \wedge E & \xrightarrow{\iota} & \Sigma^n \mathrm{THH}^\bullet(A, A) \wedge E & \xrightarrow{\simeq} & \mathrm{THH}(A, P) \wedge E \\
\uparrow & & \Big\downarrow = & & \tau \Big\downarrow \simeq \\
\| & & & & \mathrm{THH}(A, A) \wedge E \\
= & & & & \Big\downarrow = \\
\| & & \Sigma^n \mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, A^c) \wedge E & \underset{\simeq}{\xrightarrow{\tau}} & \big(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} A^c\big) \wedge E \\
\| & & \nu \Big\downarrow \simeq & & \nu \Big\downarrow \simeq \\
\| & & \Sigma^n \mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}\big(\mathbb{S}, (\Sigma^\infty(\Omega M_+))^c\big) \wedge E & \underset{\simeq}{\xrightarrow{\tau}} & \big(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (\Sigma^\infty(\Omega M_+))^c\big) \wedge E \\
\| & & \Big\uparrow \iota\wedge 1 & & \Big\uparrow \iota\wedge 1 \\
\Sigma^n \mathbb{S} \wedge E & \xrightarrow{\iota} & \Sigma^n \mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, \mathbb{S}) \wedge E & \underset{\simeq}{\xrightarrow{\tau}} & \big(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}\big) \wedge E \\
& & \Big\uparrow = & & \Big\uparrow = \\
& & \Sigma^n M^\vee \wedge E & \underset{\simeq}{\xrightarrow{\tau}} & M_+ \wedge E
\end{array}$$

A few comments about this diagram:

(1) The maps in this diagram that are labeled by a $\tau$ or a $\nu$ are induced by the respective orientations. The maps labeled by $\iota$ are induced by the units of the respective ring spectra.

(2) The reason the left side of this diagram homotopy commutes is because the vertical maps induced by both $\nu$ and $\iota$ are all maps of ring spectra. As pointed out above this is because the orientation map $\nu : A \to E$ is assumed to be a ring map.

(3) The bottom horizontal composition $\Sigma^n \mathbb{S} \wedge E \to M_+ \wedge E$ is the $E_*$-fundamental class. The reason the right side of this diagram homotopy commutes is by the naturality of the Atiyah–Klein equivalences.

(4) The homotopy orbit spectrum $\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (\Sigma^\infty(\Omega M_+))^c$ is equivalent to $\Sigma^\infty(LM_+)$, and the lower right hand vertical map

$$M_+ \wedge E \to \big(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (\Sigma^\infty(\Omega M_+))^c\big) \wedge E \simeq \Sigma^\infty(LM_+) \wedge E$$

is homotopic to the inclusion of the constant loops, and therefore factors through the homotopy fixed point set of the circle action,

$$M_+ \wedge E \to \Sigma^\infty(LM_+)^{hS^1}.$$

(5) The top horizontal composition is the cotrace map

$$\sigma \wedge 1 : \Sigma^n \mathbb{S} \wedge E \to \mathrm{THH}(A, P) \wedge E.$$

We therefore define the map $\tilde{\sigma}_E : \Sigma^n E \to \mathrm{THH}(A, A)^{hS^1} \wedge E$ to be the composition

$$\begin{aligned}\tilde{\sigma}_E : \Sigma^n \mathbb{S} \wedge E &\xrightarrow{[M]} M_+ \wedge E \to \Sigma^\infty(LM_+)^{hS^1} \wedge E \\ &\xrightarrow{\simeq} \mathrm{THH}(\Sigma^\infty(\Omega M_+), \Sigma^\infty(\Omega M_+))^{hS^1} \wedge E \xleftarrow[\nu]{\simeq} \mathrm{THH}(A, A)^{hS^1} \wedge E.\end{aligned}$$

By comments (3) and (4) above, the composite $\Sigma^n \mathbb{S} \wedge E \xrightarrow{\tilde{\sigma}_E} \mathrm{THH}(A, A)^{hS^1} \wedge E \hookrightarrow \mathrm{THH}(A, A) \wedge E$ is obtained by starting at the lower left of the diagram, going horizontally to the lower right, and then going vertically until $\mathrm{THH}(A, A) \wedge E$. And by comment (1) above and the commutativity of the diagram, this means we may conclude that the following diagram homotopy commutes:

$$\begin{array}{ccccc}\Sigma^n \mathbb{S} \wedge E & \xrightarrow{\sigma \wedge 1} & \mathrm{THH}(A, P) \wedge E & \xrightarrow[\simeq]{\tau} & \mathrm{THH}(A, A) \wedge E \\ {\scriptstyle =}\uparrow & & & & \uparrow \\ \Sigma^n \mathbb{S} \wedge E & & \xrightarrow{\quad\tilde{\sigma}_E\quad} & & \mathrm{THH}(A, A)^{hS^1} \wedge E\end{array}$$

This is what was required to show that $(u, \tilde{\sigma}_E)$ defines an orientation on the twisted Calabi–Yau structure on $A$. □

**Example.** Take $M = \mathrm{SU}(m)/\mathrm{SO}(m)$, and $f : M \to \mathrm{SU}/\mathrm{SO} \simeq B^2O$ the natural map. Then

$$\Omega f : \Omega(\mathrm{SU}(m)/\mathrm{SO}(m)) \to BO$$

is a loop map, and by Theorem 21, $A = \Omega M^{\Omega f}$ has the structure of a twisted sCY ring spectrum.

Note that $\Omega f : \Omega(\mathrm{SU}(m)/\mathrm{SO}(m)) \to BO$ induces a map of Thom spectra $A \to MO$. $\Omega f$ is an equivalence in a range, and in particular $\pi_0(A) \cong \mathbb{Z}/2$. As the unit of $A$ is 2-torsion, $A$ is 2-torsion. This is not the case if $f$ is taken to be nullhomotopic, in which case the sCY ring spectrum is $\Sigma^\infty\big(\Omega(\mathrm{SU}(m)/\mathrm{SO}(m))_+\big)$.

**4.2.** ***The image of $J$ and Lagrangian immersions of spheres.*** In this subsection we study in more detail the twisted smooth Calabi–Yau structure on Thom spectra of virtual bundles over spheres. These bundles arise naturally from the homotopy perspective from the image of the $J$-homomorphism, and from the perspective of symplectic topology from Lagrangian immersions of odd dimensional spheres into their cotangent bundles.

As discussed in [Abouzaid and Kragh 2016], Gromov's $h$-principle implies that the homotopy group $\pi_n(U)$ classifies Lagrangian immersions of $S^n$ into its cotangent bundle, $T^*S^n$, which are in the homotopy class of the zero section $S^n \hookrightarrow T^*(S^n)$. Assume that $n > 1$ and let $\alpha : S^n \to U$ represent such a homotopy class. Since $\pi_n(\mathrm{SU}) \cong \pi_n(U)$, $\alpha$ lifts to a unique (up to homotopy) map that by abuse of notation we still call $\alpha : S^n \to \mathrm{SU}$. Taking loop spaces we get a map of $A_\infty$ group-like monoids,

$$\Omega\alpha : \Omega S^n \to \Omega\,\mathrm{SU} \simeq BU.$$

The last equivalence is given by Bott periodicity. By forgetting the almost complex structure we get an $A_\infty$-map

$$\Omega\alpha : \Omega S^n \to BO.$$

By Theorem 21 above, the Thom spectrum $(\Omega S^n)^{\Omega\alpha}$ has the structure of a twisted, smooth Calabi–Yau ring spectrum. We begin with the following observation.

**Lemma 23.** *The twisted sCY ring spectrum $(\Omega S^n)^{\Omega\alpha}$ has natural orientation with respect to stable homotopy theory (that is, the generalized homology theory $\mathbb{S}_*$ represented by the sphere spectrum $\mathbb{S}$). Furthermore, this induces an orientation with respect to any generalized homology theory $E_*$ represented by a commutative ring spectrum $E$.*

*Proof.* First note that $S^n$ has a canonical stable framing. That is, it has a canonical $\mathbb{S}$-orientation. This induces an orientation with respect to any theory $E_*$. Furthermore, by the construction of the twisted sCY structure in the proof of Theorem 21, the twisting bimodule of this structure is

$$P = \mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega S^n_+)} A \wedge A^{\mathrm{op}},$$

where $A = (\Omega S^n)^{\Omega\alpha}$. Now the $\mathbb{S}$-framing of $S^n$ defines an equivalence of bimodules, $\mathbb{S} \simeq \mathbb{S}_{-\tau_M}$. Thus

$$P \simeq \mathbb{S} \wedge^L_{\Sigma^\infty(\Omega S^n_+)} A \wedge A^{\mathrm{op}}$$

but by Theorem 22 this last spectrum is equivalent to $A$ as $A$-bimodules. Thus we have an equivalence

$$u : P \wedge \mathbb{S} \xrightarrow{\simeq} A \wedge \mathbb{S}.$$

Using this identification, the cotrace element can be viewed as a class $\sigma \in \mathrm{THH}(A, A)$. To complete the construction of the $\mathbb{S}_*$-orientation we must show that $\sigma$ lifts to an element in the homotopy fixed points, $\mathrm{THH}(A, A)^{hS^1}$.

By the main result of [Blumberg et al. 2010], the topological Hochschild homology of $A = (\Omega S^n)^{\Omega\alpha}$ is equivalent as a $\Sigma^\infty(S^1_+)$-module to the Thom spectrum of a virtual bundle over the free loop space $LS^n$:

**Proposition 24** [Blumberg et al. 2010].

$$\mathrm{THH}((\Omega S^n)^{\Omega\alpha}) \simeq L(S^n)^{\ell(\alpha)},$$

*where $\ell(\alpha)$ is the virtual bundle classified by the map*

$$\ell(\alpha) : L(S^n) \xrightarrow{L\alpha} L\,\mathrm{SU} \simeq \mathrm{SU} \times \Omega\,\mathrm{SU} \xrightarrow{\text{project}} \Omega\,\mathrm{SU} \simeq BU \to BO.$$

In this composition, the equivalence $L\,\mathrm{SU} \simeq \mathrm{SU} \times \Omega\,\mathrm{SU}$ is given by the trivialization of the fibration of infinite loop spaces $\Omega\,\mathrm{SU} \to L\,\mathrm{SU} \to \mathrm{SU}$ defined by the canonical section $\mathrm{SU} \to L\,\mathrm{SU}$ given by the inclusion of constant loops, and the infinite loop structure of $L\,\mathrm{SU}$.

**Remark.** In [Blumberg et al. 2010] the map from $L\,\mathrm{SU}$ to $BO$ was described by a composition

$$L\,\mathrm{SU} \to L(\mathrm{SU}/\mathrm{SO}) \simeq \mathrm{SU}/\mathrm{SO} \times \Omega(\mathrm{SU}/\mathrm{SO}) \xrightarrow{\eta \times 1} \Omega(\mathrm{SU}/\mathrm{SO}) \times \Omega(\mathrm{SU}/\mathrm{SO}) \xrightarrow{\text{multiply}} \Omega(\mathrm{SU}/\mathrm{SO}) \simeq BO,$$

where $\eta : \mathrm{SU}/\mathrm{SO} \to \Omega(\mathrm{SU}/\mathrm{SO}) \simeq BO$ was induced by the Hopf map $S^3 \to S^2$. However the map $\eta$ becomes trivial when composed with the projection $\mathrm{SU} \to \mathrm{SU}/\mathrm{SO}$, which allows the description of $\ell(\alpha)$ given in the proposition.

Notice that the restriction to the constant loops,

$$S^n \xrightarrow{\iota} LS^n \xrightarrow{\ell(\alpha)} BO$$

is the constant map. That is, this virtual bundle is trivialized when restricted to the constant loops. But since constant loops are $S^1$-fixed points of $LS^n$, the inclusion naturally lifts to the homotopy fixed points,

$$\Sigma^\infty(S^n_+) \xrightarrow{\iota} (L(S^n)^{\ell(\alpha)})^{hS^1}.$$

Composing with the equivalence given by Proposition 24, this defines a map $\tilde{\sigma}: \Sigma^\infty(S^n_+) \to \mathrm{THH}(A, A)^{hS^1}$ that lifts the cotrace element $\sigma \in \mathrm{THH}(A, A)$.

This completes the construction of the $\mathbb{S}_*$-orientation of the sCY structure on $A = (\Omega S^n)^{\Omega\alpha}$. Given any other generalized homology theory $E_*$ represented by a commutative ring spectrum $E$, the $\mathbb{S}_*$-orientation of $(\Omega S^n)^{\Omega\alpha}$ induces an $E_*$ orientation by use of the unit $\mathbb{S} \to E$. This completes the proof of Lemma 23. $\square$

The following recasts the results of [Abouzaid and Kragh 2016] to show that topological Hochschild homology can be used as an obstruction to being able to deform a Lagrangian immersion of a sphere to a Lagrangian embedding.

**Theorem 25.** *Let $\alpha : S^n \to U$ represent a Lagrangian immersion $\phi_\alpha : S^n \to T^*S^n$ in the homotopy class of the zero section. Consider the associated twisted smooth Calabi–Yau ring spectrum $(\Omega S^n)^{\Omega(-\alpha)}$. (Here $-\alpha : S^n \to U$ is a map that represents the inverse of $\alpha$ in $\pi_n U$.) Then if $\phi_\alpha$ is Lagrangian isotopic to a Lagrangian embedding then there is an equivalence of topological Hochschild homology spectra,*

$$\mathrm{THH}((\Omega S^n)^{\Omega(-\alpha)}) \simeq \mathrm{THH}(\Sigma^\infty(\Omega S^n_+)).$$

*Proof.* Let $Q$ and $N$ be smooth, closed manifolds of the same dimension. Given an exact Lagrangian embedding $j : Q \to T^*N$, Kragh [2018] defined a virtual Maslov bundle $\nu$ on $L_0 Q$. Here $L_0$ denotes path component of the free loop space that contains constant loops. The construction, which uses notation that is different than ours, is described in section 2 of [Abouzaid and Kragh 2016]. A map of spectra

$$\psi : \Sigma^\infty(L_0 N_+) \to L_0 Q^{TN - TL \oplus \nu} \tag{45}$$

was constructed and studied. One of the main results of [Abouzaid and Kragh 2016] is that the map $\psi$ is a homotopy equivalence of spectra. The Maslov bundle $\nu$ was defined as follows. (See section 2 of [Abouzaid and Kragh 2016].) The Lagrangian embedding $Q \to T^*N$ defines a map $\tau : Q \to U/O$. Then $-\nu$ was defined to be the restriction to $L_0 Q$ of the map

$$LQ \xrightarrow{L\tau} L(U/O) \xrightarrow{\simeq} U/O \times \Omega U/O \xrightarrow{\text{project}} \Omega U/O \simeq \mathbb{Z} \times BO.$$

In considering a Lagrangian embedding or immersion $\phi_\alpha : S^n \to T^*S^n$ represented by $\alpha : S^n \to U$, then by Proposition 24 the Maslov bundle $\nu$ is just

$$-\ell(\alpha) : LS^n \to BU \to BO.$$

Thus we may conclude from (45) that if the Lagrangian immersion $\phi_\alpha$ is Lagrangian isotopic to a Lagrangian embedding, then the spectra

$$\Sigma^\infty(LS^n_+) \quad \text{and} \quad (LS^n)^{-\ell(\alpha)} = (LS^n)^{\ell(-\alpha)}$$

are equivalent.

Now the spectrum $\Sigma^\infty(LS^n_+)$ is equivalent to the topological Hochschild homology $\mathrm{THH}(\Sigma^\infty(\Omega S^n_+))$. By Proposition 24, the spectrum $(LS^n)^{\ell(-\alpha)}$ is equivalent to the topological Hochschild homology $\mathrm{THH}((\Omega S^n)^{\Omega(-\alpha)})$. The statement of the theorem now follows. □

For $k > 1$, let $\alpha_k : S^{2k+1} \to U$ be a generator of $\pi_{2k+1}(U)$, which by Bott periodicity is isomorphic to the integers. In [Abouzaid and Kragh 2016] it was proved that for $2k+1$ congruent to 1, 3, or 5 (mod 8), the Lagrangian immersion $\phi_k : S^{2k+1} \to T^*(S^{2k+1})$ represented by $\alpha_k$ is not Lagrangian isotopic to a Lagrangian embedding. We now see that this is detected by the fact the twisted smooth Calabi–Yau ring spectra $\Sigma^\infty(\Omega S^{2k+1}_+)$ and $(\Omega S^{2k+1})^{\Omega(-\alpha_k)}$ have different topological Hochschild homologies. For this we again use the fact that

$$\mathrm{THH}(\Sigma^\infty(\Omega S^{2k+1}_+)) \simeq \Sigma^\infty(LS^{2k+1}_+) \quad \text{and}$$
$$\mathrm{THH}((\Omega S^{2k+1})^{\Omega(-\alpha_k)}) \simeq (LS^{2k+1})^{\ell(-\alpha_k)}.$$

We will show that $(LS^{2k+1})^{\ell(-\alpha_k)}$ is not equivalent to $\Sigma^\infty(LS^{2k+1}_+)$ by showing that the generator $u : \mathbb{S} \to (LS^{2k+1})^{\ell(-\alpha_k)}$ of $\pi_0(LS^{2k+1})^{\ell(-\alpha_k)}) \cong \mathbb{Z}$ is not split by any map $(LS^{2k+1})^{\ell(-\alpha_k)} \to \mathbb{S}$. ($\Sigma^\infty(LS^{2k+1}_+)$ clearly admits such a splitting map.)

By the construction of $\ell(-\alpha_k)$ in [Blumberg et al. 2010] as described above, the composition

$$\Omega S^{2k+1} \hookrightarrow LS^{2k+1} \xrightarrow{\ell(-\alpha_k)} BU$$

is given by

$$\Omega(-\alpha_k) : \Omega S^{2k+1} \to BU = \{0\} \times BU \subset \mathbb{Z} \times BU.$$

This is a loop map, which by the definition of $\alpha_k$ induces an isomorphism

$$\pi_{2k}(\Omega S^{2k+1}) \to \pi_{2k} BU \cong \mathbb{Z}.$$

That is to say, if the $\iota_{2k} : S^{2k} \to \Omega S^{2k+1}$ is a generator, then the composition

$$b\alpha_k : S^{2k} \xrightarrow{\iota_{2k}} \Omega S^{2k+1} \hookrightarrow LS^{2k+1} \xrightarrow{\ell(-\alpha_k)} BU \tag{46}$$

generates the $(2k)$-th homotopy group. The Thom spectrum of this composition $(S^{2k})^{b\alpha_k}$ is equivalent to the CW spectrum

$$(S^{2k})^{b\alpha_k} \simeq \mathbb{S} \cup_{\tilde{\alpha}_k} D^{2k},$$

where the attaching map $\tilde{\alpha}_k : \Sigma^\infty S^{2k-1} \to \mathbb{S}$ is defined as follows.

Consider the composition $S^{2k} \xrightarrow{b\alpha_k} BO \xrightarrow{BJ} B\mathrm{GL}_1(\mathbb{S})$ where $BJ$ is the delooping of the $J$-homomorphism $J : O \to \mathrm{GL}_1(\mathbb{S})$. Applying the loop space defines a map $S^{2k-1} \to \mathrm{GL}_1(\mathbb{S})$. Since $S^{2k-1}$ is connected, its image lies in a single component of $\mathrm{GL}_1(\mathbb{S})$ which is equivalent to the component of the basepoint in $QS^0$. The adjoint of this map is the definition of the map $\tilde{\alpha}_k : \Sigma^\infty(S^{2k-1}) \to \mathbb{S}$. Notice, that by definition it is in the image of the $J$ homomorphism, $J : \pi_{2k-1} O \to \pi_{2k-1}(\mathbb{S})$.

