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G-symmetric monoidal categories of modules over equivariant commutative ring spectra

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We describe the multiplicative structures that arise on categories of equivariant modules over certain equivariant commutative ring spectra. Building on our previous work on $\mathcal{N}_\infty$ ring spectra, we construct categories of equivariant operadic modules over $\mathcal{N}_\infty$ rings that are structured by equivariant linear isometries operads. These categories of modules are endowed with equivariant symmetric monoidal structures, which amounts to the structure of an "incomplete Mackey functor in homotopical categories". In particular, we construct internal norms which satisfy the double coset formula. One application of the work of this paper is to provide a context in which to describe the behavior of Bousfield localization of equivariant commutative rings. We regard the work of this paper as a first step towards equivariant derived algebraic geometry.

1. Introduction

Stable homotopy theory has been revolutionized over the last twenty years by the development of symmetric monoidal categories of spectra [Elmendorf et al. 1997; Mandell et al. 2001; Hovey et al. 2000]. Commutative monoids in these categories model E_∞ ring spectra. Arguably the most important consequence of this machinery is the ability to have tractable point-set models for homotopical categories of modules over an E_∞ ring spectrum R. In the equivariant setting, analogous symmetric monoidal categories of G-spectra have been constructed, most notably the category of orthogonal G-spectra [Mandell and May 2002]. Once again commutative monoids model E_∞ ring spectra and so there are good point-set models for homotopical categories of modules over such an equivariant E_∞ ring spectrum.

Modules over a commutative ring orthogonal G-spectrum R form a "G-symmetric monoidal category" [Hill and Hopkins 2016]. Roughly speaking, for each G-set T, we have an internal norm in the category of R-modules; for an orbit G/H, the internal norm is precisely the composite of the R-relative norm $_R N_H^G$ and the forgetful functor from R-modules to $\iota_H^* R$-modules. These internal norms are compatible with disjoint unions of G-sets and with restrictions to subgroups, and if the set has a trivial action and cardinality n, then we recover the smash power functors $X \mapsto X^n$.

However, in contrast to the nonequivariant setting, there are many possible notions of E_∞ ring spectra when working over a nontrivial finite group G. The commutative monoids in orthogonal G-spectra are just one end of the spectrum of possible multiplicative structures. We previously described this situation in detail by explaining how such multiplications can be structured by $\mathcal{N}_\infty$ operads [Blumberg and Hill 2015]. Roughly speaking, just as a commutative ring is characterized by compatible multiplication maps $R^{\wedge n} \to R$ as n varies over the natural numbers, a commutative G-ring is characterized by compatible equivariant multiplication maps $R^{\wedge T} \to R$, where here T is a G-set. The $\mathcal{N}_\infty$ operads structure which such equivariant norms exist for a given commutative ring, expressed in terms of compatible families of subgroups of $G \times \Sigma_n$. Specifically, associated to an $\mathcal{N}_\infty$ operad $\mathcal{O}$ there is a coefficient system $\mathcal{C}(\mathcal{O})$ which controls the "admissible" G-sets T for which equivariant multiplications exist. The commutative monoids in orthogonal G-spectra correspond to the "complete" $\mathcal{N}_\infty$ operads which permit all norms.

In this paper, we turn to the study of the equivariant symmetric monoidal structure on categories of operadic modules associated to algebras over particular $\mathcal{N}_\infty$ operads: the linear isometries operads determined by a (possibly incomplete) G-universe U. Here the admissible sets for U will play a second role; for an $\mathcal{O}$-algebra R, the admissible sets determine additional structure on the underlying symmetric monoidal category of R-modules. Specifically, for each admissible G-set T, we have a internal norm in the category of R-modules for an $\mathcal{O}$-algebra R.

In order to describe this structure, it is convenient to instead consider the collection of categories of modules over $\iota_H^* R$, where $H \subseteq G$ is a closed subgroup of G and ι_H^* is the forgetful functor. The extra structure on the category of R-modules then is encoded in functors

$$\iota_H^* : \mathrm{Mod}_{\iota_K^* R} \to \mathrm{Mod}_{\iota_H^* R} \quad \text{and} \quad {}_{(\iota_K^* R)} N_H^K : \mathrm{Mod}_{\iota_H^* R} \to \mathrm{Mod}_{\iota_K^* R}$$

for $H \subset K \subset G$ that assemble into a kind of "incomplete Mackey functor" of homotopical categories. The internal norms arise from the composite functors $N_H^G \iota^* H(-)$, extended to arbitrary G-sets T by decomposing T into a disjoint union of orbits G/H and smashing together the corresponding composites. The compatibility conditions in particular express the double coset formula.

More precisely, we have the following functors:

Theorem 1.1. *Let G be a finite group and U a G-universe. Let R be an algebra in orthogonal G-spectra over $\mathscr{L}_U$, the linear isometries operad structured by U. Then for each $H \subset G$ there exists a symmetric monoidal model category $\mathcal{M}_{\iota_H^* R}$ of $\iota_H^* R$-modules. For each $H \subset K \subset G$ such that K/H is an admissible K-set for U, there exist homotopical functors*

$$_{(\iota_K^* R)} N_{H, \iota_H^* U}^{K, \iota_K^* U} : \mathcal{M}_{\iota_H^* R} \to \mathcal{M}_{\iota_K^* R} \quad \text{and} \quad \iota_H^* : \mathcal{M}_{\iota_K^* R} \to \mathcal{M}_{\iota_H^* R}.$$

The internal norms now arise from these functors:

Definition 1.2. Let G be a finite group and U be a G-universe. For an H-set T, writing $T = H/K_1 \amalg H/K_2 \amalg \cdots \amalg H/K_m$, define

$$_{(\iota_H^* R)} N^T : \mathcal{M}_{\iota_H^* R} \to \mathcal{M}_{\iota_H^* R}$$

by the formula

$$_{(\iota_H^* R)} N^T X = \left({}_{(\iota_H^* R)} N_{K_1}^H \iota_{K_1}^* X \right) \wedge_{\iota_H^* R} \left({}_{(\iota_H^* R)} N_{K_2}^H \iota_{K_2}^* X \right) \wedge_{\iota_H^* R} \cdots \wedge_{\iota_H^* R} \left({}_{(\iota_H^* R)} N_{K_m}^H \iota_{K_m}^* X \right).$$

More generally, define

$$_R N^T : \mathcal{M}_R \to \mathcal{M}_R$$

by the formula

$$_R N^T X = G_+ \wedge_H \left({}_{(\iota_H^* R)} N^T X \right).$$

The equivariant symmetric monoidal structure on $\mathcal{M}_R$ is encoded by the following relations between the internal norms and the forgetful functors:

Theorem 1.3. *Let G be a finite group and U be a G-universe.*

(1) *For $H_1 \subseteq H_2 \subseteq H_3 \subseteq G$, there are natural homeomorphisms*

$$N_{H_2}^{H_3} N_{H_1}^{H_2} \cong N_{H_1}^{H_3} \quad \text{and} \quad \iota_{H_1}^* \iota_{H_2}^* \cong \iota_{H_1}^*$$

that descend to the derived category,

(2) *For any H-sets T_1 and T_2, there are natural homeomorphisms $N^{T_1 \times T_2} X \cong N^{T_1} N^{T_2} X$ for each X that descend to the derived category when T_1 and T_2 are admissible, and*

(3) *For an admissible H-set T, the derived composite $\iota_K^* N^T M$ is naturally equivalent to $N^{\iota_K^* T} \iota_K^* M$.*

The last of these relations is a version of the double coset formula.

When $G = e$, the structure described by Theorem 1.3 is simply the usual symmetric monoidal structure on orthogonal spectra; the functors N^T for a set T are just the smash powers $X^{\wedge |T|}$. When U is the complete universe, this structure is precisely the G-symmetric monoidal structure on R-modules obtained by choosing a model of R that is a commutative monoid in orthogonal G-spectra.

Note that we have avoided trying to precisely formulate the notion of an incomplete Mackey functor of homotopical categories here, choosing instead to explicitly write out the structure and some of the coherences. However, if we are willing to pass to the homotopy category, we can state the following result.

Corollary 1.4. *Let R be an algebra in orthogonal G-spectra over $\mathscr{L}_U$. Let $\mathcal{B}_{G,U}$ denote the bicategory of spans of the admissible sets for $\mathscr{L}_U$. There exists a 2-functor from $\mathcal{B}_{G,U}$ to the 2-category of triangulated categories, exact functors, and natural isomorphisms that takes an admissible set G/H to $\mathcal{M}_{\iota_H^* R, \iota_H^* U}$.*

However, the coherences necessary for the definition of an incomplete Mackey functor at the level of homotopical categories is most easily handled using the formalism of ∞-categories; we expect such a treatment to come from the forthcoming work of [Barwick et al. 2016]. (See also [Bohmann and Osorno 2015] for a treatment of equivariant permutative categories from this kind of perspective. A different approach to equivariant permutative categories is described in [Guillou and May 2017].)

One of the applications of our work is the construction of strict point-set models of N_∞ ring spectra. Specifically, let $\mathbb{S}_G$ be the equivariant sphere spectrum, regarded as an $\mathscr{L}_U$ algebra. Then we have the following straightforward consequence of the proof of Theorem 1.1.

Corollary 1.5. *The category of commutative monoid objects in $\mathcal{M}_{\mathbb{S}_G}$ is equivalent to the category of N_∞ algebras structured by $\mathscr{L}_U$.*

More generally, for an N_∞ algebra R, we obtain a description of N_∞ R-algebras.

Corollary 1.6. *Let R be an N_∞ algebra structured by $\mathscr{L}_U$. The category of commutative monoid objects in $\mathcal{M}_R$ is equivalent to the category of N_∞ R-algebras structured by $\mathscr{L}_U$.*

These corollaries are particularly useful in the context of equivariant Bousfield localization. In their study of the multiplicative properties of equivariant Bousfield localization, the second author and Hopkins showed that localization of an N_∞ ring spectrum can change the universe that structures the multiplication [Hill and Hopkins 2016]. Specifically, [Hill and Hopkins 2016, Theorem 6.3] shows that a Bousfield localization L of orthogonal G-spectra takes $\mathscr{L}_U$ algebras to $\mathscr{L}_U$ algebras precisely when the category of L-acyclics is closed under norms for the indexing system determined by U. Therefore, we obtain the following result.

Theorem 1.7. *Let A be a commutative monoid in $M_{\mathbb{S}_U^G}$, where $\mathbb{S}_U^G$ denotes the sphere spectrum regarded as an $\mathscr{L}_U$ algebra in orthogonal G-spectra. Let L be a Bousfield localization functor with L-acyclics closed under norms specified by the indexing system for a universe U'. Suppose that U'' is a universe with corresponding indexing system contained in the indexing system obtained as the intersection of U and U'. Then LR is a commutative monoid object in $M_{\mathbb{S}_{U''}^G}$.*

In order to explain the restriction to $\mathcal{N}_\infty$ operads that can be modeled as linear isometries operads, we need to explain the strategy of proof for Theorem 1.1. Our approach is to adapt the strategy of EKMM [Elmendorf et al. 1997] to study operadic multiplications on G-spectra. Let $\mathcal{S}p_G$ denote the category of orthogonal G-spectra on a complete universe. Fix a different (possibly incomplete) G-universe U. Then there is a monad $\mathbb{L}_U$ on $\mathcal{S}p_G$, specified by the formula

$$X \mapsto \mathscr{L}(U, U)_+ \wedge X,$$

where $\mathscr{L}(U, U)$ is the G-space of nonequivariant linear isometries from U to U (with G acting by conjugation).

The category $\mathcal{S}p_G[\mathbb{L}_U]$ of $\mathbb{L}_U$-algebras has a model structure that is Quillen equivalent to the standard model structures on $\mathcal{S}p_G$. Moreover, it has a new symmetric monoidal product $\wedge_U$ such that the underlying orthogonal G-spectrum of $X \wedge_U Y$ is equivalent to $X \wedge Y$. But now monoids and commutative monoids for $\wedge_U$ are precisely (non)-symmetric algebras for the G-linear isometries operad for U. Just as in the category of spectra, we can restrict to the unital objects in $\mathcal{S}p_G[\mathbb{L}_U]$ to obtain a symmetric monoidal category GS_U. All of these categories can be equipped with symmetric monoidal model category structures. Using these symmetric monoidal model categories, we construct symmetric monoidal module categories for an $\mathcal{N}_\infty$ ring R structured by the G-linear isometries operad for U.

We expect that Theorem 1.1 is true more generally for any $\mathcal{N}_\infty$ operad, but it is difficult to obtain control on categories of operadic modules over operads other than the linear isometries operad using point-set techniques. In fact, a substantial part of the work of this paper involves verification of delicate point-set facts about the linear isometries operad that are simply not true for an arbitrary N_∞ operad,

just as in [Elmendorf et al. 1997]. Unfortunately, as we explain in [Blumberg and Hill 2015, Theorem 4.24], there are equivariant operads which arise from "little disks" constructions that are not equivalent to equivariant linear isometries operads for any universe. Again, we expect that it is more tractable to handle these sorts of homotopical questions in the ∞-categorical setting; specifically, working with equivariant ∞-operads structured over the nerve of distinguished subcategories of the category of finite G-sets.

One benefit of our approach to Theorem 1.1 is that our technical results about the equivariant linear isometries operad validate the multiplicative theory of the equivariant version of EKMM spectra. Although [Elmendorf et al. 1997, 0.1] famously asserts that all of the work of that volume holds mutatis mutandis when assuming that a compact Lie group G acts, verifying such a theorem requires some subtle checks about the behavior of the linear isometries operad (most notably Theorem A.9); and [Elmendorf and May 1997], which amongst other endeavors attempts to justify some of these properties, contains a critical error (in [Elmendorf and May 1997, Theorem 1.2]). As such, our work here supports prior applications of the equivariant category of S-modules, notably [Greenlees and May 1997].

Our interest in Theorem 1.1 comes in large part from examples arising from localization. As explained above, localization of an N_∞ ring spectrum can change the universe that structures the multiplication [Hill and Hopkins 2016]. In Section 6, we discuss a number of examples of this kind that arise in applications. More generally, the multiplicative behavior of localization of equivariant commutative ring spectra has significant consequences for the foundations of equivariant derived algebraic geometry.

Specifically, the possible loss of norms that occurs implies that there is not necessarily a "genuine" affine scheme associated to a commutative ring orthogonal G-spectrum when we work with the Zariski topology. Work of Nakaoka [2012] shows that something similar is true for Tambara functors: there does not exist a sheaf of Tambara functors on the Zariski site of a Tambara functor.

However, by restriction of structure, every equivariant commutative ring spectrum R is also an algebra over $\mathscr{L}_{\mathbb{R}^\infty}$, the linear isometries operad for a trivial universe. Bousfield localization always preserves the property of being an algebra over $\mathscr{L}_{\mathbb{R}^\infty}$, so in particular, we do have a sheaf of such rings in the Zariski topology. Therefore, using the work of this paper we can define equivariant derived affine schemes (and then more general derived schemes by gluing) in this fashion. More generally, Theorem 1.7 explains the situations when we can expect more general affines. We intend to return to the study of equivariant derived schemes in a subsequent paper.

As a concrete example of this circle of ideas, let $\mathcal{X} \to \mathcal{Y}$ be a Galois cover of stacks with Galois group G, and let $\mathcal{Y} \to \mathcal{M}_{\mathrm{Ell}}$ be an étale map to the moduli stack

of elliptic curves. We can evaluate the Goerss–Hopkins–Miller sheaf of topological modular forms $\mathcal{O}^{\mathrm{top}}$ on these étale maps, producing commutative ring spectra and maps

$$\mathrm{TMF}(\mathcal{Y}) \to \mathrm{TMF}(\mathcal{X}).$$

The G-action on $\mathcal{X}$ gives a G-action on $\mathrm{TMF}(\mathcal{X})$, and we can then view this as a genuine commutative equivariant ring spectrum by pushing forward to a complete universe (see [Hill and Meier 2017] for a related discussion). We would like to be able to understand the category of equivariant $\mathrm{TMF}(\mathcal{X})$-modules in algebro-geometric terms. The machinery presented in this paper is an essential tool in this endeavor, making it possible to make sense of sheaves of modules on the Zariski site.

2. Review of $\mathcal{N}_\infty$ operads

In this section, we review the framework for describing equivariant commutative ring spectra that we will work with in the paper. We refer the reader to [Blumberg and Hill 2015] for a more detailed discussion.

Let G be a finite group and let $G\mathcal{S}$ denote the category of orthogonal G-spectra structured by a complete universe and with morphisms all (not necessarily equivariant) maps. We will tacitly suppress notation for the "additive" universe implicit in the definition of $G\mathcal{S}$, as we are focused on multiplicative phenomena. Recall that the category $G\mathcal{S}$ is a complete and cocomplete closed symmetric monoidal category under the smash product $\wedge$ with unit the equivariant sphere spectrum $\mathbb{S}_G$. We will write $F(-, -)$ for the internal mapping G-spectrum in $G\mathcal{S}$. The category $G\mathcal{S}$ is enriched over based G-spaces and has tensors and cotensors; for X an object of $G\mathcal{S}$, the tensor with a based G-space A is given by the smash product $A \wedge X$ and the cotensor by the function spectrum $F(A, X)$.

The enrichment of $G\mathcal{S}$ means that we can regard operads in G-spaces as acting on objects of $G\mathcal{S}$ via the addition of a G-fixed disjoint basepoint and the tensor. Given a G-operad in spaces, recall the following definition from [Blumberg and Hill 2015, Definition 3.7].

Definition 2.1. An $\mathcal{N}_\infty$ operad is a G-operad $\mathcal{O}$ such that:

(1) The space $\mathcal{O}_0$ is G-contractible.

(2) The action of Σ_n on $\mathcal{O}_n$ is free.

(3) $\mathcal{O}_n$ is a universal space for a family $\mathcal{F}_n(\mathcal{O})$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times \{1\}$.

For any $\mathcal{N}_\infty$ operad $\mathcal{O}$, there is an associated category $\mathcal{O}\text{-Alg}$ of $\mathcal{O}$-algebras in $G\mathcal{S}$. We will be particularly interested in the algebras associated to the G-linear

isometries operads. Fix a possibly incomplete universe U of finite-dimensional G-representations; we adopt the standard convention that U contains a trivial representation and each of its finite-dimensional subrepresentations infinitely often. We do not assume any relationship between U and the "additive" universe that arises in the definition of $G\mathcal{S}$.

Definition 2.2. The G-linear isometries operad $\mathscr{L}_U$ has n-th space

$$\mathcal{L}_U(n) = \mathscr{L}(U^n, U),$$

the G-space of nonequivariant linear isometries $U^n \to U$ equipped with the conjugation action. The distinguished element $1 \in \mathcal{L}_U(1)$ is the identity map and the operad structure maps are induced by composition and direct sum.

Recall from [Blumberg and Hill 2015, Theorem 4.24] that the G-linear isometries operads do not always describe all of the possible $\mathcal{N}_\infty$ operads. Nonetheless, they do capture many examples of interest, in particular including the trivial and complete multiplicative universes.

One of the major themes of our previous study of $\mathcal{N}_\infty$ operads was that the essential structure encoded by an operad $\mathcal{O}$ is the collection of *admissible sets*. We now review the relevant definitions from [Blumberg and Hill 2015, §3].

Definition 2.3. A *symmetric monoidal coefficient system* is a contravariant functor $\underline{\mathcal{C}}$ from the orbit category of G to the category of symmetric monoidal categories and strong symmetric monoidal functors. The *value at H* of a symmetric monoidal coefficient system $\underline{\mathcal{C}}$ is $\underline{\mathcal{C}}(G/H)$, and will often be denoted $\underline{\mathcal{C}}(H)$.

The most important example of a symmetric monoidal coefficient system for us is the coefficient system of finite G-sets.

Definition 2.4. Let $\underline{\mathrm{Set}}$ be the symmetric monoidal coefficient system of finite sets. The value at H is Set^H, the category of finite H-sets and H-maps. The symmetric monoidal operation is disjoint union.

We will associate to every $\mathcal{N}_\infty$ operad a subcoefficient system of $\underline{\mathrm{Set}}$. The operadic structure gives rise to additional structure on the coefficient system.

Definition 2.5. We say that a full subsymmetric monoidal coefficient system $\mathcal{F}$ of $\underline{\mathrm{Set}}$ is *closed under self-induction* if whenever $H/K \in \mathcal{F}(H)$ and $T \in \mathcal{F}(K)$, $H \times_K T \in \mathcal{F}(H)$.

Definition 2.6. Let $\mathcal{C} \subset \mathcal{D}$ be a full subcategory. We say that $\mathcal{C}$ is a *truncation subcategory* of $\mathcal{D}$ if whenever $X \to Y$ is monic in $\mathcal{D}$ and Y is in $\mathcal{C}$, then X is also in $\mathcal{C}$. A truncation subcoefficient system of a symmetric monoidal coefficient system $\underline{\mathcal{D}}$ is a subcoefficient system that is levelwise a truncation subcategory.

In particular, for finite G-sets, truncation subcategories are subcategories that are closed under passage to subobjects and which are closed under isomorphism.

Definition 2.7. An *indexing system* is a truncation subsymmetric monoidal coefficient system $\underline{\mathcal{F}}$ of $\underline{\text{Set}}$ that contains all trivial sets and is closed under self induction and Cartesian product.

One of the main structural theorems about $\mathcal{N}_\infty$ operads [Blumberg and Hill 2015, Theorem 4.17] is that an $\mathcal{N}_\infty$ operad $\mathcal{O}$ determines an indexing system of admissible sets. This connection arises from the standard observation that subgroups Γ of $G \times \Sigma_n$ such that $\Gamma \cap (\{1\} \times \Sigma_n) = \{1\}$ arise as the graphs of homomorphisms $H \to \Sigma_n$, for some $H \subseteq G$.

3. Point-set categories of modules over an $\mathcal{N}_\infty$ algebra

In this section, we describe an approach to constructing categories of modules over $\mathcal{N}_\infty$ algebras that proceeds via a rigidification argument. Of course, for any given $\mathcal{N}_\infty$ operad $\mathcal{O}$ and an $\mathcal{O}$-algebra R in orthogonal G-spectra, we can construct a category of operadic modules over R. However, experience in the nonequivariant case teaches us that for practical work it is extremely convenient to have rigid models of such categories that are equipped with a symmetric monoidal smash product.

Specifically, for each linear isometries operad $\mathcal{O} = \mathcal{L}(U)$, we will construct a symmetric monoidal structure on a category Quillen equivalent to orthogonal G-spectra such that monoids and commutative monoids correspond to $\mathcal{O}$-algebras. We describe how to produce such a structure by adapting the techniques pioneered in the development of the EKMM category of S-modules. See also [Blumberg 2006; Blumberg et al. 2010; Kříž and May 1995; Spitzweck 2001] for other categories in which this kind of approach has been developed. We can then define modules over an $\mathcal{O}$-algebra R in the evident fashion.

We are not able to rigidify algebras and modules over $\mathcal{N}_\infty$ operads which are not equivalent to equivariant linear isometries operads. Although in these cases we can give a homotopical construction of the tensor product of operadic $\mathcal{O}$-modules in terms of the bar construction, we do not have good point-set control.

3.1. *The point-set theory of $\mathbb{L}_U$-algebras in orthogonal spectra.* We begin by discussing the point-set details of the category of algebras for a monad obtained from the first part of the equivariant linear isometries operad. Recall that $G\mathcal{S}$ denotes the category of orthogonal G-spectra structured by a complete G-universe. Since the universe implicit in the definition of $G\mathcal{S}$ does not play an essential role in what follows (once the weak equivalences are fixed), we continue to suppress this choice from the notation.

Remark 3.1. It is possible to carry out the work of this paper in the context of an incomplete additive universe on GS; the simplest case arises when the additive and multiplicative universes are the same. We leave this elaboration (and its attendant complications) to the interested reader.

Fix a (possibly incomplete) universe U. Let

$$\mathcal{L}_U(1) = \mathcal{L}(U, U)$$

denote the G-space of linear isometries $U \to U$; i.e., the space of nonequivariant linear isometries $U \to U$ equipped with the conjugation action. More generally, we write $\mathcal{L}_U(n)$ to denote $\mathcal{L}(U^n, U)$, the n-th space of the equivariant linear isometries operad.

Since GS is tensored over based G-spaces, the formula

$$X \mapsto \mathcal{L}_U(1)_+ \wedge X$$

specifies a monad $\mathbb{L}_U : GS \to GS$. The monadic structure maps are induced by the identity element $\mathrm{id}_U \in \mathcal{L}_U(1)$ and the composition

$$\mathcal{L}(U, U) \times \mathcal{L}(U, U) \to \mathcal{L}(U, U).$$

Definition 3.2. Let $GS[\mathbb{L}_U]$ denote the category of $\mathbb{L}_U$-algebras in GS.

Since the monad $\mathbb{L}_U$ has a right adjoint $F(\mathcal{L}_U(1)_+, -)$, the observation of [Elmendorf et al. 1997, I.4.3] implies that this right adjoint determines a comonad $\mathbb{L}_U^\sharp$ such that the category of coalgebras over $\mathbb{L}_U^\sharp$ is equivalent to $GS[\mathbb{L}_U]$. As a consequence we conclude the following result about the existence of limits and colimits.

Lemma 3.3. *The category $GS[\mathbb{L}_U]$ is complete and cocomplete, with limits and colimits created in GS. Similarly, $GS[\mathbb{L}_U]$ has tensors and cotensors with based G-spaces; the indexed colimits and limits are created in GS.*

The category $GS[\mathbb{L}_U]$ is equipped with mapping G-spectra $F_{GS[\mathbb{L}_U]}(-, -)$ defined by the equalizer

$$F(X, Y) \rightrightarrows F(\mathbb{L}_U X, Y),$$

where the maps are induced by the action $\mathbb{L}_U X \to X$ and the adjoint of the composite

$$(\mathbb{L}_U X) \wedge F(X, Y) \cong \mathbb{L}_U(X \wedge F(X, Y)) \to \mathbb{L}_U Y \to Y.$$

Next, we note that any orthogonal G-spectrum can be given a trivial $GS[\mathbb{L}_U]$ structure. Specifically, in addition to the free $\mathbb{L}_U$-algebra functor

$$\mathcal{L}_U(1)_+ \wedge (-) : GS \to GS[\mathbb{L}_U],$$

there is another functor $p^* : G\mathcal{S} \to G\mathcal{S}[\mathbb{L}_U]$ determined by the unique projection map $p : \mathcal{L}_U(1) \to *$; i.e., we can equip any orthogonal G-spectrum X with the trivial structure map

$$\mathcal{L}_U(1)_+ \wedge X \to (*)_+ \wedge X \cong X.$$

We will be most interested in the sphere spectrum $\mathbb{S}_G$ regarded as an $\mathbb{L}_U$-algebra in this fashion. The pullback functor is the right adjoint of a functor $Q : G\mathcal{S}[\mathbb{L}_U] \to G\mathcal{S}$ specified by the formula $QX = \mathbb{S}_G \wedge_{\Sigma^\infty_+ \mathcal{L}_U(1)} X$.

We now define a closed weak symmetric monoidal structure on $G\mathcal{S}[\mathbb{L}_U]$ with unit $\mathbb{S}_G$. (Recall that a weak symmetric monoidal category has a product and a unit satisfying all of the axioms of a symmetric monoidal category except that the unit map is not required to be an isomorphism [Elmendorf et al. 1997, II.7.1].)

Definition 3.4. Let X, Y be objects of $G\mathcal{S}[\mathbb{L}_U]$. We define the smash product $\wedge_U$ to be the coequalizer of the diagram

$$\left(\mathcal{L}_U(2) \times \mathcal{L}_U(1) \times \mathcal{L}_U(1)\right)_+ \wedge (X \wedge Y) \rightrightarrows \mathcal{L}_U(2)_+ \wedge (X \wedge Y) \to X \wedge_U Y,$$

where the maps are specified by the actions of $\mathcal{L}_U(1)_+$ on X and Y and the right action of $\mathcal{L}_U(1) \times \mathcal{L}_U(1)$ on $\mathcal{L}_U(2)$ via block sum and precomposition.

We will sometimes write this coequalizer using the notation

$$\mathcal{L}_U(2) \times_{\mathcal{L}_U(1) \times \mathcal{L}_U(1)} (X \wedge Y).$$

Here the left action of $\mathcal{L}_U(1)$ on $\mathcal{L}_U(2)$ induces a left action of $\mathcal{L}_U(1)$ on $X \wedge_U Y$ which endows it with the structure of an $\mathbb{L}_U$ algebra. As an example, when $X = \mathbb{L}_U A$ and $Y = \mathbb{L}_U B$ are free $\mathbb{L}_U$-algebras,

$$X \wedge_U Y \cong \mathcal{L}_U(2)_+ \wedge (A \wedge B). \tag{3.5}$$

Analogously, we have an internal function object in $G\mathcal{S}[\mathbb{L}_U]$ that satisfies the usual adjunction.

Definition 3.6. Let X, Y be objects of $G\mathcal{S}[\mathbb{L}_U]$. We define the mapping $\mathbb{L}_U$-spectrum $F_{\mathcal{L}_U}(X, Y)$ to be the equalizer of the diagram

$$F_{G\mathcal{S}[\mathbb{L}_U]}(\mathcal{L}_U(2)_+ \wedge X, Y) \rightrightarrows F_{G\mathcal{S}[\mathbb{L}_U]}\left(\left(\mathcal{L}_U(2) \times \mathcal{L}_U(1) \times \mathcal{L}_U(1)\right)_+ \wedge X, Y\right),$$

where the maps are induced by the action of $\mathcal{L}_U(1) \times \mathcal{L}_U(1)$ on $\mathcal{L}_U(2)$ by block sum and via the adjunction homeomorphism

$$F_{G\mathcal{S}[\mathbb{L}_U]}\left(\left(\mathcal{L}_U(2) \times \mathcal{L}_U(1) \times \mathcal{L}_U(1)\right)_+ \wedge X, Y\right)$$
$$\cong F_{G\mathcal{S}[\mathbb{L}_U]}\left((\mathcal{L}_U(2) \times \mathcal{L}_U(1))_+ \wedge X, F_{G\mathcal{S}[\mathbb{L}_U]}(\mathcal{L}_U(1)_+ \wedge \mathbb{S}_G, Y)\right)$$

along with the action $\mathscr{L}_U(1)_+ \wedge X \to X$ as well as the coaction

$$Y \to F_{G\mathcal{S}[\mathbb{L}_U]}(\mathscr{L}_U(1)_+ \wedge \mathbb{S}_G, Y).$$

In what follows, we will repeatedly make use of the fact that for any $n > 0$ and admissible set T, we can choose a G-equivariant homeomorphism $\mathbb{R}\{T\} \otimes U \to U$ (see Lemma A.1 in Appendix A). We now establish the basic properties of $\wedge_U$.

Theorem 3.7. *Let X, Y, and Z be objects of $G\mathcal{S}[\mathscr{L}_U]$. There is a natural commutativity homeomorphism*

$$\tau : X \wedge_U Y \to Y \wedge_U X$$

and a natural associativity homeomorphism

$$(X \wedge_U Y) \wedge_U Z \cong X \wedge_U (Y \wedge_U Z).$$

More generally, there is a canonical natural homeomorphism

$$X_1 \wedge_U \ldots \wedge_U X_k \cong \mathscr{L}_U(k) \times_{\underbrace{(\mathscr{L}_U(1) \times \cdots \times \mathscr{L}_U(1))}_{k}} (X_1 \wedge \cdots \wedge X_k),$$

where the left-hand side is associated in any order and the right-hand side denotes the evident coequalizer generalizing the definition of $\wedge_U$.

Proof. Commutativity is essentially immediate (see [Elmendorf et al. 1997, I.5.2]) and associativity is a consequence of the equivariant analogue of [Elmendorf et al. 1997, I.5.4], that is, the homeomorphism

$$\mathscr{L}_U(i + j) \cong \mathscr{L}_U(2) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} \mathscr{L}_U(i) \times \mathscr{L}_U(j),$$

which we prove as Lemma A.4. Associativity and the last formula now follow from the arguments for [Elmendorf et al. 1997, I.5.6]. Specifically, we have natural homeomorphisms $\mathscr{L}_U(1) \times_{\mathscr{L}_U(1)} X \cong X$ for all X in $G\mathcal{S}[\mathbb{L}_U]$, and therefore there are natural homeomorphisms

$X \wedge_U Y \wedge_U Z$
$\cong \mathscr{L}_U(2) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} \big(\mathscr{L}_U(2) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} (X \wedge Y) \big) \wedge (\mathscr{L}_U(1) \times_{\mathscr{L}_U(1)} Z)$
$\cong \big(\mathscr{L}_U(2) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} \mathscr{L}_U(2) \times \mathscr{L}_U(1) \big) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1) \times \mathscr{L}_U(1)} (X \wedge Y \wedge Z)$
$\cong \mathscr{L}_U(3) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1) \times \mathscr{L}_U(1)} (X \wedge Y \wedge Z).$ □

Next, we construct the unit map, which is a consequence of the equivariant analogue of a basic point-set property of spaces of linear isometries; see Lemma A.2.

Corollary 3.8. *There is a natural homeomorphism of $\mathbb{L}_U$-spectra*

$$\lambda : \mathbb{S}_G \wedge_U \mathbb{S}_G \cong \mathbb{S}_G$$

such that $\lambda\tau = \lambda$.

The argument for [Elmendorf et al. 1997, I.8.3] now generalizes without change to the equivariant setting:

Theorem 3.9. *Let X be an object of $G\mathcal{S}[\mathbb{L}_U]$. Then there exists a natural map*

$$\psi : \mathbb{S}_G \wedge_U X \to X$$

which is compatible with the commutativity and associativity homeomorphisms.

Combining Theorems 3.7 and 3.9, we have proved the following result.

Theorem 3.10. *The category $G\mathcal{S}[\mathbb{L}_U]$ is a closed weak symmetric monoidal category with product $\wedge_U$, unit $\mathbb{S}_G$, and function object $F_{\mathcal{L}_U}(-,-)$.*

Just as in the setting of spaces [Blumberg 2006; Blumberg et al. 2010] and spectra [Elmendorf et al. 1997], we can actually work with the closed symmetric monoidal category obtained by restricting to the unital objects.

Definition 3.11. Let $G\mathcal{S}_U$ denote the full subcategory of $G\mathcal{S}[\mathbb{L}_U]$ consisting of those objects for which $\psi : \mathbb{S}_G \wedge_U X \to X$ is a homeomorphism. For X, Y objects in $G\mathcal{S}_U$, let $F_U(X,Y)$ denote $\mathbb{S}_G \wedge_U F_{\mathcal{L}_U}(X,Y)$ and abusively denote by $X \wedge_U Y$ the coequalizer regarded as an object of $G\mathcal{S}_U$.

Corollary 3.8 implies that there is a functor $\mathbb{S}_G \wedge_U (-): G\mathcal{S}[\mathbb{L}_U] \to G\mathcal{S}_U$ which is the left adjoint to the functor $F_{\mathcal{L}_U}(\mathbb{S}_G, -)$ and the right adjoint to the forgetful functor. As a consequence, we can deduce the following result.

Proposition 3.12. *The category $G\mathcal{S}_U$ is complete and cocomplete. Colimits are created in $G\mathcal{S}[\mathbb{L}_U]$ (and hence in $G\mathcal{S}$). Limits are formed by applying $\mathbb{S}_G \wedge_U (-)$ to the limit in $G\mathcal{S}[\mathbb{L}_U]$. Similarly, $G\mathcal{S}_U$ has tensors and cotensors with based G-spaces. Tensors are created in $G\mathcal{S}[\mathbb{L}_U]$ (and hence in $G\mathcal{S}$). Cotensors are formed by applying $\mathbb{S}_G \wedge_U (-)$ to the cotensor in $G\mathcal{S}[\mathbb{L}_U]$.*

It is now straightforward to conclude the following result.

Theorem 3.13. *The category $G\mathcal{S}_U$ is a closed symmetric monoidal category with unit $\mathbb{S}_G$, product $\wedge_U$, and function object $F_U(-,-)$.*

3.2. *Point-set multiplicative change of universe functors.* A counterintuitive but useful fact about the category of orthogonal G-spectra is that the point-set change of universe functors are always symmetric monoidal equivalences of categories, although they are not always Quillen equivalences. In particular, for any universe U, there is an equivalence of categories between G-objects in the category of (nonequivariant) orthogonal spectra and orthogonal G-spectra on U. In this section, we explain the corresponding result in the context of *multiplicative* change of universe functors for the categories $G\mathcal{S}[\mathbb{L}_U]$ as U varies; the underlying (complete) additive universe that structures $G\mathcal{S}$ remains constant.

Let U and U' be G-universes, and denote by $\mathscr{L}(U, U')$ the G-space of nonequivariant linear isometries $U \to U'$, where G acts by conjugation. When $U = U'$, note that $\mathscr{L}(U, U) = \mathscr{L}_U(1)$.

Definition 3.14. Let U and U' be G-universes. We define the functor

$$\mathscr{L}\mathscr{I}_U^{U'} : GS[\mathbb{L}_U] \to GS[\mathbb{L}_{U'}]$$

by setting $\mathscr{L}\mathscr{I}_U^{U'} X$ to be the coequalizer of the diagram

$$\mathscr{L}(U, U')_+ \wedge \mathscr{L}(U, U)_+ \wedge X \rightrightarrows \mathscr{L}(U, U')_+ \wedge X,$$

where the maps are determined by the action of $\mathscr{L}_U(1)$ on X and the composition $\mathscr{L}(U, U') \times \mathscr{L}(U, U) \to \mathscr{L}(U, U')$. The action of $\mathscr{L}_{U'}(1)$ on $\mathscr{I}_U^{U'} X$ is also induced by the composition map $\mathscr{L}(U', U') \times \mathscr{L}(U, U') \to \mathscr{L}(U, U')$.

As explained in [Elmendorf and May 1997, Corollaries 1.3, 1.4], we have the following point-set result about the behavior of these functors. We include the proof here in order to make this paper more self-contained.

Theorem 3.15. *Let U and U' be G-universes. The functors $\mathscr{L}\mathscr{I}_U^{U'}$ and $\mathscr{L}\mathscr{I}_{U'}^U$ are inverse equivalences of categories between $GS[\mathbb{L}_U]$ and $GS[\mathbb{L}_{U'}]$. Both functors are strong symmetric monoidal. As a consequence, the change of universe functors descend to the categories GS_U and $GS_{U'}$.*

Proof. This result follows from the identification of the coequalizer

$$\mathscr{L}(U', U'') \times \mathscr{L}(U', U') \times \mathscr{L}(U, U') \rightrightarrows \mathscr{L}(U', U'') \times \mathscr{L}(U, U')$$

as $\mathscr{L}(U, U'')$, for any universes U, U', and U'' [Elmendorf and May 1997, Lemma 2.2] and where the maps are all induced by the composition γ. Since coequalizers in G-spaces are computed using the forgetful functor to spaces, it suffices to show that this is a coequalizer diagram of nonequivariant spaces. But in this setting, the diagram is a split coequalizer. The splitting is constructed as follows. Choose an isomorphism $s : U \to U'$ and define

$$h : \mathscr{L}(U, U'') \to \mathscr{L}(U', U'') \times \mathscr{L}(U, U')$$

and

$$k : \mathscr{L}(U', U'') \times \mathscr{L}(U, U') \to \mathscr{L}(U', U'') \times \mathscr{L}(U', U') \times \mathscr{L}(U, U')$$

via the formulas $h(f) = (f \circ s^{-1}, s)$ and $k(g', g) = (g', g \circ s^{-1}, s)$. Then $\gamma \circ h = \mathrm{id}$, $(\mathrm{id} \times \gamma) \circ k = \mathrm{id}$, and $(\gamma \times \mathrm{id}) \circ k = h \circ \gamma$. □

In particular, we have the following surprising corollary.

Corollary 3.16. *Let U be any G-universe. The categories $GS[\mathbb{L}_U]$ and GS_U are equivalent to the categories $GS[\mathbb{R}^\infty]$ and $GS_{\mathbb{R}^\infty}$ respectively.*

In our work in this paper, we will make critical use of this equivalence to establish some point-set properties of our categories $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$, notably about the multiplicative norm and the fixed-point functors. In fact, as pointed out by an anonymous referee, we could simplify some of the work of the previous section by using the fact that $G\mathcal{S}_{\mathbb{R}^\infty}$ can be described as a diagram category; the construction of the smash product, colimits, and limits is then immediate from general results about diagram categories and Corollary 3.16.

Remark 3.17. The additive version of this phenomenon was originally discovered in the context of equivariant Γ-spaces by Shimakawa [1991] and was proved for orthogonal G-spectra in [Mandell and May 2002, §V.1]. In the multiplicative setting, the use of these formulas to simplify the point-set theory for the equivariant stable category is sketched in [May 1996, §XXIII.4], in the context of the equivariant version of EKMM spectra [Elmendorf et al. 1997]; this exposition followed [Elmendorf and May 1997].

Although these facts were known to experts for a long time, the observation has become prominent after its use in the definition of the Hill–Hopkins–Ravenel multiplicative norm [Hill et al. 2016]; it is vastly simpler to define the norm directly on G-objects and use the universe only to study the homotopy theory. Another important recent application of these ideas comes from global equivariant homotopy theory; this technique is essentially required to make the point-set approach to global equivariant homotopy theory tractable [Schwede 2018]. Likely motivated by this fact, Schwede [2019] has advocated for developing the foundations of equivariant stable homotopy theory from this perspective (although on the other hand see [Mandell and May 2002, Remark V.1.9] for a contrary view).

Nonetheless, we believe that despite Corollary 3.16, it is conceptually clarifying in our work to keep track of the multiplicative universe at the point-set level. The issue is simply that we have two universes in play, the universe structuring the additive theory and the universe structuring the multiplicative theory. We believe that the approach outlined in [May 1996, §XXIII.4] works best when there is only a single universe, i.e., when the additive and multiplicative universe coincide. Moreover, when doing homotopical work, there is of course no way to avoid incorporating the universe explicitly when writing down formulas for fibrant replacement and (right) derived functors.

3.3. *Rings and modules in $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$.* We now turn to the characterization of multiplicative objects in $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$. The key observation about $\wedge_U$ is that (in direct analogy with the nonequivariant case), monoids for $\wedge_U$ are algebras over the non-Σ linear isometries operad $\mathscr{L}_U$ and commutative monoids for $\wedge_U$ are algebras over the linear isometries operad $\mathscr{L}_U$. More precisely, let $\mathbb{T}$ and $\mathbb{P}$ denote the monads structuring associative and commutative monoid objects

in $G\mathcal{S}[\mathbb{L}_U]$ respectively. Concretely, for X an object of $G\mathcal{S}[\mathbb{L}_U]$,

$$\mathbb{T}X = \bigvee_{k\geq 0} \underbrace{X \wedge_U \cdots \wedge_U X}_{k} \quad \text{and} \quad \mathbb{P}X = \bigvee_{k\geq 0} \big(\underbrace{X \wedge_U \cdots \wedge_U X}_{k}\big)/\Sigma^k,$$

where X^0 is defined to be $\mathbb{S}_G$.

Monadic algebras over the analogous monads $\mathbb{T}$ and $\mathbb{P}$ in $G\mathcal{S}_U$ are simply algebras in $G\mathcal{S}[\mathbb{L}_U]$ that are unital; this is clear for $\mathbb{T}$, and follows for $\mathbb{P}$ from the fact that colimits in $G\mathcal{S}_U$ are created in $G\mathcal{S}[\mathbb{L}_U]$. Moreover, there are functors

$$\mathbb{S}_G \wedge_U (-) \colon (G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}] \to G\mathcal{S}_U[\mathbb{T}] \quad \text{and} \quad \mathbb{S}_G \wedge_U (-) \colon (G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}] \to G\mathcal{S}_U[\mathbb{P}].$$

The next result connects the categories $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}]$ and $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}]$ of monadic algebras to categories of operadic $\mathcal{N}_\infty$ algebras [Blumberg and Hill 2015].

Theorem 3.18. *The category $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}]$ is isomorphic to the category of non-Σ $\mathscr{L}_U$-algebras in $G\mathcal{S}$. The category $(G\mathcal{S}[\mathbb{L}_U][\mathbb{P}])$ is isomorphic to the category of $\mathscr{L}_U$-algebras in $G\mathcal{S}$.*

Proof. The argument is the same as the proof of [Elmendorf et al. 1997, II.4.6], using the homeomorphism of Equation (3.5) levelwise. □

In light of the previous theorem, we will refer to monoids and commutative monoids in $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}$ as associative and commutative $\mathcal{N}_\infty$ ring orthogonal G-spectra, respectively.

Next, the arguments of [Elmendorf et al. 1997, II.7] extend to prove the following:

Theorem 3.19. *The categories $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}]$, $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}]$, $G\mathcal{S}_U[\mathbb{T}]$, and $G\mathcal{S}_U[\mathbb{P}]$ are complete and cocomplete, with limits created in $G\mathcal{S}$. The categories $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}]$ and $G\mathcal{S}_U[\mathbb{T}]$ are tensored and cotensored over based G-spaces, with cotensors created in $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$ respectively. The categories $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}]$ and $G\mathcal{S}_U[\mathbb{P}]$ are tensored and cotensored over unbased G-spaces, with cotensors created in $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$ respectively (regarding these categories as cotensored over unbased spaces via the functor that adjoins a disjoint G-fixed basepoint).*

As an aside, we note the following standard observation, which follows as usual simply by checking the universal property.

Lemma 3.20. *The symmetric monoidal product $\wedge_U$ is the coproduct on $G\mathcal{S}_U[\mathbb{P}]$.*

Finally, for any monoid or commutative monoid R, there are associated categories of (left) R-modules in $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$. Since the theory is cleanest in the case of $G\mathcal{S}_U$, we focus on the unital setting in the following discussion. The multiplication and unit maps for R give the functor $R \wedge_U (-)$ the structure of a monad on $G\mathcal{S}_U$.

Definition 3.21. Let R be an object in $G\mathcal{S}_U[\mathbb{T}]$ or $G\mathcal{S}_U[\mathbb{P}]$. The category $\mathcal{M}_{R,U}$ of R-modules in $G\mathcal{S}_U[\mathbb{P}]$ is the category of algebras for the monad $R \wedge_U (-)$ in $G\mathcal{S}_U$.

Such categories of R-modules are complete and cocomplete, with limits and colimits created in $G\mathcal{S}_U$. When R is commutative, the category of R-modules is closed symmetric monoidal with unit R and product $X \wedge_{R,U} Y$ defined as the coequalizer of the diagram

$$X \wedge_U R \wedge_U Y \rightrightarrows X \wedge_U Y,$$

where the maps are induced by the right action of R on X via the symmetry homeomorphism and the left action of R on Y. The function object is defined as the equalizer of the diagram

$$F_U(X, Y) \rightrightarrows F_U(R \wedge_U X, Y),$$

where the maps are induced by the action of R on X and the adjoint of the composite

$$R \wedge_U X \wedge_U F_U(X, Y) \to R \wedge_U Y \to Y.$$

There are also the evident categories of R-algebras and commutative R-algebras.

Definition 3.22. Let R be an object in $G\mathcal{S}_U[\mathbb{P}]$. Abusively denote by $\mathbb{T}$ and $\mathbb{P}$ the monads in $\mathcal{M}_{R,U}$ that structure monoids and commutative monoids. We refer to the categories $\mathcal{M}_{R,U}[\mathbb{T}]$ and $\mathcal{M}_{R,U}[\mathbb{P}]$ as the categories of R-algebras and commutative R-algebras respectively.

3.4. *Change of group and fixed-point functors.* In this section, we study change-of-group and fixed-point functors in the context of the categories $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$. If we are content to ignore the monoidal structure, the point-set theory of the change of group and fixed-point functors is the same as for $G\mathcal{S}$. The interaction of these functors with the action of $\mathcal{L}_U(1)$ is more subtle. Our discussion relies on observations from [Mandell and May 2002, §VI.1].

Let $\iota_H : H \to G$ be the inclusion of a subgroup. Denote by WH the quotient NH/H, where NH is the normalizer of H in G. For X an object of $G\mathcal{S}$, there is a homeomorphism

$$\iota_H^* \mathbb{L}_U X \cong \mathbb{L}_{(\iota_H^* U)}(\iota_H^* X).$$

This homeomorphism is easily seen to be compatible with the monad structure, and so we obtain a functor

$$\iota_H^* : G\mathcal{S}[\mathbb{L}_U] \to H\mathcal{S}[\mathbb{L}_{(\iota_H^* U)}],$$

where the additive universe on $H\mathcal{S}$ here is ι^* applied to the complete universe structuring $G\mathcal{S}$. Analogously, for Y an object of $H\mathcal{S}$, we have a homeomorphism

$$G_+ \wedge_H \mathbb{L}_{(\iota_H^* U)} Y \cong \mathbb{L}_U(G_+ \wedge_H Y)$$

that is compatible with the monad structure, producing a functor

$$G_+ \wedge_H (-) : H\mathcal{S}[\mathbb{L}_{(\iota_H^* U)}] \to G\mathcal{S}[\mathbb{L}_U]$$

that is the left adjoint to ι_H^*. Finally, there is also a homeomorphism

$$F_H(G, \mathbb{L}_{(\iota_H^* U)}^\sharp Y) \cong \mathbb{L}^\sharp F_H(G, Y)$$

(here recall that the comonad $\mathbb{L}^\sharp$ is described just prior to the proof of Lemma 3.3) that is compatible with the comonad structure and thus produces the right adjoint

$$F_H(G, -) : H\mathcal{S}[\mathbb{L}_{(\iota_H^* U)}] \to G\mathcal{S}[\mathbb{L}_U]$$

to ι_H^*.

Furthermore, all of these functors are compatible with the functors creating the unital objects, and so descend to functors $\iota_H^* : G\mathcal{S}_U \to H\mathcal{S}_{(\iota_H^* U)}$ and the attendant left and right adjoints.

Finally, it is evident that ι_H^* is symmetric monoidal and so it restricts to categories of monoids and commutative monoids.

Proposition 3.23. *Let H be a subgroup of G. Then there are forgetful functors*

$$\iota_H^* : (G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}] \to (H\mathcal{S}[\mathbb{L}_{\iota_H^* U}])[\mathbb{T}] \quad and \quad \iota_H^* : G\mathcal{S}_U[\mathbb{T}] \to G\mathcal{S}_{\iota_H^* U}[\mathbb{T}]$$

and

$$\iota_H^* : (G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}] \to (H\mathcal{S}[\mathbb{L}_{\iota_H^* U}])[\mathbb{P}] \quad and \quad \iota_H^* : G\mathcal{S}_U[\mathbb{P}] \to G\mathcal{S}_{\iota_H^* U}[\mathbb{P}].$$

Next, we turn to the question of the categorical fixed points. Our definition is built from the categorical fixed point functor $(-)^H$ on $G\mathcal{S}$ [Mandell and May 2002, Definition V.3.9].

Theorem 3.24. *Let H be a subgroup of G. Then the categorical H-fixed point functor on $G\mathcal{S}$ induces a lax monoidal categorical H-fixed point functor*

$$(-)^H : G\mathcal{S}[\mathbb{L}_U] \to WH\mathcal{S}[\mathbb{L}_{U^H}]$$

specified (in mild abuse of notation) by the formula

$$X^H = (\mathscr{L}\mathscr{I}_U^{U^H} X)^H,$$

where the $(-)^H$ on the right-hand side denotes the categorical fixed points in $G\mathcal{S}$. The fixed point functor has an op-lax symmetric monoidal left adjoint

$$\epsilon_H^* : WH\mathcal{S}[\mathbb{L}_{U^H}] \to G\mathcal{S}[\mathbb{L}_U],$$

which assigns to a WH-spectrum X the G-spectrum obtained by pulling back along the quotient $NH \to WH$, changing (additive) universe, and inducing up to G, and changing multiplicative universe. When H is normal, the left adjoint is strong symmetric monoidal.

Proof. Since

$$(\mathscr{L}(U^H, U^H)_+ \wedge X)^H \cong \mathscr{L}(U^H, U^H)_+ \wedge X^H,$$

the categorical H-fixed point functor restricts to a functor

$$G\mathcal{S}[\mathbb{L}_{U^H}] \to WH\mathcal{S}[\mathbb{L}_{U^H}].$$

Analogously,

$$\epsilon_H^*(\mathscr{L}(U, U)_+ \wedge Y) \cong \mathscr{L}(U, U)_+ \wedge \epsilon_H^* Y,$$

which implies that ϵ_H^* restricts to a functor from $WH\mathcal{S}[\mathbb{L}_{U^H}]$ to $G\mathcal{S}[\mathbb{L}_U]$.

Next, we consider the interaction of $(-)^H$ with the monoidal structure. Since the action of H on $\mathscr{L}(U^H, U^H)$ is trivial and $(-)^H$ is lax monoidal on $G\mathcal{S}$ [Mandell and May 2002, Proposition V.3.8], for X and Y in $G\mathcal{S}[\mathbb{L}_U]$ and $H \subseteq G$ we have a natural map

$$(\mathscr{L}_{U^H}(2) \times_{\mathscr{L}_{U^H}(1) \times \mathscr{L}_{U^H}(1)})_+ \wedge X^H \wedge Y^H \to (\mathscr{L}_{U^H}(2) \times_{\mathscr{L}_{U^H}(1) \times \mathscr{L}_{U^H}(1)})_+ \wedge (X \wedge Y)^H$$

which lands in the fixed-points

$$\left((\mathscr{L}_{U^H}(2) \times_{\mathscr{L}_{U^H}(1) \times \mathscr{L}_{U^H}(1)})_+ \wedge (X \wedge Y)\right)^H,$$

and so we deduce that $(-)^H$ is a lax symmetric monoidal functor

$$G\mathcal{S}[\mathbb{L}_U] \to WH\mathcal{S}[\mathbb{L}_{U^H}].$$

Finally, when H is normal, the left adjoint is strong symmetric monoidal since the pullback and additive change of universe are. $\qquad\square$

The situation for the geometric fixed point functor is analogous; again, we construct Φ^H on $G\mathcal{S}[\mathbb{L}_U]$ by considering the composite $\Phi^H(\mathscr{L}\mathscr{I}_U^{U^H} X)$.

Theorem 3.25. *Let H be a subgroup of G. Then there is a lax symmetric monoidal geometric H-fixed point functor*

$$\Phi^H : G\mathcal{S}[\mathbb{L}_U] \to WH\mathcal{S}[\mathbb{L}_{U^H}].$$

Proof. The compatibility of the geometric fixed points functor with $\mathscr{L}_U(1)$ action is clear. Next, once again the fact that the actions of H on $\mathscr{L}_{U^H}(2)$ and $\mathscr{L}_{U^H}(1)$ are trivial and the fact that Φ^H is lax symmetric monoidal on $G\mathcal{S}$ implies that it is lax symmetric monoidal on $G\mathcal{S}[\mathbb{L}_U]$. $\qquad\square$

3.5. *The point-set theory of the norm.* In this subsection, we construct multiplicative norm functors in the sense of [Hill et al. 2016] on the categories $G\mathcal{S}[\mathbb{L}_U]$, $G\mathcal{S}_U$, and $\mathcal{M}_{R,U}$ for R a commutative algebra in $G\mathcal{S}_U$. Fix a subgroup $H \subseteq G$ and let

$\widehat{U}$ denote an H-universe. The norm functor $N_H^G : H\mathcal{S} \to G\mathcal{S}$ is strong symmetric monoidal and so there is a natural homeomorphism

$$N_H^G(\mathscr{L}_{\widehat{U}}(1)_+ \wedge X) \cong F_H(G, \mathscr{L}_{\widehat{U}}(1))_+ \wedge N_H^G X.$$

This leads to the following definition, which can be viewed as a form of Theorem 3.7 where we have allowed the group to act on the cartesian factors.

Definition 3.26. We define the functor

$$N_{H,\widehat{U}}^{G,U} : H\mathcal{S}[\mathbb{L}_{\widehat{U}}] \to G\mathcal{S}[\mathbb{L}_U]$$

on objects X via the coequalizer of the diagram

$$\mathscr{L}(\mathrm{Ind}_H^G \widehat{U}, U)_+ \wedge F_H(G, \mathscr{L}_{\widehat{U}}(1))_+ \wedge N_H^G X \rightrightarrows \mathscr{L}(\mathrm{Ind}_H^G \widehat{U}, U)_+ \wedge N_H^G X,$$

where the left action of $\mathscr{L}_U(1)$ on $\mathscr{L}(\mathrm{Ind}_H^G \widehat{U}, U)$ provides the structure of an $\mathbb{L}_U$ algebra.

In the coequalizer, the other map is specified by the action of $F_H(G, \mathscr{L}_{\widehat{U}}(1))$ on $\mathscr{L}(\mathrm{Ind}_H^G \widehat{U}, U)$ via the map of monoids

$$I_{(-)} : F_H(G, \mathscr{L}_{\widehat{U}}(1)) \to \mathscr{L}_{\mathrm{Ind}_H^G \widehat{U}}(1)$$

given by

$$f \mapsto I_f = \big(g \otimes u \mapsto g \otimes f(g)(u)\big),$$

the target of which is underlain by the orthogonal sum of isometries and hence is an isometry.

There is an alternate characterization of $N_{H,\iota_H^* U}^{G,U}$ which can be given using the multiplicative change of universe functors.

Lemma 3.27. *There is a natural homeomorphism*

$$N_{H,\iota_H^* U}^{G,U} X \cong \mathscr{L}\mathscr{I}_{\mathbb{R}^\infty}^U \big(N_{H,\mathbb{R}^\infty}^{G,\mathbb{R}^\infty}(\mathscr{L}\mathscr{I}_{\iota_H^* U}^{\mathbb{R}^\infty} X)\big)$$

Proof. To establish the identification, we expand the right-hand side, writing $\mathbb{R}$ in place of $\mathbb{R}^\infty$ for concision:

$$\mathscr{L}(\mathbb{R}, U) \times_{\mathscr{L}(\mathbb{R},\mathbb{R})} N_{H,\mathbb{R}}^{G,\mathbb{R}} \mathscr{L}(\iota_H^* U, \mathbb{R}) \times_{\mathscr{L}(\iota_H^* U, \iota_H^* U)} X$$

$$\cong \mathscr{L}(\mathbb{R}, U) \times_{\mathscr{L}_{\mathbb{R}}(1)} \big(\mathscr{L}(\mathrm{Ind}_H^G \mathbb{R}, \mathbb{R}) \times_{F_H(G, \mathscr{L}_{\mathbb{R}}(1))} N_H^G\big(\mathscr{L}(\iota_H^* U, \mathbb{R}) \times_{\mathscr{L}_{\iota_H^* U}(1)} X\big)\big)$$

$$\cong \mathscr{L}(\mathrm{Ind}_H^G \mathbb{R}, U) \times_{F_H(G, \mathscr{L}(\mathbb{R},\mathbb{R}))} N_H^G \mathscr{L}(\iota_H^* U, \mathbb{R}) \times_{\mathscr{L}(\iota_H^* U, \iota_H^* U)} X$$

$$\cong \mathscr{L}(\mathrm{Ind}_H^G \mathbb{R}, U) \times_{F_H(G, \mathscr{L}(\mathbb{R},\mathbb{R}))} F_H(G, \mathscr{L}(\iota_H^* U, \mathbb{R})) \times_{F_H(G, \mathscr{L}(\iota_H^* U, \iota_H^* U))} N_H^G X$$

$$\cong \mathscr{L}(\mathrm{Ind}_H^G \iota_H^* U, U) \times_{F_H(G, \mathscr{L}(\iota_H^* U, \iota_H^* U))} N_H^G X$$

$$\cong N_{H,\iota_H^*}^{G,U} X.$$

In these expansions, note that we use the fact that the norm preserves reflexive coequalizers [Hill et al. 2016, Remark A.54]. $\qquad\square$

We now show that norm is strong symmetric monoidal. This can be done using Lemma 3.27, but it is convenient in the homotopical analysis to give a slightly more expansive proof that involves a bit more work with the linear isometries operad, also given in Appendix A. (In contrast, compare the proof of Theorem 3.31 below.)

Theorem 3.28. *The functor $N^{G,U}_{H,\widehat{U}}$ is strong symmetric monoidal.*

Proof. By Lemma A.3, we see that $N^{G,U}_{H,\widehat{U}}$ preserves the unit. We now compare $N^{G,U}_{H,\widehat{U}}(X \wedge_{\widehat{U}} Y)$ and $(N^{G,U}_{H,\widehat{U}}X) \wedge_U (N^{G,U}_{H,\widehat{U}}Y)$ by direct computation. By definition, we have

$$N^{G,U}_{H,\widehat{U}}(X \wedge_{\widehat{U}} Y) = \mathscr{L}(\mathrm{Ind}^G_H \widehat{U}, U) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1))} \left(N^G_H(\mathscr{L}_{\widehat{U}}(2) \times_{\mathscr{L}_{\widehat{U}}(1) \times \mathscr{L}_{\widehat{U}}(1)} (X \wedge Y)) \right).$$

Since the norm functor commutes with reflexive coequalizers and is symmetric monoidal as a functor on orthogonal H-spectra, this is isomorphic to

$$\mathscr{L}(\mathrm{Ind}^G_H \widehat{U}, U) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1))} \left(F_H(G, \mathscr{L}_{\widehat{U}}(2)) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1)) \times F_H(G,\mathscr{L}_{\widehat{U}}(1))} (N^G_H X \wedge N^G_H Y) \right).$$

Applying Corollary A.7, we rewrite this as

$$\mathscr{L}(\mathrm{Ind}^G_H \widehat{U} \oplus \mathrm{Ind}^G_H \widehat{U}, U) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1)) \times F_H(G,\mathscr{L}_{\widehat{U}}(1))} (N^G_H X \wedge N^G_H Y).$$

On the other hand, writing out $(N^{G,U}_{H,\widehat{U}}X) \wedge_U (N^{G,U}_{H,\widehat{U}}Y)$ we have

$$\mathscr{L}_U(2)$$
$$\times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} \left(\left(\mathscr{L}(\mathrm{Ind}^G_H \widehat{U}, U) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1))} N^G_H X \right) \wedge \left(\mathscr{L}(\mathrm{Ind}^G_H \widehat{U}, U) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1))} N^G_H Y \right) \right).$$

Applying Corollary A.5, we can rewrite this as

$$\mathscr{L}(\mathrm{Ind}^G_H \widehat{U} \oplus \mathrm{Ind}^G_H \widehat{U}, U) \times_{F_H(G,\mathscr{L}_{\widehat{U}}(1)) \times F_H(G,\mathscr{L}_{\widehat{U}}(1))} (N^G_H X \wedge N^G_H Y).$$

Finally, the naturality of the homeomorphisms above make it clear that the pentagon identities hold. $\qquad\square$

As a consequence of Theorem 3.28, we have the following corollary.

Corollary 3.29. *The functor $N^{G,U}_{H,\widehat{U}}$ restricts to a functor*

$$N^{G,U}_{H,\widehat{U}} : H\mathcal{S}_{\widehat{U}} \to G\mathcal{S}_U$$

which we abusively refer to with the same notation.

Remark 3.30. Lemma 3.27 can now be interpreted as the statement that the norm can be described as the indexed product on $H\mathcal{S}_{\mathbb{R}^\infty}$; this makes it clear that the norm is functorial in both the group and the input spectrum.

We now turn to establish the following adjunction on commutative ring objects. We will be predominantly interested in the case where $\widehat{U} = \iota_H^* U$, as this is relevant for describing the equivariant symmetric monoidal structures on the categories $\mathcal{M}_{R,U}$.

Theorem 3.31. *There are adjoint pairs with left adjoints*

$$N_{H,\iota_H^* U}^{G,U} : (H\mathcal{S}[\mathbb{L}_{\iota_H^* U}])[\mathbb{P}] \to (G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}] \quad and \quad N_{H,\iota_H^* U}^{G,U} : (H\mathcal{S}_{\iota_H^* U})[\mathbb{P}] \to G\mathcal{S}_U[\mathbb{P}]$$

and right adjoints

$$\iota_H^* : (G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}] \to (H\mathcal{S}[\mathbb{L}_{\iota_H^* U}])[\mathbb{P}] \quad and \quad \iota_H^* : G\mathcal{S}_U[\mathbb{P}] \to H\mathcal{S}_{\iota_H^* U}[\mathbb{P}]$$

respectively.

Proof. First, observe that the conclusion of the theorem follows immediately when $U = \mathbb{R}^\infty$: since for any G the category $G\mathcal{S}[\mathbb{L}_{\mathbb{R}^\infty}]$ is equivalent to the category of G-objects in $\mathcal{S}[\mathbb{L}_{\mathbb{R}^\infty}]$, we can apply [Hill et al. 2016, Corollary A.56]. We now use the alternate characterization of the norm from Lemma 3.27 and the fact that by Theorem 3.15 the change of universe functors are symmetric monoidal equivalences of categories. □

An immediate corollary of Theorem 3.31 is that commutative ring objects have an "internal norm" map arising from the counit of the adjunction.

Corollary 3.32. *Let R be an object in $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}]$ or $G\mathcal{S}_U[\mathbb{P}]$. Then there is a natural map*

$$N_{H,\iota_H U}^{G,U} \iota_H^* R \to R.$$

Using the counit of the adjunction of Theorem 3.31 and the absolute norm functor described in Definition 3.26, we can express the R-relative norm for a commutative ring object R in $G\mathcal{S}_U$ using base-change:

Definition 3.33. Let R be an object in $G\mathcal{S}_U[\mathbb{P}]$. We define the functor

$$_R N_{H,\iota_H^* U}^{G,U} : \mathcal{M}_{\iota_H^* R, \iota_H^* U} \to \mathcal{M}_{R,U}$$

via the formula

$$X \mapsto R \wedge_{N_{H,\iota_H^* U}^{G,U} R} N_{H,\iota_H^* U}^{G,U} X,$$

where the coequalizer is over the counit map and the map induced by the action of R on X.

It is clear from the definition and Theorem 3.28 that the R-relative norm is also a strong symmetric monoidal functor.

4. Homotopical categories of modules over an N_∞ algebra

In this section, we describe model structures on the categories $G\mathcal{S}[\mathbb{L}_U]$, $G\mathcal{S}_U$, and categories of algebras and modules over an algebra. The main goal of our efforts is to describe the derived functors of the norm and forgetful functors as a prelude to the construction of the equivariant symmetric monoidal structure.

4.1. *The homotopical theory of $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$.* We begin by quickly reviewing some of the less commonly used terminology from the theory of model categories that we will employ in the statements of results below. Recall from [Mandell et al. 2001, Definition 5.9] that a cofibrantly generated topological model structure is compactly generated if the domains of the generating cofibrations and acyclic cofibrations are compact and satisfy the "cofibration hypothesis" [Mandell et al. 2001, Cofibration Hypothesis 5.3]. Let $\mathcal{C}$ be a complete and cocomplete topologically enriched category. An h-cofibration in $\mathcal{C}$ is a map that is the analogue of a Hurewicz cofibration; i.e., a map $X \to Y$ such that the induced map $Y \cup_X (X \otimes I) \to Y \otimes I$ has a retraction. The cofibration hypothesis for a set of maps I in a model category $\mathcal{A}$ equipped with a forgetful functor $\mathcal{A} \to \mathcal{C}$ specifies that the following two conditions are satisfied.

(1) For a coproduct $A \to B$ of maps in I, in any pushout

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
$$

   in $\mathcal{A}$, the cobase change $X \to Y$ is an h-cofibration in $\mathcal{C}$.

(2) Given a sequential colimit in $\mathcal{A}$ along maps that are h-cofibrations in $\mathcal{C}$, the colimit in $\mathcal{A}$ is equal to the colimit in $\mathcal{C}$.

In order to be able to apply Bousfield localization, it is convenient to add the requirements that:

(1) The domains of the generating acyclic cofibrations are small with respect to the generating cofibrations, and

(2) the cofibrations are effective monomorphisms.

A compactly generated model category that satisfies these additional conditions is cellular [Hirschhorn 2003, Definition 12.1.1], and so admits left Bousfield localizations very generally. In mild abuse of terminology, we will use the term compactly generated to refer to a compactly generated model category that is cellular in this paper.

A model category is G-topological if it is enriched over G-spaces and satisfies the analogue of Quillen's SM7 [Mandell and May 2002, Definition III.1.14]. There

is an evident G-equivariant version of the cofibration hypothesis. The building block for our work in this section is the complete model structure on $G\mathcal{S}$ [Hill et al. 2016, Proposition B.63]. (Note that although the cited reference refers to the positive complete model structure, the existence of the complete model structure is clear.) Recall that the complete model structure has generating cofibrations given by the set of maps

$$\left\{ G_+ \wedge_H S^{-V} \wedge S^{n-1}_+ \to G_+ \wedge_H S^{-V} \wedge D^n_+ \right\},$$

where V varies over the (additive) universe, $n \geq 0$, and $H \subseteq G$.

Lemma 4.1. *The complete model structure is a compactly generated G-topological model structure.*

Proof. The discussion proving [Hill et al. 2016, Proposition B.63] establishes that the complete model structure is cofibrantly generated. Since the generating cofibrations in the complete model structure are h-cofibrations [Hill et al. 2016, Remark B.64], it is straightforward to see that the complete model structure satisfies the cofibration hypothesis. Finally, the cofibrations are effective monomorphisms since $S^{n-1}_+ \to D^n_+$ is for all $n \geq 0$, and the compactness criterion for the domain of the generating acyclics is clearly satisfied. □

Theorem 4.2. *The category $G\mathcal{S}[\mathbb{L}_U]$ is a compactly generated weak symmetric monoidal proper G-topological model category in which the weak equivalences and fibrations are detected by the forgetful functor $u : G\mathcal{S}[\mathbb{L}_U] \to G\mathcal{S}$.*

Proof. The monad $\mathbb{L}_U$ evidently satisfies the hypotheses of (the equivariant analogue of) [Mandell et al. 2001, Proposition 5.13], and so we can conclude that there is a compactly generated G-topological model structure on $G\mathcal{S}[\mathbb{L}_U]$. The proof of the unit axiom follows from the equivariant analogue of from [Elmendorf et al. 1997, XI.3.1], which holds by the same proof as in the nonequivariant case. To check the monoid axiom, observe that it suffices to check on the generating (acyclic) cofibrations, and since these are obtained by $\mathscr{L}_U(1) \wedge_+ (-)$ applied to generating (acyclic) cofibrations of $G\mathcal{S}$, the result holds since it does in $G\mathcal{S}$. Finally, it is clear that $G\mathcal{S}[\mathbb{L}_U]$ is proper. □

By construction, the adjoint pair $(\mathbb{L}_U, u)$ is a Quillen adjunction. Since $\mathscr{L}_U(1)$ is G-contractible, we can conclude that this pair induces a Quillen equivalence between $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}$.

Proposition 4.3. *The adjoint pair $(\mathbb{L}_U, u)$ forms a Quillen equivalence between $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}$.*

Although $\mathbb{L}_U$ is not strong symmetric monoidal, it is close: as a consequence of Lemma A.1, there is a homeomorphism

$$\mathbb{L}_U X \wedge_U \mathbb{L}_U Y \cong \mathbb{L}_U(X \wedge Y)$$

and more generally homeomorphisms

$$\mathbb{L}_U X_1 \wedge_U \cdots \wedge_U \mathbb{L}_U X_k \cong \mathbb{L}_U(X_1 \wedge \cdots \wedge X_k).$$

(The failure of $\mathbb{L}_U$ to be strong symmetric monoidal is a consequence of the fact that these homeomorphisms ultimately depend on choices of homeomorphisms $U^k \to U$.) On the other hand, the functor $Q(-) = \mathbb{S}_G \wedge_{\Sigma_+^\infty \mathscr{L}_U(1)} (-)$ is strong symmetric monoidal [Blumberg et al. 2010, Theorem 4.14]. As a consequence, we have the following comparison result (where here recall that p^* denotes the right adjoint to Q which gives an object of GS the trivial $\mathscr{L}_U(1)$-action).

Proposition 4.4. *The adjoint pair (Q, p^*) is a weak symmetric monoidal Quillen equivalence.*

Proof. Since p^* preserves fibrations and weak equivalences, this is clearly a Quillen adjunction. Taking $\mathbb{L}\mathbb{S}_G$ as a cofibrant replacement of the unit in GS, we compute that $Q\mathbb{L}\mathbb{S}_G \cong \mathbb{S}_G$ and so the adjunction is monoidal. Finally, evaluation of Q on the generating cofibrations makes it clear that the natural map $QX \to uX$ is a weak equivalence for cofibrant X, and so the adjunction is a Quillen equivalence. $\qquad\square$

In order to retain homotopical control over GS_U, we need to prove the equivariant analogue of [Elmendorf et al. 1997, I.8.4, XI.2.2], i.e., that the canonical unit map $\lambda : \mathbb{S}_G \wedge_U X \to X$ is always a weak equivalence. The proof of the required result follows the outline of [Elmendorf et al. 1997, I.8.5], using Theorem A.9.

Theorem 4.5. *For any X in $GS[\mathbb{L}_U]$, the unit map*

$$\lambda : \mathbb{S}_G \wedge_U X \to X$$

is a weak equivalence.

Theorem 4.5 now allows us to prove the following theorem.

Theorem 4.6. *The category GS_U is a compactly generated symmetric monoidal proper G-topological model category in which the weak equivalences are detected by the forgetful functor and the fibrations are detected by the functor $F_U(\mathbb{S}_G, -)$.*

Proof. Although $\mathbb{S}_G \wedge_U (-)$ is not a monad, the argument for [Mandell et al. 2001, Proposition 5.13] again applies. As in the corresponding proof in [Elmendorf et al. 1997, VI.4.6], consideration of the category of counital objects in $GS[\mathbb{L}_U]$ is illuminating. $\qquad\square$

Remark 4.7. By adjunction, a map $\mathbb{S}_G \wedge_U \mathbb{L}_U S^n \to X$ in GS_U is the same as a map $\mathbb{S}_G^n \to X$ in GS. As a consequence, the "internal" homotopy groups in GS_U determined by the free objects on spheres coincide with the homotopy groups on the underlying orthogonal G-spectrum.

The functor $\mathbb{S}_G \wedge_U (-) : G\mathcal{S}[\mathbb{L}_U] \to G\mathcal{S}_U$ is a Quillen left adjoint and is a symmetric monoidal functor. In fact, the following proposition is straightforward to verify.

Proposition 4.8. *The adjoint pair* $(\mathbb{S}_G \wedge_U (-), F_U(\mathbb{S}_G, -))$ *forms a weak symmetric monoidal Quillen equivalence between* $G\mathcal{S}[\mathbb{L}_U]$ *and* $G\mathcal{S}_U$.

As a consequence of these results, we have the following comparison result.

Lemma 4.9. *For cofibrant* $X, Y \in G\mathcal{S}_U$ *there is a natural equivalence*

$$X \wedge_U Y \to X \wedge Y$$

and more generally for cofibrant $\{X_1, X_2, \ldots, X_n\} \in G\mathcal{S}_U$ *there are natural equivalences*

$$X_1 \wedge_U X_2 \wedge_U \cdots \wedge_U X_n \to X_1 \wedge X_2 \wedge \cdots \wedge X_n.$$

We now turn to the study of the multiplicative structure on $G\mathcal{S}_U$. The following result explains the equivariant homotopical content of the operadic smash product $\wedge_U$.

Theorem 4.10. *Let* X *be a cofibrant object of* $G\mathcal{S}_U$. *Then there is a natural weak equivalences of* $G \times \Sigma_n$ *spectra*

$$(E_{\mathcal{F}_U} \Sigma_n)_+ \wedge X^{\wedge n} \simeq X^{\wedge_U n},$$

and a natural weak equivalence of G-*spectra*

$$(E_{\mathcal{F}_U} \Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n} \simeq X^{\wedge_U n}/\Sigma_n,$$

where here $\mathcal{F}_U$ *denotes the family of* $G \times \Sigma_n$ *specified by* U.

Proof. When X is free as an object of $G\mathcal{S}_U$ (i.e., $X = \mathbb{S}_G \wedge_U \mathbb{L}_U Y$), then the result follows immediately from Theorem 3.7 and the fact that $\mathscr{L}(U^n, U) \simeq E_{\mathcal{F}_U} \Sigma_n$. The general result now follows by inductively reducing to the free case using the filtration argument of [Hill et al. 2016, (B.117)]. □

In particular, Theorem 4.10 makes clear the way in which $G\mathcal{S}_U$ depends on the choice of U. Specifically, the $G \times \Sigma_n$-equivariant homotopy type of the n-fold $\wedge_U$ power of X is controlled by U, and is precisely the universal space for the family associated to $\mathscr{L}_U(n)$.

Corollary 4.11. *Let* $X \to X'$ *be an acyclic cofibration in* $G\mathcal{S}_U$. *Then the induced maps*

$$\mathbb{T}X \to \mathbb{T}X' \quad and \quad \mathbb{P}X \to \mathbb{P}X'$$

are weak equivalences.

Corollary 4.11 provides the essential technical input for the next theorem, which is again proved using the standard outline (e.g., see [Mandell et al. 2001, Proposition 5.13] or [Hill et al. 2016, (B.130)]).

Theorem 4.12. *The categories $G\mathcal{S}_U[\mathbb{T}]$ and $G\mathcal{S}_U[\mathbb{P}]$ are compactly generated proper G-topological model categories with weak equivalences and fibrations determined by the forgetful functor to $G\mathcal{S}_U$.*

For a fixed ring object R, we have the following relative version of the preceding theorem.

Theorem 4.13. *For an object R in $G\mathcal{S}_U[\mathbb{T}]$ or $G\mathcal{S}_U[\mathbb{P}]$, the category of R-modules in $G\mathcal{S}_U$ is a compactly generated proper G-topological model category with weak equivalences and fibrations determined by the forgetful functor to $G\mathcal{S}_U$. When R is commutative (i.e., an object in $G\mathcal{S}_U[\mathbb{P}]$), then*

(1) *the category of R-modules in $G\mathcal{S}_U$ is a compactly generated proper G-topological symmetric monoidal model category, and*

(2) *the category of R-algebras is a compactly generated proper G-topological model category.*

4.2. *The homotopical theory of change of group and fixed-point functors.* In this section, we describe how to compute the derived functors of the change-of-group and fixed-point functors described in Section 3.4. Our analysis bootstraps from the analogous theory in the setting of $G\mathcal{S}$; the following two lemmas establish that the homotopical theory for $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$ can be understood in terms of the homotopical theory for $G\mathcal{S}$.

Lemma 4.14. *Let X be an object of $G\mathcal{S}[\mathbb{L}_U]$ or $G\mathcal{S}_U$. If X is cofibrant, then the underlying orthogonal G-spectrum associated to X has the homotopy type of a cofibrant object. The analogous results hold for $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}]$ and $G\mathcal{S}_U[\mathbb{T}]$.*

Proof. This follows from inspection of the generating cells and the "cofibration hypothesis" in this context. We can assume without loss of generality that X is a cellular object. Then $X = \mathrm{colim}_n X_n$, where the colimit is sequential and along h-cofibrations. The cofibration hypothesis then implies that we can compute the colimit in the underlying category, and so it suffices to consider each X_n. Since each X_n is formed from X_{n-1} by attaching cells, the cofibration hypothesis again allows us to inductively reduce this to consideration of the generating cells, where the result is clear. $\square$

Lemma 4.15. *Let X be a fibrant object in $G\mathcal{S}[\mathbb{L}_U]$, $G\mathcal{S}[\mathbb{L}_U])[\mathbb{T}]$, or $(G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}]$. Then X is fibrant in $G\mathcal{S}$. Analogously, if X is fibrant in $G\mathcal{S}_U$, $G\mathcal{S}_U[\mathbb{T}]$, or $G\mathcal{S}_U[\mathbb{P}]$, then $F_U(\mathbb{S}_G, X)$ is fibrant in $G\mathcal{S}$.*

Proof. The statements about modules imply the statements about monoids and commutative monoids, as fibrations in the model structures on the categories of algebras are determined by the forgetful functors to $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}_U$ respectively. The first assertion is clear for $G\mathcal{S}[\mathbb{L}_U]$ since the fibrations are created by the forgetful functor to $G\mathcal{S}$. For $G\mathcal{S}_U$, the result follows from Proposition 4.8; the functor $F_U(\mathbb{S}_G, -) : G\mathcal{S}_U \to G\mathcal{S}[\mathbb{L}_U]$ is a Quillen right adjoint.   □

In order to understand the behavior of the fixed point functors on $G\mathcal{S}_U$, we need to describe the homotopical behavior of the point-set multiplicative change of universe functors. In contrast to the situation for the additive functors in orthogonal spectra, these always induce Quillen equivalences.

Proposition 4.16. *Let U and U' be G-universes. The multiplicative change of universe functors $\mathscr{LI}_U^{U'}$ are left (and right) Quillen functors that preserve weak equivalences between cofibrant objects and therefore induce Quillen equivalences between $G\mathcal{S}[\mathbb{L}_U]$ and $G\mathcal{S}[\mathbb{L}_{U'}]$ and $G\mathcal{S}_U$ and $G\mathcal{S}'_U$, respectively.*

Warning 4.17. What is not preserved by $\mathscr{LI}_U^{U'}$ is not the underlying additive homotopy theory but the multiplicative norms. Specifically, the derived functor of $\mathscr{LI}_U^{U'}$ preserves only those multiplicative norms corresponding to G-sets that are admissible in both U and U'. Put another way, these functors do not preserve the homotopical equivariant symmetric monoidal structure.

We now turn to the fixed points. The forgetful functors ι_H^* preserve all weak equivalences, and so are already derived. Their left and right adjoints can be derived by cofibrant or fibrant approximation, as a consequence of the preceding lemmas. Similarly, Proposition 4.16 implies that the (right) derived functors of the categorical fixed points can be computed by fibrant replacement and the (left) derived functors of geometric fixed points by cofibrant replacement. We summarize the situation in the following result.

Proposition 4.18. (1) *The forgetful functors ι_H^* preserve all weak equivalences on $G\mathcal{S}_U$ and $G\mathcal{S}[\mathbb{L}_U]$.*

(2) *The left adjoint $G_+ \wedge_H (-)$ to ι_H^* preserves weak equivalences between cofibrant objects on $G\mathcal{S}_U$ and $G\mathcal{S}[\mathbb{L}_U]$. The right adjoint $F_H(G, -)$ to ι_H^* preserves weak equivalences between fibrant objects on $G\mathcal{S}_U$ and $G\mathcal{S}[\mathbb{L}_U]$.*

(3) *The categorical fixed point functor $(-)^H$ preserves weak equivalences between fibrant objects in $G\mathcal{S}_U$ and $G\mathcal{S}[\mathbb{L}_U]$.*

(4) *The geometric fixed point functor Φ^H preserves weak equivalences between cofibrant objects in $G\mathcal{S}_U$ and $G\mathcal{S}[\mathbb{L}_U]$.*

Finally, we have the following result which shows that the geometric fixed-point functor is strong monoidal in the homotopical sense.

Proposition 4.19. *Let X and Y be cofibrant objects in $G\mathcal{S}[\mathbb{L}_U]$ or $G\mathcal{S}_U$. Then the natural map*

$$\Phi^H X \wedge_U \Phi^H Y \to \Phi^H(X \wedge_U Y)$$

is a weak equivalence.

Proof. First consider the case of $G\mathcal{S}[\mathbb{L}_U]$. The result follows from the result for Φ^H on $G\mathcal{S}$ [Mandell and May 2002, Proposition V.4.7] when X and Y are generating cells, since

$$\mathbb{L}_{U^H} X' \wedge_{U^H} \mathbb{L}_{U^H} Y' \cong \mathscr{L}_{U^H}(2)_+ \wedge (X' \wedge Y')$$

for any X' and Y' and WH acts trivially on $\mathscr{L}U^H(2)_+$. Since $(-) \wedge_{U^H} (-)$ preserves colimits in either variable and preserves weak equivalences between cofibrant objects, we can conclude the general statement. The case of $G\mathcal{S}_U$ follows from analogous considerations. $\square$

4.3. *The homotopical theory of the norm.* In this section, we show that the norm $N_{H,\iota_H^* U}^{G,U}$ is a homotopical functor and participates in a Quillen adjunction when restricted to commutative ring objects.

Theorem 4.20. *Let X be a cofibrant object in $G\mathcal{S}[\mathbb{L}_U]$ or $G\mathcal{S}_U$. The natural map*

$$N_{H,\iota_H^* U}^{G,U} X \to N_H^G X$$

is a weak equivalence when G/H is admissible for U.

Proof. By induction over the cellular filtration, it suffices to consider the case when X is free. In this case, we're looking at the map

$$\mathscr{L}(\mathrm{Ind}_H^G \iota_H^* U, U)_+ \wedge N_H^G X \to N_H^G X$$

given by the collapse map $\mathscr{L}(\mathrm{Ind}_H^G \iota_H^* U, U)_+ \to S^0$. Since the collapse is a G-equivalence when G/H is admissible, the result follows. $\square$

Remark 4.21. When G/H is not admissible for U, it is not clear in general what the homotopy type of $N_{H,\iota_H^* U}^{G,U}$ is. For free objects, the homotopy type is controlled by $\mathscr{L}(\mathrm{Ind}_H^G \iota_H^* U, U)$, which has no G-fixed points.

Corollary 4.22. *The functor $N_{H,\iota_H^* U}^{G,U}$ preserves weak equivalences between cofibrant objects in $H\mathcal{S}_{\iota_H^* U}$ and $H\mathcal{S}[\mathbb{L}_{\iota_H^* U}]$ when G/H is admissible for U.*

The next lemma provides homotopical control on the output of the norm functor.

Lemma 4.23. *Let X be a cofibrant object in $H\mathcal{S}[\mathbb{L}_{\widehat{U}}]$ or $H\mathcal{S}_{\widehat{U}}$. Then $N_{H,\iota_H^* U}^{G,U} X$ is cofibrant in $G\mathcal{S}[\mathbb{L}_U]$.*

Proof. Using the filtration of [Hill et al. 2016, §A.3.4], we can inductively reduce to the case when X is of the form $\mathscr{L}_{\widehat{U}}(1)_+ \wedge Y$. In this case,

$$N_{H,\iota_H^* U}^{G,U}(\mathscr{L}_{\widehat{U}}(1)_+ \wedge Y) \cong \mathscr{L}(\mathrm{Ind}_H^G \widehat{U}, U)_+ \wedge N_H^G Y.$$

Since $\mathscr{L}(\mathrm{Ind}_H^G \widehat{U}, U) \cong \mathscr{L}_1(U)$ by Lemma A.1, the result follows. $\qquad\square$

As a consequence, we have the following result about the composition of the norm functor.

Proposition 4.24. *Fix $H_1 \subseteq H_2 \subseteq G$. Let X be a cofibrant object in $H_1\mathcal{S}[\mathbb{L}_{\iota_{H_1}^* U}]$ or $H_1\mathcal{S}_{\iota_{H_1}^* U}$. Then there is a natural weak equivalence*

$$N_{H_1, \iota_{H_1}^* U}^{G,U} X \simeq N_{H_2, \iota_{H_2}^* U}^{G,U} N_{H_1, \iota_{H_1}^* U}^{H_2, \iota_{H_2}^* U} X.$$

Proof. Expanding using the definition, we have

$$N_{H_2, \iota_{H_2}^* U}^{G,U} N_{H_1, \iota_{H_1}^* U}^{H_2, \iota_{H_2}^* U} X = \mathscr{L}(\mathrm{Ind}_{H_2}^G \iota_{H_2}^* U, U) \times_{F_{H_2}(G, \mathscr{L}_{\iota_{H_2}^* U}(1))} N_{H_2}^G (N_{H_1, \iota_{H_1}^* U}^{H_2, \iota_{H_2}^* U} X)$$

and

$$N_{H_1, \iota_{H_1}^* U}^{H_2, \iota_{H_2}^* U} X = \left(\mathscr{L}(\mathrm{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U) \times_{F_{H_1}(H_2, \mathscr{L}_{\iota_{H_1}^* U}(1))} N_{H_1}^{H_2} X \right)$$

which implies that

$$N_{H_2}^G (N_{H_1, \iota_{H_1}^* U}^{H_2, \iota_{H_2}^* U} X) \cong \left(F_{H_2}(G, \mathscr{L}(\mathrm{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U)) \times_{F_{H_2}(G, \mathscr{L}_{\iota_{H_1}^* U}(1))} N_{H_1}^G X \right).$$

Next, we show that

$$\mathscr{L}(\mathrm{Ind}_{H_2}^G \iota_{H_2}^* U, U) \times_{F_{H_2}(G, \mathscr{L}_{\iota_{H_2}^* U}(1))} F_{H_2}(G, \mathscr{L}(\mathrm{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U))$$

is isomorphic to $\mathscr{L}(\mathrm{Ind}_{H_1}^G \iota_{H_1}^* U, U)$. There is an equivariant map

$$\mathscr{L}(\mathrm{Ind}_{H_2}^G \iota_{H_2}^* U, U) \times F_{H_2}(G, \mathscr{L}(\mathrm{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U)) \to \mathscr{L}(\mathrm{Ind}_{H_1}^G \iota_{H_1}^* U, U)$$

induced by composition and the natural map

$$F_{H_2}(G, \mathscr{L}(\mathrm{Ind}_{H_1}^{H_2} \iota_{H_1}^* U, \iota_{H_2}^* U)) \to \mathscr{L}(\mathrm{Ind}_{H_1}^G \iota_{H_1}^* U, \mathrm{Ind}_{H_2}^G \iota_{H_2}^* U)$$

induced by the direct sum. This map is compatible with the maps determining the coequalizer, and so it suffices to check that the underlying nonequivariant diagram is a reflexive coequalizer. This now follows from Lemma A.6. The theorem is now a consequence of the preceding homeomorphism and Lemma 4.23. $\qquad\square$

In the case of commutative monoid objects, it is straightforward to check that the adjunction involving the norm and the forgetful functor is homotopical; it is clear that ι_H^* preserves fibrations and weak equivalences.

Theorem 4.25. *The adjoint pairs*

$$N_{H, \iota_H^* U}^{G,U} : (H\mathcal{S}[\mathbb{L}_{\iota_H^* U}])[\mathbb{P}] \leftrightarrows (G\mathcal{S}[\mathbb{L}_U])[\mathbb{P}] : \iota_H^*$$

and

$$N_{H, \iota_H^* U}^{G,U} : H\mathcal{S}_{\iota_H^* U}[\mathbb{P}] \leftrightarrows G\mathcal{S}_U[\mathbb{P}] : \iota_H^*$$

are Quillen adjunction.

Note however that the derived functor of the norm $N_{H,\iota_H^* U}^{G,U}$ on commutative rings only agrees with the derived functor of the module norm when G/H is admissible for U; the following result is a consequence of the fact that the derived functor of the norm on commutative rings in orthogonal H-spectra agrees with the underlying norm [Hill et al. 2016, Remark B.148].

Proposition 4.26. *Let X be a cofibrant object in $GS[\mathbb{L}_U][\mathbb{P}]$. The natural map*

$$N_{H,\iota_H^* U}^{G,U} X \to N_H^G X$$

is a weak equivalence when G/H is admissible for U.

We now turn to the relative norm construction.

Theorem 4.27. *The functor $_R N_{H,\iota_H^* U}^{G,U}$ preserves weak equivalence between cofibrant objects in $\mathcal{M}_{\iota_H^* R, \iota_H^* U}$ when G/H is admissible for U.*

Proof. Since the R-relative norm is strong symmetric monoidal, it suffices to show that when X is cofibrant in $\mathcal{M}_{\iota_H^* R, \iota_H^* U}$, $N_{H,\iota_H^* U}^{G,U} X$ is cofibrant as an $N_{H,\iota_H^* U}^{G,U} R$ module. Once again, it suffices to check this on free objects, where it is straightforward. $\square$

5. *G*-symmetric monoidal categories of modules over an N_∞ algebra

In this section, we describe the homotopical G-symmetric monoidal structure on $\mathcal{M}_{R,U}$. More precisely, we have a $\mathscr{L}_U$-symmetric monoidal structure, where we mean an equivariant symmetric monoidal structure specified by the coefficient system of admissible sets for $\mathscr{L}_U$ [Hill and Hopkins 2016, Definition 4.4]. We characterize this structure in terms of a homotopical exponential functor

$$_R N^T : \mathcal{M}_{R,U} \to \mathcal{M}_{R,U}$$

for any admissible G-set T. We explain how this "internal norm" arises from structure on the collection of norms and forgetful functors on the categories $\mathcal{M}_{\iota_H^* R, \iota_H^* U}$ as H varies over the closed subgroups of G; these functors assemble into an incomplete Mackey functor in homotopical categories. We also explain the resulting structure on commutative monoid objects, recovering the characterizations of [Blumberg and Hill 2015, Theorem 6.11].

