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Monodromy and log geometry

Piotr Achinger and Arthur Ogus

A now classical construction due to Kato and Nakayama attaches a topological space (the "Betti realization") to a log scheme over $\mathbb{C}$. We show that in the case of a log smooth degeneration over the standard log disc, this construction allows one to recover the topology of the germ of the family from the log special fiber alone. We go on to give combinatorial formulas for the monodromy and the d_2 differentials acting on the nearby cycle complex in terms of the log structures. We also provide variants of these results for the Kummer étale topology. In the case of curves, these data are essentially equivalent to those encoded by the dual graph of a semistable degeneration, including the monodromy pairing and the Picard–Lefschetz formula.

1. Introduction

Log geometry was introduced with the purpose of studying compactification and degeneration in a wide context of geometric and arithmetic situations. For example, moduli problems usually give rise to spaces U which are not compact, and it is often desirable to construct an understandable compactification X of U. Typically the points of $D := X \setminus U$ correspond to "degenerate but decorated" versions of the objects classified by points of U. In classical language, one keeps track of the difference between X and U by remembering the sheaf of functions on X which vanish on D, a sheaf of ideals in $\mathcal{O}_X$. Log geometry takes the complementary point of view,

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encoding instead the sheaf functions on X which become invertible on U, a sheaf of multiplicative monoids in $\mathcal{O}_X$. In general, a *log scheme* is a scheme X endowed with a homomorphism of sheaves of commutative monoids $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$; it is convenient to also require that the induced map $\alpha_X^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ be an isomorphism. Thus there is a natural "exact sequence":

$$0 \to \mathcal{O}_X^* \to \mathcal{M}_X \to \overline{\mathcal{M}}_X \to 0,$$

where the quotient is a sheaf of monoids which is essentially combinatorial in nature. The resulting formalism allows one to study the properties of U locally along the complement D, and in a relative situation, provides a very appealing picture of the theory of nearby cycles. Furthermore, log structures behave well under base change, and the log structure induced on D can often be related to the "decoration" needed to define the compactified moduli problem represented by X.

In the complex analytic context, a construction of Kato and Nakayama [1999] gives a key insight into the working of log geometry. Functorially associated to any fine log analytic space X is a topological space $X_{\log}$, together with a natural proper and surjective continuous map $\tau_X : X_{\log} \to X_{\mathrm{top}}$, where X_{top} is the topological space underlying X. For each point x of X_{top}, the fiber $\tau_X^{-1}(x)$ is naturally a torsor under $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{S}^1)$. The morphism τ_X fits into a commutative diagram,

$$\begin{array}{ccc} & & X_{\log} \\ & \overset{j_{\log}}{\nearrow} & \downarrow \tau_X \\ X^*_{\mathrm{top}} & \xrightarrow[j_{\mathrm{top}}]{} & X_{\mathrm{top}} \end{array}$$

where X^* is the open set on which the log structure is trivial and $j : X^* \to X$ is the inclusion. If the log scheme X is (logarithmically) smooth over $\mathbb{C}$, then the morphism $j_{\log}$ is aspheric [Ogus 2003, 3.1.2], and in particular it induces an equivalence between the categories of locally constant sheaves on X^*_{top} and on $X_{\log}$. Thus τ_X can be viewed as a compactification of the open immersion j; it has the advantage of preserving the local homotopy theory of X^*. In particular, the behavior of a locally constant sheaf $\mathcal{F}$ on $X_{\log}$ can be studied locally over points of $X \setminus X^*$, a very agreeable way of investigating local monodromy.

We shall apply the above philosophy to study the behavior of a morphism $f : X \to Y$ of fine saturated log analytic spaces. Our goal is to exploit the log structures of X and Y to describe the topological behavior of f locally in a neighborhood of a point y of Y, especially when y is a point over which f is smooth in the logarithmic sense but singular in the classical sense. The philosophy of log geometry suggests that (a large part of) this topology can be computed just from the log fiber $X_y \to y$. For example, we show that if Y is a standard log disc and $X \to Y$ is smooth, proper, and vertical, then the germs of X_{top} and Y_{top} are homeomorphic to the (open) mapping cylinders of the maps $\tau_{X_y} : X_{y,\log} \to X_{y,\mathrm{top}}$ and $\tau_y : y_{\log} \to y_{\mathrm{top}}$ respectively, compatibly with the map $f_{\mathrm{top}} : X_{\mathrm{top}} \to Y_{\mathrm{top}}$. (See Theorem 4.1.1 for a precise statement and Conjecture 4.1.5 for a hoped for generalization.) Furthermore, it is shown in [Illusie et al. 2005, 8.5] that, in the above context, the classical complex of nearby cycles $\Psi_{X/Y}$ on $X_{y,\mathrm{top}}$ can be computed directly from the morphism of log spaces $X_y \to y$, and in fact can be identified with (a relative version of) $R\tau_{X_y *}(\mathbb{Z})$. (See Section 4 for the precise statement.) In particular, the map $X_{y,\log} \to y_{\log}$ serves as an "asymptotic approximation" to the map $X \to Y$ near y.

With the above motivation in mind, we devote our main attention to the study of a morphism $f : X \to S$, where X is a fine saturated log analytic space and S is the fine saturated split log point associated to a fine sharp monoid P. To emphasize the geometric point of view, we work mainly in the context of complex analytic geometry, describing the étale analogs of our main results in Section 6.3. We assume that f is saturated; this implies that the homomorphism $f^{\flat} : P^{\mathrm{gp}} = \overline{\mathcal{M}}_S^{\mathrm{gp}} \to \overline{\mathcal{M}}_X^{\mathrm{gp}}$ is injective and has torsion-free cokernel. The map $X_{\log} \to S_{\log}$ is a topological fibration, trivial over the universal cover $\tilde{S}_{\log}$ of $S_{\log}$, and the cohomology of $\tilde{X}_{\log} := X_{\log} \times_{S_{\log}} \tilde{S}_{\log}$ is isomorphic to the cohomology of a fiber. The fundamental group I_P of $S_{\log}$ is canonically isomorphic to $\mathrm{Hom}(P^{\mathrm{gp}}, \mathbb{Z}(1))$ and acts naturally on this cohomology and on the "nearby cycle complex" $\Psi_{X/S} := \mathrm{R}\tilde{\tau}_{X*}(\mathbb{Z})$, where $\tilde{\tau}_X : \tilde{X}_{\log} \to X_{\mathrm{top}}$ is the natural map. This situation is illustrated by the diagram

$$
\begin{array}{ccccc}
 & & \overset{\tilde{\tau}_X}{\frown} & & \\
\tilde{X}_{\log} & \longrightarrow & X_{\log} & \xrightarrow[\tau_X]{} & X_{\mathrm{top}} \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{S}_{\log} & \longrightarrow & S_{\log} & \longrightarrow & S_{\mathrm{top}} = \mathrm{pt}.
\end{array}
$$

Our first observation is that if X/S is (log) smooth, then $X/\mathbb{C}$ becomes (log) smooth when X is endowed with the idealized log structure induced from the maximal ideal of P. Theorem 4.1.6 shows that the normalization of a smooth and reduced idealized log scheme can be endowed with a natural "compactifying" log structure which makes it smooth (without idealized structure). This construction gives a canonical way of cutting our X into pieces, each of whose Betti realizations is a family of manifolds with boundary, canonically trivialized over $S_{\log}$.

We then turn to our main goal, which is to describe the topology of $\tilde{X}_{\log}$, together with the monodromy action, directly in terms of log geometry. We use the exact sequences

$$0 \to \overline{\mathcal{M}}_S^{\mathrm{gp}} \to \overline{\mathcal{M}}_X^{\mathrm{gp}} \to \mathcal{M}_{X/S}^{\mathrm{gp}} \to 0 \tag{1-0-1}$$

("log Kodaira–Spencer"), and

$$0 \to \mathbb{Z}(1) \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{M}_{X/P}^{\mathrm{gp}} \to \mathcal{M}_{X/S}^{\mathrm{gp}} \to 0, \tag{1-0-2}$$

("log Chern class"), where $\mathcal{M}_{X/P} := \mathcal{M}_X/P$, (the quotient in the category of sheaves of monoids). The sequence (1-0-2) is obtained by splicing together the two exact sequences:

$$0 \to \mathbb{Z}(1) \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0 \tag{1-0-3}$$

and

$$0 \to \mathcal{O}_X^* \to \mathcal{M}_{X/P}^{\mathrm{gp}} \to \mathcal{M}_{X/S}^{\mathrm{gp}} \to 0. \tag{1-0-4}$$

If ℓ is a global section of $\mathcal{M}_{X/S}^{\mathrm{gp}}$, its inverse image in $\mathcal{M}_{X/P}^{\mathrm{gp}}$ is an $\mathcal{O}_X^*$-torsor, which defines an invertible sheaf $\mathcal{L}_\ell$ on X. The Chern class $c_1(\mathcal{L}_\ell) \in H^2(X, \mathbb{Z}(1))$ is the image of ℓ under the morphism $H^0(X, \mathcal{M}_{X/S}) \to H^2(X, \mathbb{Z}(1))$ defined by (1-0-2).

The spectral sequence of nearby cycles reads:

$$E_2^{p,q} = H^p(X_{\mathrm{top}}, \Psi_{X/S}^q) \Rightarrow H^{p+q}(\tilde{X}_{\log}, \mathbb{Z}),$$

where $\Psi_{X/S}^q$ is the q-th cohomology sheaf of the nearby cycle complex $\Psi_{X/S}$. By (a relative version of) a theorem of Kato and Nakayama [Illusie et al. 2005, 1.5], there are natural isomorphisms:

$$\sigma_{X/S}^q : \textstyle\bigwedge^q \mathcal{M}_{X/S}^{\mathrm{gp}}(-q) \xrightarrow{\sim} \Psi_{X/S}^q. \tag{1-0-5}$$

It follows that the action of each $\gamma \in I_P$ on $\Psi_{X/S}^q$ is trivial, and hence it is also trivial on the graded groups $E_\infty^{p,q}$ associated to the filtration F of the abutment $H^{p+q}(\tilde{X}_{\log}, \mathbb{Z})$. Then $\gamma - \mathrm{id}$ maps $F^p H^{p+q}(\tilde{X}_{\log}, \mathbb{Z})$ to $F^{p+1} H^{p+q}(\tilde{X}_{\log}, \mathbb{Z})$ and induces a map

$$N_\gamma : E_\infty^{p,q} \to E_\infty^{p+1,q-1}. \tag{1-0-6}$$

We explain in Theorem 4.2.2 that (a derived category version of) this map is given by "cup product" with the extension class in $\mathrm{Ext}^1(\Psi_{X/S}^1, \mathbb{Z}) \cong \mathrm{Ext}^1(\mathcal{M}_{X/S}^{\mathrm{gp}}, \mathbb{Z}(1))$ obtained from the pushout of the log Kodaira–Spencer extension (1-0-1) along $\gamma \in \mathrm{Hom}(P^{\mathrm{gp}}, \mathbb{Z}(1))$. We present two proofs: the first, which works only in the smooth case and with $\mathbb{C}$-coefficients, is an easy argument based on a logarithmic construction of the Steenbrink complex; the second uses more complicated homological algebra techniques to prove the result with $\mathbb{Z}$-coefficients.

We also give a logarithmic formula for the d_2 differentials of the nearby cycle spectral sequence. Thanks to formula (1-0-5), these differentials can be interpreted as maps

$$H^p\big(X_{\text{top}}, \textstyle\bigwedge^q \mathcal{M}^{\text{gp}}_{X/S}(-q)\big) \to H^{p+2}\big(X_{\text{top}}, \textstyle\bigwedge^{q-1} \mathcal{M}^{\text{gp}}_{X/S}(1-q)\big).$$

Theorem 4.2.2 shows that these maps are obtained by cup-product with the derived morphism $\mathcal{M}^{\text{gp}}_{X/S} \to \mathbb{Z}(1)[2]$ obtained from the log Chern class sequence (1-0-2), up to a factor of $q!$. We do not have a formula for the higher differentials, but recall from [Illusie 1994, 2.4.4] that, in the case of a projective semistable reduction with smooth irreducible components, these higher differentials vanish.

To illustrate these techniques, we study the case in which X/S is a smooth log curve over the standard log point. In this case it is very easy to interpret our formulas explicitly in terms of the combinatorial data included in the "dual graph" which is typically attached to the nodal curve $\underline{X}$ underlying X. The log structure provides some extra information when X/S is log smooth but nonsemistable. In particular, we recover the classical Picard–Lefschetz formula, and we show that the d_2 differential in the nearby-cycle spectral sequence coincides with the differential in the chain complex computing the homology of the dual graph.

For clarity of exposition, we focus mainly on the complex analytic setting. However, one of the main strengths of log geometry is the bridge it provides between analysis and algebra and between Betti, étale, and de Rham cohomologies. For the sake of arithmetic applications, we therefore also provide a sketch of how to formulate and prove analogs of our results in the context of the Kummer étale topology. The case of p-adic cohomology looks more challenging at present.

2. Homological preliminaries

In this section, after reviewing some standard material in Section 2.1, we provide a few results in homological algebra which will be important in our study of the nearby cycle complex $\Psi_{X/S}$ together with its multiplicative structure and the monodromy action of the group I_P.

2.1. ***Notation and conventions.*** We follow the conventions of [Berthelot et al. 1982] with regard to homological algebra, particularly when it comes to signs. For simplicity, we shall work in the abelian category $\mathcal{A}$ of sheaves of modules on a ringed topological space (or more generally a ringed topos) (X, A_X). Readers will gather from our exposition that keeping track of signs presented a considerable challenge.

Shifts, cones, and distinguished triangles. If $A = (A^n, d^n : A^n \to A^{n+1})$ is a complex in an abelian category $\mathcal{A}$, the *shift* $A[k]$ of A by an integer k is the complex $(A^{n+k}, (-1)^k d^{n+k})$. We shall use the canonical identification $\mathcal{H}^n(A[k]) = \mathcal{H}^{n+k}(A)$ induced by the identity on A^{n+k}. If $u : A \to B$ is a morphism of complexes, its shift $f[k] : A[k] \to B[k]$ is given by $f^{n+k} : A^{n+k} \to B^{n+k}$ in degree n.

The *mapping cone*, written $\mathrm{Cone}(u)$ or $C(u)$, is the complex with

$$C(u)^n := B^n \oplus A^{n+1}$$

with differential $d(b, a) = (db + u(a), -da)$, which comes with a sequence of maps of complexes,

$$A \xrightarrow{u} B \xrightarrow{i} C(u) \xrightarrow{-p} A[1], \tag{2-1-1}$$

where $i(b) := (b, 0)$ and $p(b, a) := a$. This sign convention is used in [Berthelot et al. 1982] but differs from the convention used by Kashiwara and Schapira [1990, Chapter I] and many other authors. A *triangle* is a sequence of maps in $D(\mathcal{A})$ of the form $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ (abbreviated as (u, v, w)). A triangle (u, v, w) is *distinguished* if it is isomorphic in the derived category to a triangle of the form (2-1-1). Then (u, v, w) is distinguished if and only if $(v, w, -u[1])$ is distinguished. More generally, if (u, v, w) is distinguished, so is

$$\big((-1)^k u[k], (-1)^k v[k], (-1)^k w[k]\big) \cong (u[k], v[k], (-1)^k w[k]) \quad \text{for any } k \in \mathbb{Z}.$$

Total complex and tensor product. Given a double complex

$$A = \big(A^{p,q}, d_h^{p,q} : A^{p,q} \to A^{p+1,q}, d_v^{p,q} : A^{p,q} \to A^{p,q+1}\big)$$

in $\mathcal{A}$, its *total complex* is the complex $\mathrm{Tot}(A) = \big(\bigoplus_{p+q=n} A^{p,q}, d^n\big)$, where d^n is given by $d_h^{p,q} + (-1)^p d_v^{p,q}$ on $A^{p,q}$, so that the differentials form commutative squares. The tensor product $A \otimes B$ of two complexes is by definition the total complex of the double complex $(A^p \otimes B^q, d_A^p \otimes \mathrm{id}, \mathrm{id} \otimes d_B^q)$. Note that the shift functor $(-)[k]$ equals $A_X[k] \otimes (-)$, while $(-) \otimes A_X[k]$ is the "naive shift," that is, shift without sign change. Moreover, the cone $C(u)$ of a map $u : A \to B$ is the total complex of the double complex $[A \xrightarrow{u} B]$ where B is put in the zeroth column.

Truncation functors. We use the truncation functors $\tau_{\le q}$ and $\tau_{\ge q}$ (see [Beĭlinson et al. 1982, exemple 1.3.2(i)] or [Kashiwara and Schapira 1990, (1.3.12)–(1.3.13), p. 33] on the category of complexes of sheaves on X:

$$\tau_{\le q} K = [\cdots \to K^{q-1} \to \mathrm{Ker}(d^q) \to 0 \to \cdots],$$
$$\tau_{\ge q} K = [\cdots \to 0 \to \mathrm{Cok}(d^{q-1}) \to K^{q+1} \to \cdots].$$

These functors descend to the derived category $D(X)$, although they do not preserve distinguished triangles. For a pair of integers $a \le b$, we write $\tau_{[a,b]} = \tau_{\ge a}\tau_{\le b} = \tau_{\le b}\tau_{\ge a}$ and $\tau_{[a,b)} = \tau_{[a,b-1]}$. For example, $\tau_{[q,q]}K = H^q(K)[-q]$.

Proposition 2.1.1. *For each triple of integers (a, b, c) with $a < b < c$, and each complex K, there is a functorial distinguished triangle*:

$$\tau_{[a,b)}(K) \to \tau_{[a,c)}(K) \to \tau_{[b,c)}(K) \xrightarrow{\delta} \tau_{[a,b)}[1]. \tag{2-1-2}$$

The map δ above is the unique morphism making the triangle distinguished.

Proof. The natural map of complexes $\tau_{[a,b)}(K) \to \tau_{[a,c)}(K)$ is injective with cokernel

$$C := \big[\cdots \to 0 \to K^{b-1}/\ker d^{b-1} \to K^b \to \cdots \to \ker d^{c-1} \to 0 \to \cdots\big]$$

and the evident map $C \to \tau_{[b,c)}(K)$ is a quasi-isomorphism commuting with the maps from $\tau_{[a,c)}(K)$. This way we obtain the distinguished triangle (2-1-2). For the uniqueness, observe that given two such maps δ, δ', there is a map

$$\zeta : \tau_{[b,c)}(K) \to \tau_{[b,c)}(K)$$

completing (id, id) to a morphism of distinguished triangles:

$$\begin{array}{ccccccc}
\tau_{[a,b)}(K) & \longrightarrow & \tau_{[a,c)}(K) & \longrightarrow & \tau_{[b,c)}(K) & \xrightarrow{\ \delta\ } & \tau_{[a,b)}[1] \\
\| & & \| & & \downarrow{\scriptstyle \zeta} & & \| \\
\tau_{[a,b)}(K) & \longrightarrow & \tau_{[a,c)}(K) & \longrightarrow & \tau_{[b,c)}(K) & \xrightarrow[\ \delta'\ ]{} & \tau_{[a,b)}[1]
\end{array}$$

Applying the functor $\tau_{[b,c)}$ to the middle square of the above diagram, we see that $\zeta = \mathrm{id}$, and hence that $\delta = \delta'$. ☐

First order attachment maps. If K is a complex and $q \in \mathbb{Z}$, the distinguished triangle (2-1-2) for $(a, b, c) = (q-1, q, q+1)$ is

$$\mathcal{H}^{q-1}(K)[1-q] \to \tau_{[q-1,q+1)}(K) \to \mathcal{H}^q(K)[-q] \xrightarrow{\delta_K^q[-q]} \mathcal{H}^{q-1}(K)[2-q],$$

which yields a "first order attachment morphism"

$$\delta_K^q : \mathcal{H}^q(K) \to \mathcal{H}^{q-1}(K)[2], \tag{2-1-3}$$

embodying the d_2 differential of the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(K)) \Rightarrow H^{p+q}(X, K).$$

Note that $\delta_K^q[-q]$ is the unique morphism making the triangle above distinguished.

We shall need the following result, stating that the maps δ form a "derivation in the derived category."

Lemma 2.1.2. *Let A and B be complexes in the abelian category $\mathcal{A}$, and let i and j be integers such that $\mathcal{H}^i(A)$ and $\mathcal{H}^j(B)$ are flat A_X-modules. Then the following* *diagram commutes*:

$$\begin{array}{ccc} \mathcal{H}^i(A)\otimes\mathcal{H}^j(B) & \xrightarrow{\delta^i_A\otimes 1+(-1)^i\otimes\delta^j_B} & \mathcal{H}^{i-1}(A)[2]\otimes\mathcal{H}^j(B)\oplus\mathcal{H}^i(A)\otimes\mathcal{H}^{j-1}(B)[2] \\ \downarrow & & \downarrow \\ \mathcal{H}^{i+j}(A\otimes B) & \xrightarrow[\delta^{i+j}_{A\otimes B}]{} & \mathcal{H}^{i+j-1}(A\otimes B)[2] \end{array}$$

Here $A\otimes B$ denotes the derived tensor product.

Proof. Call the diagram in question $D(A,B)$ (i and j are fixed throughout). We shall first prove that $D(A,B)$ commutes if $\mathcal{H}^q(A)=0$ for $q\neq i$. Recall that δ^j_B is the unique map such that the triangle

$$\mathcal{H}^{j-1}(B)[1-j]\to\tau_{[j-1,j]}B\to\mathcal{H}^j(B)[-j]\xrightarrow{\delta^j_B[-j]}\mathcal{H}^{j-1}(B)[2-j]$$

is distinguished. Applying $A_X[-i]\otimes(-)=(-)[-i]$, we get $\delta^{i+j}_{B[-i]}=(-1)^i\delta^j_B[-i]$ under the identifications $\mathcal{H}^q(B)=\mathcal{H}^{q+i}(B[-i])$, $q=j-1,j$. This implies that $D(A_X[-i],B)$ commutes. Since

$$A\otimes(-)=\mathcal{H}^i(A)[-i]\otimes(-)=\mathcal{H}^i(A)\otimes(A_X[-i]\otimes(-)),$$

we see that $D(A,B)$ commutes as well.

Similarly if $\mathcal{H}^q(B)=0$ for $q\neq j$: $A\otimes A_X[-i]$ is the $(-i)$-th naive shift of A, preserving exactness, and we have $\delta^{i+j}_{A\otimes A_X[-i]}=\delta^i_A[-j]$ (note that the effect of naive and usual shift on maps is "the same"), so $D(A,A_X[-j])$ commutes; again, so does $D(A,B)$.

To treat the general case, note that $D(A,B)$, even if not commutative, is clearly a functor of A and B. Let $A':=\tau_{\leq i}A$ and observe that the natural map $A'\to A$ induces isomorphisms on the objects in the top row of the diagrams $D(A',B)$ and $D(A,B)$. Thus $D(A,B)$ commutes if $D(A',B)$ does, and hence we may assume that $\mathcal{H}^q(A)=0$ for $q>i$. Analogously, we can assume that $\mathcal{H}^q(B)=0$ for $q>j$.

Under these extra assumptions, the hypertor spectral sequence (see [EGA III$_2$ 1963, proposition 6.3.2])

$$E^2_{pq}=\bigoplus_{i'+j'=q}\mathrm{Tor}_{-p}(\mathcal{H}^{-i'}(A),\mathcal{H}^{-j'}(B))\Rightarrow\mathcal{H}^{-p-q}(A\otimes B)$$

shows that the vertical maps in $D(A,B)$ are isomorphisms. Let

$$u_{A,B}=(\text{right})^{-1}\circ(\text{bottom})\circ(\text{left})\quad\text{in }\ D(A,B).$$

Then $D(A,B)$ commutes if and only if $u_{A,B}=(\text{top}):=\delta^i_A\otimes 1+(-1)^i\otimes\delta^j_B$. The target of $u_{A,B}$ is a product of two terms

$$\mathcal{H}^{i-1}(A)[2]\otimes\mathcal{H}^j(B)\quad\text{and}\quad\mathcal{H}^i(A)\otimes\mathcal{H}^{j-1}(B)[2];$$

let us denote the two projections by p_A and p_B.

Let us set $A'' = \mathcal{H}^i(A)[-i]$; we know that $D(A'', B)$ commutes. This diagram reads

$$\begin{array}{ccc} \mathcal{H}^i(A)\otimes\mathcal{H}^j(B) & \xrightarrow{(-1)^i\otimes\delta^j_B} & \mathcal{H}^i(A)\otimes\mathcal{H}^{j-1}(B)[2] \\ \downarrow & & \downarrow \\ \mathcal{H}^{i+j}(A\otimes B) & \xrightarrow[\delta^{i+j}_{A'\otimes B}]{} & \mathcal{H}^{i+j-1}(A''\otimes B)[2] \end{array}$$

and the vertical maps are isomorphisms. The map between the top-right corners of $D(A, B)$ and $D(A'', B)$ induced by the canonical map $A \to A''$ is the projection p_B. It follows that $p_B \circ u_{A,B} = (-1)^i \otimes \delta^j_B$.

Similarly, considering $B'' = \mathcal{H}^j(B)[-j]$ and the canonical map $B \to B''$, and using the fact that $D(A, B'')$ commutes, we see that $p_A \circ u_{A,B} = \delta^i_A \otimes 1$. We conclude that $u_{A,B} = p_A \circ u_{A,B} + p_B \circ u_{A,B} = \delta^i_A \otimes 1 + (-1)^i \otimes \delta^j_B$, as desired. □

Maps associated to short exact sequences. Consider a short exact sequence of complexes

$$0 \to A \xrightarrow{u} B \xrightarrow{\pi} C \to 0. \tag{2-1-4}$$

The map $\tilde{\pi} : C(u) \to C$ sending (b, a) to $\pi(b)$ is a quasi-isomorphism.

Definition 2.1.3. In the above situation, $\xi_u : C \to A[1]$ is the morphism in the derived category $D(\mathcal{A})$ defined by

$$\xi_u : C \xrightarrow{\tilde{\pi}^{-1}} C(u) \xrightarrow{-p} A[1].$$

We shall also refer to ξ_u as the map corresponding to the short exact sequence (2-1-4) (rather than the injection u).

Thus the triangle

$$A \xrightarrow{u} B \xrightarrow{\pi} C \xrightarrow{\xi_u} A[1] \tag{2-1-5}$$

is distinguished, and the map $\mathcal{H}^q(\xi) : \mathcal{H}^q(C) \to \mathcal{H}^q(A[1]) = \mathcal{H}^{q+1}(A)$ agrees with the map defined by the standard diagram chase in the snake lemma. Moreover, $\xi_{-u} = -\xi_u$.

In the special case when A and B are objects of $\mathcal{A}$ concentrated in a single degree q, the map ξ_u is the unique map making the triangle (2-1-5) distinguished [Kashiwara and Schapira 1990, 10.1.11].

2.2. ***Exterior powers and Koszul complexes.*** Let us first review some relevant facts about exterior and symmetric powers. Recall that if E is a flat A_X-module and $q \geq 0$, then the exterior power $\bigwedge^q E$, the symmetric power $S^q E$, and the divided power $\Gamma^q(E)$ modules are again flat. For $i + j = q$, there are natural multiplication and comultiplication transformations:

$$\begin{aligned} \mu &: \textstyle\bigwedge^i E \otimes \bigwedge^j E \to \bigwedge^q E && \text{and} && \eta : \textstyle\bigwedge^q E \to \bigwedge^i E \otimes \bigwedge^j E, \\ \mu &: S^i E \otimes S^j E \to S^q E && \text{and} && \eta : S^q E \to S^i E \otimes S^j E, \\ \mu &: \Gamma^i E \otimes \Gamma^j E \to \Gamma^q E && \text{and} && \eta : \Gamma^q E \to \Gamma^i E \otimes \Gamma^j E. \end{aligned}$$

We shall only use the maps η with $i = 1$. In this case they are given by the formulas

$$\begin{aligned} \eta(x_1 \wedge \cdots \wedge x_q) &= \sum_i (-1)^{i-1} x_i \otimes x_1 \wedge \cdots \hat{x}_i \cdots \wedge x_q, \\ \eta(x_1 \cdots x_q) &= \sum_i x_i \otimes x_1 \cdots \hat{x}_i \cdots x_q, \\ \eta(x_1^{[q_1]} \cdots x_n^{[q_n]}) &= \sum_i x_i \otimes x_1^{[q_1]} \cdots x_i^{[q_i - 1]} \cdots x_n^{[q_n]}. \end{aligned}$$

It follows that each composition

$$\begin{gathered} \textstyle\bigwedge^q E \xrightarrow{\eta} E \otimes \bigwedge^{q-1} E \xrightarrow{\mu} \bigwedge^q E, \\ S^q E \xrightarrow{\eta} E \otimes S^{q-1} E \xrightarrow{\mu} S^q E, \\ \Gamma^q E \xrightarrow{\eta} E \otimes \Gamma^{q-1} E \xrightarrow{\mu} \Gamma^q E \end{gathered}$$

is multiplication by q. Furthermore, η is a derivation, by which we mean that the following diagram commutes:

$$\begin{array}{ccc} \bigwedge^i E \otimes \bigwedge^j E & \xrightarrow{\eta\otimes\mathrm{id},\,\mathrm{id}\otimes\eta} & \big(E \otimes \bigwedge^{i-1} E \otimes \bigwedge^j E\big) \oplus \big(\bigwedge^i E \otimes E \otimes \bigwedge^{j-1} E\big) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mathrm{id}\oplus t_i \otimes \mathrm{id}} \\ \bigwedge^{i+j} E & & \big(E \otimes \bigwedge^{i-1} E \otimes \bigwedge^j E\big) \oplus \big(E \otimes \bigwedge^i E \otimes \bigwedge^{j-1} E\big) \\ \downarrow{\scriptstyle \eta} & \swarrow{\scriptstyle \mathrm{id}\otimes\mu,\,\mathrm{id}\otimes\mu} & \\ E \otimes \bigwedge^{i+j-1} E, & & \end{array}$$

where $t_i : \bigwedge^i E \otimes E \to E \otimes \bigwedge^i E$ is $(-1)^i$ times the commutativity isomorphism for tensor products. The diagram for the symmetric and divided power products is similar (without the sign).

In fact, $\left\{\eta_q : \bigwedge^q E \to E \otimes \bigwedge^{q-1} E : q \geq 1\right\}$ is the unique derivation such that $\eta_1 = \mathrm{id}$, because the multiplication map μ is an epimorphism. This argument fails in the derived category, and we will need another argument which gives a slightly weaker result. To understand the context, let $\alpha : E \to F$ be a morphism in $D(X)$, where E is flat and concentrated in degree zero, and for each q, define α_q as the composition

$$\alpha_q : \bigwedge^q E \xrightarrow{\eta} E \otimes \bigwedge^{q-1} E \xrightarrow{\alpha\otimes\mathrm{id}} F \otimes \bigwedge^{q-1} E. \tag{2-2-1}$$

Then the family $\left\{\alpha_q : \bigwedge^q E \to F \otimes \bigwedge^{q-1} E : q \geq 1\right\}$ is a derivation in $D(X)$, in the sense that the diagram

$$\begin{array}{ccc} \bigwedge^i E \otimes \bigwedge^j E & \xrightarrow{\alpha_i\otimes\mathrm{id},\mathrm{id}\otimes\alpha_j} & \left(F \otimes \bigwedge^{i-1} E \otimes \bigwedge^j E\right) \oplus \left(\bigwedge^i E \otimes F \otimes \bigwedge^{j-1} E\right) \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mathrm{id}\oplus t_i\otimes\mathrm{id}} \\ \bigwedge^{i+j} E & & \left(F \otimes \bigwedge^{i-1} E \otimes \bigwedge^j E\right) \oplus \left(F \otimes \bigwedge^i E \otimes \bigwedge^{j-1} E\right) \\ \downarrow{\scriptstyle\alpha_{i+j}} & \swarrow{\scriptstyle\mathrm{id}\otimes\mu,\mathrm{id}\otimes\mu} & \\ F \otimes \bigwedge^{i+j-1} E & & \end{array} \tag{2-2-2}$$

commutes. We shall see that this property almost determines the maps α_q.

Proposition 2.2.1. *Let E be a flat A_X-module, let F be an object of $D(X)$, and let*

$$\left\{\alpha'_j : \bigwedge^j E \to F \otimes \bigwedge^{j-1} E : j \geq 1\right\}$$

be a family of morphisms in $D(X)$. Let $\alpha = \alpha'_1 : E \to F$, let α_q be as in (2-2-1) for $q \geq 1$, and assume that $q \in \mathbb{Z}^+$ is such that for $1 \leq j < q$, the diagrams

$$\begin{array}{ccc} E \otimes \bigwedge^j E & \xrightarrow{\alpha\otimes\mathrm{id},\mathrm{id}\otimes\alpha'_j} & \left(F \otimes \bigwedge^j E\right) \oplus \left(E \otimes F \otimes \bigwedge^{j-1} E\right) \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\mathrm{id}\oplus t\otimes\mathrm{id}} \\ \bigwedge^{j+1} E & & \left(F \otimes \bigwedge^j E\right) \oplus \left(F \otimes E \otimes \bigwedge^{j-1} E\right) \\ \downarrow{\scriptstyle\alpha'_{j+1}} & \swarrow{\scriptstyle\mathrm{id},\mathrm{id}\otimes\mu} & \\ F \otimes \bigwedge^j E & & \end{array}$$

commute, where $t : E \otimes F \to F \otimes E$ is the negative of the standard isomorphism. Then $q!\alpha'_q = q!\alpha_q$.

