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Duality relations among multiple series with three parameters

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We prove a duality relation among multiple series with three parameters. As special case of it, we obtain some new identities for multiple Hurwitz zeta values, which are relations among extensions of the multiple Hurwitz zeta value.

1. Introduction

The extended multiple zeta value (EMZV for short) is defined by the multiple series

$$\sum_{\substack{0<m_1<_{c_1}\cdots<_{c_{p-1}}m_p<\infty \\ m_i\in\mathbb{Z}}} \frac{1}{m_1^{k_1}\cdots m_p^{k_p}}, \tag{1}$$

where $1 \le p \in \mathbb{Z}$, $c_i \in \left\{\frac{1}{2}, 1\right\}$, $1 \le k_i \in \mathbb{Z}$ $(i = 1, \dots, p-1)$, $2 \le k_p \in \mathbb{Z}$, and the symbols $<_{c_i}$ $(i = 1, \dots, p-1)$ denote $<$ if $c_i = 1$ and $\le$ if $c_i = \frac{1}{2}$ (see [Ulanskiĭ 2011]). The case $c_i = 1$ $(i = 1, \dots, p-1)$ and the case $c_i = \frac{1}{2}$ $(i = 1, \dots, p-1)$ of (1) are the multiple zeta value (MZV for short) and the multiple zeta-star value (MZSV for short), respectively, which were studied by Euler [1776], Hoffman [1992], and Zagier [1994]. EMZVs were studied by Fischler and Rivoal [2016], Kawashima [2009] and Ulanskiĭ [2011]. Kawashima and Ulanskiĭ proved relations among them, and Fischler and Rivoal showed a connection between EMZVs and a solution to a Padé approximation problem involving multiple polylogarithms. Their works show that EMZVs are as useful and fruitful as MZ(S)Vs.

In [Igarashi 2015], we also studied multiple series of the extended form (1), which generalize EMZVs. In fact, we proved duality relations among them which yield numerous relations. The duality relations were proved by using a combination of the method used in [Igarashi 2012] and the calculational techniques for the Pochhammer symbol $(a)_m$ used in [Igarashi 2018]. In the present paper, we shall show that the combinative method used in [Igarashi 2015] can be applied to derive

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numerous relations among multiple series of the following type:

$$\sum_{\substack{0\le m_1<_{c_1}\cdots<_{c_{p-1}}m_p\\ m_p<_{c_p}\cdots<_{c_{p+q-1}}m_{p+q}<\infty\\ m_i\in\mathbb{Z}}} \frac{(y)_{m_p}}{(x)_{m_p}}\frac{(w)_{m_{p+q}}}{(z)_{m_{p+q}}} \times\left\{\prod_{i=1}^{p+q}\frac{1}{(m_i+x)^{a_i}(m_i+y)^{b_i}(m_i+z)^{c_i}(m_i+w)^{d_i}}\right\}, \tag{2}$$

where $1\le p\in\mathbb{Z},\ 0\le q\in\mathbb{Z},\ x,y,z,w\in\mathbb{C}\setminus\mathbb{Z}_{\le 0},\ a_i,b_i,c_i,d_i\in\mathbb{Z}$ such that

$$\sum_{i=r}^{p+q}(a_i+b_i+c_i+d_i)+\delta_r\mathrm{Re}(x-y)+\mathrm{Re}(z-w)>p+q-r+1$$

$(r=1,\ldots,p+q)$, where $\mathbb{Z}_{\le 0}:=\{0,-1,-2,\ldots\}$; $\delta_r=1$ if $r\le p$ and $\delta_r=0$ if $r>p$. (These conditions guarantee the absolute convergence of (2): see [Krattenthaler and Rivoal 2007, Lemmas 1 and 3, (3.12)].) The symbols $<_{c_i}$ $(i=1,\ldots,p+q-1)$ are the same as in (1), and the symbol $(a)_m$ denotes the Pochhammer symbol defined by $(a)_m=a(a+1)\cdots(a+m-1)$ $(1\le m\in\mathbb{Z})$ and $(a)_0=1$. The Pochhammer symbol $(a)_m$ can be expressed as $(a)_m=\Gamma(a+m)/\Gamma(a)$ by using the gamma function $\Gamma(s)$. The case $x=y=z=w=1$ and the case $x=y,\ z=w$ of (2) are EMZV and the following multiple Hurwitz zeta value (MHZV for short), respectively:

$$\sum_{\substack{0\le m_1<_{c_1}\cdots<_{c_{p-1}}m_p<\infty\\ m_i\in\mathbb{Z}}}\prod_{i=1}^{p}\frac{1}{(m_i+x)^{s_i}(m_i+z)^{t_i}}$$

$(s_i,t_i\in\mathbb{Z};\ i=1,\ldots,p)$. Since I proved the results in [Igarashi 2007], one of my research subjects has been to find the extensions of (multiple) Hurwitz zeta values which satisfy various relations as MZ(S)Vs. As I showed in [Igarashi 2018], the multiple series (2) and (3) below give such extensions. The results in [Igarashi 2018] have given me a motivation to the further research on (2) and (3). In the present paper, I prove a class of relations among (2) which gives numerous relations among extensions of MHZV: see Theorem 1 and Corollary 8 below. I remark that the partial derivatives of (2) with respect to $x,\ y,\ z,\ w$ can be expressed by $\mathbb{Z}$-linear combinations of (2). This suggests that, by partial differentiation, one relation among (2) yields further relations. The proof of Theorem 1 in the present paper gives an embodiment of this suggestion. In [Igarashi 2018] and its manuscripts (submitted in March and May 2015), I studied relations among multiple series of the types (2) and (3) by using the hypergeometric identities of [Andrews 1975, Theorem 4; Krattenthaler and Rivoal 2007, Proposition 1]. This work of mine is one of the bases of the present research.

1.1. ***Algebraic formulation.*** In a revised version of [Igarashi 2015], in order to describe the results, we followed the algebraic formulation for EMZVs given by Ulanskiĭ [2011] (see also [Hoffman 1997]). That formulation gave us concise descriptions of the results. For this reason, in the present paper, we also follow the formulation of Ulanskiĭ. The following formulation is parallel to that in the revised version of [Igarashi 2015].

Hereafter we assume that $m, n, p, q, r, k_i, k_i', m_i, r_i, s_i, M_{ij} \in \mathbb{Z}$. We consider the three noncommutative variables x_0, $x_{\frac{1}{2}}$ and x_1. For these variables, we use the expressions

$$x_1 x_0^{k_1-1} x_{c_1} x_0^{k_2-1} \cdots x_{c_{p-1}} x_0^{k_p-1} = z_1(k_1) z_{c_1}(k_2) \cdots z_{c_{p-1}}(k_p) = \prod_{i=1}^{p} z_{c_{i-1}}(k_i),$$

where $p \geq 1$ and $z_{c_{i-1}}(k_i) := x_{c_{i-1}} x_0^{k_i-1}$ $\big(c_0 = 1,\ k_p \geq 1,\ c_i \in \{\frac{1}{2}, 1\},\ k_i \geq 1;\ i = 1, \ldots, p-1\big)$. In case $p = 0$, we regard all these expressions as $1 \in \mathbb{Q}$. Here we consider the following set of monomials:

$$B^0 := \left\{ \prod_{i=1}^{p} z_{c_{i-1}}(k_i) \,\middle|\, p \geq 0,\ c_0 = 1,\ c_i \in \left\{\tfrac{1}{2}, 1\right\},\ k_i \geq 1\ (i = 1, \ldots, p-1),\ k_p \geq 2 \right\},$$

and thus $1 \in B^0$ as the case $p = 0$. We denote by V^0 the $\mathbb{Q}$-vector space with the basis B^0. For B^0, we define the evaluation map $H = H_{(x,y,z)} : B^0 \to \mathbb{C}$ by $H(1; (x, y, z)) = 1$ and

$$\begin{aligned} &H\big(z_1(k_1) z_{c_1}(k_2) \cdots z_{c_{p-1}}(k_p); (x, y, z)\big) \\ &\qquad = \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \frac{(y)_{m_p}}{(x)_{m_p+1}} \left\{ \prod_{i=1}^{p-1} \frac{1}{(m_i + z)^{k_i}} \right\} \frac{1}{(m_p + z)^{k_p - 1}}, \end{aligned} \tag{3}$$

where $x, z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $y \in \mathbb{C}$, $k_p \in \mathbb{Z}$ such that $k_p + \mathrm{Re}(x - y) > 1$ and $k_i \geq 1$ $(i = 1, \ldots, p-1)$. This map can be extended as a $\mathbb{Q}$-linear map onto the whole space V^0. Multiple series of the type (3) are studied in [Coppo 2009; Coppo and Candelpergher 2010; Emery 2004; Hasse 1930; Igarashi 2018; 2015]. See also [Igarashi 2007, Examples]. For brevity, we put $<_{c_i}^* := <_{(2c_i)^{-1}}$ $\big(c_i \in \left\{\frac{1}{2}, 1\right\}\big)$. This is the inversion of $<_{c_i}$, namely

$$<_{c_i}^* = \begin{cases} \leq & \text{if } c_i = 1, \\ < & \text{if } c_i = \frac{1}{2}. \end{cases}$$