5.1. *The G-symmetric monoidal structure on GS and $\mathcal{M}_R$.* In this subsection, we review the canonical G-symmetric monoidal structure on GS and $\mathcal{M}_R$ for R a commutative ring orthogonal G-spectrum. We begin by recalling from [Blumberg and Hill 2015, §6] the definition of the internal norm in orthogonal spectra.

Definition 5.1. Let $H \subset G$ be a subgroup. The internal norm of an orthogonal G-spectrum X is specified by the formula

$$N^{G/H} X = N_H^G \iota_H^* X.$$

For an arbitrary G-set T, we define the internal norm by decomposing T into a disjoint union of orbits $\coprod_i G/H_i$ and defining

$$N^T X = \bigwedge_i N^{G/H_i} X.$$

For example, when T is a trivial G-set, $N^T M$ is simply the smash-power of $|T|$ copies of M. Note that this definition extends in the evident way to categories of modules over a commutative ring orthogonal G-spectrum.

There is another equivalent description for this which will make the properties of the norm (summarized in Theorem 5.5 below) more transparent. If T is a finite G-set, then let $B_T G$ denote the translation category of T. This has object set T itself and the morphism set is $T \times G$ with structure maps the projection onto T and the action. Given a G-spectrum X, we have a $B_G T$-shaped diagram X^T described by $t \mapsto X$ and (t, g) acts as multiplication by g on X.

A map of finite G-sets $f : T \to S$ produces a covering category $B_T G \to B_S G$ as in [Hill et al. 2016, Definition A.24], and therefore we have an associated indexed monoidal product $f_*^\otimes$.

Proposition 5.2. *Let $p : T \to *$ be the terminal map. There is a canonical homeomorphism*

$$p_*^\otimes X^T \cong N^T(X).$$

Proof. Since both sides take disjoint unions to smash products (the left by construction and the right by definition), it suffices to construct the canonical homeomorphism when $T = G/H$. In this case, each side is then an indexed product.

Using the additive change of universe equivalence, we can work in the category of orthogonal spectra with a G-action and prove the desired equality there. In this case, both sides are the indexed product associated to $p : G/H \to *$, so it will suffice to show that the resulting diagrams are isomorphic. For the left-hand side, the diagram is the constant diagram $X^{G/H}$. For the right-hand side, the diagram is determined by choosing coset representatives and sending a coset gH to the gHg^{-1}-spectrum $g \cdot \iota_H^* X$ (where here $g \cdot Y$ for an H-spectrum Y is just the restriction along the isomorphism $gHg^{-1} \cong H$). However, we then have an equivariant isomorphism of diagrams

$$X^{G/H} \cong (gH \mapsto g \cdot \iota_H^* X)$$

which at a coset gH is simply multiplication by g. The indexed products are therefore isomorphic. $\square$

Remark 5.3. The key step in the argument is the same as the one showing that we have canonical homeomorphisms

$$F_H(G_+, i_H^* X) \cong F(G/H_+, X) \quad \text{and} \quad G_+ \wedge_H i_H^* X \cong G/H_+ \wedge X.$$

In each case, we have the same two diagrams as the one given above and then we compare the associated indexed monoidal products.

Because N_H^G, $- \wedge -$, and ι_H^* preserve weak equivalences and cofibrant objects, the internal norm is a homotopical functor.

Lemma 5.4. *For any G-set T, the internal norm*

$$N^T : G\mathcal{S} \to G\mathcal{S}$$

preserves acyclic cofibrations.

We can now recall the basic theorem establishing the G-symmetric monoidal structure on G-spectra. All of this follows easily from Appendix A of [Hill et al. 2016]; for convenience, we include details here.

Theorem 5.5. (1) *For $H_1 \subseteq H_2 \subseteq G$, there is a natural homeomorphism*

$$\iota_{H_1}^* \cong \iota_{H_1}^* \iota_{H_2}^*.$$

(2) *For G-sets T_1 and T_2, there is a natural homeomorphism*

$$N^{T_1 \times T_2} X \simeq N^{T_1} N^{T_2} X.$$

(3) *For $K \subset H$, there is a natural homeomorphism*

$$\iota_K^* N^T X \simeq N^{\iota_K^* T} \iota_K^* X.$$

Proof. The first part is obvious. For the second and third parts, we use the alternative description of $N^T X$ given by Proposition 5.2.

For the second, observe that $B_{T_1 \times T_2} G \cong B_{T_1} G \times B_{T_2} G$, and the composite of the norms is the composites of the indexed products

$$T_1 \times T_2 \to T_1 \to *.$$

The composite of the indexed products is the indexed product of the composites [Hill et al. 2016, Proposition A.29].

The third is the variant of the double coset formula here. If T is a finite G-set, then we have a pullback diagram of categories

$$
\begin{array}{ccc}
B_{G/H \times T}G & \longrightarrow & B_T G \\
\downarrow & & \downarrow \\
B_{G/H}G & \longrightarrow & BG.
\end{array}
$$

Since $G/H \times T \cong G \times_H i_H^* T$, the left-hand side of this diagram is equivalent to $B_{i_H^* T}H \to BH$. Since the map on spectra induced by pulling back along $B_{G/H}G \to BG$ is i_H^*, we conclude by [Hill et al. 2016, Proposition A.31] that

$$
i_H^* N^T X \cong N^{i_H^* T} i_H^{*X}. \qquad \square
$$

The analogue of Theorem 5.5 for modules over a commutative ring orthogonal G-spectrum R follows from the characterization of the R-relative norm via the formula

$$
{}_R N_H^G X \cong R \wedge_{N_H^G R} N_H^G X
$$

and the fact that the norm N_H^G is the left adjoint to the restriction functor ι_H^* on commutative rings. We explain in detail the argument below in the proof of Theorem 5.10.

5.2. The $\mathscr{L}_U$-symmetric monoidal structure on $G\mathcal{S}_U$ and $\mathcal{M}_{R,U}$. We now provide the analogous definitions in our context.

Definition 5.6. Given a subgroup $H \subset G$, we define the internal norm

$$
{}_R N_U^{G/H} M : \mathcal{M}_{R,U} \to \mathcal{M}_{R,U}
$$

as the composite

$$
{}_R N_U^{G/H}(-) := {}_R N_{H,\iota_H^*}^{G,U} \iota_H^*(-).
$$

We extend the internal norm to an arbitrary G-set T by decomposing T into a disjoint union of orbits $\coprod_i G/H_i$ and specifying that

$$
{}_R N_U^T M = \bigwedge_i {}_R N^{G/H_i} M.
$$

We now describe the homotopical properties of the internal norm. We begin by considering the absolute case where $R = S$.

Lemma 5.7. *Let T be an admissible G-set. The functor*

$$
N_U^T : G\mathcal{S}_U \to G\mathcal{S}_U
$$

preserves weak equivalences between cofibrant objects.

Proof. This is a consequence of the fact that ι_H^* preserves cofibrant orthogonal G-spectra [Mandell and May 2002, Lemma V.2.2], colimits, and the identification

$$\iota_H^* \mathscr{L}_U(1)_+ \wedge X \cong \mathscr{L}_{\iota_H^* U}(1)_+ \wedge (\iota_H^* X). \qquad \square$$

Furthermore, we can also identify the interaction of N^T with the cartesian product.

Lemma 5.8. *When T_1 and T_2 are admissible G-sets and M is a cofibrant object in $G\mathcal{S}[\mathbb{L}_U]$, there is a natural weak equivalence*

$$N_U^{T_1 \times T_2} M \simeq N_U^{T_1}(N_U^{T_2} M).$$

Proof. This follows from Lemma 5.7, Lemma 5.4, and Theorem 4.20. $\qquad \square$

Proposition 4.24 shows that the norm functors compose as expected, and it is clear that for $H_1 \subseteq H_2 \subseteq G$, $\iota_{H_1}^* \cong \iota_{H_1}^* \iota_{H_2}^*$.

Theorem 5.9. *Fix $K \subseteq H$, let T be an admissible H-set, and let M be a cofibrant object in $G\mathcal{S}[\mathbb{L}_U]$. The composite $\iota_K^* N_U^T M$ is naturally equivalent to $N_U^{\iota_K^* T} \iota_K^* M$.*

Proof. This again follows from Theorem 4.20 and the fact that the desired equivalence holds for the norm in orthogonal spectra. $\qquad \square$

When R is no longer necessarily the sphere, we have corresponding analogues of the preceding results; we summarize the situation in the following theorem.

Theorem 5.10. *Let R be a cofibrant object in $G\mathcal{S}_U[\mathbb{P}]$.*

(1) *The functor ${}_R N_U^T$ preserves weak equivalences between cofibrant objects.*

(2) *When T_1 and T_2 are admissible G-sets and M is a cofibrant object in $\mathcal{M}_{R,U}$, there is a natural equivalence*

$${}_R N_U^{T_1 \times T_2} M \simeq {}_R N_U^{T_1}({}_R N_U^{T_2} M).$$

(3) *For $H_1 \subseteq H_2 \subseteq G$, there is a natural homeomorphism*

$$\iota_{H_1}^* \cong \iota_{H_1}^* \iota_{H_2}^*.$$

(4) *For $K \subset H$ and T an admissible G-set, there is a natural equivalence*

$$\iota_{K\ R}^* N_U^T M \simeq {}_R N_U^{\iota_K^* T} \iota_K^* M$$

when M is a cofibrant object in $\mathcal{M}_{R,U}$.

Proof. The first of these follows from Theorem 4.27. The second is a consequence of Theorem 4.20; the proof is analogous to the proof of Lemma 5.8, along with the observation that the smash product defining the relative norm computes the derived smash product under our hypotheses. The third is immediate. For the fourth, we can leverage the absolute result as follows.

Since ι_K^* is a strong symmetric monoidal functor, we have the homeomorphisms

$$\iota_K^* {}_R N_U^T M \cong \iota_K^* (N_U^T M \wedge_{N_U^T R} R) \cong (\iota_K^* N_U^T M) \wedge_{\iota_K^* N_U^T R} \iota_K^* R.$$

By Theorem 5.9, we know that

$$\iota_K^* N_U^T M \simeq N^{\iota_K^* T} \iota_K^* M.$$

Moreover, since N_U^T is a left adjoint on commutative rings, we have an homeomorphism

$$\iota_K^* N_U^T R \cong N_{\iota_H^* U}^{\iota_H^* T} \iota_H^* R,$$

which is compatible with the counit $N_U^T R \to R$ used in the formation of the relative smash product. Since the hypotheses guarantee we are computing the derived smash product, we end up with a natural weak equivalence

$$\iota_K^* {}_R N_U^T M \simeq N^{\iota_K^* T} \iota_K^* M \wedge_{N_{\iota_H^* U}^{\iota_H^* T} \iota_H^* R} \iota_H^* R \cong {}_R N_U^{\iota_K^* T} \iota_K^* M. \qquad \square$$

5.3. The multiplicative structure on N_∞ algebras. In this subsection, we explain how the $\mathscr{L}_U$-symmetric monoidal structure on $G\mathcal{S}_U$ induces additional multiplicative structure on objects of $G\mathcal{S}_U[\mathbb{P}]$. Of course, Theorem 3.18 implies that an object of $G\mathcal{S}_U[\mathbb{P}]$ is an N_∞ algebra structured by the equivariant linear isometries operad determined by U, and [Blumberg and Hill 2015, Theorem 6.11] explains the extra structure this gives. Our purpose here is to demonstrate that this structure is essentially an immediate consequence of Theorem 5.10.

Let R be a cofibrant object of $G\mathcal{S}_U[\mathbb{P}]$. The adjunction of Theorem 4.25 yields homotopical counit maps

$$N_U^{G/H} = N_{H, \iota_H^*}^G \iota_H^* R \to R$$

for admissible G/H, which clearly induce natural maps

$$N_U^T R \to R \quad \text{and} \quad G_+ \wedge_H N_U^S \iota_K^* R \to R,$$

for admissible G-sets T and admissible $K \subseteq G$ sets S. The argument of [Blumberg and Hill 2015, Theorem 6.8] extends without change to produce a map

$$N_U^T R \to N_U^S R$$

given any G-map $f : S \to T$.

It is clear from the definition of N_U^T that the diagram

$$N_U^{S \sqcup T} R \cong N_U^S R \wedge N_U^T R \longrightarrow R \wedge R$$
$$\downarrow \qquad \qquad \swarrow$$
$$R$$

commutes. Assertion (2) of Theorem 5.10 implies that the diagram

$$
\begin{array}{ccc}
N_U^{S \times T} R \cong N_U^S N_U^T R & \longrightarrow & N_U^T R \\
\downarrow & \swarrow & \\
R & &
\end{array}
$$

commutes. Finally, assertion (4) of Theorem 5.10 implies that for any admissible sets S and T such that for some $K \subseteq G$ we have $\iota_K^* S \cong \iota_K^* T$, the diagram

$$
\begin{array}{ccc}
\iota_K^* N_U^S R \cong N_{\iota_K^* U}^{\iota_K^* S} \iota_K^* R & \xrightarrow{\ \cong\ } & N_{\iota_K^* U}^{\iota_K^* T} \iota_K^* R \cong \iota_K^* N_U^T R \\
& \searrow \qquad \swarrow & \\
& R &
\end{array}
$$

commutes. Thus, we precisely recover the characterizations of [Blumberg and Hill 2015, Theorem 6.11].

6. Examples and applications

We close with several examples in which the technology in this paper can be used to construct symmetric monoidal structures on categories of equivariant modules. The most basic example comes from algebras over the nonequivariant E_∞ operad regarded as a G-operad with trivial action. Algebras over this operad have no multiplicative norms, and so their modules cannot be described in terms of the usual symmetric monoidal model structure on G-spectra. In particular, we do not get a G-symmetric monoidal category of modules. But since this operad can be modeled by the (nonequivariant) linear isometries operad, Theorem 1.1 implies that we can produce a symmetric monoidal category of modules.

More generally, there are many examples that arise when studying smashing Bousfield localization in the equivariant setting. The examples in the first family we study in Sections 6.1 and 6.2 below (generalizing the trivial E_∞ operad) are necessary ingredients in the work of Greenlees and Shipley [2018; 2014] on monoidal equivalences between various models for rational G-spectra. The second class of examples, studied in Sections 6.3 and 6.4 below, is relevant to understanding chromatic localizations in the equivariant setting.

The phenomenon generating all of these results is the following theorem of Hill and Hopkins [2016].

Theorem 6.1. *Let $\mathcal{O}$ be an $\mathcal{N}_\infty$ operad, and let $\underline{\mathcal{C}}_\mathcal{O}$ denote the associated indexing system. Let L be a Bousfield localization on the category $G\mathcal{S}$ and let $\underline{\mathcal{Z}}$ denote the coefficient system of acyclics for L (i.e., the value at G/H is the subcategory of the homotopy category of $H\mathcal{S}$ consisting of those H-spectra which are acyclic for the*

restriction of L). Then if $\underline{\mathcal{Z}}$ is closed under the (derived) norms specified by $\underline{\mathcal{C}}_{\mathcal{O}}$, L preserves $\mathcal{O}$-algebras.

In particular, this theorem reduces questions about what structure a localization preserves to determining categorical structure on the categories of acyclics.

6.1. *Isotropic localization.* As was first observed by McClure [1996], the localization which nullifies anything induced does not preserve genuine equivariant commutative rings (e.g., algebras over the linear isometries operad for a complete universe U). In particular, $\Sigma^\infty \tilde{E}\mathcal{P}$ cannot be made into a genuine equivariant commutative ring spectrum: since the restriction to any proper subgroup of $\Sigma^\infty \tilde{E}\mathcal{P}$ is contractible, then the putative counit map determined by the commutative ring structure

$$N_H^G i_H^* \Sigma^\infty \tilde{E}\mathcal{P} \to \Sigma^\infty \tilde{E}\mathcal{P}$$

cannot be unital.

More generally, we can apply Theorem 6.1 to produce immediate strengthenings of this observation. Let $\mathcal{F}$ be a family of subgroups of G. For any $\mathcal{F}$ there exists a smashing localization $L_\mathcal{F}$ which nullifies any G-spectrum with isotropy in $\mathcal{F}$. The canonical localization sequence is then precisely the isotropy separation sequence:

$$E\mathcal{F}_+ \wedge X \to X \to \tilde{E}\mathcal{F} \wedge X.$$

Proposition 6.2. *Let $\mathcal{F}$ be a family of subgroups of G which is not the trivial family. Then $\tilde{E}\mathcal{F}$ is not a genuine equivariant commutative ring spectrum. (It is however always a naïve E_∞ ring spectrum.)*

Proof. If $\mathcal{F}$ is nontrivial, then $i_e^* \Sigma^\infty \tilde{E}\mathcal{F}$ is contractible. The argument above now shows that if $\Sigma^\infty \tilde{E}\mathcal{F}$ had a genuine equivariant commutative ring structure, then the absolute norm would factor through the zero ring; we arrive at the same contradiction as above. The final observation is always satisfied by Bousfield localizations. $\square$

Proposition 6.2 shows that the category of local objects (equivalently, the category of modules over $\Sigma^\infty \tilde{E}\mathcal{F}$) cannot be given a symmetric monoidal structure when working with the symmetric monoidal category of orthogonal G-spectra. In contrast, using Theorem 1.1 above, we can obtain a symmetric monoidal category of modules.

Corollary 6.3. *For any family $\mathcal{F}$ of subgroups of G, the category of local spectra for $L_\mathcal{F}$ can always be modeled by a symmetric monoidal category.*

More interestingly, we can describe localizations that result in richer equivariant structures on categories of local objects (i.e., modules).

Theorem 6.4. *Let $\mathcal{F}$ be a family of subgroups of G. Let $\mathscr{L}_U$ be such that for all admissible H/K and for H' in the family, the isotropy of*

$$N_K^H i_K^* (\Sigma_+^\infty H/H') = \Sigma_+^\infty \operatorname{Map}_K(H, H/H')$$

is in $\mathcal{F}$. Then $\Sigma^\infty \tilde{E}\mathcal{F}$ is a $\mathscr{L}_U$-algebra and its category of modules is a $\mathscr{L}_U$-symmetric monoidal category.

This provides a very satisfying sanity check. If N is a normal subgroup of G and if $\mathcal{F}_N$ is the family of subgroups which do not contain N, then there is a composite Quillen equivalence

$$\Sigma^\infty \tilde{E}\mathcal{F}_N\text{-Mod} \leftrightarrows (G/N)\mathcal{S},$$

where the right adjoint is essentially just the N-fixed points (e.g., see [Greenlees and Shipley 2014, Propositions 3.2, 3.3]). The target is a $\underline{\operatorname{Set}}^{G/N}$-monoidal category as recalled in Section 5.1 above. Our work can be used to promote this Quillen equivalence to a structured equivalence via the following result.

Corollary 6.5. *Let N be a normal subgroup of G, and let $\mathcal{F}_N$ denote the family of subgroups of G which do not contain N. Then the category of $\Sigma^\infty \tilde{E}\mathcal{F}_N$-modules can be modeled as a $\underline{\operatorname{Set}}^{G/N}$-symmetric monoidal category.*

6.2. *Idempotent splittings of the sphere spectrum.*

Dress studied idempotent elements in the Burnside ring and established a decomposition of the sphere spectrum $\mathbb{S}$ as the product of localizations $\mathbb{S}[e_L^{-1}]$, where e_L is a primitive idempotent corresponding to a perfect subgroup $L \subset G$. A natural problem is to describe the $\mathcal{N}_\infty$ structures on each term in the product. In his thesis, Böhme [2019] solves this problem and establishes that $\mathbb{S}[e_L^{-1}]$ is an $\mathcal{N}_\infty$ algebra structured by an operad corresponding to an explicitly described indexing system $\mathcal{O}_L$ determined by L. When $\mathcal{O}_L$ corresponds to a linear isometries operad, the main results in this paper now permit the construction of module categories over these localizations equipped with $\mathcal{O}_L$-monoidal structures.

An interesting specialization of these results shows that the idempotent splitting for the rational equivariant sphere $\mathbb{S}_\mathbb{Q}$ consists of terms which do not possess any norms, i.e., algebras over the nonequivariant E_∞ operad regarded as a G-operad. This situation is closely related to the failure of the algebraic model for rational G-spectra obtained in [Kędziorek 2017] to capture multiplicative norms, as explained in [Barnes et al. 2019].

6.3. *A nonflat arithmetic localization.*

Even various kinds of arithmetic localizations of equivariant commutative rings can have counterintuitive properties. This shows that care must be taken with the ways one might consider Zariski localizations of commutative rings. We include a basic, somewhat surprising, example here.

If $\underline{R}$ is a Green functor for a finite group G, then given any collection of elements

$$\{a_i \in \underline{R}(G/H_i) \mid i \in \mathcal{I}\},$$

we can form a new Green functor $\underline{R}[a_i^{-1}]$ which is initial amongst all Green functors under $\underline{R}$ in which all of the a_i become units [Blumberg and Hill 2018]. There are several ways to form this, but the simplest is by mirroring a classical algebra construction. Recall that the forgetful functor

$$\mathrm{u} : \mathcal{G}reen^G \to \mathcal{M}ackey^G$$

has a left-adjoint, the symmetric algebra functor Sym. Recall also that the covariant functors

$$\underline{M} \mapsto \underline{M}(T)$$

are representable for any finite G-set T, with representing object $\underline{A}_T$.

Definition 6.6 [Blumberg and Hill 2018, Definition 5.4]. For any finite G-set T, let

$$\underline{A}[x_T] = \mathrm{Sym}(\underline{A}_T).$$

The Green functors $\underline{A}[x_T]$ represent the functors

$$\underline{R} \mapsto \underline{R}(T),$$

and hence act like polynomial rings.

We restrict attention now to $G = C_2$, inverting the element 2 in the underlying ring. Here, we can take advantage of the explicit descriptions of the free Green functors from [Blumberg and Hill 2017, Lemma 3.2].

Definition 6.7. For $G = C_2$, let $\underline{A}\left[\frac{1}{2}_e\right]$ be given by the pushout in commutative Green functors

$$
\begin{array}{ccc}
\underline{A}[x_{C_2}] & \xrightarrow{\ 2x\ } & \underline{A}[x_{C_2}] \\
\downarrow{\scriptstyle 1_e} & & \downarrow \\
\underline{A} & \longrightarrow & \underline{A}\left[\frac{1}{2}_e\right].
\end{array}
$$

Here the map labeled $2x$ is the map adjoint to the element

$$2x \in \underline{A}[x_{C_2}](C_2) = \mathbb{Z}[x, \bar{x}],$$

where $\bar{x}$ is the Weyl conjugate of x. The map labeled 1_e is the map adjoint to the element

$$1 \in \underline{A}(C_2) = \mathbb{Z}.$$

Proposition 6.8. *The Green functor $\underline{A}\big[\frac{1}{2}_e\big]$ splits*:

$$\underline{A}\big[\tfrac{1}{2}_e\big] \cong \underline{I} \times \underline{\mathbb{Z}\big[\tfrac{1}{2}\big]},$$

where $\underline{I}$ is the augmentation ideal of the Burnside Mackey functor and where $\underline{\mathbb{Z}\big[\tfrac{1}{2}\big]}$ is the constant Mackey functor with value $\mathbb{Z}\big[\tfrac{1}{2}\big]$.

Proof. The underlying ring for $\underline{A}\big[\frac{1}{2}_e\big]$ is just $\mathbb{Z}\big[\frac{1}{2}\big]$, by construction. The fixed points are more interesting, however (and are non-Noetherian!). These are the subrings of

$$\mathbb{Z}\big[\tfrac{1}{2}\big][t]/t^2 - 2t$$

which give an integer when evaluated at $t = 0$. The restriction sends t to 2 and the transfer sends 1 to t. In the fixed ring, there are two orthogonal idempotents:

$$e = \tfrac{1}{2}t, \quad 1 - e,$$

and these split the fixed points into the product of rings

$$\underline{A}\big[\tfrac{1}{2}_e\big](C_2/C_2) \cong \mathbb{Z} \times \mathbb{Z}\big[\tfrac{1}{2}\big].$$

The projection onto $\mathbb{Z}$ is given by multiplication by $1 - e$, and hence this factor restricts to zero. The projection onto $\mathbb{Z}\big[\frac{1}{2}\big]$ is given by multiplication by e, and then restricts isomorphically onto $\mathbb{Z}\big[\frac{1}{2}\big]$. Similarly, the transfer of the element 1 is the element $t = 2e$, and hence lands in the factor $\mathbb{Z}\big[\frac{1}{2}\big]$. This gives our splitting in Green functors. $\square$

Remark 6.9. We can use these two idempotents to split C_2-Mackey functors provided 2 is inverted in the underlying ring. This is a weaker condition than 2 being inverted in the fixed points, and so can be viewed as a more general form of the rational splittings above.

Proposition 6.10. *There is no Tambara functor structure on $\underline{A}\big[\frac{1}{2}_e\big]$ such that the unit map $\underline{A} \to \underline{A}\big[\frac{1}{2}_e\big]$ is a map of Tambara functors.*

Proof. The element $2 \in \mathbb{Z} \cong \underline{A}(C_2)$ has norm

$$N_e^{C_2}(2) = 2 + t.$$

If the unit is a map of Tambara functors, then this maps to the element $2 + t$, which in our splitting of rings is the pair

$$(2, 4) \in \mathbb{Z} \times \mathbb{Z}\big[\tfrac{1}{2}\big].$$

Since the norm is a map of multiplicative monoids, if 2 is inverted, then this must be a unit, and we have reached a contradiction. $\square$

We can mirror all of this in spectra. The role of Sym is just the free E_∞ ring spectrum $\mathbb{P}$, and the representables $\underline{A}_T$ are just $\Sigma^\infty_+ T$. This lets us easily describe the result of inverting an element in the underlying homotopy.

Definition 6.11. Let $S^0\big[\frac{1}{2}_e\big]$ be the pushout in E_∞-ring spectra

$$
\begin{array}{ccc}
\mathbb{P}(C_{2+}) & \xrightarrow{\;2x_e\;} & \mathbb{P}(C_{2+}) \\
{\scriptstyle 1_e}\big\downarrow & & \big\downarrow \\
S^0 & \xrightarrow{\hspace{2cm}} & S^0\big[\frac{1}{2}_e\big]
\end{array}
$$

Proposition 6.12. *The E_∞-ring spectrum $S^0\big[\frac{1}{2}_e\big]$ cannot be made into a commutative ring spectrum.*

Proof. Since the sphere spectrum and $\mathbb{P}(C_2)$ are both (-1)-connected, the zeroth homotopy Green functor of $S^0\big[\frac{1}{2}_e\big]$ is just the pushout of corresponding diagram after applying $\underline{\pi}_0$ levelwise. This is the diagram in Definition 6.7. Proposition 6.10 shows that this has no Tambara functor structure, and so by work of Brun [2007], this shows that $S^0\big[\frac{1}{2}_e\big]$ cannot be a commutative ring spectrum. $\square$

However, Theorem 1.1 guarantees that we again have a good, symmetric monoidal category of modules for $S^0\big[\frac{1}{2}_e\big]$.

6.4. *Chromatic localization.* The localization in the previous section can be thought of as a localization which nullifies the spectrum $C_{2+} \wedge M(\mathbb{Z}/2)$. In other words, it is a kind of chromatic localization. Work of Balmer and Sanders [2017] and of Barthel, Hausmann, Naumann, Nikolaus, Noel, and Stapleton [Barthel et al. 2019] has (up to a small ambiguity) classified the triangulated subcategories of $G\mathcal{S}$. These are determined by the topology on the spectrum (in the sense of Balmer [2005]) of $G\mathcal{S}$: triangulated subcategories of $G\mathcal{S}$ are in bijective correspondence with Thomason subsets of the spectrum, i.e., the subsets which are a union of closed subsets with quasicompact complement. Balmer and Sanders showed that the prime ideals are exactly the inverse images under various geometric fixed points functors of the classical Devinatz–Hopkins–Smith type n-spectra.

Given a Thomason subset V, let L_V denote the associated localization nullifying the triangulated subcategory associated to V. Theorem 6.1 above specifies when L_V preserves equivariant multiplicative structures (and a complete classification of such localizations is forthcoming), so we single out a particular case of interest.

Fix a prime p such that $p\,|\,|G|$ and let $(G\mathcal{S})_p$ denote the category $G\mathcal{S}$ localized at p. Let $V_{n,G}$ denote the triangulated subcategory of $(G\mathcal{S})_p$ generated by $G_+ \wedge M(n)$, where $M(n)$ is any type n-spectrum.

Proposition 6.13. *The localization $L_{V_{n,G}}$ does not preserve genuine equivariant commutative ring spectra.*

Proof. Everything in the triangulated category $V_{n,G}$ has the property that the geometric fixed points are contractible. However, the diagonal map provides an isomorphism in the derived category

$$E \cong \Phi^G N_e^G E$$

for any spectrum E. In particular, taking

$$E = i_e^* G_+ \wedge M(n) \simeq \bigvee_{|G|} M(n)$$

shows that the geometric fixed points of the norm of the generator of the acyclics is not acyclic. □

In particular, there is little hope for any of the equivariant chromatic categories to be G-symmetric monoidal categories. Once again, Theorem 1.1 above guarantees that we can construct models that are symmetric monoidal categories, however, work of the second author builds on this in several other examples [Hill 2018].

Appendix A: The equivariant linear isometries operad

In this section, we collect some technical results about the behavior of the equivariant linear isometries operad.

Lemma A.1. *Let U be any G-universe. If T is a nonempty admissible set for $\mathscr{L}(U)$, then there is a G-equivariant homeomorphism*

$$\mathbb{R}\{T\} \otimes U \to U.$$

Proof. By definition of admissibility, for the linear isometries operad we have an equivariant embedding

$$\mathbb{R}\{T\} \otimes U \to U.$$

This implies that every isomorphism class of representations in $\mathbb{R}\{T\} \otimes U$ is contained in U. The inclusion of a trivial summand in $\mathbb{R}\{T\}$ (which exists since T is nonempty) guarantees that every irreducible representation of U is also in $\mathbb{R}\{T\} \otimes U$. □

Lemma A.2. *The orbit space $\mathscr{L}_U(2)/(\mathscr{L}_U(1) \times \mathscr{L}_U(1))$ consists of a single point. More generally, the orbit space $\mathscr{L}_U(n)/\mathscr{L}_U(1)^{\times n}$ consists of a single point.*

Proof. The right action map $\mathscr{L}_U(2) \times \mathscr{L}_U(1) \times \mathscr{L}_U(1) \to \mathscr{L}_U(2)$ is clearly a map of G-spaces. As a consequence, we can compute the orbit space as the colimit of underlying spaces, and so in this case the result follows from the nonequivariant identification of the orbit space [Elmendorf et al. 1997, I.8.1].

We deduce the general case by induction: We can use Theorem 3.7 to write

$$\mathscr{L}_U(n)/\mathscr{L}_U(1)^{\times n} \cong \big(\mathscr{L}_U(2) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} (\mathscr{L}_U(1) \times \mathscr{L}_U(n-1)) \big)/\mathscr{L}_U(1)^{\times n}.$$

Since coequalizers commute, the result for n now follows from the base case $n = 2$ and the induction hypothesis. $\qquad\square$

More generally, we have the following result.

Lemma A.3. *The orbit space* $\mathscr{L}(\mathrm{Ind}_H^G \widehat{U}, U)/F_H(G, \mathscr{L}_{\widehat{U}}(1))$ *consists of a single point.*

Proof. As in the proof of Lemma A.2, since the action map

$$\mathscr{L}(\mathrm{Ind}_H^G \widehat{U}, U) \times F_H(G, \mathscr{L}_{\widehat{U}}(1)) \to \mathscr{L}(\mathrm{Ind}_H^G \widehat{U}, U)$$

is a map of G-spaces, it suffices to compute the orbit space in terms of the colimit of the underlying spaces. In this case, we can deduce the result from Lemma A.2. $\qquad\square$

We also have a series of generalizations of [Elmendorf et al. 1997, I.5.4].

Lemma A.4. *Let T and T' be nonempty admissible sets for U. There are natural isomorphisms*

$$\mathscr{L}(\mathbb{R}\{T\} \otimes \widehat{U} \oplus \mathbb{R}\{T'\} \otimes \widehat{U}, U)$$
$$\cong \mathscr{L}_U(2) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} \big(\mathscr{L}(\mathbb{R}\{T\} \otimes \widehat{U}, U) \times \mathscr{L}(\mathbb{R}\{T'\} \otimes \widehat{U}, U) \big).$$

Proof. First, observe that it suffices to show that nonequivariantly this isomorphism arises from a reflexive coequalizer diagram. Now using Lemma A.1 to choose isomorphisms $\widehat{U} \otimes \mathbb{R}\{T'\} \cong \widehat{U}$, the required nonequivariant splittings arise just as in the proof of [Elmendorf et al. 1997, I.5.4]. $\qquad\square$

A particularly useful corollary of Lemma A.4 is the following:

Corollary A.5. *There is a natural isomorphism*

$$\mathscr{L}(\mathrm{Ind}_H^G \widehat{U} \oplus \mathrm{Ind}_H^G \widehat{U}, U) \cong \mathscr{L}_U(2) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} \big(\mathscr{L}(\mathrm{Ind}_H^G \widehat{U}, U) \times \mathscr{L}(\mathrm{Ind}_H^G \widehat{U}, U) \big).$$

We also have another kind of analogue of [Elmendorf et al. 1997, I.5.4].

Lemma A.6. *Let T be a nonempty admissible set for U. Then there is a natural isomorphism*

$$\mathscr{L}\big((\mathbb{R}\{T\} \otimes \widehat{U}) \oplus (\mathbb{R}\{T\} \otimes \widehat{U}), U \big) \cong \mathscr{L}(\mathbb{R}\{T\} \otimes \widehat{U}, U) \times_{\mathscr{L}_{\widehat{U}}(1)^T} \mathscr{L}_{\widehat{U}}(2)^T.$$

Proof. Again, the result follows by producing a reflexive coequalizer after forgetting the G-action. Specifically, we need to show that the diagram

$$\mathcal{L}(\mathbb{R}\{T\} \otimes \widehat{U}, U) \times \mathcal{L}_{\widehat{U}}(1)^T \times \mathcal{L}_{\widehat{U}}(2)^T$$

$$\Downarrow$$

$$\mathcal{L}(\mathbb{R}\{T\} \otimes \widehat{U}, U) \times \mathcal{L}_{\widehat{U}}(2)^T$$

$$\downarrow$$

$$\mathcal{L}((\mathbb{R}\{T\} \otimes \widehat{U}) \oplus (\mathbb{R}\{T\} \otimes \widehat{U}), U)$$

is a reflexive coequalizer. Choosing $|T|$ isomorphisms $h_i : \widehat{U}^2 \cong \widehat{U}$ such that the sum assembles to an isomorphism $h : \mathbb{R}\{T\} \otimes \widehat{U} \oplus \mathbb{R}\{T\} \otimes \widehat{U} \cong \mathbb{R}\{T\} \otimes \widehat{U}$, we can define the splitting map

$$\mathcal{L}\big((\mathbb{R}\{T\} \otimes \widehat{U}) \oplus (\mathbb{R}\{T\} \otimes \widehat{U}), U\big) \to \mathcal{L}(\mathbb{R}\{T\} \otimes \widehat{U}, U) \times \mathcal{L}_{\widehat{U}}(2)^T$$

via $f \mapsto (f \circ h, h_1, h_2, \ldots, h_{|T|})$. The argument now proceeds exactly as in [Elmendorf et al. 1997, I.5.4]. $\square$

This has the following corollary.

Corollary A.7. *For $H \subseteq G$, there is a natural isomorphism*

$$\mathcal{L}(\operatorname{Ind}_H^G \widehat{U} \oplus \operatorname{Ind}_H^G \widehat{U}, U) \cong \mathcal{L}(\operatorname{Ind}_H^G \widehat{U}, U) \times_{F_H(G, \mathcal{L}_{\widehat{U}}(1))} F_H(G, \mathcal{L}_{\widehat{U}}(2)).$$

Finally, we turn to the main technical theorem about the equivariant linear isometries operad that justifies the use of the unital objects. In the proof, we make use of the following standard technical lemma:

Lemma A.8. *Let X have a left H-action and right G-action which are compatible (i.e., X is an $H \times G$-space). Then the coequalizer*

$$(-) \times_H X$$

specifies a functor from the category of $G' \times H$-spaces and equivariant maps to $G' \times G$-spaces and equivariant maps.

Proof. Let Y be a $G' \times H$-space. It is clear that $Y \times_H X$ has a $G' \times G$ action inherited from the G'-action on Y and the G-action on X. Let $f : Y \to Y'$ be a map of $G' \times H$-spaces. Then there is an induced map of spaces

$$\theta_f : Y \times_H X \to Y' \times_H X$$

defined by $(y, x) \mapsto (f(y), x)$. This is a left G'-map since f is a $G' \times H$-map; $\theta_f((g'y, x)) = (f(g'y), x) = (g'f(y), x) = g'\theta_f((y, x))$. Similarly, it is a right G-map. $\square$

Theorem A.9. *For each $k > 0$, the map*

$$\gamma_k : \hat{\mathscr{L}}_U(k) = \mathscr{L}_U(2) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} (\mathscr{L}_U(0) \times \mathscr{L}_U(k)) \to \mathscr{L}_U(k)$$

induced by the operadic structure map

$$\mathscr{L}_U(2) \times \mathscr{L}_U(0) \times \mathscr{L}_U(k) \to \mathscr{L}_U(k)$$

is a homotopy equivalence of $G \times \Sigma_k$-spaces.

Proof. First, consider the case where $k = 1$. In this case, we are considering the map

$$\gamma_1 : \mathscr{L}_U(2) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} (\mathscr{L}_U(0) \times \mathscr{L}_U(1)) \to \mathscr{L}_U(1).$$

The proof of [Elmendorf et al. 1997, XI.2.2] goes through in the equivariant context to show that γ_1 is a homotopy equivalence of G-spaces. It is helpful to decompose γ_1 as follows [Mandell and May 2002, §VI.6]:

$$\mathscr{L}_U(2) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} (\mathscr{L}_U(0) \times \mathscr{L}_U(1)) \xrightarrow{\theta_1} \mathscr{L}_U(2)/\mathscr{L}_U(1) \xrightarrow{\theta_2} \mathscr{L}_U(1),$$

where $\mathscr{L}_U(2)/\mathscr{L}_U(1)$ is the orbit space for the right action of $\mathscr{L}_U(1)$ on $\mathscr{L}_U(2)$ given by $(f, h) \mapsto f \circ (h \oplus \mathrm{id})$ and equipped with the right action of $\mathscr{L}_U(1)$ specified by $([f], h) \mapsto [f \circ (\mathrm{id} \oplus h)]$, θ_2 is the restriction to the second summand, and θ_1 is specified by $(g, 0, f) \mapsto g \circ (\mathrm{id} \oplus f)$. Both maps are $G \times \mathscr{L}_U(1)$-maps, and θ_1 is a homeomorphism.

Now take $k > 1$. Then γ_k factors as the composite

$$\hat{\mathscr{L}}_U(k) \cong (\mathscr{L}_U(2)/\mathscr{L}_U(1)) \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k) \to \mathscr{L}_U(1) \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k) \cong \mathscr{L}_U(k),$$

induced by γ_1, where we are using the homeomorphism

$$\hat{\mathscr{L}}_U(k) \cong (\mathscr{L}_U(2) \times_{\mathscr{L}_U(1) \times \mathscr{L}_U(1)} (\mathscr{L}_U(0) \times \mathscr{L}_U(1))) \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k).$$

To see this, observe that $\gamma_k((g, 0, f)) = g \circ (f \oplus 0)$. On the other hand, the composite above first takes $(g, 0, f)$ to $((g \circ \mathrm{id}), f)$, then $((g \circ \mathrm{id}), f)$ to $(\theta_2(g), f)$, and finally $(\theta_2(g), f)$ to $\theta_2(g) \circ f = g \circ (f \oplus 0)$.

Since $\mathscr{L}_U(k)$ is a universal space for the family of subgroups of $G \times \Sigma_k$ prescribed by U, it suffices to show that $(\mathscr{L}_U(2)/\mathscr{L}_U(1)) \times_{\mathscr{L}_U(1)} \mathscr{L}_U(k)$ is also a universal space for the same family. To do this, we will unpack part of the proof of [Elmendorf et al. 1997, XI.2.2].

Write $U \cong U_1 \oplus U_2$ as G-spaces, where U_1 and U_2 are G-universes such that $U_1 \cong U$ and $U_2 \cong U$; we can do this by Lemma A.1. Define $\mathscr{K}(2) \subset \mathscr{L}_U(2)$ to be $\{f \mid f(\{0\} \oplus U) \subset U_2\}$, equipped with the conjugation G-action. Next, we define

$$\hat{\mathscr{K}}_1 = \mathscr{K}(2)/\mathscr{L}_U(1)$$

and we let $\mathcal{K}_1 \subset \mathcal{L}_U(1)$ be $\{f \mid f(U) \subseteq U_2\}$ with the conjugation G-action. The map θ_2 restricts to give a G-map $\hat{\mathcal{K}}_1 \to \mathcal{K}_1$ which is compatible with the action of $\mathcal{L}_U(1)$ and so by Lemma A.8 we have an induced $G \times \Sigma_k$-map

$$\hat{\mathcal{K}}_1 \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k) \to \mathcal{K}_1 \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k).$$

The nonequivariant argument in [Elmendorf et al. 1997, XI.2.2] extends to the equivariant case to show that the map $\hat{\mathcal{K}}_1 \to \mathcal{K}_1$ is a homeomorphism of G-spaces. On the other hand, we have a homeomorphism of G-spaces $\mathcal{K}_1 \cong \mathcal{L}(U, U_2) \cong \mathcal{L}_U(1)$ which is compatible with the action of $\mathcal{L}_U(1)$, and so Lemma A.8 implies that there is a composite $G \times \Sigma_k$-map

$$\mathcal{K}_1 \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k) \to \mathcal{L}_U(1) \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k) \cong \mathcal{L}_U(k)$$

which is a homeomorphism. Putting these together, we have a $G \times \Sigma_k$-map

$$\hat{\mathcal{K}}_1 \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k) \cong \mathcal{L}_U(k)$$

that is a homeomorphism.

To finish the argument, observe that the proof in [Elmendorf et al. 1997, XI.2.2] extends to the equivariant context to show that the inclusion

$$\mathcal{K}(2) \to \mathcal{L}_U(2)$$

is a G-homotopy equivalence of right $\mathcal{L}_U(1) \times \mathcal{L}_U(1)$-spaces and therefore

$$\hat{\mathcal{K}}_1 \to (\mathcal{L}_U(2)/\mathcal{L}_U(1))$$

is a G-homotopy equivalence of right $\mathcal{L}_U(1)$-spaces. As a consequence, the induced map

$$\hat{\mathcal{K}}_1 \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k) \to (\mathcal{L}_U(2)/\mathcal{L}_U(1)) \times_{\mathcal{L}_U(1)} \mathcal{L}_U(k)$$

is a $G \times \Sigma_k$-homotopy equivalence. $\square$

Appendix B: Compact Lie groups

In this appendix, we quickly outline what aspects of our work in this paper continue to hold when G is an infinite compact Lie group. Basically, all of the foundational material in this paper goes through except the results on multiplicative norms; when G is an infinite compact Lie group, norms exist only for subgroups H of finite index and hence we can only work with admissible finite sets. With this modification, the theorems of the paper remain true.

To be more precise, the work of the paper depends on various results about the linear isometries operad, mostly collected in Appendix A. Lemma A.1 holds with the same proof for finite G-sets; however, in all of our applications of Lemma A.1, this case suffices. Lemmas A.2 and A.3 hold with the same proofs; these arguments

do not rely on the finiteness of G. Lemmas A.4 and A.6 again require finite G-sets, but this suffices to conclude Lemmas A.5 and A.7, respectively. In the body of the paper, Theorem 3.15 goes through with the same proof, as does the essential Theorem A.9.

As a consequence, the work of the remainder of the paper goes through without modification in the arguments except for the material on the norm in Sections 3.5, 4.3, and 5. Here, the results on N_H^G require that G/H be a finite G-set, i.e., that the subgroups have finite index.

Acknowledgments

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Statistics of K-groups modulo p for the ring of integers of a varying quadratic number field

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For each odd prime p, we conjecture the distribution of the p-torsion subgroup of $K_{2n}(\mathcal{O}_F)$ as F ranges over real quadratic fields, or over imaginary quadratic fields. We then prove that the average size of the 3-torsion subgroup of $K_{2n}(\mathcal{O}_F)$ is as predicted by this conjecture.

1. Introduction

The original Cohen–Lenstra heuristics [1984] predicted, for each prime $p \neq 2$, the distribution of the p-primary part of $\mathrm{Cl}(F)$ as F varied over quadratic fields of a given signature. More recent work developed heuristics for other families of groups, including class groups of higher degree number fields [Cohen and Martinet 1990], Picard groups of function fields [Friedman and Washington 1989], Tate–Shafarevich groups of elliptic curves [Delaunay 2001; 2007; Delaunay and Jouhet 2014] Selmer groups of elliptic curves [Poonen and Rains 2012; Bhargava et al. 2015], and Galois groups of nonabelian unramified extensions of number fields and function fields [Boston et al. 2017; Boston and Wood 2017].

Let $\mathcal{F}$ be a number field. Let $\mathcal{O}_{\mathcal{F}}$ be the ring of integers of $\mathcal{F}$. For $m \geq 0$, the K-group $K_m(\mathcal{O}_{\mathcal{F}})$ is a finitely generated abelian group. It is finite when m is even and positive: see [Weibel 2005, Theorem 7]. Our goal is to study, for a fixed m and odd prime p, the p-torsion subgroup $K_m(\mathcal{O}_{\mathcal{F}})_p$ as $\mathcal{F}$ varies in a family of number fields, always ordered by absolute value of the discriminant. As described in Section 6, $K_m(\mathcal{O}_{\mathcal{F}})_p$ is well understood for odd m. Therefore we focus on the case $m = 2n$. Now suppose that $\mathcal{F}$ is a quadratic field F. The action of $\mathrm{Gal}(F/\mathbb{Q})$ decomposes $K_{2n}(\mathcal{O}_F)_p$ into $+$ and $-$ parts, and we will see that the $+$ part is $K_{2n}(\mathbb{Z})_p$, independent of F. Therefore we focus on the variation of the $-$ part.

A Cohen–Lenstra style heuristic will lead us to the following conjecture, involving the constants

$$\alpha_{p,u,r} := \frac{\prod_{i=r+1}^{\infty}(1-p^{-i})}{p^{r(u+r)}\prod_{i=1}^{r+u}(1-p^{-i})}$$

for nonnegative integers u and r:

Conjecture 1.1. *Fix $n \geq 1$ and an odd prime p and $r \geq 0$. As F ranges over all real (resp. imaginary) quadratic fields, $\mathrm{Prob}(\dim_{\mathbb{F}_p} K_{2n}(\mathcal{O}_F)_p^- = r)$ is given by the table entry in the row determined by the pair of residues $(n \bmod 2, \ n \bmod p-1)$ and column determined by the signature:*

residues	real	imaginary
$(0,0)$	$\frac{1}{2}\left(\frac{p}{p+1}\alpha_{p,2,r-1} + \frac{p+2}{p+1}\alpha_{p,1,r}\right)$	$\frac{1}{2}\left(\frac{p}{p+1}\alpha_{p,1,r-1} + \frac{p+2}{p+1}\alpha_{p,0,r}\right)$
$\left(0, \frac{p-1}{2}\right)$	$\frac{1}{2}\left(\frac{1}{p+1}\alpha_{p,2,r-1} + \frac{2p+1}{p+1}\alpha_{p,1,r}\right)$	$\frac{1}{2}\left(\frac{1}{p+1}\alpha_{p,1,r-1} + \frac{2p+1}{p+1}\alpha_{p,0,r}\right)$
$(0, \text{other})$	$\alpha_{p,1,r}$	$\alpha_{p,0,r}$
$\left(1, \frac{p-1}{2}\right)$	$\frac{1}{2}\left(\frac{1}{p+1}\alpha_{p,1,r-1} + \frac{2p+1}{p+1}\alpha_{p,0,r}\right)$	$\frac{1}{2}\left(\frac{1}{p+1}\alpha_{p,2,r-1} + \frac{2p+1}{p+1}\alpha_{p,1,r}\right)$
$(1, \text{other})$	$\alpha_{p,0,r}$	$\alpha_{p,1,r}$

To pass from the distribution of $\dim_{\mathbb{F}_p} K_{2n}(\mathcal{O}_F)_p^-$ to that of $\dim_{\mathbb{F}_p} K_{2n}(\mathcal{O}_F)_p$ itself, add the constant $\dim_{\mathbb{F}_p} K_{2n}(\mathbb{Z})_p$, which can be expressed in terms of a class group (see Section 5).

Conjecture 1.1 implies an average order for $K_{2n}(\mathcal{O}_F)_p$ as F varies over all real or imaginary fields (see Conjecture 8.8). We prove that this conjectured average order is correct for $p = 3$:

Theorem 1.2. *Fix $n \geq 1$. The average order of $K_{2n}(\mathcal{O}_F)_3$ as F ranges over real (resp. imaginary) quadratic fields is as follows:*

	real	imaginary
n even	$25/12$	$11/4$
n odd	$9/4$	$19/12$

Remark 1.3. By Theorem 5.1, $K_{2n}(\mathbb{Z})_3 = 0$ for all n since $\mathbb{Q}(\zeta_3)$ has class number 1. Thus $K_{2n}(\mathcal{O}_F)_3^- = K_{2n}(\mathcal{O}_F)_3$ for all n.

Remark 1.4. Theorem 1.2 is an analogue of the Davenport–Heilbronn theorem giving the average order of $\mathrm{Cl}(F)_3$ as F varies over all real or imaginary quadratic fields [Davenport and Heilbronn 1971, Theorem 3].

Remark 1.5. After this article was written, the second author proved an analogue of Theorem 1.2 for $K_{2n}(\mathcal{O}_{\mathcal{F}})_2$ as $\mathcal{F}$ varies over cubic fields [Klagsbrun 2017b].

Methods. The p-torsion subgroup G_p of a finite abelian group G has the same $\mathbb{F}_p$-dimension as $G/p := G/pG$; therefore we study $K_{2n}(\mathcal{O}_F)/p$. The latter is isomorphic to an étale cohomology group $H^2_{\text{ét}}(\mathcal{O}_F[1/p], \mu_p^{\otimes(n+1)})$, which we relate to isotypic components of the class group and Brauer group of $\mathcal{O}_E[1/p]$, where $E := F(\zeta_p)$. The Brauer group can be computed explicitly, and we develop heuristics for the class groups; combining these gives the conjectural distribution of $K_{2n}(\mathcal{O}_F)_p$.

In the case $p = 3$, the isotypic components of $\text{Cl}(\mathcal{O}_E[1/p])$ are related to $\text{Cl}(\mathcal{O}_K[1/p])$ for quadratic fields K. The *average order* of the latter class groups can be computed unconditionally by using a strategy of Davenport and Heilbronn, which we refine using recent work of Bhargava, Shankar, and Tsimerman, to control averages in subfamilies with prescribed local behavior at 3. This yields unconditional results on the average order of $K_{2n}(\mathcal{O}_F)_3$.

Prior work. As far as we know, Cohen–Lenstra style conjectures have not been proposed for K-groups before, but some results on the distribution of $K_2(\mathcal{O}_F)$ have been proved.

Guo [2009] proved that 4-ranks of $K_2(\mathcal{O}_F)$ for quadratic fields F follow a Cohen–Lenstra distribution, just as Fouvry and Klüners proved for 4-ranks of $\text{Cl}(\mathcal{O}_F)$ [Fouvry and Klüners 2007]. Studying 4-ranks is natural, since the 2-rank of $K_2(\mathcal{O}_F)$ for a quadratic field F is determined by genus theory just as the 2-rank of $\text{Cl}(\mathcal{O}_F)$ is (see [Browkin and Schinzel 1982], for example).

Similar results on the 3-ranks of $K_2(\mathcal{O}_L)$ for cyclic cubic fields L are due to Cheng, Guo, and Qin [Cheng et al. 2014]. In addition, Browkin showed that Cohen–Martinet heuristics suggest a conjecture on $\text{Prob}(3 \mid \#K_2(\mathcal{O}_F))$ as F ranges over quadratic fields of fixed signature [Browkin 2000].

Notation. If G is an abelian group and $n \geq 1$, let $G_n := \{g \in G : ng = 0\}$ and $G/n := G/nG$. For any k-representation V of a finite group G such that $\text{char } k \nmid \#G$, and for any irreducible k-representation χ of G, let V^χ be the χ-isotypic component.

Throughout the paper, p is an odd prime, $\mathcal{F}$ is an arbitrary number field, $\overline{\mathcal{F}}$ is an algebraic closure of $\mathcal{F}$, the element $\zeta_p \in \overline{\mathcal{F}}$ is a primitive p-th root of unity, and $\mathcal{E} := \mathcal{F}(\zeta_p)$. Later we specialize $\mathcal{F}$ to a quadratic field F and define $E := F(\zeta_p)$.

Let $\mathcal{O}_\mathcal{F}$ be the ring of integers of $\mathcal{F}$. If S is a finite set of places of $\mathcal{F}$ containing all the archimedean places, define the ring of S-integers $\mathcal{O}_S := \{x \in \mathcal{F} : v(x) \geq 0 \text{ for all } v \notin S\}$. Let $d_\mathcal{F} \in \mathbb{Z}$ be the discriminant of $\mathcal{F}$. Let $\mu(\mathcal{F})$ be the group of roots of unity in $\mathcal{F}$.

If $\mathcal{O}$ is a Dedekind ring, let $\text{Cl}(\mathcal{O})$ denote its class group, and let $\text{Br}(\mathcal{O})$ be its (cohomological) Brauer group, defined as $H^2_{\text{ét}}(\text{Spec } \mathcal{O}, \mathbb{G}_m)$ [Poonen 2017, Definition 6.6.4].

From now on, all cohomology is étale cohomology, and we drop the subscript ét.

2. From K-theory to class groups and Brauer groups

In this section, following Tate's argument for K_2 [1976], we relate the even K-groups to more concrete groups: class groups and Brauer groups.

Theorem 2.1 [Weibel 2005, Corollary 71]. *For any number field $\mathcal{F}$ and any $n \geq 1$,*

$$K_{2n}(\mathcal{O}_\mathcal{F})/p \simeq H^2(\mathcal{O}_\mathcal{F}[1/p], \mu_p^{\otimes(n+1)}).$$

Lemma 2.2. *There is a canonical exact sequence*

$$0 \longrightarrow \mathrm{Cl}(\mathcal{O}_\mathcal{F}[1/p])/p \longrightarrow H^2(\mathcal{O}_\mathcal{F}[1/p], \mu_p) \longrightarrow \mathrm{Br}(\mathcal{O}_\mathcal{F}[1/p])_p \longrightarrow 0.$$

Proof. Consider the exact sequence

$$1 \to \mu_p \to \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \to 1$$

of sheaves on $(\mathrm{Spec}\,\mathcal{O}_\mathcal{F}[1/p])_{\text{ét}}$. Take the associated long exact sequence of cohomology, and substitute $H^1(\mathcal{O}_\mathcal{F}[1/p], \mathbb{G}_m) = \mathrm{Pic}(\mathcal{O}_\mathcal{F}[1/p]) = \mathrm{Cl}(\mathcal{O}_\mathcal{F}[1/p])$ and $H^2(\mathcal{O}_\mathcal{F}[1/p], \mathbb{G}_m) = \mathrm{Br}(\mathcal{O}_\mathcal{F}[1/p])$. □

Lemma 2.3. *Let $\mathcal{F}'/\mathcal{F}$ be a finite Galois extension of degree prime to p. Let $i \geq 0$ and $r \in \mathbb{Z}$. Then*

$$H^i(\mathcal{O}_\mathcal{F}[1/p], \mu_p^{\otimes r}) = H^i(\mathcal{O}_{\mathcal{F}'}[1/p], \mu_p^{\otimes r})^{\mathrm{Gal}(\mathcal{F}'/\mathcal{F})}.$$

Proof. In the Hochschild–Serre spectral sequence

$$H^i(\mathrm{Gal}(\mathcal{F}'/\mathcal{F}), H^j(\mathcal{O}_{\mathcal{F}'}[1/p], \mu_p^{\otimes r})) \Longrightarrow H^{i+j}(\mathcal{O}_\mathcal{F}[1/p], \mu_p^{\otimes r}),$$

the groups $H^i(\mathrm{Gal}(\mathcal{F}'/\mathcal{F}), H^j(\mathcal{O}_{\mathcal{F}'}[1/p], \mu_p^{\otimes r}))$ for $i > 0$ are 0 because they are killed by both $\#\mathrm{Gal}(\mathcal{F}'/\mathcal{F})$ and p. □

We now specialize $\mathcal{F}'$ to $\mathcal{E} := \mathcal{F}(\zeta_p)$. The action of $\mathrm{Gal}(\mathcal{E}/\mathcal{F})$ on the p-th roots of 1 defines an injective homomorphism $\chi_1 \colon \mathrm{Gal}(\mathcal{E}/\mathcal{F}) \to (\mathbb{Z}/p\mathbb{Z})^\times$. For $m \in \mathbb{Z}$, composing χ_1 with the m-th power map on $(\mathbb{Z}/p\mathbb{Z})^\times$ yields another 1-dimensional $\mathbb{F}_p$-representation of $\mathrm{Gal}(\mathcal{E}/\mathcal{F})$; call it χ_m.

Lemma 2.4. *There is a split exact sequence*

$$0 \to (\mathrm{Cl}(\mathcal{O}_\mathcal{E}[1/p])/p)^{\chi_{-n}} \to H^2(\mathcal{O}_\mathcal{F}[1/p], \mu_p^{\otimes(n+1)}) \to \left(\mathrm{Br}(\mathcal{O}_\mathcal{E}[1/p])_p\right)^{\chi_{-n}} \to 0.$$

Proof. First, we have

$$
\begin{aligned}
H^2(\mathcal{O}_\mathcal{F}[1/p], \mu_p^{\otimes(n+1)}) &= H^2(\mathcal{O}_\mathcal{E}[1/p], \mu_p^{\otimes(n+1)})^{\mathrm{Gal}(\mathcal{E}/\mathcal{F})} \quad \text{(Lemma 2.3)} \\
&= \left(H^2(\mathcal{O}_\mathcal{E}[1/p], \mu_p) \otimes \mu_p^{\otimes n}\right)^{\mathrm{Gal}(\mathcal{E}/\mathcal{F})} \quad \text{(since } \mu_p \subset \mathcal{E}) \\
&= H^2(\mathcal{O}_\mathcal{E}[1/p], \mu_p)^{\chi_{-n}} \quad \text{(since } \mu_p^{\otimes n} \simeq \chi_n). \qquad (1)
\end{aligned}
$$

On the other hand, Lemma 2.2 for $\mathcal{E}$ yields a sequence of $\mathrm{Gal}(\mathcal{E}/\mathcal{F})$-representations

$$0 \longrightarrow \mathrm{Cl}(\mathcal{O}_{\mathcal{E}}[1/p])/p \longrightarrow H^2(\mathcal{O}_{\mathcal{E}}[1/p], \mu_p) \longrightarrow \mathrm{Br}(\mathcal{O}_{\mathcal{E}}[1/p])_p \longrightarrow 0,$$

which splits by Maschke's theorem. Take χ_{-n}-isotypic components, and substitute (1) in the middle. $\qquad\square$

Substituting Theorem 2.1 into Lemma 2.4 yields the main result of this section:

Theorem 2.5. *For each $n \geq 1$,*

$$K_{2n}(\mathcal{O}_{\mathcal{F}})/p \simeq (\mathrm{Cl}(\mathcal{O}_{\mathcal{E}}[1/p])/p)^{\chi_{-n}} \oplus \left(\mathrm{Br}(\mathcal{O}_{\mathcal{E}}[1/p])_p\right)^{\chi_{-n}}.$$

3. Even K-groups of the ring of integers of a quadratic field

Let $p^* = (-1)^{(p-1)/2} p$, so $\mathbb{Q}(\sqrt{p^*})$ is the degree 2 subfield of $\mathbb{Q}(\zeta_p)$. From now on, F is a degree 2 extension of $\mathbb{Q}$ not equal to $\mathbb{Q}(\sqrt{p^*})$. Thus $F = \mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{Q}^\times$ such that d and p^* are independent in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. Let $E = F(\zeta_p)$. Then

$$\mathrm{Gal}(E/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times \{\pm 1\}.$$

Let τ be the generator of $\mathrm{Gal}(E/\mathbb{Q}(\zeta_p)) = \{\pm 1\}$, so τ restricts to the generator of $\mathrm{Gal}(F/\mathbb{Q})$. The action of τ decomposes $K_{2n}(\mathcal{O}_F)/p$ into $+$ and $-$ eigenspaces. Let $\chi_{-n,-1} \colon G \to \mathbb{F}_p^\times$ be such that $\chi_{-n,-1}|_{\mathrm{Gal}(E/F)} = \chi_{-n}$ and $\chi_{-n,-1}(\tau) = -1$.

Theorem 3.1. *For each $n \geq 1$,*

$$(K_{2n}(\mathcal{O}_F)/p)^+ \simeq K_{2n}(\mathbb{Z})/p,$$
$$(K_{2n}(\mathcal{O}_F)/p)^- \simeq (\mathrm{Cl}(\mathcal{O}_E[1/p])/p)^{\chi_{-n,-1}} \oplus \left(\mathrm{Br}(\mathcal{O}_E[1/p])_p\right)^{\chi_{-n,-1}}.$$

Proof. To obtain the first statement, use Theorem 2.1 to rewrite each term as an étale cohomology group and apply Lemma 2.3 with $\mathcal{F}'/\mathcal{F}$ there being $F/\mathbb{Q}$. To obtain the second, take minus parts in Theorem 2.5. $\qquad\square$

4. Brauer groups

The goal of this section is to determine the rightmost term in Theorem 3.1.

Lemma 4.1 [Poonen 2017, (6.9.5)]. *Let $\mathcal{F}$ and $\mathcal{O}_S$ be in the last section on page 289. Let r_1 be the number of real places of $\mathcal{F}$. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Br}\,\mathcal{O}_S \longrightarrow \left(\tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right)^{r_1} \oplus \bigoplus_{\text{finite } v \in S} \mathbb{Q}/\mathbb{Z} \xrightarrow{\text{sum}} \mathbb{Q}/\mathbb{Z}.$$

Corollary 4.2. *We have $\mathrm{Br}(\mathcal{O}_S)_p \simeq (\mathbb{Z}/p\mathbb{Z})^{\{\text{finite } v \in S\}}_{\text{sum }0}$, where the "sum 0" subscript denotes the subgroup of elements whose sum is 0.*

Corollary 4.3. *We have $\mathrm{Br}(\mathbb{Z}[\zeta_p, 1/p])_p = 0$.*

Proof. There is only one prime above p in $\mathbb{Z}[\zeta_p]$. □

Proposition 4.4. *Let F and E be as in Section 3. Then*

$$(\mathrm{Br}(\mathcal{O}_E[1/p])_p)^{\chi_{-n},-1} = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{if } n \equiv 0 \pmod{p-1} \text{ and } d \in \mathbb{Q}_p^{\times 2}; \\ \mathbb{Z}/p\mathbb{Z}, & \text{if } n \equiv \dfrac{p-1}{2} \pmod{p-1} \text{ and } p^*d \in \mathbb{Q}_p^{\times 2}; \\ 0, & \text{in all other cases.} \end{cases}$$

Proof. The hypothesis implies that F is not the quadratic subfield $\mathbb{Q}(\sqrt{p^*})$ of $\mathbb{Q}(\zeta_p)$. Thus E is the compositum of linearly disjoint extensions $\mathbb{Q}(\zeta_p)$ and F over $\mathbb{Q}$, and $\mathrm{Gal}(E/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times \{\pm 1\}$. Let $\mathfrak{p}$ be a prime of E lying above p. Let $D \leq \mathrm{Gal}(E/\mathbb{Q})$ be the decomposition group of $\mathfrak{p}$. Since p totally ramifies in $\mathbb{Q}(\zeta_p)/\mathbb{Q}$, we have $p - 1 \mid \#D$. Let S_p be the set of primes of E lying above p, so $S_p \simeq \mathrm{Gal}(E/\mathbb{Q})/D$, which by the previous sentence is of size 1 or 2; it is 2 if and only if p splits in one of the quadratic subfields of E. These quadratic subfields are $\mathbb{Q}(\sqrt{p^*})$, F, and the field $F' = \mathbb{Q}(\sqrt{p^*d})$, but p is ramified in $\mathbb{Q}(\sqrt{p^*})$. Thus by Corollary 4.2,

$$\mathrm{Br}(\mathcal{O}_E[1/p])_p = (\mathbb{Z}/p\mathbb{Z})_{\text{sum } 0}^{S_p} = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{if } p \text{ splits in } F \text{ or } F'; \\ 0, & \text{otherwise} \end{cases}$$

as an abelian group, and it remains to determine in the first case which character it is isomorphic to. We will compute the action of $\mathrm{Gal}(E/F) \simeq (\mathbb{Z}/p\mathbb{Z})^\times$ and $\mathrm{Gal}(E/\mathbb{Q}(\zeta_p)) \simeq \{\pm 1\}$ separately.

If p splits in F (that is, $d \in \mathbb{Q}_p^{\times 2}$), then $D = \mathrm{Gal}(E/F)$, which acts trivially on $S_p \simeq \mathrm{Gal}(E/\mathbb{Q})/D$, so $\mathrm{Gal}(E/F)$ acts on $\mathrm{Br}(\mathcal{O}_E[1/p])_p$ as the trivial character χ_0. If instead p splits in F' (that is, $p^*d \in \mathbb{Q}_p^{\times 2}$), then $D \neq \mathrm{Gal}(E/F)$, so $\mathrm{Gal}(E/F)$ acts nontrivially on the two-element set $S_p \simeq \mathrm{Gal}(E/\mathbb{Q})/D$, so $\mathrm{Gal}(E/F)$ acts on $\mathrm{Br}(\mathcal{O}_E[1/p])_p$ as the character $\mathrm{Gal}(E/F) \twoheadrightarrow \{\pm 1\}$, which is $\chi_{(p-1)/2}$.

Finally, consider the action of the generator τ of $\mathrm{Gal}(E/\mathbb{Q}(\zeta_p)) \simeq \{\pm 1\}$ on $\mathrm{Br}(\mathcal{O}_E[1/p])_p$. Lemma 2.3 shows that the $+$ eigenspace is $\mathrm{Br}(\mathbb{Z}[\zeta_p, 1/p])_p$, which is 0 by Corollary 4.3. Thus $\mathrm{Br}(\mathcal{O}_E[1/p])_p$ equals its $-$ eigenspace. □

Remark 4.5. Let d be a fundamental discriminant. Then $d \in \mathbb{Q}_p^{\times 2}$ if and only if $\left(\frac{d}{p}\right) = 1$, and $p^*d \in \mathbb{Q}_p^{\times 2}$ if and only if $p \mid d$ and $\left(\frac{-d/p}{p}\right) = 1$.

5. Even K-groups of $\mathbb{Z}$

Theorem 5.1. *For each $n \geq 1$,*

$$K_{2n}(\mathbb{Z})/p \simeq \left(\mathrm{Cl}(\mathbb{Z}[\zeta_p])/p\right)^{\chi_{-n}}.$$

Proof. In Theorem 2.5 for $\mathcal{F} = \mathbb{Q}$, the Brauer term is 0 by Corollary 4.3, and $\text{Cl}(\mathbb{Z}[\zeta_p, 1/p]) = \text{Cl}(\mathbb{Z}[\zeta_p])$ since the unique prime ideal above p in $\mathbb{Z}[\zeta_p]$ is principal.   □

For $n \geq 1$, let

$$\kappa_{2n,p} := \dim_{\mathbb{F}_p} K_{2n}(\mathbb{Z})/p = \dim_{\mathbb{F}_p} \left(\text{Cl}(\mathbb{Z}[\zeta_p])/p \right)^{\chi^{-n}}.$$

Remark 5.2. Assuming Vandiver's conjecture that $p \nmid \# \text{Cl}(\mathbb{Z}[\zeta_p + \zeta_p^{-1}])$ for every prime p, the K-groups of $\mathbb{Z}$ are known; see [Weibel 2005, Section 5.9]. To state the results for even K-groups, let $B_{2k} \in \mathbb{Q}$ be the $(2k)$-th Bernoulli number, defined by

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{t^{2k}}{(2k)!}.$$

Let c_k be the numerator of $|B_{2k}/(4k)|$. Then (see [Weibel 2005, Corollary 107]):

- Vandiver's conjecture implies that $K_{4k}(\mathbb{Z}) = 0$ for all $k \geq 1$.
- For $k \geq 1$, the order of $K_{4k-2}(\mathbb{Z})$ is c_k if k is even, and $2c_k$ if k is odd; moreover, Vandiver's conjecture implies that $K_{4k-2}(\mathbb{Z})$ is cyclic.

In fact, for each prime p, Vandiver's conjecture for p implies the conclusions above for the p-primary part of the K-groups. Thus Vandiver's conjecture for an odd prime p implies that for any $n \geq 1$,

$$\kappa_{2n,p} = \begin{cases} 1 & \text{if } n = 2k - 1 \text{ and } p \mid c_k; \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, Vandiver's conjecture is known for $p < 163577856$ [Buhler and Harvey 2011].

The smallest odd prime p for which there exists n such that $p \mid \# K_{2n}(\mathbb{Z})$ is the smallest irregular prime, 37, which divides $\# K_{2n}(\mathbb{Z})$ if and only if $n \equiv 31 \pmod{36}$; thus $\kappa_{2n,37}$ is 1 if $n \equiv 31 \pmod{36}$, and 0 otherwise. Assuming Vandiver's conjecture, the smallest n such that $\# K_{2n}(\mathbb{Z})$ is divisible by an odd prime is $n = 11$: we have $K_{22}(\mathbb{Z}) \simeq \mathbb{Z}/691\mathbb{Z}$. See [Weibel 2005, Example 96] for these and other examples.

6. Odd K-groups

Proposition 6.1. *For any number field $\mathcal{F}$, positive integer i, and odd prime p, we have*

$$K_{2i-1}(\mathcal{O}_{\mathcal{F}})_p = \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{if } [\mathcal{F}(\zeta_p) : \mathcal{F}] \text{ divides } i; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Define the group

$$\mu^{(i)}(\mathcal{F}) := \{ \zeta \in \mu(\bar{\mathcal{F}}) : \sigma^i \zeta = \zeta \text{ for all } \sigma \in \text{Gal}(\bar{\mathcal{F}}/\mathcal{F}) \}.$$

For $n \geq 1$, let $\zeta_n \in \bar{\mathcal{F}}$ be a primitive n-th root of 1, and let H_n be the image of the restriction homomorphism $\mathrm{Gal}(\mathcal{F}(\zeta_n)/\mathcal{F}) \hookrightarrow \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \simeq (\mathbb{Z}/n\mathbb{Z})^{\times}$, so $\#H_n = [\mathcal{F}(\zeta_n) : \mathcal{F}]$. Then the following are equivalent:

- $\zeta_n \in \mu^{(i)}(\mathcal{F})$;
- $\sigma^i \zeta_n \equiv \zeta_n$ for all $\sigma \in \mathrm{Gal}(\bar{\mathcal{F}}/\mathcal{F})$;
- $a^i = 1$ for all $a \in H_n$.

Now suppose that n is a prime power ℓ^e for some prime ℓ. Then H_n contains a cyclic subgroup of index at most 2 (we allow the case $\ell = 2$). The last condition above implies $\#H_n | 2i$, which after multiplication by $[\mathcal{F} : \mathbb{Q}]$ becomes the statement that $[\mathcal{F}(\zeta_n) : \mathbb{Q}]$ divides $2i[\mathcal{F} : \mathbb{Q}]$, which implies that the integer $\phi(n) := [\mathbb{Q}(\zeta_n) : \mathbb{Q}]$ divides $2i[\mathcal{F} : \mathbb{Q}]$, which bounds $\phi(n)$ and hence n. Thus $\mu^{(i)}(\mathcal{F})$ contains ℓ^e-th roots of 1 for only finitely many prime powers ℓ^e, so it is finite. Define $w^{(i)}(\mathcal{F}) := \#\mu^{(i)}(\mathcal{F})$.

By Theorem 70 in [Weibel 2005], if p is an odd prime, $K_{2i-1}(\mathcal{O}_{\mathcal{F}})_p$ is $\mathbb{Z}/p\mathbb{Z}$ or 0, according to whether p divides $w^{(i)}(\mathcal{F})$ or not. The previous paragraph shows that the latter condition is equivalent to H_p being killed by i, and to $\#H_p | i$ since H_p is cyclic (a subgroup of the cyclic group $(\mathbb{Z}/p\mathbb{Z})^{\times}$). Finally, $\#H_p = [\mathcal{F}(\zeta_p) : \mathcal{F}]$. $\square$

7. Heuristics for class groups

Let A be an abelian extension of $\mathbb{Q}$. Suppose that the Galois group $G := \mathrm{Gal}(A/\mathbb{Q})$ is of exponent dividing $p - 1$. Let I_A^p be the group of fractional ideals of $\mathcal{O}_A[1/p]$, that is, the free abelian group on the set of finite primes of A not lying above p. We have the standard exact sequence of $\mathbb{Z}G$-modules

$$1 \to \mathcal{O}_A[1/p]^{\times} \to A^{\times} \to I_A^p \to \mathrm{Cl}(\mathcal{O}_A[1/p]) \to 0. \tag{2}$$

Let S be a finite set of places of $\mathbb{Q}$ including p and ∞. Let S_A be the set of places of A above S. We approximate (2) by using only S_A-units and ideals supported on S_A. Let S_A^p be the set of places of A above $S - \{p\}$. Let $\mathcal{O}_{A,S}$ be the ring of S_A-integers in A. Let $I_{A,S}^p$ be the free abelian group on the nonarchimedean places in S_A^p. If S is large enough that the finite primes in S_A^p generate $\mathrm{Cl}(\mathcal{O}_A[1/p])$, then we have an exact sequence of $\mathbb{Z}G$-modules

$$1 \to \mathcal{O}_A[1/p]^{\times} \to \mathcal{O}_{A,S}^{\times} \to I_{A,S}^p \to \mathrm{Cl}(\mathcal{O}_A[1/p]) \to 0. \tag{3}$$

Dropping the first term and tensoring with $\mathbb{F}_p$ yields an exact sequence of $\mathbb{F}_p G$-modules

$$\mathcal{O}_{A,S}^{\times}/p \to I_{A,S}^p/p \to \mathrm{Cl}(\mathcal{O}_A[1/p])/p \to 0.$$

Let χ be an irreducible $\mathbb{F}_p$-representation of G; our assumption on G guarantees that χ is 1-dimensional. Taking χ-isotypic components yields

$$\left(\mathcal{O}_{A,S}^\times/p\right)^\chi \to \left(I_{A,S}^p/p\right)^\chi \to (\mathrm{Cl}(\mathcal{O}_A[1/p])/p)^\chi \to 0. \tag{4}$$

Let $u = u(A, \chi) := \dim_{\mathbb{F}_p}\left(\mathcal{O}_A[1/p]^\times/p\right)^\chi$.

Lemma 7.1. *Assume that* $\mu_p(A)^\chi = 0$.

(a) *Let S_∞ (resp. S_p) be the set of places of A lying above ∞ (resp. p). Then*

$$u = \dim_{\mathbb{F}_p}(\mathbb{F}_p^{S_\infty})^\chi + \dim_{\mathbb{F}_p}(\mathbb{F}_p^{S_p})^\chi - \begin{cases} 1, & \text{if } \chi = 1; \\ 0, & \text{otherwise.} \end{cases}$$

(b) *We have*

$$\dim_{\mathbb{F}_p}\left(\mathcal{O}_{A,S}^\times/p\right)^\chi = \dim_{\mathbb{F}_p}\left(I_{A,S}^p/p\right)^\chi + u.$$

(c) *The quantity* $\dim_{\mathbb{F}_p}\left(I_{A,S}^p/p\right)^\chi$ *can be made arbitrarily large by choosing S appropriately.*

Proof. (a) The Dirichlet S-unit theorem implies that the abelian group $\mathcal{O}_A[1/p]^\times$ is finitely generated with torsion subgroup $\mu(A)$. Denote by M the $\mathbb{Z}G$-module $\mathcal{O}_A[1/p]^\times/\mu(A)$. Tensoring the exact sequence

$$\mu(A) \longrightarrow \mathcal{O}_A[1/p]^\times \longrightarrow M \longrightarrow 0$$

with $\mathbb{F}_p$ and taking χ-isotypic components yields

$$0 \longrightarrow \left(\mathcal{O}_A[1/p]^\times/p\right)^\chi \longrightarrow (M/p)^\chi \longrightarrow 0,$$

so $u = \dim_{\mathbb{F}_p}(M/p)^\chi$.

On the other hand, the proof of the Dirichlet S-unit theorem yields

$$M \otimes \mathbb{R} \simeq \mathcal{O}_A[1/p]^\times \otimes \mathbb{R} \simeq \left(\mathbb{R}^{S_\infty \cup S_p}\right)_{\text{sum } 0}$$

as $\mathbb{R}G$-modules. A $\mathbb{Z}_{(p)}G$-module that is free of finite rank over $\mathbb{Z}_{(p)}$ is determined by its character, so

$$M \otimes \mathbb{Z}_{(p)} \simeq \left(\mathbb{Z}_{(p)}^{S_\infty \cup S_p}\right)_{\text{sum } 0}$$

as $\mathbb{Z}_{(p)}G$-modules. Both sides are free over $\mathbb{Z}_{(p)}$, so we may tensor with $\mathbb{F}_p$ to obtain

$$M/p \simeq \left(\mathbb{F}_p^{S_\infty \cup S_p}\right)_{\text{sum } 0}$$

as $\mathbb{F}_pG$-modules. In other words, there is an exact sequence

$$0 \longrightarrow M/p \longrightarrow \mathbb{F}_p^{S_\infty} \oplus \mathbb{F}_p^{S_p} \longrightarrow \mathbb{F}_p \longrightarrow 0.$$

Taking dimensions of the χ-components yields the formula for u.

(b) The composition $G \xrightarrow{\chi} (\mathbb{Z}/p\mathbb{Z})^\times \hookrightarrow \mathbb{Z}_p^\times$ lets us identify χ with a $\mathbb{Z}_p$-representation of G. Tensor (3) with $\mathbb{Z}_p$, and take χ-isotypic components:

$$0 \to \left(\mathcal{O}_A[1/p]^\times \otimes \mathbb{Z}_p\right)^\chi \to \left(\mathcal{O}_{A,S}^\times \otimes \mathbb{Z}_p\right)^\chi \to \left(I_{A,S}^p \otimes \mathbb{Z}_p\right)^\chi$$
$$\to \left(\mathrm{Cl}(\mathcal{O}_A[1/p]) \otimes \mathbb{Z}_p\right)^\chi \to 0.$$

Since $\mu_p(A)^\chi = 0$, the first three $\mathbb{Z}_p$-modules are free; on the other hand, the last is finite as a set. Take $\mathbb{Z}_p$-ranks. If V is a $\mathbb{Z}_pG$-module such that V^χ is a free $\mathbb{Z}_p$-module of finite rank, then $\dim_{\mathbb{F}_p}(V/p)^\chi = \mathrm{rank}_{\mathbb{Z}_p} V^\chi$. This proves the formula.

(c) If S contains m rational primes that split completely in A, then $I_{A,S}^p$ contains $(\mathbb{Z}G)^m$, so $\dim_{\mathbb{F}_p}\left(I_{A,S}^p/p\right)^\chi \geq m$, and m can be chosen arbitrarily large. $\qquad\square$

Sequence (4) and Lemma 7.1(b,c) imply that $(\mathrm{Cl}(\mathcal{O}_A[1/p])/p)^\chi$ is naturally the cokernel of a linear map $\mathbb{F}_p^{m+u} \to \mathbb{F}_p^m$ for arbitrarily large m. In Section 8, we will vary (A, χ) in a family with constant u-value and conjecture that the distribution of $(\mathrm{Cl}(\mathcal{O}_A[1/p])/p)^\chi$ equals the limit as $m \to \infty$ of the distribution of the cokernel of a *random* linear map $\mathbb{F}_p^{m+u} \to \mathbb{F}_p^m$; the precise statement is Conjecture 8.2. For now, we mention that this limiting distribution and the limiting expected size of the cokernel are known:

Proposition 7.2. *Fix a prime p and an integer $u \geq 0$. For $m \geq 0$, let N be a linear map $\mathbb{F}_p^{m+u} \to \mathbb{F}_p^m$ chosen uniformly at random, and let $\mathcal{N}_{p,u,m}$ be the random variable $\dim_{\mathbb{F}_p} \mathrm{coker}(N)$. Then*

(a) *For each $r \geq 0$,*

$$\lim_{m\to\infty} \mathrm{Prob}(\mathcal{N}_{p,u,m} = r) = \alpha_{p,u,r} := \frac{\prod_{i=r+1}^\infty (1 - p^{-i})}{p^{r(u+r)} \prod_{i=1}^{r+u}(1-p^{-i})}.$$

(b) *We have $\sum_{r=0}^\infty \alpha_{p,u,r} = 1$.*

(c) *We have $\sum_{r=0}^\infty p^r \alpha_{p,u,r} = 1 + p^{-u}$.*

Proof.

(a) This is [Kovalenko and Levitskaya 1975, Theorem 1].

(b) This is the $q = 1/p$ and $\alpha = 0$ case of [Cohen and Lenstra 1984, Corollary 6.7].

(c) This is the $q = 1/p$ and $\alpha = 1$ case of [Cohen and Lenstra 1984, Corollary 6.7].
$\qquad\square$

Remark 7.3. The constant $\alpha_{p,u,r}$ appeared also in [Cohen and Lenstra 1984, Theorem 6.3], as the u-probability that a random finite abelian p-group has p-rank r. The connection between u-probabilities and coranks of random matrices was made in [Friedman and Washington 1989].

8. Heuristics for class groups and even K-groups associated to quadratic fields

Calculation of u. We now specialize Section 7 to the setting of Section 3. Thus F is $\mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{Q}^\times$ such that d and p^* are independent in $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$; by multiplying d by a square, we may assume that d is a fundamental discriminant: $d = d_F$. Also, $E := F(\zeta_p)$ and $\chi := \chi_{-n,-1}$. Define $u(E, \chi)$ as in the sentence before Lemma 7.1.

Proposition 8.1. *The value $u(E, \chi)$ is given by the following table*:

		$d > 0$	$d < 0$
n even	$n \equiv 0 \pmod{p-1}$, $d \in \mathbb{Q}_p^{\times 2}$	2	1
	$n \equiv \frac{p-1}{2} \pmod{p-1}$, $p^*d \in \mathbb{Q}_p^{\times 2}$	2	1
	all other cases	1	0
n odd	$n \equiv \frac{p-1}{2} \pmod{p-1}$, $p^*d \in \mathbb{Q}_p^{\times 2}$	1	2
	all other cases	0	1

Proof. Because $\mathrm{Gal}(E/\mathbb{Q}(\zeta_p))$ acts on $\mu_p(E)$ as $+1$, we have $\mu_p(E)^\chi = 0$, so Lemma 7.1(a) applies with $\chi = \chi_{-n,-1}$. The complex conjugation in $\mathrm{Gal}(E/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times \{\pm 1\}$ is $c := (-1, \pm 1)$, where the ± 1 is $+1$ if F is real, and -1 if F is imaginary. The G-set S_∞ is isomorphic to $G/\langle c \rangle$, so $\mathbb{F}_p^{S_\infty}$ is the c-invariant subrepresentation of the regular representation $\mathbb{F}_p G$. The multiplicity of χ in $\mathbb{F}_p G$ is 1, so the multiplicity of χ in $\mathbb{F}_p^{S_\infty}$ is 1 or 0, according to whether $\chi(c)$ is 1 or -1. By definition of $\chi_{-n,-1}$, we have $\chi(c) = (-1)^{-n} \mathrm{sgn}(d)$, so

$$\dim_{\mathbb{F}_p}\left(\mathbb{F}_p^{S_\infty}\right)^\chi = \begin{cases} 1, & \text{if } (-1)^n d > 0; \\ 0, & \text{if } (-1)^n d < 0. \end{cases}$$

Next, Corollary 4.2 implies

$$\dim_{\mathbb{F}_p}\left(\mathbb{F}_p^{S_p}\right)^\chi = \dim_{\mathbb{F}_p}(\mathrm{Br}(\mathcal{O}_E[1/p])_p)^\chi,$$

which is given by Proposition 4.4. The third term in Lemma 7.1(a) is 0 since $\chi \neq 1$. $\qquad\square$

Distribution. Suppose that $\mathscr{F}$ is a family of quadratic fields. For $X > 0$, let $\mathscr{F}_{<X}$ be the set of $F \in \mathscr{F}$ such that $|d_F| < X$. For any function $\gamma : \mathscr{F} \to \mathbb{Z}_{\geq 0}$, define the following notation:

$$\mathrm{Prob}(\gamma(F) = r) := \lim_{X \to \infty} \frac{\#\{F \in \mathscr{F}_{<X} : \gamma(F) = r\}}{\#\mathscr{F}_{<X}},$$

$$\mathrm{Average}(\gamma) := \lim_{X \to \infty} \frac{\sum_{F \in \mathscr{F}_{<X}} \gamma(F)}{\#\mathscr{F}_{<X}}.$$

Our heuristic is formalized in the following statement.

Conjecture 8.2 (distribution of class group components). *Fix one of the ten boxes (below and right of the double lines) in the table of Proposition 8.1, and fix a corresponding $n \geq 1$. Let F vary over the family $\mathscr{F}$ of quadratic fields with d satisfying the conditions defining that box, and let u be as calculated in Proposition 8.1. Then the distribution of $(\mathrm{Cl}(\mathcal{O}_E[1/p])/p)^\chi$ equals the limit as $m \to \infty$ of the distribution of a cokernel of a random linear map $\mathbb{F}_p^{m+u} \to \mathbb{F}_p^m$; by this we mean, in the notation of Proposition 7.2, that for each $r \in \mathbb{Z}_{\geq 0}$,*

$$\mathrm{Prob}\left(\dim_{\mathbb{F}_p}(\mathrm{Cl}(\mathcal{O}_E[1/p])/p)^\chi = r\right) = \lim_{m \to \infty} \mathrm{Prob}(\mathcal{N}_{p,u,m} = r),$$

which by Proposition 7.2(a) equals $\alpha_{p,u,r}$.

If Conjecture 8.2 holds, then substituting it and Proposition 4.4 (with Remark 4.5) into Theorem 3.1 yields the following:

Conjecture 8.3 (distribution of K-groups modulo p in residue classes). *Fix $n \geq 1$ and an odd prime p and $r \geq 0$. As F ranges over the quadratic fields $\mathbb{Q}(\sqrt{d})$ with d satisfying the conditions defining a box below, $\mathrm{Prob}\left(\dim_{\mathbb{F}_p}(K_{2n}(\mathcal{O}_F)/p)^- = r\right)$ is as follows:*

		$d > 0$	$d < 0$
n even	$n \equiv 0 \pmod{p-1}, d \in \mathbb{Q}_p^{\times 2}$	$\alpha_{p,2,r-1}$	$\alpha_{p,1,r-1}$
	$n \equiv \frac{p-1}{2} \pmod{p-1}, p^*d \in \mathbb{Q}_p^{\times 2}$	$\alpha_{p,2,r-1}$	$\alpha_{p,1,r-1}$
	all other cases	$\alpha_{p,1,r}$	$\alpha_{p,0,r}$
n odd	$n \equiv \frac{p-1}{2} \pmod{p-1}, p^*d \in \mathbb{Q}_p^{\times 2}$	$\alpha_{p,1,r-1}$	$\alpha_{p,2,r-1}$
	all other cases	$\alpha_{p,0,r}$	$\alpha_{p,1,r}$

The distribution of $\dim_{\mathbb{F}_p} K_{2n}(\mathcal{O}_F)/p$ is the distribution of $\dim_{\mathbb{F}_p}(K_{2n}(\mathcal{O}_F)/p)^-$ shifted by the constant $\kappa_{2n,p} = \dim_{\mathbb{F}_p} K_{2n}(\mathbb{Z})_p$ of Section 5.

In Conjecture 8.3, the family of fields was defined by specifying both the sign of d and a p-adic condition on d. To get the analogous probabilities for a larger family in which only the sign of d is specified, we can take a weighted combination of probabilities from Conjecture 8.3.

Example 8.4. Suppose that $n \equiv 0 \pmod{p-1}$ and $\mathscr{F}$ is the family of real quadratic fields F. By the first and third entries of the $d > 0$ column of the table of Conjecture 8.3, $\mathrm{Prob}\left(\dim_{\mathbb{F}_p}(K_{2n}(\mathcal{O}_F)/p)^- = r\right)$ equals

$$\mathrm{Prob}(d \in \mathbb{Q}_p^{\times 2})\, \alpha_{p,2,r-1} + \mathrm{Prob}(d \notin \mathbb{Q}_p^{\times 2})\, \alpha_{p,1,r}. \tag{5}$$

Since $p^2 \nmid d$, we have $d \in \mathbb{Q}_p^{\times 2}$ if and only if $(d \bmod p) \in \mathbb{F}_p^{\times 2}$; this condition is satisfied by d lying in $p \cdot (p-1)/2$ of the $p^2 - 1$ nonzero residue classes modulo

p^2. The discriminant d is equally likely to be in any of the $p^2 - 1$ nonzero residue classes modulo p^2, as follows, for example, from [Prachar 1958, (1)], so

$$\text{Prob}(d \in \mathbb{Q}_p^{\times 2}) = \frac{p(p-1)/2}{p^2 - 1} = \frac{p}{2p+2}.$$

Substituting this and the complementary probability into (5) yields

$$\text{Prob}\left(\dim_{\mathbb{F}_p}(K_{2n}(\mathcal{O}_F)/p)^- = r\right) = \frac{p}{2p+2}\alpha_{p,2,r-1} + \frac{p+2}{2p+2}\alpha_{p,1,r}.$$

Similar calculations show that Conjecture 8.3 implies all cases in Conjecture 8.5:

Conjecture 8.5 (distribution of K-groups modulo p; cf. Conjecture 1.1). *Fix $n \geq 1$ and an odd prime p and $r \geq 0$. As F ranges over all real (resp. imaginary) quadratic fields, $\text{Prob}\left(\dim_{\mathbb{F}_p}(K_{2n}(\mathcal{O}_F)/p)^- = r\right)$ is given by the table entry in the row determined by the pair of residues $(n \bmod 2, \ n \bmod p-1)$ and column determined by the signature:*

residues	real	imaginary
$(0,0)$	$\frac{p}{2p+2}\alpha_{p,2,r-1} + \frac{p+2}{2p+2}\alpha_{p,1,r}$	$\frac{p}{2p+2}\alpha_{p,1,r-1} + \frac{p+2}{2p+2}\alpha_{p,0,r}$
$\left(0, \frac{p-1}{2}\right)$	$\frac{1}{2p+2}\alpha_{p,2,r-1} + \frac{2p+1}{2p+2}\alpha_{p,1,r}$	$\frac{1}{2p+2}\alpha_{p,1,r-1} + \frac{2p+1}{2p+2}\alpha_{p,0,r}$
$(0, \text{other})$	$\alpha_{p,1,r}$	$\alpha_{p,0,r}$
$\left(1, \frac{p-1}{2}\right)$	$\frac{1}{2p+2}\alpha_{p,1,r-1} + \frac{2p+1}{2p+2}\alpha_{p,0,r}$	$\frac{1}{2p+2}\alpha_{p,2,r-1} + \frac{2p+1}{2p+2}\alpha_{p,1,r}$
$(1, \text{other})$	$\alpha_{p,0,r}$	$\alpha_{p,1,r}$

The distribution of $\dim_{\mathbb{F}_p} K_{2n}(\mathcal{O}_F)/p$ is the distribution of $\dim_{\mathbb{F}_p}(K_{2n}(\mathcal{O}_F)/p)^-$ shifted by $\kappa_{2n,p}$.