Proof. The statement is vacuous for $q = 1$, and we proceed by induction on q. Let $\tau_F := (\mathrm{id}, \mathrm{id} \otimes \mu) \circ (\mathrm{id} \oplus t \otimes \mathrm{id})$ in the diagram above. Then, setting $j = q - 1$, we have the following commutative diagram:

$$\begin{array}{ccccc} \bigwedge^q E & \xrightarrow{\eta} & E \otimes \bigwedge^{q-1} E & \xrightarrow{\alpha\otimes\mathrm{id},\,\mathrm{id}\otimes\alpha'_{q-1}} & \big(F \otimes \bigwedge^{q-1} E\big) \oplus \big(E \otimes F \otimes \bigwedge^{q-2} E\big) \\ & \searrow^{q} & \downarrow \mu & & \downarrow \tau_F \\ & & \bigwedge^q E & \xrightarrow[\alpha'_q]{} & F \otimes \bigwedge^{q-1} E \end{array}$$

In other words,

$$q\alpha'_q = \tau_F \circ (\alpha \otimes \mathrm{id}, \mathrm{id} \otimes \alpha'_{q-1}) \circ \eta.$$

Similarly,

$$q\alpha_q = \tau_F \circ (\alpha \otimes \mathrm{id}, \mathrm{id} \otimes \alpha_{q-1}) \circ \eta.$$

Then using the induction hypothesis, we can conclude:

$$\begin{aligned} q!\alpha'_q &= \tau_F \circ \big((q-1)!\alpha \otimes \mathrm{id}, \mathrm{id} \otimes (q-1)!\alpha'_{q-1}\big) \circ \eta \\ &= \tau_F \circ \big((q-1)!\alpha \otimes \mathrm{id}, \mathrm{id} \otimes (q-1)!\alpha_{q-1}\big) \circ \eta \\ &= q!\alpha_q. \end{aligned}$$

□

Next we discuss connecting homomorphisms, exterior powers, and Koszul complexes. Consider a short exact sequence of flat A_X-modules

$$0 \to A \xrightarrow{u} B \xrightarrow{\pi} C \to 0,$$

and the associated morphism $\xi = \xi_u : C \to A[1]$ (see Definition 2.1.3). The *Koszul filtration* is the decreasing filtration of $\bigwedge^q B$ defined by

$$K^i \bigwedge^q B = \mathrm{Im}\big(\bigwedge^i A \otimes \bigwedge^{q-i} B \xrightarrow{\wedge^i u \otimes \mathrm{id}} \bigwedge^i B \otimes \bigwedge^{q-i} B \xrightarrow{\mu} \bigwedge^q B\big).$$

There are canonical isomorphisms

$$\bigwedge^i A \otimes \bigwedge^{q-i} C \cong \mathrm{Gr}^i_K\big(\bigwedge^q B\big) \tag{2-2-3}$$

We can use this construction to give a convenient expression for the composed morphism $\xi_q : \bigwedge^q C \to A \otimes \bigwedge^{q-1} C[1]$ defined in Equation (2-2-1) above.

Proposition 2.2.2. *Let* $0 \to A \xrightarrow{u} B \xrightarrow{\pi} C \to 0$ *be an exact sequence of flat* A_X*-modules, with corresponding morphism* $\xi := \xi_u : C \to A[1]$ *in* $D(X)$*. For each* $q \in \mathbb{N}$*, let* $K^\bullet$ *be the Koszul filtration on* $\bigwedge^q B$ *defined by the inclusion* $u : A \to B$ *and consider the exact sequence*

$$0 \to A \otimes \bigwedge^{q-1} C \xrightarrow{u_q} \bigwedge^q B / K^2 \bigwedge^q B \xrightarrow{\pi_q} \bigwedge^q C \to 0$$

obtained from the filtration K *and the isomorphisms* (2-2-3) *above. Then*

$$\xi_{u_q} = \xi_q := (\xi \otimes \mathrm{id}) \circ \eta : \bigwedge^q C \to C \otimes \bigwedge^{q-1} C \to A \otimes \bigwedge^{q-1} C[1].$$

Proof. Observe that the composition

$$\textstyle\bigwedge^q B \xrightarrow{\eta} B \otimes \bigwedge^{q-1} B \xrightarrow{\mathrm{id}\otimes\pi} B \otimes \bigwedge^{q-1} C$$

annihilates $K^2 \bigwedge^q B$. Then we find the following commutative diagram, in which the rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A \otimes \bigwedge^{q-1} C & \longrightarrow & \bigwedge^q B / K^2 \bigwedge^q B & \longrightarrow & \bigwedge^q C & \longrightarrow & 0 \\ & & \Vert & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \eta} & & \\ 0 & \longrightarrow & A \otimes \bigwedge^{q-1} C & \longrightarrow & B \otimes \bigwedge^{q-1} C & \longrightarrow & C \otimes \bigwedge^{q-1} C & \longrightarrow & 0 \end{array}$$

We consequently get a commutative diagram in $D(X)$:

$$\begin{array}{ccc} \bigwedge^q C & \xrightarrow{\xi_{u_q}} & A \otimes \bigwedge^{q-1} C[1] \\ {\scriptstyle \eta}\downarrow & & \Vert \\ C \otimes \bigwedge^{q-1} C & \xrightarrow[\xi\otimes\mathrm{id}]{} & A \otimes \bigwedge^{q-1} C[1] \end{array}$$

□

Let us now recall the definition of the Koszul complex of a homomorphism (see Illusie 1971, chapitre I, 4.3.1.3; Kato and Saito 2004, 1.2.4.2]).

Definition 2.2.3. Let $u : A \to B$ be homomorphism of A_X-modules, and let $q \geq 0$. Then the q-th Koszul complex $\mathrm{Kos}^q(u)$ of u is the cochain complex whose p-th term is $\Gamma^{q-p}(A) \otimes \bigwedge^p B$ and with differential

$$\begin{array}{rcl} d^p_{u,q} : \Gamma^{q-p}(A) \otimes \bigwedge^p B & \xrightarrow{\eta\otimes\mathrm{id}} & \Gamma^{q-p-1}(A) \otimes A \otimes \bigwedge^p B \\ & & \downarrow{\scriptstyle \mathrm{id}\otimes u\otimes\mathrm{id}} \\ & & \Gamma^{q-p-1}(A) \otimes B \otimes \bigwedge^p B \xrightarrow[\mathrm{id}\otimes\mu]{} \Gamma^{q-p-1}(A) \otimes \bigwedge^{p+1} B \end{array}$$

Observe that $\mathrm{Kos}^q(u)$ (treated as a chain complex) is $\Lambda^q(u : A \to B)$ in the notation of [Kato and Saito 2004, 1.2.4.2], and is the total degree q part of $\mathrm{Kos}^{\bullet}(u)$ in the notation of [Illusie 1971, chapitre I, 4.3.1.3]. If A and B are flat, $\mathrm{Kos}^q(u)[-q]$ coincides with the derived exterior power of the complex $[A \to B]$ (placed in degrees -1 and 0), see [Kato and Saito 2004, Corollary 1.2.7]. Note that $\mathrm{Kos}^1(u)$ is the complex $[A \to B]$ in degrees 0 and 1, i.e., $\mathrm{Kos}^1(\theta) = \mathrm{Cone}(-\theta)[-1]$. If $u = \mathrm{id}_A$, its Koszul complex identifies with the divided power de Rham complex of $\Gamma^{\bullet}(A)$. In most of our applications, A_X will contain $\mathbb{Q}$, and we can and shall identify $\Gamma^q(A)$ with $S^q(A)$, the q-th symmetric power of A.

We recall the following well-known result (see [Steenbrink 1995, Lemma 1.4]):

Proposition 2.2.4. *Suppose that*

$$0 \to A \xrightarrow{u} B \to C \to 0$$

is an exact sequence of flat A_X-modules. Then the natural map

$$e_q : \mathrm{Kos}^q(u)[q] \to \textstyle\bigwedge^q C$$

is a quasi-isomorphism.

Proof. We include a proof for the convenience of the reader. The last nonzero term of the complex $\mathrm{Kos}^q(u)[q]$ is $\bigwedge^q B$ in degree 0, and the natural map $\bigwedge^q B \to \bigwedge^q C$ induces the morphism e_q. The Koszul filtration K on $\bigwedge^{\bullet} B$ makes $\mathrm{Kos}^q(u)$ a filtered complex, with

$$K^i \,\mathrm{Kos}^q(u)^n := \Gamma^{q-n}(A) \otimes K^i \textstyle\bigwedge^n B.$$

Note that the differential d of $\mathrm{Kos}^q(u)$ sends $K^i \,\mathrm{Kos}^q(u)$ to $K^{i+1} \,\mathrm{Kos}^q(u)$. Then the spectral sequence of the filtered complex $(\mathrm{Kos}^q(u), K)$ has

$$E_1^{i,j} = H^{i+j}(\mathrm{Gr}^i_K \,\mathrm{Kos}^q(u)) = \mathrm{Gr}^i_K \,\mathrm{Kos}^q(u)^{i+j} = \Gamma^{q-i-j}(A) \otimes \textstyle\bigwedge^i A \otimes \bigwedge^j C,$$

and the complex $(E_1^{\bullet,j}, d_1^{\bullet,j})$ identifies with the complex $\mathrm{Kos}(\mathrm{id}_A)^{q-j} \otimes \bigwedge^j C[-j]$, up to the sign of the differential. This complex is acyclic unless $j = q$, in which case it reduces to the single term complex $\bigwedge^q C[-q]$. It follows that the map $e_q[-q] : \mathrm{Kos}^q(u) \to \bigwedge^q C[-q]$ is a quasi-isomorphism. □

The following technical result compares the various Koszul complexes associated to u.

Proposition 2.2.5. *Let $0 \to A \xrightarrow{u} B \xrightarrow{\pi} C \to 0$ be an exact sequence of flat A_X-modules. For each $q \geq 0$, let K be the Koszul filtration of $\bigwedge^q B$ induced by u, and let*

$$u_q : A \otimes \textstyle\bigwedge^{q-1} C \to \bigwedge^q B / K^2 \bigwedge^q B$$

be as in Proposition 2.2.2.

(1) *There is a natural commutative diagram of quasi-isomorphisms*:

$$\begin{array}{ccc} \mathrm{Kos}^q(u)[q] & \xrightarrow{a_q} & \mathrm{Cone}((-1)^q u_q) \\ & \searrow^{e_q} & \downarrow b_q \\ & & \textstyle\bigwedge^q C \end{array}$$

(2) *There exist morphisms of complexes c_q and f_q as indicated below. Each of these is a quasi-isomorphism of complexes, and the resulting diagram is commutative. Hence there is a unique morphism g_q in $D(X)$ making the following*

diagram commute:

$$\begin{array}{ccccc} \mathrm{Kos}^q(u)[q] & \xrightarrow{a_q} & \mathrm{Cone}((-1)^q u_q) & \xrightarrow{b_q} & \bigwedge^q C \\ \downarrow{\scriptstyle c_q} & & & \searrow{\scriptstyle f_q} & \vdots\,{\scriptstyle g_q} \\ A \otimes \mathrm{Kos}^{q-1}(u)[q] & & \xrightarrow[\mathrm{id}\otimes e_{q-1}]{} & & A \otimes \bigwedge^{q-1} C[1] \end{array}$$

with the curved arrow $e_q : \mathrm{Kos}^q(u)[q] \to \bigwedge^q C$ along the top.

(3) *In the derived category* $D(X)$, $g_q = (-1)^{q-1}\xi_{u_q}$, *where* ξ_{u_q} *is the morphism defined by* u_q *as in Definition 2.1.3. Consequently*, g_q *is* $(-1)^{q-1}$ *times cup-product* (*on the left*) *with the morphism* ξ_u *defined by* u.

Proof. The vertical arrows in the following commutative diagram of complexes are the obvious projections. The first set of these defines the morphism of complexes a_q and the second defines the morphism b_q.

$$\begin{array}{ccccccc} \Gamma^q(A) & \longrightarrow & \cdots \Gamma^2(A) \otimes \bigwedge^{q-2} B & \longrightarrow & A \otimes \bigwedge^{q-1} B & \longrightarrow & \bigwedge^q B \\ & & \downarrow{\scriptstyle a_q} & & \downarrow{\scriptstyle a_q} & & \downarrow{\scriptstyle a_q} \\ & & 0 & \longrightarrow & A \otimes \bigwedge^{q-1} C & \xrightarrow{(-1)^q u_q} & \bigwedge^q B / K^2 \bigwedge^q B \\ & & & & & & \downarrow{\scriptstyle b_q} \\ & & & & & & \bigwedge^q C \end{array}$$

Here the top row is placed in degrees $-q$ through 0, and its differential is the Koszul differential multiplied by $(-1)^q$, and thus is the complex $\mathrm{Kos}^q(u)[q]$. The middle row is placed in degrees -1 and 0, and hence is the mapping cone of $(-1)^q u_q$. Since the sequence

$$0 \to A \otimes \bigwedge^{q-1} C \to \bigwedge^q B / K^2 \bigwedge^q B \to \bigwedge^q C \to 0$$

is exact, the map b_q is a quasi-isomorphism. We observed in Proposition 2.2.4 that e_q is a quasi-isomorphism, and it follows that a_q is also a quasi-isomorphism. This proves statement (1) of the proposition.

The morphism f_q is defined by the usual projection

$$\mathrm{Cone}((-1)^q u_q) = \left[A \otimes \bigwedge^{q-1} C \xrightarrow{(-1)^q u_q} \bigwedge^q B / K^2 \bigwedge^q B \right] \xrightarrow{p=(\mathrm{id},0)} A \otimes \bigwedge^{q-1} C.$$

The following diagram commutes, with the sign shown, because of the conventions in (2-1-1) and Definition 2.1.3:

$$\begin{array}{ccc} \mathrm{Cone}((-1)^q u_q) & \xrightarrow{\sim} & \bigwedge^q C \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle -\xi_{(-1)^q u_q}} & \\ A \otimes \bigwedge^{q-1} C & & \end{array}$$

One checks easily that the square below commutes, so that the vertical map defines a morphism of complexes $\mathrm{Kos}^q(u) \to A \otimes \mathrm{Kos}^{q-1}(u)$ whose shift is c_q:

$$\begin{array}{ccc} \Gamma^{q-n}(A) \otimes \bigwedge^n B & \xrightarrow{d_u} & \Gamma^{q-n-1}(A) \otimes \bigwedge^{n+1} B \\ {\scriptstyle \eta\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle d_{\mathrm{id}}\otimes\mathrm{id}} \\ A \otimes \Gamma^{q-n-1}(A) \otimes \bigwedge^n B & \xrightarrow[\mathrm{id}\otimes d_u]{} & A \otimes \Gamma^{q-n-2}(A) \otimes \bigwedge^{n+1} B \end{array}$$

The diagram of statement (2) in degree $q-1$ is given by the following obvious set of maps:

$$\begin{array}{ccccc} A \otimes \bigwedge^{q-1} B & \longrightarrow & A \otimes \bigwedge^{q-1} C & \longrightarrow & 0 \\ {\scriptstyle \mathrm{id}}\downarrow & & & \searrow{\scriptstyle \mathrm{id}} & \\ A \otimes \bigwedge^{q-1} B & & \longrightarrow & & A \otimes \bigwedge^{q-1} C \end{array}$$

and in degree q by

$$\begin{array}{ccccc} \bigwedge^q B & \longrightarrow & \bigwedge^q B / K^2 \bigwedge^q B & \longrightarrow & \bigwedge^q C \\ \downarrow & & & \searrow & \\ 0 & & \longrightarrow & & 0 \end{array}$$

This proves statement (2). It follows that $g_q = -\xi_{(-1)^q u_q} = (-1)^{q-1}\xi_{u_q}$ and the rest of statement (3) then follows from Proposition 2.2.2. □

2.3. ***τ-unipotent maps in the derived category.*** One frequently encounters unipotent automorphisms of objects, or more precisely, automorphisms γ of filtered objects (C, F) which induce the identity on the associated graded object $\mathrm{Gr}^\bullet_F(C)$. Then $\gamma - \mathrm{id}$ induces a map $\mathrm{Gr}^\bullet_F(C) \to \mathrm{Gr}^{\bullet-1}_F C$ which serves as an approximation to γ. For example, if γ is an automorphism of a complex C which acts as the identity on its cohomology, this construction can be applied to the canonical filtration $\tau_\leq$ of C and carries over to the derived category.

Lemma 2.3.1. *Let $\lambda : A \to B$ be a map in $D(\mathcal{A})$, and let q be an integer such that the maps $\mathcal{H}^i(\lambda) : \mathcal{H}^i(A) \to \mathcal{H}^i(B)$ are zero for $i = q-1, q$. Then there exists* *a unique morphism* $L^q_\lambda : \mathcal{H}^q(A)[-q] \to \mathcal{H}^{q-1}(B)[1-q]$ *making the following diagram commute*:

$$\begin{array}{ccc}
\tau_{[q-1,q]}(A) & \longrightarrow & \mathcal{H}^q(A)[-q] \\
{\scriptstyle \tau_{[q-1,q]}(\lambda)}\big\downarrow & & \big\downarrow{\scriptstyle L^q_\lambda} \\
\tau_{[q-1,q]}(B) & \longleftarrow & \mathcal{H}^{q-1}(B)[1-q]
\end{array}$$

The same map L^q_λ *fits into the commutative diagram*

$$\begin{array}{ccccc}
\tau_{\leq q}A & \longrightarrow & \mathcal{H}^q(A)[-q] & & \\
{\scriptstyle \tau_{\leq q}(\lambda)}\big\downarrow & \searrow & & \overset{L^q_\gamma}{\searrow} & \\
\tau_{\leq q}B & \longleftarrow & \tau_{\leq q-1}B & \longrightarrow & \mathcal{H}^{q-1}(B)[1-q]
\end{array}$$

Proof. Consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
\mathrm{Hom}(\tau_{[q]}A, \tau_{[q]}B[-1]) & \longrightarrow & \mathrm{Hom}(\tau_{[q-1,q]}A, \tau_{[q]}B[-1]) & \longrightarrow & \mathrm{Hom}(\tau_{[q-1]}A, \tau_{[q]}B[-1]) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(\tau_{[q]}A, \tau_{[q-1]}B) & \longrightarrow & \mathrm{Hom}(\tau_{[q-1,q]}A, \tau_{[q-1]}B) & \longrightarrow & \mathrm{Hom}(\tau_{[q-1]}A, \tau_{[q-1]}B) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(\tau_{[q]}A, \tau_{[q-1,q]}B) & \longrightarrow & \mathrm{Hom}(\tau_{[q-1,q]}A, \tau_{[q-1,q]}B) & \longrightarrow & \mathrm{Hom}(\tau_{[q-1]}A, \tau_{[q-1,q]}B) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(\tau_{[q]}A, \tau_{[q]}B) & \longrightarrow & \mathrm{Hom}(\tau_{[q-1,q]}A, \tau_{[q]}B) & \longrightarrow & \mathrm{Hom}(\tau_{[q-1]}A, \tau_{[q]}B)
\end{array}$$

Note that the groups in the top row and the group in the bottom right corner are zero, as $\mathrm{Hom}(X, Y) = 0$ if there exists an $n \in \mathbb{Z}$ such that $\tau_{\geq n}X = 0$ and $\tau_{\leq n-1}Y = 0$. Similarly, the left horizontal maps are injective. The first claim follows then by diagram chasing.

For the second assertion, we can first assume that $A = \tau_{\leq q}A$ and $B = \tau_{\leq q}B$. We can then reduce further to the case $A = \tau_{[q-1,q]}(A)$ and $B = \tau_{[q-1,q]}(B)$, whereupon the claim becomes identical to the first assertion. □

Proposition 2.3.2. *Let* $C \xrightarrow{i} A \xrightarrow{\lambda} B \xrightarrow{\rho} C[1]$ *be a distinguished triangle in the derived category* $D(\mathcal{A})$, *and consider the corresponding exact sequence*

$$\cdots \to \mathcal{H}^{q-1}(A) \xrightarrow{\lambda} \mathcal{H}^{q-1}(B) \xrightarrow{\rho} \mathcal{H}^q(C) \xrightarrow{i} \mathcal{H}^q(A) \xrightarrow{\lambda} \mathcal{H}^q(B) \to \cdots .$$

Assume that $\mathcal{H}^q(\lambda) = \mathcal{H}^{q-1}(\lambda) = 0$, *so that we have a short exact sequence*

$$0 \to \mathcal{H}^{q-1}(B) \xrightarrow{\rho} \mathcal{H}^q(C) \xrightarrow{i} \mathcal{H}^q(A) \to 0.$$

Let

$$\kappa^q : \mathcal{H}^q(A) \to \mathcal{H}^{q-1}(B)[1]$$

be the corresponding derived map, as in Definition 2.1.3. Then $\kappa^q = (-1)^{q-1} L^q_\lambda[q]$, *where* L^q_λ *is the map defined in Lemma 2.3.1.*

Proof. First note that if $\lambda' : A' \to B'$ satisfies $\mathcal{H}^{q-1}(\lambda') = \mathcal{H}^q(\lambda') = 0$ and there is a commutative diagram of the form

$$\begin{array}{ccc} A' & \xrightarrow{\lambda'} & B' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ A & \xrightarrow[\lambda]{} & B \end{array}$$

with the property that $\mathcal{H}^i(a)$ and $\mathcal{H}^i(b)$ are isomorphisms for $i = q-1, q$, then the proposition holds for λ' if and only if it holds for λ. Indeed, any distinguished triangle containing λ' fits into a commutative diagram

$$\begin{array}{ccccccc} C' & \xrightarrow{i'} & A' & \xrightarrow{\lambda'} & B' & \xrightarrow{\rho'} & C'[1] \\ {\scriptstyle c}\downarrow & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c[1]} \\ C & \xrightarrow[i]{} & A & \xrightarrow[\lambda]{} & B & \xrightarrow[\rho]{} & C[1] \end{array}$$

Applying the functor $\tau_{\leq q}$ leaves $\mathcal{H}^i$ unchanged for $i \leq q$, and applying $\tau_{\geq q-1}$ leaves $\mathcal{H}^i$ unchanged for $i \geq q-1$. Thus we may without loss of generality assume that $A = \tau_{[q-1,q]}(A)$ and $B = \tau_{[q-1,q]}(B)$. We have a morphism of distinguished triangles:

$$\begin{array}{ccccccc} \mathcal{H}^{q-1}(A)[1-q] & \xrightarrow{a} & A & \xrightarrow{b} & \mathcal{H}^q(A)[-q] & \longrightarrow & \mathcal{H}^{q-1}(A)[2-q] \\ {\scriptstyle \lambda=0}\downarrow & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle \lambda=0} & & \downarrow{\scriptstyle \lambda=0} \\ \mathcal{H}^{q-1}(B)[1-q] & \xrightarrow{a} & B & \xrightarrow{b} & \mathcal{H}^q(B)[-q] & \longrightarrow & \mathcal{H}^{q-1}(B)[2-q] \end{array}$$

The left map being zero by hypothesis, we have $\lambda \circ a = 0$, and hence λ factors through $b : A \to \mathcal{H}^q(A)[-q]$. It thus suffices to prove the assertion with the morphism $\mathcal{H}^q(A)[-q] \to B$ in place of λ. Similarly, since $\mathcal{H}^q(\lambda) = 0$, we may replace B by $\tau_{\leq q-1}(B)$. Thus we are reduced to the case in which $A = \mathcal{H}^q(A)[-q]$ and $B = \mathcal{H}^{q-1}(B)[1-q]$. It follows that $C = \mathcal{H}^q(C)[-q]$. Note that $\lambda = L^q_\lambda$ in this situation. Therefore we have a commutative diagram whose vertical maps are isomorphisms:

$$\begin{array}{ccccccc}
B[-1] & \xrightarrow{-\rho} & C & \xrightarrow{i} & A & \xrightarrow{\lambda} & B \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{H}^{q-1}(B)[-q] & \xrightarrow[-\rho]{} & \mathcal{H}^{q}(C)[-q] & \xrightarrow[i]{} & \mathcal{H}^{q}(A)[-q] & \xrightarrow[\lambda]{} & \mathcal{H}^{q-1}(B)[1-q]
\end{array}$$

Since the top row is distinguished, it follows that the bottom row is distinguished as well. Applying $[q]$ shows that

$$\mathcal{H}^{q-1}(B) \xrightarrow{-(-1)^q\rho} \mathcal{H}^{q}(C) \xrightarrow{(-1)^q i} \mathcal{H}^{q}(A) \xrightarrow{(-1)^q\lambda[q]} \mathcal{H}^{q-1}(B)[1]$$

is distinguished. This is isomorphic to

$$\mathcal{H}^{q-1}(B) \xrightarrow{\rho} \mathcal{H}^{q}(C) \xrightarrow{i} \mathcal{H}^{q}(A) \xrightarrow{(-1)^{q+1}\lambda[q]} \mathcal{H}^{q-1}(B)[1].$$

As we observed after Definition 2.1.3, the fact that these complexes are concentrated in a single degree implies that the last map is the unique one making the triangle distinguished. Thus $\kappa = (-1)^{q+1}\lambda[q] = (-1)^{q+1}L^q_\lambda[q]$, as desired. □

3. Logarithmic preliminaries

3.1. ***Notation and conventions.*** For the basic facts about log schemes, especially the definitions of log differentials and log smoothness, we refer to Kato's seminal paper [1989] and the forthcoming book [Ogus 2018]. Here we recall a few essential notions and constructions for the convenience of the reader.

Monoids and monoid algebras. If $(P, +, 0)$ is a commutative monoid, we denote by P^* the subgroup of units of P, by P^+ the complement of P^*, and by $\overline{P}$ the quotient of P by P^*. A monoid P is said to be *sharp* if $P^* = 0$. If R is a fixed base ring, we write $R[P]$ for the monoid algebra on P over R. This is the free R-module with basis

$$e : P \to R[P], \quad p \mapsto e^p$$

and with multiplication defined so that $e^p e^q = e^{p+q}$. The corresponding scheme $\underline{\mathsf{A}}_P := \operatorname{Spec}(R[P])$ has a natural structure of a monoid scheme. There are two natural augmentations $R[P] \to R$. The first of these, corresponding to the identity section of $\underline{\mathsf{A}}_P$, is given by the homomorphism $P \to R$ sending every element to the identity element 1 of R. The second, which we call the *vertex* of $\underline{\mathsf{A}}_P$, is defined by the homomorphism sending P^* to $1 \in R$ and P^+ to $0 \in R$. The two augmentations coincide if P is a group.

A commutative monoid P is said to be *integral* if the universal map $P \to P^{\mathrm{gp}}$ from P to a group is injective. An integral monoid P is said to be *saturated* if for every $x \in P^{\mathrm{gp}}$ such that $nx \in P$ for some positive integer n, in fact $x \in P$. A monoid is *fine* if it is integral and finitely generated and is *toric* if it is fine and saturated and P^{gp} is torsion free. An *ideal in a monoid* P is a subset of P which is stable under addition by elements of P. If J is an ideal in P, then $R[P, J]$ denotes the quotient of the monoid algebra $R[P]$ by the ideal generated by J. The complement of J in P is a basis for the underlying R-module of $R[P, J]$.

Log structures. A *prelog structure* on a ringed space $(X, \mathcal{O}_X)$ is a homomorphism α from a sheaf of commutative monoids $\mathcal{M}$ to the multiplicative monoid underlying $\mathcal{O}_X$. A *log structure* is a prelog structure such that the induced map

$$\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$$

is an isomorphism. The *trivial* log structure is the inclusion $\mathcal{O}_X^* \to \mathcal{O}_X$. A ringed spaceX endowed with a log structure α_X is referred to as a *log space.* An *idealized log space* is a log space (X, α_X) together with a sheaf of ideals $\mathcal{K}_X$ in $\mathcal{M}_X$ such that $\alpha_X(\mathcal{K}_X) = 0$ [Ogus 2003, 1.1; 2018, §III, 1.3]. A prelog structure $\alpha : P \to \mathcal{O}_X$ on a ringed space factors through a universal associated log structure $\alpha^a : P^a \to \mathcal{O}_X$. A log structure α on X is said to be *fine* (resp. fine and saturated) if locally on X there exists a fine (resp. fine and saturated) constant sheaf of monoids P and a prelog structure $P \to \mathcal{O}_X$ whose associated log structure is α. There is an evident way to form a category of log schemes, and the category of fine (resp. fine saturated) log schemes admits fiber products, although their construction is subtle. Grothendieck's deformation theory provides a geometric way to define smoothness for morphisms of log schemes, and many standard "degenerate" families become logarithmically smooth when endowed with a suitable log structure. A morphism of integral log spaces $f : X \to Y$ is *vertical* if the quotient $\mathcal{M}_{X/Y}$ of the map $f_{\log}^*(\mathcal{M}_Y) \to \mathcal{M}_X$, computed in the category of sheaves of monoids, is in fact a group. We shall use the notions of exactness, integrality, and saturatedness for morphisms of log schemes, for which we refer to the above references and also to [Tsuji 2019; Illusie et al. 2005].

If P is a commutative monoid and $\beta : P \to A$ is a homomorphism into the multiplicative monoid underlying a commutative ring A, we denote by $\operatorname{Spec}(\beta)$ the scheme $\operatorname{Spec} A$ endowed with the log structure associated to the prelog structure induced by β. In particular, if R is a fixed base ring and $P \to R[P]$ is the canonical homomorphism from P to the monoid R-algebra of P, then A_P denotes the log scheme $\operatorname{Spec}(P \to R[P])$, and if P is fine and $R = \mathbb{C}$, we write $\mathsf{A}_P^{\mathrm{an}}$ for the log analytic space associated to A_P. (If the analytic context is clear, we may just write A_P for this space.) If $v : P \to R$ is the vertex of $\underline{\mathsf{A}}_P$ (the homomorphism sending P^+ to zero and P^* to 1), the log scheme $\operatorname{Spec}(v)$ is called the *split log point* associated to P; it is called the *standard log point* when $P = \mathbb{N}$. If J is an ideal in the monoid P, we let $\mathsf{A}_{P,J}$ denote the closed idealized log subscheme of A_P

defined by the ideal J of P. The underlying scheme of $\mathsf{A}_{P,J}$ is the spectrum of the algebra $R[P, J]$, and the points of $\mathsf{A}^{\mathrm{an}}_{P,J}$ are the homomorphisms $P \to \mathbb{C}$ sending J to zero.

If P is a toric monoid, the log analytic space $\mathsf{A}^{\mathrm{an}}_P$ is a partial compactification of its dense open subset $\underline{\mathsf{A}}^*_P := \mathsf{A}^{\mathrm{an}}_{P^{\mathrm{gp}}}$, and the (logarithmic) geometry of $\mathsf{A}^{\mathrm{an}}_P$ expresses the geometry of this compactified set, a manifold with boundary. The underlying topological space of $\underline{\mathsf{A}}^*_P$ is $\mathrm{Hom}(P, \mathbb{C}^*)$, and its fundamental group I_P (the "log inertia group") will play a fundamental role in what follows.

3.2. ***Some groups and extensions associated to a monoid.*** Let us gather here the key facts and notations we shall be using. If P is a toric monoid (i.e., if P is fine and saturated and P^{gp} is torsion free) we define

$$\mathsf{T}_P := \mathrm{Hom}(P, \mathbb{S}^1), \qquad \text{where } \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\},$$
$$\mathsf{R}_P := \mathrm{Hom}(P, \mathbb{R}_{\geq}),$$
$$\text{where } \mathbb{R}_{\geq} := \big\{r \in \mathbb{R} : r \geq 0, \text{ with its multiplicative monoid law}\big\},$$
$$\mathsf{I}_P := \mathrm{Hom}(P, \mathbb{Z}(1)), \qquad \text{where } \mathbb{Z}(1) := \{2\pi i n : n \in \mathbb{Z}\} \subseteq \mathbb{C},$$
$$\mathsf{V}_P := \mathrm{Hom}(P, \mathbb{R}(1)), \qquad \text{where } \mathbb{R}(1) := \{ir : r \in \mathbb{R}\} \subseteq \mathbb{C},$$
$$\mathsf{L}_P := \{\text{affine mappings } \mathsf{I}_P \to \mathbb{Z}(1)\},$$
$$\chi : P^{\mathrm{gp}} \xrightarrow{\sim} \mathrm{Hom}(\mathsf{T}_P, \mathbb{S}^1), \quad p \mapsto \chi_p, \qquad \text{where } \chi_p(\sigma) := \sigma(p),$$
$$\tilde{\chi} : P^{\mathrm{gp}} \xrightarrow{\sim} \mathrm{Hom}(\mathsf{I}_P, \mathbb{Z}(1)) \subseteq \mathsf{L}_P, \quad p \mapsto \tilde{\chi}_p, \qquad \text{where } \tilde{\chi}_p(\gamma) := \gamma(p).$$

An affine mapping $f : \mathsf{I}_P \to \mathbb{Z}(1)$ can be written uniquely as a sum $f = f(0) + h$, where h is a homomorphism $\mathsf{I}_P \to \mathbb{Z}(1)$. Since P is toric, the map $\tilde{\chi}$ is an isomorphism, so $h = \tilde{\chi}_p$ for a unique $p \in P^{\mathrm{gp}}$. Thus the group L_P is a direct sum $\mathbb{Z}(1) \oplus P^{\mathrm{gp}}$, which we write as an exact sequence

$$0 \to \mathbb{Z}(1) \to \mathsf{L}_P \xrightarrow{\xi} P^{\mathrm{gp}} \to 0, \tag{3-2-1}$$

for reasons which will become apparent shortly.

The inclusion $\mathbb{S}^1 \to \mathbb{C}^*$ is a homotopy equivalence, and hence so is the induced map $\mathsf{T}_P \to \underline{\mathsf{A}}^*_P$, for any P. Thus the fundamental groups of $\underline{\mathsf{A}}^*_P$ and T_P can be canonically identified. The exponential mapping $\theta \mapsto e^{\theta}$ defines the universal covering space $\mathbb{R}(1) \to \mathbb{S}^1$, and there is an induced covering space $\mathsf{V}_P \to \mathsf{T}_P$. The subgroup $\mathsf{I}_P = \mathrm{Hom}(P, \mathbb{Z}(1))$ of V_P acts naturally on V_P by translation:

$$(v, \gamma) \mapsto v + \gamma \quad \text{if } v \in \mathsf{V}_P \text{ and } \gamma \in \mathsf{I}_P.$$

The induced action on T_P is trivial, and in fact I_P can be identified with the covering group of the covering $\mathsf{V}_P \to \mathsf{T}_P$, i.e., the fundamental group of T_P. (Since the group is abelian we do not need to worry about base points.) We view I_P as acting on the right on the geometric object V_P and on the left on the set of functions on V_P: if f is a function on V_P, we have

$$(\gamma f)(v) = f(v+\gamma).$$

In particular, if f is constant, then $\gamma f = f$, and if $p \in P^{\mathrm{gp}}$,

$$\gamma \tilde{\chi}_p = \tilde{\chi}_p + \gamma(p). \tag{3-2-2}$$

It follows that the set L_P of affine mappings $\mathsf{I}_P \to \mathbb{Z}(1)$ is stable under the I_P-action (3-2-2). The homomorphism $\tilde{\chi} : P^{\mathrm{gp}} \to \mathsf{L}_P$ is a canonical splitting of the exact sequence (3-2-1), but the splitting is not stable under the action of I_P, as the formula (3-2-2) shows. The formula also shows that the exact sequence (3-2-1) can be viewed as an extension of trivial I_P-modules. Any $f \in \mathsf{L}_P$ extends naturally to an affine transformation $\mathsf{V}_P \to \mathbb{R}(1)$, and in fact L_P is the smallest I_P-stable subset of the set of functions $\mathsf{V}_P \to \mathbb{R}(1)$ containing $\tilde{\chi}_p$ for all $p \in P^{\mathrm{gp}}$.

