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In this paper we import the theory of "Calabi–Yau" algebras and categories from symplectic topology and topological field theories, to the setting of spectra in stable homotopy theory. Twistings in this theory will be particularly important. There will be two types of Calabi–Yau structures in the setting of ring spectra: one that applies to compact algebras and one that applies to smooth algebras. The main application of twisted compact Calabi–Yau ring spectra that we will study is to describe, prove, and explain a certain duality phenomenon in string topology. This is a duality between the manifold string topology of Chas and Sullivan (1999) and the Lie group string topology of Chataur and Menichi (2012). This will extend and generalize work of Gruher (2007). Then, generalizing work of Cohen and Jones (2017), we show how the gauge group of the principal bundle acts on this compact Calabi–Yau structure, and we compute some explicit examples. We then extend the notion of the Calabi–Yau structure to smooth ring spectra, and prove that Thom ring spectra of (virtual) bundles over the loop space, ΩM, have this structure. In the case when M is a sphere, we will use these twisted smooth Calabi–Yau ring spectra to study Lagrangian immersions of the sphere into its cotangent bundle. We recast the work of Abouzaid and Kragh (2016) to show that the topological Hochschild homology of the Thom ring spectrum induced by the h-principle classifying map of the Lagrangian immersion detects whether that immersion can be Lagrangian isotopic to an embedding. We then compute some examples. Finally, we interpret these Calabi–Yau structures directly in terms of topological Hochschild homology and cohomology.

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Introduction

The theory of Calabi–Yau algebras and categories has proven to be very important in symplectic topology and the study of topological field theories [Costello 2007; Kontsevich and Soibelman 2009; Kontsevich and Vlassopoulos 2013; Lurie 2009; Cohen and Ganatra 2015]. One of the goals of this paper is to adapt this theory to the setting of spectra in stable homotopy theory, and to apply it to prove and explain a duality relationship between the string topology of a manifold and the string topology of a classifying space of a compact Lie group. We also use this notion to study Lagrangian immersions of spheres.

By way of background, recall that "string topology" is a term that was originally coined by Chas and Sullivan [1999] in their influential paper. In that paper, the term referred to certain algebraic properties of the homology of the loop space of a closed, oriented manifold, $H_*(LM)$, that were the result of a type of intersection theory in LM. This intersection theory came about by studying the fibration $\Omega M \to LM \xrightarrow{\mathrm{ev}} M$, where ev evaluates a loop at $1 \in S^1$. Even though the loop space is itself infinite dimensional, the intersection theory defining the string topology operations is ultimately possible because of the finite dimensionality and compactness of M, as well as the fiberwise multiplicative properties of this fibration.

Since that time, the subject has expanded considerably. An important variation of the string topology intersection theory was described by Chataur and Menichi [2012], where they defined operations on the cohomology of the loop space of the classifying space of a compact Lie group, LBG. In this setting the analogous fibration $G \to LBG \xrightarrow{\mathrm{ev}} BG$ is studied, and the intersection theory defining these operations is possible because of the compactness of G as well as the fiberwise multiplicative properties of this fibration. A theory that includes both the string topology of a manifold and that of classifying spaces was developed in the setting of stacks in [Lupercio et al. 2008; Behrend et al. 2012]. In this setting the intersection theory is done in an appropriate algebraic geometric category.

An observation that helped to shed light on this intersection theory was made by Cohen and Klein [2009] when they classified "umkehr maps" that satisfy appropriate naturality and linearity properties. This led to the observation that the ring spectrum LM^{-TM}, which was shown to realize the Chas–Sullivan loop product by Cohen and Jones [2002], can be viewed as a twisted generalized cohomology

theory evaluated on the manifold M. Specifically, if one takes the fiberwise suspension spectrum of the fibration $\Omega M \to LM \xrightarrow{\mathrm{ev}} M$, and denotes the resulting parametrized spectrum by the notation

$$\Sigma^\infty(\Omega M_+) \to \Sigma^\infty_M(LM_+) \to M,$$

then the result is a parametrized ring spectrum which defines a twisted cohomology theory $\mathcal{S}^\bullet_M$ from the category of spaces over M, $\mathcal{T}_M$, to an appropriate category of spectra. If $f : X \to M$ is an object in $\mathcal{T}_M$, then $\mathcal{S}^\bullet_M(X, f) = \Gamma_X\big(f^*(\Sigma^\infty_M(LM_+))\big)$, the spectrum of sections over X of the pullback via f of the parametrized spectrum $\Sigma^\infty_M(LM_+) \to M$. See [Cohen and Klein 2009; May and Sigurdsson 2006] for details. Since $\Sigma^\infty_M(LM_+)$ is a parametrized ring spectrum, this spectrum of sections inherits a ring spectrum structure. Moreover it was proved in [Cohen and Klein 2009] that the value of this cohomology on the identity map $\mathrm{id} : M \xrightarrow{=} M \in \mathcal{T}_M$, has the homotopy type

$$\mathcal{S}^\bullet_M(M) = \Gamma_M(\Sigma^\infty_M(LM_+)) \simeq LM^{-TM}$$

as ring spectra. This equivalence is a type of twisted Poincaré or Atiyah duality as explained in [Cohen and Klein 2009]. Moreover, one sees that the string topology intersection pairing (loop product) on $H_*(LM^{-TM}) \cong H_{*+n}(LM)$ corresponds, via this twisted Poincaré duality, to a generalized cup product pairing in the cohomology $\mathcal{S}^\bullet_M(M)$. This is a twisted generalization of a well-known phenomenon: the intersection product in $H_*(M)$ corresponds up to sign, under traditional Poincaré duality, to the cup product in $H^*(M)$.

As observed by Gruher and Salvatore [2008], the string topology product exists in the presence of any fiberwise monoid over a closed manifold, $Q \to E \to M$. Here Q is a monoid, and the bundle E comes equipped with a fiberwise product $E \times_M E \to E$ over M, consistent with the monoid structure of the fiber Q. In this case the Thom spectrum E^{-TM} is a ring spectrum. It was also observed in [Gruher and Salvatore 2008] that principal bundles $G \to P \to M$ give rise to fiberwise monoids by taking the associated adjoint bundle, $G \to P^{\mathrm{Ad}} \to M$, where $P^{\mathrm{Ad}} = P \times_G G^{\mathrm{Ad}}$. Here G^{Ad} denotes G with the left G-action given by conjugation.

As observed in [Cohen and Jones 2017], the string topology of principal bundles over manifolds can also be represented by twisted cohomology theories. The representing parametrized spectrum is the fiberwise suspension spectrum

$$\Sigma^\infty(G_+) \to \Sigma^\infty_M(P^{\mathrm{Ad}}_+) \to M.$$

Let $\mathcal{S}^\bullet_P$ denote the corresponding twisted cohomology theory. In particular,

$$\mathcal{S}^\bullet_P(M) = \Gamma_M(\Sigma^\infty_M(P^{\mathrm{Ad}}_+)) \simeq (P^{\mathrm{Ad}})^{-TM},$$

and the ring structure comes from a generalized cup product on $\Sigma^\infty_M(P^{\mathrm{Ad}}_+)^\bullet(M)$. We refer to $\mathcal{S}^\bullet_P(-)$ as the *manifold string topology* structure on the principal bundle P.

This perspective on the string topology spectrum, LM^{-TM}, or more generally $(P^{\mathrm{Ad}})^{-TM}$, in terms of the sections of a parametrized spectrum was particularly useful in [Cohen and Jones 2017], where the units of these ring spectra were studied. In particular it was shown that the gauge group $\mathcal{G}(P)$ of the principal bundle acts naturally on the string topology spectrum, and so there is a homomorphism,

$$\mathcal{G}(P) \to \mathrm{GL}_1((P^{\mathrm{Ad}})^{-TM})$$

which was studied and computed in [Cohen and Jones 2017].

The first goal of this paper is to show that there is a dual construction for the string topology of the classifying space of a compact Lie group, to investigate this duality using a stable homotopy theoretic version of compact Calabi–Yau algebras, and to compute some of its properties, including gauge symmetry.

We now state the results more precisely. Let G be a compact Lie group, and let $G \to P \to X$ be a principal G-bundle. In this, X can be any space of the homotopy type of a CW-complex. It need not be finite. In particular, an important example is the universal principal bundle $G \to EG \to BG$. As before, let $G \to P^{\mathrm{Ad}} \to X$ be the corresponding adjoint bundle. Recall that in the case of the universal bundle, $EG^{\mathrm{Ad}} \simeq LBG$.

Consider the fiberwise suspension spectrum,

$$\Sigma^\infty(G_+) \to \Sigma^\infty_M(P^{\mathrm{Ad}}_+) \to X,$$

and let $\mathcal{D}(\Sigma^\infty_M(P^{\mathrm{Ad}}_+))$ be the fiberwise Spanier–Whitehead dual, as in [May and Sigurdsson 2006]. This is a parametrized spectrum over X, whose fibers are the Spanier–Whitehead duals of the fibers of $\Sigma^\infty_M(P^{\mathrm{Ad}}_+)$:

$$G^\vee \to \mathcal{D}(\Sigma^\infty_M(P^{\mathrm{Ad}}_+)) \to X,$$

where $G^\vee = \mathrm{Map}(\Sigma^\infty(G_+), \mathbb{S})$. Here $\mathbb{S}$ denotes the sphere spectrum. Notice that $G^\vee$ is a coalgebra spectrum, with coalgebra structure dual to the ring structure on $\Sigma^\infty(G_+)$.

We denote the twisted homology theory associated to this parametrized spectrum by $\mathcal{S}^P_\bullet : \mathcal{T}_X \to \mathrm{Spectra}$. The following will be proved in Section 1.

Theorem 1. *The parametrized spectrum $\mathcal{D}(\Sigma^\infty(G_+)) \to \mathcal{D}(\Sigma^\infty_M(P^{\mathrm{Ad}}_+)) \to X$ is a weak fiberwise coalgebra spectrum satisfying the following properties.*

(1) *Let $f : Y \to X$ be an object in $\mathcal{T}_X$. Then the induced twisted homology $\mathcal{S}^P_\bullet(Y, f)$ is a weak coalgebra spectrum.*

(2) *There is an equivalence of spectra,*

$$\alpha : (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \xrightarrow{\simeq} \mathcal{S}^P_\bullet(X)$$

where $(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$ *is the Thom spectrum of minus the vertical tangent bundle* $T_{\mathrm{vert}}P^{\mathrm{Ad}} \to P^{\mathrm{Ad}}$. *Furthermore, a Pontryagin–Thom construction gives* $(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$ *a natural coproduct which is taken by* α *to the coproduct in* $\mathcal{S}^P_\bullet(X)$.

(3) *If one takes the cohomology of the coalgebra spectrum,* $H^*(\mathcal{S}^P_\bullet(Y, f); k)$ (*here the coefficients are in a field* k), *one obtains a graded algebra,*

$$H^*(\mathcal{S}^P_\bullet(Y); k) \otimes H^*(\mathcal{S}^P_\bullet(Y); k) \to H^*(\mathcal{S}^P_\bullet(Y); k)$$

which we call the ***Lie group string topology algebra of*** $f^*(P)$. *Using the equivalence in part* (2), *when the vertical tangent bundle* $T_{\mathrm{vert}} \to P^{\mathrm{Ad}}$ *is orientable, one obtains a graded algebra of degree* $-d$, *where* $d = \dim G$:

$$H^p(P^{\mathrm{Ad}}) \otimes H^q(P^{\mathrm{Ad}}) \to H^{p+q-d}(P^{\mathrm{Ad}}).$$

(4) *In the case of the universal principal bundle* $G \to EG \to BG$, *this algebra is isomorphic to the algebra structure in the string topology of the classifying space* BG (as described by Chataur and Menichi [2012]),

$$H^*(\mathcal{S}^{EG}_\bullet(BG)) \cong H^*(LBG).$$

Comments. (1) The notion of a "weak" fiberwise coalgebra spectrum will be defined in Section 1.

(2) We refer to the coalgebra spectrum $\mathcal{S}^P_\bullet(X) \simeq (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$ as the *Lie group string topology spectrum of the principal bundle* P.

(3) The equivalence $\alpha : (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \xrightarrow{\simeq} \mathcal{S}^P_\bullet(X) = \mathcal{D}(\Sigma^\infty_M(P^{\mathrm{Ad}}_+))/X$ can be viewed as a fiberwise Atiyah duality, which on the level of fibers is the classical Atiyah [1961] equivalence,

$$\alpha : G^{-TG} \simeq \Sigma^{-\mathfrak{g}}(G_+) \xrightarrow{\simeq} G^\vee,$$

where G^{-TG} is the Thom spectrum of minus the tangent bundle, which is equivariantly equivalent to the desuspension of $\Sigma^\infty(G_+)$ by the adjoint representation of G on the Lie algebra $\mathfrak{g}$.

(4) The fact that the cohomology algebra $H^*(LBG^{-T_{\mathrm{vert}}}) \cong H^{*+d}(LBG)$ is the string topology of classifying spaces was proved by Gruher [2007].

Once this theorem is established we restrict to the situation where we have a principal G-bundle over a closed manifold: $G \to P \to M$. In this case we can study

both the "manifold string topology structure" of P, that is, the twisted cohomology theory

$$\mathcal{S}_P^\bullet(M) = \Gamma_M(\Sigma_M^\infty(P_+^{\mathrm{Ad}})) \simeq (P^{\mathrm{Ad}})^{-TM}$$

as well as the "Lie group string topology structure" of P, which is to say the twisted homology theory

$$\mathcal{S}_\bullet^P(M) \xrightarrow{\simeq} (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}.$$

The following is a consequence of Theorem 1 as well as Gruher's [2007] work.

Corollary 2. *Let $G \to P \to M$ be a principal bundle, where G is a compact Lie group of dimension d and M is a closed manifold of dimension n. The string topology spectra $\mathcal{S}_P^\bullet(M) \simeq (P^{\mathrm{Ad}})^{-TM}$ and $\mathcal{S}_\bullet^P(M) \simeq (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$ are Spanier–Whitehead dual to each other, with the algebra structure of the former corresponding to the coalgebra structure of the latter under this duality. When M is oriented and the bundle T_{vert} is oriented, this gives $H_*(P^{\mathrm{Ad}})$ the structure of a Frobenius algebra of dimension $n-d$. The multiplication in this Frobenius algebra comes from the manifold string topology, and the comultiplication comes from the Lie group string topology.*

In Section 2 we will define the notion of a "twisted compact Calabi–Yau" ring spectrum ("twisted cCY"), which can be viewed as a strengthened, derived version of Frobenius algebra in the category of spectra. This definition is adapted from the notion of a "compact Calabi–Yau algebra" defined by Kontsevich and Soibelman [2009], as a way of studying two dimensional topological field theories. (We note that Kontsevich and Soibelman used different terminology for this concept.) Related notions were defined by Costello [2007] and Lurie [2009]. In these definitions, the algebra (or ring spectrum) involved must satisfy a finiteness condition called "compactness". In the spectrum setting this means that the spectrum is a perfect module over the sphere spectrum. In our definition of this structure in the setting of spectra, a key role is played by a "twisting bimodule" over the compact ring spectrum. The following is the main result of this section.

Theorem 3. *Let $G \to P \to M$ be a principal bundle with compact Lie group fiber and closed manifold base. Then the manifold string topology $\mathcal{S}_P^\bullet(M)$ naturally admits the structure of a twisted, compact Calabi–Yau ring spectrum of dimension $n-d$. The Lie group string topology spectrum $\mathcal{S}_\bullet^P(M)$ is the twisting bimodule spectrum in this structure. Moreover if E_* is a generalized homology theory with respect to which both the vertical tangent bundle $T_{\mathrm{vert}} \to P^{\mathrm{Ad}}$ and the tangent bundle $TM \to M$ are oriented, then the Calabi–Yau structure on $\mathcal{S}_P^\bullet(M)$ induces a Frobenius algebra structure on the homology of the manifold string topology, $E_*(\mathcal{S}_P^\bullet(M))$, whose dual is the homology of the Lie group string topology spectrum, $E_*(\mathcal{S}_\bullet^P(M))$.*

In [Cohen and Jones 2017] an action of the gauge group $\mathcal{G}(P)$ of the principal bundle $G \to P \to M$ on the manifold string topology spectrum $\mathcal{S}_P^\bullet(M) = (P^{\mathrm{Ad}})^{-TM}$ was described and computed. In Section 3 we use Theorems 1 and 3 to describe a similar action of $\mathcal{G}(P)$ on the Lie group string topology spectrum $\mathcal{S}_\bullet^P(M) = (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$. We also show that this gauge symmetry respects the Calabi–Yau structure. See Theorem 15 below for a precise statement. We then compute some explicit examples of this gauge symmetry.

In Section 4 we introduce the related notion of twisted *smooth* Calabi–Yau ring spectra. Smoothness is a form of smallness property different from compactness. A ring spectrum A is *smooth* if it is perfect as a bimodule over itself. That is, it is a perfect as a left $(A \wedge A^{\mathrm{op}})$-module spectrum. The spectrum notion of a "twisted sCY" structure is adapted from the notion of "sCY" algebras and categories, first defined by Kontsevich and Vlassopoulos [2013], and used by Cohen and Ganatra [2015] to compare the string topology topological field theory to the Floer symplectic field theory of cotangent bundles. In the spectral theory a twisting bimodule spectrum plays an important role. We show that this structure occurs in certain Thom spectra of virtual bundles over the based loop space of a manifold, ΩM. That is, we prove the following theorem:

Theorem 4. *Let M be a closed manifold, and $f : M \to BBO$ be a map to a delooping of BO. Here, by Bott periodicity we may take BBO to be the infinite homogeneous space* SU/SO. *Consider the induced map of loop spaces, $\Omega f : \Omega M \to BO$. Then its Thom spectrum, which we denote by $(\Omega M)^{\Omega f}$, naturally admits the structure of a twisted, smooth Calabi–Yau ring spectrum.*

Remark. When $f : M \to BBO$ is the constant map, this theorem implies that the suspension spectrum $\Sigma^\infty(\Omega M_+)$ has the structure of a twisted sCY ring spectrum. This strengthens a result of Cohen and Ganatra [2015] saying that the singular chain complex $C_*(\Omega M)$ admits the structure of a smooth Calabi–Yau differential graded algebra.

Also in Section 4, we describe how these ring spectra arise naturally in the study of Lagrangian immersions. In particular, for the case of spheres, we combine the results of Abouzaid and Kragh [2016] with those of Blumberg, Cohen and Schlichtkrull [Blumberg et al. 2010] to prove the following (see Theorem 25 for a more precise statement).