Average order. Proposition 8.1 combined with the reasoning of Section 7 (see Proposition 7.2(c), in particular) suggests the following statement:

Conjecture 8.6 (average order of class group components in residue classes). *Fix an odd prime p. The average order of $(\text{Cl}(\mathcal{O}_E[1/p])/p)^\chi$ for F ranging over the quadratic fields $\mathbb{Q}(\sqrt{d})$ satisfying the conditions defining a box below is as follows:*

		$d > 0$	$d < 0$
n even	$n \equiv 0 \pmod{p-1}, d \in \mathbb{Q}_p^{\times 2}$	$1 + p^{-2}$	$1 + p^{-1}$
	$n \equiv \frac{p-1}{2} \pmod{p-1}, p^*d \in \mathbb{Q}_p^{\times 2}$	$1 + p^{-2}$	$1 + p^{-1}$
	all other cases	$1 + p^{-1}$	2
n odd	$n \equiv \frac{p-1}{2} \pmod{p-1}, p^*d \in \mathbb{Q}_p^{\times 2}$	$1 + p^{-1}$	$1 + p^{-2}$
	all other cases	2	$1 + p^{-1}$

Conjecture 8.6, in turn, would imply the following:

Conjecture 8.7 (average order of K-groups modulo p in residue classes). *Fix $n \geq 1$ and an odd prime p. The average order of $(K_{2n}(\mathcal{O}_F)/p)^-$ for F ranging over the quadratic fields $\mathbb{Q}(\sqrt{d})$ satisfying the conditions defining a box below is as follows*:

		$d > 0$	$d < 0$
n even	$n \equiv 0 \pmod{p-1},\, d \in \mathbb{Q}_p^{\times 2}$	$p + p^{-1}$	$p + 1$
	$n \equiv \frac{p-1}{2} \pmod{p-1},\, p^* d \in \mathbb{Q}_p^{\times 2}$	$p + p^{-1}$	$p + 1$
	all other cases	$1 + p^{-1}$	2
n odd	$n \equiv \frac{p-1}{2} \pmod{p-1},\, p^* d \in \mathbb{Q}_p^{\times 2}$	$p + 1$	$p + p^{-1}$
	all other cases	2	$1 + p^{-1}$

To get the average order of $K_{2n}(\mathcal{O}_F)/p$ itself, multiply each entry by $p^{\kappa_{2n,p}}$.

Conjecture 8.7 implies Conjecture 8.8 below for the family of all quadratic fields of given signature, by taking weighted combinations of averages. For example, for $n \equiv 0 \pmod{p-1}$ and real quadratic fields (cf. Example 8.4),

$$\text{Average}\, \#(K_{2n}(\mathcal{O}_F)/p)^- = \text{Prob}(d \in \mathbb{Q}_p^{\times 2})\,(p + p^{-1}) + \text{Prob}(d \notin \mathbb{Q}_p^{\times 2})\,(1 + p^{-1})$$

$$= \frac{p}{2p+2}(p + p^{-1}) + \frac{p+2}{2p+2}(1 + p^{-1})$$

$$= \frac{p^3 + p^2 + 4p + 2}{2p^2 + 2p}.$$

Conjecture 8.8 (average order of K-groups modulo p). *Fix $n \geq 1$ and an odd prime p. The average order of $(K_{2n}(\mathcal{O}_F)/p)^-$ for F ranging over all real (resp. imaginary) quadratic fields is given in the following table by the entry in the row determined by n and column determined by the signature*:

		real	imaginary
n even	$n \equiv 0 \pmod{p-1}$	$\dfrac{p^3+p^2+4p+2}{2p^2+2p}$	$\dfrac{p^2+3p+4}{2p+2}$
	$n \equiv \frac{p-1}{2} \pmod{p-1}$	$\dfrac{3p^2+3p+2}{2p^2+2p}$	$\dfrac{5p+3}{2p+2}$
	all other cases	$\dfrac{p+1}{p}$	2
n odd	$n \equiv \frac{p-1}{2} \pmod{p-1}$	$\dfrac{5p+3}{2p+2}$	$\dfrac{3p^2+3p+2}{2p^2+2p}$
	all other cases	2	$\dfrac{p+1}{p}$

To get the distribution of the order of $K_{2n}(\mathcal{O}_F)/p$ itself, multiply each entry by $p^{\kappa_{2n,p}}$.

Remark 8.9. It is not clear that Conjectures 8.2, 8.3, and 8.5 imply Conjectures 8.6, 8.7, and 8.8, respectively. For such implications, one would need to know that the contribution to each average from the rare cases of very large class groups or K-groups is negligible.

9. The average order of even K-groups modulo 3

In this section, we prove Conjecture 8.6 for $p = 3$; then Conjectures 8.7 and 8.8 for $p = 3$ follow too; the latter becomes Theorem 1.2. For $p = 3$, the table in Conjecture 8.6 to be verified simplifies to this:

		$d > 0$	$d < 0$
n even	$d \in \mathbb{Q}_3^{\times 2}$	10/9	4/3
	$d \notin \mathbb{Q}_3^{\times 2}$	4/3	2
n odd	$-3d \in \mathbb{Q}_3^{\times 2}$	4/3	10/9
	$-3d \notin \mathbb{Q}_3^{\times 2}$	2	4/3

We have $E = F(\zeta_3)$ and $G = \mathrm{Gal}(E/\mathbb{Q}) \simeq (\mathbb{Z}/3\mathbb{Z})^\times \times \{\pm 1\}$. Let $H := \ker \chi$, which is generated by $(-1, (-1)^n)$. Let $K := E^H$, which is F if n is even, and $F' = \mathbb{Q}(\sqrt{-3d})$ if n is odd. For any $\mathbb{F}_3$-representation V of G, we have $V^H = V^G \oplus V^\chi$. Apply this to $V = \mathrm{Cl}(\mathcal{O}_E[1/3])/3$, and use Lemma 2.3 twice to obtain

$$\mathrm{Cl}(\mathcal{O}_K[1/3])/3 \simeq \mathrm{Cl}(\mathbb{Z}[1/3])/3 \oplus (\mathrm{Cl}(\mathcal{O}_E[1/3])/3)^\chi \simeq (\mathrm{Cl}(\mathcal{O}_E[1/3])/3)^\chi,$$

since $\mathrm{Cl}(\mathbb{Z}[1/3])$ is a quotient of $\mathrm{Cl}(\mathbb{Z}) = 0$. It remains to show that for each box, the average order of $\mathrm{Cl}(\mathcal{O}_K[1/3])/3$ is as given in the table above.

For fixed n, as d ranges over fundamental discriminants of fixed sign in a fixed coset of $\mathbb{Q}_3^{\times 2}$ up to some bound, the field K ranges over quadratic fields with fundamental discriminant of fixed sign in a fixed coset of $\mathbb{Q}_3^{\times 2}$ up to some bound (the same sign and coset if n is even, or sign and coset multiplied by -3 if n is odd). Thus it suffices to prove that the average order of $\mathrm{Cl}(\mathcal{O}_K[1/3])/3$ for such K is given by the table

	$d_K > 0$	$d_K < 0$
$d_K \in \mathbb{Q}_3^{\times 2}$	10/9	4/3
$d_K \notin \mathbb{Q}_3^{\times 2}$	4/3	2

Remark 9.1. There are four cosets of $\mathbb{Q}_3^{\times 2}$ in $\mathbb{Q}_3^\times$. Which coset contains a given fundamental discriminant d_K is determined by whether d_K is 1 mod 3, 2 mod 3, 3 mod 9, or 6 mod 9. In particular, $d_K \in \mathbb{Q}_3^{\times 2}$ if and only if $d_K \equiv 1 \pmod 3$.

Remark 9.2. The results in the next three subsections (through Corollary 9.10) have been generalized by the second author, yielding a computation of the average

size of $\mathrm{Cl}(\mathcal{O}_K[1/S])/3$ for an arbitrary finite set of primes S using essentially identical methods [Klagsbrun 2017a]. Similar methods have been used to compute the average size of $\mathrm{Cl}(O_K[1/p])/3$ when p splits in K [Wood 2018].

From class groups to cubic fields. To compute the average order of $\mathrm{Cl}(\mathcal{O}_K[1/3])/3$, we adapt the strategy of Davenport and Heilbronn [Davenport and Heilbronn 1971], which relies on results involving the following setup.

Let L be a degree 3 extension of $\mathbb{Q}$ whose Galois closure M has Galois group S_3. Let K be the quadratic resolvent of L, that is, the degree 2 subfield M^{A_3}.

Lemma 9.3. *The following are equivalent*:

(i) *M/K is unramified.*

(ii) *$d_L = d_K$.*

(iii) *$L/\mathbb{Q}$ is nowhere totally ramified.*

Proof. Let $\mathfrak{f}$ be the conductor of the abelian extension M/K. By [Hasse 1930, Satz 3], $d_L = N_{K/\mathbb{Q}}(\mathfrak{f})d_K$; this proves (i) $\Longleftrightarrow$ (ii). Also, d_L is an integer square times d_K (see [Hasse 1930, (1)]); this proves (ii) $\Longleftrightarrow$ (iii). Finally, [Cohen 2000, Proposition 8.4.1] yields (ii) $\Longleftrightarrow$ (iv). $\square$

Throughout this section, we consider two cubic fields to be the same if they are abstractly isomorphic (so conjugate cubic fields are counted only as one).

Theorem 9.4 (compare [Hasse 1930, Satz 7]). *Fix a quadratic field K. Then the following are naturally in bijection*:

(i) *The set of index 3 subgroups of $\mathrm{Cl}(\mathcal{O}_K)$.*

(ii) *The set of unramified $\mathbb{Z}/3\mathbb{Z}$-extensions M of K.*

(iii) *The set of cubic fields L with $d_L = d_K$.*

Proof. Class field theory gives (i) $\Longleftrightarrow$ (ii). The nontrivial element of $\mathrm{Gal}(K/\mathbb{Q})$ acts as -1 on $\mathrm{Cl}(\mathcal{O}_K)$, so each M in (ii) is an S_3-extension of $\mathbb{Q}$. The map (ii) $\Longrightarrow$ (iii) sends M to one of its cubic subfields L. The map (iii) $\Longrightarrow$ (ii) sends L to its Galois closure M. That these are bijections follows from Lemma 9.3(i) $\Longleftrightarrow$ (ii). $\square$

We are interested in $\mathrm{Cl}(\mathcal{O}_K[1/3])$ instead of $\mathrm{Cl}(\mathcal{O}_K)$, so we need the following variant.

Corollary 9.5. *Fix a quadratic field K. The following are naturally in bijection*:

(i) *The set of index 3 subgroups of $\mathrm{Cl}(\mathcal{O}_K)[1/3]$.*

(ii) *The set of unramified $\mathbb{Z}/3\mathbb{Z}$-extensions M of K such that the primes above 3 in K split completely in M/K.*

(iii) *The set of cubic fields L with $d_L = d_K$ such that if $d_K \in \mathbb{Q}_3^{\times 2}$ then 3 splits completely in $L/\mathbb{Q}$.*

Proof. The group $\mathrm{Cl}(\mathcal{O}_K)[1/3]$ is the quotient of $\mathrm{Cl}(\mathcal{O}_K)$ by the group generated by the classes of the primes $\mathfrak{p}|3$ in K. Thus we need to restrict the bijections in Theorem 9.4 to the index 3 subgroups H containing these classes. Because the class field theory isomorphism $\mathrm{Cl}(\mathcal{O}_K)/H \simeq \mathrm{Gal}(M/K)$ sends $[\mathfrak{p}]$ to $\mathrm{Frob}_{\mathfrak{p}}$, which is trivial if and only if $\mathfrak{p}$ splits in M/K, we obtain (i) $\Longleftrightarrow$ (ii). If 3 is inert or ramified in $K/\mathbb{Q}$, then the prime above 3 is of order dividing 2 in $\mathrm{Cl}(\mathcal{O}_K)$, so to require it to be in the index 3 subgroup is no condition. If 3 splits in $K/\mathbb{Q}$ (that is, $d_K \in \mathbb{Q}_3^{\times 2}$), then the primes above 3 in K split in M/K if and only if 3 splits completely in $M/\mathbb{Q}$, which is if and only if 3 splits completely in $L/\mathbb{Q}$. $\square$

Corollary 9.6. *Let K be a quadratic field.*

(a) *If $d_K \in \mathbb{Q}_3^{\times 2}$, then*

$$\#\mathrm{Cl}(\mathcal{O}_K[1/3])/3 = 2\,\#\{\text{cubic fields } L \text{ with } d_L = d_K \text{ in which 3 splits}\} + 1.$$

(b) *If $d_K \notin \mathbb{Q}_3^{\times 2}$, then*

$$\#\mathrm{Cl}(\mathcal{O}_K[1/3])/3 = 2\,\#\{\text{cubic fields } L \text{ with } d_L = d_K\} + 1.$$

Proof. For an elementary abelian 3-group V,

$$\#V = 2\,\#\{\text{index 3 subgroups of } V\} + 1.$$

Take $V = \mathrm{Cl}(\mathcal{O}_K[1/3])/3$, and apply Corollary 9.5(i) $\Longleftrightarrow$ (iii). $\square$

Counting quadratic fields. Fix a sign and a coset of $\mathbb{Q}_3^{\times 2}$ in $\mathbb{Q}_3^{\times}$; by Remark 9.1, each of the four cosets is defined by a congruence condition mod 3 or mod 9. Let $\mathcal{D}$ be the set of fundamental discriminants having this sign and lying in this coset. For $X > 0$, define $\mathcal{D}_{<X} := \{d \in \mathcal{D} : |d| < X\}$.

To compute the average of the appropriate left hand side in Corollary 9.6 as K ranges over quadratic fields with $d_K \in \mathcal{D}$, we compute the average number of cubic fields appearing in the corresponding right hand side. That is, we need the limit as $X \to \infty$ of

$$\frac{\displaystyle\sum_{K:d_K \in \mathcal{D}_{<X}} \#\{\text{cubic fields } L \text{ with } d_L = d_K \text{ such that if } d_K \in \mathbb{Q}_3^{\times 2} \text{ then 3 splits in } L\}}{\#\{\text{quadratic fields } K \text{ such that } d_K \in \mathcal{D}_{<X}\}}.$$

(6)

We first compute the denominator.

Proposition 9.7. *The number of quadratic fields K satisfying $|d_K| < X$ and prescribed sign and 3-adic congruence conditions is $\alpha_2 X/\zeta(2) + o(X)$, where α_2 is*

given by the following table:

	$d_K > 0$	$d_K < 0$
$d_K \equiv 1 \pmod 3$	3/16	3/16
$d_K \equiv 2 \pmod 3$	3/16	3/16
$d_K \equiv 3 \pmod 9$	1/16	1/16
$d_K \equiv 6 \pmod 9$	1/16	1/16

Proof. Use an elementary squarefree sieve. □

Remark 9.8. Even though many entries in the table of Proposition 9.7 coincide, it is stronger to give the asymptotics for the individual field families without merging them, and we need the stronger results.

Counting cubic fields. By Lemma 9.3(ii) $\Longleftrightarrow$ (iii), the numerator in (6) equals the number of nowhere totally ramified cubic fields L with $d_L \in \mathcal{D}_{<X}$ such that if $d_L \in \mathbb{Q}_3^{\times 2}$ then 3 splits completely in L. To compute this number, we follow the Davenport–Heilbronn approach, in the form of a refinement due to Bhargava, Shankar, and Tsimerman [Bhargava et al. 2013].

For every prime p, let $\widehat{\Sigma}_p$ be the set of maximal cubic $\mathbb{Z}_p$-orders that are not totally ramified, up to isomorphism. For $R \in \widehat{\Sigma}_p$, let $\mathrm{Disc}_p(R)$ be the power of p generating the discriminant ideal of R.

Theorem 9.9. *For each prime p (including 2), let $\Sigma_p \subseteq \widehat{\Sigma}_p$. Suppose that $\Sigma_p = \widehat{\Sigma}_p$ for all p outside a finite set $\mathcal{P}$. Define*

$$c_p := \frac{p}{p+1} \sum_{R \in \Sigma_p} \frac{1}{\mathrm{Disc}_p(R)\, |\mathrm{Aut}\, R|}.$$

(a) *The number of nowhere totally ramified totally real cubic fields L (up to isomorphism) such that $|d_L| < X$ and $\mathcal{O}_L \otimes \mathbb{Z}_p \in \Sigma_p$ for all $p \in \mathcal{P}$ is*

$$\frac{\prod_{p \in \mathcal{P}} c_p}{12 \zeta(2)} X + o(X).$$

(b) *The number of nowhere totally ramified complex cubic fields L (up to isomorphism) such that $|d_L| < X$ and $\mathcal{O}_L \otimes \mathbb{Z}_p \in \Sigma_p$ for all $p \in \mathcal{P}$ is*

$$\frac{\prod_{p \in \mathcal{P}} c_p}{4 \zeta(2)} X + o(X).$$

Proof. The definition of c_p yields

$$\frac{p-1}{p} \sum_{R \in \Sigma_p} \frac{1}{\mathrm{Disc}_p(R)\, |\mathrm{Aut}\, R|} = \left(1 - \frac{1}{p^2}\right) c_p.$$

For each $p \notin \mathcal{P}$, enumerating $\widehat{\Sigma}_p$ explicitly shows that $c_p = 1$. Substituting this into [Bhargava et al. 2013, Theorem 8] yields the result. $\square$

Corollary 9.10. *The number of nowhere totally ramified cubic fields L with $|d_L| < X$ satisfying prescribed sign and 3-adic congruence conditions below such that if $d_L \equiv 1 \pmod 3$ then 3 splits completely in L is $\alpha_3 X / \zeta(2) + o(X)$, where α_3 is given by the following table*:

	$d_L > 0$	$d_L < 0$
$d_L \equiv 1 \pmod 3$	1/96	1/32
$d_L \equiv 2 \pmod 3$	1/32	3/32
$d_L \equiv 3 \pmod 9$	1/96	1/32
$d_L \equiv 6 \pmod 9$	1/96	1/32.

Proof. We apply Theorem 9.9 with $\mathcal{P} = \{3\}$ and with Σ_3 tailored to the row. For the first row, let $\Sigma_3 := \{\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3\}$, so

$$c_3 = \frac{3}{4} \cdot \frac{1}{\mathrm{Disc}_3(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \left| \mathrm{Aut}\, (\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \right|} = \frac{3}{4} \cdot \frac{1}{6} = \frac{1}{8}.$$

For each other row, let $\Sigma_3 := \{\mathbb{Z}_3 \times \mathbb{Z}_3(\sqrt{d_L})\}$, and calculate c_3 similarly. $\square$

End of proof. Corollary 9.10 and Proposition 9.7 give the asymptotic behavior of the numerator and denominator, respectively, in (6) (see the first sentence in the previous subsection, following Remark 9.8). Thus, as $X \to \infty$, the ratio (6) tends to α_3/α_2. Following Corollary 9.6, we multiply by 2 and add 1 to obtain the average order of $\mathrm{Cl}(\mathcal{O}_K[1/3])/3$ as K varies over quadratic fields with $d_K \in \mathcal{D}$. For each signature, the answer is the same for each of the three nontrivial cosets of $\mathbb{Q}_3^{\times 2}$, so we combine them into a single entry in the table before Remark 9.1. This completes the proof of Conjecture 8.6 for $p = 3$, and hence also Conjectures 8.7 and 8.8 for $p = 3$ and Theorem 1.2.

Acknowledgments

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On p-adic vanishing cycles of log smooth families

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In this paper, we will prove that the sheaf of p-adic vanishing cycles on a regular log smooth family is generated by Milnor symbols, assuming that the base dvr contains a primitive p-th root of unity. Our result generalizes the surjectivity results of Bloch and Kato (*Inst. Hautes Études Sci. Publ. Math.* **63** (1986), 107–152) and Hyodo (*Invent. Math.* **91**:3 (1988), 543–557) to a regular log smooth case.

1. Introduction

Let K be a henselian discrete valuation field of mixed characteristic $(0, p)$, with residue field k. Let O_K be the ring of integers in K, and let X be a regular scheme which is flat of finite type over $\mathrm{Spec}(O_K)$. We consider cartesian squares of schemes

$$
\begin{array}{ccccc}
X_K & \overset{j}{\hookrightarrow} & X & \overset{i}{\hookleftarrow} & X_k \\
\downarrow & \square & \downarrow & \square & \downarrow \\
\mathrm{Spec}(K) & \hookrightarrow & \mathrm{Spec}(O_K) & \hookleftarrow & \mathrm{Spec}(k)
\end{array}
$$

The Kummer short exact sequence of étale sheaves on X_K

$$
0 \to \mu_{p^n} \to \mathscr{O}_{X_K}^{\times} \xrightarrow{\times p^n} \mathscr{O}_{X_K}^{\times} \to 0
$$

yields a long exact sequence of étale sheaves on X_k

$$
0 \to i^* j_* \mu_{p^n} \to i^* j_* \mathscr{O}_{X_K}^{\times} \xrightarrow{\times p^n} i^* j_* \mathscr{O}_{X_K}^{\times} \xrightarrow{\delta} i^* R^1 j_* \mu_{p^n} \to i^* R^1 j_* \mathscr{O}_{X_K}^{\times} \to \cdots .
$$

Since X is regular, we have $i^* R^1 j_* \mathscr{O}_{X_K}^{\times} = 0$ and the connecting map δ in this sequence induces an isomorphism

$$
i^* R^1 j_* \mu_{p^n} \cong \mathrm{Coker}\big(i^* j_* \mathscr{O}_{X_K}^{\times} \xrightarrow{\times p^n} i^* j_* \mathscr{O}_{X_K}^{\times}\big).
$$

A motivation for this note is to extend this fundamental fact to higher cohomological degrees. More precisely, we are concerned with the surjectivity of a geometric version of Tate's norm residue homomorphism

$$\varrho_{X,n}^q : \mathscr{K}_q^M / p^n \to i^* R^q j_* \mu_{p^n}^{\otimes q},$$

where $\mathscr{K}_q^M$ denotes a Milnor K-sheaf defined as a quotient of $(i^* j_* \mathscr{O}_{X_K}^\times)^{\otimes q}$ and $\mathscr{K}_q^M / p^n$ denotes the cokernel of the multiplication by p^n on $\mathscr{K}_q^M$, see Section 4 below. The sheaf $i^* R^q j_* \mu_{p^n}^{\otimes q}$ on the right hand side is the, so called, *sheaf of p-adic vanishing cycles*, which is an étale sheaf of arithmetic and geometric interest. Bloch and Kato [1986] proved that the map $\varrho_{X,n}^q$ is surjective in the case where X is smooth over $\mathrm{Spec}(O_K)$. Later, Hyodo [1988] extended this surjectivity to the case where X is a semistable family over $\mathrm{Spec}(O_K)$. These surjectivity facts play a fundamental role in a construction of p-adic period maps in the p-adic Hodge theory, see [Kurihara 1987; Kato 1987; 1994; Tsuji 1999; 2000; Yamashita and Yasuda 2014].

To state our main results more precisely, we introduce the following generalized situation with log poles. Let D be a normal crossing divisor on X which is flat over $\mathrm{Spec}(O_K)$, and let

$$\psi : U := X - (X_k \cup D) \hookrightarrow X$$

be the natural open immersion. We then have a version of symbol map with log poles

$$\varrho_{(X,D),n}^q : \mathscr{K}_q^M / p^n \to M_n^q := i^* R^q \psi_* \mu_{p^n}^{\otimes q},$$

where $\mathscr{K}_q^M$ is again a Milnor K-sheaf defined as a quotient of $(i^* \psi_* \mathscr{O}_U^\times)^{\otimes q}$, see Section 4 below. Now we state a main result of this paper, where quasilog smoothness is a generalization of log-smoothness (see Definition 5.2, Example 5.3):

Theorem 1.1 (Theorem 5.4). *If (X, D) is quasilog smooth over $\mathrm{Spec}(O_K)$ and K contains a primitive p-th root of unity, then $\varrho_{(X,D),n}^q$ is surjective for any $q \geq 2$.*

Tsuji [2000] proves an isomorphism in the derived category of étale sheaves on X_k:

$$\mathcal{S}_n(q)_{(X,D)} \cong \tau_{\leq q} i^* R\psi_* \mu_{p^n}^{\otimes q} \qquad (1.2)$$

for quasilog smooth (X, D) assuming $0 \leq q \leq p - 2$, where $\mathcal{S}_n(q)_{(X,D)}$ denotes a log syntomic complex. His strategy is to show that both hand sides in (1.2) are invariant under log blow-ups, and then to reduce his assertion to the case that X is smooth over $\mathrm{Spec}(O_K)$ (and $D = \varnothing$). This last case is due to Kurihara [1987]. We will prove Theorem 1.1 using his arguments on log blow-ups, which is the first key ingredient of our results. We have to note that Theorem 1.1 does not follow from (1.2). Indeed, it is not clear that the q-th cohomology sheaf of $\mathcal{S}_n(q)_{(X,D)}$ is generated by symbols, which is rather a consequence of (1.2) and Theorem 1.1.

To continue the outline of our proof of Theorem 1.1, we introduce a subsheaf $\mathscr{U}^1\mathscr{K}_q^M$ of $\mathscr{K}_q^M$, which is the subsheaf generated by the image of

$$i^*(1+\mathscr{I})^\times \otimes (i^*\psi_*\mathscr{O}_U^\times)^{\otimes(q-1)},$$

where $\mathscr{I}$ denotes the ideal sheaf of $\mathscr{O}_X$ defining the reduced part $Y := (X_k)_{\mathrm{red}}$ of X_k, and $(1+\mathscr{I})^\times$ means the kernel of the map $\mathscr{O}_X^\times \to i_*\mathscr{O}_Y^\times$. We will further introduce a multi-index descending filtration on $\mathscr{U}^1\mathscr{K}_q^M$, where the multi-indexes are assigned to irreducible components of X_k, see Definition 4.2. To investigate the map $\varrho_{(X,D),n}^q$, we will need to control the behavior of the sheaf

$$\mathscr{U}^1 M_1^q := \mathrm{Im}(\mathscr{U}^1\mathscr{K}_q^M \to M_1^q)$$

and a certain absolute logarithmic differential sheaf $\widetilde{\omega}_{Y,\log}^q$ under log blow-ups (see Lemma 5.10 below), which corresponds to a key computation by Hyodo [1988, Lemma (3.5)] in the semistable family case. Our second key ingredient is the computations on the multigraded quotients of the induced filtration on $\mathscr{U}^1 M_1^q$, which will be carried out by ideas of Kato, who introduced a new Cartier operator on the absolute differential modules with log poles, see Sections 3–4. By this computation on multigraded quotients, the behavior of $\mathscr{U}^1 M_1^q$ and $\widetilde{\omega}_{Y,\log}^q$ will be calculated by standard facts on the vanishing of the higher direct image of the structure sheaf under log blow-ups. We would like to mention also that the idea of our computation on multigraded quotients of $\mathscr{U}^1\mathscr{K}_q^M$ has been used in a recent joint paper of the first author with Rülling [Rülling and Saito 2018].

Throughout this paper, we will work with the Setting 1.4 stated below.

Definition 1.3. Let k be a field.

(1) *A normal crossing variety* over k is a pure-dimensional scheme Y which is separated of finite type over k and everywhere étale locally isomorphic to

$$\mathrm{Spec}\big(k[T_1,\ldots,T_N]/(T_1\cdots T_a)\big) \quad \text{for some } 1 \le a \le N = \dim(Y)+1.$$

(2) *An admissible divisor* on a normal crossing variety Y is a reduced effective Cartier divisor E such that the immersion $E \hookrightarrow Y$ is everywhere étale locally isomorphic to

$$\mathrm{Spec}\big(k[T_1,\ldots,T_N]/(T_1\cdots T_a, T_{a+1}\cdots T_{a+b})\big)$$
$$\hookrightarrow \mathrm{Spec}\big(k[T_1,\ldots,T_N]/(T_1\cdots T_a)\big)$$

for some $a,b \ge 1$ with $a+b \le N$.

Let K be a henselian discrete valuation field of characteristic 0 whose residue field k has characteristic $p > 0$. Let O_K be the integer ring of K. Unless mentioned otherwise, we do *not* assume that k is perfect. Put $B := \mathrm{Spec}(O_K)$ and $s := \mathrm{Spec}(k)$.

Setting 1.4. X is a regular scheme of finite type over B, and D is a reduced divisor on X which is flat over B (D may be empty). We put $Y := (X \times_B s)_{\mathrm{red}}$ and $U := X \smallsetminus (Y \cup D)$, and assume the following two conditions:

- *The divisor $Y \cup D$ has normal crossings on X.*
- *Y is a normal crossing variety over s, and $(D \times_B s)_{\mathrm{red}}$ is an admissible divisor on Y.*

When k is perfect, the first condition implies the second condition.

2. Absolute differential modules with log poles

Let the notation be as in Setting 1.4, and let i and ψ be as follows:

$$Y \overset{i}{\hookrightarrow} X \overset{\psi}{\longleftarrow} U = X \smallsetminus (Y \cup D).$$

Put

$$\mathscr{L} := \psi_* \mathscr{O}_U^\times \cap \mathscr{O}_X \subset \psi_* \mathscr{O}_U,$$

which we regard as a sheaf of commutative monoids by the multiplication of functions. Let

$$\alpha : i^* \mathscr{L} \to \mathscr{O}_Y$$

be the natural map of étale sheaves, where i^* denotes the topological inverse image of étale sheaves. In this section, we study the following étale sheaves.

Definition 2.1. (1) Let $\Omega^1_{Y/\mathbb{Z}}$ be the usual absolute Kähler differential sheaf on $Y_{\mathrm{ét}}$. We define the étale sheaf $\widetilde{\omega}^1_Y$ on Y as the quotient sheaf of

$$\Omega^1_{Y/\mathbb{Z}} \oplus \left(\mathscr{O}_Y \otimes_{\mathbb{Z}} i^* \psi_* \mathscr{O}_U^\times \right)$$

divided by the $\mathscr{O}_Y$-submodule generated by local sections of the form

$$(d\alpha(x), 0) - (0, \alpha(x) \otimes x) \quad \text{with } x \in i^* \mathscr{L}.$$

There is a logarithmic differential map

$$d\log : i^* \psi_* \mathscr{O}_U^\times \to \widetilde{\omega}^1_Y, \quad x \longmapsto (0, 1 \otimes x).$$

Put $\widetilde{\omega}^0_Y := \mathscr{O}_Y$ and $\widetilde{\omega}^q_Y := \bigwedge^q_{\mathscr{O}_Y} \widetilde{\omega}^1_Y$ for $q \geq 2$.

(2) We define $\widetilde{\mathscr{Z}}^q_Y$ as the kernel of $d : \widetilde{\omega}^q_Y \to \widetilde{\omega}^{q+1}_Y$, and $\widetilde{\mathscr{B}}^q_Y$ as the image of $d : \widetilde{\omega}^{q-1}_Y \to \widetilde{\omega}^q_Y$, respectively, and put

$$\widetilde{\omega}^q_{Y,\log} := \mathsf{Im}\left(d\log : (i^* \psi_* \mathscr{O}_U^\times)^{\otimes q} \to \widetilde{\omega}^q_Y \right).$$

Remark 2.2. The natural map $\mathscr{L} \to \mathscr{O}_X$ gives a log structure on X in the sense of [Kato 1989]. In terms of log schemes, the sheaf $\widetilde{\omega}^1_Y$ means the differential module

$\omega^1_{(Y,L)/\mathbb{Z}}$ defined in [loc. cit., (1.7)], where L denotes the inverse image log structure of $\mathscr{L}$ onto Y [loc. cit., (1.4)].

Theorem 2.3. (1) *The sheaf $\widetilde{\omega}^q_Y$ is locally free over $\mathscr{O}_Y$.*

(2) *There is a unique isomorphism*

$$C^{-1} : \widetilde{\omega}^q_Y \xrightarrow{\sim} \mathscr{H}^q(\widetilde{\omega}^\bullet_Y) = \widetilde{\mathscr{Z}}^q_Y / \widetilde{\mathscr{B}}^q_Y$$

sending a local section $x \cdot d\log(y_1) \wedge \cdots \wedge d\log(y_q)$ with $x \in \mathscr{O}_Y$ and each $y_i \in i^*\psi_*\mathscr{O}^\times_U$, to $x^p \cdot d\log(y_1) \wedge \cdots \wedge d\log(y_q) + \widetilde{\mathscr{B}}^q_Y$.

(3) *There is a short exact sequence on $Y_{\text{ét}}$*

$$0 \longrightarrow \widetilde{\omega}^q_{Y,\log} \longrightarrow \widetilde{\mathscr{Z}}^q_Y \xrightarrow{1-C^{-1}} \mathscr{H}^q(\widetilde{\omega}^\bullet_Y) \longrightarrow 0.$$

Proof. We first reduce the problem to the case that X is a regular semistable family over $B = \text{Spec}(O_K)$. Note that we may work étale locally. Indeed, once we prove (2) étale locally, then the isomorphisms C^{-1} patch together automatically by the uniqueness. Assume the following three conditions (see Setting 1.4):

- *X is affine, and $Y = \text{Spec}\big(k[T_1, \ldots, T_d]/(T_1 \cdots T_a)\big)$ for $1 \le {}^\exists a \le d = \dim(X)$.*

- *The irreducible components of D are regular and principal.*

- *There exists a regular sequence $t_1, \ldots, t_d$ of prime elements of $\Gamma(X, \mathscr{O}_X)$ such that t_λ lifts $T_\lambda \in \Gamma(Y, \mathscr{O}_Y)$ for $1 \le {}^\forall \lambda \le d$ and such that $t_{a+1}, \ldots, t_{a+b}$ are uniformizers of the irreducible components of D for $0 \le {}^\exists b \le d - a$.*

Let π be a prime element of O_K. We have

$$\pi = u\, t_1^{e_1}\, t_2^{e_2} \cdots t_a^{e_a} \tag{2.4}$$

for some $u \in \Gamma(X, \mathscr{O}^\times_X)$ and some $e_1, \ldots, e_a \ge 1$. Put

$$X' := \text{Spec}\big(O_K[S_1, \ldots, S_d]/(S_1 \cdots S_a - \pi)\big),$$
$$Y' := (X')_s = \text{Spec}\big(k[S_1, \ldots, S_d]/(S_1 \cdots S_a)\big),$$
$$U' := \text{Spec}\big(K[S_1, \ldots, S_d, S_{a+1}^{-1}, \ldots, S_{a+b}^{-1}]/(S_1 \cdots S_a - \pi)\big).$$

Let ψ' be the open immersion $U' \hookrightarrow X'$ and i' the closed immersion $Y' \hookrightarrow X'$, and let β be the isomorphism of schemes

$$\beta : Y \xrightarrow{\sim} Y', \qquad S_\lambda \mapsto T_\lambda \quad (1 \le \lambda \le d).$$

Put $\mathscr{K} := \text{Ker}(\mathscr{O}^\times_X \to \mathscr{O}^\times_Y)$ and $\mathscr{K}' := \text{Ker}(\mathscr{O}^\times_{X'} \to \mathscr{O}^\times_{Y'})$. By (2.4), $\psi_*\mathscr{O}^\times_U/\mathscr{O}^\times_X$ is a free abelian sheaf generated by $t_1, \ldots, t_{a+b}$. Similarly $\psi'_*\mathscr{O}^\times_{U'}/\mathscr{O}^\times_{X'}$ is a free abelian sheaf generated by $S_1, \ldots, S_{a+b}$. Hence there is an isomorphism of sheaves on $Y_{\text{ét}}$

$$\beta^*(i'^*\psi'_*\mathscr{O}^\times_{U'}/\mathscr{K}') \xrightarrow{\sim} i^*\psi_*\mathscr{O}^\times_U/\mathscr{K}$$

that extends the isomorphism $\beta^* \mathcal{O}_{Y'}^{\times} \overset{\sim}{\to} \mathcal{O}_Y^{\times}$ and sends $S_\lambda \mapsto t_\lambda$ for $1 \le \lambda \le a + b$. By this isomorphism we see that $\beta^* \widetilde{\omega}_{Y'}^q \cong \widetilde{\omega}_Y^q$. Thus we are reduced to the case that X is a regular semistable family over B.

We assume that X is a regular semistable family over B in what follows. Let ω_Y^q be the cokernel of the map

$$\widetilde{\omega}_Y^{q-1} \to \widetilde{\omega}_Y^q, \quad x \mapsto d\log(\pi) \wedge x.$$

We have a short exact sequence of complexes

$$0 \longrightarrow \omega_Y^{\bullet}[-1] \xrightarrow{\ d\log(\pi)\wedge\ } \widetilde{\omega}_Y^{\bullet} \longrightarrow \omega_Y^{\bullet} \longrightarrow 0 \qquad (2.5)$$

and a short exact sequence of the q-th cohomology sheaves for any $q \ge 0$

$$0 \longrightarrow \mathcal{H}^{q-1}(\omega_Y^{\bullet}) \xrightarrow{\ d\log(\pi)\wedge\ } \mathcal{H}^q(\widetilde{\omega}_Y^{\bullet}) \longrightarrow \mathcal{H}^q(\omega_Y^{\bullet}) \longrightarrow 0, \qquad (2.6)$$

see [Tsuji 2000, Lemma A.7] with $m = 0$. We recall here the following facts due to Tsuji [2000, Theorems A.3 and A.4] (see [Kato 1989, Proposition (3.10), Theorem (4.12)(1)]):

Fact 2.7. (1) ω_Y^q *is a locally free* $\mathcal{O}_Y$*-module, and there is a unique isomorphism*

$$C^{-1} : \omega_Y^q \overset{\sim}{\to} \mathcal{H}^q(\omega_Y^{\bullet})$$

sending a local section of the form $x \cdot d\log(y_1) \wedge \cdots \wedge d\log(y_q)$ *with* $x \in \mathcal{O}_Y$ *and each* $y_i \in i^* \psi_* \mathcal{O}_U^{\times}$, *to* $x^p \cdot d\log(y_1) \wedge \cdots \wedge d\log(y_q) + d\omega_Y^{q-1}$.

(2) *Let* V *be an open subset of* Y *which is smooth over* k *and for which* $D \times_X V$ *is empty. Then the short exact sequence* (2.5) *splits on* V *as complexes. Consequently the exact sequence* (2.6) *splits on* V, *i.e., we have*

$$\mathcal{H}^q(\widetilde{\omega}_V^{\bullet}) \overset{\sim}{\to} \mathcal{H}^q(\Omega_V^{\bullet}) \oplus \mathcal{H}^{q-1}(\Omega_V^{\bullet}).$$

(3) *There is a short exact sequence on* $Y_{\text{ét}}$

$$0 \longrightarrow \omega_{Y,\log}^q \longrightarrow \mathscr{Z}_Y^q \xrightarrow{\ 1-C^{-1}\ } \mathcal{H}^q(\omega_Y^{\bullet}) \longrightarrow 0,$$

where $\omega_{Y,\log}^q$ *is defined as* $\mathrm{Im}\big(d\log : (i^* \psi_* \mathcal{O}_U^{\times})^{\otimes q} \to \omega_Y^q\big)$.

Theorem 2.3(1) follows from (2.5) and Fact 2.7(1). We prove Theorem 2.3(2). Let V be a dense open subset of Y which is smooth over k and for which $D \times_X V$ is empty. Let σ be the open immersion $V \hookrightarrow Y$. We first show that the canonical adjunction map

$$\mathcal{H}^q(\widetilde{\omega}_Y^{\bullet}) \to \sigma_* \sigma^* \mathcal{H}^q(\widetilde{\omega}_Y^{\bullet}) \cong \sigma_* \big(\mathcal{H}^q(\Omega_V^{\bullet}) \oplus \mathcal{H}^{q-1}(\Omega_V^{\bullet})\big) \qquad (2.8)$$

is injective. Indeed by (2.6) there is a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{H}^{q-1}(\omega_Y^\bullet) & \xrightarrow{\ d\log(\pi)\wedge\ } & \mathscr{H}^q(\widetilde{\omega}_Y^\bullet) & \longrightarrow & \mathscr{H}^q(\omega_Y^\bullet) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \sigma_*\mathscr{H}^{q-1}(\Omega_V^\bullet) & \xrightarrow{\ d\log(\pi)\wedge\ } & \sigma_*\mathscr{H}^q(\widetilde{\omega}_V^\bullet) & \longrightarrow & \sigma_*\mathscr{H}^q(\Omega_V^\bullet) & \longrightarrow & 0
\end{array}
$$

(2.8)

where the vertical arrows are adjunction maps. The left and right vertical arrows are injective by Fact 2.7(1). Hence the map (2.8) is injective. We define the map $C^{-1} : \widetilde{\omega}_Y^q \to \mathscr{H}^q(\widetilde{\omega}_Y^\bullet)$ as follows. Using differential symbols, we easily see that the image of the composite map

$$
\widetilde{\omega}_Y^q \longrightarrow \sigma_*(\Omega_V^q \oplus \Omega_V^{q-1}) \xrightarrow{\ C^{-1}\ } \sigma_*\big(\mathscr{H}^q(\Omega_V^\bullet) \oplus \mathscr{H}^{q-1}(\Omega_V^\bullet)\big).
$$

is contained in $\mathscr{H}^q(\widetilde{\omega}_Y^\bullet)$. We thus obtain the map $C^{-1} : \widetilde{\omega}_Y^q \to \mathscr{H}^q(\widetilde{\omega}_Y^\bullet)$. By the construction, C^{-1} is described by the local assignment as in Theorem 2.3(2), which implies the uniqueness of C^{-1}. Moreover it is bijective by the following commutative diagram with exact rows and Fact 2.7(1):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_Y^{q-1} & \xrightarrow{\ d\log(\pi)\wedge\ } & \widetilde{\omega}_Y^q & \longrightarrow & \omega_Y^q & \longrightarrow & 0 \\
& & {\scriptstyle C^{-1}}\downarrow{\scriptstyle\wr} & & {\scriptstyle C^{-1}}\downarrow & & {\scriptstyle C^{-1}}\downarrow{\scriptstyle\wr} & & \\
0 & \longrightarrow & \mathscr{H}^{q-1}(\omega_Y^\bullet) & \xrightarrow{\ d\log(\pi)\wedge\ } & \mathscr{H}^q(\widetilde{\omega}_Y^\bullet) & \longrightarrow & \mathscr{H}^q(\omega_Y^\bullet) & \longrightarrow & 0
\end{array}
$$

This completes the proof of Theorem 2.3(2).

We prove Theorem 2.3(3). By the local presentation of C^{-1}, it is easy to see that the map $1 - C^{-1} : \widetilde{\mathscr{Z}}_Y^q \to \mathscr{H}^q(\widetilde{\omega}_Y^\bullet)$ is surjective and that its kernel contains $\widetilde{\omega}_{Y,\log}^q$. Put

$$
L := \mathsf{Ker}\big(1 - C^{-1} : \widetilde{\mathscr{Z}}_Y^q \to \mathscr{H}^q(\widetilde{\omega}_Y^\bullet)\big).
$$

We show that the natural inclusion $\widetilde{\omega}_{Y,\log}^q \hookrightarrow L$ is surjective. By (2.5), (2.6) and Fact 2.7(1), there is a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{Z}_Y^{q-1} & \xrightarrow{\ d\log(\pi)\wedge\ } & \widetilde{\mathscr{Z}}_Y^q & \longrightarrow & \mathscr{Z}_Y^q & \longrightarrow & 0 \\
& & {\scriptstyle 1-C^{-1}}\downarrow & & {\scriptstyle 1-C^{-1}}\downarrow & & {\scriptstyle 1-C^{-1}}\downarrow & & \\
0 & \longrightarrow & \mathscr{H}^{q-1}(\omega_Y^\bullet) & \xrightarrow{\ d\log(\pi)\wedge\ } & \mathscr{H}^q(\widetilde{\omega}_Y^\bullet) & \longrightarrow & \mathscr{H}^q(\omega_Y^\bullet) & \longrightarrow & 0
\end{array}
$$

By this diagram and Fact 2.7(3), the lower row of the following commutative diagram of complexes is exact:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \omega_{Y,\log}^{q-1} & \xrightarrow{d\log(\pi)\wedge} & \widetilde{\omega}_{Y,\log}^{q} & \longrightarrow & \omega_{Y,\log}^{q} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & \omega_{Y,\log}^{q-1} & \xrightarrow{d\log(\pi)\wedge} & L & \longrightarrow & \omega_{Y,\log}^{q} & \longrightarrow & 0
\end{array}
$$

Hence the middle vertical arrow is surjective, and we obtain Theorem 2.3(3). □

3. Another Cartier isomorphism

Let the notation be as in Setting 1.4. Let i and ψ be as follows:

$$
Y \xhookrightarrow{\ i\ } X \xleftarrow{\ \psi\ } U = X \smallsetminus (Y \cup D).
$$

Let $\{Y_\lambda\}_{\lambda \in \Lambda}$ be the irreducible components of Y. For $\lambda \in \Lambda$, let $\mathscr{I}_\lambda \subset \mathscr{O}_X$ be the defining ideal of Y_λ. For $\mathfrak{m} = (m_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$, put

$$
\mathscr{I}^{(\mathfrak{m})} := \prod_{\lambda \in \Lambda} \mathscr{I}_\lambda^{m_\lambda} \subset \mathscr{O}_X,
$$

where $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, 2, \ldots\}$.

Definition 3.1. (1) We endow $\mathbb{N}^\Lambda$ with a semiorder as follows. For $\mathfrak{m} = (m_\lambda)_{\lambda \in \Lambda}$ and $\mathfrak{n} = (n_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$, we say that $\mathfrak{m} \leq \mathfrak{n}$ if $m_\lambda \leq n_\lambda$ for all $\lambda \in \Lambda$.

(2) We put $\mathbf{0} := (0)_{\lambda \in \Lambda}$ and $\mathbf{1} := (1)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$.

(3) For $\mathfrak{m}, \mathfrak{l} \in \mathbb{N}^\Lambda$, we define a sheaf $\omega_{\mathfrak{m},\mathfrak{l}}^{q}$ $(q \geq 0)$ on $Y_{\text{ét}}$ as

$$
\omega_{\mathfrak{m},\mathfrak{l}}^{q} := \mathscr{I}^{(\mathfrak{m})}/\mathscr{I}^{(\mathfrak{m}+\mathfrak{l})} \otimes_{\mathscr{O}_X} \widetilde{\omega}_Y^{q},
$$

which is a locally free module over $\mathscr{O}_X/\mathscr{I}^{(\mathfrak{l})}$ if $\mathfrak{l} \leq \mathbf{1}$. We define a map $d_{\mathfrak{m}}^{q} : \mathscr{I}^{(\mathfrak{m})} \otimes_{\mathscr{O}_X} \widetilde{\omega}_Y^{q} \to \mathscr{I}^{(\mathfrak{m})} \otimes_{\mathscr{O}_X} \widetilde{\omega}_Y^{q+1}$ by the local assignment

$$
\prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \otimes \omega \mapsto \prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \otimes \left(d\omega + \sum_{\lambda \in \Lambda} m_\lambda \cdot d\log(\pi_\lambda) \wedge \omega \right) \quad (\omega \in \widetilde{\omega}_Y^{q}),
$$

where $\pi_\lambda \in \mathscr{O}_X$ denotes a local uniformizer of Y_λ for each $\lambda \in \Lambda$. This map does not depend on the choice of local uniformizers $\{\pi_\lambda\}_{\lambda \in \Lambda}$. One can easily check that $d_{\mathfrak{m}}^{q+1} \circ d_{\mathfrak{m}}^{q} = 0$ and that $d_{\mathfrak{n}}^{q}$ is compatible with $d_{\mathfrak{m}}^{q}$ for $\mathfrak{n} \geq \mathfrak{m}$. Hence $d_{\mathfrak{m}}^{q}$ induces a differential operator

$$
d = d_{\mathfrak{m},\mathfrak{l}}^{q} : \omega_{\mathfrak{m},\mathfrak{l}}^{q} \to \omega_{\mathfrak{m},\mathfrak{l}}^{q+1}.
$$

Using this d, we regard $\omega_{\mathfrak{m},\mathfrak{l}}^{\bullet} = (\omega_{\mathfrak{m},\mathfrak{l}}^{\bullet}, d)$ as a complex.

(4) We define $\mathscr{Z}^q_{\mathfrak{m},\mathfrak{l}}$ (resp. $\mathscr{B}^q_{\mathfrak{m},\mathfrak{l}}$) as the kernel of $d : \omega^q_{\mathfrak{m},\mathfrak{l}} \to \omega^{q+1}_{\mathfrak{m},\mathfrak{l}}$ (resp. the image of $d : \omega^{q-1}_{\mathfrak{m},\mathfrak{l}} \to \omega^q_{\mathfrak{m},\mathfrak{l}}$), which are étale subsheaves of $\omega^q_{\mathfrak{m},\mathfrak{l}}$.

The following result is due to Kato:

Theorem 3.2. *Let* $\mathfrak{m} = (m_\lambda)_{\lambda \in \Lambda}$ *and* $\mathfrak{l} = (\ell_\lambda)_{\lambda \in \Lambda}$ *be elements of* $\mathbb{N}^\Lambda$ *with* $\mathbf{0} \leq \mathfrak{l} \leq \mathbf{1}$. *Let* $\mathfrak{m}' \in \mathbb{N}^\Lambda$ *be the smallest element that satisfies* $p \cdot \mathfrak{m}' \geq \mathfrak{m}$, *and define* $\mathfrak{l}' = (\ell'_\lambda)_{\lambda \in \Lambda}$ *by*

$$\ell'_\lambda := \begin{cases} 1 & \text{if } \ell_\lambda = 1 \text{ and } p \mid m_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Then there is an isomorphism

$$C^{-1} : \omega^q_{\mathfrak{m}',\mathfrak{l}'} \xrightarrow{\simeq} \mathscr{H}^q(\omega^\bullet_{\mathfrak{m},\mathfrak{l}}) = \mathscr{Z}^q_{\mathfrak{m},\mathfrak{l}} / \mathscr{B}^q_{\mathfrak{m},\mathfrak{l}}, \quad x \otimes \omega \mapsto x^p \otimes \omega,$$

where x *(resp.* ω*) denotes a local section of* $\mathscr{I}^{(\mathfrak{m}')}$ *(resp.* $\widetilde{\omega}^q_{Y,\log}$*).*

We first state an immediate consequence of this theorem. For $\mu \in \Lambda$, put $\widetilde{\omega}^q_{Y_\mu} := \mathscr{O}_{Y_\mu} \otimes_{\mathscr{O}_Y} \widetilde{\omega}^q_Y$ and define $\mathfrak{d}_\mu = (\delta_{\mu,\lambda})_{\lambda \in \Lambda}$ by

$$\delta_{\mu,\lambda} := \begin{cases} 1 & (\lambda = \mu), \\ 0 & (\lambda \neq \mu). \end{cases}$$

Then we have $\omega^q_{\mathfrak{m},\mathfrak{d}_\mu} = \mathscr{I}^{(\mathfrak{m})} \otimes_{\mathscr{O}_X} \widetilde{\omega}^q_{Y_\mu}$, and Theorem 3.2 implies the following:

Corollary 3.3. (1) *The complex* $\omega^\bullet_{\mathfrak{m},\mathfrak{d}_\mu}$ *is acyclic (i.e., exact) if* $p \nmid m_\mu$.

(2) *If* $p \mid m_\mu$, *then we have an isomorphism*

$$C^{-1} : \omega^q_{\mathfrak{m}',\mathfrak{d}_\mu} \xrightarrow{\simeq} \mathscr{H}^q(\omega^\bullet_{\mathfrak{m},\mathfrak{d}_\mu}),$$

where $\mathfrak{m}' \in \mathbb{N}^\Lambda$ *is the smallest element satisfying* $p \cdot \mathfrak{m}' \geq \mathfrak{m}$.

Proof of Theorem 3.2. Let $\pi_\lambda \in \mathscr{O}_X$ be a local uniformizer of Y_λ for each $\lambda \in \Lambda$. If p divides m_λ for any $\lambda \in \Lambda$, then the map $d : \omega^q_{\mathfrak{m},\mathfrak{l}} \to \omega^{q+1}_{\mathfrak{m},\mathfrak{l}}$ sends

$$\prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \otimes \omega \quad \text{to} \quad \prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \otimes d\omega,$$

and the assertion follows from Theorem 2.3(2).

We prove the general case. Take a sequence of elements of $\mathbb{N}^\Lambda$

$$\mathfrak{m} = \mathfrak{m}_0 \leq \mathfrak{m}_1 \leq \mathfrak{m}_2 \leq \cdots \leq \mathfrak{m}_t = p \cdot \mathfrak{m}'$$

such that

$$\sum_{\lambda \in \Lambda} m_{i+1,\lambda} - \sum_{\lambda \in \Lambda} m_{i,\lambda} = 1 \quad \text{for } 0 \leq i < t,$$

where $\mathfrak{m}_i = (m_{i,\lambda})_{\lambda \in \Lambda}$ and $\mathfrak{m}_{i+1} = (m_{i+1,\lambda})_{\lambda \in \Lambda}$. For $0 \leq i \leq t$, define $\mathfrak{l}_i = (\ell_{i,\lambda})_{\lambda \in \Lambda}$ by

$$\ell_{i,\lambda} := \begin{cases} 1 & \text{if } \ell_\lambda = 1 \text{ and } m_{i,\lambda} = m_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

We have inclusions of complexes

$$\omega^\bullet_{\mathrm{m},\mathfrak{l}} = \omega^\bullet_{\mathrm{m}_0,\mathfrak{l}_0} \supset \omega^\bullet_{\mathrm{m}_1,\mathfrak{l}_1} \supset \omega^\bullet_{\mathrm{m}_2,\mathfrak{l}_2} \supset \cdots \supset \omega^\bullet_{\mathrm{m}_t,\mathfrak{l}_t} = \omega^\bullet_{p\cdot\mathrm{m}',\mathfrak{l}'},$$

and exact sequences $(0 \le i < t)$

$$0 \to \omega^\bullet_{\mathrm{m}_{i+1},\mathfrak{l}_{i+1}} \to \omega^\bullet_{\mathrm{m}_i,\mathfrak{l}_i} \to \omega^\bullet_{\mathrm{m}_i,\mathfrak{d}_\mu}/\mathscr{I}^{(\mathfrak{l}_i)}\omega^\bullet_{\mathrm{m}_i,\mathfrak{d}_\mu} \to 0,$$

where $\mu = \mu(i)$ is the unique element of Λ such that $m_{i+1,\mu} > m_{i,\mu}$ and $\mathfrak{d}_\mu$ is as we defined before Corollary 3.3. It is enough to show the following two assertions:

(A) *The complex* $\omega^\bullet_{\mathrm{m}_i,\mathfrak{d}_\mu}/\mathscr{I}^{(\mathfrak{l}_i)}\omega^\bullet_{\mathrm{m}_i,\mathfrak{d}_\mu}$ *is acyclic for* $0 \le i < t$.

(B) *There is an isomorphism* $C^{-1} : \omega^q_{\mathrm{m}',\mathfrak{l}'} \xrightarrow{\simeq} \mathscr{H}^q(\omega^\bullet_{p\cdot\mathrm{m}',\mathfrak{l}'})$.

The assertion (B) follows from the proved case of the theorem. Because $\mathscr{I}^{(\mathfrak{l}_i)}$ is locally free over $\mathscr{O}_X$ for any $0 \le i < t$, the assertion (A) is reduced to the following:

Lemma 3.4. *Let* $\mu \in \Lambda$ *and assume* $p \nmid m_\mu$. *Then the complex* $\omega^\bullet_{\mathrm{m},\mathfrak{d}_\mu}$ *is acyclic.*

We prove this lemma in what follows. Let $\widetilde{\omega}^q_{Y_\mu}$ be as before Corollary 3.3. Note that $\widetilde{\omega}^q_{Y_\mu}$ is generated by $\Omega^q_{Y_\mu}$ (usual Kähler q-forms) and q-forms of the form $d\log(\pi_\lambda) \wedge \eta$ with $\lambda \in \Lambda$ and $\eta \in \Omega^{q-1}_{Y_\mu}$. For $q \ge 1$, there is a residue homomorphism

$$\mathrm{Res}^q : \widetilde{\omega}^q_{Y_\mu} \to \widetilde{\omega}^{q-1}_{Y_\mu}$$

characterized by the following two properties:

(1) *For* $\omega \in \Omega^q_{Y_\mu}$, $\mathrm{Res}^q(\omega)$ *is zero.*

(2) *For* $\eta \in \Omega^{q-1}_{Y_\mu}$, *we have*

$$\mathrm{Res}^q(d\log(\pi_\lambda) \wedge \eta) = \begin{cases} \eta & (\lambda = \mu), \\ 0 & (\lambda \ne \mu). \end{cases}$$

Since $\omega^q_{\mathrm{m},\mathfrak{d}_\mu} = \mathscr{I}^{(\mathrm{m})} \otimes_{\mathscr{O}_X} \widetilde{\omega}^q_{Y_\mu}$, we define a residue homomorphism

$$\mathrm{Res}^q : \omega^q_{\mathrm{m},\mathfrak{d}_\mu} \to \omega^{q-1}_{\mathrm{m},\mathfrak{d}_\mu}$$

by $\mathrm{Res}^q(a \otimes \omega) := a \otimes \mathrm{Res}^q(\omega)$ for $a \in \mathscr{I}^{(\mathrm{m})}$ and $\omega \in \widetilde{\omega}^q_{Y_\mu}$. We show that

$$d\,\mathrm{Res}^q(x) + \mathrm{Res}^{q+1}(dx) = m_\mu \cdot x \quad \text{for any } x \in \omega^q_{\mathrm{m},\mathfrak{d}_\mu}, \tag{3.5}$$

which implies that $\omega^\bullet_{\mathrm{m},\mathfrak{d}_\mu}$ is acyclic if $p \nmid m_\mu$. Put

$$\xi^{\mathrm{m}} := \prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \in \prod_{\lambda \in \Lambda} \mathscr{I}_\lambda^{m_\lambda} = \mathscr{I}^{(\mathrm{m})}.$$

For $x = \xi^{\mathrm{m}} \otimes \omega$ with $\omega \in \Omega^q_{Y_\mu}$, we have $\mathrm{Res}^q(x) = 0$ and

$$\mathrm{Res}^{q+1}(dx) = \mathrm{Res}^{q+1}\big(\xi^{\mathrm{m}} \otimes \big(d\omega + \textstyle\sum_{\lambda \in \Lambda} m_\lambda \cdot d\log(\pi_\lambda) \wedge \omega\big)\big)$$

$$= \xi^{\mathrm{m}} \otimes m_\mu \cdot \omega = m_\mu \cdot x.$$

For $x = \xi^{\mathfrak{m}} \otimes d\log(\pi_\nu) \wedge \eta$ with $\eta \in \Omega_{Y_\mu}^{q-1}$ and $\nu \neq \mu$, we have $\mathrm{Res}^q(x) = 0$ and

$$\mathrm{Res}^{q+1}(dx)$$
$$= \mathrm{Res}^{q+1}\big(\xi^{\mathfrak{m}} \otimes \big(-d\log(\pi_\nu) \wedge d\eta + \textstyle\sum_{\lambda \in \Lambda} m_\lambda \cdot d\log(\pi_\lambda) \wedge d\log(\pi_\nu) \wedge \eta\big)\big)$$
$$= \xi^{\mathfrak{m}} \otimes m_\mu \cdot d\log(\pi_\nu) \wedge \eta = m_\mu \cdot x.$$

Finally for $x = \xi^{\mathfrak{m}} \otimes d\log(\pi_\mu) \wedge \eta$ with $\eta \in \Omega_{Y_\mu}^{q-1}$, we have

$$d\,\mathrm{Res}^q(x) = d(\xi^{\mathfrak{m}} \otimes \eta) = \xi^{\mathfrak{m}} \otimes \big(d\eta + \textstyle\sum_{\lambda \in \Lambda} m_\lambda \cdot d\log(\pi_\lambda) \wedge \eta\big)$$

and

$$\mathrm{Res}^{q+1}(dx)$$
$$= \mathrm{Res}^{q+1}\big(\xi^{\mathfrak{m}} \otimes \big(-d\log(\pi_\mu) \wedge d\eta + \textstyle\sum_{\lambda \in \Lambda} m_\lambda \cdot d\log(\pi_\lambda) \wedge d\log(\pi_\mu) \wedge \eta\big)\big)$$
$$= \xi^{\mathfrak{m}} \otimes \big(-d\eta - \textstyle\sum_{\lambda \neq \mu} m_\lambda \cdot d\log(\pi_\lambda) \wedge \eta\big) = -d\,\mathrm{Res}^q(x) + m_\mu \cdot x.$$

Thus we obtain (3.5), Lemma 3.4 and Theorem 3.2. $\qquad\square$

Corollary 3.6. *$\mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^q$ is generated by local sections of the following forms*:

(1) $\prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \otimes \big(d\eta + \sum_{\lambda \in \Lambda} m_\lambda \cdot d\log(\pi_\lambda) \wedge \eta\big)$ *with* $\eta \in \widetilde{\omega}_Y^{q-1}$, *where* $\pi_\lambda \in \mathscr{O}_X$ *is a local uniformizer of* Y_λ *for each* $\lambda \in \Lambda$.

(2) $x^p \otimes \omega$ *with* $x \in \mathscr{I}^{(\mathfrak{m}')}$ *and* $\omega \in \widetilde{\omega}_{Y,\log}^q$, *where* $\mathfrak{m}' \in \mathbb{N}^\Lambda$ *is the smallest element satisfying* $p \cdot \mathfrak{m}' \geq \mathfrak{m}$.

4. Structure of the sheaf $\mathscr{U}^1 M_1^q$

Let the notation be as in Setting 1.4. Let i and ψ be as follows:

$$Y \overset{i}{\hookrightarrow} X \overset{\psi}{\longleftarrow} U = X \smallsetminus (Y \cup D).$$

For $q \geq 0$ and $n \geq 1$, we define étale sheaves M_n^q and $\mathscr{K}_q^M$ on Y as

$$M_n^q := i^* R^q \psi_* \mu_{p^n}^{\otimes q} \quad \text{and} \quad \mathscr{K}_q^M := \begin{cases} \mathbb{Z} & (q = 0), \\ i^*\psi_*\mathscr{O}_U^\times & (q = 1), \\ (i^*\psi_*\mathscr{O}_U^\times)^{\otimes q}/J_q & (q \geq 2). \end{cases}$$

Here J_q for $q \geq 2$ denotes the subsheaf of $(i^*\psi_*\mathscr{O}_U^\times)^{\otimes q}$ generated by local sections $x_1 \otimes \cdots \otimes x_q$ such that $x_r + x_{r'} = 0$ or 1 for some $1 \leq r < r' \leq q$. There is a homomorphism of étale sheaves (see [Bloch and Kato 1986, 1.2])

$$\varrho_{(X,D),n}^q : \mathscr{K}_q^M/p^n \to M_n^q, \tag{4.1}$$

which is a geometric version of Tate's norm residue map. For $x_1, \ldots, x_q \in i^*\psi_*\mathscr{O}_U^\times$, we denote the image of $\{x_1, x_2, \ldots, x_q\} \in \mathscr{K}_q^M$ under (4.1) again by $\{x_1, x_2, \ldots, x_q\}$.

Definition 4.2. (1) We define $\mathscr{U}^0\mathscr{K}_q^M$ as the full-sheaf $\mathscr{K}_q^M$, and $\mathscr{U}^1\mathscr{K}_q^M$ as the subsheaf generated locally by symbols of the form

$$\{1+x, y_1, \ldots, y_{q-1}\} \quad \text{with } x \in i^*\mathscr{I} \text{ and } y_1, \ldots, y_{q-1} \in i^*\psi_*\mathscr{O}_U^\times,$$

where $\mathscr{I} \subset \mathscr{O}_X$ denotes the defining ideal of Y. We define $\mathscr{U}^0 M_n^q$ and $\mathscr{U}^1 M_n^q$ as the image of $\mathscr{U}^0\mathscr{K}_q^M$ and $\mathscr{U}^1\mathscr{K}_q^M$ under the map (4.1), respectively.

(2) For $\mathfrak{m} = (m_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$ with $\mathfrak{m} \geq 1$, we define $\mathscr{U}^{(\mathfrak{m})}\mathscr{K}_q^M$ as the subsheaf generated locally by symbols of the form

$$\{1+x, y_1, \ldots, y_{q-1}\} \quad \text{with } x \in i^*\mathscr{I}^{(\mathfrak{m})} \text{ and } y_1, \ldots, y_{q-1} \in i^*\psi_*\mathscr{O}_U^\times,$$

where $\mathscr{I}^{(\mathfrak{m})}$ is as we defined in the previous section. We define $\mathscr{U}^{(\mathfrak{m})}M_n^q$ as the image of $\mathscr{U}^{(\mathfrak{m})}\mathscr{K}_q^M$ under the map (4.1).

(3) We define $\mathfrak{e} = (e_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$ as follows. For $\lambda \in \Lambda$, let e_λ be the absolute ramification index of the discrete valuation ring $\mathscr{O}_{X,\eta_\lambda}$, where η_λ denotes the generic point of Y_λ. We put $e_\lambda' := pe_\lambda/(p-1)$ for $\lambda \in \Lambda$ and $\mathfrak{e}' := (e_\lambda')_{\lambda \in \Lambda} \in \mathbb{Q}^\Lambda$.

(4) For $\mathfrak{m} = (m_\lambda)_{\lambda \in \Lambda}$ and $\mathfrak{n} = (n_\lambda)_{\lambda \in \Lambda} \in \mathbb{Q}^\Lambda$, we say that $\mathfrak{m} < \mathfrak{n}$ (resp. $\mathfrak{m} \leq \mathfrak{n}$) if $m_\lambda < n_\lambda$ (resp. $m_\lambda \leq n_\lambda$) for any $\lambda \in \Lambda$.

The following lemma is straight-forward, and left to the reader:

Lemma 4.3. (1) *We have $\mathscr{U}^1 M_n^q = \mathscr{U}^{(1)} M_n^q$, where 1 denotes $(1)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$.*

(2) *M_n^q, $\mathscr{U}^0 M_n^q$ and $\mathscr{U}^1 M_n^q$ are contravariantly functorial in the pair (X, D).*

The main result of this section is the following, which is also due to Kato:

Theorem 4.4. *Let $\mathfrak{m}$ and $\mathfrak{l}$ be elements of $\mathbb{N}^\Lambda$.*

(1) *Assume $1 \leq \mathfrak{m} < \mathfrak{e}' + 1$ and $0 \leq \mathfrak{l} \leq 1$. For each $\lambda \in \Lambda$, assume $\ell_\lambda = 0$ if $m_\lambda \geq e_\lambda'$. Then there is an isomorphism*

$$\omega_{\mathfrak{m},\mathfrak{l}}^{q-1} / \mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1} \overset{\sim}{\longrightarrow} \mathscr{U}^{(\mathfrak{m})}M_1^q / \mathscr{U}^{(\mathfrak{m}+\mathfrak{l})}M_1^q$$

given by the local assignment

$$x \otimes d\log(y_1) \wedge \cdots \wedge d\log(y_{q-1}) \mapsto \{1+\tilde{x}, y_1, \ldots, y_{q-1}\} + \mathscr{U}^{(\mathfrak{m}+\mathfrak{l})}M_1^q$$

for $x \in i^*(\mathscr{I}^{(\mathfrak{m})}/\mathscr{I}^{(\mathfrak{m}+\mathfrak{l})})$ and $y_1, \ldots, y_{q-1} \in i^*\psi_*\mathscr{O}_U^\times$, where $\tilde{x}$ is a lift of x to $i^*\mathscr{I}^{(\mathfrak{m})}$.

(2) *If $\mathfrak{m} \geq \mathfrak{e}'$, then $\mathscr{U}^{(\mathfrak{m})}M_1^q$ is zero.*

Theorem 4.4 describes the structure of the sheaf $\mathscr{U}^1 M_1^q$ as follows.

Corollary 4.5. *Take a sequence of elements of $\mathbb{N}^\Lambda$*

$$1 = \mathfrak{m}_0 \leq \mathfrak{m}_1 \leq \mathfrak{m}_2 \leq \cdots \leq \mathfrak{m}_i \leq \cdots \leq \mathfrak{m}_t$$

satisfying the following conditions:

(a) $\mathfrak{l}_i := \mathfrak{m}_{i+1} - \mathfrak{m}_i$ *satisfies* $\mathfrak{l}_i \leq \mathbf{1}$ *for any* $i \leq t - 1$.

(b) $\mathfrak{e}' \leq \mathfrak{m}_t < \mathfrak{e}' + 1$. (*Note that such* $\mathfrak{m}_t$ *is unique in* $\mathbb{N}^\Lambda$.)

(c) *For any* (i, λ) *with* $i \leq t - 1$ *and* $m_{i,\lambda} \geq e'_\lambda$, *the* λ-*component of* $\mathfrak{l}_i$ *is zero*.

We then have

$$\mathscr{U}^{(\mathfrak{m}_i)} M_1^q / \mathscr{U}^{(\mathfrak{m}_{i+1})} M_1^q \cong \omega_{\mathfrak{m}_i, \mathfrak{l}_i}^{q-1} / \mathscr{Z}_{\mathfrak{m}_i, \mathfrak{l}_i}^{q-1} \quad \text{for } 0 \leq i \leq t - 1, \ \mathscr{U}^{(\mathfrak{m}_t)} M_1^q = 0,$$

by Theorem 4.4(1) *and* (2), *respectively*.

To prove Theorem 4.4, we need the following lemma:

Lemma 4.6. (1) *For* $\mathfrak{m}, \mathfrak{n} \in \mathbb{N}^\Lambda$, *we have* $\{\mathscr{U}^{(\mathfrak{m})} \mathscr{K}_q^M, \mathscr{U}^{(\mathfrak{n})} \mathscr{K}_{q'}^M\} \subset \mathscr{U}^{(\mathfrak{m}+\mathfrak{n})} \mathscr{K}_{q+q'}^M$.

(2) *There is a surjective homomorphism*

$$i^* \mathcal{O}_X \otimes (i^* \psi_* \mathcal{O}_U^\times)^{\otimes r} \to \widetilde{\omega}_Y^r, \quad x \otimes y_1 \otimes \cdots \otimes y_r \mapsto \bar{x} \cdot d \log(y_1) \wedge \cdots \wedge d \log(y_r),$$

where for $x \in i^* \mathcal{O}_X$, $\bar{x}$ *denotes its residue class in* $\mathcal{O}_Y$. *The kernel of this map is generated by local sections of the following forms*:

- $x \otimes y_1 \otimes \cdots \otimes y_r$ *with* $x \in i^* \mathscr{I}$ *or* $y_s \in i^*(1 + \mathscr{I})$ *for some* $1 \leq s \leq r$,
- $x \otimes y_1 \otimes \cdots \otimes y_r$ *with* $y_s = y_{s'}$ *for some* $1 \leq s < s' \leq r$,
- $\sum_{s=1}^m (x_s \otimes x_s \otimes y_1 \otimes \cdots \otimes y_{r-1}) - \sum_{t=1}^\ell (x'_t \otimes x'_t \otimes y_1 \otimes \cdots \otimes y_{m-1})$ *such that all* x_s *and* x'_t *belong to* $i^*(\mathcal{O}_X \cap \psi_* \mathcal{O}_U^\times)$ *and such that the sums* $\sum_{s=1}^m x_s$ *and* $\sum_{t=1}^\ell x'_t$ *taken in* $i^* \mathcal{O}_X$ *satisfy* $\sum_{s=1}^m x_s \equiv \sum_{t=1}^\ell x'_t \mod i^* \mathscr{I}$.

Proof. (1) This follows from the same argument as in [Bloch and Kato 1986, Lemma 4.1].

(2) Let z be a point on Y. Put $A := \mathcal{O}_{X, \bar{z}}^{\mathrm{sh}}$, $I := \mathscr{I}_{\bar{z}}$ and $L := (\psi_* \mathcal{O}_U^\times)_{\bar{z}}$. Let $A[L]$ be the free A-module over the set L. There is a surjective A-homomorphism $A[L] \to (\widetilde{\omega}_Y^1)_{\bar{z}}$ sending $a[b]$ to $\bar{a} \cdot d \log(b)$. Its kernel is the A-submodule generated by elements of the following forms:

 (i) $a[b]$ with $a \in I$ or $b \in 1 + I$,

 (ii) $[b \cdot b'] - [b] - [b']$ with $b, b' \in L$,

 (iii) $\sum_{s=1}^m a_s[a_s] - \sum_{t=1}^\ell a'_t[a'_t]$ $(a_s, a'_t \in A \cap L)$ with $\sum_{s=1}^m a_s \equiv \sum_{t=1}^\ell a'_t \mod I$.

The claim follows from this fact. The details are straight-forward and left to the reader. $\qquad \square$

Proof of Theorem 4.4. (1) By Lemma 4.6, the local assignment in the theorem gives a well-defined surjective homomorphism of sheaves

$$\rho_{\mathfrak{m}, \mathfrak{l}} : \omega_{\mathfrak{m}, \mathfrak{l}}^{q-1} \twoheadrightarrow \mathscr{U}^{(\mathfrak{m})} M_1^q / \mathscr{U}^{(\mathfrak{m}+\mathfrak{l})} M_1^q.$$

We prove $\rho_{\mathfrak{m},\mathfrak{l}}(\mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1}) = 0$ assuming $\mathfrak{m} < \mathfrak{e}'$, locally on Y. We may assume that Y_λ's are principal on X. Fix uniformizers $\pi_\lambda \in i^*\mathscr{O}_X$ of Y_λ $(\lambda \in \Lambda)$ and put

$$\xi^{\mathfrak{m}} := \prod_{\lambda \in \Lambda} \pi_\lambda^{m_\lambda} \in i^*\mathscr{I}^{(\mathfrak{m})}.$$

It is enough to show that local sections of $\mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1}$ of the forms (1) and (2) of Corollary 3.6 map to zero under $\rho_{\mathfrak{m},\mathfrak{l}}$. For $y_1 \in i^*\mathscr{O}_X^\times$ and $y_2, \ldots, y_{q-1} \in i^*\psi_*\mathscr{O}_U^\times$, we have

$$\{1 + \xi^{\mathfrak{m}} y_1, y_1, y_2, \ldots, y_{q-1}\} + \sum_{\lambda \in \Lambda} m_\lambda \cdot \{1 + \xi^{\mathfrak{m}} y_1, \pi_\lambda, y_2, \ldots, y_{q-1}\}$$
$$= \{1 + \xi^{\mathfrak{m}} y_1, \xi^{\mathfrak{m}} y_1, y_2, \ldots, y_{q-1}\}$$
$$= -\{1 + \xi^{\mathfrak{m}} y_1, -1, y_2, \ldots, y_{q-1}\} \in \mathscr{U}^{\mathfrak{m}+\mathfrak{l}}\mathscr{K}_q^M.$$

Hence $\rho_{\mathfrak{m},\mathfrak{l}}(\omega) = 0$ for

$$\omega = \xi^{\mathfrak{m}} \otimes \left(d\eta + \sum_{\lambda \in \Lambda} m_\lambda \cdot d\log(\pi_\lambda) \wedge \eta\right) \in \mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1}$$
$$\text{with } \eta = y_1 \cdot d\log(y_2) \wedge \cdots \wedge d\log(y_{q-1}) \in \widetilde{\omega}_Y^{q-2}.$$

Next let $\mathfrak{m}' \in \mathbb{N}^\Lambda$ be the smallest element that satisfies $p \cdot \mathfrak{m}' \geq \mathfrak{m}$. For $x \in i^*\mathscr{I}^{(\mathfrak{m}')}$ and $y_1, \ldots, y_{q-1} \in i^*\psi_*\mathscr{O}_U^\times$, we have

$$\{1 + x^p, y_1, \ldots, y_{q-1}\} - p \cdot \{1 + x, y_1, \ldots, y_{q-1}\} \in \mathscr{U}^{\mathfrak{m}'+\mathfrak{e}}\mathscr{K}_q^M \subset \mathscr{U}^{\mathfrak{m}+\mathfrak{l}}\mathscr{K}_q^M,$$

where we have used the assumption $\mathfrak{m} < \mathfrak{e}'$ to verify $\mathfrak{m}' + \mathfrak{e} \geq \mathfrak{m} + \mathfrak{l}$. Hence $\rho_{\mathfrak{m},\mathfrak{l}}(\omega) = 0$ for

$$\omega = x^p \otimes d\log(y_1) \wedge \cdots \wedge d\log(y_{q-1}) \in \mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1}.$$

Thus we obtain $\rho_{\mathfrak{m},\mathfrak{l}}(\mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1}) = 0$ for $\mathfrak{m} < \mathfrak{e}'$.

We prove $\rho_{\mathfrak{m},\mathfrak{l}}(\mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1}) = 0$ for $\mathfrak{m} < \mathfrak{e}' + \mathbf{1}$. Define $\mathfrak{n} = (n_\lambda)_{\lambda \in \Lambda}$ and $\mathfrak{l}' = (\ell'_\lambda)_{\lambda \in \Lambda}$ as

$$n_\lambda := \begin{cases} m_\lambda & \text{if } m_\lambda < e'_\lambda, \\ m_\lambda - 1 & \text{if } m_\lambda \geq e'_\lambda, \end{cases} \quad \text{and} \quad \ell'_\lambda := \begin{cases} \ell_\lambda & \text{if } m_\lambda < e'_\lambda, \\ 1 & \text{if } m_\lambda \geq e'_\lambda. \end{cases}$$

We have $\mathfrak{n} \leq \mathfrak{m}$, $\mathbf{1} \leq \mathfrak{n} < \mathfrak{e}'$, $\mathbf{0} \leq \mathfrak{l}' \leq \mathbf{1}$ and $\mathfrak{n} + \mathfrak{l}' = \mathfrak{m} + \mathfrak{l}$ by the assumptions on $\mathfrak{m}$ and $\mathfrak{l}$, and there is a commutative diagram

$$
\begin{array}{ccc}
\omega_{\mathfrak{m},\mathfrak{l}}^{q-1} & \lhook\joinrel\longrightarrow & \omega_{\mathfrak{n},\mathfrak{l}'}^{q-1} \\
{\scriptstyle \rho_{\mathfrak{m},\mathfrak{l}}} \downarrow & & \downarrow {\scriptstyle \rho_{\mathfrak{n},\mathfrak{l}'}} \\
\mathscr{U}^{(\mathfrak{m})}M_1^q / \mathscr{U}^{(\mathfrak{m}+\mathfrak{l})}M_1^q & \lhook\joinrel\longrightarrow & \mathscr{U}^{(\mathfrak{n})}M_1^q / \mathscr{U}^{(\mathfrak{n}+\mathfrak{l}')}M_1^q
\end{array}
$$

where the top horizontal arrow maps $\mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1}$ into $\mathscr{Z}_{\mathfrak{n},\mathfrak{l}'}^{q-1}$. Hence $\rho_{\mathfrak{m},\mathfrak{l}}(\mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1})$ is zero by the previous case and the injectivity of the bottom horizontal arrow.

It remains to prove the injectivity of the induced map

$$\bar{\rho}_{\mathfrak{m},\mathfrak{l}} : \omega_{\mathfrak{m},\mathfrak{l}}^{q-1}/\mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1} \to \mathscr{U}^{(\mathfrak{m})}M_1^q/\mathscr{U}^{(\mathfrak{m}+\mathfrak{l})}M_1^q.$$

Since $\omega_{\mathfrak{m},\mathfrak{l}}^{q-1}/\mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1}$ is a subsheaf of $\omega_{\mathfrak{m},\mathfrak{l}}^q$, the canonical adjunction map

$$\omega_{\mathfrak{m},\mathfrak{l}}^{q-1}/\mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1} \to \bigoplus_{y \in Y^0} i_{y*}i_y^*(\omega_{\mathfrak{m},\mathfrak{l}}^{q-1}/\mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1})$$

is injective by Theorem 2.3(1), where for $y \in Y^0$, i_y denotes the natural map $y \hookrightarrow Y$. Hence we may replace X with $\mathrm{Spec}(\mathcal{O}_{X,y_\mu})$ ($\mu \in \Lambda$), where y_μ denotes the generic point of Y_μ. By the definition of $d : \omega_{\mathfrak{m},\mathfrak{l}}^{q-1} \to \omega_{\mathfrak{m},\mathfrak{l}}^q$, we have

$$\omega_{\mathfrak{m},\mathfrak{l}}^{q-1}/\mathscr{Z}_{\mathfrak{m},\mathfrak{l}}^{q-1} \cong \begin{cases} \Omega_{y_\mu}^{q-1} & \text{if } p \nmid m_\mu, \ \ell_\mu = 1 \text{ and } m_\mu < e_\mu', \\ d\Omega_{y_\mu}^{q-1} \oplus d\Omega_{y_\mu}^{q-2} & \text{if } p \mid m_\mu, \ \ell_\mu = 1 \text{ and } m_\mu < e_\mu', \\ 0 & \text{otherwise}, \end{cases}$$

and the assertion follows from [Bloch and Kato 1986, Corollary 1.4.1(ii)–(iv)].

(2) For $\mathfrak{m} \in \mathbb{N}^\Lambda$ with $\mathfrak{m} \geq \mathfrak{e}'$, $1 + \mathscr{I}^{(\mathfrak{m})}$ is contained in $(1 + \mathscr{I}^{(\mathfrak{m}-\mathfrak{e})})^p$. The assertion follows from this fact. $\qquad\square$

5. Surjectivity of the symbol map

Let O_K be as in Section 1, and let π be a prime element of O_K.

Definition 5.1. For an injective morphism of monoids

$$h : \mathbb{N} \to \mathbb{N}^d, \quad 1 \mapsto (e_\lambda)_{1 \leq \lambda \leq d},$$

we define a scheme X^h and a divisor D^h on X^h as

$$X^h := \mathrm{Spec}\big(O_K[T_1, \ldots, T_d]\big/\big(\textstyle\prod_{\lambda \text{ with } e_\lambda \geq 1} T_\lambda^{e_\lambda} - \pi\big)\big)$$
$$D^h := \big\{\textstyle\prod_{\lambda \text{ with } e_\lambda = 0} T_\lambda = 0\big\} \subset X^h.$$

Put $Y^h := (X^h)_{s,\mathrm{red}}$. We define a scheme $\mathscr{Y}^h$ as $\mathrm{Spec}(k[T_1, T_2, \ldots, T_d])$ and denote the natural closed immersion $Y^h \hookrightarrow \mathscr{Y}^h$ by ι^h.

Let (X, D) be as in Setting 1.4. We introduce the following terminology:

Definition 5.2. We say that (X, D) is *quasilog smooth over* $B = \mathrm{Spec}(O_K)$, if it is, everywhere étale locally on X, isomorphic to (X^h, D^h) for some injective morphism of monoids $h : \mathbb{N} \to \mathbb{N}^d$ and a prime element $\pi \in O_K$, where d denotes $\dim(X)$.

Example 5.3. Let (X, D) be a pair as in Setting 1.4.

(1) Let $\mathcal{M}$ be the log structure on X associated with D, and let $\mathcal{N}$ be the log structure on $B = \mathrm{Spec}(O_K)$ associated with the closed point $s \in B$. If the canonical morphism $(X, \mathcal{M}) \to (B, \mathcal{N})$ of log schemes is smooth in the sense of [Kato 1989, (3.3)], then the pair (X, D) is quasilog smooth over $\mathrm{Spec}(O_K)$ in our sense. Note also that the converse is not necessarily true.

(2) As a consequence of (1), a pair (X, D) as in Setting 1.4 is quasilog smooth over $\mathrm{Spec}(O_K)$, if the multiplicities of the irreducible components of X_s are prime to p.

Theorem 5.4. *Assume that K contains a primitive p-th root of unity ζ_p, and that (X, D) is quasilog smooth over B. Then the symbol map in* (4.1),

$$\varrho^q_{(X,D),1} : \mathcal{K}^M_q / p^n \to M^q_n,$$

is surjective, and there is an isomorphism

$$M^q_1 / \mathcal{U}^1 M^q_1 \xrightarrow{\simeq} \widetilde{\omega}^q_{Y,\log}$$

fitting into a commutative diagram

$$\begin{array}{ccc}
 & \overline{\mathcal{K}^M_q / p} & \\
 \overline{\varrho^q_{(X,D),n}} \swarrow & & \searrow d\log \\
 M^q_1 / \mathcal{U}^1 M^q_1 & \xrightarrow{\ \ \simeq\ \ } & \widetilde{\omega}^q_{Y,\log}
\end{array} \tag{5.5}$$

Our proof of Theorem 5.4 will be complete in the next section. In this section, we reduce the theorem to Lemma 5.10 below. Let y be a generic point of Y, and let i_y be the natural map $y \hookrightarrow Y$. The strict henselian local ring $\mathcal{O}^{\mathrm{sh}}_{X,\bar{y}}$ is a discrete valuation ring by the regularity of X. Hence there is an isomorphism

$$i^*_y(M^q_1 / \mathcal{U}^1 M^q_1) \xrightarrow{\simeq} \Omega^q_{y,\log} \oplus \Omega^{q-1}_{y,\log} = i^*_y \widetilde{\omega}^q_{Y,\log} \tag{5.6}$$

[Bloch and Kato 1986, Lemma 5.3]. Since, by Theorem 2.3(1), $\widetilde{\omega}^q_Y$ is locally free over $\mathcal{O}_Y$, the adjunction map

$$\widetilde{\omega}^q_{Y,\log} \to \bigoplus_{y \in Y^0} i_{y*} i^*_y \widetilde{\omega}^q_{Y,\log}$$

is injective, and the isomorphism (5.6) induces a surjective map

$$\mathcal{U}^0 M^q_1 / \mathcal{U}^1 M^q_1 \longrightarrow\!\!\!\!\!\rightarrow \widetilde{\omega}^q_{Y,\log}$$

(see [Sato 2013, Lemma 2.3]). This map fits into the diagram (5.5) with $M^q_1 / \mathcal{U}^1 M^q_1$ replaced by $\mathcal{U}^0 M^q_1 / \mathcal{U}^1 M^q_1$. In what follows, put

$$M^q := M^q_1, \quad N^q := \mathrm{Ker}\big(\mathcal{U}^0 M^q / \mathcal{U}^1 M^q \to \widetilde{\omega}^q_{Y,\log}\big) \quad \text{and} \quad L^q := M^q / \mathcal{U}^0 M^q.$$

We have show that $N^q = 0$ and $L^q = 0$. Note also that once we show $L^q = 0$, we will have shown that $M_n^q = \mathscr{U}^0 M_n^q$ for all $n \geq 1$ by a standard argument as in [Bloch and Kato 1986, Corollary 6.1.1]. We first prove that $N^q = L^q = 0$ in a simple case:

Lemma 5.7. *Assume that (X, D) is quasilog smooth over B and that the underlying scheme X is smooth over B. Then we have $N^q = 0$ and $L^q = 0$ (i.e., Theorem 5.4 holds for (X, D)).*

Proof. When $D = \varnothing$, the assertion follows from a theorem of Bloch and Kato [1986, Theorem 1.4]. We proceed with the proof of the lemma by using induction on the number of the irreducible components of D. Since the problem is étale local on X and X is smooth over B by assumption, we may suppose that $X = \mathrm{Spec}(O_K[T_2, T_3, \ldots, T_d])$ and that $D = \{T_2 T_3 \cdots T_r = 0\} \subset X$, where $2 \leq r \leq d = \dim(X)$.

Fix an irreducible component V of D, which is also smooth over B. Put $D' := D - V$ as an effective Cartier divisor, and let E be the pullback of D' onto V. It is easy to see that E is a simple normal crossing divisor on V and the pair (V, E) is quasilog smooth over B and that the underlying scheme V is smooth over B. Recall that $Y := X_{s,\mathrm{red}} \ (= X_s)$ and $U := X \setminus (Y \cup D)$. Now put $Z := V_{s,\mathrm{red}} \ (= V_s)$, $U' := X \setminus (Y \cup D')$ and $W := V \setminus (Z \cup E)$, and consider a commutative diagram of schemes

We then have a commutative diagram of étale sheaves on Y whose upper row is a complex and whose lower row is exact

$$
\begin{array}{ccccc}
\mathscr{K}_{q,(X,D')}^M / p & \longrightarrow & \mathscr{K}_q^M / p & \twoheadrightarrow & \iota_* \mathscr{K}_{q-1,(V,E)}^M / p \\
{\scriptstyle \varrho_1^q} \downarrow & & {\scriptstyle \varrho_1^q} \downarrow & & {\scriptstyle \varrho_1^{q-1}} \downarrow \\
M_{(X,D')}^q & \longrightarrow & M^q & \longrightarrow & \iota_* M_{(V,E)}^{q-1}
\end{array}
\tag{5.8}
$$

Here we put $M_{(X,D')}^q := i^* R^q \psi'_* \mu_p^{\otimes q}$ and $M_{(V,E)}^{q-1} := i'^* R^{q-1} \theta_* \mu_p^{\otimes(q-1)}$, and the sheaves $\mathscr{K}_{q,(X,D')}^M$ and $\mathscr{K}_{q-1,(V,E)}^M$ are Milnor K-sheaves defined as quotients of $(i^* \psi'_* \mathscr{O}_{U'}^\times)^{\otimes q}$ and $(i'^* \theta_* \mathscr{O}_W^\times)^{\otimes(q-1)}$, respectively. The right arrow in the upper row is a boundary map of Milnor K-sheaves, which one can check to be surjective. The lower row is obtained by applying $i^* R^q \psi'_*$ to the Gysin distinguished triangle on $(U')_{\text{ét}}$,

$$\alpha_* \mu_p^{\otimes(q-1)}[-2] \to \mu_p^{\otimes q} \to R\beta_* \mu_p^{\otimes q} \to \alpha_* \mu_p^{\otimes(q-1)}[-1].$$

Now the left and the right vertical arrows in (5.8) are surjective by the induction hypothesis, and we obtain the surjectivity of $\varrho^q_{(X,D),1}$ by a simple diagram chase, that is, $L^q = 0$.