By standard calculations of Quillen and Adams, as described in [Abouzaid and Kragh 2016], for $2k+1$ congruent to 1, 3, or 5 (mod 8), the class $\tilde{\alpha}_k$ is nontrivial. Therefore there is no splitting map from $(S^{2k})^{b\alpha_k} \simeq \mathbb{S} \cup_{\tilde{\alpha}_k} D^{2k}$ to $\mathbb{S}$ that splits the generator $\mathbb{S} \to (S^{2k})^{b\alpha_k}$. By (46) there is therefore no splitting map from $(LS^{2k+1})^{\ell(-\alpha_k)}$ to $\mathbb{S}$. Thus $(LS^{2k+1})^{\ell(-\alpha_k)}$ is *not* equivalent to $\Sigma^\infty(LS^{2k+1}_+)$ and hence $\mathrm{THH}((\Omega S^{2k+1})^{\Omega(-\alpha_k)})$ is *not* homotopy equivalent to $\mathrm{THH}(\Sigma^\infty(\Omega S^{2k+1}_+))$. By Theorem 25, this implies that the Lagrangian immersion $\phi_k$ is *not* Lagrangian isotopic to a Lagrangian embedding.

## 5. A topological Hochschild (co)homology perspective

In this section, we give a topological Hochschild homology and cohomology interpretation of the Calabi–Yau structures and the dualities between the manifold string topology ring spectrum $\mathcal{S}^\bullet_P(M)$ and the Lie group string topology coalgebra spectrum $\mathcal{S}^P_\bullet(M)$.

We continue to consider a principal bundle $G \to P \xrightarrow{p} M$ where $M$ is a closed manifold of dimension $n$ and $G$ is a compact Lie group of dimension $d$. A choice of connection on the bundle $P$ defines a *holonomy* map

$$h_P : \Omega M \to G.$$

This is a map of group-like $A_\infty$ spaces, and the induced map of classifying spaces, $Bh_P : M \simeq B(\Omega M) \to BG$ classifies the bundle $P$.

We then have an induced map of ring spectra and differential graded algebras that by abuse of notation we still denote by $h_P$:

$$C_*(\Omega M) \xrightarrow{h_P} C_*(G) \quad \text{and} \quad \Sigma^\infty(\Omega M_+) \xrightarrow{h_P} \Sigma^\infty(G_+).$$

These holonomy maps therefore define bimodule structures of $C_*(G)$ over $C_*(\Omega M)$ and of $\Sigma^\infty(G_+)$ over $\Sigma^\infty(\Omega M_+)$. We can therefore study the (topological) homology of these algebras with coefficients in these bimodules. In what follows we suppress the map $h_P$ from the notation regarding these bimodules. This is somewhat justified because given any two choices of holonomy maps, the induced module structures will be equivalent. We also note that a choice of holonomy defines an inherited dual bimodule structure on the Spanier–Whitehead dual $G^\vee$ over $\Sigma^\infty(\Omega M_+)$, and similarly the cochains $C^*(G)$ inherit the dual bimodule structure over $C_*(\Omega M)$. One of the main results of this section is the following.

**Theorem 26.** *We have the following equivalences involving topological Hochschild homology* $\mathrm{THH}_\bullet$ *and topological Hochschild cohomology* $\mathrm{THH}^\bullet$.

(1) $\mathrm{THH}_\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq \Sigma^\infty(P^{\mathrm{Ad}}_+)$.

(2) $\mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq (P^{\mathrm{Ad}})^{-TM} \simeq \mathcal{S}^\bullet_P(M)$.
*This equivalence is one of ring spectra.*

(3) $\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \simeq (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \simeq \mathcal{S}^P_\bullet(M)$.
*This equivalence is one of coalgebra spectra.*

*Proof.* Given any homomorphism $\phi : H \to G$ of topological groups, one has that

$$\mathrm{THH}_\bullet\big(\Sigma^\infty(H_+), \Sigma^\infty(G_+)\big) \simeq \Sigma^\infty(EH \times_H G^{\mathrm{Ad}}),$$

where $G^{\mathrm{Ad}}$ represents the adjoint (conjugation) action of $H$ on $G$:

$$h \cdot g = \phi(h) g \phi(h)^{-1}.$$

This is because $\mathrm{THH}_*\big(\Sigma^\infty(H_+), \Sigma^\infty(G_+)\big)$ is equivalent to the suspension spectrum of the cyclic bar construction $N^{\mathrm{cy}}(H, G)$ which Waldhausen [1985] showed is equivalent to the homotopy orbit space of $H$ acting on $G$ via the conjugation action. In our case, we may think of $H$ as the based loop space $\Omega M$ by taking $H$ to be a topological group of the same $A_\infty$-homotopy type. (As we did earlier, by abuse of notation we still call this group $\Omega M$.) Then this observation says that

$$\mathrm{THH}_\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq \Sigma^\infty(E\Omega M \times_{\Omega M} G^{\mathrm{Ad}}_+) \simeq \Sigma^\infty(P^{\mathrm{Ad}}_+).$$

This proves part (1) of the theorem.

For part (2), we use the similarly well-known fact that the topological Hochschild cohomology of the suspension spectrum of a group can be described as a homotopy fixed point spectrum. That is, like above, let $\phi : H \to G$ be a homomorphism of topological groups. Then

$$\mathrm{THH}^\bullet\big(\Sigma^\infty(H_+), \Sigma^\infty(G_+)\big) \simeq \Sigma^\infty(G_+)^{h\Sigma^\infty(H_+)}, \tag{47}$$

where $\Sigma^\infty(H_+)$ acts on $\Sigma^\infty(G_+)$ via the conjugation action. Like above, we refer to this as the adjoint action and we write it as $\Sigma^\infty(G_+)^{\mathrm{Ad}}$. (See [Westerland 2008] or section 4 of [Malm 2011].)

Now, this homotopy fixed point spectrum is defined to be

$$\Sigma^\infty(G_+)^{h\Sigma^\infty(H_+)} = \mathrm{Rhom}_{\Sigma^\infty(H_+)}(\mathbb{S}, \Sigma^\infty(G_+)^{\mathrm{Ad}}).$$

So in our case we have that

$$\mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq \mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, \Sigma^\infty(G_+)^{\mathrm{Ad}}).$$

Notice that since the homotopy orbit spectrum of $\Sigma^\infty(\Omega M_+)$ acting on $\Sigma^\infty(G_+)$ via the adjoint action is $\Sigma^\infty(P_+^{\mathrm{Ad}})$, this spectrum of $\Sigma^\infty(\Omega M_+)$-equivariant morphisms is equivalent to the spectrum of sections of the parametrized spectrum $\Sigma^\infty(G_+) \to \Sigma^\infty_M((P^{\mathrm{Ad}})_+) \to M$:

$$\mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, \Sigma^\infty(G_+)^{\mathrm{Ad}}) \simeq \Gamma_M\big(\Sigma^\infty_M((P^{\mathrm{Ad}})_+)\big).$$

But this spectrum of sections is, by definition, the manifold string topology spectrum, $\mathcal{S}^\bullet_P(M)$. Furthermore, it is clear that the ring spectrum structures coincide under this equivalence. Furthermore, by the Atiyah–Poincaré duality theorem proved by Klein [2001; Cohen and Klein 2009] we have that

$$\Gamma_M(\Sigma^\infty_M((P^{\mathrm{Ad}})_+)) \simeq (P^{\mathrm{Ad}})^{-TM}$$

as ring spectra. Putting these together says that

$$\mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq \mathcal{S}^\bullet_P(M) \simeq (P^{\mathrm{Ad}})^{-TM}$$

as ring spectra. This is the statement of part (2) of the theorem.

We now consider part (3) of the theorem. The Spanier–Whitehead dual of the simplicial spectrum $\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)$ can, because of the compactness assumption of $G$, be described as the totalization of the cosimplicial spectrum given by taking the Spanier–Whitehead dual levelwise. This cosimplicial spectrum has as its spectrum of $k$-simplices, $\mathrm{Rhom}_{\mathbb{S}}\big(\Sigma^\infty(\Omega M_+)^{(k)}, \Sigma^\infty(G_+)\big)$. The coface maps and the codegeneracies are the duals of the face and degeneracy maps in the simplicial spectrum $\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)$. But this cosimplicial spectrum is exactly the cosimplicial spectrum defining the topological Hochschild cohomology spectrum, $\mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big)$. That is, we have observed that

$$\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)^\vee = \mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big).$$

This is a ring spectrum, so its Spanier–Whitehead dual, $\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)$ inherits the structure of a coalgebra spectrum. Furthermore, we know from part (2) that

$$\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)^\vee = \mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq \mathcal{S}^\bullet_P(M) \simeq (P^{\mathrm{Ad}})^{-TM}$$

as ring spectra. Thus applying Spanier–Whitehead duality and Theorems 9 and 1, we have

$$\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \simeq \mathcal{S}_\bullet^P(M) \simeq (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$$

as coalgebra spectra.

Alternatively, as in the proof of part (1) of the theorem, we have an equivalence

$$\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \simeq \mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}$$

Now, $\mathcal{S}_\bullet^P(M)$ is the homology spectrum of the spectrum over $M$ whose fiber is $G^\vee$, on which $\Omega M$ acts via (the dual of the) conjugation action. Therefore, we see that

$$\mathcal{S}_\bullet^P(M) \simeq \mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}} \simeq \mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee).$$

For all these spectra, the coproduct comes from dualizing the multiplication map $G \times G \to G$; see below for an explicit description of the coproduct on $\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}$. □

We end by observing how the twisted compact Calabi–Yau structure on $\mathcal{S}_P^\bullet(M) \simeq (P^{\mathrm{Ad}})^{-TM}$ can be understood from this Hochschild perspective.

The twisting bimodule in the twisted cCY structure on $R = \mathcal{S}_P^\bullet(M)$ is $Q = \Sigma^{d-n}(\mathcal{S}_\bullet^P(M)) \simeq \Sigma^{d-n}(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$. We first observe that the duality pairing (16) in the dimension $n-d$ twisted compact Calabi–Yau structure

$$\langle -, - \rangle : Q \wedge R \to \Sigma^{d-n}\mathbb{S}$$

can be described in terms of Hochschild theory as follows. As described above we have natural equivalences

$$R = \mathrm{THH}^\bullet(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)) \simeq \mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, \Sigma^\infty(G_+)^{\mathrm{Ad}}), \quad \text{and}$$
$$Q = \Sigma^{d-n}\,\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \simeq \Sigma^{d-n}\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}$$

We therefore have a cap product

$$\cap : Q \wedge R = \big(\Sigma^{d-n}\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}\big) \wedge \big(\mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, \Sigma^\infty(G_+)^{\mathrm{Ad}})\big) \to \Sigma^{d-n}(\Sigma^\infty(G_+))^{\mathrm{Ad}} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}. \quad (48)$$

The evaluation map $\mathrm{ev} : \Sigma^\infty(G_+) \wedge G^\vee \to \mathbb{S}$ is $\Sigma^\infty(\Omega M_+)$-invariant with respect to conjugation, and so defines a map

$$\mathrm{ev} : \Sigma^{d-n}(\Sigma^\infty(G_+))^{\mathrm{Ad}} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}} \to \Sigma^{d-n}\mathbb{S}.$$

Composing these defines the duality pairing

$$\langle -, - \rangle : Q \wedge R \to \Sigma^{d-n}\mathbb{S},$$

$$\big(\Sigma^{d-n}\,\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)\big) \wedge \big(\mathrm{THH}^\bullet(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+))\big) \xrightarrow{\cap} \Sigma^{d-n}\big((\Sigma^\infty(G_+))^{\mathrm{Ad}} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}\big) \xrightarrow{\mathrm{ev}} \Sigma^{d-n}\mathbb{S}.$$

We end by observing how the bimodule structure of $Q$ over $R$ can be understood at the topological Hochschild (co)homology level. We know that

$$Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \simeq \Sigma^{d-n}\,\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee).$$

Now, $\mathcal{S}^P_\bullet(M) \simeq \mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \simeq \mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}$ is a coalgebra spectrum, and its coproduct $\psi$ can be seen on the THH-level as follows:

$$\begin{array}{ccc} & \mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} ((G\times G)^\vee)^{\mathrm{Ad}} & \\ {}^{1\wedge\mu^\vee}\nearrow & & \nwarrow^{\simeq} \\ \mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}} & & \mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee \wedge G^\vee)^{\mathrm{Ad}} \\ & & \downarrow \Delta \\ & & (\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}) \wedge (\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}) \end{array}$$

This needs some explanation. $\mu : G \times G \to G$ is the multiplication map. $\mu^\vee : G^\vee \to (G \times G)^\vee$ is its Spanier–Whitehead dual. It is equivariant with respect to the adjoint action of $\Sigma^\infty(\Omega M_+)$ since $\mu$ is equivariant with respect to the adjoint action. The map

$$\Delta : \mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee \wedge G^\vee)^{\mathrm{Ad}} \to \big(\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}\big) \wedge \big(\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}\big)$$

is the map induced by thinking of $\Omega M$ as the diagonal subgroup of $\Omega M \times \Omega M$.

The action map $Q \wedge R \to Q$ is then homotopic to the composition

$$Q \wedge R \simeq \big(\Sigma^{d-n}\,\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)\big) \wedge \mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \xrightarrow{\psi\wedge 1} \Sigma^{d-n}\,\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \wedge \mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \wedge \mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \xrightarrow{1\wedge\langle -,-\rangle} \Sigma^{d-n}\,\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \wedge \mathbb{S} = Q.$$

The left module structure is homotopic to the analogous composition $R \wedge Q \to Q$.

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# Semiclassical approximation of the magnetic Schrödinger operator on a strip: dynamics and spectrum

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In the semiclassical regime (i.e., $\epsilon \searrow 0$), we study the effect of a slowly varying potential $V(\epsilon t, \epsilon z)$ on the magnetic Schrödinger operator $P = D_x^2 + (D_z + \mu x)^2$ on a strip $[-a, a] \times \mathbb{R}_z$. The potential $V(t, z)$ is assumed to be smooth. We derive the semiclassical dynamics and we describe the asymptotic structure of the spectrum and the resonances of the operator $P + V(\epsilon t, \epsilon z)$ for $\epsilon$ small enough. All our results depend on the eigenvalues corresponding to $D_x^2 + (\mu x + k)^2$ on $L^2([-a, a])$ with Dirichlet boundary condition.

## 1. Introduction

The quantum dynamics of an electron in a strip subject to an uniform magnetic field and an external slowly varying potential is governed by the Schrödinger operator

$$H(\epsilon) := P + V(\epsilon t, \epsilon z) = D_x^2 + (D_z + \mu x)^2 + V(\epsilon t, \epsilon z), \qquad D_\nu = \frac{1}{i}\partial_\nu, \quad \epsilon, \mu > 0,$$

where $\mu$ is proportional to the strength of the magnetic field and $\epsilon$ is a small parameter. The potential $V$ is assumed to be smooth and real valued.

The operator

$$P = D_x^2 + (D_z + \mu x)^2,$$

is defined on $\{u \in H^2(C_a);\ u|_{\partial C_a} = 0\}$, where $H^2(C_a)$ denotes the second order Sobolev space on a strip $C_a := \{(x, z) \in \mathbb{R}^2;\ -a \le x \le a\}$. The Fourier transformation with respect to $z$ reduces the spectral problem of $P$ to an analysis of the ($k$ depending) eigenvalues $E_0(k), E_1(k), \ldots$ of the Sturm-Liouville operator

$$P(k) = -\partial_x^2 + (k + \mu x)^2,$$

on the interval $[-a, a]$ with Dirichlet boundary condition at $-a$ and $a$.

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In this paper, we are interested in the asymptotic solutions of the time-dependent Schrödinger equation

$$D_t u = H(\epsilon)u, \quad u|_{t=0} = u_\epsilon(x, z), \tag{1-1}$$

as $\epsilon \searrow 0$. In particular we derive the semiclassical dynamics and we describe the asymptotic structure of the spectrum and the resonances of the operator $H(\epsilon)$ for $\epsilon$ small enough.

The hydrogen atom in a homogeneous magnetic field is a model of quantum chaos. See for example [Viehweger et al. 1990]. The spectral properties of $H(\epsilon)$ on $\mathbb{R}^2$ have been intensively studied in the last twenty years. In the case of perturbations, the Landau levels $\lambda_n(\mu) = \mu(2n+1)$ become accumulation points of the eigenvalues of $H(\epsilon)$ and the asymptotics of the function counting the number of the eigenvalues lying in a neighborhood of $\lambda_n(\mu)$ have been examined by many authors in different aspects. For recent results, the reader may consult [Gérard and Łaba 2002; Ivrii 2018; Fournais and Helffer 2010].

The spectrum of $P$ on a bounded domain $\Omega \subset \mathbb{R}^2$ were considered by many others. In particular the asymptotic behavior of the bottom of the spectrum of $P$ as $\mu$ tends to infinity has been treated for different geometry of $\Omega$ (see [Fournais and Helffer 2010]). In the case where $\Omega$ is the semiinfinite plane or the disk, the WKB approximations of the energies and the eigenfunctions are obtained in [Spehner et al. 1998; Bonnaillie-Noël et al. 2016].

S. De Bièvre and J. F. Pulé [1999] studied the perturbed operator $H(1)$ on the half plane with Dirichlet boundary condition. They showed that the spectrum of $H(1)$ is purely absolutely continuous in a spectral interval of size $\gamma\mu$ (for some $\gamma < 1$) between the Landau levels of the operator $P$. A similar problem has been considered in [Briet et al. 2008; 2009; Bony et al. 2009] for $H(1)$ on a strip $C_a$. Moreover the behavior of the spectral shift function near the thresholds $E_i(0)$ was studied in [Briet et al. 2008].

In this work, by the WKB method we construct nontrivial asymptotic solutions of (1-1) (see Theorem 3.1). From the eikonal equation, we derive the classical effective Hamiltonian corresponding to (1-1). In particular we show that the equations of motion in the $z$-direction are given by $\dot{z} = -\partial_k E_l(k)$, $\dot{k} = \partial_z V(s, z)$. These WKB approximate solutions fail at the so called turning points. In such neighborhoods, where the semiclassical approximation fails, we use the semiclassical Airy equation to describe the solution of (1-1). Next the connection of the two solutions in the matching regions leads to the Bohr–Sommerfeld quantization conditions. In Section 5 we use these quantization conditions to determine asymptotically the eigenvalues and the resonances of $H(\epsilon)$ for $\epsilon$ small enough. Particular attention will be paid to the asymptotic behavior of the spectrum near the thresholds of $P$.

The paper is organized as follows: Section 2 is devoted to the study of the operator $P(k)$ on the interval $[-a, a]$. In Section 3, we construct the approximate solutions of (1-1). In Section 4, we study the concept of a turning point $t_i$ for equations of the form (1-1). We describe also the asymptotic behavior as $\epsilon \searrow 0$ of solutions in a neighborhood of $t_i$, and we derive the Bohr–Sommerfeld quantization conditions.

## 2. The unperturbed Hamiltonian

Consider the $2D$ Schrödinger operator with constant magnetic field in the strip $C_a$:

$$P = D_x^2 + (D_z + \mu x)^2.$$

The operator $P$ is unitarily equivalent to

$$\mathcal{F} P \mathcal{F}^* = \int_{\mathbb{R}}^{\oplus} P(k)\, dk, \tag{2-1}$$

where $\mathcal{F}$ is the partial Fourier transform with respect to $z$,

$$(\mathcal{F}u)(x,k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-izk} u(x,z)\, dz,$$

and

$$P(k) = D_x^2 + (k + \mu x)^2,$$

is the operator defined on $\mathcal{H}_a := \{u \in H^2([-a,a]);\ u(-a) = u(a) = 0\}$. We begin with a general result on such operators.

