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The Markov sequence problem for the Jacobi polynomials and on the simplex

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The Markov sequence problem aims at the description of possible eigenvalues of symmetric Markov operators with some given orthonormal basis as eigenvector decomposition. A fundamental tool for their description is the hypergroup property. We first present the general Markov sequence problem and provide the classical examples, most of them associated with the classical families of orthogonal polynomials. We then concentrate on the hypergroup property, and provide a general method to obtain it, inspired by a fundamental work of Carlen, Geronimo and Loss. Using this technique and a few properties of diffusion operators having polynomial eigenvectors, we then provide a simplified proof of the hypergroup property for the Jacobi polynomials (Gasper's theorem) on the unit interval. We finally investigate various generalizations of this property for the family of Dirichlet laws on the simplex.

1. Introduction

In this paper, we are interested in the Markov sequence problem and the related hypergroup property, and concentrate in particular on Beta measures on the interval and on Dirichlet measures on the simplex.

The general Markov sequence problem may be stated as follows: given a unit orthonormal $\mathcal{L}^2(\mu)$ basis $\{f_0 = \mathbf{1}, f_1, \dots, f_n, \dots\}$ on some probability space $(E, \mathcal{E}, \mu)$, one aims at the description of all sequences (λ_n), such that the linear operator K defined through $K(f_n) = \lambda_n f_n$ is a Markov operator, that is satisfies $K(\mathbf{1}) = \mathbf{1}$ and is positivity preserving. Since the first property amounts to $\lambda_0 = 1$, the problem is reduced to studying the positivity preserving property.

This problem arises in many areas, particularly in statistics, special function theory, orthogonal polynomials theory and so on (see, among many others, [Bakry et al. 2014; Bakry and Zribi 2017; Bochner 1954; Carlen et al. 2011; Connett and Schwartz 1990; Gasper 1971; 1972; Lasser 1983; Sarmanov and Bratoeva 1967]).

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The aim of this paper is to describe these Markov sequences for the family of Jacobi polynomials and their extension to some families of polynomials in many variables on the simplex $\{x_i \geq 0; \sum_{i=1}^n x_i \leq 1\} \subset \mathbb{R}^n$, orthogonal for the Dirichlet measures

$$C_{p_1 \cdots p_{n+1}} x_1^{p_1/2-1} \cdots x_n^{p_n/2-1} (1 - x_1 - \cdots x_n)^{p_{n+1}/2-1}\, dx_1 \cdots dx_n,$$

where $p_i > 0$, $i = 1, \ldots, n+1$. (The choice for this parametrization will be explained below).

These Dirichlet measures again play an important rôle in many areas (statistics, probability, mathematical biology, etc., see, for example, [Balakrishnan 2003; Gelman et al. 2004; Letac 2012; Letac and Massam 1998]), and are natural generalizations of Beta measures on $(-1, 1)$, associated with the Jacobi polynomials. For the Beta measure, we shall revisit the fundamental result of Gasper through a method introduced by Carlen, Geronimo and Loss [Carlen et al. 2011], and our aim is to use this technique to propose some extensions to the Dirichlet measures.

The Markov sequence set shares some basic generic properties, whatever the space E and the basis $\mathcal{F}$. We refer to [Bakry and Huet 2008] for further details.

As we already mentioned, since $f_0 = \mathbf{1}$, $\lambda_0 = 1$. Moreover, it is easily seen that for any n, $|\lambda_n| \leq 1$.

The set of Markov sequences is a convex set (a convex combination of sequences corresponds to the same convex combination of the associated Markov operators), and is closed under pointwise convergence on the sequences. Therefore, through Choquet's representation theorem, the description of all Markov sequences amounts to the description of the extremal ones.

Moreover, it is also stable under pointwise multiplication (which corresponds to the composition of the associated Markov operators).

Let us mention a few classical results concerning the Markov sequence problem.

(1) Hermite polynomials. The Hermite polynomials are the orthogonal polynomials for the Gaussian measure on $\mathbb{R}$, that is $\mu(dx) = (1/\sqrt{2\pi})e^{-x^2/2}\, dx$. Sarmanov and Bratoeva [1967] proved that, for any Markov sequence, there exists a probability measure ν on $[-1, 1]$ such that $\lambda_n = \int_{-1}^1 x^n\, \nu(dx)$. In other words, the extremal Markov sequences are of the form $\lambda_n = e^{-nt}$ for some $t \geq 0$, or $(-1)^n e^{-nt}$, for some $t \geq 0$. The sequence (e^{-nt}) corresponds to a well known family of Markov operators K_t, namely the heat kernel associated with the Ornstein–Uhlenbeck operator. Indeed, $K_t = e^{tL}$, where $L(f)(x) = f'' - xf'$. This family of Markov kernels is known as the Ornstein–Uhlenbeck semigroup and there is a large literature devoted to it (see for example [Bakry et al. 2014; Gross 1975; 2006; Meyer 1982]). Moreover, the sequence $\lambda_n = (-1)^n$ corresponds to the symmetry $K(f)(x) = f(-x)$, so that those two operations generate all Markov sequences.

(2) Ultraspherical polynomials. The ultraspherical polynomials (P_n^α) form the family of orthogonal polynomials for $C_\alpha(1-x^2)^\alpha\, dx$, the ultraspherical probability measure on $(-1, 1)$, where $\alpha > -1$ is some real parameter and C_α the normalizing constant. Then, Bochner's theorem [1954] (see also [Bochner 1956; 1979; Lasser 1983]) asserts that a sequence (λ_n) is a Markov sequence for this basis if and only if there exists a probability measure ν on $(-1, 1)$ such that

$$\lambda_n = \int_{-1}^{1} \frac{P_n^\alpha(x)}{P_n^\alpha(1)}\, \nu(dx).$$

Indeed, at least formally, Sarmanov and Bratoeva's theorem may be deduced from Bochner's one, through a limiting procedure known as the Poincaré ansatz, that is considering the scaling of ultraspherical probability on $(-a, a)$ and letting a go to infinity. But the method followed in [Sarmanov and Bratoeva 1967] is completely different.

(3) Jacobi polynomials. Gasper's theorem [1970; 1971; 1972] concerns the Beta measures $C_{a,b}(1-x)^\alpha(1+x)^\beta\, dx$ on $(-1, 1)$, where $\alpha, \beta > -1$. As before, the basis is chosen to be the sequence of orthogonal polynomials for this measure, which are the Jacobi polynomials $P_n^{\alpha,\beta}$. Then, provided $\beta \ge \alpha \ge \frac{1}{2}$, a sequence (λ_n) is a Markov sequence for this family if and only if there exists a probability measure μ on $(-1, 1)$ such that, for any $n \in \mathbb{N}$,

$$\lambda_n = \int_{-1}^{1} \frac{P_n^{\alpha,\beta}(x)}{P_n^{\alpha,\beta}(1)}\, \nu(dx).$$

This example looks very close to the previous one, but is considerably more difficult. In Section 3 we shall come back to this result, which is central in our study.

(4) Eigenvectors of Sturm–Liouville operators. Another remarkable result in this direction is the Achour–Trimèche theorem, which may be stated as follows. Consider the interval $[-1, 1]$, and a probability measure μ on it, with a smooth density ρ, that we suppose bounded for simplicity ($0 < c \le \rho \le C < \infty$). Then, consider the diffusion operator $L(f) = f'' + \frac{\rho'}{\rho} f'$, which is symmetric in $\mathcal{L}^2(\mu)$. We choose as $\mathcal{L}^2(\mu)$ basis (f_n) the one formed by the eigenvectors of L with Neumann boundary condition, such that $f_0 = \mathbf{1}$. Then, provided that $\log \rho$ is concave and symmetric, for any Markov sequence (λ_n) associated with this family (f_n), there exists some probability measure ν on $(-1, 1)$ such that $\lambda_n = \int_{-1}^{1} f_n(x)/f_n(1)\, \nu(dx)$. Although not stated as presented here in [Achour and Trimèche 1979] or in the book [Bloom and Heyer 1995], one may find this result in [Bakry and Huet 2008].

This situation, where the extremal values for the Markov sequence problem are given by the values $f_n(x)/f_n(x_0)$ for some point x_0, appears in a number of situations. This property is described in [Bakry and Huet 2008], where it is called the hypergroup property at the point x_0, and is developed in Section 2. In particular, it is proven in [Bakry and Huet 2008] that, in the finite set case, the point x_0 must be of minimal mass for the measure μ. The sole exception in the above list is that of Hermite polynomials, which is in fact a degenerate case where the point x_0 is $+\infty$.

Although Gasper's result looks like a simple generalization of Bochner's one, which itself is a consequence of Achour and Trimèche's one, and contains as a limiting case the Hermite polynomial sequence, the proof of it is absolutely not straightforward. It has been considerably simplified by Carlen, Geronimo and Loss [Carlen et al. 2011] by a technique which we shall expose below in full generality, and is also used in [Bakry and Zribi 2017] for the corresponding question for the family of orthogonal polynomials associated to the A_2 root system. We provide here a further simplified proof of the proof of [Carlen et al. 2011]. It relies on the construction of some symmetric diffusion operator having polynomial eigenvectors in some 3 dimensional space.

Moreover, we study this Markov sequence problem for the most direct extensions of the Beta measures, which are the above mentioned Dirichlet measures on the simplex.

The paper is organized as follows. In Section 2, we introduce the hypergroup property, which is closely related to the Markov sequence problem. This is a property of some bases of $\mathcal{L}^2(\mu)$ which provides automatically the answer to the Markov sequence problem. In Section 3, we concentrate on the case of Jacobi polynomials, for which the hypergroup property holds true, thanks to Gasper's theorem. In particular, we present the Carlen–Geronimo–Loss method, which provides in the geometric case a simplified proof of Gasper's theorem. With the help of some basic results on diffusion processes with polynomial eigenvectors, we then provide a simplified proof of Gasper's theorem in the nongeometric situation, following the scheme of Carlen–Geronimo–Loss, and which avoids any tedious computation. Finally, in Section 4, we introduce the Dirichlet measure on the simplex, and the natural generalization of the Jacobi polynomials. Although the situation is much more complicated, and despite the fact that the hypergroup property is much harder to investigate, we provide some bases having the hypergroup property, and, for the generalized Jacobi polynomials, we provide a description of Markov sequences, but only for Markov operators which strongly commute with the operator for which these generalized Jacobi polynomials are eigenvectors.

2. The hypergroup property: general description

Hypergroups appear in the literature as a natural extension of the notion of locally compact groups, where the convolution of two Dirac masses is a probability measure and no longer a Dirac mass. For example, this happens naturally when one looks at the convolution of class functions in a group.

The hypergroup property (denoted HGP) as described in [Bakry and Huet 2008] is just a simplification of this theory, basically valid in the previous situation in the compact setting, and appears as a key tool in many subjects like probability, statistics, statistical mechanics, coding theory and algorithms, reversible Markov chain, etc., see [Bakry and Huet 2008].

The hypergroup property concerns some properties of a unit $\mathcal{L}^2(\mu)$ orthonormal basis on a probability space $(E, \mathcal{E}, \mu)$, which carries the answer to the Markov sequence problem, as in the above described examples. Consider indeed a probability space $(E, \mathcal{E}, \mu)$, where E is a topological space, $\mathcal{E}$ is the Borel σ-field, μ a probability measure. On this space is given an orthonormal basis $\mathcal{F} = (f_0, f_1, \dots, f_n, \dots)$ for $\mathcal{L}^2(\mu)$, where we suppose that $f_0 = \mathbf{1}$. For everything to make sense, we shall require that the functions f_n are continuous.