We also define the evaluation map

$$H^*_{(\{r_i\}_{i=1}^n; \{s_i\}_{i=1}^q)} = H^*_{(\{r_i\}_{i=1}^n; \{s_i\}_{i=1}^q), (x,y,z)} : B^0 \to \mathbb{C}$$

by $H^*_{(\{r_i\}_{i=1}^n;\{s_i\}_{i=1}^q)}(1;(x,y,z))=1$ and

$$
\begin{aligned}
&H^*_{(\{r_i\}_{i=1}^n;\{s_i\}_{i=1}^q)}\bigl(z_1(k_1)z_{c_1}(k_2)\cdots z_{c_{q-1}}(k_q);(x,y,z)\bigr)\\
&\quad=\sum_{\substack{0\le M_{11}<\cdots<M_{1n}<m_1\\ m_1<_{c_1}M_{21}\le\cdots\le M_{2s_2}<^*_{c_2}m_2\\ \vdots\\ m_{i-1}<_{c_{i-1}}M_{i1}\le\cdots\le M_{is_i}<^*_{c_i}m_i\\ \vdots\\ m_{q-1}<_{c_{q-1}}M_{q1}\le\cdots\le M_{qs_q}<^*_{c_q}m_q<\infty}}\frac{(x)_{m_1}}{m_1!}\frac{(y)_{m_q+1}}{(z)_{m_q+1}}\\
&\qquad\times\left(\prod_{i=1}^n\frac{1}{(M_{1i}+x)(M_{1i}+z)^{r_i}}\right)\frac{1}{(m_1+y)^{k_1}(m_1+z)^{s_1}}\\
&\qquad\times\left\{\prod_{i=2}^q\left(\prod_{j=1}^{s_i}\frac{1}{M_{ij}+z}\right)\frac{1}{(m_i+y)^{k_i}}\right\},
\end{aligned}\tag{4}
$$

where $n,r_i\ge 0$ $(i=1,\dots,n)$; $q\ge 1$, $s_i\ge 0$ $(i=1,\dots,q)$, $c_q=1$; $x,y,z\in\mathbb{C}\setminus\mathbb{Z}_{\le 0}$ such that $\mathrm{Re}(2-x-y+z)$, $\mathrm{Re}(1-y+z)>0$. We regard $\{a_i\}_{i=m}^{m-1}$ as the empty set $\varnothing$. In case $s_i=0$, we regard the inequalities $m_{i-1}<_{c_{i-1}}M_{i1}\le\cdots\le M_{is_i}<^*_{c_i}m_i$ under the summation sign in (4) as $m_{i-1}<_{c_{i-1}}m_i$. For example, the case $n=0$ and $s_i=0$ $(i=1,\dots,q)$ of (4) becomes

$$
\begin{aligned}
&H^*_{(\varnothing;\{0\}_{i=1}^q)}\bigl(z_1(k_1)z_{c_1}(k_2)\cdots z_{c_{q-1}}(k_q);(x,y,z)\bigr)\\
&\qquad=\sum_{0\le m_1<_{c_1}\cdots<_{c_{q-1}}m_q<\infty}\frac{(x)_{m_1}}{m_1!}\frac{(y)_{m_q+1}}{(z)_{m_q+1}}\left\{\prod_{i=1}^q\frac{1}{(m_i+y)^{k_i}}\right\}\\
&\qquad=:H^*\bigl(z_1(k_1)z_{c_1}(k_2)\cdots z_{c_{q-1}}(k_q);(x,y,z)\bigr).
\end{aligned}
$$

This map can also be extended as a $\mathbb{Q}$-linear map onto the whole space V^0.

We define the map $\sigma_r^b:B^0\to V^0$ by $\sigma_r^b(1)=1$ and

$$
\begin{aligned}
&\sigma_r^b\bigl(z_1(k_1)z_{c_1}(k_2)\cdots z_{c_{p-1}}(k_p)\bigr)\\
&\qquad=\sum_{\substack{r_1+\cdots+r_p=r\\ r_i\ge 0}}\left\{\prod_{i=1}^{p-1}\binom{k_i+r_i-1}{r_i}\right\}\binom{k_p+r_p-2}{r_p}\prod_{i=1}^p z_{c_{i-1}}(k_i+r_i),
\end{aligned}
$$

where $r\ge 0$. This map corresponds to the partial derivative of (3) with respect to z. Following [Ulanskiĭ 2011, p. 106], we also define the dual map $\tau:B^0\to B^0$ by $\tau(1)=1$ and $\tau(x_1x_{e_1}\cdots x_{e_{n-1}}x_0)=x_1x_{1-e_{n-1}}\cdots x_{1-e_1}x_0$, where $n\ge 1$ and $e_i\in\bigl\{0,\frac{1}{2},1\bigr\}$ $(i=1,\dots,n-1)$. We call $\tau(v)$ the dual of v. By the definition, it is easy to see that $\tau^2(v)=v$. The dual $\tau(v)$ can also be written as $\tau(v)=\prod_{i=1}^q z_{c'_{i-1}}(k'_i)$, where $q \geq 0,\ c'_0 = 1,\ c'_i \in \left\{\frac{1}{2}, 1\right\},\ k'_i \geq 1\ (i = 1, \ldots, q-1),\ k'_q \geq 2$. Hereafter we assume this expression for $\tau(v)$. The maps σ_r^b and τ can be extended as $\mathbb{Q}$-linear maps from the whole space V^0 to V^0.

1.2. ***Main theorem and its examples.*** We define the symbol $\varepsilon(c_i)$ by

$$\varepsilon(c_i) = \begin{cases} 1 & \text{if } c_i = 1, \\ 0 & \text{if } c_i \neq 1. \end{cases}$$

For brevity, we put

$$\begin{aligned} (\boldsymbol{r}^n, \boldsymbol{s}^q, \varepsilon(c'_1)) &:= \left(\{r_i\}_{i=1}^n; \varepsilon(c'_1)s_1, \{s_i\}_{i=2}^q\right), \\ (\alpha, \beta, \gamma)^\tau &:= \left(1 - \beta + \gamma, \alpha - \beta + 1, \alpha - \beta + \gamma\right). \end{aligned}$$

The main theorem is as follows:

Theorem 1. *Let $v \in B^0$, and let $\tau(v)$ be its dual. Then the identity*

$$\begin{aligned} &H(\sigma_r^b(v); (\alpha, \beta, \gamma)) \\ &= \sum_{n=0}^{r} (1-\alpha)^n \sum_{\substack{\sum_{i=0}^n r_i + \varepsilon(c'_1)s_1 + \sum_{i=2}^q s_i = r \\ r_i \geq 1\ (i=1,\ldots,n) \\ r_0, s_i \geq 0\ (i=1,\ldots,q)}} G^{(r_0)}(\alpha, \beta, \gamma) H^*_{(\boldsymbol{r}^n, \boldsymbol{s}^q, \varepsilon(c'_1))}(\tau(v); (\alpha, \beta, \gamma)^\tau) \end{aligned} \tag{5}$$

holds for all $r \geq 0,\ \alpha, \beta, \gamma \in \mathbb{C}$ such that $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta)$, $\mathrm{Re}(\gamma)$, $\mathrm{Re}(1 - \beta + \gamma)$, $\mathrm{Re}(\alpha - \beta + 1)$, $\mathrm{Re}(\alpha - \beta + \gamma) > 0$, *where c'_1 and q are those of the dual* $\tau(v) = \prod_{i=1}^q z_{c'_{i-1}}(k'_i)$. (For the definition of $G^{(r_0)}(\alpha, \beta, \gamma)$, see Section 2.)

