Nilpotence theorems via homological residue fields

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We prove nilpotence theorems in tensor-triangulated categories using suitable Gabriel quotients of the module category, and discuss examples.

1. Introduction

For the average Joe, and the median Jane, *the* nilpotence theorem refers to a result in stable homotopy theory, conjectured by Ravenel and proved by Devinatz, Hopkins and Smith in their famous work on chromatic theory [Devinatz et al. 1988; Hopkins and Smith 1998]. One form of the result says that a map between finite spectra which gets annihilated by all Morava K-theories must be tensor-nilpotent. Under Hopkins's [1987] impetus, these ideas soon expanded beyond topology. Neeman [1992] in commutative algebra and Thomason [1997] in algebraic geometry proved nilpotence theorems for maps in derived categories of perfect complexes, using ordinary residue fields instead of Morava K-theories. Benson, Carlson and Rickard [Benson et al. 1997] led the charge into yet another area, namely modular representation theory of finite groups, where the appropriate "residue fields" turned out to be shifted cyclic subgroups, and later π-points [Friedlander and Pevtsova 2007]. As further areas kept joining the fray, expectations rose of a unified treatment applicable to every tensor-triangulated category in nature. In this vein, Mathew [2017, Theorem 4.14(b)] proved an abstract nilpotence theorem via E_∞-rings in ∞-categories over the field $\mathbb{Q}$. However, this rationality assumption is a severe restriction, incompatible with the chromatic joys of topological Joe and the positive characteristic tastes of modular Jane. Here, we prove abstract nilpotence theorems, integrally and without ∞-categories. For instance, Corollary 4.5 says:

1.1. Theorem. *Let $f : x \to y$ be a morphism in an essentially small, rigid tensor-triangulated category $\mathcal{K}$. If we have $\bar{\mathrm{h}}(f) = 0$ for every homological residue field $\bar{\mathrm{h}} : \mathcal{K} \to \bar{\mathcal{A}}$, then there exists $n \geq 1$ such that $f^{\otimes n} = 0$ in $\mathcal{K}$.*

We need to explain the *homological residue fields* that appear in this statement. Their purpose is to encapsulate the key features of Morava *K*-theories, ordinary residue fields, shifted cyclic subgroups, etc., from an abstract point of view. In other words, when first meeting a tensor-triangulated category $\mathcal{K}$, we would like to extract its "residue fields" without knowing intimate details about $\mathcal{K}$, as we are used to extracting residue fields $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ from any commutative ring R, without much knowledge about R beyond its propensity to harbor prime ideals $\mathfrak{p} \in \mathrm{Spec}(R)$. We investigated this question of "tensor-triangular fields" in the recent joint work [Balmer et al. 2019], with Krause and Stevenson. Although the present article can be read independently, we refer to that prequel for motivation, background, justification, and a couple of lemmas. In retrospect, our nilpotence theorems further validate the ideas introduced in [Balmer et al. 2019].

In a nutshell, as we do not know how to produce residue fields within triangulated categories, we consider instead homological tensor-functors to *abelian* categories

$$\bar{\mathrm{h}} = \bar{\mathrm{h}}_{\mathcal{B}} : \mathcal{K} \hookrightarrow \mathrm{mod}\text{-}\mathcal{K} \twoheadrightarrow (\mathrm{mod}\text{-}\mathcal{K})/\mathcal{B}$$

composed of the Yoneda embedding $\mathrm{h} : \mathcal{K} \hookrightarrow \mathrm{mod}\text{-}\mathcal{K}$ into the Freyd envelope of $\mathcal{K}$ (Remark 2.6) followed by the Gabriel quotient $Q_{\mathcal{B}} : \mathrm{mod}\text{-}\mathcal{K} \twoheadrightarrow (\mathrm{mod}\text{-}\mathcal{K})/\mathcal{B}$ with respect to any *maximal* $\otimes$-ideal Serre subcategory $\mathcal{B}$. Thus Theorem 1.1 can be rephrased as follows:

> *If a morphism $f : x \to y$ in $\mathcal{K}$ is annihilated by $\bar{\mathrm{h}}_{\mathcal{B}} : \mathcal{K} \to (\mathrm{mod}\text{-}\mathcal{K})/\mathcal{B}$ for every maximal Serre $\otimes$-ideal $\mathcal{B} \subset \mathrm{mod}\text{-}\mathcal{K}$, then f is $\otimes$-nilpotent in $\mathcal{K}$.*

At first, it might be counterintuitive to only invoke *maximal* $\otimes$-ideals $\mathcal{B}$, instead of some kind of more general "*prime*" $\otimes$-ideals of $\mathrm{mod}\text{-}\mathcal{K}$, but we explain in Section 3 why this notion covers all points of the triangular spectrum $\mathrm{Spc}(\mathcal{K})$ of $\mathcal{K}$, not just the closed points. We also explain in Remark 3.10 how the above homological residue fields correspond to the local constructions proposed in [Balmer et al. 2019, §4].

As a matter of fact, in the examples, there exist alternative formulations of the nilpotence theorem. And the same holds here. Most notably, if our triangulated category $\mathcal{K}$ sits inside a "big" one, $\mathcal{K} \subset \mathcal{T}$, as the compact objects $\mathcal{K} = \mathcal{T}^c$ of a so-called "rigidly compactly generated" tensor-triangulated category $\mathcal{T}$ (Remark 4.6), we expect a nilpotence theorem for maps $f : x \to Y$ with compact source $x \in \mathcal{K}$ but *arbitrary* target Y in $\mathcal{T}$. This flavor of nilpotence theorem is Corollary 4.7.

In order to handle such generalizations, we consider the big Grothendieck category $\mathcal{A} := \mathrm{Mod}\text{-}\mathcal{K}$ of *all* right $\mathcal{K}$-modules (Notation 2.5), not just the subcategory of finitely presented ones that is the Freyd envelope $\mathcal{A}^{\mathrm{fp}} = \mathrm{mod}\text{-}\mathcal{K}$. When $\mathcal{K} = \mathcal{T}^c$, the big category $\mathcal{T}$ still admits a so-called "restricted-Yoneda" functor $\mathrm{h} : \mathcal{T} \to \mathrm{Mod}\text{-}\mathcal{K}$ (Remark 4.6). Every maximal Serre $\otimes$-ideal $\mathcal{B} \subset \mathcal{A}^{\mathrm{fp}}$ generates a localizing (Serre)

$\otimes$-ideal $\langle \mathcal{B} \rangle$ of $\mathcal{A}$ and we can consider the corresponding "big" Gabriel quotient $\bar{\mathcal{A}} := \mathcal{A}/\langle \mathcal{B} \rangle$. Composing with restricted-Yoneda, we obtain a homological $\otimes$-functor $\bar{h}_{\mathcal{B}} : \mathcal{T} \to \bar{\mathcal{A}}$ on the "big" category $\mathcal{T}$, extending the one on $\mathcal{K}$:

$$
\begin{array}{c}
\overbrace{\phantom{\mathcal{K} = \mathcal{T}^c \hookrightarrow \mathcal{A}^{\mathrm{fp}} = \mathrm{mod}\text{-}\mathcal{K} \xrightarrow{Q_{\mathcal{B}}} \bar{\mathcal{A}}^{\mathrm{fp}}}}^{\bar{h}_{\mathcal{B}}} \\
\mathcal{K} = \mathcal{T}^c \overset{h}{\hookrightarrow} \mathcal{A}^{\mathrm{fp}} = \mathrm{mod}\text{-}\mathcal{K} \xrightarrow{Q_{\mathcal{B}}} \bar{\mathcal{A}}^{\mathrm{fp}} = (\mathrm{mod}\text{-}\mathcal{K})/\mathcal{B} \\
\cap \qquad\qquad \cap \qquad\qquad\qquad \cap \\
\mathcal{T} \xrightarrow{h} \mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K} \xrightarrow{Q_{\mathcal{B}}} \bar{\mathcal{A}} = (\mathrm{Mod}\text{-}\mathcal{K})/\langle \mathcal{B} \rangle \\
\underbrace{\phantom{\mathcal{T} \xrightarrow{h} \mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K} \xrightarrow{Q_{\mathcal{B}}} \bar{\mathcal{A}}}}_{\bar{h}_{\mathcal{B}}}
\end{array}
$$

Thanks to [Balmer et al. 2018, Proposition A.14], the image $h(Y)$ of every object Y in the big category $\mathcal{T}$ remains $\otimes$-*flat* in the module category $\mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K}$, meaning that the functor $h(Y) \otimes - : \mathcal{A} \to \mathcal{A}$ is exact. In fact, this $\otimes$-flatness plays an important role in the proof of the nilpotence theorem. In particular, Corollary 4.2 tells us:

1.2. Theorem. *Let $f : h(x) \to F$ be a morphism in $\mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K}$, for $x \in \mathcal{K}$. Suppose that the $\mathcal{K}$-module F is $\otimes$-flat and that $Q_{\mathcal{B}}(f) = 0$ in $\bar{\mathcal{A}} = \mathcal{A}/\langle \mathcal{B} \rangle$, for every maximal Serre $\otimes$-ideal $\mathcal{B} \subset \mathrm{mod}\text{-}\mathcal{K}$. Then f is $\otimes$-nilpotent in $\mathrm{Mod}\text{-}\mathcal{K}$.*

All these statements are corollaries of our most general nilpotence theorem (Theorem 4.1), which further involves a "parameter" à la Thomason [1997], i.e., a closed subset $W \subseteq \mathrm{Spc}(\mathcal{K})$ of the spectrum on which we test the vanishing of f.

Finally, in Section 5, we classify those homological residue fields in examples. For brevity, let us pack three theorems into one:

1.3. Theorem. *There exists a surjection $\phi : \mathrm{Spc}^h(\mathcal{K}) \twoheadrightarrow \mathrm{Spc}(\mathcal{K})$ from the set of maximal Serre $\otimes$-ideals $\mathcal{B} \subset \mathrm{mod}\text{-}\mathcal{K}$ to the triangular spectrum of $\mathcal{K}$. Moreover, it is a bijection for each of the following tensor-triangulated categories $\mathcal{K}$:*

(a) *Let X be a quasicompact and quasiseparated scheme and $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ its derived category of perfect complexes. (Corollary 5.11.)*

(b) *Let G be a compact Lie group and $\mathcal{K} = \mathrm{SH}(G)^c$ the G-equivariant stable homotopy category of finite genuine G-spectra. In particular, this holds for $\mathcal{K} = \mathrm{SH}^c$ the usual stable homotopy category. (Corollary 5.10.)*

(c) *Let G be a finite group scheme over a field k and $\mathcal{K} = \mathrm{stab}(kG)$ its stable module category of finite-dimensional kG-modules modulo projectives. (Example 5.13.)*

We caution that the above does *not* give a new proof of the nilpotence theorems known in those examples, except perhaps for modular representation theory (c), as we discuss further in Remark 5.14. Indeed, the above results rely on existing

classification results, which themselves often rely on a form of nilpotence theorem.
These results should rather be read as a converse to our nilpotence theorems via ho-
mological residue fields: if a collection of homological residue fields detects nilpo-
tence then that collection contains all homological residue fields (Theorem 5.4).