Then, as mentioned earlier, the Markov sequence problem aims at the description of all sequences (λ_n), with $\lambda_0 = 0$ such that the (unique) operator such $K(f_n) = \lambda_n f_n$ is a Markov operator, that is $K(\mathbf{1}) = \mathbf{1}$ and $f \geq 0 \Longrightarrow K(f) \geq 0$.

We already mentioned that the set of all Markov sequences is a compact set (under the pointwise convergence), and convex. Therefore, the description of all Markov sequences is reduced to the description of its extremal points.

Under very generic properties of the probability space, any Markov operator K may be represented as

$$K(f)(x) = \int f(y) K(x, dy),$$

where $K(x, dy)$ is a Markov transition kernel, that is, for each x, $K(x, \cdot)$ is a probability measure on E, and, for any $A \in \mathcal{E}$, $x \mapsto K(x, A)$ is measurable. Moreover, as soon as $\sum_n \lambda_n^2 < \infty$, then the operator is Hilbert–Schmidt, and the kernel $K(x, dy)$ has a density with respect to the measure μ, that is $K(x, dy) = k(x, y)\, \mu(dy)$, where

$$k(x, y) = \sum_n \lambda_n f_n(x) f_n(y),$$

where it is easily seen that the series converges in $\mathcal{L}^2(E^2, \mu \otimes \mu)$.

Then, as soon as $\lambda_0 = 1$ and $\sum_n \lambda_n^2 < \infty$, the Markov property amounts to checking that the function $k(x, y) = \sum_n \lambda_n f_n(x) f_n(y)$ is nonnegative. However, since every function f_n oscillates as soon as $n \geq 1$, since it satisfies $\int_E f_n(x) \mu(dx) = 0$, it is in general not at all easy to obtain this positivity property from the previous representation.

In [Bakry and Huet 2008], the semigroup property is introduced as follows:

Definition 2.1. The family $\mathcal{F}$ has the hypergroup property at the point x_0 if for any $x \in E$, the sequence $\lambda_n(x) = f_n(x)/f_n(x_0)$ is a Markov sequence.

The main consequence of [Bakry and Huet 2008], is that, when the hypergroup property holds at some point x_0, then the sequences $f_n(x)/f_n(x_0)$ form the set of extremal sequences, and therefore, in this situation, for any Markov operator K, there exists a probability measure ν_K on E such that

$$\lambda_n = \int_E \frac{f_n(x)}{f_n(x_0)}\, \nu_K(dx).$$

In the examples described in Section 1, this is the case for ultraspherical polynomials, for the Jacobi polynomials, and, for the basis of Neumann eigenvectors of Sturm–Liouville operators, as soon as the reference measure is log-concave and symmetric.

The hypergroup property may be restated (in some more or less formal way however) into the following: for any $(x, y, z) \in E^3$,

$$k(x, y, z) = \sum_i \frac{f_i(x) f_i(y) f_i(z)}{f_i(x_0)} \geq 0. \tag{2-1}$$

But it may happen that this series is not convergent in $\mathcal{L}^2(E^3, \mu \otimes \mu \otimes \mu)$, and that the formal measure $k(x, y, z)\, \mu(dz)$ is not even absolutely continuous with respect to the measure μ. Anyhow, one may describe, at least formally, the convolution $\mu_1 * \mu_2$ of two probability measures μ_1 and μ_2 as the measure μ_3 with density with respect to μ equal to $\int k(x, y, z)\, d\mu_1(x)\, d\mu_2(y)$, and then the measure $k(x, y, z)\, d\mu(z)$ appears as the convolutions of the Dirac masses in x and y. Then, again formally, one has

$$\int f_n(x)(\mu_1 * \mu_2)(dx) = \frac{1}{f_n(x_0)} \int f_n\, d\mu_1 \int f_n\, d\mu_2.$$

We can extend this convolution to all pairs of measures by bilinearity and from measures to functions by identifying f to the measure $f\, d\mu$. With this in mind, the link with the usual theory of hypergroups is easily done.

Another aspect of the 3 variable kernel $k(x, y, z)$ is that it allows some product formulas. Likewise, if we introduce the probability kernel

$$K(x, y, dz) = \sum_n \frac{f_n(x) f_n(y) f_n(z)}{f_n(x_0)} \mu(dz) = k(x, y, z)\mu(dz),$$

one may see that for each n, the function f_n satisfies the product formula

$$\frac{f_n(x) f_n(y)}{f_n(x_0)} = \int_E f_n(z) K(x, y, dz).$$

In practice, for all this to make sense, it is useful to have at disposal a family $\rho_n(t)$ of Markov sequences such that, for any $t > 0$, $\sum_n \rho_n^2(t) < \infty$, and which converges pointwise to 1 as $t \to 0$. Then, one applies all the previous formal computations to the Markov sequences $\rho_n(t) f_n(x)/f_n(x_0)$, and let t go to 0. In general, and in particular in the models studied below, this sequence $\rho_n(t)$ is provided by some adapted heat kernel.

An interesting aspect of the hypergroup property is its stability under tensorization. Namely,

Proposition 2.2. *Assume that $(E_1, \mathcal{E}_1, \mu_1)$ and $(E_2, \mathcal{E}_2, \mu_2)$ are two probability spaces on which there exist two unit orthonormal bases $(f_0 = \mathbf{1}, f_1, \dots, f_n, \dots)$ and $(g_0 = \mathbf{1}, g_1, \dots, g_p, \dots)$, satisfying the hypergroup property at points $x_0 \in E_1$ and $y_0 \in E_2$, respectively. Then, on the product space $(E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2, \mu_1 \otimes \mu_2)$, the unit orthonormal basis $(f_n(x)g_p(y), n, p \geq 0)$ satisfies the hypergroup property at the point (x_0, y_0).*

Proof. This is straightforward. If $K_1^x(x_1, dx_2)$ is a Markov kernel on E_1 with eigenvectors f_n associated with the eigenvalue $f_n(x)/f_n(x_0)$, and $K_2^y(y_1, dy_2)$ is a Markov kernel on E_2 with eigenvectors g_p associated with the eigenvalue $g_p(y)/g_p(y_0)$, then the product kernel $K_1^x \otimes K_2^y$ has eigenvectors $f_n(x_1)g_p(y_1)$ with associated eigenvalue $(f_n(x)/f_n(x_0))g_p(y)/g_p(y_0)$. □

Let us finally mention that this HGP property may be seen as the dual of the GKS property, named after Griffiths and Kelly and Sherman [1968], who described the so called GKS inequality in statistical mechanics, and assert that the product of two elements of the $\mathcal{L}^2(\mu)$ basis may be expressed as a linear combination of the elements of the basis with nonnegative coefficients (see [Bakry and Echerbault 1996]). However, we do not dispose at the moment of any efficient scheme similar to the one of [Carlen et al. 2011] to obtain this last property.

3. Gasper's theorem

3A. ***Jacobi Polynomials.*** As mentioned earlier, Gasper's theorem is the statement that the hypergroup property is valid for the family of Jacobi polynomials. One may find many proofs of it in the literature (see for example [Bakry and Huet 2008; Carlen et al. 2011; Connett and Schwartz 1990; Gasper 1970; 1971; 1972; Flensted-Jensen and Koornwinder 1979; Koornwinder 1974; 1977]). It plays an important role in many areas, even for example in the proof of Bieberbach conjecture, see [de Branges 1985].

As described in the introduction (and with a small change in the notation that will be justified later), the Beta measure $\beta_{p,q}(dx)$ on $(-1, 1)$ is defined as

$$\beta_{p,q}(dx) = C_{p,q}(1-x)^{\frac{1}{2}p-1}(1+x)^{\frac{1}{2}q-1}\,dx,$$

where p and q are positive and $C_{p,q}$ is the normalizing constant which makes $\beta_{p,q}$ a probability measure. In what follows, we find it convenient to move everything on (0, 1) through $x \mapsto \frac{1}{2}(1+x)$, so that the Beta measure is now, with another normalizing constant,

$$\beta_{p,q}(dx) = C_{p,q} x^{p/2-1}(1-x)^{q/2-1}\, dx.$$

The Jacobi polynomials are then defined as the unique family of orthogonal polynomials associated with $\beta_{p,q}$ and positive dominant coefficient. We shall denote by $P_n^{p,q}(x)$ the Jacobi polynomial of degree n.

The Jacobi polynomials are also the eigenvectors of the Jacobi operator on (0, 1)

$$J_{p,q} = x(1-x)\frac{d^2}{dx^2} + \Big[\frac{q}{2} - \Big(\frac{q+p}{2}\Big)x\Big]\frac{d}{dx} \tag{3-1}$$

with eigenvalue equal to $\lambda_n = -n\big(n + \frac{1}{2}(p+q) - 1\big)$, see [Bakry et al. 2014] for example. The specificity of these polynomials is that they represent the unique family of orthogonal polynomials in dimension 1 (together with their limiting cases, the Laguerre and Hermite polynomials) that are simultaneously the eigenvectors of diffusion operators, that is elliptic second order differential operators with no zero order terms (see [Bakry and Mazet 2003]).

Through a simple change of variables, $P_n^{p,q}(\cos^2(t))$ are the eigenvectors of the Sturm–Liouville operator

$$\frac{d^2}{dt^2} + ((q-1)\cot(t) - (p-1)\tan(t))\frac{d}{dt} \quad \text{on } [0,\pi],$$

with Neumann boundary condition, which is symmetric with respect of the measure $\sin^{q-1}(t)\cos^{p-1}(t)\, dt$.

Under this form, one may check that the density of the measure is log-concave as soon as $p, q > 1$, and is symmetric under the change $x \mapsto \pi - x$ whenever $p = q$. So that, after a translation of $-\pi/2$, the latter case enters in the scope of Achour–Trimèche theorem. However, this is not the case when $p \neq q$.

For this family, we have

Theorem 3.1 (Gasper). *Let $p, q > 0$. Then, the hypergroup property holds for the family of Jacobi polynomials at the point $x_0 = 1$ if and only if $q \geq p \geq 1$.*

As already mentioned in the introduction, Gasper's theorem is indeed an extension of a previous theorem due to Bochner [1954], which deals with the symmetric case $p = q$, that is the case of ultraspherical (or Gegenbauer) polynomials. However, although the arguments for the symmetric case are quite easy to follow, the proofs of Gasper's theorem remained quite complicated, up to the paper [Carlen et al. 2011], which provided an illuminating argument that we shall briefly recall below in Section 3B.

Moreover, in the case $p = q$, letting p go to ∞, scaling x to $x/\sqrt{p}$, then the measure $\mu_{p,p}$ converges to the Gaussian measure, the Jacobi polynomials converge to Hermite ones, and $\frac{2}{p} J_{p,p}$ converges to the Hermite operator. With this in mind, Sarmanov and Bratoeva's result may be seen again as a limiting case of Bochner's theorem.

In the Jacobi polynomials case, it is worth observing that the set of parameters for which the hypergroup property is valid is closed. Later on, Lemma 3.2 will allow us to restrict to cases where the auxiliary measures used in the proof have smooth densities.

Lemma 3.2. *If the hypergroup property for the Jacobi polynomials* $(P_n^{p_k,q_k})$ *holds true for a sequence* (p_k, q_k) *converging to* (p, q), *then it holds for* (p, q).

Proof. The family of orthogonal polynomials $P_n^{p,q}$ is obviously continuous in the parameters (p, q). The hypergroup property may be stated as the fact that the operator $K(x)$ with eigenvalues $P_n^{p,q}(x)/P_n^{p,q}(1)$ is positivity preserving. But this may be checked on polynomials, since any positive function may be approximated by positive polynomials, and any positive polynomial is a sum of squared polynomials. Therefore, it is enough to check that for any polynomial Q with degree K, one has $K(Q^2) \geq 0$.