The identity (5) gives numerous relations among (2) which contain identities for MHZVs and EMZVs as special cases. For example, we consider the monomial $v_0 := \left\{\prod_{i=1}^{p-1} z_{c_{i-1}}(1)\right\} z_{c_{p-1}}(2)$ $\left(c_0 = 1,\ c_i \in \left\{\frac{1}{2}, 1\right\};\ i = 1, \ldots, p-1\right)$. This has the dual

$$\tau(v_0) = x_1 x_{1-c_{p-1}} \cdots x_{1-c_1} x_0 = z_1(k'_1) \left\{\prod_{i=2}^{q} z_{\frac{1}{2}}(k'_i)\right\},$$

where $q, k'_i \geq 1\ (i = 1, \ldots, q-1),\ k'_q \geq 2$. Taking $v = v_0$ in (5), we get the identity

$$\begin{aligned} &\sum_{\substack{r_1 + \cdots + r_p = r \\ r_i \geq 0}} H\left(\left\{\prod_{i=1}^{p-1} z_{c_{i-1}}(1 + r_i)\right\} z_{c_{p-1}}(2 + r_p); (\alpha, \beta, \gamma)\right) \\ &\qquad = \sum_{n=0}^{r} (1-\alpha)^n \sum_{\substack{\sum_{i=0}^n r_i + \varepsilon(q)s_1 + \sum_{i=2}^q s_i = r \\ r_i \geq 1\ (i=1,\ldots,n) \\ r_0, s_i \geq 0\ (i=1,\ldots,q)}} G^{(r_0)}(\alpha, \beta, \gamma) \\ &\qquad\qquad \times H^*_{(\boldsymbol{r}^n, \boldsymbol{s}^q, \varepsilon(q))}\left(z_1(k'_1)\left\{\prod_{i=2}^{q} z_{\frac{1}{2}}(k'_i)\right\}; (\alpha, \beta, \gamma)^\tau\right). \end{aligned} \tag{6}$$

The sum on the left-hand side of (6) has the same form as that for the sum formula for MZVs: see (8) below. By the definition of $G^{(r_0)}(\alpha, \beta, \gamma)$ in Section 2, we see that $G^{(r_0)}(\alpha, \alpha, \alpha) = 0$ $(r_0 \geq 1)$ and $G^{(0)}(\alpha, \alpha, \alpha) = 1$. Therefore, taking $\alpha = \beta = \gamma$ and $c_i = 1$ $(i = 1, \ldots, p-1)$ in (6), we get the following sum formula for MHZVs:

$$\sum_{\substack{r_1+\cdots+r_p=r \\ r_i \geq 0}} \zeta\left(\left\{\prod_{i=1}^{p-1} z_1(1+r_i)\right\} z_1(2+r_p); \alpha\right) = \sum_{n=0}^{r} (1-\alpha)^n \sum_{\substack{\sum_{i=1}^{n} r_i + s_1 = r \\ r_i \geq 1\,(i=1,\ldots,n) \\ s_1 \geq 0}} H^{*}_{(\{r_i\}_{i=1}^{n}; s_1)}(z_1(p+1); (1, 1, \alpha)) \tag{7}$$

for $p \geq 1,\ r \geq 0,\ \alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. Here

$$\zeta(v; \alpha) := H(v; (\alpha, \alpha, \alpha)) = \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \prod_{i=1}^{p} \frac{1}{(m_i+\alpha)^{k_i}}$$

$\left(v = \prod_{i=1}^{p} z_{c_{i-1}}(k_i) \in B^0\right)$. If $v = \prod_{i=1}^{p} z_1(k_i)$, this becomes the usual MHZV, i.e., the case $<_{c_i} = <$ $(i = 1, \ldots, p-1)$ of $\zeta(v; \alpha)$. The identity (7) was shown in [Igarashi 2018, **(R3)**] as an explicit expression for the identity (2) in [Igarashi 2007]. The identity (2) in [Igarashi 2007] is one of the bases of my research on MHZVs. For other identities for MHZVs which can be derived from Theorem 1, see Corollary 8 below. Finally, taking $\alpha = 1$ in (7), we get the sum formula for MZVs, which may be one of the basic relations among MZVs:

$$\sum_{\substack{r_1+\cdots+r_p=r \\ r_i \geq 0}} \zeta\left(\left\{\prod_{i=1}^{p-1} z_1(1+r_i)\right\} z_1(2+r_p)\right) = \zeta(z_1(p+1+r)) \tag{8}$$

for $p \geq 1$ and $r \geq 0$ ([Granville 1997], Zagier (unpublished)). Here $\zeta(v) := \zeta(v; 1)$ $(v \in B^0)$, which is EMZV. If $v = \prod_{i=1}^{p} z_1(k_i)$, this becomes the usual MZV. The case $\alpha = \beta = \gamma$ and the case $\alpha = \beta = \gamma = 1$ of (6) are extensions of (7) and (8), respectively. For another example of Theorem 1, taking $\alpha = \beta = \gamma = 1$ in (5), we can get the following relation among EMZVs:

$$\sum_{\substack{r_1+\cdots+r_p=r \\ r_i \geq 0}} \left\{\prod_{i=1}^{p-1} \binom{k_i+r_i-1}{r_i}\right\} \binom{k_p+r_p-2}{r_p} \zeta\left(\prod_{i=1}^{p} z_{c_{i-1}}(k_i+r_i)\right) = \sum_{\substack{s_1+\cdots+s_q=r \\ s_i \geq 0}} \sum_{\substack{0<m_1<_{c_1} M_{21} \leq \cdots \leq M_{2s_2} <^{*}_{c_2} m_2 \\ \vdots \\ m_{i-1}<_{c_{i-1}} M_{i1} \leq \cdots \leq M_{is_i} <^{*}_{c_i} m_i \\ \vdots \\ m_{q-1}<_{c_{q-1}} M_{q1} \leq \cdots \leq M_{qs_q} <^{*}_{c_q} m_q < \infty}} \frac{1}{m_1^{k'_1+\varepsilon(c'_1)s_1}} \left\{\prod_{i=2}^{q} \left(\prod_{j=1}^{s_i} \frac{1}{M_{ij}}\right) \frac{1}{m_i^{k'_i}}\right\} \tag{9}$$

for $r \geq 0$. The identity (9) gives numerous relations among EMZVs. In fact, in [Igarashi 2015], I showed that the identity (9) is equivalent to a certain new extension of Ohno's relation for MZVs [Ohno 1999, Theorem 1]: Ohno's relation is a large class of relations among MZVs. By the above examples, we see that Theorem 1 contains various interesting relations for multiple series. See also Remark 11 below.

We explain the idea of the proof of Theorem 1. It is proved by using a symmetry of (3) with respect to the parameters x, y and z. Indeed we derive Theorem 1 from the duality formula (12) below, which has a symmetry with respect to the parameters α, β and X, by partial differentiation. The symmetry of (12) can be found by applying a change of variables to an iterated integral representation of (3) (see (10) below and the proof of (12)). It is important that the change of variables also brings about a change of the positions of the parameters of (3) (compare, e.g., the positions of X on both sides of (12)). Consequently partial differential operators act on each side of (12) in different ways, and this gives the identity in Theorem 1. (See also [Igarashi 2012].) For the factor $G^{(r_0)}(\alpha, \beta, \gamma)$ in Theorem 1, which is a partial differential coefficient of the gamma factor in (12), we prove its two explicit expressions (see Lemma 6 below). Those expressions can be used for deriving various relations among (2) from Theorem 1 (see Remarks 7 and 11 below).

I remark that Theorem 1 is a variation of my former results written in [Igarashi 2015], but it yields some new identities for MHZVs, which are different from those in [Igarashi 2015]: see Corollary 8 and Remark 10 below. For example, taking $\alpha = \beta = \gamma$ in Theorem 1, we get the following new identity for MHZVs:

$$\zeta(\sigma_r^b(v); \alpha) = \sum_{n=0}^{r} (1-\alpha)^n \sum_{\substack{\sum_{i=1}^n r_i + \varepsilon(c_1') s_1 + \sum_{i=2}^q s_i = r \\ r_i \geq 1 \, (i=1,\dots,n) \\ s_i \geq 0 \, (i=1,\dots,q)}} H^*_{(\boldsymbol{r}^n, \boldsymbol{s}^q, \varepsilon(c_1'))}(\tau(v); (1, 1, \alpha))$$

for all $r \geq 0$, $\alpha \in \mathbb{C}$ such that $\mathrm{Re}(\alpha) > 0$, where $\zeta(\sigma_r^b(v); \alpha) := H(\sigma_r^b(v); (\alpha, \alpha, \alpha))$. This identity gives an extension of (7). See also Corollary 8(i) and Remark 10 below.

2. Proof of Theorem 1

In this section, we prove Theorem 1. The proof is a variation of the proofs of the results in [Igarashi 2015], therefore several proofs in this section overlap those in [Igarashi 2015] (see also [Igarashi 2012, Section 2]). However we shall not omit the details for the sake of keeping the present paper self-contained. The calculational techniques for the Pochhammer symbol $(a)_m$ used in the present paper are based on those used in [Igarashi 2018].