2. Background and notation

2.1. Hypothesis. Throughout the paper, we denote by $\mathcal{K}$ an essentially small
tensor-triangulated category and by $\mathbb{1} \in \mathcal{K}$ its $\otimes$-unit. We often assume $\mathcal{K}$ *rigid*,
in the sense recalled in Remark 2.4 below. See details in [Balmer 2005, §1] or
[Balmer 2010, §1].

2.2. Examples. Such $\mathcal{K}$ include the usual suspects: in topology $\mathcal{K} = \mathrm{SH}^c$ the
stable homotopy category of finite spectra; in algebraic geometry $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X) =$
$\mathrm{D}_{\mathrm{Qcoh}}(\mathcal{O}_X\text{-Mod})^c$ the derived category of perfect complexes over a scheme X
which is assumed quasicompact and quasiseparated (e.g., a noetherian, or an affine
one); in modular representation theory $\mathcal{K} = \mathrm{stab}(kG) = \mathrm{Stab}(kG)^c$ the stable cate-
gory of finite-dimensional k-linear representations of a finite group G over a field
k of characteristic dividing the order of G. But there are many more examples:
equivariant versions, categories of motives, KK-categories of C^*-algebras, etc.,
etc.

2.3. Remark. We use the *triangular spectrum* $\mathrm{Spc}(\mathcal{K}) = \{\mathcal{P} \subset \mathcal{K} \mid \mathcal{P} \text{ is a prime}\}$
where a proper thick $\otimes$-ideal $\mathcal{P} \subsetneq \mathcal{K}$ is called a *(triangular) prime* if $x \otimes y \in \mathcal{P}$
implies $x \in \mathcal{P}$ or $y \in \mathcal{P}$. The *support* of an object $x \in \mathcal{K}$ is the closed subset
$\mathrm{supp}(x) := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid x \notin \mathcal{P}\} = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid x \text{ is nonzero in } \mathcal{K}/\mathcal{P}\}$. These are
exactly the so-called *Thomason closed* subsets of $\mathrm{Spc}(\mathcal{K})$, i.e., those closed subsets
$Z \subseteq \mathrm{Spc}(\mathcal{K})$ with quasicompact open complement $\mathrm{Spc}(\mathcal{K}) \smallsetminus Z$, by [Balmer 2005,
Proposition 2.14].

2.4. Remark. Rigidity of $\mathcal{K}$ will play an important role in the proof of the main
Theorem 4.1. Rigidity means that every object $x \in \mathcal{K}$ is strongly dualizable, hence
admits a dual $x^{\vee} \in \mathcal{K}$ with an adjunction $\mathrm{Hom}_{\mathcal{K}}(x \otimes y, z) \cong \mathrm{Hom}_{\mathcal{K}}(y, x^{\vee} \otimes z)$.
In particular, the unit-counit relation forces $x \otimes \eta_x : x \to x \otimes x^{\vee} \otimes x$ to be a split
monomorphism where $\eta_x : \mathbb{1} \to x^{\vee} \otimes x$ is the unit of the adjunction. This implies
that x is a direct summand of a $\otimes$-multiple of $x^{\otimes n}$ for any $n \geq 1$. It follows that if
a map f satisfies $(f \otimes x)^{\otimes n} = 0$ then $f^{\otimes n} \otimes x = 0$ as well.

2.5. Notation. The Grothendieck abelian category Mod-$\mathcal{K}$ of right $\mathcal{K}$-modules,
i.e., additive functors $M : \mathcal{K}^{\mathrm{op}} \to \mathrm{Ab}$, receives $\mathcal{K}$ via the Yoneda embedding,
denoted

$$\mathrm{h} : \mathcal{K} \hookrightarrow \mathrm{Mod}\text{-}\mathcal{K} = \mathrm{Add}(\mathcal{K}^{\mathrm{op}}, \mathrm{Ab}), \quad x \mapsto \hat{x} := \mathrm{Hom}_{\mathcal{K}}(-, x), \quad f \mapsto \hat{f}.$$

2.6. Remark. Let us recall some standard facts about $\mathcal{K}$-modules. See details in [Balmer et al. 2018, Appendix A]. By Day convolution, the category $\mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K}$ admits a tensor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ which is colimit-preserving (in particular *right-exact*) in each variable and which makes $\mathrm{h} : \mathcal{K} \to \mathcal{A}$ a tensor functor: $\widehat{x \otimes y} \cong \hat{x} \otimes \hat{y}$. Hence h preserves rigidity, so $\hat{x}$ will be rigid in $\mathcal{A}$ whenever x is in $\mathcal{K}$. Moreover, the object $\hat{x} \in \mathcal{A}$ is finitely presented projective and $\otimes$-flat in $\mathcal{A}$. The tensor subcategory

$$\mathcal{A}^{\mathrm{fp}} = \mathrm{mod}\text{-}\mathcal{K} \subset \mathrm{Mod}\text{-}\mathcal{K} = \mathcal{A}$$

of finitely presented objects is itself abelian and is nothing but the classical *Freyd envelope* of $\mathcal{K}$, see [Neeman 2001, Chapter 5]. Recall that $\mathrm{h} : \mathcal{K} \hookrightarrow \mathrm{mod}\text{-}\mathcal{K}$ is the universal homological functor out of $\mathcal{K}$, and that a functor from a triangulated category to an abelian category is *homological* if it maps exact triangles to exact sequences. Every object of $\mathcal{A}$ is a filtered colimit of finitely presented ones. In short, $\mathcal{A}$ is a *locally coherent* Grothendieck category.

2.7. Remark. Given a Serre subcategory $\mathcal{B} \subseteq \mathcal{A}^{\mathrm{fp}}$ we can form $\langle \mathcal{B} \rangle$, or $\vec{\mathcal{B}}$, the localizing subcategory of $\mathcal{A}$ generated by $\mathcal{B}$. The subcategory $\langle \mathcal{B} \rangle$ is the smallest Serre subcategory containing $\mathcal{B}$ and closed under coproducts; it consists of all (filtered) colimits in $\mathcal{A}$ of objects of $\mathcal{B}$. For instance $\langle \mathcal{A}^{\mathrm{fp}} \rangle = \mathcal{A}$ and it follows that if $\mathcal{B}$ is $\otimes$-ideal in $\mathcal{A}^{\mathrm{fp}}$ then so is $\langle \mathcal{B} \rangle$ in $\mathcal{A}$. We denote the corresponding Gabriel [1962] quotient by

$$Q_{\mathcal{B}} : \mathcal{A} \twoheadrightarrow \mathcal{A}/\langle \mathcal{B} \rangle.$$

We recall that $\langle \mathcal{B} \rangle$ is also locally coherent with $\langle \mathcal{B} \rangle^{\mathrm{fp}} = \mathcal{B}$ and so is the quotient $\bar{\mathcal{A}}$ with $\bar{\mathcal{A}}^{\mathrm{fp}} \cong \mathcal{A}^{\mathrm{fp}}/\mathcal{B}$. When $\mathcal{B}$ is $\otimes$-ideal then $\bar{\mathcal{A}}$ inherits a unique tensor structure turning $Q_{\mathcal{B}} : \mathcal{A} \twoheadrightarrow \bar{\mathcal{A}}$ into a tensor functor, which preserves $\otimes$-flat objects. All this remains true without assuming $\mathcal{K}$ rigid. See details in [Balmer et al. 2019, §2].

2.8. Remark. For $\mathcal{K}$ rigid, consider the special case of the adjunction

$$\mathrm{Hom}_{\mathcal{K}}(x, y) \cong \mathrm{Hom}_{\mathcal{K}}(\mathbb{1}, x^{\vee} \otimes y).$$

Under this isomorphism, if $f : x \to y$ corresponds to $g : \mathbb{1} \to x^{\vee} \otimes y$ then for $n \geq 1$ the morphism $f^{\otimes n} : x^{\otimes n} \to y^{\otimes n}$ corresponds to $g^{\otimes n} : \mathbb{1} \to (x^{\vee} \otimes y)^{\otimes n}$ under the analogous isomorphism $\mathrm{Hom}_{\mathcal{K}}(x^{\otimes n}, y^{\otimes n}) \cong \mathrm{Hom}_{\mathcal{K}}(\mathbb{1}, (x^{\otimes n})^{\vee} \otimes y^{\otimes n}) \cong \mathrm{Hom}_{\mathcal{K}}(\mathbb{1}, (x^{\vee} \otimes y)^{\otimes n})$. In particular, f is $\otimes$-nilpotent if and only if g is. Note that the above observation only uses that x is rigid in a tensor category and does not use that y itself is rigid. We can therefore also use this trick for any morphism $f : \hat{x} \to M$ in the module category $\mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K}$, as long as x comes from $\mathcal{K}$.

We shall need the following folklore result about modules and localization:

2.9. Proposition. *Let $\mathcal{J} \subset \mathcal{K}$ be a thick $\otimes$-ideal and let $q : \mathcal{K} \to \mathcal{L}$ be the corresponding Verdier quotient $\mathcal{K} \twoheadrightarrow \mathcal{K}/\mathcal{J}$ or its idempotent completion $\mathcal{K} \to (\mathcal{K}/\mathcal{J})^{\natural}$.*

Consider the left Kan extension $q_! : \text{Mod-}\mathcal{K} \to \text{Mod-}\mathcal{L}$, left-adjoint to restriction $q^* : \text{Mod-}\mathcal{L} \to \text{Mod-}\mathcal{K}$ along q. Then $q_!$ is a localization identifying $\text{Mod-}\mathcal{L}$ as the Gabriel quotient of $\text{Mod-}\mathcal{K}$ by $\text{Ker}(q_!) = \langle \text{h}(\mathcal{J}) \rangle$ the localizing subcategory generated by $\text{h}(\mathcal{J})$. The localization $q_!$ restricts to a localization $q_! : \text{mod-}\mathcal{K} \twoheadrightarrow \text{mod-}\mathcal{L}$ on finitely presented objects, identifying $\text{mod-}\mathcal{L}$ as the quotient of $\text{mod-}\mathcal{K}$ by $\text{Ker}(q_!)^{\text{fp}}$ which is also the Serre envelope of $\text{h}(\mathcal{J})$ in $\text{mod-}\mathcal{K}$.