But this translates into

$$K(Q^2)(y) = \int Q^2(z) \sum_{r=1}^{2k} \frac{P_r^{p,q}(x)}{P_r^{p,q}(1)} P_r^{p,q}(y) P_r^{p,q}(z)\, \mu_{p,q}(dz),$$

since Q^2 is orthogonal to $P_r^{p,q}$ for any $r > 2K$.

The polynomial Q being fixed, this property is obviously satisfied in the limit (p, q) as soon as it holds for a sequence (p_k, q_k). □

An important feature of the Jacobi operator is that, when p and q are integers, there is a natural interpretation of it through the unit sphere in dimension $p+q-1$. Then, the Jacobi operator (3-1) may be seen as an image of the spherical Laplace operator.

Indeed, if one considers the unit sphere $\mathbb{S}^{p+q-1} \subset \mathbb{R}^{p+q}$, there is a diffusion operator on it, namely the spherical Laplace operator $\Delta^{\mathbb{S}^{p+q-1}}$, which commutes to rotations and is unique up to scaling. If one considers the function

$$\mathbb{R}^{p+q} \to (0, 1), \quad \boldsymbol{x} = (x_1, \dots x_{p+q}) \mapsto y = \sum_{i=1}^{p} x_i^2,$$

one has, for any smooth function $f : (-1, 1) \to \mathbb{R}$,

$$\Delta^{\mathbb{S}^{p+q-1}}(f(y)) = 4J_{p,q}(f)(y). \tag{3-2}$$

As such, the Jacobi operator $J_{p,q}$ appears, as announced above, as an image of the spherical Laplace operator, and this remark is the key tool in the Carlen–Geronimo–Loss method to obtain the hypergroup property in this geometric case.

3B. ***The Carlen–Geronimo–Loss method.*** The Carlen–Geronimo–Loss scheme appears to be a quite general method to obtain the hypergroup property in various contexts (see for example [Bakry and Zribi 2017]).

Recall that we consider some probability space $(E, \mathcal{E}, \mu)$ on which we have a $\mathcal{L}^2(\mu)$ orthonormal basis $\mathcal{F} = (f_0 = 1, f_1, \dots, f_n, \dots)$. As before, in order for everything to make sense, we shall assume that E is a topological space, that $\mathcal{E}$ is the Borel sigma-algebra, and that all the functions f_i are continuous.

We assume that we have some dense linear subspace $\mathcal{A}$ in $\mathcal{L}^2(\mu)$, containing all the functions (f_n) of the basis $\mathcal{F}$, and a symmetric operator $\mathrm{L} : \mathcal{A} \to \mathcal{A}$. The basis $\mathcal{F}$ is formed of eigenvectors of L, that is $\mathrm{L}(f_n) = \rho_n f_n$, for some real sequence (ρ_n). In our example, $\mathcal{A}$ will be the space of polynomials.

We assume that there is an auxiliary topological space $(E_1, \mathcal{E}_1, \mu_1)$, endowed also with a dense subspace $\mathcal{A}_1 \subset \mathcal{L}^2(\mu_1)$, and another symmetric operator $\mathrm{L}_1 : \mathcal{A}_1 \to \mathcal{A}_1$. Moreover, there exists a continuous map $\pi : E_1 \to E$, and another continuous map $\phi : E_1 \to E_1$, with properties described in Theorem 3.3. We assume that the image of μ_1 under π is μ. For a function $f : E \to \mathbb{R}$, we denote by $\pi(f) : E_1 \to \mathbb{R}$ the function $\pi(f)(y) = f(\pi(y))$. Similarly, for a function $g : E_1 \to \mathbb{R}$, we denote $\phi(g)(y) = g(\phi(y))$. We also assume that $f \in \mathcal{A} \Longrightarrow \pi(f) \in \mathcal{A}_1$ and similarly $g \in \mathcal{A}_1 \Longrightarrow \phi(g) \in \mathcal{A}_1$.

Theorem 3.3. *Assume the following*:

(1) *For each n, the eigenspace of* L *associated with the eigenvalue ρ_n is one dimensional.*

(2) $\pi \mathrm{L} = \mathrm{L}_1 \pi$.

(3) $\phi \mathrm{L}_1 = \mathrm{L}_1 \phi$.

(4) *For two points x_0 and x in E, if Y is a random variable with values in E_1 with law μ_1, then the conditional law of $\pi(\phi(Y))$ given that $\pi(Y) = x_0$ is a Dirac mass at x.*

Then, the sequence $f_n(x)/f_n(x_0)$ is a Markov sequence for the basis (f_n). (If $f_n(x_0) = 0$, then the conclusion is that we also have $f_n(x) = 0$).

Remark 3.4. Point (4) requires a bit of explanation. Indeed, we assume that the probability measure μ_1 has a regular decomposition $\mu_1(dy) = \nu_x(dy)\, \mu(dx)$, where the measure $\nu_x(dy)$ has support the set $\pi(y) = x$, which means that, for any bounded measurable function $h : E_1 \to \mathbb{R}$,

$$\int_{E_1} h(y)\mu_1(dy) = \int_E \left(\int_{\{\mu(y)=x\}} h(\pi(y))\, \nu_x(dy) \right) \mu(dx),$$

and that the map $x \mapsto \nu_x$ is continuous. This allows us to make sense of ν_x for any $x \in E$ (since in general, those measures ν_x are just defined μ-almost everywhere). Then the hypothesis (4) asserts that the image measure through $\pi\phi$ of the measure ν_{x_0} is a Dirac mass δ_x.

Proof. Although the proof of this theorem is more or less implicit in [Carlen et al. 2011], and fully developed in [Bakry and Zribi 2017], we provide a sketch of it for completeness.

We denote $\langle f, g\rangle$ the scalar product in $\mathcal{L}^2(\mu)$ and $\langle f, g\rangle_1$ the scalar product in $\mathcal{L}^2(\mu_1)$.

We consider the correlation operator K defined on bounded Borel functions $f : E \to \mathbb{R}$ as

$$K(f)(x) = \mathbb{E}(\phi(\pi(f))(Y)/\pi(Y) = x),$$

where Y is a random variable with law μ_1. It is clearly a Markov operator. We shall see that $K(f_n) = \mu_n f_n$, where $\mu_n = f_n(x)/f_n(x_0)$.

The main remark is that the hypotheses imply that K commutes with L. Indeed, the operator K is entirely determined by the following property, which is just a rephrasing of the definition of a conditional expectation:

$$\text{for all } f, g \in \mathcal{A}, \quad \langle K(f), g\rangle = \langle \phi\pi(f), \pi g\rangle_1. \tag{3-3}$$

Indeed, using the measure decomposition introduced in Remark 3.4, one may introduces the operator π^*, such that

$$\pi^*(h)(x) = \mathbb{E}\big(h(Y)/\pi(Y) = x\big) = \int_{\{\pi(y)=x\}} h(y)\, \nu_x(dy),$$

the operator K may be written as $K = \pi^*\phi\pi$.

Then, for any pair $(f, g) \in \mathcal{A}$, we have

$$\begin{aligned}\langle \mathrm{L}K(f), g\rangle &= \langle K(f), \mathrm{L}g\rangle = \langle \phi\pi(f), \pi\mathrm{L}(g)\rangle_1 = \langle \phi\pi(f), \mathrm{L}_1\pi(g)\rangle_1 \\ &= \langle \mathrm{L}_1\phi\pi(f), \pi(g)\rangle_1 = \langle \phi\mathrm{L}_1\pi(f), \pi(g)\rangle_1 = \langle \phi\pi\mathrm{L}(f), \pi(g)\rangle_1 \\ &= \langle K\mathrm{L}(f), g\rangle,\end{aligned}$$

which proves the commutation property between K and L.

Therefore, if f_n is an eigenvector of L, with eigenvalue ρ_n, then $K(f_n)$ is again an eigenvector of L with the same eigenvalue. Since the eigenspaces of L are one dimensional, $K(f_n) = \mu_n f_n$ for some sequence (μ_n), which is therefore a Markov sequence.

Looking at the values at the point x_0, we get

$$f_n(x) = \mu_n f_n(x_0),$$

from which the conclusion follows. □

Corollary 3.5. *Under the hypothesis of Theorem 3.3, if, for any $x \in E$, there exists a map $\phi_x : E_1 \to E_1$ satisfying point (3) and such that the conditional law of $\pi\phi_x(Y)$ given $\pi(Y) = x_0$ is a Dirac mass at x, then the hypergroup property holds at x_0.*

Proof. It is an immediate consequence of Theorem 3.3. Indeed, if such happens, $f_n(x_0) \neq 0$, since otherwise one would get $f_n = 0$ everywhere, which may not be true for an element of a basis. □

With this in mind, Gasper's theorem in the geometric case follows easily. Of course, in this context, the auxiliary space E_1 is $\mathbb{S}^{p+q-1}$, L_1 is the spherical Laplace operator, and the map π is the map $\boldsymbol{x} \mapsto y = \sum_{i=1}^{p} x_i^2$ described in Section 3A.

The maps ϕ are as follows: since $p \le q$, for some point $\boldsymbol{x} = (x_1, \dots, x_{p+q}) \in \mathbb{R}^{p+q}$, we extract $\boldsymbol{x}_1 = (x_1, \dots, x_p)$, $\boldsymbol{x}_2 = (x_{p+1}, \dots, x_{2p})$ and $\boldsymbol{x}_3 = (x_{2p+1}, \dots, x_{p+q})$ (the last one may be empty). Then, for $\theta \in [0, 2\pi]$, $\phi_\theta(\boldsymbol{x}) = (\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{x}_3)$, where

$$\boldsymbol{y}_1 = \cos(\theta)\boldsymbol{x}_1 + \sin(\theta)\boldsymbol{x}_2, \quad \boldsymbol{y}_2 = -\sin(\theta)\boldsymbol{x}_1 + \cos(\theta)\boldsymbol{x}_2. \tag{3-4}$$

Then, $\boldsymbol{x} \mapsto \phi_\theta(\boldsymbol{x})$ is a rotation in $\mathbb{R}^{p+q}$, and as such commutes with the spherical Laplace operator.

Then, it remains to observe that whenever $\pi(\boldsymbol{x}) = 1$, then $\boldsymbol{x}_2 = \boldsymbol{x}_3 = 0$, so that $\pi(\phi_\theta(\boldsymbol{x})) = \cos^2(\theta)$. Then, the conditional law property is satisfied (with $x = \cos^2(\theta)$ and $x_0 = 1$), and therefore we obtain the hypergroup property in this case.

To extend this proof to the general case, we shall require a few concepts from the general diffusion theory.

3C. ***Symmetric diffusions and orthogonal polynomials.*** Most of the material presented here is borrowed from [Bakry et al. 2014] for the general situation, and from [Bakry et al. 2013] for the particular case where orthogonal polynomials come into play.

A diffusion operator in an open set $\Omega \subset \mathbb{R}^d$ is a second order semielliptic differential operator with no zero order terms. As such, it may be written in a given system of coordinates as

$$\mathrm{L}(f)(x) = \sum_{ij} g^{ij}(x)\partial^2_{ij} f + \sum_i b^i(x)\partial_i f, \tag{3-5}$$

where, here and in what follows, the coefficients $g^{ij}(x)$ and $b^i(x)$ are assumed to be smooth (indeed, for our purpose, they always will be polynomials in the variables

(x_i) which are the coordinates of the point x). The matrix $g = (g^{ij})$ is always symmetric and, in this paper, positive definite in Ω (that is our operator L is indeed elliptic).