We first prove some lemmas. We define the symbol $\omega_{e_i}(t)$ $\left(e_i \in \left\{0, \frac{1}{2}, 1\right\}\right)$ by

$$\omega_0(t) = \frac{1}{t}, \quad \omega_{\frac{1}{2}}(t) = \frac{1}{t(1-t)}, \quad \omega_1(t) = \frac{1}{1-t}.$$

The multiple series (3) has the following iterated integral representation, which is one of the most important ingredients of the proof of Theorem 1:

Lemma 2. *Let $n \geq 1$ and $e_i \in \left\{0, \frac{1}{2}, 1\right\}$ $(i = 1, \ldots, n-1)$. Then the identity*

$$\begin{aligned}
&H(x_1 x_{e_1} \cdots x_{e_{n-1}} x_0; (\alpha, \beta, X)) \\
&= \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta+1)} \\
&\qquad \times \int \cdots \int_{0<t_0<\cdots<t_n<1} t_0^{X-1} \omega_1(t_0) \left\{ \prod_{i=1}^{n-1} \omega_{e_i}(t_i) \right\} \omega_0(t_n) t_n^{\beta-X} (1-t_n)^{\alpha-\beta} \, \mathrm{d}t_0 \cdots \mathrm{d}t_n \qquad (10)
\end{aligned}$$

holds for all $\alpha, \beta, X \in \mathbb{C}$ such that $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(X), \mathrm{Re}(\alpha-\beta+1) > 0$, *where $\Gamma(s)$ is the gamma function.*

Proof. Using $\omega_0(t)$ and $\omega_{c_i}(t)$ $\left(c_i \in \left\{\frac{1}{2}, 1\right\}\right)$, we can rewrite the integrand as

$$\omega_1 \left\{ \prod_{i=1}^{n-1} \omega_{e_i} \right\} \omega_0 = \prod_{i=1}^{p} \omega_{c_{i-1}} \omega_0^{k_i - 1},$$

where $p \geq 1,\ c_0 = 1,\ c_i \in \{\frac{1}{2}, 1\},\ k_i \geq 1\ (i = 1, \ldots, p-1),\ k_p \geq 2$. This gives the following expression for the iterated integral in (10), which we denote by I:

$$I = \int \cdots \int_{\substack{0<t_{11}<\cdots<t_{1k_1} \\ \vdots \\ <t_{i1}<\cdots<t_{ik_i} \\ \vdots \\ <t_{p1}<\cdots<t_{pk_p}<1}} t_{11}^{X-1} \left\{ \prod_{i=1}^{p} \omega_{c_{i-1}}(t_{i1}) \left(\prod_{j=2}^{k_i} \omega_0(t_{ij}) \right) \right\} t_{pk_p}^{\beta-X} (1-t_{pk_p})^{\alpha-\beta} \times \left(\prod_{i=1}^{p} \prod_{j=1}^{k_i} \mathrm{d}t_{ij} \right). \qquad (11)$$

Applying the expansions $(1-t_{i1})^{-1} = \sum_{m=0}^{\infty} t_{i1}^m$ $(i = 1, \ldots, p)$ to the integrand in (11) and integrating term by term, we get the identities

$$\begin{aligned}
I &= \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \left\{ \prod_{i=1}^{p-1} \frac{1}{(m_i+X)^{k_i}} \right\} \frac{1}{(m_p+X)^{k_p-1}} \int_0^1 (1-t_{pk_p})^{\alpha-\beta} t_{pk_p}^{\beta+m_p-1} \, \mathrm{d}t_{pk_p} \\
&= \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \left\{ \prod_{i=1}^{p-1} \frac{1}{(m_i+X)^{k_i}} \right\} \frac{1}{(m_p+X)^{k_p-1}} \frac{\Gamma(\alpha-\beta+1)\Gamma(\beta+m_p)}{\Gamma(\alpha+m_p+1)} \\
&= \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \left\{ \prod_{i=1}^{p-1} \frac{1}{(m_i+X)^{k_i}} \right\} \frac{1}{(m_p+X)^{k_p-1}} \frac{(\beta)_{m_p}}{(\alpha)_{m_p+1}} \frac{\Gamma(\alpha-\beta+1)\Gamma(\beta)}{\Gamma(\alpha)},
\end{aligned}$$

so

$$I = \frac{\Gamma(\alpha-\beta+1)\Gamma(\beta)}{\Gamma(\alpha)} H\bigl(x_1 x_0^{k_1-1} x_{c_1} x_0^{k_2-1} \cdots x_{c_{p-1}} x_0^{k_p-1}; (\alpha, \beta, X)\bigr)$$

for $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(X), \mathrm{Re}(\alpha-\beta+1) > 0$. We can also get the following expression for the above monomial:

$$x_1 x_0^{k_1-1} x_{c_1} x_0^{k_2-1} \cdots x_{c_{p-1}} x_0^{k_p-1} = x_1 x_{e_1} \cdots x_{e_{n-1}} x_0,$$

where $e_i \in \bigl\{0, \frac{1}{2}, 1\bigr\}$ $(i = 1, \dots, n-1)$, therefore we get (10). □

Lemma 2 gives the following duality formula for (3):

Lemma 3 (Duality formula). *Let $v \in B^0$, and let $\tau(v)$ be its dual. Then the identity*

$$H(v; (\alpha, \beta, X)) = \frac{\Gamma(\alpha)}{\Gamma(\beta)} \frac{\Gamma(X)}{\Gamma(\alpha-\beta+X)} H^*(\tau(v); (\alpha, \beta, X)^\tau) \tag{12}$$

holds for all $\alpha, \beta, X \in \mathbb{C}$ such that

$$\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(X), \mathrm{Re}(\alpha-\beta+1), \mathrm{Re}(\alpha-\beta+X) > 0.$$

Proof. The assertion follows by applying the change of variables $t_i = 1 - u_{n-i}$ $(i = 0, 1, \dots, n)$ to the iterated integral on the right-hand side of (10). (This change of variables was used in [Zagier 1994, p. 510] for MZVs.) □

Remark 4. The cases $\alpha = \beta = X = 1$ of (10) and (12) are Ulanskiĭ's results on EMZVs [Ulanskiĭ 2011, Corollary 2 and Theorem 1]. The case $\alpha = \beta$ and $v = z_1(1)^{p-1} z_1(2)$ $(p \geq 1)$ of (12) was used in [Igarashi 2007] to prove a sum formula for MHZVs:

$$\sum_{0 \leq m_1 < \cdots < m_p < \infty} \left(\prod_{i=1}^{p} \frac{1}{m_i + X} \right) \frac{1}{m_p + \alpha} = \sum_{m=0}^{\infty} \frac{(1-\alpha+X)_m}{(X)_{m+1}} \frac{1}{(m+1)^p}.$$

To use the partial differentiation, we need the following lemma:

Lemma 5. *Let $v = \prod_{i=1}^{q} z_{c_{i-1}}(k_i) \in B^0$, and let $x, y, z \in \mathbb{C}$ with* $\mathrm{Re}(x)$, $\mathrm{Re}(y)$, $\mathrm{Re}(z) > 0$. *Then the multiple series*

$$H^*(v; (x+Y, y, z+Y)) = \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{q-1}} m_q < \infty} \frac{(x+Y)_{m_1}}{m_1!} \frac{(y)_{m_q+1}}{(z+Y)_{m_q+1}} \left\{ \prod_{i=1}^{q} \frac{1}{(m_i+y)^{k_i}} \right\} \tag{13}$$

converges uniformly in $\{Y \in \mathbb{C} \mid |Y| \leq \varepsilon\}$, where $\varepsilon \in \mathbb{R}$ such that

$$0 < \varepsilon < \min\bigl\{\mathrm{Re}(x), \mathrm{Re}(z), 1 - \mathrm{Re}(x+y-z)/2, 1 - \mathrm{Re}(y-z)\bigr\}.$$

Proof. By using the expression $(a)_m = \Gamma(a+m)/\Gamma(a)$, the Pochhammer symbol $(a)_m$ can be estimated as follows:

$$\begin{aligned}|(a)_m| &= \left|\frac{\Gamma(a+m)}{\Gamma(a)}\right| = \left|\frac{1}{\Gamma(a)}\int_0^\infty e^{-t}t^{a+m-1}\,\mathrm{d}t\right| \\ &\le \frac{1}{|\Gamma(a)|}\int_0^\infty e^{-t}t^{\mathrm{Re}(a)+m-1}\,\mathrm{d}t = \frac{\Gamma(\mathrm{Re}(a)+m)}{|\Gamma(a)|} = \frac{\Gamma(\mathrm{Re}(a))}{|\Gamma(a)|}(\mathrm{Re}(a))_m \\ &\le C_1\frac{\Gamma(\mathrm{Re}(a))}{|\Gamma(a)|}(m+1)^{\mathrm{Re}(a)}\end{aligned}$$

for $m \ge 0$, $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$, where C_1 is a positive constant which does not depend on m. (The last inequality above follows by applying Stirling's formula for $\Gamma(s)$ to $(\mathrm{Re}(a))_m$.) Using this estimate and $|(a)_m| \ge (\mathrm{Re}(a))_m > 0$ for $a \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$, we get the estimate

$$\begin{aligned}&\left|\frac{(x+Y)_{m_1}}{m_1!}\frac{(y)_{m_q+1}}{(z+Y)_{m_q+1}}\right| \\ &\qquad\le \frac{\Gamma(\mathrm{Re}(x+Y))\Gamma(\mathrm{Re}(y))}{|\Gamma(x+Y)\Gamma(y)|}\frac{(\mathrm{Re}(x+Y))_{m_1}}{m_1!}\frac{(\mathrm{Re}(y))_{m_q+1}}{(\mathrm{Re}(z+Y))_{m_q+1}} \\ &\qquad\le C_2\frac{(\mathrm{Re}(x)+\varepsilon)_{m_1}}{m_1!}\frac{(\mathrm{Re}(y))_{m_q+1}}{(\mathrm{Re}(z)-\varepsilon)_{m_q+1}} \\ &\qquad\le C_3\frac{1}{(m_1+1)^{1-\mathrm{Re}(x)-\varepsilon}}\frac{1}{(m_q+1)^{-\mathrm{Re}(y-z)-\varepsilon}}\end{aligned}$$

for all $Y \in \{Y \in \mathbb{C} \mid |Y| \le \varepsilon\}$, where C_2 and C_3 are positive constants which do not depend on Y. By this estimate, we can take the absolutely convergent multiple series

$$\sum_{0\le m_1<_{c_1}\cdots<_{c_{q-1}}m_q<\infty}\frac{1}{(m_1+1)^{1-\mathrm{Re}(x)-\varepsilon}}\frac{1}{(m_q+1)^{-\mathrm{Re}(y-z)-\varepsilon}}\left\{\prod_{i=1}^{q}\frac{1}{(m_i+\mathrm{Re}(y))^{k_i}}\right\}$$

as a majorant of (13), and this implies the uniform convergence of (13). For conditions on the absolute convergence of the above majorant, see those for (2). □