Theorem 5. *Associated to a Lagrangian immersion $\phi : S^n \to T^*S^n$ there is a loop map $\Omega\alpha_\phi : \Omega S^n \to BU$. If the Lagrangian immersion ϕ is Lagrangian isotopic to a Lagrangian embedding, then there is an equivalence of topological Hochschild homology spectra,*

$$\mathrm{THH}((\Omega S^n)^{\Omega\alpha_\phi}) \simeq \mathrm{THH}(\Sigma^\infty(\Omega S^n_+)).$$

We then use this theorem, together with homotopy theoretic results about the image of the J-homomorphism, to recast results in [Abouzaid and Kragh 2016] giving examples of Lagrangian immersions of spheres that are not Lagrangian isotopic to embeddings, but *are* smoothly isotopic to embeddings.

Finally, in Section 5 we describe this structure from the perspective of topological Hochschild (co)homology. More specifically, let $G \to P \to M$ be a smooth principal bundle, where G is a compact Lie group and M is a smooth, closed manifold. Let $h : \Omega M \to G$ be the holonomy of a connection on P. This induces a map of ring spectra $h : \Sigma^\infty(\Omega M_+) \to \Sigma^\infty(G_+)$. Thus h defines bimodule structures on $\Sigma^\infty(G_+)$ over $\Sigma^\infty(\Omega M_+)$. The main result of this section is the following:

Theorem 6. *We have the following equivalences involving topological Hochschild homology* $\mathrm{THH}_\bullet$ *and topological Hochschild cohomology* $\mathrm{THH}^\bullet$.

(1) $\mathrm{THH}_\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq \Sigma^\infty(P_+^{\mathrm{Ad}})$

(2) $\mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq (P^{\mathrm{Ad}})^{-TM} \simeq \mathcal{S}_P^\bullet(M)$.

This equivalence is one of ring spectra.

(3) $\mathrm{THH}_\bullet\big(\Sigma^\infty(\Omega M_+), G^\vee\big) \simeq (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \simeq \mathcal{S}_\bullet^P(M)$.

This equivalence is one of coalgebra spectra.

We end by describing the twisted Calabi–Yau structure on the string topology spectrum from the perspective of these topological Hochschild homology spectra. A consequence of the resulting duality properties is the following:

Corollary 7. *If M is oriented and the bundle T_{vert} is oriented, there is a nondegenerate bilinear form on Hochschild homology,*

$$HH_*(C_*(\Omega M), C_*(G)) \times HH_*(C_*(\Omega M), C_*(G)) \to k.$$

That is, this Hochschild homology space is self dual.

1. A twisted homology theory representing Lie group string topology

The goal of this section is to describe Lie group string topology as a twisted generalized homology theory, and to prove Theorem 1. The main issue in proving this theorem is to describe a parametrized form of Atiyah duality. We begin by recalling the specific map yielding the Atiyah duality between the Thom spectrum of minus the tangent bundle of a closed manifold M, and the Spanier–Whitehead dual of M [Atiyah 1961; Cohen 2004].

Let M^n be a closed n-dimensional manifold and $e : M \hookrightarrow \mathbb{R}^k$ be an embedding into Euclidean space with normal bundle $\eta_e \to M$. By the tubular neighborhood theorem, for sufficiently small $\epsilon > 0$, the open set $\nu_\epsilon(e) \subset \mathbb{R}^k$ consisting of points within a distance of ϵ of $e(M)$ can be identified with the total space η_e.

Consider the map

$$\alpha : (\mathbb{R}^k - \nu_\epsilon(e)) \times M \to \mathbb{R}^k - B_\epsilon(0) \simeq S^{k-1},$$
$$(v, y) \mapsto v - e(y), \tag{1}$$

where $B_\epsilon(0)$ is the open ball of radius ϵ. This map induces the Alexander duality isomorphism

$$\tilde{H}_q(\mathbb{R}^k - e(M)) \cong \tilde{H}_q(\mathbb{R}^k - \nu_\epsilon(e)) \xrightarrow{\cong} \tilde{H}^{k-q-1}(M).$$

Atiyah duality [1961] is induced by the same map,

$$M^{\eta_e} \wedge M_+ \cong (\mathbb{R}^k \times M)/\big((\mathbb{R}^k - \nu_\epsilon(e)) \times M\big) \to \mathbb{R}^k/(\mathbb{R}^k - B_\epsilon(0)) \cong S^k,$$
$$(v, y) \mapsto v - e(y). \tag{2}$$

The adjoint of this map gives a map from the Thom space of η_e to the mapping space, $\alpha : M^{\eta_e} \to \mathrm{Map}(M, S^k)$ which defines the Atiyah duality equivalence of spectra,

$$\alpha : M^{-TM} \to \mathrm{Map}(M, \mathbb{S}). \tag{3}$$

Here this notation refers to the mapping spectrum between the suspension spectrum $\Sigma^\infty(M_+)$ to the sphere spectrum $\mathbb{S}$. This is the Spanier–Whitehead dual of M, and will be denoted by $M^\vee$. Indeed in [Cohen 2004] the first author constructed a symmetric ring spectrum (without unit), M^{-TM}. The k-th space of this spectrum is equivalent, through a range of dimensions that increases with k, to the Thom space M^{η_e} and is constructed by allowing the embeddings and the choices of ϵ to vary. The k-th space of the mapping spectrum $\mathrm{Map}(M, \mathbb{S})$ has the homotopy type of $\mathrm{Map}(M, S^k)$. It was shown in [Cohen 2004] that the map α induces an equivalence of symmetric ring spectra. We refer the reader to [Cohen 2004] for details.

We now pass to the parametrized setting. Our goal is to describe a parametrized form of this Atiyah duality equivalence. Let $G \to P \to X$ be a principal bundle with compact Lie group fiber. By the fiberwise duality theorem of May and Sigurdsson (Theorem 15.1.1 of [May and Sigurdsson 2006]), the parametrized suspension spectrum $\Sigma^\infty(G_+) \to \Sigma^\infty_X(P_+^{\mathrm{Ad}}) \to X$ is (fiberwise) dualizable because each fiber spectrum is dualizable. This in turn is because every fiber spectrum is equivalent to $\Sigma^\infty(G_+)$, which is dualizable since G is a compact manifold. The parametrized Spanier–Whitehead dual is what we called $G^\vee \to \mathcal{D}(\Sigma^\infty_X(P_+^{\mathrm{Ad}})) \to X$ in the introduction. The construction in [May and Sigurdsson 2006] is quite general. In this particular case, however, we will describe this fiberwise dual explicitly.

The spectra we work with will be orthogonal spectra, and when we describe a group action, we use $\mathrm{RO}(G)$-indexed orthogonal spectra. We refer the reader to [Mandell and May 2002] for details.

Recall that $P^{\mathrm{Ad}} = P \times_G G^{\mathrm{Ad}}$. Let V be a finite dimensional orthogonal representation of G, and let $S^V = V \cup \infty$ be the one-point compactification where the G-action fixes ∞. The conjugation action of G on itself defines an action of G on $\mathrm{Map}(G, S^V)$,

$$g \cdot \phi : G \to S^V, \quad h \mapsto g\phi(g^{-1}hg). \tag{4}$$

This defines an $\mathrm{RO}(G)$-graded G-spectrum, which we call $G^\vee$.

We define the parametrized spectrum $\mathcal{D}(\Sigma_X^\infty(P_+^{\mathrm{Ad}}))$ as an $\mathrm{RO}(G)$-graded spectrum. For a representation W, the W-space is defined to be

$$\mathcal{D}(\Sigma_X^\infty(P_+^{\mathrm{Ad}}))_W = P \times_G \mathrm{Map}(G, S^W) \tag{5}$$

which fibers over $X = P/G$ with fiber $\mathrm{Map}(G, S^W) = \mathrm{Map}(G, S^k)$, where $k = \dim W$. The fiberwise suspension by a representation U is given by

$$\Sigma_X^U\big(\mathcal{D}(\Sigma_X^\infty(P_+^{\mathrm{Ad}}))_W\big) = P \times_G S^U \wedge \mathrm{Map}(G, S^W)$$

and the structure map $\epsilon_U : \Sigma_X^U\big(\mathcal{D}(\Sigma_X^\infty(P_+^{\mathrm{Ad}}))_W\big) \to \mathcal{D}(\Sigma_X^\infty(P_+^{\mathrm{Ad}}))_{W\oplus U}$ is induced by the G-equivariant map

$$\epsilon_U : S^U \wedge \mathrm{Map}(G, S^W) \to \mathrm{Map}(G, S^{W\oplus U}), \quad \epsilon_U(t \wedge \phi)(g) = \phi(g) \wedge t.$$

Notice that since the multiplication map $G \times G \to G$ is equivariant with respect to the adjoint action (the action on $G \times G$ is diagonal), the induced map $G^\vee \to (G \times G)^\vee$ is also equivariant. Along with the natural weak equivalence $G^\vee \wedge G^\vee \to (G \times G)^\vee$, this induces a weak fiberwise coalgebra structure on the parametrized spectrum $\mathcal{D}(\Sigma_X^\infty(P_+^{\mathrm{Ad}}))$.

By a "weak fiberwise coalgebra" structure on a parametrized spectrum, we simply mean the following.

Definition. A parametrized spectrum $E \to \mathcal{E} \to X$ is a *weak fiberwise coalgebra* if there is a "comultiplication" map $\gamma : \mathcal{E} \to \mathcal{E} \wedge_X \mathcal{E}$ and a "counit" $\eta : \mathcal{E} \to \mathbb{S}_X$ in the category of parametrized spectra over X, that satisfy the usual coassociativity and counit properties up to homotopy. No coherence conditions on the homotopies are assumed. Here $\mathbb{S} \to \mathbb{S}_X \to X$ is the parametrized sphere spectrum. Namely the n-th space of $\mathbb{S}_X$ is $X \times S^n$.

Notice that given a fiberwise coalgebra spectrum $E \to \mathcal{E} \to X$, then for any object $f : Y \to X$ in $\mathcal{T}_X$, the twisted homology spectrum $\mathcal{E}_\bullet(Y, f) = \mathcal{E}/X$ is an ordinary coalgebra spectrum.

The source of the parametrized Atiyah duality map is a parametrized Thom spectrum. More precisely, let $e : G \subset V$ be an equivariant embedding of G with its conjugation action into a finite dimensional G-representation V. Let $k = \dim V$. Let $\nu_\epsilon(e)$ be an equivariant tubular neighborhood as above. It is equivariantly

diffeomorphic to the normal bundle $\eta_V \to G$. (We are suppressing the embedding e from the notation.) We let $\eta_V^{\mathrm{vert}} \to P^{\mathrm{Ad}}$ be the vector bundle

$$\eta_V^{\mathrm{vert}} = P \times_G \eta_V \to P \times_G G^{\mathrm{Ad}} = P^{\mathrm{Ad}} \tag{6}$$

The fiberwise Thom space of this bundle is homeomorphic to the fiberwise one-point compactification of the tubular neighborhood,

$$P \times_G G^{\eta_V} \cong P \times_G (\nu_\epsilon(e) \cup \infty).$$

Notice also that there is a map from the fiberwise suspension

$$\epsilon_W : \Sigma_X^W (P \times_G G^{\eta_V}) = P \times_G (S^W \wedge G^{\eta_V}) \xrightarrow{\simeq} P \times_G G^{\eta_V \oplus W}$$

This data defines an $\mathrm{RO}(G)$-graded parametrized spectrum $P \times_G G^{-TG}$ over X whose W-th space is $\Omega^V(P \times_G G^{\eta_V \oplus W})$. Here, for a representation U, Ω^U refers to the U-fold loop space, $\mathrm{Map}_\bullet(S^U, -)$.

Furthermore, the Atiyah duality map described above defines a map $\alpha : G^{\eta_V} \cong \nu_\epsilon(e) \cup \infty \to \mathrm{Map}(G, S^V)$. This map is equivariant, and so defines Atiyah duality maps $\overline{\alpha_V} : P \times_G G^{\eta_V} \to P \times_G \mathrm{Map}(G, S^V)$. These maps respect the spectrum structure maps and so this proves the following:

Lemma 8. *The maps $\overline{\alpha_k}$ define an equivalence of parametrized spectra over X,*

$$\bar{\alpha} : P \times_G G^{-TG} \xrightarrow{\simeq} \mathcal{D}(\Sigma_X^\infty (P_+^{\mathrm{Ad}})).$$

The map $\bar{\alpha}$ therefore defines an equivalence of the generalized twisted homology theories these parametrized spectra represent. Given an object $f : Y \to X$ in $\mathcal{T}_X$, the twisted homology theory that the parametrized spectrum $\mathcal{D}(\Sigma_X^\infty (P_+^{\mathrm{Ad}}))$ represents is what we called $\mathcal{S}_\bullet^P(Y, f)$ in the introduction. The twisted homology theory that the parametrized spectrum $P \times_G G^{-TG}$ represents is given by the (ordinary) spectrum $f^*(P)_+ \wedge_G G^{-TG}$, which is the Thom spectrum of the virtual bundle $-f^*(T_{\mathrm{vert}})$ over $f^*(P^{\mathrm{Ad}})$, where $T_{\mathrm{vert}} \to P^{\mathrm{Ad}}$ is the vertical tangent bundle. Applying this to X itself, we have the equivalence of spectra,

$$\bar{\alpha} : (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \xrightarrow{\simeq} \mathcal{S}_\bullet^P(X) \tag{7}$$

Now recall that given any map $\phi : M \to N$ between closed manifolds, the Pontryagin–Thom construction defines a map $\tau_g : N^{-TN} \to M^{-TM}$ making the following diagram of spectra homotopy commute:

$$\begin{array}{ccc} N^{-TN} & \xrightarrow{\tau_g} & M^{-TM} \\ \alpha \downarrow \simeq & & \simeq \downarrow \alpha \\ N^\vee & \xrightarrow{g^\vee} & M^\vee \end{array} \tag{8}$$

Applying this to the multiplication map $\mu : G \times G \to G$, we get a homotopy commutative diagram

$$\begin{tikzcd} G^{-TG} \arrow[r, "\tau_\mu"] \arrow[d, "\alpha"', "\simeq"] & G^{-TG} \wedge G^{-TG} \arrow[d, "\alpha", "\simeq"'] \\ G^\vee \arrow[r, "\mu^\vee"] & G^\vee \wedge G^\vee \end{tikzcd}$$

Given the adjoint action of G on itself, and the diagonal adjoint action of G on $G \times G$, the multiplication map $\mu : G \times G \to G$ is equivariant. Therefore there is an induced fiberwise coproduct on the parametrized spectrum $P \times_G G^{-TG}$, as there is on $\mathcal{D}(\Sigma_X^\infty(P_+^{\mathrm{Ad}}))$.

We now verify that the induced map $\bar{\alpha} : P \times_G G^{-TG} \xrightarrow{\simeq} \mathcal{D}(\Sigma_X^\infty(P_+^{\mathrm{Ad}}))$ preserves these coproducts. We do this by studying the definition of the maps involved more carefully.

Toward this end let $e : G \subset V$ be an equivariant embedding of G with its conjugation action into a finite dimensional G-representation V, as above. Let $k = \dim V$. We then have an induced composition of equivariant embeddings,

$$G \times G \xrightarrow{\mu \times e \times e} G \times V \times V \xrightarrow{e \times 1 \times 1} V \times V \times V. \tag{9}$$

Recall that the tangent bundle of G has an equivariant trivialization $TG \cong G \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra with its adjoint action. Differentiating $e : G \hookrightarrow V$ at the identity gives a linear equivariant embedding $\mathfrak{g} \hookrightarrow V$. We let $\mathfrak{g}^\perp$ be the orthogonal complement with its induced action.

The total space of the normal bundle of $G \times V \times V \xrightarrow{e \times 1 \times 1} V \times V \times V$ is clearly equivariantly isomorphic to $G \times \mathfrak{g}^\perp \times V \times V$. We perform the Pontryagin–Thom construction on the induced (equivariant) embedding of the restriction of the total space

$$(\mu \times e \times e)^*(G \times \mathfrak{g}^\perp \times V \times V) \hookrightarrow G \times \mathfrak{g}^\perp \times V \times V.$$

This is a codimension $2k - d$ embedding. The Pontryagin–Thom construction gives an equivariant map

$$\tau_\mu^{V \times V \times V} : G_+ \wedge S^{\mathfrak{g}^\perp} \wedge S^V \wedge S^V \to (G \times G)_+ \wedge S^{\mathfrak{g}^\perp} \wedge S^{\mathfrak{g}^\perp} \wedge S^V,$$

or equivalently,

$$\tau_\mu^{V \times V \times V} : G^{\eta_e} \wedge S^V \wedge S^V \to G^{\eta_e} \wedge G^{\eta_e} \wedge S^V.$$

This defines the map

$$\tau_\mu^{V \times V \times V} : P \times_G (G^{\eta_e} \wedge S^V \wedge S^V) \to P \times_G (G^{\eta_e} \wedge G^{\eta_e} \wedge S^V).$$

Similarly, the Atiyah duality map, which as discussed above is defined via a Pontryagin–Thom collapse, is an equivariant map

$$\alpha : G^{\eta_e} \wedge S^V \wedge S^V \to \mathrm{Map}(G, S^V) \wedge S^V \wedge S^V,$$

which induces a map

$$\alpha : P \times_G (G^{\eta_e} \wedge S^V \wedge S^V) \to P \times_G (\mathrm{Map}(G, S^V) \wedge S^V \wedge S^V).$$

The compatibility of these Pontryagin–Thom maps yields that the following diagram commutes:

$$\begin{tikzcd} P \times_G (G^{\eta_e} \wedge S^V \wedge S^V) \arrow[r, "\tau_\mu^{V\times V\times V}"] \arrow[d, "\alpha"'] & P \times_G (G^{\eta_e} \wedge G^{\eta_e} \wedge S^V) \arrow[d, "\alpha"] \\ P \times_G \big(\mathrm{Map}(G, S^V) \wedge S^V \wedge S^V\big) \arrow[r, "\mu^\vee"] & P \times_G \big(\mathrm{Map}(G \times G, S^V \wedge S^V) \wedge S^V\big) \end{tikzcd} \tag{10}$$

Passing to spectra, this says that the following diagram of parametrized spectra over X homotopy commutes:

$$\begin{tikzcd} P \times_G G^{-TG} \arrow[r, "\tau_\mu"] \arrow[d, "\alpha"', "\simeq"] & P \times_G G^{-TG} \wedge G^{-TG} \arrow[d, "\simeq"', "\alpha"] \\ P \times_G G^\vee \arrow[r, "\mu^\vee"] & P \times_G (G^\vee \wedge G^\vee) \end{tikzcd} \tag{11}$$

Or, written with the notation used above, the following diagram of parametrized spectra over X homotopy commutes:

$$\begin{tikzcd} P \times_G G^{-TG} \arrow[r, "\tau_\mu"] \arrow[d, "\bar\alpha"', "\simeq"] & P \times_G G^{-TG} \wedge_X P \times_G G^{-TG} \arrow[d, "\simeq"', "\bar\alpha"] \\ \mathcal{D}(\Sigma_X^\infty(P_+^{\mathrm{Ad}})) \arrow[r, "\mu^\vee"] & \mathcal{D}(\Sigma_X^\infty(P_+^{\mathrm{Ad}})) \wedge_X \mathcal{D}(\Sigma_X^\infty(P_+^{\mathrm{Ad}})) \end{tikzcd} \tag{12}$$

In other words, the induced map

$$\alpha : (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \xRightarrow{\simeq} \mathcal{S}_\bullet^P(X)$$

respects coproducts up to homotopy.