We are interested in the case where these operators are symmetric with respect to some measure $\mu(dx)$, which has a smooth positive density $\rho(x)$ with respect to the Lebesgue measure. That is, for any pair (f, g) of smooth functions $\Omega \to \mathbb{R}$, compactly supported in Ω, we require that

$$\int_\Omega \mathrm{L}(f)(x)g(x)\rho(x)\,dx = \int_\Omega f(x)\mathrm{L}(g)(x)\rho(x)\,dx. \tag{3-6}$$

For this to happen, a necessary and sufficient condition is that

$$\text{for all } i = 1, \dots, d, \quad b^i(x) = \sum_j \partial_j g^{ij}(x) + \sum_j g^{ij}(x)\partial_j \log(\rho)(x), \tag{3-7}$$

since, by integration by parts

$$\int_\Omega \mathrm{L}(f)(x)g(x)\rho(x)\,dx = -\int_\Omega g^{ij}\partial_i f\,\partial_j g\rho\,dx + \int_\Omega g\partial_i f[b_i - r_i]\rho\,dx, \tag{3-8}$$

where $r_i(x) = \sum_j \partial_j g^{ij}(x) + \sum_j g^{ij}(x)\partial_j \log(\rho)(x)$.

Such a measure is often called a reversible measure. It is unique in general, up to a multiplicative constant.

We then see that the coefficients b^i are entirely determined by the second order terms g^{ij} and by the density $\rho(x)$.

Moreover, let us introduce the carré du champ

$$\Gamma(f, g) = \tfrac{1}{2}(\mathrm{L}(fg) - f\mathrm{L}(g) - g\mathrm{L}(f)).$$

We have

$$\Gamma(f, g) = \sum_{ij} g^{ij}(x)\partial_i f\,\partial_j g,$$

and this bilinear operator characterizes the second order terms (g^{ij}) of the operator L. We have $g^{ij}(x) = \Gamma(x_i, x_j)$, and, when the operator L is symmetric, for any pair of smooth compactly supported functions (f, g), we have

$$\int_\Omega \mathrm{L}(f)g\rho(x)\,dx = -\int_\Omega \Gamma(f, g)\rho(x)\,dx. \tag{3-9}$$

This is the integration by parts formula.

Moreover, the operator Γ allows us to describe the so-called "change of variable formula," which is a way to describe in a general setting second order differential operators with no zero order terms. More precisely, when $f_1, \dots, f_q$ are smooth functions $\Omega \to \mathbb{R}$, then, for any smooth function $\Phi : \mathbb{R}^q \to \mathbb{R}$, one has

$$\mathrm{L}(\Phi(f_1, \dots, f_q)) = \sum_{ij} \Gamma(f_i, f_j)\partial^2_{ij}\Phi(f_1, \dots, f_q) + \sum_i \mathrm{L}(f_i)\partial_i\Phi(f_1, \dots, f_q). \quad (3\text{-}10)$$

It is also worth observing that Γ is a bilinear operator which is first order in each of its variables, which translates into

$$\Gamma\big(\Phi_1(f_1, \dots, f_q), \Phi_2(f_1, \dots, f_q)\big) = \sum_{ij} \Gamma(f_i, f_j)\partial_i\Phi_1(f_1, \dots, f_q)\partial_j\Phi_2(f_1, \dots, f_q). \quad (3\text{-}11)$$

From this, one sees that in order to describe locally a symmetric diffusion operator, it is enough to describe in some coordinate basis $(x_1, \dots, x_d)$ the quantities $\Gamma(x_i, x_j)$ and either ρ, or the functions $\mathrm{L}(x_i) = b^i(x)$ provided they satisfy Equation (3-7) for some ρ.

It is not necessary to restrict diffusion operators to open sets in $\mathbb{R}^d$. One may as well consider operators defined on smooth manifolds (and quite often compact manifolds such as spheres), or closed sets with boundaries. Then, the operator may be described through Equation (3-5) in any local system of coordinates, and formula (3-10) allows one to change coordinates to obtain a coherent system. However, when considering such operators on manifolds with boundaries, one has in general to describe to which functions one may apply the integration by parts formula (3-9). This is done in general through the prescription of the so called "boundary conditions" (such as Neumann or Dirichlet). In what follows, we shall require the possibility to apply this formula to any polynomial (and even any restriction to Ω of any smooth function defined in a neighborhood of Ω), and this requires some extra conditions concerning the behavior of the matrix (g^{ij}) at the boundary. Indeed, the fundamental property for that (assuming that the boundary is piecewise smooth) is that, for any regular point x_0 of the boundary, the normal unit vector belongs to the kernel of the matrix (g^{ij}): in this situation, the extra term in the integration by parts formula (3-9), coming from the boundary term in Stokes formula, vanishes (see [Bakry et al. 2013], for example). It is easily seen that this condition is also sufficient.

This is what is hidden indeed in the boundary equation (3-12) below, which is the translation of this property when the boundary is described through some algebraic equation (see [Bakry et al. 2013]).

A key feature is the notion of image of a diffusion operator L_1 on some set E_1. This is the basic tool to construct new diffusion operators L on a set E and maps $\pi : E_1 \to E$ such that $\pi\mathrm{L} = \mathrm{L}_1\pi$, as in Theorem 3.3.

Let E_1 be some space on which we have a diffusion operator L_1 and d applications $x_1, \dots, x_d : E_1 \to \mathbb{R}$. Consider the map $\pi : E_1 \to E \subset \mathbb{R}^d$, $\pi(y) = (x_1, \dots, x_d)$. Then, assume that for any i, $\mathrm{L}_1(x_i) = B^i(\pi)$, and for any $i, j = 1, \dots, d$, one has $\Gamma(x_i, x_j) = G^{ij}(\pi)$ for some functions B^i and G^{ij}, $E \to \mathbb{R}$. We say in this situation that we have a closed system. Then, the operator

$$\mathrm{L} = \sum_{ij} G^{ij} \partial^2_{ij} + \sum_i B^i \partial_i$$

defined on E is such that $\mathrm{L}_1 \pi = \pi \mathrm{L}$ (this is just the translation of Equation (3-10)). Moreover, L is a diffusion operator which is symmetric as soon as L_1 is, with reversible measure which is the image through π of the reversible measure μ_1 of L_1. In this situation, we say that L is the image of L_1 through π, or that L_1 projects onto L through π. An example of this is the case of the spherical Laplace operator $\Delta^{\mathbb{S}^{p+q-1}}$ which projects (up to the factor 4) onto the Jacobi operator through the map $\boldsymbol{y} = (x_1, \dots, x_{p+q}) \mapsto x = \sum_i x_i^2$ as described in Equation (3-2), so that the Beta measure $\beta_{p,q}$ is the image measure of the uniform measure on the sphere through this projection.

As mentioned above, the symmetry identity (3-6) is not enough for our purpose. We shall require it to be valid for pair of polynomials, when the symmetry property is only stated for compactly supported functions. In what follows, we shall be concerned with symmetric diffusion operators which may be diagonalized in a basis of orthogonal polynomials. That is, for every $n \geq 0$, there exists a basis of the space of polynomials in d variables with degree less than n, and which are at the same time eigenvectors for L. When this happens, we say that (Ω, Γ, ρ) is a polynomial model, and Ω is a polynomial domain.

When the set Ω is bounded with a piecewise $\mathcal{C}^1$ boundary, this requires the boundary of Ω to be an algebraic set and also some extra algebraic condition relating the boundary and the coefficients g^{ij}, called the boundary equation, see [Bakry et al. 2013].

More precisely, the boundary $\partial\Omega$ is included in an algebraic set $\{P_1 \cdots P_k = 0\}$, where P_i are real polynomials, which are irreducible in the complex field. Here, we assume that $P_1 \cdots P_k = 0$ is the reduced equation of the boundary, that is:

(1) For each regular point $x \in \partial\Omega$, there exists a neighborhood $\mathcal{V}(x)$ which contains x and a unique i such that $\mathcal{V}(x) \cap \partial\Omega = \mathcal{V}(x) \cap \{P_i = 0\}$.

(2) For $i = 1, \dots, k$, there exist a regular point $x \in \partial\Omega$ such that $P_i(x) = 0$.

Then, following [Bakry et al. 2013], bounded polynomial models are characterized by the following:

(1) For any $i, j = 1, \dots, d$, $g^{ij}(x)$ is a polynomial with degree at most 2.

(2) For any $i = 1, \dots, d$, $b^i(x)$ is a polynomial with degree at most 1.

(3) For any $i = 1, \dots, d$ and any $q = 1, \dots, k$, there exists a polynomial $L_{i,q}$ with degree at most 1 such that

$$\sum_j g^{ij} \partial_j \log P_q = L_{i,q}. \tag{3-12}$$

(We call this last Equation (3-12) the boundary equation).

As a consequence of the previous, each polynomial P_q is a factor of the polynomial $\det(g^{ij})$ (of degree at most $2d$). Moreover, every polynomial P_q has a constant sign on the open set Ω and we may decide that they are all positive on it. Beyond this, provided (g^{ij}) satisfies the boundary equation (3-12), for any choice of parameters $a_1, \dots, a_k$ such that $P_1^{a_1} \cdots P_k^{a_k}$ is integrable on Ω, the density measure

$$\rho(x) = C_{a_1 \cdots a_k} P_1^{a_1} \cdots P_k^{a_k}, \tag{3-13}$$

where $C_{a_1 \cdots a_k}$ is the normalizing constant, is such that (Ω, Γ, ρ) is a polynomial model.

Indeed, for the integration by parts formula to be true for a pair of polynomial functions, and thanks to the boundary equation (3-12), one may allow the parameters a_i in Equation (3-13) to be negative, as soon as $a_i > -1$, which is anyway a necessary condition for the measure $\rho(x)\,dx$ to be finite on Ω.

Sometimes one needs to extend those polynomial models using weighted degrees, that is deciding that the degree of a monomial $x_1^{p_1} \cdots x_d^{p_d}$ is $\sum_i n_i p_i$, where $n_1, \dots, n_d$ are some positive integers. All the picture remains valid, except that g^{ij} must have degree $n_i + n_j$ and b^i must have degree n_i. We call the sequence $(n_1, \dots, n_d)$ the weights of the polynomial model.

It is worth observing that whenever (Ω, Γ, ρ) is a polynomial model, and when we have a closed system $(y_1, \dots, y_q)$ where the functions y_i are polynomials, then the image model is again a polynomial model. But the degree may change. For example, if one starts from a polynomial model with the usual degree (that is $n_i = 1$ for any i), and if the degree of y_i is n_i, then we get a polynomial degree with weights $n_1, \dots, n_d$. Of course, one may always reduce to the case where the degrees have no common factor.

3D. ***A proof of Gasper's theorem in the general case.*** In this section, we extend the proof of Gasper's theorem provided in Section 3B which was valid only in the geometric case (that is when p and q are integers) to the general case. For this, we need to construct a model $(E_1, \mathrm{L}_1, \mu_1)$, with the adapted functions $\pi : E_1 \to E$ and $\phi_\theta : E_1 \to E_1$ with the properties required in Theorem 3.3. The key observation is that, in the geometric picture, one just requires the knowledge of $\|\boldsymbol{x}_1\|^2$, $\|\boldsymbol{x}_2\|^2$ and the scalar product $\boldsymbol{x}_1 \cdot \boldsymbol{x}_2$ to describe the action of the rotations ϕ_θ on $\|\boldsymbol{x}_1\|^2$.