For the gamma factor in (12), we put

$$G^{(n)}(\alpha,\beta,\gamma) := \frac{\Gamma(\alpha)}{\Gamma(\beta)}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial X^n}\left(\frac{\Gamma(X)}{\Gamma(\alpha-\beta+X)}\right)\Bigg|_{X=\gamma},$$

where $n \ge 0$, $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\alpha, \gamma, \alpha-\beta+\gamma \notin \mathbb{Z}_{\le 0}$, which appears in Theorem 1. We prove two explicit expressions for $G^{(n)}(\alpha,\beta,\gamma)$. For brevity, we also put

$$P^{(n)}(z,w) := \frac{1}{n!}\frac{\partial^n}{\partial Y^n}\bigl(\psi(z-Y)-\psi(w-Y)\bigr)\Big|_{Y=0}$$
$$= \begin{cases} (z-w)\sum_{m=0}^{\infty}(m+z)^{-1}(m+w)^{-1} & \text{if } n=0,\\ -\sum_{m=0}^{\infty}(m+z)^{-n-1}+\sum_{m=0}^{\infty}(m+w)^{-n-1} & \text{if } n\geq 1,\end{cases}$$

where $z, w \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$ and $\psi(z) := \Gamma'(z)/\Gamma(z)$ is the digamma function. This identity is an immediate consequence of the following two properties of the digamma function:

$$\psi(z)-\psi(w) = (z-w)\sum_{m=0}^{\infty}(m+z)^{-1}(m+w)^{-1}$$

$(z, w \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0})$ and $\psi^{(n)}(z) = (-1)^{n+1}n!\sum_{m=0}^{\infty}(m+z)^{-n-1}$ $(n\geq 1)$, where $\psi^{(n)}(z)$ is the n-th derivative of $\psi(z)$. (For the property of the digamma function, see, e.g., [Srivastava and Choi 2001, Section 1.2].) Then we can prove the following:

Lemma 6. *Let $n \geq 1$. Then $G^{(n)}(\alpha,\beta,\gamma)$ has the following two expressions*:

(i)

$$G^{(n)}(\alpha,\beta,\gamma) = \frac{\Gamma(\alpha)}{\Gamma(\beta)}\frac{\Gamma(\gamma)}{\Gamma(\alpha-\beta+\gamma)}\sum_{i=1}^{n}\sum_{0=l_0<l_1<\cdots<l_{i-1}<l_i=n}\prod_{j=1}^{i} l_j^{-1}P^{(l_j-l_{j-1}-1)}(\alpha-\beta+\gamma,\gamma) \tag{14}$$

for all $\alpha,\beta,\gamma\in\mathbb{C}$ such that $\alpha,\gamma,\alpha-\beta+\gamma\notin\mathbb{Z}_{\leq 0}$.

(ii)

$$G^{(n)}(\alpha,\beta,\gamma) = \frac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\alpha-\beta)}\sum_{m=0}^{\infty}\frac{(\beta-\alpha+1)_m}{m!}\frac{1}{(m+\gamma)^{n+1}} \tag{15}$$

for all $\alpha,\beta,\gamma\in\mathbb{C}$ such that $\alpha\notin\mathbb{Z}_{\leq 0}$, $\mathrm{Re}(\gamma), \mathrm{Re}(\alpha-\beta+1)>0$: *this single series is $H^*\bigl(z_1(n+1);(\beta-\alpha+1,\gamma,\gamma)\bigr)$.*

Proof. The expression (14) can be proved in the same way as in [Igarashi 2018, Proof of Lemma 2.18]. The expression (15) can be proved as follows: The quotient $\Gamma(X)/\Gamma(\alpha-\beta+X)$ can be expressed as

$$\frac{\Gamma(X)}{\Gamma(\alpha-\beta+X)} = \frac{\alpha-\beta+X}{\Gamma(\alpha-\beta+1)}\frac{\Gamma(X)\Gamma(\alpha-\beta+1)}{\Gamma(\alpha-\beta+X+1)} = \frac{\alpha-\beta+X}{\Gamma(\alpha-\beta+1)}\int_0^1 t^{X-1}(1-t)^{\alpha-\beta}\,\mathrm{d}t$$
$$= \frac{\alpha-\beta+X}{\Gamma(\alpha-\beta+1)}\int_0^1\sum_{m=0}^{\infty}\frac{(\beta-\alpha)_m}{m!}t^{m+X-1}\,\mathrm{d}t$$
$$= \frac{\alpha-\beta+X}{\Gamma(\alpha-\beta+1)}\sum_{m=0}^{\infty}\frac{(\beta-\alpha)_m}{m!}\frac{1}{m+X}$$

for $\mathrm{Re}(X), \mathrm{Re}(\alpha-\beta+1)>0$. Using the last expression above, which is a special case of Gauss' formula for the hypergeometric series ${}_2F_1(a,b;c;1)$, we get the identities

$$\begin{aligned}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial X^n}\left(\frac{\Gamma(X)}{\Gamma(\alpha-\beta+X)}\right)\bigg|_{X=\gamma}\\ &=\frac{1}{\Gamma(\alpha-\beta+1)}\sum_{m=0}^{\infty}\frac{(\beta-\alpha)_m}{m!}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial X^n}\left(\frac{\alpha-\beta+X}{m+X}\right)\bigg|_{X=\gamma}\\ &=\frac{1}{\Gamma(\alpha-\beta+1)}\sum_{m=0}^{\infty}\frac{(\beta-\alpha)_m}{m!}\left(\frac{\alpha-\beta+\gamma}{(m+\gamma)^{n+1}}-\frac{1}{(m+\gamma)^n}\right)\\ &=\frac{1}{\Gamma(\alpha-\beta)}\sum_{m=0}^{\infty}\frac{(\beta-\alpha+1)_m}{m!}\frac{1}{(m+\gamma)^{n+1}}\end{aligned}$$

for $n\geq 1,\ \mathrm{Re}(\gamma), \mathrm{Re}(\alpha-\beta+1)>0$. Thus we get (15). □

Remark 7. Both (14) and (15) can be applied to (5), in particular, to evaluating (5) in terms of MHZVs: see Corollary 8(ii) below. The expression (15) is simpler than (14).

Proof of Theorem 1. Let

$$v=\prod_{i=1}^{p} z_{c_{i-1}}(k_i)\in B^0 \quad \text{with the dual} \quad \tau(v)=\prod_{i=1}^{q} z_{c'_{i-1}}(k'_i).$$

For brevity, we put $a_0:=1-\beta$ and $b_0:=\alpha-\beta$. Differentiating the left-hand side of (12) r times with respect to X at $X=\gamma$ ($\mathrm{Re}(\gamma)>0$) and by the Leibniz rule, we get the left-hand side of (5).