This completes the proof of parts (1) and (2) of Theorem 1. Part (3) of Theorem 1 follows from part (1) and the Thom isomorphism applied to the vertical tangent bundle $T_{\mathrm{vert}} \to P^{\mathrm{Ad}}$. The algebra structure on $H^*(P^{\mathrm{Ad}})$ was discovered first by Gruher [2007]. The main point of part (1) of Theorem 1 is that it realizes the work of Gruher on the level of parametrized spectra and the induced twisted homology

theory. In [Gruher 2007] it was shown that in the case of the universal bundle

$$G \to EG \to BG,$$

the algebra structure on $H^*(P^{\mathrm{Ad}} \simeq LBG)$ (or equivalently the coalgebra structure on $H_*(LBG)$) is isomorphic to the Lie group string topology algebra of Chataur and Menichi [2012]. This completes the proof of Theorem 1.

2. Twisted, compact Calabi–Yau ring spectra and the duality between manifold and Lie group string topology

The goal of this section is to study duality phenomena in the string topology of a principal bundle $G \to P \to M$, where G is a compact, d-dimensional Lie group, and M is a closed, n-dimensional manifold. More specifically, our goal is to study the duality between the manifold string topology and the Lie group string topology in this setting. To do this we describe the notion of "twisted, compact Calabi–Yau ring spectra" and show how the string topology of such a principal bundle has this structure. This notion is a lifting to the category of spectra of the notion of "Calabi–Yau" algebras and categories as defined by Costello [2007], Kontsevich and his collaborators [Kontsevich and Soibelman 2009; Kontsevich and Vlassopoulos 2013], Lurie [2009], and Cohen and Ganatra [2015].

Our first result is the following:

Theorem 9. *For a principal bundle $G \to P \to M$ where G a compact Lie group and M is a closed manifold, the manifold string topology spectrum $\mathcal{S}^\bullet_P(M)$ and the Lie group string topology spectrum $\mathcal{S}^P_\bullet(M)$ are Spanier–Whitehead dual. Under this duality the ring spectrum structure of $\mathcal{S}^\bullet_P(M)$ corresponds to the coalgebra structure of $\mathcal{S}^P_\bullet(M)$.*

Proof. Recall from [Cohen and Jones 2017] (and restated in the introduction above), the manifold string topology $\mathcal{S}^\bullet_P$ is the twisted cohomology theory corresponding to the fiberwise suspension spectrum $\Sigma^\infty(G_+) \to \Sigma^\infty_M(P^{\mathrm{Ad}}_+) \to M$. Using a particular version of Poincaré duality proven by Klein [2001] (called "Atiyah duality" in this paper), Cohen and Klein [2009] showed that

$$\mathcal{S}^\bullet_P(M) = \Gamma_M(\Sigma^\infty_M(P^{\mathrm{Ad}}_+)) \simeq (P^{\mathrm{Ad}})^{-TM},$$

and the ring structure comes from a generalized cup product in this (twisted) cohomology theory arising from the fiberwise ring structure of this parametrized spectrum.

Furthermore, Theorem 1 above states that the Lie group string topology $\mathcal{S}^P_\bullet$ is the twisted homology theory corresponding to the fiberwise Spanier–Whitehead dual spectrum, $G^\vee \to \mathcal{D}(\Sigma^\infty_M(P^{\mathrm{Ad}}_+)) \to M$. It was also shown that this is a fiberwise

coalgebra spectrum whose coalgebra structure is (fiberwise) Spanier–Whitehead dual to the ring structure of the parametrized spectrum

$$\Sigma^\infty(G_+) \to \Sigma^\infty_M(P^{\mathrm{Ad}}_+) \to M.$$

Finally it was shown that there is a coproduct-preserving equivalence of spectra,

$$(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \xrightarrow{\simeq} \mathcal{S}^P_\bullet(M).$$

We remark that the fact that the Thom spectra

$$(P^{\mathrm{Ad}})^{-TM} \simeq \mathcal{S}^\bullet_P(M) \quad \text{and} \quad (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \simeq \mathcal{S}^P_\bullet(M)$$

are Spanier–Whitehead dual follows from classical Atiyah [1961] duality.

This completes the proof. □

Remark. When M is oriented and the bundle T_{vert} is oriented, one can apply the two Thom isomorphisms,

$$H_*(P^{\mathrm{Ad}}) \cong H_{*-n}((P^{\mathrm{Ad}})^{-TM}) \quad \text{and} \quad H_*(P^{\mathrm{Ad}}) \cong H_{*-d}((P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}).$$

The Spanier–Whitehead duality above then yields a Frobenius algebra structure on $H_*(P^{\mathrm{Ad}})$ as discovered by Gruher [2007].

We now strengthen this result by proving that in this situation (i.e., a principal bundle $G \to P \to M$, where G is a compact Lie group and M is a closed, smooth manifold), the spectrum $(P^{\mathrm{Ad}})^{-TM}$ is a "twisted, compact Calabi–Yau ring spectrum." The notions of Calabi–Yau differential graded algebras or A_∞ algebras or (higher) categories were introduced in [Costello 2007; Lurie 2009; Kontsevich and Soibelman 2009; Kontsevich and Vlassopoulos 2013] because of their connections with two-dimensional topological field theories. This notion can be viewed as a derived version of a Frobenius algebra. This will be made precise in Proposition 11 below. In this paper we lift these ideas to the category of spectra, where we must deal with "twisted" versions of these notions in order to get many interesting examples. We actually introduce two versions of twisted Calabi–Yau ring spectra: a compact version and a smooth version. This follows the ideas of Kontsevich and his collaborators [Kontsevich and Soibelman 2009; Kontsevich and Vlassopoulos 2013], who worked with A_∞ algebras over a field of characteristic zero, and of Cohen and Ganatra [2015] who worked with A_∞-algebras or categories over arbitrary fields.

We begin with the notion of a "twisted, compact, Calabi–Yau ring spectrum." Recall that a compact E_1- ring spectrum R is one that is perfect as an $\mathbb{S}$-module.

Definition. A *twisted, compact Calabi–Yau ring spectrum* (or *twisted* cCY) of dimension n is a triple (R, Q, t), where R is a compact ring spectrum, Q is an

R-bimodule which is compact as an $\mathbb{S}$-module, and has the same $\mathbb{Z}/2$-homology as R:

$$Q \wedge H\mathbb{Z}/2 \simeq R \wedge H\mathbb{Z}/2,$$

as R-bimodules.

We refer to Q as the "twisting" bimodule. If $Q = R$ we say that R has *trivial* twisting. The map

$$t : \mathrm{THH}(R;\, Q) \to \Sigma^{-n}\mathbb{S}$$

is a map of spectra we call the n-dimensional *trace map* that has the following duality property: The pairing defined by the composition

$$\langle -, -\rangle : R \wedge Q \xrightarrow{\mu} Q \hookrightarrow \mathrm{THH}(R;\, Q) \xrightarrow{t} \Sigma^{-n}\mathbb{S}$$

is nondegenerate in the sense that the adjoint $R \to \Sigma^{-n} Q^{\vee}$ is an equivalence of R-bimodule spectra. Here $\mu : R \wedge Q \to Q$ is the module structure, $Q \hookrightarrow \mathrm{THH}(R;\, Q)$ is the inclusion of the spectrum of zero simplices, and $Q^{\vee}$ is the Spanier–Whitehead dual of Q, which exists because of the compactness assumption.

The following observation is an immediate consequence of the definition.

Proposition 10. *Let (R, Q, t) be a twisted compact Calabi–Yau ring spectrum. Then the duality between R and Q defined by the nondegenerate pairing $\langle -, -\rangle$ defines a coalgebra structure on the twisting bimodule Q, whose coproduct is Spanier–Whitehead dual to the product in the ring structure R.*

The main applications of compact Calabi–Yau ring spectra occur in the presence of orientations. We now define what we mean by this. In our discussion on orientations, we use the S^1-action on topological Hochschild homology given by the cyclic bar construction; a realization of a cyclic spectrum has a circle action. See [Dwyer et al. 1985; Angeltveit et al. 2014].

Definition. Let (R, Q, t) be a twisted, cCY ring spectrum of dimension n, and let E be a ring spectrum representing a homology theory E_*. An E_**-orientation* of (R, Q, t) is a pair $(u, \tilde{t}_E)$, where

$$u : Q \wedge E \xrightarrow{\simeq} R \wedge E$$

is an equivalence of the E_*-homology spectra as R-bimodules. Here R acts trivially on E.

$$\tilde{t}_E : \mathrm{THH}(R;\ R)_{hS^1} \wedge E \to \Sigma^{-n} E$$

is an E-module map from the homotopy orbit spectrum of the S^1-action induced by the cyclic structure, which factorizes the trace map t in E-homology. That is,

the induced trace map $t_E = t \wedge 1 : \mathrm{THH}(R;\ Q) \wedge E \to \Sigma^{-n}\mathbb{S} \wedge E$ is homotopic to the composition

$$t_E : \mathrm{THH}(R;\ Q) \wedge E \xrightarrow{u} \mathrm{THH}(R;\ R) \wedge E \xrightarrow{\text{project}} \mathrm{THH}(R;\ R)_{hS^1} \wedge E, \tag{13}$$

$$\xrightarrow{\tilde{t}_E} \Sigma^{-n} E. \tag{14}$$

When $E = Hk$, the Eilenberg–MacLane spectrum for a field k, then a twisted, compact Calabi–Yau ring spectrum (R, Q, t) together with an Hk-orientation $(u, \tilde{t}_{Hk})$ defines a compact Calabi–Yau algebra structure on the singular chains with k-coefficients, $C_*(R; k)$, as defined in [Kontsevich and Vlassopoulos 2013; Cohen and Ganatra 2015].

The following gives a precise relation between twisted cCY ring spectra and Frobenius algebras.

Proposition 11. *Let (R, Q, t) be a twisted* cCY *ring spectrum of dimension n, and let E be a ring spectrum representing a homology theory E_* with respect to which (R, Q, t) has orientation $(u, \tilde{t}_E)$. Then $R \wedge E$ is a Frobenius algebra over E of dimension n. That is, the pairing*

$$\langle -, - \rangle : (R \wedge E) \wedge (R \wedge E) \xrightarrow{\text{multiply}} R \wedge E \xrightarrow{\iota} \mathrm{THH}(R \wedge E; R \wedge E)$$
$$\xrightarrow{\text{project}} \mathrm{THH}(R \wedge E; R \wedge E)_{hS^1} \xrightarrow{\tilde{t}_E} \Sigma^{-n} E \tag{15}$$

is a nondegenerate pairing of E-modules. Here $\iota : R \wedge E \hookrightarrow \mathrm{THH}(R \wedge E; R \wedge E)$ is the inclusion of the spectrum of 0*-simplices. "Nondegeneracy" means that the adjoint of this pairing,*

$$R \wedge E \to \mathrm{Rhom}_E(R \wedge E, \Sigma^{-n} E)$$

is an equivalence of E-modules.

Proof. It is easily checked from the definition of orientation that the pairing $\langle -, - \rangle$ defined above is homotopic to the composition

$$(R \wedge E) \wedge (R \wedge E) \xrightarrow{1 \wedge u^{-1}} (R \wedge E) \wedge (Q \wedge E) \xrightarrow{\mu} Q \wedge E \hookrightarrow \mathrm{THH}(R; Q) \wedge E$$
$$\xrightarrow{t_E = t \wedge 1} \Sigma^{-n}\mathbb{S} \wedge E.$$

But this pairing is nondegenerate by the definition of the twisted Calabi–Yau structure. □

We now give two important examples of twisted cCY ring spectra.

Example. The first example shows how ordinary Poincaré or Atiyah duality fits the definition of twisted compact Calabi–Yau ring spectra.

Proposition 12. *Let M be a closed n-dimensional manifold. Then its Spanier–Whitehead dual, $M^\vee$, which, by Atiyah duality is equivalent to M^{-TM}, comes naturally equipped with the structure of a twisted* cCY *ring spectrum of dimension n.*

Proof. The suspension spectrum $\Sigma^\infty(M_+)$ can be viewed as a $M^\vee$ bimodule in the usual way. Notice that since, by Atiyah duality, $M^\vee$ is equivalent to M^{-TM}, then the Thom isomorphism gives

$$H_*(\Sigma^\infty(M_+); \mathbb{Z}/2) \cong H_{*-n}(M^\vee; \mathbb{Z}/2).$$

So we let $R = M^\vee$ and let the twisting bimodule $Q = \Sigma^{-n}\Sigma^\infty(M_+)$, which we simply denote $\Sigma^{-n}(M_+)$.

In order to define the n-dimensional trace map on $\mathrm{THH}(R; Q)$, we first study its homotopy type. This is a simplicial object in finite type spectra. That is, for each k, the spectrum of k-simplices is a spectrum of finite type. For such a simplicial spectrum $\mathbb{X}_\bullet$ we define its Spanier–Whitehead dual $\mathbb{X}^\vee$ to be the totalization of the cosimplicial spectrum whose spectrum of k-simplices is the Spanier–Whitehead dual $\mathbb{X}_k^\vee = \mathrm{Map}(\mathbb{X}_k, \mathbb{S})$. We then have the following result.

Lemma 13. *For M a closed n-manifold, $R = M^\vee$ and $Q = \Sigma^{-n}(M_+)$, the Spanier–Whitehead dual of* $\mathrm{THH}(R; Q)$ *is given by*

$$\mathrm{THH}(R; Q)^\vee \simeq \Sigma^n L M^{-TM}.$$

Proof. Note that

$$\mathrm{THH}(R; Q)_k = R^{(k)} \wedge Q \simeq (M^k)^\vee \wedge \Sigma^\infty(M_+) \wedge S^{-n}.$$

Therefore in the cosimplicial spectrum $\mathrm{THH}(R; Q)^\vee$, the spectrum of k-simplices is equivalent to

$$\mathrm{THH}(R; Q)_k^\vee \simeq \Sigma^\infty(M_+^k) \wedge M^\vee \wedge S^n \simeq \Sigma^\infty(M_+^k) \wedge M^{-TM} \wedge S^n.$$

Under this equivalence, the coface maps are determined by the coalgebra structure of $\Sigma^\infty(M_+)$ defined by the diagonal map of M, as well as the bicomodule structure of M^{-TM}. This structure is a special case of the bicomodule structure of any Thom spectrum of a bundle (or spherical fibration) ζ over a space X over the coalgebra $\Sigma^\infty(X_+)$. The bicomodule structure is given by the maps of Thom spectra that are induced by the diagonal maps of the base,

$$X^\zeta \to X_+ \wedge X^\zeta \quad \text{and} \quad X^\zeta \to X^\zeta \wedge X_+.$$

This cosimplicial spectrum is the n-fold suspension of the cosimplicial spectrum studied in [Cohen and Jones 2002] where it was shown to have totalization equivalent to LM^{-TM}. □

Remark. Notice that the inclusion of the spectrum of zero simplices,

$$\Sigma^{-n}(M_+) \hookrightarrow \mathrm{THH}(R;\, Q)$$

is Spanier–Whitehead dual to the map

$$\Sigma^n LM^{-TM} \xrightarrow{\text{eval.}} \Sigma^n M^{-TM} \simeq \Sigma^n M^\vee$$

induced on Thom spectra by the usual evaluation fibration $LM \to M$.

One way of thinking of the n-dimensional trace map $t : \mathrm{THH}(R;\, Q) \to \Sigma^{-n}\mathbb{S}$ is that it is Spanier–Whitehead dual to the n-fold suspension of the unit map in the ring structure of LM^{-TM}:

$$\Sigma^n \mathbb{S} \to \Sigma^n LM^{-TM}.$$

More concretely, notice that the augmentation map of R,

$$\epsilon : R = M^\vee \to \mathbb{S}$$

and the map induced by sending all of M to the nonbase point

$$p : \Sigma^{-n}(M_+) \to \Sigma^{-n}\mathbb{S}$$

define a map

$$t : \mathrm{THH}(R;\, Q) = \mathrm{THH}(M^\vee;\, \Sigma^{-n}(M_+)) \xrightarrow{(\epsilon,\, p)} \mathrm{THH}(\mathbb{S};\, \Sigma^{-n}\mathbb{S}) = \Sigma^{-n}\mathbb{S}.$$

The reader can now check that the composition

$$M^\vee \wedge \Sigma^{-n}(M_+) \xrightarrow{\mu} \Sigma^{-n}(M_+) \to \mathrm{THH}(M^\vee;\, \Sigma^{-n}(M_+)) \xrightarrow{t} \Sigma^{-n}\mathbb{S}$$

is simply the n-fold desuspension of the duality map, and therefore is nondegenerate. This proves that $(M^\vee,\, \Sigma^{-n}(M_+),\, t)$ is a twisted, compact Calabi–Yau ring spectrum of dimension n. □

We now consider orientations. Let E be any ring spectrum representing a generalized homology theory with respect to which M is oriented. The Thom isomorphism then defines an equivalence

$$u : \Sigma^{-n}(M_+) \wedge E \xrightarrow{\simeq} M^{-TM} \wedge E \simeq M^\vee \wedge E$$

which is clearly an equivalence of $M^\vee$-bimodules. Again consider the augmentation map $\epsilon : M^\vee \to \mathbb{S}$. Now the orientation induces a Thom class map

$$\tau : M^\vee \simeq M^{-TM} \to \Sigma^{-n} E.$$

These maps define a composition

$$\tilde{t}_E : \mathrm{THH}(M^\vee; M^\vee)_{hS^1} \wedge E \xrightarrow{(\epsilon,\tau)} \mathrm{THH}(\mathbb{S}; \mathbb{S})_{hS^1} \wedge \Sigma^{-n} E \simeq \Sigma^{-n}(BS^1_+) \wedge E \xrightarrow{p\wedge 1} \Sigma^{-n} E,$$

where $p : BS^1_+ \to S^0$ is the projection map.

We leave it to the reader to check that the composition

$$\mathrm{THH}(M^\vee, \Sigma^{-n}(M_+)) \wedge E \xrightarrow{u} \mathrm{THH}(M^\vee, M^\vee) \wedge E \xrightarrow{\text{projection}} \mathrm{THH}(M^\vee, M^\vee)_{hS^1} \wedge E \xrightarrow{\tilde{t}_E} \Sigma^{-n} E$$

is equivalent to $t \wedge 1 : \mathrm{THH}(M^\vee, \Sigma^\infty(M_+)) \wedge E \to \Sigma^{-n} \wedge \mathbb{S} \wedge E$. This proves that the pair $(u, \tilde{t}_E)$ defines an orientation of the twisted cCY structure on $M^\vee$ with respect to E.