For this, we first observe the action of the spherical Laplace operator on those variables. Following [Bakry et al. 2014], the spherical Laplace operator in dimension d may be described through its action on the coordinates, that is considering the restrictions of the various coordinates $x_1, \dots, x_{d+1}$ to the spheres as functions $\mathbb{S}^d \to \mathbb{R}$. Then, we get

$$\Gamma^{\mathbb{S}}(x_i, x_j) = \delta_{ij} - x_i x_j, \quad \Delta^{\mathbb{S}^d}(x_i) = -d x_i. \tag{3-14}$$

It is worth observing that $\Gamma^{\mathbb{S}}$ does not depend on the dimension d. The image through $\Delta^{\mathbb{S}^d}$ of a polynomial in the variables x_i with degree less than n is again a polynomial in the variables x_i with degree less than n. From this, it is easily seen that whenever we have a closed system made of polynomials, then the image of $\Delta^{\mathbb{S}^d}$ through this system is a polynomial model.

Now fix d large enough and, for $p \le [d/2]$, consider the 3 variables $\mathbb{S}^d \to \mathbb{R}$ defined as

$$X = \sum_{i=1}^{p} x_i^2, \quad Y = \sum_{i=p+1}^{2p} x_i^2, \quad U = \sum_{i=1}^{p} x_i x_{i+p}.$$

With the help of the change of variables formulas (3-10) and (3-11), we get

$$\begin{gathered}
\Gamma^{\mathbb{S}}(X,X) = 4X(1-X), \quad \Gamma^{\mathbb{S}}(Y,Y) = 4Y(1-Y), \\
\Gamma^{\mathbb{S}}(U,U) = X + Y - 4U^2, \\
\Gamma^{\mathbb{S}}(X,Y) = -4XY, \quad \Gamma^{\mathbb{S}}(X,U) = -4XU + 2U, \\
\Gamma^{\mathbb{S}}(Y,U) = -4YU + 2U, \\
\Delta^{\mathbb{S}^d}(X) = -2(d+1)X + 2p, \quad \Delta^{\mathbb{S}^d}(Y) = -2(d+1)Y + 2p, \\
\Delta^{\mathbb{S}^d}(U) = -2(d+1)U,
\end{gathered} \tag{3-15}$$

which shows that the triple (X, Y, U) forms a closed system for the spherical Laplace operator. (We omit the parameter d in $\Gamma^{\mathbb{S}}$ since it does not depend on the dimension d.)

It is worth observing that X itself is a closed subsystem of this closed system (and the image of the spherical Laplace operator is nothing other than the Jacobi operator, up to some affine transformation on the variable and scaling). Such is $\{X, Y\}$, but neither $\{U\}$ or $\{X, U\}$, for example.

Let us consider the image of the sphere under $x \mapsto (X, Y, U)$. It is a polynomial domain in $\mathbb{R}^3$ with boundary equation $\{(1 - X - Y)(XY - U^2) = 0\}$.

The image of $\mathbb{S}^d$ through the map (X, Y, U) is therefore a polynomial model, with domain E_1 being the bounded set which is the connected component in $\mathbb{R}^3$ of the complement of the set $\{(1-X-Y)(XY-U^2)=0\}$ which contains for example the point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right)$. Observe that the boundary Equation (3-12) is automatically satisfied for this model. Indeed, since the spherical operator may be diagonalized in a basis of orthogonal polynomials in the variable (x_i) (the eigenvectors are the restrictions to the sphere of the harmonic homogeneous polynomials in dimension $d+1$), and one sees that the eigenvectors of this operator are nothing other than those polynomial eigenvectors which depend only on the variables X, Y, U.

The Γ operator is given in these coordinates by the matrix

$$G = (G^{ij}) := \begin{pmatrix} 4X(1-X) & -4XY & -4XU+2U \\ -4XY & 4Y(1-Y) & -4YU+2U \\ -4XU+2U & -4YU+2U & X+Y-4U^2 \end{pmatrix}, \tag{3-16}$$

and one may check (but, as already mentioned, this is automatic) that the two polynomials $1-X-Y$ and $XY-U^2$ satisfy the boundary equation (3-12). The reversible measure has density (up to a normalizing constant) $(1-X-Y)^a(XY-U^2)^b$, where the coefficients a and b may be computed through Equation (3-7). Then, we get

$$a = \frac{d-1}{2} - p, \quad b = \frac{p-3}{2},$$

Now, this diffusion operator again projects, up to a factor 4, on the Jacobi operator $J_{p,q}$ through the map $(X, Y, U) \mapsto X$, whenever $d = p+q-1$.

We may now consider this polynomial model (E_1, Γ) with a new measure with density $\rho(X, Y, U) = C(1-X-Y)^a(XY-U^2)^b$, where now a and b are real numbers.

It is easily seen that this measure is integrable on the domain E_1 as soon as $a > -1$ and $b > -1$. Setting $a = (q-p)/2 - 1$ and $b = (p-3)/2$, this requires $q > p > 1$, where now p and q are no longer integers but again real numbers.

As described in Section 3C, this provides a diffusion operator according to formula (3-5). The second order terms are provided by the matrix (3-16), and the first order coefficients may be computed explicitly through formula (3-7), with density $\rho = (1-X-Y)^a(XY-U^2)^b$ where, for given $q > p > 1$, we have $a = (q-p)/2-1$ and $b = (p-3)/2$.

More explicitly, one gets for the first order terms, exactly as in (3-15),

$$\begin{gathered} \mathrm{L}_1(X) = -2(p+q)X + 2p, \quad \mathrm{L}_1(Y) = -2(p+q)Y + 2p, \\ \mathrm{L}_1(U) = -2(p+q)U. \end{gathered} \tag{3-17}$$

The symmetry of the operators on a pair of polynomials is then insured by the fact that the first order coefficients b^i are chosen according to formula (3-7), and the fact that the boundary equation (3-12) is satisfied for the two factors $P_1(X, Y, U) = 1-X-Y$ and $P_2(X, Y, U) = XY-U^2$.

We get in such a way a model (E_1, Γ_1, μ_1) which projects through the map $\pi : (X, Y, U) \mapsto X$ on $4J_{pq}$, where J_{pq} is the Jacobi operator defined in Equation (3-1) (it is obvious: the variable X alone forms a closed system).

To complete the picture, it remains to describe the operators $\Phi_\theta : E_1 \to E_1$ which commute with L_1. From the geometric picture, when p and q are integers, one may describe the action of the rotations Φ_θ defined in Equation (3-4). We get $\Phi_\theta(X) = A(X, Y, U)$, where A is the linear operator with matrix

$$\begin{pmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\sin(\theta)\cos(\theta) \\ \sin^2(\theta) & \cos^2(\theta) & -2\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & \cos(\theta)\sin(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{pmatrix}. \tag{3-18}$$

To check that it commutes with L_1, and following Section 3C, it is enough to check its action on the variables X, Y, U for L_1 and Γ. For example, writing $\Phi_\theta(X, Y, U) = (X_\theta, Y_\theta, U_\theta)$, and $\Gamma(X, Y) = G(X, Y) := -4XY$, one has to check that $\Gamma(X_\theta, Y_\theta) = -4X_\theta Y_\theta$ (there are 6 such formulas to check), and also, with $\mathrm{L}_1(X) = -2(p+q)X + 2p$, that $\mathrm{L}_1(X_\theta)) = -2(p+q)X_\theta + 2p$ (3 formulas to check).

The property for Γ comes from the geometric picture (the action of Γ on (X, Y, U) does not depend on the parameters p and q). As for the action of L_1, it may be checked directly, from $X_\theta = \cos^2(\theta)X + \sin^2(\theta)Y + 2\sin(\theta)\cos(\theta)U$, using (3-17).

As before, the point x_0 is 1. Whenever $\pi(X, Y, U) = 1$, then $(X, Y, U) = (1, 0, 0)$ and $\pi\Phi_\theta(1, 0, 0) = \cos^2(\theta)$.

This completes the proof of Gasper's theorem in the case $q > p > 1$. The general case $q \geq p \geq 1$ comes from Lemma 3.2.

Remark 3.6. If one considers the kernel $K_\theta(f)(\xi) = \mathrm{E}\big(f(\pi(R_\theta Z))/\pi(Z) = \xi\big)$, the previous representation allows one to compute it explicitly through some integral expression. However, the result is quite complicated, but one may check that the kernel $K_\theta(\xi, dy)$ has support $[0, (\sqrt{\xi}\cos\theta + \sqrt{1-\xi}\sin\theta)^2]$.

4. Dirichlet laws and diffusion processes on the simplex

4A. ***Dirichlet laws, and a first basis with the HGP property.*** The d-dimensional simplex $\mathbb{D}_d$ is the set of points $(x_1, \dots, x_d) \in \mathbb{R}^d$ such that, for all $i = 1, \dots, d$, $x_i \geq 0$ and such that $\sum_{i=1}^d x_i \leq 1$. In what follows, it will be convenient to set $x_{d+1} = 1 - \sum_{i=1}^d x_i$, so that $x_{d+1} \geq 0$ and $\sum_1^{d+1} x_i = 1$.

The Dirichlet laws $\mu_{d,\boldsymbol{p}}$ depend on a multi-index real parameter $\boldsymbol{p} = \{p_1, \dots, p_{d+1}\}$, where $p_i > 0$, $i = 1, \dots, d+1$, are probability measures on $\mathbb{D}_d$ with densities with respect to the Lebesgue measure $dx_1 \cdots dx_d$ of the form

$$C_{d,\boldsymbol{p}} x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d} x_{d+1}^{a_{d+1}},$$

where for $i = 1, \dots, d+1$, $a_i = \frac{p_i}{2} - 1$. The normalizing constant

$$C_{d,\boldsymbol{p}} = \frac{\Gamma\left(\sum_{i=1}^{d+1} a_i + d + 1\right)}{\prod_{i=1}^{d+1} \Gamma(a_i + 1)},$$

where Γ is the Euler function, which ensures that $\mu_{d,\boldsymbol{p}}$ is a probability. The choice of the parameters p_i instead of $a_i = \frac{p_i}{2} - 1$, similar to the choice made for Beta measures, comes from geometric considerations which will be described below.

Dirichlet measures appear as extensions the Beta measures on the interval. It turns out that the simplex is a polynomial domain as described in Section 3C, so that the Dirichlet laws are the natural measures associated to it, the boundary of the domain having reduced equation $x_1 \cdots x_d(1 - x_1 - \cdots - x_d) = 0$.

When the parameters p_i are integers, this Dirichlet law is the image measure of the uniform measure on the unit sphere in $\mathbb{R}^n$, with $n = \sum_1^{d+1} p_i$. Indeed, consider some partition of $\{1, \dots, n\}$ in sets $I_1, \dots, I_{d+1}$ with respective size $p_1, \dots, p_{d+1}$. Then, for $(y_1, \dots, y_n) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, consider the variables $x_i = \sum_{j \in I_i} y_j^2$. Then $(x_1, \dots, x_d) \in \mathbb{D}_d$, and the image measure of the uniform measure on the sphere through the map $y \mapsto (x_1, \dots, x_d)$ is $\mu_{d,\boldsymbol{p}}$. This will be obvious later on when we shall identify some diffusion operator on $\mathbb{D}_d$ with reversible measure $\mu_{d,\boldsymbol{p}}$ as the image of the spherical Laplace operator, as are the Beta measures on $[0, 1]$.

It is worth observing that the change of variables $x_i \mapsto 1 - x_{d+1}$ allows one to exchange the parameters p_i and p_{d+1}, so that one may order the parameters p_i, $i = 1, \dots, d+1$, in whichever order desired.