The right-hand side of (5) can be proved as follows: We remark that the identity

$$(b_0+X)_{m_1+\varepsilon(c'_1)}=(b_0+X)_{m_1}(m_1+b_0+X)^{\varepsilon(c'_1)}$$

holds, because $\varepsilon(c_i)\in\{0,1\}$. Using this identity, we get the identities

$$\begin{aligned}\frac{(a_0+X)_{m_1}}{(b_0+X)_{m_q+1}}&=\frac{(a_0+X)_{m_1}}{(b_0+X)_{m_1}}\frac{(b_0+X)_{m_1}}{(b_0+X)_{m_q+1}}\\ &=\frac{(a_0+X)_{m_1}}{(b_0+X)_{m_1}}\frac{1}{(m_1+b_0+X)^{\varepsilon(c'_1)}}\left(\prod_{i=2}^{q}\frac{(b_0+X)_{m_{i-1}+\varepsilon(c'_{i-1})}}{(b_0+X)_{m_i+\varepsilon(c'_i)}}\right)\end{aligned}\tag{16}$$

for $m_1,\dots,m_q\in\mathbb{Z}$ such that $0\leq m_1<_{c'_1}\cdots<_{c'_{q-1}} m_q$, where $c'_q=1$. We calculate the partial differential coefficients of each factor on the right-hand side of (16). By direct calculation, we get the identities

$$\begin{aligned}
&\frac{(b_0+\gamma)_{m_1}}{(a_0+\gamma)_{m_1}}\frac{(a_0+X)_{m_1}}{(b_0+X)_{m_1}} \\
&\qquad = \prod_{n=0}^{m_1-1} \frac{(b_0+\gamma+n)(a_0+X+n)}{(a_0+\gamma+n)(b_0+X+n)} \\
&\qquad = \prod_{n=0}^{m_1-1} \left(1+\frac{(1-\alpha)(\gamma-X)}{(a_0+\gamma+n)(b_0+X+n)}\right) \\
&\qquad = \sum_{n=0}^{m_1} \sum_{0\le M_{11}<\cdots<M_{1n}<m_1} \prod_{i=1}^{n} \frac{(1-\alpha)(\gamma-X)}{(a_0+\gamma+M_{1i})(b_0+X+M_{1i})} \qquad (17)
\end{aligned}$$

for $m_1 \ge 0$. Using (17), we can get the identity

$$\begin{aligned}
&\frac{(-1)^{s_0}}{s_0!}\frac{\partial^{s_0}}{\partial X^{s_0}}\left(\frac{(a_0+X)_{m_1}}{(b_0+X)_{m_1}}\right)\Bigg|_{X=\gamma} \\
&\quad = \frac{(a_0+\gamma)_{m_1}}{(b_0+\gamma)_{m_1}} \sum_{n=0}^{m_1} \sum_{\substack{\sum_{i=1}^{n} r_i=s_0 \\ r_i\ge 1}} (1-\alpha)^n \\
&\qquad\qquad \times \sum_{0\le M_{11}<\cdots<M_{1n}<m_1} \prod_{i=1}^{n} \frac{1}{(a_0+\gamma+M_{1i})(b_0+\gamma+M_{1i})^{r_i}} \qquad (18)
\end{aligned}$$

for $m_1, s_0 \ge 0$. The partial differential coefficients of other factors can be calculated as follows:

$$\begin{aligned}
&\frac{(-1)^{s_i}}{s_i!}\frac{\partial^{s_i}}{\partial X^{s_i}}\left(\frac{(b_0+X)_{m_{i-1}+\varepsilon(c'_{i-1})}}{(b_0+X)_{m_i+\varepsilon(c'_i)}}\right)\Bigg|_{X=\gamma} \\
&\quad = \frac{(b_0+\gamma)_{m_{i-1}+\varepsilon(c'_{i-1})}}{(b_0+\gamma)_{m_i+\varepsilon(c'_i)}} \sum_{m_{i-1}+\varepsilon(c'_{i-1})\le M_{i1}\le\cdots\le M_{is_i}<m_i+\varepsilon(c'_i)} \prod_{j=1}^{s_i} \frac{1}{M_{ij}+b_0+\gamma} \\
&\quad = \frac{(b_0+\gamma)_{m_{i-1}+\varepsilon(c'_{i-1})}}{(b_0+\gamma)_{m_i+\varepsilon(c'_i)}} \sum_{m_{i-1}<_{c'_{i-1}} M_{i1}\le\cdots\le M_{is_i} <^*_{c'_i} m_i} \prod_{j=1}^{s_i} \frac{1}{M_{ij}+b_0+\gamma} \qquad (19)
\end{aligned}$$

for $s_i \ge 0$ ($i = 2, \dots, q$): By the definitions of the symbols $<_{c_i}$, $<^*_{c_i}$ and $\varepsilon(c_i)$, the inequalities $m_{i-1}+\varepsilon(c'_{i-1}) \le M_{i1}$ and $M_{is_i} < m_i+\varepsilon(c'_i)$ under the summation sign in (19) can be rewritten as $m_{i-1} <_{c'_{i-1}} M_{i1}$ and $M_{is_i} <^*_{c'_i} m_i$, respectively. Indeed these can be verified directly. For example, if $c'_i = 1$, then we get $\varepsilon(c'_i) = 1$ and $<^*_{c'_i} = \le$. Therefore the inequality $M_{is_i} < m_i + \varepsilon(c'_i) = m_i + 1$ can be rewritten as

$M_{is_i} <^*_{c'_i} m_i$. Using (19) and the Leibniz rule, we get the identities

$$
\begin{aligned}
&\frac{(-1)^r}{r!}\frac{\partial^r}{\partial X^r}\left(\frac{1}{(m_1+b_0+X)^{\varepsilon(c'_1)}}\left(\prod_{i=2}^{q}\frac{(b_0+X)_{m_{i-1}+\varepsilon(c'_{i-1})}}{(b_0+X)_{m_i+\varepsilon(c'_i)}}\right)\right)\bigg|_{X=\gamma}\\
&=\sum_{\substack{s_1+\cdots+s_q=r\\ s_i\geq 0}}\frac{\binom{s_1+\varepsilon(c'_1)-1}{s_1}}{(m_1+b_0+\gamma)^{s_1+\varepsilon(c'_1)}}\\
&\quad\times\left(\prod_{i=2}^{q}\frac{(b_0+\gamma)_{m_{i-1}+\varepsilon(c'_{i-1})}}{(b_0+\gamma)_{m_i+\varepsilon(c'_i)}}\sum_{m_{i-1}<_{c'_{i-1}}M_{i1}\leq\cdots\leq M_{is_i}<^*_{c'_i}m_i}\;\prod_{j=1}^{s_i}\frac{1}{M_{ij}+b_0+\gamma}\right)\\
&=\sum_{\substack{s_1+\cdots+s_q=r\\ s_i\geq 0}}\frac{\binom{s_1+\varepsilon(c'_1)-1}{s_1}}{(m_1+b_0+\gamma)^{s_1}}\\
&\qquad\times\frac{(b_0+\gamma)_{m_1}}{(b_0+\gamma)_{m_q+1}}\prod_{i=2}^{q}\left(\sum_{m_{i-1}<_{c'_{i-1}}M_{i1}\leq\cdots\leq M_{is_i}<^*_{c'_i}m_i}\;\prod_{j=1}^{s_i}\frac{1}{M_{ij}+b_0+\gamma}\right)\\
&=\sum_{\substack{\varepsilon(c'_1)s_1+\sum_{i=2}^{q}s_i=r\\ s_j\geq 0}}\frac{1}{(m_1+b_0+\gamma)^{\varepsilon(c'_1)s_1}}\\
&\qquad\times\frac{(b_0+\gamma)_{m_1}}{(b_0+\gamma)_{m_q+1}}\prod_{i=2}^{q}\left(\sum_{m_{i-1}<_{c'_{i-1}}M_{i1}\leq\cdots\leq M_{is_i}<^*_{c'_i}m_i}\;\prod_{j=1}^{s_i}\frac{1}{M_{ij}+b_0+\gamma}\right)
\end{aligned}\tag{20}
$$

for $r\geq 0$. The last equality sign of (20) comes from the identity

$$
\binom{s_1+\varepsilon(c'_1)-1}{s_1}=\begin{cases}1 & \text{if } \varepsilon(c'_1)=s_1=0 \text{ or } \varepsilon(c'_1)=1,\ s_1\geq 0,\\ 0 & \text{if } \varepsilon(c'_1)=0,\ s_1\geq 1.\end{cases}
$$

Letting

$$
\begin{aligned}
&g(\{m_i\}_{i=1}^{q};\{r_i\}_{i=1}^{n};\{s_i\}_{i=1}^{q})\\
&=\sum_{\substack{0\leq M_{11}<\cdots<M_{1n}<m_1\\ m_1<_{c'_1}M_{21}\leq\cdots\leq M_{2s_2}<^*_{c'_2}m_2\\ \vdots\\ m_{i-1}<_{c'_{i-1}}M_{i1}\leq\cdots\leq M_{is_i}<^*_{c'_i}m_i\\ \vdots\\ m_{q-1}<_{c'_{q-1}}M_{q1}\leq\cdots\leq M_{qs_q}<^*_{c'_q}m_q}}\frac{(a_0+\gamma)_{m_1}}{m_1!}\frac{(b_0+1)_{m_q+1}}{(b_0+\gamma)_{m_q+1}}\\
&\qquad\times\left(\prod_{i=1}^{n}\frac{1}{(M_{1i}+a_0+\gamma)(M_{1i}+b_0+\gamma)^{r_i}}\right)\frac{1}{(m_1+b_0+1)^{k'_1}(m_1+b_0+\gamma)^{\varepsilon(c'_1)s_1}}\\
&\qquad\qquad\times\left\{\prod_{i=2}^{q}\left(\prod_{j=1}^{s_i}\frac{1}{M_{ij}+b_0+\gamma}\right)\frac{1}{(m_i+b_0+1)^{k'_i}}\right\},
\end{aligned}
$$