The dual of the extension (3-2-1) has an important geometric interpretation. Consider the group algebra $\mathbb{Z}[\mathsf{I}_P]$ with basis $e : \mathsf{I}_P \to \mathbb{Z}[\mathsf{I}_P]$. It is equipped with a right action of I_P defined by $e^\delta\gamma = e^{\delta+\gamma}$. Its augmentation ideal J is generated by elements of the form $e^\delta - 1$ for $\delta \in \mathsf{I}_P$ and is stable under the action of I_P. The induced action on J/J^2 is trivial, and there is an isomorphism of abelian groups:

$$\lambda : \mathsf{I}_P \to J/J^2, \quad \gamma \mapsto [e^\gamma - e^0]. \tag{3-2-3}$$

Identifying I_P with J/J^2, we have a split exact sequence of I_P-modules:

$$0 \to \mathsf{I}_P \to \mathbb{Z}[\mathsf{I}_P]/J^2 \to \mathbb{Z} \to 0, \tag{3-2-4}$$

where the action of I_P on I_P and on $\mathbb{Z}$ is trivial.

Proposition 3.2.1. *There is a natural isomorphism*

$$\mathsf{L}_P \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}[\mathsf{I}_P]/J^2, \mathbb{Z}(1)),$$

compatible with the structures of extensions (3-2-1) and (3-2-4) and the (left) actions of I_P. The boundary map ∂ arising from the extension (3-2-1)

$$\partial : P^{\mathrm{gp}} \to H^1(\mathsf{I}_P, \mathbb{Z}(1)) \cong \mathrm{Hom}(\mathsf{I}_P, \mathbb{Z}(1)) \cong P^{\mathrm{gp}}$$

is the identity.

Proof. Since $\mathbb{Z}[\mathsf{I}_P]$ is the free abelian group with basis I_P, the map $f \to h_f$ from the set of functions $f : \mathsf{I}_P \to \mathbb{Z}(1)$ to the set of homomorphisms $\mathbb{Z}[\mathsf{I}_P] \to \mathbb{Z}(1)$ is an isomorphism, compatible with the natural left actions of I_P. If $f : \mathsf{I}_P \to \mathbb{Z}(1)$, then h_f annihilates J if and only if $f(\gamma) = f(0)$ for every γ, i.e., if and only if $f \in \mathbb{Z}(1) \subseteq L_P$. Furthermore, h_f annihilates J^2 if and only if for every pair of elements γ, δ of I_P,

$$h_f((e^\delta - 1)(e^\gamma - 1)) = 0,$$

i.e., if and only if

$$f(\delta+\gamma) - f(\delta) - f(\gamma) + f(0) = 0.$$

But this holds if and only if $f(\delta+\gamma) - f(0) = f(\delta) - f(0) + f(\gamma) - f(0)$, i.e., if and only if $f - f(0)$ belongs to $\tilde{\chi}(P^{\mathrm{gp}})$, i.e., if and only if $f \in \mathsf{L}_P$.

To check the claim about the boundary map ∂, and in particular its sign, we must clarify our conventions. If Γ is a group and E is a Γ-module, then we view $H^1(\Gamma, E)$ as the set of isomorphism classes of E-torsors in the category of Γ-sets. If the action of Γ on E is trivial and L is such a torsor, then for any $\ell \in L$ and any $\gamma \in \Gamma$, the element $\phi_{L,\gamma} := \gamma(\ell) - \ell$ is independent of the choice of ℓ, the mapping $\gamma \to \phi_{L,\gamma}$ is a homomorphism $\phi_L : \Gamma \to E$, and the correspondence $L \mapsto \phi_L$ is the isomorphism

$$\phi : H^1(\Gamma, E) \to \mathrm{Hom}(\Gamma, E). \tag{3-2-5}$$

To verify the claim, let p be an element of P^{gp}. Then $\partial(p) \in H^1(\mathsf{I}_P, \mathbb{Z}(1))$ is the $\mathbb{Z}(1)$-torsor of all $f \in L_P$ whose image under $\xi : \mathsf{L}_P \to P^{\mathrm{gp}}$ is p. Choose any such f, and write $f = f(0) + \tilde{\chi}_p$. Then if $\gamma \in \mathsf{I}_P$, we have $\gamma(f) = f + \gamma(p)$, and thus

$$\partial(p) \mapsto \phi_L(\gamma) = \gamma(f) - f = \gamma(p) = \tilde{\chi}_p(\gamma).$$

This equality verifies our claim. □

3.3. ***Betti realizations of log schemes.*** Since the Betti realization $X_{\log}$ of an fs-log analytic space X plays a crucial role here, we briefly review its construction. As a set, $X_{\log}$ consists of pairs (x, σ), where x is a point of X and σ is a homomorphism of monoids making the following diagram commute:

$$\begin{array}{ccc} \mathcal{O}^*_{X,x} & \longrightarrow & \mathcal{M}_{X,x} \\ {\scriptstyle f\mapsto f(x)}\downarrow & & \vdots\,{\scriptstyle \sigma} \\ \mathbb{C}^* & \xrightarrow[\arg]{} & \mathbb{S}^1 \end{array}$$

The map $\tau_X : X_{\log} \to X$ sends (x, σ) to x. A (local) section m of $\mathcal{M}_X$ gives rise to a (local) function $\arg(m) : X_{\log} \to \mathbb{S}^1$, and the topology on $X_{\log}$ is the weak topology coming from the map τ_X and these functions. The map τ_X is proper, and for $x \in X$, the fiber $\tau_X^{-1}(x)$ is naturally a torsor under the group $\mathsf{T}_{X,x} := \mathrm{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X,x}, \mathbb{S}^1)$. Thus the fiber is connected if and only if $\overline{\mathcal{M}}^{\mathrm{gp}}_{X,x}$ is torsion free, and if this is the case, the fundamental group $\mathsf{I}_{X,x}$ of the fiber is canonically isomorphic to $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{Z}(1))$. The map $\tau_X : X_{\log} \to X_{\mathrm{top}}$ is characterized by the property that for every topological space T, the set of morphisms $T \to X_{\log}$ identifies with the set of pairs (p, c), where $p : T \to X_{\mathrm{top}}$ is a continuous map and $c : p^{-1}(\mathcal{M}_X^{\mathrm{gp}}) \to \mathbb{S}^1_T$ is a homomorphism from $p^{-1}(\mathcal{M}_X^{\mathrm{gp}})$ to the sheaf $\mathbb{S}^1_T$ of continuous $\mathbb{S}^1$-valued functions on T such that $c(f) = \arg(f)$ for all $f \in p^{-1}(\mathcal{O}_X^*)$.

When $X = \mathsf{A}_{P,J}$, the construction of $S_{\log}$ can be understood easily as the introduction of "polar coordinates." The multiplication map $\mathbb{R}_{\geq} \times \mathbb{S}^1 \to \mathbb{C}$ maps polar coordinates to the standard complex coordinate. Let $\mathsf{R}_{P,J} := \{\rho \in \mathsf{R}_P : \rho(J) = 0\}$. Multiplication induces a natural surjection:

$$\tau : \mathsf{R}_{P,J} \times \mathsf{T}_P \to \mathsf{A}^{\mathrm{an}}_{P,J}, \quad (\rho, \sigma) \mapsto \rho\sigma. \tag{3-3-1}$$

Then $\mathsf{A}^{\log}_{P,J} \cong \mathsf{R}_{P,J} \times \mathsf{T}_P$, and τ corresponds to the canonical map $\tau_{\mathsf{A}^{\log}_{P,J}}$. The exponential map induces a universal covering

$$\eta : \tilde{\mathsf{A}}^{\log}_{P,J} := \mathsf{R}_{P,J} \times \mathsf{V}_P \to \mathsf{A}^{\log}_{P,J}, \tag{3-3-2}$$

whose covering group identifies naturally with I_P. Thus the group I_P is also the fundamental group of $\mathsf{A}^{\log}_{P,J}$.

Remark 3.3.1. If X is a smooth curve endowed with the compactifying log structure induced by the complement of a point x, then $\tau_X : X_{\log} \to X_{\mathrm{top}}$ is the "real oriented blow-up"of X at x, and there is a natural bijection between $\tau_X^{-1}(x)$ and the set of "real tangent directions" $(T_x\underline{X} \setminus \{0\})/\mathbb{R}_>$ at x. Below we provide a more general and robust identification of this kind.

If X is any log analytic space and q is a global section of $\overline{\mathcal{M}}_X^{\mathrm{gp}}$, let $\mathcal{L}_q^*$ denote the sheaf of sections of $\mathcal{M}_X^{\mathrm{gp}}$ which map to q. This sheaf has a natural structure of an $\mathcal{O}_X^*$-torsor, and we let $\mathcal{L}_q$ denote the corresponding invertible sheaf of $\mathcal{O}_X$-modules and $\mathcal{L}_q^\vee$ its dual. A local section m of $\mathcal{L}_q^*$ defines a local generator for the invertible sheaf $\mathcal{L}_q$. If (x, σ) is a point of $X_{\log}$ and m is a local section of $\mathcal{L}_q^*$, let $m(x)$ be the value of m in the one-dimensional $\mathbb{C}$-vector space $\mathcal{L}_q(x)$ and let $\phi_m \in \mathcal{L}_q^\vee(x)$ be the unique linear map $\mathcal{L}(x) \to \mathbb{C}$ taking $m(x)$ to $\sigma(m)$. If m' is another local section of $\mathcal{L}_q^*$, there is a unique local section u of $\mathcal{O}_X^*$ such that $m' = um$, and then $\phi_{m'} = |u(x)|^{-1}\phi_m$. Indeed,

$$\begin{aligned}\phi_{m'}(m(x)) &= u(x)^{-1}\phi_{m'}(m'(x)) = u(x)^{-1}\sigma(m') \\ &= u(x)^{-1}\arg(u(x))\sigma(m) = |u(x)|^{-1}\phi_m(m(x)).\end{aligned}$$

Thus ϕ'_m and ϕ_m have the same image in the quotient of $\mathcal{L}_q^\vee(x)$ by the action of $\mathbb{R}_>$. This quotient corresponds to the set of directions in the one-dimensional complex vector space $\mathcal{L}_q^\vee(x)$. If L is any one-dimensional complex vector space, it seems reasonable to denote the quotient $L/\mathbb{R}_>$ by $\mathbb{S}^1(L)$. Thus we see that there is a natural map: $\beta : \tau_X^{-1}(x) \to \mathbb{S}^1(\mathcal{L}_q^\vee(x))$. The source of this continuous map is a torsor under $\mathsf{T}_{X,x} = \mathrm{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X,x} . \mathbb{S}^1)$ and its target is naturally a torsor under $\mathbb{S}^1$. One verifies immediately that if $\zeta \in \mathrm{Hom}(\overline{\mathcal{M}}_X^{\mathrm{gp}}, \mathbb{S}^1)$ and $\sigma \in \tau_X^{-1}(x)$, then $\beta(\zeta\sigma) = \zeta(q)\beta(\sigma)$.

When α_X is the log structure coming from a divisor D on X, the divisor D gives rise to a global section q of $\overline{\mathcal{M}}_X$, the invertible sheaf $\mathcal{L}_q$ is the ideal sheaf defining D, and $\mathcal{L}_q^\vee(x)$ is the normal bundle to D at x. In particular, if X is a curve, then $\mathcal{L}_q^\vee(x) \cong T_x(\underline{X})$ and $\mathbb{S}^1(\mathcal{L}_q^\vee(x))$ is the aforementioned space $(T_x(\underline{X}) \setminus \{0\})/\mathbb{R}_>$ of real tangent directions at x.

On the space $X_{\log}$ one can make sense of logarithms of sections of $\mathcal{M}_X$. There is an exact sequence of abelian sheaves

$$0 \to \mathbb{Z}(1) \to \mathcal{L}_X \xrightarrow{\pi} \tau_X^{-1}(\mathcal{M}_X^{\text{gp}}) \to 0, \tag{3-3-3}$$

where $\mathcal{L}_X$ is defined via the Cartesian diagram

$$\begin{array}{ccc}
\mathcal{L}_X & \longrightarrow & \mathbb{R}(1)_{X_{\log}} \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\exp} \\
\tau_X^{-1}(\mathcal{M}_X^{\text{gp}}) & \xrightarrow[\arg]{} & \mathbb{S}^1_{X_{\log}}
\end{array}$$

There is also a homomorphism

$$\epsilon : \tau_X^{-1}(\mathcal{O}_X) \to \mathcal{L}_X, \quad f \mapsto (\exp f, \operatorname{Im}(f)), \tag{3-3-4}$$

and the sequence

$$0 \to \tau_X^{-1}(\mathcal{O}_X) \to \mathcal{L}_X \to \tau_X^{-1}(\overline{\mathcal{M}}_X^{\text{gp}}) \to 0 \tag{3-3-5}$$

is exact. When the log structure on X is trivial the map ϵ is an isomorphism, and the exact sequence (3-3-3), called the "logarithmic exponential sequence" reduces to the usual exponential sequence on X.

We can make this construction explicit in a special "charted" case.

Proposition 3.3.2. *Let $X := \mathsf{A}^{\text{an}}_{P,J}$, where J is an ideal in a sharp toric monoid P, let $\eta : \tilde{X}_{\log} \to X_{\log}$ be the covering (3-3-2), and let $\tilde{\tau}_X := \tau_X \circ \eta$. Then on $\tilde{X}$, the pullback*

$$0 \to \mathbb{Z}(1) \to \mathcal{L}_P \to P^{\text{gp}} \to 0,$$

of the extension (3-3-3) along the natural map $P^{\text{gp}} \to \tilde{\tau}_X^{-1}(\mathcal{M}_X^{\text{gp}})$ identifies with the sheafification of the extension (3-2-1). This identification is compatible with the actions of I_P.

Proof. It is enough to find a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & \mathsf{L}_P & \longrightarrow & P^{\text{gp}} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}(1) & \longrightarrow & \eta_S^*(\mathcal{L}_S) & \longrightarrow & \tilde{\tau}_S^{-1}(\mathcal{M}_S^{\text{gp}}) & \longrightarrow & 0
\end{array}$$

We define a map $\mathsf{L}_P \to \eta_S^*(\mathcal{L}_S)$ as follows. Every $f \in \mathsf{L}_P$ can be written uniquely as $f = f(0) + \tilde{\chi}_p$, where $p \in P^{\mathrm{gp}}$ and $f(0) \in \mathbb{Z}(1)$. Let (ρ, θ) be a point of $\tilde{S}_{\log}$, with image $(\rho, \sigma) \in S_{\log}$. Then the pair $(f(\theta), p) \in \mathbb{R}(1) \times P^{\mathrm{gp}}$ defines an element of $\mathcal{L}_{S,(\rho,\sigma)}$, because

$$\sigma(p) = \exp(\theta(p)) = \exp(\tilde{\chi}_p(\theta)) = \exp(f(\theta) - f(0)) = \exp(f(\theta)),$$

since $f(0) \in \mathbb{Z}(1)$. □

4. Logarithmic degeneration

4.1. ***Log germs and log fibers.*** We begin with an illustration of the philosophy that the local geometry of a suitable morphism can be computed from its log fibers. We use the following notation and terminology. If $\tau : X' \to X$ is a continuous map of topological spaces, then $\mathrm{Cyl}(\tau)$ is the (*open*) *mapping cylinder* of τ, defined as the pushout in the diagram

$$\begin{array}{ccc} X' & \longrightarrow & X' \times [0, \infty) \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \longrightarrow & \mathrm{Cyl}(\tau) \end{array}$$

where the top horizontal arrow is the embedding sending $x' \in X'$ to $(x', 0)$. In $\mathrm{Cyl}(\tau)$, the point $(x', 0)$ becomes identified with the point $\tau(x)$. A commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle\tau_X}\downarrow & & \downarrow{\scriptstyle\tau_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

induces a mapping $\mathrm{Cyl}_f : \mathrm{Cyl}(\tau_X) \to \mathrm{Cyl}(\tau_Y)$.

Theorem 4.1.1. *Let $f : X \to Y$ be a morphism of fine saturated log analytic spaces, where Y is an open neighborhood of the origin v of the standard log disc $\mathsf{A}_{\mathbb{N}}$. Assume that f is proper, smooth, and vertical. Then after Y is replaced by a possibly smaller neighborhood of v, there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Cyl}(\tau_{X_v}) & \xrightarrow{\sim} & X_{\mathrm{top}} \\ {\scriptstyle\mathrm{Cyl}_{f_v}}\downarrow & & \downarrow{\scriptstyle f_{\mathrm{top}}} \\ \mathrm{Cyl}(\tau_v) & \xrightarrow{\sim} & Y_{\mathrm{top}} \end{array}$$

in which the horizontal arrows are isomorphisms. (The arrows are neither unique nor canonical, and depend on a choice of a trivialization of a fibration; see Lemma 4.1.3.)

Proof. Note that since the stalks of $\overline{\mathcal{M}}_Y$ are either 0 or $\mathbb{N}$, the morphism f is automatically exact. We may assume that $Y = \{z \in \mathbb{C} : |z| < \epsilon\}$ for some $\epsilon > 0$. Then $Y_{\log} \cong v_{\log} \times [0, \epsilon) \cong \mathbb{S}^1 \times [0, \infty)$. With this identification, the map τ_Y is just the collapsing map shrinking $\mathbb{S}^1 \times 0$ to a point, and hence induces a homeomorphism $\mathrm{Cyl}(\tau_v) \to Y_{\mathrm{top}}$. The following lemma generalizes this construction.

Lemma 4.1.2. *Let X be a fine log analytic space and let X^+ be the closed subspace of X on which the log structure is nontrivial, endowed with the induced log structure. Then the diagram*

$$\begin{array}{ccc} X^+_{\log} & \longrightarrow & X_{\log} \\ {\scriptstyle \tau_{X'}}\big\downarrow & & \big\downarrow{\scriptstyle \tau_X} \\ X^+_{\mathrm{top}} & \longrightarrow & X_{\mathrm{top}} \end{array}$$

is cocartesian as well as cartesian.

Proof. The diagram is cartesian because formation of $X_{\log}$ is compatible with strict base change. To see that it is cocartesian, observe that since τ_X is surjective and proper, X_{top} has the quotient topology induced from $X_{\log}$. Since τ_X is an isomorphism over $X_{\mathrm{top}} \setminus X^+_{\mathrm{top}}$, the equivalence relation defining τ_X is generated by the equivalence relation defining τ_{X^+}. It follows that the square is a pushout, i.e., is cocartesian. □

Let Y be an open disc as above and fix an identification $Y_{\log} \cong \mathbb{S}^1 \times [0, \infty)$.

Lemma 4.1.3. *Let $f : X \to Y$ be a smooth and proper morphism of fine saturated log analytic spaces, where Y is an open log disc as above. Then there exist a homeomorphism $X_{v,\log} \times [0, \infty) \to X_{\log}$ and a commutative diagram*

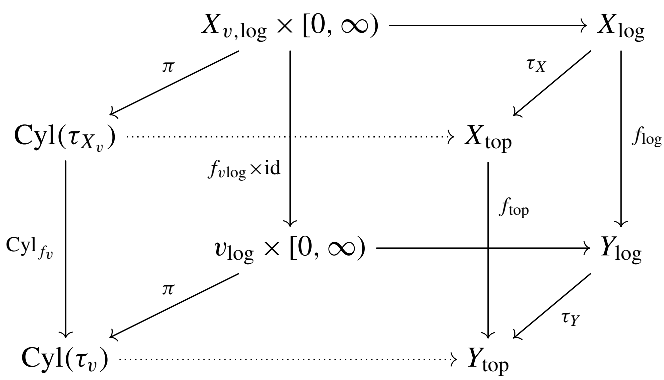
$$\begin{array}{ccc} X_{v,\log} \times [0, \infty) & \longrightarrow & X_{\log} \\ {\scriptstyle f_{v,\log} \times \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle f_{\log}} \\ v_{\log} \times [0, \infty) & \xrightarrow{\sim} & Y_{\log} \end{array}$$

where the restrictions of the horizontal arrows to $X_{v,\log} \times 0$ and $v_{\log} \times 0$ are the inclusions.

Proof. Since Y is a log disc, the morphism f is automatically exact. Then by [Nakayama and Ogus 2010, 5.1], the map $f_{\log}$ is a topological fiber bundle, and since $Y_{\log}$ is connected, all fibers are homeomorphic. Let $r : Y_{\log} = \mathbb{S}^1 \times [0, \epsilon) \to v_{\log}$ be the obvious projection and let $i : v_{\log} \to Y_{\log}$ be the embedding at 0. Then $f_{v,\log} \times \mathrm{id}$ identifies with the pullback of $f_{\log}$ along ir. The space of isomorphisms of fibrations $f_{v,\log} \times \mathrm{id} \to f_{\log}$ is a principal G-bundle, where G is the group of automorphisms of the fiber, endowed with the compact open topology. Since ir

is homotopic to the identity, it follows from [Husemoller 1994, IV, 9.8] that this principal G-bundle is trivial, proving the lemma. $\square$

The diagram of Lemma 4.1.3 forms the rear square of the following diagram:



The map π, from the definition of the mapping cylinder, is part of the pushout diagram which identifies a point $(x_{\log}, 0)$ with $\tau_{X_v}(x) \in X_{v,\mathrm{top}} \subseteq X_{\mathrm{top}}$, and the existence of the dotted arrows follows. Because the morphism f is vertical, the subset X^+ of X where the log structure is nontrivial is just X_v, and Lemma 4.1.2 tells us that the morphism τ_X is also a pushout making the same identifications. Thus the horizontal arrows are homeomorphisms, and Theorem 4.1.1 follows. $\square$

Remark 4.1.4. Although we shall not go into details here, let us mention that the same result, with the same proof, holds if $X \to Y$ is only relatively smooth, as defined in [Nakayama and Ogus 2010].

More generally, suppose that P is a sharp toric monoid and that Y is a neighborhood of the vertex v of A_P. Note that v has a neighborhood basis of sets of the form $V_P := \{y \in \mathsf{A}_P : |y| \in V\}$, where V ranges over the open neighborhoods of the vertex of R_P. If $f : X \to V_P$ is a morphism of log spaces, let $g : X \to V := |f|$, and note that $X_{v,\log} = g^{-1}(0) = (\tau_Y \circ f_{\log})^{-1}(v)$. For each $x \in X_v$, the fiber $\tau_X^{-1}(x)$ is a torsor under the action of $\mathsf{T}_{X,x} := \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{S}^1)$. For $\rho \in V \subseteq \mathsf{R}_P$, let $F(\rho) := \rho^{-1}(\mathbb{R}_>)$, a face of P, and let $G(\rho)$ be the face of $\overline{\mathcal{M}}_{X,x}$ generated by the image of $F(\rho)$ in $\overline{\mathcal{M}}_{X,x}$ via the homomorphism $f_x^\flat : P \to \overline{\mathcal{M}}_{X,x}$. Then we set

$$\mathsf{T}_{Y,\rho} := \mathrm{Hom}(P/F(\rho), \mathbb{S}^1) \subseteq \mathsf{T}_P,$$
$$\mathsf{T}_{X_{x,\rho}} := \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}/G(\rho), \mathbb{S}^1) \subseteq \mathsf{T}_{X,x}\,.$$

There is a natural map $\mathsf{T}_{X_{x,\rho}} \to \mathsf{T}_{Y,\rho}$ induced by $f_x^\flat$.

Conjecture 4.1.5. *With the notation of the previous paragraph, let $f : X \to Y = V_P$ be a smooth proper and exact morphism of fine saturated log analytic spaces. Then,* *after possibly shrinking V, there is a commutative diagram*

$$\begin{array}{ccc} X_{v,\log} \times V & \longrightarrow & X_{\text{top}} \\ \downarrow{\scriptstyle f_{v,\log}\times \mathrm{id}} & & \downarrow{\scriptstyle f_{\text{top}}} \\ v_{\log} \times V \cong Y_{\log} & \longrightarrow & Y_{\text{top}} \end{array}$$

where the bottom arrow is (*the restriction of*) *the map* τ_Y (3-3-1), *and the top arrow is the quotient map which identifies* (x_1, ρ_1) *and* (x_2, ρ_2) *if and only if*:

(1) $\rho_1 = \rho_2$,

(2) $\tau_X(x_1) = \tau_X(x_2)$,

(3) x_1 *and* x_2 *are in the same orbit under the action of* $\mathsf{T}_{X_{\tau(x_i)}}(\rho)$ *on* $\tau_X^{-1}(\tau(x_i))$.

In particular, *the log fiber* $f_v : X_v \to v$ *determines* f *topologically in a neighborhood of* v.

This conjecture is suggested by Remark 2.6 of [Nakayama and Ogus 2010], which implies that such a structure theorem holds locally on X.

Motivated by the above philosophy, we now turn to a more careful study of log schemes which are smooth over a log point S. We shall see that the normalization of such a scheme provides a canonical way of cutting it into pieces, each of whose Betti realizations is a manifold with boundary and is canonically trivialized over $S_{\log}$. In fact this cutting process works more generally, for ideally smooth log schemes.

Theorem 4.1.6. *Let X be a fine*, *smooth*, *and saturated idealized log scheme over a field k such that $\mathcal{K}_X \subseteq \mathcal{M}_X$ is a sheaf of radical ideals. Let $\epsilon : \underline{X}' \to \underline{X}$ be the normalization of the underlying scheme $\underline{X}$.*

(1) *The set U of points x such that $\mathcal{K}_{X,\bar{x}} = \mathcal{M}^+_{X,\bar{x}}$ for some* (*equivalently every*) *geometric point $\bar{x}$ over x is an open and dense subset of X. Its underlying scheme $\underline{U}$ is smooth over k*, *and its complement Y is defined by a coherent sheaf of ideals $\mathcal{J}$ in $\mathcal{M}_X$.*

(2) *The log scheme X' obtained by endowing $\underline{X}'$ with the compactifying log structure associated to the open subset $\epsilon^{-1}(\underline{U})$ is fine*, *saturated*, *and smooth over k.*

(3) *Let X'' be the log scheme obtained by endowing $\underline{X}'$ with the log structure induced from X. There exists a unique morphism $h : X'' \to X'$ such that $\underline{h}$ is the identity. The homomorphism $h^\flat : \mathcal{M}_{X'} \to \mathcal{M}_{X''}$ is injective and identifies $\mathcal{M}_{X'}$ with a sheaf of faces in $\mathcal{M}_{X''}$*, *and the quotient $\mathcal{M}_{X''/X'}$ is a locally constant sheaf of fine sharp monoids.*

Proof. All these statements can be checked étale locally on X. Thus we may assume that there exists a chart $\beta : (Q, K) \to (\mathcal{M}_X, \mathcal{K}_X)$ for X, where (Q, K) is a fine saturated idealized monoid, where the order of Q^{gp} is invertible in k, and where the corresponding morphism $b : \underline{X} \to \underline{\mathsf{A}}_{(Q,K)}$ is étale [Ogus 2018, IV, 3.3.5]. Thanks to the existence of the chart, we can work with ordinary points instead of geometric points. Furthermore we may, after a further localization, assume that the chart is local at some point x of X, so that the map $\overline{Q} \to \overline{\mathcal{M}}_{X,x}$ is an isomorphism. Since $\mathcal{K}_{X,x}$ is a radical ideal, it follow that the same is true of K. Statements (1)–(3) are stable under étale localization, so we are reduced to proving them when $X = \mathsf{A}_{(Q,K)}$.

Since K is a radical ideal of Q, it is the intersection of a finite number of primes $\mathfrak{p}_1, \dots, \mathfrak{p}_r$, and we may assume that each $\mathfrak{p}_i$ minimal among those ideals containing K [Ogus 2018, I, 2.1.13]. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the remaining prime ideals of Q which contain K and let

$$J := \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s.$$

If $x \in X$, let $\beta_x : Q \to \mathcal{M}_{X,x}$ be the homomorphism induced by β and let $\mathfrak{q}_x := \beta_x^{-1}(\mathcal{M}^+_{X,x})$. Then $x \in Y$ if and only if $K_{\mathfrak{q}_x} \subsetneq Q^+_{\mathfrak{q}_x}$, which is the case if and only if $\mathfrak{q}_x = \mathfrak{q}_i$ for some i, or equivalently, if and only if $J \subseteq \mathfrak{q}_x$. Thus Y is the closed subscheme of X defined by the coherent sheaf of ideals $\tilde{J}$ associated to J.

To see that U is dense, observe that the irreducible components of X are defined by the prime ideals $\mathfrak{p}_i$ of Q above. Let ζ_i be the generic point of the irreducible component corresponding to $\mathfrak{p}_i$. Then $K_{\mathfrak{p}_i} = Q^+_{\mathfrak{p}_i}$, so $\zeta_i \in U$. It follows that U is dense in X. To see that $\underline{U}$ is smooth, let x be a point of U and replace β by its localization at x. Then it follows from the definition of U that $K = Q^+$ and hence that $\underline{\mathsf{A}}_{Q,K} \cong \underline{\mathsf{A}}_{Q^*}$ which is indeed smooth over k.

To prove statement (2) we continue to assume that $X = \mathsf{A}_{(Q,K)}$. For each minimal $\mathfrak{p}_i$ over K, let F_i be the corresponding face. Then $\underline{\mathsf{A}}_{Q,\mathfrak{p}_i} \cong \underline{\mathsf{A}}_{F_i}$. Since Q is saturated, so is each F_i, and hence each scheme $X_{F_i} := \underline{\mathsf{A}}_{F_i}$ is normal. Thus the disjoint union $\bigsqcup\{\underline{\mathsf{A}}_{F_i}\}$ is the normalization of $\underline{\mathsf{A}}_{Q,K}$. A point x' of $\underline{X}_{F_i}$ lies in $\epsilon^{-1}(U)$ if and only if its image in $\underline{\mathsf{A}}_{F_i}$ lies in $\underline{\mathsf{A}}_{F_i^{\mathrm{gp}}}$. It follows from [Ogus 2018, III, 1.9.5] that the compactifying log structure on $\underline{X}_{F_i}$ is coherent, charted by F, and hence from [Ogus 2018, IV, 3.1.7] that the resulting log scheme X'_{F_i} is smooth over k. Thus X'/k is smooth. This completes the proof of statements (1) and (2).

To define the morphism h, it will be convenient to first introduce an auxiliary log structure. Let $\mathcal{O}'_{X'} \subseteq \mathcal{O}_{X'} = \mathcal{O}_{X''}$ be the sheaf of nonzero divisors in $\mathcal{O}_{X'}$ and let $\mathcal{M}'$ be its inverse image in $\mathcal{M}_{X''}$ via the map $\alpha_{X''} : \mathcal{M}_{X''} \to \mathcal{O}_{X'}$. Then $\mathcal{M}'$ is a sheaf of faces in $\mathcal{M}_{X''}$, and the induced map $\alpha' : \mathcal{M}' \to \mathcal{O}_{X'}$ is a log structure on $\underline{X}'$. If $\bar{x}'$ is a geometric point of U', then $\mathcal{K}_{X',\bar{x}'} = \mathcal{M}^+_{X',\bar{x}'}$, so the map $\mathcal{M}^+_{X'',\bar{x}'} \to \mathcal{O}_{X'_{\bar{x}'}}$ is zero. Hence $\mathcal{M}'_{\bar{x}'} = \mathcal{O}^*_{X',\bar{x}'}$, and thus α' is trivial on U'. It follows that there is a natural morphism from α' to the compactifying log structure $\alpha_{U'/X'}$. We check that this morphism is an isomorphism at each point x' of X'. Since both log structures are trivial if $x' \in U'$, we may assume that $x' \in Y' := X' \setminus U'$ and that X admits a chart β as above, local at $\epsilon(x')$. Let F be the face of Q such that $x' \in X_F$. If $q \in \mathfrak{p} := Q \setminus F$, then $\alpha_{X''}(\beta(q))$ vanishes in $\mathcal{O}_{X_F}$. If on the other hand $q \in F$, then $\beta(q)$ is a nonzero divisor on X_F Thus $\beta^{-1}(\mathcal{M}'_{x'}) = F$. Since F is a chart for $\mathcal{M}_{X'}$ and $\mathcal{M}'_{x'}$ is a face of $\mathcal{M}_{X'',x'}$, it follows that the map $\mathcal{M}'_{x'} \to \mathcal{M}_{X',x'}$ is an isomorphism. The inverse of this isomorphism followed by the inclusion $\mathcal{M}' \to \mathcal{M}_{X''}$ defines a morphism of log structures $\alpha' \to \alpha_{X''}$ and hence a morphism of log schemes $h : X'' \to X'$ with $\underline{h} = \text{id}$.

To prove that h is unique, note that since $\mathcal{M}' \to \mathcal{M}_{X'}$ is an isomorphism and $\alpha_{X'}$ is injective, the homomorphism $\alpha' : \mathcal{M} \to \mathcal{O}_{X'}$ is also injective. Let $h' : X'' \to X'$ be any morphism of log schemes with $\underline{h}' = \text{id}$ and let m be a local section of $\mathcal{M}_{X'}$. Then $\alpha_{X''}(h'^{\flat}(m)) = \alpha_{X'}(m)$ is a nonzero divisor in $\mathcal{O}_{X'}$, so the homomorphism $h'^{\flat} : \mathcal{M}_{X'} \to \mathcal{M}_{X''}$ necessarily factors through $\mathcal{M}'$. Since $\alpha' \circ h'^{\flat} = \alpha_{X'}$ and α' is injective, necessarily $h'^{\flat} = h^{\flat}$.

We have already observed that the image $\mathcal{M}'$ of $h^{\flat}$ is a sheaf of faces of $\mathcal{M}_{X''}$, and it follows that the quotient monoid $\mathcal{M}_{X''/X'}$ is sharp. To check that it is locally constant, we may assume that X admits a chart as above and work on the subscheme X''_F of X'' defined by a face F as above. Then $\beta'' : Q \to \mathcal{M}_{X''}$ is a chart for $\mathcal{M}_{X''}$. Assume that β'' is local at a point x'' of X'' and that ξ'' is a generization of x''. Then $\mathcal{M}_{X''/X',x''} = Q/F$ and $\mathcal{M}_{X''/X',\xi''} = Q_G/F_G$, where $G := \beta_{\xi''}^{-1}(\mathcal{O}^*_{X'',\xi''})$. Since $G \subseteq F$, the cospecialization map

$$Q/F \to Q_G/F_G$$

is an isomorphism. It follows that $\mathcal{M}_{X''/X'}$ is (locally) constant. □

Let us now return to the case of smooth log schemes over a log point.