The change of variables $x_i = y_i(1 - x_1)$, for $i = 2, \dots, d$ transforms the measure $\mu_{d,\boldsymbol{p}}$ into a product measure $\beta_{p_1, n-p_1}(dx_1) \otimes \mu_{d-1,\boldsymbol{q}}(dy_2 \cdots dy_d)$, where $n = \sum_1^{d+1} p_i$, and $\boldsymbol{q} = \{p_2, \dots, p_{d+1}\}$. Iterating the procedure, one may transform the Dirichlet measure into a product of Beta measures on $[0, 1]^d$:

$$\beta_{p_1, n-p_1} \otimes \beta_{p_2, n-p_1-p_2} \otimes \cdots \otimes \beta_{p_d, n-n_1-\cdots-p_d}.$$

We may now choose a basis for $\mathcal{L}^2(\mathbb{D}_d, \mu_{d,\boldsymbol{p}})$ made of products of Jacobi polynomials associated to each of the factors (to be more precise, the image of these products under the inverse change of variables which maps $[0, 1]^d$ to $\mathbb{D}_d$). Now, provided that, for $i = 1, \dots, d+1$, $p_i \geq 1$, one may apply Gasper's theorem and the tensorization procedure of Proposition 2.2, and therefore get the hypergroup property for this basis.

Observe that this procedure depends on the choice of the ordering in the parameters $p_1, \dots, p_{d+1}$, so that one may construct in this way many different bases. But these bases are not the most natural direct extensions of the Jacobi polynomial bases on the simplex. In particular, in the coordinates $(x_1, \dots, x_d)$, they do not appear as polynomials, but as rational functions. On the other hand, on the simplex and for the Dirichlet measures, there are many choices of polynomial bases which are the natural extensions of the Jacobi polynomials, as we shall see in the next paragraph.

4B. ***Diffusion operators on the simplex having polynomial eigenvectors.*** To describe the diffusion processes which may be diagonalized in a system of orthogonal polynomials on the simplex, we have just to describe their carré du champ Γ, since the measure is given. It is a special feature of the simplex that there are many such Γ structures which answer the question, beyond the mere scaling factor, and this situation is very peculiar (in the dimension 2 classification of [Bakry et al. 2013], only the simplex, the circle, and a particular case of the double parabola have this property).

The various Γ operators on the simplex such that $(\mathbb{D}_d, \Gamma, \mu_{d,\boldsymbol{p}})$ are a polynomial model have been described for example in [Li 2019]. They depend on a symmetric parameter matrix $\boldsymbol{A}$ with entries A_{rs} as follows

$$g^{rs} := \Gamma_{\boldsymbol{A}}(x_r, x_s) = -A_{rs}x_r x_s + \delta_{rs}x_r \sum_{k=1}^{d+1} A_{rk}x_k, \quad 1 \le r \le s \le d, \tag{4-1}$$

where $A_{rs} = A_{sr},\ 1 \le r \le s \le d+1$ are nonnegative real parameters. The operator is elliptic on the simplex as soon as, for every $r \neq s$, $A_{rs} \neq 0$. One should check that the value of A_{ii} plays no role in the definition of $\Gamma_{\boldsymbol{A}}$, and we shall set $A_{ii} = 0$.

For this operator, and for the Dirichlet measure $\mu_{d,\boldsymbol{p}}$, one has

$$\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}(x_i) = \frac{1}{2}\sum_{k=1}^{d+1} A_{ik}(x_k p_i - x_i p_k).$$

One may check the validity of the boundary Equation (3-12), that is the fact that $\sum_{i=1}^{d} g^{ij}\partial_j \log P_p$ is an affine function for every boundary polynomial $P_p = x_1, \dots, x_{d+1}$.

Indeed, for $k = 1, \dots, d+1$, one has

$$\sum_{j=1}^{d} g^{ij}\partial_j \log x_k = -A_{ik}x_i + \sum_{q=1}^{d+1} A_{iq}x_q.$$

It is worth it to write $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ as

$$\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}} = \sum_{i<j} A_{ij}\mathrm{L}_{ij,\boldsymbol{p}},$$

where $\mathrm{L}_{ij,\boldsymbol{p}}$ has a carré du champ Γ_{ij} with

$$\Gamma_{ij}(x_r, x_s) = x_i x_j[\delta_{rs}(\delta_{ri} + \delta_{rj}) - (\delta_{ri}\delta_{sj} + \delta_{rj}\delta_{si})] \tag{4-2}$$

and

$$\mathrm{L}_{ij,\boldsymbol{p}}(x_r) = \tfrac{1}{2}(\delta_{ri} - \delta_{rj})(x_j p_i - x_i p_j). \tag{4-3}$$

In the case where all the A_{pq} are set to 1 (let us denote this matrix $\mathbf{1}$), and when the parameters p_i are integers, there is a natural interpretation for this operator coming from the spherical Laplace operator in dimension $n = \sum_{i=1}^{d+1} p_i$, that is for the sphere imbedded in $\mathbb{R}^n$.

Indeed, let n be an integer and, as in the previous Section 4A, consider the $n-1$ dimensional spherical Laplace operator acting on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, defined through the equation $\sum_1^n y_i^2 = 1$. Let us look at a partition of the index set $\{1, \dots, n\}$ into $d+1$ disjoint sets $I_1, \dots, I_{d+1}$ with respective sizes $p_1, \dots, p_{d+1}$, and as before the variables $x_j = \sum_{i \in I_j} y_i^2$. As already observed, the map $y \in \mathbb{S}^{n-1} \mapsto (x_1, \dots, x_d)$ maps the sphere onto the simplex $\mathbb{D}_d$.

Moreover, following Equation (3-14), we see that

$$\Gamma^{\mathbb{S}}(x_i, x_j) = 4(\delta_{ij} x_i - x_i x_j), \quad \Delta^{\mathbb{S}^{n-1}}(x_i) = 2(p_i - n x_i). \tag{4-4}$$

The variables $(x_1, \dots, x_d)$ form a closed system, and we see that those formulas are the one obtained for $4\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$. This first shows that the Dirichlet measure $\mu_{d,\boldsymbol{p}}$ is the image of the uniform measure on the sphere through this map, as mentioned earlier. One may therefore address the question of the hypergroup property for the family of orthogonal polynomials which are the eigenvectors of this operator, following the same path. Unfortunately, it turns out that the eigenspaces for $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ are not one dimensional.

Indeed, consider a polynomial eigenvector of degree k, and look at the action of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ on its highest degree term $x_{\boldsymbol{k}} := x_1^{k_1} \cdots x_d^{k_d}$, where $k = \sum_1^d k_i$. The highest degree term of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}(x_{\boldsymbol{k}})$ is

$$-k\Big(k + \frac{n-2}{2}\Big) x_{\boldsymbol{k}},$$

so that the corresponding eigenvalue is $\nu_{\boldsymbol{k}} = -k\big(k + \frac{n-2}{2}\big)$, which depends only on $k = \sum_1^d k_i$. The corresponding eigenspace has then dimension $\binom{k+d-1}{k}$. However, for this operator, one may follow the scheme of [Carlen et al. 2011] and construct a new space E_1 (the sphere in the geometric case), with a symmetric diffusion operator L_1 on it, together with maps $\pi : E_1 \to \mathbb{D}_d$ and $\phi : E_1 \to E_1$ with the properties that $\pi \mathrm{L} = \mathrm{L}_1 \pi$, $\phi \mathrm{L}_1 = \mathrm{L}_1 \phi$, together with the conditional law property at the point $(1, 0, \dots, 0)$. But the fundamental property that the eigenspaces of L are one dimensional is missing, and the analysis of Markov sequences is therefore much more delicate.

Indeed, following the scheme of the proof of Gasper's theorem, one may first concentrate on the geometric case. To understand the difficulty, let us also concentrate on the case $d = 2$. In this situation, one has 3 integer parameters $p_1 \le p_2 \le p_3$, and, setting $n = p_1 + p_2 + p_3$, we look at the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Then, one considers three subsets I_1, I_2, I_3 of $\{1, \dots, n\}$, with respective sizes p_1, p_2, p_3 and three vectors $\boldsymbol{x}_1 = (y_i, i \in I_1)$, $z_2 = (y_i, i \in I_2)$ and $z_3 = (y_i, i \in I_3)$. Moreover, we split I_2 and I_3 into disjoint sets $I_2 = J_2 \cup K_2$, $I_3 = J_3 \cup K_3$, with $|J_2| = |J_3| = p_1$. Then, we consider the vectors $\boldsymbol{x}_2 = (y_i, i \in J_2)$, $\boldsymbol{y}_3 = (y_i, i \in J_3)$, $\boldsymbol{y}_2 = (y_i, i \in K_2)$ and $\boldsymbol{y}_3 = (y_i, i \in K_3)$.

We consider now the variables $x_i = \|\boldsymbol{x}_i\|^2$, $i = 1, 2, 3$, and $y_i = \|z_i\|^2$, $i = 2, 3$. Moreover, we look at the variables $u_{ij} = \boldsymbol{x}_i \cdot \boldsymbol{x}_j$, $1 \le i < j \le 3$. For simplicity, we stick to the case where $p_1 < p_2 \le p_3$, and, observing that $y_3 = 1 - x_1 - x_2 - x_3 - y_2$, we are left to the 7 variables

$$(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23}).$$

It happens that these 7 variables form a closed system for the spherical Laplace operator, and we obtain some operator L_7 on some bounded polynomial domain $\Omega_7 \subset \mathbb{R}^7$. Moreover, the operator $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ is the image of L_7 under the map

$$\pi_1 : \Omega_7 \to \mathbb{D}_2, \quad (x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23}) \mapsto (x_1, x_2 + y_2).$$

Let us denote by π_2 the projection from the sphere onto Ω_7, and $\pi : \mathbb{S}^{n-1} \to \mathbb{D}_2$, $\pi = \pi_1 \pi_2$.

One then may consider the full $O(3)$ group acting in a horizontal way on the triple of vectors $(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$. For example the plane rotations R_θ^{ij}, $1 \le i < j \le 3$:

$$R_\theta^{ij}(\boldsymbol{x}_i, \boldsymbol{x}_j) = (\cos\theta \boldsymbol{x}_i + \sin\theta \boldsymbol{x}_j, -\sin\theta \boldsymbol{x}_i + \cos\theta \boldsymbol{x}_j). \tag{4-5}$$

For any of these horizontal rotations R, there exists some point x_R in the simplex such that whenever $\pi(Y) = (1, 0)$, then $\pi R(Y) = x_R$ (that is $x_R = \pi R(1, 0, \dots, 0)$). One may see that for any point $x \in \mathbb{D}_d$, there exists such horizontal rotation $R \in \mathrm{SO}(3)$ such that $x_R = x$.

One may immediately see the action of these rotations on the variables

$$(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23}),$$

as we did in dimension 1.

In order to apply the one dimensional scheme, one may expect to find a common orthonormal base in the eigenspaces of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ in which the correlation operators $K_R(f)(x) = \mathbb{E}(\pi R f(Y)/\pi(Y) = x)$, where Y is uniformly distributed on the sphere, are jointly diagonalizable. (Observe that $R \mapsto K_R$ is not a representation of $O(3)$.) We shall see that it is impossible. Indeed, if such were the case, they would commute with each other. But this is not the case, as shown next in Proposition 4.1. For this, we just concentrate on the plane rotations R_θ^{ij} (4-5) and their conditional expectations $K_\theta^{ij}(f)(x) = \mathbb{E}(\pi R_\theta^{ij} f(Y)/\pi(Y) = x)$.