Remark. The above discussion together with Proposition 11 implies that if M^n is an oriented closed manifold, $M^\vee \wedge H\mathbb{Z}$ is a Frobenius algebra over the Eilenberg–MacLane spectrum $H\mathbb{Z}$. Using the Atiyah duality equivalence $M^\vee \simeq M^{-TM}$ we see that $M^{-TM} \wedge H\mathbb{Z} \simeq \Sigma^{-n}(M_+ \wedge H\mathbb{Z})$ is a Frobenius algebra. The multiplication reflects the classical intersection product on the level of chains, $C_{*+n}(M; \mathbb{Z})$. The comultiplication comes from the diagonal, $M \to M \times M$.

Example. The following example supplies the main ingredient for the proof of Theorem 3 as stated in the introduction.

Proposition 14. *Let $G \to P \to M$ be a principal bundle where G is a compact Lie group of dimension d and M is a closed manifold of dimension n. Then the manifold string topology ring spectrum $R = \mathcal{S}^\bullet_P(M) \simeq (P^{\mathrm{Ad}})^{-TM}$ naturally admits the structure of a twisted, compact Calabi–Yau ring spectrum of dimension $n - d$.*

Proof. We need to produce the twisting module Q and a trace map

$$t : \mathrm{THH}(R; Q) \to \Sigma^{d-n}\mathbb{S}.$$

For the twisting module we take the Lie group string topology spectrum $Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \simeq \Sigma^{d-n}(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$. The fact that R and Q have isomorphic mod-2 homology follows from the Thom isomorphism. The fact that Q is indeed an R-bimodule follows from the Spanier–Whitehead duality of $R = \mathcal{S}^\bullet_P(M)$ and $\Sigma^{n-d} Q = \mathcal{S}^P_\bullet(M)$ established in Theorem 9, reflecting Gruher's work [2007]. The bimodule structure of Q over R is then the dual of the bimodule structure of R over itself.

Notice also that $R = \mathcal{S}^\bullet_P(M) \simeq (P^{\mathrm{Ad}})^{-TM}$ has an augmentation $\epsilon : R \to \mathbb{S}$. To see this, consider the following diagram, which we view as a map of bundles.

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \{\mathrm{id}\} \arrow[d] \\ P^{\mathrm{Ad}} \arrow[r] \arrow[d] & M \arrow[d, "="] \\ M \arrow[r, "="] & M \end{tikzcd}$$

This defines a map of twisted cohomology ring spectra

$$\mathcal{S}^\bullet_P(M) \to \mathcal{S}^\bullet_M(M)$$

or equivalently,

$$\Gamma_M(\Sigma^\infty_M(P^{\mathrm{Ad}}_+)) \to \mathrm{Map}(\Sigma^\infty(M_+), \mathbb{S}) = M^\vee.$$

The augmentation is then given by

$$\epsilon : R = \mathcal{S}^\bullet_P(M) \to M^\vee \to \mathbb{S},$$

where the second map in this composition is the augmentation of $M^\vee \to \mathbb{S}$.

Notice that the above diagram also defines a map of bimodules, induced on the homology spectra,

$$Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \simeq \Sigma^{d-n}(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \to \Sigma^{d-n}\mathcal{S}^M_\bullet(M) = \Sigma^{d-n}(M_+).$$

Composing this map with the projection $p : \Sigma^{d-n}(M_+) \to \Sigma^{d-n}\mathbb{S}$ defines a map $u : Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \to \Sigma^{d-n}\mathbb{S}$. Putting these maps together gives a map of topological Hochschild homologies,

$$t : \mathrm{THH}(R, Q) \xrightarrow{(\epsilon, u)} \mathrm{THH}(\mathbb{S}, \Sigma^{d-n}\mathbb{S}) = \Sigma^{d-n}\mathbb{S}.$$

We leave it to the reader to verify that the pairing defined by the composition

$$\langle -, - \rangle : R \wedge Q = \mathcal{S}^\bullet_P(M) \wedge \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \xrightarrow{\mu} Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \hookrightarrow \mathrm{THH}(R, Q) \xrightarrow{t} \Sigma^{d-n}\mathbb{S} \quad (16)$$

is the duality map given by Theorem 9 above. It is therefore nondegenerate. This proves that the triple $\left(\mathcal{S}^\bullet_P(M), \Sigma^{d-n}\mathcal{S}^P_\bullet(M), t\right)$ is a twisted compact Calabi–Yau ring spectrum of dimension $n - d$. □

Notice that Propositions 14, 10, and 11 imply both Corollary 2 and Theorem 3 as stated in the introduction.

3. Gauge symmetry

In this section we continue considering a principal bundle $G \to P \xrightarrow{p} M$, where G is a compact Lie group of dimension d, and M^n is a closed manifold of dimension n.

Recall that the *gauge group* $\mathcal{G}(P)$ of the bundle P is the group of G-equivariant bundle automorphisms of P living over the identity of M. Said another way, let $G \to \mathcal{A}ut^G(P) \to M$ be the fibration whose fiber over $x \in M$ is the group of G-equivariant automorphisms of the fiber $p^{-1}(x)$. This bundle is a fiberwise group, and the gauge group is the group of sections

$$\mathcal{G}(P) = \Gamma_M(\mathcal{A}ut^G(P)).$$

Now a standard exercise shows that the bundle $\mathcal{A}ut^G(P)$ is isomorphic to the adjoint bundle $G \to P^{\mathrm{Ad}} \to M$. Thus we may identify

$$\mathcal{G}(P) = \Gamma_M(P^{\mathrm{Ad}}).$$

In [Cohen and Jones 2017] a fiberwise stabilization map was defined and studied:

$$\begin{aligned}\rho : \Sigma^\infty(\mathcal{G}(P)_+) = \Sigma^\infty(\Gamma_M(P^{\mathrm{Ad}})_+) \\ \to \Gamma_M(\Sigma^\infty(P^{\mathrm{Ad}}_+)) \simeq (P^{\mathrm{Ad}})^{-TM} = \mathcal{S}^\bullet_P(M).\end{aligned} \tag{17}$$

The map ρ is a map of ring spectra and also defines a map to the group of units of the (manifold) string topology ring spectrum

$$\rho : \mathcal{G}(P) \to \mathrm{GL}_1(\mathcal{S}^\bullet_P(M)). \tag{18}$$

In [Cohen and Jones 2017] this map was studied and computed in several important cases. Now recall from Proposition 14 that in the twisted compact Calabi–Yau structure of $\mathcal{S}^\bullet_P(M)$, the twisting bimodule is given by a suspension of the Lie group string topology spectrum, $Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \simeq \Sigma^{d-n}(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$. In particular, the Lie group string topology spectrum $\mathcal{S}^P_\bullet(M)$ inherits a coalgebra structure. One of the goals of this section is to show that there is a similarly defined and compatible gauge symmetry on this spectrum. We also show how these actions are related, and describe different perspectives on this action. We then compute two examples of this gauge symmetry.

Theorem 15. *The twisting bimodule structure on the Lie group string topology spectrum, $Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M)$ has a natural action of the gauge group $\mathcal{G}(P)$. That is, $\Sigma^{d-n}\mathcal{S}^P_\bullet(M)$ is a module spectrum over the ring spectrum $\Sigma^\infty(\mathcal{G}(P)_+)$. Furthermore, this action is compatible with the gauge symmetry on the manifold string topology spectrum $R = \mathcal{S}^\bullet_P(M)$ via its twisted Calabi–Yau duality pairing*

$$\langle -, - \rangle : R \wedge Q = \mathcal{S}^\bullet_P(M) \wedge \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \to \Sigma^{d-n}\mathbb{S}$$

as defined in the proof of Proposition 14 (*see Equation* (16)). *That is*, *the adjoint equivalence*

$$\mathcal{S}_P^\bullet(M) \xrightarrow{\simeq} \mathcal{S}_\bullet^P(M)^\vee$$

is equivariant with respect to the gauge symmetry of these spectra.

Proof. Recall that $\mathcal{S}_\bullet^P(M)$ is the generalized homology associated to the parametrized spectrum

$$G^\vee \to \mathcal{D}(\Sigma_M^\infty(P_+^{\mathrm{Ad}})) \to M.$$

We may take $\mathcal{D}(\Sigma_M^\infty(P_+^{\mathrm{Ad}}))$ to be the parametrized spectrum whose k-th space over M is the fibration

$$\mathrm{Map}(G, S^k) \to P \times_G \mathrm{Map}(G, S^k) \to M$$

where the action of G on $\mathrm{Map}(G, S^k)$ is the dual of the adjoint action as described in (4), with trivial G-action on S^k. This is because the spaces $\mathrm{Map}(G, S^k)$ with this action form the underlying naive G-spectrum of $G^\vee$, on which the homotopy theory of $G^\vee$ as a $\Sigma^\infty(G_+)$-module is determined.

This fibration has a canonical section

$$\sigma : M = P \times_G \mathrm{point} = P \times_G \epsilon \hookrightarrow P \times_G \mathrm{Map}(G, S^k).$$

Here $\epsilon : G \to S^k$ is the constant map at the basepoint $(1, 0, \ldots, 0) \in S^k$. Then the k-th space of the generalized homology spectrum

$$\mathcal{S}_\bullet^P(M) = \mathcal{D}(\Sigma_M^\infty(P_+^{\mathrm{Ad}}))/\sigma(M)$$

is given by

$$P_+ \wedge_G \mathrm{Map}(G, S^k).$$

The structure maps are given by

$$\Sigma\big(P_+ \wedge_G \mathrm{Map}(G, S^k)\big) = P_+ \wedge_G \Sigma(\mathrm{Map}(G, S^k)) \xrightarrow{1\wedge s} P_+ \wedge_G \mathrm{Map}(G, S^{k+1}),$$

where $s : \Sigma(\mathrm{Map}(G, S^k)) \to \mathrm{Map}(G, \Sigma S^k)$ is given by $s(t, \phi)(g) = t \wedge \phi(g)$.

Now the bundle $p : P \times_G \mathrm{Map}(G, S^k) \to M$ is $\mathcal{G}(P)$-equivariant with respect to the following action. Let $\phi \in \mathcal{G}(P) = \Gamma_M(P^{\mathrm{Ad}})$, and let $(y, \theta) \in P \times \mathrm{Map}(G, S^k)$ represent an element in $P \times_G \mathrm{Map}(G, S^k)$. Then

$$\phi \cdot (y, \theta) = (y, h \cdot \theta), \tag{19}$$

where $h \in G$ is the unique element such that $\phi(p(y, \theta)) \in P \times_G G^{\mathrm{Ad}}$ is represented by $(y, h) \in P \times G$.

One can check that this action is well-defined, and that the section $\sigma(M = P \times_G \epsilon)$ consists of fixed points of this action. It therefore descends to a $\mathcal{G}(P)$-action on $P_+ \wedge_G \mathrm{Map}(G, S^k)$.

These actions (one for each k) clearly respect the structure maps and therefore define an action of $\mathcal{G}(P)$ on the spectrum $\mathcal{D}(\Sigma^\infty_M(P^{\mathrm{Ad}}_+))/\sigma(M) = \mathcal{S}^P_\bullet(M)$.

Now, as seen in Corollary 2, the Lie group string topology spectrum $\mathcal{S}^P_\bullet(M)$ is Spanier–Whitehead dual to the manifold string topology spectrum $\mathcal{S}^\bullet_P(M) = \Gamma_M(\Sigma^\infty_M(P^{\mathrm{Ad}}_+))$. The action of the gauge group on this ring spectrum is given by the stabilization representation (17), (18), and it is immediate that the gauge symmetry defined on the Lie group string topology spectrum $\mathcal{S}^P_\bullet(M)$ in (19) is the dual action. This implies that with respect to the twisted Calabi–Yau duality pairing

$$\langle -, - \rangle : R \wedge Q = \mathcal{S}^\bullet_P(M) \wedge \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \to \Sigma^{d-n}\mathbb{S}$$

the corresponding adjoint equivalence

$$\mathcal{S}^\bullet_P(M) \xrightarrow{\simeq} \mathcal{S}^P_\bullet(M)^\vee$$

is equivariant with respect to the gauge symmetry of these spectra. □

We now study examples of this gauge symmetry and describe this symmetry from different perspectives.

Example. Consider the $U(1)$ Hopf bundle

$$U(1) \to P = S^{2n+1} \to \mathbb{CP}^n.$$

Since $U(1)$ is abelian, the adjoint bundle P^{Ad} is trivial,

$$U(1) \to \mathbb{CP}^n \times U(1) \to \mathbb{CP}^n.$$

Therefore the gauge group is given by the mapping group,

$$\mathcal{G}(P) = \mathrm{Map}(\mathbb{CP}^n, U(1)).$$

Also, the fiberwise suspension spectrum $\Sigma^\infty(U(1)_+) \to \Sigma^\infty_{\mathbb{CP}^n}(P^{\mathrm{Ad}}_+) \to \mathbb{CP}^n$ is given by the trivially parametrized spectrum

$$\Sigma^\infty(U(1)_+) \to \mathbb{CP}^n_+ \wedge \Sigma^\infty(U(1)_+) \to \mathbb{CP}^n.$$

(By the *trivially parametrized spectrum* $\mathbb{CP}^n_+ \wedge \Sigma^\infty(U(1)_+)$ we mean the parametrized spectrum whose k-th space is the trivial fibration $\mathbb{CP}^n \times \Sigma^k(U(1)_+) \to \mathbb{CP}^n$.)

The twisted cohomology theory this parametrized spectrum represents is therefore actually untwisted, and so the defining spectrum of sections is the mapping spectrum,

$$R = \mathcal{S}^\bullet_P(\mathbb{CP}^n) = \mathrm{Map}\big(\Sigma^\infty(\mathbb{CP}^n), \Sigma^\infty(U(1)_+)\big).$$

This is an E_∞-ring spectrum because the source of the mapping spectrum is an E_∞-coalgebra spectrum and the target is an E_∞-ring spectrum.

The action of the gauge group $\mathcal{G}(P) = \mathrm{Map}(\mathbb{CP}^n, U(1))$ is then given by the map of ring spectra

$$\Sigma^\infty(\mathcal{G}(P)_+) = \Sigma^\infty\big(\mathrm{Map}(\mathbb{CP}^n, U(1))_+\big) \xrightarrow{\sigma} \mathrm{Map}\big(\Sigma^\infty(\mathbb{CP}^n), \Sigma^\infty(U(1)_+)\big) = \mathcal{S}^\bullet_P(\mathbb{CP}^n) = R, \quad (20)$$

where σ is the obvious stabilization map. The role of stabilization in understanding gauge symmetry on manifold string topology spectra was studied in general in [Cohen and Jones 2017].

Now consider the gauge symmetry on the bimodule $Q = \Sigma^{1-2n}\mathcal{S}^P_\bullet(\mathbb{CP}^n) = \Sigma^{1-2n}(S^{2n+1}_+ \wedge_{U(1)} U(1)^\vee)$ where the action of $U(1)$ on the Spanier–Whitehead dual $U(1)^\vee$ is (the dual of) the conjugation action. Again, since $U(1)$ is abelian this action is trivial, so

$$Q = \Sigma^{1-2n}(\mathbb{CP}^n_+ \wedge U(1)^\vee).$$

Of course $U(1) \cong S^1$, so $U(1)^\vee \simeq \Sigma^\infty(S^{-1} \vee S^0)$. By Spanier–Whitehead duality, the action of the gauge group $\mathcal{G}(P)$ is given by composing the stabilization map described above (20)

$$\sigma : \Sigma^\infty(\mathcal{G}(P)_+) \to R$$

with the R-bimodule action on the desuspension of its dual, Q, as described in the proof of Proposition 14.

Before we move on to another example, we consider the action of the gauge group on the level of Thom spectra. The point is that the Calabi–Yau ring spectrum in question, $R \simeq (P^{\mathrm{Ad}})^{-TM}$, and the twisting bimodule, $Q \simeq \Sigma^{d-n}(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$, are both Thom spectra. To understand the induced gauge symmetry on these Thom spectra, we first observe that the gauge group actually acts on the space P^{Ad}, and the actions on the Thom spectra are induced from it.

Let $G \to P \xrightarrow{p} M$ be a principal bundle with M a closed n-manifold and G a compact Lie group of dimension d. By abuse of notation we also call the projection map of the induced bundle $p : P^{\mathrm{Ad}} \to M$. Let $\phi \in \mathcal{G}(P) = \Gamma_M(P^{\mathrm{Ad}})$. Since ϕ is a section of P^{Ad}, for $y \in P^{\mathrm{Ad}}$, $\phi(p(y))$ and y live in the same fiber over M. That is, $p(\phi(p(y)) = p(y) \in M$. Thus the pair, $(\phi(p(y)), y)$ lies in the fiber product $P^{\mathrm{Ad}} \times_M P^{\mathrm{Ad}}$. Since P^{Ad} is a fiberwise group, we can compose with the fiberwise multiplication $\mu : P^{\mathrm{Ad}} \times_M P^{\mathrm{Ad}} \to P^{\mathrm{Ad}}$ to produce an element $\phi \cdot y = \mu(\phi(p(y)), y) \in P^{\mathrm{Ad}}$. The map

$$\mathcal{G}(P) \times P^{\mathrm{Ad}} \to P^{\mathrm{Ad}}, \quad (\phi, y) \mapsto \phi \cdot y$$

defines an action of the gauge group on P^{Ad}. This in fact defines a $\mathcal{G}(P)$-equivariance on the fiber bundle, $G \to P^{\mathrm{Ad}} \to M$. That is, the following diagram commutes:

$$\begin{tikzcd} \mathcal{G}(P) \times P^{\mathrm{Ad}} \arrow[r, "\cdot"] \arrow[d] & P^{\mathrm{Ad}} \arrow[d, "p"'] \\ M \arrow[r, "="] & M \end{tikzcd} \tag{21}$$

where the left vertical arrow composes the projection map $\mathcal{G}(P) \times P^{\mathrm{Ad}} \to P^{\mathrm{Ad}}$ with the bundle map $P^{\mathrm{Ad}} \to M$. Therefore this action induces an action on any (virtual) vector bundle over P^{Ad} that is pulled back from a bundle over M. In particular, on the level of Thom spectra, there is an induced action

$$\mathcal{G}(P)_+ \wedge (P^{\mathrm{Ad}})^{-TM} \to (P^{\mathrm{Ad}})^{-TM}.$$

This is easily seen to be equivalent to the action of $\mathcal{G}(P)$ on $\mathcal{S}^{\bullet}_P(M) \simeq (P^{\mathrm{Ad}})^{-TM}$ described above.

We now observe that the gauge symmetry on the Lie group string topology spectrum $\mathcal{S}^P_{\bullet}(M) \simeq (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$ described above can also be viewed in terms of the space level action of $\mathcal{G}(P)$ on P^{Ad}. This is a consequence of the following observation.