Proof. The fact that $q_!$ is a localization follows from [Krause 2005, Theorem 4.4 and §3]. The left Kan extension $q_!(M)$ is defined as $\text{colim}_{\alpha: \hat{x} \to M} q_!(\hat{x})$ with $q_!(\hat{x}) = \widehat{q(x)}$. To identify the kernel of $q_! : \text{Mod-}\mathcal{K} \to \text{Mod-}\mathcal{L}$, since $\text{h}(\mathcal{J}) \subseteq \text{Ker}(q_!)^{\text{fp}}$ is clear, it suffices to show $\text{Ker}(q_!) \subseteq \langle \text{h}(\mathcal{J}) \rangle$. For every $M \in \text{Ker}(q_!)$, using that $\widehat{q(x)}$ is finitely presented projective for all $x \in \mathcal{K}$ together with faithfulness of Yoneda $\mathcal{L} \hookrightarrow \text{Mod-}\mathcal{L}$, one shows that every morphism $\alpha : \hat{x} \to M$ with $x \in \mathcal{K}$ factors via a morphism $\hat{\beta} : \hat{x} \to \hat{y}$ where $q(\beta)$ vanishes in $\mathcal{K}/\mathcal{J}$, meaning that the morphism $\beta : x \to y$ in $\mathcal{K}$ factors via an object of $\mathcal{J}$. In short, every morphism $\hat{x} \to M$ factors via an object in $\text{h}(\mathcal{J})$ which implies that M belongs to the localizing subcategory $\langle \text{h}(\mathcal{J}) \rangle$. The $\otimes$-properties are then easily added onto this purely abelian picture. $\square$

3. Homological primes and homological residue fields

Let $\mathcal{K}$ be a tensor-triangulated category as in Hypothesis 2.1.

3.1. Definition. A *(coherent) homological prime* for $\mathcal{K}$ is a *maximal* proper Serre $\otimes$-ideal subcategory $\mathcal{B} \subset \mathcal{A}^{\text{fp}} = \text{mod-}\mathcal{K}$ of the Freyd envelope of $\mathcal{K}$. The *homological residue field* corresponding to $\mathcal{B}$ is the functor constructed as follows

$$\bar{\text{h}}_{\mathcal{B}} = Q_{\mathcal{B}} \circ \text{h}: \quad \mathcal{K} \overset{\text{h}}{\hookrightarrow} \mathcal{A} = \text{Mod-}\mathcal{K} \overset{Q_{\mathcal{B}}}{\twoheadrightarrow} \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B}) := \frac{\text{Mod-}\mathcal{K}}{\langle \mathcal{B} \rangle},$$

$$x \longmapsto \quad \hat{x} \quad \longmapsto \quad \bar{x}.$$

The functor $\bar{\text{h}}_{\mathcal{B}} : \mathcal{K} \to \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})^{\text{fp}}$ is a (strong) monoidal homological functor (Remark 2.6), that lands in the finitely presented subcategory

$$\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})^{\text{fp}} \cong (\text{mod-}\mathcal{K})/\mathcal{B}.$$

By construction, the tensor-abelian category $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})^{\text{fp}}$ has only the trivial Serre $\otimes$-ideals, 0 and $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})^{\text{fp}}$ itself. These homological residue fields are truly the same as the homological $\otimes$-functors constructed in [Balmer et al. 2019, §4], up to a little paradigm change that we explain in Remark 3.10 below.

3.2. Remark. Since $\mathcal{K}$ is essentially small, its Freyd envelope, $\text{mod-}\mathcal{K}$, admits only a *set* of Serre subcategories. So we can apply Zorn to construct homological primes and homological residue fields as soon as $\mathcal{K} \neq 0$. Contrary to what happens with commutative rings, these *maximal* Serre $\otimes$-ideals are not only picking up

"closed points" as one could first fear. In fact, they "live" above *every* prime of the triangular spectrum $\mathrm{Spc}(\mathcal{K})$ of $\mathcal{K}$ (Remark 2.3). First, let us explain the relationship.

3.3. Proposition. *Let $\mathcal{B}$ be a homological prime with homological residue field $\bar{\mathrm{h}}_{\mathcal{B}} : \mathcal{K} \to \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$. Then $\mathcal{P}(\mathcal{B}) := \mathrm{Ker}(\bar{\mathrm{h}}_{\mathcal{B}}) = \mathrm{h}^{-1}(\mathcal{B})$ is a triangular prime of $\mathcal{K}$.*

Proof. Since Yoneda $\mathrm{h} : \mathcal{K} \to \mathrm{mod}\text{-}\mathcal{K}$ is homological and (strong) monoidal, the preimage $\mathcal{P}(\mathcal{B}) = \mathrm{h}^{-1}(\mathcal{B})$ is a proper thick $\otimes$-ideal of $\mathcal{K}$. To see that $\mathcal{P}(\mathcal{B})$ is prime, let $x, y \in \mathcal{K}$ with $x \otimes y \in \mathcal{P}(\mathcal{B})$ and $x \notin \mathcal{P}(\mathcal{B})$ and let us show that $y \in \mathcal{P}(\mathcal{B})$. Consider the $\otimes$-ideal $\mathcal{C} = \{M \in \mathrm{mod}\text{-}\mathcal{K} \mid \hat{x} \otimes M \in \mathcal{B}\}$. It is Serre by flatness of $\hat{x}$ and the assumption $x \notin \mathcal{P}(\mathcal{B})$ implies $\mathcal{B} \subseteq \mathcal{C} \neq \mathrm{mod}\text{-}\mathcal{K}$. Therefore $\mathcal{C} = \mathcal{B}$ by maximality of $\mathcal{B}$ and we get $y \in \mathrm{h}^{-1}(\mathcal{C}) = \mathrm{h}^{-1}(\mathcal{B}) = \mathcal{P}(\mathcal{B})$. $\square$

3.4. Remark. We can push the analogy with the triangular spectrum $\mathrm{Spc}(\mathcal{K})$ a little further by considering the set $\mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ of all homological primes:

$$\mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) = \left\{ \mathcal{B} \subset \mathrm{mod}\text{-}\mathcal{K} \mid \mathcal{B} \text{ is a maximal Serre } \otimes\text{-ideal} \right\}.$$

We call it the *homological spectrum of* $\mathcal{K}$ and equip it with a topology having as basis of closed subsets the following subsets $\mathrm{supp}^{\mathrm{h}}(x)$, one for every $x \in \mathcal{K}$:

$$\mathrm{supp}^{\mathrm{h}}(x) := \{\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \mid \hat{x} \notin \mathcal{B}\} = \{\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \mid \bar{x} \neq 0 \text{ in } \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})\}.$$

One can verify that this pair $(\mathrm{Spc}^{\mathrm{h}}(\mathcal{K}), \mathrm{supp}^{\mathrm{h}})$ is a *support data* on $\mathcal{K}$, in the sense of [Balmer 2005]. Hence there exists a unique continuous map

$$\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$$

such that $\mathrm{supp}^{\mathrm{h}}(x) = \phi^{-1}(\mathrm{supp}(x))$ for every $x \in \mathcal{K}$. The explicit formula for ϕ ([Balmer 2005, Theorem 3.2]) shows that ϕ is exactly the map $\mathcal{B} \mapsto \mathrm{h}^{-1}(\mathcal{B})$ of Proposition 3.3. We prove in Corollary 3.9 below that this comparison map is surjective, at least for $\mathcal{K}$ rigid. In fact, there are many known examples where ϕ is bijective. See Section 5.

3.5. Definition. It will be convenient to say that a homological prime $\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ *lives over* a given subset $W \subseteq \mathrm{Spc}(\mathcal{K})$ of the triangular spectrum if the prime $\mathcal{P}(\mathcal{B}) = \mathrm{h}^{-1}(\mathcal{B})$ of Proposition 3.3 belongs to W. By extension, we shall also say in that case that the corresponding homological residue field $\bar{\mathrm{h}} = \bar{\mathrm{h}}_{\mathcal{B}}$ *lives over* W.

In order to show surjectivity of ϕ, we derive from Proposition 2.9 the following:

3.6. Corollary. *Let $\mathcal{J} \subset \mathcal{K}$ be a thick $\otimes$-ideal. With notation as in Proposition 2.9, there is an inclusion-preserving one-to-one correspondence $\mathcal{C} \mapsto (q_!)^{-1}(\mathcal{C})$ between (maximal) Serre $\otimes$-ideals $\mathcal{C}$ of $\mathrm{mod}\text{-}\mathcal{L}$ and the (maximal) Serre $\otimes$-ideals $\mathcal{B}$ of $\mathrm{mod}\text{-}\mathcal{K}$ which contain $\mathrm{h}(\mathcal{J})$; the inverse is given by $\mathcal{B} \mapsto q_!(\mathcal{B})$. If $\mathcal{C}$ corresponds*

to $\mathcal{B}$ then the residue categories are canonically equivalent, $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B}) \cong \bar{\mathcal{A}}(\mathcal{L}; \mathcal{C})$, in such a way that the following diagram commutes up to canonical isomorphism:

$$
\begin{array}{ccccccc}
 & & \overset{\bar{\mathrm{h}}_{\mathcal{B}}}{\displaystyle\xrightarrow{\hspace{6cm}}} & & & & \\
\mathcal{K} & \overset{\mathrm{h}}{\hookrightarrow} \text{mod-}\mathcal{K} & \longrightarrow & \text{Mod-}\mathcal{K} & \overset{Q_{\mathcal{B}}}{\twoheadrightarrow} & \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B}) \\
\Big\downarrow q & \Big\downarrow q_! & & \Big\downarrow q_! & & \Big\| \cong \\
\mathcal{L} & \overset{\mathrm{h}}{\hookrightarrow} \text{mod-}\mathcal{L} & \longrightarrow & \text{Mod-}\mathcal{L} & \overset{Q_{\mathcal{C}}}{\twoheadrightarrow} & \bar{\mathcal{A}}(\mathcal{L}; \mathcal{C}) \\
 & & \underset{\bar{\mathrm{h}}_{\mathcal{C}}}{\displaystyle\xrightarrow{\hspace{6cm}}} & & & &
\end{array}
$$

Proof. Standard "third isomorphism theorem" about ideals in a quotient.   □

3.7. Remark. In the notation of Remark 3.4, the above Corollary 3.6 can be rephrased as saying that, for $\mathcal{L} = \mathcal{K}/\mathcal{J}$ or $(\mathcal{K}/\mathcal{J})^\natural$, the map $\mathrm{Spc}^{\mathrm{h}}(q_!) : \mathcal{C} \mapsto (q_!)^{-1}\mathcal{C}$ yields a homeomorphism between $\mathrm{Spc}^{\mathrm{h}}(\mathcal{L})$ and the subspace

$$\{\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \mid \mathcal{P}(\mathcal{B}) \supseteq \mathcal{J}\}$$

of $\mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$. In other words, the following commutative diagram is cartesian:

$$
\begin{array}{ccc}
\mathrm{Spc}^{\mathrm{h}}(\mathcal{L}) & \overset{\mathrm{Spc}^{\mathrm{h}}(q_!)}{\hookrightarrow} & \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \\
\Big\downarrow \phi & & \Big\downarrow \phi \\
\mathrm{Spc}(\mathcal{L}) & \overset{\mathrm{Spc}(q)}{\hookrightarrow} & \mathrm{Spc}(\mathcal{K})
\end{array}
$$

In the terminology of Definition 3.5, the homological primes (or the residue fields) of $\mathcal{L} = \mathcal{K}/\mathcal{J}$ or $\mathcal{L} = (\mathcal{K}/\mathcal{J})^\natural$ canonically correspond to those of $\mathcal{K}$ which live "above" the subset $W(\mathcal{J}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{J} \subseteq \mathcal{P}\} \cong \mathrm{Spc}(\mathcal{L})$ of $\mathrm{Spc}(\mathcal{K})$.