Corollary 4.1.7. *Let $f : X \to S$ be a smooth and saturated morphism from a fine saturated log scheme to the log point* $\operatorname{Spec}(P \to k)$, *where P is a fine saturated and sharp monoid. Let $\underline{\epsilon} : \underline{X}' \to \underline{X}$ be the normalization of the underlying scheme $\underline{X}$.*

(1) *The set $U := \{x \in X : \mathcal{M}_{X/S,\bar{x}} = 0\}$ is a dense open subset of X. Its underlying scheme $\underline{U}$ is smooth over* $\mathbb{C}$, *and ϵ induces an isomorphism*

$$\underline{U}' := \underline{\epsilon}^{-1}(\underline{U}) \to \underline{U}.$$

(2) *The log scheme X' obtained by endowing $\underline{X}'$ with the compactifying log structure associated to the open subset $\underline{U}'$ is fine, saturated and smooth over* $\mathbb{C}$.

(3) *Let X'' be the log scheme obtained by endowing $\underline{X}'$ with the log structure induced from X. There exist a unique morphism $h : X'' \to X'$ such that $\underline{h}$ is the identity.*

(4) *The homomorphism* $f^\flat$ *induces an isomorphism* $P \to \mathcal{M}_{X''/X'}$. *Thus there is a unique homomorphism* $\rho : \mathcal{M}_{X''} \to P_{X'}$ *such that* $\rho \circ f''^\flat = \mathrm{id}$. *The homomorphism* $h^\flat : \mathcal{M}_{X'} \to \mathcal{M}_{X''}$ *induces isomorphisms* $\mathcal{M}_{X'} \cong \rho^{-1}(0)$ *and* $\overline{\mathcal{M}}^{\mathrm{gp}}_{X'} \to \epsilon^*(\mathcal{M}^{\mathrm{gp}}_{X/S})$.

Proof. Let $\mathcal{K}_X \subseteq \mathcal{M}_X$ be the sheaf of ideals generated by $f^\flat(\mathcal{M}^+_S)$. Since X/S is saturated, this is a radical sheaf of ideals of $\mathcal{M}_X$ [Ogus 2018, I, 4.8.14]. Since $X \to S$ is smooth, so is the base changed map $(X, \mathcal{K}_X) \to (S, \mathcal{M}_S)$, and since $(S, \mathcal{M}^+_S) \to \underline{S}$ is smooth, it follows that $(X, \mathcal{K}_X) \to \underline{S}$ is smooth. Note that if $x \in U$, then $P^{\mathrm{gp}} \to \mathcal{M}^{\mathrm{gp}}_{X,\bar{x}}$ is an isomorphism, and since f is exact, it follows that $P = \mathcal{M}_{X,\bar{x}}$ and hence that $\mathcal{K}_{X,\bar{x}} = \mathcal{M}^+_{X,\bar{x}}$. Conversely, if $\mathcal{K}_{X,\bar{x}} = \mathcal{M}^+_{X,\bar{x}}$, then P^+ and $\mathcal{M}^+_{X,\bar{x}}$ both have height zero, and since f is saturated it follows from statement (2) of [Ogus 2018, I, 4.8.14] that $P \to \mathcal{M}_{X,\bar{x}}$ is an isomorphism and hence that $\mathcal{M}_{X/S} = 0$. Thus the open set U defined here is the same as the set U defined in Theorem 4.1.6. Hence statements (1), (2), and (3) follow from that result.

We check that the map $P \to \mathcal{M}_{X''/X'}$ is an isomorphism locally on X, with the aid of a chart as in the proof of Theorem 4.1.6. Then $\mathcal{M}_{X''/X'} = Q/F$, where F is the face corresponding to a minimal prime $\mathfrak{p}$ of the ideal K of Q generated by P^+. Then Q/F and P have the same dimension, so $\mathfrak{p} \subseteq Q$ and $P^+ \subseteq P$ have the same height. Then it follows from (2) of [Ogus 2018, 4.18.4] that the homomorphism $P \to Q/F$ is an isomorphism. It remains only to prove that the map $\overline{\mathcal{M}}^{\mathrm{gp}}_{X'} \to \epsilon^*(\mathcal{M}^{\mathrm{gp}}_{X/S})$ is an isomorphism. We have a commutative diagram

$$\begin{array}{ccccc} & & \epsilon^*(\mathcal{M}^{\mathrm{gp}}_{X/S}) & & \\ & \nearrow & \uparrow & & \\ \overline{\mathcal{M}}^{\mathrm{gp}}_{X'} & \longrightarrow & \overline{\mathcal{M}}^{\mathrm{gp}}_{X''} & \longrightarrow & \mathcal{M}^{\mathrm{gp}}_{X''/X'} \\ & & \uparrow & \nearrow_{\sim} & \\ & & f'^{-1}(\overline{\mathcal{M}}^{\mathrm{gp}}_S) & & \end{array} \qquad (4\text{-}1\text{-}1)$$

The rows and columns of this diagram are short exact sequences, and the diagonal map on the bottom right is an isomorphism. It follows that the diagonal map on the top left is also an isomorphism. □

Proposition 4.1.8. *With the hypotheses of Corollary 4.1.7, let* $g : X'' \to X' \times S$ *be the morphism induced by* $f \circ \epsilon$ *and* h. *The morphism of underlying schemes* $\underline{g}$ *is an isomorphism, and* $g^\flat$ *induces an isomorphism of abelian sheaves*: $g^{\flat\mathrm{gp}}$:

$\mathcal{M}^{\mathrm{gp}}_{X'\times S'} \to \mathcal{M}^{\mathrm{gp}}_{X''}$. *The horizontal arrows in the commutative diagram*

$$\begin{array}{ccc} X''_{\log} & \xrightarrow{g_{\log}} & X'_{\log} \times S_{\log} \\ {\scriptstyle \tau_{X''}}\downarrow & & \downarrow {\scriptstyle \mathrm{pr}\circ\tau_{X'}} \\ X''_{\mathrm{top}} & \xrightarrow{g_{\mathrm{top}}} & X'_{\mathrm{top}} \end{array}$$

are isomorphisms.

Proof. Since $\underline{h}$ is an isomorphism and $\underline{S}$ is a point, the morphism $\underline{g}$ is also an isomorphism. Since $\overline{\mathcal{M}}_{X'\times S} = \overline{\mathcal{M}}_{X'} \oplus P_{X'}$ and $P_{X'} \cong \mathcal{M}_{X''/X'}$, it follows from the horizontal exact sequence in diagram (4-1-1) that the homomorphism $\overline{\mathcal{M}}^{\mathrm{gp}}_{X'\times S} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{X''}$ is an isomorphism, and hence the same is true of $g^{\flat\mathrm{gp}}$. It follows that $g_{\log}$ is bijective, and since it is proper, it is a homeomorphism. □

The corollary below shows that the fibration $X_{\log} \to S_{\log} \cong \mathsf{T}_P$ can be cut into pieces (the connected components of $X'_{\log}$), each of which is a trivial fibration whose fiber is a manifold with boundary, in a canonical way. We shall make the gluing data needed to undo the cuts more explicit in the case of curves; see Section 7.1.

Corollary 4.1.9. *With the hypotheses of Corollary 4.1.7, there is a natural commutative diagram*

$$\begin{array}{ccc} X'_{\log} \times \mathsf{T}_P & \xrightarrow{p} & X_{\log} \\ & {\scriptstyle \mathrm{pr}}\searrow & \downarrow {\scriptstyle f_{\log}} \\ & & \mathsf{T}_P \end{array}$$

where $X'_{\log}$ is a topological manifold with boundary and where p is a proper surjective morphism with finite fibers and is an isomorphism over $U_{\log}$.

Proof. Let $p := \epsilon_{\log} \circ g^{-1}_{\log}$, which is proper and surjective and has finite fibers. Recall from [Nakayama and Ogus 2010, 2.14] that $X'_{\log}$ is a topological manifold with boundary, and that its boundary is $Y'_{\log}$. □

4.2. ***Log nearby cycles.*** Let $f : X \to S$ be a morphism of fine saturated log schemes, where S is the split log point associated to a sharp monoid P. We assume that for every $x \in X$, the map $P^{\mathrm{gp}} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{X,x}$ is injective, and that the quotient group $\overline{\mathcal{M}}^{\mathrm{gp}}_{X/S.x}$ is torsion free. These assumptions hold if, for example, f is smooth and saturated. We form the following commutative diagram:

$$\begin{array}{ccccc}
\tilde{X}_{\log} & \xrightarrow{\eta_X} & X_{\log} & & \\
\downarrow{\scriptstyle \tilde{\tau}_{X/S}} & & \downarrow{\scriptstyle \tau_{X/S}} & \searrow{\scriptstyle \tau_X} & \\
X_{\tilde{S},\log} & \xrightarrow{\eta} & X_{S,\log} & \xrightarrow{\tau_{X_S}} & X_{\mathrm{top}} \\
\downarrow{\scriptstyle \tilde{f}_S^{\log}} & & \downarrow{\scriptstyle f_S^{\log}} & & \downarrow{\scriptstyle f_{\mathrm{top}}} \\
\tilde{S}_{\log} & \xrightarrow{\eta_S} & S_{\log} & \xrightarrow{\tau_S} & S_{\mathrm{top}}
\end{array}
\qquad \text{(with } \tilde{\tau}_X : \tilde{X}_{\log} \to X_{\mathrm{top}},\ \tilde{f} : \tilde{X}_{\log} \to \tilde{S}_{\log},\ \tilde{\tau}_S : \tilde{S}_{\log} \to S_{\mathrm{top}}) \tag{4-2-1}$$

where the squares are Cartesian. Thus $S_{\log} \cong \mathsf{T}_P$, $\tilde{S}_{\log} \cong \mathsf{V}_P$, $X_{\tilde{S},\log} = X_{\mathrm{top}} \times \tilde{S}_{\log}$, and $\tilde{X}_{\log} = X_{\log} \times_{S_{\log}} \tilde{S}_{\log}$. We let $\tilde{\tau}_X := \tau_X \circ \eta_X$, $\tilde{\tau}_{X_S} := \tau_{X_S} \circ \eta$, and $\tilde{\tau}_S := \tau_X \circ \eta_S$, so that we have the diagram

$$\begin{array}{ccccc}
\tilde{X}_{\log} & \xrightarrow{\tilde{\tau}_{X/S}} & X_{\mathrm{top}} \times \mathsf{V}_P & \xrightarrow{\cong} & X_{\tilde{S},\log} \\
& {\scriptstyle \tilde{\tau}_X}\searrow & \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \tilde{\tau}_{X_S}} & \\
& & X_{\mathrm{top}}. & &
\end{array} \tag{4-2-2}$$

The logarithmic inertia group I_P acts on $\tilde{S}_{\log}$ over S_{top} and hence also on $\tilde{X}_{\log}$ over X_{top}. Our goal is to describe the cohomology of $\tilde{X}_{\log}$, together with its I_P-action, using this diagram and the log structures on X and S. We set

$$\Psi^q_{X/S} := R^q \tilde{\tau}_{X*}\mathbb{Z}, \quad (\text{resp.} \quad \Psi_{X/S} := R\tilde{\tau}_{X*}\mathbb{Z}),$$

viewed as a sheaf (resp. object in the derived category of sheaves) of $\mathbb{Z}[\mathsf{I}_P]$-modules on X_{top}. When S is the standard log point and f is obtained by base change from a smooth proper morphism over the standard log disk, the complex $\Psi_{X/S}$ can be identified with the usual complex of nearby cycles, as was proved in [Illusie et al. 2005, 8.3]. Then $H^*(\tilde{X}_{\log}, \mathbb{Z}) \cong H^*(X_{\mathrm{top}}, \Psi_{X/S})$, and there is the (Leray) spectral sequence

$$E_2^{p,q} = H^p(X_{\mathrm{top}}, \Psi^q_{X/S}) \Longrightarrow H^{p+q}(\tilde{X}_{\log}, \mathbb{Z}).$$

Our first ingredient is the following computation of the cohomology sheaves $\Psi^q_{X/S}$.

Theorem 4.2.1 [Kato and Nakayama 1999, Lemma 1.4]. *Let $f : X \to S$ be a saturated morphism of log schemes, where X is fine and saturated and S is the* *split log point over $\mathbb{C}$ associated to a fine sharp monoid P. Then on the topological space X_{top} associated to X, there are canonical isomorphisms*

$$\sigma^q : \bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S}(-q) \xrightarrow{\sim} \Psi^q_{X/S} \tag{4-2-3}$$

for all q. In particular, the logarithmic inertia group I_P acts trivially on $\Psi^q_{X/S}$.

Proof. The construction of these isomorphisms depends on the logarithmic exponential sequence (3-3-3) on $X_{\log}$. In the absolute case it is shown in [Kato and Nakayama 1999] that the boundary map associated to (3-3-3) induces a homomorphism $\overline{\mathcal{M}}^{\mathrm{gp}}_X \to R^1\tau_{X*}(\mathbb{Z}(1))$, and then one finds by cup-product the homomorphisms σ^q for all $q \geq 0$. These can be seen to be isomorphisms by using the proper base change theorem to reduce to the case in which X is a log point.

The argument in our relative setting is similar. Let $\mathcal{M}_{X/P}$ be the quotient of the sheaf of monoids $\mathcal{M}_X$ by P. Since $P^{\mathrm{gp}} \to \overline{\mathcal{M}}^{\mathrm{gp}}_X$ is injective, the sequence

$$0 \to \mathcal{O}^*_X \to \mathcal{M}^{\mathrm{gp}}_{X/P} \to \mathcal{M}^{\mathrm{gp}}_{X/S} \to 0 \tag{4-2-4}$$

is exact. The homomorphism $P^{\mathrm{gp}} \to f^{-1}(\mathcal{M}^{\mathrm{gp}}_S) \to \mathcal{M}^{\mathrm{gp}}_X$ does not lift to $\mathcal{L}_X$ on $X_{\log}$, but the map $\tilde{\chi} : P^{\mathrm{gp}} \to \mathcal{L}_S$ (defined at the beginning of Section 3.2) defines such a lifting on $\tilde{S}_{\log}$ and hence also on $\tilde{X}_{\log}$. Letting $\mathcal{L}_{X/P}$ be the quotient of $\mathcal{L}_X$ by $\tilde{\chi}(P^{\mathrm{gp}})$, we find an exact sequence:

$$0 \to \mathbb{Z}(1) \to \mathcal{L}_{X/P} \to \tilde{\tau}_X^{-1}(\mathcal{M}^{\mathrm{gp}}_{X/P}) \to 0 \tag{4-2-5}$$

The boundary map associated with this sequence produces a map

$$\mathcal{M}^{\mathrm{gp}}_{X/P} \to R^1\tilde{\tau}_{X*}(\mathbb{Z}(1))$$

which factors through $\mathcal{M}^{\mathrm{gp}}_{X/S}$ because, locally on X, the inclusion $\mathcal{O}^*_X \to \mathcal{M}_{X/P}$ factors through $\tilde{\tau}_{X*}(\mathcal{L}_{X/P})$. Then cup product induces maps $\bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S} \to R^q\tilde{\tau}_{X*}(\mathbb{Z}(1))$ for all q, which we can check are isomorphisms on the stalks. The map $\tilde{\tau}_{X/S}$ is proper, and its fiber over a point (x, v) of $X_{\mathrm{top}} \times \mathsf{V}_P$ is a torsor under $\mathrm{Hom}(\mathcal{M}_{X/S,x}, \mathbb{S}^1)$. It follows that the maps $\bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S,x} \to (R^q\tilde{\tau}_{X/S*}(\mathbb{Z}(1))_{(x,v)}$ are isomorphisms. In particular, the sheaves $R^q\tilde{\tau}_{X/S*}(\mathbb{Z}(1))$ are locally constant along the fibers of $\pi : X_{\mathrm{top}} \times \mathsf{V}_P \to \mathsf{V}_P$. Then it follows from [Kashiwara and Schapira 1990, 2.7.8] that the map $\pi^* R\pi_*(R\tilde{\tau}_{X/S*}(\mathbb{Z}(1)) \to R\tilde{\tau}_{X/S*}(\mathbb{Z}(1))$ is an isomorphism. Thus the maps $(R\tilde{\tau}_{X*}(\mathbb{Z}(1))_x \to (R\tilde{\tau}_{X/S*}(\mathbb{Z}(1))_{(x,v)}$ are isomorphisms, and the result follows. □

Our goal is to use the Leray spectral sequence for the morphism $\tilde{\tau}_X$ to describe the cohomology of $\tilde{X}_{\log}$ together with its monodromy action. In fact it is convenient to work on the level of complexes, in the derived category. The "first order attachment maps" defined in Section 2.1 are maps

$$\delta^q : \Psi^q_{X/S} \to \Psi^{q-1}_{X/S}[2].$$

On the other hand, the "log Chern class" sequence (1-0-2) defines a morphism

$$\mathrm{ch}_{X/S} : \mathcal{M}^{\mathrm{gp}}_{X/S} \to \mathbb{Z}(1)[2]$$

and hence for all $q \geq 0$, maps

$$\mathrm{ch}^q_{X/S} : \bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S} \to \bigwedge^{q-1} \mathcal{M}^{\mathrm{gp}}_{X/S}(1)[2],$$

defined as the composition

$$\begin{aligned}\bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S} &\xrightarrow{\eta} \mathcal{M}^{\mathrm{gp}}_{X/S} \otimes \bigwedge^{q-1} \mathcal{M}^{\mathrm{gp}}_{X/S} \\ &\xrightarrow{\mathrm{ch}_{X/S} \otimes \mathrm{id}} \mathbb{Z}(1)[2] \otimes \bigwedge^{q-1} \mathcal{M}^{\mathrm{gp}}_{X/S} \cong \bigwedge^{q-1} \mathcal{M}^{\mathrm{gp}}_{X/S}(1)[2],\end{aligned}$$

where η is the comultiplication map as defined in Section 2.2. We show below that the maps δ^q and $\mathrm{ch}^q_{X/S}$ agree, at least after multiplication by $q!$.

To describe the monodromy action ρ of I_P on $\Psi_{X/S}$, observe that, since each $\gamma \in \mathsf{I}_P$ acts trivially on $\Psi^q_{X/S}$, the endomorphism $\lambda_\gamma := \rho_\gamma - \mathrm{id}$ of $\Psi_{X/S}$ annihilates $\Psi^q_{X/S}$ and hence induces maps (see Section 2.3)

$$\lambda^q_\gamma : \Psi^q_{X/S} \to \Psi^{q-1}_{X/S}[1].$$

On the other hand, the pushout of the "log Kodaira–Spencer" sequence (1-0-1) along $\gamma : P^{\mathrm{gp}} \to \mathbb{Z}(1)$ is a sequence

$$0 \to \mathbb{Z}(1) \to \overline{\mathcal{M}}^{\mathrm{gp}}_{X,\gamma} \to \mathcal{M}^{\mathrm{gp}}_{X/S} \to 0.$$

The stalk of this sequence at each point of X is a splittable sequence of finitely generated free abelian groups, so the exterior power construction of Section 2.2 provides a sequence

$$0 \to \bigwedge^{q-1} \mathcal{M}^{\mathrm{gp}}_{X/S}(1) \to \bigwedge^q \overline{\mathcal{M}}^{\mathrm{gp}}_X / K^2 \bigwedge^q \overline{\mathcal{M}}^{\mathrm{gp}}_X \to \bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S} \to 0,$$

which gives rise to a morphism in the derived category

$$\kappa^q_\gamma : \bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S} \to \bigwedge^{q-1} \mathcal{M}^{\mathrm{gp}}_{X/S}(1)[1]. \tag{4-2-6}$$

Recall from Proposition 2.2.2 that κ_γ is "cup product with κ," that is, that $\kappa_\gamma = (\mathrm{id} \otimes \kappa) \circ (\mathrm{id} \otimes \gamma) \circ \eta$. We show below that this morphism agrees with the monodromy morphism λ^q_γ up to sign. We shall provide a version of this result for the étale topology in Theorem 6.3.4. A similar formula, in the context of a semistable reduction and étale cohomology, is at least implicit in statement (4) of a result of T. Saito [2003, 2.5].

Theorem 4.2.2. *Let S be the split log point associated to a fine sharp and saturated monoid P and let $f : X \to S$ be a saturated morphism of fine saturated log analytic spaces.*

(1) *For each $q \geq 0$, the following diagram commutes*:

$$\begin{array}{ccc} \bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S}(-q) & \xrightarrow{\mathrm{ch}^q_{X/S}} & \bigwedge^{q-1}\mathcal{M}^{\mathrm{gp}}_{X/S}(1-q)[2] \\ {\scriptstyle q!\sigma^q}\downarrow & & \downarrow{\scriptstyle q!\sigma^{q-1}} \\ \Psi^q_{X/S} & \xrightarrow[\delta^q]{} & \Psi^{q-1}_{X/S}[2] \end{array}$$

(2) *For each $q \geq 0$ and each $\gamma \in \mathsf{I}_P$, the following diagram commutes*:

$$\begin{array}{ccc} \bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S}(-q) & \xrightarrow{\kappa^q_\gamma} & \bigwedge^{q-1}\mathcal{M}^{\mathrm{gp}}_{X/S}(1-q)[1] \\ {\scriptstyle \sigma^q}\downarrow & & \downarrow{\scriptstyle \sigma^{q-1}} \\ \Psi^q_{X/S} & \xrightarrow[(-1)^{q-1}\lambda^q_\gamma]{} & \Psi^{q-1}_{X/S}[1] \end{array}$$

Proof. The main ingredient in the proof of statement (1) is the quasi-isomorphism

$$\left[\mathcal{O}_X \xrightarrow{\exp} \mathcal{M}^{\mathrm{gp}}_{X/P}\right] \xrightarrow{\sim} \tau_{\leq 1}\Psi_{X/S}(1), \tag{4-2-7}$$

which is obtained as follows. The exact sequence (4-2-5) defines an isomorphism in $D^+(\tilde{X}, \mathbb{Z})$

$$\mathbb{Z}(1) \xrightarrow{\sim} \left[\mathcal{L}_{X/P} \to \tilde{\tau}_X^{-1}(\mathcal{M}^{\mathrm{gp}}_{X/P})\right],$$

and there is an evident morphism of complexes,

$$\left[\tilde{\tau}_X^{-1}(\mathcal{O}_X) \to \tilde{\tau}_X^{-1}(\mathcal{M}^{\mathrm{gp}}_{X/P})\right] \to \left[\mathcal{L}_{X/P} \to \tilde{\tau}_X^{-1}(\mathcal{M}^{\mathrm{gp}}_{X/P})\right],$$

defined by the homomorphism $\epsilon : \tau_X^{-1}(\mathcal{O}_X) \to \mathcal{L}_X$ (3-3-4). Using these two morphisms and adjunction, we find a morphism

$$\left[\mathcal{O}_X \xrightarrow{\exp} \mathcal{M}^{\mathrm{gp}}_{X/P}\right] \to R\tilde{\tau}_X(\mathbb{Z}(1)) := \Psi_{X/S}(1).$$

Since this morphism induces an isomorphism on cohomology sheaves in degrees 0 and 1, it induces a quasi-isomorphism after the application of the truncation functor $\tau_{\leq 1}$. This is the quasi-isomorphism (4-2-7). Since the map δ^1 of the complex $\left(\mathcal{O}_X \xrightarrow{\exp} \mathcal{M}^{\mathrm{gp}}_{X/P}\right)$ is precisely the map $\mathrm{ch}_{X/S}$, we see that the diagram in statement (1) commutes when $q = 1$.

To deduce the general case, we use induction and the multiplicative structure on cohomology. Let $E := \mathcal{M}^{\mathrm{gp}}_{X/S}(-1)$ and let $F := \mathbb{Z}[2]$. Using the isomorphisms σ^q, we can view δ^q as a morphism $\bigwedge^q E \to \bigwedge^{q-1} E[2] = F \otimes \bigwedge^{q-1} E$. Lemma 2.1.2 asserts that the family of maps δ^q form a derivation in the sense that diagram (2-2-2) commutes. Then by the definition of $\mathrm{ch}^q_{X/S}$, it follows from Proposition 2.2.1 that $q!\,\mathrm{ch}^q_{X/S} = q!\delta^q$ for all q.

We defer the proof of the monodromy formula described in statement (2) to Section 5 (with complex coefficients) and Section 6 (the general case). □

5. Monodromy and the Steenbrink complex

Our goal in this section is to extend Steenbrink's formula (4-2-7) for $\tau_{\leq 1}\Psi_{X/S}$ to all of $\Psi_{X/S}$. We shall see that there is a very natural logarithmic generalization of the classical Steenbrink complex [1975/76, §2.6] which computes the logarithmic nearby cycle complex $\mathbb{C}\otimes\Psi_{X/S}$. The advantage of this complex is that it is a canonical differential graded algebra with an explicit action of I_P, from which it is straightforward to prove the monodromy formula of Theorem 4.2.2 (tensored with $\mathbb{C}$). Since the construction is based on logarithmic de Rham cohomology, we require that X/S be (ideally) smooth. Note that once we have tensored with $\mathbb{C}$, there is no point in keeping track of the Tate twist, since there is a canonical isomorphism $\mathbb{C}(1) \xrightarrow{\sim} \mathbb{C}$.

5.1. ***Logarithmic construction of the Steenbrink complex.*** Steenbrink's original construction, which took place in the context of a semistable family of analytic varieties over a complex disc with parameter z, was obtained by formally adjoining the powers of $\log z$ to the complex of differential forms with log poles. Our construction is based on the logarithmic de Rham complex on $X_{\log}$ constructed in [Kato and Nakayama 1999, §3.5].

Let us begin by recalling Kato's construction of the logarithmic de Rham complex on X [Kato 1989; Illusie et al. 2005]. If $f : X \to Y$ is a morphism of log analytic spaces, the sheaf of logarithmic differentials $\Omega^1_{X/Y}$ is the universal target of a pair of maps

$$d : \mathcal{O}_X \to \Omega^1_{X/Y}, \quad d\log : \mathcal{M}_X^{\mathrm{gp}} \to \Omega^1_{X/Y},$$

where d is a derivation relative to Y, where $d\log$ is a homomorphism of abelian sheaves annihilating the image of $\mathcal{M}_Y^{\mathrm{gp}}$, and where $d\alpha_X(m) = \alpha_X(m)d\log(m)$ for every local section m of $\mathcal{M}_X$. The sheaf $\Omega^1_{X/Y}$ is locally free if f is a smooth morphism of (possibly idealized) log spaces. Then $\Omega^i_{X/S} := \bigwedge^i \Omega^1_{X/Y}$, and there is a natural way to make $\bigoplus \Omega^i_{X/Y}$ into a complex satisfying the usual derivation rules and such that $d\circ d\log = 0$. In particular the map $d\log : \mathcal{M}_X^{\mathrm{gp}} \to \Omega^1_{X/Y}$ factors through the sheaf of closed one-forms, and one finds maps

$$\sigma_{\mathrm{DR}} : \mathbb{C}\otimes \bigwedge^i \mathcal{M}^{\mathrm{gp}}_{X/Y} \to \mathcal{H}^i(\Omega^{\bullet}_{X/Y}). \tag{5-1-1}$$

When $S = \mathbb{C}$ (with trivial log structure) and $X/\mathbb{C}$ is ideally log smooth, these maps fit into a commutative diagram of isomorphisms (see [Kato and Nakayama 1999, Proposition 4.6] and its proof):

$$\begin{array}{ccc} \mathbb{C}\otimes\bigwedge^i \overline{\mathcal{M}}_X^{\mathrm{gp}} & \xrightarrow{\sigma_{\mathrm{DR}}} & \mathcal{H}^i(\Omega^{\bullet}_{X/\mathbb{C}}) \\ & \searrow^{\sigma} & \uparrow \\ & & R^i\tau_{X*}(\mathbb{C}) \end{array}$$

As explained in [Kato and Nakayama 1999, §3.2], to obtain the de Rham complex on $X_{\log}$, one begins with the construction of the universal sheaf of $\tau_X^{-1}(\mathcal{O}_X)$-algebras $\mathcal{O}_X^{\log}$ which fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_X & \longrightarrow & \mathcal{O}_X^{\log} \\ & \nwarrow^{\epsilon} & \uparrow \\ & & \tau_X^{-1}(\mathcal{O}_X). \end{array}$$

This sheaf of $\tau_X^{-1}(\mathcal{O}_X)$ modules admits a unique integrable connection

$$d:\mathcal{O}_X^{\log}\to\mathcal{O}_X^{\log}\otimes_{\tau_X^{-1}(\mathcal{O}_X)}\tau_X^{-1}(\Omega^1_{X/\mathbb{C}})$$

such that $d(\ell)=d\pi(\ell)$ (see (3-3-3)) for each section ℓ of $\mathcal{L}_X$ and which is compatible with the multiplicative structure of $\mathcal{O}_X^{\log}$. The de Rham complex of this connection is a complex whose terms are sheaves of $\mathcal{O}_X^{\log}$-modules on $X_{\log}$, denoted by $\Omega^{\bullet,\log}_{X/\mathbb{C}}$. In particular, $\Omega^{i,\log}_{X/\mathbb{C}}:=\mathcal{O}_X^{\log}\otimes_{\tau_X^{-1}(\mathcal{O}_X)}\tau_X^{-1}(\Omega^i_{X/\mathbb{C}})$

When S is the split log point associated to a fine sharp saturated monoid P, the sheaf $\mathcal{O}_S^{\log}$ on the torus $S_{\log}\cong\mathsf{T}_P$ is locally constant, and hence is determined by $\Gamma(\tilde{S}_{\log},\eta^*(\mathcal{O}_S^{\log}))$ together with its natural action of I_P. These data are easy to describe explicitly. The structure sheaf $\mathcal{O}_S$ is $\mathbb{C}$ and $\Omega^1_{S/\mathbb{C}}$ is $\mathbb{C}\otimes P^{\mathrm{gp}}$. Twisting the exact sequence (3-2-1) yields the sequence

$$0\to\mathbb{Z}\to\mathsf{L}_P(-1)\to P^{\mathrm{gp}}(-1)\to 0.$$

For each n, the map $\mathbb{Z}\to\mathsf{L}_P(-1)$ induces a map $S^{n-1}(\mathsf{L}_P(-1))\to S^n(\mathsf{L}_P(-1))$, and we let

$$\tilde{\mathcal{O}}_P^{\log}:=\varinjlim S^n(\mathsf{L}_P(-1)).$$

The action of I_P on L_P induces an action on $\tilde{\mathcal{O}}_P^{\log}$, compatible with its ring structure. Let $N_n\tilde{\mathcal{O}}_P$ denote the image of the map $S^n(\mathsf{L}_P(-1))\to\varinjlim S^n(\mathsf{L}_P(-1))=\tilde{\mathcal{O}}_P^{\log}$. Then $N_{\bullet}$ defines an I_P-invariant filtration on $\tilde{\mathcal{O}}_P^{\log}$. The action of I_P on $\mathrm{Gr}_n^N\tilde{\mathcal{O}}_P^{\log}\cong S^n(P^{\mathrm{gp}}(-1))$ is trivial and thus the action on $\tilde{\mathcal{O}}_P^{\log}$ is unipotent.

The splitting $\tilde{\chi}$ defines a splitting $P^{\mathrm{gp}}(-1)\to\mathsf{L}_P(-1)$ and thus an isomorphism

$$\tilde{\mathcal{O}}_P^{\log}\cong\bigoplus_n\mathrm{Gr}_n^N\tilde{\mathcal{O}}_P^{\log}\cong\bigoplus_n S^n(P^{\mathrm{gp}}(-1));$$

this isomorphism is "canonical" but not I_P-equivariant.

For $\gamma \in \mathsf{I}_P$, denote by ρ_γ the corresponding automorphism of $\tilde{\mathcal{O}}_P^{\log}$, and let

$$\lambda_\gamma := \log(\rho_\gamma) := \sum_i (-1)^{i+1}(\rho_\gamma - \mathrm{id})^i/i. \tag{5-1-2}$$

The above formula defines, a priori, an endomorphism of $\mathbb{Q}\otimes\tilde{\mathcal{O}}_P^{\log}$, but, as we shall soon see, in fact this endomorphism preserves $\tilde{\mathcal{O}}_P^{\log}$.

Claim 5.1.1. *For $\gamma \in \mathsf{I}_P = \mathrm{Hom}(P^{\mathrm{gp}}(-1), \mathbb{Z})$, the endomorphism λ_γ of $\mathbb{Q}\otimes\tilde{\mathcal{O}}_P^{\log}$ defined above is given by interior multiplication with γ:*

$$\tilde{\mathcal{O}}_P^{\log} \cong S^{\bullet}(P^{\mathrm{gp}}(-1)) \xrightarrow{\eta} P^{\mathrm{gp}}(-1)\otimes S^{\bullet}(P^{\mathrm{gp}}(-1)) \xrightarrow{\gamma\otimes\mathrm{id}} S^{\bullet}(P^{\mathrm{gp}}(-1)) \cong \tilde{\mathcal{O}}_P^{\log},$$

where η is the map defined in Section 2.2. The subspace $N_n\tilde{\mathcal{O}}_P^{\log}$ of $\tilde{\mathcal{O}}_P^{\log}$ is the annihilator of the ideal J^{n+1} of the group algebra $\mathbb{Z}[\mathsf{I}_P]$.

Proof. Let $V := \mathbb{Q}\otimes P^{\mathrm{gp}}(-1)$ and let ϕ be an element of $\mathrm{Hom}(V, \mathbb{Q})$. Interior multiplication by ϕ is the unique derivation λ of the algebra $S^{\bullet}V$ such that $\lambda(v) = \phi(v)$ for all $v \in V$. There is also a unique automorphism ρ of $S^{\bullet}V$ such that $\rho(v) = v + \phi(v)$ for all $v \in V$. We claim that $\lambda = \log\rho$, or, equivalently, that $\rho = \exp\lambda$. (These are well-defined because $\rho - \mathrm{id}$ and λ are locally nilpotent.) Since λ is a derivation of $S^{\bullet}V$, we have

$$\lambda^k(ab)/k! = \sum_{i+j=k} (\lambda^i a/i!)(\lambda^j b/j!),$$

hence

$$\exp(\lambda)(ab) = \sum_k \lambda^k(ab)/k! = \Bigl(\sum_i \lambda^i(a)/i!\Bigr)\Bigl(\sum_j \lambda^j(b)/j!\Bigr)$$
$$= \exp(\lambda(a))\cdot\exp(\lambda(b)).$$

Thus $\exp\lambda$ is an automorphism of the algebra $S^{\bullet}V$. Since it sends v to $v+\phi(v)$, it agrees with ρ, as claimed.