To prove that $N^q = 0$, we consider the following commutative diagram of sheaves on $Y_{\text{ét}}$, whose middle row is exact and whose other rows and columns are complexes:

$$
\begin{array}{ccccccc}
\mathscr{U}^1 M^q_{(X,D')} & \longrightarrow & \mathscr{U}^1 M^q & \xrightarrow{r_1} & \iota_* \mathscr{U}^1 M^{q-1}_{(V,E)} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
M^q_{(X,D')} & \longrightarrow & M^q & \xrightarrow{r_2} & \iota_* M^{q-1}_{(V,E)} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow \widetilde{\omega}^q_{Y',\log} & \longrightarrow & \widetilde{\omega}^q_{Y,\log} & \xrightarrow{r_3} & \iota_* \widetilde{\omega}^{q-1}_{Z,\log} & \longrightarrow & 0
\end{array}
$$

where $\widetilde{\omega}^q_{Y',\log}$ (resp. $\widetilde{\omega}^{q-1}_{Z,\log}$) denotes the logarithmic differential sheaf defined for the pair (X, D') (resp. (V, E)); the arrow r_1 denotes the map induced by r_2, and r_3 denotes a residue map of logarithmic differential sheaves. By the smoothness of X over B, we have

$$
\widetilde{\omega}^q_{Y,\log} \cong \Omega^q_Y(\log D_Y)_{\log} \oplus \Omega^{q-1}_Y(\log D_Y)_{\log} \quad (D_Y := D \cap Y)
$$

and similar presentations for $\widetilde{\omega}^q_{Y',\log}$ and $\widetilde{\omega}^{q-1}_{Z,\log}$. By this fact and a simple variant of the purity of logarithmic differential sheaves [Shiho 2007, Theorem 3.2], we see that the bottom row is exact. On the other hand, one can easily check that the sequence

$$
(0 \longrightarrow)\, \mathscr{U}^m M^q_{(X,D')} \longrightarrow \mathscr{U}^m M^q \xrightarrow{r_1} \iota_* \mathscr{U}^m M^{q-1}_{(V,E)} \longrightarrow 0
$$

is exact for $1 \le m \le e' := pe/(p-1)$ by Theorem 4.4 and descending induction on m. Hence the top row in the above diagram is also exact. Now the left and the right columns are exact by the induction hypothesis, and the middle column is exact as well by a simple diagram chase, which shows that $N^q = 0$. $\qquad \square$

We now reduce the general case of Theorem 5.4 to Lemma 5.10 below. Fix an arbitrary point $x \in Y$. We show the stalks $(N^q)_{\bar{x}}$ and $(L^q)_{\bar{x}}$ are zero by induction on $c := \text{codim}_Y(x)$. If $c = 0$, then $(N^q)_{\bar{x}} = (L^q)_{\bar{x}} = 0$ by (5.6). In what follows, assume $c \ge 1$ and the following induction hypothesis:

(1) $(N^q)_{\bar{y}} = (L^q)_{\bar{y}} = 0$ *for any $q \ge 0$ and any $y \in Y$ of codimension $\le c - 1$.*

Since the problem is étale local, we may assume $(X, D) = (X^h, D^h)$ for an injective morphism $h : \mathbb{N} \hookrightarrow \mathbb{N}^d$ of monoids $(d = \dim(X))$. Sorting the components of $\mathbb{N}^d$ if necessarily, we assume the following two conditions:

(2) *The first component of $h(1) \in \mathbb{N}^d$ is nonzero.*

(3) *The composite map*

$$Y \overset{\iota^h}{\hookrightarrow} \mathscr{Y} = \mathrm{Spec}(k[T_1, \ldots, T_d]) \longrightarrow \mathrm{Spec}(k[T_{c+2}, \ldots, T_d])$$

sends x to the generic point of $\mathrm{Spec}(k[T_{c+2}, \ldots, T_d])$ *[Tsuji 2000, Lemma 5.3].*

Following the idea of Tsuji [2000, proof of Theorem 5.1], we decompose $h : \mathbb{N} \to \mathbb{N}^d$ into a sequence of morphisms of monoids

$$h : \mathbb{N} \xrightarrow{\ h^0\ } \mathbb{N}^d \xrightarrow{\ \kappa^1\ } \mathbb{N}^d \xrightarrow{\ \kappa^2\ } \cdots \xrightarrow{\ \kappa^r\ } \mathbb{N}^d$$

which satisfies the following two conditions:

(4) $h^0(1) = (e, \overbrace{0, \ldots, 0}^{c\text{-copies}}, *, \ldots, *)$ *for some $e \neq 0$ (see (2)).*

(5) *For $1 \le t \le d$, let $\epsilon_t \in \mathbb{N}^d$ be the element whose t-th component is 1 and whose other components are 0. Then for $1 \le \nu \le r$, κ^ν sends ϵ_t $(1 \le t \le d)$ to*

$$\begin{cases} \epsilon_t & (t \neq m), \\ \epsilon_m + \epsilon_n & (t = m), \end{cases} \quad \text{for some } m \neq n \text{ with } 1 \le m \le c+1,\ 1 \le n \le c+1.$$

Put $h^\nu := \kappa^\nu \kappa^{\nu-1} \cdots \kappa^1 h^0$ and $^\nu X := X^{h^\nu}$, and let f^ν be the morphism induced by κ^ν:

$$f^\nu : {}^\nu X \to {}^{\nu-1}X.$$

We further fix some notation. Put $^\nu Y := Y^{h^\nu} = ({}^\nu X)_{s,\mathrm{red}}$, and let $x^\nu \in {}^\nu Y$ be the image of $x \in Y$ under the composite

$$Y = {}^r Y \xrightarrow{\ g^r\ } {}^{r-1}Y \xrightarrow{\ g^{r-1}\ } \cdots \xrightarrow{\ g^{\nu+1}\ } {}^\nu Y,$$

where $g^\nu : {}^\nu Y \to {}^{\nu-1}Y$ denotes the morphism induced by f^ν. For $0 \le \nu \le r$, let σ^ν be the composite map

$${}^\nu Y \overset{\iota^\nu}{\hookrightarrow} \mathscr{Y}^{h^\nu} = \mathrm{Spec}(k[T_1, \ldots, T_d]) \to \mathrm{Spec}(k[T_{c+2}, \ldots, T_d]),$$

where ι^ν denotes ι^{h^ν}. Since $\sigma^\nu = \sigma^{\nu-1} g^\nu$ by (5), the point $\sigma^\nu(x^\nu)$ is the generic point of $\mathrm{Spec}(k[T_{c+2}, \ldots, T_d])$ for any $0 \le \nu \le r$ by (3). This implies the following:

(6) *For any $0 \le \nu \le r$, x^ν has codimension c on $^\nu Y$. Consequently, x^ν is a closed point of $(g^\nu)^{-1}(x^{\nu-1})$.*

We also need the following fact (see [Tsuji 2000, Lemmas 3.2 and 3.4]):

(7) *For $1 \leq \nu \leq r$, f^ν factors as*

$$^\nu X \lhook\joinrel\longrightarrow \overline{^\nu X} \xrightarrow{\ \overline{f^\nu}\ } {^{\nu-1}X},$$

where the left arrow is an open immersion and $\overline{f^\nu}$ is the blow-up at the closed subscheme $\{T_m = T_n = 0\} \subset {^{\nu-1}X}$. The fibers of $\overline{f^\nu}$ have dimension at most one.

Put

$$^\nu D := D^{h_\nu} \quad \text{and} \quad {^\nu U} := {^\nu X} \smallsetminus ({^\nu Y} \cup {^\nu D}),$$

and define the sheaves ${_\nu M^q}$, ${_\nu L^q}$ and ${_\nu N^q}$ on ${^\nu Y_{\text{ét}}}$ for the diagram

$$^\nu Y \lhook\joinrel\longrightarrow {^\nu X} \longleftarrow\joinrel\rhook {^\nu U}$$

in the same way as for M^q, L^q and N^q on $Y_{\text{ét}}$, respectively. In what follows, we prove

$$({_\nu L^q})_{\overline{x^\nu}} = ({_\nu N^q})_{\overline{x^\nu}} = 0$$

by induction on $0 \leq \nu \leq r$. We first note:

Lemma 5.9. $\qquad\qquad ({_0 L^q})_{\overline{x^0}} = 0 \quad \text{and} \quad ({_0 N^q})_{\overline{x^0}} = 0.$

Proof. By the assumption (4), the pair $({^0 X}, {^0 D})$ is, étale locally around x^0, isomorphic to a quasilog smooth pair (X', D') over a henselian discrete valuation ring A' of mixed characteristic such that X' is smooth over A', see the first isomorphism on page 559 of [Tsuji 2000]. Hence the assertion follows from Lemma 5.7. $\qquad\square$

Assume $\nu \geq 1$ and the following induction hypothesis:

(8) $\qquad\qquad ({_{\nu-1} L^q})_{\overline{x^{\nu-1}}} = 0 \quad \text{and} \quad ({_{\nu-1} N^q})_{\overline{x^{\nu-1}}} = 0.$

We change the notation slightly and put

$$\begin{cases} X := \operatorname{Spec}(\mathcal{O}^{\text{sh}}_{\nu-1X, \overline{x^{\nu-1}}}), \\ Y := X_{s,\text{red}}, \\ D := \operatorname{Spec}(\mathcal{O}^{\text{sh}}_{\nu-1D, \overline{x^{\nu-1}}}), \\ U := X \smallsetminus (Y \cup D), \end{cases} \quad \text{and} \quad \begin{cases} X' := \overline{^\nu X} \times_{\nu-1X} \operatorname{Spec}(\mathcal{O}^{\text{sh}}_{\nu-1X, \overline{x^{\nu-1}}}), \\ Y' := (X')_{s,\text{red}}, \\ D' := \overline{^\nu D} \times_{\overline{^\nu X}} X', \\ U' := X' \smallsetminus (Y' \cup D'), \end{cases}$$

for simplicity. Here $\overline{^\nu D}$ denotes the closure of $(\overline{f^\nu})^{-1}({^\nu D})_{\text{red}} \smallsetminus (\overline{^\nu X})_{s,\text{red}} \subset \overline{^\nu X}$. Note that ${^\nu D} = \overline{^\nu D} \times_{\overline{^\nu X}} {^\nu X}$. Let $i' : Y' \hookrightarrow X'$ and $\psi' : U' \hookrightarrow X'$ be the canonical closed and open immersions, respectively. We define étale sheaves M^q, L^q and N^q on Y' as

$$M^q := i'^* R^q \psi'_* \mu_p^{\otimes q}, \quad N^q := \operatorname{Ker}\!\big(\mathcal{U}^0 M^q / \mathcal{U}^1 M^q \twoheadrightarrow \widetilde{\omega}^q_{Y',\log}\big), \quad L^q := M^q / \mathcal{U}^0 M^q.$$

In view of (6), once we prove N^q and L^q are zero, we will finish the induction on ν and c. We will prove the following lemma in the next section:

Lemma 5.10 (cf. [Hyodo 1988, Lemma (3.5)]). *For any $t \geq 0$, we have*

$$\Gamma(Y, \widetilde{\omega}_{Y,\log}^t) \cong \Gamma(Y', \widetilde{\omega}_{Y',\log}^t), \quad H^1(Y', \mathscr{U}^1 M^t) = H^1(Y', \widetilde{\omega}_{Y',\log}^t) = 0.$$

We prove here that N^q and L^q are zero admitting this lemma. Noting that $\mu_p \cong \mathbb{Z}/p\mathbb{Z}$ on U' by the assumption on K, we compute the Leray spectral sequence

$$E_2^{a,b} = H^a(Y', M^b) \Longrightarrow H^{a+b}(U', \mu_p^{\otimes q}) \cong H^{a+b}(U, \mu_p^{\otimes q}),$$

where we have used the proper base-change theorem [SGA 4_3 1973, XII.5.2] for the identification $H^a(Y', M^b) \cong H^a(X', R^b \psi_*' \mu_p^{\otimes q})$ and also used the fact that $\overline{f^\nu}$ induces an isomorphism $U' \cong U$. Since $\mathrm{cd}_p(Y') \leq 1$ by (7), this spectral sequence yields a short exact sequence

$$0 \to H^1(Y', M^{q-1}) \to H^q(U, \mu_p^{\otimes q}) \to \Gamma(Y', M^q) \to 0.$$

Because L^t and N^t are skyscraper sheaves on Y' for any $t \geq 0$ by the induction hypothesis (1) for X', both $H^1(Y', M^{q-1})$ and $H^1(Y', \mathscr{U}^0 M^q)$ are zero by Lemma 5.10. Hence there is a commutative diagram whose lower row is exact

$$
\begin{array}{ccc}
\mathscr{U}^0 H^q(U, \mu_p^{\otimes q}) & =\!=\!= & H^q(U, \mu_p^{\otimes q}) \\
\downarrow & & \downarrow \wr \\
0 \longrightarrow \Gamma(Y', \mathscr{U}^0 M^q) \longrightarrow \Gamma(Y', M^q) & \longrightarrow & \Gamma(Y', L^q) \longrightarrow 0
\end{array}
$$

where $\mathscr{U}^\bullet H^q(U, \mu_p^{\otimes q})$ ($\bullet = 0, 1$) denotes the filtration on the stalk of the sheaf of p-adic vanishing cycles on Y (see Definition 4.2(1) and Lemma 4.3(2)), and the upper equality follows from the induction hypothesis (8). This diagram shows that the skyscraper sheaf L^q is zero. We next show that N^q is zero. Put

$$\mathrm{gr}_{\mathscr{U}}^0 M^q := \mathscr{U}^0 M^q / \mathscr{U}^1 M^q = M^q / \mathscr{U}^1 M^q.$$

Since N^q is skyscraper by (1), there is an exact sequence

$$0 \to \Gamma(Y', N^q) \to \Gamma(Y', \mathrm{gr}_{\mathscr{U}}^0 M^q) \xrightarrow{\alpha} \Gamma(Y', \widetilde{\omega}_{Y',\log}^q) \to 0$$

and a commutative diagram with exact rows (see Lemma 4.3(2))

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{U}^1 H^q(U, \mu_p^{\otimes q}) & \longrightarrow & H^q(U, \mu_p^{\otimes q}) & \longrightarrow & \Gamma(Y, \widetilde{\omega}_{Y,\log}^q) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \wr & & \beta \downarrow & & \\
0 & \longrightarrow & \Gamma(Y', \mathscr{U}^1 M^q) & \longrightarrow & \Gamma(Y', M^q) & \longrightarrow & \Gamma(Y', \mathrm{gr}_{\mathscr{U}}^0 M^q) & \longrightarrow & 0
\end{array}
$$

Here the upper row is exact by the induction hypothesis (8), and the lower row is exact by Lemma 5.10. The arrow β denotes the map induced by the left square, and the middle vertical arrow is bijective by the proof of the vanishing of L^q. Now

this diagram shows that α is bijective, because β is surjective and $\alpha\beta$ is bijective by Lemma 5.10. Hence the skyscraper sheaf N^q is zero. Thus the induction on ν and c is complete and we obtain Theorem 5.4, assuming Lemma 5.10.

6. Proof of Lemma 5.10

In this section we prove Lemma 5.10 to finish the proof of Theorem 5.4. Let the notation be as in Setting 1.4. Assume that

$$X = \mathrm{Spec}\big(O_K[T_1, \ldots, T_d]/(T_1^{e_1} \cdots T_a^{e_a} - \pi)\big) \quad (e_1, \ldots, e_a \geq 1, \ d = \dim(X)),$$

$$D = \{T_{a+1} \cdots T_d = 0\} \subset X,$$

where π is a prime element of O_K and D is empty if $a = d$. Let

$$f : X' \to X$$

be the blow-up at the regular closed subscheme $\{T_b = T_c = 0\} \subset X$ with $1 \leq b < c \leq d$. Put $Y' := (X')_{s,\mathrm{red}}$. We define a reduced normal crossing divisor D' on X' as

$$D' := \overline{f^{-1}(D)_{\mathrm{red}} \smallsetminus Y'} \subset X'$$

and define the sheaves $\widetilde{\omega}^q_{Y',\log}$ and $M^q_{1,X'}$ on $(Y')_{\mathrm{\acute et}}$ in the same way as for $\widetilde{\omega}^q_{Y,\log}$ and M^q_1 on $Y_{\mathrm{\acute et}}$. Let

$$g : Y' \to Y$$

be the morphism induced by f. Lemma 5.10 follows from:

Lemma 6.1. *Let $D(Y_{\mathrm{\acute et}})$ be the derived category of complexes of étale sheaves on Y.*

(1) $$\widetilde{\omega}^q_{Y,\log} \xrightarrow{\simeq} Rg_*\widetilde{\omega}^q_{Y',\log} \quad \text{in } D(Y_{\mathrm{\acute et}}) \text{ for any } q \geq 0.$$

(2) $$R^1 g_*(\mathscr{U}^1 M^q_{1,X'}) = 0 \quad \text{for any } q \geq 0.$$

We introduce some notation that will be useful throughout the proof of Lemma 6.1. For $\lambda \in \Lambda := \{1, 2, \ldots, a\}$, let Y_λ be the closed subset $\{T_\lambda = 0\} \subset X$ endowed with the reduced subscheme structure, which is an irreducible component of Y. Let $\{Y'_\lambda\}_{\lambda \in \Lambda'}$ be the irreducible components of Y'. We have

$$\Lambda' = \begin{cases} \Lambda \cup \{\infty\} & \text{if } b \leq a, \\ \Lambda & \text{if } a < b, \end{cases}$$

where Y'_λ for $\lambda \in \Lambda$ means the strict transform of Y_λ and Y'_∞ is the exceptional fiber of f. For $\mathfrak{m} = (m_\lambda)_{\lambda \in \Lambda} \in \mathbb{N}^\Lambda$, we define $f^*\mathfrak{m} = (n_\lambda)_{\lambda \in \Lambda'} \in \mathbb{N}^{\Lambda'}$ as

$$n_\lambda = \begin{cases} m_\lambda & \text{if } \lambda \in \Lambda, \\ m_b & \text{if } b \leq a < c \text{ and } \lambda = \infty, \\ m_b + m_c & \text{if } c \leq a \text{ and } \lambda = \infty. \end{cases}$$

For $\lambda \in \Lambda'$, let $\mathscr{I}'_\lambda \subset \mathcal{O}_{X'}$ be the defining ideal of Y'_λ. For $\mathfrak{n} \in \mathbb{N}^{\Lambda'}$, we define $\mathscr{I}^{(\mathfrak{n})}_{X'} \subset \mathcal{O}_{X'}$ in the same way as for $\mathscr{I}^{(\bullet)} \subset \mathcal{O}_X$, see Section 3. We have

$$f^* \mathscr{I}^{(\mathfrak{m})} = \mathscr{I}^{(f^*\mathfrak{m})}_{X'}. \tag{6.2}$$

We will often write $\mathscr{I} \subset \mathcal{O}_X$ and $\mathscr{I}' \subset \mathcal{O}_{X'}$ for the defining ideals of Y and Y', respectively.

Sublemma 6.3. (1) *We have $g^* \widetilde{\omega}^q_Y \xrightarrow{\sim} \widetilde{\omega}^q_{Y'}$ for any $q \geq 0$, where g^* denotes the inverse image of coherent sheaves.*

(2) *Assume that $c \leq a$, and let $\mathfrak{n} = (n_\lambda)_{\lambda \in \Lambda'} \in \mathbb{N}^{\Lambda'}$ be an arbitrary element with $n_b + n_c = n_\infty + 1$. Then we have $Rg_*(\mathscr{I}^{(\mathfrak{n})}_{X'} / \mathscr{I}^{(\mathfrak{n}+\mathfrak{d}_\infty)}_{X'}) = 0$ in $D(Y_{\text{ét}})$. (See the definitions before Corollary 3.3 for $\mathfrak{d}_\infty \in \mathbb{N}^{\Lambda'}$.)*

(3) *We have $\mathcal{O}_Y \xrightarrow{\sim} Rg_* \mathcal{O}_{Y'}$ in $D(Y_{\text{ét}})$.*

Proof of Sublemma 6.3. (1) Let $(X, \mathscr{L})$ and $(X', \mathscr{L}')$ be the log schemes associated with the pairs (X, D) and (X', D'), respectively (see Remark 2.2), and let L (resp. L') be the inverse image log structure onto Y (resp. onto Y'). The morphism $(Y', L') \to (Y, L)$ of log schemes induced by f is étale in the sense of [Kato 1989, (3.3)] by [loc. cit., Theorem (3.5)]. Hence the assertion follows from [loc. cit., Proposition 3.12].

(2) When $c \leq a$, Y'_∞ is isomorphic to the trivial $\mathbb{P}^1$-bundle over $Y_b \cap Y_c$, and $Y'_\infty \cap Y'_b$ and $Y'_\infty \cap Y'_c$ are relative hyperplane sections of Y'_∞ over $Y_b \cap Y_c$. Since $(\mathscr{I}'_\infty)^{n_\infty} / (\mathscr{I}'_\infty)^{n_\infty+1} \cong \mathcal{O}(n_\infty)$ on Y'_∞, we have

$$\mathscr{I}^{(\mathfrak{n})}_{X'} / \mathscr{I}^{(\mathfrak{n}+\mathfrak{d}_\infty)}_{X'} \cong (\mathscr{I}'_b)^{n_b} \otimes_{\mathcal{O}_{X'}} (\mathscr{I}'_c)^{n_c} \otimes_{\mathcal{O}_{X'}} \left((\mathscr{I}'_\infty)^{n_\infty} / (\mathscr{I}'_\infty)^{n_\infty+1} \right) \cong \mathcal{O}(-1)$$

on Y'_∞, where we have used the fact that $\mathscr{I}'_\lambda$ is principal for $\lambda \neq b, c, \infty$ and the assumption that $n_b + n_c = n_\infty + 1$. Hence we obtain the assertion by a standard fact on the cohomology of projective lines [Hartshorne 1977, Chapter III Theorem 5.1].

(3) Noting that $f : X' \to X$ is the blow-up along a closed subscheme of X defined by a regular sequence and that $\mathscr{I} = T_0 T_1 \cdots T_a \mathcal{O}_X$ is a free $\mathcal{O}_X$-module (of rank 1), we have

$$\mathcal{O}_X \xrightarrow{\sim} Rf_* \mathcal{O}_{X'} \quad \text{and} \quad \mathscr{I} \xrightarrow{\sim} Rf_* f^* \mathscr{I} \quad \text{in } D(X_{\text{ét}})$$

by the theorem on formal functions [Hartshorne 1977, Chapter III Theorem 11.1] and the cohomology of projective lines. Our task is to check

$$Rf_*(\mathscr{I}' / f^* \mathscr{I}) = 0 \quad \text{in } D(X_{\text{ét}}). \tag{6.4}$$

Note that $\mathscr{I}' = \mathscr{I}^{(\mathbf{1})}_{X'}$ and that $f^* \mathscr{I} = \mathscr{I}^{(f^*\mathbf{1})}_{X'}$ by (6.2). If $a < c$, then we have $f^* \mathbf{1} = \mathbf{1}$ in $\mathbb{N}^{\Lambda'}$ and the assertion (6.4) is obvious. If $c \leq a$, then we have

$$f^* \mathbf{1} = \mathbf{1} + \mathfrak{d}_\infty \quad \text{in } \mathbb{N}^{\Lambda'} \tag{6.5}$$

and the assertion follows from Sublemma 6.3(2). □

Proof of Lemma 6.1. (1) By Theorem 2.3(2) and (3), it is enough to show that

(a) $\widetilde{\omega}_Y^q \xrightarrow{\sim} Rg_*\widetilde{\omega}_{Y'}^q$ and   (b) $\widetilde{\mathscr{Z}}_Y^q \xrightarrow{\sim} Rg_*\widetilde{\mathscr{Z}}_{Y'}^q$ in $D(Y_{\text{ét}})$.

We first show (a). Since $\widetilde{\omega}_Y^q$ is locally free over $\mathscr{O}_Y$ by Theorem 2.3(1), we have

$$Rg_*\widetilde{\omega}_{Y'}^q \cong Rg_*g^*\widetilde{\omega}_Y^q \cong Rg_*(\mathscr{O}_{Y'} \otimes_{\mathscr{O}_{Y'}} g^*\widetilde{\omega}_Y^q) \cong (Rg_*\mathscr{O}_{Y'}) \otimes_{\mathscr{O}_Y} \widetilde{\omega}_Y^q \cong \widetilde{\omega}_Y^q$$

by the projection formula and Sublemma 6.3(3). We next show (b). The case $q = 0$ follows from (a) and the isomorphism $\widetilde{\mathscr{Z}}_Y^0 = (\mathscr{O}_Y)^p \cong \mathscr{O}_Y$. We proceed with the proof by using induction on q. There is a commutative diagram with distinguished rows in $D(Y_{\text{ét}})$:

$$
\begin{array}{ccccccc}
\widetilde{\mathscr{Z}}_Y^q & \longrightarrow & \widetilde{\omega}_Y^q & \xrightarrow{\,d\,} & \widetilde{\mathscr{B}}_Y^{q+1} & \longrightarrow & \widetilde{\mathscr{Z}}_Y^q[1] \\
\downarrow & & \wr\downarrow & & \downarrow & & \downarrow \\
Rg_*\widetilde{\mathscr{Z}}_{Y'}^q & \longrightarrow & Rg_*\widetilde{\omega}_{Y'}^q & \xrightarrow{\,d\,} & Rg_*\widetilde{\mathscr{B}}_{Y'}^{q+1} & \longrightarrow & Rg_*\widetilde{\mathscr{Z}}_{Y'}^q[1]
\end{array}
$$

By Theorem 2.3(2), there is another commutative diagram with distinguished rows in $D(Y_{\text{ét}})$:

$$
\begin{array}{ccccccc}
\widetilde{\mathscr{B}}_Y^{q+1} & \longrightarrow & \widetilde{\mathscr{Z}}_Y^{q+1} & \xrightarrow{\,C\,} & \widetilde{\omega}_Y^{q+1} & \longrightarrow & \widetilde{\mathscr{B}}_Y^{q+1}[1] \\
\downarrow & & \downarrow & & \wr\downarrow & & \downarrow \\
Rg_*\widetilde{\mathscr{B}}_{Y'}^{q+1} & \longrightarrow & Rg_*\widetilde{\mathscr{Z}}_{Y'}^{q+1} & \xrightarrow{\,C\,} & Rg_*\widetilde{\omega}_{Y'}^{q+1} & \longrightarrow & Rg_*\widetilde{\mathscr{B}}_{Y'}^{q+1}[1]
\end{array}
$$

where C denotes the inverse of the isomorphism C^{-1}. The induction on q works by these diagrams.

(2) We first show

$$R^1 g_*(\mathscr{U}^{(f^*\mathbf{1})}M_{1,X'}^q) = 0. \tag{6.6}$$

Take a sequence of elements of $\mathbb{N}^\Lambda$

$$\mathbf{1} = \mathfrak{m}_0 \leq \mathfrak{m}_1 \leq \mathfrak{m}_2 \leq \cdots \leq \mathfrak{m}_i \leq \cdots \leq \mathfrak{m}_t$$

satisfying the following conditions:

(i) If $i \leq t - 1$, then $\mathfrak{l}_i := \mathfrak{m}_{i+1} - \mathfrak{m}_i$ agrees with $\mathfrak{d}_\mu$ for some $\mu = \mu(i) \in \Lambda$.

(ii) We have $\mathfrak{m}_i \geq \mathfrak{e}'$ (in $\mathbb{Q}^\Lambda$) if and only if $i = t$.

See the definitions before Corollary 3.3 for $\mathfrak{d}_\mu$, and see Definition 4.2(3) for $\mathfrak{e}' \in \mathbb{Q}^\Lambda$. We define $\mathfrak{E}' \in \mathbb{Q}^{\Lambda'}$ for X' in the same way as we defined $\mathfrak{e}'$ for X. For $\mathfrak{n}, \mathfrak{n}' \in \mathbb{N}^{\Lambda'}$

with $\mathbf{0} \leq \mathfrak{n}' \leq \mathbf{1}$, we define the sheaf $\omega^{q-1}_{\mathfrak{n},\mathfrak{n}',X'}$ on Y' in the same way as for $\omega^{q-1}_{*,*}$ on Y. By the choice of the above $\mathfrak{m}_i$'s and the fact that $f^*\mathfrak{e}' = \mathfrak{E}'$, we see that

(i') For $i \leq t - 1$, we have $f^*\mathfrak{l}_i = f^*\mathfrak{m}_{i+1} - f^*\mathfrak{m}_i \in \mathbb{N}^{\Lambda'}$ and $\mathbf{0} \leq f^*\mathfrak{l}_i \leq \mathbf{1}$.

(ii') We have $f^*\mathfrak{m}_i \geq \mathfrak{E}'$ (in $\mathbb{Q}^{\Lambda'}$) if and only if $i = t$.

Hence by Theorem 4.4 for X', we have

$$\mathscr{U}^{(f^*\mathfrak{m}_i)}M^q_{1,X'}/\mathscr{U}^{(f^*\mathfrak{m}_{i+1})}M^q_{1,X'} \cong \omega^{q-1}_{f^*\mathfrak{m}_i,f^*\mathfrak{l}_i,X'}/\mathscr{Z}^{q-1}_{f^*\mathfrak{m}_i,f^*\mathfrak{l}_i,X'} \quad \text{for } 0 \leq i \leq t-1,$$
$$\mathscr{U}^{(f^*\mathfrak{m}_t)}M^q_{1,X'} = 0,$$

and we are reduced to showing that

$$R^1 g_*\big(\omega^{q-1}_{f^*\mathfrak{m},f^*\mathfrak{l},X'}/\mathscr{Z}^{q-1}_{f^*\mathfrak{m},f^*\mathfrak{l},X'}\big) = 0 \tag{6.7}$$

for $\mathfrak{m}, \mathfrak{l} \in \mathbb{N}^{\Lambda}$ with $\mathfrak{m} \geq \mathbf{1}$ and $\mathfrak{l} = \mathfrak{d}_\mu$ for some $\mu \in \Lambda$. We prove (6.7). By (6.2), there is a short exact sequence on $Y'_{\text{ét}}$

$$0 \to f^*\mathscr{I}^{(\mathfrak{m}+\mathfrak{l})} \otimes_{\mathscr{O}_{X'}} \widetilde{\omega}^{q-1}_{Y'} \to f^*\mathscr{I}^{(\mathfrak{m})} \otimes_{\mathscr{O}_{X'}} \widetilde{\omega}^{q-1}_{Y'} \to \omega^{q-1}_{f^*\mathfrak{m},f^*\mathfrak{l},X'} \to 0.$$

Since $\mathscr{I}^{(\mathfrak{m}+\mathfrak{l})}$ and $\mathscr{I}^{(\mathfrak{m})}$ are locally free $\mathscr{O}_X$-modules, we obtain

$$\omega^{q-1}_{\mathfrak{m},\mathfrak{l}} \xrightarrow{\sim} Rg_*\omega^{q-1}_{f^*\mathfrak{m},f^*\mathfrak{l},X'}$$

by applying Rf_* to the above short exact sequence and the projection formula together with the claim (a) in the proof of Lemma 6.1(1). In particular we obtain $R^1 g_*\omega^{q-1}_{f^*\mathfrak{m},f^*\mathfrak{l},X'} = 0$ and (6.7), because $R^1 g_*$ is right exact for p-torsion sheaves for the reason of the dimension of the fibers of g [SGA 4_3 1973, X.5.2, XII.5.2]. Thus we obtain (6.6).

By (6.6), it remains to check

$$R^1 g_*\big(\mathscr{U}^1 M^q_{1,X'}/\mathscr{U}^{(f^*\mathbf{1})}M^q_{1,X'}\big) = 0 \tag{6.8}$$

If $a < c$, then we have $f^*\mathbf{1} = \mathbf{1}$ in $\mathbb{N}^{\Lambda'}$ and the assertion is obvious. As for the case $c \leq a$, we have

$$\mathscr{U}^1 M^q_{1,X'}/\mathscr{U}^{(f^*\mathbf{1})}M^q_{1,X'} \cong \omega^{q-1}_{\mathbf{1},\mathfrak{d}_\infty,X'}/\mathscr{Z}^{q-1}_{\mathbf{1},\mathfrak{d}_\infty,X'}$$

by (6.5) and Theorem 4.4. We have

$$Rg_*\omega^{q-1}_{\mathbf{1},\mathfrak{d}_\infty,X'} \cong Rg_*\big(\mathscr{I}^{(\mathbf{1})}_{X'}/\mathscr{I}^{(f^*\mathbf{1})}_{X'} \otimes_{\mathscr{O}_{Y'}} g^*\widetilde{\omega}^{q-1}_Y\big) \cong Rg_*(\mathscr{I}^{(\mathbf{1})}_{X'}/\mathscr{I}^{(f^*\mathbf{1})}_{X'}) \otimes_{\mathscr{O}_Y} \widetilde{\omega}^{q-1}_Y = 0$$

by (6.4). Thus we obtain $R^1 g_*\omega^{q-1}_{\mathbf{1},\mathfrak{d}_\infty,X'} = 0$ and (6.8), noting that $R^1 g_*$ is right exact for p-torsion sheaves. This completes the proof of Lemmas 6.1 and 5.10, and Theorem 5.4. $\qquad\qquad\square$

Remark 6.9. By Lemma 5.10 and the proof of Theorem 5.4 (see the last diagram of Section 5), we have $\mathscr{U}^1 M_1^q \cong g_*(\mathscr{U}^1 M_{1,X'}^q)$ as sheaves on $Y_{\text{ét}}$ for any $q \geq 0$ (under the notation of this section). Consequently, we have

$$\mathscr{U}^1 M_1^q \xrightarrow{\sim} Rg_*(\mathscr{U}^1 M_{1,X'}^q) \quad \text{in } D(Y_{\text{ét}})$$

by Lemma 6.1(2) and the fact that $R^s g_*(\mathscr{U}^1 M_{1,X'}^q) = 0$ for $s \geq 2$, see [SGA 4$_3$ 1973, X.5.2, XII.5.2].

Acknowledgments

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Tame multiplicity and conductor
for local Galois representations

Colin J. Bushnell and Guy Henniart

Let F be a non-Archimedean locally compact field of residual characteristic p. Let σ be an irreducible smooth representation of the absolute Weil group $\mathcal{W}_F$ of F and $\mathrm{sw}(\sigma)$ the Swan exponent of σ. Assume $\mathrm{sw}(\sigma) \geqslant 1$. Let $\mathcal{I}_F$ be the inertia subgroup of $\mathcal{W}_F$ and $\mathcal{P}_F$ the wild inertia subgroup. There is an essentially unique, finite, cyclic group Σ, of order prime to p, such that $\sigma(\mathcal{I}_F) = \Sigma \sigma(\mathcal{P}_F)$. In response to a query of Mark Reeder, we show that the multiplicity in σ of any character of Σ is bounded by $\mathrm{sw}(\sigma)$.

1. Introduction

1.1. Let F be a non-Archimedean, locally compact field of residual characteristic p. Let $\bar{F}$ be a separable algebraic closure of F and $\mathcal{W}_F$ the Weil group of $\bar{F}/F$. Write $\mathcal{I}_F$ for the inertia subgroup of $\mathcal{W}_F$ and $\mathcal{P}_F$ for the wild inertia subgroup.

Let σ be an irreducible, smooth, complex representation of $\mathcal{W}_F$. Thus $I = \sigma(\mathcal{I}_F)$ and $P = \sigma(\mathcal{P}_F)$ are finite groups, with P being the unique p-Sylow subgroup of I. The quotient I/P is cyclic, of order prime to p. It follows readily that there is a subgroup Σ of P such that the quotient map $I \to I/P$ induces an isomorphism $\Sigma \to I/P$. Thus $\Sigma \cap P = 1$ and $I = \Sigma P$. Moreover, the subgroup Σ, satisfying these conditions, is uniquely determined up to conjugation by an element of P.

(See, for instance, [Gorenstein 2012, Chapter 6, Theorem 4.1] for a full discussion.)
Define the *tame multiplicity* $m(\sigma)$ of σ by

$$m(\sigma) = \max_{\chi} \dim \operatorname{Hom}_\Sigma(\chi, \sigma),$$

where χ ranges over the group $\widehat{\Sigma}$ of linear characters of Σ. The integer $m(\sigma)$ does
not depend on the choice of Σ and, in all cases, $m(\sigma) \geqslant 1$.

Let $\operatorname{sw}(\sigma)$ be the Swan exponent of σ. We prove:

Tame multiplicity theorem. *Let σ be an irreducible smooth representation of $\mathcal{W}_F$.
If $\operatorname{sw}(\sigma) > 0$, then*

$$m(\sigma) \leqslant \operatorname{sw}(\sigma). \tag{1-1-1}$$

In particular, the space of Σ-fixed points in σ has dimension at most $\operatorname{sw}(\sigma)$.

Mark Reeder [2018] gives compelling reasons for being interested in the in-
variant $m(\sigma)$ and the inequality (1-1-1). He proves the theorem when σ is either
essentially tame or of epipelagic type, in the sense that $\operatorname{sw}(\sigma) = 1$. This paper is
written in response to his query as to whether it might hold in general.

Remark 1.1.1. A couple of cases can be dispatched straightaway.

(1) If $\operatorname{sw}(\sigma) = 0$, then $\Sigma = I$ and σ is induced from a tamely ramified character
    of $\mathcal{W}_K$, where K/F is unramified. It follows that $m(\sigma) = 1$.

(2) If $\dim \sigma = 1$ and $\operatorname{sw}(\sigma) \geqslant 1$, then $m(\sigma) = 1$ for trivial reasons.

1.2. This is, obviously, a "small conductor" problem: certainly $m(\sigma) \leqslant \dim \sigma$
while, for the vast majority of representations σ, one has $\dim \sigma < \operatorname{sw}(\sigma)$. On the
other hand, if $\operatorname{sw}(\sigma) = 1$ then $m(\sigma) = 1$ [Bushnell and Henniart 2014; Reeder
2018]. It is the contrast between these two extremes that dictates the flavour of the
paper. In many cases, rather coarse estimates should suffice to give the result but,
in others, delicacy is likely to be required.

The small-conductor aspect suggests that primitive representations σ must play
a central role. At first glance, one might hope to prove the theorem for primitive
representations and then proceed by induction. That light-hearted approach falls at
the first hurdle. If one tries to calculate $m(\sigma)$ directly from the description of σ in
Koch's structure theory [1977], the combinatorics rapidly get out of hand. Further,
we have an uncertain grasp of the relation between Koch's description of σ and the
value of $\operatorname{sw}(\sigma)$. Examples show that there is sometimes no room for any sloppiness
in the estimates.

More positively, there is a strong lower bound for $\operatorname{sw}(\sigma)$ in [Henniart 1980].
On the other side, help comes from a rather different source. Glauberman's gen-
eral theory [1968] of character correspondences for finite groups, as developed in
Isaacs' book [2006], leads to an exact and manageable formula for $m(\sigma)$, but only

for a restricted class of primitive representations σ. To outline this, we need some terminology.

Let τ be an irreducible representation of $\mathcal{W}_F$. Say that τ is *absolutely ramified* if it factors through $\mathrm{Gal}(E/F)$, where E/F is a finite, totally ramified field extension. Let σ be primitive and absolutely ramified, viewed as a faithful representation of $G = \mathrm{Gal}(E/F)$. Let $\mathrm{Gal}(E/K)$ be the centre of G and let T/F be the maximal tame subextension of E/F. We may reduce to the case where $\mathrm{Gal}(E/T)$ is a p-group, and therefore the wild inertia subgroup of G. Let Σ be a complement of $\mathrm{Gal}(E/T)$ in G. For the purposes of this introduction, say that σ is Σ-*homogeneous* if the G-centralizer of any nontrivial element of Σ is $\Sigma\,\mathrm{Gal}(E/K)$. If σ is Σ-homogeneous, then [Isaacs 2006] gives an exact formula for the character $\mathrm{tr}\,\sigma \mid \Sigma$.

If σ is absolutely ramified and Σ-homogeneous, comparison of the character formula with the conductor bound of [Henniart 1980] yields the theorem. This is a case in which $m(\sigma)$ can be close to $\mathrm{sw}(\sigma)$ (see Section 4.5). More generally, an absolutely ramified primitive representation is essentially a tensor product of homogeneous ones. A relatively relaxed estimate then gives the theorem in this case.

For the third step, we prove the theorem when σ is absolutely ramified (but not necessarily primitive). We can assume that σ is an induced representation $\mathrm{Ind}_{K/F}\,\tau$, where τ is an absolutely ramified representation of $\mathcal{W}_K$ with $K \neq F$ and $m(\tau) \leqslant \mathrm{sw}(\tau)$. A standard property asserts that $\mathrm{sw}(\sigma) = \mathrm{sw}(\tau) + w_{K/F}\dim\tau$, where $w_{K/F}$ is the *wild exponent* of the extension K/F. The relation between $m(\sigma)$ and $m(\tau)$ is group-theoretic in nature, so we have to estimate the arithmetic quantity $w_{K/F}$ in group-theoretic terms. A rather coarse argument suffices. It shows that, relative to induction of representations, $\mathrm{sw}(\sigma)$ grows much more quickly that $m(\sigma)$ and so justifies the initial emphasis on primitive representations. From there on, the general case of the theorem follows easily.

1.3. The paper is arranged as follows. The necessary material from finite group theory is assembled in Section 2. In Section 3, we review some properties of primitive representations leading to the conductor estimate of [Henniart 1980, théorème 1.8]. We give a complete proof of that result. It uses the same ideas as [Henniart 1980] but, in the present limited context, they can be expressed more succinctly and transparently. Section 4 is the heart of the argument, proving the theorem for "Σ-homogeneous", absolutely ramified, primitive representations, as sketched above. Section 5 is the group-theoretic estimate of the wild exponent, and Section 6 completes the proof.

2. Group-theoretic preliminaries

We gather some techniques from the representation theory of finite groups. This section has its own scheme of notation.

2.1. We consider a special class of finite p-groups, using the terminology of [Bushnell and Henniart 2019].

Definition 2.1.1. Let P be a finite p-group with centre $Z \neq P$. It is called *H-cyclic* if it satisfies the following conditions.

(1) The centre Z is cyclic, and

(2) the quotient $V = P/Z$ is elementary abelian.

For convenience, we summarize the main properties of these groups, following the account in [Bushnell and Henniart 2019]. For $x, y \in P$, the commutator $[x, y]$ lies in the centre Z and satisfies $[x, y]^p = [x^p, y] = 1$.

We think of the elementary abelian p-group V as a vector space over the field $\mathbb{F}_p$ of p elements. Let ζ be a faithful character of Z. The commutator pairing

$$(x, y) \mapsto \zeta(xyx^{-1}y^{-1}), \quad x, y \in P, \tag{2-1-1}$$

takes its values in the group $\boldsymbol{\mu}_p(\mathbb{C})$ of complex p-th roots of unity. Composing with a fixed isomorphism $\boldsymbol{\mu}_p(\mathbb{C}) \to \mathbb{F}_p$, the pairing (2-1-1) induces an alternating bilinear form

$$h_\zeta : V \times V \to \mathbb{F}_p. \tag{2-1-2}$$

Because Z is the centre of P, the form h_ζ is nondegenerate. Consequently, V has p^{2r} elements, for an integer $r \geqslant 1$.

A subspace W of V is a *Lagrangian subspace* of the alternating space (V, h_ζ) if it has exactly p^r elements and $h_\zeta(w_1, w_2) = 0$ for all $w_1, w_2 \in W$.

Lemma 2.1.2. *There is a unique irreducible representation τ of P such that $\tau \mid Z$ contains ζ. It has the following properties.*

(1) *The representation τ is faithful, it satisfies $\dim \tau = p^r$, and $\tau \mid Z$ is a multiple of ζ.*

(2) *Let W be a Lagrangian subspace of (V, h_ζ) with inverse image $\widetilde{W}$ in P. The group $\widetilde{W}$ is abelian and the character ζ of Z admits extension to a character ζ^W of $\widetilde{W}$. For any such ζ^W, one has*

$$\tau \cong \mathrm{Ind}_{\widetilde{W}}^P \zeta^W.$$

Proof. See [Bushnell and Henniart 2019, 8.1 Proposition]. $\square$

Remark 2.1.3. A finite p-group is called *extra special of class* 2 if it is H-cyclic and its centre has order p (see [Gorenstein 2012, p. 183]). More generally, let P be H-cyclic with centre Z. Since the representation τ of the lemma is faithful, one may identify P with $\tau(P)$. One can then follow Rigby's argument [1960, Theorem 2] to show that P is the central product of the finite cyclic p-group Z and an extra special p-group of class 2.

Corrigendum. In the preamble to [Bushnell and Henniart 2019, §8.1], we assert that an H-cyclic group is extra special of class 2. The arguments of [Bushnell and Henniart 2019, Section 8] are conducted axiomatically, so this error has no effect on the results or their proofs. In particular, the lemma above remains valid.

2.2. Let P be a finite, H-cyclic p-group with centre Z, write $V = P/Z$ and let $|V| = p^{2r}$. Let ζ be a faithful character of Z. We introduce another element of structure.

Definition 2.2.1. Let S be a cyclic group of automorphisms of P, such that:

(1) the order $|S|$ of S is not divisible by p, and

(2) S acts trivially on Z.

Because of condition (2), the action of S on P fixes the commutator pairing (2-1-2), so the $\mathbb{F}_p$-representation of S provided by V is *symplectic*. We consider a specific representation of the semidirect product $G = S \ltimes P$.

Lemma 2.2.2. *Let $G = S \ltimes P$ and let τ be the unique irreducible representation of G such that $\tau \mid Z$ contains ζ.*

(1) *There exists a unique representation $\tilde{\tau}$ of G such that $\tilde{\tau} \mid P \cong \tau$ and $\det \tilde{\tau}(s) = 1$, for all $s \in S$.*

(2) *An irreducible representation ρ of G satisfies $\rho \mid P \cong \tau$ if and only if there is a character χ of $S = G/P$ such that $\rho \cong \chi \otimes \tilde{\tau}$.*

Proof. This is a pleasant exercise, written out in [Bushnell and Fröhlich 1983, (8.4.1) Proposition]. For a very general result of this kind, see [Isaacs 2006, 13.3 Lemma]. □

Under certain circumstances, one can write down the character $\operatorname{tr} \tilde{\tau}$ of $\tilde{\tau}$ on elements of S.

Proposition 2.2.3. *Suppose that, for every $s \in S$, $s \neq 1$, the G-centralizer of s is SZ. There is then a constant $\epsilon = \pm 1$ and a character μ of S, such that $\mu^2 = 1$, with the following properties.*

(1) $\operatorname{tr} \tilde{\tau}(sz) = \epsilon \mu(s)\zeta(z)$, *for $s \in S$, $s \neq 1$, and $z \in Z$.*

(2) $p^r - \epsilon = k|S|$, *for an integer k.*

(3) $\operatorname{tr} \tilde{\tau} \mid S = k\mathrm{R}_S + \epsilon \mu$, *where R_S is the character of the regular representation of S.*

(4) *The character μ is nontrivial if and only if $|S|$ is even and k is odd.*

Proof. This is a special case of [Isaacs 2006, Theorem 13.32]: to translate the notation, our τ is χ in [Isaacs 2006], while $\tilde{\tau}$ is $\hat{\chi}$ and ζ is β. Otherwise, conventions are the same. □

Remark 2.2.4. (1) The formulas in the proposition show that the character $\operatorname{tr} \tilde{\tau} \mid S$ of S is determined by the group orders $|S|$ and $|V|$. Indeed, if $|S| \geqslant 3$, the invariants k, ϵ, μ are individually determined by the group orders. When $|S| = 2$, the character is determined but the invariants are not. In all cases, the character $\operatorname{tr} \tilde{\tau} \mid S$ depends only on the *linear* $\mathbb{F}_p$-representation of S afforded by V.

(2) Let S' be a cyclic group, of order prime to p, equipped with a surjective homomorphism $S' \to S$. One may inflate $\tilde{\tau}$ to a representation $\tilde{\tau}'$ of $S' \ltimes P$ and then use the proposition to write down $\operatorname{tr} \tilde{\tau}' \mid S'$.

Character computations of this sort feature in [Bushnell and Fröhlich 1983], especially (8.6.1) Theorem, and have been widely used. However, the account in [Bushnell and Fröhlich 1983] deals only with the case where the symplectic $\mathbb{F}_p S$-representation P/Z is indecomposable. The proposition gives the exact formula for a wider class of cases. It neatly avoids an estimation process at a point where absolute precision is essential (see Sections 4.4 and 4.5 below).

2.3. Suppose that the space V of Section 2.1 decomposes as a direct sum of nonzero subspaces V_1, V_2, orthogonal with respect to the alternating form (2-1-2), say

$$V = V_1 \perp V_2. \tag{2-3-1}$$

Let P_i be the inverse image of V_i in P. The commutator group $[P_1, P_2]$ is trivial, that is, P_1 commutes with P_2. Moreover, each P_i is H-cyclic with centre Z.

The obvious map $P_1 \times P_2 \to P$ is a surjective homomorphism with kernel $\{(z, z^{-1}) : z \in Z\}$. That is, P is the central product of its subgroups P_1, P_2. As in Lemma 2.1.2, the group P_i admits a unique irreducible representation τ_i containing the character ζ of Z. The representation $\tau_1 \otimes \tau_2$ factors through the quotient map $\pi : P_1 \times P_2 \to P$ and so $\tau \circ \pi \cong \tau_1 \otimes \tau_2$: one may reasonably write

$$\tau = \tau_1 \otimes \tau_2. \tag{2-3-2}$$

2.4. Let S be a cyclic group of automorphisms of P, as in Definition 2.2.1, and suppose that the factors V_i in (2-3-1) are S-invariant. It follows that the subgroups P_i of P are normalized by S. Let S_i be the image of S in $\operatorname{Aut} P_i$. Following the procedure of Lemma 2.2.2, we form the representation $\tilde{\tau}_i$ of $S_i \ltimes P_i$. We inflate $\tilde{\tau}_i$ to a representation $\tilde{\tau}_i^S$ of $S \ltimes P_i$. We can equally set $\tau = \tau_1 \otimes \tau_2$ as in (2-3-2) and extend it to a representation $\tilde{\tau}$ of $S \ltimes P$ as before. We then have

$$\operatorname{tr} \tilde{\tau}(s) = \operatorname{tr} \tilde{\tau}_1^S(s) \cdot \operatorname{tr} \tilde{\tau}_2^S(s), \quad s \in S. \tag{2-4-1}$$

3. Conductor estimate for primitive representations

We give a lower bound, in terms of ramification structure, for the Swan exponent of a certain class of representations of the Weil group. Before we start, we lay down some notation and conventions to remain in force for the rest of the paper.

Notation and conventions.

(1) Let $\mathcal{W}_F$ be the Weil group of a chosen separable closure $\bar{F}/F$. When speaking of a "representation of $\mathcal{W}_F$" we mean a "smooth complex representation of $\mathcal{W}_F$". Let $\mathcal{I}_F$ be the inertia subgroup of $\mathcal{W}_F$ and $\mathcal{P}_F$ the wild inertia subgroup.

(2) Let $\mathfrak{p}_F$ be the maximal ideal of the discrete valuation ring in F. If $k \geqslant 1$ is an integer, then U_F^k is the unit group $1+\mathfrak{p}_F^k$. The residue field of F is $\Bbbk_F$.

(3) We use the conventions of [Serre 1968] when dealing with ramification groups, their numberings and the Herbrand functions φ, ψ.

3.1. An irreducible representation σ of $\mathcal{W}_F$ is called *primitive* if $\dim \sigma > 1$ and if σ is not induced from a representation of $\mathcal{W}_K$, where K/F is a finite field extension with $K \neq F$.

Hypothesis. For the rest of this section, we suppose that the representation σ is primitive.

The restriction $\sigma \mid \mathcal{P}_F$ is then irreducible and the finite p-group $\sigma(\mathcal{P}_F)$ is H-cyclic [Rigby 1960, Theorem 1]. Consequently, $\dim \sigma = p^r$, for some $r \geqslant 1$. Let $\bar{\sigma}$ be the projective representation defined by σ and set $\mathcal{W}_K = \mathrm{Ker}\,\bar{\sigma}$. In particular, $\sigma(\mathcal{W}_K)$ is the centre of $\sigma(\mathcal{W}_F)$, so $\sigma \mid \mathcal{W}_K$ is a multiple of a character ζ_σ of $\mathcal{W}_K$.

Let T/F be the maximal tamely ramified subextension of K/F. The group $\Delta = \mathrm{Gal}(K/T)$ is elementary abelian of order p^{2r}. Since $\sigma(\mathcal{P}_F)$ is H-cyclic, the pairing

$$(x, y) \mapsto \zeta_\sigma(xyx^{-1}y^{-1}), \quad x, y \in \mathcal{W}_T,$$

induces a bilinear form

$$h_\sigma : \Delta \times \Delta \to \mathbb{F}_p \tag{3-1-1}$$

that is alternating and nondegenerate. The natural action of $\Theta = \mathrm{Gal}(T/F)$ on Δ fixes h_σ, so (Δ, h_σ) provides a *symplectic* representation of Θ over the field $\mathbb{F}_p$. A crucial point is the following:

Proposition 3.1.1 [Koch 1977, Theorem 4.1]. *The symplectic $\mathbb{F}_p$-representation of Θ on Δ is Θ-anisotropic, in that Δ has no nonzero Θ-subspace on which h_σ is identically zero.*

It is usually convenient to impose a further normalization.

Lemma 3.1.2. *There is a tamely ramified character χ of $\mathcal{W}_F$, such that the representation $\sigma' = \chi \otimes \sigma$ has the following properties.*

(1) *The kernel of σ' is of the form $\mathcal{W}_E$, where E/K is finite, cyclic and totally wildly ramified.*

(2) *The order of the character $\zeta_{\sigma'}$ is finite and a power of p, with $\mathcal{W}_E = \mathrm{Ker}\,\zeta_{\sigma'}$.*

Proof. We construct the character χ in stages. First, there is an unramified character χ_1 of $\mathcal{W}_F$ such that the representation $\sigma_1 = \chi_1 \otimes \sigma$ has finite image. The character $\det \sigma_1$ therefore has finite order. There exists a character χ_2 of $\mathcal{W}_F$, of finite order relatively prime to p, such that $\chi_2^{p^r} \det \sigma_1$ has finite p-power order. In particular, χ_2 is tamely ramified. Set $\sigma_2 = \chi_2 \otimes \sigma_1$, so that $\det \sigma_2$ has finite p-power order. The restriction of σ_2 to $\mathcal{W}_K$ is a multiple of the character $\zeta_2 = \zeta_\sigma \cdot \chi_2 \chi_1 \mid \mathcal{W}_K$. By construction, ζ_2 has finite p-power order.

Let $\mathrm{Ker}\,\zeta_2 = \mathcal{W}_{E_2}$. Thus E_2/K is a finite, cyclic p-extension. Viewing ζ_2 as a character of $K^\times$ via class field theory, the extension E_2/K is totally ramified if and only if $\zeta_2(K^\times) = \zeta_2(U_K)$ or, equivalently, there is a Frobenius element ϕ of $\mathcal{W}_K$ such that $\zeta_2(\phi) = 1$. So, suppose we have a Frobenius ϕ for which $\zeta_2(\phi) \neq 1$. There is an unramified character ψ of $\mathcal{W}_K$, of finite, p-power order, such that $\psi\zeta_2(\phi) = 1$. This character ψ is the restriction of an unramified character χ_3 of $\mathcal{W}_F$ of finite, p-power order. Write $\zeta_3 = \chi_3\zeta_2$ and $\mathcal{W}_E = \mathrm{Ker}\,\zeta_3$. The extension E/K is cyclic and totally ramified of p-power degree. Moreover, $\mathcal{W}_E = \mathrm{Ker}\,\sigma_3$, where $\sigma_3 = \chi_3 \otimes \sigma_2$, and all assertions have been proved for $\sigma' = \sigma_3$. $\qquad\square$

Remark 3.1.3. Replacing σ by σ' has no effect on the pairing h_σ or the fields K, T. The Tame multiplicity theorem holds for σ if and only if it holds for σ'.

3.2. In the notation of Section 3.1, we analyze the symplectic $\mathbb{F}_p$-representation of Θ provided by Δ. Let $J_{K/T}$ be the set of ramification jumps of K/T, in the upper numbering. Since K/T is abelian, these jumps are positive integers, by the Hasse–Arf theorem [Serre 1968, V théorème 1]. Observe that, for a real number $x \geqslant 0$, the ramification group Δ^x is an $\mathbb{F}_p\Theta$-subspace of Δ.

Proposition 3.2.1. *Let $j \in J_{K/T}$.*

(1) *The restriction of h_σ to Δ^j is nondegenerate.*

(2) *If W^j denotes the h_σ-orthogonal complement of Δ^{1+j} in Δ^j, then Δ is the orthogonal sum of the spaces W^j, $j \in J_{K/T}$.*

Proof. If X is a subspace of Δ, let $X^\perp$ be its h_σ-orthogonal complement in Δ. For an integer $j \geqslant 1$, the radical of the alternating form $h_\sigma \mid \Delta^j \times \Delta^j$ is $\Delta^j \cap (\Delta^j)^\perp$. This is an $\mathbb{F}_p\Theta$-subspace of Δ on which h_σ is null. Since h_σ is Θ-anisotropic, $\Delta^j \cap (\Delta^j)^\perp = 0$ whence (1) follows.

If $j \in J_{K/T}$, then Δ^{1+j} is trivial or equal to $\Delta^{j'}$, where j' is the least element of $J_{K/T}$ strictly greater than j. Assertion (2) now follows from (1). $\qquad\square$

3.3. We continue with the notation of Sections 3.1 and 3.2 to establish a lower bound on the Swan exponent $\mathrm{sw}(\sigma)$.

First, we specify a family of Lagrangian subspaces of the alternating space (Δ, h_σ). For each $j \in J_{K/T}$, let $\Xi(j)$ be a Lagrangian subspace of the nondegenerate space W^j. The various $\Xi(j)$ are mutually orthogonal, and so $\Xi = \sum_j \Xi(j)$ is Lagrangian. A Lagrangian subspace of this form will be called J-*split*.

Theorem 3.3.1. *Let Ξ be a J-split Lagrangian subspace of Δ. If $K^\Xi = L$, then $J_{L/T} = J_{K/T}$. If j_∞ is the largest element of $J_{L/T}$ and $e(T|F) = e$ then*

$$e \, \mathrm{sw}(\sigma) \geqslant \psi_{L/T}(j_\infty) + p^r j_\infty \geqslant (1 + p^r) j_\infty. \tag{3-3-1}$$

Proof. We may assume, without loss, that the representation σ has been normalized as in Lemma 3.1.2. In particular, $\mathrm{Ker}\,\sigma = \mathcal{W}_E$, where E/K is cyclic and totally wildly ramified. The extension E/F is Galois.

By construction, the extensions K/T and L/T have the same jumps, $J_{L/T} = J_{K/T}$. Let $\widetilde{\Delta} = \mathrm{Gal}(E/T)$, $\widetilde{\Xi} = \mathrm{Gal}(E/L)$. Since Ξ is a Lagrangian subspace of Δ, the extension E/L is abelian and totally wildly ramified. The Artin reciprocity isomorphism therefore induces a surjective homomorphism

$$a_L : U_L^1 \to \widetilde{\Xi} = \mathrm{Gal}(E/L).$$

Let $x \in \Delta^{j_\infty} \cap \Xi$ be nontrivial, and choose $y \in \Delta^{j_\infty}$ such that $h_\sigma(x, y) \neq 0$. We have $\Delta^{j_\infty} = \Delta_{k_\infty}$, where $k_\infty = \psi_{K/T}(j_\infty)$, and so x is an element of $\Delta_{k_\infty} \cap \Xi = \Xi_{k_\infty}$. However,

$$\Xi_{k_\infty} = \Xi^{\varphi_{K/L}(k_\infty)} = \Xi^{\psi_{L/T}(j_\infty)},$$

as follows from the transitivity relation $\psi_{K/T} = \psi_{K/L} \circ \psi_{L/T}$. Choose an inverse image $\tilde{x}$ of x in $\widetilde{\Xi}^{\psi_{L/T}(j_\infty)}$. As Galois operator on E therefore, we have $\tilde{x} = a_L(v)$, for some $\psi_{L/T}(j_\infty)$-unit v of L (by the higher ramification theorem of local class field theory [Serre 1968, XV théorème 1 corollaire 3]).

On the other hand, y acts on L as an element of

$$(\widetilde{\Delta}/\widetilde{\Xi})^{j_\infty} = (\Delta/\Xi)^{j_\infty} = (\Delta/\Xi)_{\psi_{L/T}(j_\infty)}.$$

The definition of the lower ramification sequence implies that, if $z \in U_L^k$, for some $k \geqslant 1$, then z^y/z is a $(k + \psi_{L/T}(j_\infty))$-unit of L.

Choose an inverse image $\tilde{y}$ of y in $\widetilde{\Delta}^{j_\infty}$. Therefore

$$\tilde{y}^{-1}\tilde{x}\tilde{y}\tilde{x}^{-1} = a_L(v^{\tilde{y}}v^{-1}) = a_L(u),$$

where $u = v^{\tilde{y}}v^{-1} = v^y v^{-1}$ is a $2\psi_{L/T}(j_\infty)$-unit of L.

Set $\sigma \mid \mathcal{W}_T = \tau$. The representation τ is irreducible. Since Ξ is Lagrangian, τ is induced from a character ϕ of $\mathcal{W}_L$ extending the character ζ_σ of $\mathcal{W}_K$ (Lemma 2.1.2). By construction, $\zeta_\sigma[y^{-1}, x] = \zeta_\sigma[\tilde{y}^{-1}, \tilde{x}] \neq 1$. So, if we view ϕ as a character

of $L^\times$ via class field theory, it is nontrivial on $2\psi_{L/T}(j_\infty)$-units of L. That is, $\mathrm{sw}(\phi) \geqslant 2\psi_{L/T}(j_\infty)$. Let $w_{L/T}$ be the wild exponent of the extension L/T (see (5-1-1) below). The standard induction formula reads

$$\mathrm{sw}(\tau) = \mathrm{sw}(\phi) + w_{L/T} \geqslant 2\psi_{L/T}(j_\infty) + w_{L/T}.$$

Since $[L:T] = p^r$ and j_∞ is the largest jump of L/T, we have

$$\psi_{L/T}(j_\infty) = p^r j_\infty - w_{L/T}$$

by [Bushnell and Henniart 2019, 1.6 Proposition]. It follows that

$$\mathrm{sw}(\tau) \geqslant \psi_{L/T}(j_\infty) + p^r j_\infty.$$

The Herbrand function satisfies $\psi_{L/T}(x) \geqslant x$, for all $x \geqslant 0$, so we further have

$$\mathrm{sw}(\tau) \geqslant \psi_{L/T}(j_\infty) + p^r j_\infty \geqslant (1+p^r) j_\infty.$$

Since $\mathrm{sw}(\tau) = e\,\mathrm{sw}(\sigma)$, we are done. $\square$

3.4. Theorem 3.3.1, and its proof, apply unchanged in greater generality. We shall not use the fact here, but this is a convenient place to record it. Suppose only that the irreducible representation σ is H-cyclic, in the sense of [Bushnell and Henniart 2019]: this means that $\sigma \mid \mathcal{P}_F$ is irreducible and that the finite p-group $\sigma(\mathcal{P}_F)$ is H-cyclic in the sense of Section 2.1. We can use all the same notation relative to σ. The inequalities (3-3-1) then hold, *provided the alternating form h_σ is nondegenerate on Δ^{j_∞}.*

4. Certain primitive representations

In this section, we prove the Tame multiplicity theorem for a certain class of primitive representations of $\mathcal{W}_F$.

4.1. Let σ be a primitive irreducible representation of $\mathcal{W}_F$. Say that σ is called *absolutely ramified* if the associated projective representation $\bar\sigma$ factors through a finite Galois group $\mathrm{Gal}(L/F)$ for which L/F is totally ramified.

Theorem 4.1.1. *If σ is an irreducible, primitive, absolutely ramified representation of $\mathcal{W}_F$, then $m(\sigma) \leqslant \mathrm{sw}(\sigma)$.*

The proof will occupy the rest of the section.

4.2. We normalize σ as permitted by Lemma 3.1.2 and use the notation developed in Section 3.1. Thus $\mathrm{Ker}\,\bar\sigma = \mathcal{W}_K$, where K/F is totally ramified. Let T/F be the maximal tame subextension of K/F. In addition, $\mathrm{Ker}\,\sigma = \mathcal{W}_E$ where E/K is cyclic and totally wildly ramified.

Set $\Gamma = \mathrm{Gal}(K/F)$, $\Delta = \mathrm{Gal}(K/T)$ and $\Theta = \mathrm{Gal}(T/F)$. Therefore Δ is elementary abelian of order $p^{2r} = (\dim \sigma)^2$ and Θ is cyclic of order prime to p. The restriction of σ to $\mathcal{W}_K$ is a multiple of a character ζ_σ and the group $\widetilde{\Delta} = \mathrm{Gal}(E/T)$ is an H-cyclic p-group with centre $\mathrm{Gal}(E/K)$. The subgroup Δ admits a complement Σ in Γ. Thus $\Sigma \cap \Delta = \{1\}$ and $\Gamma = \Sigma \Delta$. Restriction of operators induces an isomorphism $\Sigma \cong \Theta$. In particular, Σ is cyclic of order $e = e(T|F)$.

Let h_σ be the commutator pairing as in (3-1-1). The pair (Δ, h_σ) affords an anisotropic, symplectic $\mathbb{F}_p$-representation of Σ, of dimension $2r$. We review the classification of such representations, following [Bushnell and Fröhlich 1983].

Choose an algebraic closure $\overline{\mathbb{F}}_p/\mathbb{F}_p$, and write $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \Omega$. Let $\chi : \Sigma \to \overline{\mathbb{F}}_p^\times$ be a homomorphism and let $\mathbb{F}_p(\chi)$ be the field generated by the values $\chi(s)$, $s \in \Sigma$. The group Σ acts on $\mathbb{F}_p(\chi)$ via the character χ, that is,

$$s : x \mapsto \chi(s)x, \quad s \in \Sigma, \ x \in \mathbb{F}_p(\chi).$$

The group Ω acts on $\mathrm{Hom}(\Sigma, \overline{\mathbb{F}}_p)$ in a natural way. The map $\chi \mapsto \mathbb{F}_p(\chi)$ then induces a bijection between $\Omega \backslash \mathrm{Hom}(\Sigma, \overline{\mathbb{F}}_p)$ and the set of isomorphism classes of irreducible $\mathbb{F}_p$-representations of Σ.

Proposition 4.2.1. (1) *For $\chi \in \mathrm{Hom}(\Sigma, \overline{\mathbb{F}}_p^\times)$, the following conditions are equivalent.*

    (a) *The representation $\mathbb{F}_p(\chi)$ is symplectic, that is, it admits a nondegenerate, Σ-invariant, alternating form.*

    (b) *The character χ^{-1} is Ω-conjugate, but not equal, to χ.*

    (c) *The field $\mathbb{F}_p(\chi)$ satisfies $[\mathbb{F}_p(\chi) : \mathbb{F}_p] = p^{2d}$, for an integer $d \geqslant 1$, and $\chi(\Sigma)$ is contained in the subgroup of $\mathbb{F}_p(\chi)^\times$ of order $1+p^d$.*

(2) *Suppose that $\mathbb{F}_p(\chi)$ is symplectic. Any nonzero Σ-invariant alternating form on $\mathbb{F}_p(\chi)$ is Σ-anisotropic. Any two such forms are Σ-isometric.*

(3) *A finite $\mathbb{F}_p$-representation U of Σ provides a symplectic anisotropic representation of Σ if and only if there exist $\chi_j \in \mathrm{Hom}(\Sigma, \overline{\mathbb{F}}_p^\times)$, $1 \leqslant j \leqslant r$, such that*

    (a) *each $\mathbb{F}_p(\chi_j)$ is symplectic;*

    (b) *if $i \neq j$, then χ_i is not Ω-conjugate to χ_j;*

    (c) *$U = \bigoplus_{1 \leqslant j \leqslant r} \mathbb{F}_p(\chi_j)$.*

The proposition is taken from Section 8.2 of [Bushnell and Fröhlich 1983]. It may equally be viewed as an instance of the more general classification in [Koch 1977, Theorem 8.1], although some effort of translation would be required.

Remark 4.2.2. If χ has order a and satisfies the conditions in part (1) of the proposition, then

(a) $a \geqslant 3$, and

(b) the integer d is the least for which $1+p^d$ is divisible by a.

4.3. In the same situation, we analyze the symplectic $\mathbb{F}_p$-representation of Σ on $\Delta = \mathrm{Gal}(K/T)$. Following Proposition 4.2.1(2), it is only the structure of the *linear* $\mathbb{F}_p\Sigma$-representation Δ that need concern us.

Recall that $J_{K/T}$ is the set of (upper) ramification jumps of K/T. For $j \in J_{K/T}$, define W^j as in Proposition 3.2.1.

Proposition 4.3.1. *For all $j \in J_{K/T}$, the $\mathbb{F}_p\Sigma$-space W^j is irreducible.*

Proof. Let $k \in \mathbb{Z}$, $k \geqslant 1$. The group Δ^k is the image of the unit group U_T^k under the Artin reciprocity map $T^\times \to \Delta = \mathrm{Gal}(K/T)$. This map is Σ-equivariant and W^j, $j \in J_{K/T}$, is so realized as a Σ-quotient of U_T^j/U_T^{1+j}.

The natural action of $\Sigma = \mathrm{Gal}(T/F)$ on $\mathfrak{p}_T/\mathfrak{p}_T^2$ is given by a faithful character $\theta : \Sigma \to \Bbbk_F^\times$. The natural action on $\mathfrak{p}_T^j/\mathfrak{p}_T^{1+j}$, $j \geqslant 1$, is therefore implemented by θ^j. The character θ^j induces an algebra homomorphism $\mathbb{F}_p\Sigma \to \Bbbk_F$, the image of which is necessarily a subfield of $\Bbbk_F$. The $\mathbb{F}_p\Sigma$-module $U_T^j/U_T^{1+j} \cong \mathfrak{p}_T^j/\mathfrak{p}_T^{1+j}$ is therefore *isotypic*. However, W^j is anisotropic, so Proposition 4.2.1(3) implies that W^j is a direct sum of mutually inequivalent irreducible $\mathbb{F}_p\Sigma$-modules. It is therefore irreducible, as required. $\qquad\square$

We underline some points made in the preceding proof.

Corollary 4.3.2. *Let $j \in J_{K/T}$.*

(1) *The symplectic $\mathbb{F}_p$-representation W^j of Σ is equivalent to $\mathbb{F}_p(\theta^j)$.*

(2) *Let $e = e(T|F) = |\Sigma|$. An element $s \in \Sigma$ has a nontrivial fixed point in W^j if and only if $s^{\gcd(e,j)} = 1$.*

4.4. Twisting σ with a character of $\mathrm{Gal}(T/F)$ has no effect on the assertion to be proved. We therefore assume that the character $\det \sigma$ is trivial on Σ: this puts us in the situation of Proposition 2.2.3.

The orthogonal decomposition $\Delta = \sum_{j \in J_{K/T}} W^j$ implies a canonical realization of the $\mathbb{F}_p\Sigma$-module W^j as a subspace of Δ: let $\widetilde{W}^j$ be its inverse image in $\widetilde{\Delta}$. The construction outlined in Sections 2.3 and 2.4 gives a representation σ^j of $\Sigma\widetilde{W}^j$ and a tensor decomposition

$$\sigma = \bigotimes_{j \in J_{K/T}} \sigma^j.$$

We may choose the factors σ^j so that each character $\det \sigma^j \mid \Sigma$ is trivial. As in Corollary 4.3.2(2), Σ acts on W^j via its quotient of order $e_j = e/(e, j)$, and that action is faithful.

Definition 4.4.1. Let A be the set of positive divisors a of e of the form $e_j = e/(e, j)$, for some $j \in J_{K/T}$. For $a \in A$, set

$$\sigma_a = \bigotimes_{\substack{j \in J_{K/T} \\ a = e_j}} \sigma^j.$$

Observe that $e = |\Sigma|$ is the lcm of the elements of A. Note also that a factor σ_a may have several ramification jumps: this possibility is *not* excluded by Proposition 4.2.1.

We work in the ring $\mathbb{Z}\widehat{\Sigma}$ of virtual characters of Σ. The elements of $\mathbb{Z}\widehat{\Sigma}$ are thus the formal linear combinations

$$c = \sum_{\chi \in \widehat{\Sigma}} c_\chi \chi$$

in which the coefficients c_χ lie in $\mathbb{Z}$. Let $\mathbb{N}\widehat{\Sigma}$ be the "order" consisting of those $c \in \mathbb{Z}\widehat{\Sigma}$ for which the coefficients c_χ are all nonnegative. For $a, b \in \mathbb{Z}\widehat{\Sigma}$, we write $a \geqslant b$ when $a - b \in \mathbb{N}\widehat{\Sigma}$. We also use the relation $\geqslant$ to compare elements of $\mathbb{Q}\widehat{\Sigma}$ in the obvious way.

Definition 4.4.2. Let $a \in A$.

(1) Let q_a be the least power of p such that $1+q_a$ is divisible by a and define $\ell(a)$ as the number of $j \in J_{K/T}$ for which $a = e_j$.

(2) Define the positive integer k_a by

$$a k_a = \left(q_a^{\ell(a)} - (-1)^{\ell(a)} \right).$$

(3) Let μ_a denote the trivial character of Σ if a is odd or k_a is even. Otherwise, let $\mu_a \in \widehat{\Sigma}$ have order 2.

(4) Let $\rho_a \in \mathbb{Z}\widehat{\Sigma}$ be the sum of characters ϕ of Σ such that $\phi^a = 1$, and define

$$R_a = k_a \rho_a + (-1)^{\ell(a)} \mu_a \in \mathbb{Z}\widehat{\Sigma}. \tag{4-4-1}$$

Proposition 4.4.3. *If $a \in A$, then*

$$\sigma_a \mid \Sigma = R_a \tag{4-4-2}$$

and, moreover,

$$\sigma \mid \Sigma = \prod_{a \in A} R_a, \tag{4-4-3}$$

the product being taken in $\mathbb{Z}\widehat{\Sigma}$.

Proof. This follows directly from Proposition 2.2.3 and (2-4-1). $\qquad\square$

4.5. We treat a special case of Theorem 4.1.1, working directly from (4-4-2).

Proposition 4.5.1. *If the set A has exactly one element, then $m(\sigma) \leqslant \mathrm{sw}(\sigma)$.*

Proof. The lcm of the elements of A is e, so $A = \{e\}$. Moreover, $q_e^{\ell(e)} = p^r = \dim \sigma$.

Suppose first that $\ell(e)$ is odd, so that $m(\sigma) = k_e = (p^r+1)/e$. We have to show that $p^r+1 \leqslant e\, \mathrm{sw}(\sigma)$. By Theorem 3.3.1, $e\, \mathrm{sw}(\sigma) \geqslant (p^r+1) j_\infty$, where j_∞ is the

largest element of $J_{K/T}$. As j_∞ is a positive integer (see Section 3.2), so

$$e \, \mathrm{sw}(\sigma) \geqslant (p^r + 1) j_\infty \geqslant (p^r + 1),$$

as required. Suppose, on the other hand, that $\ell(e)$ is even. In this case, e divides $p^r - 1$, so $e \leqslant p^r - 1$ and

$$em(\sigma) = p^r - 1 + e \leqslant 2(p^r - 1).$$

On the other hand, since $\ell(e)$ is even, $j_\infty \geqslant 2$. Therefore

$$em(\sigma) \leqslant 2(p^r - 1) < (p^r + 1) j_\infty \leqslant e \, \mathrm{sw}(\sigma).$$

This completes the proof.   □

We reflect briefly on the proof of this proposition.

Corollary 4.5.2. *In the situation of the proposition, if $m(\sigma) = \mathrm{sw}(\sigma)$ then $J_{K/T} = \{1\}$.*

Proof. If $\ell(e)$ is even then, as we have just seen, $m(\sigma) < \mathrm{sw}(\sigma)$, so suppose $\ell(e)$ is odd. If $\ell(e) \neq 1$, then $j_\infty \geqslant 3$ and

$$em(\sigma) = 1 + p^r < (1 + p^r) j_\infty \leqslant e \, \mathrm{sw}(\sigma).$$

So, we assume $\ell(e) = 1$ and $J_{K/T} = \{j_\infty\}$. In this case, if $j_\infty > 1$ then $m(\sigma) < \mathrm{sw}(\sigma)$.
                                                                             □

4.6. Assume now that A has at least two elements. We can make some simplifying approximations. The expressions (4-4-1) and (4-4-2) imply

$$\sigma_a \mid \Sigma = R_a \leqslant q_a^{\ell(a)} d_a \frac{\rho_a}{a}, \tag{4-6-1}$$

where

$$d_a = \begin{cases} 1 + q_a^{-\ell(a)} & \text{if } \ell(a) \text{ is odd,} \\ 1 + (a-1) q_a^{-\ell(a)} & \text{if } \ell(a) \text{ is even.} \end{cases} \tag{4-6-2}$$

If $a, b \in A$, then

$$\frac{\rho_a}{a} \frac{\rho_b}{b} = \frac{\rho_c}{c},$$

where c is the lcm of a and b. So, taking the product over $a \in A$, we get

$$\sigma \mid \Sigma \leqslant \dim \sigma \prod_{a \in A} d_a \frac{\rho_e}{e},$$

whence

$$m(\sigma) \leqslant \frac{\dim \sigma}{e} \prod_{a \in A} d_a. \tag{4-6-3}$$

So, we are reduced to proving:

Proposition 4.6.1. *If $|A| \geqslant 2$, then*

$$\prod_{a \in A} d_a \leqslant \frac{e \operatorname{sw}(\sigma)}{\dim \sigma}. \tag{4-6-4}$$

Proof. Let j_∞ be the largest element of $J_{K/T}$. Let $\varXi$ be a J-split Lagrangian subspace of the symplectic space $\varDelta$ as in Section 3.3. Let L be the fixed field of $\varXi$ and recall that $J_{K/T}$ is equal to the set $J_{L/T}$ of jumps of L/T.

Lemma 4.6.2. *Suppose that A has at least two elements. If $a \in A$, then $a \geqslant 3$ and $d_a \leqslant a/(a-1)$.*

Proof. For the first assertion, see Remark 4.2.2(a). For the second, suppose first that $\ell(a)$ is even, whence $d_a = 1 + (a-1)q_a^{-\ell(a)}$. By definition, a divides $1 + q_a$ whence $a - 1 \leqslant q_a$. Therefore

$$d_a \leqslant 1 + q_a^{-1} \leqslant 1 + (a-1)^{-1} = a/(a-1).$$

The case of $\ell(a)$ odd is similar but even easier. $\qquad\square$

It follows that

$$\prod_{a \in A} d_a \leqslant \prod_{a \in A} \frac{a}{a-1}.$$

Let $t \geqslant 2$ be the number of elements of A. The largest element of A is therefore, at least, $t+2$ and all elements are $\geqslant 3$. Consequently,

$$\prod_{a \in A} d_a \leqslant \frac{3}{2}\frac{4}{3} \cdots \frac{t+2}{t+1} \leqslant \frac{t+2}{2} \leqslant t.$$

Lemma 4.6.3. *If h denotes the number of jumps of L/T, then*

$$h < p^{-r}\psi_{L/T}(j_\infty) + j_\infty.$$

Proof. The jumps of the abelian extension L/T are positive integers, so $h \leqslant j_\infty$ and $h - j_\infty \leqslant 0 < p^{-r}\psi_{L/T}(j_\infty)$, as required. $\qquad\square$

Assembling these relations and applying (3-3-1), we get

$$\prod_{a \in A} d_a \leqslant t \leqslant h < e \operatorname{sw}(\sigma)/p^r,$$

as required to complete the proof of the proposition. $\qquad\square$

This finishes the proof of Theorem 4.1.1. $\qquad\square$

5. An estimate of the different

Preliminary to the proof of the general case of the main theorem, we make an estimate of the wild exponent $w_{K/F}$ of a class of finite extensions K/F. It is not remotely sharp (see Example 5.2.4) but is adequate for our purposes.

5.1. Let K/F be a finite separable extension, with $K \subset \bar{F}$. The *wild exponent* $w_{K/F}$ of K/F is

$$w_{K/F} = d_{K/F} + 1 - e(K|F) = \mathrm{sw}(\mathrm{Ind}_{K/F} 1_K), \qquad (5\text{-}1\text{-}1)$$

where $d_{K/F}$ is the exponent of the different of K/F and 1_K denotes the trivial character of $\mathcal{W}_K$.

5.2. Let E/F be a finite, totally ramified, Galois extension. Set $\mathrm{Gal}(E/F) = \Gamma$ and let Δ be the wild inertia subgroup of Γ. As in Section 1.1, Δ is the unique p-Sylow subgroup of Γ and admits a complement Σ in Γ. In particular, Σ is cyclic of order prime to p.

Proposition 5.2.1. *Let Φ be a subgroup of Γ, such that the index $(\Gamma : \Phi)$ is a power of p. If K is the fixed field E^{Φ} of Φ in E, then*

$$w_{K/F} \geqslant |\Phi\backslash\Gamma/\Sigma| - 1.$$

Proof. Recall that any two choices of the complement Σ are conjugate in Γ. The assertion is therefore independent of the choice of Σ.

If $\Phi = \Gamma$ there is nothing to prove, so we assume otherwise.

Lemma 5.2.2. *Let Ξ be a normal subgroup of Γ such that $\Xi \subset \Phi$ and let $f : \Gamma \to \Gamma/\Xi$ be the quotient map.*

(1) *The group $f(\Sigma)$ is a complement of $f(\Delta)$ in Γ/Ξ.*

(2) *The map f induces a Σ-equivariant bijection $\Phi\backslash\Gamma \to f(\Phi)\backslash f(\Gamma)$, and hence a bijection $\Phi\backslash\Gamma/\Sigma \to f(\Phi)\backslash f(\Gamma)/f(\Sigma)$.*

Proof. Straightforward. $\qquad\qquad\square$

Continue with Ξ as in the lemma. If we replace E by E^{Ξ}, the extension K/F is unchanged. The effect of the lemma is to show that, if the proposition holds for the configuration $F \subset K \subset E^{\Xi}$, then it holds for $F \subset K \subset E$. We may choose Ξ so that E^{Ξ}/F is a normal closure of K/F. It is therefore enough to prove the proposition under the assumption that E/F is a normal closure of K/F. We henceforward assume this to be the case.

Lemma 5.2.3. *Let Θ be the smallest nontrivial ramification subgroup of Γ. The group Θ is elementary abelian and central in Δ. It is not contained in Φ.*

Proof. The first assertions are given by [Serre 1968, IV Propositions 7 and 10]. If Θ were contained in Φ then E^Θ/F would be a normal extension containing K/F and such that $[E^\Theta : F] < [E : F]$. Since E/F is a normal closure of K/F, this is impossible. $\qquad\square$

Suppose for the moment that $\Gamma = \Phi\Theta$ or, equivalently, that $\Delta = (\Phi \cap \Delta)\Theta$. As Θ is central in Δ, so $\Phi \cap \Delta$ is normal in Δ and $\Delta/\Phi \cap \Delta$ is abelian. Let 1_Φ denote the trivial character of Φ, and similarly for other groups. The Mackey formula gives the relations

$$\mathrm{Ind}_\Phi^\Gamma 1_\Phi \mid \Delta = \mathrm{Ind}_{\Phi\cap\Delta}^\Delta 1_{\Phi\cap\Delta}, \qquad (5\text{-}2\text{-}1)$$

$$\mathrm{Ind}_\Phi^\Gamma 1_\Phi \mid \Theta = \mathrm{Ind}_{\Phi\cap\Theta}^\Theta 1_{\Phi\cap\Theta}. \qquad (5\text{-}2\text{-}2)$$

The restriction (5-2-2) is the direct sum of all characters χ of $\Phi\cap\Theta\backslash\Theta$. Any such character χ extends uniquely to a character χ_Δ of Δ trivial on $\Phi\cap\Delta$: one puts $\chi_\Delta(hr) = \chi(r)$, $h \in \Phi\cap\Delta$, $r \in \Theta$. Consequently,

$$\mathrm{Ind}_{\Phi\cap\Delta}^\Delta 1_{\Phi\cap\Delta} = \sum_{\chi\in(\Phi\cap\Theta\backslash\Theta)^\wedge} \chi_\Delta.$$

If Γ_χ denotes the Γ-centralizer of $\chi_\Delta \in (\Delta/\Phi\cap\Delta)^\wedge$, then $\Gamma_\chi = \Sigma_\chi\Delta$, where Σ_χ is the Σ-centralizer of χ_Δ (or, equivalently, of χ). Consequently, there is a unique character χ_Σ of Γ_χ that extends χ_Δ and is trivial on Σ_χ. Therefore

$$\mathrm{Ind}_\Phi^\Gamma 1_\Phi = \sum_{\chi\in\Sigma\backslash(\Phi\cap\Theta\backslash\Theta)^\wedge} \sum_{\eta\in(\Gamma_\chi/\Delta)^\wedge} \mathrm{Ind}_{\Gamma_\chi}^\Gamma \eta\chi_\Sigma.$$

We calculate the contribution of each term here to the exponent $\mathrm{sw}(\mathrm{Ind}_\Phi^\Gamma 1_\Phi) = w_{K/F}$.

If χ is trivial, then $\Gamma_\chi = \Gamma$ and we get a contribution of 0. Otherwise, $\mathrm{Ind}_{\Gamma_\chi}^\Gamma \eta\chi_\Sigma$ has Swan exponent at least 1, whence

$$w_{K/F} \geqslant \sum_{\substack{\chi\in\Sigma\backslash(\Phi\cap\Theta\backslash\Theta)^\wedge \\ \chi\neq 1}} (\Gamma_\chi : \Delta).$$

However, $|\Sigma\backslash(\Phi \cap \Theta\backslash\Theta)^\wedge| = |\Phi\backslash\Gamma/\Sigma|$, and so $w_{K/F} \geqslant |\Phi\backslash\Gamma/\Sigma| - 1$ in this case.

Example 5.2.4. Remark here that, once the trivial character χ is excluded, all groups Γ_χ are the same: they depend only on the denominator of j, where $\Theta = \Gamma^j \neq \Gamma^{j+\epsilon}$, $\epsilon > 0$. All characters $\eta\chi_\Sigma$ have the same slope, namely j. The index $(\Gamma : \Gamma_\chi)$, for $\chi \neq 1$, is the gcd of $|\Sigma|$ and the denominator of j. So, for $\chi \neq 1$, the inner sum has Swan exponent $j(\Gamma : \Gamma_\chi)(\Gamma_\chi : \Delta) = j|\Sigma|$. Therefore

$$\mathrm{sw}(\mathrm{Ind}_\Phi^\Gamma 1_\Phi) = w_{K/F} = j|\Sigma|(|\Phi\backslash\Gamma/\Sigma| - 1). \qquad (5\text{-}2\text{-}3)$$

We return to the proof of Proposition 5.2.1, assuming now that $\Phi\Theta \neq \Gamma$. Since the index $(\Gamma : \Phi)$ is a power of p, the group Φ contains a conjugate of Σ. Following the remark at the beginning of the proof, we may assume that $\Sigma \subset \Phi$.

Let $L = E^{\Phi\Theta}$. The first case above gives

$$w_{K/L} \geqslant |\Phi\backslash\Phi\Theta/\Sigma| - 1.$$

By induction on $[K{:}F] = (\Gamma : \Phi)$, we likewise have

$$w_{L/F} \geqslant \big|f(\Phi)\backslash f(\Gamma)/f(\Sigma)\big| - 1,$$

where $f : \Gamma \to \Gamma/\Theta$ is the quotient map. On the other hand,

$$w_{K/F} = w_{K/L} + [K{:}L]w_{L/F},$$

so

$$w_{K/F} \geqslant |\Phi\backslash\Phi\Theta/\Sigma| - 1 + [K{:}L]\big(\big|f(\Phi)\backslash f(\Gamma)/f(\Sigma)\big| - 1\big).$$

Under the canonical surjection $\bar{f} : \Phi\backslash\Gamma/S \to f(\Phi)\backslash f(\Gamma)/f(\Sigma)$ induced by the quotient map $f : \Gamma \to \Gamma/\Theta$, the fibre of the trivial coset $f(\Phi) = f(\Phi)f(\Sigma)$ is precisely $\Phi\backslash\Phi\Theta/\Sigma$. On the other hand, let $x = f(g) \notin f(\Phi)$. The fibre, under $\bar{f}$, of $f(\Phi)xf(\Sigma)$ is contained in $\Phi g\Sigma$. This comprises at most $[K{:}L]$ double cosets $\Phi g\Sigma$, whence the result follows. $\qquad\square$

6. Proof of the main theorem

We prove the Tame multiplicity theorem in the general case. Let σ be an irreducible representation of $\mathcal{W}_F$ that is *not tamely ramified*: see Remark 1.1.1(1). Since the assertion of the theorem is unaffected by tensoring σ with an unramified character of $\mathcal{W}_F$, we may treat σ as a representation of $\Gamma = \mathrm{Gal}(E/F)$, where E/F is finite. Let Γ_0, Γ_1 be respectively the inertia and the wild inertia subgroups of Γ, and similarly for other finite Galois groups.

6.1. Let σ be an irreducible representation of $\Gamma = \mathrm{Gal}(E/F)$, with $\mathrm{sw}(\sigma) > 0$. Let Σ be a complement of Γ_1 in Γ_0.

Proposition 6.1.1. *If σ is absolutely ramified, that is, if E/F is totally ramified, then $m(\sigma) \leqslant \mathrm{sw}(\sigma)$.*

Proof. If σ is primitive or of dimension one, the result holds by Theorem 4.1.1 or Remark 1.1.1(2) respectively. We therefore suppose otherwise: there is a proper subgroup Δ of Γ and an irreducible representation τ of Δ such that $\sigma = \mathrm{Ind}_\Delta^\Gamma \tau$. The representation τ is absolutely ramified and, by induction on dimension, we may assume that $m(\tau) \leqslant \mathrm{sw}(\tau)$.

Suppose first that Δ may be chosen to contain Γ_1. Thus $\Gamma = \Gamma_0 = \Sigma \Delta$, and Δ is a normal subgroup of Γ. The Mackey formula gives

$$\sigma \mid \Sigma = \mathrm{Ind}_{\Sigma \cap \Delta}^{\Sigma} \tau \mid \Sigma \cap \Delta = \tau \mid \Sigma,$$

whence $m(\sigma) = m(\tau)$. As E^{Δ}/F is tamely ramified, so $\mathrm{sw}(\sigma) = \mathrm{sw}(\tau)$ and we are done in this case.

We therefore assume that σ cannot be induced from a proper subgroup of $\Gamma = \Gamma_0$ that contains Γ_1. Since Γ/Γ_1 is cyclic, the restriction $\sigma \mid \Gamma_1$ is irreducible. In particular, $\dim \sigma$ is a power of p. It follows that, if σ is induced from a representation τ of a proper subgroup Δ of Γ, then $(\Gamma{:}\Delta)$ is a power of p and, if $K = E^{\Delta}$, the extension K/F is totally wildly ramified. We have $m(\tau) \leqslant \mathrm{sw}(\tau)$, and

$$\mathrm{sw}(\sigma) = \mathrm{sw}(\tau) + w_{K/F} \dim \tau. \tag{6-1-1}$$

We apply Proposition 5.2.1. We adjust our choice of Σ, via conjugation by an element of Γ_1, to achieve $\Sigma \subset \Delta$. Let χ be a character of Σ. In the Mackey expansion

$$\sigma \mid \Sigma = \sum_{g \in \Delta \backslash \Gamma / \Sigma} \mathrm{Ind}_{g^{-1}\Delta g \cap \Sigma}^{\Sigma} (\tau^g \mid g^{-1}\Delta g \cap \Sigma).$$

the trivial double coset gives the term $\tau \mid \Sigma$, in which χ occurs with multiplicity at most $m(\tau)$. The contribution from a nontrivial double coset contains χ with multiplicity at most $\dim \tau$ so, overall,

$$m(\sigma) \leqslant m(\tau) + (|\Delta \backslash \Gamma / \Sigma| - 1) \dim \tau. \tag{6-1-2}$$

Comparing (6-1-1) with (6-1-2), Proposition 5.2.1 implies

$$m(\sigma) \leqslant \mathrm{sw}(\tau) + w_{K/F} \dim \tau = \mathrm{sw}(\sigma),$$

as required. □

We return to a more general situation.

Corollary 6.1.2. *Let E/F be a finite Galois extension and let σ be an irreducible representation of $\Gamma = \mathrm{Gal}(E/F)$, with $\mathrm{sw}(\sigma) > 0$. If $\sigma \mid \Gamma_0$ is irreducible, then $m(\sigma) \leqslant \mathrm{sw}(\sigma)$.*

Proof. The representation $\sigma_0 = \sigma \mid \Gamma_0$ is irreducible and absolutely ramified. The proposition gives $m(\sigma_0) \leqslant \mathrm{sw}(\sigma_0)$. However, since $\Sigma \subset \Gamma_0$, we have $m(\sigma_0) = m(\sigma)$. On the other hand, $\mathrm{sw}(\sigma_0) = \mathrm{sw}(\sigma)$, since E^{Γ_0}/F is unramified. □

Example 6.1.3. Example 2 of [Bushnell and Henniart 2017, §8.5] is interesting in this context. Suppose that $p = 2$ and that F contains a primitive cube root of unity. The construction in [Bushnell and Henniart 2017] yields a primitive representation σ of dimension 8, with $\mathrm{sw}(\sigma) = 3$ and a unique ramification jump. (In the notation of Theorem 3.3.1, this jump is j_{∞} and it has value 1.) If $\mathrm{Ker}\,\bar{\sigma} = \mathcal{W}_K$, and T/F

is the maximal tame subextension of K/F, then $[T{:}F] = 9$ and $e(T|F) = 3$. In particular, σ is not absolutely ramified. If T_0/F is the maximal unramified subextension of T/F, the restriction $\sigma \mid \mathcal{W}_{T_0}$ is irreducible but not primitive. A simple counting argument gives $m(\sigma) = 3 = \mathrm{sw}(\sigma)$.