Let us prove an analogue of [Balmer 2005, Lemma 2.2], an old tensor-triangular friend:

3.8. Lemma. *Suppose that $\mathcal{K}$ is rigid. Let $\mathcal{S} \subset \mathcal{K}$ be a $\otimes$-multiplicative class of objects (i.e., $\mathbb{1} \in \mathcal{S} \supseteq \mathcal{S} \otimes \mathcal{S}$) and let $\mathcal{B}_0 \subset \mathcal{A}^{\mathrm{fp}} = \text{mod-}\mathcal{K}$ be a Serre $\otimes$-ideal which avoids $\mathcal{S}$, that is, $\mathcal{B}_0 \cap \mathrm{h}(\mathcal{S}) = \varnothing$. Then there exists $\mathcal{B} \subset \mathcal{A}^{\mathrm{fp}}$ a maximal Serre $\otimes$-ideal such that $\mathcal{B}_0 \subseteq \mathcal{B}$ and $\mathcal{B}$ still avoids $\mathcal{S}$.*

Proof. By Zorn, there exists $\mathcal{B}$ maximal among the Serre $\otimes$-ideals which avoid $\mathcal{S}$ and contain $\mathcal{B}_0$. So we have $\mathcal{B} \supseteq \mathcal{B}_0$ and $\mathcal{B} \cap \mathrm{h}(\mathcal{S}) = \varnothing$ and we are left to prove that $\mathcal{B}$ is plain maximal in $\mathcal{A}^{\mathrm{fp}}$. Consider $\mathcal{B}' := \{M \in \mathcal{A}^{\mathrm{fp}} \mid \hat{x} \otimes M \in \mathcal{B} \text{ for some } x \in \mathcal{S}\}$. Since $\mathcal{S}$ is $\otimes$-multiplicative and each $\hat{x}$ is $\otimes$-flat, the subcategory $\mathcal{B}' \subsetneq \mathcal{A}^{\mathrm{fp}}$ is a Serre $\otimes$-ideal avoiding $\mathcal{S}$ and containing $\mathcal{B}_0$. By maximality of $\mathcal{B}$ among those, the relation $\mathcal{B} \subseteq \mathcal{B}'$ forces $\mathcal{B} = \mathcal{B}'$. In particular, $M = \ker(\hat{\eta}_x : \hat{\mathbb{1}} \to \hat{x}^\vee \otimes \hat{x})$ belongs to $\mathcal{B}$ for every $x \in \mathcal{S}$ since $\hat{x} \otimes M = 0$ by rigidity (Remark 2.4). Let us show that $\mathcal{B} \subset \mathcal{A}^{\mathrm{fp}}$ is a maximal Serre $\otimes$-ideal by showing that a strictly bigger Serre $\otimes$-ideal

$\mathcal{C} \supsetneq \mathcal{B}$ of $\mathcal{A}^{\mathrm{fp}}$ must be $\mathcal{A}^{\mathrm{fp}}$ itself. Since $\mathcal{B}$ is maximal among those avoiding $\mathcal{S}$ and containing $\mathcal{B}_0$, such a strictly bigger $\mathcal{C}$ cannot avoid $\mathcal{S}$. Therefore $\mathcal{C}$ contains some $\hat{x}$ for $x \in \mathcal{S}$ and, by the above discussion, we also have $\ker(\hat{\eta}_x : \hat{\mathbb{1}} \to \hat{x}^\vee \otimes \hat{x}) \in \mathcal{B} \subseteq \mathcal{C}$. So in the exact sequence $0 \to \ker(\hat{\eta}_x) \to \hat{\mathbb{1}} \to \hat{x}^\vee \otimes \hat{x}$ we have $\ker(\hat{\eta}_x)$ and $\hat{x}$ in $\mathcal{C}$. This forces $\hat{\mathbb{1}} \in \mathcal{C}$ by Serritude and therefore $\mathcal{C} = \mathcal{A}^{\mathrm{fp}}$ as wanted. $\square$

3.9. Corollary. *Suppose that $\mathcal{K}$ is rigid. Then the map $\mathcal{B} \mapsto \mathcal{P}(\mathcal{B})$ from homological primes to triangular primes as in Proposition 3.3 (i.e., the comparison map $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ of Remark 3.4) is surjective. That is, every triangular prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ is of the form $\mathcal{P} = \mathrm{h}^{-1}(\mathcal{B})$ for some maximal Serre $\otimes$-ideal $\mathcal{B}$ in* mod-$\mathcal{K}$.

Proof. Consider the quotient $\mathcal{K}/\mathcal{P}$ (or its idempotent completion $\mathcal{K}_\mathcal{P} := (\mathcal{K}/\mathcal{P})^\natural$). The map $\mathrm{Spc}(q) : \mathrm{Spc}(\mathcal{K}/\mathcal{P}) \hookrightarrow \mathrm{Spc}(\mathcal{K})$ sends 0 to $q^{-1}(0) = \mathcal{P}$. So, by Corollary 3.6 applied to $\mathcal{J} = \mathcal{P}$, it suffices to prove the result for $\mathcal{P} = 0$. In that case, $\mathcal{K}$ is *local*, meaning that 0 is a prime: $x \otimes y = 0 \Rightarrow x = 0$ or $y = 0$. We conclude by Lemma 3.8 for $\mathcal{B}_0 = 0$ and $\mathcal{S} = \mathcal{K} \smallsetminus \{0\}$ which is $\otimes$-multiplicative because $\mathcal{K}$ is local.[1] $\square$

3.10. Remark. There are a few differences between our approach to homological residue fields and the treatment in [Balmer et al. 2019, §4]. First, the whole [Balmer et al. 2019] is written for a "big" (i.e., rigidly compactly generated) tensor-triangulated category $\mathcal{T}$ and the modules are taken over its rigid-compact objects $\mathcal{K} := \mathcal{T}^c$. This restriction is unimportant, certainly as far as most examples are concerned.

Another difference is that, in [loc. cit.], we focused on a *local* category in the sense that $\mathrm{Spc}(\mathcal{K})$ is a local space, i.e., has a unique closed point $\mathcal{M} = 0$. We then considered quotients of the module category $\mathcal{A} \twoheadrightarrow \mathcal{A}/\langle \mathcal{B} \rangle$ for $\mathcal{B} \subseteq \mathcal{A}^{\mathrm{fp}}$ maximal *among those which meet $\mathcal{K}$ trivially*, i.e., $\mathcal{B} \cap \mathrm{h}(\mathcal{K}) = \{0\}$. This property means that the homological prime $\mathcal{B}$ lives *above the closed point* of $\mathrm{Spc}(\mathcal{K})$ in the sense of Definition 3.5. Equivalently, it means that the functor $\bar{\mathrm{h}}_\mathcal{B} : \mathcal{K} \to \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$ is conservative, i.e., detects isomorphisms. All these properties are reminiscent of commutative algebra, where the residue field of a local ring R is indeed conservative on perfect complexes and maps the unique prime of the field to the closed point of $\mathrm{Spec}(R)$.

Continuing the analogy with commutative algebra, when dealing with a *global* (i.e., not necessarily local) category $\mathcal{K}$, we can analyze it one prime at a time. For each $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ we can consider the local category $\mathcal{K}_\mathcal{P} = (\mathcal{K}/\mathcal{P})^\natural$. By Corollary 3.6 we can identify the homological primes $\mathcal{C}$ of this local category $\mathcal{K}_\mathcal{P}$ with a subset of those of the global category. Requesting that the local prime $\mathcal{C}$ lives "above the closed point" of $\mathrm{Spc}(\mathcal{K}_\mathcal{P})$ as we did in [Balmer et al. 2019] amounts to

---

[1]The argument was already used in [Balmer et al. 2019, Corollary 4.9], which in turn inspired Lemma 3.8.

requesting that the corresponding global prime $\mathcal{B} = (q_!)^{-1}(\mathcal{C})$ lives exactly above the point $\mathcal{P}$ in $\mathrm{Spc}(\mathcal{K})$.

In other words, in Definition 3.1 we are considering all homological residue fields $\bar{\mathrm{h}}_{\mathcal{B}} : \mathcal{K} \to \bar{A}(\mathcal{K}; \mathcal{B})$ at once but we can also regroup them according to the associated triangular primes $\mathrm{h}^{-1}(\mathcal{B})$, in which case we obtain the constructions of [Balmer et al. 2019] for the local category $\mathcal{K}_{\mathcal{P}}$.

4. The Nilpotence Theorems

In this section, we assume $\mathcal{K}$ rigid.

Let us prove Theorem 1.2, in a strong form "with parameter". Recall that an object F in a tensor abelian category A is $\otimes$-*flat* if $F \otimes - : A \to A$ is exact.