Proposition 4.1. *The operators K_θ^{12} and K_ϕ^{13} do not commute with each other.*

Proof. The operators K_θ^{ij} are not easy to describe. We may look at the easier operators $S_{ij} = \partial_\theta K^{ij}_{|\theta=0}$ But we shall see that those operators vanish identically. We may therefore compute $R_{ij} = \partial_\theta^2 K^{ij}_{|\theta=0}$.

To compute these operators S_{ij} and K_{ij} on the simplex, for the pairs $(1,2)$, $(1,3)$, $(2,3)$, we observe that for two bounded polynomial functions $f(x,y)$ and $g(x,y)$ on $\mathbb{D}_2$, up to a constant 2, we have

$$\langle S_{12}(f), g\rangle = 2\int_{\mathbb{S}^{n-1}} u_{12}(\partial_1 f - \partial_2 f)(\pi(\mathbf{y}))g(\pi\mathbf{y})\,d\mathbf{y},$$

where $\pi(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23}) = (x_1, x_2+y_2)$. Thus

$$S_{12}(f) = 2s_{12}(x)(\partial_i f - \partial_j f), \quad \text{where } s_{12}(x) = \mathbb{E}\big(u_{12}(\mathbf{y})/\pi(\mathbf{y}) = (x_1, x_2+y_2)\big),$$

which is 0 by symmetry, and

$$\langle K_{12} f, g\rangle = \int_{\mathbb{S}^{n-1}} \big(2(x_2-x_1)(\partial_1 f - \partial_2 f) + 4u_{12}^2(\partial_1-\partial_2)^2 f\big)\pi(\mathbf{y})g(\pi\mathbf{y})\,d\mathbf{y}.$$

Thus

$$K_{12}(f) = 2k_{12}(\partial_1-\partial_2)f + 4t_{12}(\partial_1-\partial_2)^2 f,$$

where

$$k_{12}(x,y) = \mathbb{E}\big(x_1 - x_2/(x_1, x_2+y_2) = (x,y)\big),$$
$$t_{12}(x,y) = \mathbb{E}\big(u_{12}^2(Y)/\pi(Y) = (x_1, x_2+y_2) = (x,y)\big).$$

For the operators S_{13} and K_{13}, we may perform a similar computation, and obtain a similar computation:

$$K_{13}(f) = 2k_{13}(\partial_1 f - \partial_2 f) + 4t_{13}(\partial_1-\partial_2)^2 f,$$

with

$$k_{13}(x,y) = \mathbb{E}\big(x_1 - x_3/(x_1, x_2+y_2) = (x,y)\big),$$
$$t_{13}(x,y) = \mathbb{E}\big(u_{13}^2(Y)/(x_1, x_2+y_2) = (x,y)\big),$$

and for K_{23}, we obtain

$$K_{23}(f) = 2k_{23}\partial_2 f + 4t_{23}\partial_2^2 f,$$

with

$$k_{23}(x,y) = \mathbb{E}\big(x_2 - x_3/(x_1, x_2+y_2) = (x,y)\big),$$
$$t_{23}(x,y) = \mathbb{E}\big(u_{23}^2/(x_1, x_2+y_2) = (x,y)\big),$$

It remains to compute these conditional laws.

Following the computations of Section 3D, we may compute the law of the set of variables $(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23})$ under the uniform measure on the sphere through the action of the spherical Laplace operator $\Delta_{\mathbb{S}^{n-1}}$ on these variables. The Gamma operator acts on the variables as

$$\Gamma(x_p, x_q) = 4x_p(\delta_{pq} - x_q), \quad \Gamma(x_i, y_2) = -4x_i y_2, \quad \Gamma(y_2, y_2) = 4y_2(1-y_2),$$

while

$$\Gamma(y_2, u_{ij}) = -4y_2u_{ij}, \quad \Gamma(x_i, u_{lk}) = -4x_iu_{kl} + 2\delta_{il}u_{ik} + 2\delta_{ik}u_{il},$$
$$\Gamma(u_{ij}, u_{kl}) = -4u_{ij}u_{kl} + \delta_{ik}u_{jl} + \delta_{il}u_{jk} + \delta_{jk}u_{il} + \delta_{jl}u_{ik}.$$

where, in the last formulas, u_{ii} stands for x_i. Moreover, with $n = p_1 + p_2 + p_3$, we have

$$\Delta_{\mathbb{S}^{n-1}}(x_i) = -2nx_i + 2p_1, \quad \Delta_{\mathbb{S}^{n-1}}(y_2) = -2ny_i + 2(p_2 - p_1),$$
$$\Delta_{\mathbb{S}^{n-1}}(u_{ij}) = -2nu_{ij}.$$

Then, the image measure of the sphere is the reversible measure for this operator, that we compute through Equation (3-7). Up to some normalizing constant, we may compute the density through formula (3-7). In order to compute this density with respect to the product measure $dx_1\, dx_2\, dx_3\, dy_1\, du_{12}\, du_{13}\, du_{23}$, we introduce

$$F_1 = x_1x_2x_3 + 2u_{12}u_{13}u_{23} - x_1u_{23}^2 - x_2u_{13}^2 - x_3u_{12}^2,$$
$$F_2 = 1 - x_1 - x_2 - x_3 - y_2$$

Observe that F_1 is the determinant of the Gram matrix associated with the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$.

Rewriting the variables $(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23})$ as $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in this order, (to have a more compact presentation of what follows), we get, with $G_{ij} = \frac{1}{4}\Gamma(x_i, x_j)$,

$$\begin{aligned}
\textstyle\sum_j \partial_j G_{ij} &= 2 - 8x_i, && i = 1, 2, 3,\\
\textstyle\sum_j \partial_j G_{4j} &= 1 - 8x_4, &&\\
\textstyle\sum_j \partial_j G_{ij} &= -8x_i, && i = 5, 6, 7,\\
\textstyle\sum_j G_{ij}\partial_j \log F_1 &= 1 - 3x_i, && i = 1, 2, 3,\\
\textstyle\sum_j G_{ij}\partial_j \log F_1 &= -3x_i, && i = 4, 5, 6, 7,\\
\textstyle\sum_i G_{ij}\partial_j \log F_2 &= -x_i, && i = 1, \dots, 7,\\
\textstyle\sum_i G_{ij}\partial_j \log x_4 &= -x_i + \delta_{i4}, && i = 1, \dots, 7.
\end{aligned}$$

In the end, through formula (3-7), we are able to compute the density of the measure, which is, up to some normalizing constant

$$\rho = F_1^{\alpha} F_2^{\beta} y_2^{\gamma},$$

with

$$\alpha = \frac{p_1}{2} - 2, \quad \beta = \frac{n - p_2}{2} - p_1 - 1, \quad \gamma = \frac{p_2 - p_1}{2} - 1.$$

Observe that the equation $F_1F_2y_2 = 0$ is indeed the reduced equation of the set Ω_7.

To compute the conditional law, it is worthwhile to change variables in order to transform the measure $\rho(x)\,dx$ into a product measure. For this, we set

$$u_{ij} = \sqrt{x_i x_j}\sigma_{ij}, \quad y_2 = z - x_2, x_2 = uz, \quad x_3 = v(1 - x_1 - z),$$

so that the measure becomes a product measure, of the form

$$\mu(dx_1, dz)\beta_1(du)\beta_2(dv)\gamma(d\sigma_{12}, d\sigma_{23}, d\sigma_{13}),$$

where μ is, as expected, the Dirichlet law in dimension 2, $\mu_{2,(p_1,p_2)}$.

With this in mind, it is easy to check that we have

$$\begin{aligned} k_{12} &= 2(x - a_1 y), & t_{12} &= b_1 xy, \\ k_{13} &= 2(x - a_2(1 - x - y)), & t_{13} &= b_2 x(1 - x - y), \\ k_{23} &= 2(a_3 y - a_4(1 - x - y)), & t_{23} &= b_3 y(1 - x - y), \end{aligned}$$

for some constants a_i, b_j that we are not going to identify directly, but where we may assert that $b_i > 0$, for example. (Indeed, knowing that those differential operators K_{ij} must commute with $\mathrm{L}_{2,\boldsymbol{p}}$ allows one to compute them up to some constant.)

Now, if one wants to see that these operators do not commute, we may look at $\big[\frac{1}{b_1}K_{12}, \frac{1}{b_3}K_{13}\big]$, for example. This is a third order operator whose leading term is $2(1 - x - y)(x - y)(\partial_1 - \partial_2)^3$, which clearly does not vanish. □

Remark 4.2. For any horizontal rotation R, the associated kernel

$$K_R(f)(x) = \mathbb{E}\big(f(\pi(R\boldsymbol{x}))/\pi(\boldsymbol{x}) = x\big)$$

leaves invariant all the eigenspaces of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$. But the question of their action on this space remains completely open. In particular, one may ask if any Markov operator which commutes with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ is a mixture of such conditional expectations of rotations K_R.

We now concentrate on the operators $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$. We shall show that in the generic case (that is for some dense set for the parameters A_{ij} and p_i), their eigenspaces are one dimensional.

There is still a geometric interpretation for them, in the geometric case $p_i \in \mathbb{N}$, as we shall see below. And this geometric interpretation allows us to use the same space E_1 with the projection $\pi : E_1 \to \mathbb{D}_d$, which may be extended to the general case $p_i \notin \mathbb{N}$ as we did in Section 3D. But the problem now is that the horizontal rotations do not commute with the lift of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ to the geometric model. Therefore, we may not apply the Carlen–Geronimo–Loss scheme to them.

The geometric interpretation of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ that we present now is inspired from [Li 2019], where a similar interpretation is carried out for the matrix simplex. In $\mathbb{R}^n$, consider the infinitesimal rotations in the coordinate plane (i, j), $D_{ij} = y_i\partial_j - y_j\partial_i$.

Consider now as before a partition $\{I_1, \dots, I_{d+1}\}$ of the set $\{1, \dots, n\}$, where $|I_i| = p_i$. For $i \neq j$ consider the following second order diffusion operator on the sphere $\mathbb{S}^{n-1}$:

$$\Delta_{ij} = \sum_{p\in I_i,\, q\in I_j} D^2_{pq}.$$

The action of Δ_{ij}, and its associated carré du champ Γ_{ij} on the variables $x_r = \sum_{p\in I_r} y_p^2$ and $x_s = \sum_{p\in I_s} y_p^2$ is as follows.

Proposition 4.3.
$$\Gamma_{ij}(x_r, x_s) = 4[\delta_{rs}x_i x_j(\delta_{ri}+\delta_{rj}) - (\delta_{ri}\delta_{sj}+\delta_{rj}\delta_{si})x_r x_s],$$
$$\Delta_{ij}(x_r) = 2(\delta_{ir}-\delta_{jr})(x_j p_i - x_i p_j).$$

Proof. We start by the computation of this action on the variables $y_p, y_q : \mathbb{S}^{n-1} \to \mathbb{R}$.

$$\begin{aligned} \Delta_{i,j}(y_p) &= -y_p(\mathbf{1}_{p\in I_j} p_i + \mathbf{1}_{p\in I_i} p_j),\\ \Gamma_{i,j}(y_p, y_q) &= \delta_{pq}(\mathbf{1}_{p\in I_i}x_j + \mathbf{1}_{p\in I_j}x_i) - y_p y_q(\mathbf{1}_{p\in I_i}\mathbf{1}_{q\in I_j} + \mathbf{1}_{p\in I_j}\mathbf{1}_{q\in I_i}), \end{aligned} \tag{4-6}$$

where $\mathbf{1}_{p\in A}$ stands for $1_A(p)$, the indicator function of the set A. From this, using the change of variable formula (3-11), we get

$$\Gamma_{i,j}(x_p, x_q) = 4x_i x_j[\delta_{pq}(\delta_{pi}+\delta_{pj}) - (\delta_{pi}\delta_{qj}+\delta_{pj}\delta_{qi})].$$

In the same way, we obtain the formula for $\Delta_{ij}(x_r)$ using formula (3-10). □

As a corollary, and comparing with formulae (4-2) and (4-3), we get:

Corollary 4.4. *The operator* $4\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ *is the image of the operator* $\sum_{i<j} A_{ij}\Delta_{ij}$ *through the map* $y \mapsto (x_1, \dots, x_d)$ *which maps* $\mathbb{S}^{n-1}$ *onto* $\mathbb{D}_d$, *where* $n = \sum_{i=1}^{d+1} p_i$.