6.2. We complete the proof of the Tame multiplicity theorem. Let σ be an irreducible representation of the finite group $\Gamma = \mathrm{Gal}(E/F)$ with $\mathrm{sw}(\sigma) > 0$. Let Σ be a complement of Γ_1 in Γ_0. If $\sigma \mid \Gamma_0$ is irreducible, the theorem is Corollary 6.1.2. We therefore assume otherwise, so there exist a proper subgroup Δ of Γ containing Γ_0 and an irreducible representation τ of Δ that induces σ. We choose Δ minimal with respect to this property, so that $\tau \mid \Delta_0$ is irreducible. By Corollary 6.1.2, $m(\tau) \leqslant \mathrm{sw}(\tau)$ while

$$\mathrm{sw}(\sigma) = (\Gamma{:}\Delta)\,\mathrm{sw}(\tau). \tag{6-2-1}$$

As $\Delta_0 = \Gamma_0$ and $\Delta_1 = \Gamma_1$, so Σ is also a complement of Δ_1 in Δ_0. Applying the standard Mackey formula, we get

$$\sigma \mid \Sigma = \sum_{g \in \Delta \backslash \Gamma / \Sigma} \mathrm{Ind}^{\Sigma}_{g^{-1}\Delta g \cap \Sigma}(\tau^g \mid g^{-1}\Delta g \cap \Sigma).$$

We have $\Gamma_0 = \Sigma\Gamma_1 \subset \Delta$, while any Γ-conjugate of Σ is contained in Δ. The canonical map $\Delta \backslash \Gamma \to \Delta \backslash \Gamma / \Sigma$ is therefore bijective. Consider the expression

$$\sigma \mid \Sigma = \sum_{g \in \Delta \backslash \Gamma} \tau^g \mid \Sigma.$$

If χ is a character of Σ, the multiplicity of χ in τ^g is that of $\chi^{g^{-1}}$ in τ, whence at most $m(\tau)$. We conclude that $m(\sigma) \leqslant (\Gamma{:}\Delta)m(\tau)$. Since $m(\tau) \leqslant \mathrm{sw}(\tau)$, the desired relation $m(\sigma) \leqslant \mathrm{sw}(\sigma)$ follows from (6-2-1). $\qquad\square$

Acknowledgement

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Nilpotence theorems via homological residue fields

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We prove nilpotence theorems in tensor-triangulated categories using suitable
Gabriel quotients of the module category, and discuss examples.

1. Introduction

For the average Joe, and the median Jane, *the* nilpotence theorem refers to a result in
stable homotopy theory, conjectured by Ravenel and proved by Devinatz, Hopkins
and Smith in their famous work on chromatic theory [Devinatz et al. 1988; Hopkins
and Smith 1998]. One form of the result says that a map between finite spectra
which gets annihilated by all Morava K-theories must be tensor-nilpotent. Under
Hopkins's [1987] impetus, these ideas soon expanded beyond topology. Neeman
[1992] in commutative algebra and Thomason [1997] in algebraic geometry proved
nilpotence theorems for maps in derived categories of perfect complexes, using or-
dinary residue fields instead of Morava K-theories. Benson, Carlson and Rickard
[Benson et al. 1997] led the charge into yet another area, namely modular repre-
sentation theory of finite groups, where the appropriate "residue fields" turned out
to be shifted cyclic subgroups, and later π-points [Friedlander and Pevtsova 2007].
As further areas kept joining the fray, expectations rose of a unified treatment
applicable to every tensor-triangulated category in nature. In this vein, Mathew
[2017, Theorem 4.14(b)] proved an abstract nilpotence theorem via E_∞-rings in
∞-categories over the field $\mathbb{Q}$. However, this rationality assumption is a severe
restriction, incompatible with the chromatic joys of topological Joe and the positive
characteristic tastes of modular Jane. Here, we prove abstract nilpotence theorems,
integrally and without ∞-categories. For instance, Corollary 4.5 says:

1.1. Theorem. *Let* $f : x \to y$ *be a morphism in an essentially small, rigid tensor-
triangulated category* $\mathcal{K}$. *If we have* $\bar{\mathrm{h}}(f) = 0$ *for every homological residue field*
$\bar{\mathrm{h}} : \mathcal{K} \to \bar{\mathcal{A}}$, *then there exists* $n \geq 1$ *such that* $f^{\otimes n} = 0$ *in* $\mathcal{K}$.

We need to explain the *homological residue fields* that appear in this statement. Their purpose is to encapsulate the key features of Morava K-theories, ordinary residue fields, shifted cyclic subgroups, etc., from an abstract point of view. In other words, when first meeting a tensor-triangulated category $\mathcal{K}$, we would like to extract its "residue fields" without knowing intimate details about $\mathcal{K}$, as we are used to extracting residue fields $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ from any commutative ring R, without much knowledge about R beyond its propensity to harbor prime ideals $\mathfrak{p} \in \mathrm{Spec}(R)$. We investigated this question of "tensor-triangular fields" in the recent joint work [Balmer et al. 2019], with Krause and Stevenson. Although the present article can be read independently, we refer to that prequel for motivation, background, justification, and a couple of lemmas. In retrospect, our nilpotence theorems further validate the ideas introduced in [Balmer et al. 2019].

In a nutshell, as we do not know how to produce residue fields within triangulated categories, we consider instead homological tensor-functors to *abelian* categories

$$\bar{\mathrm{h}} = \bar{\mathrm{h}}_\mathcal{B} : \mathcal{K} \hookrightarrow \mathrm{mod}\text{-}\mathcal{K} \twoheadrightarrow (\mathrm{mod}\text{-}\mathcal{K})/\mathcal{B}$$

composed of the Yoneda embedding $\mathrm{h} : \mathcal{K} \hookrightarrow \mathrm{mod}\text{-}\mathcal{K}$ into the Freyd envelope of $\mathcal{K}$ (Remark 2.6) followed by the Gabriel quotient $Q_\mathcal{B} : \mathrm{mod}\text{-}\mathcal{K} \twoheadrightarrow (\mathrm{mod}\text{-}\mathcal{K})/\mathcal{B}$ with respect to any *maximal* $\otimes$-ideal Serre subcategory $\mathcal{B}$. Thus Theorem 1.1 can be rephrased as follows:

> *If a morphism $f : x \to y$ in $\mathcal{K}$ is annihilated by $\bar{\mathrm{h}}_\mathcal{B} : \mathcal{K} \to (\mathrm{mod}\text{-}\mathcal{K})/\mathcal{B}$ for every maximal Serre $\otimes$-ideal $\mathcal{B} \subset \mathrm{mod}\text{-}\mathcal{K}$, then f is $\otimes$-nilpotent in $\mathcal{K}$.*

At first, it might be counterintuitive to only invoke *maximal* $\otimes$-ideals $\mathcal{B}$, instead of some kind of more general "*prime*" $\otimes$-ideals of $\mathrm{mod}\text{-}\mathcal{K}$, but we explain in Section 3 why this notion covers all points of the triangular spectrum $\mathrm{Spc}(\mathcal{K})$ of $\mathcal{K}$, not just the closed points. We also explain in Remark 3.10 how the above homological residue fields correspond to the local constructions proposed in [Balmer et al. 2019, §4].

As a matter of fact, in the examples, there exist alternative formulations of the nilpotence theorem. And the same holds here. Most notably, if our triangulated category $\mathcal{K}$ sits inside a "big" one, $\mathcal{K} \subset \mathcal{T}$, as the compact objects $\mathcal{K} = \mathcal{T}^c$ of a so-called "rigidly compactly generated" tensor-triangulated category $\mathcal{T}$ (Remark 4.6), we expect a nilpotence theorem for maps $f : x \to Y$ with compact source $x \in \mathcal{K}$ but *arbitrary* target Y in $\mathcal{T}$. This flavor of nilpotence theorem is Corollary 4.7.

In order to handle such generalizations, we consider the big Grothendieck category $\mathcal{A} := \mathrm{Mod}\text{-}\mathcal{K}$ of *all* right $\mathcal{K}$-modules (Notation 2.5), not just the subcategory of finitely presented ones that is the Freyd envelope $\mathcal{A}^{\mathrm{fp}} = \mathrm{mod}\text{-}\mathcal{K}$. When $\mathcal{K} = \mathcal{T}^c$, the big category $\mathcal{T}$ still admits a so-called "restricted-Yoneda" functor $\mathrm{h} : \mathcal{T} \to \mathrm{Mod}\text{-}\mathcal{K}$ (Remark 4.6). Every maximal Serre $\otimes$-ideal $\mathcal{B} \subset \mathcal{A}^{\mathrm{fp}}$ generates a localizing (Serre)

$\otimes$-ideal $\langle\mathcal{B}\rangle$ of $\mathcal{A}$ and we can consider the corresponding "big" Gabriel quotient $\bar{\mathcal{A}} := \mathcal{A}/\langle\mathcal{B}\rangle$. Composing with restricted-Yoneda, we obtain a homological $\otimes$-functor $\bar{\mathrm{h}}_{\mathcal{B}} : \mathcal{T} \to \bar{\mathcal{A}}$ on the "big" category $\mathcal{T}$, extending the one on $\mathcal{K}$:

$$
\begin{array}{ccccccc}
 & & & & \bar{\mathrm{h}}_{\mathcal{B}} & & \\
 & & \overbrace{\hspace{7cm}} & & & & \\
\mathcal{K} = \mathcal{T}^c & \overset{\mathrm{h}}{\hookrightarrow} & \mathcal{A}^{\mathrm{fp}} = \mathrm{mod}\text{-}\mathcal{K} & \overset{Q_{\mathcal{B}}}{\twoheadrightarrow} & \bar{\mathcal{A}}^{\mathrm{fp}} = (\mathrm{mod}\text{-}\mathcal{K})/\mathcal{B} \\
\cap & & \cap & & \cap \\
\mathcal{T} & \overset{\mathrm{h}}{\longrightarrow} & \mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K} & \overset{Q_{\mathcal{B}}}{\twoheadrightarrow} & \bar{\mathcal{A}} = (\mathrm{Mod}\text{-}\mathcal{K})/\langle\mathcal{B}\rangle \\
 & & \underbrace{\hspace{7cm}} & & & & \\
 & & & & \bar{\mathrm{h}}_{\mathcal{B}} & &
\end{array}
$$

Thanks to [Balmer et al. 2018, Proposition A.14], the image $\mathrm{h}(Y)$ of every object Y in the big category $\mathcal{T}$ remains $\otimes$-*flat* in the module category $\mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K}$, meaning that the functor $\mathrm{h}(Y) \otimes - : \mathcal{A} \to \mathcal{A}$ is exact. In fact, this $\otimes$-flatness plays an important role in the proof of the nilpotence theorem. In particular, Corollary 4.2 tells us:

1.2. Theorem. *Let* $f : \mathrm{h}(x) \to F$ *be a morphism in* $\mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K}$, *for* $x \in \mathcal{K}$. *Suppose that the $\mathcal{K}$-module F is $\otimes$-flat and that $Q_{\mathcal{B}}(f) = 0$ in $\bar{\mathcal{A}} = \mathcal{A}/\langle\mathcal{B}\rangle$, for every maximal Serre $\otimes$-ideal $\mathcal{B} \subset \mathrm{mod}\text{-}\mathcal{K}$. Then f is $\otimes$-nilpotent in* $\mathrm{Mod}\text{-}\mathcal{K}$.

All these statements are corollaries of our most general nilpotence theorem (Theorem 4.1), which further involves a "parameter" à la Thomason [1997], i.e., a closed subset $W \subseteq \mathrm{Spc}(\mathcal{K})$ of the spectrum on which we test the vanishing of f.

Finally, in Section 5, we classify those homological residue fields in examples. For brevity, let us pack three theorems into one:

1.3. Theorem. *There exists a surjection $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \twoheadrightarrow \mathrm{Spc}(\mathcal{K})$ from the set of maximal Serre $\otimes$-ideals $\mathcal{B} \subset \mathrm{mod}\text{-}\mathcal{K}$ to the triangular spectrum of $\mathcal{K}$. Moreover, it is a bijection for each of the following tensor-triangulated categories $\mathcal{K}$:*

  (a) *Let X be a quasicompact and quasiseparated scheme and $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ its derived category of perfect complexes. (Corollary 5.11.)*

  (b) *Let G be a compact Lie group and $\mathcal{K} = \mathrm{SH}(G)^c$ the G-equivariant stable homotopy category of finite genuine G-spectra. In particular, this holds for $\mathcal{K} = \mathrm{SH}^c$ the usual stable homotopy category. (Corollary 5.10.)*

  (c) *Let G be a finite group scheme over a field k and $\mathcal{K} = \mathrm{stab}(kG)$ its stable module category of finite-dimensional kG-modules modulo projectives. (Example 5.13.)*

We caution that the above does *not* give a new proof of the nilpotence theorems known in those examples, except perhaps for modular representation theory (c), as we discuss further in Remark 5.14. Indeed, the above results rely on existing

classification results, which themselves often rely on a form of nilpotence theorem. These results should rather be read as a converse to our nilpotence theorems via homological residue fields: if a collection of homological residue fields detects nilpotence then that collection contains all homological residue fields (Theorem 5.4).

2. Background and notation

2.1. Hypothesis. Throughout the paper, we denote by $\mathcal{K}$ an essentially small tensor-triangulated category and by $\mathbb{1} \in \mathcal{K}$ its $\otimes$-unit. We often assume $\mathcal{K}$ *rigid*, in the sense recalled in Remark 2.4 below. See details in [Balmer 2005, §1] or [Balmer 2010, §1].

2.2. Examples. Such $\mathcal{K}$ include the usual suspects: in topology $\mathcal{K} = \mathrm{SH}^c$ the stable homotopy category of finite spectra; in algebraic geometry $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X) = \mathrm{D}_{\mathrm{Qcoh}}(\mathcal{O}_X\text{-Mod})^c$ the derived category of perfect complexes over a scheme X which is assumed quasicompact and quasiseparated (e.g., a noetherian, or an affine one); in modular representation theory $\mathcal{K} = \mathrm{stab}(kG) = \mathrm{Stab}(kG)^c$ the stable category of finite-dimensional k-linear representations of a finite group G over a field k of characteristic dividing the order of G. But there are many more examples: equivariant versions, categories of motives, KK-categories of C^*-algebras, etc., etc.

2.3. Remark. We use the *triangular spectrum* $\mathrm{Spc}(\mathcal{K}) = \{\mathcal{P} \subset \mathcal{K} \mid \mathcal{P} \text{ is a prime}\}$ where a proper thick $\otimes$-ideal $\mathcal{P} \subsetneq \mathcal{K}$ is called a *(triangular) prime* if $x \otimes y \in \mathcal{P}$ implies $x \in \mathcal{P}$ or $y \in \mathcal{P}$. The *support* of an object $x \in \mathcal{K}$ is the closed subset $\mathrm{supp}(x) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid x \notin \mathcal{P}\} = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid x \text{ is nonzero in } \mathcal{K}/\mathcal{P}\}$. These are exactly the so-called *Thomason closed* subsets of $\mathrm{Spc}(\mathcal{K})$, i.e., those closed subsets $Z \subseteq \mathrm{Spc}(\mathcal{K})$ with quasicompact open complement $\mathrm{Spc}(\mathcal{K}) \smallsetminus Z$, by [Balmer 2005, Proposition 2.14].

2.4. Remark. Rigidity of $\mathcal{K}$ will play an important role in the proof of the main Theorem 4.1. Rigidity means that every object $x \in \mathcal{K}$ is strongly dualizable, hence admits a dual $x^\vee \in \mathcal{K}$ with an adjunction $\mathrm{Hom}_{\mathcal{K}}(x \otimes y, z) \cong \mathrm{Hom}_{\mathcal{K}}(y, x^\vee \otimes z)$. In particular, the unit-counit relation forces $x \otimes \eta_x : x \to x \otimes x^\vee \otimes x$ to be a split monomorphism where $\eta_x : \mathbb{1} \to x^\vee \otimes x$ is the unit of the adjunction. This implies that x is a direct summand of a $\otimes$-multiple of $x^{\otimes n}$ for any $n \geq 1$. It follows that if a map f satisfies $(f \otimes x)^{\otimes n} = 0$ then $f^{\otimes n} \otimes x = 0$ as well.

2.5. Notation. The Grothendieck abelian category Mod-$\mathcal{K}$ of right $\mathcal{K}$-modules, i.e., additive functors $M : \mathcal{K}^{\mathrm{op}} \to \mathrm{Ab}$, receives $\mathcal{K}$ via the Yoneda embedding, denoted

$$\mathrm{h} : \mathcal{K} \hookrightarrow \mathrm{Mod\text{-}}\mathcal{K} = \mathrm{Add}(\mathcal{K}^{\mathrm{op}}, \mathrm{Ab}), \quad x \mapsto \hat{x} := \mathrm{Hom}_{\mathcal{K}}(-, x), \quad f \mapsto \hat{f}.$$

2.6. Remark. Let us recall some standard facts about $\mathcal{K}$-modules. See details in [Balmer et al. 2018, Appendix A]. By Day convolution, the category $\mathcal{A} = \text{Mod-}\mathcal{K}$ admits a tensor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is colimit-preserving (in particular *right-exact*) in each variable and which makes $h : \mathcal{K} \to \mathcal{A}$ a tensor functor: $\widehat{x \otimes y} \cong \hat{x} \otimes \hat{y}$. Hence h preserves rigidity, so $\hat{x}$ will be rigid in $\mathcal{A}$ whenever x is in $\mathcal{K}$. Moreover, the object $\hat{x} \in \mathcal{A}$ is finitely presented projective and $\otimes$-flat in $\mathcal{A}$. The tensor subcategory

$$\mathcal{A}^{\text{fp}} = \text{mod-}\mathcal{K} \subset \text{Mod-}\mathcal{K} = \mathcal{A}$$

of finitely presented objects is itself abelian and is nothing but the classical *Freyd envelope* of $\mathcal{K}$, see [Neeman 2001, Chapter 5]. Recall that $h : \mathcal{K} \hookrightarrow \text{mod-}\mathcal{K}$ is the universal homological functor out of $\mathcal{K}$, and that a functor from a triangulated category to an abelian category is *homological* if it maps exact triangles to exact sequences. Every object of $\mathcal{A}$ is a filtered colimit of finitely presented ones. In short, $\mathcal{A}$ is a *locally coherent* Grothendieck category.

2.7. Remark. Given a Serre subcategory $\mathcal{B} \subseteq \mathcal{A}^{\text{fp}}$ we can form $\langle \mathcal{B} \rangle$, or $\vec{\mathcal{B}}$, the localizing subcategory of $\mathcal{A}$ generated by $\mathcal{B}$. The subcategory $\langle \mathcal{B} \rangle$ is the smallest Serre subcategory containing $\mathcal{B}$ and closed under coproducts; it consists of all (filtered) colimits in $\mathcal{A}$ of objects of $\mathcal{B}$. For instance $\langle \mathcal{A}^{\text{fp}} \rangle = \mathcal{A}$ and it follows that if $\mathcal{B}$ is $\otimes$-ideal in $\mathcal{A}^{\text{fp}}$ then so is $\langle \mathcal{B} \rangle$ in $\mathcal{A}$. We denote the corresponding Gabriel [1962] quotient by

$$Q_{\mathcal{B}} : \mathcal{A} \twoheadrightarrow \mathcal{A}/\langle \mathcal{B} \rangle.$$

We recall that $\langle \mathcal{B} \rangle$ is also locally coherent with $\langle \mathcal{B} \rangle^{\text{fp}} = \mathcal{B}$ and so is the quotient $\bar{\mathcal{A}}$ with $\bar{\mathcal{A}}^{\text{fp}} \cong \mathcal{A}^{\text{fp}}/\mathcal{B}$. When $\mathcal{B}$ is $\otimes$-ideal then $\bar{\mathcal{A}}$ inherits a unique tensor structure turning $Q_{\mathcal{B}} : \mathcal{A} \twoheadrightarrow \bar{\mathcal{A}}$ into a tensor functor, which preserves $\otimes$-flat objects. All this remains true without assuming $\mathcal{K}$ rigid. See details in [Balmer et al. 2019, §2].

2.8. Remark. For $\mathcal{K}$ rigid, consider the special case of the adjunction

$$\text{Hom}_{\mathcal{K}}(x, y) \cong \text{Hom}_{\mathcal{K}}(\mathbb{1}, x^{\vee} \otimes y).$$

Under this isomorphism, if $f : x \to y$ corresponds to $g : \mathbb{1} \to x^{\vee} \otimes y$ then for $n \geq 1$ the morphism $f^{\otimes n} : x^{\otimes n} \to y^{\otimes n}$ corresponds to $g^{\otimes n} : \mathbb{1} \to (x^{\vee} \otimes y)^{\otimes n}$ under the analogous isomorphism $\text{Hom}_{\mathcal{K}}(x^{\otimes n}, y^{\otimes n}) \cong \text{Hom}_{\mathcal{K}}(\mathbb{1}, (x^{\otimes n})^{\vee} \otimes y^{\otimes n})$ $\cong \text{Hom}_{\mathcal{K}}(\mathbb{1}, (x^{\vee} \otimes y)^{\otimes n})$. In particular, f is $\otimes$-nilpotent if and only if g is. Note that the above observation only uses that x is rigid in a tensor category and does not use that y itself is rigid. We can therefore also use this trick for any morphism $f : \hat{x} \to M$ in the module category $\mathcal{A} = \text{Mod-}\mathcal{K}$, as long as x comes from $\mathcal{K}$.

We shall need the following folklore result about modules and localization:

2.9. Proposition. *Let $\mathcal{J} \subset \mathcal{K}$ be a thick $\otimes$-ideal and let $q : \mathcal{K} \to \mathcal{L}$ be the corresponding Verdier quotient $\mathcal{K} \twoheadrightarrow \mathcal{K}/\mathcal{J}$ or its idempotent completion $\mathcal{K} \to (\mathcal{K}/\mathcal{J})^{\natural}$.*

Consider the left Kan extension $q_! : \text{Mod-}\mathcal{K} \to \text{Mod-}\mathcal{L}$, left-adjoint to restriction $q^* : \text{Mod-}\mathcal{L} \to \text{Mod-}\mathcal{K}$ along q. Then $q_!$ is a localization identifying $\text{Mod-}\mathcal{L}$ as the Gabriel quotient of $\text{Mod-}\mathcal{K}$ by $\text{Ker}(q_!) = \langle h(\mathcal{J}) \rangle$ the localizing subcategory generated by $h(\mathcal{J})$. The localization $q_!$ restricts to a localization $q_! : \text{mod-}\mathcal{K} \twoheadrightarrow \text{mod-}\mathcal{L}$ on finitely presented objects, identifying $\text{mod-}\mathcal{L}$ as the quotient of $\text{mod-}\mathcal{K}$ by $\text{Ker}(q_!)^{\text{fp}}$ which is also the Serre envelope of $h(\mathcal{J})$ in $\text{mod-}\mathcal{K}$.

Proof. The fact that $q_!$ is a localization follows from [Krause 2005, Theorem 4.4 and §3]. The left Kan extension $q_!(M)$ is defined as $\text{colim}_{\alpha: \hat{x} \to M} q_!(\hat{x})$ with $q_!(\hat{x}) = \widehat{q(x)}$. To identify the kernel of $q_! : \text{Mod-}\mathcal{K} \to \text{Mod-}\mathcal{L}$, since $h(\mathcal{J}) \subseteq \text{Ker}(q_!)^{\text{fp}}$ is clear, it suffices to show $\text{Ker}(q_!) \subseteq \langle h(\mathcal{J}) \rangle$. For every $M \in \text{Ker}(q_!)$, using that $\widehat{q(x)}$ is finitely presented projective for all $x \in \mathcal{K}$ together with faithfulness of Yoneda $\mathcal{L} \hookrightarrow \text{Mod-}\mathcal{L}$, one shows that every morphism $\alpha : \hat{x} \to M$ with $x \in \mathcal{K}$ factors via a morphism $\hat{\beta} : \hat{x} \to \hat{y}$ where $q(\beta)$ vanishes in $\mathcal{K}/\mathcal{J}$, meaning that the morphism $\beta : x \to y$ in $\mathcal{K}$ factors via an object of $\mathcal{J}$. In short, every morphism $\hat{x} \to M$ factors via an object in $h(\mathcal{J})$ which implies that M belongs to the localizing subcategory $\langle h(\mathcal{J}) \rangle$. The $\otimes$-properties are then easily added onto this purely abelian picture. $\square$

3. Homological primes and homological residue fields

Let $\mathcal{K}$ be a tensor-triangulated category as in Hypothesis 2.1.

3.1. Definition. A *(coherent) homological prime* for $\mathcal{K}$ is a *maximal* proper Serre $\otimes$-ideal subcategory $\mathcal{B} \subset \mathcal{A}^{\text{fp}} = \text{mod-}\mathcal{K}$ of the Freyd envelope of $\mathcal{K}$. The *homological residue field* corresponding to $\mathcal{B}$ is the functor constructed as follows

$$\bar{h}_{\mathcal{B}} = Q_{\mathcal{B}} \circ h : \quad \mathcal{K} \overset{h}{\hookrightarrow} \mathcal{A} = \text{Mod-}\mathcal{K} \xrightarrow{Q_{\mathcal{B}}} \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B}) := \frac{\text{Mod-}\mathcal{K}}{\langle \mathcal{B} \rangle},$$

$$x \longmapsto \quad \hat{x} \quad \longmapsto \quad \bar{x}.$$

The functor $\bar{h}_{\mathcal{B}} : \mathcal{K} \to \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})^{\text{fp}}$ is a (strong) monoidal homological functor (Remark 2.6), that lands in the finitely presented subcategory

$$\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})^{\text{fp}} \cong (\text{mod-}\mathcal{K})/\mathcal{B}.$$

By construction, the tensor-abelian category $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})^{\text{fp}}$ has only the trivial Serre $\otimes$-ideals, 0 and $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})^{\text{fp}}$ itself. These homological residue fields are truly the same as the homological $\otimes$-functors constructed in [Balmer et al. 2019, §4], up to a little paradigm change that we explain in Remark 3.10 below.

3.2. Remark. Since $\mathcal{K}$ is essentially small, its Freyd envelope, $\text{mod-}\mathcal{K}$, admits only a *set* of Serre subcategories. So we can apply Zorn to construct homological primes and homological residue fields as soon as $\mathcal{K} \neq 0$. Contrary to what happens with commutative rings, these *maximal* Serre $\otimes$-ideals are not only picking up

"closed points" as one could first fear. In fact, they "live" above *every* prime of the triangular spectrum $\mathrm{Spc}(\mathcal{K})$ of $\mathcal{K}$ (Remark 2.3). First, let us explain the relationship.

3.3. Proposition. *Let* $\mathcal{B}$ *be a homological prime with homological residue field* $\bar{\mathrm{h}}_{\mathcal{B}} : \mathcal{K} \to \bar{A}(\mathcal{K}; \mathcal{B})$. *Then* $\mathcal{P}(\mathcal{B}) := \mathrm{Ker}(\bar{\mathrm{h}}_{\mathcal{B}}) = \mathrm{h}^{-1}(\mathcal{B})$ *is a triangular prime of* $\mathcal{K}$.

Proof. Since Yoneda $\mathrm{h} : \mathcal{K} \to \mathrm{mod}\text{-}\mathcal{K}$ is homological and (strong) monoidal, the preimage $\mathcal{P}(\mathcal{B}) = \mathrm{h}^{-1}(\mathcal{B})$ is a proper thick $\otimes$-ideal of $\mathcal{K}$. To see that $\mathcal{P}(\mathcal{B})$ is prime, let $x, y \in \mathcal{K}$ with $x \otimes y \in \mathcal{P}(\mathcal{B})$ and $x \notin \mathcal{P}(\mathcal{B})$ and let us show that $y \in \mathcal{P}(\mathcal{B})$. Consider the $\otimes$-ideal $\mathcal{C} = \{M \in \mathrm{mod}\text{-}\mathcal{K} \mid \hat{x} \otimes M \in \mathcal{B}\}$. It is Serre by flatness of $\hat{x}$ and the assumption $x \notin \mathcal{P}(\mathcal{B})$ implies $\mathcal{B} \subseteq \mathcal{C} \neq \mathrm{mod}\text{-}\mathcal{K}$. Therefore $\mathcal{C} = \mathcal{B}$ by maximality of $\mathcal{B}$ and we get $y \in \mathrm{h}^{-1}(\mathcal{C}) = \mathrm{h}^{-1}(\mathcal{B}) = \mathcal{P}(\mathcal{B})$. $\qquad\square$

3.4. Remark. We can push the analogy with the triangular spectrum $\mathrm{Spc}(\mathcal{K})$ a little further by considering the set $\mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ of all homological primes:

$$\mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) = \left\{ \mathcal{B} \subset \mathrm{mod}\text{-}\mathcal{K} \mid \mathcal{B} \text{ is a maximal Serre } \otimes\text{-ideal} \right\}.$$

We call it the *homological spectrum of* $\mathcal{K}$ and equip it with a topology having as basis of closed subsets the following subsets $\mathrm{supp}^{\mathrm{h}}(x)$, one for every $x \in \mathcal{K}$:

$$\mathrm{supp}^{\mathrm{h}}(x) := \{\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \mid \hat{x} \notin \mathcal{B}\} = \{\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \mid \bar{x} \neq 0 \text{ in } \bar{A}(\mathcal{K}; \mathcal{B})\}.$$

One can verify that this pair $(\mathrm{Spc}^{\mathrm{h}}(\mathcal{K}), \mathrm{supp}^{\mathrm{h}})$ is a *support data* on $\mathcal{K}$, in the sense of [Balmer 2005]. Hence there exists a unique continuous map

$$\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$$

such that $\mathrm{supp}^{\mathrm{h}}(x) = \phi^{-1}(\mathrm{supp}(x))$ for every $x \in \mathcal{K}$. The explicit formula for ϕ ([Balmer 2005, Theorem 3.2]) shows that ϕ is exactly the map $\mathcal{B} \mapsto \mathrm{h}^{-1}(\mathcal{B})$ of Proposition 3.3. We prove in Corollary 3.9 below that this comparison map is surjective, at least for $\mathcal{K}$ rigid. In fact, there are many known examples where ϕ is bijective. See Section 5.

3.5. Definition. It will be convenient to say that a homological prime $\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ *lives over* a given subset $W \subseteq \mathrm{Spc}(\mathcal{K})$ of the triangular spectrum if the prime $\mathcal{P}(\mathcal{B}) = \mathrm{h}^{-1}(\mathcal{B})$ of Proposition 3.3 belongs to W. By extension, we shall also say in that case that the corresponding homological residue field $\bar{\mathrm{h}} = \bar{\mathrm{h}}_{\mathcal{B}}$ *lives over* W.

In order to show surjectivity of ϕ, we derive from Proposition 2.9 the following:

3.6. Corollary. *Let* $\mathcal{J} \subset \mathcal{K}$ *be a thick* $\otimes$-*ideal. With notation as in Proposition 2.9, there is an inclusion-preserving one-to-one correspondence* $\mathcal{C} \mapsto (q_!)^{-1}(\mathcal{C})$ *between (maximal) Serre* $\otimes$-*ideals* $\mathcal{C}$ *of* $\mathrm{mod}\text{-}\mathcal{L}$ *and the (maximal) Serre* $\otimes$-*ideals* $\mathcal{B}$ *of* $\mathrm{mod}\text{-}\mathcal{K}$ *which contain* $\mathrm{h}(\mathcal{J})$; *the inverse is given by* $\mathcal{B} \mapsto q_!(\mathcal{B})$. *If* $\mathcal{C}$ *corresponds*

*to $\mathcal{B}$ then the residue categories are canonically equivalent, $\bar{A}(\mathcal{K}; \mathcal{B}) \cong \bar{A}(\mathcal{L}; \mathcal{C})$,
in such a way that the following diagram commutes up to canonical isomorphism:*

$$
\begin{array}{ccccccc}
 & & & & & \bar{h}_{\mathcal{B}} & \\
 & & & & & & \\
\mathcal{K} & \overset{h}{\hookrightarrow} & \mathrm{mod}\text{-}\mathcal{K} & \hookrightarrow & \mathrm{Mod}\text{-}\mathcal{K} & \overset{Q_{\mathcal{B}}}{\twoheadrightarrow} & \bar{A}(\mathcal{K}; \mathcal{B}) \\
{\scriptstyle q}\downarrow & & {\scriptstyle q_!}\downarrow & & {\scriptstyle q_!}\downarrow & & \Vert{\scriptstyle \cong} \\
\mathcal{L} & \overset{h}{\hookrightarrow} & \mathrm{mod}\text{-}\mathcal{L} & \hookrightarrow & \mathrm{Mod}\text{-}\mathcal{L} & \overset{Q_{\mathcal{C}}}{\twoheadrightarrow} & \bar{A}(\mathcal{L}; \mathcal{C}) \\
 & & & & & & \\
 & & & \bar{h}_{\mathcal{C}} & & &
\end{array}
$$

Proof. Standard "third isomorphism theorem" about ideals in a quotient.   □

3.7. Remark. In the notation of Remark 3.4, the above Corollary 3.6 can be rephrased as saying that, for $\mathcal{L} = \mathcal{K}/\mathcal{J}$ or $(\mathcal{K}/\mathcal{J})^\natural$, the map $\mathrm{Spc}^h(q_!) : \mathcal{C} \mapsto (q_!)^{-1}\mathcal{C}$ yields a homeomorphism between $\mathrm{Spc}^h(\mathcal{L})$ and the subspace

$$\{\mathcal{B} \in \mathrm{Spc}^h(\mathcal{K}) \mid \mathcal{P}(\mathcal{B}) \supseteq \mathcal{J}\}$$

of $\mathrm{Spc}^h(\mathcal{K})$. In other words, the following commutative diagram is cartesian:

$$
\begin{array}{ccc}
\mathrm{Spc}^h(\mathcal{L}) & \overset{\mathrm{Spc}^h(q_!)}{\hookrightarrow} & \mathrm{Spc}^h(\mathcal{K}) \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\
\mathrm{Spc}(\mathcal{L}) & \overset{\mathrm{Spc}(q)}{\hookrightarrow} & \mathrm{Spc}(\mathcal{K})
\end{array}
$$

In the terminology of Definition 3.5, the homological primes (or the residue fields) of $\mathcal{L} = \mathcal{K}/\mathcal{J}$ or $\mathcal{L} = (\mathcal{K}/\mathcal{J})^\natural$ canonically correspond to those of $\mathcal{K}$ which live "above" the subset $W(\mathcal{J}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{J} \subseteq \mathcal{P}\} \cong \mathrm{Spc}(\mathcal{L})$ of $\mathrm{Spc}(\mathcal{K})$.

Let us prove an analogue of [Balmer 2005, Lemma 2.2], an old tensor-triangular friend:

3.8. Lemma. *Suppose that $\mathcal{K}$ is rigid. Let $\mathcal{S} \subset \mathcal{K}$ be a $\otimes$-multiplicative class of objects (i.e., $\mathbb{1} \in \mathcal{S} \supseteq \mathcal{S} \otimes \mathcal{S}$) and let $\mathcal{B}_0 \subset \mathcal{A}^{\mathrm{fp}} = \mathrm{mod}\text{-}\mathcal{K}$ be a Serre $\otimes$-ideal which avoids $\mathcal{S}$, that is, $\mathcal{B}_0 \cap h(\mathcal{S}) = \varnothing$. Then there exists $\mathcal{B} \subset \mathcal{A}^{\mathrm{fp}}$ a maximal Serre $\otimes$-ideal such that $\mathcal{B}_0 \subseteq \mathcal{B}$ and $\mathcal{B}$ still avoids $\mathcal{S}$.*

Proof. By Zorn, there exists $\mathcal{B}$ maximal among the Serre $\otimes$-ideals which avoid $\mathcal{S}$ and contain $\mathcal{B}_0$. So we have $\mathcal{B} \supseteq \mathcal{B}_0$ and $\mathcal{B} \cap h(\mathcal{S}) = \varnothing$ and we are left to prove that $\mathcal{B}$ is plain maximal in $\mathcal{A}^{\mathrm{fp}}$. Consider $\mathcal{B}' := \{M \in \mathcal{A}^{\mathrm{fp}} \mid \hat{x} \otimes M \in \mathcal{B} \text{ for some } x \in \mathcal{S}\}$. Since $\mathcal{S}$ is $\otimes$-multiplicative and each $\hat{x}$ is $\otimes$-flat, the subcategory $\mathcal{B}' \subsetneqq \mathcal{A}^{\mathrm{fp}}$ is a Serre $\otimes$-ideal avoiding $\mathcal{S}$ and containing $\mathcal{B}_0$. By maximality of $\mathcal{B}$ among those, the relation $\mathcal{B} \subseteq \mathcal{B}'$ forces $\mathcal{B} = \mathcal{B}'$. In particular, $M = \ker(\hat{\eta}_x : \hat{\mathbb{1}} \to \hat{x}^\vee \otimes \hat{x})$ belongs to $\mathcal{B}$ for every $x \in \mathcal{S}$ since $\hat{x} \otimes M = 0$ by rigidity (Remark 2.4). Let us show that $\mathcal{B} \subset \mathcal{A}^{\mathrm{fp}}$ is a maximal Serre $\otimes$-ideal by showing that a strictly bigger Serre $\otimes$-ideal

$\mathcal{C} \supsetneq \mathcal{B}$ of $\mathcal{A}^{\mathrm{fp}}$ must be $\mathcal{A}^{\mathrm{fp}}$ itself. Since $\mathcal{B}$ is maximal among those avoiding $\mathcal{S}$ and containing $\mathcal{B}_0$, such a strictly bigger $\mathcal{C}$ cannot avoid $\mathcal{S}$. Therefore $\mathcal{C}$ contains some $\hat{x}$ for $x \in \mathcal{S}$ and, by the above discussion, we also have $\ker(\hat{\eta}_x : \hat{\mathbb{1}} \to \hat{x}^{\vee} \otimes \hat{x}) \in \mathcal{B} \subseteq \mathcal{C}$. So in the exact sequence $0 \to \ker(\hat{\eta}_x) \to \hat{\mathbb{1}} \to \hat{x}^{\vee} \otimes \hat{x}$ we have $\ker(\hat{\eta}_x)$ and $\hat{x}$ in $\mathcal{C}$. This forces $\hat{\mathbb{1}} \in \mathcal{C}$ by Serritude and therefore $\mathcal{C} = \mathcal{A}^{\mathrm{fp}}$ as wanted. $\qquad\square$

3.9. Corollary. *Suppose that $\mathcal{K}$ is rigid. Then the map $\mathcal{B} \mapsto \mathcal{P}(\mathcal{B})$ from homological primes to triangular primes as in Proposition 3.3 (i.e., the comparison map $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ of Remark 3.4) is surjective. That is, every triangular prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ is of the form $\mathcal{P} = \mathrm{h}^{-1}(\mathcal{B})$ for some maximal Serre $\otimes$-ideal $\mathcal{B}$ in $\mathrm{mod}\text{-}\mathcal{K}$.*

Proof. Consider the quotient $\mathcal{K}/\mathcal{P}$ (or its idempotent completion $\mathcal{K}_{\mathcal{P}} := (\mathcal{K}/\mathcal{P})^{\natural}$). The map $\mathrm{Spc}(q) : \mathrm{Spc}(\mathcal{K}/\mathcal{P}) \hookrightarrow \mathrm{Spc}(\mathcal{K})$ sends 0 to $q^{-1}(0) = \mathcal{P}$. So, by Corollary 3.6 applied to $\mathcal{J} = \mathcal{P}$, it suffices to prove the result for $\mathcal{P} = 0$. In that case, $\mathcal{K}$ is *local*, meaning that 0 is a prime: $x \otimes y = 0 \Rightarrow x = 0$ or $y = 0$. We conclude by Lemma 3.8 for $\mathcal{B}_0 = 0$ and $\mathcal{S} = \mathcal{K} \smallsetminus \{0\}$ which is $\otimes$-multiplicative because $\mathcal{K}$ is local.[1] $\qquad\square$

3.10. Remark. There are a few differences between our approach to homological residue fields and the treatment in [Balmer et al. 2019, §4]. First, the whole [Balmer et al. 2019] is written for a "big" (i.e., rigidly compactly generated) tensor-triangulated category $\mathcal{T}$ and the modules are taken over its rigid-compact objects $\mathcal{K} := \mathcal{T}^c$. This restriction is unimportant, certainly as far as most examples are concerned.

Another difference is that, in [loc. cit.], we focused on a *local* category in the sense that $\mathrm{Spc}(\mathcal{K})$ is a local space, i.e., has a unique closed point $\mathcal{M} = 0$. We then considered quotients of the module category $\mathcal{A} \twoheadrightarrow \mathcal{A}/\langle\mathcal{B}\rangle$ for $\mathcal{B} \subseteq \mathcal{A}^{\mathrm{fp}}$ maximal *among those which meet $\mathcal{K}$ trivially*, i.e., $\mathcal{B} \cap \mathrm{h}(\mathcal{K}) = \{0\}$. This property means that the homological prime $\mathcal{B}$ lives *above the closed point* of $\mathrm{Spc}(\mathcal{K})$ in the sense of Definition 3.5. Equivalently, it means that the functor $\bar{\mathrm{h}}_{\mathcal{B}} : \mathcal{K} \to \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$ is conservative, i.e., detects isomorphisms. All these properties are reminiscent of commutative algebra, where the residue field of a local ring R is indeed conservative on perfect complexes and maps the unique prime of the field to the closed point of $\mathrm{Spec}(R)$.

Continuing the analogy with commutative algebra, when dealing with a *global* (i.e., not necessarily local) category $\mathcal{K}$, we can analyze it one prime at a time. For each $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ we can consider the local category $\mathcal{K}_{\mathcal{P}} = (\mathcal{K}/\mathcal{P})^{\natural}$. By Corollary 3.6 we can identify the homological primes $\mathcal{C}$ of this local category $\mathcal{K}_{\mathcal{P}}$ with a subset of those of the global category. Requesting that the local prime $\mathcal{C}$ lives "above the closed point" of $\mathrm{Spc}(\mathcal{K}_{\mathcal{P}})$ as we did in [Balmer et al. 2019] amounts to

---

[1]The argument was already used in [Balmer et al. 2019, Corollary 4.9], which in turn inspired Lemma 3.8.

requesting that the corresponding global prime $\mathcal{B} = (q_!)^{-1}(\mathcal{C})$ lives exactly above the point $\mathcal{P}$ in $\mathrm{Spc}(\mathcal{K})$.

In other words, in Definition 3.1 we are considering all homological residue fields $\bar{h}_{\mathcal{B}} : \mathcal{K} \to \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$ at once but we can also regroup them according to the associated triangular primes $h^{-1}(\mathcal{B})$, in which case we obtain the constructions of [Balmer et al. 2019] for the local category $\mathcal{K}_{\mathcal{P}}$.

4. The Nilpotence Theorems

In this section, we assume $\mathcal{K}$ rigid.

Let us prove Theorem 1.2, in a strong form "with parameter". Recall that an object F in a tensor abelian category $\mathcal{A}$ is ⊗-*flat* if $F \otimes - : \mathcal{A} \to \mathcal{A}$ is exact.

4.1. Theorem. *Let $\mathcal{K}$ be an essentially small, rigid tensor-triangulated category* (Hypothesis 2.1) *and $W \subseteq \mathrm{Spc}(\mathcal{K})$ a closed subset* (the *"parameter"*). *Let $f : \hat{x} \to F$ be a morphism in $\mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K}$ satisfying the following hypotheses:*

(i) *The source of f comes from an object $x \in \mathcal{K}$ via Yoneda, as indicated above.*

(ii) *Its target F is ⊗-flat in $\mathcal{A}$.*

(iii) *The morphism f vanishes in every homological residue field over W in the following sense: For every maximal Serre ⊗-ideal $\mathcal{B} \subset \mathrm{mod}\text{-}\mathcal{K}$ living over W* (Definition 3.5) *we have $Q_{\mathcal{B}}(f) = 0$ in $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B}) = (\mathrm{Mod}\text{-}\mathcal{K})/\langle \mathcal{B} \rangle$.*

Then:

(a) *There exist an object $s \in \mathcal{K}$ such that $\mathrm{supp}(s) \supseteq W$ and an integer $n \geq 1$ such that $f^{\otimes n} \otimes \hat{s} = 0$ in $\mathcal{A}$.*

(b) *For any object s as above, if we let $Z = \mathrm{supp}(s) \supseteq W$, then for every $z \in \mathcal{K}_Z = \{z \in \mathcal{K} \mid \mathrm{supp}(z) \subseteq Z\}$ there exists $m \geq 1$ with $f^{\otimes m} \otimes \hat{z} = 0$.*

Proof. By Remark 2.8, we can and shall assume that $x = \mathbb{1}$. So $f : \hat{\mathbb{1}} \to F$. Consider

$$\mathcal{S} := \{s \in \mathcal{K} \mid \mathrm{supp}(s) \supseteq W\}.$$

This is a ⊗-multiplicative class of objects of $\mathcal{K}$ since

$$\mathrm{supp}(s_1 \otimes s_2) = \mathrm{supp}(s_1) \cap \mathrm{supp}(s_2).$$

Since W is closed and $\{\mathrm{supp}(s)\}_{s \in \mathcal{K}}$ is a basis of closed subsets, we have

$$\bigcap_{s \in \mathcal{S}} \mathrm{supp}(s) = W.$$

On the other hand, consider the following subcategory of finitely presented $\mathcal{K}$-modules

$$\mathcal{B}_0 := \left\{ M \in \mathcal{A}^{\mathrm{fp}} \mid f^{\otimes n} \otimes M = 0 \text{ in } \mathcal{A} \text{ for some } n \geq 1 \right\}.$$

Note that $\mathcal{B}_0$ is a Serre $\otimes$-ideal. This uses that F is $\otimes$-flat in $\mathcal{A}$ and was already proved in [Balmer et al. 2019, Lemma 4.17]. In particular, when we prove that $\mathcal{B}_0$ is closed under extension, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence in $\mathcal{A}^{\mathrm{fp}}$ and if $f^{\otimes n_1} \otimes M_1 = 0$ and $f^{\otimes n_3} \otimes M_3 = 0$ then we show that $f^{\otimes(n_1+n_3)} \otimes M_2 = 0$. This is the place where nilpotence is needed, as opposed to mere vanishing.

If $\mathcal{B}_0$ meets $\mathrm{h}(\mathcal{S})$ we obtain the conclusion of part (a). Suppose ab absurdo, that $\mathcal{B}_0 \cap \mathrm{h}(\mathcal{S}) = \varnothing$. By Lemma 3.8 there exists a homological prime $\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ containing $\mathcal{B}_0$ and still avoiding $\mathcal{S}$. The latter property $\mathcal{B} \cap \mathrm{h}(\mathcal{S}) = \varnothing$ means that the triangular prime $\mathrm{h}^{-1}(\mathcal{B})$ belongs to the subset $\{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{P} \cap \mathcal{S} = \varnothing\} = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid s \notin \mathcal{P}, \; \forall\, s \in \mathcal{S}\} = \bigcap_{s \in \mathcal{S}} \mathrm{supp}(s) = W$, as we proved above from the definition of $\mathcal{S}$. So $\mathcal{B}$ lives over W and we trigger hypothesis (iii) for that $\mathcal{B}$, namely that we have $Q_{\mathcal{B}}(f) = 0$ in $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$.

Consider now the kernel of $f : \hat{\mathbb{1}} \to F$ in $\mathcal{A}$ and the commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(f) & \xrightarrow{\;\;j\;\;} & \hat{\mathbb{1}} & \xrightarrow{\;\;f\;\;} & F \\
& & {\scriptstyle f\otimes 1}\big\downarrow & & {\scriptstyle f}\big\downarrow & & {\scriptstyle f\otimes 1}\big\downarrow \\
0 & \longrightarrow & F \otimes \ker(f) & \xrightarrow{\;1\otimes j\;} & F & \xrightarrow{\;1\otimes f\;} & F \otimes F
\end{array}
$$

whose first row is the obvious one; the diagram is obtained by tensoring that first row with $f : \hat{\mathbb{1}} \to F$ itself (on the left) and using that $\hat{\mathbb{1}}$ is the $\otimes$-unit in $\mathcal{A}$. It is essential here that the source of f is $\hat{\mathbb{1}}$ (which we achieved through rigidity), and not some random object. Indeed, the diagonal of the left-hand square is now $f \circ j = 0$. Since the lower row is exact (F $\otimes$-flat again), we conclude that $f \otimes \ker(f) = 0$. We cannot jump to the conclusion that $\ker(f) \in \mathcal{B}$ since $\ker(f)$ is not finitely presented. However, $\ker(f) \rightarrowtail \hat{\mathbb{1}}$ is a subobject of a finitely presented object, hence it is the *union* of its finitely presented subobjects as in [Balmer et al. 2018, Lemma 3.9], i.e.,

$$
\ker(f) = \operatorname*{colim}_{\substack{M \rightarrowtail \ker(f) \\ M \in \mathcal{A}^{\mathrm{fp}}}} M.
$$

For any such $i : M \rightarrowtail \ker(f)$ with $M \in \mathcal{A}^{\mathrm{fp}}$, we have a commutative square obtained by tensoring $i : M \rightarrowtail \ker(f)$ with $f : \hat{\mathbb{1}} \to F$:

$$
\begin{array}{ccc}
M & \xrightarrow{\quad i \quad} & \ker(f) \\
{\scriptstyle f\otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle f\otimes 1=0} \\
F \otimes M & \xrightarrow{\;1\otimes i\;} & F \otimes \ker(f)
\end{array}
$$

Note that the bottom map remains a monomorphism because F is $\otimes$-flat. The vanishing of the right-hand vertical map, proved above, gives us $f \otimes M = 0$, which means $M \in \mathcal{B}_0 \subseteq \mathcal{B}$. It follows that $\ker(f)$ is a colimit of objects $M \in \mathcal{B}$ and

therefore belongs to $\langle \mathcal{B} \rangle$. Applying the exact functor $Q_\mathcal{B} : \mathcal{A} \twoheadrightarrow \bar{\mathcal{A}} = \mathcal{A}/\langle \mathcal{B} \rangle$ to the morphism f, we have just proved that $Q_\mathcal{B}(f)$ has trivial kernel, i.e., it is a monomorphism $Q_\mathcal{B}(\hat{\mathbb{1}}) \rightarrowtail Q_\mathcal{B}(F)$ in $\bar{\mathcal{A}}$. But this monomorphism $Q_\mathcal{B}(f)$ is also zero by assumption (iii) on f that we triggered in the first part of the proof. This forces $Q_\mathcal{B}(\hat{\mathbb{1}}) = 0$ and thus $\hat{\mathbb{1}} \in \mathcal{B}$, a contradiction.

This finishes the proof of part (a). To deduce (b), we use the fact that

$$\mathcal{B}_0 = \left\{ M \in \mathcal{A}^{\mathrm{fp}} \mid f^{\otimes m} \otimes M = 0 \text{ for some } m \geq 1 \right\}$$

is a Serre $\otimes$-ideal, as we saw above, by [Balmer et al. 2019, Lemma 4.17]. Therefore $\mathrm{h}^{-1}(\mathcal{B}_0)$ is a thick $\otimes$-ideal of $\mathcal{K}$ and so the fact that $\mathrm{h}^{-1}(\mathcal{B}_0)$ contains s implies that it contains the whole thick $\otimes$-ideal of $\mathcal{K}$ generated by s, namely exactly $\mathcal{K}_Z$ where $Z = \mathrm{supp}(s)$, see [Balmer 2005, §4]. $\square$

We can then deduce the form announced in Theorem 1.2:

4.2. Corollary. *Let* $f : \hat{x} \to F$ *be a morphism in* $\mathcal{A} = \mathrm{Mod\text{-}}\mathcal{K}$, *with* $x \in \mathcal{K}$ *and with* F $\otimes$-*flat in* $\mathcal{A}$. *Suppose that for every homological prime* $\mathcal{B} \subset \mathrm{mod\text{-}}\mathcal{K}$ *we have* $Q_\mathcal{B}(f) = 0$ *in* $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$. *Then there exists* $m \geq 1$ *such that* $f^{\otimes m} = 0$ *in* $\mathcal{A}$.

Proof. Apply Theorem 4.1 for $W = \mathrm{Spc}(\mathcal{K})$. Thus the subset $Z \supseteq W$ of part (b) must be $Z = \mathrm{Spc}(\mathcal{K})$, and we can take $z = \mathbb{1}$. $\square$

4.3. Remark. We stress that the "parameter" W in Theorem 4.1 is more flexible than the "parameter" in [Thomason 1997, Theorem 3.8], where the closed subset W is supposed to be the support of some object, i.e., a Thomason closed subset (Remark 2.3). In that case, any object s with $\mathrm{supp}(s) = W$ will do, as follows from Theorem 4.1(b).

Of course, the easiest source of $\otimes$-flat objects in $\mathrm{Mod\text{-}}\mathcal{K}$ is the Yoneda embedding:

4.4. Corollary. *Let* $f : x \to y$ *be a morphism in* $\mathcal{K}$ *and* $W \subseteq \mathrm{Spc}(\mathcal{K})$ *be a closed subset such that* $\bar{\mathrm{h}}_\mathcal{B}(f) = 0$ *for every homological residue field corresponding to a homological prime* $\mathcal{B}$ *living above* W (*Definition 3.5*). *Then there exists a Thomason closed subset* $Z \supseteq W$ (*Remark 2.3*) *with the property that for every* $z \in \mathcal{K}$ *such that* $\mathrm{supp}(z) \subseteq Z$ *there exists* $n \geq 1$ *with* $f^{\otimes n} \otimes z = 0$ *in* $\mathcal{K}$. *In particular, this holds for some* z *with* $\mathrm{supp}(z) = Z \supseteq W$.

Proof. This is Theorem 4.1 for $F = \hat{y}$, which is $\otimes$-flat, combined with faithfulness of Yoneda $\mathrm{h} : \mathcal{K} \hookrightarrow \mathcal{A}$ to bring the conclusion back into $\mathcal{K}$. $\square$

This specializes to the flagship nilpotence theorem (Theorem 1.1):

4.5. Corollary. *Let* $f : x \to y$ *be a morphism in* $\mathcal{K}$ *such that* $\bar{\mathrm{h}}(f) = 0$ *for every homological residue field* $\bar{\mathrm{h}}$ *of* $\mathcal{K}$. *Then there exists* $n \geq 1$ *with* $f^{\otimes n} = 0$ *in* $\mathcal{K}$.

Proof. Corollary 4.4 for $W = \mathrm{Spc}(\mathcal{K})$, hence $Z = \mathrm{Spc}(\mathcal{K})$, and $z = \mathbb{1}$. $\square$

4.6. Remark. Many of our examples of tensor-triangulated categories $\mathcal{K}$, if not all, appear as the compact-rigid objects $\mathcal{K} = \mathcal{T}^c$ in some compactly rigidly generated "big" tensor-triangulated category $\mathcal{T}$. See [Balmer et al. 2019, Hypothesis 0.1]. In that case, we have a *restricted-Yoneda* functor which extends $h : \mathcal{K} \hookrightarrow \mathcal{A} = \text{Mod-}\mathcal{K}$ to the whole of $\mathcal{T}$:

$$
\begin{array}{ccc}
\mathcal{K} = \mathcal{T}^c & \xrightarrow{\ h\ } & \mathcal{A}^{\text{fp}} = \text{mod-}\mathcal{K} \\
\big\downarrow & & \big\downarrow \\
\mathcal{T} & \xrightarrow{\ h\ } & \mathcal{A} = \text{Mod-}\mathcal{K} \\
X & \longmapsto & \hat{X} := \text{Hom}_{\mathcal{T}}(- , X)
\end{array}
$$

Note that $h : \mathcal{T} \to \mathcal{A}$ is not faithful anymore (it kills the so-called *phantom maps*). However, it is faithful for maps *out of* a compact, by the usual Yoneda lemma, that is, $\text{Hom}_{\mathcal{T}}(x, Y) \to \text{Hom}_{\mathcal{A}}(\hat{x}, \hat{Y})$ is bijective as soon as $x \in \mathcal{K}$. We prove in [Balmer et al. 2018, Proposition A.14] that every $\hat{Y}$ remains $\otimes$-flat in Mod-$\mathcal{K}$, even for $Y \in \mathcal{T}$ noncompact.

For every homological prime $\mathcal{B} \in \text{Spc}^{\text{h}}(\mathcal{K})$ we can still compose restricted-Yoneda $\mathcal{T} \to \mathcal{A}$ with $Q_{\mathcal{B}} : \mathcal{A} \twoheadrightarrow \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$. We obtain a well-defined homological residue field on the whole "big" category $\mathcal{T}$, that we still denote

$$
\bar{h}_{\mathcal{B}} : \mathcal{T} \to \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B}).
$$

This remains a homological tensor-functor. Compare [Balmer et al. 2019, Theorem 1.6]. Note that we may use these functors to define a support for big objects in $\mathcal{T}$, as will be investigated elsewhere.

We can finally unpack our nilpotence theorems in that special case:

4.7. Corollary. *Let $\mathcal{T}$ be a rigidly compactly generated "big" tensor-triangulated category and $\mathcal{K} = \mathcal{T}^c$ as in Remark 4.6. Let $f : x \to Y$ be a morphism in $\mathcal{T}$ with $x \in \mathcal{K}$ compact and Y arbitrary.*

(a) *Suppose that $\bar{h}_{\mathcal{B}}(f) = 0$ in $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$ for every homological residue field $\bar{h}_{\mathcal{B}}$ of Remark 4.6, for every homological prime $\mathcal{B} \subset \text{mod-}\mathcal{K}$ of Definition 3.1. Then we have $f^{\otimes n} = 0$ in $\mathcal{T}$ for some $n \geq 1$.*

(b) *Suppose that $\bar{h}_{\mathcal{B}}(f) = 0$ in $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$ for every homological residue field over a closed subset $W \subseteq \text{Spc}(\mathcal{K})$ (Definition 3.5). Then there exists a Thomason closed subset $Z \subseteq \text{Spc}(\mathcal{K})$ such that $Z \supseteq W$ and such that for every $z \in \mathcal{K}_Z = \{z \in \mathcal{K} \mid \text{supp}(z) \subseteq Z\}$ we have $f^{\otimes n} \otimes z = 0$ for some $n \geq 1$. In particular, this holds for some z with $\text{supp}(z) = Z \supseteq W$ (Remark 2.3).*

Proof. This follows from Theorem 4.1 applied to $F = \hat{Y}$, together with the partial faithfulness of restricted-Yoneda explained in Remark 4.6. $\qquad\square$

5. Examples

For this section, we keep the setting of Remark 4.6, that is, $\mathcal{T}$ is a "big" tensor-triangulated category generated by its subcategory $\mathcal{K} = \mathcal{T}^c$ of rigid-compact objects.

5.1. Remark. We recall some of the tools developed in [Balmer et al. 2019, §3]. Let $\mathcal{B}$ be a proper Serre $\otimes$-ideal in $\mathcal{A}^{\mathrm{fp}} = \mathrm{mod}\text{-}\mathcal{K}$. Picking an injective hull of the unit $\bar{\mathbb{1}}$ in the quotient $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$ yields a (pure-injective) object $E_{\mathcal{B}}$ in $\mathcal{T}$ such that

$$(5.2) \qquad \langle \mathcal{B} \rangle = \mathrm{Ker}(\hat{E}_{\mathcal{B}} \otimes -) = \{M \in \mathrm{Mod}\text{-}\mathcal{K} \mid \hat{E}_{\mathcal{B}} \otimes M = 0\}.$$

In fact the object $E_{\mathcal{B}}$ is a *weak ring* in $\mathcal{T}$, i.e., it comes with a map $\eta_{\mathcal{B}} : \mathbb{1} \to E_{\mathcal{B}}$ such that $E_{\mathcal{B}} \otimes \eta_{\mathcal{B}} : E_{\mathcal{B}} \to E_{\mathcal{B}} \otimes E_{\mathcal{B}}$ is a split monomorphism. A retraction $E_{\mathcal{B}} \otimes E_{\mathcal{B}} \to E_{\mathcal{B}}$ of this monomorphism can be viewed as a (nonassociative) multiplication on $E_{\mathcal{B}}$ for which $\eta_{\mathcal{B}} : \mathbb{1} \to E_{\mathcal{B}}$ is a right unit. In any case, one important property of $\eta_{\mathcal{B}}$ is that it cannot be $\otimes$-nilpotent, for otherwise $E_{\mathcal{B}}$ would be a retract of zero, hence zero, forcing $\mathcal{B} = \mathcal{A}^{\mathrm{fp}}$.

Another important feature of the objects $E_{\mathcal{B}}$, for $\mathcal{B}$ maximal, is the following:

5.3. Proposition. *For distinct $\mathcal{B} \neq \mathcal{B}'$ in $\mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ we have $E_{\mathcal{B}} \otimes E_{\mathcal{B}'} = 0$.*

Proof. This is already in [Balmer et al. 2019, Corollary 4.12], at least in the local setting. Let us recall the idea, which is easy. In $\mathcal{A}^{\mathrm{fp}} = \mathrm{mod}\text{-}\mathcal{K}$, the kernel

$$\mathrm{Ker}(\hat{E}_{\mathcal{B}} \otimes \hat{E}_{\mathcal{B}'} \otimes -)$$

is a Serre $\otimes$-ideal containing both $\mathcal{B}$ and $\mathcal{B}'$, which are maximal. If that kernel was a proper subcategory it would then be equal to both $\mathcal{B}$ and $\mathcal{B}'$, thus forcing the forbidden $\mathcal{B} = \mathcal{B}'$. So this kernel is not proper, i.e., it contains the unit $\hat{\mathbb{1}}$. This reads $\hat{E}_{\mathcal{B}} \otimes \hat{E}_{\mathcal{B}'} = 0$. Now restricted-Yoneda $\mathrm{h} : \mathcal{T} \to \mathrm{Mod}\text{-}\mathcal{K}$ is a conservative $\otimes$-functor, hence $E_{\mathcal{B}} \otimes E_{\mathcal{B}'} = 0$ as claimed. $\qquad \square$

Let us now prove the converse to Corollary 4.7.

5.4. Theorem. *Let $\mathcal{T}$ be a "big" tensor-triangulated category with $\mathcal{K} = \mathcal{T}^c$, as in Remark 4.6. Consider a family $\mathcal{F} \subseteq \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ of points in the homological spectrum. Suppose that the corresponding functors $\bar{\mathrm{h}}_{\mathcal{B}} : \mathcal{T} \to \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$ collectively detect $\otimes$-nilpotence in the following sense: If $f : x \to Y$ in $\mathcal{T}$ is such that $x \in \mathcal{T}^c$ and $\bar{\mathrm{h}}_{\mathcal{B}}(f) = 0$ for all $\mathcal{B} \in \mathcal{F}$, then $f^{\otimes n} = 0$ for $n \gg 1$. Then we have $\mathcal{F} = \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$.*

Proof. Suppose that $\mathcal{F} \neq \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$. Then there exists $\mathcal{B}' \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ which does not belong to $\mathcal{F}$. By Proposition 5.3 we have $E_{\mathcal{B}} \otimes E_{\mathcal{B}'} = 0$ for all $\mathcal{B} \in \mathcal{F}$. Consider the map $\eta_{\mathcal{B}'} : \mathbb{1} \to E_{\mathcal{B}'}$ as in Remark 5.1. We have therefore proved that $\bar{\mathrm{h}}_{\mathcal{B}}(\eta_{\mathcal{B}'}) = 0$ for all $\mathcal{B} \in \mathcal{F}$ for the obvious reason that its target object, $E_{\mathcal{B}'}$, goes to zero along all $\bar{\mathrm{h}}_{\mathcal{B}}$. On the other hand, we have seen in Remark 5.1 that $\eta_{\mathcal{B}'}$ cannot be $\otimes$-nilpotent. Hence a proper family $\mathcal{F} \subsetneq \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ cannot detect $\otimes$-nilpotence. $\qquad \square$

5.5. Remark. The proof shows that it is enough to assume the property that the family $\{\bar{h}_{\mathcal{B}}\}_{\mathcal{B} \in \mathcal{F}}$ detects the vanishing of objects $Y \in \mathcal{T}$. Compare Remark 5.12.

Here is the picture we will observe in several examples:

5.6. Theorem. *Let $\mathcal{T}$ be a "big" tensor-triangulated category and $\mathcal{K} = \mathcal{T}^c$ as in Remark 4.6. Suppose given for every point $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ of the triangular spectrum, a coproduct-preserving, homological and (strong) monoidal functor*

$$H_{\mathcal{P}} : \mathcal{T} \to \mathcal{A}_{\mathcal{P}}$$

with values in a tensor-abelian category $\mathcal{A}_{\mathcal{P}}$ and satisfying the following properties:

(1) *For each $\mathcal{P}$, the target $\mathcal{A}_{\mathcal{P}}$ is a locally coherent* (Remark 2.6) *Grothendieck category with colimit-preserving tensor; the subcategory $\mathcal{A}_{\mathcal{P}}^{\mathrm{fp}}$ of finitely presented objects is* simple *in the sense that its only Serre $\otimes$-ideals are 0 and $\mathcal{A}_{\mathcal{P}}^{\mathrm{fp}} \neq 0$.*

(2) *The functor $H_{\mathcal{P}} : \mathcal{T} \to \mathcal{A}_{\mathcal{P}}$ maps compacts to finitely presented: $H_{\mathcal{P}}(\mathcal{K}) \subseteq \mathcal{A}_{\mathcal{P}}^{\mathrm{fp}}$. Furthermore, it maps every $X \in \mathcal{T}$ to a $\otimes$-flat object in $\mathcal{A}_{\mathcal{P}}$. Finally the thick $\otimes$-ideal $\mathrm{Ker}(H_{\mathcal{P}}) \cap \mathcal{K} = \{x \in \mathcal{K} \mid H_{\mathcal{P}}(x) = 0\}$ is equal to $\mathcal{P}$.*

(3) *The family $\{H_{\mathcal{P}}\}_{\mathcal{P} \in \mathrm{Spc}(\mathcal{K})}$ detects $\otimes$-nilpotence of maps $f : x \to Y$ in $\mathcal{T}$, with $x \in \mathcal{K}$ compact: If $H_{\mathcal{P}}(f) = 0$ for all $\mathcal{P}$ then $f^{\otimes n} = 0$ for $n \gg 1$.*

Then the comparison map $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ of Proposition 3.3 is a bijection.

Proof. The core of the proof amounts to the kernel of each $H_{\mathcal{P}}$ defining an element of $\mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$. More precisely, let us fix $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ for the moment and denote by

$$\hat{H}_{\mathcal{P}} : \mathcal{A} \to \mathcal{A}_{\mathcal{P}}$$

the unique coproduct-preserving *exact* functor that extends $H_{\mathcal{P}}$ to $\mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K}$, that is, such that the following diagram commutes:

(5.7)

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\;h\;} & \mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K} \\
& {\scriptstyle H_{\mathcal{P}}} \searrow & \big\downarrow {\scriptstyle \hat{H}_{\mathcal{P}}} \\
& & \mathcal{A}_{\mathcal{P}}
\end{array}
$$

The existence of such an $\hat{H}_{\mathcal{P}}$ was established by Krause [2000, Corollary 2.4]. It is not hard to show that $\hat{H}_{\mathcal{P}}$ is also monoidal. At the very least, for every $M \in \mathcal{A}$, we can find $g : Y \to Z$ in $\mathcal{T}$ such that $M = \mathrm{im}(\hat{g})$ and then exactness of $\hat{H}_{\mathcal{P}}$ gives

(5.8) $$\hat{H}_{\mathcal{P}}(\hat{X} \otimes M) \cong H_{\mathcal{P}}(X) \otimes \hat{H}_{\mathcal{P}}(M)$$

for every $X \in \mathcal{T}$ and $M \in \mathcal{A}$. Consider now the kernel of $\hat{H}_{\mathcal{P}}$ in $\mathcal{A}$. The exact functor $\hat{H}_{\mathcal{P}} : \mathcal{A} \to \mathcal{A}_{\mathcal{P}}$ preserves coproducts and finitely presented objects because of (2). It then follows by a general property of abelian categories that $\mathrm{Ker}(H_{\mathcal{P}})$ is generated by its finitely presented objects (see [Balmer et al. 2019, Proposition A.6]). In other words, if we define a Serre $\otimes$-ideal of $\mathcal{A}^{\mathrm{fp}}$ as follows:

$$\mathcal{B}(\mathcal{P}) := \mathrm{Ker}(\hat{H}_{\mathcal{P}}) \cap \mathcal{A}^{\mathrm{fp}} = \{M \in \mathrm{mod}\text{-}\mathcal{K} \mid \hat{H}_{\mathcal{P}}(M) = 0\},$$

then we have $\mathrm{Ker}(\hat{H}_{\mathcal{P}}) = \langle \mathcal{B}(\mathcal{P}) \rangle$. The fact that $\mathcal{A}_{\mathcal{P}} \neq 0$ tells us that $\mathcal{B} := \mathcal{B}(\mathcal{P})$ is proper. We claim that it is maximal. Let $\mathcal{B}'$ be a strictly larger Serre $\otimes$-ideal

$$\mathcal{B} = \mathcal{B}(\mathcal{P}) \subsetneq \mathcal{B}' \subseteq \mathcal{A}^{\mathrm{fp}}.$$

We want to prove that $\mathcal{B}' = \mathcal{A}^{\mathrm{fp}}$. Choose $M \in \mathcal{B}'$ which does not belong to $\mathcal{B}$. Let us now invoke the objects $E_{\mathcal{B}}$ and $E_{\mathcal{B}'}$ of $\mathcal{T}$, as in Remark 5.1, so that

$$\langle \mathcal{B} \rangle = \mathrm{Ker}(\hat{E}_{\mathcal{B}} \otimes -) \quad \text{and} \quad \langle \mathcal{B}' \rangle = \mathrm{Ker}(\hat{E}_{\mathcal{B}'} \otimes -).$$

We then have $\hat{E}_{\mathcal{B}'} \otimes M = 0$ because $M \in \mathcal{B}'$. Hence by (5.8), we have

$$H_{\mathcal{P}}(E_{\mathcal{B}'}) \otimes \hat{H}_{\mathcal{P}}(M) = 0.$$

On the other hand, we have $\hat{H}_{\mathcal{P}}(M) \neq 0$ because $M \notin \mathcal{B}$. Since $H_{\mathcal{P}}(E_{\mathcal{B}'})$ is $\otimes$-flat in $\mathcal{A}_{\mathcal{P}}$, we can consider the Serre $\otimes$-ideal

$$\mathrm{Ker}(H_{\mathcal{P}}(E_{\mathcal{B}'}) \otimes -) \cap \mathcal{A}_{\mathcal{P}}^{\mathrm{fp}}$$

of $\mathcal{A}_{\mathcal{P}}^{\mathrm{fp}}$. We just proved that it contains a nonzero object, namely $\hat{H}_{\mathcal{P}}(M)$. By the "simplicity" of $\mathcal{A}_{\mathcal{P}}^{\mathrm{fp}}$, we get that $\mathrm{Ker}(H_{\mathcal{P}}(E_{\mathcal{B}'}) \otimes -) \cap \mathcal{A}^{\mathrm{fp}}$ must be the whole of $\mathcal{A}_{\mathcal{P}}^{\mathrm{fp}}$. This means that $H_{\mathcal{P}}(E_{\mathcal{B}'}) = 0$, or in other words, $\hat{E}_{\mathcal{B}'} \in \mathrm{Ker}(\hat{H}_{\mathcal{P}}) = \langle \mathcal{B} \rangle = \mathrm{Ker}(\hat{E}_{\mathcal{B}} \otimes -)$. We have thus proved

$$(5.9) \qquad\qquad \hat{E}_{\mathcal{B}} \otimes \hat{E}_{\mathcal{B}'} = 0.$$

Consider now the exact sequence in $\mathcal{A}$ associated to the morphism $\eta_{\mathcal{B}} : \mathbb{1} \to E_{\mathcal{B}}$

$$0 \to I_{\mathcal{B}} \to \hat{\mathbb{1}} \xrightarrow{\hat{\eta}_{\mathcal{B}}} \hat{E}_{\mathcal{B}}.$$

Since $E_{\mathcal{B}} \otimes \eta_{\mathcal{B}}$ is a split monomorphism, we have $I_{\mathcal{B}} \in \mathrm{Ker}(\hat{E}_{\mathcal{B}} \otimes -) = \langle \mathcal{B} \rangle \subseteq \langle \mathcal{B}' \rangle$ and therefore $I_{\mathcal{B}} \otimes \hat{E}_{\mathcal{B}'} = 0$. Combined with (5.9) we see from the above exact sequence that $\hat{\mathbb{1}}$ is also killed by $\hat{E}_{\mathcal{B}'}$, that is $\hat{E}_{\mathcal{B}'} = 0$, or $\mathcal{B}' = \mathcal{A}^{\mathrm{fp}}$ as claimed.

In summary, we have now shown that $\mathcal{B}(\mathcal{P}) = \mathrm{Ker}(\hat{H}_{\mathcal{P}}) \cap \mathcal{A}^{\mathrm{fp}}$ belongs to the homogeneous spectrum $\mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$. We see that $\phi(\mathcal{B}(\mathcal{P})) = \mathcal{P}$, by the last assumption in (2). Finally, we need to relate the functor $H_{\mathcal{P}}$ with the homological residue field $\bar{\mathrm{h}}_{\mathcal{B}}$ for $\mathcal{B} := \mathcal{B}(\mathcal{P})$. This is now easy. From (5.7) and the fact that $\langle \mathcal{B} \rangle = \mathrm{Ker}(\hat{H}_{\mathcal{P}})$ we can further factor $\hat{H}_{\mathcal{P}}$ by first modding out this kernel. We obtain a unique

exact functor $\bar{H}_{\mathcal{P}} : \bar{A}(\mathcal{K}; \mathcal{B}) \to \mathcal{A}_{\mathcal{P}}$ making the right-hand triangle in the following diagram commute:

$$
\begin{array}{c}
\bar{\mathrm{h}}_{\mathcal{B}} \\
\mathcal{T} \xrightarrow{\mathrm{h}} \mathcal{A} \xrightarrow{Q_{\mathcal{B}}} \bar{A}(\mathcal{K}; \mathcal{B}) = \mathrm{mod}\text{-}\mathcal{K}/\langle\mathcal{B}\rangle \\
H_{\mathcal{P}} \searrow \quad \downarrow \hat{H}_{\mathcal{P}} \quad \swarrow \bar{H}_{\mathcal{P}} \\
\mathcal{A}_{\mathcal{P}}
\end{array}
$$

The left-hand triangle was already in (5.7). The top "triangle" commutes by definition of $\bar{\mathrm{h}}_{\mathcal{B}} : \mathcal{T} \to \bar{A}(\mathcal{K}; \mathcal{B})$. Expanding the notation $\mathcal{B} = \mathcal{B}(\mathcal{P})$, we have factored each $H_{\mathcal{P}} : \mathcal{T} \to \mathcal{A}_{\mathcal{P}}$ via a homological residue field $\bar{\mathrm{h}}_{\mathcal{B}(\mathcal{P})}$ as follows:

$$H_{\mathcal{P}} = \bar{H}_{\mathcal{P}} \circ \bar{\mathrm{h}}_{\mathcal{B}(\mathcal{P})}.$$

We now claim that the family

$$\mathcal{F} := \{\mathcal{B}(\mathcal{P}) \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{K})\}$$

satisfies the hypothesis of Theorem 5.4, in other words that the family of functors $\{\bar{\mathrm{h}}_{\mathcal{B}(\mathcal{P})}\}_{\mathcal{P} \in \mathrm{Spc}(\mathcal{K})}$ detects $\otimes$-nilpotence of maps $f : x \to Y$ in $\mathcal{T}$ with $x \in \mathcal{K}$ compact. Indeed if $\bar{\mathrm{h}}_{\mathcal{B}(\mathcal{P})}(f) = 0$ then $H_{\mathcal{P}}(f) = \bar{H}_{\mathcal{P}} \circ \bar{\mathrm{h}}_{\mathcal{B}(\mathcal{P})}(f) = 0$ by the above factorization. If this holds for all $\mathcal{P}$, we conclude by (3) that $f^{\otimes n} = 0$ for $n \gg 1$. So Theorem 5.4 tells us that this family $\mathcal{F} = \{\mathcal{B}(\mathcal{P}) \mid \mathcal{P} \in \mathrm{Spc}(\mathcal{K})\}$ is the whole $\mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$.

In conclusion, we have constructed a set-theoretic section

$$\sigma : \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}), \quad \mathcal{P} \mapsto \mathcal{B}(\mathcal{P})$$

of $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \twoheadrightarrow \mathrm{Spc}(\mathcal{K})$ and we just proved that $\mathrm{im}(\sigma) = \mathcal{F} = \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$. In other words, the surjection ϕ admits a surjective section, i.e., ϕ is a bijection. $\square$

We can now use known nilpotence-detecting families in examples, to prove that $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ is bijective.

5.10. Corollary. *Let $\mathcal{T} = \mathrm{SH}$ be the stable homotopy category and $\mathcal{K} = \mathrm{SH}^c$. Then $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ is a bijection. More generally, let G be a compact Lie group and $\mathcal{T} = \mathrm{SH}(G)$ the G-equivariant stable homotopy category of genuine G-spectra, and $\mathcal{K} = \mathrm{SH}(G)^c$. Then $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ is a bijection.*

Proof. In the case of $\mathcal{T} = \mathrm{SH}$, this relies on [Devinatz et al. 1988; Hopkins and Smith 1998]. As explained in [Balmer 2010, §9], the spectrum consists of points $\mathcal{P}(p, n)$ for each prime number p and each "chromatic height" $1 \leq n \leq \infty$, with the collision $\mathcal{P}(0) := \mathcal{P}(p, 1) = \mathrm{SH}^{\mathrm{tor}}$ for all p. This prime $\mathcal{P}(0)$ is the kernel of rational homology $H\mathbb{Q} \otimes - : \mathrm{SH}^c \to \mathrm{D}^{\mathrm{b}}(\mathbb{Q}) \cong \mathbb{Q}\text{-}\mathrm{Mod}_\bullet$. The other primes $\mathcal{P}(p, n)$

for $n \geq 2$ are given as the kernels of Morava K-theory $K(p, n-1)_\bullet$, which are homological functors

$$K(p, n)_\bullet : \mathrm{SH} \to \mathcal{A}_{p,n} := \mathbb{F}_p[v_n^{\pm 1}]\text{-Mod}_\bullet$$

for $1 \leq n < \infty$ with v_n of degree $2(p^n - 1)$, and $K(p, \infty)_\bullet : \mathrm{SH} \to \mathbb{F}_p\text{-Mod}_\bullet$ is mod-p homology. The target categories are graded modules over (graded) fields and the Morava K-theories satisfy Künneth formulas, which amounts to say that they are monoidal functors when $\mathcal{A}_{p,n}$ is equipped with the graded tensor product. See [Ravenel 1992]. The reader can now verify conditions (1)–(3) of Theorem 5.6. The crucial (3) is the original nilpotence theorem [Devinatz et al. 1988].

For $\mathcal{T} = \mathrm{SH}(G)$ and $\mathcal{K} = \mathrm{SH}(G)^c$, the description of the set $\mathrm{Spc}(\mathcal{K})$ and the relevant nilpotence theorem was achieved for finite groups in [Balmer and Sanders 2017], and more recently for arbitrary compact Lie groups in [Barthel et al. 2018]. Specifically, there is exactly one prime $\mathcal{P}(H, p, n) = (\Phi^H)^{-1}(\mathcal{P}(p, n))$ in $\mathrm{Spc}(\mathcal{K})$ for every conjugacy class of closed subgroups $H \leq G$ and for every "chromatic" prime $\mathcal{P}(p, n)$ as above; here $\Phi^H : \mathrm{SH}(G) \to \mathrm{SH}$ is the geometric H-fixed point functor, which is tensor-triangulated. The relevant homology theories are simply these Φ^H composed with the nonequivariant Morava K-theories. So conditions (1)–(2) in Theorem 5.6 are easy to verify. The relevant nilpotence theorem giving us (3) can be found in [Barthel et al. 2018, Theorem 3.12] (or [Balmer and Sanders 2017, Theorem 4.15] for finite groups, a result also obtained earlier by N. Strickland). $\square$

5.11. Corollary. *Let X be a quasicompact and quasiseparated scheme and $\mathcal{T} = \mathrm{D}(X)$ the derived category of $\mathcal{O}_X$-modules with quasicoherent homology. Here $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ is the category of perfect complexes, the spectrum $\mathrm{Spc}(\mathcal{K}) \cong |X|$ is the underlying space of X and the map $\phi : \mathrm{Spc}^h(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ is a bijection.*

Proof. The homological functors of Theorem 5.6 are simply the classical residue fields $\kappa(x) \otimes_{\mathcal{O}_X}^{\mathrm{L}} - : \mathrm{D}(X) \to \mathrm{D}(\kappa(x)) \cong \kappa(x)\text{-Mod}_\bullet$ at each $x \in |X|$, where of course $\kappa(x)$ is the residue fields of the local ring $\mathcal{O}_{X,x}$. Here, the relevant nilpotence theorem is due to Thomason [1997, Theorem 3.6]. $\square$

5.12. Remark. There is a simpler proof of the above when X is noetherian, following the pattern of the next example. It is worth noting that when X is not noetherian, even for $|X| = *$, the residue fields do not detect vanishing of objects. See [Neeman 2000].

5.13. Example. Let G be a finite group scheme over a field k and $\mathcal{T} = \mathrm{Stab}(kG)$ the category of k-linear representations of G modulo projectives. See [Benson et al. 2018]. Here $\mathcal{K} = \mathrm{stab}(kG)$ is the stable category of finite-dimensional representations modulo projectives, $\mathrm{Spc}(\mathcal{K}) \cong \mathrm{Proj}(\mathrm{H}^\bullet(G, k))$ is the so-called projective support variety, and the map $\phi : \mathrm{Spc}^h(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ is again bijective. The method

of proof is different, for there is no known homology theories capturing the points of $\mathrm{Spc}(\mathcal{K})$ *and satisfying a Künneth formula*. Indeed, points are detected by equivalence classes of so-called π-*points*, following [Friedlander and Pevtsova 2007], but these functors are *not* monoidal!

Instead, we can use the fact that localizing subcategories of $\mathcal{T}$ are classified by subsets of $\mathrm{Spc}(\mathcal{K})$ in this case, a nontrivial result that can be found in [Benson et al. 2018, §10]. In such situations, the property $E_{\mathcal{B}} \otimes E_{\mathcal{B}'} = 0$ isolated in Proposition 5.3, for $\mathcal{B} \neq \mathcal{B}'$ in $\mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ can be used to show that $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ is injective. Indeed, if $\phi(\mathcal{B}) = \phi(\mathcal{B}') =: \mathcal{P}$, we can use minimality of the localizing category $\mathcal{T}_{\mathcal{P}}$ supported at the point $\mathcal{P}$ to show that $E_{\mathcal{B}} \otimes E_{\mathcal{B}'} = 0$ forces $E_{\mathcal{B}} = 0$ or $E_{\mathcal{B}'} = 0$ which is absurd. This argument can already be found in [Balmer et al. 2019, Corollary 4.26].

5.14. Remark. We note that the above does *not* use a nilpotence theorem for $\mathcal{T} = \mathrm{Stab}(kG)$. To the best of the author's knowledge, there is no such result in the literature, the probable reason being that π-points (or shifted cyclic subgroups) are not monoidal. Thanks to the present work, we now know that there exists for every point $\mathcal{P} \in \mathrm{Spc}(\mathrm{stab}(kG)) \cong \mathrm{Proj}(\mathrm{H}^{\bullet}(G, k))$ a unique homological *tensor* functor

$$\bar{\mathrm{h}}_{\mathcal{B}(\mathcal{P})} : \mathrm{Stab}(kG) \to \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B}(\mathcal{P}))$$

to a "simple" Grothendieck tensor category (in the sense of (1) in Theorem 5.6), whose kernel on compacts $\mathcal{K} = \mathcal{T}^c = \mathrm{stab}(kG)$ is exactly $\mathcal{P}$. And we know that the family $\{\bar{\mathrm{h}}_{\mathcal{B}(\mathcal{P})}\}_{\mathcal{P} \in \mathrm{Spc}(\mathcal{K})}$ detects tensor-nilpotence.

5.15. Remark. In view of the avalanche of examples where $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ is a bijection, it takes nerves of steel not to conjecture that this property holds for all tensor-triangulated categories.

Acknowledgments

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Finite-dimensional reduction
of a supercritical exponent equation

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We present a finite-dimensional reduction for a supercritical exponent PDE. We
reduce the existence of a solution of the problem

$$-\Delta u = K|u|^{4/(n-2)+\varepsilon}u \quad \text{in } \Omega \quad (\text{with } \varepsilon > 0), \qquad u = 0 \quad \text{on } \partial\Omega,$$

to finding a critical point of a function defined in some set $\mathcal{V} \subset \mathbb{R}^N \times \mathbb{R}^N \times \Omega^N$.

1. Introduction and main results

We study this semilinear elliptic problem with supercritical nonlinearity:

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} -\Delta u = K|u|^{p-1+\varepsilon}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 3$, ε is a real parameter, $p+1 = 2n/(n-2)$ is the critical Sobolev exponent for the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ and K is a C^2 positive function on $\bar{\Omega}$.

If $\varepsilon = 0$, $K \equiv 1$ and $\Omega = \mathbb{R}^n$ (equivalently S^n), this problem is known as the Yamabe problem. When $K \neq 1$, it becomes the scalar curvature problem. These problems concern finding a new metric g conformal to the standard one g_0 (namely $g = u^{4/(n-2)}g_0$) so that its scalar curvature is K. These problems have been extensively studied in the last decades. Taking $K \equiv 1$, Pokhozaev [1965] proved that if Ω is a strictly star-shaped domain then no solution of (P_ε) exists for each $\varepsilon \geq 0$. In contrast, Kazdan and Warner [1975] showed that if Ω is a radially symmetric annulus, there exists a radial positive solution to (P_ε) for each $\varepsilon > 1 - p$. For $\varepsilon = 0$, a sufficient condition for the existence of positive solution is that Ω has nontrivial topology (in a suitable sense) as showed Bahri and Coron [1988]. This condition is sufficient but not necessary as shown by some examples of contractible domains where (P_0) has positive solutions [Dancer 1988; Ding 1989; Passaseo 1989]. For $\varepsilon > 0$, the nontriviality of the domain is neither a sufficient nor a necessary condition. In fact, nonexistence results hold for some $\varepsilon > 0$ in

some nontrivial domains [Passaseo 1993; 1995] while an arbitrary large number of solutions can be obtained in some contractible domains for $\varepsilon > 0$ [Passaseo 1998].

If $\varepsilon < 0$, then (P$_\varepsilon$) becomes a subcritical problem because the embedding

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega) \quad (\text{with } q < 2n/(n-2))$$

is compact and therefore the associated variational functional satisfies the Palais–Smale condition. Hence, for each $\varepsilon < 0$, there exist solutions u_ε of (P$_\varepsilon$). Many authors were interested to understand the behavior of the solutions u_ε when ε tends to 0.

It is known that, if (P$_0$) has no solution, the solutions u_ε with $\varepsilon < 0$ have to blow up. In fact, many results are devoted to understanding the asymptotic behavior of u_ε. Now, from [Brezis and Coron 1985; Lions 1985; Struwe 1984], it is well known that, given a positive solution u_ε of (P$_\varepsilon$) (with $\varepsilon < 0$), then, if u_ε blows up, it has to be of the form

$$u_\varepsilon = u_0 + \sum_{i=1}^k K(a_i)^{(2-n)/4} P\delta_{(a_i, \lambda_i)} + v, \tag{1-1}$$

where u_0 is a solution of (P$_0$) and, as $\varepsilon \to 0$, $\lambda_i d(a_i, \partial\Omega) \to \infty$, $\|v\| \to 0$ and $\langle P\delta_{(a_i, \lambda_i)}, P\delta_{(a_j, \lambda_j)} \rangle \to 0$ for $i \neq j$ and where the $\delta_{(a, \lambda)}$ are the family of positive solutions of $-\Delta u = u^{(n+2)/(n-2)}$ in $\mathbb{R}^n$ and $P\delta_{(a, \lambda)}$ is its projection onto $H_0^1(\Omega)$, that is, it satisfies $\Delta P\delta_{(a, \lambda)} = \Delta\delta_{(a, \lambda)}$ in Ω and $P\delta_{(a, \lambda)} = 0$ on $\partial\Omega$. Once this asymptotic behavior of u_ε is proved, the authors were interested to characterize the blow up points a_i and the blow up speeds λ_i. Furthermore, they tried to construct solutions of (P$_\varepsilon$) having the obtained properties. For $K \equiv 1$, this program is done in [Han 1991; Rey 1989] concerning the least energy solutions and in [Bahri et al. 1995; Rey 1999], for a general case. For $K \neq 1$, it is proved that each concentration interior point a_i has to converge to a critical point of K [Aubin 1998; Bahri 1989]. Moreover, if a_i converges to $\bar{a} \in \partial\Omega$ then we have $(\partial K/\partial \nu)(\bar{a}) \geq 0$ [Aubin 1998]. For these kinds of solutions, in [Cao and Peng 2003], taking $\Omega = B(O, 1)$ and $K = |x|^\gamma$, the authors constructed solutions with one bubble which converge to a point on $\partial\Omega$. Recently, in [Dávila et al. 2017], the authors constructed positive solutions (with many bubbles) which blow up at one point on the boundary for the dimension $3 \leq n \leq 6$ (assuming that $\partial K/\partial \nu > 0$ at a critical point of $K_{|\partial\Omega}$). Furthermore, they removed the restriction on the dimensions if the solutions are with one bubble.

Concerning changing sign solutions, in [Pistoia and Weth 2007] (with $K \equiv 1$), the authors constructed solutions having the behavior of bubbles over bubbles $u_\varepsilon = \sum(-1)^i P\delta_{(a_i, \lambda_i)} + v$ with $a_i \to a^* \in \Omega$ for each i and $\lambda_i/\lambda_{i+1} \to 0$. This result was extended by other authors concerning other equations and assumptions.

Concerning $\varepsilon > 0$, many results in this direction are nonexistence results, see [Ben Ayed et al. 2003; Ben Ayed and Ould Bouh 2008; Ould Bouh 2012; Passaseo 1993; 1995]. However, in [del Pino et al. 2004; Khenissy and Rey 2004], the authors proved the existence of solutions u_ε which blow up at two points using some topological assumptions and arguments. Furthermore, in [del Pino et al. 2002; 2004], it is established that a positive solution (which looks like the sum of two bubbles) to (P$_\varepsilon$) exists for $K \equiv 1$ and any small $\varepsilon > 0$ if Ω is a smooth domain exhibiting a small hole. We notice that in the most of the existence results, when u_ε blows up, it becomes as sum of bubbles $P\delta_{a_i, \lambda_i}$. However, in [Musso and Wei 2016], the authors used a sign-changing solution of the Yamabe equation on $\mathbb{R}^n$, denoted by Q, and they constructed a solution u_ε which looks like two copies of the solution Q (using the invariance of Q by translation and dilatation).

Recall that (P$_\varepsilon$) is a variational problem. Its associated variational structure is defined on $H_0^1(\Omega)$ ($H_0^1(\Omega) \cap L^{p+1+\varepsilon}(\Omega)$ if $\varepsilon > 0$) by

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1+\varepsilon} \int_\Omega K|u|^{p+1+\varepsilon}. \qquad (1\text{-}2)$$

Let $N \in \mathbb{N} \setminus \{0\}$ and let $\mu > 0$ be a small constant, we define the sets

$$\mathcal{V}(N, \mu) := \Big\{ (\alpha, \lambda, a) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega^N : |\alpha_i^{4/(n-2)} K(a_i) - 1| < \mu,$$
$$\lambda_i d(a_i, \partial\Omega) > \mu^{-1}, \ \varepsilon_{ij} < \mu, \ \varepsilon \ln \lambda_i < \mu, \ \forall i \leq N \Big\}, \quad (1\text{-}3)$$

$$V(N, \mu) := \Big\{ \sum_{i=1}^N \alpha_i \gamma_i P\delta_{(a_i, \lambda_i)} + v : \gamma_i \in \{-1, 1\}, \ \|v\| < \mu,$$
$$(\alpha, \lambda, a) \in \mathcal{V}(N, \mu) \Big\}, \quad (1\text{-}4)$$

where $\varepsilon_{ij} = (\lambda_i/\lambda_j + \lambda_j/\lambda_i + \lambda_i\lambda_j|a_i - a_j|^2)^{(2-n)/2}$ and $d_i := d(a_i, \partial\Omega)$. The ε_{ij} represents the interaction between two bubbles $P\delta_{(a_i, \lambda_i)}$ and $P\delta_{(a_j, \lambda_j)}$. In $V(N, \mu)$, these parameters are small, this implies that the bubbles $P\delta_{(a_i, \lambda_i)}$ are almost orthogonal. We remark that, in [Bahri 1989; Bahri and Coron 1988; Bahri et al. 1995; Rey 1999; Struwe 1984], the authors studied positive solutions and therefore they defined $V(N, \mu)$ with $\gamma_i = 1$ for each i. This set is very important when studying the asymptotic behavior of the positive solutions u_ε, for $\varepsilon < 0$. In fact, it is well known that, when $u_\varepsilon \rightharpoonup 0$, it has to enter in $V(N, \mu)$ (with $\gamma_i = 1$ for each i since $u_\varepsilon > 0$). Moreover, for $\varepsilon = 0$, the lack of compactness of J occurs in this set.