4.1. Theorem. *Let $\mathcal{K}$ be an essentially small, rigid tensor-triangulated category* (Hypothesis 2.1) *and $W \subseteq \mathrm{Spc}(\mathcal{K})$ a closed subset* (*the "parameter"*). *Let $f : \hat{x} \to F$ be a morphism in $A = \mathrm{Mod}\text{-}\mathcal{K}$ satisfying the following hypotheses*:

(i) *The source of f comes from an object $x \in \mathcal{K}$ via Yoneda, as indicated above.*

(ii) *Its target F is $\otimes$-flat in A.*

(iii) *The morphism f vanishes in every homological residue field over W in the following sense*: *For every maximal Serre $\otimes$-ideal $\mathcal{B} \subset \mathrm{mod}\text{-}\mathcal{K}$ living over W* (Definition 3.5) *we have $Q_{\mathcal{B}}(f) = 0$ in $\bar{A}(\mathcal{K}; \mathcal{B}) = (\mathrm{Mod}\text{-}\mathcal{K})/\langle \mathcal{B} \rangle$.*

Then:

(a) *There exist an object $s \in \mathcal{K}$ such that $\mathrm{supp}(s) \supseteq W$ and an integer $n \geq 1$ such that $f^{\otimes n} \otimes \hat{s} = 0$ in A.*

(b) *For any object s as above, if we let $Z = \mathrm{supp}(s) \supseteq W$, then for every $z \in \mathcal{K}_Z = \{z \in \mathcal{K} \mid \mathrm{supp}(z) \subseteq Z\}$ there exists $m \geq 1$ with $f^{\otimes m} \otimes \hat{z} = 0$.*

Proof. By Remark 2.8, we can and shall assume that $x = \mathbb{1}$. So $f : \hat{\mathbb{1}} \to F$. Consider

$$\mathcal{S} := \{s \in \mathcal{K} \mid \mathrm{supp}(s) \supseteq W\}.$$

This is a $\otimes$-multiplicative class of objects of $\mathcal{K}$ since

$$\mathrm{supp}(s_1 \otimes s_2) = \mathrm{supp}(s_1) \cap \mathrm{supp}(s_2).$$

Since W is closed and $\{\mathrm{supp}(s)\}_{s \in \mathcal{K}}$ is a basis of closed subsets, we have

$$\bigcap_{s \in \mathcal{S}} \mathrm{supp}(s) = W.$$

On the other hand, consider the following subcategory of finitely presented $\mathcal{K}$-modules

$$\mathcal{B}_0 := \left\{ M \in A^{\mathrm{fp}} \mid f^{\otimes n} \otimes M = 0 \text{ in } A \text{ for some } n \geq 1 \right\}.$$

Note that $\mathcal{B}_0$ is a Serre $\otimes$-ideal. This uses that F is $\otimes$-flat in $\mathcal{A}$ and was already proved in [Balmer et al. 2019, Lemma 4.17]. In particular, when we prove that $\mathcal{B}_0$ is closed under extension, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence in $\mathcal{A}^{\mathrm{fp}}$ and if $f^{\otimes n_1} \otimes M_1 = 0$ and $f^{\otimes n_3} \otimes M_3 = 0$ then we show that $f^{\otimes(n_1+n_3)} \otimes M_2 = 0$. This is the place where nilpotence is needed, as opposed to mere vanishing.

If $\mathcal{B}_0$ meets $\mathrm{h}(\mathcal{S})$ we obtain the conclusion of part (a). Suppose ab absurdo, that $\mathcal{B}_0 \cap \mathrm{h}(\mathcal{S}) = \varnothing$. By Lemma 3.8 there exists a homological prime $\mathcal{B} \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ containing $\mathcal{B}_0$ and still avoiding $\mathcal{S}$. The latter property $\mathcal{B} \cap \mathrm{h}(\mathcal{S}) = \varnothing$ means that the triangular prime $\mathrm{h}^{-1}(\mathcal{B})$ belongs to the subset $\{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{P} \cap \mathcal{S} = \varnothing\} = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid s \notin \mathcal{P}, \ \forall\, s \in \mathcal{S}\} = \bigcap_{s \in \mathcal{S}} \mathrm{supp}(s) = W$, as we proved above from the definition of $\mathcal{S}$. So $\mathcal{B}$ lives over W and we trigger hypothesis (iii) for that $\mathcal{B}$, namely that we have $Q_{\mathcal{B}}(f) = 0$ in $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$.

Consider now the kernel of $f : \hat{\mathbb{1}} \to F$ in $\mathcal{A}$ and the commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker(f) & \xrightarrow{\ j\ } & \hat{\mathbb{1}} & \xrightarrow{\ f\ } & F \\
& & \Big\downarrow{\scriptstyle f\otimes 1} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f\otimes 1} \\
0 & \longrightarrow & F \otimes \ker(f) & \xrightarrow{\ 1\otimes j\ } & F & \xrightarrow{\ 1\otimes f\ } & F \otimes F
\end{array}
$$

whose first row is the obvious one; the diagram is obtained by tensoring that first row with $f : \hat{\mathbb{1}} \to F$ itself (on the left) and using that $\hat{\mathbb{1}}$ is the $\otimes$-unit in $\mathcal{A}$. It is essential here that the source of f is $\hat{\mathbb{1}}$ (which we achieved through rigidity), and not some random object. Indeed, the diagonal of the left-hand square is now $f \circ j = 0$. Since the lower row is exact (F $\otimes$-flat again), we conclude that $f \otimes \ker(f) = 0$. We cannot jump to the conclusion that $\ker(f) \in \mathcal{B}$ since $\ker(f)$ is not finitely presented. However, $\ker(f) \rightarrowtail \hat{\mathbb{1}}$ is a subobject of a finitely presented object, hence it is the *union* of its finitely presented subobjects as in [Balmer et al. 2018, Lemma 3.9], i.e.,

$$
\ker(f) = \operatornamewithlimits{colim}_{\substack{M \rightarrowtail \ker(f) \\ M \in \mathcal{A}^{\mathrm{fp}}}} M.
$$

For any such $i : M \rightarrowtail \ker(f)$ with $M \in \mathcal{A}^{\mathrm{fp}}$, we have a commutative square obtained by tensoring $i : M \rightarrowtail \ker(f)$ with $f : \hat{\mathbb{1}} \to F$:

$$
\begin{array}{ccc}
M & \xrightarrow{\ \ i\ \ } & \ker(f) \\
\Big\downarrow{\scriptstyle f\otimes 1} & & \Big\downarrow{\scriptstyle f\otimes 1=0} \\
F \otimes M & \xrightarrow{\ 1\otimes i\ } & F \otimes \ker(f)
\end{array}
$$

Note that the bottom map remains a monomorphism because F is $\otimes$-flat. The vanishing of the right-hand vertical map, proved above, gives us $f \otimes M = 0$, which means $M \in \mathcal{B}_0 \subseteq \mathcal{B}$. It follows that $\ker(f)$ is a colimit of objects $M \in \mathcal{B}$ and

therefore belongs to $\langle \mathcal{B} \rangle$. Applying the exact functor $Q_{\mathcal{B}} : \mathcal{A} \twoheadrightarrow \bar{\mathcal{A}} = \mathcal{A}/\langle \mathcal{B} \rangle$ to the morphism f, we have just proved that $Q_{\mathcal{B}}(f)$ has trivial kernel, i.e., it is a monomorphism $Q_{\mathcal{B}}(\hat{\mathbb{1}}) \rightarrowtail Q_{\mathcal{B}}(F)$ in $\bar{\mathcal{A}}$. But this monomorphism $Q_{\mathcal{B}}(f)$ is also zero by assumption (iii) on f that we triggered in the first part of the proof. This forces $Q_{\mathcal{B}}(\hat{\mathbb{1}}) = 0$ and thus $\hat{\mathbb{1}} \in \mathcal{B}$, a contradiction.

This finishes the proof of part (a). To deduce (b), we use the fact that

$$\mathcal{B}_0 = \left\{ M \in \mathcal{A}^{\mathrm{fp}} \mid f^{\otimes m} \otimes M = 0 \text{ for some } m \geq 1 \right\}$$

is a Serre $\otimes$-ideal, as we saw above, by [Balmer et al. 2019, Lemma 4.17]. Therefore $\mathrm{h}^{-1}(\mathcal{B}_0)$ is a thick $\otimes$-ideal of $\mathcal{K}$ and so the fact that $\mathrm{h}^{-1}(\mathcal{B}_0)$ contains s implies that it contains the whole thick $\otimes$-ideal of $\mathcal{K}$ generated by s, namely exactly $\mathcal{K}_Z$ where $Z = \mathrm{supp}(s)$, see [Balmer 2005, §4]. □

We can then deduce the form announced in Theorem 1.2:

4.2. Corollary. *Let $f : \hat{x} \to F$ be a morphism in $\mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K}$, with $x \in \mathcal{K}$ and with $F \otimes$-flat in $\mathcal{A}$. Suppose that for every homological prime $\mathcal{B} \subset \mathrm{mod}\text{-}\mathcal{K}$ we have $Q_{\mathcal{B}}(f) = 0$ in $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$. Then there exists $m \geq 1$ such that $f^{\otimes m} = 0$ in $\mathcal{A}$.*

Proof. Apply Theorem 4.1 for $W = \mathrm{Spc}(\mathcal{K})$. Thus the subset $Z \supseteq W$ of part (b) must be $Z = \mathrm{Spc}(\mathcal{K})$, and we can take $z = \mathbb{1}$. □

4.3. Remark. We stress that the "parameter" W in Theorem 4.1 is more flexible than the "parameter" in [Thomason 1997, Theorem 3.8], where the closed subset W is supposed to be the support of some object, i.e., a Thomason closed subset (Remark 2.3). In that case, any object s with $\mathrm{supp}(s) = W$ will do, as follows from Theorem 4.1(b).