**Theorem 2.1.** *The operator $P(k)$ has a simple discrete spectrum i.e., $\sigma(P(k)) = \bigcup_{j=1}^{\infty}\{E_j(k)\}$ with $E_1(k) < E_2(k) < E_3(k) < \cdots$. Moreover, for every $j$, $E_j(k)$ is an even real analytic function in $k$, with the following properties*:

$$k E_j'(k) > 0, \quad k \neq 0 \qquad \textit{and} \qquad E_j'(0) = 0, \quad E_j''(0) > 0, \tag{2-2}$$

$$E_j(k) = E_j(0) + \sum_{i=1}^{\infty} a_{j,i} k^{2i}, \quad (k \to 0), \quad a_{j,1} > 0, \tag{2-3}$$

$$E_j(k) = k^2 - 2a\mu k + \nu_j (2\mu k)^{\frac{2}{3}} + \mathcal{O}(1), \quad k \to +\infty, \tag{2-4}$$

*where $0 < \nu_1 < \nu_2 < \cdots < \nu_j < \cdots$ are the eigenvalues of the operator $M = D_x^2 + x$ on $\mathbb{R}^+$. Here $E_j(0)$ are the eigenvalues of the operator $D_x^2 + \mu^2 x^2$. In particular, $E_j(0) \sim (2j-1)\mu$ for strong magnetic field (i.e., $\mu$ large enough), and $E_j(0) \sim (j\pi)^2/a^2 + (\mu^2/a^2)\big(\frac{1}{3} - 1/(2\pi^2 j^2)\big)$ for weak magnetic field (i.e., $\mu \ll 1$). The normalized eigenfunctions $\Psi_j(\cdot, k)$ corresponding to $E_j(k)$ can be* *chosen real-valued analytic with respect to k satisfying*:

$$\textit{for all } p \in \mathbb{N}, \textit{ there exists } C_p \textit{ such that } \int_{-a}^{a} \big(\partial_k^p \Psi(x,k)\big)^2\,dx \le C_p. \tag{2-5}$$

*Proof.* From the Sturm–Liouville theory (see for instance [Marchenko 1986]), it is well known that $P(k)$ has a simple discrete spectrum: $E_1(k) < E_2(k) < \cdots$. The change of variable $x \mapsto -x$ shows that $E_l(k) = E_l(-k)$. Since the eigenvalues are simple, ordinary perturbation theory shows that $E_l(k)$ (and the corresponding eigenfunction) are analytic functions in $k$ (see [Kato 1966; Reed and Simon 1978]). The estimate (2-2) is proved by [Geĭler and Senatorov 1997] in a more general setting (see [Geĭler and Senatorov 1997, Theorem 2]). Formula (2-3) follows from the fact that $E_j(k)$ is an even real analytic function with $E_j''(0) > 0$. The asymptotic behavior of $E_j(0)$ for $\mu$ small enough (resp. large enough) follows from the perturbation theory (resp. semiclassical analysis).

To prove (2-4) it suffices to study the operator[1] $D_x^2 + 2\mu x k + k^2$. Replacing $x$ by $t = \mu(x+a)$ and rescaling $t \mapsto t/\lambda$ (with $\lambda = (2\mu k)^{\frac{1}{3}}$) we transform $\widetilde{H}(k)$ into

$$\lambda^2(D_t^2 + t) - 2a\mu k + 2k^2 : L^2([0, 2\lambda\mu a]) \to L^2([0, 2\lambda\mu a]),$$

which yields (2-4) since[2] $\lambda \to +\infty$ as $k \to +\infty$.

The only point remaining concerns the estimate (2-5). Let $\Psi_n(\,\cdot\,,k)$ be the normalized real-valued[3] analytic function corresponding to $E_n(k)$. Since $\Psi_n$ is real and $\|\Psi_n(\,\cdot\,,k)\| = 1$, it follows that

$$\frac{\partial}{\partial k}\int_{-a}^{a} \Psi_n(x,k)^2\,dx = 0 = 2\int_{-a}^{a} \Psi_n(x,k)\frac{\partial}{\partial k}\Psi_n(x,k)\,dx. \tag{2-6}$$

Put $\widehat{P}(k) = D_x^2 + 2xk + x^2$, and let $\Gamma_n$ be a simple closed contour around $E_n(k) - k^2$ such that $\operatorname{dist}\big(\Gamma_n, \sigma(\widehat{P}(k))\big) \ge C > 0$ uniformly on $k$. Let $\Pi_n(k)$ be the orthogonal projection onto $\Psi_n(\,\cdot\,,k)$:

$$\Pi_n(k) = \frac{1}{2\pi i}\int_{\Gamma_n} (\widehat{P}(k) - z)^{-1}\,dz = \langle\,\cdot\,, \Psi_n(\,\cdot\,,k)\rangle \Psi_n(x,k). \tag{2-7}$$

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[1]By the min-max principle the spectrum of $D_x^2 + 2kx + k^2$ and $P(k)$ differ by a constant for $k$ large enough.

[2]The eigenvalues of the Airy equation, $(D_t^2 + (t - \nu_j))u(t) = 0$, on $L^2([0,\mu])$ with Dirichlet condition $u(0) = u(\mu) = 0$ are the solutions of the equation

$$\operatorname{Ai}(-\nu_j) = \operatorname{Bi}(-\nu_j)\frac{\operatorname{Ai}(-\nu_j + \mu)}{\operatorname{Bi}(-\nu_j + \mu)}. \tag{E}$$

Here $\operatorname{Ai}(x)$ is the Airy function and $\operatorname{Bi}(x) = \operatorname{Ai}(e^{2\pi i/3}x)$. Since the right hand side of (E) tends to zero as $\mu$ tends to $+\infty$, $-\nu_j$ are approximated by the zeros of the Airy function.

[3]Since $D_x^2 + (x+k)^2 = \overline{D_x^2 + (x+k)^2}$, $\Psi_n(x,k)$ can be chosen real-valued.

From (2-6) we deduce that $\Pi_n(k)\partial_k\Psi_n(x,k)=0$. Combining this with the fact that $\Pi_n(k)\Psi_n(x,k)=\Psi_n(x,k)$ and using (2-7), we get

$$\begin{aligned}\partial_k\Psi_n(x,k)&=\partial_k\Pi_n(k)\Psi_n(x,k)\\&=\frac{1}{2\pi i}\int_{\Gamma_n}(\widehat{P}(k)-z)^{-1}2x(\widehat{P}(k)-z)^{-1}\,dz\,\Psi_n(x,k),\end{aligned}\tag{2-8}$$

which yields

$$\|\partial_k\Psi_n(\,\cdot\,,k)\|=\mathcal{O}(1)\|\Psi_n(\,\cdot\,,k)\|=\mathcal{O}(1).$$

We now proceed by induction using (2-8). □

## 3. The perturbed Hamiltonian

For the simplicity of the notation we take $\mu=1$. As stated in the introduction, we consider the time-dependent Schrödinger equation with perturbed potentials:

$$\big[D_t-H(\epsilon)\big]u=0,\quad u=u(t,x,z,\epsilon).\tag{3-1}$$

With the change of variables

$$s=\epsilon t\quad\text{(adiabatic scale)}\qquad\text{and}\qquad y=\epsilon z\quad\text{(long spacial scale)},$$

Equation (3-1) becomes

$$\big[\epsilon D_s-\widehat{H}(\epsilon)\big]v=0,\quad v=v(s,x,y,\epsilon),\tag{3-2}$$

where

$$\widehat{H}(\epsilon):=D_x^2+(\epsilon D_y+x)^2+V(s,y).$$

Now if $\widehat{H}(\epsilon)$ is regarded as an $\epsilon$-pseudodifferential operator on $(s,y)$ with operator-valued symbol, one looks for a local solution of the form

$$v(s,x,y,\epsilon)=e^{i\phi(s,y)/\epsilon}m(s,x,y,\epsilon),\tag{3-3}$$

$$m(s,x,y;\epsilon)=m_0(s,x,y)+\epsilon m_1(s,x,y)+\cdots.\tag{3-4}$$

Substituting (3-3) into (3-2) and collecting terms which are the same order in $\epsilon$, we get

$$\begin{aligned}&e^{-i\phi(s,y)/\epsilon}\big[\epsilon D_s-\widehat{H}(\epsilon)\big]v\\&\quad=\big[\partial_s\phi-P(\phi'_y(s,y))-V(s,y)\big]m+\big[\epsilon D_s-\partial_kP(\phi'_y(s,y))\epsilon D_y+i\epsilon(\Delta\phi)\big]m\\&\qquad\qquad+\epsilon^2\Delta m\\&\quad=c_0(s,x,y)+\epsilon c_1(s,x,y)+\cdots+\epsilon^{N+2}c_{N+2}(s,x,y,\epsilon),\end{aligned}\tag{3-5}$$

with

$$c_0(s,x,y) = \big[\phi'_s - P(\phi'_y) - V(s,y)\big]m_0, \tag{3-6}$$

$$c_1(s,x,y) = Km_0 + \big[\phi'_s - P(\phi'_y) - V(s,y)\big]m_1, \tag{3-7}$$

and for $j = 2, 3, \dots, N+2$,

$$c_j(s,x,y) = Km_{j-1} + \Delta_y m_{j-2} + \big[\phi'_s - P(\phi'_y) - V(s,y)\big]m_j. \tag{3-8}$$

Here

$$K = i\big[\partial_k P(\phi'_y)\partial_y + \phi''_{yy} - \partial_s\big], \tag{3-9}$$

and

$$\phi'_y = \partial_y\phi(s,y), \quad \phi''_{yy} = \partial^2_{yy}\phi(s,y), \quad \partial_k P(k) = 2(k+x). \tag{3-10}$$

Notice that, when $\phi$ is real-valued, (3-3) is the standard ansatz of geometric optics. In the construction of geometric optics solutions one requires that

$$c_0(s,x,y) = 0, \tag{3-11}$$

$$c_j(s,x,y) = 0, \quad j = 1, 2, \cdots. \tag{3-12}$$

***Eikonal equation and semiclassical dynamics.*** From now on we fix $l$, and we let $\Psi_l(\,\cdot\,,k)$ be the normalized eigenfunction corresponding to $E_l(k)$:

$$P(k)\Psi_l(\,\cdot\,,k) = E_l(k)\Psi_l(\,\cdot\,,k), \quad \int_{-a}^{a} \Psi_l(x,k)^2\,dx = 1. \tag{3-13}$$

By Theorem 2.1, the function $k \to \Psi_l(\,\cdot\,,k)$ can be chosen real analytic.

Equations (3-11) and (3-6) tell us that for all $s, y, m_0(s,\,\cdot\,,y)$ is an eigenfunction of $P(\phi'_y)$ with eigenvalue $\partial_s\phi - V(s,y)$. Hence, we can satisfy (3-11) by choosing

$$\phi'_s = E_l(\phi'_y) + V(s,y), \quad \text{(eikonal equation)} \tag{3-14}$$

and setting

$$m_0(s,x,y) = f_0(s,y)\Psi_l(x,\phi'_y). \tag{3-15}$$

Since the eikonal equation is derived from the "effective Hamiltonian"

$$G(s,\sigma,y,k) = \sigma - E_l(k) - V(s,y),$$

we see that equation of motion in the $y$-direction are

$$\dot{s} = 1, \quad \dot{\sigma} = \partial_s V(s,y), \quad \dot{y} = -\partial_k E_l(k), \quad \dot{k} = \partial_y V(s,y). \tag{3-16}$$

By applying the classical "method of characteristics," one can solve the eikonal equation (3-14) at least for small $s$. From now on we assume that $\phi$ is constructed.

***Propagation of the amplitude.*** For simplicity we ignore the dependence of the following operators and functions on $\phi'_y$ and we write $P$, $\partial_k P$, $E_l$, $\partial_k E_l$, $\Psi$ and $f_0 \Psi_l$ instead of $P(\phi'_y)$, $\partial_k P(\phi'_y)$, $E_l(\phi'_y)$, $\partial_k E_l(\phi'_y)$, $\Psi_l(x, \phi'_y)$ and $f_0(s, y)\Psi_l(x, \phi'_y)$.

By the Fredholm alternative in $L^2([-a, a])$, we can solve (3-12) for $j = 1$ if and only if the first term of the right hand side of (3-7) is orthogonal to

$$\ker\bigl[P - (\phi'_s - V(y))\bigr] = \ker\bigl[P - E_l\bigr] = \mathrm{Vect}(\Psi_l),$$

where we have used (3-14) and (3-15). In view of (3-7) and (3-8) this is equivalent to

$$\bigl\langle [\partial_k P \partial_y + \phi''_y - \partial_s](f_0 \Psi_l),\ \Psi_l \bigr\rangle = 0.$$

We conclude from (2-6) that $\langle \partial_s \Psi_l, \Psi_l \rangle = 0$, hence that

$$\langle \partial_k P \Psi_l, \Psi_l \rangle \partial_y f_0 - \partial_s f_0 + \bigl[\langle \partial_k P \partial_y \Psi_l, \Psi_l \rangle + \phi''_y\bigr] f_0 = 0. \tag{3-17}$$

Taking the derivative with respect to $k$ in (3-13),

$$\bigl[P(k) - E_l(k)\bigr]\partial_k \Psi(\,\cdot\,, k) = \bigl[\partial_k E_l(k) - \partial_k P(k)\bigr]\Psi_l(\,\cdot\,, k), \tag{3-18}$$

and taking the inner product with $\Psi_l$ and using again (3-13) we get

$$\partial_k E_l(k) = \langle (\partial_k P)\Psi_l, \Psi_l \rangle = 2\int_{-a}^{a} (x + k)\Psi_l(x, k)^2 \, dx. \tag{3-19}$$

Next, taking the derivative with respect to $y$ of $\partial_k E_l = \partial_k E_l(\phi'_y)$, we obtain

$$\partial_y \cdot \partial_k E_l = 2\phi''_y + 2\langle (\partial_k P)\partial_y(\Psi_l), \Psi_l \rangle. \tag{3-20}$$

Substituting (3-19) and (3-20) into the left hand side of (3-17), we get the transport equation for $f_0$:

$$\partial_k E_l \partial_y f_0 - \partial_s f_0 + \tfrac{1}{2}\bigl[\partial_y \cdot \partial_k E_l\bigr] f_0 = 0. \tag{3-21}$$

Assuming that $\phi(s, y)$ is selected and let $U_s : y = y(0) \to y(s)$ be the flow on the configuration space $\mathbb{R}_y$ corresponding to

$$\dot{y}(s) = -\partial_k E_l\bigl(\phi'_y(s, y(s))\bigr).$$

It is well known that

$$-\partial_y \cdot \partial_k E_l\bigl(\phi'_y(s, y(s))\bigr) = \frac{d}{ds} \log\left|\frac{\partial y(s)}{\partial y}\right|.$$

Along $y(s)$ the differential equation (3-21) takes the form

$$\frac{d}{ds}\bigl[f_0(s, y(s))\bigr] + \left[\frac{1}{2}\frac{d}{ds} \log\left|\frac{\partial y(s)}{\partial y}\right|\right] f_0 = 0, \tag{3-22}$$

which yields a kind of energy conservation

$$\frac{d}{ds}\left[|f_0|^2\left|\frac{\partial y(s)}{\partial y}\right|\right]=0. \tag{3-23}$$

Thus, $\int_{\mathbb{R}}|f_0(s,y)|^2\,dy$ does not depend on $s$ and consequently

$$\int_{-a}^{a}\int_{\mathbb{R}_y}|u(s,x,y,\epsilon)|^2\,dx\,dy=\int_{-a}^{a}\int_{\mathbb{R}_y}|u(0,x,y,\epsilon)|^2\,dx\,dy+\mathcal{O}(\epsilon).$$

We now derive the transport equation for $m_1(s,x,y)$. Like (3-7), Equation (3-8) can be solved for $j=2$ if and only

$$\big\langle i\big[\partial_k P\partial_y+\phi_y''-\partial_s\big]m_1+\Delta_y m_0,\ \Psi_l\big\rangle=0. \tag{3-24}$$

Writing

$$m_1(s,x,y)=f_1(s,y)\Psi_l(x,\phi_y')+m_1^{\perp}(s,x,y) \tag{3-25}$$

with

$$\langle\Psi_l(\,\cdot\,,\phi_y'),\ m_1^{\perp}\rangle=0.$$

According to (3-7) and (3-25), the term $m_1^{\perp}$ is given by

$$m_1^{\perp}=-\big[\phi_s'-P(\phi_y')-V(s,y)\big]^{-1}(Km_0). \tag{3-26}$$

Inserting (3-25) in (3-24) and using (3-26) we see that $f_1(s,y)$ satisfies an inhomogeneous version of the transport equation (4-3):

$$\partial_k E_l\partial_y f_1-\partial_s f_1+\tfrac{1}{2}[\partial_y\cdot\partial_k E_l]f_1=-\big\langle[\partial_k P\partial_y+\phi_y''-\partial_s]m_1^{\perp}+i\Delta_y m_0,\ \Psi_l\big\rangle. \tag{3-27}$$

We repeat this process (by solving the transport equation with a right-hand side) and get explicitly all the terms $m_j$ (at least for $s$ small). This gives a solution of (3-1) modulo $\mathcal{O}(\epsilon^\infty)$. Consequently, we have proved:

**Theorem 3.1.** *Given $N\in\mathbb{N}$, $\phi\in C^\infty(\mathbb{R})$ and $f\in C_0^\infty(\mathbb{R})$. There exists $T,\epsilon_0>0$ and an approximate solution*

$$v(s,x,y;\epsilon)=m_0(s,x,y)+\epsilon m_1(s,x,y)+\cdots+\epsilon^N m_N(s,x,y)$$

*such that for all $|s|<T$ and $\epsilon\in\,]0,\epsilon_0[$ we have*:

$$\|v(0,x,y;\epsilon)-e^{i\phi(y)/\epsilon}\Psi_l(x,\phi'(y))\|=\mathcal{O}(\epsilon),$$
$$\|(\epsilon D_s-\widehat{H}(\epsilon))v\|=\mathcal{O}_N(\epsilon^N),$$

*with*

$$m_0(s,x,y)=e^{i\phi(s,y)/\epsilon}f_0(s,y)\Psi_l(x,\partial_y\phi(s,y)),$$

*where $\phi(s,y)$ and $f_0(s,y)$ are solutions of* (3-14) *and* (3-21) *respectively with initial condition $f_0(0,y)=f(y)$ and $\phi(0,y)=\phi(y)$.*

However, as is well known, if we try to construct WKB-solutions globally (that is in some large given region), $\phi$ may develop singularities at "caustic" and the transport equations then become undefined. The consideration of these difficulties, beginning with [Keller 1958; Maslov and Fedoriuk 1981], lead to the development of the theory of Fourier integral operators as given by Hörmander [1971]. Since here the problem is reduced to study a one-dimensional Hamiltonian in the $y$-direction, we will use in the next section the standard semiclassical techniques based on the Airy function.

## 4. Quantization conditions

Recall that $H(\epsilon)$ and $\widehat{H}(\epsilon)$ have the same spectrum, since they are unitarily equivalent by a change of variable (see (3-1) and (3-2)). Hence, in this section we will be concerned with the spectrum of the operator $\widehat{H}(\epsilon)$. From now on we assume that $V$ is time independent (i.e., $V(y) := V(t, y)$).