Thus, for $\varepsilon < 0$, many authors tried to construct positive solutions of (P$_\varepsilon$) which belong to $V(N, \mu)$ with $\gamma_i = 1$ for each i (see [Bahri et al. 1995; Han 1991; Rey 1989; 1999]). The proofs of these results are based on the finite-dimensional reduction. In fact, from [Bahri 1989; Bahri and Coron 1988], we know that: for $\varepsilon \leq 0$,

given $\underline{u} := \sum \alpha_i P\delta_{(a_i,\lambda_i)} \in V(N,\mu)$, we have

$$J(\underline{u}+v) = J(\underline{u}) + f(v) + \tfrac{1}{2}Q(v) + o(\|v\|^2), \quad \text{for all } v \in E^\perp, \qquad (1\text{-}5)$$

where $f(\,\cdot\,)$ is a linear continuous form, $Q(\,\cdot\,)$ is a positive definite quadratic form and

$$E := \text{span}\big\{ P\delta_{(a_i,\lambda_i)},\ \partial P\delta_{(a_i,\lambda_i)}/\partial\lambda_i,\ \partial P\delta_{(a_i,\lambda_i)}/\partial(a_i)_k,\quad i \le N,\ k \le n \big\}. \quad (1\text{-}6)$$

Therefore, it is easy to deduce the existence of a unique $\bar{v}_{(\alpha,\lambda,a)} \in E^\perp$ such that

$$J(\underline{u}+\bar{v}_{(\alpha,\lambda,a)}) = \min\{J(\underline{u}+v) : v \in E^\perp, \|v\| < \mu\}. \qquad (1\text{-}7)$$

Hence, the existence of a positive critical point of J will be reduced to find a critical point of the function

$$I_+ : (\alpha,\lambda,a) \in \mathcal{V}(N,\mu) \longmapsto J\Big(\sum \alpha_i P\delta_{(a_i,\lambda_i)} + \bar{v}_{(\alpha,\lambda,a)}\Big).$$

This program was established in [Bahri et al. 1995] and many authors adapted the proof in other situations. Furthermore, some authors tried to construct sign-changing solutions following the same ideas. For this purpose, they added some parameters $\gamma_i \in \{-1,1\}$ before the bubbles $P\delta_{(a_i,\lambda_i)}$. Thus, given $(\gamma_1,\ldots,\gamma_N) \in \{-1,1\}^N$, they extended the function I_+ by introducing the function

$$I : (\alpha,\lambda,a) \in \mathcal{V}(N,\mu) \longmapsto J\Big(\sum \alpha_i \gamma_i P\delta_{(a_i,\lambda_i)} + \bar{v}_{(\alpha,\lambda,a)}\Big). \qquad (1\text{-}8)$$

The main part of this paper is to prove analogous results of [Bahri 1989] and [Bahri and Coron 1988]. In our case, since we take $\varepsilon > 0$, we lose the continuous embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$. Moreover, it is important to notice that the functional J is not a continuous function and therefore, we are not able to prove the expansion (1-5). The difficulty is essentially to control the integral $\int |v|^{p+1+\varepsilon}$. To overcome this problem, we use some ideas originally introduced by Bahri and Xu [2007] to control the remainder function v by means of the bubbles $\delta_{(a_i,\lambda_i)}$. More precisely, we prove:

Theorem 1.1. *Let μ be a small positive constant and, for $N \in \mathbb{N}$, let $(\alpha,\lambda,a) \in \mathcal{V}(N,\mu)$ (defined in (1-3)). For $i = 1,\ldots,N$, let $\gamma_i \in \{-1,1\}$ and we denote $\underline{u} := \sum_{i=1}^N \alpha_i \gamma_i P\delta_{(a_i,\lambda_i)}$. Then, for $\varepsilon > 0$ small enough, there exists a unique function $\bar{v}_{(\alpha,\lambda,a)} \in E^\perp$ satisfying*

$$-\Delta\bar{v}_{(\alpha,\lambda,a)} = K|\underline{u}+\bar{v}_{(\alpha,\lambda,a)}|^{p-1+\varepsilon}(\underline{u}+\bar{v}_{(\alpha,\lambda,a)}) - \sum_{i=1}^N \alpha_i\gamma_i\delta_{(a_i,\lambda_i)}^p$$

$$- \sum_{i=1}^N \Delta\bigg(A_i P\delta_{(a_i,\lambda_i)} + B_i\lambda_i\frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial\lambda_i} + \sum_{k=1}^n C_{ik}\frac{1}{\lambda_i}\frac{\partial P\delta_{(a_i,\lambda_i)}}{\partial(a_i)_k}\bigg),$$

where $(A, B, C) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{nN}$ *are some constants with* $|A| + |B| + |C|$ *small* (*see* (2-6) *for its estimate*).

The map $(\alpha, \lambda, a) \mapsto \bar{v}_{(\alpha,\lambda,a)}$ *is a* C^1-*function. Moreover*:

(i) $\|\bar{v}_{(\alpha,\lambda,a)}\|$

$$\leq M_2\left(\varepsilon + \sum\left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right)\right)$$

$$+ M_2 \begin{cases} \sum_i 1/(\lambda_i d_i)^{n-2} + \sum_{i \neq j} \varepsilon_{ij} \ln(\varepsilon_{ij}^{-1})^{(n-2)/n} & \text{if } n \leq 5, \\ \sum_i \ln(\lambda_i d_i)/(\lambda_i d_i)^4 + \sum_{i \neq j} \varepsilon_{ij} \ln(\varepsilon_{ij}^{-1})^{\frac{2}{3}} & \text{if } n = 6, \\ \sum_i 1/(\lambda_i d_i)^{(n+2)/2} + \sum_{i \neq j} \varepsilon_{ij}^{(n+2)/(2(n-2))} \ln(\varepsilon_{ij}^{-1})^{(n+2)/(2n)} & \text{if } n \geq 7. \end{cases}$$

(ii) *Let* β *be a small positive constant. Choosing* μ *small enough* ($\mu^{\sigma_0} \leq \beta$ *for some positive constant* σ_0), *we have*

$$|\bar{v}_{(\alpha,\lambda,a)}(x)| \leq \beta \sum_{i=1}^N \delta_{(a_i,\lambda_i)}(x) \quad \text{for each } x \in \Omega.$$

(iii) $|\bar{v}_{(\alpha,\lambda,a)}|_\infty^\varepsilon \leq M_1$.

Here M_1 *and* M_2 *are positive constants independent of* ε *and* (α, λ, a).

Remark. Claim (ii) gives a pointwise estimate of the function $\bar{v}_{(\alpha,\lambda,a)}$. A more precise estimate is given in Proposition 2.4. Furthermore, claim (i) of Proposition 2.4 is not optimal. In fact, as in [Bahri and Xu 2007], we can improve the small coefficient which depends on (α, λ, a) and ε (the new coefficient has to be well chosen). In fact, we can improve some estimates (for example (2-4)).

The first application of this theorem is to remove the assumption "$|u_\varepsilon|^\varepsilon$ *is bounded*" in [Ben Ayed and Ould Bouh 2008; Ould Bouh 2012]. This theorem implies that the analog of the equation (E_v) in [Bahri et al. 1995] is satisfied. Hence to find solutions of (P_ε), with $\varepsilon > 0$, it remains to find critical points of I, defined in (1-8), that is, we need to prove that the analogs of the equations (E_α), (E_λ) and (E_a) are satisfied. These equations can be obtained by taking $\partial I/\partial \alpha_i = 0$, $\partial I/\partial \lambda_i = 0$ and $\partial I/\partial (a_i)_k = 0$. Hence, this result will allow us to adapt the constructions done in, for examples, [Bahri et al. 1995; Ben Ayed and Ghoudi 2008; Dávila et al. 2017; Pistoia and Weth 2007] to find solutions of (P_ε), with $\varepsilon > 0$, which blow up as $\varepsilon \to 0$. This kind of result will be presented in works to come.

Proposition 1.2. *Let* $(\alpha, \lambda, a) \in \mathcal{V}(N, \mu)$ *and* $\gamma_i \in \{-1, 1\}$ *for* $i = 1, \ldots, N$. *We set* $\underline{u} := \sum \alpha_i \gamma_i P \delta_{(a_i,\lambda_i)}$ *and* $\bar{u} := \underline{u} + \bar{v}_{(\alpha,\lambda,a)}$. *Then* $\bar{u}$ *is a solution of* (P_ε) *if and only if* (α, λ, a) *is a critical point of* I, *that is, if and only if for each* $i = 1, \ldots, N$ *and* $j = 1, \ldots, n$, *the following hold*:

(E_{α_i}) $\langle \underline{u}, P\delta_{(a_i,\lambda_i)} \rangle - \int_\Omega K|\bar{u}|^{p-1+\varepsilon} \bar{u} P\delta_{(a_i,\lambda_i)} = 0.$

(E_{λ_i}) $\langle \underline{u}, \alpha_i \gamma_i \partial P\delta_{(a_i,\lambda_i)} / \partial \lambda_i \rangle - \int_\Omega K|\bar{u}|^{p-1+\varepsilon} \bar{u} \alpha_i \gamma_i \partial P\delta_{(a_i,\lambda_i)} / \partial \lambda_i$

$$= B_i \langle \bar{v}, \lambda_i \partial^2 P\delta_i / \partial \lambda_i^2 \rangle + \sum_{k=1}^n C_{ik} \langle \bar{v}, \lambda_i^{-1} \partial^2 P\delta_i / \partial(a_i)_k \partial \lambda_i \rangle.$$

$(E_{(a_i)_j})$ $\langle \underline{u}, \alpha_i \gamma_i \partial P\delta_{(a_i,\lambda_i)} / \partial(a_i)_j \rangle - \int_\Omega K|\bar{u}|^{p-1+\varepsilon} \bar{u} \alpha_i \gamma_i \partial P\delta_{(a_i,\lambda_i)} / \partial(a_i)_j$

$$= B_i \langle \bar{v}, \lambda_i \partial^2 P\delta_i / \partial \lambda_i \partial(a_i)_j \rangle + \sum_{k=1}^n C_{ik} \langle \bar{v}, \lambda_i^{-1} \partial^2 P\delta_i / \partial(a_i)_k \partial(a_i)_j \rangle.$$

Notice that this system is similar to the one found in [Bahri et al. 1995] for the subcritical case. For this reason, we believe that the constructions in that paper (and other results) can be extended in our case, possibly with some changes in the assumptions.

This paper is organized as follows: Section 2 is devoted to proving our results, and we collect some known results in the Appendix.

2. Proof of the results

Let $(\alpha, \lambda, a) \in \mathcal{V}(N, \mu)$ and let $\underline{u} := \sum_{i=1}^N \alpha_i \gamma_i P\delta_{(a_i,\lambda_i)}$. For sake of simplicity, we will write δ_i and $P\delta_i$ instead of $\delta_{(a_i,\lambda_i)}$ and $P\delta_{(a_i,\lambda_i)}$ respectively.

We start this section by a lemma that will be used to estimate $\|\bar{v}\|$.

Lemma 2.1. *Let* $(\alpha, \lambda, a) \in \mathcal{V}(N, \mu)$ *and* $\underline{u} := \sum \alpha_i \gamma_i P\delta_i$. *For each* $v \in E^\perp$, *we have*

$$\int_\Omega K|\underline{u}|^{p-1+\varepsilon} \underline{u} v = \sum \alpha_i^{p+\varepsilon} \gamma_i \int_\Omega P\delta_i^{p+\varepsilon} v + O\left(\|v\|\left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right)\right)$$

$$+ \begin{cases} O\left(\|v\|\left(\varepsilon_{ij}^{(n+2)/(2(n-2))} \ln(\varepsilon_{ij}^{-1})^{(n+2)/(2n)}\right)\right) & \text{if } n \geq 6, \\ O\left(\|v\|\left(\varepsilon_{ij} \ln(\varepsilon_{ij}^{-1})^{(n-2)/n}\right)\right) & \text{if } n \leq 5. \end{cases}$$

Proof. Observe that

$$\int_\Omega K|\underline{u}|^{p-1+\varepsilon} \underline{u} v = \sum \alpha_i^{p+\varepsilon} \gamma_i \int_\Omega K P\delta_i^{p+\varepsilon} v + O\left(\sum_{i \neq j} \int \delta_i^{p-1} \inf(\delta_i, \delta_j)|v|\right).$$

For the first integral, we have

$$\int_\Omega K P\delta_i^{p+\varepsilon} v = \int_\Omega P\delta_i^{p+\varepsilon} v + O\left(\int_\Omega \left(|\nabla K(a_i)||x - a_i| + |x - a_i|^2\right)\delta_i^p |v|\right)$$

$$= \int_\Omega P\delta_i^{p+\varepsilon} v + O\left(\|v\|\left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right)\right).$$

Concerning the last one, for $n \leq 5$ (in this case, $p - 1 > 1$), by using Holder's inequality, we get

$$\int \delta_i^{p-1} \inf(\delta_i, \delta_j)|v| \leq \int \delta_i^{p-1} \delta_j |v| \leq c\|v\| \varepsilon_{ij} \ln(\varepsilon_{ij}^{-1})^{(n-2)/n},$$

and for $n \geq 6$ (in this case, $p - 1 \leq 1$), by using Holder's inequality, we get

$$\int \delta_i^{p-1} \inf(\delta_i, \delta_j)|v| \leq \int (\delta_i \delta_j)^{(n+2)/(2(n-2))}|v|$$

$$\leq \|v\| \big(\varepsilon_{ij}^{n/(n-2)} \ln(\varepsilon_{ij}^{-1})\big)^{(n+2)/(2n)}.$$

The result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.2. *Let G be the Green's function with Dirichlet boundary condition. Then, for each $y \in \Omega$ and for each $h \in L^{2n/(n-2)}(\Omega)$, we have*

$$\int_\Omega \delta_{(a,\lambda)}^4(x)|h(x)|G(x,y)\,dx \leq c\|h\|_{L^{2n/(n-2)}} \delta_{(a,\lambda)}(y) \quad \text{for } n = 3,$$

$$\int_\Omega \delta_{(a,\lambda)}^{n/(n-2)}(x)|h(x)|^{2/(n-2)}G(x,y)\,dx \leq c\|h\|_{L^{2n/(n-2)}}^{2/(n-2)} \delta_{(a,\lambda)}(y) \quad \text{for } n \geq 4.$$

Proof. We notice that this lemma is proved in [Dammak 2017, Lemma 2.10] for $n = 4, 5, 6$ and the proof can be extended for the dimensions $n \geq 4$. In fact, let $n \geq 4$ and $y \in \Omega$, to prove the second claim of the lemma, two cases may occur.

1st case: $\lambda|y - a| \geq 2$. We remark that, in this case we have

$$(\lambda|y - a|^2)^{(2-n)/2} \leq c\delta_{(a,\lambda)}(y).$$

Observe that, for $x \in B(y, 1/\lambda)$, we get $|a - x| \geq |a - y|/2$ and therefore we obtain $\delta_{(a,\lambda)}(x) \leq c\delta_{(a,\lambda)}(y)$. Hence, using the Holder's inequality, we get

$$I_1 := \int_{B(y,1/\lambda)} \delta_{(a,\lambda)}^{n/(n-2)}(x)|h(x)|^{2/(n-2)}G(x,y)\,dx$$

$$\leq c\delta_{(a,\lambda)}^{n/(n-2)}(y) \int_{B(y,1/\lambda)} \frac{|h(x)|^{2/(n-2)}}{|y - x|^{n-2}}\,dx$$

$$\leq c\delta_{(a,\lambda)}^{n/(n-2)}(y)\|h\|^{2/(n-2)}\frac{c}{\lambda}$$

$$\leq c\|h\|^{2/(n-2)}\delta_{(a,\lambda)}(y).$$

However, for $x \in B(a, |y - a|/2)$, we have $|x - y| \geq |a - y|/2$ and therefore $G(x, y) \leq c/|y - a|^{n-2}$ and

$$I_2 := \int_{B(a,|y-a|/2)} \delta_{(a,\lambda)}^{n/(n-2)}(x)|h(x)|^{2/(n-2)} G(x, y)\, dx$$

$$\leq \frac{c}{|y - a|^{n-2}} \int |h|^{2/(n-2)} \delta_{(a,\lambda)}^{n/(n-2)}$$

$$\leq \frac{c}{|y - a|^{n-2}} \|h\|^{2/(n-2)} \frac{1}{\lambda^{(n-2)/2}}$$

$$\leq c\|h\|^{2/(n-2)} \delta_{(a,\lambda)}(y).$$

Now, for $x \in \Omega' := \Omega \setminus (B(y, 1/\lambda) \cup B(a, |y - a|/2))$, it is easy to get $G(x, y) \leq 1/|x - y|^{n-2} \leq c\lambda^{(n-2)/2}\delta_{(y,\lambda)}(x)$ and $|a - x| \geq |a - y|/2$ which implies that $\delta_{(a,\lambda)}(x) \leq c\delta_{(a,\lambda)}(y)$. Hence we obtain

$$I_3 := \int_{\Omega'} \delta_{(a,\lambda)}^{n/(n-2)}(x)|h(x)|^{2/(n-2)} G(x, y)\, dx$$

$$\leq c\delta_{(a,\lambda)}(y) \int_{\Omega'} \frac{|h|^{2/(n-2)}}{|x - y|^{n-2}} \delta_{(a,\lambda)}^{2/(n-2)}(x)$$

$$\leq c\lambda^{(n-2)/2}\delta_{(a,\lambda)}(y) \int_{\Omega'} |h|^{2/(n-2)} \delta_{(a,\lambda)}^{2/(n-2)}(x)\delta_{(y,\lambda)}(x)$$

$$\leq c\lambda^{(n-2)/2}\delta_{(a,\lambda)}(y) \int_{\Omega'} |h|^{2/(n-2)} \left(\delta_{(a,\lambda)}^{n/(n-2)} + \delta_{(y,\lambda)}^{n/(n-2)}\right)$$

$$\leq c\|h\|^{2/(n-2)} \delta_{(a,\lambda)}(y).$$

Thus, we derive that

$$\int_{\Omega} \delta_{(a,\lambda)}^{n/(n-2)}(x)|h(x)|^{2/(n-2)} G(x, y)\, dx = I_1 + I_2 + I_3 \leq c\|h\|^{2/(n-2)} \delta_{(a,\lambda)}(y),$$

which completes the proof in this case.

<u>2nd case:</u> $\lambda|y - a| \leq 2$. Note that, in this case, we have $\lambda^{(n-2)/2} \leq c\delta_{(a,\lambda)}(y)$. Observe that

$$L_1 := \int_{B(y,4/\lambda)} \delta_{(a,\lambda)}^{n/(n-2)}(x)|h(x)|^{2/(n-2)} G(x, y)\, dx$$

$$\leq \lambda^{n/2} \int_{B(y,4/\lambda)} \frac{|h|^{2/(n-2)}}{|x - y|^{n-2}}$$

$$\leq c\lambda^{n/2}\|h\|^{2/(n-2)} \frac{c}{\lambda}$$

$$\leq c\|h\|^{2/(n-2)} \delta_{(a,\lambda)}(y).$$

Now, for $x \in \Omega'' := \Omega \setminus B(y, 4/\lambda)$, we get $|x - y| \geq c|a - x|$ and therefore $G(x, y) \leq |x - y|^{2-n} \leq c\lambda^{(n-2)/2}\delta_{(a,\lambda)}(x)$. Thus we derive that

$$L_2 := \int_{\Omega''} \delta_{(a,\lambda)}^{n/(n-2)}(x)|h(x)|^{2/(n-2)} G(x, y) \, dx$$

$$\leq c\lambda^{(n-2)/2} \int |h|^{2/(n-2)} \delta_{(a,\lambda)}^{(2n-2)/(n-2)}(x)$$

$$\leq c\|h\|^{2/(n-2)}\delta_{(a,\lambda)}(y).$$

Hence we obtain

$$\int_\Omega \delta_{(a,\lambda)}^{n/(n-2)}(x)|h(x)|^{2/(n-2)} G(x, y) \, dx = L_1 + L_2 \leq c\|h\|^{2/(n-2)}\delta_{(a,\lambda)}(y),$$

which completes the proof in the second case.

The proof of the second claim (that is for $n \geq 4$) is completed.

For the first claim, that is for $n = 3$, the proof can be obtain by following the previous proof step by step. Hence, we will omit it. $\square$

Let $(\varphi_1, \ldots, \varphi_{N(n+2)})$ be an orthonormal basis of E (which is defined in (1-6)). We consider the PDE

(P) $\quad \begin{cases} -\Delta v = g\left(\min\{|v|, \beta \sum \delta_i\} \operatorname{sign} v\right) \\ \qquad - \sum_{i=1}^{N(n+2)} \left(\int g\left(\min\{|v|, \beta \sum \delta_i\} \operatorname{sign} v\right)\varphi_i\right)(-\Delta\varphi_i) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$

where β is a small positive constant and

$$g(w) := K|\underline{u} + w|^{p-1+\varepsilon}(\underline{u} + w) - \sum \alpha_i \gamma_i \delta_i^p. \tag{2-1}$$

This kind of PDE was originally introduced in [Bahri and Xu 2007] to control the remainder function v by means of the bubbles. The estimate there is very precise compared to our result: we can deduce from it that $|\bar{v}(x)| \leq c \sum \varepsilon_{kj}\delta_j(x)$, where c is a positive constant (this estimate is done for $n = 3$ and $\Omega = \mathbb{R}^3$ concerning the sign-changing Yamabe type problem: $\Delta u + u^5 = 0$ in $\mathbb{R}^3$). Bakri and Xu also considered sign-changing solutions ω of the Yamabe problem instead of $\delta_{(a,\lambda)}$. For more details, see Proposition C, page 14 of [Bahri and Xu 2007].

In our work, we need only to justify that $|v|_\infty^\varepsilon$ is bounded to be able to use the function I defined in (1-8) which we need to be a C^2-function.

To simplify the presentation, for a function $w \in H_0^1(\Omega)$, we set

$$w^* := \min\{|w|, \beta \sum \delta_i\} \operatorname{sign} w.$$

It is easy to see that $|g(v^*)| \le c \sum \delta_i^p$ and therefore (P) has at least one solution, which we will denote by $\bar{v}_{(\alpha,\lambda,a)}$.

Lemma 2.3. *The solution $\bar{v}_{(\alpha,\lambda,a)}$ of* (P) *satisfies*

(i) $\bar{v}_{(\alpha,\lambda,a)} \in E^{\perp}$,

(ii) $\|\bar{v}_{(\alpha,\lambda,a)}\|$

$$\le c\left(\varepsilon + \sum \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right)$$

$$+ c \begin{cases} \sum_i 1/(\lambda_i d_i)^{n-2} + \sum_{i\ne j} \varepsilon_{ij} \ln(\varepsilon_{ij}^{-1})^{(n-2)/n} & \text{if } n \le 5, \\[2mm] \sum_i \ln(\lambda_i d_i)/(\lambda_i d_i)^4 + \sum_{i\ne j} \varepsilon_{ij} \ln(\varepsilon_{ij}^{-1})^{\frac{2}{3}} & \text{if } n = 6, \\[2mm] \sum_i 1/(\lambda_i d_i)^{(n+2)/2} + \sum_{i\ne j} \varepsilon_{ij}^{(n+2)/(2(n-2))} \ln(\varepsilon_{ij}^{-1})^{(n+2)/(2n)} & \text{if } n \ge 7, \end{cases}$$

(iii) $\left| \int_{\Omega} g\left(\min\left\{|\bar{v}_{(\alpha,\lambda,a)}|, \beta \sum \delta_i\right\} \operatorname{sign} \bar{v}_{(\alpha,\lambda,a)}\right) \varphi_i \right| \le c \, \Xi(\varepsilon, \alpha, \lambda, a),$

where

$$\Xi(\varepsilon, \alpha, \lambda, a) := \sum \left(\left|\alpha_j^{p-1} K(a_j) - 1\right| + \varepsilon \ln \lambda_j + \frac{1}{\lambda_j} + \frac{1}{(\lambda_j d_j)^{(n-2)/2}} + \varepsilon_{kr} + \|\bar{v}\| \right).$$

Proof. In the sequel, for sake of simplicity, we will write $\bar{v}$ instead of $\bar{v}_{(\alpha,\lambda,a)}$. For the first claim, we multiply (P) by φ_i and we integrate, we get $\langle \bar{v}, \varphi_i \rangle = 0$ for each i. Thus, the proof of claim (i) follows.

Concerning (ii), multiplying (P) by $\bar{v}$ and integrating on Ω, using claim (i), we get

$$\|\bar{v}\|^2 = \int_{\Omega} K |\underline{u} + \bar{v}^*|^{p-1+\varepsilon}(\underline{u} + \bar{v}^*)\bar{v}$$

$$= \int_{\Omega} K |\underline{u}|^{p-1+\varepsilon}\underline{u}\bar{v} + (p+\varepsilon)\int_{\Omega} K |\underline{u}|^{p-1+\varepsilon}\bar{v}^*\bar{v} + O\left(\int |\bar{v}^*|^p |\bar{v}|\right)$$

$$+ O\left(\int |\underline{u}|^{p-2+\varepsilon}(\bar{v}^*)^2\bar{v} \quad (\text{if } n \le 6)\right)$$

$$\le \int K |\underline{u}|^{p-1+\varepsilon}\underline{u}\bar{v} + p\sum \alpha_i^{p-1+\varepsilon} \int K P \delta_i^{p-1+\varepsilon}\bar{v}^2 + \sum_{i\ne j} O\left(\delta_i^{p-2} \inf(\delta_i, \delta_j)\bar{v}^2\right)$$

$$+ \sum O\left(\left(\beta\int \delta_i^{p-1}\bar{v}^2 \quad (\text{if } n \le 6)\right) + (\beta^{p-1} + \varepsilon)\int \delta_i^{p-1}\bar{v}^2\right)$$

$$\le \int K |\underline{u}|^{p-1+\varepsilon}\underline{u}\bar{v} + p\sum \alpha_i^{p-1+\varepsilon} \int K P \delta_i^{p-1+\varepsilon}\bar{v}^2 + c\|\bar{v}\|^2(\beta + \beta^{p-1} + o(1)).$$

Now, observe that

$$\alpha_i^{p-1+\varepsilon} \int K P \delta_i^{p-1+\varepsilon} \bar{v}^2$$

$$= \alpha_i^{p-1+\varepsilon} K(a_i) \int \delta_i^{p-1} \bar{v}^2 + O\left(\int \left(|x - a_i| \delta_i^{p-1} + \delta_i^{p-2} \theta_i \right) \bar{v}^2 + \int \delta_i^{p-1} |\delta_i^\varepsilon - 1| \bar{v}^2 \right)$$

$$= \int \delta_i^{p-1} \bar{v}^2 + o(\|\bar{v}\|^2),$$

by using the fact that $|\alpha_i^{p-1} K(a_i) - 1| \le \mu$, $|\delta_i^\varepsilon - 1| \le c\varepsilon \ln \lambda_i \le c\mu$ (μ is a small positive constant introduced in Theorem 1.1) and where $\theta_i := \delta_i - P\delta_i$. Therefore, we get

$$\|\bar{v}\|^2 - p \sum_{i=1}^N \int \delta_i^{p-1} \bar{v}^2 - c \|\bar{v}\|^2 \left(\beta + \beta^{p-1} + o(1) \right) \le \int K |\underline{u}|^{p-1+\varepsilon} \underline{u} \bar{v}.$$

Claim (ii) follows from Lemmas 2.1, A.1 and A.2 (by choosing β small so that $c(\beta + \beta^{p-1}) \le \beta_0/4$, where β_0 is defined in Lemma A.2).

It remains to prove (iii). Let $\bar{v}^* := \min\{|\bar{v}|, \beta \sum \delta_i\}$ sign $\bar{v}$, we have

$$g(\bar{v}^*) = K |\underline{u} + \bar{v}^*|^{p-1+\varepsilon} (\underline{u} + \bar{v}^*) - \sum \alpha_i \gamma_i \delta_i^p$$

$$= K |\underline{u}|^{p-1+\varepsilon} \underline{u} + \mathcal{O}\left(|\underline{u}|^{p-1+\varepsilon} |\bar{v}^*| + |\bar{v}^*|^{p+\varepsilon} \right) - \sum \alpha_i \gamma_i \delta_i^p$$

$$= K \sum \alpha_i^{p+\varepsilon} \gamma_i P \delta_i^{p+\varepsilon} + \mathcal{O}\left(\sum_{i \ne j} \delta_i^{p-1} (\delta_j + |\bar{v}^*|) + |\bar{v}^*|^{p+\varepsilon} \right) - \sum \alpha_i \gamma_i \delta_i^p$$

$$= \sum K(a_i) \alpha_i^{p+\varepsilon} \gamma_i \delta_i^{p+\varepsilon} + \mathcal{O}\left(\sum |x - a_i| \delta_i^p + \sum \delta_i^{p-1} \theta_i \right)$$

$$+ \mathcal{O}\left(\sum_{i \ne j} \delta_i^{p-1} \delta_j + \sum \delta_i^{p-1} |\bar{v}^*| + |\bar{v}^*|^{p+\varepsilon} \right) - \sum \alpha_i \gamma_i \delta_i^p, \quad (2\text{-}2)$$

where $\theta_i := \delta_i - P\delta_i$. Note that $|\alpha_i^\varepsilon \delta_i^\varepsilon - 1| \le c\varepsilon \ln \lambda_i$. Therefore we get

$$K(a_i) \alpha_i^{p+\varepsilon} \gamma_i \delta_i^{p+\varepsilon} - \alpha_i \gamma_i \delta_i^p = \alpha_i \gamma_i \delta_i^p \left(\alpha_i^\varepsilon (\alpha_i^{p-1} K(a_i)) \delta_i^\varepsilon - 1 \right)$$

$$= \mathcal{O}\left(\left(\varepsilon \ln \lambda_i + |\alpha_i^{p-1} K(a_i) - 1| \right) \delta_i^p \right). \quad (2\text{-}3)$$

Hence, since $\int \delta_k^{p+1} \le c$ and $\|\varphi_k\| = 1$ for each k, by using the Holder's inequality, easy computations imply

$$\left| \int g(\bar{v}^*) \varphi_i \right| \le c \sum \left(\varepsilon \ln \lambda_j + |\alpha_j^{p-1} K(a_j) - 1| + \frac{1}{\lambda_j} + \|\theta_j\| + \sum \varepsilon_{kr} + \|\bar{v}\| \right).$$

The proof of claim (iii) follows from the fact that $\|\theta_i\| \le c/(\lambda_i d_i)^{(n-2)/2}$ (see [Bahri 1989]). $\square$

Proposition 2.4. *The solution $\bar{v}$ of* (P) *satisfies*

(i)
$$|\bar{v}(y)| \le c \begin{cases} (\Xi(\varepsilon, \alpha, \lambda, a) + \Xi_1) \sum \delta_i(y) & \text{if } n \ge 4, \\ (\Xi(\varepsilon, \alpha, \lambda, a) + \Xi_2) \sum \delta_i(y) & \text{if } n = 3, \end{cases}$$

where $\Xi(\varepsilon, \alpha, \lambda, a)$ is defined in Lemma 2.3 and

$$\Xi_1 := \sum \frac{1}{\lambda_i d_i} + \varepsilon_{ij}^{1/(n-2)} \ln^{1/n}(\varepsilon_{ij}^{-1}) + \frac{\ln^{1/n} \lambda_i}{\lambda_i} + \beta^{(n-4)/(n-2)} \|\bar{v}\|^{2/(n-2)}$$

$$\Xi_2 := \sum \frac{1}{\sqrt{\lambda_i d_i}} + \varepsilon_{ij}^{\frac{1}{2}} \ln^{\frac{1}{6}}(\varepsilon_{ij}^{-1}) + \frac{1}{\sqrt{\lambda_i}} + \|\bar{v}\|.$$

Choosing μ (introduced in Theorem 1.1) small with respect to β ($\mu^{\sigma_0} \le \beta$ for some positive constant σ_0), we derive that

(ii)
$$|\bar{v}(y)| \le \beta \sum \delta_{(a_i, \lambda_i)}(y) \quad \text{for each } y \in \Omega.$$

Proof. Let G be the Green's function for the Laplace operator with Dirichlet condition on the boundary. For $y \in \Omega$, it satisfies $-\Delta G(y, \cdot) = c'\delta_y$, where δ_y denotes the Dirac mass at the point y, $c' := 1/(n(n-2)\omega_n)$ and ω_n denotes the volume of unit ball of $\mathbb{R}^n$.

Since $\bar{v}$ is a solution of (P), then, for each $y \in \Omega$, we have

$$\bar{v}(y) = \frac{1}{c'} \int_\Omega g(\bar{v}^*)(x) G(x, y) \, dx - \sum \left(\int_\Omega g(\bar{v}^*)(x) \varphi_k(x) dx \right) \varphi_k(y).$$

Using claim (iii) of Lemma 2.3 and the fact that $\varphi_k \in E$ (which implies, by easy computations, that $\varphi_k(y) \le c \sum \delta_i(y)$), we derive that

$$\left(\int_\Omega g(\bar{v}^*)(x) \varphi_k(x) \, dx \right) \varphi_k(y) = \mathcal{O}\left(\Xi(\varepsilon, \alpha, \lambda, a) \sum \delta_i(y) \right).$$

Concerning the other integral, using (2-2) and (2-3) we derive

$$\int_\Omega g(\bar{v}^*)(x) G(x, y) \, dx$$

$$= \sum \mathcal{O}\left(\left(|\alpha_i^{p-1} K(a_i) - 1| + \varepsilon \ln \lambda_i \right) \int_\Omega G(x, y) \delta_i^p(x) \right.$$

$$+ \int_\Omega G(x, y) |x - a_i| \delta_i^p + \int_\Omega G(x, y) \delta_i^{p-1} \theta_i + \int_\Omega G(x, y) \delta_i^{p-1} |\bar{v}^*|$$

$$\left. + \int_\Omega G(x, y) |\bar{v}^*|^{p+\varepsilon} + \sum_{i \ne j} \int_\Omega G(x, y) \delta_i^{p-1} \delta_j \right).$$

Observe that

$$\int_\Omega G(x, y) \delta_i^p(x) = c' P\delta_i(y),$$

and, for $n \geq 4$, using the second claim of Lemma 2.2, easy computations imply that

$$\int \delta_i^{4/(n-2)} \theta_i \, G(x, y) \leq \int \delta_i^{n/(n-2)} \theta_i^{2/(n-2)} G(x, y) \leq c \|\theta_i\|^{2/(n-2)} \delta_i(y)$$

$$\leq \frac{1}{\lambda_i d_i} \delta_i(y),$$

$$\int \delta_i^{4/(n-2)} \delta_j \, G(x, y) \leq \int (\delta_i^{n/(n-2)} + \delta_j^{n/(n-2)})(\sqrt{\delta_i \delta_j})^{2/(n-2)} G(x, y) \qquad (2\text{-}4)$$

$$\leq \varepsilon_{ij}^{1/(n-2)} \ln^{1/n}(\varepsilon_{ij}^{-1})(\delta_i(y) + \delta_j(y)),$$

$$\int \delta_i^{(n+2)/(n-2)} |x - a_i| \, G(x, y) = \int \delta_i^{n/(n-2)} (\delta_i |x - a_i|^{(n-2)/2})^{2/(n-2)} G(x, y)$$

$$\leq (\ln^{1/n} \lambda_i / \lambda_i) \delta_i(y),$$

and, for $n = 3$, using the first claim of Lemma 2.2, we have

$$\int \delta_i^4 \theta_i \, G(x, y) \leq c \|\theta_i\| \delta_i(y) \leq c \delta_i(y)/\sqrt{\lambda_i d_i},$$

$$\int \delta_i^4 \delta_j \, G(x, y) \leq \int \delta_i^{\frac{7}{2}} \delta_j^{\frac{1}{2}} (\delta_i \delta_j)^{\frac{1}{2}} G(x, y)$$

$$\leq \int (\delta_i^4 + \delta_j^4)(\delta_i \delta_j)^{\frac{1}{2}} G(x, y)$$

$$\leq c \varepsilon_{ij}^{\frac{1}{2}} \ln^{\frac{1}{6}}(\varepsilon_{ij}^{-1})(\delta_i(y) + \delta_j(y)),$$

$$\int \delta_i^5 |x - a_i| \, G(x, y) = \int \delta_i^4 (\delta_i |x - a_i|) G(x, y) \leq \delta_i(y)/\sqrt{\lambda_i}.$$

Concerning the other integrals, recall that $|\bar{v}^*| = \min\{|\bar{v}|, \beta \sum \delta_i\}$, thus we get, for $n \geq 4$,

$$\int \delta_i^{4/(n-2)} |\bar{v}^*| \, G(x, y) \leq \beta^{(n-4)/(n-2)} \sum \int \delta_j^{n/(n-2)} |\bar{v}|^{2/(n-2)} G(x, y)$$

$$\leq c \beta^{(n-4)/(n-2)} \|\bar{v}\|^{2/(n-2)} \sum \delta_j(y),$$

$$\int_\Omega |\bar{v}^*|^{p+\varepsilon} G(x, y) \leq \beta^{n/(n-2)+\varepsilon} \sum \int \delta_j^{n/(n-2)} |\bar{v}|^{2/(n-2)} G(x, y)$$

$$\leq c \beta^{n/(n-2)} \|\bar{v}\|^{2/(n-2)} \sum \delta_j(y),$$

and for $n = 3$ we have

$$\int \delta_i^4 |\bar{v}^*| \, G(x, y) \leq \int \delta_i^4 |\bar{v}| G(x, y) \leq c \|\bar{v}\| \sum \delta_j(y),$$

$$\int_\Omega |\bar{v}^*|^{p+\varepsilon} G(x, y) \, dx \leq \beta^{4+\varepsilon} \sum \int \delta_j^4 |\bar{v}| G(x, y) \leq c \beta^4 \|\bar{v}\| \sum \delta_j(y),$$

by using Lemma 2.2. Combining the previous estimates, the proof of claim (i) follows.

Concerning claim (ii), since $(\alpha, \lambda, a) \in \mathcal{V}(N, \mu)$ (see its definition in (1-3)), we derive that all the terms involved in $\Xi(\varepsilon, \alpha, \lambda, a)$, Ξ_1 and Ξ_2 are dominated by μ^σ (for some positive constant σ). Hence, there exists a positive constant σ_0 such that $|\bar{v}(y)| \leq \mu^{\sigma_0} \sum \delta_j(y)$. Thus, choosing μ small enough with respect to β ($\mu^{\sigma_0} \leq \beta$) we derive the desired estimated.   □

Corollary 2.5. *The solution $\bar{v}$ of* (P) *satisfies*

$$(\mathrm{P}_v) \qquad \begin{cases} -\Delta \bar{v} = g(\bar{v}) - \sum_{i=1}^{N(n+2)} \left(\int_\Omega g(\bar{v}) \varphi_i \right)(-\Delta \varphi_i) & in\ \Omega, \\ \qquad \bar{v} = 0 & on\ \partial\Omega. \end{cases}$$

Proof. This follows from Proposition 2.4, which implies that $\bar{v}^* = \bar{v}$.   □

Lemma 2.6. *Let $(\alpha, \lambda, a) \in \mathcal{V}(N, \mu)$. Then, there exists a unique solution $\bar{v}_{(\alpha,\lambda,a)}$ of* (P_v).

Proof. Assume that there exist two different solutions $\bar{v}_1$ and $\bar{v}_2$ of (P_v). By claim (ii) of Lemma 2.3, we have that $\|\bar{v}_i\|$ is small for $i = 1, 2$. Furthermore, since $\bar{v}_i$ belongs to $E^\perp$ (by claim (i) of Lemma 2.3), it is easy to derive that

$$\|\bar{v}_1 - \bar{v}_2\|^2 = \int_\Omega (g(\bar{v}_1) - g(\bar{v}_2))(\bar{v}_1 - \bar{v}_2).$$

Note that

$$g(\bar{v}_1) - g(\bar{v}_2)$$
$$= K|\underline{u} + \bar{v}_1|^{p-1+\varepsilon}(\underline{u} + \bar{v}_1) - K|\underline{u} + \bar{v}_2|^{p-1+\varepsilon}(\underline{u} + \bar{v}_2)$$
$$= K|w + (\bar{v}_1 - \bar{v}_2)|^{p-1+\varepsilon}(w + (\bar{v}_1 - \bar{v}_2)) - K|w|^{p-1+\varepsilon}w \qquad \text{(with } w = \underline{u} + \bar{v}_2)$$
$$= (p+\varepsilon)K|w|^{p-1+\varepsilon}(\bar{v}_1 - \bar{v}_2) + O\big(|(\bar{v}_1 - \bar{v}_2)|^{p+\varepsilon} + |w|^{p-2+\varepsilon}|(\bar{v}_1 - \bar{v}_2)|^2\big) \quad \text{(if } n \leq 6)$$

by using the formula

$$\begin{cases} |a + x|^\alpha(a + x) - |a|^\alpha a = (\alpha + 1)|a|^\alpha x + O(|a|^{\alpha-1}x^2 + |x|^{\alpha+1}) & \text{(if } \alpha \geq 1), \\ |a + x|^\alpha(a + x) - |a|^\alpha a = (\alpha + 1)|a|^\alpha x + O(|x|^{\alpha+1}) & \text{(if } 0 < \alpha < 1). \end{cases}$$

Thus we obtain

$$\|\bar{v}_1 - \bar{v}_2\|^2 = (p + \varepsilon) \int K|\underline{u} + \bar{v}_2|^{p-1+\varepsilon}|(\bar{v}_1 - \bar{v}_2)|^2$$

$$+ O\left(\int |(\bar{v}_1 - \bar{v}_2)|^{p+1+\varepsilon} + \int |\underline{u} + \bar{v}_2|^{p-2+\varepsilon}|(\bar{v}_1 - \bar{v}_2)|^3 \quad \text{(if } n \leq 6)\right)$$

$$= p \int K |\underline{u}|^{p-1+\varepsilon} (\bar{v}_1 - \bar{v}_2)^2 + o(\|\bar{v}_1 - \bar{v}_2\|^2)$$

$$= p \sum \alpha_i^{p-1+\varepsilon} K(a_i) \int P\delta_i^{p-1+\varepsilon} (\bar{v}_1 - \bar{v}_2)^2 + o(\|\bar{v}_1 - \bar{v}_2\|^2)$$

$$= p \sum \int \delta_i^{p-1} (\bar{v}_1 - \bar{v}_2)^2 + o(\|\bar{v}_1 - \bar{v}_2\|^2),$$

which implies that

$$Q(\bar{v}_1 - \bar{v}_2) = o(\|\bar{v}_1 - \bar{v}_2\|^2), \tag{2-5}$$

where Q is defined in Lemma A.2. Notice that Q is a positive definite quadratic form on $E^\perp$ (by Lemma A.2) and $\bar{v}_1 - \bar{v}_2 \in E^\perp$. Hence (2-5) gives a contradiction. Thus, the proof follows. □

Proof of Theorem 1.1. From Corollary 2.5 and Lemma 2.6, we derive that the problem (P_v) has a unique solution $\bar{v}$. Recall that $(\varphi_1, \ldots, \varphi_{N(n+2)})$ is an orthonormal basis of E. Therefore, each φ_k can be written as a linear combination of the functions $P\delta_i$, $\lambda_i \partial P\delta_i / \partial \lambda_i$ and $\lambda_i^{-1} \partial P\delta_i / \partial (a_i)_j$. Thus we obtain that

$$\sum_{i=1}^{N} \left(\int_\Omega g(\bar{v}) \varphi_i \right) \varphi_i = \sum_{i=1}^{N} \left(A_i P\delta_i + B_i \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} + \sum_{k=1}^{n} \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_k} \right).$$

Now, using claim (iii) of Lemma 2.3, we derive that

$$|A| + |B| + |C| := \sum_{i=1}^{N} (|A_i| + |B_i| + \sum_{k=1}^{n} |C_{ik}|) \le c \Xi(\varepsilon, \alpha, \lambda, a), \tag{2-6}$$

which implies that it is very small (since $\Xi(\varepsilon, \alpha, \lambda, a) \le \mu^\sigma$ for some positive constant σ). Finally, the proof of the first statement of the theorem follows from the fact that $\bar{v}$ is a solution of (P_v) and (2-1).

Concerning the second statement, claim (i) follows from (ii) of Lemma 2.3 and claim (ii) follows from Proposition 2.4. Finally, claim (iii) follows from claim (ii) and the facts that $\delta_i^\varepsilon \le 1 + c\varepsilon \ln \lambda_i$ and $\varepsilon \ln \lambda_i$ is very small since (α, λ, a) is in $\mathcal{V}(N, \mu)$ (see (1-3)).

Hence the proof of the theorem is completed. □

Proof of Proposition 1.2. Notice that, the restriction of J on the space $\mathcal{H} := H_0^1(\Omega) \cap \{u : |u|_\infty^\varepsilon \le M\}$, for a fixed constant M, is a C^1-functional. In fact, for $u, h \in \mathcal{H}$, it is easy to see that

$$J(u + h) = J(u) + \langle u, h \rangle - \int K |u|^{p-1+\varepsilon} uh + O(\|h\|^2)$$

which implies that $J_{|\mathcal{H}}$ is a C^1 functional and we have

$$DJ(u)(h) = \langle u, h \rangle - \int K |u|^{p-1+\varepsilon} uh.$$

Note that, for each $(\alpha, \lambda, a) \in \mathcal{V}(N, \mu)$, we have

$$\bar{u} := \sum \alpha_i \gamma_i P\delta_i + \bar{v} \in \mathcal{H}.$$

Hence, I (defined in (1-8)) is a C^1 function on $\mathcal{V}(N, \mu)$ and therefore, $(\alpha, \lambda, a) \in \mathcal{V}(N, \mu)$ is a critical point of I if and only if

$$\begin{cases} (E_{\alpha_i}): & 0 = (\partial I/\partial \alpha_i)(\alpha, \lambda, a) = \langle \bar{u}, \partial \bar{u}/\partial \alpha_i \rangle - \int K |\bar{u}|^{p-1+\varepsilon} \bar{u} \partial \bar{u}/\partial \alpha_i, \\ (E_{\lambda_i}): & 0 = (\partial I/\partial \lambda_i)(\alpha, \lambda, a) = \langle \bar{u}, \partial \bar{u}/\partial \lambda_i \rangle - \int K |\bar{u}|^{p-1+\varepsilon} \bar{u} \partial \bar{u}/\partial \lambda_i, \\ (E_{(a_i)_k}): & 0 = (\partial I/\partial (a_i)_k)(\alpha, \lambda, a) = \langle \bar{u}, \partial \bar{u}/\partial (a_i)_k \rangle - \int K |\bar{u}|^{p-1+\varepsilon} \bar{u} \partial \bar{u}/\partial (a_i)_k. \end{cases}$$

Recall that $\bar{u} := \underline{u} + \bar{v} := \sum \alpha_j \gamma_j P\delta_j + \bar{v}$. Thus we derive that

$$\partial \bar{u}/\partial \alpha_i = \gamma_i P\delta_i + \partial \bar{v}/\partial \alpha_i,$$

$$\partial \bar{u}/\partial \lambda_i = \alpha_i \gamma_i \partial P\delta_i/\partial \lambda_i + \partial \bar{v}/\partial \lambda_i, \quad \text{and}$$

$$\partial \bar{u}/\partial (a_i)_k = \alpha_i \gamma_i \partial P\delta_i/\partial (a_i)_k + \partial \bar{v}/\partial (a_i)_k.$$

Now, we will focus on the equation (E_{λ_i}) and the other ones can be obtained in the same way. Observe that, (E_{λ_i}) is equivalent to

$$0 = \left\langle \underline{u}, \alpha_i \gamma_i \frac{\partial P\delta_i}{\partial \lambda_i} \right\rangle - \int K |\bar{u}|^{p-1+\varepsilon} \bar{u} \alpha_i \gamma_i \frac{\partial P\delta_i}{\partial \lambda_i}$$

$$+ \left\langle \bar{u}, \frac{\partial \bar{v}}{\partial \lambda_i} \right\rangle - \int K |\bar{u}|^{p-1+\varepsilon} \bar{u} \frac{\partial \bar{v}}{\partial \lambda_i} + \left\langle \bar{v}, \alpha_i \gamma_i \frac{\partial P\delta_i}{\partial \lambda_i} \right\rangle. \quad (2\text{-}7)$$

Since $\bar{v} \in E^{\perp}$, we derive that the last term is zero. Now, recall that, from Theorem 1.1, we have

$$-\Delta \bar{u} = K |\bar{u}|^{p-1+\varepsilon} \bar{u} - \sum_{i=1}^{N} \Delta \left(A_i P\delta_i + B_i \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} + \sum_{k=1}^{n} C_{ik} \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial (a_i)_k} \right)$$

which implies that

$$\left\langle \bar{u}, \frac{\partial \bar{v}}{\partial \lambda_i} \right\rangle - \int K |\bar{u}|^{p-1+\varepsilon} \bar{u} \frac{\partial \bar{v}}{\partial \lambda_i}$$

$$= \sum_{j=1}^{N} \left\langle A_j P\delta_j + B_j \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j} + \sum_{k=1}^{n} C_{jk} \frac{1}{\lambda_j} \frac{\partial P\delta_j}{\partial (a_j)_k}, \frac{\partial \bar{v}}{\partial \lambda_i} \right\rangle. \quad (2\text{-}8)$$

Using again the fact that $\bar{v} \in E^\perp$, we get

$$\langle \bar{v}, P\delta_i \rangle = 0 \quad \Rightarrow \quad \langle \partial \bar{v}/\partial \lambda_i, P\delta_i \rangle + \langle \bar{v}, \partial P\delta_i/\partial \lambda_i \rangle = 0$$

$$\Rightarrow \quad \langle \partial \bar{v}/\partial \lambda_i, P\delta_i \rangle = 0,$$

$$\langle \bar{v}, P\delta_j \rangle = 0 \quad \Rightarrow \quad \langle \partial \bar{v}/\partial \lambda_i, P\delta_j \rangle = 0 \quad \text{for each } j \neq i,$$

$$\langle \bar{v}, \partial P\delta_i/\partial \lambda_i \rangle = 0 \quad \Rightarrow \quad \langle \partial \bar{v}/\partial \lambda_i, \partial P\delta_i/\partial \lambda_i \rangle = -\langle \bar{v}, \partial^2 P\delta_i/\partial \lambda_i^2 \rangle,$$

$$\langle \bar{v}, \partial P\delta_j/\partial \lambda_j \rangle = 0 \quad \Rightarrow \quad \langle \partial \bar{v}/\partial \lambda_i, \partial P\delta_j/\partial \lambda_j \rangle = 0 \quad \text{for each } j \neq i,$$

$$\langle \bar{v}, \partial P\delta_i/\partial (a_i)_l \rangle = 0 \quad \Rightarrow \quad \langle \partial \bar{v}/\partial \lambda_i, \partial P\delta_i/\partial (a_i)_l \rangle = -\langle \bar{v}, \partial^2 P\delta_i/\partial (a_i)_l \partial \lambda_i \rangle,$$

$$\langle \bar{v}, \partial P\delta_j/\partial (a_j)_l \rangle = 0 \quad \Rightarrow \quad \langle \partial \bar{v}/\partial \lambda_i, \partial P\delta_j/\partial (a_j)_l \rangle = 0 \quad \text{for each } j \neq i.$$

Therefore we obtain

$$\left\langle \bar{u}, \frac{\partial \bar{v}}{\partial \lambda_i} \right\rangle - \int K|\bar{u}|^{p-1+\varepsilon} \bar{u} \frac{\partial \bar{v}}{\partial \lambda_i}$$

$$= -B_i \left\langle \bar{v}, \lambda_i \frac{\partial^2 P\delta_i}{\partial \lambda_i^2} \right\rangle - \sum_{k=1}^{n} C_{ik} \left\langle \bar{v}, \frac{1}{\lambda_i} \frac{\partial^2 P\delta_i}{\partial (a_i)_k \partial \lambda_i} \right\rangle. \quad (2\text{-}9)$$

Combining (2-7), (2-9) and the fact that $\bar{v} \in E^\perp$ (which implies that the last term of (2-7) is zero), the estimate of the equation (E_{λ_i}) follows.

The equation $(E_{(a_i)_k})$ can be obtain exactly in the same way by taking $\partial P\delta_i/\partial (a_i)_k$ and $\partial \bar{v}/\partial (a_i)_k$ instead of $\partial P\delta_i/\partial \lambda_i$ and $\partial \bar{v}/\partial \lambda_i$ respectively. Concerning the equation (E_{α_i}), we notice that $\partial \bar{v}/\partial \alpha_i$ belongs to $E^\perp$ (following the computations of the previous system) and therefore the terms in the right hand side of the analog of (2-8) are zero. This completes the proof. $\qquad\square$

Appendix

In this appendix, we collect two known results. The first one is extracted from [Ben Ayed et al. 2003] (see Equation (2.26)).

Lemma A.1. *Let $v \in H_0^1(\Omega)$ be such that $\langle v, P\delta_{a_i,\lambda_i} \rangle = 0$. Then we derive that*

$$\left| \int_\Omega P\delta_{(a_i,\lambda_i)}^{p+\varepsilon} v \right| \leq C\varepsilon \|v\| + \begin{cases} C\|v\| 1/(\lambda_i d_i)^{\min\{n-2,(n+2)/2\}} & \text{if } n \neq 6, \\ C\|v\| \ln(\lambda_i d_i)/(\lambda_i d_i)^4 & \text{if } n = 6, \end{cases}$$

where $d_i := d(a_i, \partial\Omega)$ and C is a positive constant independent of ε, a_i and λ_i.

The second lemma is extracted from [Bahri 1989, Proposition 3.1]

Lemma A.2. *There exists a positive constant β_0 such that*

$$Q(v) := \|v\|^2 - \frac{n+2}{n-2} \sum_{i=1}^{N} \int_\Omega \delta_i^{4/(n-2)} v^2 \geq \beta_0 \|v\|^2 \quad \text{for each } v \in E^\perp,$$

where E is defined in (1-6).

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Potentially good reduction loci
of Shimura varieties

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We give a notion of the potentially good reduction locus of a Shimura variety. It consists of the points which should be related with motives having potentially good reductions in some sense. We show the existence of such locus for a Shimura variety of preabelian type. Further, we construct a partition of the adic space associated to a Shimura variety of preabelian type, which is expected to describe degenerations of motives. Using this partition, we prove that the cohomology of the potentially good reduction locus is isomorphic to the cohomology of a Shimura variety up to nonsupercuspidal parts.

1. Introduction

Let (G, X) be a Shimura datum, and $\mathrm{Sh}_K(G, X)$ the Shimura variety attached to (G, X) and a compact open subgroup K of $G(\mathbb{A}^\infty)$. It is known to be defined over a number field E, called the reflex field, which is canonically determined by (G, X). We fix a prime number p and a place v of E above p, and write E_v for the completion of E at v. The main theme of this article is the "potentially good reduction locus" of $\mathrm{Sh}_K(G, X)_{E_v} = \mathrm{Sh}_K(G, X) \otimes_E E_v$.

To explain what this locus is, let us first assume that (G, X) is of PEL type, in which case $\mathrm{Sh}_K(G, X)$ parametrizes abelian varieties with additional PEL structures. We denote by $\mathcal{A}$ the universal abelian scheme over $\mathrm{Sh}_K(G, X)$. If moreover the PEL datum is unramified at p and $K = K_{p,0}K^p$ where $K_{p,0}$ is hyperspecial, by extending the moduli problem to $\mathcal{O}_{E_v}$, we can obtain a good integral model $\mathcal{S}_{K^p}$ of $\mathrm{Sh}_K(G, X)$ over $\mathcal{O}_{E_v}$ (see [Kottwitz 1992b]). This model is quite important in the study of the ℓ-adic cohomology of Shimura varieties; see [Kottwitz 1992a] for instance. Let us denote by $\mathcal{S}_{K^p}^\wedge$ the formal completion of $\mathcal{S}_{K^p}$ along its special fiber, and by $\mathcal{S}_{K^p}^{\wedge\mathrm{rig}}$ the rigid generic fiber of it. Then, $\mathcal{S}_{K^p}^{\wedge\mathrm{rig}}$ is naturally identified with a quasicompact rigid-analytic open subset of $\mathrm{Sh}_K(G, X)_{E_v}$. For a finite extension F of E_v, an F-valued point x of $\mathrm{Sh}_K(G, X)_{E_v}$ lies in $\mathcal{S}_{K^p}^{\wedge\mathrm{rig}}$ if and only if the abelian variety $\mathcal{A}_x$ over F has (potentially) good reduction. In this sense, $\mathcal{S}_{K^p}^{\wedge\mathrm{rig}}$

can be considered as the locus over which $\mathcal{A}$ has (potentially) good reduction. By this reason, we will write $\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{E_v} = \mathscr{S}^{\wedge \mathrm{rig}}_{K^p}$, and call it the potentially good reduction locus. We also have a rigid-analytic open subspace $\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{E_v}$ of $\mathrm{Sh}_K(G, X)_{E_v}$ for a compact open subgroup K whose p-part is smaller than $K_{p,0}$, by taking the inverse image. The ℓ-adic cohomology of $\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{E_v}$ can be computed by using the cohomology of the nearby cycle complex, provided that $\mathrm{Sh}_K(G, X)_{E_v}$ has a suitable integral model over $\mathcal{O}_{E_v}$.

In this paper, we will introduce the notion of the potentially good reduction locus for a general Shimura variety. The rough idea is as follows. Let us fix a prime number ℓ. In the PEL type case, let $\mathcal{L}$ be the ℓ-adic automorphic étale sheaf on $\mathrm{Sh}_K(G, X)$ attached to the standard representation of G. Then, for a finite extension F of E_v and an F-valued point x of $\mathrm{Sh}_K(G, X)$, the stalk $\mathcal{L}_{\bar{x}}$ can be identified with the rational ℓ-adic Tate module $V_\ell \mathcal{A}_{\bar{x}}$. By the Neron–Ogg–Shafarevich criterion, we conclude that $\mathcal{A}_x$ has potentially good reduction if and only if $\mathcal{L}_{\bar{x}}$ is potentially unramified (resp. potentially crystalline) when $\ell \neq p$ (resp. $\ell = p$). This observation urges us to define in the general case that an F-valued point x of $\mathrm{Sh}_K(G, X)_{E_v}$ is of potentially good reduction if $\mathcal{L}_{\bar{x}}$ is potentially unramified/crystalline for every automorphic étale sheaf $\mathcal{L}$. Actually in the paper, we look at the torsor over x obtained as the pull-back of $\varprojlim_{K' \subset K} \mathrm{Sh}_{K'}(G, X)_{E_v} \to \mathrm{Sh}_K(G, X)_{E_v}$, which is more concise but essentially equivalent to the above way by the Tannakian duality. Our potentially good reduction locus is defined as a quasicompact open subset of the adic space $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$ attached to $\mathrm{Sh}_K(G, X)_{E_v}$, whose F-valued points consist of those of potentially good reduction for every F. It is unique, if exists. We will show the existence of the potentially good reduction locus $\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{E_v}$ when the Shimura datum (G, X) is of preabelian type. Recall that (G, X) is said to be of preabelian type if there exists a Shimura datum (G', X') of Hodge type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$. This class contains almost all Shimura data in practice. As in [Deligne 1979, Introduction] and [Milne 2005, §9], a Shimura variety is believed to have a moduli interpretation by motives, if the weight homomorphism for (G, X) is defined over the rational number field. The subset $\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{E_v}$ is expected to parametrize motives with potentially good reduction at v.

We are also interested in what happens outside the locus $\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{E_v}$. In the PEL type case, degenerations of abelian varieties occur; if a Shimura variety parametrizes motives, then degenerations of motives should occur. Based on this observation, we will construct a partition of $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$ into finitely many locally closed constructible subsets labeled by conjugacy classes of certain kind of adelic parabolic subgroups of G, so that the piece corresponding to G equals $\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{E_v}$. It is closely related to the theory of integral toroidal compactifications. Actually, in the PEL type case, we may also use the integral toroidal

compactification developed in [Lan 2013] to construct our partition (see [Imai and Mieda 2013, §7]); there should be some more cases to which the method in [Imai and Mieda 2013, §7] can be applied (for example, [Madapusi Pera 2019]). However, our argument here is almost totally rigid-geometric, and requires only the existence of the integral toroidal compactifications of the Siegel modular varieties with hyperspecial level at p ([Faltings and Chai 1990]) as an input from the integral theory. Note also that our partition is independent of any choice, unlike the toroidal compactification that depends on the choice of a cone decomposition.

By using the partition above, we can compare the ℓ-adic cohomology of the tower $\{\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{\bar{E}_v}\}_K$ and that of $\{\mathrm{Sh}_K(G, X)\}_K$. We assume that (G, V) is of preabelian type and satisfies the condition SV6 in [Milne 2005, p. 311]. Let ℓ be a prime number different from p, and $\mathcal{L}$ an ℓ-adic automorphic étale sheaf on $\mathrm{Sh}_K(G, X)$ corresponding to an algebraic representation of G^c over $\overline{\mathbb{Q}}_\ell$, where G^c is the quotient of G defined in [Milne 1990, p. 347]. The statement is as follows:

Theorem 6.1. *In the kernel and the cokernel of the natural map*

$$\varinjlim_K H^i_c(\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{\bar{E}_v}, \mathcal{L}) \to \varinjlim_K H^i_c(\mathrm{Sh}_K(G, X)_{\bar{E}_v}, \mathcal{L}),$$

no irreducible supercuspidal representation of $G(\mathbb{Q}_{p'})$ appears as a subquotient for any prime number p'.

Recall that an irreducible smooth representation of $G(\mathbb{Q}_{p'})$ is said to be supercuspidal if it does not appear as a subquotient of the parabolically induced representations from any proper parabolic subgroup. Loosely speaking, this theorem is a consequence of the observation that the partition of the complement $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v} \setminus \mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{E_v}$ is "geometrically induced" from proper parabolic subgroups of $G(\mathbb{Q}_{p'})$. It will be worth noting that our method is totally geometric, so that it is also valid in the torsion coefficient case. See Theorem 6.12 for an analogue for the $\overline{\mathbb{F}}_\ell$-coefficients.

We have already mentioned that in the PEL type case the ℓ-adic cohomology $H^i_c(\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{\bar{E}_v}, \mathcal{L})$ can be computed as the cohomology of a nearby cycle complex. Hence, in this case the theorem above says that the nearby cycle cohomology is isomorphic to the compactly supported cohomology up to nonsupercuspidal representations. This result is useful, since it connects the cohomology of Shimura varieties and that of Rapoport–Zink spaces; see Section 7.3 for a simple example in this direction. Recently, during the preparation of this article, Lan and Stroh obtained a stronger result that the nearby cycle cohomology is isomorphic to the compactly supported cohomology in the cases where reasonable integral toroidal compactifications exist (see [Lan and Stroh 2018]). However, we have decided to include our weaker result in this paper, since the argument is totally different.

We sketch the outline of this paper. In Section 2, we consider Galois representations of a p-adic field with values in a general connected reductive group G. Under some condition, we attach a parabolic subgroup of G to such a representation. In Section 3, we give some preliminary results on adic spaces and semi-abelian schemes. In Section 4, we recall some notation and results on Shimura varieties. In Section 5, we construct a partition of the adic space associated to a Shimura variety of preabelian type by using results obtained in Section 2. The potentially good reduction locus is introduced here, as a piece of the constructed partition. In Section 6, we prove the theorem comparing the cohomology of potentially good reduction loci with that of Shimura varieties. In Section 7, we specialize our results to Shimura varieties of PEL type, and discuss a simple application.

Notation. Put $\widehat{\mathbb{Z}} = \prod_{\text{prime } p} \mathbb{Z}_p$ and $\mathbb{A}^\infty = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. For a prime p, put

$$\widehat{\mathbb{Z}}^p = \prod_{\text{prime } p' \neq p} \mathbb{Z}_{p'}$$

and $\mathbb{A}^{\infty,p} = \widehat{\mathbb{Z}}^p \otimes_{\mathbb{Z}} \mathbb{Q}$. More generally, for a finite set of primes S, we put $\mathbb{A}_S = \prod_{\ell \in S} \mathbb{Q}_\ell$, $\widehat{\mathbb{Z}}^S = \prod_{\text{prime } p \notin S} \mathbb{Z}_p$ and $\mathbb{A}^{\infty,S} = \widehat{\mathbb{Z}}^S \otimes_{\mathbb{Z}} \mathbb{Q}$.

For a scheme X over a field F and an extension field L of F, we write X_L for the base change of X to L. Similar notation will be used for adic spaces.

For an algebraic group G, let $Z(G)$ denote the center of G, and $G^{\text{ad}} = G/Z(G)$ the adjoint group of G. For a field L over which G is defined, we write $\mathbf{Rep}_L(G)$ for the Tannakian category of finite-dimensional algebraic representations of G over L.

Every sheaf and cohomology are considered in the étale topology.

2. Preliminaries on Galois representations

In this section, fix a p-adic field F and its algebraic closure $\overline{F}$. Let ℓ be a prime number and G a connected reductive group over $\overline{\mathbb{Q}}_\ell$. Consider a continuous homomorphism $\phi\colon \text{Gal}(\overline{F}/F) \to G(\overline{\mathbb{Q}}_\ell)$.

Definition 2.1. (i) Assume that $\ell \neq p$. We say that ϕ is potentially unramified if $\xi \circ \phi$ is potentially unramified for any $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$.

(ii) Assume that $\ell = p$. We say that ϕ is potentially crystalline if $\xi \circ \phi$ is potentially crystalline for any $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_p}(G)$.

(iii) Assume that $\ell = p$. We say that ϕ is de Rham if $\xi \circ \phi$ is de Rham (or equivalently, potentially semistable) for any $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_p}(G)$.

To measure how far ϕ is from potentially unramified or potentially crystalline, we consider the monodromy filtration on $\xi \circ \phi$ for each $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$. First, we

assume that $\ell \neq p$. Then, for each $(\xi, V_\xi) \in \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$, we obtain the ℓ-adic representation $(\xi \circ \phi, V_\xi)$ of $\mathrm{Gal}(\overline{F}/F)$ and its monodromy filtration $M_\bullet V_\xi$.

Lemma 2.2. *Assume that $\ell \neq p$.*

(i) *The stabilizer P_ξ of the filtration $M_\bullet V_\xi \subset V_\xi$ is a parabolic subgroup of G.*

(ii) *If $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$ is faithful, then P_ξ stabilizes $M_\bullet V_{\xi'}$ for every $\xi' \in \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$. In particular P_ξ for faithful ξ is independent of ξ. We write P_ϕ for this P_ξ.*

(iii) *The homomorphism ϕ is potentially unramified if and only if $P_\phi = G$.*

(iv) *For the composite $\phi^{\mathrm{ad}} \colon \mathrm{Gal}(\overline{F}/F) \xrightarrow{\phi} G(\overline{\mathbb{Q}}_\ell) \to G^{\mathrm{ad}}(\overline{\mathbb{Q}}_\ell)$, we have $P_{\phi^{\mathrm{ad}}} = P_\phi^{\mathrm{ad}}$, where P_ϕ^{ad} denotes the image of P_ϕ in G^{ad}.*

(v) *For a finite extension F' of F contained in $\overline{F}$, we put $\phi' = \phi|_{\mathrm{Gal}(\overline{F}/F')}$. Then we have $P_{\phi'} = P_\phi$.*

Proof. The assertion (i) follows from [Kisin 2010, Lemma 1.1.1, Lemma 1.1.3], since $V_\xi \mapsto M_\bullet V_\xi$ gives a filtration on the Tannakian category $\mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$.

Let us prove (ii). For integers $m, m' \geq 0$, the monodromy filtration on $V_\xi^{\otimes m} \otimes V_\xi^{\vee \otimes m'}$ can be written by using $M_\bullet V_\xi$ (see [Deligne 1980, Proposition 1.6.9]). Therefore it is stable under P_ξ. As every representation $(\xi', V_{\xi'})$ of G appears as a direct summand of $V_\xi^{\otimes m} \otimes V_\xi^{\vee \otimes m'}$ for some integers $m, m' \geq 0$, the filtration $M_\bullet V_{\xi'}$ is also preserved by P_ξ. This concludes the proof.

For (iii), note that ϕ is potentially unramified if and only if the monodromy operator N on V_ξ is zero for every $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_\ell}(G)$. If this condition is satisfied, we have $M_i V_\xi = 0$ for $i < 0$ and $M_i V_\xi = V_\xi$ for $i \geq 0$, thus $P_\phi = G$. Conversely assume that $P_\phi = G$, in other words, $M_i V_\xi$ is G-stable for every ξ and i. It suffices to show that $N = 0$ on V_ξ for each irreducible representation ξ of G. Since $M_i V_\xi$ is G-stable, there exists a unique integer i_0 such that $M_i V_\xi = 0$ ($i < i_0$) and $M_i V_\xi = V_\xi$ ($i \geq i_0$). Hence we have $N(V_\xi) = N(M_{i_0} V_\xi) \subset M_{i_0 - 2} V_\xi = 0$, as desired.

We prove (iv). Clearly we have $P_\phi^{\mathrm{ad}} \subset P_{\phi^{\mathrm{ad}}}$. For the reverse inclusion, take $g \in G(\overline{\mathbb{Q}}_\ell)$ which is mapped into $P_{\phi^{\mathrm{ad}}}(\overline{\mathbb{Q}}_\ell)$ under $G(\overline{\mathbb{Q}}_\ell) \to G^{\mathrm{ad}}(\overline{\mathbb{Q}}_\ell)$. It suffices to show that g stabilizes $M_\bullet V_\xi$ for each irreducible representation (ξ, V_ξ) of G. Put $W = V_\xi \otimes V_\xi^\vee$. Since ξ is irreducible, the center of G acts trivially on W. Therefore W can be regarded as a representation of G^{ad}. In particular, the monodromy filtration $M_\bullet W$ on W is stable under the action of g.

Let j_0 be the minimal integer such that $M_{j_0} V_\xi^\vee = V_\xi^\vee$. Fix an integer i_0. Then we have $M_{i_0 + j_0} W = \sum_{i+j = i_0 + j_0} M_i V_\xi \otimes M_j V_\xi^\vee$. Note that $M_{i_0} V_\xi$ can be recovered from $M_{i_0 + j_0} W$ by

$$M_{i_0} V_\xi = \bigcap_{f \in \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(V_\xi^\vee, \overline{\mathbb{Q}}_\ell) \setminus \{0\}} (\mathrm{id} \otimes f)(M_{i_0 + j_0} W). \qquad (*)$$

Since $M_{i_0+j_0}W = g(M_{i_0+j_0}W) = \sum_{i+j=i_0+j_0} g(M_i V_\xi) \otimes g(M_j V_\xi^\vee)$, the right hand side of $(*)$ is also equal to $g(M_{i_0} V_\xi)$. Hence we conclude that $M_\bullet V_\xi$ is stable under g.

The (v) is clear, since the monodromy filtration $M_\bullet V_\xi$ does not change after restricting ϕ to $\mathrm{Gal}(\overline{F}/F')$. $\qquad\square$

Next we consider the case $\ell = p$. We shall introduce the notion of the monodromy filtration on a p-adic Galois representation. Let L be a finite extension of $\mathbb{Q}_p$ and V a finite-dimensional de Rham L-representation of $\mathrm{Gal}(\overline{F}/F)$. We regard V as a $\mathbb{Q}_p$-representation of $\mathrm{Gal}(\overline{F}/F)$ and consider $D_{\mathrm{pst}}(V)$, where D_{pst} is the functor introduced in [Fontaine 1994, §5.6]. If we write $\mathbb{Q}_p^{\mathrm{ur}}$ for the maximal unramified extension of $\mathbb{Q}_p$ contained in $\overline{F}$, $D_{\mathrm{pst}}(V)$ is an $L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{ur}}$-module equipped with several structures. Among them, we have the monodromy operator on $D_{\mathrm{pst}}(V)$, from which the monodromy filtration $M_\bullet D_{\mathrm{pst}}(V)$ on $D_{\mathrm{pst}}(V)$ is naturally induced.

Definition 2.3. Let V be a de Rham L-representation of $\mathrm{Gal}(\overline{F}/F)$. We say that V has the monodromy filtration if there exists a $\mathrm{Gal}(\overline{F}/F)$-stable $\mathbb{Q}_p$-subspace $M_i V$ of V such that $D_{\mathrm{pst}}(M_i V) = M_i D_{\mathrm{pst}}(V)$ for each i. Such a subspace $M_i V$ is unique and stable under the action of L if it exists, thanks to the fact that D_{pst} is fully faithful.

Now, let V be a finite-dimensional de Rham $\overline{\mathbb{Q}}_p$-representation of $\mathrm{Gal}(\overline{F}/F)$. We can find a subfield L of $\overline{\mathbb{Q}}_p$ which is finite over $\mathbb{Q}_p$ and a $\mathrm{Gal}(\overline{F}/F)$-stable L-subspace V_L of V such that $V_L \otimes_L \overline{\mathbb{Q}}_p = V$. The L-representation V_L is de Rham.

Definition 2.4. The condition that V_L has the monodromy filtration is independent of the choice of L and V_L. If it is the case, we say that V has the monodromy filtration, and put $M_i V = M_i V_L \otimes_L \overline{\mathbb{Q}}_p$, which is easily seen to be independent of L and V_L. We call $M_\bullet V$ the monodromy filtration of V.

Lemma 2.5. *Let V, W be finite-dimensional $\overline{\mathbb{Q}}_p$-representations of $\mathrm{Gal}(\overline{F}/F)$ which are de Rham and have the monodromy filtrations.*

(i) *Let V' be a direct summand of V as a $\mathrm{Gal}(\overline{F}/F)$-representation. Then V' is de Rham and has the monodromy filtration.*

(ii) *The representations $V \otimes W$ and $V^\vee$ are de Rham and have the monodromy filtrations. The monodromy filtration $M_\bullet(V \otimes W)$ (resp. $M_\bullet(V^\vee)$) is given by $M_n(V \otimes W) = \sum_{i+j=n} M_i V \otimes M_j W$ (resp. $M_n(V^\vee) = (V/M_{-n-1}V)^\vee$).*

Proof. Let L be a finite extension of $\mathbb{Q}_p$. We have only to consider L-representations in place of $\overline{\mathbb{Q}}_p$-representations. In the following, let V, W be finite-dimensional

L-representations of $\mathrm{Gal}(\overline{F}/F)$ which are de Rham and have the monodromy filtrations.

We prove (i). We write $V = V' \oplus V''$. Then we have $D_{\mathrm{pst}}(V) = D_{\mathrm{pst}}(V') \oplus D_{\mathrm{pst}}(V'')$ and $M_i D_{\mathrm{pst}}(V) = M_i D_{\mathrm{pst}}(V') \oplus M_i D_{\mathrm{pst}}(V'')$. Since the essential image of the functor D_{pst} is stable under direct factors (see [Fontaine 1994, Théorème 5.6.7]), there exists a $\mathrm{Gal}(\overline{F}/F)$-stable subspace $M_i V'$ of V' such that $D_{\mathrm{pst}}(M_i V') = M_i D_{\mathrm{pst}}(V')$. Hence V' has the monodromy filtration.

Next consider (ii). It is known that $V \otimes_{\mathbb{Q}_p} W$ is a de Rham representation, and $D_{\mathrm{pst}}(V \otimes_{\mathbb{Q}_p} W) = D_{\mathrm{pst}}(V) \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} D_{\mathrm{pst}}(W)$. The monodromy filtration on $D_{\mathrm{pst}}(V \otimes_{\mathbb{Q}_p} W)$ is given by

$$M_n D_{\mathrm{pst}}(V \otimes_{\mathbb{Q}_p} W) = \sum_{i+j=n} M_i D_{\mathrm{pst}}(V) \otimes_{\mathbb{Q}_p^{\mathrm{ur}}} M_j D_{\mathrm{pst}}(W).$$

Thus, if we put $M_n(V \otimes_{\mathbb{Q}_p} W) = \sum_{i+j=n} M_i V \otimes_{\mathbb{Q}_p} M_j W$, we have $D_{\mathrm{pst}}(M_n(V \otimes_{\mathbb{Q}_p} W)) = M_n D_{\mathrm{pst}}(V \otimes_{\mathbb{Q}_p} W)$ by the exactness of D_{pst}. Hence $V \otimes_{\mathbb{Q}_p} W$ has the monodromy filtration.

Let $e \in L \otimes_{\mathbb{Q}_p} L$ denote the idempotent corresponding to the diagonal component $\mathrm{Spec}\, L \hookrightarrow \mathrm{Spec}(L \otimes_{\mathbb{Q}_p} L)$. Then, we have $e(V \otimes_{\mathbb{Q}_p} W) = V \otimes_L W$. In particular $V \otimes_L W$ is a direct summand of $V \otimes_{\mathbb{Q}_p} W$. Therefore, by (i), $V \otimes_L W$ is de Rham and has the monodromy filtration. Clearly, the monodromy filtration on $V \otimes_L W = e(V \otimes_{\mathbb{Q}_p} W)$ is given by

$$M_n(V \otimes_L W) = e M_n(V \otimes_{\mathbb{Q}_p} W) = \sum_{i+j=n} e(M_i V \otimes_{\mathbb{Q}_p} M_j W) = \sum_{i+j=n} M_i V \otimes_L M_j W.$$

The dual $V^\vee$ can be treated similarly. $\qquad\square$

Lemma 2.6. *Let V be a finite-dimensional $\overline{\mathbb{Q}}_p$-representation of $\mathrm{Gal}(\overline{F}/F)$. For a finite extension F' of F contained in $\overline{F}$, we put $V' = V|_{\mathrm{Gal}(\overline{F}/F')}$. Then, V is de Rham and has the monodromy filtration if and only if so is V'. Moreover we have $M_\bullet V' = (M_\bullet V)|_{\mathrm{Gal}(\overline{F}/F')}$.*

Proof. As in the proof of Lemma 2.5, we may replace V by an L-representation of $\mathrm{Gal}(\overline{F}/F)$, where L is a finite extension of $\mathbb{Q}_p$. By definition, V is de Rham if and only if V' is de Rham. Suppose that V and V' are de Rham. We have $D_{\mathrm{pst}}(V') = D_{\mathrm{pst}}(V)|_{\mathrm{Gal}(\overline{F}/F')}$. Therefore, if V has the monodromy filtration $M_\bullet V$, then V' has the monodromy filtration $(M_\bullet V)|_{\mathrm{Gal}(\overline{F}/F')}$. Conversely, assume that V' has the monodromy filtration $M_\bullet V'$. Since $D_{\mathrm{pst}}(M_i V') = M_i D_{\mathrm{pst}}(V') = M_i D_{\mathrm{pst}}(V) \subset D_{\mathrm{pst}}(V)$ is stable under $\mathrm{Gal}(\overline{F}/F)$, so is $M_i V' \subset V' = V$. Therefore, $M_\bullet V'$ gives the monodromy filtration of V. This concludes the proof. $\qquad\square$

Now, let $\phi\colon \mathrm{Gal}(\overline{F}/F) \to G(\overline{\mathbb{Q}}_p)$ be a continuous homomorphism, as in the beginning of this section.

Definition 2.7. Assume that ϕ is de Rham. We say that ϕ has the monodromy filtration if V_ξ has the monodromy filtration for every $\xi \in \mathbf{Rep}_{\overline{\mathbb{Q}}_p}(G)$.

Lemma 2.8. *Assume that there exists a faithful algebraic representation (ξ, V_ξ) of G such that V_ξ is de Rham and has the monodromy filtration. Then ϕ is de Rham and has the monodromy filtration.*

Proof. Thanks to Lemma 2.5, we can use the same argument as in the proof of Lemma 2.2 (ii). □

Lemma 2.9. *Assume that ϕ is de Rham and has the monodromy filtration.*

(i) *The stabilizer P_ξ of the filtration $M_\bullet V_\xi \subset V_\xi$ is a parabolic subgroup of G.*

(ii) *If ξ is faithful, then P_ξ stabilizes $M_\bullet V_{\xi'}$ for every representation ξ'. In particular P_ξ for faithful ξ is independent of ξ. We write P_ϕ for this P_ξ.*

(iii) *The homomorphism ϕ is potentially crystalline if and only if $P_\phi = G$.*

(iv) *The composite $\phi^{\mathrm{ad}} \colon \mathrm{Gal}(\overline{F}/F) \xrightarrow{\phi} G(\overline{\mathbb{Q}}_p) \to G^{\mathrm{ad}}(\overline{\mathbb{Q}}_p)$ is de Rham and has the monodromy filtration. Moreover we have $P_{\phi^{\mathrm{ad}}} = P_\phi^{\mathrm{ad}}$.*

Proof. This can be proved in the same way as Lemma 2.2. □

Lemma 2.10. *For a finite extension F' of F contained in $\overline{F}$, $\phi' = \phi|_{\mathrm{Gal}(\overline{F}/F')}$ is de Rham and has the monodromy filtration if and only if so is ϕ. Moreover, if the above conditions are satisfied, we have $P_{\phi'} = P_\phi$.*

Proof. This is an immediate consequence of Lemma 2.6. □

Corollary 2.11. *Let ℓ be a prime number.*

(i) *Assume that $\ell \neq p$. Then ϕ is potentially unramified if and only if ϕ^{ad} is potentially unramified.*

(ii) *Assume that $\ell = p$, ϕ is de Rham and has the monodromy filtration. Then ϕ is potentially crystalline if and only if ϕ^{ad} is potentially crystalline.*

Proof. The first assertion follows from Lemma 2.2 (iii), (iv), and the second from Lemma 2.9 (iii), (iv). □

Remark 2.12. In fact, Corollary 2.11 (ii) holds without assuming that ϕ has the monodromy filtration (we have only to consider the monodromy filtration on the image of D_{pst}). However we do not need this fact later.

Remark 2.13. Let ℓ be a prime number, and assume that ϕ is de Rham and has the monodromy filtration if $\ell = p$. If G is defined over $\mathbb{Q}_\ell$ and the image of ϕ is contained in $G(\mathbb{Q}_\ell)$, the parabolic subgroup P_ϕ is defined over $\mathbb{Q}_\ell$. Indeed, we can take a faithful representation ξ which is defined over $\mathbb{Q}_\ell$, and then the monodromy filtration $M_\bullet V_\xi$ is also defined over $\mathbb{Q}_\ell$.

3. Rigid geometry and semi-abelian schemes

3.1. *Notation for adic spaces.* Throughout this paper, we will use the framework of adic spaces introduced by Huber [1993; 1994; 1996]. Here we recall some notation briefly.

Let S be a noetherian scheme and S_0 a closed subscheme of S. We denote the formal completion of S along S_0 by $\mathcal{S}$. Put $\mathcal{S}^{\mathrm{rig}} = t(\mathcal{S})_a$, where $t(\mathcal{S})$ is the adic space associated to $\mathcal{S}$ (*cf.* [Huber 1994, §4]) and $t(\mathcal{S})_a$ denotes the open adic subspace of $t(\mathcal{S})$ consisting of analytic points. It is a quasicompact analytic adic space.

Let X be a scheme of finite type over S. Put $X_0 = X \times_S S_0$ and denote the formal completion of X along X_0 by $\widehat{X}$. Then we can construct an adic space $\widehat{X}^{\mathrm{rig}}$ in the same way as $\mathcal{S}^{\mathrm{rig}}$. The induced morphism $\widehat{X}^{\mathrm{rig}} \to \mathcal{S}^{\mathrm{rig}}$ is of finite type. On the other hand, we can construct another adic space $X \times_S \mathcal{S}^{\mathrm{rig}}$. Indeed, since we have morphisms of locally ringed spaces $(\mathcal{S}^{\mathrm{rig}}, \mathcal{O}_{\mathcal{S}^{\mathrm{rig}}}) \to (t(\mathcal{S}), \mathcal{O}_{t(\mathcal{S})}) \to (S, \mathcal{O}_S)$ (for the second one, see [Huber 1994, Remark 4.6(iv)]), we can make the fiber product $X \times_S \mathcal{S}^{\mathrm{rig}}$ in the sense of [Huber 1994, Proposition 3.8]. For simplicity, we write X^{ad} for $X \times_S \mathcal{S}^{\mathrm{rig}}$, though it depends on (S, S_0). Since the morphism $\mathcal{S}^{\mathrm{rig}} \to S$ factors through $S^0 = S \setminus S_0$, we have $X \times_S \mathcal{S}^{\mathrm{rig}} = (X \times_S S^0) \times_{S^0} \mathcal{S}^{\mathrm{rig}}$. In particular, X^{ad} depends only on $X \times_S S^0$. The natural morphism $X^{\mathrm{ad}} \to \mathcal{S}^{\mathrm{rig}}$ is locally of finite type, but not necessarily quasicompact; see the following example.

Example 3.1. Let R be a complete discrete valuation ring and F its fraction field. Consider the case where $S = \operatorname{Spec} R$ and S_0 is the closed point of S. Then, for an S-scheme X of finite type, X^{ad} can be regarded as the rigid space over F associated to a scheme $X \times_S \operatorname{Spec} F$ over F. For example, $(\mathbb{A}_S^1)^{\mathrm{ad}} = (\mathbb{A}_F^1)^{\mathrm{ad}}$ is the rigid-analytic affine line over F and thus is not quasicompact. On the other hand, $(\widehat{\mathbb{A}}_S^1)^{\mathrm{rig}}$ is the unit disc "$|z| \le 1$" in $(\mathbb{A}_F^1)^{\mathrm{ad}}$, which is quasicompact.

Lemma 3.2. *The functors $X \mapsto \widehat{X}^{\mathrm{rig}}$ and $X \mapsto X^{\mathrm{ad}}$ commute with fiber products.*

Proof. For the functor $X \mapsto \widehat{X}^{\mathrm{rig}}$, it can be checked easily (*cf.* [Mieda 2006, Lemma 3.4] and [Mieda 2014, Proof of Lemma 4.4 (v)]). Consider the functor $X \mapsto X^{\mathrm{ad}}$. Let $Y \to X \leftarrow Z$ be a diagram of S-schemes of finite type. What we should prove is

$$(Y \times_X Z) \times_S \mathcal{S}^{\mathrm{rig}} \cong (Y \times_S \mathcal{S}^{\mathrm{rig}}) \times_{X \times_S \mathcal{S}^{\mathrm{rig}}} (Z \times_S \mathcal{S}^{\mathrm{rig}}).$$

It is not totally automatic, since $Y \times_X Z$ is not a fiber product in the category of locally ringed spaces. It follows from the fact that morphisms of locally ringed spaces $\operatorname{Spa}(A, A^+) \to \operatorname{Spec} B$ for a complete affinoid ring (A, A^+) and a ring B correspond bijectively to ring homomorphisms $B \to A$ (this fact is used implicitly in [Huber 1994, Remark 4.6(iv)] to define $t(\mathcal{S}) \to S$). $\square$

Let us compare $\widehat{X}^{\mathrm{rig}}$ and X^{ad}; by the commutative diagram

$$
\begin{array}{ccc}
\widehat{X}^{\mathrm{rig}} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathcal{S}^{\mathrm{rig}} & \longrightarrow & S
\end{array}
$$

and the universality of the fiber product $X \times_S \mathcal{S}^{\mathrm{rig}}$, we have a natural morphism $\widehat{X}^{\mathrm{rig}} \to X^{\mathrm{ad}}$.

Lemma 3.3. (i) *If X is separated over S, $\widehat{X}^{\mathrm{rig}} \to X^{\mathrm{ad}}$ is an open immersion.*

(ii) *If X is proper over S, $\widehat{X}^{\mathrm{rig}} \to X^{\mathrm{ad}}$ is an isomorphism.*

Proof. See [Huber 1994, Remark 4.6(iv)]. □

Remark 3.4. Let $f \colon S' \to S$ be a morphism of finite type and $S'_0 = S' \times_S S_0$. We denote by $\mathcal{S}'^{\mathrm{rig}}$ the formal completion of S' along S'_0. Then, all constructions above are compatible with the base change by f. More precisely, for a scheme X of finite type over S, we have $(X \times_S S')^{\wedge\mathrm{rig}} \cong \widehat{X}^{\mathrm{rig}} \times_{\mathcal{S}^{\mathrm{rig}}} \mathcal{S}'^{\mathrm{rig}}$ and $(X \times_S S')^{S'\text{-ad}} \cong X^{\mathrm{ad}} \times_{\mathcal{S}^{\mathrm{rig}}} \mathcal{S}'^{\mathrm{rig}}$. Here $(-)^{S'\text{-ad}}$ denotes the functor $(-)^{\mathrm{ad}}$ for the base (S', S'_0), namely, $(-)^{S'\text{-ad}} = (-) \times_{S'} \mathcal{S}'^{\mathrm{rig}}$.

In the remaining part of this subsection, assume that S is the spectrum of a complete discrete valuation ring R and S_0 is the closed point of S. For a scheme of finite type X over S, we have a natural morphism of locally and topologically ringed spaces $(t(\widehat{X}), \mathcal{O}^+_{t(\widehat{X})}) \to (\widehat{X}, \mathcal{O}_{\widehat{X}})$ (*cf.* [Huber 1994, Proposition 4.1]). Note that the underlying continuous map $t(\widehat{X}) \to X_0$ is different from the map $t(\widehat{X}) \to X$ considered above. We denote the composite $\widehat{X}^{\mathrm{rig}} \hookrightarrow t(\widehat{X}) \to X_0$ by $\mathrm{sp}_{\widehat{X}}$, or simply by sp.

Let Y be a closed subscheme of X_0 and $\mathcal{X}$ the formal completion of X along Y. Then we can consider the generic fiber $t(\mathcal{X})_\eta = S^0 \times_S t(\mathcal{X})$ of the adic space $t(\mathcal{X})$. This is so-called the rigid generic fiber of $\mathcal{X}$ due to Raynaud and Berthelot, in the context of adic spaces. If $Y = X_0$, then $t(\mathcal{X})_\eta = \widehat{X}^{\mathrm{rig}}$.

Lemma 3.5. *The natural morphism $t(\mathcal{X})_\eta \to \widehat{X}^{\mathrm{rig}}$ induced from $\mathcal{X} \to \widehat{X}$ is an open immersion. Its image coincides with $\mathrm{sp}^{-1}(Y)^\circ$, where $(-)^\circ$ denotes the interior in $\widehat{X}^{\mathrm{rig}}$.*

Proof. See [Huber 1998b, Lemma 3.13 i)]. □

Let X be an adic space locally of finite type over $\mathcal{S}^{\mathrm{rig}} = \mathrm{Spa}(F, R)$, where F denotes the fraction field of R. For a point x of X, we write κ_x and κ_x^+ for the residue field and the valuation ring at x, respectively. We say that $x \in X$ is classical if κ_x is a finite extension of F. We denote the set of classical points of X by $X(\mathrm{cl})$. Further, for a subset Y of X, we put $Y(\mathrm{cl}) = X(\mathrm{cl}) \cap Y$.

Lemma 3.6. *Let X be an adic space locally of finite type over* $\mathrm{Spa}(F, R)$.

(i) *For constructible subsets L_1, L_2 of X (see [Huber 1996, 1.1.13]), we have $L_1 \subset L_2$ if and only if $L_1(\mathrm{cl}) \subset L_2(\mathrm{cl})$. In particular, $L_1 = L_2$ if and only if $L_1(\mathrm{cl}) = L_2(\mathrm{cl})$.*

(ii) *For a constructible subset L, we write L^- (resp. L°) for the closure (resp. interior) of L in X. Then we have $L(\mathrm{cl}) = L^-(\mathrm{cl}) = L^\circ(\mathrm{cl})$.*

Proof. For (i), it suffices to show that $L_1(\mathrm{cl}) \subset L_2(\mathrm{cl})$ implies $L_1 \subset L_2$. Put $L = L_1 \setminus L_2 = L_1 \cap (X \setminus L_2)$. It is a constructible subset of X satisfying $L(\mathrm{cl}) = \varnothing$. Let U be an arbitrary affinoid open subset of X . Then, we have $(U \cap L)(\mathrm{cl}) = \varnothing$. Therefore, [Huber 1993, Corollary 4.3] tells us that $U \cap L = \varnothing$. Now we conclude that $L = \varnothing$, that is, $L_1 \subset L_2$.

For (ii), it suffices to prove that $L(\mathrm{cl}) = L^-(\mathrm{cl})$. Take $x \in L^- \setminus L$ and an affinoid open neighborhood U of x. Then x lies in the closure of $U \cap L$ in U. Since $U \cap L$ is a constructible subset of the spectral space U, by [Hochster 1969, Corollary of Theorem 1], there exists $y \in U \cap L$ such that $x \in \{y\}^-$. Therefore, by [Huber 1996, Lemma 1.1.10 ii)], the valuation v_x attached to x is not rank 1. In particular x is not classical. Hence we have $L(\mathrm{cl}) = L^-(\mathrm{cl})$, as desired. $\square$

The following basic lemma is also used in Section 5.