Of course, the easiest source of $\otimes$-flat objects in $\mathrm{Mod}\text{-}\mathcal{K}$ is the Yoneda embedding:

4.4. Corollary. *Let $f : x \to y$ be a morphism in $\mathcal{K}$ and $W \subseteq \mathrm{Spc}(\mathcal{K})$ be a closed subset such that $\bar{\mathrm{h}}_{\mathcal{B}}(f) = 0$ for every homological residue field corresponding to a homological prime $\mathcal{B}$ living above W (Definition 3.5). Then there exists a Thomason closed subset $Z \supseteq W$ (Remark 2.3) with the property that for every $z \in \mathcal{K}$ such that $\mathrm{supp}(z) \subseteq Z$ there exists $n \geq 1$ with $f^{\otimes n} \otimes z = 0$ in $\mathcal{K}$. In particular, this holds for some z with $\mathrm{supp}(z) = Z \supseteq W$.*

Proof. This is Theorem 4.1 for $F = \hat{y}$, which is $\otimes$-flat, combined with faithfulness of Yoneda $\mathrm{h} : \mathcal{K} \hookrightarrow \mathcal{A}$ to bring the conclusion back into $\mathcal{K}$. □

This specializes to the flagship nilpotence theorem (Theorem 1.1):

4.5. Corollary. *Let $f : x \to y$ be a morphism in $\mathcal{K}$ such that $\bar{\mathrm{h}}(f) = 0$ for every homological residue field $\bar{\mathrm{h}}$ of $\mathcal{K}$. Then there exists $n \geq 1$ with $f^{\otimes n} = 0$ in $\mathcal{K}$.*

Proof. Corollary 4.4 for $W = \mathrm{Spc}(\mathcal{K})$, hence $Z = \mathrm{Spc}(\mathcal{K})$, and $z = \mathbb{1}$. □

4.6. Remark. Many of our examples of tensor-triangulated categories $\mathcal{K}$, if not all, appear as the compact-rigid objects $\mathcal{K} = \mathcal{T}^c$ in some compactly rigidly generated "big" tensor-triangulated category $\mathcal{T}$. See [Balmer et al. 2019, Hypothesis 0.1]. In that case, we have a *restricted-Yoneda* functor which extends $h : \mathcal{K} \hookrightarrow \mathcal{A} = \text{Mod-}\mathcal{K}$ to the whole of $\mathcal{T}$:

$$
\begin{array}{ccc}
\mathcal{K} = \mathcal{T}^c & \xrightarrow{\;\;h\;\;} & \mathcal{A}^{\text{fp}} = \text{mod-}\mathcal{K} \\
\downarrow & & \downarrow \\
\mathcal{T} & \xrightarrow{\;\;h\;\;} & \mathcal{A} = \text{Mod-}\mathcal{K} \\
X & \longmapsto & \hat{X} := \text{Hom}_{\mathcal{T}}(-\,,X)
\end{array}
$$

Note that $h : \mathcal{T} \to \mathcal{A}$ is not faithful anymore (it kills the so-called *phantom maps*). However, it is faithful for maps *out of* a compact, by the usual Yoneda lemma, that is, $\text{Hom}_{\mathcal{T}}(x, Y) \to \text{Hom}_{\mathcal{A}}(\hat{x}, \hat{Y})$ is bijective as soon as $x \in \mathcal{K}$. We prove in [Balmer et al. 2018, Proposition A.14] that every $\hat{Y}$ remains $\otimes$-flat in Mod-$\mathcal{K}$, even for $Y \in \mathcal{T}$ noncompact.

For every homological prime $\mathcal{B} \in \text{Spc}^h(\mathcal{K})$ we can still compose restricted-Yoneda $\mathcal{T} \to \mathcal{A}$ with $Q_{\mathcal{B}} : \mathcal{A} \twoheadrightarrow \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$. We obtain a well-defined homological residue field on the whole "big" category $\mathcal{T}$, that we still denote

$$
\bar{h}_{\mathcal{B}} : \mathcal{T} \to \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B}).
$$

This remains a homological tensor-functor. Compare [Balmer et al. 2019, Theorem 1.6]. Note that we may use these functors to define a support for big objects in $\mathcal{T}$, as will be investigated elsewhere.

We can finally unpack our nilpotence theorems in that special case:

4.7. Corollary. *Let $\mathcal{T}$ be a rigidly compactly generated "big" tensor-triangulated category and $\mathcal{K} = \mathcal{T}^c$ as in Remark 4.6. Let $f : x \to Y$ be a morphism in $\mathcal{T}$ with $x \in \mathcal{K}$ compact and Y arbitrary.*

(a) *Suppose that $\bar{h}_{\mathcal{B}}(f) = 0$ in $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$ for every homological residue field $\bar{h}_{\mathcal{B}}$ of Remark 4.6, for every homological prime $\mathcal{B} \subset \text{mod-}\mathcal{K}$ of Definition 3.1. Then we have $f^{\otimes n} = 0$ in $\mathcal{T}$ for some $n \geq 1$.*

(b) *Suppose that $\bar{h}_{\mathcal{B}}(f) = 0$ in $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$ for every homological residue field over a closed subset $W \subseteq \text{Spc}(\mathcal{K})$ (Definition 3.5). Then there exists a Thomason closed subset $Z \subseteq \text{Spc}(\mathcal{K})$ such that $Z \supseteq W$ and such that for every $z \in \mathcal{K}_Z = \{z \in \mathcal{K} \mid \text{supp}(z) \subseteq Z\}$ we have $f^{\otimes n} \otimes z = 0$ for some $n \geq 1$. In particular, this holds for some z with $\text{supp}(z) = Z \supseteq W$ (Remark 2.3).*

Proof. This follows from Theorem 4.1 applied to $F = \hat{Y}$, together with the partial faithfulness of restricted-Yoneda explained in Remark 4.6. $\qquad\square$

5. Examples

For this section, we keep the setting of Remark 4.6, that is, $\mathcal{T}$ is a "big" tensor-triangulated category generated by its subcategory $\mathcal{K} = \mathcal{T}^c$ of rigid-compact objects.

5.1. Remark. We recall some of the tools developed in [Balmer et al. 2019, §3]. Let $\mathcal{B}$ be a proper Serre $\otimes$-ideal in $\mathcal{A}^{\mathrm{fp}} = \mathrm{mod}\text{-}\mathcal{K}$. Picking an injective hull of the unit $\hat{\mathbb{1}}$ in the quotient $\bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$ yields a (pure-injective) object $E_{\mathcal{B}}$ in $\mathcal{T}$ such that

$$(5.2) \qquad \langle \mathcal{B} \rangle = \mathrm{Ker}(\hat{E}_{\mathcal{B}} \otimes -) = \{ M \in \mathrm{Mod}\text{-}\mathcal{K} \mid \hat{E}_{\mathcal{B}} \otimes M = 0 \}.$$

In fact the object $E_{\mathcal{B}}$ is a *weak ring* in $\mathcal{T}$, i.e., it comes with a map $\eta_{\mathcal{B}} : \mathbb{1} \to E_{\mathcal{B}}$ such that $E_{\mathcal{B}} \otimes \eta_{\mathcal{B}} : E_{\mathcal{B}} \to E_{\mathcal{B}} \otimes E_{\mathcal{B}}$ is a split monomorphism. A retraction $E_{\mathcal{B}} \otimes E_{\mathcal{B}} \to E_{\mathcal{B}}$ of this monomorphism can be viewed as a (nonassociative) multiplication on $E_{\mathcal{B}}$ for which $\eta_{\mathcal{B}} : \mathbb{1} \to E_{\mathcal{B}}$ is a right unit. In any case, one important property of $\eta_{\mathcal{B}}$ is that it cannot be $\otimes$-nilpotent, for otherwise $E_{\mathcal{B}}$ would be a retract of zero, hence zero, forcing $\mathcal{B} = \mathcal{A}^{\mathrm{fp}}$.

Another important feature of the objects $E_{\mathcal{B}}$, for $\mathcal{B}$ maximal, is the following:

5.3. Proposition. *For distinct $\mathcal{B} \neq \mathcal{B}'$ in $\mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ we have $E_{\mathcal{B}} \otimes E_{\mathcal{B}'} = 0$.*

Proof. This is already in [Balmer et al. 2019, Corollary 4.12], at least in the local setting. Let us recall the idea, which is easy. In $\mathcal{A}^{\mathrm{fp}} = \mathrm{mod}\text{-}\mathcal{K}$, the kernel

$$\mathrm{Ker}(\hat{E}_{\mathcal{B}} \otimes \hat{E}_{\mathcal{B}'} \otimes -)$$

is a Serre $\otimes$-ideal containing both $\mathcal{B}$ and $\mathcal{B}'$, which are maximal. If that kernel was a proper subcategory it would then be equal to both $\mathcal{B}$ and $\mathcal{B}'$, thus forcing the forbidden $\mathcal{B} = \mathcal{B}'$. So this kernel is not proper, i.e., it contains the unit $\hat{\mathbb{1}}$. This reads $\hat{E}_{\mathcal{B}} \otimes \hat{E}_{\mathcal{B}'} = 0$. Now restricted-Yoneda h : $\mathcal{T} \to \mathrm{Mod}\text{-}\mathcal{K}$ is a conservative $\otimes$-functor, hence $E_{\mathcal{B}} \otimes E_{\mathcal{B}'} = 0$ as claimed. $\qquad\square$

Let us now prove the converse to Corollary 4.7.

5.4. Theorem. *Let $\mathcal{T}$ be a "big" tensor-triangulated category with $\mathcal{K} = \mathcal{T}^c$, as in Remark 4.6. Consider a family $\mathcal{F} \subseteq \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ of points in the homological spectrum. Suppose that the corresponding functors $\bar{\mathrm{h}}_{\mathcal{B}} : \mathcal{T} \to \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B})$ collectively detect $\otimes$-nilpotence in the following sense: If $f : x \to Y$ in $\mathcal{T}$ is such that $x \in \mathcal{T}^c$ and $\bar{\mathrm{h}}_{\mathcal{B}}(f) = 0$ for all $\mathcal{B} \in \mathcal{F}$, then $f^{\otimes n} = 0$ for $n \gg 1$. Then we have $\mathcal{F} = \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$.*

Proof. Suppose that $\mathcal{F} \neq \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$. Then there exists $\mathcal{B}' \in \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ which does not belong to $\mathcal{F}$. By Proposition 5.3 we have $E_{\mathcal{B}} \otimes E_{\mathcal{B}'} = 0$ for all $\mathcal{B} \in \mathcal{F}$. Consider the map $\eta_{\mathcal{B}'} : \mathbb{1} \to E_{\mathcal{B}'}$ as in Remark 5.1. We have therefore proved that $\bar{\mathrm{h}}_{\mathcal{B}}(\eta_{\mathcal{B}'}) = 0$ for all $\mathcal{B} \in \mathcal{F}$ for the obvious reason that its target object, $E_{\mathcal{B}'}$, goes to zero along all $\bar{\mathrm{h}}_{\mathcal{B}}$. On the other hand, we have seen in Remark 5.1 that $\eta_{\mathcal{B}'}$ cannot be $\otimes$-nilpotent. Hence a proper family $\mathcal{F} \subsetneq \mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$ cannot detect $\otimes$-nilpotence. $\qquad\square$

5.5. Remark. The proof shows that it is enough to assume the property that the family $\{\bar{\mathrm{h}}_{\mathcal{B}}\}_{\mathcal{B}\in\mathcal{F}}$ detects the vanishing of objects $Y \in \mathcal{T}$. Compare Remark 5.12.