Fix an energy $e$, and consider the stationary equation

$$(\widehat{H}(\epsilon) - e)w = 0, \quad w = e^{i\phi(y)/\epsilon}\big(m_0(x, y) + \epsilon m_1(x, y) + \cdots\big). \tag{4-1}$$

Clearly, $f$ is a solution of (4-1) if and only if $v(x, y, s, \epsilon) = e^{ise/\epsilon} w$ is a solution of (3-2). In particular, the eikonal and transport equations corresponding to (4-1) are

$$e = E_l(\phi'(y)) + V(y), \tag{4-2}$$

$$\partial_y f_0 + \frac{1}{2}\left[\frac{\partial_y \cdot \partial_k E_l}{\partial_k E_l}\right] f_0 = 0. \tag{4-3}$$

Let $\Sigma_e^l = \{(y, k) \in \mathbb{R} \times \mathbb{C}, E_l(k) + V(y) = e\}$ be the isoenergy curve. Recalling that $k = 0$ is the only critical point of $k \mapsto E_l(k)$. Assume that

$$V'(y) \neq 0 \text{ on the set of turning points } \Gamma_e^l := \big\{y \in \mathbb{R};\ V(y) = e - E_l(0)\big\}. \tag{4-4}$$

Thus, $\Gamma_e^l$ is a discrete set: $\Gamma_e^l = \{\cdots < y_{-1} < y_0 < y_1 < \cdots\}$. Each finite interval; $[y_j, y_{j+1}]$ is covered by a closed finite branch $\gamma_e$ of $\Sigma_e$, which consists of two regular branches $\gamma_+, \gamma_-$:

$$\gamma_+ : k = \tau(y), \quad \gamma_- : k = -\tau(y).$$

(*Classical allowed region*). Consider an interval $[y_j, y_{j+1}]$ covered by a real closed branch $\gamma_e$ of $\Sigma_e$. The construction of Section 3 and (4-3) give us two solutions $w, \overline{w}$ of (4-1) such that

$$w = e^{i\phi(y)/\epsilon}(m_0 + \epsilon m_1 + \cdots), \quad \text{with } \phi(y) = \int^y \tau(t)\, dt, \tag{4-5}$$

and

$$m_0(x,y) = C_0 \frac{1}{|\partial_k E_l(\phi'_y(y))|^{\frac{1}{2}}} \Psi_l(x, \phi'_y(y)).$$

(*Classical forbidden region*). In the regions $]y_{j-1}, y_j[$ and $]y_{j+1}, y_{j+2}[$, which are classically forbidden, we can also construct a solution of the form (4-5). But now, since in $\Sigma_e^l$ the number $k$ is complex, the phases $\phi(y)$ are purely imaginary. We denote the corresponding solution by $g_1$. From $g_1$ we can construct a linearly independent other solution $g_2$ by the change $k \to -k$, (we recall that $E_l(k)$ is an even function). The solutions $g_1$ and $g_2$ in this regions are decreasing and increasing exponential functions. Notice that, the turning points separate the projections of the real and complex branches of the isoenergy curve to the $y$-axis. As indicated above, in vicinities of the turning points $y_j$ the semiclassical approximations $w$, $\overline{w}$, $g_1$ and $g_2$ are note defined, since $\phi'_y(y_j) = 0$ and $\partial_k E_l(\phi'_y(y_j)) = 0$. To describe the solutions near $y_j$ we use the standard semiclassical Airy equation. More precisely, near the turning point $y_j$ we replace the variable $y$ by the new one $\tilde{y} = \epsilon^{-\frac{2}{3}}(y - y_j)$, and we consider instead of (4-1) the equation

$$\big[D_x^2 + (\epsilon^{\frac{1}{3}} D_{\tilde{y}} + x)^2 + V(y_j + \epsilon^{\frac{2}{3}} \tilde{y}) - e\big] w(x, \tilde{y}; \epsilon) = 0, \tag{4-6}$$

with

$$w(x, \tilde{y}; \epsilon) = \sum_{l \geq 0} \epsilon^{l/\epsilon} m_l(x, \tilde{y}). \tag{4-7}$$

Expanding the operator in the left hand side of (4-6) in powers of $\epsilon^{\frac{1}{3}}$ and substituting into (4-7), we obtain

$$\big[D_x^2 + x^2 + V(y_j) - e\big] m_0(x, \tilde{y}) = 0, \tag{4-8}$$

$$\big[D_x^2 + x^2 + V(y_j) - e\big] m_1(x, \tilde{y}) = -2x D_z m_0(x, \tilde{y}), \tag{4-9}$$

$$\begin{aligned}\big[D_x^2 + x^2 + V(y_j) - e\big] m_2(x, \tilde{y}) = {} & -2x D_{\tilde{y}} m_1(x, \tilde{y}) \\ & - \big[D_{\tilde{y}}^2 + V'(y_j)\tilde{y}\big] m_0(x, \tilde{y}).\end{aligned} \tag{4-10}$$

Since $E_l(0) + V(y_j) = e$, it follows from (4-8) that

$$m_0(x, \tilde{y}) = N(\tilde{y}) \Psi_l(x, 0). \tag{4-11}$$

Notice that $x \mapsto \psi_l(x, 0)^2$ is an even function, hence the right hand side of (4-9) is orthogonal to $\Psi_l(x, 0)$. We conclude from the Fredholm alternative that (4-9) is always soluble and its solution is given by

$$m_1(x, \tilde{y}) = B(\tilde{y}) \Psi_l(x, 0) + i N'(\tilde{y}) \big[D_x^2 + x^2 + V(y_j) - e\big]^{-1} (2x \Psi_l(x, 0)).$$

Next, applying (3-18) to $k=0$, and recalling that $\partial_k P(0)=2x$ and $\partial_k E_l(0)=0$, we deduce that

$$\big[D_x^2+x^2+V(y_j)-e\big]^{-1}(2x\Psi_l(x,0))=-\partial_k\Psi_l(x,0).$$

Consequently,

$$m_1(x,\tilde{y})=B(\tilde{y})\Psi_l(x,0)-iN'(\tilde{y})\partial_k\Psi_l(x,0). \tag{4-12}$$

The right hand side of (4-10) can be written as

$$2ixB'(\tilde{y})\Psi_l(x,0)+2ixN''(\tilde{y})\partial_k\Psi_l(x,0) + \big[-N''(\tilde{y})+V'(y_j)\tilde{y}N(\tilde{y})\big]\Psi_l(x,0). \tag{4-13}$$

Combining this with fact that $x\Psi_l(x,0)$ is orthogonal to $\Psi_l(x,0)$, we deduce that Equation (4-10) has a solution if and only if

$$-N''(\tilde{y})+V'(y_j)\tilde{y}N(\tilde{y})-2N''(\tilde{y})\int_{-a}^{a}x\Psi_l(x,0)\partial_k\Psi_l(x,0)\,dx=0. \tag{4-14}$$

On the other hand, it follows from (3-19) that

$$\kappa_l^{-1}:=\tfrac{1}{2}\partial_k^2E_l(0)=1+\int_{-a}^{a}2x\Psi_l(x,0)\partial_k\Psi_l(x,0)\,dx, \tag{4-15}$$

which together with (4-14) yields the following Airy equation for $N(\tilde{y})$:

$$-N''(\tilde{y})+\eta_l\tilde{y}N(\tilde{y})=0,\qquad \eta_l:=\kappa_lV'(y_j). \tag{4-16}$$

$$m_0(x,\tilde{y})=\big[C_3\,\mathrm{Ai}(\eta_l^{\frac{1}{3}}\tilde{y})+C_4\,\mathrm{Ai}(\eta_l^{\frac{1}{3}}e^{i2\pi/3}\tilde{y})\big]\Psi_l(x,0). \tag{4-17}$$

Thus, the leading term of the series (4-7) is given by (4-11) where for $N(\tilde{y})$ we can choose an arbitrary solution of (4-16). All the remaining terms of (4-7) can easily be constructed. This gives a solution near the turning points.

It now remains to construct a global approximate solution to Equation (4-1). For the wave function to be square-integrable, we must take only the exponentially decaying solutions $(g_{1,j}, g_{1,j+1})$ in the two classically forbidden regions $]y_{j-1}, y_j[$ and $]y_{j+1}, y_{j+2}[$. These must then connect properly through the turning points $y_j$ to the classically allowed region. Let us fix a solution in the allowed region $]y_j, y_{j+1}[$:

$$v(x,y;\epsilon)=(C_1e^{i\phi(y)/\epsilon}+C_2e^{-i\phi(y)/\epsilon})\left(\frac{1}{|\partial_kE_l(\phi_y'(y))|^{\frac{1}{2}}}\Psi_l(x,\phi_y'(y))+\mathcal{O}(\epsilon)\right).$$

Next, it is possible to find a condition under which $v(x,y;\epsilon)$ satisfies the following property: the continuation of $v$ through the turning points $y_j$ and $y_{j+1}$ by means of solutions of the form (4-17) leads to solutions of the form $g_{1,j}$, $g_{1,j+1}$.

This condition, called the "Bohr–Sommerfeld quantization condition," has the following form:

$$\int_{\gamma_e} k\,dy = \pi(2n + \operatorname{ind}(\gamma_e)/2)\epsilon + \sum_{n\geq 2} \omega_n \epsilon^n, \tag{4-18}$$

where $(\omega_n)$ is some sequence of 1-form and $\operatorname{ind}(\gamma_e)$ is the Maslov index of $\gamma_e$ (see [Maslov and Fedoriuk 1981]).

This condition, which can be considered as a condition on the spectral parameter $e$, plays a crucial role in calculation of eigenvalues or resonances (see the next section).

**Remark.** Consider the $\epsilon$-pseudodifferential operator

$$H^l_{\text{eff}}(\epsilon) = E_l(\epsilon D_y) + V(y) - e.$$

The equations (4-2) and (4-3) are exactly the eikonal and transport equations in the construction of asymptotic solutions to $H^l_{\text{eff}}(\epsilon)u = \mathcal{O}(\epsilon^2)\|u\|^2$. The operator $H^l_{\text{eff}}(\epsilon)$ is called[4] the effective Hamiltonian of order $\mathcal{O}(\epsilon^2)$ corresponding to $\widehat{H}(\epsilon)$ near $e$. By using the Feshbach method (or Grushin problem) see [Dimassi and Sjöstrand 1999], we can construct an effective Hamiltonian of any order $\mathcal{O}(\epsilon^N)$ corresponding to $\widehat{H}(\epsilon)$ (see [Dimassi 1993; Martinez 1991a]).

## 5. Asymptotic behavior of the spectrum

According to (2-1), (2-4) and (2-3), we have

$$\sigma(P) = \sigma_{\text{ac}}(P) = \bigcup_{j=1} \bigcup_{k\in\mathbb{R}} E_j(k) = [E_1(0), +\infty[,$$

and $E_j(0)$, $j = 1, 2, \ldots$ are thresholds in $\sigma(P)$.

It is known that the spectrum of the perturbed operator $\widehat{H}(\epsilon)$ depends on the asymptotic behavior of $V$ at infinity. Here we distinguish two definite types of these asymptotics:

$$V(y) \to +\infty, \quad y \to \infty, \tag{A}$$

$$V(y) \to 0, \quad y \to \infty. \tag{B}$$

We refer to the remark on page 211 for other type of asymptotics.

*Case* (A). If $V(y) \to +\infty$ as $y \to \infty$, the spectrum of $\widehat{H}(\epsilon)$ is simple and purely discrete. For instance, let us assume:

[4]An effective Hamiltonian of order $\mathcal{O}(\epsilon^N)$ is a Hamiltonian that acts in a reduced space (here $L^2(\mathbb{R}_y)$ instead of $L^2([-a, a] \times \mathbb{R}_y)$) and only describes a part of the spectrum of the true Hamiltonian $H(\epsilon)$ modulo an error term $\mathcal{O}(\epsilon^N)$.

**Assumption A.** A nondegenerate minimum occurs at $y = 0$ and $V(y)$ is strictly increasing when $y > 0$ and strictly decreasing when $y < 0$.

Then the discrete spectrum of $\widehat{H}(\epsilon)$ is included in $[E_1(0) + V(0), +\infty[$.

For $e \in \,]E_1(0) + V(0), E_2(0) + V(0)[$, the isoenergy curve $\Sigma_e^1$ has a nontrivial real branch $\gamma_e^1$ which is oval. The projection of $\gamma_e^1$ on the $y$-axis is $[y_1, y_2]$, where $\{y_1, y_2\}$ are the unique turning points of $\Sigma_e^1$. The quantization condition (4-18) leads to the asymptotic description of the allowed discrete spectrum near $e$. The corresponding eigenfunction are localized asymptotically in $[y_1, y_2]$. For $l \geq 2$, the real branches of the isoenergy curve $\Sigma_e^l$ are absent.

Fix $e \in \,]E_N(0)+V(0), E_{N+1}(0)+V(0)[$. For $l \in \{1, 2, \dots, N\}$ the real branch $\gamma_e^l$ of the isoenergy curve $\Sigma_e^l$ is oval. The projection of $\gamma_e^l$ on the $y$-axis is $[y_{1,l}, y_{2,l}]$, where $\{y_{1,l}, y_{2,l}\}$ are the unique turning points of $\Sigma_e^l$. The quantization condition for each curve $\gamma_e^l$,

$$\int_{\gamma_e^l} k\,dy = \pi(2n + \operatorname{ind}(\gamma_e^l)/2)\epsilon + \sum_{n \geq 2} \omega_n^l \epsilon^n, \tag{5-1}$$

gives a set of eigenvalues $e_n^l(\epsilon) \sim \sum_j a_{j,n,l}\epsilon^j$. The corresponding eigenfunction are concentrated in $[y_{1,l}, y_{2,l}]$. For $l \geq N + 1$, the real branches of the isoenergy curve $\Sigma_e^l$ are absent.

Now let us treat the general case (i.e., without the monotonicity assumption). Fix $e \in I := \,]E_1(0) + \inf_{x\in\mathbb{R}} V(x), +\infty[$. We recall that the spectrum of $\widehat{H}(\epsilon)$ is discrete and included in $\bar{I}$. The isoenergy curve $\Sigma_e^l$ has a finite real branches $\gamma_e^{l,j}$, $j = 1, \dots, N(l)$. Under the assumption (4-4), the curve $\gamma_e^{l,j}$ is oval. The quantization condition (4-18) for each curve $\gamma_e^{l,j}$, $j = 1, \dots, N(l)$ gives the description of the asymptotic behavior with respect to $\epsilon$ of the corresponding eigenvalue. If all these asymptotic series are different they give the description of the asymptotic properties of the eigenvalues. However, if some of these series are equal (this happens for example if $V$ is even) we have to take into account the interaction effects to describe their splitting. The interaction between two series corresponding to different real branches of the isoenergy curve can be estimated in terms of tunneling through all the intervals separating the asymptotic supports of the eigenfunctions and covered by the complex branches of the isoenergy curve. This interaction leads to exponentially small displacement of the eigenvalues. Thus, the spectrum of $\widehat{H}(\epsilon)$ can be described essentially as the simple unification of the contributions of the separate $\gamma_e^l$.

Finally, let us study the bottom of the spectrum (i.e., $e = E_1(0) + \inf_{y\in\mathbb{R}} V(y)$). Without any loss of generality we may assume that $\inf_{y\in\mathbb{R}} V(y) = V(0)$ with $V'(0) = 0$, $V''(0) > 0$ and $V(y) > 0$ for all $y \neq 0$. In this case, the isoenergy curve $\Sigma_e^1$ is reduced to a single point $X_0 = (0, 0)$ and $\Sigma_e^l = \varnothing$ for $l \geq 2$. Therefore, the approximate solutions of the Equation (4-1) are localized in $\{y = 0\}$. In the Appendix, we follow the standard construction of approximate solutions near a nondegenerate minimum of the semiclassical Schrödinger operator (see [Dimassi and Sjöstrand 1999, Chapter 3]). More precisely, denote $\kappa_m := \sqrt{E_1''(0)V''(0)}\big(m + \frac{1}{2}\big)$, $m \in \mathbb{N}$. We have:

**Theorem 5.1.** *Fix $C_0$ in $]\kappa_N, \kappa_{N+1}[$. For $\epsilon$ small enough the operator $\widehat{H}(\epsilon)$ has exactly $N$ eigenvalues $e_1(\epsilon), \dots, e_N(\epsilon)$ in $]-\infty, E_1(0) + V(0) + C_0\epsilon[$. Moreover, for $j \in \{1, 2, \dots, N\}$, the following asymptotics hold*:

$$e_j(\epsilon) = E_1(0) + V(0) + \kappa_j \epsilon + \sum_{l=2}^{\infty} c_{j,l}\epsilon^l.$$

*Case* (B). Assume that $V$ tends to zero at infinity. By the Weyl criterion the essential spectrum of $\widehat{H}(\epsilon)$ and $P$ are the same, and coincide with $[E_1(0), +\infty[$. In $]-\infty, E_1(0)[$ we have a discrete spectrum caused by the potential $V$. This part of the spectrum (except near $E_1(0)$) can be studied as above. If $e \to E_1(0)$, the isoenergy curve becomes infinite. In particular, the total number of eigenvalues near $E_1(0)$ can be infinite. In this case, the asymptotic behavior of eigenvalues near $E_1(0)$ is the same as for the operator $-(E_1''(0)/2)\epsilon^2 \frac{\partial^2}{\partial y^2} + V(y)$ on $L^2(\mathbb{R})$.

To investigate the effect of $V$ on the continuous spectrum of $\widehat{H}(\epsilon)$, it is natural to study the resonances. One can treat the resonances by two different ways. If the potential $V$ is analytic in some neighborhood of the real axis, one can consider the resonances as the eigenvalues of the spectrally deformed Hamiltonian [Hislop and Sigal 1996]. If the analytical continuation of $V$ is impossible, the resonances can be considered as a poles of the meromorphic continuation of the kernel of the resolvent $(\widehat{H}(\epsilon) - z)^{-1}$ on some weighted $L^2$ space (see [Helffer and Martinez 1987]).

Let us assume that $V$ satisfies the assumption A with $v_0 := -V(0) > 0$. The structure of the isoenergy curve $\Sigma_e^l$ depends on the correlation of $v_0$ and $E_{j+1}(0) - E_j(0)$. For simplicity let us assume that

$$v_0 < \min\big(E_2(0) - E_1(0), E_3(0) - E_2(0)\big).$$

If $e \in\, ]E_1(0), E_2(0) + V(0)[$, then the isoenergy curve $\Sigma_e^1$ is unbounded, and its projection on the $y$-axis is $\mathbb{R}$. The real branches of the isoenergy curve $\Sigma_e^l$ are absent for $l \geq 2$. Thus, the quantization conditions (4-18) lose their meaning and for each $e$ there are two eigenfunctions of the continuous spectrum which correspond to two separated unbounded parts $\gamma_-$ and $\gamma_+$ of $\Sigma_e^1$ (see Section 4).

If $e = E_2(0) + V(0)$, then $\Sigma_e^1$ is unbounded and $\Sigma_e^2 = \{(0, 0)\}$. Real branches of the isoenergy curve $\Sigma_e^l$ are absent for $l \geq 3$, so the contribution of these branches to the spectrum is empty asymptotically. As indicated above the real branch of $\Sigma_e^1$

are two separated unbounded curves $\gamma_-$ and $\gamma_+$. Thus the set $\Sigma_e^1$ is nontrapping[5] and then the operator $K_1$ does not produce resonances near $e$. The operator $K_2$ has a discrete spectrum near $e$ which is resonances of the operator $\widehat{H}(\epsilon)$. More precisely, let

$$K_2 = \tfrac{1}{2}\big(E_2''(0)\epsilon^2 D_y^2 + V''(0)y^2\big),$$

and let $\zeta_1\epsilon < \zeta_2\epsilon \cdots < \zeta_j\epsilon < \cdots$ be the eigenvalues of $K_2$. As in the proof of Theorem 5.1 we have:

**Theorem 5.2.** *Fix $C_0$ in $]\zeta_N, \zeta_{N+1}[$. For $\epsilon$ small enough the operator $\widehat{H}(\epsilon)$ has exactly $N$ resonances $z_1(\epsilon), \dots, z_N(\epsilon)$ in the disk*

$$D(e, C_0\epsilon) := \{z \in \mathbb{C};\ |z - e| < C_0\epsilon\}.$$

*Moreover, for $j \in \{1, 2, \dots, N\}$, the following asymptotics hold*:

$$z_j(\epsilon) = E_2(0) + V(0) + \kappa_j\epsilon + \sum_{l=2}^{\infty} \epsilon^l d_{j,l}. \tag{5-2}$$

Formula (5-2) shows that $\Im z_j(\epsilon) = \mathcal{O}(\epsilon^\infty)$. Assuming that $V$ is analytic in a complex conic neighborhood of the real axis we can show as in [Martinez 1991b] that $\Im z_j(\epsilon) = \mathcal{O}(e^{-C/\epsilon})$ for some positive constant $C$. We cannot exclude the existence of embedded eigenvalues as the case $\mu = 0$ (i.e., $\widehat{H}(\epsilon) = D_x^2 + \epsilon^2 D_y^2 + V(y)$) shows. So in this paper we make no distinction between real eigenvalues and resonances.