Lemma 3.7. *Let $f : X \to Y$ be a quasicompact quasiseparated étale morphism between adic spaces.*

(i) *For a constructible subset L of X, the image $f(L)$ is a constructible subset of Y.*

(ii) *For a locally closed subset L of X satisfying $f^{-1}(f(L)) = L$, the image $f(L)$ is a locally closed subset of Y.*

Proof. The assertion (i) can be proved in the same way as [Huber 1996, (1) in the proof of Lemma 2.7.4]. We recall the argument for reader's convenience. We may assume that X and Y are quasicompact and quasiseparated. Fix $y \in f(L)$. Let Λ denote the set of constructible subsets of Y containing y. We have $\bigcap_{W \in \Lambda} W = \{y\}$, as Y is a spectral space. Since $f^{-1}(y)$ is a finite discrete subset of X, there exists a quasicompact open subset U of X such that $U \cap f^{-1}(y) = L \cap f^{-1}(y)$. Then we have $U \cap \bigcap_{W \in \Lambda} f^{-1}(W) = L \cap \bigcap_{W \in \Lambda} f^{-1}(W)$. By the quasicompactness of X with respect to the constructible topology, there exists $W \in \Lambda$ such that $U \cap f^{-1}(W) = L \cap f^{-1}(W)$. Put $V_y = U \cap f^{-1}(W) = L \cap f^{-1}(W)$, which is a constructible subset of X. Since f is étale, $f(U)$ is a quasicompact open subset of Y. Therefore $f(V_y) = f(U) \cap W$ is a constructible subset of Y.

Since $L \cap f^{-1}(y) \subset V_y \subset L$, we have $L = \bigcup_{y \in f(L)} V_y$. On the other hand, L is quasicompact under the constructible topology of X. Therefore, there exist finitely

many points $y_1, \ldots, y_m \in f(L)$ such that $L = \bigcup_{i=1}^{m} V_{y_i}$. Now we conclude that $f(L) = \bigcup_{i=1}^{m} f(V_{y_i})$ is a constructible subset of Y, as desired.

Next we consider (ii). Since L is locally closed, it can be written in the form $U \cap W$, where U is an open subset of X and W is a closed subset of X. Note that $L^- \subset W$, thus $U \cap L^- = L$. For simplicity we write $L' = f(L)$. Since f is an open map, we can check that $f^{-1}(L')^- = f^{-1}(L'^-)$. Therefore we obtain

$$L = U \cap L^- = U \cap f^{-1}(L')^- = U \cap f^{-1}(L'^-)$$

and $f(L) = f(U) \cap L'^-$. As f is étale, $f(U)$ is open, hence $f(L)$ is locally closed. $\square$

3.2. Etale sheaves associated to semi-abelian schemes.

We continue to use the notation introduced in the beginning of the previous subsection. Let U be an open subscheme of $S^0 = S \setminus S_0$ and ℓ a prime number invertible on U. Fix an integer $m > 0$.

Let G be a semi-abelian scheme over S. Namely, G is a separated smooth commutative group scheme over S such that each fiber G_s of G at $s \in S$ is an extension of an abelian variety A_s by a torus T_s. We denote the relative dimension of G over S by d. Assume the following:

- The rank of T_s (called the toric rank of G_s) with $s \in S_0$ is a constant r.

- $G_U = G \times_S U$ is an abelian scheme.

Under the first condition, it is known that $G_0 = G \times_S S_0$ is globally an extension

$$0 \to T_0 \to G_0 \to A_0 \to 0,$$

where T_0 is a torus of rank r over S_0 and A_0 is an abelian scheme over S_0 ([Faltings and Chai 1990, Chapter I, Corollary 2.11]).

Let us consider two group spaces $\widehat{G}^{\mathrm{rig}}[\ell^m]_{U^{\mathrm{ad}}}$ and $G^{\mathrm{ad}}[\ell^m]_{U^{\mathrm{ad}}}$ over U^{ad}, where $(-)_{U^{\mathrm{ad}}}$ denotes the restriction to U^{ad}.

Lemma 3.8. *The adic space $G^{\mathrm{ad}}[\ell^m]_{U^{\mathrm{ad}}}$ is finite étale of degree ℓ^{2dm} over U^{ad}.*

Proof. By Lemma 3.2, we have $G^{\mathrm{ad}}[\ell^m]_{U^{\mathrm{ad}}} = (G_U[\ell^m]) \times_U U^{\mathrm{ad}}$. Since $G_U[\ell^m]$ is finite étale of degree ℓ^{2dm} over U, $G^{\mathrm{ad}}[\ell^m]_{U^{\mathrm{ad}}}$ is finite étale of degree ℓ^{2dm} over U^{ad} (see [Huber 1996, Corollary 1.7.3 i)]). $\square$

Lemma 3.9. *The adic space $\widehat{G}^{\mathrm{rig}}[\ell^m]_{U^{\mathrm{ad}}}$ is finite étale of degree $\ell^{(2d-r)m}$ over U^{ad}.*

Proof. We may assume that $S = \operatorname{Spec} R$ is affine. Let $I \subset R$ be the defining ideal of S_0. By replacing R by its I-adic completion, we can reduce to the case where R is I-adically complete. Put $S_i = \operatorname{Spec} R/I^{i+1}$ and $G_i = G \times_S S_i$.

By [SGA 3_{II} 1970, Exposé IX, Théorème 3.6, Théorème 3.6 bis], the exact sequence

$$0 \to T_0 \to G_0 \to A_0 \to 0$$

can be lifted canonically to an exact sequence

$$0 \to T_i \to G_i \to A_i \to 0$$

over S_i, where T_i is a torus over S_i and A_i is an abelian scheme over S_i (see [Lan 2013, §3.3.3]). Let $\widehat{T} = \varinjlim_i T_i$ and $\widehat{A} = \varinjlim_i A_i$ be associated formal groups over $\mathcal{S}$. Then $\widehat{G}$ is an extension of $\widehat{A}$ by $\widehat{T}$.

By taking ℓ^m-torsion points, we get an exact sequence

$$0 \to \widehat{T}[\ell^m] \to \widehat{G}[\ell^m] \to \widehat{A}[\ell^m] \to 0$$

of formal groups over $\mathcal{S}$. Since $\widehat{G}^{\mathrm{rig}}[\ell^m] \cong (\widehat{G}[\ell^m])^{\mathrm{rig}}$, it suffices to see that $(\widehat{T}[\ell^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}}$ (resp. $(\widehat{A}[\ell^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}}$) is finite étale of degree ℓ^{rm} (resp. $\ell^{2(d-r)m}$) over U^{ad}.

First we consider $(\widehat{T}[\ell^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}}$. Since $\widehat{T}[\ell^m] = \varinjlim_i(T_i[\ell^m])$, it is finite flat over $\mathcal{S} = \mathrm{Spf}\, R$. Therefore there exists a finite flat R-algebra R' such that $\widehat{T}[\ell^m] = \mathrm{Spf}\, R'$. Moreover, a scheme $T' = \mathrm{Spec}\, R'$ is naturally equipped with a structure of a commutative group scheme over $S = \mathrm{Spec}\, R$. Since T' is killed by ℓ^m and p is invertible on U, $T'_U = T' \times_S U$ is a finite étale group scheme over U. By Lemma 3.3 (ii), we have $(\widehat{T}[\ell^m])^{\mathrm{rig}} = (T')^{\wedge \mathrm{rig}} = T'^{\mathrm{ad}} = T' \times_S \mathcal{S}^{\mathrm{rig}}$. Therefore $(\widehat{T}[\ell^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}} = T'_U \times_U U^{\mathrm{ad}}$ is finite étale over U^{ad} (see [Huber 1996, Corollary 1.7.3 i)]). Its degree is clearly ℓ^{rm}.

The same argument also works for $(\widehat{A}[\ell^m])^{\mathrm{rig}}_{U^{\mathrm{ad}}}$. □

By Lemma 3.8 and Lemma 3.9, we may regard $\widehat{G}^{\mathrm{rig}}[\ell^m]_{U^{\mathrm{ad}}}$ and $G^{\mathrm{ad}}[\ell^m]_{U^{\mathrm{ad}}}$ as locally constant constructible sheaves over U^{ad}. Since we have a natural open immersion $\widehat{G}^{\mathrm{rig}} \hookrightarrow G^{\mathrm{ad}}$, $\widehat{G}^{\mathrm{rig}}[\ell^m]_{U^{\mathrm{ad}}}$ is a subsheaf of $G^{\mathrm{ad}}[\ell^m]_{U^{\mathrm{ad}}}$.

Remark 3.10. In the setting of Remark 3.4, the construction above is clearly compatible with the base change by $f \colon S' \to S$.

In the remaining part of this subsection, we consider the case where $S = \mathrm{Spec}\, R$ is the spectrum of a complete discrete valuation ring R, S_0 is the closed point of S and $U = S^0 = S \setminus S_0$. Let $\overline{\eta}$ be a geometric point lying over the unique point of U^{ad}.

As in the proof of Lemma 3.9, $\widehat{G}$ is an extension

$$0 \to \widehat{T} \to \widehat{G} \to \widehat{A} \to 0$$

of a formal group $\widehat{A}$ by $\widehat{T}$. Therefore, we have $\mathbb{Z}/\ell^m\mathbb{Z}$-submodules

$$\widehat{T}^{\mathrm{rig}}[\ell^m]_{\overline{\eta}} \subset \widehat{G}^{\mathrm{rig}}[\ell^m]_{\overline{\eta}} \subset G^{\mathrm{ad}}[\ell^m]_{\overline{\eta}}.$$

By taking inverse limit and tensoring with $\mathbb{Q}_\ell$, we have

$$T_\ell \widehat{T}^{\mathrm{rig}}_{\overline{\eta}} \subset T_\ell \widehat{G}^{\mathrm{rig}}_{\overline{\eta}} \subset T_\ell G^{\mathrm{ad}}_{\overline{\eta}}, \quad V_\ell \widehat{T}^{\mathrm{rig}}_{\overline{\eta}} \subset V_\ell \widehat{G}^{\mathrm{rig}}_{\overline{\eta}} \subset V_\ell G^{\mathrm{ad}}_{\overline{\eta}},$$

where we put $V_\ell(-) = T_\ell(-) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. By Lemma 3.9 and its proof, we can deduce that $\dim_{\mathbb{Q}_\ell} V_\ell \widehat{T}_{\overline{\eta}}^{\mathrm{rig}} = r$ and $\dim_{\mathbb{Q}_\ell} V_\ell \widehat{G}_{\overline{\eta}}^{\mathrm{rig}} = 2d - r$.

Proposition 3.11. *Assume that the fraction field F of R is a finite extension of $\mathbb{Q}_p$. For a filtration*

$$0 \subset V_\ell \widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \subset V_\ell \widehat{G}_{\overline{\eta}}^{\mathrm{rig}} \subset V_\ell G_{\overline{\eta}}^{\mathrm{ad}} = V_\ell G_{\overline{\eta}},$$

we have the following:

  (i) *If $\ell \neq p$, the above filtration is the weight filtration of $V_\ell G_{\overline{\eta}}$.*

  (ii) *If $\ell = p$, then the above filtration is a filtration as semistable representations of $\mathrm{Gal}(\overline{F}/F)$. Further, this filtration induces the weight filtration on $D_{\mathrm{st}}(V_p G_{\overline{\eta}})$.*

Proof. We give a proof of (ii). We can show (i) similarly. Let λ be a polarization of G_U. Then an alternating bilinear pairing

$$\langle \,, \, \rangle_\lambda \colon V_p G_{\overline{\eta}} \times V_p G_{\overline{\eta}} \to \mathbb{Q}_p(1)$$

is induced. First, we will prove that $(V_p \widehat{G}_{\overline{\eta}}^{\mathrm{rig}})^\perp = V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}}$. Since

$$\dim_{\mathbb{Q}_p} V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}} + \dim_{\mathbb{Q}_p} V_p \widehat{G}_{\overline{\eta}}^{\mathrm{rig}} = r + (2d - r) = 2d = \dim_{\mathbb{Q}_p} V_p G_{\overline{\eta}},$$

it is sufficient to prove that $V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \subset (V_p \widehat{G}_{\overline{\eta}}^{\mathrm{rig}})^\perp$. Namely, we should prove that the homomorphism $V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V_p \widehat{G}_{\overline{\eta}}^{\mathrm{rig}} \to \mathbb{Q}_p(1)$ induced by $\langle \,, \, \rangle_\lambda$ is zero.

Since $V_p G_{\overline{\eta}}$ is a semistable representation of $\mathrm{Gal}(\overline{F}/F)$, so are $V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}}$ and $V_p \widehat{G}_{\overline{\eta}}^{\mathrm{rig}}$. We denote the residue field of F by κ_F and put $q = \#\kappa_F$. Consider the action of $\varphi^{[\kappa_F : \mathbb{F}_p]}$ on $D_{\mathrm{st}}(V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}})$ and $D_{\mathrm{st}}(V_p \widehat{A}_{\overline{\eta}}^{\mathrm{rig}})$. By [SGA 3$_{\mathrm{II}}$ 1970, Exposé X, Théorème 3.2], $\widehat{T}$ can be algebraized into a torus T over S. Then we have $V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \cong V_p T_{\overline{\eta}}$. Therefore every eigenvalue of $\varphi^{[\kappa_F : \mathbb{F}_p]}$ on $D_{\mathrm{st}}(V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}})$ is a Weil q^{-2}-number (for the definition of Weil numbers, see [Taylor and Yoshida 2007, p. 471]). Similarly, by [Lan 2013, Proposition 3.3.3.6, Remark 3.3.3.9], $\widehat{A}$ can be algebraized into an abelian scheme A over S, and we have $V_p \widehat{A}_{\overline{\eta}}^{\mathrm{rig}} \cong V_p A_{\overline{\eta}}$. By the Weil conjecture for the crystalline cohomology of abelian varieties, every eigenvalue of $\varphi^{[\kappa_F : \mathbb{F}_p]}$ on $D_{\mathrm{st}}(V_p \widehat{A}_{\overline{\eta}}^{\mathrm{rig}})$ is a Weil q^{-1}-number. Therefore, every eigenvalue of $\varphi^{[\kappa_F : \mathbb{F}_p]}$ on $D_{\mathrm{st}}(V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V_p \widehat{G}_{\overline{\eta}}^{\mathrm{rig}})$ is either a Weil q^{-4}-number or a Weil q^{-3}-number. On the other hand, every eigenvalue of $\varphi^{[\kappa_F : \mathbb{F}_p]}$ on $D_{\mathrm{st}}(\mathbb{Q}_p(1))$ is equal to q^{-1}, which is a Weil q^{-2}-number. Hence any φ-homomorphism $D_{\mathrm{st}}(V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V_p \widehat{G}_{\overline{\eta}}^{\mathrm{rig}}) \to D_{\mathrm{st}}(\mathbb{Q}_p(1))$ is zero. Since the functor D_{st} is fully faithful, any $\mathrm{Gal}(\overline{F}/F)$-equivariant homomorphism

$$V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V_p \widehat{G}_{\overline{\eta}}^{\mathrm{rig}} \to \mathbb{Q}_p(1)$$

is zero. Hence, we have $(V_p \widehat{G}_{\overline{\eta}}^{\mathrm{rig}})^{\perp} = V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}}$. Then we have a perfect pairing

$$V_p \widehat{T}_{\overline{\eta}}^{\mathrm{rig}} \times (V_p G_{\overline{\eta}} / V_p \widehat{G}_{\overline{\eta}}^{\mathrm{rig}}) \to \mathbb{Q}_p(1).$$

The claim follows from the above arguments and this perfect pairing. □

Corollary 3.12. *The semistable representation $V_p G_{\overline{\eta}}$ of $\mathrm{Gal}(\overline{F}/F)$ has the monodromy filtration in the sense of Definition 2.3.*

Proof. It is well-known that in this case the weight filtration and the monodromy filtration on $D_{\mathrm{pst}}(V_p G_{\eta}) = D_{\mathrm{st}}(V_p G_{\eta})$ coincide up to shift. Therefore the claim follows from Proposition 3.11 (ii). □

Remark 3.13. Actually, the extension $0 \to \widehat{T} \to \widehat{G} \to \widehat{A} \to 0$ considered above can be algebraized; namely, there exists an exact sequence

$$0 \to T \to G^{\natural} \to A \to 0$$

of commutative group schemes over S, where T and A are as in the proof of Proposition 3.11, such that its formal completion along the special fiber is isomorphic to the extension above (see [Lan 2013, Proposition 3.3.3.6, Remark 3.3.3.9]). Such an extension is called the Raynaud extension associated to G.

Our construction above is related to the Raynaud extension in the following way. First, we have a natural isomorphism $\widehat{G}^{\mathrm{rig}}[\ell^m]_{\overline{\eta}} \xrightarrow{\cong} (G^{\natural})^{\mathrm{ad}}[\ell^m]_{\overline{\eta}}$, which is induced from an open immersion $\widehat{G}^{\mathrm{rig}} \cong (\widehat{G^{\natural}})^{\mathrm{rig}} \hookrightarrow (G^{\natural})^{\mathrm{ad}}$ (see Lemma 3.3 (i)). Moreover, the image of $\widehat{G}^{\mathrm{rig}}[\ell^m]_{\overline{\eta}} \hookrightarrow G^{\mathrm{ad}}[\ell^m]_{\overline{\eta}}$ coincides with the image of the map $G^{\natural}[\ell^m]_{\overline{\eta}} \to G[\ell^m]_{\overline{\eta}}$ in [Lan 2013, Corollary 4.5.3.12].

4. Shimura varieties

4.1. *Notation on Shimura varieties.* Let (G, X) be a Shimura datum, and $E(G, X)$ the reflex field of (G, X). We simply write E for $E(G, X)$ if there is no risk of confusion. There is the canonical model over E of the Shimura variety for (G, X), which we denote by $\{\mathrm{Sh}_K(G, X)\}_{K \subset G(\mathbb{A}^{\infty})}$. Let $K \subset G(\mathbb{A}^{\infty})$ be a compact open subgroup, which is always supposed to be small enough so that $\mathrm{Sh}_K(G, X)$ becomes a scheme.

4.2. *Siegel modular varieties.* Let $(V, \langle \, , \, \rangle)$ be a symplectic space of dimension $2n$ over $\mathbb{Q}$, and L a self-dual $\mathbb{Z}$-lattice of V. Let $(\mathrm{GSp}_{2n}, X_{2n})$ be the Shimura datum associated to $(V, \langle \, , \, \rangle)$. Then the Shimura variety for $(\mathrm{GSp}_{2n}, X_{2n})$ is called the Siegel modular variety. In this case the reflex field $E(\mathrm{GSp}_{2n}, X_{2n})$ equals $\mathbb{Q}$. We put

$$K(N) = \mathrm{Ker}(\mathrm{GSp}_{2n}(\widehat{\mathbb{Z}}) \to \mathrm{GSp}_{2n}(\widehat{\mathbb{Z}}/N\widehat{\mathbb{Z}}))$$

for $N \geq 1$, and

$$K_{\ell,m} = \mathrm{Ker}(\mathrm{GSp}_{2n}(\mathbb{Z}_\ell) \to \mathrm{GSp}_{2n}(\mathbb{Z}/\ell^m\mathbb{Z}))$$

for a prime number ℓ and $m \geq 0$.

We recall a moduli interpretation of $\mathrm{Sh}_K(\mathrm{GSp}_{2n}, X_{2n})$ using integral level structures. For simplicity, we assume that $K = K(N)$ with $N \geq 3$. We consider the functor from the category of $\mathbb{Q}$-schemes to the category of sets, that associates S to the set of isomorphism classes of triples (A, λ, η), where

- A is an abelian scheme over S,

- $\lambda \colon A \to A^\vee$ is a principal polarization, and

- $\eta \colon L/NL \xrightarrow{\cong} A[N]$ is a symplectic similitude.

This functor is represented by $\mathrm{Sh}_K(\mathrm{GSp}_{2n}, X_{2n})$ (see [Deligne 1971, 4.16]).

There is another moduli interpretation using rational level structures. Let S be a connected Noetherian scheme over $\mathbb{Q}$, and fix a geometric point $\bar{s}$ of S. We put

$$T^\infty(-) = \prod_\ell T_\ell(-), \quad V^\infty(-) = T^\infty(-) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where ℓ in the product ranges over all prime numbers. Then, S-valued points of $\mathrm{Sh}_K(\mathrm{GSp}_{2n}, X_{2n})$ correspond to the isogeny classes of triples $(A, \lambda, \eta K)$, where

- A is an abelian scheme over S,

- $\lambda \colon A \to A^\vee$ is a $\mathbb{Q}$-polarization, and

- ηK is a $\pi_1(S, \bar{s})$-invariant K-orbit of symplectic similitudes $V_{\mathbb{A}^\infty} \xrightarrow{\cong} V^\infty A$.

Using this description, the Hecke action of $g \in \mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ can be described as

$$\mathrm{Sh}_K \to \mathrm{Sh}_{g^{-1}Kg}; \ [(A, \lambda, \eta K)] \mapsto [(A, \lambda, (\eta \circ g)g^{-1}Kg)].$$

See [Deligne 1971, 4.12] for the relation between two moduli interpretations.

Assume that $K = K_{p,0}K^p$ with a compact open subgroup K^p of $\mathrm{GSp}_{2n}(\widehat{\mathbb{Z}}^p)$. Then $\mathrm{Sh}_K(\mathrm{GSp}_{2n}, X_{2n})$ has a natural integral model $\mathscr{S}_{K^p}$ over $\mathbb{Z}_p$ constructed as a moduli space of principally polarized abelian schemes with level structures (*cf.* [Mumford et al. 1994, Chapter 7, §3]). Let $\mathcal{A}$ denote the universal abelian scheme on $\mathscr{S}_{K^p}$.

Thanks to a work of Faltings and Chai [Faltings and Chai 1990], we have a toroidal compactification $\mathscr{S}_{K^p}^{\mathrm{tor}}$ of $\mathscr{S}_{K^p}$ over $\mathbb{Z}_p$. We have a semi-abelian scheme on $\mathscr{S}_{K^p}^{\mathrm{tor}}$ extending $\mathcal{A}$ on $\mathscr{S}_{K^p}$, for which we write the same symbol $\mathcal{A}$.

4.3. *Shimura varieties of Hodge type.* In this subsection, we assume that (G, X) is of Hodge type. We take an embedding $i \colon (G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ of Shimura data. For a compact open subgroup $\widetilde{K}$ of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K, we have a natural morphism $\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})$ to the Siegel modular variety, which is known to be a closed immersion if $\widetilde{K}$ is small enough. We shall recall a moduli interpretation of $\mathbb{C}$-points of $\mathrm{Sh}_K(G, X)$. Let V be the standard representation of GSp_{2n}. By [Deligne et al. 1982, I, Proposition 3.1], there exists a finite collection of tensors $(s_\alpha)_{\alpha \in J'}$ with $s_\alpha \in V^{m_\alpha} \otimes V^{\vee m'_\alpha}$ such that G equals the pointwise stabilizer of $(s_\alpha)_{\alpha \in J'}$ in GSp_{2n}. We put $J = J' \amalg \{0\}$, $m_0 = m'_0 = 1$ and let s_0 be the symplectic form $\langle \, , \, \rangle \in V \otimes V^\vee$ on V.

Proposition 4.1. *A $\mathbb{C}$-valued point of $\mathrm{Sh}_K(G, X)$ corresponds to the isogeny class of triples $(A, (t_\alpha)_{\alpha \in J}, \eta K)$, where*

- *A is an abelian variety over $\mathbb{C}$,*
- *$(t_\alpha)_{\alpha \in J}$ with $t_\alpha \in H_1(A, \mathbb{Q})^{m_\alpha} \otimes H_1(A, \mathbb{Q})^{\vee m'_\alpha}$ is a finite collection of Hodge cycles on A (see [Deligne et al. 1982, V, §2]) such that $\pm t_0$ is a polarization of the rational Hodge structure $H_1(A, \mathbb{Q})$,*
- *ηK is a K-orbit of $\mathbb{A}^\infty$-linear isomorphisms $V_{\mathbb{A}^\infty} \xrightarrow{\cong} V^\infty A$ which send s_0 to a $(\mathbb{A}^\infty)^\times$-multiple of t_0 and s_α with $\alpha \in J'$ to t_α,*

satisfying the following condition $(*)$:

$(*)$ *there exists an isomorphism $\eta_{\mathbb{Q}} \colon V \xrightarrow{\cong} H_1(A, \mathbb{Q})$ such that $\eta_{\mathbb{Q}}^{-1}$ sends t_0 to a $\mathbb{Q}^\times$-multiple of s_0, t_α with $\alpha \in J'$ to s_α, and the Hodge structure on $H_1(A, \mathbb{Q})$ to a Hodge structure on V induced by an element of X and the embedding $i \colon G \hookrightarrow \mathrm{GSp}_{2n}$.*

For a proof, see [Milne 2005, Theorem 7.4].

Lemma 4.2. *Let F be a p-adic field containing the reflex field E, and x an F-valued point of $\mathrm{Sh}_K(G, X)$. Choose an algebraic closure $\overline{F}$ of F and denote by $\overline{x}$ the corresponding geometric point over x.*

We take an isomorphism $\iota \colon \overline{F} \xrightarrow{\cong} \mathbb{C}$ over E, and write $\iota\overline{x}$ for the $\mathbb{C}$-valued point of $\mathrm{Sh}_K(G, X)$ determined by $\overline{x}$ and ι. Let $(A, (t_\alpha)_{\alpha \in J}, \eta K)$ be a triple in the isogeny class corresponding to $\iota\overline{x}$ such that $A = \mathcal{A}_{\overline{x}} \otimes_{\overline{F}, \iota} \mathbb{C}$. Here $\mathcal{A}_{\overline{x}}$ is the abelian variety corresponding to the image of $\overline{x}$ in the Siegel modular variety. Let us choose a representative η of ηK. Under ι, it corresponds to a trivialization of the K-torsor $\pi_K^{-1}(\overline{x})$ on $\overline{x}$, where π_K denotes the natural map $\varprojlim_{K' \subset K} \mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$.

For a prime number ℓ, let $\mathcal{L}_{\mathrm{Std} \circ i, \ell}$ be the $\mathbb{Q}_\ell$-sheaf on $\mathrm{Sh}_K(G, X)$ corresponding to the representation $\mathrm{Std} \circ i$ of G on V. For the stalk $\mathcal{L}_{\mathrm{Std} \circ i, \ell, \overline{x}}$, the following hold:

(i) *We have a canonical $\mathrm{Gal}(\overline{F}/F)$-equivariant isomorphism $\mathcal{L}_{\mathrm{Std} \circ i, \ell, \overline{x}} \cong V_\ell \mathcal{A}_{\overline{x}}$.*

(ii) *Each trivialization of the K-torsor $\pi_K^{-1}(\overline{x})$ determines an isomorphism $V_{\mathbb{Q}_\ell} \xrightarrow{\cong} \mathcal{L}_{\mathrm{Std}\circ i, \ell, \overline{x}}$. The isomorphism given by the trivialization corresponding to the chosen representative η equals the composite of $V_{\mathbb{Q}_\ell} \xrightarrow[\cong]{\eta_\ell} V_\ell A \xrightarrow[\cong]{\iota^{-1}} V_\ell \mathcal{A}_{\overline{x}} \overset{(i)}{\cong} \mathcal{L}_{\mathrm{Std}\circ i, \ell, \overline{x}}$, where η_ℓ denotes the ℓ-part of η.*

Proof. The first assertion is essentially a statement for the Siegel case, which is well-known. The second can be checked directly by working over $\mathbb{C}$.　　□

4.4. *Shimura varieties of preabelian type.*

Definition 4.3. A Shimura datum (G, X) is said to be of preabelian type if there exists a Shimura datum (G', X') of Hodge type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$. If a Shimura data is of preabelian type, the associated Shimura variety is said to be of preabelian type (*cf.* [Vasiu 1999, p. 402]).

Lemma 4.4. *Assume that (G, X) is of preabelian type. We take a Shimura datum (G', X') of Hodge type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$. Let K'' be a compact open subgroup of $G^{\mathrm{ad}}(\mathbb{A}^\infty)$ which contains the image of K under the map $G(\mathbb{A}^\infty) \to G^{\mathrm{ad}}(\mathbb{A}^\infty)$. We regard it as a compact open subgroup of $G'^{\mathrm{ad}}(\mathbb{A}^\infty)$ by the isomorphism $G^{\mathrm{ad}} \cong G'^{\mathrm{ad}}$. Then there exist a compact open subgroup $K' \subset G'(\mathbb{A}^\infty)$ and $g_1, \ldots, g_m \in G'^{\mathrm{ad}}(\mathbb{A}^\infty)$ such that the following hold:*

(i) *The morphism $(G', X') \to (G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$ and the conjugation by g_i induces the morphism*

$$f_i \colon \mathrm{Sh}_{K'}(G', X') \to \mathrm{Sh}_{g_i^{-1}K''g_i}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}}) \to \mathrm{Sh}_{K''}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$$

for each i.

(ii) *The morphism*

$$\coprod_{1 \le i \le m} f_i \colon \coprod_{1 \le i \le m} \mathrm{Sh}_{K'}(G', X') \to \mathrm{Sh}_{K''}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$$

is surjective.

Proof. This follows from the definition of Shimura varieties of preabelian type and the fact that Hecke action is transitive on the connected components of a Shimura variety.　　□

5. Partition of Shimura varieties

5.1. *Partition of classical points.* We fix a prime number p and a finite place v of E above p. We write $\mathcal{O}_v$ for the ring of integers of E_v.

Throughout the paper, we assume that a compact open subgroup K of $G(\mathbb{A}^\infty)$ is small enough so that the following conditions are satisfied:

- The morphism $\pi_K \colon \varprojlim_{K' \subset K} \mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$ is a torsor under the quotient K_{Sh} of K by a closed subgroup of $K \cap Z(G)(\mathbb{A}^\infty)$ (cf. [Milne 2005, Theorem 5.28]).

- If the Shimura datum (G, X) satisfies the condition SV5 in [Milne 2005, p. 311], then K_{Sh} equals K. Note that a Shimura datum of Hodge type satisfies SV5.

Definition 5.1. Let x be a classical point of $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$, and $\overline{\kappa}_x$ an algebraic closure of κ_x. We write $\overline{x}$ for the geometric point corresponding to $\overline{\kappa}_x$.

By taking the pull-back of $\pi_K \colon \varprojlim_{K' \subset K} \mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$, we obtain a K_{Sh}-torsor $\pi_K^{-1}(x)$ on x. This torsor and its trivialization η over $\overline{x}$ give rise to a continuous homomorphism $\phi_{x,\eta} \colon \mathrm{Gal}(\overline{\kappa}_x/\kappa_x) \to K_{\mathrm{Sh}}$. If we change η, the homomorphism $\phi_{x,\eta}$ changes by a K_{Sh}-conjugation.

(i) We write $\phi^{\mathrm{ad}}_{x,\eta}$ for the composite $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x) \xrightarrow{\phi_{x,\eta}} K_{\mathrm{Sh}} \to G^{\mathrm{ad}}(\mathbb{A}^\infty)$. If we change η, the homomorphism $\phi^{\mathrm{ad}}_{x,\eta}$ changes by a K^{ad}-conjugation, where K^{ad} denotes the image of K in $G^{\mathrm{ad}}(\mathbb{A}^\infty)$. When we are only interested in the K^{ad}-conjugacy class of $\phi^{\mathrm{ad}}_{x,\eta}$, we often drop the subscript η and simply write ϕ^{ad}_x for $\phi^{\mathrm{ad}}_{x,\eta}$.

  For a prime number ℓ, we denote by $\phi^{\mathrm{ad}}_{x,\eta,\ell}$ the composite of $\phi^{\mathrm{ad}}_{x,\eta}$ and the projection $G^{\mathrm{ad}}(\mathbb{A}^\infty) \to G^{\mathrm{ad}}(\mathbb{Q}_\ell)$.

(ii) Assume that (G, X) satisfies the condition SV5. Then we write $\phi_{x,\eta}$ for the composite $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x) \xrightarrow{\phi_{x,\eta}} K \to G(\mathbb{A}^\infty)$. As in (i), we often write ϕ_x for $\phi_{x,\eta}$, which is well-defined up to K-conjugacy.

  For a prime number ℓ, we define $\phi_{x,\eta,\ell}$ similarly.

Remark 5.2. The homomorphism $\phi^{\mathrm{ad}}_{x,\eta,\ell}$ is related to ℓ-adic automorphic étale sheaves on $\mathrm{Sh}_K(G, X)$ as follows. Let (ξ, V_ξ) be a finite-dimensional algebraic representation of G over $\overline{\mathbb{Q}}_\ell$ such that $\mathrm{Ker}\,\xi$ contains $\mathrm{Ker}(K \to K_{\mathrm{Sh}})$. Then, we have an associated smooth $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\xi$ on $\mathrm{Sh}_K(G, X)$ (cf. [Milne 1990, Remark III.6.1]). As in Lemma 4.2 (ii), the trivialization η of $\pi^{-1}(\overline{x})$ induces an isomorphism $\mathcal{L}_{\xi,\overline{x}} \cong V_\xi$. Hence we obtain an ℓ-adic Galois representation $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x) \to \mathrm{GL}(\mathcal{L}_{\xi,\overline{x}}) \cong \mathrm{GL}(V_\xi)$.

(i) If ξ factors through G^{ad}, it is equal to the composite

$$\mathrm{Gal}(\overline{\kappa}_x/\kappa_x) \xrightarrow{\phi^{\mathrm{ad}}_{x,\eta,\ell}} G^{\mathrm{ad}}(\mathbb{Q}_\ell) \xrightarrow{\xi} \mathrm{GL}(V_\xi).$$

(ii) If (G, X) satisfies SV5 (hence any ξ is allowable), it is equal to the composite
$$\mathrm{Gal}(\overline{\kappa}_x/\kappa_x) \xrightarrow{\phi_{x,\eta,\ell}} G(\mathbb{Q}_\ell) \xrightarrow{\xi} \mathrm{GL}(V_\xi).$$

The following proposition can be checked easily.

Proposition 5.3. *Let $(G, X) \to (G', X')$ be a morphism of Shimura data such that $Z(G)$ is mapped into $Z(G')$. Let $K \subset G(\mathbb{A}^\infty)$ and $K' \subset G'(\mathbb{A}^\infty)$ be compact open subgroups such that K is mapped into K'. For $x \in \mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl})$, we write x' for the image of x under the induced morphism $\mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(G', X')$. Then the diagram*

$$
\begin{array}{ccc}
\mathrm{Gal}(\overline{\kappa}_x/\kappa_x) & \xrightarrow{\phi^{\mathrm{ad}}_x} & G^{\mathrm{ad}}(\mathbb{A}^\infty) \\
\downarrow & & \downarrow \\
\mathrm{Gal}(\overline{\kappa}_{x'}/\kappa_{x'}) & \xrightarrow{\phi^{\mathrm{ad}}_{x'}} & G'^{\mathrm{ad}}(\mathbb{A}^\infty)
\end{array}
$$

is commutative up to K'^{ad}-conjugacy, where K'^{ad} denotes the image of K' in $G'^{\mathrm{ad}}(\mathbb{A}^\infty)$.

Proposition 5.4. *Assume that (G, X) is of preabelian type.*

(i) *For $x \in \mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl})$, $\phi^{\mathrm{ad}}_{x,p}$ is de Rham and has the monodromy filtration.*

(ii) *Assume that (G, X) is of Hodge type. Then, for $x \in \mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl})$, $\phi_{x,p}$ is de Rham and has the monodromy filtration.*

Proof. By Lemma 2.9 (iv), Lemma 2.10, Lemma 4.4 and Proposition 5.3, the assertion (i) is reduced to (ii). We prove (ii). Take an embedding $i : (G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum and a compact open subgroup $\widetilde{K} = \widetilde{K}_p \widetilde{K}^p$ of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K. By shrinking K, we may assume that $\widetilde{K}_p \subset \widetilde{K}_{p,0} = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$ and $\widetilde{K}^p$ is small enough. Then, the morphism

$$
\mathrm{Spec}\,\kappa_x \to \mathrm{Sh}_K(G, X)_{E_v} \to \mathrm{Sh}_{\widetilde{K}_{p,0}\widetilde{K}^p}(\mathrm{GSp}_{2n}, X_{2n})_{\mathbb{Q}_p} = \mathscr{S}_{\widetilde{K}^p, \mathbb{Q}_p}
$$

uniquely extends to $\mathrm{Spec}\,\kappa_x^+ \to \mathscr{S}^{\mathrm{tor}}_{\widetilde{K}^p}$. Let $\mathcal{A}_{\kappa_x^+}$ denote the pull-back of the universal semi-abelian scheme $\mathcal{A}$ by this morphism. It extends the abelian variety $\mathcal{A}_x$ over κ_x. Therefore, the representation $V_p \mathcal{A}_{\overline{x}}$ of $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x)$ is semistable and has the monodromy filtration by Corollary 3.12.

Let $\mathrm{Std} : \mathrm{GSp}_{2n} \to \mathrm{GL}(V)$ denote the standard representation of GSp_{2n}. By Lemma 4.2 (i), we have a $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x)$-equivariant isomorphism $\mathcal{L}_{\mathrm{Std}\circ i, \overline{x}, p} \cong V_p \mathcal{A}_{\overline{x}}$. We fix a trivialization η of the K-torsor $\pi_K^{-1}(x)$ over $\overline{x}$. By the isomorphism $V_{\mathbb{Q}_p} \cong \mathcal{L}_{\mathrm{Std}\circ i, \overline{x}, p}$ induced from η, we regard $V_{\mathbb{Q}_p}$ as a representation of $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x)$. As in Remark 5.2 (ii), it is isomorphic to $\mathrm{Std} \circ i \circ \phi_{x,\eta,p}$. Summing up, we obtain a $\mathrm{Gal}(\overline{\kappa}_x/\kappa_x)$-equivariant isomorphism $\mathrm{Std} \circ i \circ \phi_{x,\eta,p} \cong V_p \mathcal{A}_{\overline{x}}$. Therefore, we conclude that $\mathrm{Std} \circ i \circ \phi_{x,\eta,p}$ is semistable (hence de Rham) and has the monodromy filtration. Since $\mathrm{Std} \circ i$ is a faithful representation of G, $\phi_{x,\eta,p}$ is de Rham and has the monodromy filtration by Lemma 2.8. This completes the proof. $\square$

Remark 5.5. Assume that (G, X) satisfies the condition SV6 in [Milne 2005, p. 312]. Recently, Liu and Zhu announced a result that the p-adic sheaf $\mathcal{L}_{\xi, x}$ is de

Rham for any finite-dimensional algebraic representation ξ of G^c over $\overline{\mathbb{Q}}_p$, where G^c is the quotient of G defined in [Milne 1990, p. 347] (*cf.* [Liu and Zhu 2017, Theorem 1.2]). This implies that $\phi_{x,p}^{\mathrm{ad}}$ is de Rham. We do not use this remark later.

In the sequel, we assume that (G, X) is of preabelian type. Take a finite nonempty set of primes S such that $K = K_S K^S$, where K_S is a compact open subgroup of $G(\mathbb{A}_S)$ and K^S is a hyperspecial compact open subgroup of $G(\mathbb{A}^{\infty,S})$. We write $\mathcal{P}_{G,S}(K_S)$ for the set of K_S-conjugacy classes of $\mathbb{A}_S$-parabolic subgroups of G.

Let η be a trivialization of $\pi_K^{-1}(x)$ over $\overline{x}$. By Proposition 5.4 and the results in Section 2, we can attach to $\phi_{x,\eta,\ell}^{\mathrm{ad}}$ the $\mathbb{Q}_\ell$-parabolic subgroup $P_{\phi_{x,\eta,\ell}^{\mathrm{ad}}}$ of G^{ad} for each $\ell \in S$ and $x \in \mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}(\mathrm{cl})$. By taking the product with respect to ℓ, we obtain an $\mathbb{A}_S$-parabolic subgroup of G^{ad}. It is easy to observe that the K_S^{ad}-conjugacy class $[\prod_{\ell \in S} P_{\phi_{x,\eta,\ell}^{\mathrm{ad}}}] \in \mathcal{P}_{G^{\mathrm{ad}},S}(K_S^{\mathrm{ad}})$ is independent of the choice of η. Note that the natural map $\mathcal{P}_{G,S}(K_S) \to \mathcal{P}_{G^{\mathrm{ad}},S}(K_S^{\mathrm{ad}})$; $[P] \mapsto [P^{\mathrm{ad}}]$ is bijective.

Definition 5.6. Let $[P_{x,S}] \in \mathcal{P}_{G,S}(K_S)$ be the K_S-conjugacy class that is mapped to $[\prod_{\ell \in S} P_{\phi_{x,\eta,\ell}^{\mathrm{ad}}}]$ under the bijection $\mathcal{P}_{G,S}(K_S) \to \mathcal{P}_{G^{\mathrm{ad}},S}(K_S^{\mathrm{ad}})$.

Remark 5.7. If the Shimura datum (G, X) satisfies the condition SV5, we can define $[P_{x,S}]$ directly by using $\phi_{x,\eta}$. These two ways give the same result by Lemma 2.2 (iv) and Lemma 2.9 (iv).

By the proof of Proposition 5.4, we obtain the following description of $[P_{x,S}]$ in the Hodge type case.

Corollary 5.8. *Let (G, X) be a Shimura datum of Hodge type with an embedding $(G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. Assume that $K_p \subset \mathrm{GSp}_{2n}(\mathbb{Z}_p)$ and K^p is small enough. For $x \in \mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}(\mathrm{cl})$, fix an isomorphism $\iota \colon \overline{\kappa}_x \xrightarrow{\cong} \mathbb{C}$ and let $(A, (t_\alpha), \eta K)$ be a triple in the isogeny class corresponding to the $\mathbb{C}$-point $\iota\overline{x}$ of $\mathrm{Sh}_K(G, X)$ such that $A = \mathcal{A}_{\overline{x}} \otimes_{\overline{\kappa}_x,\iota} \mathbb{C}$.*

(i) *The abelian variety $\mathcal{A}_x$ over κ_x extends to a semi-abelian scheme $\mathcal{A}_{\kappa_x^+}$ over κ_x^+. For $\ell \in S$, the monodromy filtration $M_\bullet V_\ell \mathcal{A}_{\overline{x}}$ on $V_\ell \mathcal{A}_{\overline{x}}$ is a shift of the filtration in Proposition 3.11.*

(ii) *Fix an arbitrary representative $\eta \colon V_{\mathbb{A}^\infty} \xrightarrow{\cong} V^\infty A$ of the K-orbit ηK. For $\ell \in S$, consider the filtration $(\eta_\ell^{-1} \circ \iota)(M_\bullet V_\ell \mathcal{A}_{\overline{x}})$ on $V_{\mathbb{Q}_\ell}$ obtained as the inverse image of $M_\bullet V_\ell \mathcal{A}_{\overline{x}}$ under $V_{\mathbb{Q}_\ell} \xrightarrow[\cong]{\eta_\ell} V_\ell A \xrightarrow[\cong]{\iota^{-1}} V_\ell \mathcal{A}_{\overline{x}}$. Then, this filtration is $G_{\mathbb{Q}_\ell}$-split in the sense of [Kisin 2010, (1.1.2)]. Moreover, if we write $P_{x,\eta,\ell}$ for the stabilizer of this filtration, the K_S-conjugacy class $[\prod_{\ell \in S} P_{x,\eta,\ell}]$ equals $[P_{x,S}]$.*

(iii) *If $(G, X) = (\mathrm{GSp}_{2n}, X_{2n})$, then $P_{x,\eta,\ell}$ in (ii) is the stabilizer of a totally isotropic subspace of $V_{\mathbb{Q}_\ell}$ whose dimension equals the toric rank of the special fiber of $\mathcal{A}_{\kappa_x^+}$.*

Proof. The assertion (i) follows from the proofs of Proposition 5.4 and Corollary 3.12.

We prove (ii). The choice of η gives a trivialization of the K-torsor $\pi_K^{-1}(x)$ over $\bar{x}$, which is denoted by the same symbol η. By the argument in the proof of Proposition 5.4, we have $\mathrm{Gal}(\bar{\kappa}_x/\kappa_x)$-equivariant isomorphisms

$$\mathrm{Std} \circ i \circ \phi_{x,\eta,\ell} = V_{\mathbb{Q}_\ell} \overset{\eta}{\cong} \mathcal{L}_{\mathrm{Std}\circ i, \bar{x}, \ell} \cong V_\ell \mathcal{A}_{\bar{x}}.$$

By Lemma 4.2 (ii), their composite is equal to

$$\mathrm{Std} \circ i \circ \phi_{x,\eta,\ell} = V_{\mathbb{Q}_\ell} \overset{\eta_\ell}{\underset{\cong}{\longrightarrow}} V_\ell A \overset{\iota^{-1}}{\underset{\cong}{\longrightarrow}} V_\ell \mathcal{A}_{\bar{x}}.$$

Hence the filtration $(\eta_\ell^{-1} \circ \iota)(M_\bullet V_\ell \mathcal{A}_{\bar{x}})$ equals the monodromy filtration $M_\bullet V_{\mathbb{Q}_\ell}$ on $V_{\mathbb{Q}_\ell}$ with respect to the action of $\mathrm{Gal}(\bar{\kappa}_x/\kappa_x)$ by $\mathrm{Std} \circ i \circ \phi_{x,\eta,\ell}$. Since the monodromy filtration of $\mathrm{Std} \circ i \circ \phi_{x,\eta,\ell}$ extends to a filtration on the Tannakian category $\mathbf{Rep}_{\mathbb{Q}_\ell}(G_{\mathbb{Q}_\ell})$, we conclude that $M_\bullet V_{\mathbb{Q}_\ell}$ is $G_{\mathbb{Q}_\ell}$-split by [Kisin 2010, Lemma 1.1.3]. Further, by Lemma 2.2 (ii) and Lemma 2.9 (ii), we have $P_{x,\eta,\ell} = P_{\phi_{x,\eta,\ell}}$. Therefore we have $[P_{x,S}] = [\prod_{x \in S} P_{x,\eta,\ell}]$ by Remark 5.7.

The (iii) follows from Corollary 3.12 and the equality $(V_p \widehat{G}_{\bar{\eta}}^{\mathrm{rig}})^\perp = V_p \widehat{T}_{\bar{\eta}}^{\mathrm{rig}}$ (and its ℓ-adic version) in the proof of Proposition 3.11. $\square$

Next we will show that the ℓ-part of $P_{x,S}$ is independent of $\ell \in S$ in some sense. To state the result, we need some preparation.

Definition 5.9. (*cf.* [Pink 1989, 4.5 Definition]) Let $G^{\mathrm{ad}} = G_1 \times \cdots \times G_r$ be a decomposition into $\mathbb{Q}$-simple factors. We say that a parabolic subgroup P of G is an admissible $\mathbb{Q}$-parabolic subgroup if there exists a parabolic subgroup P_i of G_i for each i such that P is the inverse image of $P_1 \times \cdots \times P_r$ and P_i is either equal to G_i or a maximal $\mathbb{Q}$-parabolic subgroup of G_i for each i. We write $\mathcal{P}_{G,\mathbb{Q}}$ for the set of $G(\mathbb{Q})$-conjugacy classes of admissible $\mathbb{Q}$-parabolic subgroups of G.

An admissible $\mathbb{A}^\infty$-parabolic subgroup means a parabolic subgroup of $G_{\mathbb{A}^\infty}$ which is $G(\mathbb{A}^\infty)$-conjugate to an admissible $\mathbb{Q}$-parabolic subgroup of G. Let $\mathcal{P}_G(K)$ denote the set of K-conjugacy classes of admissible $\mathbb{A}^\infty$-parabolic subgroups of G. Further, we write $\mathcal{P}_{G,\mathbb{A}^\infty}$ for the set of $G(\mathbb{A}^\infty)$-conjugacy classes of admissible $\mathbb{A}^\infty$-parabolic subgroups of G. We have a natural map $\mathcal{P}_G(K) \to \mathcal{P}_{G,\mathbb{A}^\infty}$.

Lemma 5.10.   (i) *The natural map $\mathcal{P}_G(K) \to \mathcal{P}_{G,S}(K_S)$ is injective.*

(ii) *The set $\mathcal{P}_G(K)$ is finite.*

(iii) *We take a hyperspecial compact open subgroup K''^S of $G^{\mathrm{ad}}(\mathbb{A}^{\infty,S})$ containing the image of K^S, and put $K'' = K_S^{\mathrm{ad}} K''^S$, which is a compact open subgroup of $G^{\mathrm{ad}}(\mathbb{A}^\infty)$. Then, the natural map $\mathcal{P}_G(K) \to \mathcal{P}_{G^{\mathrm{ad}}}(K'')$ is bijective.*

Proof. Fix a minimal parabolic subgroup P_0 of G. For an admissible $\mathbb{Q}$-parabolic subgroup P containing P_0, we write $\mathcal{P}_G(K)_P$ for the subset of $\mathcal{P}_G(K)$ consisting of K-conjugacy classes which are $G(\mathbb{A}^\infty)$-conjugate to P. Then, we have a bijection $K \backslash G(\mathbb{A}^\infty) / P(\mathbb{A}^\infty) \xrightarrow{\cong} \mathcal{P}_G(K)_P$ given by $KgP(\mathbb{A}^\infty) \mapsto gPg^{-1}$.

Let us prove (i). For an admissible $\mathbb{Q}$-parabolic subgroup P containing P_0, we have

$$K \backslash G(\mathbb{A}^\infty) / P(\mathbb{A}^\infty) = K_S \backslash G(\mathbb{A}_S) / P(\mathbb{A}_S) \times K^S \backslash G(\mathbb{A}^S) / P(\mathbb{A}^S)$$
$$= K_S \backslash G(\mathbb{A}_S) / P(\mathbb{A}_S)$$

by the Iwasawa decomposition $K^S P_0(\mathbb{A}^S) = G(\mathbb{A}^S)$. This implies that the composite $\mathcal{P}_G(K)_P \hookrightarrow \mathcal{P}_G(K) \to \mathcal{P}_{G,S}(K_S)$ is injective. It suffices to show that the images of $\mathcal{P}_G(K)_P$ and $\mathcal{P}_G(K)_{P'}$ in $\mathcal{P}_{G,S}(K_S)$ are disjoint, where P and P' are distinct admissible $\mathbb{Q}$-parabolic subgroups containing P_0. If the images of $\mathcal{P}_G(K)_P$ and $\mathcal{P}_G(K)_{P'}$ intersect, then $P_{\mathbb{Q}_\ell}$ and $P'_{\mathbb{Q}_\ell}$ are $G(\mathbb{Q}_\ell)$-conjugate for each $\ell \in S$. By [Borel and Tits 1965, Théorème 4.13], this means that P and P' are $G(\mathbb{Q})$-conjugate. Since they contain P_0, they are equal. Note that in particular we have $\mathcal{P}_G(K)_P \cap \mathcal{P}_G(K)_{P'} = \varnothing$. Hence $\mathcal{P}_G(K)$ equals $\coprod_{P \supset P_0} \mathcal{P}_G(K)_P$, where P runs through admissible $\mathbb{Q}$-parabolic subgroups of G containing P_0.

Next we prove (ii). It suffices to show that $\mathcal{P}_G(K)_P$ is a finite set for each admissible $\mathbb{Q}$-parabolic subgroup P of G containing P_0. Since

$$\mathcal{P}_G(K)_P \cong K \backslash G(\mathbb{A}^\infty) / P(\mathbb{A}^\infty) \cong K_S \backslash G(\mathbb{A}_S) / P(\mathbb{A}_S),$$

it suffices to show that $K_S \backslash G(\mathbb{A}_S) / P(\mathbb{A}_S)$ is a finite set. Let K_S^0 be the product of special compact open subgroups of $G(\mathbb{Q}_\ell)$ for $\ell \in S$. Then we have $K_S^0 P_0(\mathbb{A}_S) = G(\mathbb{A}_S)$. By shrinking K_S, we may assume that $K_S \subset K_S^0$. Then the map $K_S \backslash K_S^0 \to K_S \backslash G(\mathbb{A}_S) / P(\mathbb{A}_S)$ is surjective, hence $K_S \backslash G(\mathbb{A}_S) / P(\mathbb{A}_S)$ is finite.

Finally we prove (iii). Note that admissible $\mathbb{Q}$-parabolic subgroups of G containing P_0 are in bijection with those of G^{ad} containing P_0^{ad}. Therefore, we have only to show that $\mathcal{P}_G(K)_P \to \mathcal{P}_{G^{\mathrm{ad}}}(K'')_{P^{\mathrm{ad}}}$ is bijective for every admissible $\mathbb{Q}$-parabolic subgroup of G containing P_0. Further, it is equivalent to the bijectivity of

$$K_S \backslash G(\mathbb{A}_S) / P(\mathbb{A}_S) \xrightarrow{(*)} K_S^{\mathrm{ad}} \backslash G^{\mathrm{ad}}(\mathbb{A}_S) / P^{\mathrm{ad}}(\mathbb{A}_S).$$

By [Springer 1998, 15.1.4], we have

$$G(\mathbb{Q}_\ell) / P(\mathbb{Q}_\ell) \cong (G_{\mathbb{Q}_\ell} / P_{\mathbb{Q}_\ell})(\mathbb{Q}_\ell) \cong (G_{\mathbb{Q}_\ell}^{\mathrm{ad}} / P_{\mathbb{Q}_\ell}^{\mathrm{ad}})(\mathbb{Q}_\ell) \cong G^{\mathrm{ad}}(\mathbb{Q}_\ell) / P^{\mathrm{ad}}(\mathbb{Q}_\ell)$$

for each $\ell \in S$. Therefore the map $G(\mathbb{A}_S)/P(\mathbb{A}_S) \to G^{\mathrm{ad}}(\mathbb{A}_S)/P^{\mathrm{ad}}(\mathbb{A}_S)$ is bijective. The bijectivity of $(*)$ easily follows from it.   □

By the proof above, we also obtain the following:

Corollary 5.11. *The natural map $\mathcal{P}_{G,\mathbb{Q}} \to \mathcal{P}_{G,\mathbb{A}^\infty}$ is a bijection. In particular, for a compact open subgroup K of $G(\mathbb{A}^\infty)$, we have a natural map $\mathcal{P}_G(K) \to \mathcal{P}_{G,\mathbb{Q}}$.*

Proof. We use the notation in the proof of Lemma 5.10. By definition, the natural map $\mathcal{P}_{G,\mathbb{Q}} \to \mathcal{P}_{G,\mathbb{A}^\infty}$ is surjective. We shall show that it is injective. Take two admissible $\mathbb{Q}$-parabolic subgroups P_1, P_2 of G containing P_0. If P_1 and P_2 are $G(\mathbb{A}^\infty)$-conjugate, then $[P_1] \in \mathcal{P}_G(K)_{P_1} \cap \mathcal{P}_G(K)_{P_2}$ for every compact open subgroup K of $G(\mathbb{A}^\infty)$. By the proof of Lemma 5.10, it implies that $P_1 = P_2$. This completes the proof.   □

Proposition 5.12. *For $x \in \mathrm{Sh}_K(G,X)^{\mathrm{ad}}_{E_v}(\mathrm{cl})$, there uniquely exists an element $[P_x] \in \mathcal{P}_G(K)$ which is mapped to $[P_{x,S}]$ under the injection $\mathcal{P}_G(K) \hookrightarrow \mathcal{P}_{G,S}(K_S)$ in Lemma 5.10 (i). It is independent of S.*

To prove this proposition, we use the following lemma.

Lemma 5.13. *Let (G,X) be a Shimura datum of Hodge type with an embedding $i: (G,X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. Recall that V denotes the standard representation of GSp_{2n}.*

Let W be a totally isotropic subspace of V, and define a filtration $W_\bullet V$ on V as follows:

$$W_0 V = V, \quad W_{-1}V = W^\perp, \quad W_{-2}V = W, \quad W_{-3}V = 0.$$

We write P for the stabilizer of $W_\bullet V$ in G.

Let L be a field of characteristic 0. Assume that the filtration $W_\bullet V \otimes_\mathbb{Q} L$ on $V_L = V \otimes_\mathbb{Q} L$ is G_L-split in the sense of [Kisin 2010, (1.1.2)]. Then, P is an admissible $\mathbb{Q}$-parabolic subgroup of G. Further, if we write P' for the stabilizer of $W_\bullet V$ in GSp_{2n}, we have $P' = i_* P$ in the notation of [Madapusi Pera 2019, 2.1.28]. Namely, the cocharacter of GSp_{2n} associated to P' (as in [Pink 1989, 4.1]) is equal to $i \circ \lambda$, where λ is the cocharacter of G associated to P.

Proof. We write U for the subgroup of P consisting of elements acting on $\mathrm{gr}^W_\bullet V$ trivially, and ν for the cocharacter $\mathbb{G}_m \to \mathrm{GL}(\mathrm{gr}^W_\bullet V)$ determined from the grading on $\mathrm{gr}^W_\bullet V$.

By [Kisin 2010, Lemma 1.1.1], P_L is a parabolic subgroup of G_L, U_L is the unipotent radical of P_L, and the cocharacter $\nu_L: \mathbb{G}_m \to \mathrm{GL}(\mathrm{gr}^W_\bullet V \otimes_\mathbb{Q} L)$ over L factors through the closed subgroup P_L/U_L. Therefore, we conclude that P is a parabolic subgroup of G, U is the unipotent radical of P, and the cocharacter $\nu: \mathbb{G}_m \to \mathrm{GL}(\mathrm{gr}^W_\bullet V)$ factors through P/U. This means that the filtration $W_\bullet V$ is G-split by [Kisin 2010, Lemma 1.1.1]. Take a cocharacter $w: \mathbb{G}_m \to G$ over

$\mathbb{Q}$ which induces the filtration $W_\bullet V$ on V. It induces a filtration on the Tannakian category $\mathbf{Rep}_\mathbb{Q}(G)$. Let us prove that this filtration is Cayley in the sense of [Milne 1990, V, Definition 2.3]. Take an arbitrary element $h \in X^+$. By [Milne 1990, IV, Example 1.1(c)] (*cf.* [Brylinski 1983, 4.2.1]), $W_\bullet V$ and h give a mixed Hodge structure on V. Therefore, [Milne 1990, IV, Proposition 1.3] tells us that w and h define a mixed Hodge structure on V_ξ for all objects (ξ, V_ξ) of $\mathbf{Rep}_\mathbb{Q}(G)$. Hence the filtration induced from w is Cayley, as desired.

Now, by applying [Milne 1990, V, Proposition 2.4] to each simple factor of $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) = (G_1, X_1) \times \cdots \times (G_r, X_r)$, we conclude that P is admissible (we use the same argument as in the proof of Lemma 2.2 (iv) to pass to the adjoint group). The equality $P' = i_* P$ is proved in [Pink 1989, 4.16]. □

Proof of Proposition 5.12. Only the existence of $[P_x]$ requires a proof. By Lemma 2.2 (v), Lemma 2.10, Lemma 4.4, Proposition 5.3 and Lemma 5.10 (iii), we may assume that (G, X) is of Hodge type. We use the notation in Section 4.3. By shrinking K, we may assume that $K_p \subset \widetilde{K}_{p,0} = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$ and K^p is small enough. We use the notation in Corollary 5.8. Let $M_x = (G^\sharp, \underline{Y} \to G^\sharp_{\kappa_x})$ be the degeneration datum corresponding to $\mathcal{A}_{\kappa_x^+}$ under the functor M in [Faltings and Chai 1990, Chapter III, Corollary 7.2]. It gives a 1-motive $M_{\overline{x}}$ over $\overline{\kappa}_x$ (*cf.* [Deligne 1974, §10.1]). For each prime $\ell \in S$, the ℓ-adic realization $H_1(M_{\overline{x}}, \mathbb{Q}_\ell)$ of $M_{\overline{x}}$ is identified with $V_\ell \mathcal{A}_{\overline{x}}$, and equipped with the weight filtration $W_{\bullet, \overline{x}, \ell}$, which coincides with the monodromy filtration $M_\bullet V_\ell \mathcal{A}_{\overline{x}}$ on $V_\ell \mathcal{A}_{\overline{x}}$ up to a shift.

The 1-motive $M_{\overline{x}}$ and the fixed isomorphism $\iota: \overline{\kappa}_x \xrightarrow{\cong} \mathbb{C}$ gives rise to a 1-motive $M_{\iota\overline{x}}$ over $\mathbb{C}$. Its Betti realization $H_1(M_{\iota\overline{x}}, \mathbb{Q})$ is naturally isomorphic to $H_1(A, \mathbb{Q})$ (recall that $(A, (t_\alpha)_{\alpha \in J}, \eta K)$ denotes the triple corresponding to the $\mathbb{C}$-point $\iota\overline{x}$). We denote the weight filtration on it by $W_{\bullet, \iota\overline{x}, \mathbb{Q}}$. It is known that $W_{-2, \iota\overline{x}, \mathbb{Q}}$ is totally isotropic and $W_{-1, \iota\overline{x}, \mathbb{Q}} = W^\perp_{-2, \iota\overline{x}, \mathbb{Q}}$ with respect to the polarization $\pm t_0$ on A. For $\ell \in S$, we write ε_ℓ for the comparison isomorphism $H_1(A, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{Q}_\ell \xrightarrow{\cong} V_\ell A$. The composite

$$H_1(A, \mathbb{Q}) \otimes_\mathbb{Q} \mathbb{Q}_\ell \xrightarrow[\cong]{\varepsilon_\ell} V_\ell A \xrightarrow[\cong]{\iota^{-1}} V_\ell \mathcal{A}_{\overline{x}} = H_1(M_{\overline{x}}, \mathbb{Q}_\ell)$$

carries the filtration $W_{\bullet, \iota\overline{x}, \mathbb{Q}} \otimes_\mathbb{Q} \mathbb{Q}_\ell$ onto $W_{\bullet, \overline{x}, \ell}$ defined above.

We take a representative $\eta: V_{\mathbb{A}^\infty} \xrightarrow{\cong} V^\infty A$ of the K-orbit ηK and an isomorphism $\eta_\mathbb{Q}: V \xrightarrow{\cong} H_1(A, \mathbb{Q})$ as in the condition $(*)$ of Proposition 4.1. For each prime ℓ, we can easily observe that $g_\ell = \eta_\ell^{-1} \circ \varepsilon_\ell \circ (\eta_\mathbb{Q} \otimes \mathbb{Q}_\ell)$ preserves the tensors $(s_\alpha)_{\alpha \in J'}$. Therefore g_ℓ lies in $G(\mathbb{Q}_\ell)$.

We put $W_\bullet V = \eta_{\mathbb{Q}}^{-1}(W_{\bullet,\iota\overline{x},\mathbb{Q}})$ (it depends on the choice of $\overline{x}$, ι and $\eta_{\mathbb{Q}}$), and denote by P' the stabilizer of $W_\bullet V$ in G. For $\ell \in S$, we have

$$(\eta_\ell^{-1} \circ \iota)(W_{\bullet,\overline{x},\ell}) = (\eta_\ell^{-1} \circ \varepsilon_\ell)(W_{\bullet,\iota\overline{x},\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) = g_\ell(W_\bullet V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell).$$

By Corollary 5.8 (ii), $g_\ell(W_\bullet V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$ is $G_{\mathbb{Q}_\ell}$-split. Hence $W_\bullet V \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ is also $G_{\mathbb{Q}_\ell}$-split. Therefore, Lemma 5.13 tells us that P' is an admissible $\mathbb{Q}$-parabolic subgroup of G. Further, by the above equality, $P_{x,\eta,\ell}$ in Corollary 5.8 (ii) is equal to $g_\ell P'_{\mathbb{Q}_\ell} g_\ell^{-1}$.

Set $g = (g_\ell)_{\ell \in S} \times 1 \in G(\mathbb{A}^\infty) = G(\mathbb{A}_S) \times G(\mathbb{A}^{\infty,S})$. Then $P_x = g P'_{\mathbb{A}^\infty} g^{-1}$ is an admissible $\mathbb{A}^\infty$-parabolic subgroup of G, and its image $[P_x]$ in $\mathcal{P}_K(G)$ is mapped to $[\prod_{\ell \in S} P_{x,\eta,\ell}] = [P_{x,S}]$ under the injection $\mathcal{P}_G(K) \hookrightarrow \mathcal{P}_{G,S}(K_S)$ (for the last equality, see Corollary 5.8 (ii)). This completes the proof. $\square$

Corollary 5.14. *For $x \in \mathrm{Sh}_K(G,X)_{E_v}^{\mathrm{ad}}(\mathrm{cl})$ and a prime number $\ell \neq p$, the following are equivalent*:

(i) $\phi_{x,\ell}^{\mathrm{ad}}$ *is potentially unramified.*

(ii) $\phi_{x,p}^{\mathrm{ad}}$ *is potentially crystalline.*

(iii) $[P_x] = [G]$.

If the above conditions are satisfied, we say that x is of potentially good reduction.

Proof. By Lemma 2.2 (iii), $\phi_{x,\ell}^{\mathrm{ad}}$ is potentially unramified if and only if $P_{\phi_{x,\ell}^{\mathrm{ad}}} = G_{\mathbb{Q}_\ell}^{\mathrm{ad}}$. By definition this is clearly equivalent to $[P_x] = [G]$. Similarly we can prove the equivalence of (ii) and (iii). $\square$

Definition 5.15. We denote the map $\mathrm{Sh}_K(G,X)_{E_v}^{\mathrm{ad}}(\mathrm{cl}) \to \mathcal{P}_G(K)$; $x \mapsto [P_x]$ by Φ_K.

Proposition 5.16. *Let (G, X) be a Shimura datum of preabelian type.*

(i) *The map Φ_K is Hecke-equivariant in the following sense. Let K, K' be compact open subgroups of $G(\mathbb{A}^\infty)$, and g an element of $G(\mathbb{A}^\infty)$ such that $g^{-1}Kg \subset K'$. Then the diagram*

$$\begin{array}{ccc}
\mathrm{Sh}_K(G,X)_{E_v}^{\mathrm{ad}}(\mathrm{cl}) & \xrightarrow{\Phi_K} & \mathcal{P}_G(K) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle [P] \mapsto [g^{-1}Pg]} \\
\mathrm{Sh}_{K'}(G,X)_{E_v}^{\mathrm{ad}}(\mathrm{cl}) & \xrightarrow{\Phi_{K'}} & \mathcal{P}_G(K')
\end{array}$$

is commutative.

(ii) *Let K be a compact open subgroup of $G(\mathbb{A}^\infty)$, and K'' a compact open subgroup of $G^{\mathrm{ad}}(\mathbb{A}^\infty)$ containing the image of K. We write E^{ad} for the reflex field of $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ and v^{ad} the place of E^{ad} below v. Then the diagram*

$$
\begin{array}{ccc}
\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl}) & \xrightarrow{\ \Phi_K\ } & \mathcal{P}_G(K) \\
\downarrow & & \downarrow{\scriptstyle [P] \mapsto [P^{\mathrm{ad}}]} \\
\mathrm{Sh}_{K''}(G^{\mathrm{ad}}, X^{\mathrm{ad}})^{\mathrm{ad}}_{E^{\mathrm{ad}}_{v^{\mathrm{ad}}}}(\mathrm{cl}) & \xrightarrow{\ \Phi_{K''}\ } & \mathcal{P}_{G^{\mathrm{ad}}}(K'')
\end{array}
$$

is commutative.

(iii) *Assume that (G, X) is of Hodge type, and let $i : (G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ be an embedding into a Siegel Shimura datum. Let K be a compact open subgroup of $G(\mathbb{A}^\infty)$, and $\widetilde{K}$ a compact open subgroup of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K. Then the diagram*

$$
\begin{array}{ccc}
\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl}) & \xrightarrow{\ \Phi_K\ } & \mathcal{P}_G(K) \\
\downarrow & & \downarrow{\scriptstyle [P] \mapsto [i_* P]} \\
\mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})^{\mathrm{ad}}_{\mathbb{Q}_p}(\mathrm{cl}) & \xrightarrow{\ \Phi_{\widetilde{K}}\ } & \mathcal{P}_{\mathrm{GSp}_{2n}}(\widetilde{K})
\end{array}
$$

is commutative.

Proof. The assertions (i) and (ii) are immediate consequences of Proposition 5.3 and Lemma 5.10 (i). The (iii) follows from Lemma 5.13 and the construction of $[P_x]$ in the proof of Proposition 5.12. $\qquad\square$

In the remaining part of this section, we will prove that the map Φ_K comes from a partition of $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$ into locally closed constructible subsets.

Theorem 5.17. *For each $[P] \in \mathcal{P}_G(K)$, there uniquely exists a locally closed constructible subset $C_{[P]}$ of $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$ such that*

$$
C_{[P]}(\mathrm{cl}) = \{x \in \mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl}) \mid \Phi_K(x) = [P]\}.
$$

Furthermore, the subset $C_{[G]}$ is open and quasicompact.

Remark 5.18. The subsets $\{C_{[P]}\}_{[P] \in \mathcal{P}_G(K)}$ in Theorem 5.17 are mutually disjoint and cover $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$. Indeed, by Lemma 3.6 (i), it can be checked at the level of classical points, which is obvious.

Theorem 5.17 will be proved in Section 5.3. Admitting this theorem, we have the following definition.

Definition 5.19. We put $\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{E_v} = C_{[G]}$, and call it the *potentially good reduction locus* of $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$. It is a quasicompact open subset of $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$ characterized by the following property:

- for $x \in \mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}(\mathrm{cl})$, x lies in $\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{E_v}$ if and only if x is of potentially good reduction in the sense of Corollary 5.14.

Example 5.20. When $\mathrm{Sh}_K(G, X)$ is proper over E, G has no proper parabolic subgroup defined over $\mathbb{Q}$. Therefore, we have $\mathcal{P}_G(K) = \{[G]\}$ and $\mathrm{Sh}_K(G, X)^{\mathrm{pg}}_{E_v} = \mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$.

5.2. *Partition in the Siegel case.* In this section, we give a proof of Theorem 5.17 in the Siegel case. We use the notation in Section 4.2. In particular, recall that $(V, \langle\ ,\ \rangle)$ is a symplectic space of dimension $2n$ over $\mathbb{Q}$ and L is a self-dual $\mathbb{Z}$-lattice of V. For simplicity, we write $\mathcal{S}_K$ for $\mathrm{Sh}_K(\mathrm{GSp}_{2n}, X_{2n})^{\mathrm{ad}}_{\mathbb{Q}_p}$.

It is well-known that conjugacy classes of maximal parabolic subgroups of GSp_{2n} are parametrized by integers $0 \leq r \leq n$; the class corresponding to r consists of parabolic subgroups obtained as stabilizers of r-dimensional totally isotropic subspaces of V. Namely, $\mathcal{P}_{\mathrm{GSp}_{2n}, \mathbb{Q}} \cong \{0, 1, \ldots, n\}$ under the notation in Definition 5.9. We write $\mathcal{P}_{\mathrm{GSp}_{2n}}(K)_r$ for the inverse image of r under the map

$$\mathcal{P}_{\mathrm{GSp}_{2n}}(K) \to \mathcal{P}_{\mathrm{GSp}_{2n}, \mathbb{Q}} \cong \{0, 1, \ldots, n\}$$

(see Corollary 5.11). Clearly we have $\mathcal{P}_{\mathrm{GSp}_{2n}}(K) = \bigsqcup_{0 \leq r \leq n} \mathcal{P}_{\mathrm{GSp}_{2n}}(K)_r$. We put $\mathcal{P}_{\mathrm{GSp}_{2n}}(K)_{\leq r} = \bigsqcup_{r' \leq r} \mathcal{P}_{\mathrm{GSp}_{2n}}(K)_{r'}$. Note that $\mathcal{P}_{\mathrm{GSp}_{2n}}(K)_0 = \mathcal{P}_{\mathrm{GSp}_{2n}}(K)_{\leq 0} = \{[G]\}$.

Proposition 5.21. *There uniquely exists a constructible open subset $\mathcal{S}_{K,]\leq r[}$ of $\mathcal{S}_K$ such that $x \in \mathcal{S}_K(\mathrm{cl})$ belongs to $\mathcal{S}_{K,]\leq r[}$ if and only if $\Phi_K(x) \in \mathcal{P}_{\mathrm{GSp}_{2n}}(K)_{\leq r}$. Moreover, $\mathcal{S}_{K,]\leq 0[}$ is quasicompact.*

Proof. By Proposition 5.16 (i), we may shrink K freely. Therefore, we may assume that K_p is contained in $K_{p,0} = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$.

For $r \geq 0$, let $\mathscr{S}^{\mathrm{tor}}_{K^p, \leq r}$ be the subset of $\mathscr{S}^{\mathrm{tor}}_{K^p}$ consisting of $x \in \mathscr{S}^{\mathrm{tor}}_{K^p}$ such that the toric rank of the semi-abelian variety $\mathcal{A}_x$ is at most r. By [Lan 2013, Lemma 3.3.1.4], it is an open subset of $\mathscr{S}^{\mathrm{tor}}_{K^p}$.

Since $\mathscr{S}^{\mathrm{tor}}_{K^p}$ is proper over $\mathbb{Z}_p$, we may consider the specialization map

$$\mathrm{sp} \colon (\mathscr{S}^{\mathrm{tor}}_{K^p, \mathbb{Q}_p})^{\mathrm{ad}} = (\mathscr{S}^{\mathrm{tor}}_{K^p})^{\mathrm{ad}} = (\mathscr{S}^{\mathrm{tor}}_{K^p})^{\wedge \mathrm{rig}} \to \mathscr{S}^{\mathrm{tor}}_{K^p, \mathbb{F}_p}$$

introduced in Section 3.1 (for the second equality, see Lemma 3.3 (ii)). For an integer $r \geq 0$, we put $\mathcal{S}_{K_{p,0}K^p,]\leq r[} = \mathrm{sp}^{-1}(\mathscr{S}^{\mathrm{tor}}_{K^p, \leq r, \mathbb{F}_p}) \cap \mathscr{S}^{\mathrm{ad}}_{K^p, \mathbb{Q}_p}$. It is a constructible open subset of $\mathscr{S}^{\mathrm{ad}}_{K^p, \mathbb{Q}_p} = \mathcal{S}_{K_{p,0}K^p}$. Further, let $\mathcal{S}_{K,]\leq r[}$ be the inverse image of $\mathcal{S}_{K_{p,0}K^p,]\leq r[}$ under the natural morphism $\mathcal{S}_K \to \mathcal{S}_{K_{p,0}K^p}$. By Corollary 5.8 (iii), it satisfies the desired property. Since $\mathcal{S}_{K_{p,0}K^p,]\leq 0[} = \mathrm{sp}^{-1}(\mathscr{S}_{K^p, \mathbb{F}_p})$, it is quasicompact. Hence $\mathcal{S}_{K,]\leq 0[}$ is also quasicompact. $\square$

For $0 \leq r \leq n$, we put $\mathcal{S}_{K,]r[} = \mathcal{S}_{K,] \leq r[} \setminus \mathcal{S}_{K,] \leq r-1[}$, where $\mathcal{S}_{K,] \leq -1[}$ means $\varnothing$. It is a locally closed constructible subset of $\mathcal{S}_K$.

Lemma 5.22. *The set $\pi_0(\mathcal{S}_{K,]r[})$ of connected components of $\mathcal{S}_{K,]r[}$ is finite, and consists of locally closed constructible subsets of $\mathcal{S}_K$. Hence $\mathcal{S}_{K,]r[}$ is topologically the disjoint union of elements of $\pi_0(\mathcal{S}_{K,]r[})$.*

Proof. By shrinking K, we may assume that K_p is contained in $K_{p,0} = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$. Then the claim is the case $X = \mathscr{S}^{\mathrm{tor}}_{K^p, \mathbb{Q}_p}$, $U = \mathscr{S}_{K^p, \mathbb{Q}_p}$, $U' = \mathrm{Sh}_K(\mathrm{GSp}_{2n}, X_{2n})_{\mathbb{Q}_p}$, and $L = \mathrm{sp}^{-1}(\mathscr{S}^{\mathrm{tor}}_{K^p, \leq r, \mathbb{F}_p}) \setminus \mathrm{sp}^{-1}(\mathscr{S}^{\mathrm{tor}}_{K^p, \leq r-1, \mathbb{F}_p})$ of the subsequent general lemma. $\square$

Lemma 5.23. *Let F be a p-adic field. Let X be a purely d-dimensional proper smooth scheme over F, and Y a closed subscheme of X whose dimension is less than d. We put $U = X \setminus Y$, and consider a finite étale surjection $f : U' \to U$.*

For a locally closed constructible subset L of X^{ad}, we put $L_{U'} = (f^{\mathrm{ad}})^{-1}(L \cap U^{\mathrm{ad}})$. Then, the set $\pi_0(L_{U'})$ is finite, and consists of locally closed constructible subsets of U'^{ad}. In particular, $L_{U'}$ is topologically the disjoint union of elements of $\pi_0(L_{U'})$.

Proof. Let X' be the normalization of X in U'. By the resolution of singularities, there exists a purely d-dimensional proper smooth scheme X'' over F and a proper birational morphism $\phi : X'' \to X'$ which induces an isomorphism $\phi^{-1}(U') \xrightarrow{\cong} U'$. Let us denote the composite $X'' \xrightarrow{\phi} X' \to X$ by ϕ'. By replacing X, Y, L with X'', $X'' \times_X Y$, $(\phi'^{\mathrm{ad}})^{-1}(L)$ respectively, we may assume that $U' = U$.

We denote by L° (resp. L_U°) the interior of L (resp. L_U) in X^{ad} (resp. U^{ad}). Clearly we have $L_U^{\circ} = L^{\circ} \setminus Y^{\mathrm{ad}}$. We fix a prime number $\ell \neq p$, and consider the following commutative diagram:

$$
\begin{array}{ccc}
H^0(L_{\overline{F}}, \mathbb{F}_\ell) & \longrightarrow & H^0(L_{\overline{F}}^{\circ}, \mathbb{F}_\ell) \\
\downarrow {\scriptstyle (1)} & & \downarrow {\scriptstyle (2)} \\
H^0(L_{U, \overline{F}}, \mathbb{F}_\ell) & \longrightarrow & H^0(L_{U, \overline{F}}^{\circ}, \mathbb{F}_\ell).
\end{array}
$$

Since X^{ad} is proper of finite type over $\mathrm{Spa}(F, \mathcal{O}_F)$, $H^0(L_{\overline{F}}, \mathbb{F}_\ell)$ is a finite-dimensional $\mathbb{F}_\ell$-vector space by [Huber 1998b, Proposition 3.16 i)]. Therefore, to show the finiteness of $\pi_0(L_U)$, it suffices to prove that the map (1) is an isomorphism. On the other hand, by [Huber 1998b, Theorem 3.7], the horizontal maps are isomorphisms. Hence it suffices to prove that (2) is an isomorphism. Note that, by [Huber 1998c, Lemma 1.3 iii)], L° and L_U° are taut, and then $L^{\circ} \cap Y^{\mathrm{ad}}$ is also taut. Therefore we can consider the compactly supported cohomology of these spaces. Since $\dim(L^{\circ} \cap Y^{\mathrm{ad}}) < d$, we have $H_c^{2d-1}((L^{\circ} \cap Y^{\mathrm{ad}})_{\overline{F}}, \mathbb{F}_\ell) = H_c^{2d}((L^{\circ} \cap Y^{\mathrm{ad}})_{\overline{F}}, \mathbb{F}_\ell) = 0$.

This implies that the natural map $H^{2d}_c(L^\circ_{U,\overline{F}}, \mathbb{F}_\ell) \to H^{2d}_c(L^\circ_{\overline{F}}, \mathbb{F}_\ell)$ is an isomorphism. By the Poincaré duality, we conclude that the map (2) is an isomorphism.

By the finiteness of $\pi_0(L_U)$, every element C of $\pi_0(L_U)$ is an open and closed subset of L_U. Since L is locally closed constructible, so is C. $\qquad\square$

Lemma 5.24. *There uniquely exists a map* $\Psi_K \colon \pi_0(\mathcal{S}_{K,]r[}) \to \mathcal{P}_G(K)$ *satisfying the following*: *for* $C \in \pi_0(\mathcal{S}_{K,]r[})$ *and* $x \in C(\mathrm{cl})$, *we have* $\Psi_K(C) = \Phi_K(x)$.

Proof. Let $C \in \pi_0(\mathcal{S}_{K,]r[})$. Then, by Lemma 5.22 and Lemma 3.6 (i), we have $C(\mathrm{cl}) \neq \varnothing$. Therefore, it suffices to show that $\Phi_K(x)$ is independent of the choice of $x \in C(\mathrm{cl})$. By Proposition 5.16 (i), we may assume that K_p is contained in $K_{p,0} = \mathrm{GSp}_{2n}(\mathbb{Z}_p)$. Recall that in this case $\mathcal{S}_{K,]\leq r[}$ is obtained as the inverse image under $\mathcal{S}_K \to \mathcal{S}_{K_{p,0}K^p}$ of $\mathcal{S}_{K_{p,0}K^p,]\leq r[}$, which is equal to $\mathrm{sp}^{-1}(\mathscr{S}^{\mathrm{tor}}_{K^p,\leq r,\mathbb{F}_p}) \cap \mathscr{S}^{\mathrm{ad}}_{K^p,\mathbb{Q}_p}$ (see the proof of Proposition 5.21).

Let $\mathcal{S}^\circ_{K,]r[}$ be the interior of $\mathcal{S}_{K,]r[}$ in $\mathcal{S}_K$. Then, the inverse image of $\mathcal{S}^\circ_{K_{p,0}K^p,]r[}$ under $\mathcal{S}_K \to \mathcal{S}_{K_{p,0}K^p}$ equals $\mathcal{S}^\circ_{K,]r[}$. We write C° for the interior of C in $\mathcal{S}_K$. It is connected by [Huber 1998b, Theorem 3.7], and included in $\mathcal{S}^\circ_{K,]r[}$. Since C is constructible in $\mathcal{S}_K$, we have $C(\mathrm{cl}) = C^\circ(\mathrm{cl})$ by Lemma 3.6 (ii). Hence it suffices to show that $\Phi_K(x)$ is independent of the choice of $x \in C^\circ(\mathrm{cl})$.

We apply the construction introduced in Section 3.2 to the case where $S = \mathscr{S}^{\mathrm{tor}}_{K^p,\leq r}$, $S_0 = \mathscr{S}^{\mathrm{tor}}_{K^p,r,\mathbb{F}_p}$, $U = \mathscr{S}_{K^p,\mathbb{Q}_p}$ and $G = \mathcal{A}|_{\mathscr{S}^{\mathrm{tor}}_{K^p,\leq r}}$. We write $\widehat{\mathcal{T}}$ for the corresponding $\widehat{T}$. By Lemma 3.5, $U^{\mathrm{ad}} = U \times_S t(S)_a = \mathcal{S}^\circ_{K_{p,0}K^p,]r[}$ in this case. Therefore, for each $m \geq 0$ and a prime ℓ, we have three locally constant constructible sheaves

$$\widehat{\mathcal{T}}^{\mathrm{rig}}[\ell^m]_{\mathcal{S}^\circ_{K_{p,0}K^p,]r[}} \subset \widehat{\mathcal{A}}^{\mathrm{rig}}[\ell^m]_{\mathcal{S}^\circ_{K_{p,0}K^p,]r[}} \subset \mathcal{A}^{\mathrm{ad}}[\ell^m]_{\mathcal{S}^\circ_{K_{p,0}K^p,]r[}}.$$

We put

$$\mathcal{V}_\ell = \left(\varprojlim_m \mathcal{A}^{\mathrm{ad}}[\ell^m]_{\mathcal{S}^\circ_{K_{p,0}K^p,]r[}}|_{C^\circ}\right) \otimes \mathbb{Q}_\ell, \quad \mathcal{F}_\ell = \left(\varprojlim_m \widehat{\mathcal{A}}^{\mathrm{rig}}[\ell^m]_{\mathcal{S}^\circ_{K_{p,0}K^p,]r[}}|_{C^\circ}\right) \otimes \mathbb{Q}_\ell,$$

$$\mathcal{T}_\ell = \left(\varprojlim_m \widehat{\mathcal{T}}^{\mathrm{rig}}[\ell^m]_{\mathcal{S}^\circ_{K_{p,0}K^p,]r[}}|_{C^\circ}\right) \otimes \mathbb{Q}_\ell.$$

They are smooth ℓ-adic sheaves over C°.

Now we use the moduli interpretation with rational level structures of $\mathcal{S}_K$. Fix a geometric point $\overline{x}_0$ of C°, and let ηK be the $\pi_1(C^\circ, \overline{x}_0)$-invariant K-orbit of isomorphisms $V_{\mathbb{A}^\infty} \xrightarrow{\cong} V^\infty \mathcal{A}_{\overline{x}_0}$ corresponding to the universal level structure on $\mathcal{A}|_{C^\circ}$. For $x \in C^\circ(\mathrm{cl})$, the rational K-level structure on $\mathcal{A}_x$ corresponding to x itself is obtained in the following manner. Fix a geometric point $\overline{x}$ lying over x. Since C° is connected, there exists an isomorphism $\pi_1(C^\circ, \overline{x}_0) \to \pi_1(C^\circ, \overline{x})$, which is canonical up to $\pi_1(C^\circ, \overline{x}_0)$-conjugacy. If we fix such an isomorphism, for a smooth sheaf $\mathcal{G}$ on C°, we have a functorial isomorphism $\mathcal{G}_{\overline{x}_0} \xrightarrow{\cong} \mathcal{G}_{\overline{x}}$ compatible

with the π_1-actions. In particular, the smooth sheaf $(\varprojlim_N \mathcal{A}^{\mathrm{ad}}[N]_{C^\circ}) \otimes \mathbb{Q}$ determines an isomorphism $V^\infty \mathcal{A}_{\overline{x}_0} \xrightarrow{\cong} V^\infty \mathcal{A}_{\overline{x}}$. By composing it with each element of ηK, we obtain a K-orbit of isomorphisms $V_{\mathbb{A}^\infty} \xrightarrow{\cong} V^\infty \mathcal{A}_{\overline{x}}$, which turns out to be $\pi_1(C^\circ, \overline{x})$-invariant. Since the action of $\pi_1(x, \overline{x})$ on $V^\infty \mathcal{A}_{\overline{x}}$ factors through $\pi_1(x, \overline{x}) \to \pi_1(C^\circ, \overline{x})$, this orbit gives a rational K-level structure on $\mathcal{A}_x$.

Fix a representative η of ηK and write η_x for the composite of η and the isomorphism $V^\infty \mathcal{A}_{\overline{x}_0} \xrightarrow{\cong} V^\infty \mathcal{A}_{\overline{x}}$. We take a prime number ℓ and consider the ℓ-part

$$\eta_{x,\ell} \colon V_{\mathbb{Q}_\ell} \underset{\eta_\ell}{\xrightarrow{\cong}} V_\ell \mathcal{A}_{\overline{x}_0} \underset{(*)}{\xrightarrow{\cong}} V_\ell \mathcal{A}_{\overline{x}}.$$

of η_x. Note that the isomorphism $(*)$ is given by the smooth ℓ-adic sheaf $\mathcal{V}_\ell$ introduced above. Moreover, by Corollary 5.8 (i), Proposition 3.11, Corollary 3.12 and its proof, the monodromy filtrations on $V_\ell \mathcal{A}_{\overline{x}_0}$ and $V_\ell \mathcal{A}_{\overline{x}}$ are given by

$$0 \subset \mathcal{T}_{\ell,\overline{x}_0} \subset \mathcal{F}_{\ell,\overline{x}_0} \subset \mathcal{V}_{\ell,\overline{x}_0}, \quad 0 \subset \mathcal{T}_{\ell,\overline{x}} \subset \mathcal{F}_{\ell,\overline{x}} \subset \mathcal{V}_{\ell,\overline{x}},$$

respectively. Since $\mathcal{F}_\ell$ and $\mathcal{T}_\ell$ are smooth sheaves, the isomorphism $(*)$ carries the first filtration to the second. Hence we have $\eta_{x,\ell}^{-1}(M_\bullet V_\ell \mathcal{A}_{\overline{x}}) = \eta_\ell^{-1}(M_\bullet V_\ell \mathcal{A}_{\overline{x}_0})$, which is independent of $x \in C^\circ(\mathrm{cl})$. Therefore, the parabolic subgroup $P_{x, \iota_x \circ \eta_x, \ell}$ in Corollary 5.8 (ii), where $\iota_x \colon \overline{\kappa}_x \xrightarrow{\cong} \mathbb{C}$ is a fixed isomorphism, is also independent of x. By Corollary 5.8 (ii) and Lemma 5.10 (i), we conclude that $\Phi_K(x) = [P_x] \in \mathcal{P}_{\mathrm{GSp}_{2n}}(K)$ is independent of x. $\qquad\square$

Proof of Theorem 5.17 for the Siegel case. For $[P] \in \mathcal{P}_{\mathrm{GSp}_{2n}}(K)$, take $0 \le r \le n$ such that $[P]$ lies in $\mathcal{P}_{\mathrm{GSp}_{2n}}(K)_r$. We put

$$C_{[P]} = \bigcup_{\substack{C \in \pi_0(\mathcal{S}_{K,]r[}), \\ \Psi_K(C) = [P]}} C.$$

It is a constructible subset of $\mathcal{S}_K$ by Lemma 5.22. Since each $C \in \pi_0(\mathcal{S}_{K,]r[})$ is open in $\mathcal{S}_{K,]r[}$, $C_{[P]}$ is a locally closed subset of $\mathcal{S}_K$. We can also check that $x \in \mathcal{S}_K(\mathrm{cl})$ lies in $C_{[P]}$ if and only if $\Phi_K(x) = [P]$. We have already checked in Proposition 5.21 that $C_{[G]} = \mathcal{S}_{K,]\le 0[}$ is quasicompact open. $\qquad\square$

Corollary 5.25. *For $[P] \in \mathcal{P}_G(K)_r$ and $[P'] \in \mathcal{P}_G(K)_{r'}$, assume that $r > r'$ or $r = r'$ and $[P] \ne [P']$. Then we have $C_{[P]}^- \cap C_{[P']} = \varnothing$, where $C_{[P]}^-$ denotes the closure of $C_{[P]}$ in $\mathcal{S}_K$.*

Proof. First assume that $r > r'$. Since the complement $\mathcal{S}_{K,]\le r'[}^c$ of $\mathcal{S}_{K,]\le r'[}$ is a closed subset of $\mathcal{S}_K$, we have

$$C_{[P]}^- \cap C_{[P']} \subset \mathcal{S}_{K,]\le r'[}^c \cap \mathcal{S}_{K,]\le r'[} = \varnothing.$$

Next assume that $r = r'$ and $[P] \neq [P']$. By construction, $C_{[P]}$ is closed in $\mathcal{S}_{K,]r[}$. Therefore, we have

$$C_{[P]}^- \cap C_{[P']} = (C_{[P]}^- \cap \mathcal{S}_{K,]r[}) \cap C_{[P']} = C_{[P]} \cap C_{[P']} = \varnothing.$$

For the last equality, see Remark 5.18. □

5.3. *Existence of partition.* In this section, we complete the proof of Theorem 5.17 by reducing to the Siegel case. First we consider the Hodge type case.

Lemma 5.26. *Let (G, X) be a Shimura datum of Hodge type with an embedding $i \colon (G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. For a compact open subgroup K of $G(\mathbb{A}^\infty)$, there exists a compact open subgroup $\widetilde{K}$ of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K such that the map $\mathcal{P}_G(K) \to \mathcal{P}_{\mathrm{GSp}_{2n}}(\widetilde{K}); [P] \mapsto [i_*P]$ is injective.*

Proof. It suffices to prove the following claim:

> for $[P_1], [P_2] \in \mathcal{P}_G(K)$ with $[P_1] \neq [P_2]$, there exists a compact open subgroup $\widetilde{K}_{[P_1],[P_2]}$ of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K such that $[i_*P_1] \neq [i_*P_2]$ in $\mathcal{P}_{\mathrm{GSp}_{2n}}(\widetilde{K}_{[P_1],[P_2]})$.

Indeed, the intersection of $\widetilde{K}_{[P_1],[P_2]}$ for all pairs $([P_1], [P_2])$ with $[P_1] \neq [P_2]$ satisfies the desired condition.