Here is the picture we will observe in several examples:

5.6. Theorem. *Let $\mathcal{T}$ be a "big" tensor-triangulated category and $\mathcal{K} = \mathcal{T}^c$ as in Remark 4.6. Suppose given for every point $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ of the triangular spectrum, a coproduct-preserving, homological and (strong) monoidal functor*

$$H_{\mathcal{P}} : \mathcal{T} \to \mathcal{A}_{\mathcal{P}}$$

with values in a tensor-abelian category $\mathcal{A}_{\mathcal{P}}$ and satisfying the following properties:

(1) *For each $\mathcal{P}$, the target $\mathcal{A}_{\mathcal{P}}$ is a locally coherent (Remark 2.6) Grothendieck category with colimit-preserving tensor; the subcategory $\mathcal{A}_{\mathcal{P}}^{\mathrm{fp}}$ of finitely presented objects is* simple *in the sense that its only Serre $\otimes$-ideals are 0 and $\mathcal{A}_{\mathcal{P}}^{\mathrm{fp}} \neq 0$.*

(2) *The functor $H_{\mathcal{P}} : \mathcal{T} \to \mathcal{A}_{\mathcal{P}}$ maps compacts to finitely presented: $H_{\mathcal{P}}(\mathcal{K}) \subseteq \mathcal{A}_{\mathcal{P}}^{\mathrm{fp}}$. Furthermore, it maps every $X \in \mathcal{T}$ to a $\otimes$-flat object in $\mathcal{A}_{\mathcal{P}}$. Finally the thick $\otimes$-ideal $\mathrm{Ker}(H_{\mathcal{P}}) \cap \mathcal{K} = \{x \in \mathcal{K} \mid H_{\mathcal{P}}(x) = 0\}$ is equal to $\mathcal{P}$.*

(3) *The family $\{H_{\mathcal{P}}\}_{\mathcal{P}\in\mathrm{Spc}(\mathcal{K})}$ detects $\otimes$-nilpotence of maps $f : x \to Y$ in $\mathcal{T}$, with $x \in \mathcal{K}$ compact: If $H_{\mathcal{P}}(f) = 0$ for all $\mathcal{P}$ then $f^{\otimes n} = 0$ for $n \gg 1$.*

Then the comparison map $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ of Proposition 3.3 is a bijection.

Proof. The core of the proof amounts to the kernel of each $H_{\mathcal{P}}$ defining an element of $\mathrm{Spc}^{\mathrm{h}}(\mathcal{K})$. More precisely, let us fix $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ for the moment and denote by

$$\hat{H}_{\mathcal{P}} : \mathcal{A} \to \mathcal{A}_{\mathcal{P}}$$

the unique coproduct-preserving *exact* functor that extends $H_{\mathcal{P}}$ to $\mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K}$, that is, such that the following diagram commutes:

(5.7)

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\mathrm{h}} & \mathcal{A} = \mathrm{Mod}\text{-}\mathcal{K} \\
 & {\scriptstyle H_{\mathcal{P}}} \searrow & \downarrow {\scriptstyle \hat{H}_{\mathcal{P}}} \\
 & & \mathcal{A}_{\mathcal{P}}
\end{array}
$$

The existence of such an $\hat{H}_{\mathcal{P}}$ was established by Krause [2000, Corollary 2.4]. It is not hard to show that $\hat{H}_{\mathcal{P}}$ is also monoidal. At the very least, for every $M \in \mathcal{A}$, we can find $g : Y \to Z$ in $\mathcal{T}$ such that $M = \mathrm{im}(\hat{g})$ and then exactness of $\hat{H}_{\mathcal{P}}$ gives

(5.8) $$\hat{H}_{\mathcal{P}}(\hat{X} \otimes M) \cong H_{\mathcal{P}}(X) \otimes \hat{H}_{\mathcal{P}}(M)$$

for every $X \in \mathcal{T}$ and $M \in \mathcal{A}$. Consider now the kernel of $\hat{H}_\mathcal{P}$ in $\mathcal{A}$. The exact functor $\hat{H}_\mathcal{P} : \mathcal{A} \to \mathcal{A}_\mathcal{P}$ preserves coproducts and finitely presented objects because of (2). It then follows by a general property of abelian categories that $\text{Ker}(H_\mathcal{P})$ is generated by its finitely presented objects (see [Balmer et al. 2019, Proposition A.6]). In other words, if we define a Serre $\otimes$-ideal of $\mathcal{A}^{\text{fp}}$ as follows:

$$\mathcal{B}(\mathcal{P}) := \text{Ker}(\hat{H}_\mathcal{P}) \cap \mathcal{A}^{\text{fp}} = \{M \in \text{mod-}\mathcal{K} \mid \hat{H}_\mathcal{P}(M) = 0\},$$

then we have $\text{Ker}(\hat{H}_\mathcal{P}) = \langle \mathcal{B}(\mathcal{P}) \rangle$. The fact that $\mathcal{A}_\mathcal{P} \neq 0$ tells us that $\mathcal{B} := \mathcal{B}(\mathcal{P})$ is proper. We claim that it is maximal. Let $\mathcal{B}'$ be a strictly larger Serre $\otimes$-ideal

$$\mathcal{B} = \mathcal{B}(\mathcal{P}) \subsetneq \mathcal{B}' \subseteq \mathcal{A}^{\text{fp}}.$$

We want to prove that $\mathcal{B}' = \mathcal{A}^{\text{fp}}$. Choose $M \in \mathcal{B}'$ which does not belong to $\mathcal{B}$. Let us now invoke the objects $E_\mathcal{B}$ and $E_{\mathcal{B}'}$ of $\mathcal{T}$, as in Remark 5.1, so that

$$\langle \mathcal{B} \rangle = \text{Ker}(\hat{E}_\mathcal{B} \otimes -) \quad \text{and} \quad \langle \mathcal{B}' \rangle = \text{Ker}(\hat{E}_{\mathcal{B}'} \otimes -).$$

We then have $\hat{E}_{\mathcal{B}'} \otimes M = 0$ because $M \in \mathcal{B}'$. Hence by (5.8), we have

$$H_\mathcal{P}(E_{\mathcal{B}'}) \otimes \hat{H}_\mathcal{P}(M) = 0.$$

On the other hand, we have $\hat{H}_\mathcal{P}(M) \neq 0$ because $M \notin \mathcal{B}$. Since $H_\mathcal{P}(E_{\mathcal{B}'})$ is $\otimes$-flat in $\mathcal{A}_\mathcal{P}$, we can consider the Serre $\otimes$-ideal

$$\text{Ker}(H_\mathcal{P}(E_{\mathcal{B}'}) \otimes -) \cap \mathcal{A}_\mathcal{P}^{\text{fp}}$$

of $\mathcal{A}_\mathcal{P}^{\text{fp}}$. We just proved that it contains a nonzero object, namely $\hat{H}_\mathcal{P}(M)$. By the "simplicity" of $\mathcal{A}_\mathcal{P}^{\text{fp}}$, we get that $\text{Ker}(H_\mathcal{P}(E_{\mathcal{B}'}) \otimes -) \cap \mathcal{A}^{\text{fp}}$ must be the whole of $\mathcal{A}_\mathcal{P}^{\text{fp}}$. This means that $H_\mathcal{P}(E_{\mathcal{B}'}) = 0$, or in other words, $\hat{E}_{\mathcal{B}'} \in \text{Ker}(\hat{H}_\mathcal{P}) = \langle \mathcal{B} \rangle = \text{Ker}(\hat{E}_\mathcal{B} \otimes -)$. We have thus proved

$$(5.9) \qquad\qquad\qquad \hat{E}_\mathcal{B} \otimes \hat{E}_{\mathcal{B}'} = 0.$$

Consider now the exact sequence in $\mathcal{A}$ associated to the morphism $\eta_\mathcal{B} : \mathbb{1} \to E_\mathcal{B}$

$$0 \to I_\mathcal{B} \to \hat{\mathbb{1}} \xrightarrow{\hat{\eta}_\mathcal{B}} \hat{E}_\mathcal{B}.$$

Since $E_\mathcal{B} \otimes \eta_\mathcal{B}$ is a split monomorphism, we have $I_\mathcal{B} \in \text{Ker}(\hat{E}_\mathcal{B} \otimes -) = \langle \mathcal{B} \rangle \subseteq \langle \mathcal{B}' \rangle$ and therefore $I_\mathcal{B} \otimes \hat{E}_{\mathcal{B}'} = 0$. Combined with (5.9) we see from the above exact sequence that $\hat{\mathbb{1}}$ is also killed by $\hat{E}_{\mathcal{B}'}$, that is $\hat{E}_{\mathcal{B}'} = 0$, or $\mathcal{B}' = \mathcal{A}^{\text{fp}}$ as claimed.

In summary, we have now shown that $\mathcal{B}(\mathcal{P}) = \text{Ker}(\hat{H}_\mathcal{P}) \cap \mathcal{A}^{\text{fp}}$ belongs to the homogeneous spectrum $\text{Spc}^{\text{h}}(\mathcal{K})$. We see that $\phi(\mathcal{B}(\mathcal{P})) = \mathcal{P}$, by the last assumption in (2). Finally, we need to relate the functor $H_\mathcal{P}$ with the homological residue field $\bar{h}_\mathcal{B}$ for $\mathcal{B} := \mathcal{B}(\mathcal{P})$. This is now easy. From (5.7) and the fact that $\langle \mathcal{B} \rangle = \text{Ker}(\hat{H}_\mathcal{P})$ we can further factor $\hat{H}_\mathcal{P}$ by first modding out this kernel. We obtain a unique

exact functor $\bar{H}_{\mathcal{P}} : \bar{A}(\mathcal{K}; \mathcal{B}) \to \mathcal{A}_{\mathcal{P}}$ making the right-hand triangle in the following diagram commute:

$$
\begin{array}{ccccc}
& & \overset{\bar{h}_{\mathcal{B}}}{\xrightarrow{\hspace{4cm}}} & & \\
\mathcal{T} & \overset{h}{\longrightarrow} \mathcal{A} & \overset{Q_{\mathcal{B}}}{\twoheadrightarrow} & \bar{A}(\mathcal{K}; \mathcal{B}) = \text{mod-}\mathcal{K}/\langle \mathcal{B} \rangle \\
& {\scriptstyle H_{\mathcal{P}}} \searrow \quad \downarrow {\scriptstyle \hat{H}_{\mathcal{P}}} & & \swarrow {\scriptstyle \bar{H}_{\mathcal{P}}} \\
& \mathcal{A}_{\mathcal{P}} & &
\end{array}
$$
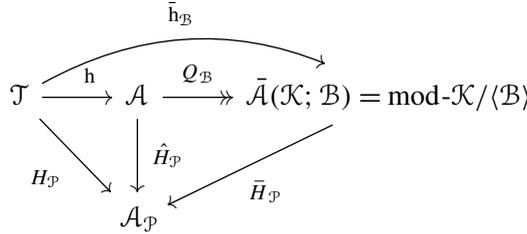
The left-hand triangle was already in (5.7). The top "triangle" commutes by definition of $\bar{h}_{\mathcal{B}} : \mathcal{T} \to \bar{A}(\mathcal{K}; \mathcal{B})$. Expanding the notation $\mathcal{B} = \mathcal{B}(\mathcal{P})$, we have factored each $H_{\mathcal{P}} : \mathcal{T} \to \mathcal{A}_{\mathcal{P}}$ via a homological residue field $\bar{h}_{\mathcal{B}(\mathcal{P})}$ as follows:

$$
H_{\mathcal{P}} = \bar{H}_{\mathcal{P}} \circ \bar{h}_{\mathcal{B}(\mathcal{P})}.
$$