Next, fix $e \in\, ]E_2(0) + V(0), E_2(0)[$. The isoenergy curve $\Sigma_e^1$ is unbounded, the real branch $\gamma_e$ of $\Sigma_e^2$ is oval and the real branches of $\Sigma_e^l$ are absent for $l \ge 3$. Again the branch $\Sigma_e^1$ does not produce resonances near $e$, and the quantization condition (4-18) corresponding to $\gamma_e$ gives the asymptotic expansion in powers of $\epsilon$ of the real part of some resonances $z_j(\epsilon)$. The study of the resonances of $H(\epsilon)$ near $E_2(0)$ is related to the study of the operator $-(E_2''(0)/2)\epsilon^2 \frac{\partial^2}{\partial y^2} + V(y)$ on $L^2(\mathbb{R})$.

**Remark.** (1) Our results hold for more general potential $V$. In particular, one can consider the case where $V$ depends on the variable $x$ (i.e., $V = V(x, \epsilon t, \epsilon z)$). In this case the results depend on the eigenvalues $G_l(s, k, y)$ corresponding to $D_x^2 + (\mu x + k)^2 + V(x, s, y)$ on $L^2([-a, a])$ with Dirichlet boundary condition.

(2) The asymptotic behavior of the spectrum of $\widehat{H}(\epsilon)$ in the case where $V$ is periodic will be treated elsewhere. The case of the plane was considered in [Brüning et al. 2002] for strong magnetic field ($\mu$ is large enough).

[5] $\Sigma_e^1$ is nontrapping for the classical Hamiltonian $p(y; k) = E_1(k) + V(y)$ if for all $(y, k) \in \Sigma_e^1$, $|\exp(tH_p(y, k)| \to \infty$ when $t \to \infty$.

(3) If $V(y)-y$ tends to zero at infinity then the spectrum of $\widehat{H}(\epsilon)$ covers the real axis. In this case to study the resonances we can use the analytic deformation. More precisely, set $W(y)=V(y)-y$, and suppose that $W$ admits a holomorphic extension into the domain $\Gamma_\delta := \{z\in\mathbb{C};\ |\Im z|<\delta\}$ for some $\delta>0$. We also assume that $W(z)$ tends to zero uniformly on $z\in\Gamma_\delta$.

For real $\theta$ the operator $\widehat{H}(\epsilon)$ is unitarily equivalent to

$$\widehat{H}_\theta(\epsilon) := D_x^2 + (\epsilon D_y + x)^2 + y + \theta + W(y+\theta).$$

The above assumption on $W$ implies that $\widehat{H}_\theta(\epsilon)_{\theta\in\Gamma_\delta}$ is an analytic family of type-A in the sense of Kato [1966]. Fix $\theta=-i\nu$ with $\nu>0$. According to the Weyl criterion, we have

$$\sigma_{\text{ess}}(\widehat{H}_{-i\nu}(\epsilon)) = \sigma(\widehat{H}_{-i\nu}(\epsilon) - W(y-i\delta)) = \mathbb{R} - i\nu.$$

Thus on the upper half plane $\{z\in\mathbb{C};\ \Im z > -\nu\}$, the operator $\widehat{H}_{-i\nu}(\epsilon)$ has discrete eigenvalues of finite multiplicities. These eigenvalues are the resonances of $\widehat{H}(\epsilon)$.

## Appendix: Sketch of proof of Theorem 5.1

Recalling that $\Sigma_e^1 = \{(0,0)\}$ and $\Sigma_e^l = \varnothing$ for $l\neq 1$. Thus the approximate solutions $w$ of (4-1) are localized in $y=0$. Therefore, we want to find

$$w = e^{i\phi(y)/\epsilon}\big(m_0(x,y) + \epsilon m_1(x,y) + \cdots\big), \quad \text{and} \quad e = e_0 + \epsilon e_1 + \epsilon^2 e_2 + \cdots$$

solutions of (4-1) near $y=0$, with $\Re(i\phi(y))>0$ for $y\neq 0$ and $\Re(i\phi(0))=0$.

From (3-21), (3-27), (4-2) and (4-3) we have:

$$E_l(\phi'(y)) + V(y) = e_0, \tag{A-1}$$

$$(\mathcal{L} - ie_1) f_0 = 0, \tag{A-2}$$

$$(\mathcal{L} - ie_1) f_1 = -\big\langle [\partial_k P \partial_y + \phi_y''] m_1^\perp + i\Delta_y m_0, \Psi_l \big\rangle - ie_2 f_0, \tag{A-3}$$

and for[6] $j=2,3,\ldots$

$$(\mathcal{L} - ie_1) f_j = F(y) - ie_j f_0. \tag{A-4}$$

Here

$$\mathcal{L} := \partial_k E_l(\phi'(y))\partial_y + \tfrac{1}{2}\phi''(y)\partial_k^2 E_l(\phi'(y)). \tag{A-5}$$

Since $k\to E_1(k)$ is an even real analytic function, the same is true for $E_1(ik)$, and it follows from (2-3) that

$$E_1(ik) - E_1(0) = -\frac{E_1''(0)}{2}k^2 + \mathcal{O}(k^4). \tag{A-6}$$

[6] $F$ is a function given by the preceding equations depending on $m_0,\ldots,m_{j-1}$ and $m_j^\perp$.

Combining this with the fact that

$$V(y) = V(0) + \frac{V''(0)}{2} y^2 + \mathcal{O}(y^3), \tag{A-7}$$

we deduce that there exists a unique real valued function $\phi_0(y)$ such that $y\phi_0(y) > 0$ for $y \neq 0$ and

$$E_l(i\phi_0'(y)) + V(y) = e_0 = E_l(0) + V(0), \quad \phi'(0) = 0, \tag{A-8}$$

for $y$ small enough. From (A-6) and (A-7), we have

$$\phi_0(y) = \frac{1}{2}\sqrt{\frac{V''(0)}{E_1''(0)}} y^2 + \mathcal{O}(y^3). \tag{A-9}$$

Next, replacing $\phi$ by $i\phi_0$ in (A-5) and using (A-8), we obtain

$$\mathcal{L} = i(yc_0(y)\partial_y + c_1(y)), \tag{A-10}$$

where $c_0(y)$ and $c_1(y)$ are real valued function with $c_0(0) = \sqrt{E_1''(0)V''(0)}$ and $c_1(0) = \frac{1}{2}\sqrt{E_1''(0)V''(0)}$.

We now turn to Equation (A-2). We look for a solution of (A-2) in the form $f_0(y) = y^m g_0(y)$ with $g_0(0) = 1$. An easy computation shows that

$$(mc_0(y) + c_1(y) - e_1)g_0(y) + yg_0'(y) = 0.$$

Hence $mc_0(0) + c_1(0) - e_1 = 0$, since $g_0(0) = 1$. This gives the allowed values of $e_1$:

$$e_1 = mc_0(0) + c_1(0) = \sqrt{E_1''(0)V''(0)}\left(m + \tfrac{1}{2}\right) =: \kappa_m, \quad m \in \mathbb{N}. \tag{A-11}$$

In this case $g_0(y)$ is uniquely determined by

$$(yc_0(y)\partial_y + mc_0(y) + c_1(y) - \lambda_1)g_0(y) = 0, \quad g_0(0) = 1. \tag{A-12}$$

From now on we fix $e_1 = mc_0(0) + c_1(0)$ and $f_0(y) = y^m g_0(y)$ with $g_0(0) = 1$. Let us solve Equation (A-3) for the unknown $(e_2, f_1)$. Applying Taylor's formula to the first term of the right hand side of (A-3) and using (A-10), we see that $(e_2, f_1)$ is a solution of the following equation

$$\big(yc_0(y)\partial_y + c_1(y) - e_1\big)f_1 = \sum_{j=0}^{m-1} \gamma_j y^j + y^m(k(y) + e_2 g_0(y)). \tag{A-13}$$

Put $f_1(y) = \sum_{j=0}^{m-1} \nu_j y^j + y^m g_1(y)$. Using again Taylor's formula for $c_0(y)$ and $c_1(y)$ at $y = 0$, and equating the coefficients of $y^j$ in both sides of (A-13) we get

$$(c_1(0) - e_1)\nu_0 = \gamma_0, \tag{A-14}$$

and by induction for $j \in \{1, \dots, m-1\}$,

$$\big(jc_0(0) + c_1(0) - e_1\big)\nu_j + F(\nu_0, \dots, \nu_{j-1}) = \gamma_j. \tag{A-15}$$

By (A-11), $jc_0(0) + c_1(0) - e_1 \neq 0$ for $j = 0, 1, 2, \dots, m-1$. Thus, the equations (A-14) and (A-15) uniquely determine $\nu_j$. This gives the polynomial $p_{m-1}(y) := \sum_{j=0}^{m-1} \nu_j y^j$. Similarly, comparing the coefficient of $y^m$ on both sides of (A-13), we see that $e_2$ is given by

$$e_2 = -k(0) + \sum_{j=1}^{m-1} \frac{(m-j)}{j!} c_0^{(j)}(0)\nu_{m-j} + \frac{1}{m!}\partial_y^m (c_1 p_{m-1})(0),$$

and thus, $g_1$ satisfies

$$(yc_0(y)\partial_y + mc_0(y) + c_1(y) - e_1)g_1(y) = r(y) + k(y) + e_2 g_0(y). \tag{A-16}$$

Here $r(y)$ only depends on $c_0(y)$, $c_1(y)$ and $p_{m-1}(y)$ with $r(0) + k(0) + e_2 = 0$. Combining this with (A-11) and (A-15) we see that $g_1(y)$ is uniquely determined by adding $g_1(0) = 0$.

We can now proceed analogously to construct $(e_j, f_{j-1})$ for $j = 2, 3, \dots$. This yields Theorem 5.1.

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# Duality relations among multiple series with three parameters

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We prove a duality relation among multiple series with three parameters. As special case of it, we obtain some new identities for multiple Hurwitz zeta values, which are relations among extensions of the multiple Hurwitz zeta value.

## 1. Introduction

The extended multiple zeta value (EMZV for short) is defined by the multiple series

$$\sum_{\substack{0<m_1<_{c_1}\cdots<_{c_{p-1}}m_p<\infty \\ m_i\in\mathbb{Z}}} \frac{1}{m_1^{k_1}\cdots m_p^{k_p}}, \tag{1}$$

where $1 \le p \in \mathbb{Z}$, $c_i \in \left\{\frac{1}{2}, 1\right\}$, $1 \le k_i \in \mathbb{Z}$ $(i = 1, \dots, p-1)$, $2 \le k_p \in \mathbb{Z}$, and the symbols $<_{c_i}$ $(i = 1, \dots, p-1)$ denote $<$ if $c_i = 1$ and $\le$ if $c_i = \frac{1}{2}$ (see [Ulanskiĭ 2011]). The case $c_i = 1$ $(i = 1, \dots, p-1)$ and the case $c_i = \frac{1}{2}$ $(i = 1, \dots, p-1)$ of (1) are the multiple zeta value (MZV for short) and the multiple zeta-star value (MZSV for short), respectively, which were studied by Euler [1776], Hoffman [1992], and Zagier [1994]. EMZVs were studied by Fischler and Rivoal [2016], Kawashima [2009] and Ulanskiĭ [2011]. Kawashima and Ulanskiĭ proved relations among them, and Fischler and Rivoal showed a connection between EMZVs and a solution to a Padé approximation problem involving multiple polylogarithms. Their works show that EMZVs are as useful and fruitful as MZ(S)Vs.

In [Igarashi 2015], we also studied multiple series of the extended form (1), which generalize EMZVs. In fact, we proved duality relations among them which yield numerous relations. The duality relations were proved by using a combination of the method used in [Igarashi 2012] and the calculational techniques for the Pochhammer symbol $(a)_m$ used in [Igarashi 2018]. In the present paper, we shall show that the combinative method used in [Igarashi 2015] can be applied to derive

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numerous relations among multiple series of the following type:

$$\sum_{\substack{0\le m_1<_{c_1}\cdots<_{c_{p-1}}m_p\\ m_p<_{c_p}\cdots<_{c_{p+q-1}}m_{p+q}<\infty\\ m_i\in\mathbb{Z}}} \frac{(y)_{m_p}}{(x)_{m_p}}\frac{(w)_{m_{p+q}}}{(z)_{m_{p+q}}} \times\left\{\prod_{i=1}^{p+q}\frac{1}{(m_i+x)^{a_i}(m_i+y)^{b_i}(m_i+z)^{c_i}(m_i+w)^{d_i}}\right\}, \tag{2}$$

where $1\le p\in\mathbb{Z},\ 0\le q\in\mathbb{Z},\ x,y,z,w\in\mathbb{C}\setminus\mathbb{Z}_{\le 0},\ a_i,b_i,c_i,d_i\in\mathbb{Z}$ such that

$$\sum_{i=r}^{p+q}(a_i+b_i+c_i+d_i)+\delta_r\mathrm{Re}(x-y)+\mathrm{Re}(z-w)>p+q-r+1$$

$(r=1,\ldots,p+q)$, where $\mathbb{Z}_{\le 0}:=\{0,-1,-2,\ldots\}$; $\delta_r=1$ if $r\le p$ and $\delta_r=0$ if $r>p$. (These conditions guarantee the absolute convergence of (2): see [Krattenthaler and Rivoal 2007, Lemmas 1 and 3, (3.12)].) The symbols $<_{c_i}$ $(i=1,\ldots,p+q-1)$ are the same as in (1), and the symbol $(a)_m$ denotes the Pochhammer symbol defined by $(a)_m=a(a+1)\cdots(a+m-1)$ $(1\le m\in\mathbb{Z})$ and $(a)_0=1$. The Pochhammer symbol $(a)_m$ can be expressed as $(a)_m=\Gamma(a+m)/\Gamma(a)$ by using the gamma function $\Gamma(s)$. The case $x=y=z=w=1$ and the case $x=y,\ z=w$ of (2) are EMZV and the following multiple Hurwitz zeta value (MHZV for short), respectively:

$$\sum_{\substack{0\le m_1<_{c_1}\cdots<_{c_{p-1}}m_p<\infty\\ m_i\in\mathbb{Z}}}\prod_{i=1}^{p}\frac{1}{(m_i+x)^{s_i}(m_i+z)^{t_i}}$$

$(s_i,t_i\in\mathbb{Z};\ i=1,\ldots,p)$. Since I proved the results in [Igarashi 2007], one of my research subjects has been to find the extensions of (multiple) Hurwitz zeta values which satisfy various relations as MZ(S)Vs. As I showed in [Igarashi 2018], the multiple series (2) and (3) below give such extensions. The results in [Igarashi 2018] have given me a motivation to the further research on (2) and (3). In the present paper, I prove a class of relations among (2) which gives numerous relations among extensions of MHZV: see Theorem 1 and Corollary 8 below. I remark that the partial derivatives of (2) with respect to $x$, $y$, $z$, $w$ can be expressed by $\mathbb{Z}$-linear combinations of (2). This suggests that, by partial differentiation, one relation among (2) yields further relations. The proof of Theorem 1 in the present paper gives an embodiment of this suggestion. In [Igarashi 2018] and its manuscripts (submitted in March and May 2015), I studied relations among multiple series of the types (2) and (3) by using the hypergeometric identities of [Andrews 1975, Theorem 4; Krattenthaler and Rivoal 2007, Proposition 1]. This work of mine is one of the bases of the present research.

**1.1.** ***Algebraic formulation.*** In a revised version of [Igarashi 2015], in order to describe the results, we followed the algebraic formulation for EMZVs given by Ulanskiĭ [2011] (see also [Hoffman 1997]). That formulation gave us concise descriptions of the results. For this reason, in the present paper, we also follow the formulation of Ulanskiĭ. The following formulation is parallel to that in the revised version of [Igarashi 2015].

Hereafter we assume that $m, n, p, q, r, k_i, k'_i, m_i, r_i, s_i, M_{ij} \in \mathbb{Z}$. We consider the three noncommutative variables $x_0$, $x_{\frac{1}{2}}$ and $x_1$. For these variables, we use the expressions

$$x_1 x_0^{k_1-1} x_{c_1} x_0^{k_2-1} \cdots x_{c_{p-1}} x_0^{k_p-1} = z_1(k_1) z_{c_1}(k_2) \cdots z_{c_{p-1}}(k_p) = \prod_{i=1}^{p} z_{c_{i-1}}(k_i),$$

where $p \geq 1$ and $z_{c_{i-1}}(k_i) := x_{c_{i-1}} x_0^{k_i-1}$ $\bigl(c_0 = 1,\ k_p \geq 1,\ c_i \in \{\frac{1}{2}, 1\},\ k_i \geq 1;\ i = 1, \dots, p-1\bigr)$. In case $p = 0$, we regard all these expressions as $1 \in \mathbb{Q}$. Here we consider the following set of monomials:

$$B^0 := \left\{ \prod_{i=1}^{p} z_{c_{i-1}}(k_i) \,\middle|\, p \geq 0,\ c_0 = 1,\ c_i \in \bigl\{\tfrac{1}{2}, 1\bigr\},\ k_i \geq 1\ (i = 1, \dots, p-1),\ k_p \geq 2 \right\},$$

and thus $1 \in B^0$ as the case $p = 0$. We denote by $V^0$ the $\mathbb{Q}$-vector space with the basis $B^0$. For $B^0$, we define the evaluation map $H = H_{(x,y,z)} : B^0 \to \mathbb{C}$ by $H(1; (x, y, z)) = 1$ and

$$H\bigl(z_1(k_1) z_{c_1}(k_2) \cdots z_{c_{p-1}}(k_p); (x, y, z)\bigr) = \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \frac{(y)_{m_p}}{(x)_{m_p+1}} \left\{ \prod_{i=1}^{p-1} \frac{1}{(m_i + z)^{k_i}} \right\} \frac{1}{(m_p + z)^{k_p - 1}}, \quad (3)$$

where $x, z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $y \in \mathbb{C}$, $k_p \in \mathbb{Z}$ such that $k_p + \mathrm{Re}(x - y) > 1$ and $k_i \geq 1$ $(i = 1, \dots, p-1)$. This map can be extended as a $\mathbb{Q}$-linear map onto the whole space $V^0$. Multiple series of the type (3) are studied in [Coppo 2009; Coppo and Candelpergher 2010; Emery 2004; Hasse 1930; Igarashi 2018; 2015]. See also [Igarashi 2007, Examples]. For brevity, we put $<^*_{c_i} := <_{(2c_i)^{-1}}$ $\bigl(c_i \in \bigl\{\frac{1}{2}, 1\bigr\}\bigr)$. This is the inversion of $<_{c_i}$, namely

$$<^*_{c_i} = \begin{cases} \leq & \text{if } c_i = 1, \\ < & \text{if } c_i = \frac{1}{2}. \end{cases}$$

We also define the evaluation map

$$H^*_{(\{r_i\}_{i=1}^n; \{s_i\}_{i=1}^q)} = H^*_{(\{r_i\}_{i=1}^n; \{s_i\}_{i=1}^q), (x,y,z)} : B^0 \to \mathbb{C}$$

by $H^*_{(\{r_i\}_{i=1}^n;\{s_i\}_{i=1}^q)}(1;(x,y,z))=1$ and

$$H^*_{(\{r_i\}_{i=1}^n;\{s_i\}_{i=1}^q)}\big(z_1(k_1)z_{c_1}(k_2)\cdots z_{c_{q-1}}(k_q);(x,y,z)\big)$$
$$=\sum_{\substack{0\le M_{11}<\cdots<M_{1n}<m_1\\ m_1<_{c_1}M_{21}\le\cdots\le M_{2s_2}<^*_{c_2}m_2\\ \vdots\\ m_{i-1}<_{c_{i-1}}M_{i1}\le\cdots\le M_{is_i}<^*_{c_i}m_i\\ \vdots\\ m_{q-1}<_{c_{q-1}}M_{q1}\le\cdots\le M_{qs_q}<^*_{c_q}m_q<\infty}}\frac{(x)_{m_1}}{m_1!}\frac{(y)_{m_q+1}}{(z)_{m_q+1}}$$
$$\times\left(\prod_{i=1}^n\frac{1}{(M_{1i}+x)(M_{1i}+z)^{r_i}}\right)\frac{1}{(m_1+y)^{k_1}(m_1+z)^{s_1}}$$
$$\times\left\{\prod_{i=2}^q\left(\prod_{j=1}^{s_i}\frac{1}{M_{ij}+z}\right)\frac{1}{(m_i+y)^{k_i}}\right\},\quad (4)$$

where $n,r_i\ge0$ $(i=1,\ldots,n)$; $q\ge1$, $s_i\ge0$ $(i=1,\ldots,q)$, $c_q=1$; $x,y,z\in\mathbb{C}\setminus\mathbb{Z}_{\le0}$ such that $\mathrm{Re}(2-x-y+z)$, $\mathrm{Re}(1-y+z)>0$. We regard $\{a_i\}_{i=m}^{m-1}$ as the empty set $\varnothing$. In case $s_i=0$, we regard the inequalities $m_{i-1}<_{c_{i-1}}M_{i1}\le\cdots\le M_{is_i}<^*_{c_i}m_i$ under the summation sign in (4) as $m_{i-1}<_{c_{i-1}}m_i$. For example, the case $n=0$ and $s_i=0$ $(i=1,\ldots,q)$ of (4) becomes

$$H^*_{(\varnothing;\{0\}_{i=1}^q)}\big(z_1(k_1)z_{c_1}(k_2)\cdots z_{c_{q-1}}(k_q);(x,y,z)\big)$$
$$=\sum_{0\le m_1<_{c_1}\cdots<_{c_{q-1}}m_q<\infty}\frac{(x)_{m_1}}{m_1!}\frac{(y)_{m_q+1}}{(z)_{m_q+1}}\left\{\prod_{i=1}^q\frac{1}{(m_i+y)^{k_i}}\right\}$$
$$=:H^*\big(z_1(k_1)z_{c_1}(k_2)\cdots z_{c_{q-1}}(k_q);(x,y,z)\big).$$

This map can also be extended as a $\mathbb{Q}$-linear map onto the whole space $V^0$.