Proposition 16. *Let* $\mathrm{Act} : \mathcal{G}(P) \times P^{\mathrm{Ad}} \to P^{\mathrm{Ad}}$ *be the action map described above. Then there is an isomorphism of virtual bundles over* $\mathcal{G}(P) \times P^{\mathrm{Ad}}$,

$$\mathcal{G}(P) \times -T_{\mathrm{vert}} P^{\mathrm{Ad}} \xrightarrow{\cong} \mathrm{Act}^*(-T_{\mathrm{vert}} P^{\mathrm{Ad}}).$$

Proof. We first observe that the commutativity of diagram (21) says that there is an isomorphism of vector bundles over $\mathcal{G}(P) \times P^{\mathrm{Ad}}$,

$$\mathcal{G}(P) \times p^*(TM) \cong \mathrm{Act}^*(p^*(TM)).$$

Notice also that there is an isomorphism of vector bundles,

$$D : \mathcal{G}(P) \times TP^{\mathrm{Ad}} \xrightarrow{\cong} \mathrm{Act}^*(TP^{\mathrm{Ad}}),$$

where $TP^{\mathrm{Ad}} \to P^{\mathrm{Ad}}$ is the tangent bundle. The isomorphism is given by differentiation of the action. Now notice that there is an induced isomorphism of virtual bundles, which by abuse of notation we call $-D : \mathcal{G}(P) \times -TP^{\mathrm{Ad}} \xrightarrow{\cong} \mathrm{Act}^*(-TP^{\mathrm{Ad}})$. This is defined by the composition of isomorphisms

$$\begin{aligned} \mathrm{Act}^*(-TP^{\mathrm{Ad}}) &= -\,\mathrm{Act}^*(TP^{\mathrm{Ad}}) \\ &\cong -(\mathcal{G}(P) \times TP^{\mathrm{Ad}}) \quad \text{by the above,} \\ &= \mathcal{G}(P) \times -TP^{\mathrm{Ad}}. \end{aligned}$$

Now, using the fact that

$$-T_{\mathrm{vert}}P^{\mathrm{Ad}} \cong -TP^{\mathrm{Ad}} \oplus p^*(TM)$$

we have that

$$\begin{aligned}\mathrm{Act}^*(-T_{\mathrm{vert}}P^{\mathrm{Ad}}) &\cong \mathrm{Act}^*(-TP^{\mathrm{Ad}}) \oplus \mathrm{Act}^*(p^*(TM)) \\ &\cong \mathcal{G}(P) \times \big(-TP^{\mathrm{Ad}} \oplus p^*(TM)\big) \quad \text{by the above} \\ &\cong \mathcal{G}(P) \times -T_{\mathrm{vert}}P^{\mathrm{Ad}}. \qquad \square\end{aligned}$$

The last explicit example of this gauge symmetry will be one that was studied initially in [Cohen and Jones 2017].

Example. Consider the principal SU(2)-bundle over an oriented 4-dimensional sphere,

$$\mathrm{SU}(2) \to P_k \to S^4 \tag{22}$$

having second Chern class $c_2(P_k) = k \in H^4(S^4) \cong \mathbb{Z}$.

In this case we restrict our attention to the *based* gauge group $\mathcal{G}^b(P_k)$ which is defined to be the kernel of the homomorphism,

$$\mathcal{G}(P_k) \to \mathrm{SU}(2), \quad \phi \mapsto \phi(\infty).$$

Here we are thinking of S^4 as the one point compactification, $\mathbb{R}^4 \cup \infty$.

In the case $k = 1$ the (based) gauge symmetry on the manifold string topology ring spectrum $\mathcal{S}^{\bullet}_{P_k}(S^4)$ was studied in [Cohen and Jones 2017]. We now observe that the argument presented in [Cohen and Jones 2017] quickly extends to P_k for all k, and we then show how it gives an understanding of the gauge symmetry of the Lie group string topology spectrum $\mathcal{S}^{P_k}_{\bullet}(S^4)$ as well.

As has become standard notation, given a ring spectrum R, let $\mathrm{GL}_1(R)$ denote the "group of units" of R. More precisely, $\mathrm{GL}_1(R)$ is defined so that the following diagram of spaces is homotopy cartesian:

$$\begin{tikzcd} \mathrm{GL}_1(R) \arrow[r] \arrow[d] & \Omega^\infty(R) \arrow[d, "\text{components}"] \\ \pi_0(R)^\times \arrow[r, hook] & \pi_0(R) \end{tikzcd} \tag{23}$$

Here $\pi_0(R)$ is the discrete ring of components and $\pi_0(R)^\times$ is its group of units.

In other words, $\mathrm{GL}_1(R)$ consists of those path components of the zero space $\Omega^\infty(R)$ consisting of homotopy invertible elements. An action of a group G on a ring spectrum R via R-module automorphisms is induced by an A_∞-morphism ("representation")

$$\rho : G \to \mathrm{GL}_1(R).$$

(See for example [Lind 2016]). To understand the gauge symmetry on the manifold string topology spectrum $\mathcal{S}^{\bullet}_{P_k}(S^4)$ we therefore want to describe the representation

$$\mathcal{G}(P_k) \to \mathrm{GL}_1(\mathcal{S}^{\bullet}_{P_k}(S^4)). \tag{24}$$

Now as observed in [Cohen and Jones 2017], the group-like monoid $\mathrm{GL}_1(\mathcal{S}^{\bullet}_{P_k}(S^4))$ is equivalent to the grouplike monoid $\mathrm{hAut}(\Sigma^{\infty}_{S^4}((P_k)_+)$ of homotopy automorphisms of the parametrized spectrum $\Sigma^{\infty}_{S^4}((P_k)_+)$.

To understand this monoid of homotopy automorphisms, note that given any ring spectrum R and parametrized R-line bundle $\mathcal{E}$ over M, there is a fibration sequence

$$\mathrm{hAut}^b(\mathcal{E}) \to \mathrm{hAut}(\mathcal{E}) \xrightarrow{\mathrm{ev}} \mathrm{hAut}^R(\mathcal{E}_{x_0}) = \mathrm{GL}_1(R) \tag{25}$$

where the map ev evaluates an automorphism on the fiber over the basepoint $x_0 \in M$. The fiber is $\mathrm{hAut}^b(R)$, which is the A_∞ group-like monoid of based homotopy automorphisms. This is the subgroup of $\mathrm{hAut}(\mathcal{E})$ consisting of those homotopy automorphisms that are equal to the identity on the fiber spectrum at the basepoint $\mathcal{E}_{x_0}$.

Putting these facts together yields a fibration sequence of group-like monoids,

$$\mathrm{hAut}^b(\Sigma^{\infty}_{S^4}((P_k)_+)) \to \mathrm{GL}_1(\mathcal{S}^{\bullet}_{P_k}(S^4)) \to \mathrm{GL}_1(\Sigma^{\infty}(\mathrm{SU}(2)_+)). \tag{26}$$

As was done in [Cohen and Jones 2017], we observe that since $\mathrm{SU}(2) \cong S^3$, the defining diagram (23) becomes, in the case of $R = \Sigma^{\infty}(\mathrm{SU}(2)_+)$,

$$\begin{tikzcd}
\mathrm{GL}_1(\Sigma^{\infty}(\mathrm{SU}(2)_+)) \arrow[r] \arrow[d] & Q(S^3_+) \arrow[d, "\text{components}"] \\
\pm 1 \arrow[r, hook] & \mathbb{Z}
\end{tikzcd}$$

That is, $\mathrm{GL}_1(\Sigma^{\infty}(\mathrm{SU}(2)_+))$ consists of two path components of the infinite loop space $Q(S^3_+)$ corresponding to the units $\pm 1 \in \mathbb{Z} \cong \pi_0(Q(S^3_+))$. We denote this space by $Q_{\pm 1}(S^3_+)$. Therefore fibration (26) has base space $Q_{\pm 1}(S^3_+)$. We now examine the homotopy type of the fiber, $\mathrm{hAut}^b(\Sigma^{\infty}_{S^4}((P_k)_+))$.

By one of the main results of [Cohen and Jones 2017] (Theorem 3), there is an equivalence

$$\begin{aligned}
&\mathrm{hAut}^b(\Sigma^{\infty}_{S^4}((P_k)_+)) \\
&\qquad \xrightarrow{\simeq} \Omega\,\mathrm{Map}^b_k\big(S^4, B\mathrm{GL}_1(\Sigma^{\infty}(\mathrm{SU}(2)_+))\big) = \Omega^4_k\,\mathrm{GL}_1(\Sigma^{\infty}(\mathrm{SU}(2)_+)),
\end{aligned}$$

where Map^b_k denotes the path component of the based mapping space corresponding to

$$k \in \mathbb{Z} = \pi_0\big(\mathrm{Map}^b\big(S^4, B\mathrm{GL}_1(\Sigma^{\infty}(\mathrm{SU}(2)_+))\big)\big).$$

Similarly Ω^4_k denotes the corresponding path component in $\Omega^4\,\mathrm{GL}_1(\Sigma^\infty(\mathrm{SU}(2)_+))$. Now, since $\Omega^4\,\mathrm{GL}_1(\Sigma^\infty(\mathrm{SU}(2)_+))$ is a group-like monoid, all of its path components are homotopy equivalent. So we therefore have the following result, which gives a good understanding of the group of units of the manifold string topology spectrum of the principal bundle $\mathrm{SU}(2) \to P_k \to S^4$.

Lemma 17. *For any k, there is an equivalence of group-like monoids,*

$$\phi_k : \mathrm{hAut}^b\big(\Sigma^\infty_{S^4}((P_k)_+)\big) \xrightarrow{\simeq} \Omega^4 Q(S^3_+).$$

Furthermore, there are homotopy fibration sequences of group-like monoids

$$\Omega^4 Q(S^3_+) \xrightarrow{\iota_k} \mathrm{GL}_1(\mathcal{S}^\bullet_{P_k}(S^4)) \xrightarrow{q_k} Q_{\pm 1}(S^3_+).$$

In order to understand the representation $\rho_k : \mathcal{G}^b(P_k) \to \mathrm{GL}_1(\mathcal{S}^\bullet_{P_k}(S^4))$ describing the gauge symmetry of the manifold string topology spectrum, we now consider the homotopy type of the based gauge group $\mathcal{G}^b(P_k)$. Again, for $k = 1$ this was done in [Cohen and Jones 2017], and we simply adapt the argument there to apply to all k.

By a basic result on the topology of gauge groups proved by Atiyah and Bott [1983], we have that

$$\mathcal{G}^b(P_k) \simeq \Omega\,\mathrm{Map}^b_k(S^4, B\,\mathrm{SU}(2)),$$

where, as above, Map^b_k denotes the path component of degree k based maps. This (based) loop space is equivalent to $\Omega\Omega^3_k\,\mathrm{SU}(2) = \Omega\Omega^3_k S^3$. Since $\Omega^3 S^3$ is a group-like monoid, all of its path components are equivalent, so we have the following.

Lemma 18. *For any k, there is an equivalence of group-like monoids,*

$$\psi_k : \mathcal{G}^b(P_k) \xrightarrow{\simeq} \Omega^4 S^3.$$

By Proposition 5 of [Cohen and Jones 2017], one knows that given any principal bundle over a manifold, $G \to P \to M$, the action of the gauge group (and therefore the based gauge group) on the manifold string topology spectrum $\mathcal{S}^\bullet_P(M)$ is defined by the representation given by the stabilization map

$$\mathcal{G}^b(P) \xrightarrow{\rho} \mathrm{GL}_1(\mathcal{S}^\bullet_P(M)), \quad \Omega\,\mathrm{Map}^b_P(M, BG) \xrightarrow{\sigma} \Omega\,\mathrm{Map}^b_P(M, B\mathrm{GL}_1(\Sigma^\infty(G_+)),$$

where σ is induced by the natural inclusion $G \hookrightarrow \mathrm{GL}_1(\Sigma^\infty(G_+))$. Here Map^b_P denotes the path component of the based mapping space that classifies the bundle P.

In the case of $\mathrm{SU}(2) \to P_k \to S^4$ then Lemma 17 says that the representation $\rho_k : \mathcal{G}^b(P_k) \to \mathrm{GL}_1(\mathcal{S}^\bullet_{P_k}(S^4))$ is given by the stabilization map

$$\Omega^4 S^3 \xrightarrow{\sigma} \Omega^4 Q(S^3_+) \xrightarrow{\iota_k} \mathrm{GL}_1(\mathcal{S}^\bullet_{P_k}(S^4)), \tag{27}$$

where σ is induced by the map $u_k : S^3 \to Q(S^3_+) \simeq Q(S^3) \times QS^0$ that sends S^3 to a generator of $\pi_3 Q(S^3) \cong \mathbb{Z}$ cross the basepoint of the component $Q_k(S^0)$.

Finally, notice that given any compact ring spectrum R, the group of units $\mathrm{GL}_1(R)$ acts on its Spanier–Whitehead dual $R^\vee$ by the dual action of $\mathrm{GL}_1(R)$ on R. Given the Spanier–Whitehead duality between the manifold string topology spectrum $\mathcal{S}^\bullet_{P_k}(S^4) \simeq (P_k^{\mathrm{Ad}})^{-TS^4}$ and the Lie group string topology spectrum $\mathcal{S}^{P_k}_\bullet(S^4) \simeq (P_k^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$, (27) describes the action of the based gauge group $\mathcal{G}^b(P_k)$ on the Lie group string topology spectrum as well.

To end this section, we point out that Proposition 11 and the above analysis of gauge symmetry implies the following.

Theorem 19. *Let $G \to P \to M$ be a principal bundle over a closed n-manifold M, with G a d-dimensional compact Lie group. Let E be any ring spectrum with respect to which the compact Calabi–Yau structure on $\mathcal{S}^\bullet_P(M)$ given in Proposition 14 is oriented. Then the homology $E_*(\mathcal{S}^\bullet_P(M))$ is a Frobenius algebra over the homology of the gauge group, $E_*(\mathcal{G}(P))$. That is, the following conditions hold*:

- *The homology algebra structure of the manifold string topology ring spectrum $E_*(\mathcal{S}^\bullet_P(M))$ carries the structure of an algebra over $E_*(\mathcal{G}(P))$.*
- *The homology coalgebra structure of the Lie group string topology coalgebra spectrum $E_*(\mathcal{S}^P_\bullet(M))$ is a module over $E_*(\mathcal{G}(P))$.*
- *The duality homomorphism defined by the Frobenius algebra structure induced from the compact Calabi–Yau structure,*

$$E_*(\mathcal{S}^\bullet_P(M)) \xrightarrow{\cong} E_{n-d-*}(\mathcal{S}^P_\bullet(M))^*$$

is an isomorphism of $E_*(\mathcal{G}(P))$ modules.

4. Twisted, smooth Calabi–Yau ring spectra, Thom ring spectra, and Lagrangian immersions of spheres

We now turn to the notion of twisted Calabi–Yau structures for *smooth* ring spectra.

Recall that a *smooth* ring spectrum A is one that is perfect as an A-bimodule. That is, it is perfect as a left $A \wedge A^{\mathrm{op}}$- module. Given a smooth ring spectrum A, let $A^!$ be its "bimodule dual". That is,

$$A^! = \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A, A \wedge A^{\mathrm{op}}).$$

A cap product pairing between Hochschild homology and cohomology can then be defined using the fact that $\mathrm{THH}(A, Q) \simeq A \wedge^L_{A\wedge A^{\mathrm{op}}} Q$. Given any A-bimodules P and Q, this pairing is given by

$$\cap : \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A, P) \wedge (A \wedge^L_{A\wedge A^{\mathrm{op}}} Q) \to P \wedge^L_{A\wedge A^{\mathrm{op}}} Q.$$

When $P = A \wedge A^{\mathrm{op}}$, then one can take a cap product with respect to a map $\rho : \mathbb{S} \to A \wedge_{A \wedge A^{\mathrm{op}}} Q$ to obtain a map

$$\cap\rho : A^! \to Q. \tag{28}$$

Definition. A *twisted, smooth Calabi–Yau ring spectrum* (*twisted* sCY) of dimension n is a triple (A, P, σ), where A is a smooth ring spectrum and P is a smooth A-bimodule that has the same $\mathbb{Z}/2$-homology as A:

$$P \wedge H\mathbb{Z}/2 \simeq A \wedge H\mathbb{Z}/2$$

as A-bimodules.

We refer to P as the "twisting" bimodule. If $P = A$ we say that A has *trivial* twisting. The map of spectra we call the *n-dimensional cotrace map*,

$$\sigma : \Sigma^n \mathbb{S} \to \mathrm{THH}(A,\ P),$$

has the following duality property: the induced cap product pairing

$$\cap\sigma : A^! = \mathrm{Rhom}_{A \wedge A^{\mathrm{op}}}(A, A \wedge A^{\mathrm{op}}) \to \Sigma^{-n} P$$

is an equivalence of A-bimodule spectra.

Note. Given a graded module P over a ring R let $P[-n]$ denote the desuspension $\Sigma^{-n}(P)$.

Like in the compact case, in most applications a twisted, smooth Calabi–Yau spectrum is reduced over a homology theory with respect to which the twisting becomes trivialized, or "oriented." We now make this precise.

Definition. Let (A, P, σ) be a twisted, sCY ring spectrum of dimension n, and let E be a ring spectrum representing a homology theory E_*. An *E_*-orientation* of (A, P, σ) is a pair $(u, \tilde{\sigma}_E)$, where

$$u : P \wedge E \xrightarrow{\simeq} A \wedge E$$

is an equivalence of E-module spectra as A-bimodules. Here A acts trivially on E. $\tilde{\sigma}_E : \Sigma^n E \to \mathrm{THH}(A,\ A)^{hS^1} \wedge E$ is a map to the E-homology of the homotopy fixed point spectrum of the S^1-action induced by the cyclic structure, which factorizes the cotrace map σ in E_*-homology. That is, the following diagram homotopy commutes:

$$\begin{tikzcd}
\Sigma^n E \arrow[r, "\tilde{\sigma}_E"] \arrow[d, "\sigma \wedge 1"'] & \mathrm{THH}(A,\ A)^{hS^1} \wedge E \arrow[d, "j"] \\
\mathrm{THH}(A,\ P) \wedge E \arrow[r, "\simeq", "u_*"'] & \mathrm{THH}(A, A) \wedge E
\end{tikzcd}$$

Here j is the natural inclusion of the homotopy fixed points.

Notice that when $E = Hk$, the Eilenberg–MacLane spectrum for a field k, then a twisted, smooth Calabi–Yau spectrum (A, τ, σ) together with an Hk-orientation $(u, \tilde{\sigma}_{Hk})$ defines a smooth Calabi–Yau algebra structure on the singular chains with k-coefficients, $C_*(A; k)$, as defined in [Kontsevich and Vlassopoulos 2013; Cohen and Ganatra 2015].

4.1. ***Thom spectra of virtual bundles over the loop space of a manifold.*** We now consider important examples of a twisted, smooth Calabi–Yau spectra. These are Thom ring spectra of virtual bundles over ΩM, for M a closed manifold.