If $v_1, v_2, \dots, v_n$ is a sequence of elements of V, then

$$\begin{aligned}\rho(v_1v_2\cdots v_n) &= (v_1+\phi(v_1))(v_2+\phi(v_2))\cdots(v_n+\phi(v_n))\\ &= v_1v_2\cdots v_n + \sum_i \phi(v_i)v_1\cdots\hat{v}_i\cdots v_n + R,\end{aligned}$$

where the symbol $\hat{v}_i$ means that the i-th element is omitted and where $R \in N_{n-2}S^nV$. In particular, $\rho - \mathrm{id}$ maps $N_nS^{\bullet}V$ to $N_{n-1}S^{\bullet}V$ and acts on $\mathrm{Gr}_n S^{\bullet}V \cong S^nV$ as interior multiplication by ϕ. Since $\mathrm{Gr}^N S^{\bullet}(P^{\mathrm{gp}}(-1))$ is torsion free, the analogous results hold for $S^{\bullet}(P^{\mathrm{gp}}(-1))$. The augmentation ideal J of the group algebra $\mathbb{Z}[I_P]$ is generated by elements of the form $\gamma - 1$, and it follows that J takes $N_n\tilde{\mathcal{O}}_P^{\log}$ to $N_{n-1}\tilde{\mathcal{O}}_P^{\log}$ and hence that J^{n+1} annihilates $N_n\tilde{\mathcal{O}}_P^{\log}$. Moreover, the natural map $S^n \mathsf{I}_P \to J^n/J^{n+1}$ is an isomorphism and identifies the pairing $J^n/J^{n+1} \times \mathrm{Gr}^N_n \tilde{\mathcal{O}}^{\log}_P \to \mathbb{Z}$ with the standard pairing $S^n \mathsf{I}_P \times S^n P^{\mathrm{gp}}(1) \to \mathbb{Z}$. Since this pairing is nondegenerate over $\mathbb{Q}$, it follows that $N_n \tilde{\mathcal{O}}^{\log}_P$ is the annihilator of J^{n+1}. □

The map $\tilde{\chi} : P^{\mathrm{gp}} \to \mathsf{L}_P$ defines a homomorphism $P^{\mathrm{gp}} \to \eta_S^*(\mathcal{L}_S)$ and hence also a homomorphism $P^{\mathrm{gp}} \to \tilde{\tau}_{S*}\eta_S^*(\mathcal{O}^{\log}_S)$. In fact, one checks easily that the induced map

$$\mathbb{C} \otimes \tilde{\mathcal{O}}^{\log}_P \to \Gamma(\tilde{S}, \eta_S^*(\mathcal{O}^{\log}_S)) \tag{5-1-3}$$

is an isomorphism, compatible with the action of I_P. The map $d : \tilde{\mathcal{O}}^{\log}_S \to \tilde{\mathcal{O}}^{\log}_S \otimes \Omega^1_{S/\mathbb{C}}$ identifies with the map

$$\eta : \mathbb{C} \otimes S^{\bullet} P^{\mathrm{gp}} \to \mathbb{C} \otimes S^{\bullet} P^{\mathrm{gp}} \otimes P^{\mathrm{gp}} : p_1 \cdots p_n \mapsto \sum_i p_1 \cdots \hat{p}_i \cdots p_n \otimes p_i, \tag{5-1-4}$$

and the action of $\gamma \in \mathsf{I}_P$ on $\tilde{\mathcal{O}}^{\log}_P$ is given by the unique ring homomorphism taking $p \otimes 1$ to $p \otimes 1 + \gamma(p)$.

More generally, suppose that x is a point of a fine saturated log analytic space X. Let $Q := \overline{\mathcal{M}}_{X,x}$ and choose a splitting of the map $\mathcal{M}_{X,x} \to Q$. This splitting induces an isomorphism $\tau_X^{-1}(x) \cong \mathsf{T}_Q$, which admits a universal cover $\mathsf{V}_Q \to \mathsf{T}_Q$. An element q of Q^{gp} defines a function $\mathsf{V}_Q \to \mathbb{R}(1)$ which in fact is a global section of the pullback of $\mathcal{L}_X \subseteq \mathcal{O}^{\log}_X$ to V_Q. Since $\mathcal{O}^{\log}_X$ is a sheaf of rings, there is an induced ring homomorphism: $S^{\bullet}(Q^{\mathrm{gp}}) \to \Gamma(\mathsf{V}_Q, \mathcal{O}^{\log}_X)$. These constructions result in the Proposition 5.1.2 below. For more details, we refer again to [Kato and Nakayama 1999, 3.3; Ogus 2003, 3.3.4; 2018, V, §3.3].

Proposition 5.1.2. *Let x be a point of fine saturated log analytic space X. Then a choice of a splitting $\mathcal{M}_{X,x} \to Q := \overline{\mathcal{M}}_{X,x}$ yields*:

(1) *an isomorphism*: $\tau_X^{-1}(x) \xrightarrow{\sim} \mathsf{T}_Q := \mathrm{Hom}(Q, \mathbb{S}^1)$,

(2) *a universal cover*: $\mathsf{V}_Q := \mathrm{Hom}(Q, \mathbb{R}(1)) \to \tau_X^{-1}(x)$,

(3) *for each i, an isomorphism* $\Omega^i_{X,x} \otimes S^{\bullet} Q^{\mathrm{gp}} \xrightarrow{\sim} \Gamma(\mathsf{V}_Q, \tilde{\eta}_x^{-1}(\Omega^{i,\log}_X))$, *where*

$$\tilde{\eta}_x : V_Q \to T_Q \to \tau_X^{-1}(x) \to X_{\log}$$

is the natural map.

If $\gamma \in \mathsf{I}_P := \mathrm{Hom}(P^{\mathrm{gp}}, \mathbb{Z}(1))$ then the action of ρ_γ on $\Gamma(\mathsf{V}_X, \mathcal{O}^{\log}_X)$ is given by $\exp(\lambda_\gamma)$, *where λ_γ is interior multiplication by γ.* □

Since $\mathbb{C} \otimes \tilde{\mathcal{O}}^{\log}_P$ is a module with connection on the log point S, its pull-back $f^*(\tilde{\mathcal{O}}^{\log}_P)$ to X has an induced connection $f^*(\mathbb{C} \otimes \tilde{\mathcal{O}}^{\log}_P) \to \tilde{\mathcal{O}}^{\log}_P \otimes \Omega^1_{X/\mathbb{C}}$.

In the following definition and theorem we use the notation of diagrams (4-2-1) and (4-2-2), and if $\mathcal{F}$ is a sheaf on $X_{\log}$ (resp. $S_{\log}$), we write $\tilde{\mathcal{F}}$ for its pullback to $\tilde{X}_{\log}$ (resp. $\tilde{S}_{\log}$).

Definition 5.1.3. Let $f : X \to S$ be a smooth morphism of fine saturated log analytic spaces over the split log point S associated to a fine sharp monoid P. The *Steenbrink complex* of X/S is the de Rham complex

$$K^\bullet_{X/S} := \tilde{\mathcal{O}}^{\log}_P \otimes_{\mathbb{Z}} \Omega^\bullet_{X/\mathbb{C}} = \tilde{\tau}_{X_S *}\big(\tilde{f}^{\log *}_S(\tilde{\mathcal{O}}^{\log}_S) \otimes \tilde{\tau}^*_{X_S}(\Omega^\bullet_{X/\mathbb{C}})\big)$$

of the $\mathcal{O}_X$-module with connection $f^*\tilde{\mathcal{O}}^{\log}_S$, given by

$$S^\bullet P^{\mathrm{gp}} \otimes \mathcal{O}_X \to S^\bullet P^{\mathrm{gp}} \otimes \Omega^1_{X/\mathbb{C}}, \quad p \mapsto p \otimes d\log p$$

endowed with its natural I_P-action.

Theorem 5.1.4. *Let S be the split log point associated to a fine sharp and saturated monoid P and let $f : X \to S$ be a smooth saturated morphism of fine saturated log analytic spaces. Let $\tilde{\Omega}^{\bullet,\log}_{X/\mathbb{C}} := \eta^*_X(\Omega^{\bullet,\log}_{X/\mathbb{C}})$ on $\tilde{X}_{\log} := X_{\log} \times_{S_{\log}} \tilde{S}_{\log}$. Then in the derived category $D^+(X_{\mathrm{top}}, \mathbb{C}[\mathsf{I}_P])$ of complexes of sheaves of $\mathbb{C}[\mathsf{I}_P]$-modules on X_{top}, there are natural isomorphisms*

$$R\tilde{\tau}_{X*}(\mathbb{C}) \xrightarrow{\sim} R\tilde{\tau}_{X*}(\tilde{\Omega}^{\bullet,\log}_{X/\mathbb{C}}) \xleftarrow{\sim} K^\bullet_{X/S}.$$

Proof. It is proved in [Kato and Nakayama 1999, 3.8] that, on the space $X_{\log}$, the natural map

$$\mathbb{C} \to \Omega^{\bullet,\log}_{X/\mathbb{C}}$$

is a quasi-isomorphism. Its pullback via η_X is a quasi-isomorphism

$$\mathbb{C} \to \eta^*_X(\Omega^{\bullet,\log}_{X/\mathbb{C}}) = \tilde{\Omega}^{\bullet,\log}_{X/\mathbb{C}}$$

on $\tilde{X}_{\log}$, invariant under the action of I_P. Applying the derived functor $R\tilde{\tau}_{X*}$, we obtain the isomorphism

$$R\tilde{\tau}_{X*}(\mathbb{C}) \xrightarrow{\sim} R\tilde{\tau}_{X*}(\tilde{\Omega}^{\bullet,\log}_{X/\mathbb{C}})$$

in the theorem.

The natural map $f^{-1}_{\log}(\mathcal{O}^{\log}_S) \to \mathcal{O}^{\log}_X$ induces a map

$$\tilde{f}^*(\tilde{\mathcal{O}}^{\log}_S) \otimes \tilde{\tau}^*_X(\Omega^\bullet_{X/\mathbb{C}}) \to \tilde{\Omega}^{\bullet,\log}_{X/\mathbb{C}},$$

and hence by adjunction a map

$$\tilde{\mathcal{O}}^{\log}_P \otimes \Omega^\bullet_{X/\mathbb{C}} \to R\tilde{\tau}_{X*}(\tilde{\Omega}^{\bullet,\log}_{X/\mathbb{C}}).$$

The lemma below shows that this map is an isomorphism and completes the proof of the theorem. □

Lemma 5.1.5. *The terms of the complex $\tilde{\Omega}^{\bullet,\log}_{\tilde{X}}$ are acyclic for $\tilde{\tau}_{X*}$, and for each q the natural map*

$$K^q_{X/S} \to \tilde{\tau}_{X*}(\tilde{\Omega}^{q,\log}_{X/\mathbb{C}})$$

is an isomorphism.

Proof. The morphism $\tau_{X/S}$ in diagram (4-2-1) is proper and the left upper square is Cartesian, and hence $\tilde{\tau}_{X/S}$ is also proper. Let $\tilde{x} = (x, \theta)$ be a point in $\tilde{X}_S^{\log} \cong X_{\mathrm{top}} \times \mathsf{V}_P$. By the proper base change theorem, the natural map

$$(R^i \tilde{\tau}_{X/S*} \tilde{\Omega}^{q,\log}_{X/\mathbb{C}})_{\tilde{x}} \to H^i(\tilde{\tau}^{-1}_{X/S}(\tilde{x}), \tilde{\Omega}^{q,\log}_{X/\mathbb{C}})$$

is an isomorphism. (Here the term on the right means the i-th cohomology of the sheaf-theoretic restriction of $\tilde{\Omega}^{q,\log}_{X/\mathbb{C}}$ to the fiber.) The fiber $\tilde{\tau}^{-1}_{X/S}(\tilde{x})$ is a torsor under the group

$$\mathsf{T}_{X/S,x} := \mathrm{Hom}(\overline{\mathcal{M}}_{X/S,x}\mathbb{S}^1) \subseteq \mathsf{T}_{X,x} := \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{S}^1).$$

Hence the fiber is homeomorphic to this torus, and $\tilde{\Omega}^{q,\log}_{X/\mathbb{C}}$ is locally constant on the fiber, as follows from Proposition 5.1.2. Since the fiber is a $K(\pi, 1)$, its cohomology can be calculated as group cohomology. More precisely, view x as a log point (with its log structure inherited from X), so that we have a morphism of log points $x \to S$ and hence a morphism: $x_{\log} \to S_{\log}$. Then a choice of a point $\bar{x}$ of $\tau_X^{-1}(x)$ allows us to make identifications

$$\tau_X^{-1}(x) \cong x_{\log} \cong \mathsf{T}_{X,x} \quad \text{and} \quad \tilde{\tau}^{-1}_{X/S}(\bar{x}) \cong \mathsf{T}_{X/S,x}\,.$$

The second torus has a universal cover $\mathsf{V}_{X/S,x} := \mathrm{Hom}(\overline{\mathcal{M}}_{X/S,x}, \mathbb{R}(1))$, and every locally constant sheaf $\mathcal{F}$ on $\mathsf{T}_{X/S,x}$ is constant when pulled back to this cover, so the natural map $\Gamma(\mathsf{V}_{X/S,x}, \mathcal{F}) \to \mathcal{F}_{\bar{x}}$ is an isomorphism. These groups have a natural action of the covering group $\mathsf{I}_{X/S,x} = \mathrm{Hom}(\mathcal{M}^{\mathrm{gp}}_{X/S,x}, \mathbb{Z}(1))$. Then

$$H^i(\tilde{\tau}^{-1}_{X/S}(\tilde{x}), \mathcal{F}) \cong H^i(\mathsf{I}_{X/S,x}, \mathcal{F}_{\bar{x}}).$$

In our case, we have

$$\tilde{\Omega}^{q,\log}_{X/\mathbb{C},\bar{x}} = \mathcal{O}^{\log}_{X,\bar{x}} \otimes \Omega^q_{X/\mathbb{C},x} \cong S^{\bullet}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X,x}) \otimes \Omega^q_{X/\mathbb{C},x}.$$

Choosing a splitting of $P^{\mathrm{gp}} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{X,x}$, we can write

$$S^{\bullet}\overline{\mathcal{M}}^{\mathrm{gp}}_{X,x} \cong S^{\bullet}P^{\mathrm{gp}} \otimes S^{\bullet}\mathcal{M}^{\mathrm{gp}}_{X/S,x},$$

compatibly with the action of $\mathsf{I}_{X/S,x}$. Let $V := \mathbb{C} \otimes \mathcal{M}^{\mathrm{gp}}_{X/S.x}$, and for $\gamma \in I_{X/S,x} \subseteq \mathrm{Hom}(V, \mathbb{C})$, let λ_γ denote interior multiplication by γ on $S^{\bullet}V$. An analog of Claim 5.1.1 shows that $\rho_\gamma = \exp \lambda_\theta$. Then a standard calculation shows that

$$H^i(\mathsf{I}_{X/S,x}, \mathbb{C} \otimes S^{\bullet}\mathcal{M}^{\mathrm{gp}}_{X/S,x}) \cong \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

Here is one way to carry out this calculation. As we have seen, the representation $(S^{\bullet}V, \rho)$ of $I_{X/S,x}$ is the exponential of the locally nilpotent Higgs field $\lambda : S^{\bullet}V \to S^{\bullet}V \otimes V$ given by the exterior derivative. It follows from [Ogus 2003, 1.44] that one can use Higgs cohomology to calculate the group cohomology of such locally unipotent representations. In our case the Higgs complex of λ identifies with the de Rham complex of the symmetric algebra $S^{\bullet}V$, and the result follows.

We conclude that $H^i(\tilde{\tau}^{-1}_{X/S}(\tilde{x}), \tilde{\Omega}^{q,\log}_{X/\mathbb{C}})$ vanishes if $i > 0$, and that the natural map

$$\tilde{\mathcal{O}}^{\log}_P \otimes \Omega^{q,\log}_{X/\mathbb{C},x} \to H^0(\tilde{\tau}^{-1}_{X/S}(\tilde{x}), \tilde{\Omega}^{q,\log}_{X/\mathbb{C}})$$

is an isomorphism. Then the proper base change theorem implies that $R^i\tilde{\tau}_{X/S*}\Omega^{q,\log}_{\tilde{X}/\mathbb{C}}$ vanishes for $i > 0$ and that the natural map

$$\tilde{\tau}^*_{X_S}(\tilde{\mathcal{O}}^{\log}_P \otimes \Omega^q_{X/\mathbb{C}}) \to \tilde{\tau}_{X/S*}(\Omega^{q,\log}_{\tilde{X}/\mathbb{C}}).$$

is an isomorphism. But the map $\tilde{\tau}_{X_S}$ is just the projection $X_{\mathrm{top}} \times \mathsf{V}_P \to X$, so for any abelian sheaf $\mathcal{F}$ on X_{top}, $R^i\tilde{\tau}_{X_S*}\tilde{\tau}^*_{X_S}\mathcal{F} = 0$ and $\mathcal{F} \cong \tilde{\tau}_{X_S*}\tilde{\tau}^*_{X_S}\mathcal{F}$, by [Kashiwara and Schapira 1990, 2.7.8]. Since $\tilde{\tau}_X = \tilde{\tau}_{X_S} \circ \tilde{\tau}_{X/S}$, we conclude that $R^i\tilde{\tau}_{X*}(\tilde{\Omega}^{q,\log}_{X/\mathbb{C}})$ vanishes if $i > 0$ and that the natural map $\tilde{\mathcal{O}}^{\log}_S \otimes \Omega^q_{X/\mathbb{C}} \to \tilde{\tau_X}_*(\Omega^{q,\log}_{X/\mathbb{C}})$ is an isomorphism. The lemma follows. □

Corollary 5.1.6. *In the situation of Theorem 5.1.4, the maps* σ_{DR} (5-1-1) *factor through isomorphisms*

$$\mathbb{C} \otimes \bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S} \xrightarrow{\sim} \mathcal{H}^q(K^{\bullet}_{X/S}).$$

Proof. There is an evident inclusion $\Omega^{\bullet}_{X/\mathbb{C}} \to K^{\bullet}_{X/S}$, and hence we find natural maps

$$\mathbb{C} \otimes \overline{\mathcal{M}}^{\mathrm{gp}}_X \to \mathcal{H}^1(\Omega^{\bullet}_{X/\mathbb{C}}) \to \mathcal{H}^1(K^{\bullet}_{X/S}).$$

It follows from the formula (5-1-4) that the image of each element of P^{gp} becomes exact in $K^1_{X/S}$, and hence this composed map factors through $\mathbb{C} \otimes \mathcal{M}^{\mathrm{gp}}_{X/S}$. The maps in the statement of the corollary are then obtained by cup product. We now have a commutative diagram

$$\begin{array}{ccc} \mathbb{C} \otimes \bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S} & \longrightarrow & \mathcal{H}^q(K^{\bullet}_{X/S}) \\ & \searrow^{\tilde{\sigma}} & \downarrow \cong \\ & & R^q\tilde{\tau}_{X*}(\mathbb{C}), \end{array}$$

where the vertical arrow is the isomorphism coming from Theorem 5.1.4. Since $\tilde{\sigma}$ is an isomorphism by Theorem 4.2.1, the horizontal arrow is also an isomorphism. □

5.2. ***Monodromy and the canonical filtration.*** The filtration $N_{\bullet}$ of $\tilde{\mathcal{O}}_S^{\log}$ is stable under I_P and the connection and hence induces a filtration of the complex $K^{\bullet}_{X/S}$. Claim 5.1.1 shows that N_n corresponds to the n-th level of the "kernel" filtration defined by the monodromy action on the complex $K^{\bullet}_{X/S}$. We shall see that this filtration coincides up to quasi-isomorphism with the canonical filtration $\tau_{\leq}$. Since we prefer to work with decreasing filtrations, we set

$$N^k K^q_{X/S} := (N_{-k}\tilde{\mathcal{O}}_S^{\log}) \otimes \Omega^q_{X/\mathbb{C}},$$

and

$$T^k K^q_{X/S} := \begin{cases} K^q_{X/S} & \text{if } k \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $N^0 K^{\bullet}_{X/S} = \Omega^{\bullet}_{X/S}$ and $N^1 K^{\bullet}_{X/S} = 0$, so $N^i K^{\bullet}_{X/S} \subseteq T^i K^{\bullet}_{X/S}$ for all i, that is, the filtration $N^{\bullet}$ is finer than the filtration $T^{\bullet}$.

Recall from [Deligne 1971, 1.3.3] that if F is a filtration of a complex $K^{\bullet}$, the "filtration décalée" $\tilde{F}$ is the filtration of $K^{\bullet}$ defined by

$$\tilde{F}^i K^n := \{x \in F^{i+n} K^n : dx \in F^{i+n+1} K^n\}.$$

Then there are natural maps

$$E_0^{i,n-i}(K^{\bullet}, \tilde{F}) = \mathrm{Gr}^i_{\tilde{F}} K^n \to \mathcal{H}^n(\mathrm{Gr}^{i+n}_F K^{\bullet}) = E_1^{i+n,-i}(K^{\bullet}, F)$$

inducing quasi-isomorphisms

$$(E_0^{i,\bullet}(K^{\bullet}, \tilde{F}), d_0^{i,\bullet}) \to (E_1^{\bullet+i,-i}(K^{\bullet}, F), d_1^{\bullet+i,-i}).$$

This equation says that the natural maps (induced by the identity map of $K^{\bullet}$), are quasi-isomorphisms

$$(E_0^{-q,\bullet}(K^{\bullet}, \tilde{F}), d_0^{-q,\bullet}) \to (E_1^{\bullet,q}(K^{\bullet}, F), d_1^{\bullet,q})[q]' \tag{5-2-1}$$

where the symbol $[q]'$ means the *naive shift* of the complex (which does not change the sign of the differential). More generally, there are isomorphisms of spectral sequences, after a suitable renumbering [Deligne 1971, 1.3.4]:

$$(E_r^{\bullet,\bullet}(K^{\bullet}, \tilde{F}), d_r^{\bullet,\bullet}) \to (E_{r+1}^{\bullet,\bullet}(K^{\bullet}, F), d_{r+1}^{\bullet,\bullet}).$$

Let $\tilde{N}^{\bullet}$ denote the filtration décalée of $N^{\bullet}$, and similarly for $T^{\bullet}$; note that $\tilde{T}^i = \tau_{\leq -i}$, the "filtration canonique." Since the filtration $N^{\bullet}$ is finer than $T^{\bullet}$, the filtration $\tilde{N}$ is finer than the filtration $\tilde{T}^{\bullet}$, and we find a morphism of filtered complexes

$$(K^{\bullet}_{X/S}, \tilde{N}^{\bullet}) \to (K^{\bullet}_{X/S}, \tilde{T}^{\bullet}). \tag{5-2-2}$$

Theorem 5.2.1. *Let $f : X \to S$ be a smooth and saturated morphism of fine saturated log analytic spaces, where S is the split log point associated to a sharp toric monoid. Then there are natural filtered quasi-isomorphisms*

$$(K^{\bullet}_{X/S}, \tilde{N}^{\bullet}) \xrightarrow{\sim} (K^{\bullet}_{X/S}, \tilde{T}^{\bullet}) \xleftarrow{\sim} (\Psi_{X/S}, \tilde{T}^{\bullet}).$$

The existence of the second filtered quasi-isomorphism of the theorem follows from the canonicity of the filtration $\tilde{T}$ and Theorem 5.1.4. The proof that the first arrow is a filtered quasi-isomorphism is a consequence of the following more precise result.

Recall from Definition 2.2.3 that associated to the homomorphism

$$\theta : \mathbb{C} \otimes \overline{\mathcal{M}}_{S^{\mathrm{gp}}} \to \mathbb{C} \otimes \overline{\mathcal{M}}^{\mathrm{gp}}_X$$

we have for each q a complex $\mathrm{Kos}^{q,\bullet}(\theta)$ and whose n-th term is given by

$$\mathrm{Kos}^{n,q}(\theta) = \mathbb{C} \otimes S^{q-n} \overline{\mathcal{M}}_{S^{\mathrm{gp}}} \otimes \Lambda^n \overline{\mathcal{M}}^{\mathrm{gp}}_X.$$

Theorem 5.2.2. *Let $f : X \to S$ be as in Theorem 5.1.4, let $K^{\bullet}_{X/S}$ be the Steenbrink complex on X_{top}, and let*

$$0 \to \mathbb{C} \otimes \overline{\mathcal{M}}^{\mathrm{gp}}_S \xrightarrow{\theta} \mathbb{C} \otimes \overline{\mathcal{M}}^{\mathrm{gp}}_X \xrightarrow{\pi} \mathbb{C} \otimes \overline{\mathcal{M}}^{\mathrm{gp}}_{X/S} \to 0$$

be the exact sequence of sheaves of $\mathbb{C}$-vectors spaces on X obtained by tensoring the log Kodaira–Spencer sequence (1-0-1) with $\mathbb{C}$.

(1) *For each $q \geq 0$, there are natural morphisms of complexes*:

$$\mathrm{Gr}^{-q}_{\tilde{N}} K^{\bullet}_{X/S} \xrightarrow{\sim} E_1^{\bullet,q}(K^{\bullet}_{X/S}, N)[-q]' \xrightarrow{\cong} \mathrm{Kos}^{\bullet}_q(\theta) \xrightarrow{\sim} \mathbb{C} \otimes \bigwedge^q \overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}[-q],$$

where the first and last maps are quasi-isomorphisms and the second map is an isomorphism. (The notation $[-q]'$ means the naive shift of the complex, and $\mathrm{Kos}^{\bullet}_q$ is the complex defined in Definition 2.2.3.)

(2) *The morphism of spectral sequences induced by the map of filtered complexes $(K^{\bullet}_{X/S}, N^{\bullet}) \to (K^{\bullet}_{X/S}, T^{\bullet})$ is an isomorphism at the E_2-level and beyond.*

(3) *The map of filtered complexes $(K^{\bullet}_{X/S}, \tilde{N}^{\bullet}) \to (K^{\bullet}_{X/S}, \tilde{T}^{\bullet})$ is a filtered quasi-isomorphism.*

Proof. The first arrow in (1) is the general construction of Deligne as expressed in Equation (5-2-1). It follows from the definitions that

$$E_0^{-p,q}(K^{\bullet}_{X/S}, N) = \mathrm{Gr}^{-p}_N K^{q-p}_{X/S} \cong S^p \overline{\mathcal{M}}^{\mathrm{gp}}_S \otimes \Omega^{q-p}_{X/\mathbb{C}}.$$

Since the elements of $S^p \overline{\mathcal{M}}^{\mathrm{gp}}_S$ are horizontal sections of $\mathrm{Gr}^{-p}_N \tilde{\mathcal{O}}^{\log}_S$, the differential $d_0^{p,\bullet}$ of the complex $E_0^{p,\bullet}(K^{\bullet}_{X/S}, N)$ can be identified with the identity map of $S^p\overline{\mathcal{M}}_S^{\text{gp}}$ tensored with the differential of $\Omega^\bullet_{X/\mathbb{C}}$. Then the isomorphism (5-1-1) allows us to write:

$$E_1^{-p,q}(K^\bullet_{X/S}, N) = \begin{cases} S^p\overline{\mathcal{M}}_S^{\text{gp}} \otimes \mathcal{H}^{q-p}(\Omega^\bullet_{X/\mathbb{C}}) \cong \mathbb{C}\otimes S^p\overline{\mathcal{M}}_S^{\text{gp}} \otimes \bigwedge^{q-p}\overline{\mathcal{M}}_X^{\text{gp}} & \text{if } 0 \le p \le q, \\ 0 & \text{otherwise.} \end{cases}$$

The isomorphism appearing above is the identity of $S^p\overline{\mathcal{M}}_S^{\text{gp}}$ tensored with the isomorphism (5-1-1). The differential $d_1^{-p,q}$ becomes identified with a map

$$\begin{array}{ccc} \mathbb{C}\otimes S^p\overline{\mathcal{M}}_S^{\text{gp}} \otimes \bigwedge^{q-p}\overline{\mathcal{M}}_X^{\text{gp}} & \longrightarrow & \mathbb{C}\otimes S^{p-1}\overline{\mathcal{M}}_S^{\text{gp}} \otimes \bigwedge^{q-p+1}\overline{\mathcal{M}}_X^{\text{gp}} \\ \downarrow & & \downarrow \\ \operatorname{Kos}^{q-p,q}(\theta) & \longrightarrow & \operatorname{Kos}^{q-p+1,q}(\theta) \end{array}$$

It follows from formula (5-1-4) that this differential is indeed the Koszul differential. Thus we have found the isomorphism

$$E_1^{\bullet,q}(K^\bullet_{X/S}, N)[-q]' \xrightarrow{\cong} \operatorname{Kos}^{\bullet,q}(\theta).$$

The quasi-isomorphism $\operatorname{Kos}^{\bullet,q}(\theta) \xrightarrow{\sim} \bigwedge^q\overline{\mathcal{M}}_{X/S}^{\text{gp}}[-q]$ comes from Proposition 2.2.4. This completes the proof of statement (1) of the theorem.

We have natural maps of filtered complexes

$$(K^\bullet_{X/S}, N^\bullet) \to (K^\bullet_{X/S}, T^\bullet), \quad \text{hence also} \quad (K^\bullet_{X/S}, \tilde{N}^\bullet) \to (K^\bullet_{X/S}, \tilde{T}^\bullet).$$

These maps produce the map of spectral sequences in statement (2). Consider the spectral sequence associated to the filtered complex $(K^\bullet_{X/S}, T^\bullet)$, in the category of abelian sheaves. We have

$$E_0^{-p,q}(K^\bullet_{X/S}, T) = \operatorname{Gr}_T^{-p} K^{q-p}_{X/S} = \begin{cases} K^{q-p}_{X/S} & \text{if } p = 0, \\ 0 & \text{otherwise,} \end{cases}$$

hence an isomorphism of complexes,

$$(E_0^{0,\bullet}, d) \cong K^\bullet_{X/S},$$

and of cohomology groups,

$$E_1^{-p,q}(K_{X/S}, T) = \begin{cases} \mathcal{H}^q(K^\bullet_{X/S}) & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the complex $E_1^{\bullet,q}(K, T)$ is isomorphic to the sheaf $\mathcal{H}^q(K_{X/S})$, viewed as a complex in degree zero, and the spectral sequence degenerates at E_1. Then

$$E_\infty^{0,q}(K, T) = E_1^{0,q}(K, T) = \mathcal{H}^q(K^\bullet_{X/S}) \cong R^q\tilde{\tau}_X(\mathbb{C}) \cong \bigwedge^q\mathbb{C}\otimes\overline{\mathcal{M}}_{X/S}^{\text{gp}},$$

by Corollary 5.1.6. Since the maps are all natural, statement (2) of Theorem 5.2.2 follows.

Using the naturality of the maps in (5-2-1), we find for every i a commutative diagram of complexes:

$$\begin{array}{ccc} \mathrm{Gr}^{-i}_{\tilde N} K^\bullet_{X/S} & \longrightarrow & \mathrm{Gr}^{-i}_{\tilde T} K^\bullet_{X/S} \\ \downarrow & & \downarrow \\ E_1^{\bullet,-i}(K^\bullet_{X/S}, N) & \longrightarrow & E_1^{\bullet,-i}(K^\bullet_{X/S}, T) \end{array}$$

Since the bottom horizontal arrow is a quasi-isomorphism and the vertical arrows are quasi-isomorphisms, it follows that the top horizontal arrow is also a quasi-isomorphism. Since $\tilde T^i$ and $\tilde N^i$ both vanish for $i > 0$, it follows by induction that for every i, the map $\tilde N^i K^\bullet_{X/S} \to \tilde T^i K^\bullet_{X/S}$ is a quasi-isomorphism. □

Combining the above results with our study of Koszul complexes in Section 2.2, we can now give our first proof of the monodromy formula in Theorem 4.2.2 after tensoring with $\mathbb{C}$.

Any $\gamma \in \mathsf{I}_P$ induces a homomorphism $\overline{\mathcal{M}}^{\mathrm{gp}}_S \to \mathbb{C}$, which we denote also by γ. By Proposition 5.1.2 the action of $\tilde\lambda_\gamma := \log(\rho_\gamma)$ on $\tilde{\mathcal{O}}^{\log}_S \cong S^\bullet \overline{\mathcal{M}}^{\mathrm{gp}}_S$ corresponds to interior multiplication by γ. Thus for every i, λ_γ maps $N^{-i} K^\bullet_{X/S}$ to $N^{1-i} K^\bullet_{X/S}$ and hence $\tilde N^{-i} K^\bullet_{X/S}$ to $\tilde N^{1-i} K^\bullet_{X/S}$. We need to compute the induced map

$$\tilde\lambda^i_\gamma : \mathrm{Gr}^{-i}_{\tilde N} K^\bullet_{X/S} \to \mathrm{Gr}^{1-i}_{\tilde N} K^\bullet_{X/S}.$$

Using the quasi-isomorphism of statement (1) of Theorem 5.2.2, we can identify this as the map

$$\gamma_i : \mathrm{Kos}^\bullet_i(\theta) \to \mathrm{Kos}^\bullet_{i-1}(\theta)$$

which in degree n is the composition

$$\begin{array}{ccc} \mathbb{C} \otimes S^{i-n} \overline{\mathcal{M}}^{\mathrm{gp}}_S \otimes \bigwedge^n \overline{\mathcal{M}}^{\mathrm{gp}}_X & \xrightarrow{\eta\otimes\mathrm{id}} & \mathbb{C} \otimes \overline{\mathcal{M}}^{\mathrm{gp}}_S \otimes S^{i-n-1} \overline{\mathcal{M}}^{\mathrm{gp}}_S \otimes \bigwedge^n \overline{\mathcal{M}}^{\mathrm{gp}}_X \\ & \searrow & \downarrow{\scriptstyle \gamma\otimes\mathrm{id}} \\ & & \mathbb{C} \otimes S^{i-n-1} \overline{\mathcal{M}}^{\mathrm{gp}}_S \otimes \bigwedge^n \overline{\mathcal{M}}^{\mathrm{gp}}_X, \end{array}$$

where η is the map defined at beginning of Section 2.2. In other words, our map is the composition of the morphism

$$c_q : \mathrm{Kos}^{\bullet,q}(\theta) \to \overline{\mathcal{M}}^{\mathrm{gp}}_S \otimes \mathrm{Kos}^{\bullet,q-1}(\theta),$$

constructed in Proposition 2.2.5 with $\gamma \otimes \mathrm{id}$. We thus find a commutative diagram in the derived category:

$$\begin{array}{ccccccc}
\Psi^q_{X/S} & \xleftarrow{\sim} & \mathrm{Gr}^{-q}_{\tilde{N}} K^\bullet_{X/S}[q] & \xrightarrow{\sim} & \mathrm{Kos}^\bullet_q(\theta)[q] & \xrightarrow{\sim} & \bigwedge^q \overline{\mathcal{M}}^{\mathrm{gp}}_{X/S} \\
\big\downarrow{\scriptstyle \lambda^q_\gamma} & & \big\downarrow{\scriptstyle \tilde{\lambda}^q_\gamma} & & \big\downarrow{\scriptstyle (\gamma\otimes\mathrm{id})\circ c_q} & & \big\downarrow{\scriptstyle g_q} \\
\Psi^{q-1}_{X/S}[1] & \xleftarrow{\sim} & \mathrm{Gr}^{1-q}_{\tilde{N}} K^\bullet_{X/S}[q] & \xrightarrow{\sim} & \mathrm{Kos}^\bullet_{q-1}(\theta)[q] & \xrightarrow{\sim} & \bigwedge^{q-1} \overline{\mathcal{M}}^{\mathrm{gp}}_{X/S}[1]
\end{array}$$

The horizontal arrows in the leftmost square come from Theorem 5.2.1 and those in the remaining squares come from statement (1) of Theorem 5.2.2. Statement (3) of Proposition 2.2.5 shows that $g_q = (-1)^{q-1}\kappa_\gamma$, and statement (2) of Theorem 4.2.2, tensored with $\mathbb{C}$, follows.