We define the map $\sigma_r^b:B^0\to V^0$ by $\sigma_r^b(1)=1$ and

$$\sigma_r^b\big(z_1(k_1)z_{c_1}(k_2)\cdots z_{c_{p-1}}(k_p)\big)$$
$$=\sum_{\substack{r_1+\cdots+r_p=r\\ r_i\ge0}}\left\{\prod_{i=1}^{p-1}\binom{k_i+r_i-1}{r_i}\right\}\binom{k_p+r_p-2}{r_p}\prod_{i=1}^p z_{c_{i-1}}(k_i+r_i),$$

where $r\ge0$. This map corresponds to the partial derivative of (3) with respect to $z$. Following [Ulanskiĭ 2011, p. 106], we also define the dual map $\tau:B^0\to B^0$ by $\tau(1)=1$ and $\tau(x_1x_{e_1}\cdots x_{e_{n-1}}x_0)=x_1x_{1-e_{n-1}}\cdots x_{1-e_1}x_0$, where $n\ge1$ and $e_i\in\{0,\frac12,1\}$ $(i=1,\ldots,n-1)$. We call $\tau(v)$ the dual of $v$. By the definition, it is easy to see that $\tau^2(v)=v$. The dual $\tau(v)$ can also be written as $\tau(v)=\prod_{i=1}^q z_{c'_{i-1}}(k'_i)$, where $q \geq 0,\ c'_0 = 1,\ c'_i \in \left\{\frac{1}{2}, 1\right\},\ k'_i \geq 1\ (i = 1, \dots, q-1), k'_q \geq 2$. Hereafter we assume this expression for $\tau(v)$. The maps $\sigma_r^b$ and $\tau$ can be extended as $\mathbb{Q}$-linear maps from the whole space $V^0$ to $V^0$.

**1.2.** ***Main theorem and its examples.*** We define the symbol $\varepsilon(c_i)$ by

$$\varepsilon(c_i) = \begin{cases} 1 & \text{if } c_i = 1, \\ 0 & \text{if } c_i \neq 1. \end{cases}$$

For brevity, we put

$$(\boldsymbol{r}^n, \boldsymbol{s}^q, \varepsilon(c'_1)) := \left(\{r_i\}_{i=1}^n; \varepsilon(c'_1)s_1, \{s_i\}_{i=2}^q\right),$$
$$(\alpha, \beta, \gamma)^\tau := \left(1 - \beta + \gamma, \alpha - \beta + 1, \alpha - \beta + \gamma\right).$$

The main theorem is as follows:

**Theorem 1.** *Let $v \in B^0$, and let $\tau(v)$ be its dual. Then the identity*

$$H(\sigma_r^b(v); (\alpha, \beta, \gamma)) = \sum_{n=0}^{r} (1-\alpha)^n \sum_{\substack{\sum_{i=0}^n r_i + \varepsilon(c'_1)s_1 + \sum_{i=2}^q s_i = r \\ r_i \geq 1\ (i=1,\dots,n) \\ r_0, s_i \geq 0\ (i=1,\dots,q)}} G^{(r_0)}(\alpha, \beta, \gamma) H^*_{(\boldsymbol{r}^n, \boldsymbol{s}^q, \varepsilon(c'_1))}(\tau(v); (\alpha, \beta, \gamma)^\tau) \tag{5}$$

*holds for all $r \geq 0,\ \alpha, \beta, \gamma \in \mathbb{C}$ such that* $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta)$, $\mathrm{Re}(\gamma)$, $\mathrm{Re}(1 - \beta + \gamma)$, $\mathrm{Re}(\alpha - \beta + 1)$, $\mathrm{Re}(\alpha - \beta + \gamma) > 0$, *where $c'_1$ and $q$ are those of the dual $\tau(v) = \prod_{i=1}^q z_{c'_{i-1}}(k'_i)$.* (For the definition of $G^{(r_0)}(\alpha, \beta, \gamma)$, see Section 2.)

The identity (5) gives numerous relations among (2) which contain identities for MHZVs and EMZVs as special cases. For example, we consider the monomial $v_0 := \left\{\prod_{i=1}^{p-1} z_{c_{i-1}}(1)\right\} z_{c_{p-1}}(2)$ $\left(c_0 = 1,\ c_i \in \left\{\frac{1}{2}, 1\right\};\ i = 1, \dots, p-1\right)$. This has the dual

$$\tau(v_0) = x_1 x_{1-c_{p-1}} \cdots x_{1-c_1} x_0 = z_1(k'_1) \left\{\prod_{i=2}^{q} z_{\frac{1}{2}}(k'_i)\right\},$$

where $q, k'_i \geq 1\ (i = 1, \dots, q-1),\ k'_q \geq 2$. Taking $v = v_0$ in (5), we get the identity

$$\sum_{\substack{r_1 + \cdots + r_p = r \\ r_i \geq 0}} H\left(\left\{\prod_{i=1}^{p-1} z_{c_{i-1}}(1 + r_i)\right\} z_{c_{p-1}}(2 + r_p); (\alpha, \beta, \gamma)\right)$$
$$= \sum_{n=0}^{r} (1-\alpha)^n \sum_{\substack{\sum_{i=0}^n r_i + \varepsilon(q)s_1 + \sum_{i=2}^q s_i = r \\ r_i \geq 1\ (i=1,\dots,n) \\ r_0, s_i \geq 0\ (i=1,\dots,q)}} G^{(r_0)}(\alpha, \beta, \gamma)$$
$$\times H^*_{(\boldsymbol{r}^n, \boldsymbol{s}^q, \varepsilon(q))}\left(z_1(k'_1)\left\{\prod_{i=2}^{q} z_{\frac{1}{2}}(k'_i)\right\}; (\alpha, \beta, \gamma)^\tau\right). \tag{6}$$

The sum on the left-hand side of (6) has the same form as that for the sum formula for MZVs: see (8) below. By the definition of $G^{(r_0)}(\alpha, \beta, \gamma)$ in Section 2, we see that $G^{(r_0)}(\alpha, \alpha, \alpha) = 0$ $(r_0 \geq 1)$ and $G^{(0)}(\alpha, \alpha, \alpha) = 1$. Therefore, taking $\alpha = \beta = \gamma$ and $c_i = 1$ $(i = 1, \ldots, p-1)$ in (6), we get the following sum formula for MHZVs:

$$\sum_{\substack{r_1+\cdots+r_p=r\\ r_i\geq 0}} \zeta\Bigg(\Bigg\{\prod_{i=1}^{p-1} z_1(1+r_i)\Bigg\} z_1(2+r_p); \alpha\Bigg) = \sum_{n=0}^{r}(1-\alpha)^n \sum_{\substack{\sum_{i=1}^n r_i+s_1=r\\ r_i\geq 1\ (i=1,\ldots,n)\\ s_1\geq 0}} H^*_{(\{r_i\}_{i=1}^n; s_1)}(z_1(p+1); (1, 1, \alpha)) \quad (7)$$

for $p \geq 1$, $r \geq 0$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. Here

$$\zeta(v; \alpha) := H(v; (\alpha, \alpha, \alpha)) = \sum_{0\leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \prod_{i=1}^{p} \frac{1}{(m_i+\alpha)^{k_i}}$$

$\big(v = \prod_{i=1}^p z_{c_{i-1}}(k_i) \in B^0\big)$. If $v = \prod_{i=1}^p z_1(k_i)$, this becomes the usual MHZV, i.e., the case $<_{c_i} = <$ $(i = 1, \ldots, p-1)$ of $\zeta(v; \alpha)$. The identity (7) was shown in [Igarashi 2018, **(R3)**] as an explicit expression for the identity (2) in [Igarashi 2007]. The identity (2) in [Igarashi 2007] is one of the bases of my research on MHZVs. For other identities for MHZVs which can be derived from Theorem 1, see Corollary 8 below. Finally, taking $\alpha = 1$ in (7), we get the sum formula for MZVs, which may be one of the basic relations among MZVs:

$$\sum_{\substack{r_1+\cdots+r_p=r\\ r_i\geq 0}} \zeta\Bigg(\Bigg\{\prod_{i=1}^{p-1} z_1(1+r_i)\Bigg\} z_1(2+r_p)\Bigg) = \zeta(z_1(p+1+r)) \quad (8)$$

for $p \geq 1$ and $r \geq 0$ ([Granville 1997], Zagier (unpublished)). Here $\zeta(v) := \zeta(v; 1)$ $(v \in B^0)$, which is EMZV. If $v = \prod_{i=1}^p z_1(k_i)$, this becomes the usual MZV. The case $\alpha = \beta = \gamma$ and the case $\alpha = \beta = \gamma = 1$ of (6) are extensions of (7) and (8), respectively. For another example of Theorem 1, taking $\alpha = \beta = \gamma = 1$ in (5), we can get the following relation among EMZVs:

$$\sum_{\substack{r_1+\cdots+r_p=r\\ r_i\geq 0}} \Bigg\{\prod_{i=1}^{p-1}\binom{k_i+r_i-1}{r_i}\Bigg\}\binom{k_p+r_p-2}{r_p}\zeta\Bigg(\prod_{i=1}^{p} z_{c_{i-1}}(k_i+r_i)\Bigg)$$
$$= \sum_{\substack{s_1+\cdots+s_q=r\\ s_i\geq 0}} \sum_{\substack{0<m_1<_{c_1} M_{21}\leq\cdots\leq M_{2s_2} <^*_{c_2} m_2\\ \vdots \\ m_{i-1}<_{c_{i-1}} M_{i1}\leq\cdots\leq M_{is_i} <^*_{c_i} m_i\\ \vdots\\ m_{q-1}<_{c_{q-1}} M_{q1}\leq\cdots\leq M_{qs_q} <^*_{c_q} m_q<\infty}} \frac{1}{m_1^{k'_1+\varepsilon(c'_1)s_1}}\Bigg\{\prod_{i=2}^{q}\Bigg(\prod_{j=1}^{s_i}\frac{1}{M_{ij}}\Bigg)\frac{1}{m_i^{k'_i}}\Bigg\} \quad (9)$$

for $r \geq 0$. The identity (9) gives numerous relations among EMZVs. In fact, in [Igarashi 2015], I showed that the identity (9) is equivalent to a certain new extension of Ohno's relation for MZVs [Ohno 1999, Theorem 1]: Ohno's relation is a large class of relations among MZVs. By the above examples, we see that Theorem 1 contains various interesting relations for multiple series. See also Remark 11 below.

We explain the idea of the proof of Theorem 1. It is proved by using a symmetry of (3) with respect to the parameters $x$, $y$ and $z$. Indeed we derive Theorem 1 from the duality formula (12) below, which has a symmetry with respect to the parameters $\alpha$, $\beta$ and $X$, by partial differentiation. The symmetry of (12) can be found by applying a change of variables to an iterated integral representation of (3) (see (10) below and the proof of (12)). It is important that the change of variables also brings about a change of the positions of the parameters of (3) (compare, e.g., the positions of $X$ on both sides of (12)). Consequently partial differential operators act on each side of (12) in different ways, and this gives the identity in Theorem 1. (See also [Igarashi 2012].) For the factor $G^{(r_0)}(\alpha, \beta, \gamma)$ in Theorem 1, which is a partial differential coefficient of the gamma factor in (12), we prove its two explicit expressions (see Lemma 6 below). Those expressions can be used for deriving various relations among (2) from Theorem 1 (see Remarks 7 and 11 below).

I remark that Theorem 1 is a variation of my former results written in [Igarashi 2015], but it yields some new identities for MHZVs, which are different from those in [Igarashi 2015]: see Corollary 8 and Remark 10 below. For example, taking $\alpha = \beta = \gamma$ in Theorem 1, we get the following new identity for MHZVs:

$$\zeta(\sigma_r^b(v); \alpha) = \sum_{n=0}^{r} (1-\alpha)^n \sum_{\substack{\sum_{i=1}^{n} r_i + \varepsilon(c_1')s_1 + \sum_{i=2}^{q} s_i = r \\ r_i \geq 1\ (i=1,\dots,n) \\ s_i \geq 0\ (i=1,\dots,q)}} H^*_{(\boldsymbol{r}^n, \boldsymbol{s}^q, \varepsilon(c_1'))}(\tau(v); (1, 1, \alpha))$$

for all $r \geq 0$, $\alpha \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > 0$, where $\zeta(\sigma_r^b(v); \alpha) := H(\sigma_r^b(v); (\alpha, \alpha, \alpha))$. This identity gives an extension of (7). See also Corollary 8(i) and Remark 10 below.

## 2. Proof of Theorem 1

In this section, we prove Theorem 1. The proof is a variation of the proofs of the results in [Igarashi 2015], therefore several proofs in this section overlap those in [Igarashi 2015] (see also [Igarashi 2012, Section 2]). However we shall not omit the details for the sake of keeping the present paper self-contained. The calculational techniques for the Pochhammer symbol $(a)_m$ used in the present paper are based on those used in [Igarashi 2018].

We first prove some lemmas. We define the symbol $\omega_{e_i}(t)$ $\left(e_i \in \left\{0, \frac{1}{2}, 1\right\}\right)$ by

$$\omega_0(t) = \frac{1}{t}, \quad \omega_{\frac{1}{2}}(t) = \frac{1}{t(1-t)}, \quad \omega_1(t) = \frac{1}{1-t}.$$

The multiple series (3) has the following iterated integral representation, which is one of the most important ingredients of the proof of Theorem 1:

**Lemma 2.** *Let* $n \geq 1$ *and* $e_i \in \left\{0, \frac{1}{2}, 1\right\}$ $(i = 1, \ldots, n-1)$. *Then the identity*

$$\begin{aligned} &H(x_1 x_{e_1} \cdots x_{e_{n-1}} x_0; (\alpha, \beta, X)) \\ &= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta+1)} \\ &\quad \times \underset{0<t_0<\cdots<t_n<1}{\int\cdots\int} t_0^{X-1} \omega_1(t_0) \left\{\prod_{i=1}^{n-1} \omega_{e_i}(t_i)\right\} \omega_0(t_n) t_n^{\beta-X} (1-t_n)^{\alpha-\beta}\, \mathrm{d}t_0 \cdots \mathrm{d}t_n \end{aligned} \tag{10}$$

*holds for all* $\alpha, \beta, X \in \mathbb{C}$ *such that* $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(X), \mathrm{Re}(\alpha-\beta+1) > 0$, *where* $\Gamma(s)$ *is the gamma function.*

*Proof.* Using $\omega_0(t)$ and $\omega_{c_i}(t)$ $\left(c_i \in \left\{\frac{1}{2}, 1\right\}\right)$, we can rewrite the integrand as

$$\omega_1 \left\{\prod_{i=1}^{n-1} \omega_{e_i}\right\} \omega_0 = \prod_{i=1}^{p} \omega_{c_{i-1}} \omega_0^{k_i - 1},$$

where $p \geq 1$, $c_0 = 1$, $c_i \in \{\frac{1}{2}, 1\}$, $k_i \geq 1$ $(i = 1, \ldots, p-1)$, $k_p \geq 2$. This gives the following expression for the iterated integral in (10), which we denote by $I$:

$$I = \underset{\substack{0<t_{11}<\cdots<t_{1k_1} \\ \vdots \\ <t_{i1}<\cdots<t_{ik_i} \\ \vdots \\ <t_{p1}<\cdots<t_{pk_p}<1}}{\int\cdots\int} t_{11}^{X-1} \left\{\prod_{i=1}^{p} \omega_{c_{i-1}}(t_{i1}) \left(\prod_{j=2}^{k_i} \omega_0(t_{ij})\right)\right\} t_{pk_p}^{\beta-X} (1-t_{pk_p})^{\alpha-\beta} \times \left(\prod_{i=1}^{p} \prod_{j=1}^{k_i} \mathrm{d}t_{ij}\right). \tag{11}$$

Applying the expansions $(1-t_{i1})^{-1} = \sum_{m=0}^{\infty} t_{i1}^m$ $(i = 1, \ldots, p)$ to the integrand in (11) and integrating term by term, we get the identities

$$\begin{aligned} I &= \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \left\{\prod_{i=1}^{p-1} \frac{1}{(m_i+X)^{k_i}}\right\} \frac{1}{(m_p+X)^{k_p-1}} \int_0^1 (1-t_{pk_p})^{\alpha-\beta} t_{pk_p}^{\beta+m_p-1}\, \mathrm{d}t_{pk_p} \\ &= \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \left\{\prod_{i=1}^{p-1} \frac{1}{(m_i+X)^{k_i}}\right\} \frac{1}{(m_p+X)^{k_p-1}} \frac{\Gamma(\alpha-\beta+1)\Gamma(\beta+m_p)}{\Gamma(\alpha+m_p+1)} \\ &= \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \left\{\prod_{i=1}^{p-1} \frac{1}{(m_i+X)^{k_i}}\right\} \frac{1}{(m_p+X)^{k_p-1}} \frac{(\beta)_{m_p}}{(\alpha)_{m_p+1}} \frac{\Gamma(\alpha-\beta+1)\Gamma(\beta)}{\Gamma(\alpha)}, \end{aligned}$$

so

$$I = \frac{\Gamma(\alpha-\beta+1)\Gamma(\beta)}{\Gamma(\alpha)} H\bigl(x_1 x_0^{k_1-1} x_{c_1} x_0^{k_2-1} \cdots x_{c_{p-1}} x_0^{k_p-1}; (\alpha, \beta, X)\bigr)$$

for $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(X), \mathrm{Re}(\alpha-\beta+1) > 0$. We can also get the following expression for the above monomial:

$$x_1 x_0^{k_1-1} x_{c_1} x_0^{k_2-1} \cdots x_{c_{p-1}} x_0^{k_p-1} = x_1 x_{e_1} \cdots x_{e_{n-1}} x_0,$$

where $e_i \in \bigl\{0, \frac{1}{2}, 1\bigr\}$ $(i = 1, \ldots, n-1)$, therefore we get (10). □

Lemma 2 gives the following duality formula for (3):