We begin by studying the Thom spectrum of the trivial bundle over ΩM, namely the suspension spectrum $\Sigma^\infty(\Omega M_+)$. The following generalizes the chain complex analogue proven by Cohen and Ganatra [2015].

Theorem 20. *Let M be a closed manifold of dimension n. Then the suspension spectrum of its based loop space, $\Sigma^\infty(\Omega M_+)$, can be given the structure of a twisted, smooth Calabi–Yau ring spectrum of dimension n.*

Proof. In order to give $A = \Sigma^\infty(\Omega M_+)$ a twisted sCY structure, we need to define a twisting bimodule and a cotrace map. Consider the virtual bundle $-TM$ over M. The associated virtual spherical fibration is classified by a map $B_{-\tau_M} : M \to B\mathrm{GL}_1(\mathbb{S})$. By taking the loop of this map and applying suspension spectra we get a map of ring spectra

$$-\tau_M : A = \Sigma^\infty(\Omega M_+) \to \Sigma^\infty(\mathrm{GL}_1(\mathbb{S})). \tag{29}$$

This defines a $\Sigma^\infty(\Omega M_+)$-bimodule structure on the sphere spectrum. We let $\mathbb{S}_{-\tau_M}$ be the sphere spectrum with this bimodule structure, and we define

$$P = \mathbb{S}_{-\tau_M} \wedge \Sigma^\infty(\Omega M_+) = \mathbb{S}_{-\tau_M} \wedge A$$

to be the induced bimodule. Here $\mathbb{S}_{-\tau_M} \wedge \Sigma^\infty(\Omega M_+)$ is given the diagonal A-bimodule structure.

P will be the twisting bimodule. To describe the n-dimensional cotrace map, we first need the following observation.

Replace ΩM by its Kan loop group, which by abuse of notation we still write as ΩM. Consider the ΩM-space $(\Omega M \times \Omega M)^{\mathrm{Ad}}$, which is $\Omega M \times \Omega M$ with the adjoint action of ΩM defined by

$$g \cdot (h_1, h_2) = (gh_1, h_2 g^{-1}).$$

If E is the homotopy orbit space of this action, $E = \text{point} \times^L_{\Omega M} (\Omega M \times \Omega M)^{\mathrm{Ad}}$, then we have a homotopy fiber sequence (i.e., each successive three terms form a

homotopy fibration)

$$\Omega M \xrightarrow{\tilde{\Delta}} \Omega M \times \Omega M \to E \xrightarrow{\pi} M, \tag{30}$$

where $\tilde{\Delta}(g) = (g, g^{-1})$. Note that $(\Omega M \times \Omega M)^{\mathrm{Ad}} \cong \Omega M \times \Omega M$, where the latter has ΩM-action given by $\gamma(a, b) = (\gamma a, b)$. This isomorphism is defined by sending (a, b) to (a, ba). Hence $E \simeq \Omega M$, and $\pi : E \to M$ is nullhomotopic.

The Thom spectrum of the pull-back virtual bundle $\pi^*(-TM)$, which we denote by E^{-TM}, is given by

$$\Sigma^{-n}\mathbb{S}_{-\tau_M} \wedge^L_{\Omega M} \Sigma^\infty(\Omega M_+) \wedge \Sigma^\infty(\Omega M_+) = \Sigma^{-n}\mathbb{S}_{-\tau_M} \wedge^L_A (A \wedge A^{\mathrm{op}})^{\mathrm{Ad}}.$$

This is an A-bimodule, with action given by right multiplication of $A \wedge A^{\mathrm{op}}$ on itself. The isomorphism $(\Omega M \times \Omega M)^{\mathrm{Ad}} \cong \Omega M \times \Omega M$ above gives an ΩM-equivariant equivalence between the corresponding suspension spectra, and hence

$$h : E^{-TM} = \Sigma^{-n}\, \mathbb{S}_{-\tau_M} \wedge^L_A (A \wedge A^{\mathrm{op}})^{\mathrm{Ad}} \xrightarrow{\simeq} \Sigma^{-n}\mathbb{S}_{-\tau_M} \wedge A = P[-n]. \tag{31}$$

as A-bimodules.

We therefore have an equivalence

$$A \wedge^L_{A\wedge A^{\mathrm{op}}} \left(\mathbb{S}_{-\tau_M} \wedge^L_A (A \wedge A^{\mathrm{op}})^{\mathrm{Ad}}\right) \xrightarrow[\simeq]{h} A \wedge^L_{A\wedge A^{\mathrm{op}}} (\mathbb{S}_{-\tau_M} \wedge A) = \mathrm{THH}(A,\ P).$$

Now the map $\tilde{\Delta} : \Omega M \to \Omega M \times \Omega M$ defines a ring map on the level of suspension spectra, which by abuse of notation we still call $\tilde{\Delta}$,

$$\tilde{\Delta} : A \to A \wedge A^{\mathrm{op}}.$$

We then get a change-of-rings equivalence,

$$\phi : A^{\mathrm{Ad}} \wedge^L_A \mathbb{S}_{-\tau_M} \xrightarrow{\simeq} A \wedge^L_{A\wedge A^{\mathrm{op}}} \left((A \wedge A^{\mathrm{op}})^{\mathrm{Ad}} \wedge^L_A \mathbb{S}_{-\tau_M}\right).$$

Consider the unit map $u : \mathbb{S} \to \Sigma^\infty(\Omega M_+) = A$. This defines a map

$$u : \mathbb{S} \wedge^L_A \mathbb{S}_{-\tau_M} \to A^{\mathrm{Ad}} \wedge^L_A \mathbb{S}_{-\tau_M}.$$

Now $\mathbb{S} \wedge^L_A \mathbb{S}_{-\tau_M}$ is the Thom spectrum $\Sigma^n(M^{-TM})$. Thus there is a Pontryagin–Thom map $\gamma : \Sigma^n\mathbb{S} \to \Sigma^n(M^{-TM}) = \mathbb{S} \wedge^L_A \mathbb{S}_{-\tau_M}$. This can be viewed as the n-fold suspension of the unit map of the Spanier–Whitehead dual,

$$\mathbb{S} \to M^\vee \simeq M^{-TM}.$$

We can now define the n-dimensional cotrace map $\sigma : \Sigma^n\mathbb{S} \to \mathrm{THH}(A,\ P)$ to be the composition

$$\begin{aligned}\sigma : \Sigma^n\mathbb{S} \xrightarrow{\gamma} \mathbb{S} \wedge^L_A \mathbb{S}_{-\tau_M} \xrightarrow{u} A^{\mathrm{Ad}} \wedge^L_A \mathbb{S}_{-\tau_M} \xrightarrow{\phi} A \wedge^L_{A\wedge A^{\mathrm{op}}} \left((A \wedge A^{\mathrm{op}})^{\mathrm{Ad}} \wedge^L_A \mathbb{S}_{-\tau_M}\right)\\ \xrightarrow{h} A \wedge^L_{A\wedge A^{\mathrm{op}}} (\mathbb{S}_{-\tau_M} \wedge A) = \mathrm{THH}(A,\ P).\end{aligned} \tag{32}$$

To show σ is a valid cotrace map we need to check that it satisfies the required duality condition. Namely, we need to show that the cap product,

$$\cap\sigma : A^! = \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A,\ A\wedge A^{\mathrm{op}}) \to \Sigma^{-n}\mathbb{S}_{-\tau_M}\wedge A = P[-n] \tag{33}$$

is an equivalence of A-bimodules. This map was constructed to be an A-bimodule map, so it suffices to check that it is an ordinary weak equivalence.

In order to do this, we study the homotopy type of $\mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A,\ A\wedge A^{\mathrm{op}})$, where $A = \Sigma^\infty(\Omega M_+)$. Notice that since A is a connective Hopf algebra (in the weak sense), we have an equivalence

$$\mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A,\ A\wedge A^{\mathrm{op}}) \simeq \mathrm{Rhom}_A(\mathbb{S},\ (A\wedge A^{\mathrm{op}})^{\mathrm{Ad}}).$$

Consider again the homotopy fibration $\Omega M\times\Omega M\to E\to M$, and its fiberwise suspension spectrum. This is the parametrized spectrum

$$\Sigma^\infty(\Omega M_+)\wedge\Sigma^\infty(\Omega M_+)\to\Sigma^\infty_M(E_+)\to M.$$

The spectrum of sections of this parametrized spectrum is the spectrum whose homotopy type we are trying to compute:

$$\Gamma_M(\Sigma^\infty_M(E_+)) \simeq \mathrm{Rhom}_A(\mathbb{S},\ (A\wedge A^{\mathrm{op}})^{\mathrm{Ad}}) \simeq \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A,\ A\wedge A^{\mathrm{op}}).$$

If we let $\mathcal{E}^\bullet$ be the twisted equivariant cohomology theory represented by the parametrized spectrum $\Sigma^\infty_M(E_+)$, then what we are trying to compute is $\mathcal{E}^\bullet(M)$. But by twisted Poincaré duality theorem of Klein [2001] as described in [Cohen and Klein 2009], this twisted cohomology spectrum is given by the Thom spectrum

$$\mathcal{E}^\bullet(M) = \Gamma_M(\Sigma^\infty_M(E_+)) \simeq E^{-TM}.$$

Now as seen in (31) above, there is an equivalence

$$h : E^{-TM} \xrightarrow{\simeq} \Sigma^{-n}\mathbb{S}_{-\tau_M}\wedge A.$$

Putting these together gives an equivalence,

$$A^! = \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A,\ A\wedge A^{\mathrm{op}}) = \mathcal{E}^\bullet(M) \simeq E^{-TM} \xrightarrow{h} \Sigma^{-n}\mathbb{S}_{-\tau_M}\wedge A = P[-n].$$

We leave it to the reader to check that the cap product map

$$\cap\sigma : A^! \to P[-n]$$

induces such an equivalence.

Given this, the proof that $(\Sigma^\infty(\Omega M_+), -\tau_M, \sigma)$ is a twisted, smooth Calabi–Yau ring spectrum is complete. $\square$

Orientations on this twisted, smooth Calabi–Yau ring spectrum will be addressed in a more general context, in the setting of Thom spectra of virtual bundles over ΩM.

We now generalize the above to the setting of Thom ring spectra. Namely, let $\Omega f : \Omega M \to B\mathrm{GL}_1(\mathbb{S})$ be a loop map. That is, it is obtained by applying the based loop functor to a map $f : M \to B(B\mathrm{GL}_1(\mathbb{S}))$. Let $\Omega M^{\Omega f}$ denote the Thom spectrum of Ωf. By a theorem of Lewis, $\Omega M^{\Omega f}$ is a ring spectrum. Let E be any commutative (E_∞) ring spectrum. As is usual, we say that a virtual bundle $\omega : X \to B\mathrm{GL}_1(\mathbb{S})$ is E-orientable if there is a "Thom class" $\tau : X^\omega \to E$ such that the composition

$$\theta_\tau : X^\omega \wedge E \xrightarrow{\Delta\wedge 1} X_+ \wedge X^\omega \wedge E \xrightarrow{1\wedge\tau\wedge 1} X_+ \wedge E \wedge E \xrightarrow{1\wedge\text{multiply}} X_+ \wedge E \tag{34}$$

is an equivalence (the "Thom isomorphism"). Here $\Delta : X^\omega \to X_+ \wedge X^\omega$ is the map of Thom spectra induced by the diagonal map $X \to X \times X$. Again, as is usual, we define an E-orientation of a manifold M to be an E-orientation of its tangent bundle τ_M or equivalently of $-\tau_M$. An E-orientation of a loop map $\Omega f : \Omega Y \to B\mathrm{GL}_1(\mathbb{S})$ has the additional requirement that the Thom class $\tau : (\Omega Y)^{\Omega f} \to E$ be a map of ring spectra. In this case notice that the orientation equivalence θ_τ in (34) is an equivalence of ring spectra.

Theorem 21. *$A = \Omega M^{\Omega f}$ naturally has the structure of a twisted* sCY *ring spectrum of dimension n. Furthermore, suppose E is a commutative ring spectrum. Then an E orientation $\tau : M^{-TM} \to E$ of M and an E-orientation $A = \Omega M^{\Omega f} \to E$ together induce an E-orientation on the* sCY *structure on A.*

Remark. This theorem readily generalizes to the setting of generalized Thom spectra of maps to $B\mathrm{GL}_1(R)$, where R is a commutative ring spectrum.

Proof. Let $\mathbb{S}_{-\tau_M}$ denote the sphere spectrum viewed as a $\Sigma^\infty(\Omega M_+)$-module as above. We first observe that by the twisted Poincaré duality theorem of Klein [2001] as described in [Cohen and Klein 2009] (see also [Dwyer et al. 2006; Malm 2011]), there is an equivalence of $\Sigma^\infty(\Omega M_+)$-modules

$$\mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}\big(\mathbb{S}, \Sigma^\infty(\Omega M_+)\big) \simeq \Sigma^{-n}\mathbb{S}_{-\tau_M} \tag{35}$$

Here $\Sigma^\infty(\Omega M_+)$ acts on itself by left multiplication. The reason this equivalence holds is the following. The left-hand side describes the section spectrum of the fiberwise suspension spectrum of the path-loop fibration, $\Omega M \to P(M) \to M$. By Klein's theorem on Poincaré duality for parametrized spectra, the section spectrum (i.e., twisted cohomology spectrum) associated to this parametrized spectrum is equivalent to a twisting by $-TM$ of the homology spectrum,

$$\Sigma^\infty(\Omega M_+) \wedge^L_{\Sigma^\infty(\Omega M_+)} \Sigma^{-n}\mathbb{S}_{-\tau_M} \simeq \Sigma^{-n}\mathbb{S}_{-\tau_M}.$$

This equivalence is given by cap product with the Pontryagin–Thom class

$$t_M : \mathbb{S} \to M^{-TM} = \Sigma^{-n}\big(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}_{-\tau_M}\big).$$

Consider now the map of ring spectra $\Sigma^\infty(\Omega M_+) \to A \wedge A^{\mathrm{op}}$, induced on Thom spectra by the map $\tilde{\Delta} : \Omega M \to \Omega M \times \Omega M$ of (30). The source of this map is indeed $\Sigma^\infty(\Omega M_+)$, as the composite

$$\Omega M \xrightarrow{\tilde{\Delta}} \Omega M \times \Omega M \xrightarrow{\Omega f \times \Omega f} B\mathrm{GL}_1(\mathbb{S}) \times B\mathrm{GL}_1(\mathbb{S}) \xrightarrow{\text{multiply}} B\mathrm{GL}_1(\mathbb{S})$$

is null homotopic. The following is a result of the second author [Klang 2018, Theorem 5.1].

Theorem 22. *Under this action of $\Sigma^\infty(\Omega M_+)$ on $A \wedge A^{\mathrm{op}}$, there is an equivalence of $A \wedge A^{\mathrm{op}}$-modules*

$$A \simeq \mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (A \wedge A^{\mathrm{op}}) \tag{36}$$

Here the module structure on the right hand side is by right action on $A \wedge A^{\mathrm{op}}$.

Notice that $\mathbb{S}$ is a perfect $\Sigma^\infty(\Omega M_+)$-module since M is assumed to be compact. That is, $\mathbb{S}$ is equivalent to a retract of a finite $\Sigma^\infty(\Omega M_+)$-module. Applying $(-) \wedge^L_{\Sigma^\infty(\Omega M_+)} (A \wedge A^{\mathrm{op}})$, we can conclude from Theorem 22 that A is a retract of a finite $A \wedge A^{\mathrm{op}}$-module, hence a perfect $A \wedge A^{\mathrm{op}}$-module. That is, A is a smooth ring spectrum. Also because $\mathbb{S}$ is perfect as a $\Sigma^\infty(\Omega M_+)$-module, we can apply $(-) \wedge^L_{\Sigma^\infty(\Omega M_+)} (A \wedge A^{\mathrm{op}})$ to both sides of the equivalence (35) to obtain

$$\mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}\big(\mathbb{S}, A \wedge A^{\mathrm{op}}\big) \simeq \Sigma^{-n}\big(\mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} A \wedge A^{\mathrm{op}}\big). \tag{37}$$

We now take our twisting bimodule to be $P = \mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} A \wedge A^{\mathrm{op}}$. Notice that if the original map f is null homotopic, this agrees with the twisting bimodule given in the proof of Theorem 20. Furthermore, by the Thom isomorphism for $-TM$,

$$H\mathbb{Z}/2 \wedge P \simeq H\mathbb{Z}/2 \wedge A.$$

This is an equivalence of A-bimodules because the equivalence in Theorem 22 is. Moreover, by the above theorem,

$$\begin{aligned}\mathrm{Rhom}_{A \wedge A^{\mathrm{op}}}(A, A \wedge A^{\mathrm{op}}) &\simeq \mathrm{Rhom}_{A \wedge A^{\mathrm{op}}}\big((A \wedge A^{\mathrm{op}}) \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S},\ A \wedge A^{\mathrm{op}}\big)\\ &\simeq \mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, A \wedge A^{\mathrm{op}}).\end{aligned}$$

So the equivalence (37) becomes

$$A^! = \mathrm{Rhom}_{A \wedge A^{\mathrm{op}}}(A, A \wedge A^{\mathrm{op}}) \simeq \Sigma^{-n} P. \tag{38}$$

Our goal is to show that this equivalence is given by taking the cap product with an appropriate n-dimensional cotrace map $\sigma : \Sigma^n \mathbb{S} \to \mathrm{THH}(A, P)$ which we now define.

Since A is smooth,

$$\begin{aligned}\mathrm{THH}(A, P) &= \Sigma^n \operatorname{Rhom}_{A\wedge A^{\mathrm{op}}}(A, A \wedge A^{\mathrm{op}}) \wedge^L_{A\wedge A^{\mathrm{op}}} A \simeq \Sigma^n \operatorname{Rhom}_{A\wedge A^{\mathrm{op}}}(A, A) \\ &\simeq \Sigma^n \mathrm{THH}^\bullet(A, A), \end{aligned} \tag{39}$$

where the last quantity is the topological Hochschild cohomology.

The inverse equivalence also has a natural description. Consider the $\Sigma^\infty(\Omega M_+)$-module structure on $A = (\Omega M)^{\Omega f}$ given by the generalized conjugation action defined to be the pull-back of the $A\wedge A^{\mathrm{op}}$-action on A to $\Sigma^\infty(\Omega M_+)$ via the ring map $\Sigma^\infty(\Omega M_+) \to A \wedge A^{\mathrm{op}}$ defined above. This action was studied in detail by Klang [2018]. In [Klang 2018] the second author showed that there are equivalences,

$$\mathrm{THH}(A, A) \simeq \mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} A^c, \quad \mathrm{THH}^\bullet(A, A) \simeq \operatorname{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, A^c), \tag{40}$$

where A^c denotes the algebra A with this generalized conjugation action of $\Sigma^\infty(\Omega M_+)$.