and using (16), (18) and (20), we get the identity

$$\frac{(-1)^r}{r!}\frac{\partial^r}{\partial X^r}\biggl(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\frac{\Gamma(X)}{\Gamma(b_0+X)}\frac{(a_0+X)_{m_1}}{m_1!}\frac{(b_0+1)_{m_q+1}}{(b_0+X)_{m_q+1}}\biggl\{\prod_{i=1}^{q}\frac{1}{(m_i+b_0+1)^{k'_i}}\biggr\}\biggr)\bigg|_{X=\gamma}$$

$$= \sum_{\substack{r_0+s_0+\varepsilon(c'_1)s_1+\sum_{i=2}^q s_i=r\\ r_0,s_j\geq 0}}\ \sum_{n=0}^{m_1}\ \sum_{\substack{\sum_{i=1}^n r_i=s_0\\ r_i\geq 1}} (1-\alpha)^n G^{(r_0)}(\alpha,\beta,\gamma) \times g(\{m_i\}_{i=1}^q;\{r_i\}_{i=1}^n;\{s_i\}_{i=1}^q) \quad (21)$$

for $r\geq 0$ and $m_1,\ldots,m_q\in\mathbb{Z}$ such that $0\leq m_1<_{c'_1}\cdots<_{c'_{q-1}} m_q$, where $c'_q=1$. Here we impose the conditions

$$\mathrm{Re}(\beta),\mathrm{Re}(\gamma),\mathrm{Re}(1-\beta+\gamma),\mathrm{Re}(\alpha-\beta+1),\mathrm{Re}(\alpha-\beta+\gamma)>0.$$

Then, taking $x=1-\beta+\gamma,\ y=\alpha-\beta+1$ and $z=\alpha-\beta+\gamma$ in Lemma 5, we see that the multiple series on the right-hand side of (12) converges uniformly in $\{X\in\mathbb{C}\mid |X-\gamma|\leq\varepsilon\}$, where $\varepsilon\in\mathbb{R}$ such that

$$0<\varepsilon<\min\bigl\{\mathrm{Re}(1-\beta+\gamma),\mathrm{Re}(\alpha-\beta+\gamma),\mathrm{Re}(\beta/2),\mathrm{Re}(\gamma)\bigr\}.$$

Thus, differentiating term by term and using (21), we can get the following identity for the right-hand side of (12):

$$\frac{(-1)^r}{r!}\frac{\partial^r}{\partial X^r}\biggl(\frac{\Gamma(\alpha)}{\Gamma(\beta)}\frac{\Gamma(X)}{\Gamma(\alpha-\beta+X)}H^*\bigl(\tau(v);(\alpha,\beta,X)^\tau\bigr)\biggr)\bigg|_{X=\gamma}$$

$$= \sum_{\substack{r_0+s_0+\varepsilon(c'_1)s_1+\sum_{i=2}^q s_i=r\\ r_0,s_j\geq 0}}\ \sum_{n=0}^{s_0}\ \sum_{\substack{\sum_{i=1}^n r_i=s_0\\ r_i\geq 1}} (1-\alpha)^n G^{(r_0)}(\alpha,\beta,\gamma) \times H^*_{(\boldsymbol{r}^n,\boldsymbol{s}^q,\varepsilon(c'_1))}\bigl(\tau(v);(\alpha,\beta,\gamma)^\tau\bigr) \quad (22)$$

for $r\geq 0$, $\mathrm{Re}(\alpha)$, and

$$\mathrm{Re}(\beta),\mathrm{Re}(\gamma),\mathrm{Re}(1-\beta+\gamma),\mathrm{Re}(\alpha-\beta+1),\mathrm{Re}(\alpha-\beta+\gamma)>0.$$

Finally we verify that the right-hand side of (22) is the same as that of (5). For brevity, we put the summand on the right-hand side of (22) in $h(n;\{r_i\}_{i=0}^n;\{s_i\}_{i=1}^q)$. Then the right-hand side of (22) can be rewritten as

$$\sum_{\substack{r_0+s_0+\varepsilon(c'_1)s_1+\sum_{i=2}^q s_i=r\\ r_0,s_j\geq 0}}\ \sum_{n=0}^{s_0}\ \sum_{\substack{\sum_{i=1}^n r_i=s_0\\ r_i\geq 1}} h(n;\{r_i\}_{i=0}^n;\{s_i\}_{i=1}^q)$$

$$=\sum_{s_0=0}^{r}\sum_{n=0}^{s_0}\ \sum_{\substack{r_0+\varepsilon(c'_1)s_1+\sum_{i=2}^q s_i=r-s_0\\ r_0,s_j\geq 0}}\ \sum_{\substack{\sum_{i=1}^n r_i=s_0\\ r_i\geq 1}} h(n;\{r_i\}_{i=0}^n;\{s_i\}_{i=1}^q)$$

$$= \sum_{n=0}^{r} \sum_{s_0=n}^{r} \sum_{\substack{r_0+\varepsilon(c_1')s_1+\sum_{i=2}^{q} s_i=r-s_0 \\ r_0, s_j \geq 0}} \sum_{\substack{\sum_{i=1}^{n} r_i=s_0 \\ r_i \geq 1}} h(n; \{r_i\}_{i=0}^{n}; \{s_i\}_{i=1}^{q})$$

$$= \sum_{n=0}^{r} \sum_{\substack{r_0+s_0+\varepsilon(c_1')s_1+\sum_{i=2}^{q} s_i=r-n \\ r_0, s_j \geq 0}} \sum_{\substack{\sum_{i=1}^{n} r_i=s_0+n \\ r_i \geq 1}} h(n; \{r_i\}_{i=0}^{n}; \{s_i\}_{i=1}^{q})$$

$$= \sum_{n=0}^{r} \sum_{\substack{\sum_{i=0}^{n} r_i+\varepsilon(c_1')s_1+\sum_{i=2}^{q} s_i=r \\ r_i \geq 1\,(i=1,\ldots,n) \\ r_0, s_i \geq 0\,(i=1,\ldots,q)}} h(n; \{r_i\}_{i=0}^{n}; \{s_i\}_{i=1}^{q}).$$

The last rewrite is exactly the same as the right-hand side of (5). This completes the proof of Theorem 1. □

For $v = \prod_{i=1}^{p} z_{c_{i-1}}(k_i) \in B^0$, we put

$$\begin{aligned} \zeta(v; (x, z)) &:= H(v; (x, x, z)) \\ &= \sum_{0 \leq m_1 <_{c_1} \cdots <_{c_{p-1}} m_p < \infty} \left(\prod_{i=1}^{p-1} \frac{1}{(m_i+z)^{k_i}} \right) \frac{1}{(m_p+x)(m_p+z)^{k_p-1}} \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\zeta^*_{(\{r_i\}_{i=1}^{n}; \{s_i\}_{i=1}^{p})}(v; y) \\ &:= H^*_{(\{r_i\}_{i=1}^{n}; \{s_i\}_{i=1}^{p})}(v; (1, y, y)) \\ &= \sum_{\substack{0 \leq M_{11} < \cdots < M_{1n} < m_1 \\ m_1 <_{c_1} M_{21} \leq \cdots \leq M_{2s_2} <^*_{c_2} m_2 \\ \vdots \\ m_{i-1} <_{c_{i-1}} M_{i1} \leq \cdots \leq M_{is_i} <^*_{c_i} m_i \\ \vdots \\ m_{p-1} <_{c_{p-1}} M_{p1} \leq \cdots \leq M_{ps_p} <^*_{c_p} m_p < \infty}} \left(\prod_{i=1}^{n} \frac{1}{(M_{1i}+1)(M_{1i}+y)^{r_i}} \right) \frac{1}{(m_1+y)^{k_1+s_1}} \\ &\qquad \times \left\{ \prod_{i=2}^{p} \left(\prod_{j=1}^{s_i} \frac{1}{M_{ij}+y} \right) \frac{1}{(m_i+y)^{k_i}} \right\}. \end{aligned}$$

These multiple series are MHZVs. We can derive the following two identities for these MHZVs from Theorem 1:

Corollary 8. *Let $v \in B^0$, and let $\tau(v)$ be its dual. Then the following two identities hold*:

(i)

$$\zeta(\sigma_r^b(v); (\alpha, \gamma)) \\ = \sum_{n=0}^{r} (1-\alpha)^n \sum_{\substack{\sum_{i=1}^n r_i + \varepsilon(c_1') s_1 + \sum_{i=2}^q s_i = r \\ r_i \geq 1\, (i=1,\dots,n) \\ s_i \geq 0\, (i=1,\dots,q)}} H^*_{(\boldsymbol{r}^n, \boldsymbol{s}^q, \varepsilon(c_1'))}(\tau(v); (\alpha, \alpha, \gamma)^\tau) \tag{24}$$

for all $r \geq 0$, $\alpha, \gamma \in \mathbb{C}$ *such that*

$$\mathrm{Re}(\alpha), \mathrm{Re}(\gamma), \mathrm{Re}(1-\alpha+\gamma) > 0,$$

where $\zeta(\sigma_r^b(v); (x, z)) := H(\sigma_r^b(v); (x, x, z))$.