**Lemma 3** (Duality formula). *Let $v \in B^0$, and let $\tau(v)$ be its dual. Then the identity*

$$H(v; (\alpha, \beta, X)) = \frac{\Gamma(\alpha)}{\Gamma(\beta)} \frac{\Gamma(X)}{\Gamma(\alpha-\beta+X)} H^*(\tau(v); (\alpha, \beta, X)^\tau) \tag{12}$$

*holds for all $\alpha, \beta, X \in \mathbb{C}$ such that*

$$\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(X), \mathrm{Re}(\alpha-\beta+1), \mathrm{Re}(\alpha-\beta+X) > 0.$$

*Proof.* The assertion follows by applying the change of variables $t_i = 1 - u_{n-i}$ $(i = 0, 1, \ldots, n)$ to the iterated integral on the right-hand side of (10). (This change of variables was used in [Zagier 1994, p. 510] for MZVs.) □

**Remark 4.** The cases $\alpha = \beta = X = 1$ of (10) and (12) are Ulanskiĭ's results on EMZVs [Ulanskiĭ 2011, Corollary 2 and Theorem 1]. The case $\alpha = \beta$ and $v = z_1(1)^{p-1} z_1(2)$ $(p \geq 1)$ of (12) was used in [Igarashi 2007] to prove a sum formula for MHZVs:

$$\sum_{0 \leq m_1 < \cdots < m_p < \infty} \Biggl(\prod_{i=1}^{p} \frac{1}{m_i + X}\Biggr) \frac{1}{m_p + \alpha} = \sum_{m=0}^{\infty} \frac{(1-\alpha+X)_m}{(X)_{m+1}} \frac{1}{(m+1)^p}.$$

To use the partial differentiation, we need the following lemma:

**Lemma 5.** *Let $v = \prod_{i=1}^{q} z_{c_{i-1}}(k_i) \in B^0$, and let $x, y, z \in \mathbb{C}$ with* $\mathrm{Re}(x)$, $\mathrm{Re}(y)$, $\mathrm{Re}(z) > 0$. *Then the multiple series*

$$H^*(v; (x+Y, y, z+Y)) = \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{q-1}} m_q < \infty} \frac{(x+Y)_{m_1}}{m_1!} \frac{(y)_{m_q+1}}{(z+Y)_{m_q+1}} \Biggl\{\prod_{i=1}^{q} \frac{1}{(m_i+y)^{k_i}}\Biggr\} \tag{13}$$

*converges uniformly in $\{Y \in \mathbb{C} \mid |Y| \leq \varepsilon\}$, where $\varepsilon \in \mathbb{R}$ such that*

$$0 < \varepsilon < \min\bigl\{\mathrm{Re}(x), \mathrm{Re}(z), 1 - \mathrm{Re}(x+y-z)/2, 1 - \mathrm{Re}(y-z)\bigr\}.$$

*Proof.* By using the expression $(a)_m = \Gamma(a+m)/\Gamma(a)$, the Pochhammer symbol $(a)_m$ can be estimated as follows:

$$\begin{aligned}
|(a)_m| &= \left|\frac{\Gamma(a+m)}{\Gamma(a)}\right| = \left|\frac{1}{\Gamma(a)}\int_0^\infty e^{-t}t^{a+m-1}\,\mathrm{d}t\right| \\
&\le \frac{1}{|\Gamma(a)|}\int_0^\infty e^{-t}t^{\mathrm{Re}(a)+m-1}\,\mathrm{d}t = \frac{\Gamma(\mathrm{Re}(a)+m)}{|\Gamma(a)|} = \frac{\Gamma(\mathrm{Re}(a))}{|\Gamma(a)|}(\mathrm{Re}(a))_m \\
&\le C_1\frac{\Gamma(\mathrm{Re}(a))}{|\Gamma(a)|}(m+1)^{\mathrm{Re}(a)}
\end{aligned}$$

for $m \ge 0,\ a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$, where $C_1$ is a positive constant which does not depend on $m$. (The last inequality above follows by applying Stirling's formula for $\Gamma(s)$ to $(\mathrm{Re}(a))_m$.) Using this estimate and $|(a)_m| \ge (\mathrm{Re}(a))_m > 0$ for $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$, we get the estimate

$$\begin{aligned}
&\left|\frac{(x+Y)_{m_1}}{m_1!}\frac{(y)_{m_q+1}}{(z+Y)_{m_q+1}}\right| \\
&\qquad \le \frac{\Gamma(\mathrm{Re}(x+Y))\Gamma(\mathrm{Re}(y))}{|\Gamma(x+Y)\Gamma(y)|}\frac{(\mathrm{Re}(x+Y))_{m_1}}{m_1!}\frac{(\mathrm{Re}(y))_{m_q+1}}{(\mathrm{Re}(z+Y))_{m_q+1}} \\
&\qquad \le C_2\frac{(\mathrm{Re}(x)+\varepsilon)_{m_1}}{m_1!}\frac{(\mathrm{Re}(y))_{m_q+1}}{(\mathrm{Re}(z)-\varepsilon)_{m_q+1}} \\
&\qquad \le C_3\frac{1}{(m_1+1)^{1-\mathrm{Re}(x)-\varepsilon}}\frac{1}{(m_q+1)^{-\mathrm{Re}(y-z)-\varepsilon}}
\end{aligned}$$

for all $Y \in \{Y \in \mathbb{C} \mid |Y| \le \varepsilon\}$, where $C_2$ and $C_3$ are positive constants which do not depend on $Y$. By this estimate, we can take the absolutely convergent multiple series

$$\sum_{0\le m_1<_{c_1}\cdots<_{c_{q-1}}m_q<\infty}\frac{1}{(m_1+1)^{1-\mathrm{Re}(x)-\varepsilon}}\frac{1}{(m_q+1)^{-\mathrm{Re}(y-z)-\varepsilon}}\left\{\prod_{i=1}^{q}\frac{1}{(m_i+\mathrm{Re}(y))^{k_i}}\right\}$$

as a majorant of (13), and this implies the uniform convergence of (13). For conditions on the absolute convergence of the above majorant, see those for (2). □

For the gamma factor in (12), we put

$$G^{(n)}(\alpha,\beta,\gamma) := \frac{\Gamma(\alpha)}{\Gamma(\beta)}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial X^n}\left(\frac{\Gamma(X)}{\Gamma(\alpha-\beta+X)}\right)\bigg|_{X=\gamma},$$

where $n \ge 0,\ \alpha,\beta,\gamma \in \mathbb{C}$ such that $\alpha,\gamma,\alpha-\beta+\gamma \notin \mathbb{Z}_{\le 0}$, which appears in Theorem 1. We prove two explicit expressions for $G^{(n)}(\alpha,\beta,\gamma)$. For brevity, we also put

$$\begin{aligned}P^{(n)}(z,w) &:= \frac{1}{n!}\frac{\partial^n}{\partial Y^n}\big(\psi(z-Y)-\psi(w-Y)\big)\Big|_{Y=0}\\ &= \begin{cases}(z-w)\sum_{m=0}^{\infty}(m+z)^{-1}(m+w)^{-1} & \text{if } n=0,\\ -\sum_{m=0}^{\infty}(m+z)^{-n-1}+\sum_{m=0}^{\infty}(m+w)^{-n-1} & \text{if } n\geq 1,\end{cases}\end{aligned}$$

where $z, w \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ and $\psi(z) := \Gamma'(z)/\Gamma(z)$ is the digamma function. This identity is an immediate consequence of the following two properties of the digamma function:

$$\psi(z)-\psi(w) = (z-w)\sum_{m=0}^{\infty}(m+z)^{-1}(m+w)^{-1}$$

$(z, w \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0})$ and $\psi^{(n)}(z) = (-1)^{n+1}n!\sum_{m=0}^{\infty}(m+z)^{-n-1}$ $(n\geq 1)$, where $\psi^{(n)}(z)$ is the $n$-th derivative of $\psi(z)$. (For the property of the digamma function, see, e.g., [Srivastava and Choi 2001, Section 1.2].) Then we can prove the following:

**Lemma 6.** *Let $n \geq 1$. Then $G^{(n)}(\alpha,\beta,\gamma)$ has the following two expressions*:

(i)

$$\begin{aligned}&G^{(n)}(\alpha,\beta,\gamma)\\ &\quad= \frac{\Gamma(\alpha)}{\Gamma(\beta)}\frac{\Gamma(\gamma)}{\Gamma(\alpha-\beta+\gamma)}\sum_{i=1}^{n}\sum_{0=l_0<l_1<\cdots<l_{i-1}<l_i=n}\prod_{j=1}^{i} l_j^{-1}P^{(l_j-l_{j-1}-1)}(\alpha-\beta+\gamma,\gamma) \qquad (14)\end{aligned}$$

*for all $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha, \gamma, \alpha-\beta+\gamma \notin \mathbb{Z}_{\leq 0}$.*

(ii)

$$G^{(n)}(\alpha,\beta,\gamma) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\sum_{m=0}^{\infty}\frac{(\beta-\alpha+1)_m}{m!}\frac{1}{(m+\gamma)^{n+1}} \qquad (15)$$

*for all $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha \notin \mathbb{Z}_{\leq 0}$,* $\mathrm{Re}(\gamma), \mathrm{Re}(\alpha-\beta+1) > 0$: *this single series is $H^*\big(z_1(n+1); (\beta-\alpha+1,\gamma,\gamma)\big)$.*

*Proof.* The expression (14) can be proved in the same way as in [Igarashi 2018, Proof of Lemma 2.18]. The expression (15) can be proved as follows: The quotient $\Gamma(X)/\Gamma(\alpha-\beta+X)$ can be expressed as

$$\begin{aligned}\frac{\Gamma(X)}{\Gamma(\alpha-\beta+X)} &= \frac{\alpha-\beta+X}{\Gamma(\alpha-\beta+1)}\frac{\Gamma(X)\Gamma(\alpha-\beta+1)}{\Gamma(\alpha-\beta+X+1)} = \frac{\alpha-\beta+X}{\Gamma(\alpha-\beta+1)}\int_0^1 t^{X-1}(1-t)^{\alpha-\beta}\,\mathrm{d}t\\ &= \frac{\alpha-\beta+X}{\Gamma(\alpha-\beta+1)}\int_0^1\sum_{m=0}^{\infty}\frac{(\beta-\alpha)_m}{m!}t^{m+X-1}\,\mathrm{d}t\\ &= \frac{\alpha-\beta+X}{\Gamma(\alpha-\beta+1)}\sum_{m=0}^{\infty}\frac{(\beta-\alpha)_m}{m!}\frac{1}{m+X}\end{aligned}$$

for $\mathrm{Re}(X), \mathrm{Re}(\alpha-\beta+1)>0$. Using the last expression above, which is a special case of Gauss' formula for the hypergeometric series ${}_2F_1(a, b; c; 1)$, we get the identities

$$\begin{aligned}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial X^n}\left(\frac{\Gamma(X)}{\Gamma(\alpha-\beta+X)}\right)\Big|_{X=\gamma} &= \frac{1}{\Gamma(\alpha-\beta+1)}\sum_{m=0}^{\infty}\frac{(\beta-\alpha)_m}{m!}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial X^n}\left(\frac{\alpha-\beta+X}{m+X}\right)\Big|_{X=\gamma}\\ &= \frac{1}{\Gamma(\alpha-\beta+1)}\sum_{m=0}^{\infty}\frac{(\beta-\alpha)_m}{m!}\left(\frac{\alpha-\beta+\gamma}{(m+\gamma)^{n+1}}-\frac{1}{(m+\gamma)^n}\right)\\ &= \frac{1}{\Gamma(\alpha-\beta)}\sum_{m=0}^{\infty}\frac{(\beta-\alpha+1)_m}{m!}\frac{1}{(m+\gamma)^{n+1}}\end{aligned}$$

for $n\geq 1$, $\mathrm{Re}(\gamma), \mathrm{Re}(\alpha-\beta+1)>0$. Thus we get (15). □

**Remark 7.** Both (14) and (15) can be applied to (5), in particular, to evaluating (5) in terms of MHZVs: see Corollary 8(ii) below. The expression (15) is simpler than (14).

*Proof of Theorem 1.* Let

$$v=\prod_{i=1}^{p} z_{c_{i-1}}(k_i)\in B^0 \quad \text{with the dual} \quad \tau(v)=\prod_{i=1}^{q} z_{c'_{i-1}}(k'_i).$$

For brevity, we put $a_0 := 1-\beta$ and $b_0 := \alpha-\beta$. Differentiating the left-hand side of (12) $r$ times with respect to $X$ at $X=\gamma$ $(\mathrm{Re}(\gamma)>0)$ and by the Leibniz rule, we get the left-hand side of (5).

The right-hand side of (5) can be proved as follows: We remark that the identity

$$(b_0+X)_{m_1+\varepsilon(c'_1)}=(b_0+X)_{m_1}(m_1+b_0+X)^{\varepsilon(c'_1)}$$

holds, because $\varepsilon(c_i)\in\{0, 1\}$. Using this identity, we get the identities

$$\begin{aligned}\frac{(a_0+X)_{m_1}}{(b_0+X)_{m_q+1}} &= \frac{(a_0+X)_{m_1}}{(b_0+X)_{m_1}}\frac{(b_0+X)_{m_1}}{(b_0+X)_{m_q+1}}\\ &= \frac{(a_0+X)_{m_1}}{(b_0+X)_{m_1}}\frac{1}{(m_1+b_0+X)^{\varepsilon(c'_1)}}\left(\prod_{i=2}^{q}\frac{(b_0+X)_{m_{i-1}+\varepsilon(c'_{i-1})}}{(b_0+X)_{m_i+\varepsilon(c'_i)}}\right)\end{aligned} \tag{16}$$

for $m_1, \ldots, m_q\in\mathbb{Z}$ such that $0\leq m_1 <_{c'_1} \cdots <_{c'_{q-1}} m_q$, where $c'_q=1$. We calculate the partial differential coefficients of each factor on the right-hand side of (16). By direct calculation, we get the identities

$$
\begin{aligned}
\frac{(b_0+\gamma)_{m_1}}{(a_0+\gamma)_{m_1}}\frac{(a_0+X)_{m_1}}{(b_0+X)_{m_1}}
&=\prod_{n=0}^{m_1-1}\frac{(b_0+\gamma+n)(a_0+X+n)}{(a_0+\gamma+n)(b_0+X+n)}\\
&=\prod_{n=0}^{m_1-1}\left(1+\frac{(1-\alpha)(\gamma-X)}{(a_0+\gamma+n)(b_0+X+n)}\right)\\
&=\sum_{n=0}^{m_1}\sum_{0\le M_{11}<\cdots<M_{1n}<m_1}\prod_{i=1}^{n}\frac{(1-\alpha)(\gamma-X)}{(a_0+\gamma+M_{1i})(b_0+X+M_{1i})} \qquad (17)
\end{aligned}
$$

for $m_1\ge 0$. Using (17), we can get the identity

$$
\begin{aligned}
&\frac{(-1)^{s_0}}{s_0!}\frac{\partial^{s_0}}{\partial X^{s_0}}\left(\frac{(a_0+X)_{m_1}}{(b_0+X)_{m_1}}\right)\Bigg|_{X=\gamma}\\
&\quad=\frac{(a_0+\gamma)_{m_1}}{(b_0+\gamma)_{m_1}}\sum_{n=0}^{m_1}\sum_{\substack{\sum_{i=1}^{n} r_i=s_0\\ r_i\ge 1}}(1-\alpha)^n\\
&\qquad\times\sum_{0\le M_{11}<\cdots<M_{1n}<m_1}\prod_{i=1}^{n}\frac{1}{(a_0+\gamma+M_{1i})(b_0+\gamma+M_{1i})^{r_i}} \qquad (18)
\end{aligned}
$$

for $m_1, s_0\ge 0$. The partial differential coefficients of other factors can be calculated as follows:

$$
\begin{aligned}
&\frac{(-1)^{s_i}}{s_i!}\frac{\partial^{s_i}}{\partial X^{s_i}}\left(\frac{(b_0+X)_{m_{i-1}+\varepsilon(c'_{i-1})}}{(b_0+X)_{m_i+\varepsilon(c'_i)}}\right)\Bigg|_{X=\gamma}\\
&\quad=\frac{(b_0+\gamma)_{m_{i-1}+\varepsilon(c'_{i-1})}}{(b_0+\gamma)_{m_i+\varepsilon(c'_i)}}\sum_{m_{i-1}+\varepsilon(c'_{i-1})\le M_{i1}\le\cdots\le M_{is_i}<m_i+\varepsilon(c'_i)}\prod_{j=1}^{s_i}\frac{1}{M_{ij}+b_0+\gamma}\\
&\quad=\frac{(b_0+\gamma)_{m_{i-1}+\varepsilon(c'_{i-1})}}{(b_0+\gamma)_{m_i+\varepsilon(c'_i)}}\sum_{m_{i-1}<_{c'_{i-1}}M_{i1}\le\cdots\le M_{is_i}<^*_{c'_i}m_i}\prod_{j=1}^{s_i}\frac{1}{M_{ij}+b_0+\gamma} \qquad (19)
\end{aligned}
$$

for $s_i\ge 0$ $(i=2,\dots,q)$: By the definitions of the symbols $<_{c_i}$, $<^*_{c_i}$ and $\varepsilon(c_i)$, the inequalities $m_{i-1}+\varepsilon(c'_{i-1})\le M_{i1}$ and $M_{is_i}<m_i+\varepsilon(c'_i)$ under the summation sign in (19) can be rewritten as $m_{i-1}<_{c'_{i-1}}M_{i1}$ and $M_{is_i}<^*_{c'_i}m_i$, respectively. Indeed these can be verified directly. For example, if $c'_i=1$, then we get $\varepsilon(c'_i)=1$ and $<^*_{c'_i}=\le$. Therefore the inequality $M_{is_i}<m_i+\varepsilon(c'_i)=m_i+1$ can be rewritten as

$M_{is_i} <^*_{c'_i} m_i$. Using (19) and the Leibniz rule, we get the identities

$$
\begin{aligned}
&\frac{(-1)^r}{r!}\frac{\partial^r}{\partial X^r}\left(\frac{1}{(m_1+b_0+X)^{\varepsilon(c'_1)}}\left(\prod_{i=2}^{q}\frac{(b_0+X)_{m_{i-1}+\varepsilon(c'_{i-1})}}{(b_0+X)_{m_i+\varepsilon(c'_i)}}\right)\right)\Bigg|_{X=\gamma}\\
&=\sum_{\substack{s_1+\cdots+s_q=r\\ s_i\ge 0}}\frac{\binom{s_1+\varepsilon(c'_1)-1}{s_1}}{(m_1+b_0+\gamma)^{s_1+\varepsilon(c'_1)}}\\
&\quad\times\left(\prod_{i=2}^{q}\frac{(b_0+\gamma)_{m_{i-1}+\varepsilon(c'_{i-1})}}{(b_0+\gamma)_{m_i+\varepsilon(c'_i)}}\sum_{m_{i-1}<_{c'_{i-1}}M_{i1}\le\cdots\le M_{is_i}<^*_{c'_i}m_i}\ \prod_{j=1}^{s_i}\frac{1}{M_{ij}+b_0+\gamma}\right)\\
&=\sum_{\substack{s_1+\cdots+s_q=r\\ s_i\ge 0}}\frac{\binom{s_1+\varepsilon(c'_1)-1}{s_1}}{(m_1+b_0+\gamma)^{s_1}}\\
&\qquad\times\frac{(b_0+\gamma)_{m_1}}{(b_0+\gamma)_{m_q+1}}\prod_{i=2}^{q}\left(\sum_{m_{i-1}<_{c'_{i-1}}M_{i1}\le\cdots\le M_{is_i}<^*_{c'_i}m_i}\ \prod_{j=1}^{s_i}\frac{1}{M_{ij}+b_0+\gamma}\right)\\
&=\sum_{\substack{\varepsilon(c'_1)s_1+\sum_{i=2}^{q}s_i=r\\ s_j\ge 0}}\frac{1}{(m_1+b_0+\gamma)^{\varepsilon(c'_1)s_1}}\\
&\qquad\times\frac{(b_0+\gamma)_{m_1}}{(b_0+\gamma)_{m_q+1}}\prod_{i=2}^{q}\left(\sum_{m_{i-1}<_{c'_{i-1}}M_{i1}\le\cdots\le M_{is_i}<^*_{c'_i}m_i}\ \prod_{j=1}^{s_i}\frac{1}{M_{ij}+b_0+\gamma}\right)
\end{aligned}
\tag{20}
$$

for $r \ge 0$. The last equality sign of (20) comes from the identity

$$
\binom{s_1+\varepsilon(c'_1)-1}{s_1}=\begin{cases}1 & \text{if } \varepsilon(c'_1)=s_1=0 \ \text{ or } \ \varepsilon(c'_1)=1,\ s_1\ge 0,\\ 0 & \text{if } \varepsilon(c'_1)=0,\ s_1\ge 1.\end{cases}
$$