Fix a compact open subgroup $\widetilde{K}_0$ of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K. Take representatives P_1, P_2 of $[P_1]$, $[P_2]$, respectively. We put $Z = \{g \in \widetilde{K}_0 \mid g(i_*P_1)g^{-1} = i_*P_2\}$, which is clearly a closed subset of $\widetilde{K}_0$. Therefore, a subset KZ of $\widetilde{K}_0$ is compact, hence closed. We prove that $1 \notin KZ$. If $1 \in KZ$, there exists $k \in K$ such that $k(i_*P_1)k^{-1} = i_*P_2$. Taking intersections with G, we obtain $kP_1k^{-1} = P_2$, which contradicts the assumption $[P_1] \neq [P_2]$. Therefore, we can find a compact open normal subgroup $\widetilde{K}_1$ of $\widetilde{K}_0$ such that $\widetilde{K}_1 \cap KZ = \varnothing$. Then, $\widetilde{K}_{[P_1],[P_2]} = K\widetilde{K}_1$ is a compact open subgroup of $\mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ satisfying $\widetilde{K}_{[P_1],[P_2]} \cap Z = \varnothing$. This concludes the proof. □

Proof of Theorem 5.17 for the Hodge type case. We assume that (G, X) is of Hodge type, and take an embedding $i \colon (G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. Further, we take a compact open subgroup $\widetilde{K} \subset \mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ as in Lemma 5.26.

Let $[P] \in \mathcal{P}_G(K)$. Since Theorem 5.17 is known for the Siegel case, we have a locally closed constructible subset $C_{[i_*P]}$ of $\mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})_{\mathbb{Q}_p}^{\mathrm{ad}}$. Let $C_{[P]}$ be the inverse image of $C_{[i_*P]}$ in $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$. Then, $C_{[P]}$ satisfies the desired condition by Proposition 5.16 (iii). The subset $C_{[G]}$ is open and quasicompact, since $C_{[\mathrm{GSp}_{2n}]} = \mathcal{S}_{\widetilde{K},]\leq 0[}$ is open and quasicompact. □

The following lemma is the Hodge type version of Corollary 5.25.

Lemma 5.27. *Let (G, X) be a Shimura datum of Hodge type. Take an embedding $i : (G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. For a compact open subgroup K of $G(\mathbb{A}^\infty)$ and an integer $0 \leq r \leq n$, we write $\mathcal{P}_G(K)_r$ for the inverse image of r under the composite*

$$\mathcal{P}_G(K) \to \mathcal{P}_{G,\mathbb{Q}} \xrightarrow{i_*} \mathcal{P}_{\mathrm{GSp}_{2n},\mathbb{Q}} \cong \{0, 1, \ldots, n\}.$$

For $[P] \in \mathcal{P}_G(K)_r$ and $[P'] \in \mathcal{P}_G(K)_{r'}$, assume that $r > r'$ or $r = r'$ and $[P] \neq [P']$. Then we have $C^-_{[P]} \cap C_{[P']} = \varnothing$, where $C^-_{[P]}$ denotes the closure of $C_{[P]}$ in $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v}$.

In particular, if $[P], [P']$ are distinct elements of $\mathcal{P}_G(K)$ such that P is $G(\mathbb{A}^\infty)$-conjugate to P', then we have $C^-_{[P]} \cap C_{[P']} = \varnothing$.

Proof. We take $\widetilde{K}$ as in Lemma 5.26. By definition, we have $[i_* P] \in \mathcal{P}_{\mathrm{GSp}_{2n}}(\widetilde{K})_r$ and $[i_* P'] \in \mathcal{P}_{\mathrm{GSp}_{2n}}(\widetilde{K})_{r'}$. Since the map $\mathcal{P}_G(K) \to \mathcal{P}_{\mathrm{GSp}_{2n}}(\widetilde{K})$ is injective, we have $[i_* P] \neq [i_* P']$ if $r = r'$. Hence Corollary 5.25 tells us that $C^-_{[i_* P]} \cap C_{[i_* P']} = \varnothing$.

Since the natural morphism $\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v} \to \mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})^{\mathrm{ad}}_{\mathbb{Q}_p}$ maps $C_{[P]}$ and $C_{[P']}$ into $C_{[i_* P]}$ and $C_{[i_* P']}$, respectively (see the construction of $C_{[P]}$ in the proof of Theorem 5.17), the set $C^-_{[P]} \cap C_{[P']}$ is mapped into $C^-_{[i_* P]} \cap C_{[i_* P']} = \varnothing$. Therefore we conclude that $C^-_{[P]} \cap C_{[P']} = \varnothing$.

The last claim follows from the observation that if P and P' are $G(\mathbb{A}^\infty)$-conjugate, then $r = r'$. $\qquad\square$

Now we can prove Theorem 5.17 for the preabelian type case.

Proof of Theorem 5.17. We choose a compact open subgroup K'' of $G^{\mathrm{ad}}(\mathbb{A}^\infty)$ in such a way as in Lemma 5.10 (iii), and use the notation in Lemma 4.4. We write E' for the reflex field of (G', X'), and choose a place v' of E' above v^{ad} (for the definition of v^{ad}, see Proposition 5.16 (ii)).

We have natural maps

$$\mathcal{P}_{G'}(K') \to \mathcal{P}_{G'^{\mathrm{ad}}}(g_i^{-1} K'' g_i) \xrightarrow{g_i} \mathcal{P}_{G'^{\mathrm{ad}}}(K'') \cong \mathcal{P}_{G^{\mathrm{ad}}}(K'') \cong \mathcal{P}_G(K)$$

for each i (see Lemma 5.10 (iii)). Let S_i be the inverse image of $[P] \in \mathcal{P}_G(K)$ under this map. We put $C^{\mathrm{ad}}_{[P]} = \bigcup_{1 \leq i \leq m} f_i(\bigcup_{[Q] \in S_i} C_{[Q]})$. Since $C_{[Q]}$ for each $[Q] \in S_i$ is constructible, Lemma 3.7 (i) tells us that $C^{\mathrm{ad}}_{[P]}$ is constructible. Let us prove that $C^{\mathrm{ad}}_{[P]}$ is locally closed. By Lemma 3.6 (i), Proposition 5.16 (i), (ii) and the constructibility of $C^{\mathrm{ad}}_{[P]}$, we can check that the inverse image of $C^{\mathrm{ad}}_{[P]}$ under $\coprod_{1 \leq i \leq m} f_i$ equals $\coprod_{1 \leq i \leq m} \bigcup_{[Q] \in S_i} C_{[Q]}$. Therefore, Lemma 3.7 (ii) tells us that it suffices to prove that $\bigcup_{[Q] \in S_i} C_{[Q]}$ is locally closed. Note that we have the

commutative diagram

$$
\begin{array}{ccccccccc}
\mathcal{P}_{G'}(K') & \longrightarrow & \mathcal{P}_{G'^{\mathrm{ad}}}(g_i^{-1}K''g_i) & \xrightarrow{g_i} & \mathcal{P}_{G'^{\mathrm{ad}}}(K'') & \xleftrightarrow{\cong} & \mathcal{P}_{G^{\mathrm{ad}}}(K'') & \xleftarrow{\cong} & \mathcal{P}_G(K) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{P}_{G',\mathbb{Q}} & \xrightarrow{\cong} & \mathcal{P}_{G'^{\mathrm{ad}},\mathbb{Q}} & = & \mathcal{P}_{G'^{\mathrm{ad}},\mathbb{Q}} & \xleftrightarrow{\cong} & \mathcal{P}_{G^{\mathrm{ad}},\mathbb{Q}} & \xleftarrow{\cong} & \mathcal{P}_{G,\mathbb{Q}},
\end{array}
$$

where the vertical arrows are the maps in Corollary 5.11. From this diagram we can observe that all elements in S_i are $G'(\mathbb{A}^\infty)$-conjugate. Now Lemma 5.27 tells us that the closure $C_{[Q]}^-$ of $C_{[Q]}$ for $[Q] \in S_i$ does not intersect $\bigcup_{[Q'] \in S_i \setminus \{[Q]\}} C_{[Q']}$. Therefore $C_{[Q]}$ is closed (hence open) in $\bigcup_{[Q'] \in S_i} C_{[Q']}$, from which we conclude that $\bigcup_{[Q] \in S_i} C_{[Q]}$ is locally closed in $\mathrm{Sh}_{K'}(G', X')_{E'_{v'}}^{\mathrm{ad}}$, as desired.

Let $C_{[P]}$ be the inverse image of $C_{[P]}^{\mathrm{ad}}$ under the map

$$
\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}} \to \mathrm{Sh}_{K''}(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{E_{v^{\mathrm{ad}}}^{\mathrm{ad}}}^{\mathrm{ad}} \cong \mathrm{Sh}_{K''}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})_{E_{v^{\mathrm{ad}}}^{\mathrm{ad}}}^{\mathrm{ad}}
$$

(here E^{ad} and v^{ad} are as in Proposition 5.16 (ii)). Then $C_{[P]}$ satisfies the desired condition by Proposition 5.16 (i), (ii).

If $[P] = [G]$, we have $S_i = \{[G']\}$. Hence the openness and the quasicompactness of $C_{[G]}$ follows from those of $C_{[G']}$. $\qquad\square$

Lemma 5.28. *Let (G, X) be a Shimura datum of preabelian type. Take a Shimura datum (G', X') of Hodge type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$, and an embedding $i : (G', X') \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. For a compact open subgroup K of $G(\mathbb{A}^\infty)$ and an integer $0 \le r \le n$, we write $\mathcal{P}_G(K)_r$ for the inverse image of r under the composite*

$$
\mathcal{P}_G(K) \to \mathcal{P}_{G,\mathbb{Q}} \cong \mathcal{P}_{G^{\mathrm{ad}},\mathbb{Q}} \cong \mathcal{P}_{G'^{\mathrm{ad}},\mathbb{Q}} \cong \mathcal{P}_{G',\mathbb{Q}} \xrightarrow{i_*} \mathcal{P}_{\mathrm{GSp}_{2n},\mathbb{Q}} \cong \{0, 1, \dots, n\}.
$$

For $[P] \in \mathcal{P}_G(K)_r$ and $[P'] \in \mathcal{P}_G(K)_{r'}$, assume that $r > r'$ or $r = r'$ and $[P] \neq [P']$. Then we have $C_{[P]}^- \cap C_{[P']} = \varnothing$, where $C_{[P]}^-$ denotes the closure of $C_{[P]}$ in $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$.

In particular, if $[P], [P']$ are distinct elements of $\mathcal{P}_G(K)$ such that P is $G(\mathbb{A}^\infty)$-conjugate to P', then we have $C_{[P]}^- \cap C_{[P']} = \varnothing$.

Proof. We use the same notation as in the proof of Theorem 5.17 above, and denote the morphism $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}} \to \mathrm{Sh}_{K''}(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{E_{v^{\mathrm{ad}}}^{\mathrm{ad}}}^{\mathrm{ad}} \cong \mathrm{Sh}_{K''}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})_{E_{v^{\mathrm{ad}}}^{\mathrm{ad}}}^{\mathrm{ad}}$ by h.

If a point x belongs to $C_{[P]}^- \cap C_{[P']}$, we can find $y \in C_{[P]}$ specializing to x (see [Hochster 1969, Corollary of Theorem 1]). By the construction of $C_{[P]}$, there exist $1 \le i \le m$, $[Q] \in S_i$ and $y' \in C_{[Q]}$ such that $h(y) = f_i(y')$. Since f_i is finite, there exists $x' \in \mathrm{Sh}_{K'}(G', X')_{E'_{v'}}^{\mathrm{ad}}$ which is a specialization of y' and mapped to $h(x)$ by f_i. By Remark 5.18, x' belongs to $C_{[Q'']}$ for a unique $[Q''] \in \mathcal{P}_{K'}(G')$. Take r''

such that $[Q'']$ lies in $\mathcal{P}_{K'}(G')_{r''}$. Since $x' \in C_{[Q]}^- \cap C_{[Q'']}$, Lemma 5.27 tells us that we have either $r < r''$ or $[Q] = [Q'']$.

We write $[P'']$ for the image of $[Q'']$ under the composite

$$\mathcal{P}_{G'}(K') \to \mathcal{P}_{G'^{\mathrm{ad}}}(g_i^{-1}K''g_i) \xrightarrow{g_i} \mathcal{P}_{G'^{\mathrm{ad}}}(K'') \cong \mathcal{P}_{G^{\mathrm{ad}}}(K'') \cong \mathcal{P}_G(K).$$

It belongs to $\mathcal{P}_G(K)_{r''}$. Since $h(x) = f_i(x') \in C_{[P'']}^{\mathrm{ad}}$, the point x lies in $C_{[P'']}$. Hence Remark 5.18 tells us that $[P'] = [P'']$. In particular we have $r' = r''$, which implies $[Q] = [Q'']$. Therefore we have $[P] = [P''] = [P']$, which contradicts the assumption on $[P']$. $\qquad\square$

Corollary 5.29. *Let (G, X) be a Shimura datum of preabelian type. Let K and K' be compact open subgroups of $G(\mathbb{A}^\infty)$ and $g \in G(\mathbb{A}^\infty)$ with $g^{-1}Kg \subset K'$. For an element $[P']$ of $\mathcal{P}_G(K')$, the inverse image of $C_{[P']}$ under the Hecke action $g\colon \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K'}(G, X)$ is equal to*

$$\coprod_{\substack{[P] \in \mathcal{P}_G(K), \\ [g^{-1}Pg]=[P'] \text{ in } \mathcal{P}_G(K')}} C_{[P]}$$

as topological spaces.

In particular, for $[P] \in \mathcal{P}_K(G)$, $C_{[P]}$ is mapped to $C_{[g^{-1}Pg]}$ under the Hecke action by g.

Proof. First note that both $g^{-1}(C_{[P']})$ and $\bigcup_{[P] \in \mathcal{P}_K(G), [g^{-1}Pg]=[P']} C_{[P]}$ are constructible subsets and have the same set of classical points by Proposition 5.16 (i). Therefore, by Lemma 3.6 (i), they are equal. The union $\bigcup_{[P] \in \mathcal{P}_K(G), [g^{-1}Pg]=[P']} C_{[P]}$ is set-theoretically disjoint by Remark 5.18. Let $[P_1]$, $[P_2]$ be distinct elements of $\mathcal{P}_G(K)$ such that $[g^{-1}P_1 g] = [g^{-1}P_2 g]$ in $\mathcal{P}_{K'}(G)$. Lemma 5.28 tells us that $C_{[P_1]}^-$ and $C_{[P_2]}$ are disjoint. This implies that $C_{[P_1]}$ is closed (hence open) in $\bigcup_{[P] \in \mathcal{P}_K(G), [g^{-1}Pg]=[P']} C_{[P]}$. Now the proof is complete. $\qquad\square$

The following lemma will be used in the next section.

Lemma 5.30. *Let the notation be as in Lemma 5.28. Put $\mathcal{P}_G(K)_{\le r} = \bigcup_{r' \le r} \mathcal{P}_G(K)_{r'}$. Then, $U_{r,K} = \bigcup_{[P] \in \mathcal{P}_G(K)_{\le r}} C_{[P]}$ is a constructible open subset of $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$, and $U_{r,K} \setminus U_{r-1,K}$ equals $\coprod_{[P] \in \mathcal{P}_G(K)_r} C_{[P]}$ as topological spaces (here we put $U_{-1,K} = \varnothing$).*

Proof. By Lemma 5.28, the set $\bigcup_{r' > r} \bigcup_{[P] \in \mathcal{P}_G(K)_{r'}} C_{[P]}$ is closed in $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$. Therefore, by Remark 5.18, $U_{r,K}$ is open in $\mathrm{Sh}_K(G, X)_{E_v}^{\mathrm{ad}}$. Since $C_{[P]}$ is constructible for every $[P]$, the subset $U_{r,K}$ is also constructible.

The claim $U_{r,K} \setminus U_{r-1,K} = \coprod_{[P] \in \mathcal{P}_G(K)_r} C_{[P]}$ follows from Lemma 5.28 by the same argument as in the proof of Corollary 5.29. $\qquad\square$

Remark 5.31. If (G, X) is of Hodge type (namely, $(G, X) = (G', X')$), we can also construct $U_{r,K}$ in the following way. Take a compact open subgroup $\widetilde{K} \subset \mathrm{GSp}_{2n}(\mathbb{A}^\infty)$ containing K. Then, $U_{r,K}$ equals the inverse image of $\mathcal{S}_{\widetilde{K},]\leq r[}$ under

$$\mathrm{Sh}_K(G, X)^{\mathrm{ad}}_{E_v} \to \mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})^{\mathrm{ad}}_{\mathbb{Q}_p} = \mathcal{S}_{\widetilde{K}}.$$

This is an immediate consequence of Proposition 5.16 (iii) and Proposition 5.21.

6. Cohomology of Shimura varieties

6.1. *Comparison of cohomology.* We continue to assume that (G, X) is of pre-abelian type. For simplicity, we further assume that (G, X) satisfies SV6 in [Milne 2005, p. 311]. We simply write Sh_K for $\mathrm{Sh}_K(G, X)$, if there is no risk of confusion. Fix a prime ℓ which is different from p. Let G^c be the quotient of G defined in [Milne 1990, p. 347], and ξ an algebraic representation of G^c on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Then we have the associated $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\xi$ on Sh_K (see [Milne 1990, III, §6]). Moreover, $\mathcal{L}_\xi$ is equivariant with respect to the Hecke action.

Let p' be a prime number. Let us fix a compact open subgroup $K^{p'}$ of $G(\widehat{\mathbb{Z}}^{p'})$. We consider the compactly supported cohomology

$$H^i_c(\mathrm{Sh}_{\infty, K^{p'}, \bar{E}_v}, \mathcal{L}_\xi) = \varinjlim_{K_{p'}} H^i_c(\mathrm{Sh}_{K_{p'} K^{p'}, \bar{E}_v}, \mathcal{L}_\xi).$$

The group $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\bar{E}_v/E_v)$ acts on it in a natural way. By this action, $H^i_c(\mathrm{Sh}_{\infty, K^{p'}, \bar{E}_v}, \mathcal{L}_\xi)$ becomes an admissible/continuous representation of $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\bar{E}_v/E_v)$ in the sense of [Harris and Taylor 2001, §I.2].

The group $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\bar{E}_v/E_v)$ naturally acts also on

$$H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty, K^{p'}, \bar{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi) = \varinjlim_{K_{p'}} H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{K_{p'} K^{p'}, \bar{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi).$$

See [Huber 1998a, §1] for the definition of the compactly supported ℓ-adic cohomology for adic spaces. It gives an admissible/continuous representation of $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\bar{E}_v/E_v)$ (*cf.* Lemma 6.6 in the next subsection).

Here we use the notation in [Harris and Taylor 2001, §I.2]. Let H be a locally profinite group. For an admissible/continuous representation V of $H \times \mathrm{Gal}(\bar{E}_v/E_v)$ over $\overline{\mathbb{Q}}_\ell$ and an irreducible admissible representation π of H, put $V[\pi] = \bigoplus_\sigma \sigma^{\oplus m_{\pi \boxtimes \sigma}}$, where σ runs through finite-dimensional irreducible continuous $\overline{\mathbb{Q}}_\ell$-representations of $\mathrm{Gal}(\bar{E}_v/E_v)$ and $m_{\pi \boxtimes \sigma}$ denotes the coefficient of $[\pi \boxtimes \sigma]$ in the image of V in the Grothendieck group considered in [Harris and Taylor 2001, §I.2]. It is a semisimple continuous representation of $\mathrm{Gal}(\bar{E}_v/E_v)$.

Theorem 6.1. *The kernel and the cokernel of the canonical homomorphism*

$$H_c^i(\mathrm{Sh}^{\mathrm{pg}}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}) \to H_c^i(\mathrm{Sh}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi)$$

are noncuspidal, namely, they have no supercuspidal subquotient of $G(\mathbb{Q}_{p'})$. In particular, for an irreducible supercuspidal representation π of $G(\mathbb{Q}_{p'})$, we have an isomorphism

$$H_c^i(\mathrm{Sh}^{\mathrm{pg}}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}})[\pi] \cong H_c^i(\mathrm{Sh}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi)[\pi].$$

This theorem will be proved in Section 6.2.

Remark 6.2. Let $H^i(\mathrm{Sh}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi) = \varinjlim_{K_{p'}} H^i(\mathrm{Sh}_{K_{p'}K^{p'},\bar{E}_v}, \mathcal{L}_\xi)$ be the ordinary cohomology of our Shimura variety. This is also an admissible/continuous representation of $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\bar{E}_v/E_v)$. By using the minimal compactification of Sh_K and its natural stratification (*cf.* [Pink 1992, §3.7]), it is easy to see that the kernel and the cokernel of the canonical homomorphism

$$H_c^i(\mathrm{Sh}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi) \to H^i(\mathrm{Sh}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi)$$

are noncuspidal as $G(\mathbb{Q}_{p'})$-representations (in fact, we can use the similar argument as in the next subsection). Therefore, the kernel and the cokernel of the composite

$$H_c^i(\mathrm{Sh}^{\mathrm{pg}}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}) \to H_c^i(\mathrm{Sh}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi) \to H^i(\mathrm{Sh}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi)$$

are noncuspidal by Theorem 6.1.

Remark 6.3. Let $IH^i(\mathrm{Sh}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi) = \varinjlim_{K_{p'}} H^i(\mathrm{Sh}^{\min}_{K_{p'}K^{p'},\bar{E}_v}, j_{!*}\mathcal{L}_\xi)$ be the intersection cohomology of our Shimura variety, where $j\colon \mathrm{Sh}_{K_{p'}K^{p'}} \hookrightarrow \mathrm{Sh}^{\min}_{K_{p'}K^{p'}}$ denotes the minimal compactification of $\mathrm{Sh}_{K_{p'}K^{p'}}$. Then, as in the previous remark, it is easy to see that the kernel and the cokernel of the canonical homomorphism

$$H_c^i(\mathrm{Sh}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi) \to IH^i(\mathrm{Sh}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi)$$

are noncuspidal. Therefore, by Theorem 6.1, we have an isomorphism

$$H_c^i(\mathrm{Sh}^{\mathrm{pg}}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}})[\pi] \cong IH^i(\mathrm{Sh}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi)[\pi]$$

for an irreducible supercuspidal representation π of $G(\mathbb{Q}_{p'})$.

Corollary 6.4. *We put*

$$H_c^i(\mathrm{Sh}_{\infty,\bar{E}_v}, \mathcal{L}_\xi) = \varinjlim_{K^{p'}} H_c^i(\mathrm{Sh}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi), \quad H_c^i(\mathrm{Sh}^{\mathrm{pg}}_{\infty,\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}})$$

$$= \varinjlim_{K^{p'}} H_c^i(\mathrm{Sh}^{\mathrm{pg}}_{\infty,K^{p'},\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}).$$

These are admissible/continuous $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\bar{E}_v/E_v)$-representations.

Let Π be an irreducible admissible representation of $G(\mathbb{A}^\infty)$. Assume that there exists a prime p' such that $\Pi_{p'}$ is a supercuspidal representation of $G(\mathbb{Q}_{p'})$. Then, Π does not appear as a subquotient of the kernel or the cokernel of the canonical homomorphism $H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty, \bar{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi) \to H^i_c(\mathrm{Sh}_{\infty, \bar{E}_v}, \mathcal{L}_\xi)$. In particular, we have an isomorphism of $\mathrm{Gal}(\bar{E}_v/E_v)$-representations

$$H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty, \bar{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi)[\Pi] \cong H^i_c(\mathrm{Sh}_{\infty, \bar{E}_v}, \mathcal{L}_\xi)[\Pi].$$

Proof. We take a compact open subgroup $K^{p'} \subset G(\widehat{\mathbb{Z}}^{p'})$ such that $\Pi^{K^{p'}} \neq 0$. If Π appears as a subquotient of the kernel or the cokernel of $H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty, \bar{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi) \to H^i_c(\mathrm{Sh}_{\infty, \bar{E}_v}, \mathcal{L}_\xi)$, then $\Pi_{p'}$ appears as a subquotient of the kernel or the cokernel of $H^i_c(\mathrm{Sh}^{\mathrm{pg}}_{\infty, K^{p'}, \bar{E}_v}, \mathcal{L}^{\mathrm{ad}}_\xi) \to H^i_c(\mathrm{Sh}_{\infty, K^{p'}, \bar{E}_v}, \mathcal{L}_\xi)$ (*cf.* Lemma 6.6 in the next subsection). This contradicts Theorem 6.1. $\square$

6.2. *Proof of Theorem 6.1.* Let K be a compact open subgroup of $G(\mathbb{A}^\infty)$. We regard $C_{[P]}$ for $[P] \in \mathcal{P}_G(K)$ as a pseudo-adic space (*cf.* [Huber 1996, §1.10]). See [Huber 1998a, Proposition 2.6(i)] for the definition of the compactly supported ℓ-adic cohomology of pseudo-adic spaces.

Proposition 6.5. *Let $[P]$ be an element of $\mathcal{P}_G(K)$.*

 (i) *For a constructible ℓ-adic sheaf $\mathcal{F} = (\mathcal{F}_n) \otimes \overline{\mathbb{Q}}_\ell$ on $\mathrm{Sh}_{K, \bar{E}_v}$, $H^i_c(C_{[P], \bar{E}_v}, \mathcal{F}^{\mathrm{ad}})$ is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space.*

 (ii) *For a constructible $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaf $\mathcal{F}$ on $\mathrm{Sh}_{K, \bar{E}_v}$, $H^i_c(C_{[P], \bar{E}_v}, \mathcal{F}^{\mathrm{ad}})$ is a finitely generated $\mathbb{Z}/\ell^n\mathbb{Z}$-module.*

Before proving this proposition, we note the following general lemmas.

Lemma 6.6. *Let k be an algebraically closed nonarchimedean field, and X an adic space locally of finite type, separated and taut over k. Let L be a locally closed constructible subset of X, which is regarded as a pseudo-adic space. Let $\pi : X' \to X$ be a finite étale Galois covering with Galois group H. We put $L' = \pi^{-1}(L)$. For an ℓ-adic sheaf $\mathcal{F} = (\mathcal{F}_n) \otimes \overline{\mathbb{Q}}_\ell$ on X, the natural map*

$$H^i_c(L, \mathcal{F}) \to H^i_c(L', \pi^* \mathcal{F})^H$$

is an isomorphism.

Proof. This lemma might be well-known, but we include its proof for reader's convenience. We put $\mathcal{F}'_n = \pi_* \pi^* \mathcal{F}_n$ and $\mathcal{F}' = (\mathcal{F}'_n) \otimes \overline{\mathbb{Q}}_\ell$. The group H acts on $\mathcal{F}'_n$, and we have $(\mathcal{F}'_n)^H = \mathcal{F}_n$. Consider the map $\psi = \sum_{h \in H} h : \mathcal{F}'_n \to \mathcal{F}_n$. The composite $\mathcal{F}_n \hookrightarrow \mathcal{F}'_n \xrightarrow{\psi} \mathcal{F}_n$ equals the multiplication by $\#H$. By taking the

cohomology, we have a commutative diagram

$$\begin{array}{ccccc} H_c^i\big(L, (\mathcal{F}_n)_n\big) \otimes \overline{\mathbb{Q}}_\ell & \longrightarrow & H_c^i\big(L, (\mathcal{F}'_n)_n\big) \otimes \overline{\mathbb{Q}}_\ell & \xrightarrow{H_c^i(\psi)} & H_c^i\big(L, (\mathcal{F}_n)_n\big) \otimes \overline{\mathbb{Q}}_\ell \\ & \searrow{\scriptstyle \pi^*} & \| & & \downarrow{\scriptstyle \pi^*} \\ & & H_c^i\big(L', (\pi^*\mathcal{F}_n)_n\big) \otimes \overline{\mathbb{Q}}_\ell & \xrightarrow{\sum_{h \in H} h} & H_c^i\big(L', (\pi^*\mathcal{F}_n)_n\big) \otimes \overline{\mathbb{Q}}_\ell. \end{array}$$

The composite of two upper horizontal arrows is the multiplication by $\#H$, which is an isomorphism. Therefore π^* is injective and $H_c^i(\psi)$ is surjective. The surjectivity of $H_c^i(\psi)$ implies that the image of π^* is equal to that of $\sum_{h \in H} h$, that is, the H-invariant part of $H_c^i(L', (\pi^*\mathcal{F}_n)_n) \otimes \overline{\mathbb{Q}}_\ell$. □

Remark 6.7. By the same method, we can also prove that the natural map

$$\varprojlim_n H_c^i(L, \mathcal{F}_n) \otimes \overline{\mathbb{Q}}_\ell \to \Big(\varprojlim_n H_c^i(L', \pi^*\mathcal{F}_n) \otimes \overline{\mathbb{Q}}_\ell \Big)^H$$

is an isomorphism.

Lemma 6.8. *Let k be an algebraically closed nonarchimedean field. Let X be an adic space locally of finite type, separated and taut over k, and U a constructible open subset of X. Set $Z = X \setminus U$, which is regarded as a pseudo-adic space. For an ℓ-adic sheaf $\mathcal{F} = (\mathcal{F}_n) \otimes \overline{\mathbb{Q}}_\ell$ over X, we have a long exact sequence*

$$\cdots \to H_c^i(U, \mathcal{F}) \to H_c^i(X, \mathcal{F}) \to H_c^i(Z, \mathcal{F}) \to H_c^{i+1}(U, \mathcal{F}) \to \cdots .$$

Proof. Since the open immersion $j : U \hookrightarrow X$ is quasicompact, we have $H_c^i(U, \mathcal{F}) = H_c^i(X, j_! j^* \mathcal{F})$ by the definition of the compactly supported cohomology. Therefore, the claim follows from [Huber 1998a, Proposition 2.6(i)]. □

Proof of Proposition 6.5. We consider (i). Note that Corollary 5.29 and Lemma 6.6 enable us to shrink K arbitrarily.

First consider the Hodge type case. Take an embedding $(G, X) \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura datum. We have a constructible open subset $U_{r,K}$ for each $0 \le r \le n$ by Lemma 5.30. By the long exact sequence

$$\cdots \to H_c^i(U_{r-1,K,\bar{E}_v}, \mathcal{F}) \to H_c^i(U_{r,K,\bar{E}_v}, \mathcal{F}) \to H_c^i(U_{r,K,\bar{E}_v} \setminus U_{r-1,K,\bar{E}_v}, \mathcal{F})$$
$$\to H_c^{i+1}(U_{r-1,K,\bar{E}_v}, \mathcal{F}) \to \cdots$$

(see Lemma 6.8) and Lemma 5.30, it suffices to show that $H_c^i(U_{r,K,\bar{E}_v}, \mathcal{F})$ is finite-dimensional for each $0 \le r \le n$.

Take a compact open subgroup $\widetilde{K}$ of $G(\mathbb{A}^\infty)$ so that we have a natural embedding $\mathrm{Sh}_K(G, X) \hookrightarrow \mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})_E$. By shrinking K, we may assume that $\widetilde{K} = \widetilde{K}_p \widetilde{K}^p$ with $\widetilde{K}_p \subset \mathrm{GSp}_{2n}(\mathbb{Z}_p)$. Let $\mathscr{S}_{\widetilde{K}}^{\mathrm{nor}}$ be the normalization of $\mathscr{S}_{\widetilde{K}_p}^{\mathrm{tor}}$ in $\mathrm{Sh}_{\widetilde{K}}(\mathrm{GSp}_{2n}, X_{2n})_{\mathbb{Q}_p}$, and $\mathscr{S}_{\widetilde{K}, \le r, \mathbb{F}_p}^{\mathrm{nor}}$ the inverse image of $\mathscr{S}_{\widetilde{K}^p, \le r, \mathbb{F}_p}^{\mathrm{tor}}$ in $\mathscr{S}_{\widetilde{K}}^{\mathrm{nor}}$. We

write Y for the closure of $\mathrm{Sh}_K(G, X)_{E_v}$ in $\mathscr{S}^{\mathrm{nor}}_{\widetilde{K}, E_v}$ and put $Z = Y \setminus \mathrm{Sh}_K(G, X)_{E_v}$. Let V denote the inverse image of $\mathscr{S}^{\mathrm{nor}}_{\widetilde{K}, \leq r, \mathbb{F}_p}$ under the composite

$$Y^{\mathrm{ad}} \hookrightarrow (\mathscr{S}^{\mathrm{nor}}_{\widetilde{K}})^{\mathrm{ad}}_{E_v} \to (\mathscr{S}^{\mathrm{nor}}_{\widetilde{K}})^{\mathrm{ad}}_{\mathbb{Q}_p} = (\mathscr{S}^{\mathrm{nor}}_{\widetilde{K}})^{\wedge \mathrm{rig}} \xrightarrow{\mathrm{sp}} \mathscr{S}^{\mathrm{nor}}_{\widetilde{K}, \mathbb{F}_p},$$

which is a quasicompact open subset of Y^{ad}. Note that $V \setminus (V \cap Z^{\mathrm{ad}}) = U_{r,K}$ by Remark 5.31. Therefore, [Huber 1998a, Theorem 3.3(i)] tells us that $H^i_c(U_{r,K,\bar{E}_v}, \mathcal{F})$ is finite-dimensional. This completes the proof in the Hodge type case.

Next we consider the preabelian type case. We choose a compact open subgroup of K'' of $G^{\mathrm{ad}}(\mathbb{A}^\infty)$ in such a way as in Lemma 5.10 (iii), and use the notation in Lemma 4.4. Then, for $[P] \in \mathcal{P}_K(G)$, the inverse image of $C_{[P^{\mathrm{ad}}]}$ under $\pi \colon \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{K''}(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is equal to $C_{[P]}$. Therefore, by pushing forward sheaves by π, we may assume that $G = G'^{\mathrm{ad}}$ (note that $(\pi_* \mathcal{F}_n)^{\mathrm{ad}} = \pi^{\mathrm{ad}}_* \mathcal{F}^{\mathrm{ad}}_n$ by [Huber 1996, Theorem 3.7.2]). Since the Hecke action is transitive on the connected components of $\mathrm{Sh}_K(G, X)_{\bar{E}_v}$, we may work on a connected component $\mathrm{Sh}_K(G, X)^0_{\bar{E}_v}$ of $\mathrm{Sh}_K(G, X)_{\bar{E}_v}$ which is a quotient of a connected component $\mathrm{Sh}_{K'}(G', X')^0_{\bar{E}_v}$ of $\mathrm{Sh}_{K'}(G', X')_{\bar{E}_v}$ by a free action of a finite group H for some K'. By Corollary 5.29, the inverse image of $C_{[P], \bar{E}_v} \cap \mathrm{Sh}_K(G, X)^{0,\mathrm{ad}}_{\bar{E}_v}$ for $[P] \in \mathcal{P}_K(G)$ under

$$f \colon \mathrm{Sh}_{K'}(G', X')^{0,\mathrm{ad}}_{\bar{E}_v} \to \mathrm{Sh}_K(G, X)^{0,\mathrm{ad}}_{\bar{E}_v}$$

equals

$$\coprod_{[P'] \in \mathcal{P}_{K'}(G'), [P'] \mapsto [P]} C_{[P'], \bar{E}_v} \cap \mathrm{Sh}_{K'}(G', X')^{0,\mathrm{ad}}_{\bar{E}_v}.$$

Since $H^i_c(C_{[P'], \bar{E}_v}, f^* \mathcal{F})$ is finite-dimensional, so is

$$H^i_c\big(f^{-1}(C_{[P], \bar{E}_v} \cap \mathrm{Sh}_K(G, X)^{0,\mathrm{ad}}_{\bar{E}_v}), f^* \mathcal{F}\big).$$

By Lemma 6.6, $H^i_c(C_{[P], \bar{E}_v} \cap \mathrm{Sh}_K(G, X)^{0,\mathrm{ad}}_{\bar{E}_v}, \mathcal{F})$ is equal to the H-invariant part of the above, hence finite-dimensional. This concludes the proof of (i).

The assertion (ii) can be proved in the same way, by using the Hochschild–Serre spectral sequence in place of Lemma 6.6 when taking a quotient. $\square$

Remark 6.9. By the same method and Remark 6.7, we can also prove that the natural map $H^i_c(C_{[P], \bar{E}_v}, \mathcal{F}^{\mathrm{ad}}) \to \varprojlim_n H^i_c(C_{[P], \bar{E}_v}, \mathcal{F}^{\mathrm{ad}}_n) \otimes \overline{\mathbb{Q}}_\ell$ is an isomorphism. However, we do not need this fact.

Now let $K^{p'}$ be as in Section 6.1, $K_{p'}$ a compact open subgroup of $G(\mathbb{Q}_{p'})$, and $K = K_{p'} K^{p'}$. Take a Shimura datum (G', X') of Hodge type such that $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$, and an embedding $i \colon (G', X') \hookrightarrow (\mathrm{GSp}_{2n}, X_{2n})$ into a Siegel Shimura

datum. Then, as in Lemma 5.28 and Lemma 5.30, we obtain an increasing sequence

$$\{[G]\} = \mathcal{P}_G(K)_{\leq 0} \subset \mathcal{P}_G(K)_{\leq 1} \subset \cdots \subset \mathcal{P}_G(K)_{\leq n} = \mathcal{P}_G(K)$$

of subsets of $\mathcal{P}_G(K)$. Note that $\mathcal{P}_G(K)_r = \mathcal{P}_G(K)_{\leq r} \setminus \mathcal{P}_G(K)_{\leq r-1}$ is a union of fibers of the natural map $\mathcal{P}_G(K) \to \mathcal{P}_{G,\mathbb{A}^\infty}$. Therefore, by refining the sequence above, we can find an increasing sequence

$$\{[G]\} = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_m = \mathcal{P}_G(K)$$

of subsets of $\mathcal{P}_G(K)$ satisfying the following conditions:

- For every $0 \leq r \leq n$, there exists $0 \leq j \leq m$ such that $S_j = \mathcal{P}_G(K)_{\leq r}$.
- For $[P_1], [P_2] \in \mathcal{P}_G(K)$, P_1 and P_2 are conjugate by $G(\mathbb{Q}_{p'})$ if and only if $[P_1], [P_2] \in S_j \setminus S_{j-1}$ for some $0 \leq j \leq m$ (here we put $S_{-1} = \varnothing$).

For $0 \leq j \leq m$, we put $T_{j,K} = \bigcup_{[P] \in S_j} C_{[P]}$. By Lemma 5.30, it is a constructible open subset of $\mathrm{Sh}_{K,E_v}^{\mathrm{ad}}$. Further, we put $Z_{j,K} = T_{j,K} \setminus T_{j-1,K}$ ($T_{-1,K}$ is regarded as $\varnothing$). Lemma 5.30 tells us that $Z_{j,K} = \bigsqcup_{[P] \in S_j \setminus S_{j-1}} C_{[P]}$ as topological spaces. For a compact open subgroup $K_{p'}' \subset K_{p'}$, we put $K' = K_{p'}' K^{p'}$, and let S_j' be the inverse image of S_j under the natural map $\mathcal{P}_G(K') \to \mathcal{P}_G(K)$. The sequence $\{S_j'\}_{0 \leq j \leq m}$ satisfies the same conditions as $\{S_j\}_{0 \leq j \leq m}$ does. Therefore, we can define $T_{j,K'}$ and $Z_{j,K'}$ in the same way as $T_{j,K}$ and $Z_{j,K}$. Note that $T_{j,K'}$ (resp. $Z_{j,K'}$) is the inverse image of $T_{j,K}$ (resp. $Z_{j,K}$) under $\mathrm{Sh}_{K',E_v}^{\mathrm{ad}} \to \mathrm{Sh}_{K,E_v}^{\mathrm{ad}}$. We put

$$V_{\leq j}^i = \varinjlim_{K_{p'}'} H_c^i(T_{j,K',\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}), \qquad V_j^i = \varinjlim_{K_{p'}'} H_c^i(Z_{j,K',\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}),$$

where $K_{p'}'$ runs through compact open subgroups of $K_{p'}$.

Lemma 6.10. (i) *The group $G(\mathbb{Q}_{p'})$ naturally acts on $V_{\leq j}^i$ and V_j^i, and these are admissible $G(\mathbb{Q}_{p'})$-representations.*

(ii) *We have the following long exact sequence of $G(\mathbb{Q}_{p'})$-modules*:

$$\cdots \to V_{\leq j-1}^i \to V_{\leq j}^i \to V_j^i \to V_{\leq j-1}^{i+1} \to \cdots.$$

Proof. By Corollary 5.29, the group $G(\mathbb{Q}_{p'})$ acts on $V_{\leq j}^i$ and V_j^i. By Lemma 6.8, we have a long exact sequence as in (ii), which is obviously $G(\mathbb{Q}_{p'})$-equivariant.

Let us prove (i). Clearly $V_{\leq j}^i$ and V_j^i are smooth $G(\mathbb{Q}_{p'})$-representations. We will show the admissibility of them. Take a compact open subgroup $K_{p'}$ of $G(\mathbb{Q}_{p'})$ and its open normal subgroup $K_{p'}'$, and put $K = K_{p'}K^{p'}$, $K' = K_{p'}'K^{p'}$. Then, we have

$$H_c^i(Z_{j,K',\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}})^{K_{p'}} = H_c^i(Z_{j,K,\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}})$$

by Lemma 6.6. Taking the inductive limit with respect to $K'_{p'}$, we have

$$(V_j^i)^{K_{p'}} = H_c^i(Z_{j,K,\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}) = \bigoplus_{[P] \in S_j \setminus S_{j-1}} H_c^i(C_{[P],\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}).$$

By Proposition 6.5 (i), it is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Therefore we conclude that V_j^i is an admissible representation of $G(\mathbb{Q}_{p'})$. By the long exact sequence in (ii) and the obvious identity $V_0^i = V_{\leq 0}^i$, we can see the admissibility of $V_{\leq j}^i$ inductively. $\qquad\square$

Proposition 6.11. *We take a representative P_j of an element of $S_j \setminus S_{j-1}$. For a compact open subgroup $K'_{p'} \subset K_{p'}$, let $[P_j]_{K'}$ denote the class of P_j in $\mathcal{P}_G(K')$. We put $W_j^i = \varinjlim_{K'_{p'}} H_c^i(C_{[P_j]_{K'},\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}})$, where $K'_{p'}$ runs through compact open subgroups of $K_{p'}$.*

  (i) *We have a natural smooth action of $P_j(\mathbb{Q}_{p'})$ on W_j^i.*

 (ii) *We have a natural $G(\mathbb{Q}_{p'})$-equivariant isomorphism $V_j^i \cong \mathrm{Ind}_{P_j(\mathbb{Q}_{p'})}^{G(\mathbb{Q}_{p'})} W_j^i$.*

(iii) *The $G(\mathbb{Q}_{p'})$-representation V_j^i is noncuspidal.*

Proof. The claim (i) is clear from Corollary 5.29. Let us prove (ii). We follow the proof of [Ito and Mieda 2010, Proposition 5.20]. By the Frobenius reciprocity, we have a homomorphism of $G(\mathbb{Q}_{p'})$-modules $\mathrm{Ind}_{P_j(\mathbb{Q}_{p'})}^{G(\mathbb{Q}_{p'})} W_j^i \to V_j^i$. We shall observe that this is bijective. We take a special maximal compact subgroup $K_{p'}^0$ of $G(\mathbb{Q}_{p'})$. For a compact open subgroup $K'_{p'} \subset K_{p'}$ which is normal in $K_{p'}^0$, we have

$$
\begin{aligned}
H_c^i(Z_{j,K',\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}) &\overset{(1)}{=} \bigoplus_{[P] \in S'_j \setminus S'_{j-1}} H_c^i(C_{[P],\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}) \\
&\overset{(2)}{=} \bigoplus_{g \in K'_{p'} \backslash G(\mathbb{Q}_{p'}) / P_j(\mathbb{Q}_{p'})} H_c^i(C_{[gP_jg^{-1}]_{K'},\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}) \\
&\overset{(3)}{=} \bigoplus_{g \in K'_{p'} \backslash K_{p'}^0 / P_j(\mathbb{Q}_{p'}) \cap K_{p'}^0} H_c^i(C_{[gP_jg^{-1}]_{K'},\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}) \\
&\overset{(4)}{\cong} \mathrm{Ind}_{(P_j(\mathbb{Q}_{p'}) \cap K_{p'}^0)/(P_j(\mathbb{Q}_{p'}) \cap K'_{p'})}^{K_{p'}^0/K'_{p'}} H_c^i(C_{[P_j]_{K'},\bar{E}_v}, \mathcal{L}_\xi^{\mathrm{ad}}).
\end{aligned}
$$

Here (1) follows from $Z_{j,K'} = \coprod_{[P] \in S'_j \setminus S'_{j-1}} C_{[P]}$ mentioned before, (2) from the definitions of S'_j and P_j, and (3) from the Iwasawa decomposition $G(\mathbb{Q}_{p'}) = P_j(\mathbb{Q}_{p'})K_{p'}^0$. The isomorphism (4) is a consequence of Corollary 5.29 and [Boyer

1999, Lemme 13.2]. By taking the inductive limit, we obtain $K_{p'}^0$-isomorphisms

$$V_j^i \cong \operatorname{Ind}_{P_j(\mathbb{Q}_{p'}) \cap K_{p'}^0}^{K_{p'}^0} W_j^i \overset{\cong}{\leftarrow} \operatorname{Ind}_{P_j(\mathbb{Q}_{p'})}^{G(\mathbb{Q}_{p'})} W_j^i$$

(the second map is an isomorphism by the Iwasawa decomposition). By the proof of [Boyer 1999, Lemme 13.2], it is easy to see that the first isomorphism above is nothing but the $K_{p'}^0$-homomorphism obtained by the Frobenius reciprocity for $P_j(\mathbb{Q}_{p'}) \cap K_{p'}^0 \subset K_{p'}^0$. Therefore the composite of the two isomorphisms above coincides with the $G(\mathbb{Q}_{p'})$-homomorphism introduced at the beginning of our proof of (ii). Thus we conclude the proof of (ii).

Finally consider (iii). By (ii), we have only to prove that the unipotent radical of $P_j(\mathbb{Q}_{p'})$ acts trivially on W_j^i. By [Boyer 1999, Lemme 13.2.3], it suffices to prove that W_j^i is an admissible $P_j(\mathbb{Q}_{p'})$-representation. For any compact open subgroup $K_{p'}'$ of $K_{p'}^0$, the vector space $(W_j^i)^{P_j(\mathbb{Q}_{p'}) \cap K_{p'}'}$ is a subspace of $(\operatorname{Ind}_{P_j(\mathbb{Q}_{p'})}^{G(\mathbb{Q}_{p'})} W_j^i)^{K_{p'}'}$. By (ii) and Lemma 6.10 (i), $(\operatorname{Ind}_{P_j(\mathbb{Q}_{p'})}^{G(\mathbb{Q}_{p'})} W_j^i)^{K_{p'}'} \cong (V_j^i)^{K_{p'}'}$ is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space. Hence W_j^i is an admissible $P_j(\mathbb{Q}_{p'})$-representation, as desired. $\square$

Proof of Theorem 6.1. The claim follows from Lemma 6.10 (ii) and Proposition 6.11 (iii), because $V_{\leq 0}^i = H_c^i(\operatorname{Sh}_{\infty, K^{p'}, \overline{E}_v}^{\mathrm{pg}}, \mathcal{L}_\xi^{\mathrm{ad}})$ and $V_{\leq m}^i = H_c^i(\operatorname{Sh}_{\infty, K^{p'}, \overline{E}_v}, \mathcal{L}_\xi)$. $\square$

6.3. *Torsion coefficients.* Since our method of proving Theorem 6.1 is totally geometric, we may also obtain an analogous result for ℓ-torsion coefficients. For simplicity, we will only consider a constant coefficient $\overline{\mathbb{F}}_\ell$. We assume that $p' \neq \ell$, and put

$$H_c^i(\operatorname{Sh}_{\infty, K^{p'}, \overline{E}_v}, \overline{\mathbb{F}}_\ell) = \varinjlim_{K_{p'}} H_c^i(\operatorname{Sh}_{K_{p'} K^{p'}, \overline{E}_v}, \overline{\mathbb{F}}_\ell),$$

$$H_c^i(\operatorname{Sh}_{\infty, K^{p'}, \overline{E}_v}^{\mathrm{pg}}, \overline{\mathbb{F}}_\ell) = \varinjlim_{K_{p'}} H_c^i(\operatorname{Sh}_{K_{p'} K^{p'}, \overline{E}_v}^{\mathrm{pg}}, \overline{\mathbb{F}}_\ell).$$

They are naturally endowed with actions of $G(\mathbb{Q}_{p'}) \times \operatorname{Gal}(\overline{E}_v/E_v)$. They are admissible/continuous $G(\mathbb{Q}_{p'}) \times \operatorname{Gal}(\overline{E}_v/E_v)$-representations; note that we have

$$H_c^i(\operatorname{Sh}_{\infty, K^{p'}, \overline{E}_v}, \overline{\mathbb{F}}_\ell)^{K_{p'}} = H_c^i(\operatorname{Sh}_{K_{p'} K^{p'}, \overline{E}_v}, \overline{\mathbb{F}}_\ell),$$

$$H_c^i(\operatorname{Sh}_{\infty, K^{p'}, \overline{E}_v}^{\mathrm{pg}}, \overline{\mathbb{F}}_\ell)^{K_{p'}} = H_c^i(\operatorname{Sh}_{K_{p'} K^{p'}, \overline{E}_v}^{\mathrm{pg}}, \overline{\mathbb{F}}_\ell),$$

if $K_{p'}$ is a pro-p' group (*cf.* [Mieda 2010, Proposition 2.5]).

The following theorem can be proved in exactly the same way as Theorem 6.1 (we use Proposition 6.5 (ii) in place of Proposition 6.5 (i)).

Theorem 6.12. *We assume that $p' \neq \ell$. The kernel and the cokernel of the canonical homomorphism*

$$H_c^i(\mathrm{Sh}^{\mathrm{pg}}_{\infty, K^{p'}, \bar{E}_v}, \overline{\mathbb{F}}_\ell) \to H_c^i(\mathrm{Sh}_{\infty, K^{p'}, \bar{E}_v}, \overline{\mathbb{F}}_\ell)$$

have no supercuspidal subquotient of $G(\mathbb{Q}_{p'})$. (For the definition of supercuspidal representations over $\overline{\mathbb{F}}_\ell$, see [Vignéras 1996, II.2.5].)

7. PEL type case

7.1. *Notation for Shimura varieties of PEL type.* In this section, we are interested in Shimura varieties of PEL type considered in [Kottwitz 1992b, §5] (see also [Lan 2013, §1.4]). We recall it briefly. Fix a prime p. Consider a 6-tuple $(B, \mathcal{O}_B, *, V, L, \langle \, , \, \rangle)$, where

- B is a finite-dimensional simple $\mathbb{Q}$-algebra such that $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a product of matrix algebras over unramified extensions of $\mathbb{Q}_p$,

- $\mathcal{O}_B$ is an order of B whose p-adic completion is a maximal order of $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$,

- $*$ is a positive involution of B (namely, an involution such that $\mathrm{Tr}(bb^*) > 0$ for every nonzero $b \in B$) which preserves $\mathcal{O}_B$,

- V is a nonzero finite B-module,

- L is a $\mathbb{Z}$-lattice of V preserved by $\mathcal{O}_B$, and

- $\langle \, , \, \rangle \colon V \times V \to \mathbb{Q}$ is a nondegenerate alternating $*$-Hermitian pairing with respect to the B-action such that $\langle x, y \rangle \in \mathbb{Z}$ for every $x, y \in L$, and that $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a self-dual lattice of $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

From $(B, V, \langle \, , \, \rangle)$, we define a simple $\mathbb{Q}$-algebra $C = \mathrm{End}_B(V)$ with a unique involution # satisfying $\langle cv, w \rangle = \langle v, c^\# w \rangle$ for every $c \in C$ and $v, w \in V$. Moreover we define an algebraic group G over $\mathbb{Q}$ by

$$G(R) = \{g \in (C \otimes_{\mathbb{Q}} R)^\times \mid gg^\# \in R^\times\}$$

for every $\mathbb{Q}$-algebra R. The condition $gg^\# \in R^\times$ is equivalent to the existence of $c(g) \in R^\times$ such that $\langle gv, gw \rangle = c(g)\langle v, w \rangle$ for every $v, w \in V \otimes_{\mathbb{Q}} R$. By the presence of the lattice L, G can be naturally extended to a group scheme over $\mathbb{Z}$, which is also denoted by the same symbol G.

Consider an $\mathbb{R}$-algebra homomorphism $h \colon \mathbb{C} \to C \otimes_{\mathbb{Q}} \mathbb{R}$ preserving involutions (on $\mathbb{C}$, we consider the complex conjugation) such that the symmetric real-valued bilinear form $(v, w) \mapsto \langle v, h(i)w \rangle$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$ is positive definite. Such a 7-tuple $(B, \mathcal{O}_B, *, V, L, \langle \, , \, \rangle, h)$ is said to be an unramified integral PEL datum. Note that the map h induces a homomorphism $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$ of algebraic groups over $\mathbb{R}$, which is also denoted by h.

Let F be the center of B and F^+ the subfield of F consisting of elements fixed by $*$. The existence of h tells us that $N = [F : F^+](\dim_F C)^{1/2}/2$ is an integer. An unramified integral PEL datum falls into the following three types:

type (A): $[F : F^+] = 2$.

type (C): $[F : F^+] = 1$ and $C \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to a product of $M_{2N}(\mathbb{R})$.

type (D): $[F : F^+] = 1$ and $C \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to a product of $M_N(\mathbb{H})$.

For simplicity, we will exclude the type (D) case.

Using $h \colon \mathbb{C} \to C \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow C \otimes_{\mathbb{Q}} \mathbb{C}$, we can decompose the $B \otimes_{\mathbb{Q}} \mathbb{C}$-module $V \otimes_{\mathbb{Q}} \mathbb{C}$ as $V \otimes_{\mathbb{Q}} \mathbb{C} = V_1 \oplus V_2$, where V_1 (resp. V_2) is the subspace of $V \otimes_{\mathbb{Q}} \mathbb{C}$ on which $h(z)$ acts by z (resp. $\bar{z}$) for every $z \in \mathbb{C}$. We denote by E the field of definition of the isomorphism class of the $B \otimes_{\mathbb{Q}} \mathbb{C}$-module V_1, and call it the reflex field. It is a subfield of $\mathbb{C}$ which is finite over $\mathbb{Q}$.

In the sequel, we fix an unramified integral PEL datum $(B, \mathcal{O}_B, *, V, L, \langle\ ,\ \rangle, h)$. For a compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, consider the functor $\mathscr{S}_{K^p}$ from the category of $\mathcal{O}_{E,(p)}$-schemes to the category of sets, that associates S to the set of isomorphism classes of quadruples (A, i, λ, η^p), where

- A is an abelian scheme over S,
- $\lambda \colon A \to A^\vee$ is a prime-to-p polarization,
- $i \colon \mathcal{O}_B \to \mathrm{End}_S(A)$ is an algebra homomorphism such that $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$ for every $b \in \mathcal{O}_B$,
- η^p is a level-K^p structure of (A, i, λ) of type $(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p, \langle\ ,\ \rangle)$ in the sense of [Lan 2013, Definition 1.3.7.6],

satisfying the equality of polynomials $\det_{\mathcal{O}_S}(b; \mathrm{Lie}\, A) = \det_E(b; V_1)$ in the sense of [Kottwitz 1992b, §5]. Recall that two quadruples (A, i, λ, η^p) and $(A', i', \lambda', \eta'^p)$ are said to be isomorphic if there exists an isomorphism $f \colon A \to A'$ of abelian schemes such that

- $\lambda = f^\vee \circ \lambda' \circ f$,
- $f \circ i(b) = i'(b) \circ f$ for every $b \in \mathcal{O}_B$,
- and $f \circ \eta^p = \eta'^p$ in the sense of [Lan 2013, Definition 1.4.1.4].

If K^p is neat (*cf.* [Lan 2013, Definition 1.4.1.8]), the functor $\mathscr{S}_{K^p}$ is represented by a quasiprojective smooth $\mathcal{O}_{E,(p)}$-scheme (see [Lan 2013, Corollary 7.2.3.9]), which is also denoted by $\mathscr{S}_{K^p}$. Here we will call it a Shimura variety of PEL type. The group $G(\mathbb{A}^{\infty,p})$ naturally acts on the tower of schemes $(\mathscr{S}_{K^p})_{K^p \subset G(\widehat{\mathbb{Z}}^p)}$ as Hecke correspondences.

Let ℓ be a prime number different from p. For an algebraic representation ξ of G on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space, we can define a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\xi$ on Sh_K (see [Milne 1990, III, §6]). It is equivariant with respect to the Hecke action.

Remark 7.1. (i) Our definition of $\mathscr{S}_{K^p}$, due to [Lan 2013], is slightly different from that in [Kottwitz 1992b], but they give the same moduli space. See [Lan 2013, Proposition 1.4.3.4].

(ii) Let us recall the relation between $\mathscr{S}_{K^p}$ and Shimura varieties in Section 4. See [Kottwitz 1992b, §8] for detail. Let X denote the $G(\mathbb{R})$-orbit of the homomorphism $h\colon \operatorname{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$. Then, the pair (G, X) forms a Shimura datum, and $\mathscr{S}_{K^p,E}$ is isomorphic to a disjoint union of $\#\ker^1(\mathbb{Q}, G)$ copies of $\operatorname{Sh}_{G(\mathbb{Z}_p)K^p}(G, X)$. In the cases of type (C) or type (A) with N even, it is known that $\ker^1(\mathbb{Q}, G) = 1$.

So far in this section, we have only considered level structures which are prime to p. Now we add p^m-level structures on the universal abelian scheme of the generic fiber $\mathscr{S}_{K^p,E}$. Let $\mathscr{S}_{m,K^p,E}$ be the scheme over $\mathscr{S}_{K^p,E}$ classifying principal level-m structures (*cf.* [Lan 2013, Definition 1.3.6.2]) of the universal object $(\mathcal{A}, i^{\mathrm{univ}}, \lambda^{\mathrm{univ}})$ over $\mathscr{S}_{K^p,E}$. We denote the structure morphism $\mathscr{S}_{m,K^p,E} \to \mathscr{S}_{K^p,E}$ by pr_m, which is finite and étale. We write $\mathcal{L}_{\xi,m}$ or $\mathcal{L}_\xi$ for the inverse image of $\mathcal{L}_\xi$ by pr_m.

Let $K_{p,m}$ be the compact open subgroup of $G(\mathbb{Q}_p)$ defined as the kernel of $G(\mathbb{Z}_p) \to G(\mathbb{Z}/p^m\mathbb{Z})$. Then $\mathscr{S}_{m,K^p,E}$ coincides with a disjoint union of the Shimura variety $\operatorname{Sh}_{K_{p,m}K^p}(G, X)$, where we use the notation in Remark 7.1 (ii).

7.2. *Compactly supported cohomology and nearby cycle cohomology.* Fix a place v of E over p. We write E_v for the completion of E at v, $\mathcal{O}_v$ for the ring of integers of E_v, and κ_v the residue field of $\mathcal{O}_v$. We put $\mathscr{S}_{K^p,\eta} = \mathscr{S}_{K^p,E_v}$, $\mathscr{S}_{K^p,\overline{\eta}} = \mathscr{S}_{K^p,\overline{E}_v}$, $\mathscr{S}_{K^p,v} = \mathscr{S}_{K^p,\kappa_v}$ and $\mathscr{S}_{K^p,\overline{v}} = \mathscr{S}_{K^p,\overline{\kappa}_v}$. Further, for $m \geq 0$ we set $\mathscr{S}_{m,K^p,\eta} = \mathscr{S}_{m,K^p,E} \otimes_E E_v$ and $\mathscr{S}_{m,K^p,\overline{\eta}} = \mathscr{S}_{m,K^p,E} \otimes_E \overline{E}_v$.

Let p' be a prime number, and $K^{p'} \subset G(\widehat{\mathbb{Z}}^{p'})$ a compact open subgroup. If $p' \neq p$, we assume that $K^{p'} = K_{p,m_0}K^{p',p}$ for some $m_0 \geq 0$ and compact open subgroup $K^{p',p}$ of $G(\widehat{\mathbb{Z}}^{p',p})$. We put

$$H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{\eta}}, \mathcal{L}_\xi) = \begin{cases} \varinjlim_m H_c^i(\mathscr{S}_{m,K^p,\overline{\eta}}, \mathcal{L}_\xi) & \text{if } p' = p, \\ \varinjlim_{K_{p'}} H_c^i(\mathscr{S}_{m_0,K_{p'}K^{p',p},\overline{\eta}}, \mathcal{L}_\xi) & \text{if } p' \neq p, \end{cases}$$

which is an admissible/continuous $G(\mathbb{Q}_{p'}) \times \operatorname{Gal}(\overline{E}_v/E_v)$-representation. We are also interested in the nearby cycle cohomology defined as follows:

$$H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{v}}, R\psi\mathcal{L}_\xi) = \begin{cases} \varinjlim_m H_c^i\big(\mathscr{S}_{K^p,\overline{v}}, R\psi(\mathrm{pr}_{m*}\mathcal{L}_\xi)\big) & \text{if } p' = p, \\ \varinjlim_{K_{p'}} H_c^i\big(\mathscr{S}_{K_{p'}K^{p',p},\overline{v}}, R\psi(\mathrm{pr}_{m_0*}\mathcal{L}_\xi)\big) & \text{if } p' \neq p. \end{cases}$$

Obviously the group $\operatorname{Gal}(\overline{E}_v/E_v)$ acts on it. The following lemma gives an action of $G(\mathbb{Q}_{p'})$ on $H_c^i(\mathscr{S}_{\infty,K^{p'},\overline{v}}, R\psi\mathcal{L}_\xi)$.

Lemma 7.2. *We have a natural action of $G(\mathbb{Q}_{p'})$ on $H_c^i(\mathscr{S}_{\infty,K^{p'},\bar{v}}, R\psi\mathcal{L}_\xi)$. By this action, $H_c^i(\mathscr{S}_{\infty,K^{p'},\bar{v}}, R\psi\mathcal{L}_\xi)$ becomes an admissible/continuous $G(\mathbb{Q}_{p'}) \times \mathrm{Gal}(\bar{E}_v/E_v)$-representation.*

Proof. We show the claim in the case $p' = p$. The other cases are easier. To ease notation, we omit the subscript K^p.

As in [Mantovan 2005, §6], we can construct a tower $(\mathscr{S}_m)_{m\geq0}$ of schemes over $\mathcal{O}_v$ with finite transition maps such that $\mathscr{S}_m$ gives an integral model of $\mathscr{S}_{m,\eta}$ and $\mathscr{S}_0 = \mathscr{S}$. In this situation, we have

$$H_c^i\big(\mathscr{S}_{\bar{v}}, R\psi(\mathrm{pr}_{m*}\mathcal{L}_\xi)\big) = H_c^i(\mathscr{S}_{m,\bar{v}}, R\psi\mathcal{L}_\xi),$$

where $\mathscr{S}_{m,\bar{v}} = \mathscr{S}_m \otimes_{\mathcal{O}_v} \bar{\kappa}_v$.

We put $G^+(\mathbb{Q}_p) = \{g \in G(\mathbb{Q}_p) \mid g^{-1}L_p \subset L_p\}$. For $g \in G^+(\mathbb{Q}_p)$, let $e(g)$ be the minimal nonnegative integer such that $gL_p \subset p^{-e(g)}L_p$. Then we can construct a tower $(\mathscr{S}_{m,g})_{m\geq e(g)}$ of schemes over $\mathcal{O}_v$ and two morphisms

$$\mathrm{pr}\colon \mathscr{S}_{m,g} \to \mathscr{S}_m, \quad [g]\colon \mathscr{S}_{m,g} \to \mathscr{S}_{m-e(g)}$$

which are compatible with the transition maps. It is known that these are proper morphisms, pr induces an isomorphism on the generic fibers, and $[g]$ induces the Hecke action of g on the generic fibers (see [Mantovan 2005, Proposition 16, Proposition 17]). In particular, we have a canonical cohomological correspondence (*cf.* [SGA 5 1977, Exposé III], [Fujiwara 1997])

$$c_g\colon [g]_\eta^*\mathcal{L}_{\xi,m-e(g)} \xrightarrow{\cong} \mathrm{pr}_\eta^*\mathcal{L}_{\xi,m} = R\,\mathrm{pr}_\eta^!\mathcal{L}_{\xi,m}.$$

Let

$$R\psi(c_g)\colon [g]_{\bar{v}}^*R\psi\mathcal{L}_{\xi,m-e(g)} \to R\,\mathrm{pr}_{\bar{v}}^!R\psi\mathcal{L}_{\xi,m}$$

be the specialization of c_g (*cf.* [Fujiwara 1997, §1.5], [Ito and Mieda 2010, §6]). Since $[g]_{\bar{v}}$ is proper, this induces a homomorphism

$$H_c^i(\mathscr{S}_{m-e(g),\bar{v}}, R\psi\mathcal{L}_\xi) \xrightarrow{H_c^i(R\psi(c_g))} H_c^i(\mathscr{S}_{m,\bar{v}}, R\psi\mathcal{L}_\xi).$$

Taking the inductive limit, we get

$$\gamma_g\colon H_c^i(\mathscr{S}_{\infty,K^p,\bar{v}}, R\psi\mathcal{L}_\xi) \to H_c^i(\mathscr{S}_{\infty,K^p,\bar{v}}, R\psi\mathcal{L}_\xi).$$

From an obvious relation $c_{gg'} = c_g \circ g^*c_{g'}$ for $g, g' \in G^+(\mathbb{Q}_p)$, we deduce $\gamma_{gg'} = \gamma_g \circ \gamma_{g'}$ (*cf.* [Ito and Mieda 2010, Corollary 6.3]). On the other hand, by [Mantovan 2005, Proposition 16 (3), Proposition 17 (3)], $\gamma_{p^{-1}}$ is the identity. Since $G(\mathbb{Q}_p)$ is generated by $G^+(\mathbb{Q}_p)$ and p as a monoid, we can extend γ_g to the whole $G(\mathbb{Q}_p)$. By [Mantovan 2005, Proposition 16 (4)], the restriction of this action to $K_{p,0} = G(\mathbb{Z}_p)$ coincides with the inductive limit of the natural action of $K_{p,0}$ on

$H_c^i(\mathscr{S}_{m,\bar{v}}, R\psi\mathcal{L}_\xi)$. In particular, it is a smooth action. Furthermore, for integers $m' \geq m \geq 1$, we have

$$H_c^i(\mathscr{S}_{m',\bar{v}}, R\psi\mathcal{L}_\xi)^{K_{p,m}/K_{p,m'}} = H_c^i(\mathscr{S}_{m,\bar{v}}, R\psi\mathcal{L}_\xi)$$

(see [Mieda 2010, Proposition 2.5]). Taking inductive limit, we obtain

$$H_c^i(\mathscr{S}_{\infty,K^p,\bar{v}}, R\psi\mathcal{L}_\xi)^{K_{p,m}} = H_c^i(\mathscr{S}_{m,\bar{v}}, R\psi\mathcal{L}_\xi).$$

This implies that $H_c^i(\mathscr{S}_{\infty,K^p,\bar{v}}, R\psi\mathcal{L}_\xi)$ is an admissible/continuous representation of $G(\mathbb{Q}_p) \times \mathrm{Gal}(\bar{E}_v/E_v)$. $\qquad\square$

Corollary 7.3. *The kernel and the cokernel of the canonical homomorphism*

$$H_c^i(\mathscr{S}_{\infty,K^{p'},\bar{v}}, R\psi\mathcal{L}_\xi) \to H_c^i(\mathscr{S}_{\infty,K^{p'},\bar{\eta}}, \mathcal{L}_\xi)$$

(*cf.* [SGA 7$_{\mathrm{II}}$ 1973, Exposé XIII, (2.1.7.3)]) *are noncuspidal. In particular, for an irreducible supercuspidal representation π of $G(\mathbb{Q}_{p'})$, we have an isomorphism*

$$H_c^i(\mathscr{S}_{\infty,K^{p'},\bar{\eta}}, \mathcal{L}_\xi)[\pi] \cong H_c^i(\mathscr{S}_{\infty,K^{p'},\bar{v}}, R\psi\mathcal{L}_\xi)[\pi].$$

Similar results hold for the coefficient $\overline{\mathbb{F}}_\ell$ for a prime number $\ell \neq p, p'$.

Proof. Analogues of Theorem 5.17 and Theorem 6.1 are also valid in the PEL type case in this section by Remark 7.1 (ii). Let $\mathscr{S}_{K^p,E_v}^{\mathrm{pg}}$ be the potentially good reduction locus of $\mathscr{S}_{K^p,E_v}^{\mathrm{ad}}$. As in the Siegel case, $\mathscr{S}_{K^p,E_v}^{\mathrm{pg}}$ coincides with the rigid generic fiber of the completion of $\mathscr{S}_{K^p,\mathcal{O}_v}$ along the special fiber. Hence, we have an isomorphism

$$H_c^i(\mathscr{S}_{K^p,\bar{E}_v}^{\mathrm{pg}}, \mathrm{pr}_{m*}\mathcal{L}_\xi^{\mathrm{ad}}) \cong H_c^i(\mathscr{S}_{K^p,\bar{v}}, R\psi(\mathrm{pr}_{m*}\mathcal{L}_\xi))$$

for any nonnegative integer m by [Huber 1996, Theorem 3.7.2, Theorem 5.7.6] and [Huber 1998a, Theorem 3.1]. Taking inductive limits, we have an isomorphism

$$H_c^i(\mathscr{S}_{\infty,K^{p'},\bar{E}_v}^{\mathrm{pg}}, \mathcal{L}_\xi^{\mathrm{ad}}) \cong H_c^i(\mathscr{S}_{\infty,K^{p'},\bar{v}}, R\psi\mathcal{L}_\xi).$$

Hence the claim follows from the analogue of Theorem 6.1. $\qquad\square$

Remark 7.4. (i) The case where $p' \neq p$ in Corollary 7.3 was previously obtained by Tetsushi Ito and the second author. In that case, we can use minimal compactifications over $\mathcal{O}_v$ to show the claim.

(ii) In [Lan and Stroh 2018], Lan and Stroh obtained a stronger result that the canonical homomorphism in Corollary 7.3 is in fact an isomorphism. Their method is totally different from ours.

7.3. *Example.* In this subsection, we give a very simple application of Corollary 7.3. Proofs in this subsection are rather sketchy, since the technique is more or less well-known.

Here we consider the Shimura variety for $GU(1, n-1)$ over $\mathbb{Q}$. Let F be an imaginary quadratic extension of $\mathbb{Q}$ and $\mathrm{Spl}_{F/\mathbb{Q}}$ the set of rational primes over which $F/\mathbb{Q}$ splits. We fix a field embedding $F \hookrightarrow \mathbb{C}$ and regard F as a subfield of $\mathbb{C}$. For an integer $n \geq 2$, consider the integral PEL datum $(B, \mathcal{O}_B, *, V, L, \langle \, , \, \rangle, h)$ as follows:

- $B = F$, $\mathcal{O}_B = \mathcal{O}_F$ and $*$ is the unique nontrivial element of $\mathrm{Gal}(F/\mathbb{Q})$.
- $V = F^n$ and $L = \mathcal{O}_F^n$.
- $\langle \, , \, \rangle \colon V \times V \to \mathbb{Q}$ is an alternating pairing satisfying the following conditions:
  - $\langle x, y \rangle \in \mathbb{Z}$ for every $x, y \in L$,
  - $\langle bx, y \rangle = \langle x, b^* y \rangle$ for every $x, y \in V$ and $b \in F$, and
  - $G_{\mathbb{R}} \cong GU(1, n-1)$ (for the definition of G, see Section 7.1).
- $h \colon \mathbb{C} \to \mathrm{End}_F(V) \otimes_{\mathbb{Q}} \mathbb{R} \cong M_n(\mathbb{C})$ is given by $z \mapsto \begin{pmatrix} z & 0 \\ 0 & \bar{z} I_{n-1} \end{pmatrix}$, where the last isomorphism is induced by the fixed embedding $F \hookrightarrow \mathbb{C}$.

In this case, the reflex field E is equal to F. To a neat compact open subgroup K of $G(\widehat{\mathbb{Z}})$, we can attach the Shimura variety Sh_K of PEL type, which is not proper over $\mathrm{Spec}\, F$.

Put $\Sigma = \{p \in \mathrm{Spl}_{F/\mathbb{Q}} \mid L_p = L_p^{\perp}\}$. Then our integral Shimura datum is unramified at every $p \in \Sigma$. Moreover, for such p, $G_{\mathbb{Q}_p}$ is isomorphic to $\mathrm{GL}_n(\mathbb{Q}_p) \times \mathrm{GL}_1(\mathbb{Q}_p)$ (*cf.* [Fargues 2004, §1.2.3]). If $K = K_{p,0} K^p$ for some compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, we have $\mathrm{Sh}_K = \mathscr{S}_{K^p} \otimes_{\mathcal{O}_{F,(p)}} F$, where $\mathscr{S}_{K^p}$ is the moduli space introduced in Section 7.1.

Let us fix a prime number ℓ. We put

$$H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell) = \varinjlim_K H_c^i(\mathrm{Sh}_K \otimes_F \overline{F}, \overline{\mathbb{Q}}_\ell).$$

It is an admissible/continuous $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{F}/F)$-representation over $\overline{\mathbb{Q}}_\ell$.

Theorem 7.5. *Let Π be an irreducible admissible representation of $G(\mathbb{A}^\infty)$ over $\overline{\mathbb{Q}}_\ell$. Assume that there exists a prime $p \in \Sigma$ such that Π_p is a supercuspidal representation of $G(\mathbb{Q}_p)$. Then $H_c^i(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell)[\Pi] = 0$ unless $i = n - 1$.*

Remark 7.6. For proper Shimura varieties, an analogous result is known [Clozel 1991; Harris and Taylor 2001, Corollary IV.2.7]. It would be possible to give an "automorphic" proof of Theorem 7.5 by using results in [Morel 2010]. However, the authors think that our proof, consisting of purely local arguments, is simpler and has some importance.

Proof. Let ℓ' be another prime number and fix an isomorphism of fields $\iota \colon \overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_{\ell'}$. Then ι induces an isomorphism $H^i_c(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell)[\Pi] \cong H^i_c(\mathrm{Sh}, \overline{\mathbb{Q}}_{\ell'})[\iota\Pi]$, where $\iota\Pi$ is the representation of $G(\mathbb{A}^\infty)$ over $\overline{\mathbb{Q}}_{\ell'}$ induced by Π and ι. It is easy to observe that Π_p is supercuspidal if and only if $(\iota\Pi)_p$ is supercuspidal. Therefore, we can change our ℓ freely, and thus we can assume that there exists a prime $p \in \Sigma \setminus \{\ell\}$ such that Π_p is supercuspidal. Fix such p and take a place v of F lying over p. Then, for an integer $m \geq 0$ and a neat compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, $\mathrm{Sh}_{K_{p,m}K^p} \otimes_F F_v$ is isomorphic to $\mathscr{S}_{m,K^p,\eta}$ introduced in Section 7.2. Therefore we have an isomorphism

$$H^i_c(\mathrm{Sh}, \overline{\mathbb{Q}}_\ell) \cong \varinjlim_{m,K^p} H^i_c(\mathscr{S}_{m,K^p,\overline{\eta}}, \overline{\mathbb{Q}}_\ell) = \varinjlim_{K^p} H^i_c(\mathscr{S}_{\infty,K^p,\overline{\eta}}, \overline{\mathbb{Q}}_\ell).$$

Thus it suffices to show that $H^i_c(\mathscr{S}_{\infty,K^p,\overline{\eta}}, \overline{\mathbb{Q}}_\ell)[\pi] = 0$ for a supercuspidal representation π of $G(\mathbb{Q}_p)$, a neat compact open subgroup K^p, and an integer $i \neq n-1$. By Corollary 7.3, it is equivalent to showing that $H^i_c(\mathscr{S}_{\infty,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell)[\pi] = 0$.

For an integer $h \geq 0$, let $\mathscr{S}^{[h]}_{K^p,v}$ be the reduced closed subscheme of $\mathscr{S}_{K^p,v}$ consisting of points x such that the étale rank of $\mathcal{A}_x[v^\infty]$ is less than or equal to h (*cf.* [Harris and Taylor 2001, p. 111]), where $\mathcal{A}$ denotes the universal abelian scheme over $\mathscr{S}_{K^p}$. Put $\mathscr{S}^{(h)}_{K^p,v} = \mathscr{S}^{[h]}_{K^p,v} \setminus \mathscr{S}^{[h-1]}_{K^p,v}$. Our proof of the theorem is divided into the subsequent two lemmas. $\square$

Lemma 7.7. *For every supercuspidal representation π of $G(\mathbb{Q}_p)$, we have*

$$H^i_c(\mathscr{S}_{\infty,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell)[\pi] = \left(\varinjlim_m H^i_c\left(\mathscr{S}^{[0]}_{K^p,\overline{v}}, (R\psi\,\mathrm{pr}_{m*}\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}^{[0]}_{K^p,\overline{v}}}\right)\right)[\pi].$$

Proof. First recall that $\mathscr{S}_{m,K^p,\eta}$ has a good integral model over $\mathcal{O}_v$. For an integer $m \geq 0$, consider the functor from the category of $\mathcal{O}_v$-schemes to the category of sets, that associates S to the set of isomorphism classes of 6-tuples $(A, i, \lambda, \eta^p, \eta_v, \eta_{p,0})$, where

- $[(A, i, \lambda, \eta^p)] \in \mathscr{S}_{K^p}(S)$,

- $\eta_v \colon L \otimes_{\mathbb{Z}} (v^{-m}\mathcal{O}_v/\mathcal{O}_v) \to A[v^m]$ is a Drinfeld v^m-level structure (*cf.* [Harris and Taylor 2001, II.2]), and

- $\eta_{p,0} \colon p^{-m}\mathbb{Z}/\mathbb{Z} \to \mu_{p^m,S}$ is a Drinfeld p^m-level structure.

Then it is easy to see that this functor is represented by a scheme $\mathscr{S}_{m,K^p}$ which is finite over $\mathscr{S}_{K^p}$. Moreover the generic fiber of $\mathscr{S}_{m,K^p}$ can be naturally identified with $\mathscr{S}_{m,K^p,\eta}$ (*cf.* the moduli problem $\mathfrak{X}'_U$ introduced in [Harris and Taylor 2001, p. 92]). As in [Harris and Taylor 2001, III.4], we can extend the Hecke action of $G(\mathbb{Q}_p)$ on $(\mathscr{S}_{m,K^p,\eta})_{m\geq 0}$ to the tower $(\mathscr{S}_{m,K^p})_{m\geq 0}$. We have a $G(\mathbb{Q}_p)$-equivariant

isomorphism

$$H_c^i(\mathscr{S}_{\infty,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell) \cong \varinjlim_m H_c^i(\mathscr{S}_{m,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell).$$

Let us denote by $\mathscr{S}_{m,K^p,v}^{[h]}$ (resp. $\mathscr{S}_{m,K^p,v}^{(h)}$) the inverse image of $\mathscr{S}_{K^p,v}^{[h]}$ (resp. $\mathscr{S}_{K^p,v}^{(h)}$) under $\mathscr{S}_{m,K^p} \to \mathscr{S}_{K^p}$. For an integer $h \geq 0$, it is easy to observe that

$$\varinjlim_m H_c^i\big(\mathscr{S}_{K^p,\overline{v}}^{[h]}, (R\psi\, \mathrm{pr}_{m*}\,\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{K^p,\overline{v}}^{[h]}}\big) \cong \varinjlim_m H_c^i\big(\mathscr{S}_{m,K^p,\overline{v}}^{[h]}, (R\psi\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{m,K^p,\overline{v}}^{[h]}}\big),$$

$$\varinjlim_m H_c^i\big(\mathscr{S}_{K^p,\overline{v}}^{(h)}, (R\psi\, \mathrm{pr}_{m*}\,\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{K^p,\overline{v}}^{(h)}}\big) \cong \varinjlim_m H_c^i\big(\mathscr{S}_{m,K^p,\overline{v}}^{(h)}, (R\psi\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{m,K^p,\overline{v}}^{(h)}}\big)$$

and that they are admissible $G(\mathbb{Q}_p)$-representations. Moreover, by considering the kernel of the universal Drinfeld v^m-level structure η_v^{univ}, we can decompose $\mathscr{S}_{m,K^p,v}^{(h)}$ into finitely many open and closed subsets indexed by the set consisting of direct summands of $L \otimes_{\mathbb{Z}} (v^{-m}\mathcal{O}_v/\mathcal{O}_v)$ with rank $n-h$ (cf. [Boyer 1999, Définition 10.4.1, Proposition 10.4.2] and [Ito and Mieda 2010, Definition 5.1, Lemma 5.3]). Using this partition, when $h > 0$, we can prove that the $G(\mathbb{Q}_p)$-representation

$$\varinjlim_m H_c^i\big(\mathscr{S}_{m,K^p,\overline{v}}^{(h)}, (R\psi\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{m,K^p,\overline{v}}^{(h)}}\big)$$

is parabolically induced from a proper parabolic subgroup of $G(\mathbb{Q}_p)$. Therefore, by the same argument as in the proof of Theorem 6.1, we can conclude that the kernel and the cokernel of

$$\varinjlim_m H_c^i(\mathscr{S}_{m,K^p,\overline{v}}, R\psi\overline{\mathbb{Q}}_\ell) \to \varinjlim_m H_c^i\big(\mathscr{S}_{m,K^p,\overline{v}}^{[0]}, (R\psi\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{m,K^p,\overline{v}}^{[0]}}\big)$$

are noncuspidal. This completes the proof of the lemma. □

Lemma 7.8. *Let π be a supercuspidal representation of $G(\mathbb{Q}_p)$. If $i \neq n-1$, we have*

$$\Big(\varinjlim_m H_c^i\big(\mathscr{S}_{K^p,\overline{v}}^{[0]}, (R\psi\, \mathrm{pr}_{m*}\,\overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{K^p,\overline{v}}^{[0]}}\big)\Big)[\pi] = 0.$$

Proof. Let $\mu_h\colon \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$ be the homomorphism of algebraic groups over $\mathbb{C}$ defined as the composite of

$$\mathbb{G}_{m,\mathbb{C}} \xrightarrow{z \mapsto (z,1)} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \overset{(*)}{\cong} (\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_{m,\mathbb{C}}) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{h_{\mathbb{C}}} G_{\mathbb{C}},$$

where $(*)$ is given by $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\cong} \mathbb{C} \times \mathbb{C}$; $a \otimes b \mapsto (ab, a\bar{b})$. Fix an isomorphism of fields $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ and denote by $\mu\colon \mathbb{G}_{m,\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$ the induced cocharacter of $G_{\overline{\mathbb{Q}}_p}$. Let b be a unique basic element of $B(G_{\mathbb{Q}_p}, \mu)$ (for the definition of $B(G, \mu)$, we refer to [Fargues 2004, §2.1.1]), and denote by $\mathcal{M}$ the Rapoport–Zink space associated to the local unramified PEL datum $(F \otimes_{\mathbb{Q}} \mathbb{Q}_p, \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p, *, V_p, L_p, \langle\,,\,\rangle, b, \mu)$ (cf. [Fargues 2004, §2.3.5]). The Rapoport–Zink space $\mathcal{M}$ is equipped with an action

of the group $J(\mathbb{Q}_p)$, where J denotes the algebraic group over $\mathbb{Q}_p$ associated to b (*cf.* [Rapoport and Zink 1996, Proposition 1.12]). By [Fargues 2004, §2.3.7.1], $\mathcal{M}$ is isomorphic to $\mathcal{M}_{\mathrm{LT}} \times \mathbb{Q}_p^\times / \mathbb{Z}_p^\times$, where $\mathcal{M}_{\mathrm{LT}}$ is the Lubin–Tate space for $\mathrm{GL}_n(\mathbb{Q}_p)$. Furthermore, $J(\mathbb{Q}_p)$ is isomorphic to $D^\times \times \mathbb{Q}_p^\times$, where D denotes the central division algebra over $\mathbb{Q}_p$ with invariant $1/n$. The action of $J(\mathbb{Q}_p)$ on $\mathcal{M}$ is identified with the well-known action of $D^\times \times \mathbb{Q}_p^\times$ on $\mathcal{M}_{\mathrm{LT}} \times \mathbb{Q}_p^\times / \mathbb{Z}_p^\times$.

By the p-adic uniformization theorem of Rapoport–Zink [Rapoport and Zink 1996, Theorem 6.30; Fargues 2004, Corollaire 3.1.9], we have an isomorphism

$$\coprod_{\ker^1(\mathbb{Q}, G)} I(\mathbb{Q}) \backslash \mathcal{M} \times G(\mathbb{A}^{\infty, p}) / K^p \cong \mathscr{S}_{K^p}^\wedge,$$

where I is an algebraic group over $\mathbb{Q}$ satisfying $I(\mathbb{A}^\infty) \cong J(\mathbb{Q}_p) \times G(\mathbb{A}^{\infty, p})$ and $\mathscr{S}_{K^p}^\wedge$ denotes the formal completion of $\mathscr{S}_{K^p} \otimes_{\mathcal{O}_v} W(\overline{\mathbb{F}}_p)$ along $\mathscr{S}_{K^p, \overline{v}}^{[0]}$, the basic locus of $\mathscr{S}_{K^p, \overline{v}}$. By this isomorphism, we know that $\mathscr{S}_{K^p, \overline{v}}^{[0]}$, which coincides with $\mathscr{S}_{K^p}^\wedge$ as topological spaces, consists of finitely many closed points; indeed, the left hand side of the isomorphism above is a finite disjoint union of formal schemes of the form $\Gamma \backslash \mathcal{M}$, where $\Gamma \subset J(\mathbb{Q}_p)$ is a discrete cocompact subgroup (*cf.* [Fargues 2004, Lemme 3.1.7]). Therefore, by [Berkovich 1996, Theorem 3.1], we have an isomorphism

$$H_c^i\big(\mathscr{S}_{K^p, \overline{v}}^{[0]}, (R\psi \, \mathrm{pr}_{m*} \overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{K^p, \overline{v}}^{[0]}}\big) = H^i\big(\mathscr{S}_{K^p, \overline{v}}^{[0]}, (R\psi \, \mathrm{pr}_{m*} \overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{K^p, \overline{v}}^{[0]}}\big)$$
$$\cong H^i\big(\mathscr{S}_{m, K^p, \overline{\eta}}(b), \overline{\mathbb{Q}}_\ell\big),$$

where $\mathscr{S}_{m, K^p, \overline{\eta}}(b) = \mathrm{pr}_m^{-1}(\mathrm{sp}^{-1}(\mathscr{S}_{K^p, v}^{[0]})^\circ)_{\overline{F}_v}$.

Now we use the Hochschild–Serre spectral sequence (see [Fargues 2004, Théorème 4.5.12])

$$E_2^{r,s} = \varinjlim_m \mathrm{Ext}_{J(\mathbb{Q}_p)\text{-smooth}}^r\big(H_c^{2(n-1)-s}(\mathcal{M}_{K_{p,m}}, \overline{\mathbb{Q}}_\ell)(n-1), \mathcal{A}(I)_{\mathbf{1}}^{K^p}\big)$$
$$\Rightarrow \varinjlim_m H^{r+s}\big(\mathscr{S}_{m, K^p, \overline{\eta}}(b), \overline{\mathbb{Q}}_\ell\big),$$

where $\mathcal{M}_{K_{p,m}}$ is the Rapoport–Zink space of level $K_{p,m}$, and $\mathcal{A}(I)_{\mathbf{1}}$ is the space of automorphic forms on $I(\mathbb{A}^\infty)$ (see [Fargues 2004, Définition 4.5.8] for detail). Since $J(\mathbb{Q}_p) = D^\times \times \mathbb{Q}_p^\times$ is anisotropic modulo center, it is easy to see that $E_2^{r,s} = 0$ unless $r = 0$. If $r = 0$, we have

$$E_2^{0,s} = \varinjlim_m \mathrm{Hom}_{J(\mathbb{Q}_p)}\big(H_c^{2(n-1)-s}(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell)(n-1), \mathcal{A}(I)_{\mathbf{1}}^{K^p}\big)^{K_{p,m}},$$

where we put $H_c^i(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell) = \varinjlim_m H_c^i(\mathcal{M}_{K_{p,m}}, \overline{\mathbb{Q}}_\ell)$.

By [Mieda 2010], the $G(\mathbb{Q}_p)$-representation $H_c^{2(n-1)-s}(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell)(n-1)$ has nonzero supercuspidal part only if $s = n-1$. Indeed, for an irreducible supercuspidal representation $\pi = \pi_1 \otimes \chi$ of $G(\mathbb{Q}_p) = \mathrm{GL}_n(\mathbb{Q}_p) \times \mathrm{GL}_1(\mathbb{Q}_p)$, where π_1 is an irreducible supercuspidal representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ and χ is a character of $\mathrm{GL}_1(\mathbb{Q}_p)$, we have

$$H_c^i(\mathcal{M}_\infty, \overline{\mathbb{Q}}_\ell)[\pi] = H_c^i(\mathcal{M}_{\mathrm{LT},\infty}, \overline{\mathbb{Q}}_\ell)[\pi_1] \otimes \chi,$$

as we see in [Fargues 2004, p. 168]. Therefore $E_2^{0,s}$ has a supercuspidal subquotient only if $s = n-1$.

Hence we conclude that

$$\varinjlim_m H_c^i\big(\mathscr{S}_{K^p, \overline{v}}^{[0]}, (R\psi\, \mathrm{pr}_{m*}\, \overline{\mathbb{Q}}_\ell)|_{\mathscr{S}_{K^p, \overline{v}}^{[0]}}\big) \cong \varinjlim_m H^i\big(\mathscr{S}_{m, K^p, \overline{\eta}}(b), \overline{\mathbb{Q}}_\ell\big)$$

has nonzero supercuspidal part only if $i = n-1$. $\qquad\square$

We also have a similar result for the torsion coefficient case. For a neat compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, we put

$$H_c^i(\mathrm{Sh}_{K^p}, \overline{\mathbb{F}}_\ell) = \varinjlim_m H_c^i(\mathrm{Sh}_{K_{p,m}K^p} \otimes_F \overline{F}, \overline{\mathbb{F}}_\ell).$$

It is an admissible/continuous $G(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{F}/F)$-representation over $\overline{\mathbb{F}}_\ell$.

Theorem 7.9. *Let p be a prime in $\Sigma \setminus \{\ell\}$ and π an irreducible supercuspidal $\overline{\mathbb{F}}_\ell$-representation of $G(\mathbb{Q}_p)$. Then, for every neat compact open subgroup K^p of $G(\widehat{\mathbb{Z}}^p)$, we have $H_c^i(\mathrm{Sh}_{K^p}, \overline{\mathbb{F}}_\ell)[\pi] = 0$ unless $i = n-1$.*

Remark 7.10. (i) Theorem 7.9 for proper Shimura varieties is due to Shin [Shin 2015]. His method, using Mantovan's formula, is slightly different from ours. The nonproper cases are also covered in his paper using results in our paper.

(ii) Using the result in [Dat 2012], it is possible to describe the action of $W_{\mathbb{Q}_p}$ on $H_c^{n-1}(\mathrm{Sh}_{K^p}, \overline{\mathbb{F}}_\ell)[\pi]$ by means of the mod-ℓ local Langlands correspondence. Such study has also been carried out by Shin when the Shimura variety is proper.

Proof. Almost all arguments in the proof of Theorem 7.5 work well. The only one point which should be modified is about the vanishing of the supercuspidal part of $E_2^{r,s}$ for $(r, s) \neq (0, n-1)$ in the proof of Lemma 7.8; note that an irreducible $\overline{\mathbb{F}}_\ell$-representation of $J(\mathbb{Q}_p)$, being supercuspidal, is not necessarily injective in the category of smooth $\overline{\mathbb{F}}_\ell$-representations of $J(\mathbb{Q}_p)$ with the fixed central character. For this point, we can use the same argument as that by Shin (see [Shin 2015, §3.2]), in which he uses the vanishing of the supercuspidal part $H_c^i(\mathcal{M}_{\mathrm{LT},\infty}, \overline{\mathbb{F}}_\ell)_{\mathrm{sc}}$ for $i \neq n-1$ (*cf.* [Dat 2012, proof of Proposition 3.1.1, Remarque 3.1.5]) and the

projectivity of the $D^\times$-representation $H_c^{n-1}(\mathcal{M}_{\mathrm{LT},\infty}, \overline{\mathbb{F}}_\ell)_{\mathrm{sc}}$ (*cf.* [Dat 2012, §3.2.2, Remarque iii)]). $\hfill\square$

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