6. Proof of the integral monodromy formula

We present a proof of the monodromy formula Theorem 4.2.2(2) with integral coefficients. In contrast with the proof with complex coefficients presented in the previous section, this one uses more abstract homological algebra; not only does this method work with $\mathbb{Z}$-coefficients in the complex analytic context, it can be adapted to the algebraic category, using the Kummer étale topology, as we shall see in Section 6.3.

6.1. ***Group cohomology.*** Our proof of the monodromy formula with integral coefficients is hampered by the fact that we have no convenient explicit complex of sheaves of I_P-modules representing $\Psi_{X/S}$. Instead we will need some abstract arguments in homological algebra, which require some preparation. Recall that the *cocone* $\mathrm{Cone}'(u)$ of a morphism u is the shift by -1 of the cone $\mathrm{Cone}(u)$ of u, so that there is a distinguished triangle:

$$\mathrm{Cone}'(u) \to A \xrightarrow{u} B \to \mathrm{Cone}'(u)[1].$$

In other words, $\mathrm{Cone}'(u)$ is the total complex of the double complex $[A \xrightarrow{-u} B]$ where A is put in the 0-th column (that is, $\mathrm{Fibre}(-u)$ in the notation of [Saito 2003]). Explicitly, $\mathrm{Cone}'(u)^n = A^n \oplus B^{n-1}$, $d(a,b) = (da, -u(a) - db)$, $\mathrm{Cone}'(u) \to A$ maps (a,b) to a and $B \to \mathrm{Cone}'(u)[1]$ maps b to $(0,b)$.

Let X be a topological space and I a group. We identify the (abelian) category of sheaves of I-modules on X with the category of sheaves of R-modules on X, where R is the group ring $\mathbb{Z}[\mathsf{I}]$. The functor $\underline{\Gamma}_{\mathsf{I}}$ which takes an object to its sheaf of I-invariants identifies with the functor $\mathcal{H}om(\mathbb{Z}, -)$, where $\mathbb{Z} \cong R/J$ and J is the augmentation ideal of R.

Now suppose that I is free of rank one, with a chosen generator γ. Then $\lambda := e^\gamma - 1$ (see Section 3.1) is a generator of the ideal J, and we have an exact sequence of sheaves of R-modules,

$$0 \to R \xrightarrow{\lambda} R \to \mathbb{Z} \to 0,$$

which defines a quasi-isomorphism $C_{\bullet} \xrightarrow{\sim} \mathbb{Z}$, where $C_{\bullet}$ is the complex $[R \xrightarrow{\lambda} R]$ in degrees -1 and 0. The functor $\mathcal{H}om(R, -)$ is exact, and hence the functor $\underline{\Gamma}_{\mathbb{I}}$ can be identified with the functor $\mathcal{H}om(C_{\bullet}, -)$. The R-linear dual of $C_{\bullet}$ is the complex

$$C^{\bullet} := \left[R \xrightarrow{-\lambda} R\right],$$

(see [Berthelot et al. 1982, 0.3.3.2] for the sign change) in degrees 0 and 1, and for any complex $K^{\bullet}$ of sheaves of $\mathbb{I}$-modules,

$$C(K^{\bullet}) := \mathcal{H}om_R(C_{\bullet}, K^{\bullet}) \cong C^{\bullet} \otimes_R K^{\bullet} \tag{6-1-1}$$

is a representative for $\mathrm{R}\underline{\Gamma}_{\mathbb{I}}(K^{\bullet})$. Note that

$$C^q(K^{\bullet}) = K^q \oplus K^{q-1}, \quad d(x, y) = (dx, -\lambda x - dy),$$

and thus $C(K^{\bullet})$ is the cocone of the morphism $\lambda : K^{\bullet} \to K^{\bullet}$.

In particular, $C^{\bullet} = C(R^{\bullet})$, where, $R^{\bullet}$ is the complex consisting of R placed in degree zero, and we have a quasi-isomorphism

$$\epsilon : C^{\bullet} \xrightarrow{\sim} \mathbb{Z}[-1] \quad \text{given by the augmentation } R \to \mathbb{Z} \text{ in degree one.}$$

Proposition 6.1.1 (Compare with [Rapoport and Zink 1982, §1]). *Let $\mathbb{I}$ be a free abelian group of rank one, with generator γ, let $R := \mathbb{Z}[\mathbb{I}]$, and let $C^{\bullet}$ be the complex (6-1-1) above. For an object $K^{\bullet}$ of the derived category $D_{\mathbb{I}}(X)$ of sheaves of $\mathbb{I}$-modules on a topological space X, let $C(K^{\bullet}) := C^{\bullet} \otimes_R K^{\bullet}$.*

(1) *There are natural isomorphisms*

$$C(K^{\bullet}) \cong R\mathcal{H}om_{\mathbb{I}}(\mathbb{Z}, K^{\bullet}) \cong \mathrm{R}\underline{\Gamma}_{\mathbb{I}}(K^{\bullet})$$

and a distinguished triangle

$$C(K^{\bullet}) \xrightarrow{a} K^{\bullet} \xrightarrow{\lambda} K^{\bullet} \xrightarrow{b} C(K^{\bullet})[1]. \tag{6-1-2}$$

(2) *Let $\partial : \mathbb{Z} \to \mathbb{Z}[1]$ denote the morphism defined by the exact sequence (3-2-4)*

$$0 \to \mathbb{Z} \to R/J^2 \to \mathbb{Z} \to 0$$

(*the first map sends* 1 *to the class of* λ). *Then $b \circ a = \partial \otimes \mathrm{id} : C(K^{\bullet}) \to C(K^{\bullet})[1]$.*

(3) *There are natural exact sequences*

$$\cdots \to R^q\underline{\Gamma}_{\mathbb{I}}(K^{\bullet}) \xrightarrow{a} \mathcal{H}^q(K^{\bullet}) \xrightarrow{\lambda} \mathcal{H}^q(K^{\bullet}) \xrightarrow{b} R^{q+1}\underline{\Gamma}_{\mathbb{I}}(K^{\bullet}) \to \cdots$$

and

$$0 \to R^1\underline{\Gamma}_{\mathbb{I}}(\mathcal{H}^{q-1}(K^\bullet)) \xrightarrow{b} R^q\underline{\Gamma}_{\mathbb{I}}(K^\bullet) \xrightarrow{a} \underline{\Gamma}_{\mathbb{I}}(\mathcal{H}^q(K^\bullet)) \to 0.$$

(4) *If the action of $\mathbb{I}$ on $\mathcal{H}^q(K^\bullet)$ is trivial, a and b induce canonical isomorphisms*

$$\Gamma_{\mathbb{I}}(\mathcal{H}^q(K^\bullet)) \cong \mathcal{H}^q(K^\bullet), \quad \textit{and} \quad \mathcal{H}^q(K^\bullet) \cong R^1\Gamma_{\mathbb{I}}(\mathcal{H}^q(K^\bullet)).$$

Proof. We have already explained statement (1) in the preceding paragraphs (the distinguished triangle expresses the fact that $C(K^\bullet)$ is the cocone of $\lambda : K^\bullet \to K^\bullet$). Since $C(K^\bullet) \cong C(R) \otimes K^\bullet$ and the distinguished triangle in (1) for $K^\bullet$ is obtained by tensoring the triangle for R with $K^\bullet$, it will suffice to prove (2) when $K^\bullet = R$. In this case, $a : C^\bullet \to R$ is given by the identity map in degree 0, and $b : R \to C^\bullet[1]$ is given by the identity map in degree 0. Thus $b \circ a : C^\bullet \to C^\bullet[1]$ is the map

$$\begin{array}{ccccc} 0 & \longrightarrow & R & \xrightarrow{-\lambda} & R \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ R & \xrightarrow{\lambda} & R & \longrightarrow & 0 \end{array}$$

Composing with the quasi-isomorphism $\epsilon[1]$, we find that $\epsilon[1] \circ b \circ a$ is given by

$$\begin{array}{ccc} R & \xrightarrow{-\lambda} & R \\ {\scriptstyle \mathrm{aug}}\downarrow & & \\ \mathbb{Z} & & \end{array}$$

The pushout of the exact sequence $0 \to R \to R \to \mathbb{Z} \to 0$ along $R \to \mathbb{Z}$ is the sequence (3-2-4). It follows that the morphism

$$b \circ a : C^\bullet \to C^\bullet[1]$$

is the same as the morphism $\partial : \mathbb{Z} \to \mathbb{Z}[1]$ defined by that sequence. This proves statement (2).

Since $C(K^\bullet) = \mathrm{R}\underline{\Gamma}_{\mathbb{I}}(K^\bullet)$, the first sequence of statement (3) is just the cohomology sequence associated with the distinguished triangle in (1); the second sequence follows from the first and the fact that for any $\mathbb{I}$ module E, $\Gamma_{\mathbb{I}}(E) \cong \mathrm{Ker}(\lambda)$ and $\mathrm{R}^1\Gamma_{\mathbb{I}}(E) \cong \mathrm{Cok}(\lambda)$. Statement (4) follows, since in this case $\lambda = 0$. □

6.2. ***Proof of the monodromy formula.*** We now turn to the proof of the integral version of statement (2) of Theorem 4.2.2. Recall that $\Psi_{X/S} = R\tilde{\tau}_{X*}\mathbb{Z}$ (see (4-2-1)); let us also set $\Psi_X = R\tau_{X*}\mathbb{Z}$ and $\Psi_S = R\tau_{S*}\mathbb{Z}$. We begin with the following observation, which is a consequence of the functoriality of the maps σ as defined in Theorem 4.2.1.

Lemma 6.2.1. *The following diagram with exact rows commutes*:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^*\overline{\mathcal{M}}_S^{\mathrm{gp}}(-1) & \longrightarrow & \overline{\mathcal{M}}_X^{\mathrm{gp}}(-1) & \longrightarrow & \mathcal{M}_{X/S}^{\mathrm{gp}}(-1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^*\sigma_S} & & \downarrow{\scriptstyle \sigma_X} & & \downarrow{\scriptstyle \sigma_{X/S}} & & \\ 0 & \longrightarrow & f^*\Psi_S^1 & \longrightarrow & \Psi_X^1 & \longrightarrow & \Psi_{X/S}^1 & \longrightarrow & 0 \end{array} \tag{6-2-1}$$

Consequently one has a commutative diagram in the derived category

$$\begin{array}{ccc} \mathcal{M}_{X/S}^{\mathrm{gp}}(-1) & \xrightarrow{E} & f^*\overline{\mathcal{M}}_S^{\mathrm{gp}}(-1)[1] = P^{\mathrm{gp}}(-1)[1] \\ \downarrow & & \downarrow \\ \Psi_{X/S} & \xrightarrow[F]{} & f^*\Psi_S^1[1] \end{array}$$

□

We will achieve our goal by establishing the commutativity of the following diagram:

$$\begin{array}{ccccc} \bigwedge^q \mathcal{M} & \xrightarrow{E_q} & P^{\mathrm{gp}}(-1)[1] \otimes \bigwedge^{q-1}\mathcal{M} & \xrightarrow{\gamma\otimes 1} & \bigwedge^{q-1}\mathcal{M}[1] \\ \downarrow{\scriptstyle \wedge^q\sigma} & (1) & \downarrow & (2) & \downarrow{\scriptstyle \wedge^{q-1}\sigma} \\ \bigwedge^q \Psi_{X/S}^1 & \xrightarrow[F_q]{} & f^*\Psi_S[1] \otimes \bigwedge^{q-1}\Psi_{X/S}^1 & \xrightarrow[\gamma\otimes 1]{} & \bigwedge^{q-1}\Psi_{X/S}^1[1] \\ & & \text{(4)} \quad G_q(\gamma) \ (\text{curved arrow from } \textstyle\bigwedge^q \Psi_{X/S}^1 \text{ to } \bigwedge^{q-1}\Psi_{X/S}^1[1]) & & \\ \downarrow{\scriptstyle \text{mult.}} & & (3) & & \downarrow{\scriptstyle \text{mult.}} \\ \Psi_{X/S}^q & & \xrightarrow[(-1)^{q-1}L_\lambda^q]{} & & \Psi_{X/S}^{q-1}[1] \end{array}$$

Here we have written $\mathcal{M}$ as a shorthand for $\mathcal{M}_{X/S}^{\mathrm{gp}}(-1)$ and $\gamma : f^*\Psi_S^1 \to \mathbb{Z}$ for the pullback by f of $\Psi_S^1 = P^{\mathrm{gp}}(-1) \xrightarrow{\gamma} \mathbb{Z}$. The maps E_q, F_q, and $G_q(\gamma)$ are defined by applying the q-th exterior power construction $\xi \mapsto \xi_q$ of Section 2.2 to the extensions E, F, and $G(\gamma)$, respectively. Here the extension $G(\gamma) : \Psi_{X/S}^1 \to \mathbb{Z}[1]$ is defined by the exact sequence (6-2-4) below. Thus the commutativity of the larger outer rectangle in this diagram is the desired formula (2) of Theorem 4.2.2. We prove this commutativity by checking the interior cells (1) through (4).

(1) This square commutes by functoriality of the maps ξ_q defined in Section 2.2 and Lemma 6.2.1.

(2) It suffices to check the commutativity when $q = 1$, in which case it follows from the definition of the map $\gamma : \Psi^1_S \to \mathbb{Z}$.

(3) We let the $\mathbb{I}$ be the subgroup of I_P generated by γ and work in the category of $\mathbb{I}$-modules. Applying (1) of Proposition 6.1.1, we find a distinguished triangle:

$$R\Gamma_{\mathbb{I}}(\Psi_{X/S}) \xrightarrow{a} \Psi_{X/S} \xrightarrow{\gamma-1} \Psi_{X/S} \xrightarrow{b} R\Gamma_{\mathbb{I}}(\Psi_{X/S})[1]. \tag{6-2-2}$$

Since γ acts trivially on the $\Psi^q_{X/S}$, the long cohomology exact sequence of the above triangle yields a short exact sequence

$$0 \to \Psi^{q-1}_{X/S} \xrightarrow{b^q} R^q\Gamma_{\mathbb{I}}(\Psi_{X/S}) \xrightarrow{a^q} \Psi^q_{X/S} \to 0. \tag{6-2-3}$$

When $q = 1$, the exact sequence (6-2-3) reduces to

$$0 \to \mathbb{Z} \xrightarrow{\beta:=b^1} R^1\Gamma_{\mathbb{I}}(\Psi_{X/S}) \xrightarrow{\alpha:=a^1} \Psi^1_{X/S} \to 0, \tag{6-2-4}$$

where $\beta(1)$ is the image of the class $\theta \in R^1\Gamma_{\mathbb{I}}(\mathbb{Z})$ in $R^1\Gamma_{\mathbb{I}}(\Psi_{X/S})$. Applying the exterior power construction of Section 2.2, one obtains for each $q \geq 1$ an exact sequence

$$0 \to \textstyle\bigwedge^{q-1}\Psi^1_{X/S} \xrightarrow{\beta^q} \bigwedge^q R^1\Gamma_{\mathbb{I}}(\Psi_{X/S}) \xrightarrow{\alpha^q} \bigwedge^q \Psi^1_{X/S} \to 0,$$

where β^q is deduced from cup product with θ on the left. We assemble the arrows α^{q-1} and β^q to form the top row of the following diagram, and the arrows a^{q-1} and b^q to form the bottom row:

$$\begin{array}{ccccc}
\bigwedge^{q-1} R^1\Gamma_{\mathbb{I}}(\Psi_{X/S}) & \xrightarrow{\alpha^{q-1}} & \bigwedge^{q-1}\Psi^1_{X/S} & \xrightarrow{\beta^q} & \bigwedge^q R^1\Gamma_{\mathbb{I}}(\Psi_{X/S}) \\
\downarrow{\scriptstyle \text{mult.}} & & \downarrow{\scriptstyle \text{mult.}} & & \downarrow{\scriptstyle \text{mult.}} \\
R^{q-1}\Gamma_{\mathbb{I}}(\Psi_{X/S}) & \xrightarrow[a^{q-1}]{} & \Psi^{q-1}_{X/S} & \xrightarrow[b^q]{} & R^q\Gamma_{\mathbb{I}}(\Psi_{X/S})
\end{array}$$

The maps a and α are the restriction maps on group cohomology from $\mathbb{I}$ to the zero group, and hence commute with cup product, so that the left square commutes. By (2) of Proposition 6.1.1, the composition $b^q \circ a^{q-1}$ is given by cup product on the left with the morphism $\theta : \mathbb{Z} \to \mathbb{Z}[1]$ defined by the fundamental extension (3-2-4). By the above discussion, the same is true for $\beta^q \circ \alpha^{q-1}$. Since the vertical maps are also defined by cup product, we see that the outer rectangle commutes. As the map α^{q-1} is surjective, we deduce that the right square also commutes.

Putting these squares alongside each other in the opposite order, we get a commutative diagram with exact rows:

$$\begin{CD}
0 @>>> \bigwedge^{q-1}\Psi^1_{X/S} @>{\beta^q}>> \bigwedge^q R^1\Gamma_{\mathbb{I}}(\Psi_{X/S}) @>{\alpha^q}>> \bigwedge^q \Psi^1_{X/S} @>>> 0\\
@. @V{\text{mult.}}VV @V{\text{mult.}}VV @V{\text{mult.}}VV @.\\
0 @>>> \Psi^{q-1}_{X/S} @>{b^q}>> R^q\Gamma_{\mathbb{I}}(\Psi_{X/S}) @>{a^q}>> \Psi^q_{X/S} @>>> 0
\end{CD}$$

Taking the maps in the derived category corresponding to these extensions gives a commutative square

$$\begin{CD}
\bigwedge^q \Psi^1_{X/S} @>>> \bigwedge^{q-1}\Psi^1_{X/S}[1]\\
@V{\text{mult.}}VV @VV{\text{mult.}}V\\
\Psi^q_{X/S} @>>> \Psi^{q-1}_{X/S}[1]
\end{CD}$$

Proposition 2.3.2 applied to the triangle (6-2-2) implies that the bottom arrow is $\kappa^q = (-1)^{q-1}L^q_\lambda[q]$, while the top arrow is $G_q(\gamma)$ by definition. It follows that cell (3) commutes.

(4) Once again we can reduce to the case $q = 1$ by functoriality of the construction of Section 2.2. Consider the action of $\mathbb{I}$ on $\Psi_{X/S}$ via γ. It is enough to establish the commutativity of the diagram

$$\begin{CD}
0 @>>> f^*\Psi^1_S @>>> \Psi^1_X @>>> \Psi^1_{X/S} @>>> 0\\
@. @V{\gamma}VV @V{\phi}VV @| @.\\
0 @>>> \mathbb{Z} @>>> R^1\Gamma_{\mathbb{I}}(\Psi_{X/S}) @>>> \Psi^1_{X/S} @>>> 0
\end{CD} \tag{6-2-5}$$

Here ϕ is the restriction map

$$\Psi^1_X = R^1\Gamma_{I_P}(\Psi_{X/S}) \to R^1\Gamma_{\mathbb{I}}(\Psi_{X/S}) \quad \text{along } \gamma : \mathbb{Z} \to I.$$

Indeed, the top extension being $F : \Psi^1_{X/S} \to f^*\Psi^1_S[1]$, the bottom extension (which is the pushout of the top extension by γ) is $\gamma \circ F : \Psi^1_{X/S} \to \mathbb{Z}[1]$. On the other hand, as we saw in Proposition 2.3.2, the bottom extension corresponds to $L^1_\lambda :$ $\Psi^1_{X/S} \to \mathbb{Z}[1]$.

The right square of (6-2-5) commutes by functoriality of restriction maps

$$R\Gamma_I(-) \to R\Gamma_{\mathbb{Z}}(-) \to R\Gamma_0(-) = (-).$$

The left square is isomorphic to

$$\begin{array}{ccc} H^1(\mathsf{I}_P, \mathcal{H}^0(\Psi_{X/S})) & \longrightarrow & H^1(\mathsf{I}_P, \Psi_{X/S}) \\ \downarrow & & \downarrow \\ H^1(\mathbb{I}, \mathcal{H}^0(\Psi_{X/S})) & \longrightarrow & H^1(\mathbb{I}, \Psi_{X/S}) \end{array}$$

which commutes by functoriality of the maps $H^1(G, \mathcal{H}^0(-)) \to H^1(G, (-))$ with respect to G.

6.3. ***Étale cohomology.*** The results of Sections 4.2 and 6 have natural algebraic analogs, due to Fujiwara, Kato, and Nakayama [Nakayama 1997], obtained by replacing the space $X_{\log}$ with the Kummer-étale topos $X_{\text{két}}$, and the (logarithmic) exponential sequence (3-3-3) with the (logarithmic) Kummer sequence(s). We refer the reader to [Illusie 2002] for a survey of the Kummer étale cohomology.

The algebraic version of our setup is as follows: we fix an algebraically closed field k and work in the category of fine and saturated log schemes locally of finite type over k. We fix an integer $N > 1$ invertible in k and use $\Lambda = \mathbb{Z}/N\mathbb{Z}$ as a coefficient ring. We define $\Lambda(1) = \mu_N(k)$, $\Lambda(n) = \Lambda(1)^{\otimes n}$ for $n \geq 0$, $\Lambda(n) = \Lambda(-n)^\vee$ for $n \leq 0$; for a Λ-module M, $M(n)$ denotes $M \otimes \Lambda(n)$.

We start by considering a single fs log scheme X. We denote by $\varepsilon : X_{\text{két}} \to X_{\text{ét}}$ the projection morphism (here $X_{\text{ét}}$ is the étale site of the underlying scheme). The sheaf of monoids $\mathcal{M}_X$ on $X_{\text{ét}}$ extends naturally to a sheaf $\mathcal{M}_X^{\text{két}}$ on $X_{\text{két}}$ associating $\Gamma(Y_{\text{ét}}, \mathcal{M}_Y)$ to a Kummer étale $Y \to X$; we have a natural homomorphism $\varepsilon^*\mathcal{M}_X \to \mathcal{M}_X^{\text{két}}$. The *logarithmic Kummer sequence* is the exact sequence of sheaves on $X_{\text{két}}$

$$0 \to \Lambda(1) \to \mathcal{M}_X^{\text{két},\text{gp}} \xrightarrow{N} \mathcal{M}_X^{\text{két},\text{gp}} \to 0. \tag{6-3-1}$$

Applying the projection ε_* yields a homomorphism

$$\sigma_0 : \mathcal{M}_X^{\text{gp}} \to \varepsilon_*\varepsilon^*\mathcal{M}_X^{\text{gp}} \to \varepsilon_*\mathcal{M}_X^{\text{két},\text{gp}} \to R^1\varepsilon_*\Lambda(1).$$

Theorem 6.3.1 [Kato and Nakayama 1999, Theorem 2.4; Illusie et al. 2005, Theorem 5.2]. *The map σ_0 factors through $\overline{\mathcal{M}}_X^{\text{gp}}$, inducing an isomorphism*

$$\sigma : \overline{\mathcal{M}}_X^{\text{gp}} \otimes \Lambda(-1) \to R^1\varepsilon_*\Lambda$$

and, by cup product, isomorphisms

$$\sigma^q : \textstyle\bigwedge^q \overline{\mathcal{M}}_X^{\text{gp}} \otimes \Lambda(-q) \xrightarrow{\sim} R^q\varepsilon_*\Lambda.$$

We now turn to the relative situation. The base S is a fine and saturated split log point associated to a fine sharp monoid P (that is, $S = \operatorname{Spec}(P \to k)$). Consider the inductive system $\tilde{P}$ of all injective maps $\phi : P \to Q$ into a sharp fs monoid Q such that the cokernel of ϕ^{gp} is torsion of order invertible in k, and let $\tilde{S} = \operatorname{Spec}(\tilde{P} \to k)$. Let I_P be the automorphism group of $\tilde{S}$ over S (the logarithmic inertia group of S); we have a natural identification $\mathsf{I}_P \cong \operatorname{Hom}(P^{gp}, \hat{\mathbb{Z}}'(1))$ where $\hat{\mathbb{Z}}'(1) = \varprojlim_N \mu_N(k)$ [Illusie 2002, 4.7(a)]. We can identify the topos $S_{két}$ with the classifying topos of I_P.

We consider an fs log scheme X locally of finite type over k and a saturated morphism $f : X \to S$. We define $\tilde{X} = X \times_S \tilde{S}$ (fiber product in the category of systems of fs log schemes). We denote the projections $\varepsilon : X_{két} \to X_{ét}$ and $\tilde{\varepsilon} : \tilde{X}_{két} \to \tilde{X}_{ét} = X_{ét}$.

Lemma 6.3.2. *The sequence of étale sheaves on X*

$$0 \to \overline{\mathcal{M}}^{gp}_S \otimes \Lambda \to \overline{\mathcal{M}}^{gp}_X \otimes \Lambda \to \overline{\mathcal{M}}^{gp}_{\tilde{X}} \otimes \Lambda \to 0$$

is exact, yielding an identification $\overline{\mathcal{M}}^{gp}_{\tilde{X}} \otimes \Lambda \cong \mathcal{M}^{gp}_{X/S} \otimes \Lambda$.

Proof. Note first that since $X \to S$ is saturated, the square

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \tilde{S} & \longrightarrow & S \end{array}$$

is cartesian in the category of (systems of) log schemes, and in particular the corresponding diagram of underlying schemes is cartesian, i.e., $\tilde{X} \cong X$ as schemes. Let $\bar{x}$ be a geometric point of X. We have pushout squares

$$\begin{array}{ccc} P & \longrightarrow & \overline{\mathcal{M}}_{X,\bar{x}} \\ \downarrow & & \downarrow \\ \tilde{P} & \longrightarrow & \overline{\mathcal{M}}_{\tilde{X},\bar{x}} \end{array} \quad \text{and} \quad \begin{array}{ccc} P^{gp} & \longrightarrow & \overline{\mathcal{M}}^{gp}_{X,\bar{x}} \\ \downarrow & & \downarrow \\ \tilde{P}^{gp} & \longrightarrow & \overline{\mathcal{M}}^{gp}_{\tilde{X},\bar{x}} \end{array}$$

and therefore also a pushout square

$$\begin{array}{ccc} P^{gp} \otimes \Lambda & \longrightarrow & \overline{\mathcal{M}}^{gp}_{X,\bar{x}} \otimes \Lambda \\ \downarrow & & \downarrow \\ \tilde{P}^{gp} \otimes \Lambda & \longrightarrow & \overline{\mathcal{M}}^{gp}_{\tilde{X},\bar{x}} \otimes \Lambda \end{array}$$

But $\tilde{P}^{gp}$ is N-divisible for all N invertible in k, so $\tilde{P}^{gp} \otimes \Lambda = 0$, yielding the desired exactness. □

The complex of nearby cycles is the complex $\Psi_{X/S} := R\tilde{\varepsilon}_* \Lambda$ of discrete I_P-modules on $X_{ét}$. Its cohomology is described by an analog of Theorem 4.2.1:

Theorem 6.3.3. *There are canonical isomorphisms*

$$\sigma^q : \bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S} \otimes \Lambda(-q) \xrightarrow{\sim} \Psi^q_{X/S}$$

for all q. In particular, the logarithmic inertia group I_P acts trivially on $\Psi^q_{X/S}$.

Proof. This follows from Theorem 6.3.1 for $\tilde{X}$, using the identifications $X_{\text{ét}} = \tilde{X}_{\text{ét}}$ and $\mathcal{M}^{\mathrm{gp}}_{X/S} \otimes \Lambda \cong \bar{\mathcal{M}}^{\mathrm{gp}}_{\tilde{X}} \otimes \Lambda$. □

As before, we denote by $\lambda^q_\gamma : \Psi^q_{X/S} \to \Psi^{q-1}_{X/S}[1]$ the map induced by $\gamma - 1 : \Psi_{X/S} \to \Psi_{X/S}$. The usual Kummer sequence on $X_{\text{ét}}$ yields a map $\mathcal{O}^{\mathrm{gp}}_X \to \Lambda(1)[1]$, which composed with the map $\mathcal{M}^{\mathrm{gp}}_{X/S} \to \mathcal{O}^*_X[1]$ coming from the extension (4-2-4) yields a map $\mathrm{ch}_{X/S} : \mathcal{M}^{\mathrm{gp}}_{X/S} \to \Lambda(1)[2]$.

With these in place, we can state the étale analog of Theorem 4.2.2.

Theorem 6.3.4. *Let $f : X \to S$ be as above. Then*:

(1) *For each $q \geq 0$, the following diagram commutes*:

$$\begin{array}{ccc}
\bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S} \otimes \Lambda(-q) & \xrightarrow{\mathrm{ch}^q_{X/S}} & \bigwedge^{q-1} \mathcal{M}^{\mathrm{gp}}_{X/S} \otimes \Lambda(1-q)[2] \\
\Big\downarrow {\scriptstyle q!\sigma^q} & & \Big\downarrow {\scriptstyle q!\sigma^{q-1}} \\
\Psi^q_{X/S} & \xrightarrow[\delta^q]{} & \Psi^{q-1}_{X/S}[2]
\end{array}$$

(2) *For each $q \geq 0$ and each $\gamma \in \mathsf{I}_P$, the following diagram commutes*:

$$\begin{array}{ccc}
\bigwedge^q \mathcal{M}^{\mathrm{gp}}_{X/S} \otimes \Lambda(-q) & \xrightarrow{\kappa^q_\gamma} & \bigwedge^{q-1} \mathcal{M}^{\mathrm{gp}}_{X/S} \otimes \Lambda(1-q)[1] \\
\Big\downarrow {\scriptstyle \sigma^q} & & \Big\downarrow {\scriptstyle \sigma^{q-1}} \\
\Psi^q_{X/S} & \xrightarrow[(-1)^{q-1}\lambda^q_\gamma]{} & \Psi^{q-1}_{X/S}[1]
\end{array}$$

where κ^q_γ is as in Proposition 2.3.2.

Proof. The proof of (1) relies on the following analog of the isomorphism (4-2-7). The exact sequence (6-3-1) provides a quasi-isomorphism on $\tilde{X}_{\text{két}}$

$$\Lambda(1) \xrightarrow{\sim} \big[\mathcal{M}^{\text{két,gp}}_{\tilde{X}} \to \mathcal{M}^{\text{két,gp}}_{\tilde{X}}\big]$$

and the morphism of complexes

$$\big[\tilde{\varepsilon}^* \mathcal{M}^{\mathrm{gp}}_{\tilde{X}} \xrightarrow{N} \tilde{\varepsilon}^* \mathcal{M}^{\mathrm{gp}}_{\tilde{X}}\big] \to \big[\mathcal{M}^{\text{két,gp}}_{\tilde{X}} \xrightarrow{N} \mathcal{M}^{\text{két,gp}}_{\tilde{X}}\big]$$

induces by adjunction a morphism

$$\big[\tilde{\varepsilon}^* \mathcal{M}^{\mathrm{gp}}_{\tilde{X}} \xrightarrow{N} \tilde{\varepsilon}^* \mathcal{M}^{\mathrm{gp}}_{\tilde{X}}\big] \to \Psi_{X/S}(1).$$

This morphism induces an isomorphism $\left[\tilde{\varepsilon}^*\mathcal{M}^{\mathrm{gp}}_{\tilde{X}} \xrightarrow{N} \tilde{\varepsilon}^*\mathcal{M}^{\mathrm{gp}}_{\tilde{X}}\right] \cong \tau_{\leq 1}\Psi_{X/S}$, our analog of (4-2-7). Then assertion (1) follows exactly as before. The proof of (2) follows the lines of our second proof of the analogous assertion in Section 6.2. We omit the details. □

7. Curves

The goal of the present section is to illustrate Theorems 4.2.2 and 4.1.6 for curves. We shall attempt to convince our readers that the combinatorics arising from the log structures are essentially equivalent to the data usually expressed in terms of the "dual graph" of a degenerate curve, for example in [SGA 7_{I} 1972, IX, 12.3.7]. In particular, we show how the classical Picard–Lefschetz formula for curves can be derived from our monodromy formula. In this section we work over the field $\mathbb{C}$ of complex numbers.

7.1. ***Log curves and their normalizations.*** Our exposition is based on F. Kato's study [2000] of the moduli of log curves and their relation to the classical theories. Let us recall his basic notions.

Definition 7.1.1. Let S be a fine saturated and locally noetherian log scheme. A *log curve* over S is a smooth, finite type, and saturated morphism $f : X \to S$ of fine saturated log schemes such that every geometric fiber of $\underline{f} : \underline{X} \to \underline{S}$ has pure dimension one.

Kato requires that $\underline{X}$ be connected, a condition we have dropped from our definition. If $\underline{X}/\mathbb{C}$ is a smooth curve and $\underline{Y}$ is a finite set of closed points of $\underline{X}$, then the compactifying log structure associated with the open subset $\underline{X} \setminus \underline{Y}$ of $\underline{X}$ is fine and saturated, and the resulting log scheme is a log curve over $\mathbb{C}$. In fact, every log curve over $\mathbb{C}$ arises in this way, so that to give a log curve over $\mathbb{C}$ is equivalent to giving a smooth curve with a set (not a sequence) of marked points.

For simplicity, we concentrate on the case of vertical log curves over the standard log point $S := \mathrm{Spec}(\mathbb{N} \to \mathbb{C})$. Then a morphism of fine saturated log schemes $X \to S$ is automatically integral [Kato 1989, 4.4], and if it is smooth, it is saturated if and only if its fibers are reduced [Tsuji 2019, II.4.2; Ogus 2018, IV, 4.3.6]. Since X/S is vertical, the sheaf $\mathcal{M}_{X/S} := \mathcal{M}_X/f^*\mathcal{M}_S$ is in fact a sheaf of groups. Corollary 4.1.7 says that the set $Y := \{x \in X : \mathcal{M}_{X/S} \neq 0\}$ is closed in X, that its complement U is open and dense, and that the underlying scheme $\underline{U}$ of U is smooth. In fact Kato's analysis of log curves gives the following detailed local description of X/S.