Letting

$$
\begin{aligned}
&g(\{m_i\}_{i=1}^{q};\{r_i\}_{i=1}^{n};\{s_i\}_{i=1}^{q})\\
&=\sum_{\substack{0\le M_{11}<\cdots<M_{1n}<m_1\\ m_1<_{c'_1}M_{21}\le\cdots\le M_{2s_2}<^*_{c'_2}m_2\\ \vdots\\ m_{i-1}<_{c'_{i-1}}M_{i1}\le\cdots\le M_{is_i}<^*_{c'_i}m_i\\ \vdots\\ m_{q-1}<_{c'_{q-1}}M_{q1}\le\cdots\le M_{qs_q}<^*_{c'_q}m_q}}\frac{(a_0+\gamma)_{m_1}}{m_1!}\frac{(b_0+1)_{m_q+1}}{(b_0+\gamma)_{m_q+1}}\\
&\qquad\times\left(\prod_{i=1}^{n}\frac{1}{(M_{1i}+a_0+\gamma)(M_{1i}+b_0+\gamma)^{r_i}}\right)\frac{1}{(m_1+b_0+1)^{k'_1}(m_1+b_0+\gamma)^{\varepsilon(c'_1)s_1}}\\
&\qquad\qquad\times\left\{\prod_{i=2}^{q}\left(\prod_{j=1}^{s_i}\frac{1}{M_{ij}+b_0+\gamma}\right)\frac{1}{(m_i+b_0+1)^{k'_i}}\right\},
\end{aligned}
$$

and using (16), (18) and (20), we get the identity

$$\frac{(-1)^r}{r!}\frac{\partial^r}{\partial X^r}\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\frac{\Gamma(X)}{\Gamma(b_0+X)}\frac{(a_0+X)_{m_1}}{m_1!}\frac{(b_0+1)_{m_q+1}}{(b_0+X)_{m_q+1}}\left\{\prod_{i=1}^{q}\frac{1}{(m_i+b_0+1)^{k_i'}}\right\}\right)\Bigg|_{X=\gamma}$$
$$=\sum_{\substack{r_0+s_0+\varepsilon(c_1')s_1+\sum_{i=2}^{q}s_i=r\\ r_0,s_j\geq 0}}\sum_{n=0}^{m_1}\sum_{\substack{\sum_{i=1}^{n}r_i=s_0\\ r_i\geq 1}}(1-\alpha)^n G^{(r_0)}(\alpha,\beta,\gamma)$$
$$\times g(\{m_i\}_{i=1}^{q};\{r_i\}_{i=1}^{n};\{s_i\}_{i=1}^{q}) \quad (21)$$

for $r\geq 0$ and $m_1,\dots,m_q\in\mathbb{Z}$ such that $0\leq m_1 <_{c_1'} \cdots <_{c_{q-1}'} m_q$, where $c_q'=1$. Here we impose the conditions

$$\mathrm{Re}(\beta),\mathrm{Re}(\gamma),\mathrm{Re}(1-\beta+\gamma),\mathrm{Re}(\alpha-\beta+1),\mathrm{Re}(\alpha-\beta+\gamma)>0.$$

Then, taking $x=1-\beta+\gamma,\ y=\alpha-\beta+1$ and $z=\alpha-\beta+\gamma$ in Lemma 5, we see that the multiple series on the right-hand side of (12) converges uniformly in $\{X\in\mathbb{C}\mid |X-\gamma|\leq\varepsilon\}$, where $\varepsilon\in\mathbb{R}$ such that

$$0<\varepsilon<\min\big\{\mathrm{Re}(1-\beta+\gamma),\mathrm{Re}(\alpha-\beta+\gamma),\mathrm{Re}(\beta/2),\mathrm{Re}(\gamma)\big\}.$$

Thus, differentiating term by term and using (21), we can get the following identity for the right-hand side of (12):

$$\frac{(-1)^r}{r!}\frac{\partial^r}{\partial X^r}\left(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\frac{\Gamma(X)}{\Gamma(\alpha-\beta+X)}H^*\big(\tau(v);(\alpha,\beta,X)^\tau\big)\right)\Bigg|_{X=\gamma}$$
$$=\sum_{\substack{r_0+s_0+\varepsilon(c_1')s_1+\sum_{i=2}^{q}s_i=r\\ r_0,s_j\geq 0}}\sum_{n=0}^{s_0}\sum_{\substack{\sum_{i=1}^{n}r_i=s_0\\ r_i\geq 1}}(1-\alpha)^n G^{(r_0)}(\alpha,\beta,\gamma)$$
$$\times H^*_{(\boldsymbol{r}^n,\boldsymbol{s}^q,\varepsilon(c_1'))}\big(\tau(v);(\alpha,\beta,\gamma)^\tau\big) \quad (22)$$

for $r\geq 0$, $\mathrm{Re}(\alpha)$, and

$$\mathrm{Re}(\beta),\mathrm{Re}(\gamma),\mathrm{Re}(1-\beta+\gamma),\mathrm{Re}(\alpha-\beta+1),\mathrm{Re}(\alpha-\beta+\gamma)>0.$$

Finally we verify that the right-hand side of (22) is the same as that of (5). For brevity, we put the summand on the right-hand side of (22) in $h(n;\{r_i\}_{i=0}^{n};\{s_i\}_{i=1}^{q})$. Then the right-hand side of (22) can be rewritten as

$$\sum_{\substack{r_0+s_0+\varepsilon(c_1')s_1+\sum_{i=2}^{q}s_i=r\\ r_0,s_j\geq 0}}\sum_{n=0}^{s_0}\sum_{\substack{\sum_{i=1}^{n}r_i=s_0\\ r_i\geq 1}}h(n;\{r_i\}_{i=0}^{n};\{s_i\}_{i=1}^{q})$$
$$=\sum_{s_0=0}^{r}\sum_{n=0}^{s_0}\sum_{\substack{r_0+\varepsilon(c_1')s_1+\sum_{i=2}^{q}s_i=r-s_0\\ r_0,s_j\geq 0}}\sum_{\substack{\sum_{i=1}^{n}r_i=s_0\\ r_i\geq 1}}h(n;\{r_i\}_{i=0}^{n};\{s_i\}_{i=1}^{q})$$

$$= \sum_{n=0}^{r} \sum_{s_0=n}^{r} \sum_{\substack{r_0+\varepsilon(c_1')s_1+\sum_{i=2}^{q} s_i = r-s_0 \\ r_0, s_j \geq 0}} \sum_{\substack{\sum_{i=1}^{n} r_i = s_0 \\ r_i \geq 1}} h(n; \{r_i\}_{i=0}^{n}; \{s_i\}_{i=1}^{q})$$

$$= \sum_{n=0}^{r} \sum_{\substack{r_0+s_0+\varepsilon(c_1')s_1+\sum_{i=2}^{q} s_i = r-n \\ r_0, s_j \geq 0}} \sum_{\substack{\sum_{i=1}^{n} r_i = s_0+n \\ r_i \geq 1}} h(n; \{r_i\}_{i=0}^{n}; \{s_i\}_{i=1}^{q})$$

$$= \sum_{n=0}^{r} \sum_{\substack{\sum_{i=0}^{n} r_i+\varepsilon(c_1')s_1+\sum_{i=2}^{q} s_i = r \\ r_i \geq 1\ (i=1,\dots,n) \\ r_0, s_i \geq 0\ (i=1,\dots,q)}} h(n; \{r_i\}_{i=0}^{n}; \{s_i\}_{i=1}^{q}).$$

The last rewrite is exactly the same as the right-hand side of (5). This completes the proof of Theorem 1. □

For $v = \prod_{i=1}^{p} z_{c_{i-1}}(k_i) \in B^0$, we put

$$\begin{aligned} \zeta(v; (x, z)) &:= H(v; (x, x, z)) \\ &= \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \left( \prod_{i=1}^{p-1} \frac{1}{(m_i + z)^{k_i}} \right) \frac{1}{(m_p + x)(m_p + z)^{k_p - 1}} \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\zeta^*_{(\{r_i\}_{i=1}^{n}; \{s_i\}_{i=1}^{p})}(v; y) \\ &:= H^*_{(\{r_i\}_{i=1}^{n}; \{s_i\}_{i=1}^{p})}(v; (1, y, y)) \\ &= \sum_{\substack{0 \leq M_{11} < \cdots < M_{1n} < m_1 \\ m_1 <_{c_1} M_{21} \leq \cdots \leq M_{2s_2} <^*_{c_2} m_2 \\ \vdots \\ m_{i-1} <_{c_{i-1}} M_{i1} \leq \cdots \leq M_{is_i} <^*_{c_i} m_i \\ \vdots \\ m_{p-1} <_{c_{p-1}} M_{p1} \leq \cdots \leq M_{ps_p} <^*_{c_p} m_p < \infty}} \left( \prod_{i=1}^{n} \frac{1}{(M_{1i}+1)(M_{1i}+y)^{r_i}} \right) \frac{1}{(m_1+y)^{k_1+s_1}} \\ &\qquad \times \left\{ \prod_{i=2}^{p} \left( \prod_{j=1}^{s_i} \frac{1}{M_{ij}+y} \right) \frac{1}{(m_i+y)^{k_i}} \right\}. \end{aligned}$$

These multiple series are MHZVs. We can derive the following two identities for these MHZVs from Theorem 1:

**Corollary 8.** *Let $v \in B^0$, and let $\tau(v)$ be its dual. Then the following two identities hold*:

(i)

$$\zeta(\sigma_r^b(v);(\alpha,\gamma)) = \sum_{n=0}^{r}(1-\alpha)^n \sum_{\substack{\sum_{i=1}^{n} r_i+\varepsilon(c_1')s_1+\sum_{i=2}^{q} s_i=r \\ r_i\ge 1\ (i=1,\dots,n) \\ s_i\ge 0\ (i=1,\dots,q)}} H^*_{(\boldsymbol{r}^n,\boldsymbol{s}^q,\varepsilon(c_1'))}(\tau(v);(\alpha,\alpha,\gamma)^\tau) \quad (24)$$

*for all* $r\ge 0$, $\alpha,\gamma\in\mathbb{C}$ *such that*

$$\mathrm{Re}(\alpha),\mathrm{Re}(\gamma),\mathrm{Re}(1-\alpha+\gamma)>0,$$

*where* $\zeta(\sigma_r^b(v);(x,z)) := H(\sigma_r^b(v);(x,x,z))$.

(ii)

$$H(\sigma_r^b(v);(\alpha,1,1)) = \sum_{n=0}^{r}(1-\alpha)^n \sum_{\substack{\sum_{i=0}^{n} r_i+\varepsilon(c_1')s_1+\sum_{i=2}^{q} s_i=r \\ r_i\ge 1\ (i=1,\dots,n) \\ r_0,s_i\ge 0\ (i=1,\dots,q)}} G^{(r_0)}(\alpha,1,1)\zeta^*_{(\boldsymbol{r}^n,\boldsymbol{s}^q,\varepsilon(c_1'))}(\tau(v);\alpha) \quad (25)$$

*for all* $r\ge 0$, $\alpha\in\mathbb{C}$ *with* $\mathrm{Re}(\alpha)>0$. *The factor* $G^{(r_0)}(\alpha,1,1)$ $(r_0\ge 1)$ *becomes* $(\alpha-1)\zeta\big(z_1(1)^{r_0-1}z_1(2);(\alpha,1)\big)$ *or a* $\mathbb{Q}$*-polynomial of*

$$(1-\alpha)^c\sum_{m=0}^{\infty}(m+1)^{-a}(m+\alpha)^{-b}$$

$(a,b,c\in\mathbb{Z}$ *such that* $a,b,c\ge 0,\ a+b>1)$, *therefore the right-hand side of* (25) *can be expressed by MHZVs and* $(1-\alpha)^n$ $(n\ge 0)$.

*Proof.* By the definition of $G^{(n)}(\alpha,\beta,\gamma)$, we see that $G^{(n)}(\alpha,\alpha,\gamma)=0$ $(n\ge 1)$ and $G^{(0)}(\alpha,\alpha,\gamma)=1$. By this fact and taking $\alpha=\beta$ in Theorem 1, we get (24). The identity (25) can be derived from Theorem 1 by taking $\beta=\gamma=1$. In this case, Lemma 6(i) shows that the factor $G^{(r_0)}(\alpha,1,1)$ $(r_0\ge 1)$ becomes a $\mathbb{Q}$-polynomial of $(1-\alpha)^c\sum_{m=0}^{\infty}(m+1)^{-a}(m+\alpha)^{-b}$ $(a,b,c\in\mathbb{Z},\ a,b,c\ge 0,\ a+b>1)$. On the other hand, Lemma 6(ii) gives the expressions

$$\begin{aligned} G^{(r_0)}(\alpha,1,1) &= (\alpha-1)\sum_{m=0}^{\infty}\frac{(2-\alpha)_m}{m!}\frac{1}{(m+1)^{r_0+1}} \\ &= (\alpha-1)H^*\big(z_1(r_0+1);(2-\alpha,1,1)\big) \end{aligned}$$

for $r_0 \geq 1$, $\operatorname{Re}(\alpha) > 0$. Further, this single series can be rewritten as

$$\begin{aligned} H^*\big(z_1(r_0+1); (2-\alpha, 1, 1)\big) &= H^*\big(z_1(r_0+1); (\alpha, \alpha, 1)^\tau\big) \\ &= H\big(z_1(1)^{r_0-1}z_1(2); (\alpha, \alpha, 1)\big) \quad \text{(by Lemma 3)} \\ &= \zeta\big(z_1(1)^{r_0-1}z_1(2); (\alpha, 1)\big) \qquad \text{(by (23)).} \end{aligned}$$

Thus we get $G^{(r_0)}(\alpha, 1, 1) = (\alpha-1)\zeta\big(z_1(1)^{r_0-1}z_1(2); (\alpha, 1)\big)$ $(r_0 \geq 1,\ \operatorname{Re}(\alpha) > 0)$, and this completes the proof of the assertion for $G^{(r_0)}(\alpha, 1, 1)$ stated in (ii). □

**Remark 9.** The identity

$$\sum_{m=0}^{\infty} \frac{(2-\alpha)_m}{m!} \frac{1}{(m+1)^{p+1}} = \sum_{0 \leq m_1 < \cdots < m_p < \infty} \left(\prod_{i=1}^{p} \frac{1}{m_i+1}\right) \frac{1}{m_p+\alpha}$$

($p \geq 1$, $\operatorname{Re}(\alpha) > 0$), which we used in the proof of Corollary 8(ii), is proved by Hoffman [1992, Section 4] by using a theorem of Mordell. He used this identity to prove a duality formula for MZVs [Hoffman 1992, Theorem 4.4].

**Remark 10.** I give a remark on a connection between (24) and a former result of mine written in [Igarashi 2015]. Let $v = \prod_{i=1}^{p} z_{c_{i-1}}(k_i) \in B^0$, and let $\tau(v) = \prod_{i=1}^{q} z_{c'_{i-1}}(k'_i)$ be its dual. In [Igarashi 2015], I proved the following new identity for MHZVs:

$$\sum_{\substack{r_1+\cdots+r_p=r \\ r_i \geq 0}} \left\{\prod_{i=1}^{p} \binom{k_i+r_i-1}{r_i}\right\} \zeta\left(\prod_{i=1}^{p} z_{c_{i-1}}(k_i+r_i); \alpha\right) = \sum_{\substack{s_1+\cdots+s_q=r \\ s_i \geq 0}} H^*_{(\{s_i\}_{i=1}^{q})}(\tau(v); \alpha) \tag{26}$$

for all $r \geq 0$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$, where $\zeta(v; \alpha)$ is the extension of MHZV defined under (7) and

$$H^*_{(\{s_i\}_{i=1}^{p})}(v; \alpha) := \sum_{\substack{0=m_0 \leq M_{11} \leq \cdots \leq M_{1s_1} <^*_{c_1} m_1 \\ \vdots \\ m_{i-1} <_{c_{i-1}} M_{i1} \leq \cdots \leq M_{is_i} <^*_{c_i} m_i \\ \vdots \\ m_{p-1} <_{c_{p-1}} M_{p1} \leq \cdots \leq M_{ps_p} <^*_{c_p} m_p < \infty}} \frac{(m_p+1)!}{(\alpha)_{m_p+1}} \left\{\prod_{i=1}^{p}\left(\prod_{j=1}^{s_i} \frac{1}{M_{ij}+\alpha}\right) \frac{1}{(m_i+1)^{k_i}}\right\}$$

($p \geq 1$, $s_i \geq 0$ $(i = 1, \dots, p)$, $c_p = 1$, $\alpha \in \mathbb{C}$ with $\operatorname{Re}(\alpha) > 0$). By the identity $\binom{k_p+r_p-1}{r_p} = \binom{k_p+r_p-2}{r_p} + \frac{r_p}{k_p-1}\binom{k_p+r_p-2}{r_p}$, we see that the sum on the left-hand side of (24) with $\alpha = \gamma$ is a partial sum of that of (26), and therefore the identity (24) with $\alpha = \gamma$ is a decomposition of (26). The factors $(1-\alpha)^n$ $(n = 0, 1, \dots, r)$ in

(24) explicitly indicate vanishing terms at $\alpha = 1$ of the decomposition: compare the case $\alpha = 1$ of the right-hand side of (24) with that of (26). The identity (24) gives a two-parameter extension of (7) also. The identities for MHZVs (24), (25) and (26) are relations among extensions of MHZV, namely the multiple series (2).

**Remark 11.** As is shown in the present paper, the gamma factor in (12) contributes to deriving various relations among (2) from (12). Here we give another example of this sort: By dividing both sides of (12) by the gamma factor, the identity (12) can be modified as

$$\frac{\Gamma(\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha-\beta+X)}{\Gamma(X)}H(v;(\alpha,\beta,X)) = H^*(\tau(v);(\alpha,\beta,X)^\tau). \tag{27}$$

In the same way as in the proof of Theorem 1, we can derive the following inversion formula for (5) from (27):

$$\sum_{r_0=0}^{r} G^{(r_0)}(\beta,\alpha,\alpha-\beta+\gamma)H(\sigma^b_{r-r_0}(v);(\alpha,\beta,\gamma)) \\ = \sum_{n=0}^{r}(1-\alpha)^n \sum_{\substack{\sum_{i=1}^n r_i+\varepsilon(c'_1)s_1+\sum_{i=2}^q s_i=r \\ r_i\ge 1\ (i=1,\dots,n) \\ s_i\ge 0\ (i=1,\dots,q)}} H^*_{(\boldsymbol{r}^n,\boldsymbol{s}^q,\varepsilon(c'_1))}(\tau(v);(\alpha,\beta,\gamma)^\tau)$$

for all $v \in B^0$, $r \ge 0$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that

$$\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma), \mathrm{Re}(1-\beta+\gamma), \mathrm{Re}(\alpha-\beta+1), \mathrm{Re}(\alpha-\beta+\gamma) > 0,$$

where $\tau(v) = \prod_{i=1}^{q} z_{c'_{i-1}}(k'_i)$ is the dual of $v$. This identity also yields numerous relations among (2). For this kind of application of gamma factors, see also [Igarashi 2018, Theorem 2.17] and its proof.

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