Recall the cap product operation

$$\begin{aligned}&\big(\operatorname{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, A^c)\big) \wedge \big(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}_{-\tau_M}\big) \\ &\qquad\qquad \xrightarrow{\cap} \mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} A^c \simeq \mathrm{THH}(A, P). \end{aligned} \tag{41}$$

Now $\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}_{-\tau_M} \simeq \Sigma^n M^{-TM}$, so there is a Pontryagin–Thom map $S^n \xrightarrow{t_M} \Sigma^n M^{-TM}$ which corresponds to the unit $\iota \in M^\vee$ under the Atiyah-equivalence of the Spanier–Whitehead dual of a manifold $M^\vee$ and its Thom spectrum M^{-TM}. We then get an induced equivalence

$$\begin{aligned}&\mathrm{THH}^\bullet(A, A) \wedge S^n \simeq \big(\operatorname{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, A^c)\big) \wedge S^n \\ &\qquad \xrightarrow{1\wedge t_M} \big(\operatorname{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, A^c)\big) \wedge \big(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}_{-\tau_M}\big) \\ &\qquad\qquad \xrightarrow{\cap} \mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} A^c \simeq \mathrm{THH}(A, P). \end{aligned} \tag{42}$$

This is the inverse to the equivalence (39).

Now, $\mathrm{THH}^\bullet(A, A)$ is an E_2-ring spectrum; see, for example, McClure and Smith's [2002] solution of the Deligne conjecture. Let $\iota : \mathbb{S} \to \mathrm{THH}^\bullet(A, A)$ be the unit. Alternatively, recall that $\mathrm{THH}^\bullet(A, A)$ is the spectrum of A-bimodule maps $A \to A$; ι corresponds to the identity map $\mathrm{id} : A \to A$, a characterization which does not rely on the multiplicative structure on $\mathrm{THH}^\bullet(A, A)$. The map ι allows us to define our n-dimensional cotrace map as

$$\sigma : \Sigma^n \mathbb{S} \xrightarrow{\Sigma^n \iota} \Sigma^n \mathrm{THH}^\bullet(A, A) \simeq \mathrm{THH}(A, P). \tag{43}$$

Clearly taking cap product

$$\cap\sigma : A^! = \mathrm{Rhom}_{A\wedge A^{\mathrm{op}}}(A, A\wedge A^{\mathrm{op}}) \to \Sigma^{-n} P$$

defines the equivalence given in (38). This then proves that (A, P, σ) is a twisted, smooth Calabi–Yau ring spectrum.

Let E be any commutative ring spectrum satisfying the hypotheses of the theorem. An E_*-orientation of M is given by a Thom class $\tau : M^{-TM} \to E$ and induces an equivalence (34) $\theta_\tau : \Sigma^n M^{-TM} \wedge E \xrightarrow{\simeq} M_+ \wedge E$ as in (34). This can be also be written as an equivalence

$$\theta_\tau : (\mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}) \wedge E \xrightarrow{\simeq} (\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}) \wedge E.$$

Given such an orientation $\tau : E \to M$ and an orientation $\nu : A = (\Omega M)^{\Omega f} \to E$ we describe a resulting E_*-orientation $(u, \tilde{\sigma}_E)$ of the sCY structure on $A = (\Omega M)^{\Omega f}$.

Notice that the E_* orientation τ of M induces an equivalence

$$\theta_{\tau,R} : (\mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} R) \wedge E \xrightarrow{\simeq} (\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} R) \wedge E$$

for any left $\Sigma^\infty(\Omega M_+)$-module R. Now take $R = A \wedge A^{\mathrm{op}}$ with the $\Sigma^\infty(\Omega M_+)$-action defined by the ring homomorphism $\Sigma^\infty(\Omega M_+) \to A \wedge A^{\mathrm{op}}$ described above. We then define

$$\begin{aligned} u = \theta_{\tau, A\wedge A^{\mathrm{op}}} : P \wedge E &= (\mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega M_+)} A \wedge A^{\mathrm{op}}) \wedge E \\ &\xrightarrow{\simeq} (\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} A \wedge A^{\mathrm{op}}) \wedge E = A \wedge E. \end{aligned} \tag{44}$$

We now define the map $\tilde{\sigma}_E : \mathbb{S} \wedge E \to \mathrm{THH}(A, A)^{hS^1} \wedge E$ needed for the orientation of the sCY-structure.

Again, let $\tau : M^{-TM} \to E$ and $\nu : A = (\Omega M)^{\Omega f} \to E$ be orientations of M and Ωf respectively. Consider the following homotopy commutative diagram.

$$\begin{tikzcd}
\Sigma^n \mathbb{S} \wedge E \arrow[r, "\iota"] & \Sigma^n \mathrm{THH}^\bullet(A, A) \wedge E \arrow[rr, "\simeq"] \arrow[d, "="'] & & \mathrm{THH}(A, P) \wedge E \arrow[d, "\tau"', "\simeq"] \\
& & & \mathrm{THH}(A, A) \wedge E \arrow[d, "="] \\
& \Sigma^n \mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, A^c) \wedge E \arrow[rr, "\tau", "\simeq"'] \arrow[d, "\nu"', "\simeq"] & & \left(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} A^c\right) \wedge E \arrow[d, "\nu"', "\simeq"] \\
& \Sigma^n \mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}\left(\mathbb{S}, (\Sigma^\infty(\Omega M_+))^c\right) \wedge E \arrow[rr, "\tau", "\simeq"'] & & \left(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (\Sigma^\infty(\Omega M_+))^c\right) \wedge E \\
\Sigma^n \mathbb{S} \wedge E \arrow[r, "\iota"] \arrow[uuu, "="] & \Sigma^n \mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, \mathbb{S}) \wedge E \arrow[rr, "\tau", "\simeq"'] \arrow[u, "\iota\wedge 1"'] & & \left(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} \mathbb{S}\right) \wedge E \arrow[u, "\iota\wedge 1"'] \\
& \Sigma^n M^\vee \wedge E \arrow[u, "="] \arrow[rr, "\tau", "\simeq"'] & & M_+ \wedge E \arrow[u, "="]
\end{tikzcd}$$

A few comments about this diagram:

(1) The maps in this diagram that are labeled by a τ or a ν are induced by the respective orientations. The maps labeled by ι are induced by the units of the respective ring spectra.

(2) The reason the left side of this diagram homotopy commutes is because the vertical maps induced by both ν and ι are all maps of ring spectra. As pointed out above this is because the orientation map $\nu : A \to E$ is assumed to be a ring map.

(3) The bottom horizontal composition $\Sigma^n \mathbb{S} \wedge E \to M_+ \wedge E$ is the E_*-fundamental class. The reason the right side of this diagram homotopy commutes is by the naturality of the Atiyah–Klein equivalences.

(4) The homotopy orbit spectrum $\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (\Sigma^\infty(\Omega M_+))^c$ is equivalent to $\Sigma^\infty(LM_+)$, and the lower right hand vertical map

$$M_+ \wedge E \to \left(\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (\Sigma^\infty(\Omega M_+))^c\right) \wedge E \simeq \Sigma^\infty(LM_+) \wedge E$$

is homotopic to the inclusion of the constant loops, and therefore factors through the homotopy fixed point set of the circle action,

$$M_+ \wedge E \to \Sigma^\infty(LM_+)^{hS^1}.$$

(5) The top horizontal composition is the cotrace map

$$\sigma \wedge 1 : \Sigma^n \mathbb{S} \wedge E \to \mathrm{THH}(A, P) \wedge E.$$

We therefore define the map $\tilde{\sigma}_E : \Sigma^n E \to \mathrm{THH}(A, A)^{hS^1} \wedge E$ to be the composition

$$\begin{aligned}\tilde{\sigma}_E : \Sigma^n \mathbb{S} \wedge E \xrightarrow{[M]} M_+ \wedge E \to \Sigma^\infty(LM_+)^{hS^1} \wedge E \\ \xrightarrow{\simeq} \mathrm{THH}(\Sigma^\infty(\Omega M_+), \Sigma^\infty(\Omega M_+))^{hS^1} \wedge E \xleftarrow[\nu]{\simeq} \mathrm{THH}(A, A)^{hS^1} \wedge E.\end{aligned}$$

By comments (3) and (4) above, the composite $\Sigma^n \mathbb{S} \wedge E \xrightarrow{\tilde{\sigma}_E} \mathrm{THH}(A, A)^{hS^1} \wedge E \hookrightarrow \mathrm{THH}(A, A) \wedge E$ is obtained by starting at the lower left of the diagram, going horizontally to the lower right, and then going vertically until $\mathrm{THH}(A, A) \wedge E$. And by comment (1) above and the commutativity of the diagram, this means we may conclude that the following diagram homotopy commutes:

$$\begin{tikzcd}
\Sigma^n \mathbb{S} \wedge E \arrow[r, "\sigma \wedge 1"] & \mathrm{THH}(A, P) \wedge E \arrow[r, "\tau", "\simeq"'] & \mathrm{THH}(A, A) \wedge E \\
\Sigma^n \mathbb{S} \wedge E \arrow[u, "="] \arrow[rr, "\tilde{\sigma}_E"] & & \mathrm{THH}(A, A)^{hS^1} \wedge E \arrow[u]
\end{tikzcd}$$

This is what was required to show that $(u, \tilde{\sigma}_E)$ defines an orientation on the twisted Calabi–Yau structure on A. $\square$

Example. Take $M = \mathrm{SU}(m)/\mathrm{SO}(m)$, and $f : M \to \mathrm{SU}/\mathrm{SO} \simeq B^2 O$ the natural map. Then

$$\Omega f : \Omega(\mathrm{SU}(m)/\mathrm{SO}(m)) \to BO$$

is a loop map, and by Theorem 21, $A = \Omega M^{\Omega f}$ has the structure of a twisted sCY ring spectrum.

Note that $\Omega f : \Omega(\mathrm{SU}(m)/\mathrm{SO}(m)) \to BO$ induces a map of Thom spectra $A \to MO$. Ωf is an equivalence in a range, and in particular $\pi_0(A) \cong \mathbb{Z}/2$. As the unit of A is 2-torsion, A is 2-torsion. This is not the case if f is taken to be nullhomotopic, in which case the sCY ring spectrum is $\Sigma^\infty\big(\Omega(\mathrm{SU}(m)/\mathrm{SO}(m))_+\big)$.

4.2. ***The image of J and Lagrangian immersions of spheres.*** In this subsection we study in more detail the twisted smooth Calabi–Yau structure on Thom spectra of virtual bundles over spheres. These bundles arise naturally from the homotopy perspective from the image of the J-homomorphism, and from the perspective of symplectic topology from Lagrangian immersions of odd dimensional spheres into their cotangent bundles.

As discussed in [Abouzaid and Kragh 2016], Gromov's h-principle implies that the homotopy group $\pi_n(U)$ classifies Lagrangian immersions of S^n into its cotangent bundle, T^*S^n, which are in the homotopy class of the zero section $S^n \hookrightarrow T^*(S^n)$. Assume that $n > 1$ and let $\alpha : S^n \to U$ represent such a homotopy class. Since $\pi_n(\mathrm{SU}) \cong \pi_n(U)$, α lifts to a unique (up to homotopy) map that by abuse of notation we still call $\alpha : S^n \to \mathrm{SU}$. Taking loop spaces we get a map of A_∞ group-like monoids,

$$\Omega\alpha : \Omega S^n \to \Omega\,\mathrm{SU} \simeq BU.$$

The last equivalence is given by Bott periodicity. By forgetting the almost complex structure we get an A_∞-map

$$\Omega\alpha : \Omega S^n \to BO.$$

By Theorem 21 above, the Thom spectrum $(\Omega S^n)^{\Omega\alpha}$ has the structure of a twisted, smooth Calabi–Yau ring spectrum. We begin with the following observation.

Lemma 23. *The twisted sCY ring spectrum $(\Omega S^n)^{\Omega\alpha}$ has natural orientation with respect to stable homotopy theory (that is, the generalized homology theory $\mathbb{S}_*$ represented by the sphere spectrum $\mathbb{S}$). Furthermore, this induces an orientation with respect to any generalized homology theory E_* represented by a commutative ring spectrum E.*

Proof. First note that S^n has a canonical stable framing. That is, it has a canonical $\mathbb{S}$-orientation. This induces an orientation with respect to any theory E_*. Furthermore, by the construction of the twisted sCY structure in the proof of Theorem 21, the twisting bimodule of this structure is

$$P = \mathbb{S}_{-\tau_M} \wedge^L_{\Sigma^\infty(\Omega S^n_+)} A \wedge A^{\mathrm{op}},$$

where $A = (\Omega S^n)^{\Omega\alpha}$. Now the $\mathbb{S}$-framing of S^n defines an equivalence of bimodules, $\mathbb{S} \simeq \mathbb{S}_{-\tau_M}$. Thus

$$P \simeq \mathbb{S} \wedge^L_{\Sigma^\infty(\Omega S^n_+)} A \wedge A^{\mathrm{op}}$$

but by Theorem 22 this last spectrum is equivalent to A as A-bimodules. Thus we have an equivalence

$$u : P \wedge \mathbb{S} \xrightarrow{\simeq} A \wedge \mathbb{S}.$$

Using this identification, the cotrace element can be viewed as a class $\sigma \in \mathrm{THH}(A, A)$. To complete the construction of the $\mathbb{S}_*$-orientation we must show that σ lifts to an element in the homotopy fixed points, $\mathrm{THH}(A, A)^{hS^1}$.

By the main result of [Blumberg et al. 2010], the topological Hochschild homology of $A = (\Omega S^n)^{\Omega\alpha}$ is equivalent as a $\Sigma^\infty(S^1_+)$-module to the Thom spectrum of a virtual bundle over the free loop space LS^n:

Proposition 24 [Blumberg et al. 2010].

$$\mathrm{THH}((\Omega S^n)^{\Omega\alpha}) \simeq L(S^n)^{\ell(\alpha)},$$

where $\ell(\alpha)$ is the virtual bundle classified by the map

$$\ell(\alpha) : L(S^n) \xrightarrow{L\alpha} L\,\mathrm{SU} \simeq \mathrm{SU} \times \Omega\,\mathrm{SU} \xrightarrow{\text{project}} \Omega\,\mathrm{SU} \simeq BU \to BO.$$

In this composition, the equivalence $L\,\mathrm{SU} \simeq \mathrm{SU} \times \Omega\,\mathrm{SU}$ is given by the trivialization of the fibration of infinite loop spaces $\Omega\,\mathrm{SU} \to L\,\mathrm{SU} \to \mathrm{SU}$ defined by the canonical section $\mathrm{SU} \to L\,\mathrm{SU}$ given by the inclusion of constant loops, and the infinite loop structure of $L\,\mathrm{SU}$.

Remark. In [Blumberg et al. 2010] the map from $L\,\mathrm{SU}$ to BO was described by a composition

$$L\,\mathrm{SU} \to L(\mathrm{SU}/\mathrm{SO}) \simeq \mathrm{SU}/\mathrm{SO} \times \Omega(\mathrm{SU}/\mathrm{SO}) \xrightarrow{\eta\times 1} \Omega(\mathrm{SU}/\mathrm{SO}) \times \Omega(\mathrm{SU}/\mathrm{SO}) \xrightarrow{\text{multiply}} \Omega(\mathrm{SU}/\mathrm{SO}) \simeq BO,$$

where $\eta : \mathrm{SU}/\mathrm{SO} \to \Omega(\mathrm{SU}/\mathrm{SO}) \simeq BO$ was induced by the Hopf map $S^3 \to S^2$. However the map η becomes trivial when composed with the projection $\mathrm{SU} \to \mathrm{SU}/\mathrm{SO}$, which allows the description of $\ell(\alpha)$ given in the proposition.

Notice that the restriction to the constant loops,

$$S^n \xrightarrow{\iota} LS^n \xrightarrow{\ell(\alpha)} BO$$

is the constant map. That is, this virtual bundle is trivialized when restricted to the constant loops. But since constant loops are S^1-fixed points of LS^n, the inclusion naturally lifts to the homotopy fixed points,

$$\Sigma^\infty(S^n_+) \xrightarrow{\iota} (L(S^n)^{\ell(\alpha)})^{hS^1}.$$

Composing with the equivalence given by Proposition 24, this defines a map $\tilde{\sigma} : \Sigma^\infty(S^n_+) \to \mathrm{THH}(A, A)^{hS^1}$ that lifts the cotrace element $\sigma \in \mathrm{THH}(A, A)$.

This completes the construction of the $\mathbb{S}_*$-orientation of the sCY structure on $A = (\Omega S^n)^{\Omega\alpha}$. Given any other generalized homology theory E_* represented by a commutative ring spectrum E, the $\mathbb{S}_*$-orientation of $(\Omega S^n)^{\Omega\alpha}$ induces an E_* orientation by use of the unit $\mathbb{S} \to E$. This completes the proof of Lemma 23. □

The following recasts the results of [Abouzaid and Kragh 2016] to show that topological Hochschild homology can be used as an obstruction to being able to deform a Lagrangian immersion of a sphere to a Lagrangian embedding.

Theorem 25. *Let $\alpha : S^n \to U$ represent a Lagrangian immersion $\phi_\alpha : S^n \to T^*S^n$ in the homotopy class of the zero section. Consider the associated twisted smooth Calabi–Yau ring spectrum $(\Omega S^n)^{\Omega(-\alpha)}$. (Here $-\alpha : S^n \to U$ is a map that represents the inverse of α in $\pi_n U$.) Then if ϕ_α is Lagrangian isotopic to a Lagrangian embedding then there is an equivalence of topological Hochschild homology spectra,*

$$\mathrm{THH}((\Omega S^n)^{\Omega(-\alpha)}) \simeq \mathrm{THH}(\Sigma^\infty(\Omega S^n_+)).$$

Proof. Let Q and N be smooth, closed manifolds of the same dimension. Given an exact Lagrangian embedding $j : Q \to T^*N$, Kragh [2018] defined a virtual Maslov bundle ν on $L_0 Q$. Here L_0 denotes path component of the free loop space that contains constant loops. The construction, which uses notation that is different than ours, is described in section 2 of [Abouzaid and Kragh 2016]. A map of spectra

$$\psi : \Sigma^\infty(L_0 N_+) \to L_0 Q^{TN - TL \oplus \nu} \tag{45}$$

was constructed and studied. One of the main results of [Abouzaid and Kragh 2016] is that the map ψ is a homotopy equivalence of spectra. The Maslov bundle ν was defined as follows. (See section 2 of [Abouzaid and Kragh 2016].) The Lagrangian embedding $Q \to T^*N$ defines a map $\tau : Q \to U/O$. Then $-\nu$ was defined to be the restriction to $L_0 Q$ of the map

$$LQ \xrightarrow{L\tau} L(U/O) \xrightarrow{\simeq} U/O \times \Omega U/O \xrightarrow{\text{project}} \Omega U/O \simeq \mathbb{Z} \times BO.$$

In considering a Lagrangian embedding or immersion $\phi_\alpha : S^n \to T^*S^n$ represented by $\alpha : S^n \to U$, then by Proposition 24 the Maslov bundle ν is just

$$-\ell(\alpha) : LS^n \to BU \to BO.$$

Thus we may conclude from (45) that if the Lagrangian immersion ϕ_α is Lagrangian isotopic to a Lagrangian embedding, then the spectra

$$\Sigma^\infty(LS^n_+) \quad \text{and} \quad (LS^n)^{-\ell(\alpha)} = (LS^n)^{\ell(-\alpha)}$$

are equivalent.