We now claim that the family

$$
\mathcal{F} := \{\mathcal{B}(\mathcal{P}) \mid \mathcal{P} \in \text{Spc}(\mathcal{K})\}
$$

satisfies the hypothesis of Theorem 5.4, in other words that the family of functors $\{\bar{h}_{\mathcal{B}(\mathcal{P})}\}_{\mathcal{P} \in \text{Spc}(\mathcal{K})}$ detects $\otimes$-nilpotence of maps $f : x \to Y$ in $\mathcal{T}$ with $x \in \mathcal{K}$ compact. Indeed if $\bar{h}_{\mathcal{B}(\mathcal{P})}(f) = 0$ then $H_{\mathcal{P}}(f) = \bar{H}_{\mathcal{P}} \circ \bar{h}_{\mathcal{B}(\mathcal{P})}(f) = 0$ by the above factorization. If this holds for all $\mathcal{P}$, we conclude by (3) that $f^{\otimes n} = 0$ for $n \gg 1$. So Theorem 5.4 tells us that this family $\mathcal{F} = \{\mathcal{B}(\mathcal{P}) \mid \mathcal{P} \in \text{Spc}(\mathcal{K})\}$ is the whole $\text{Spc}^{\text{h}}(\mathcal{K})$.

In conclusion, we have constructed a set-theoretic section

$$
\sigma : \text{Spc}(\mathcal{K}) \to \text{Spc}^{\text{h}}(\mathcal{K}), \quad \mathcal{P} \mapsto \mathcal{B}(\mathcal{P})
$$

of $\phi : \text{Spc}^{\text{h}}(\mathcal{K}) \twoheadrightarrow \text{Spc}(\mathcal{K})$ and we just proved that $\text{im}(\sigma) = \mathcal{F} = \text{Spc}^{\text{h}}(\mathcal{K})$. In other words, the surjection ϕ admits a surjective section, i.e., ϕ is a bijection. $\square$

We can now use known nilpotence-detecting families in examples, to prove that $\phi : \text{Spc}^{\text{h}}(\mathcal{K}) \to \text{Spc}(\mathcal{K})$ is bijective.

5.10. Corollary. *Let $\mathcal{T} = \text{SH}$ be the stable homotopy category and $\mathcal{K} = \text{SH}^c$. Then $\phi : \text{Spc}^{\text{h}}(\mathcal{K}) \to \text{Spc}(\mathcal{K})$ is a bijection. More generally, let G be a compact Lie group and $\mathcal{T} = \text{SH}(G)$ the G-equivariant stable homotopy category of genuine G-spectra, and $\mathcal{K} = \text{SH}(G)^c$. Then $\phi : \text{Spc}^{\text{h}}(\mathcal{K}) \to \text{Spc}(\mathcal{K})$ is a bijection.*

Proof. In the case of $\mathcal{T} = \text{SH}$, this relies on [Devinatz et al. 1988; Hopkins and Smith 1998]. As explained in [Balmer 2010, §9], the spectrum consists of points $\mathcal{P}(p, n)$ for each prime number p and each "chromatic height" $1 \leq n \leq \infty$, with the collision $\mathcal{P}(0) := \mathcal{P}(p, 1) = \text{SH}^{\text{tor}}$ for all p. This prime $\mathcal{P}(0)$ is the kernel of rational homology $H\mathbb{Q} \otimes - : \text{SH}^c \to \text{D}^{\text{b}}(\mathbb{Q}) \cong \mathbb{Q}\text{-Mod}_{\bullet}$. The other primes $\mathcal{P}(p, n)$

for $n \geq 2$ are given as the kernels of Morava K-theory $K(p, n - 1)_\bullet$, which are homological functors

$$K(p, n)_\bullet : \mathrm{SH} \to \mathcal{A}_{p,n} := \mathbb{F}_p[v_n^{\pm 1}]\text{-Mod}_\bullet$$

for $1 \leq n < \infty$ with v_n of degree $2(p^n - 1)$, and $K(p, \infty)_\bullet : \mathrm{SH} \to \mathbb{F}_p\text{-Mod}_\bullet$ is mod-p homology. The target categories are graded modules over (graded) fields and the Morava K-theories satisfy Künneth formulas, which amounts to say that they are monoidal functors when $\mathcal{A}_{p,n}$ is equipped with the graded tensor product. See [Ravenel 1992]. The reader can now verify conditions (1)–(3) of Theorem 5.6. The crucial (3) is the original nilpotence theorem [Devinatz et al. 1988].

For $\mathcal{T} = \mathrm{SH}(G)$ and $\mathcal{K} = \mathrm{SH}(G)^c$, the description of the set $\mathrm{Spc}(\mathcal{K})$ and the relevant nilpotence theorem was achieved for finite groups in [Balmer and Sanders 2017], and more recently for arbitrary compact Lie groups in [Barthel et al. 2018]. Specifically, there is exactly one prime $\mathcal{P}(H, p, n) = (\Phi^H)^{-1}(\mathcal{P}(p, n))$ in $\mathrm{Spc}(\mathcal{K})$ for every conjugacy class of closed subgroups $H \leq G$ and for every "chromatic" prime $\mathcal{P}(p, n)$ as above; here $\Phi^H : \mathrm{SH}(G) \to \mathrm{SH}$ is the geometric H-fixed point functor, which is tensor-triangulated. The relevant homology theories are simply these Φ^H composed with the nonequivariant Morava K-theories. So conditions (1)–(2) in Theorem 5.6 are easy to verify. The relevant nilpotence theorem giving us (3) can be found in [Barthel et al. 2018, Theorem 3.12] (or [Balmer and Sanders 2017, Theorem 4.15] for finite groups, a result also obtained earlier by N. Strickland). □

5.11. Corollary. *Let X be a quasicompact and quasiseparated scheme and $\mathcal{T} = \mathrm{D}(X)$ the derived category of $\mathcal{O}_X$-modules with quasicoherent homology. Here $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ is the category of perfect complexes, the spectrum $\mathrm{Spc}(\mathcal{K}) \cong |X|$ is the underlying space of X and the map $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ is a bijection.*

Proof. The homological functors of Theorem 5.6 are simply the classical residue fields $\kappa(x) \otimes_{\mathcal{O}_X}^{\mathrm{L}} - : \mathrm{D}(X) \to \mathrm{D}(\kappa(x)) \cong \kappa(x)\text{-Mod}_\bullet$ at each $x \in |X|$, where of course $\kappa(x)$ is the residue fields of the local ring $\mathcal{O}_{X,x}$. Here, the relevant nilpotence theorem is due to Thomason [1997, Theorem 3.6]. □

5.12. Remark. There is a simpler proof of the above when X is noetherian, following the pattern of the next example. It is worth noting that when X is not noetherian, even for $|X| = *$, the residue fields do not detect vanishing of objects. See [Neeman 2000].

5.13. Example. Let G be a finite group scheme over a field k and $\mathcal{T} = \mathrm{Stab}(kG)$ the category of k-linear representations of G modulo projectives. See [Benson et al. 2018]. Here $\mathcal{K} = \mathrm{stab}(kG)$ is the stable category of finite-dimensional representations modulo projectives, $\mathrm{Spc}(\mathcal{K}) \cong \mathrm{Proj}(\mathrm{H}^\bullet(G, k))$ is the so-called projective support variety, and the map $\phi : \mathrm{Spc}^{\mathrm{h}}(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ is again bijective. The method

of proof is different, for there is no known homology theories capturing the points of $\mathrm{Spc}(\mathcal{K})$ *and satisfying a Künneth formula*. Indeed, points are detected by equivalence classes of so-called π-*points*, following [Friedlander and Pevtsova 2007], but these functors are *not* monoidal!

Instead, we can use the fact that localizing subcategories of $\mathcal{T}$ are classified by subsets of $\mathrm{Spc}(\mathcal{K})$ in this case, a nontrivial result that can be found in [Benson et al. 2018, §10]. In such situations, the property $E_{\mathcal{B}} \otimes E_{\mathcal{B}'} = 0$ isolated in Proposition 5.3, for $\mathcal{B} \neq \mathcal{B}'$ in $\mathrm{Spc}^h(\mathcal{K})$ can be used to show that $\phi : \mathrm{Spc}^h(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ is injective. Indeed, if $\phi(\mathcal{B}) = \phi(\mathcal{B}') =: \mathcal{P}$, we can use minimality of the localizing category $\mathcal{T}_{\mathcal{P}}$ supported at the point $\mathcal{P}$ to show that $E_{\mathcal{B}} \otimes E_{\mathcal{B}'} = 0$ forces $E_{\mathcal{B}} = 0$ or $E_{\mathcal{B}'} = 0$ which is absurd. This argument can already be found in [Balmer et al. 2019, Corollary 4.26].

5.14. Remark. We note that the above does *not* use a nilpotence theorem for $\mathcal{T} = \mathrm{Stab}(kG)$. To the best of the author's knowledge, there is no such result in the literature, the probable reason being that π-points (or shifted cyclic subgroups) are not monoidal. Thanks to the present work, we now know that there exists for every point $\mathcal{P} \in \mathrm{Spc}(\mathrm{stab}(kG)) \cong \mathrm{Proj}(\mathrm{H}^\bullet(G, k))$ a unique homological *tensor* functor

$$\bar{\mathrm{h}}_{\mathcal{B}(\mathcal{P})} : \mathrm{Stab}(kG) \to \bar{\mathcal{A}}(\mathcal{K}; \mathcal{B}(\mathcal{P}))$$

to a "simple" Grothendieck tensor category (in the sense of (1) in Theorem 5.6), whose kernel on compacts $\mathcal{K} = \mathcal{T}^c = \mathrm{stab}(kG)$ is exactly $\mathcal{P}$. And we know that the family $\{\bar{\mathrm{h}}_{\mathcal{B}(\mathcal{P})}\}_{\mathcal{P} \in \mathrm{Spc}(\mathcal{K})}$ detects tensor-nilpotence.

5.15. Remark. In view of the avalanche of examples where $\phi : \mathrm{Spc}^h(\mathcal{K}) \to \mathrm{Spc}(\mathcal{K})$ is a bijection, it takes nerves of steel not to conjecture that this property holds for all tensor-triangulated categories.

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