Remark 4.5. In view of Equation (4-4), it is worth observing that the spherical Laplace operator may be written as $\sum_{i\leq j}\Delta_{ij}$. Therefore, comparing with Corollary 4.4, we see that what is missing is the operator $\sum_i \Delta_{ii}$, where

$$\Delta_{ii} = \sum_{p<q,\, p\in I_i,\, q\in I_i} D^2_{pq}.$$

But it is easily seen that the action of Δ_{ii} on the variables x_p vanishes: $\Gamma_{ii}(x_p, x_q) = \Delta_{ii}(x_p) = 0$.

It is also worth observing that one may split some subset I_i into two subsets I_{i_1} and I_{i_2}. More precisely, suppose that we have a partition $\{I_1, \dots, I_{d+1})$ of $\{1, \dots, n\}$ and that we split say I_1 into two disjoint sets $I_{1a} \cup I_{1b}$. Then we may consider a new operator on $\mathbb{D}_{d+1}$ $\mathrm{L}_{\boldsymbol{A}_1,\boldsymbol{a}_1}$, for some matrix $\boldsymbol{A}_1$ and some vector $\boldsymbol{a}_1$. Then, provided that for any $j > 1$, $A_{1a,j} = A_{1b,j} = A_{1j}$, the image of $\mathrm{L}_{\boldsymbol{A}_1,\boldsymbol{a}_1}$ on $\mathbb{D}_d$ under the map $(x_{1a}, x_{1b}, x_2, \dots, x_d) \mapsto (x_{1a}+x_{1b}, x_2, \dots, x_d)$ is $\mathrm{L}_{\boldsymbol{A},\boldsymbol{a}}$, where $\boldsymbol{a} = (a_{1a}+a_{1b}, a_2, \dots, a_d)$.

Of course, the same reasoning applies for any parameter i instead of 1.

For the sake of completeness, we show below that the eigenspaces of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ have dimension 1 in the generic case.

Proposition 4.6. *For a dense set for the parameters A_{ij} and p_i, the eigenspaces of the operator $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ are one dimensional.*

Proof. Since the space $\mathcal{P}_n$ of polynomials with total degree n is preserved by $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$, one may concentrate on its action on $\mathcal{P}_n$. To understand the eigenvalues of this restriction, which do not come from the restriction to $\mathcal{H}_{n-1}$, it is enough to look at the restriction of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ to homogeneous polynomial of degree n, and consider for such polynomial P, the degree-n homogeneous part of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}(P)$.

Then, the eigenvalues of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ are the eigenvalues of this linear operator, represented by some matrix $M_{n,\boldsymbol{A},\boldsymbol{p}}$ in the natural basis of these homogeneous polynomials $e_{k_1,\dots,k_d} = \{x_1^{k_1}\cdots x_d^{k_d},\ \sum_i k_i = n\}$. We shall see that for each n, there exists a dense subset Ω_n of parameters (even with a complementary with Lebesgue measure 0) such that the eigenvalues of $M_{n\boldsymbol{A},\boldsymbol{p}}$ are all distinct for this parameters. Then, on $\bigcap_n \Omega_n$, which is dense by Baire's theorem, all the eigenvalues of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ are distinct.

To assert that the eigenvalues of $M_{n,\boldsymbol{A},\boldsymbol{p}}$ are distinct, it is enough to check that the characteristic polynomial has distinct roots, or in other words that its discriminant does not vanish. But the discriminant is a polynomial in the coefficients of the characteristic polynomial, which themselves are polynomials in the entries of the matrix, which themselves are polynomials in the variables A_{ij} and p_i. Therefore, there exists some polynomial Q in the variables A_{ij}, p_i, depending on the degree n, such that, if $Q \neq 0$, all the eigenvalues of $M_{n,\boldsymbol{A},\boldsymbol{p}}$ are distinct.

It remains to show that Q does not vanish identically, that is that there exists some choice of the parameters A_{ij} and p_i for which the eigenvalues are distinct.

Let us choose the matrix A_{ij} such that $A_{ij} = A_{i(d+1)}$ for $j > i$. Then, if we order the elements of the basis $\{e_{k_1,\dots,k_d},\ \sum_1^d k_i = n\}$ according to their inverse lexicographic order of $(k_1,\dots,k_{d-1})$ (so that $(n,\dots,0,0)$ is the lowest term), then one may check that all the elements of $M_{n,\boldsymbol{A},\boldsymbol{p}}$ which are above the diagonal vanish. Then, the eigenvalues of $M_{n,\boldsymbol{A},\boldsymbol{p}}$ are the diagonal elements. On the diagonal, the coefficient corresponding to $e_{k_1,\dots,k_d}$ is

$$-\sum_{i\neq j} k_i k_j A_{ij} - \sum_i k_i(k_i-1)A_{i,d+1} + \tfrac{1}{2}\sum_i k_i\Big(A_{i,d+1}p_i - \sum_{k=1}^{d+1} A_{ik}p_k\Big).$$

With the choice that we made, for $i \neq j$, $A_{i,j} = a_{\min(i,j)}$ for some sequence a_i, $i = 1,\dots,d$. Then, it is not hard to see that there exists a choice for the sequences a_i, $i = 1,\dots,d$ and p_i, $i = 1,\dots,d+1$ for which all these terms are different, for all the sequences of integers $(k_1,\dots,k_d)$ such that $\sum_1^d k_i = n$. $\square$

4C. ***Representations of Markov sequences.*** In what follows, we restrict ourselves to the case where all the coefficients A_{ij}, $i \neq j$ are set to 1. Since the eigenspaces E_n are not one dimensional, we also restrict our attention to the study of Markov operators which have constant eigenvalues on the space E_n. That is, instead of looking at Markov operators which commute with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$, we look at Markov operators which are functions of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$. We say that such a Markov operator strongly commutes with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$

Observe first that, for any choice of a strict subset $I \subset \{1, \dots, d+1\}$, the projection $\pi : \mathbb{D}_d \to [0, 1]$, $\pi(x) = \sum_{i \in I} x_i$ maps the Dirichlet law $\mu_{d,\boldsymbol{p}}$ on the Beta measure $\beta_{q,n-q}$, where $q = \sum_{i \in I} p_i$ and $n = \sum_1^{d+1} p_i$. (We recall that by convention, $x_{d+1} = 1 - \sum_1^d x_i$). As usual, for any function $f : [0, 1] \to \mathbb{R}$, we denote $\pi f : \mathbb{D}_d \to \mathbb{R}$ the function $\pi f(y) = f(\pi(y))$. Then, with the Jacobi operator $J_{q,n-q} = \mathrm{L}_{1,q,n-q}$, one has

$$\pi J_{q,n-q} = \mathrm{L}_{\mathbf{1},\boldsymbol{p}} \pi,$$

as may be checked directly and easily, computing $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}\pi(x)$ and $\Gamma_{\mathbf{1},\boldsymbol{p}}(\pi(x), \pi(x))$.

Now, the eigenvalues of $J_{p,n-q}$ and $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ are the same (namely $-k\big(k + \frac{n-2}{2}\big)$, acting on polynomials of degree k). In other words, any eigenspace for $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ contains an eigenvector of the form $P(\pi(x))$.

Now, let K be a Markov operator on $\mathbb{D}_d$ which strongly commutes with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$, with eigenvalue μ_k on E_k. For a Jacobi polynomial P_k, $K(\pi P_k) = \mu_k \pi P_k$. Therefore, for any polynomial P defined on [0, 1], one sees that $K(\pi P) = \pi Q$, for some uniquely defined polynomial Q. This allows one to define a new Markov operator K_1 on [0, 1] through its action on polynomials as $K(\pi P) = \pi K_1(P)$. It is clear that K_1 commutes with $J_{q,n-q}$.

If μ_k is the eigenvalue of K on the eigenspace E_k of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$, then, for any Jacobi polynomial with degree k, $K_1(P) = \mu_k P$. One may now apply Gasper's theorem and we have obtained:

Proposition 4.7. *Let K be a Markov operator on $\mathbb{D}_d$ which strongly commutes with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$, and let (μ_k) be the sequence of its eigenvalues on the eigenspace E_k of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$. Choose $I \subset \{1, \dots, d+1\}$, $I \neq \{1, \dots, d+1\}$, and let $q = \sum_{i \in I} p_i$, and $n = \sum_1^{d+1} p_i$. Then, there exists a probability measure ν on $[0, 1]$ such that, for any $k \in \mathbb{N}$*

$$\mu_k = \int_0^1 \frac{P_k^{q,n-q}(x)}{P_n^{q,n-q}(x_0)} \, \nu(dx),$$

where $P_k^{p,n-q}$ is the Jacobi polynomial with degree k for the measure $\beta_{q,n-q}$, and $x_0 = 0$ or $x_0 = 1$ according to $p \leq n - q$ or not.

Remarks 4.8. (1) Contrary to the one dimensional case, it is not true in general that for any probability measure ν on [0, 1], the associated sequence μ_n may be the sequence of eigenvalues of a Markov operator. Indeed, if such were the case, then for some value of $q = \sum_{i\in I} p_i$, one would have that the sequence $P_k^{q,n-q}(x)/P_k^{q,n-q}(1)$ is such a strong Markov sequence. Choosing another value of q, say q_1, associated to another subset I_1 of $\{1, \dots, d+1\}$, one would therefore get some measure $\nu(x, dy)$ on $[0, 1]$ such that

$$\frac{P_k^{q,n-q}(x)}{P_k^{q,n-q}(1)} = \int_0^1 \frac{P_k^{q_1,n-q_1}(y)}{P_k^{q_1,n-q_1}(1)}\,\nu(x, dy).$$

Repeating the operation with $P_k^{q_1,n-q_1}(y)/P_k^{q_1,n-q_1}(1)$ and another measure $\nu_1(y, dz)$, one would get

$$\frac{P_k^{q,n-q}(x)}{P_k^{q,n-q}(1)} = \int \frac{P_k^{q,n-q}(z)}{P_k^{q,n-q}(1)}\,\nu_2(x, dz),$$

where $\nu_2(x, dz) = \int \nu(x, dy)\,\nu_1(y, dz)$.

Then, $\nu_2(x, dz)$ is the Dirac mass in x. As a consequence, for $\nu(x, dy)$ almost every y, $\nu_1(y, dz)$ is a Dirac mass in some point $h(y)$, and moreover this point is constant. This is clearly wrong, since the Jacobi polynomials for different values of the parameters do not coincide.

(2) In view of Theorem 3.3, in order to obtain the true hypergroup representation, that is the set of extremal points for Markov which strongly commutes with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$, it would be enough to produce the associated space E_1 and the corresponding operations π and ϕ such that the associated correlation operator $K(f) = \mathbb{E}\big(f(\phi\pi f(Y))/\pi(Y) = x\big)$ strongly commutes with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$. Even in the geometric case, when the parameters p_i are integers, it does not seem to be the case for the horizontal rotations described in (3-4).

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