Theorem 7.1.2 (F. Kato). *Let $f : X \to S$ be a vertical log curve over the standard log point S and let x be a closed point of $\underline{X}$.*

(1) *The underlying scheme* $\underline{X}$ *is smooth at* x *if and only if there is an isomorphism* $\overline{\mathcal{M}}_{X,x} \cong \mathbb{N}$. *If this is the case, there exist an étale neighborhood* V *of* x *and a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{g} & S \times \mathsf{A}_{\mathbb{Z}} \\ & \searrow & \downarrow \\ & & S \end{array}$$

where g *is strict and étale.*

(2) *The underlying scheme* $\underline{X}$ *is singular at* x *if and only if there exist an integer* n *and an isomorphism* $\overline{\mathcal{M}}_{X,x} \cong Q_n$, *where* Q_n *is the monoid given by generators* q_1, q_2, q *satisfying the relation* $q_1 + q_2 = nq$. *In this case there exist an étale neighborhood* V *of* x *and a commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{g} & \mathsf{A}_{Q_n,J} \\ & \searrow & \downarrow \mathsf{A}_\theta \\ & & S \end{array}$$

where g *is strict and étale, where* J *is the ideal of* Q_n *generated by* q, *and where* $\theta : \mathbb{N} \to Q_n$ *is the homomorphism sending* 1 *to* q. □

Proof. For the convenience of the reader we give an indication of the proofs, using the point of view developed in Corollary 4.1.7. We saw there that the set $U := \{x \in X : \overline{\mathcal{M}}_{X,x} = \mathbb{N}\}$ is open in X. Moreover $\underline{U}$ is smooth over $\mathbb{C}$, so it can be covered by open sets V each of which admits an étale map $U \to \mathbb{G}_m = \mathsf{A}_{\mathbb{Z}}$. Since the morphism $U \to \underline{U} \times S$ is an isomorphism, we find a diagram as in case (1) of the theorem.

Suppose on the other hand that $x \in Y := X \setminus U$. Since the sheaf of groups $\mathcal{M}^{\mathrm{gp}}_{X/S}$ is torsion free, one sees from [Ogus 2018, IV, 3.3.1] that in a neighborhood V of x, there exists a chart for f which is neat and smooth at x. That is, there exist a fine saturated monoid Q, an injective homomorphism $\theta : \mathbb{N} \to Q$, and a map $V \to \mathsf{A}_Q$ such that induced map $V \to S \times_{\mathsf{A}_{\mathbb{N}}} \mathsf{A}_Q$ is smooth and such that the homomorphism $Q^{\mathrm{gp}}/\mathbb{Z} \to \mathcal{M}_{X/S,x}$ is an isomorphism. By [Ogus 2018, III, 2.4.5], the chart $Q \to \mathcal{M}_X$ is also neat at x. Let J be the ideal of Q generated by $q := \theta(1)$. Then $S \times_{\mathsf{A}_{\mathbb{N}}} \mathsf{A}_Q = \mathsf{A}_{(Q,J)}$. Since θ is vertical, J is the interior ideal of Q, and the set of minimal primes containing it is the set of height one primes of Q. Thus the dimension of $\underline{\mathsf{A}}_{(Q,J)}$ is the dimension of $\underline{\mathsf{A}}_F$, where F is any facet of Q. This dimension is the rank of F^{gp}; if it is zero, then Q^{gp} has rank at most one, hence $\mathbb{N} \to Q$ is an isomorphism, contradicting our assumption that $x \in Y$. Thus Q^{gp} has rank at least two. Since $\underline{V}$ has dimension one and is smooth over $\underline{\mathsf{A}}_{(Q,J)}$, it follows that in fact F^{gp} has rank one, that Q^{gp} has rank two, and that $\underline{V}$ is étale over $\underline{\mathsf{A}}_{(Q,J)}$. Since $Q^{\mathrm{gp}}/\mathbb{Z} \cong \mathcal{M}^{\mathrm{gp}}_{X/S}$ and Q^* maps to zero in $\mathcal{M}^{\mathrm{gp}}_{X/S}$ it follows that Q is sharp of dimension two, and hence has exactly two faces F_1 and F_2. Each of these is canonically isomorphic to $\mathbb{N}$; let q_i be the unique generator of F_i. Since θ is saturated, every element of Q can be written uniquely in the form $nq + m_i q_i$, where $n, m_i \in \mathbb{N}$ and $i \in \{1, 2\}$ [Ogus 2018, I, 4.8.14]. Writing $q_1 + q_2$ in this form, we see that necessarily $m_i = 0$ (otherwise q would belong to a proper face). Thus Q is generated by q_1 and q_2, subject to the relation $nq = q_1 + q_2$ for some $n > 0$. Since Q has dimension two, it is necessarily isomorphic to Q_n, and since the chart $Q \to \mathcal{M}_X$ is neat at x, in fact $Q_n \cong \overline{\mathcal{M}}_{X,x}$. Finally, we note that $\underline{X}$ is singular at x, so U is precisely the smooth locus of $\underline{X}$. □

Let us remark that the isomorphism $\overline{\mathcal{M}}_{X,x} \cong \mathbb{N}$ in (1) is unique, since the monoid $\mathbb{N}$ has no nontrivial automorphisms. In case (2), the integer n is unique, and there are exactly two isomorphisms $\overline{\mathcal{M}}_{X,x} \cong Q_n$, since Q_n has a unique nontrivial automorphism, which exchanges q_1 and q_2.

Thanks to Kato's result, we can give the following more explicit version of Corollary 4.1.7 in the cases of log curves. Since we are working over the standard (split) log point S, we have a map $\mathbb{N} \to \mathcal{M}_S \to \mathcal{M}_X$, and we let $\mathcal{M}_{X/\mathbb{N}} := \mathcal{M}_X/\mathbb{N}$.

Proposition 7.1.3. *Let X/S be a vertical log curve over the standard log point, let $\epsilon : \underline{X}' \to \underline{X}$ be its normalization, and let $X'/\mathbb{C}$ (resp. X'') be the log curve obtained by endowing $\underline{X}'$ with the compactifying log structure associated to the open embedding $U' \to U$ (resp. with the induced log structure from X).*

(1) *There is a unique morphism of log schemes $h : X'' \to X'$ which is the identity on the underlying schemes.*

(2) *The maps $\overline{\mathcal{M}}^{\mathrm{gp}}_{X'} \to \epsilon^{-1}(\mathcal{M}_{X/S})$ and $\mathcal{M}^{\mathrm{gp}}_{X'} \to \epsilon^*(\mathcal{M}_{X/\mathbb{N}})$ induced by h are isomorphisms, where $\epsilon^*(\mathcal{M}_{X/\mathbb{N}}) := \mathcal{O}^*_{X'} \times_{\epsilon^{-1}(\mathcal{O}^*_X)} \epsilon^{-1}(\mathcal{M}_{X/\mathbb{N}})$.*

(3) *Let $X''' := X' \times_{\underline{S}} S$, and let $g : X'' \to X'''$ be the map induced by $f \circ \epsilon$ and h. Then the morphism g identifies X'' with a strict log transform of X''', i.e., the closure of U' in the log blowup of X''' along a coherent sheaf of ideals of $\mathcal{M}_{X'''}$,* (made explicit below).

Proof. Statement (3) of Corollary 4.1.7 implies statement (1) of Proposition 7.1.3, statement (4) implies that h induces an isomorphism $\theta : \overline{\mathcal{M}}^{\mathrm{gp}}_{X'} \to \epsilon^{-1}(\mathcal{M}_{X/S})$, and it follows that $\mathcal{M}^{\mathrm{gp}}_{X'} \to \epsilon^*(\mathcal{M}_{X/P})$ is an isomorphism, since this map is a morphism of $\mathcal{O}^*_{X'}$ torsors over θ. This proves statements (1) and (2); we should remark that they are quite simple to prove directly in the case of curves, because the normalization $\underline{X}'$ of $\underline{X}$ is smooth.

Our proof of (3) will include an explicit description of a sheaf of ideals defining the blowup. For each point y' of Y', let n be the integer such that $\overline{\mathcal{M}}_{X,\epsilon(y')} \cong Q_n$, let $\mathcal{K}_{y'}$ be the ideal of $\mathcal{M}_{X''',y'}$ generated by $\mathcal{M}^+_{X',y'}$ and $n\mathcal{M}^+_S$, and let $\mathcal{K} :=$

$\bigcap\{\mathcal{K}_{y'} : y' \in Y'\}$, a coherent sheaf of ideals in $\mathcal{M}_{X'''}$. Observe that the ideal of $\mathcal{M}_{X''}$ generated by $\mathcal{K}$ is invertible. This is clear at points x' of U'. If $y' \in Y'$, the ideal $\mathcal{K}_{y'}$ is generated by the images of q_2 and nq, and in Q_n the ideal (q_2, nq) is generated by q_2, since $nq = q_1 + q_2$. Thus the map $X'' \to X'''$ factors through the log blowup [Nizioł 2006, 4.2]. A chart for X''' near y' is given by $\mathbb{N} \oplus \mathbb{N}$ mapping $(1, 0)$ to q_2 and $(0, 1)$ to q, and the log blowup of the ideal (q_2, nq) has a standard affine cover consisting of two open sets. The first is obtained by adjoining $nq - q_2$, and the corresponding monoid is Q_n; and the second by adjoining $q_2 - nq$. The closure of U' is contained in the first affine piece, so we can ignore the second. Thus the induced map is indeed an isomorphism as described. □

Proposition 7.1.3 shows that one can recover the log curve X''/S directly from the log curve X' together with the function $\nu : Y' \to \mathbb{Z}^+$ taking a point y' to the number n such that $\overline{\mathcal{M}}_{\epsilon(y')} \cong Q_n$. In fact there are additional data at our disposal, as the following proposition shows.

Proposition 7.1.4. *Let X/S be a vertical log curve over the standard log point and let $X'/\mathbb{C}$ be the corresponding log curve over $\mathbb{C}$ as described in Proposition 7.1.3. Then $X'/\mathbb{C}$ is naturally equipped with the following additional data.*

(1) *A fixed point free involution ι of $\underline{Y}'$.*

(2) *A mapping $\nu : \underline{Y}' \to \mathbb{N}$ such that $\nu(y') = \nu(\iota(y'))$ for every $y' \in Y'$.*

(3) *A trivialization of the invertible sheaf $N_{\underline{Y}'/\underline{X}'} \otimes \iota^*(N_{\underline{Y}'/\underline{X}'})$ on $\underline{Y}'$.*

Proof. These data arise as follows. Each fiber of the map $\underline{\epsilon} : \underline{Y}' \to \underline{Y}$ has cardinality two, and hence there is a unique involution ι of $\underline{Y}'$ which interchanges the points in each fiber. The function ν is defined as above: $\nu(y')$ is the integer n such that $\overline{\mathcal{M}}_{X,\epsilon(y')} \cong Q_n$. To obtain the trivialization in (3), let y' be a point of Y' and let $y := \epsilon(y')$. Recall from Remark 3.3.1 that if X is a fine log space and $\overline{m} \in \Gamma(X, \overline{\mathcal{M}}_X)$, there is an associated invertible sheaf $\mathcal{L}_{\overline{m}}$ whose local generators are the sections of $\mathcal{M}_X$ mapping to $\overline{m}$. Observe that, since the log point S is equipped with a splitting $\overline{\mathcal{M}}_S \to \mathcal{M}_S$, there is a canonical generator m_S of the invertible sheaf $\mathcal{L}_{1,S}$ on S. Let us use the notation of the proof of Proposition 7.1.3. Endow Y with the log structure from X and choose a point y of Y. The choice of a chart at y defines sections m_1 and m_2 of $\mathcal{M}_{Y,y}$, whose images $\overline{m}_1$ and $\overline{m}_2$ in $\Gamma(y, \overline{\mathcal{M}}_{Y,y})$ are independent of the choice of the chart and define one-dimensional vector spaces $\mathcal{L}_{\overline{m}_i}$. The equality $\overline{m}_1 + \overline{m}_2 = n\overline{f}^\flat(1)$ induces an isomorphism

$$\mathcal{L}_{\overline{m}_1} \otimes \mathcal{L}_{\overline{m}_2} \cong \mathcal{L}^n_{1,s} \cong \mathbb{C}.$$

As we have seen, the element m_2 corresponds to a generator of the ideal of the point y'_1 in $\underline{X}'_1$, so there is a canonical isomorphism $\epsilon^*(\mathcal{L}_{\overline{m}_2}) \cong N^{-1}_{\underline{Y}'/\underline{X}',y'_1}$; similarly

$\epsilon^*(\mathcal{L}_{\overline{m}_1}) \cong N^{-1}_{\underline{Y}'/\underline{X}',y_2}$. Thus we find isomorphisms

$$N_{\underline{Y}'/\underline{X}',y'_1} \otimes (\iota^* N_{\underline{Y}'/\underline{X}'})_{y'_1} \cong N_{\underline{Y}'/\underline{X}',y'_1} \otimes N_{\underline{Y}'/\underline{X},y'_2} \cong \epsilon^*(\mathcal{L}_{\overline{m}_2})^{-1} \otimes \epsilon^*(\mathcal{L}_{\overline{m}_1})^{-1} \cong \mathbb{C}. \qquad \square$$

In fact the data in Proposition 7.1.4 are enough to reconstruct the original log curve X/S. (For an analogous result in the context of semistable reduction, see [Ogus 2018, III, Proposition 1.8.8].) Rather that write out the proof, let us explain how one can construct the fibration $X_{\log} \to \mathbb{S}^1$ directly from X' together with these data.

It will be notationally convenient for us to extend ι to a set-theoretic involution on all of $\underline{X}'$, acting as the identity on $\underline{U}'$. If $y' \in Y'$ and v is a nonzero element of $N_{y'/\underline{X}'}$, let $\iota(v)$ be the element of $N_{\iota(y')/\underline{X}'}$ which is dual to v with respect to the pairing defined by (3) above. Then $\iota(\lambda v) = \lambda^{-1}\iota(v)$ for all v. Note that since $\overline{\mathcal{M}}_{X',y'} = \mathbb{N}$ for every $y' \in Y'$ and vanishes otherwise, we have a natural set-theoretic action of $\mathbb{S}^1$ on $X'_{\log}$ covering the identity of X'. Thus the following sets of data are equivalent:

(1) a trivialization of $N_{\underline{Y}'/\underline{X}'} \otimes \iota^*(N_{\underline{Y}'/\underline{X}'})$;

(2) an involution ι of $N_{\underline{Y}'/\underline{X}'}$, covering the involution ι of Y', such that $\iota(\lambda v) = \lambda^{-1}\iota(v)$ for $\lambda \in \mathbb{C}^*$ and $v \in N_{Y'/X}$;

(3) an involution of ι of $\mathbb{S}^1(N_{\underline{Y}'/\underline{X}'})$ (the circle bundle of $N_{\underline{Y}'/\underline{X}'}$), covering the involution ι of Y', such that $\iota(\lambda(v)) = \lambda^{-1}(\iota(v))$ for $\lambda \in \mathbb{S}^1$ and $v \in \mathbb{S}^1(N_{\underline{Y}'/\underline{X}'})$;

(4) an involution ι of $X'_{\log}$ such that $\tau_{X'}(\iota(x'_{\log})) = \iota(\tau_{X'}(x'_{\log}))$ and $\iota(\zeta x'_{\log}) = \zeta^{-1}\iota(x'_{\log})$ for $\zeta \in \mathbb{S}^1$ and $x'_{\log} \in X'_{\log}$.

The data in (3) and (4) are equivalent thanks to Remark 3.3.1. We should also point out that these data are unique up to (nonunique) isomorphism.

Proposition 7.1.5. *Let X/S be a log curve and let X' and ι be as above. Let $\nu(x'_{\log}) := \nu(\epsilon(\tau_{X'}(x'_{\log})))$ and define ι on $X'_{\log} \times \mathbb{S}^1$ by*

$$\iota(x'_{\log}, \zeta) := (\zeta^{\nu(x'_{\log})}\iota(x'_{\log}), \zeta).$$

Then $X_{\log}$ is the quotient of $X'_{\log} \times \mathbb{S}^1$ by the equivalence relation generated by ι.

Proof. Let y be a point of Y and let $\epsilon^{-1}(y) := \{y'_1, y'_2\}$. We can check the formula with the aid of charts, using again the notation of the proof of Proposition 7.1.3. Then $\overline{\mathcal{M}}^{\mathrm{gp}}_{X'',y'_1} \cong \overline{\mathcal{M}}^{\mathrm{gp}}_{X',y'_1} \oplus \overline{\mathcal{M}}^{\mathrm{gp}}_S$ is free with basis $\overline{m}_2, \overline{m}$ and $\overline{\mathcal{M}}_{X'',y'_2} \cong \overline{\mathcal{M}}^{\mathrm{gp}}_{X',y'_2} \oplus \overline{\mathcal{M}}^{\mathrm{gp}}_S$ is free with basis $\overline{m}_1, \overline{m}$. We have isomorphisms

$$\overline{\mathcal{M}}^{\mathrm{gp}}_{X'',y''_1} \longleftarrow \overline{\mathcal{M}}^{\mathrm{gp}}_{X,y} \longrightarrow \overline{\mathcal{M}}^{\mathrm{gp}}_{X'',y''_2}$$

sending $\overline{m}$ to $\overline{m}$ and $\overline{m}_2$ to $\overline{m}_1 = n\overline{m} - \overline{m}_2$. Then the formula follows immediately. $\square$

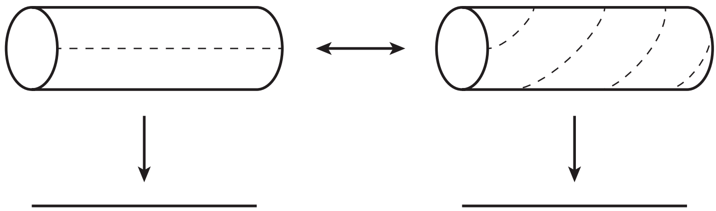


Figure 1. Gluing log fibers.

This gluing map is compatible with the map $X_{\log} \to \mathbb{S}^1$. Figure 1 illustrates the restriction of this gluing to $\tau^{-1}(y) \to \mathbb{S}^1$, when pulled back to via the exponential map $[0, 2\pi i] \to \mathbb{S}^1$. The reader may recognize the gluing map as a Dehn twist. It appears here as gluing data, not monodromy. It is of course also possible to see the monodromy from this point of view as well, using a chart for X at a point y of Y. Since this description is well-known but not functorial, we shall not develop it here.

7.2. ***Log combinatorics and the dual graph.*** Proposition 7.1.3 and the data of Proposition 7.1.4 will enable us to give a combinatorial description of the sheaf $\mathcal{M}_{X/S}$ on X. In fact there are two ways to do this, each playing its own role and each related to the "dual graph" associated to the underlying nodal curve of X/S.

We begin with the following elementary construction. Let B be a finite set with an involution ι and let $\epsilon : B \to E$ be its orbit space. There are two natural exact sequences of $\mathbb{Z}[\iota]$-modules:

$$0 \to \mathbb{Z}_{E/B} \xrightarrow{i} \mathbb{Z}^B \xrightarrow{s} \mathbb{Z}^E \to 0, \quad 0 \to \mathbb{Z}^E \xrightarrow{j} \mathbb{Z}^B \xrightarrow{p} \mathbb{Z}_{B/E} \to 0. \tag{7-2-1}$$

The map s in the first sequence sends a basis vector b of $\mathbb{Z}^B$ to the basis vector $\epsilon(b)$ of $\mathbb{Z}^E$, and i is the kernel of s. The map j in the second sequence sends a basis vector e to $\sum\{b \in \epsilon^{-1}(e)\}$, and p is the cokernel of j. These two sequences are naturally dual to each other, and in particular $\mathbb{Z}_{E/B}$ and $\mathbb{Z}_{B/E}$ are naturally dual. For each $b \in B$, let $d_b := b - \iota(b) \in \mathbb{Z}_{E/B}$ and $p_b := p(b) \in \mathbb{Z}_{B/E}$. Then $\pm d_b$ (resp. $\pm p_b$) depends only on $\epsilon(b)$. There is a well-defined isomorphism of $\mathbb{Z}[\iota]$-modules defined by

$$t : \mathbb{Z}_{B/E} \to \mathbb{Z}_{E/B}, \quad p_b := p(b) \mapsto d_b := b - \iota(b). \tag{7-2-2}$$

The resulting duality $\mathbb{Z}_{B/E} \times \mathbb{Z}_{B/E} \to \mathbb{Z}$ is positive definite, and the set of classes of elements $\{p(b) : b \in B\}$ forms an orthonormal basis.

We apply these constructions to the involution ι of $\underline{Y}'$ and regard $\epsilon : \underline{Y}' \to \underline{Y}$ as the orbit space of this action. The construction of $\mathbb{Z}_{Y/Y'}$ and $\mathbb{Z}_{Y'/Y}$ is compatible with localization on Y and hence these form sheaves of groups on Y. Since we are assuming that X/S is vertical, ι is fixed point free. As we shall see, there are natural identifications of the sheaf $\mathcal{M}_{X/S}$ both with $\mathbb{Z}_{Y/Y'}$ and with $\mathbb{Z}_{Y'/Y}$. We begin with the former.

Because $\alpha_{X'}$ is the compactifying log structure associated to the set of marked points Y', there are natural isomorphisms of sheaves of monoids on $\underline{X}'$:

$$\overline{\mathcal{M}}_{X'} \cong \underline{\Gamma}_{Y'}(\mathrm{Div}^{+}_{\underline{X}'}) \cong \mathbb{N}_{Y'}. \tag{7-2-3}$$

Combining this identification with the isomorphism $\epsilon^{-1}(\mathcal{M}_{X/S}) \cong \overline{\mathcal{M}}^{\mathrm{gp}}_{X'}$ of statement (2) of Proposition 7.1.3, we find an isomorphism $\epsilon^{-1}(\mathcal{M}_{X/S}) \cong \mathbb{Z}_{\underline{Y}'}$, and hence an injection

$$\psi : \mathcal{M}_{X/S} \to \epsilon_* \epsilon^{-1}(\mathcal{M}_{X/S}) \cong \epsilon_*(\mathbb{Z}_{\underline{Y}'}).$$

Proposition 7.2.1. *The inclusion ψ defined above fits into an exact sequence*

$$0 \to \mathcal{M}_{X/S} \xrightarrow{\psi} \epsilon_* \mathbb{Z}_{\underline{Y}'} \xrightarrow{s} \mathbb{Z}_{\underline{Y}} \to 0,$$

and hence induces an isomorphism

$$\psi_{X/S} : \mathcal{M}_{X/S} \xrightarrow{\sim} \mathbb{Z}_{Y/Y'}.$$

If $\ell \in \Gamma(X, \mathcal{M}_{X/S})$ and $\mathcal{L}_\ell$ is the corresponding invertible sheaf on X coming from the exact sequence (1-0-4), then $\epsilon^*(\mathcal{L}_\ell) \cong \mathcal{O}_{\underline{X}'}(-\psi(\ell))$.

Proof. Since the maps ψ and s are already defined globally, it is enough to check that the sequence is exact at each point y of Y. We work in a charted neighborhood of a point $y \in Y$ as in the proof of Proposition 7.1.3, using the notation there. Then $\mathcal{M}_{X/S,y}$ is the free abelian group generated by the image ℓ_2 of q_2, and $\ell_1 = -\ell_2$. The pullback t_2' of t_2 to $\underline{X}'$ is a local coordinate near y_1' and defines a generator for $\overline{\mathcal{M}}_{X',y_1'}$ mapping to $1_{y_1'} \in \mathbb{Z}_{Y'}$. The analogous formulas hold near y_2', and hence $\psi(\ell_2) = 1_{y_1'} - 1_{y_2'}$. This implies that $s \circ \psi = 0$ and that the sequence is exact. Furthermore, it follows from Proposition 7.1.3 that $\epsilon^*(\mathcal{L}_\ell) = \mathcal{L}_{\overline{m}'}$, where $\overline{m}' \in \Gamma(X', \overline{\mathcal{M}}^{\mathrm{gp}}_{X'})$ corresponds to $\ell \in \Gamma(X, \mathcal{M}_{X/S})$ via the isomorphism $\overline{\mathcal{M}}^{\mathrm{gp}}_{X'} \to \mathcal{M}^{\mathrm{gp}}_{X/S}$ in statement (2) of that proposition. The sheaf $\mathcal{L}_{\overline{m}'}$ is the ideal sheaf of the divisor D corresponding to $\overline{m}'$, i.e., $\mathcal{O}_{X'}(-D)$, and $D = \psi(\ell)$. □

The relationship between $\mathcal{M}_{X/S}$ and $\mathbb{Z}_{Y'/Y}$ is more subtle and involves the integers $\nu(y)$. First consider the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_X & \longrightarrow & \epsilon_*(\mathbb{Z}_{X'}) & \longrightarrow & \mathbb{Z}_{X'/X} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{res}} & & \\
0 & \longrightarrow & \mathbb{Z}_Y & \longrightarrow & \epsilon_*(\mathbb{Z}_{Y'}) & \longrightarrow & \mathbb{Z}_{Y'/Y} & \longrightarrow & 0
\end{array} \tag{7-2-4}$$

where $\mathbb{Z}_{X'/X}$ is by definition the cokernel in the top row. As ϵ is an isomorphism over U, the right vertical map is an isomorphism and we will allow ourselves to identify its source and target without further comment. Note that since $\underline{X}'/\mathbb{C}$ is smooth, the natural map $\mathbb{Z}_{X'} \to j'_*(\mathbb{Z}_U)$ is an isomorphism, hence $\epsilon_*(\mathbb{Z}_{X'}) \cong j_*(\mathbb{Z}_U)$, and the top row of the above diagram can be viewed as an exact sequence:

$$0 \to \mathbb{Z}_X \to j_*(\mathbb{Z}_U) \to \mathbb{Z}_{X'/X} \to 0. \tag{7-2-5}$$

Since X/S is vertical, it follows from Theorem 7.1.2 that Y is precisely the support of $\mathcal{M}_{X/S}$ and that the map $\mathbb{Z} \to \overline{\mathcal{M}}^{\mathrm{gp}}_X$ is an isomorphism on U. Thus there is a natural map

$$\phi_X : \overline{\mathcal{M}}^{\mathrm{gp}}_X \to j_* j^*(\overline{\mathcal{M}}^{\mathrm{gp}}_X) \cong j_*(\mathbb{Z}_U) \cong \epsilon_*(\mathbb{Z}_{X'}). \tag{7-2-6}$$

In fact, the map ϕ_X is the adjoint of the homomorphism

$$\bar{\rho}^{\mathrm{gp}} : \overline{\mathcal{M}}^{\mathrm{gp}}_{X''} \cong \epsilon^{-1}(\overline{\mathcal{M}}^{\mathrm{gp}}_X) \to \mathbb{Z}_{X'}$$

deduced from the homomorphism

$$\rho : \mathcal{M}_{X''} \to P_{X'} = \mathbb{N}_{X'}$$

defined in (4) of Corollary 4.1.7.

Proposition 7.2.2. *Let X/S be a vertical log curve. The homomorphisms $\psi_{X/S}$ of Proposition 7.2.1 and ϕ_X defined above fit into a commutative diagram with exact rows*:

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & f^{-1}\overline{\mathcal{M}}^{\mathrm{gp}}_S & \longrightarrow & \overline{\mathcal{M}}^{\mathrm{gp}}_X & \xrightarrow{\ \pi\ } & \mathcal{M}_{X/S} \longrightarrow 0 \\
 & & & \swarrow{\scriptstyle \mathrm{id}} & & \swarrow{\scriptstyle \mathrm{id}} & & \swarrow{\scriptstyle \psi_{X/S}} & \\
0 & \longrightarrow & f^{-1}\overline{\mathcal{M}}^{\mathrm{gp}}_S & \longrightarrow & \overline{\mathcal{M}}^{\mathrm{gp}}_X & \cdots\to & \mathbb{Z}_{Y/Y'} & \xrightarrow{\ \phi_{X/S}\ } & 0 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle c_{X/S}} & & \\
0 & \longrightarrow & \mathbb{Z}_X & \longrightarrow & \epsilon_*(\mathbb{Z}_{X'}) & \xrightarrow{\ p\ } & \mathbb{Z}_{Y'/Y} & \longrightarrow & 0
\end{array} \tag{7-2-7}$$

where $c_{X/S}$ is the map sending $d_{y'}$ to $-\nu(y')p_{y'}$ for every $y' \in Y'$.

Proof. We compute the stalk of the map ϕ_X at a point x of X. If x belongs to U, the maps

$$\mathbb{Z} \to \overline{\mathcal{M}}^{\mathrm{gp}}_{X,x} \quad \text{and} \quad \overline{\mathcal{M}}^{\mathrm{gp}}_{X,x} \to \epsilon_*(\mathbb{Z}'_X)_x$$

are isomorphisms, and hence so is ϕ_X. If x belongs to Y, we call it y and work in a neighborhood as in the proof of Proposition 7.1.3. Then $\underline{X}$ is the analytic space associated to $\mathrm{Spec}(\mathbb{C}[x_1, x_2]/(x_1x_2))$, endowed with the log structure associated to the homomorphism $\beta : Q_n \to \mathbb{C}[x_1, x_2]/(x_1x_2)$ sending q_i to x_i and q to 0. The point $y := x$ is defined by $x_1 = x_2 = 0$, and has a basis of neighborhoods W

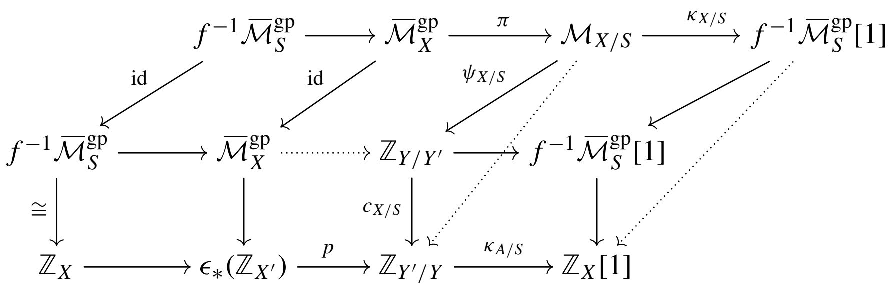
defined by $|x_i| < \epsilon$. On the connected component $W_1 \cap U$ of $W \cap U$, the coordinate x_1 vanishes and x_2 becomes a unit. Let $\overline{m}_i$ (resp. $\overline{m}$) be the image of q_i (resp. of q) in $\overline{\mathcal{M}}_X$. The stalk of $\epsilon_*(\mathbb{Z}_{X'}) \cong j_*(U)$ at x is free with basis (b_1, b_2), where b_i is the germ of the characteristic function of $W_i \cap U$ at x. The isomorphism $\mathrm{res} : \epsilon_*(\mathbb{Z}_{X'})_y \xrightarrow{\sim} \epsilon_*(\mathbb{Z}_{Y'})_y$ takes b_i to the basis element y'_i. The restriction of the sheaf $\overline{\mathcal{M}}_X$ to $W_1 \cap U$ is constant and freely generated by $\overline{m}_{|W_1 \cap U}$, while $\overline{m}_{1|W_1 \cap U} = \nu(y)\overline{m}_{|W_1 \cap U}$ and $\overline{m}_{2|W_1 \cap U} = 0$. Thus $\phi_X(\overline{m}_i) = \nu(y) b_i$ and $\phi_X(\overline{m}) = b_1 + b_2$. In particular, $p(\phi(\overline{m}_i)) = \nu(y) p(y'_i)$. On the other hand, we saw in the proof of Proposition 7.2.1 that $\psi_{X/S}(\pi(\overline{m}_1)) = y'_2 - y'_1 = -d_{y'_1} \in \mathbb{Z}_{Y/Y'}$. Thus

$$c_{X/S}(\psi_{X/S}(\pi(\overline{m}_1))) = c_{X/S}(-d_{y'_1}) = \nu(x) p_{y'_1} = p(\phi_X(\overline{m}_1)). \qquad \square$$

Since $\psi_{X/S}$ is an isomorphism, the middle row of the diagram (7-2-7) above contains the same information as the top row, a.k.a. the log Kodaira–Spencer sequence. Furthermore, the bottom row identifies with the exact sequence (7-2-5). The following corollary relates the corresponding derived morphisms of these sequences.

Corollary 7.2.3. *Let $\kappa_{X/S} : \mathcal{M}^{\mathrm{gp}}_{X/S} \to \mathbb{Z}[1]$ be the morphism associated to the log Kodaira–Spencer sequence (1-0-1) and let $\kappa_{A/S} : \mathbb{Z}_{X'/X} \to \mathbb{Z}[1]$ be the morphism associated to the exact sequence (7-2-5). Then $\kappa_{X/S} = \kappa_{A/S} \circ c_{X/S} \circ \psi_{X/S}$.*

Proof. The diagram (7-2-7) of exact sequences yields a diagram of distinguished triangles:



The arrows on the right are all identifications, and the formula in the corollary follows. $\square$

Remark 7.2.4. The sheaf $\mathbb{Z}_{X'/X}$ can be naturally identified with $\mathcal{H}^1_Y(\mathbb{Z})$. In fact there are two such natural identifications differing by sign. The first identification is the boundary map $\delta : \mathcal{Q} = \mathcal{H}^0_Y(\mathbb{Z}_{X'/X}) \to \mathcal{H}^1_Y(\mathbb{Z})$ in the long exact sequence obtained by applying the cohomological δ-functor $\mathcal{H}^*_Y(-)$ to the short exact sequence (7-2-5). It is an isomorphism because $\mathcal{H}^i_Y(j_*\mathbb{Z}_U) = 0$ for $i = 0, 1$. To define the second, recall that, by the construction of local cohomology, there is a canonical exact sequence

$$0 \to H^0_Y(X,\mathbb{Z}) \to H^0(X,\mathbb{Z}) \to H^0(U,\mathbb{Z}) \to H^1_Y(X,\mathbb{Z}) \to H^1(X,\mathbb{Z}) \to \cdots,$$

compatible with restriction to open subsets $V \subseteq X$. In our situation, $H^0_Y(V,\mathbb{Z}) = 0$ for all V and $H^1(V) = 0$ for a neighborhood basis of every point of X. Replacing X by V and U by $V \cap U$ for varying open V and sheafifying yields a map $j_*(\mathbb{Z}_U) \to \mathcal{H}^1_Y(\mathbb{Z})$ which factors through an isomorphism $\delta' : \mathbb{Z}_{X'/X} \xrightarrow{\sim} \mathcal{H}^1_Y(\mathbb{Z})$. It follows from [SGA 4½ 1977, Cycle 1.1.5, p. 132]) that $\delta = -\delta'$.