Now the spectrum $\Sigma^\infty(LS^n_+)$ is equivalent to the topological Hochschild homology $\mathrm{THH}(\Sigma^\infty(\Omega S^n_+))$. By Proposition 24, the spectrum $(LS^n)^{\ell(-\alpha)}$ is equivalent to the topological Hochschild homology $\mathrm{THH}((\Omega S^n)^{\Omega(-\alpha)})$. The statement of the theorem now follows. □

For $k > 1$, let $\alpha_k : S^{2k+1} \to U$ be a generator of $\pi_{2k+1}(U)$, which by Bott periodicity is isomorphic to the integers. In [Abouzaid and Kragh 2016] it was proved that for $2k+1$ congruent to 1, 3, or 5 (mod 8), the Lagrangian immersion $\phi_k : S^{2k+1} \to T^*(S^{2k+1})$ represented by α_k is not Lagrangian isotopic to a Lagrangian embedding. We now see that this is detected by the fact the twisted smooth Calabi–Yau ring spectra $\Sigma^\infty(\Omega S^{2k+1}_+)$ and $(\Omega S^{2k+1})^{\Omega(-\alpha_k)}$ have different topological Hochschild homologies. For this we again use the fact that

$$\mathrm{THH}(\Sigma^\infty(\Omega S^{2k+1}_+)) \simeq \Sigma^\infty(LS^{2k+1}_+) \quad \text{and}$$
$$\mathrm{THH}((\Omega S^{2k+1})^{\Omega(-\alpha_k)}) \simeq (LS^{2k+1})^{\ell(-\alpha_k)}.$$

We will show that $(LS^{2k+1})^{\ell(-\alpha_k)}$ is not equivalent to $\Sigma^\infty(LS^{2k+1}_+)$ by showing that the generator $u : \mathbb{S} \to (LS^{2k+1})^{\ell(-\alpha_k)}$ of $\pi_0(LS^{2k+1})^{\ell(-\alpha_k)}) \cong \mathbb{Z}$ is not split by any map $(LS^{2k+1})^{\ell(-\alpha_k)} \to \mathbb{S}$. ($\Sigma^\infty(LS^{2k+1}_+)$ clearly admits such a splitting map.)

By the construction of $\ell(-\alpha_k)$ in [Blumberg et al. 2010] as described above, the composition

$$\Omega S^{2k+1} \hookrightarrow LS^{2k+1} \xrightarrow{\ell(-\alpha_k)} BU$$

is given by

$$\Omega(-\alpha_k) : \Omega S^{2k+1} \to BU = \{0\} \times BU \subset \mathbb{Z} \times BU.$$

This is a loop map, which by the definition of α_k induces an isomorphism

$$\pi_{2k}(\Omega S^{2k+1}) \to \pi_{2k} BU \cong \mathbb{Z}.$$

That is to say, if the $\iota_{2k} : S^{2k} \to \Omega S^{2k+1}$ is a generator, then the composition

$$b\alpha_k : S^{2k} \xrightarrow{\iota_{2k}} \Omega S^{2k+1} \hookrightarrow LS^{2k+1} \xrightarrow{\ell(-\alpha_k)} BU \tag{46}$$

generates the $(2k)$-th homotopy group. The Thom spectrum of this composition $(S^{2k})^{b\alpha_k}$ is equivalent to the CW spectrum

$$(S^{2k})^{b\alpha_k} \simeq \mathbb{S} \cup_{\tilde{\alpha}_k} D^{2k},$$

where the attaching map $\tilde{\alpha}_k : \Sigma^\infty S^{2k-1} \to \mathbb{S}$ is defined as follows.

Consider the composition $S^{2k} \xrightarrow{b\alpha_k} BO \xrightarrow{BJ} B\mathrm{GL}_1(\mathbb{S})$ where BJ is the delooping of the J-homomorphism $J : O \to \mathrm{GL}_1(\mathbb{S})$. Applying the loop space defines a map $S^{2k-1} \to \mathrm{GL}_1(\mathbb{S})$. Since S^{2k-1} is connected, its image lies in a single component of $\mathrm{GL}_1(\mathbb{S})$ which is equivalent to the component of the basepoint in QS^0. The adjoint of this map is the definition of the map $\tilde{\alpha}_k : \Sigma^\infty(S^{2k-1}) \to \mathbb{S}$. Notice, that by definition it is in the image of the J homomorphism, $J : \pi_{2k-1}O \to \pi_{2k-1}(\mathbb{S})$.

By standard calculations of Quillen and Adams, as described in [Abouzaid and Kragh 2016], for $2k+1$ congruent to 1, 3, or 5 (mod 8), the class $\tilde{\alpha}_k$ is nontrivial. Therefore there is no splitting map from $(S^{2k})^{b\alpha_k} \simeq \mathbb{S} \cup_{\tilde{\alpha}_k} D^{2k}$ to $\mathbb{S}$ that splits the generator $\mathbb{S} \to (S^{2k})^{b\alpha_k}$. By (46) there is therefore no splitting map from $(LS^{2k+1})^{\ell(-\alpha_k)}$ to $\mathbb{S}$. Thus $(LS^{2k+1})^{\ell(-\alpha_k)}$ is *not* equivalent to $\Sigma^\infty(LS^{2k+1}_+)$ and hence $\mathrm{THH}((\Omega S^{2k+1})^{\Omega(-\alpha_k)})$ is *not* homotopy equivalent to $\mathrm{THH}(\Sigma^\infty(\Omega S^{2k+1}_+))$. By Theorem 25, this implies that the Lagrangian immersion ϕ_k is *not* Lagrangian isotopic to a Lagrangian embedding.

5. A topological Hochschild (co)homology perspective

In this section, we give a topological Hochschild homology and cohomology interpretation of the Calabi–Yau structures and the dualities between the manifold string topology ring spectrum $\mathcal{S}^\bullet_P(M)$ and the Lie group string topology coalgebra spectrum $\mathcal{S}^P_\bullet(M)$.

We continue to consider a principal bundle $G \to P \xrightarrow{p} M$ where M is a closed manifold of dimension n and G is a compact Lie group of dimension d. A choice of connection on the bundle P defines a *holonomy* map

$$h_P : \Omega M \to G.$$

This is a map of group-like A_∞ spaces, and the induced map of classifying spaces, $Bh_P : M \simeq B(\Omega M) \to BG$ classifies the bundle P.

We then have an induced map of ring spectra and differential graded algebras that by abuse of notation we still denote by h_P:

$$C_*(\Omega M) \xrightarrow{h_P} C_*(G) \quad \text{and} \quad \Sigma^\infty(\Omega M_+) \xrightarrow{h_P} \Sigma^\infty(G_+).$$

These holonomy maps therefore define bimodule structures of $C_*(G)$ over $C_*(\Omega M)$ and of $\Sigma^\infty(G_+)$ over $\Sigma^\infty(\Omega M_+)$. We can therefore study the (topological) homology of these algebras with coefficients in these bimodules. In what follows

we suppress the map h_P from the notation regarding these bimodules. This is somewhat justified because given any two choices of holonomy maps, the induced module structures will be equivalent. We also note that a choice of holonomy defines an inherited dual bimodule structure on the Spanier–Whitehead dual $G^\vee$ over $\Sigma^\infty(\Omega M_+)$, and similarly the cochains $C^*(G)$ inherit the dual bimodule structure over $C_*(\Omega M)$. One of the main results of this section is the following.

Theorem 26. *We have the following equivalences involving topological Hochschild homology* $\mathrm{THH}_\bullet$ *and topological Hochschild cohomology* $\mathrm{THH}^\bullet$.

(1) $\mathrm{THH}_\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq \Sigma^\infty(P_+^{\mathrm{Ad}})$.

(2) $\mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq (P^{\mathrm{Ad}})^{-TM} \simeq \mathcal{S}_P^\bullet(M)$.
This equivalence is one of ring spectra.

(3) $\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \simeq (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}} \simeq \mathcal{S}_\bullet^P(M)$.
This equivalence is one of coalgebra spectra.

Proof. Given any homomorphism $\phi : H \to G$ of topological groups, one has that

$$\mathrm{THH}_\bullet\big(\Sigma^\infty(H_+), \Sigma^\infty(G_+)\big) \simeq \Sigma^\infty(EH \times_H G^{\mathrm{Ad}}),$$

where G^{Ad} represents the adjoint (conjugation) action of H on G:

$$h \cdot g = \phi(h) g \phi(h)^{-1}.$$

This is because $\mathrm{THH}_*\big(\Sigma^\infty(H_+), \Sigma^\infty(G_+)\big)$ is equivalent to the suspension spectrum of the cyclic bar construction $N^{\mathrm{cy}}(H, G)$ which Waldhausen [1985] showed is equivalent to the homotopy orbit space of H acting on G via the conjugation action. In our case, we may think of H as the based loop space ΩM by taking H to be a topological group of the same A_∞-homotopy type. (As we did earlier, by abuse of notation we still call this group ΩM.) Then this observation says that

$$\mathrm{THH}_\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq \Sigma^\infty(E\Omega M \times_{\Omega M} G_+^{\mathrm{Ad}}) \simeq \Sigma^\infty(P_+^{\mathrm{Ad}}).$$

This proves part (1) of the theorem.

For part (2), we use the similarly well-known fact that the topological Hochschild cohomology of the suspension spectrum of a group can be described as a homotopy fixed point spectrum. That is, like above, let $\phi : H \to G$ be a homomorphism of topological groups. Then

$$\mathrm{THH}^\bullet\big(\Sigma^\infty(H_+), \Sigma^\infty(G_+)\big) \simeq \Sigma^\infty(G_+)^{h\Sigma^\infty(H_+)}, \tag{47}$$

where $\Sigma^\infty(H_+)$ acts on $\Sigma^\infty(G_+)$ via the conjugation action. Like above, we refer to this as the adjoint action and we write it as $\Sigma^\infty(G_+)^{\mathrm{Ad}}$. (See [Westerland 2008] or section 4 of [Malm 2011].)

Now, this homotopy fixed point spectrum is defined to be

$$\Sigma^\infty(G_+)^{h\Sigma^\infty(H_+)} = \mathrm{Rhom}_{\Sigma^\infty(H_+)}(\mathbb{S}, \Sigma^\infty(G_+)^{\mathrm{Ad}}).$$

So in our case we have that

$$\mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq \mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, \Sigma^\infty(G_+)^{\mathrm{Ad}}).$$

Notice that since the homotopy orbit spectrum of $\Sigma^\infty(\Omega M_+)$ acting on $\Sigma^\infty(G_+)$ via the adjoint action is $\Sigma^\infty(P_+^{\mathrm{Ad}})$, this spectrum of $\Sigma^\infty(\Omega M_+)$-equivariant morphisms is equivalent to the spectrum of sections of the parametrized spectrum $\Sigma^\infty(G_+) \to \Sigma^\infty_M((P^{\mathrm{Ad}})_+) \to M$:

$$\mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, \Sigma^\infty(G_+)^{\mathrm{Ad}}) \simeq \Gamma_M\big(\Sigma^\infty_M((P^{\mathrm{Ad}})_+)\big).$$

But this spectrum of sections is, by definition, the manifold string topology spectrum, $\mathcal{S}^\bullet_P(M)$. Furthermore, it is clear that the ring spectrum structures coincide under this equivalence. Furthermore, by the Atiyah–Poincaré duality theorem proved by Klein [2001; Cohen and Klein 2009] we have that

$$\Gamma_M(\Sigma^\infty_M((P^{\mathrm{Ad}})_+)) \simeq (P^{\mathrm{Ad}})^{-TM}$$

as ring spectra. Putting these together says that

$$\mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq \mathcal{S}^\bullet_P(M) \simeq (P^{\mathrm{Ad}})^{-TM}$$

as ring spectra. This is the statement of part (2) of the theorem.

We now consider part (3) of the theorem. The Spanier–Whitehead dual of the simplicial spectrum $\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)$ can, because of the compactness assumption of G, be described as the totalization of the cosimplicial spectrum given by taking the Spanier–Whitehead dual levelwise. This cosimplicial spectrum has as its spectrum of k-simplices, $\mathrm{Rhom}_{\mathbb{S}}\big(\Sigma^\infty(\Omega M_+)^{(k)}, \Sigma^\infty(G_+)\big)$. The coface maps and the codegeneracies are the duals of the face and degeneracy maps in the simplicial spectrum $\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)$. But this cosimplicial spectrum is exactly the cosimplicial spectrum defining the topological Hochschild cohomology spectrum, $\mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big)$. That is, we have observed that

$$\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)^\vee = \mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big).$$

This is a ring spectrum, so its Spanier–Whitehead dual, $\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)$ inherits the structure of a coalgebra spectrum. Furthermore, we know from part (2) that

$$\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)^\vee = \mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \simeq \mathcal{S}^\bullet_P(M) \simeq (P^{\mathrm{Ad}})^{-TM}$$

as ring spectra. Thus applying Spanier–Whitehead duality and Theorems 9 and 1, we have

$$\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \simeq \mathcal{S}^P_\bullet(M) \simeq (P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$$

as coalgebra spectra.

Alternatively, as in the proof of part (1) of the theorem, we have an equivalence

$$\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \simeq \mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}$$

Now, $\mathcal{S}^P_\bullet(M)$ is the homology spectrum of the spectrum over M whose fiber is $G^\vee$, on which ΩM acts via (the dual of the) conjugation action. Therefore, we see that

$$\mathcal{S}^P_\bullet(M) \simeq \mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}} \simeq \mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee).$$

For all these spectra, the coproduct comes from dualizing the multiplication map $G \times G \to G$; see below for an explicit description of the coproduct on $\mathbb{S} \wedge^L_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}$. □

We end by observing how the twisted compact Calabi–Yau structure on $\mathcal{S}^\bullet_P(M) \simeq (P^{\mathrm{Ad}})^{-TM}$ can be understood from this Hochschild perspective.

The twisting bimodule in the twisted cCY structure on $R = \mathcal{S}^\bullet_P(M)$ is $Q = \Sigma^{d-n}(\mathcal{S}^P_\bullet(M)) \simeq \Sigma^{d-n}(P^{\mathrm{Ad}})^{-T_{\mathrm{vert}}}$. We first observe that the duality pairing (16) in the dimension $n-d$ twisted compact Calabi–Yau structure

$$\langle -, - \rangle : Q \wedge R \to \Sigma^{d-n}\mathbb{S}$$

can be described in terms of Hochschild theory as follows. As described above we have natural equivalences

$$R = \mathrm{THH}^\bullet(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)) \simeq \mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, \Sigma^\infty(G_+)^{\mathrm{Ad}}), \quad \text{and}$$
$$Q = \Sigma^{d-n}\,\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \simeq \Sigma^{d-n}\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}$$

We therefore have a cap product

$$\begin{aligned}\cap : Q \wedge R = \big(\Sigma^{d-n}\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}\big) \wedge \big(\mathrm{Rhom}_{\Sigma^\infty(\Omega M_+)}(\mathbb{S}, \Sigma^\infty(G_+)^{\mathrm{Ad}})\big)\\ \to \Sigma^{d-n}(\Sigma^\infty(G_+))^{\mathrm{Ad}} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}. \quad (48)\end{aligned}$$

The evaluation map $\mathrm{ev} : \Sigma^\infty(G_+) \wedge G^\vee \to \mathbb{S}$ is $\Sigma^\infty(\Omega M_+)$-invariant with respect to conjugation, and so defines a map

$$\mathrm{ev} : \Sigma^{d-n}(\Sigma^\infty(G_+))^{\mathrm{Ad}} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}} \to \Sigma^{d-n}\mathbb{S}.$$

Composing these defines the duality pairing

$$\langle -, - \rangle : Q \wedge R \to \Sigma^{d-n}\mathbb{S},$$

$$\big(\Sigma^{d-n}\,\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)\big) \wedge \big(\mathrm{THH}^\bullet(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+))\big) \xrightarrow{\cap}$$
$$\Sigma^{d-n}\big((\Sigma^\infty(G_+))^{\mathrm{Ad}} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}\big) \xrightarrow{\mathrm{ev}} \Sigma^{d-n}\mathbb{S}.$$

We end by observing how the bimodule structure of Q over R can be understood at the topological Hochschild (co)homology level. We know that

$$Q = \Sigma^{d-n}\mathcal{S}^P_\bullet(M) \simeq \Sigma^{d-n}\,\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee).$$

Now, $\mathcal{S}^P_\bullet(M) \simeq \mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \simeq \mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}$ is a coalgebra spectrum, and its coproduct ψ can be seen on the THH-level as follows:

$$\begin{tikzcd}
 & \mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} ((G \times G)^\vee)^{\mathrm{Ad}} & \\
\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}} \arrow[ur, "1\wedge\mu^\vee"] & & \mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee \wedge G^\vee)^{\mathrm{Ad}} \arrow[ul, "\simeq"'] \arrow[d, "\Delta"] \\
 & & (\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}) \wedge (\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}})
\end{tikzcd}$$

This needs some explanation. $\mu : G \times G \to G$ is the multiplication map. $\mu^\vee : G^\vee \to (G \times G)^\vee$ is its Spanier–Whitehead dual. It is equivariant with respect to the adjoint action of $\Sigma^\infty(\Omega M_+)$ since μ is equivariant with respect to the adjoint action. The map

$$\Delta : \mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee \wedge G^\vee)^{\mathrm{Ad}} \to \big(\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}\big) \wedge \big(\mathbb{S} \wedge_{\Sigma^\infty(\Omega M_+)} (G^\vee)^{\mathrm{Ad}}\big)$$

is the map induced by thinking of ΩM as the diagonal subgroup of $\Omega M \times \Omega M$.

The action map $Q \wedge R \to Q$ is then homotopic to the composition

$$Q \wedge R \simeq \big(\Sigma^{d-n}\,\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee)\big) \wedge \mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big) \xrightarrow{\psi\wedge 1}$$
$$\Sigma^{d-n}\,\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \wedge \mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \wedge \mathrm{THH}^\bullet\big(\Sigma^\infty(\Omega M_+), \Sigma^\infty(G_+)\big)$$
$$\xrightarrow{1\wedge\langle -,-\rangle} \Sigma^{d-n}\,\mathrm{THH}_\bullet(\Sigma^\infty(\Omega M_+), G^\vee) \wedge \mathbb{S} = Q.$$

The left module structure is homotopic to the analogous composition $R \wedge Q \to Q$.

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