(ii)

$$H(\sigma_r^b(v); (\alpha, 1, 1)) \\ = \sum_{n=0}^{r} (1-\alpha)^n \sum_{\substack{\sum_{i=0}^n r_i + \varepsilon(c_1') s_1 + \sum_{i=2}^q s_i = r \\ r_i \geq 1\, (i=1,\dots,n) \\ r_0, s_i \geq 0\, (i=1,\dots,q)}} G^{(r_0)}(\alpha, 1, 1) \zeta^*_{(\boldsymbol{r}^n, \boldsymbol{s}^q, \varepsilon(c_1'))}(\tau(v); \alpha) \tag{25}$$

for all $r \geq 0$, $\alpha \in \mathbb{C}$ *with* $\mathrm{Re}(\alpha) > 0$. *The factor* $G^{(r_0)}(\alpha, 1, 1)$ $(r_0 \geq 1)$ *becomes* $(\alpha - 1)\zeta\big(z_1(1)^{r_0-1} z_1(2); (\alpha, 1)\big)$ *or a* $\mathbb{Q}$*-polynomial of*

$$(1-\alpha)^c \sum_{m=0}^{\infty} (m+1)^{-a} (m+\alpha)^{-b}$$

$(a, b, c \in \mathbb{Z}$ *such that* $a, b, c \geq 0,\ a + b > 1)$, *therefore the right-hand side of* (25) *can be expressed by MHZVs and* $(1-\alpha)^n$ $(n \geq 0)$.

Proof. By the definition of $G^{(n)}(\alpha, \beta, \gamma)$, we see that $G^{(n)}(\alpha, \alpha, \gamma) = 0$ $(n \geq 1)$ and $G^{(0)}(\alpha, \alpha, \gamma) = 1$. By this fact and taking $\alpha = \beta$ in Theorem 1, we get (24). The identity (25) can be derived from Theorem 1 by taking $\beta = \gamma = 1$. In this case, Lemma 6(i) shows that the factor $G^{(r_0)}(\alpha, 1, 1)$ $(r_0 \geq 1)$ becomes a $\mathbb{Q}$-polynomial of $(1-\alpha)^c \sum_{m=0}^{\infty} (m+1)^{-a} (m+\alpha)^{-b}$ $(a, b, c \in \mathbb{Z},\ a, b, c \geq 0,\ a + b > 1)$. On the other hand, Lemma 6(ii) gives the expressions

$$\begin{aligned} G^{(r_0)}(\alpha, 1, 1) &= (\alpha - 1) \sum_{m=0}^{\infty} \frac{(2-\alpha)_m}{m!} \frac{1}{(m+1)^{r_0+1}} \\ &= (\alpha - 1) H^*\big(z_1(r_0 + 1); (2 - \alpha, 1, 1)\big) \end{aligned}$$

for $r_0 \geq 1$, $\mathrm{Re}(\alpha) > 0$. Further, this single series can be rewritten as

$$\begin{aligned} H^*\big(z_1(r_0+1);\,(2-\alpha,1,1)\big) &= H^*\big(z_1(r_0+1);\,(\alpha,\alpha,1)^{\tau}\big) \\ &= H\big(z_1(1)^{r_0-1}z_1(2);\,(\alpha,\alpha,1)\big) \quad \text{(by Lemma 3)} \\ &= \zeta\big(z_1(1)^{r_0-1}z_1(2);\,(\alpha,1)\big) \qquad \text{(by (23))}. \end{aligned}$$

Thus we get $G^{(r_0)}(\alpha,1,1) = (\alpha-1)\zeta\big(z_1(1)^{r_0-1}z_1(2);\,(\alpha,1)\big)$ $(r_0 \geq 1,\ \mathrm{Re}(\alpha) > 0)$, and this completes the proof of the assertion for $G^{(r_0)}(\alpha,1,1)$ stated in (ii). $\square$

Remark 9. The identity

$$\sum_{m=0}^{\infty} \frac{(2-\alpha)_m}{m!} \frac{1}{(m+1)^{p+1}} = \sum_{0 \leq m_1 < \cdots < m_p < \infty} \left(\prod_{i=1}^{p} \frac{1}{m_i+1} \right) \frac{1}{m_p+\alpha}$$

$(p \geq 1,\ \mathrm{Re}(\alpha) > 0)$, which we used in the proof of Corollary 8(ii), is proved by Hoffman [1992, Section 4] by using a theorem of Mordell. He used this identity to prove a duality formula for MZVs [Hoffman 1992, Theorem 4.4].

Remark 10. I give a remark on a connection between (24) and a former result of mine written in [Igarashi 2015]. Let $v = \prod_{i=1}^{p} z_{c_{i-1}}(k_i) \in B^0$, and let $\tau(v) = \prod_{i=1}^{q} z_{c'_{i-1}}(k'_i)$ be its dual. In [Igarashi 2015], I proved the following new identity for MHZVs:

$$\sum_{\substack{r_1+\cdots+r_p=r \\ r_i \geq 0}} \left\{ \prod_{i=1}^{p} \binom{k_i+r_i-1}{r_i} \right\} \zeta\left(\prod_{i=1}^{p} z_{c_{i-1}}(k_i+r_i);\,\alpha \right) = \sum_{\substack{s_1+\cdots+s_q=r \\ s_i \geq 0}} H^*_{(\{s_i\}_{i=1}^{q})}(\tau(v);\,\alpha) \tag{26}$$

for all $r \geq 0$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, where $\zeta(v;\alpha)$ is the extension of MHZV defined under (7) and

$$H^*_{(\{s_i\}_{i=1}^{p})}(v;\alpha) := \sum_{\substack{0=m_0 \leq M_{11} \leq \cdots \leq M_{1s_1} <^*_{c_1} m_1 \\ \vdots \\ m_{i-1} <_{c_{i-1}} M_{i1} \leq \cdots \leq M_{is_i} <^*_{c_i} m_i \\ \vdots \\ m_{p-1} <_{c_{p-1}} M_{p1} \leq \cdots \leq M_{ps_p} <^*_{c_p} m_p < \infty}} \frac{(m_p+1)!}{(\alpha)_{m_p+1}} \left\{ \prod_{i=1}^{p} \left(\prod_{j=1}^{s_i} \frac{1}{M_{ij}+\alpha} \right) \frac{1}{(m_i+1)^{k_i}} \right\}$$

$(p \geq 1,\ s_i \geq 0\ (i=1,\ldots,p),\ c_p = 1,\ \alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0)$. By the identity $\binom{k_p+r_p-1}{r_p} = \binom{k_p+r_p-2}{r_p} + \frac{r_p}{k_p-1}\binom{k_p+r_p-2}{r_p}$, we see that the sum on the left-hand side of (24) with $\alpha = \gamma$ is a partial sum of that of (26), and therefore the identity (24) with $\alpha = \gamma$ is a decomposition of (26). The factors $(1-\alpha)^n$ $(n = 0, 1, \ldots, r)$ in

(24) explicitly indicate vanishing terms at $\alpha = 1$ of the decomposition: compare the case $\alpha = 1$ of the right-hand side of (24) with that of (26). The identity (24) gives a two-parameter extension of (7) also. The identities for MHZVs (24), (25) and (26) are relations among extensions of MHZV, namely the multiple series (2).

Remark 11. As is shown in the present paper, the gamma factor in (12) contributes to deriving various relations among (2) from (12). Here we give another example of this sort: By dividing both sides of (12) by the gamma factor, the identity (12) can be modified as

$$\frac{\Gamma(\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha-\beta+X)}{\Gamma(X)}H(v;(\alpha,\beta,X)) = H^*(\tau(v);(\alpha,\beta,X)^\tau). \tag{27}$$

In the same way as in the proof of Theorem 1, we can derive the following inversion formula for (5) from (27):

$$\sum_{r_0=0}^{r} G^{(r_0)}(\beta,\alpha,\alpha-\beta+\gamma)H(\sigma^b_{r-r_0}(v);(\alpha,\beta,\gamma)) \\ = \sum_{n=0}^{r}(1-\alpha)^n \sum_{\substack{\sum_{i=1}^n r_i+\varepsilon(c_1')s_1+\sum_{i=2}^q s_i=r \\ r_i\geq 1\,(i=1,\dots,n) \\ s_i\geq 0\,(i=1,\dots,q)}} H^*_{(\boldsymbol{r}^n,\boldsymbol{s}^q,\varepsilon(c_1'))}(\tau(v);(\alpha,\beta,\gamma)^\tau)$$

for all $v \in B^0,\ r \geq 0,\ \alpha,\beta,\gamma \in \mathbb{C}$ such that

$$\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma), \mathrm{Re}(1-\beta+\gamma), \mathrm{Re}(\alpha-\beta+1), \mathrm{Re}(\alpha-\beta+\gamma) > 0,$$

where $\tau(v) = \prod_{i=1}^q z_{c'_{i-1}}(k'_i)$ is the dual of v. This identity also yields numerous relations among (2). For this kind of application of gamma factors, see also [Igarashi 2018, Theorem 2.17] and its proof.

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