We shall see that there is a very natural connection between the log structures associated to a log curve over the standard log point and the "dual graph" of the underlying marked nodal curve. The precise meaning of this graph seems to vary from author to author; the original and most precise definition we have found is due to Grothendieck [SGA $7_{\rm I}$ 1972, IX, 12.3.7]. We use the following variant, corresponding to what some authors call an "unoriented multigraph."

Definition 7.2.5. A *graph* Γ consists of two mappings between finite sets: $\epsilon : B \to E$ and $\zeta : B \to V$, where for each $e \in E$, the cardinality of $\epsilon^{-1}(e)$ is either one or two. A morphism of graphs $\Gamma_1 \to \Gamma_2$ consists of morphisms $f_B : B_1 \to B_2$, $f_E : E_1 \to E_2$ and $f_V : V_1 \to V_2$ compatible with ϵ_i and ζ_i in the evident sense.

The set V is the set of "vertices" of Γ, the set E is the set of "edges" of Γ, and the set B is the set of "endpoints" of the edges of Γ. For each edge e, the set $\epsilon^{-1}(e)$ is the set of endpoints of the edge e, and for each $b \in B$, $\zeta(b)$ is the vertex of Γ corresponding to the endpoint b. There is a natural involution $b \mapsto \iota(b)$ of B, defined so that for each $b \in B$, $\epsilon^{-1}(\epsilon(b)) = \{b, \iota(b)\}$. The notion of a graph could equivalently be defined as a map $\zeta : B \to V$ together with an involution of B; the map $\epsilon : B \to E$ is then just the projection to the orbit space of the involution.

Definition 7.2.6. Let $\underline{X}$ be a nodal curve. The *dual graph* $\Gamma(\underline{X})$ of $\underline{X}$ consists of the following data:

(1) V is the set of irreducible components of $\underline{X}$, or equivalently, the set of connected components of the normalization $\underline{X}'$ of $\underline{X}$.

(2) E is the set Y of nodes of $\underline{X}$.

(3) $B := \epsilon^{-1}(E)$, the inverse image of E in the normalization $\underline{X}'$ of $\underline{X}$.

(4) $\zeta : B \to V$ is the map taking a point x' in $\underline{X}'$ to the connected component of $\underline{X}'$ containing it.

The involution of the graph of a nodal curve is fixed point free, since each $\epsilon^{-1}(y)$ has exactly two elements. A morphism of nodal curves $f : \underline{X}_1 \to \underline{X}_2$ induces a morphism of graphs provided that f takes each node of $\underline{X}_1$ to a node of $\underline{X}_2$.

Definition 7.2.7. Let Γ be a graph in the sense of Definition 7.2.5. Suppose that ι is fixed-point free, so that each $\epsilon^{-1}(e)$ has cardinality two.

(1) $C_\bullet(\Gamma)$ is the chain complex $C_1(\Gamma) \to C_0(\Gamma)$:

$$\mathbb{Z}_{E/B} \xrightarrow{d_1} \mathbb{Z}^V,$$

where d_1 is the composition $\mathbb{Z}_{E/B} \xrightarrow{i} \mathbb{Z}^B \xrightarrow{\zeta_*} \mathbb{Z}^V$, where i is as shown in (7-2-1), and where ζ_* sends b to $\zeta(b)$.

(2) $C^\bullet(\Gamma)$ is the cochain complex $C^0(\Gamma) \to C^1(\Gamma)$:

$$\mathbb{Z}^V \xrightarrow{d_0} \mathbb{Z}_{B/E},$$

where d_0 is the composition $\mathbb{Z}^V \xrightarrow{\zeta^*} \mathbb{Z}^B \xrightarrow{p} \mathbb{Z}_{B/E}$, where p is as shown in (7-2-1), and where $\zeta^*(v) = \sum\{b : \zeta(b) = v\}$.

(3) $\langle -, - \rangle : C_i(\Gamma) \times C^i(\Gamma) \to \mathbb{Z}$ is the (perfect) pairing induced by the evident bases for $\mathbb{Z}^B$ and $\mathbb{Z}^V$,

$(-|-) : C^1(\Gamma) \times C^1(\Gamma) \to \mathbb{Z}$ is the (perfect) pairing defined by $\langle -, - \rangle$ and the isomorphism $t : C^1(\Gamma) \to C_1(\Gamma)$ (7-2-2).

It is clear from the definitions that the complexes $C_\bullet(\Gamma)$ that a morphism of graphs $f : \Gamma_1 \to \Gamma_2$ induces morphisms of complexes

$$C_\bullet(f) : C_\bullet(\Gamma_1) \to C_\bullet(\Gamma_2) \quad \text{and} \quad C^\bullet(f) : C^\bullet(\Gamma_2) \to C^\bullet(\Gamma_1),$$

compatible with d_1 and d^0.

The proposition below is of course well-known. We explain it here because our constructions are somewhat nonstandard. Statement (3) explains the relationship between the pairings we have defined and intersection multiplicities.

Proposition 7.2.8. *Let Γ be a finite graph such that $\epsilon^{-1}(e)$ has cardinality two for every $e \in E$. Let $C_\bullet(\Gamma)$ and $C^\bullet(\Gamma)$ be the complexes defined in Definition 7.2.7, and let $H_*(\Gamma)$ and $H^*(\Gamma)$ the corresponding (co)homology groups. For each pair of elements (v, w) in V, let*

$$E(v, w) := \epsilon(\zeta^{-1}(v)) \cap \epsilon(\zeta^{-1}(w)) \subseteq E$$

and let $e(v, w)$ be the cardinality of $E(v, w)$.

(1) *The homomorphisms $d_1 : C_1(\Gamma) \to C_0(\Gamma)$ and $d^0 : C^0(\Gamma) \to C^1(\Gamma)$ are adjoints, with respect to the pairings defined above.*

(2) *The groups $H_*(\Gamma)$ and $H^*(\Gamma)$ are torsion free, and the inner product on $C_1(\Gamma)$ (resp. on $C_0(\Gamma)$) defines a perfect pairing $\langle -, - \rangle$ between $H^1(\Gamma)$ and $H_1(\Gamma)$ (resp. between $H_0(\Gamma)$ and $H^0(\Gamma)$). In fact, $H_0(\Gamma)$ identifies with the*

free abelian group on $V/\sim$, *where* $\sim$ *is the equivalence relation generated by the set of pairs* (v, v') *such that* $E(v, v') \neq \varnothing$.

(3) *For each* $v \in V$,

$$d_1(t(d^0(v))) = \sum_{v' \neq v} e(v, v')(v - v'),$$

and

$$(d^0(v)|d^0(w)) = \begin{cases} -e(v, w) & \text{if } v \neq w, \\ \sum_{v' \neq v} e(v, v') & \text{if } v = w. \end{cases}$$

(4) *Let* $h^i(\Gamma)$ *denote the rank of* $H^i(\Gamma)$ *and let* $\chi(\Gamma) := h^0(\Gamma) - h^1(\Gamma)$. *Then*

$$\chi(\Gamma) = |V| - |E|.$$

Proof. Statement (1) is clear from the construction, since d_0 is dual to d^1 and ζ_* is dual to ζ^*.

To prove (2), observe that each equivalence class of E defines a subgraph of Γ, that Γ is the disjoint union of these subgraphs, and that the complex $C_\bullet(\Gamma)$ is the direct sum of the corresponding complexes. Thus we are reduced to proving (2) when there is only one such equivalence class. There is a natural augmentation $\alpha : \mathbb{Z}^V \to \mathbb{Z}$ sending each basis vector v to 1, and if $b \in B$, $\alpha\big(d_1(b - \iota(b))\big) = \alpha(\zeta(b) - \zeta(\iota(b))) = 0$, so d_1 factors through $\mathrm{Ker}(\alpha)$. Thus it will suffice to prove that d_1 maps surjectively to this kernel. Choose some $v_0 \in V$; then

$$\{v - v_0 : v \in V, v \neq v_0\}$$

is a basis for $\mathrm{Ker}(\alpha)$. Say (v, v') is a pair of distinct elements of V and $E(v, v') \neq \varnothing$. Choose $e \in \epsilon(\zeta^{-1}(v)) \cap \epsilon(\zeta^{-1}(v'))$ and $b \in \epsilon^{-1}(e) \cap \zeta^{-1}(v)$. Then necessarily $\zeta(\iota(b)) = v'$, so $d_1(b - \iota(b)) = v - v'$. Since any two elements of E are equivalent, given any $v \in V$, there is a sequence $(v_0, v_1, \ldots, v_n)$ with each $v_{i-1} \sim v_i$, and for each such pair choose b_i with $d_1(b_i - \iota(b_i)) = v_i - v_{i-1}$. Then $d_1(b_1 + \cdots + b_n) = v_n - v_0$.

It follows that $H_0(\Gamma)$ is torsion free. Then the duality statement follows from the fact that d^0 is dual to d_1.

The formulas for d_1 and d^0 imply that for $b \in B$ and $v \in V$,

$$d_1(d_b) = \zeta(b) - \zeta(\iota(b)), \quad d^0(v) = \sum_{b \in \zeta^{-1}(v)} p_b.$$

Hence if v in V,

$$d_1(t(d^0(v))) = d_1\Bigg(\sum_{b \in \zeta^{-1}(v)} d_b\Bigg), = \sum_{b \in \zeta^{-1}(v)} \zeta(b) - \zeta(\iota(b)).$$

But if $b \in \zeta^{-1}(v)$,

$$\zeta(b) - \zeta(\iota(b)) = \begin{cases} v - \zeta(\iota(b)) & \text{if } \zeta(b) \neq \zeta(\iota(b)), \\ 0 & \text{otherwise.} \end{cases}$$

For each $v' \in V \setminus \{v\}$, the map ϵ induces a bijection from $\{b \in \zeta^{-1}(v) : \zeta(\iota(b)) = v'\}$ to $E(v, v')$. Thus

$$\sum_{b \in \zeta^{-1}(v)} \zeta(b) - \zeta(\iota(b)) = \sum_{v' \in V} e(v, v')v - e(v, v')v'$$

and the first formula of (3) follows. Then

$$(d^0 v | d^0 w) = (d_1(t(d^0(v)))|w) = \sum_{v' \neq v} e(v, v')(v|w) - \sum_{v' \neq v} e(v, v')(v'|w),$$

and the second formula follows. Statement (4) is immediate. □

The geometric meaning of the cochain complex of a nodal curve is straightforward and well-known.

Proposition 7.2.9. *Let $\underline{X}/\mathbb{C}$ be a nodal curve and let $\Gamma(\underline{X})$ be its dual graph. Then there is a commutative diagram*

$$\begin{array}{ccc} H^0(\underline{X}', \mathbb{Z}) & \xrightarrow{A} & H^0(\underline{X}, \mathbb{Z}_{X'/X}) \\ \cong \downarrow & & \downarrow \cong \\ C^0(\Gamma(\underline{X})) & \xrightarrow[d^0]{} & C^1(\Gamma(\underline{X})) \end{array}$$

where the homomorphism A comes from the map also denoted by A in the short exact sequence

$$0 \to \mathbb{Z}_X \to \epsilon_*(\mathbb{Z}_{X'}) \xrightarrow{A} \mathbb{Z}_{X'/X} \to 0. \tag{7-2-8}$$

Consequently there is an exact sequence

$$0 \to H^1(\Gamma(\underline{X})) \to H^1(\underline{X}, \mathbb{Z}) \to H^1(\underline{X}', \mathbb{Z}) \to 0. \tag{7-2-9}$$

Proof. The commutative diagram is an immediate consequence of the definitions. The cohomology sequence attached to the exact sequence (7-2-8) reads

$$0 \to H^0(\underline{X}, \mathbb{Z}) \to H^0(\underline{X}', \mathbb{Z}) \xrightarrow{A} H^0(\underline{X}, \mathbb{Z}_{X'/X}) \to H^1(\underline{X}, \mathbb{Z}) \to H^1(\underline{X}', \mathbb{Z}) \to 0,$$

and the sequence (7-2-9) follows immediately. □

Note that $H^1(\underline{X}', \mathbb{Z})$ vanishes if and only if each irreducible component of $\underline{X}$ is rational, a typical situation.

7.3. ***The nearby cycles spectral sequence.*** We now consider the associated morphism $f_{\log} : X_{\log} \to S_{\log}$. Our goal is to use the nearby cycles diagram (4-2-1) and Theorem 4.2.2 to describe the general fiber X_η of $f_{\log}$, together with its monodromy action.

Theorem 7.3.1. *Let $f : X \to S$ be a vertical log curve over the standard log point S. The morphism*

$$f_{\log} : X_{\log} \to S_{\log} = \mathbb{S}^1$$

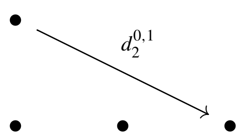
is a topological submersion whose fibers are topological manifolds of real dimension 2. If f is proper and $\underline{X}$ is connected, then the morphism $f_{\log}$ is a locally trivial fibration, its general fiber X_η is compact, connected, and orientable, and its genus is $1 + g(\underline{X}') + h^0(Y) - h^0(\underline{X}')$.

Proof. The first statement is proved in [Nakayama and Ogus 2010], although it is much more elementary over a log point as here. Suppose f is proper. Then so is $f_{\log}$, and it follows that X_η is compact. Its orientability is proved in [Nakayama and Ogus 2010].

To compute the cohomology of X_η, observe that since the fibration $\tilde{X}_{\log} \to \mathbb{R}(1)$ is necessarily trivial, X_η and $\tilde{X}_{\log}$ have the same homotopy type, and in particular their homology groups are isomorphic. The spectral sequence of nearby cycles for the sheaf $\mathbb{Z}(1)$ on $\tilde{X}_{\log}$ reads

$$E_2^{p,q}(1) = H^p(X, \Psi^q_{X/S}(1)) \Longrightarrow H^{p+q}(\tilde{X}_{\log}, \mathbb{Z}(1)).$$

Theorem 4.2.1 defines an isomorphism $\sigma : \mathcal{M}_{X/S} \xrightarrow{\sim} \Psi^1_{X/S}(1)$, and Proposition 7.2.1 an isomorphism $\psi_{X/S} : \mathcal{M}_{X/S} \xrightarrow{\sim} \mathbb{Z}_{Y/Y'}$. These sheaves are supported on the zero dimensional space Y, and $\Psi^q_{X/S}(1)$ vanishes for $q > 1$ Since X has (real) dimension 2, the only possible nonzero terms and arrows in the spectral sequence are



Hence $E_\infty^{1,0}(1) = E_2^{1,0}(1) = H^1(\underline{X}, \mathbb{Z}(1))$, and there is an exact sequence

$$0 \to E_\infty^{0,1}(1) \to H^0(\underline{X}, \mathbb{Z}_{Y/Y'}) \xrightarrow{d_2^{0,1}} H^2(\underline{X}, \mathbb{Z}(1)) \to H^2(\tilde{X}_{\log}, \mathbb{Z}(1)) \to 0. \quad (7\text{-}3\text{-}1)$$

Since the normalization map ϵ is proper and an isomorphism outside Y', it induces an isomorphism

$$H^2(\underline{X}, \mathbb{Z}(1)) \xrightarrow{\sim} H^2(\underline{X}', \mathbb{Z}(1)).$$

Since $\underline{X}'$ is a smooth compact complex analytic manifold of dimension 1, the trace map induces a canonical isomorphism: $H^2(\underline{X}', \mathbb{Z}(1)) \cong H^0(\underline{X}', \mathbb{Z})$. Combining this isomorphism with the one above, we obtain an isomorphism

$$\mathrm{tr}' : H^2(\underline{X}, \mathbb{Z}(1)) \xrightarrow{\sim} H^0(\underline{X}', \mathbb{Z}). \tag{7-3-2}$$

Lemma 7.3.2. *Let X/S be a proper, connected, and vertical log curve over the standard log point, and let $\underline{X}$ be its underlying nodal curve. Then the Betti numbers of $\Gamma(\underline{X})$, of X, and of the general fiber X_η of the fibration $X_{\log} \to \mathbb{S}^1$, are given by the following formulas*:

$$h^1(\Gamma(\underline{X})) = 1 - h^0(\underline{X}') + h^0(Y),$$
$$h^1(\underline{X}) = h^1(\Gamma(\underline{X})) + h^1(\underline{X}'),$$
$$h^1(X_\eta) = h_1(\Gamma(\underline{X})) + h^1(\underline{X}).$$

Proof. The first formula follows from (4) of Proposition 7.2.8 and the definition of $\Gamma(\underline{X})$. The second formula follows from the exact sequence (7-2-9). For the third formula, observe that $H^2(\tilde{X}_{\log}, \mathbb{Z}(1))$ has rank one, since $\tilde{X}_{\log}$ has the same homotopy type as X_η, which is a compact two-dimensional orientable manifold. It then follows from the exact sequence (7-3-1) that the rank $e_\infty^{0,1}(1)$ of $E_\infty^{0,1}(1)$ is given by

$$\begin{aligned} e_\infty^{0,1}(1) &= h^0(\underline{X}, \mathbb{Z}_{Y/Y'}) - h^2(\underline{X}, \mathbb{Z}(1)) + 1 = h^0(\underline{X}, \mathbb{Z}_{Y/Y'}) - h^0(\underline{X}', \mathbb{Z}) + 1 \\ &= h^0(\underline{X}, \mathbb{Z}_{Y/Y'}) - h^0(\underline{X}', \mathbb{Z}) + 1 = |E(\Gamma(\underline{X})| - |V(\Gamma(\underline{X})| + 1 \\ &= 1 - \chi(\Gamma(\underline{X})) \\ &= h_1(\Gamma(\underline{X})). \end{aligned}$$

Then $h^1(X_\eta) = e_\infty^{0,1}(1) + e_\infty^{1,0}(1) = h_1(\Gamma(\underline{X})) + h^1(\underline{X})$. □

Combining the formulas of the lemma, we find

$$h^1(X_\eta) = h_1(\Gamma) + h^1(\Gamma) + h^1(\underline{X}') = 2 - 2h^0(\underline{X}') + 2h^0(Y) + 2g(\underline{X}'),$$

and hence $g(X_\eta) = 1 - h^0(\underline{X}') + h^0(Y) + g(\underline{X}')$. □

The following more precise result shows that the differential in the nearby spectral sequence can be identified with the differential in the chain complex $C_\bullet$ attached to the dual graph of $\underline{X}$.

Proposition 7.3.3. *Let X/S be a proper and vertical log curve over the standard log point and let $\underline{X}$ be its underlying nodal curve. Then the following diagram* *commutes*:

$$
\begin{array}{ccccc}
 & & H^0(X, \Psi^1_{X/S}(1)) & \xrightarrow{-d_2^{0,1}} & H^2(\underline{X}, \mathbb{Z}(1)) \\
 & \overset{\cong}{\nearrow} & \sigma \uparrow \cong & & \cong \downarrow \mathrm{tr}' \\
H^0(X, \mathbb{Z}_{Y/Y'}) & \underset{\cong}{\xleftarrow{\psi_{X/S}}} & H^0(\underline{X}, \mathcal{M}_{X/S}) & \longrightarrow & H^0(\underline{X}', \mathbb{Z}) \\
 & \underset{\cong}{\searrow} & \downarrow \cong & & \downarrow \cong \\
 & & C_1(\Gamma(\underline{X})) & \xrightarrow{d_1} & C_0(\Gamma(\underline{X}))
\end{array}
$$

Consequently there is a canonical isomorphism $E_\infty^{0,1}(1) \cong H_1(\Gamma(\underline{X}))$ *and hence an exact sequence*

$$0 \to H^1(\underline{X}, \mathbb{Z}(1)) \to H^1(\tilde{X}, \mathbb{Z}(1)) \to H_1(\Gamma(\underline{X})) \to 0. \tag{7-3-3}$$

Proof. The commutativity of this diagram follows from Proposition 7.2.1 and statement (1) of Theorem 4.2.2. To write out the proof in detail, we use the notation of the proof of that result. It suffices to check what happens to each basis element of the free abelian group $H^0(X, \mathcal{M}_{X/S})$. Let y be a point of Y and let m_1 and m_2 be the elements of $\mathcal{M}_{X,y}$ as in the proof of Proposition 7.1.4, with images ℓ_1 and ℓ_2 in $\Gamma(X, \mathcal{M}_{X/S})$. Then $\ell_1 = -\ell_2$ is a typical basis element of $H^0(X, \mathcal{M}_{X/S})$. Theorem 4.2.2 says that $d_2^{0,1}(\ell_1)$ is the Chern class $c_1(\mathcal{L}_{\ell_1})$ of $\mathcal{L}_{\ell_1}$, where $\mathcal{L}_{\ell_1}$ is the invertible sheaf on X coming from the exact sequence (1-0-4). Then

$$\epsilon^*(c_1(\mathcal{L}_{\ell_1})) = c_1(\epsilon^*(\mathcal{L}_{\ell_1})) = c_1(\mathcal{O}_{X'}(-\psi(\ell_1)),$$

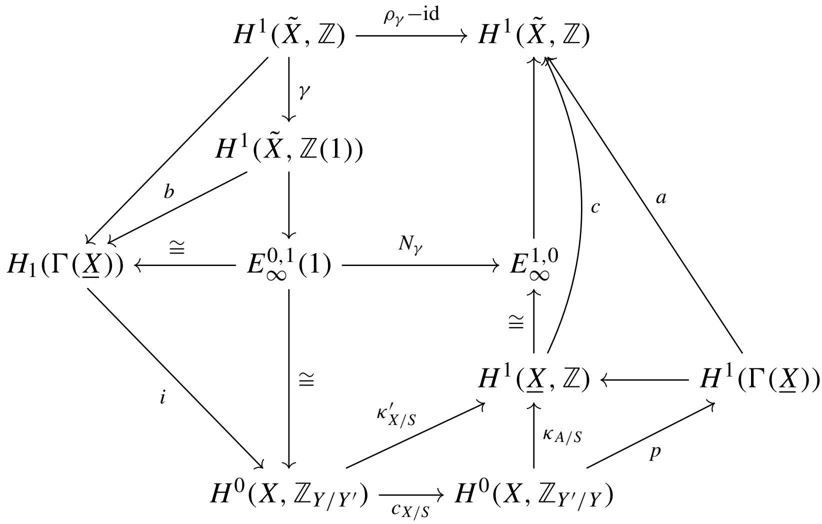
by Proposition 7.2.1. But if p is a point of the (smooth) curve $\underline{X}'$, then $\mathbf{tr}\big(c_1(\mathcal{O}_{X'}(D))\big)$ is the basis element of $H^0(\underline{X}', \mathbb{Z})$ corresponding the connected component of $\underline{X}'$ containing p. The corresponding generator of $C_0(\Gamma)$ is precisely $\zeta(p)$. This proves that the diagram commutes. □

7.4. ***Monodromy and the Picard–Lefschetz formula.*** We can now compute the monodromy action on $H^1(\tilde{X}, \mathbb{Z})$.

Theorem 7.4.1. *Let X/S be a log curve over the standard log point. Choose $\gamma \in \mathsf{I}_{\mathbb{N}} = \mathbb{Z}(1)$, let ρ_γ be the corresponding automorphism of $H^1(\tilde{X}, \mathbb{Z})$, and let $N_\gamma : E_\infty^{0,1} \to E_\infty^{1,0}$ be the map induced by $\rho_\gamma - \mathrm{id}$ (see (1-0-6)). Let*

$$\kappa'_{X/S} := \kappa_{X/S} \circ \psi_{X/S}^{-1} : \mathbb{Z}_{Y/Y'} \to \mathcal{M}_{X/S} \to \mathbb{Z}_X[1].$$

Then there is a commutative diagram:



Proof. Applying H^1 to the commutative diagram defining λ^1_γ

$$\begin{array}{ccc} \Psi_{X/S} & \longrightarrow & \Psi^1_{X/S}[-1] \\ {\scriptstyle \rho_\gamma - \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \lambda^1_\gamma[-1]} \\ \Psi_{X/S} & \longleftarrow & \Psi^0_{X/S} \end{array}$$

yields a commutative diagram

$$\begin{array}{ccc} H^1(\tilde{X}, \mathbb{Z}) & \longrightarrow & H^0(\underline{X}, R^1\tilde{\tau}_*\mathbb{Z}) \\ {\scriptstyle \rho_\gamma - \mathrm{id}}\downarrow & & \downarrow \\ H^1(\tilde{X}, \mathbb{Z}) & \longleftarrow & H^1(\underline{X}, R^0\tilde{\tau}_*\mathbb{Z}). \end{array}$$

Thanks to the identifications

$$H^0(X, R^1\tilde{\tau}_*\mathbb{Z}(1)) \cong E_2^{0,1}(1) \cong H^0(X, \mathbb{Z}_{Y/Y'}),$$
$$H^1(X, R^0\tilde{\tau}_*\mathbb{Z}) = H^1(\underline{X}, \mathbb{Z}) = E_\infty^{1,0},$$

our monodromy formula from Theorem 4.2.2(2) shows that the following diagram commutes:

$$\begin{array}{ccccc} H^1(\tilde{X}, \mathbb{Z}) & \xrightarrow{\gamma} & H^1(\tilde{X}, \mathbb{Z}(1)) & \longrightarrow & H^0(\underline{X}, \mathbb{Z}_{Y/Y'}) \\ {\scriptstyle \rho_\gamma - \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \kappa'_{X/S}} \\ H^1(\tilde{X}, \mathbb{Z}) & & \xleftarrow[\quad c \quad]{} & & H^1(\underline{X}, \mathbb{Z}). \end{array}$$

The rest of the big diagram commutes by the preceding discussion of the dual graph. ☐

Remark 7.4.2. The dual of the exact sequence (7-3-3) can be written

$$0 \to H_1(\Gamma(\underline{X}))^\vee \to H_1(\tilde{X}, \mathbb{Z}(-1)) \to H_1(\underline{X}, \mathbb{Z}(-1)) \to 0,$$

so that the elements of $H_1(\Gamma(\underline{X}))^\vee \cong H^1(\Gamma(\underline{X}))$ can be interpreted as *vanishing cycles* on $\tilde{X}$. The exact sequence (7-2-9) shows that they can also be interpreted as *vanishing cocycles* on $\underline{X}$.

The monodromy formula expressed by Theorem 7.4.1 can be made more explicit in terms of vanishing cycles. For each node $y \in Y$, choose a branch $y' \in \epsilon^{-1}(y)$ and note that $\pm p_{y'} \in \Gamma(X, \mathbb{Z}_{Y'/Y})$ depends only on y. Write $\langle -, - \rangle$ for the pairing $\mathbb{Z}_{Y/Y'} \times \mathbb{Z}_{Y'/Y} \to \mathbb{Z}$ and let $h_y : \mathbb{Z}_{Y/Y'} \to \mathbb{Z}_{Y'/Y}$ be the map $\langle -, p_{y'} \rangle p_{y'}$. Then h_y depends only on y and not on y', and, by Proposition 7.2.2, we can write

$$c_{X/S} = \sum_y -\nu(y) h_y = \sum_y -\nu(y) \langle -, p_{y'} \rangle p_{y'}.$$

(The map $c_{X/S}$ above encodes the "monodromy pairing" of Grothendieck, see [SGA 7_{I} 1972, IX, §9 and 12.3]). Then the composition

$$H^1(\tilde{X}, \mathbb{Z}) \xrightarrow{b \circ \gamma} H_1(\Gamma) \xrightarrow{p \circ c_{X/S} \circ i} H^1(\Gamma) \xrightarrow{a} H^1(\tilde{X}, \mathbb{Z})$$

is the map sending an element x to $\sum_y -\nu(y) \langle b \circ \gamma(x), p_y \rangle a(p_y)$. The following formula is then immediate.

Corollary 7.4.3. *If $\gamma \in \mathsf{I}_P$ and $x \in H^1(\tilde{X}, \mathbb{Z})$,*

$$\rho_\gamma(x) = x - \sum_y \nu(y) \langle b \circ \gamma(x), p_y \rangle a(p_y).$$ ☐

When all $\nu(y) = 1$, the formula of Corollary 7.4.3 is the standard Picard–Lefschetz formula [SGA 7_{II} 1973, exposé XV]. To verify this, we must check the compatibility of the pairing $\langle -, - \rangle$ used above with the standard pairing on cohomology. As usual the determination of signs is delicate; we give a (somewhat heuristic) argument below.

Recall that we have a proper fibration $\tilde{X} \to \mathbb{R}(1)$, and hence for all i, $H^i(\tilde{X}, \mathbb{Z}) \cong H^i(\tilde{X}_0)$, where $\tilde{X}_0$ is the fiber of $\tilde{X} \to \mathbb{R}_{\geq}$ over zero (equivalently, the fiber of $X_{\log} \to \mathbb{S}^1$ over 1). Thus we can replace $\tilde{X}$ by $\tilde{X}_0$ in the diagrams above. Since $\tilde{X}_0$ is a compact manifold, whose orientation sheaf identifies with $\mathbb{Z}(1)$ [Nakayama and Ogus 2010], we have a perfect pairing

$$(-|-) : H^1(\tilde{X}_0, \mathbb{Z}(1)) \times H^1(\tilde{X}_0, \mathbb{Z}) \to H^2(\tilde{X}_0, \mathbb{Z}(1)) \xrightarrow{\mathbf{tr}} \mathbb{Z},$$

defined by cup-product and trace map. For each y, let $v_y := a(\delta_y) \in H^1(\tilde{X}_0, \mathbb{Z})$. Then the usual Picard–Lefschetz formula [SGA 7_{II} 1973, exposé XV, théorème 3.4] reads

$$\rho_\gamma(x) = x - \sum_y \nu(y)(\gamma(x)|v_y)v_y. \tag{7-4-1}$$

As we shall see from Proposition 7.4.4 below, for $x \in H^1(\tilde{X}_0, \mathbb{Z}(1))$ and $y \in Y$,

$$\langle b(x), \delta_y\rangle = (x|a(\delta_y)).$$

Thus Corollary 7.4.3 implies the Picard–Lefschetz formula (7-4-1).

Proposition 7.4.4. *The maps*

$$a : H^1_Y(X, \mathbb{Z}) \to H^1(\tilde{X}_0, \mathbb{Z}) \quad \textit{and} \quad b : H^1(\tilde{X}_0, \mathbb{Z}(1)) \to H^1_Y(X, \mathbb{Z})$$

of the diagram in Theorem 7.4.1 are mutually dual, where we use the standard cup-product and trace map pairing,

$$H^1(\tilde{X}_0, \mathbb{Z}(1)) \otimes H^1(\tilde{X}_0, \mathbb{Z}) \to H^2(\tilde{X}_0, \mathbb{Z}(1)) \xrightarrow{\text{tr}} \mathbb{Z},$$

and the form $(-|-)$ *of Definition 7.2.7 on* $H^1_Y(X, \mathbb{Z}) \cong C^1(\Gamma)$.

Proof. We start by reducing to the local case. Since we will have to deal with nonproper X, we need to modify the map a slightly, letting

$$a : H^1_Y(X, \mathbb{Z}) \to H^1_c(X, \mathbb{Z}) \to H^1_c(\tilde{X}_0, \mathbb{Z}),$$

where the first map is induced by the natural transformation $\Gamma_Y \to \Gamma_c$ (defined because Y is proper), and the other map is pull-back by $\tilde{\tau}_0 : \tilde{X}_0 \to X$ (defined because $\tilde{\tau}_0$ is proper). Note that a is well-defined in the situation when X is not proper, and that it coincides with a defined previously in case X is proper. Moreover, the map b makes sense for nonproper X, and both maps are functorial with respect to (exact) open immersions in the following sense: if $j : U \to X$ is an open immersion, then the following squares commute:

$$\begin{CD}
H^1_Y(X, \mathbb{Z}) @>{a}>> H^1_c(\tilde{X}_0, \mathbb{Z}) \\
@A{j_*}AA @AA{j_*}A \\
H^1_{Y\cap U}(U, \mathbb{Z}) @>>{a}> H^1_c(\tilde{U}_0, \mathbb{Z})
\end{CD}
\qquad\qquad
\begin{CD}
H^1(\tilde{X}_0, \mathbb{Z}(1)) @>{b}>> H^1_Y(X, \mathbb{Z}) \\
@V{j^*}VV @VV{j^*}V \\
H^1(\tilde{U}_0, \mathbb{Z}(1)) @>>{b}> H^1_{Y\cap U}(U, \mathbb{Z})
\end{CD}$$

The two pairings in question are similarly functorial. Recall that

$$H^1_Y(X, \mathbb{Z}) = \bigoplus_{y\in Y} H^1_{\{y\}}(X, \mathbb{Z})$$

is an orthogonal decomposition. Let $a_y,\ b_y$ be the compositions

$$a_y : H^1_{\{y\}}(X,\mathbb{Z}) \to H^1_Y(X,\mathbb{Z}) \xrightarrow{a} H^1_c(\tilde{X}_0,\mathbb{Z}),$$
$$b_y : H^1(\tilde{X}_0,\mathbb{Z}(1)) \to H^1_Y(X,\mathbb{Z}) \to H^1_{\{y\}}(X,\mathbb{Z}).$$

To check that a and b are mutually dual, it suffices to check that a_y and b_y are mutually dual for all $y \in Y$. Fix $y \in Y$, and let U be a standard neighborhood of y. The functoriality of a and b discussed above implies that it suffices to prove the proposition for $X = U$.

We henceforth assume that $X = \{(x_1, x_2) : x_1x_2 = 0\}$. So $Y = \{y\}$, $y = (0,0)$, and $X = X_1 \cup X_2$ where $X_i = \{x_i = 0\}$. The choice of ordering of the branches at y yields generators of the three groups in question as follows. First, the class of X_1 (treated as a section of $j_*\mathbb{Z}_U$, where $U = X \setminus Y$) gives a generator u of $H^1_Y(X,\mathbb{Z})$. Second, the loop in the one-point compactification of $\tilde{X}_0$ going from the point at infinity through X_2 and then X_1 gives a basis of its fundamental group, and hence a basis element v of $H^1_c(\tilde{X}_0,\mathbb{Z})$. Finally, identifying the circle $\tilde{Y}_0 = \tilde{\tau}_0^{-1}(y) = \{(\phi_1,\phi_2) \in \mathbb{S}^1 : \phi_1\phi_2 = 1\}$ with the unit circle in X_1 via the map $(\phi_1,\phi_2) \mapsto \phi_1$ yields a generator w of $H^1(\tilde{X}_0,\mathbb{Z}(1)) \cong H^1(\tilde{Y}_0,\mathbb{Z}(1))$.

The assertion of the proposition will now follow from three claims:

(1) $a(u) = v$,

(2) $b(w) = -u$,

(3) $\langle v, w\rangle = 1$.

To check the first claim, note that we have a similarly defined basis element v' of $H^1_c(X,\mathbb{Z})$ which pulls back to v. Let $\gamma : \mathbb{R} \cup \{\infty\} \to X \cup \{\infty\}$ be a loop representing v', sending 0 to y. Pull-back via γ reduces the question to Lemma 7.4.5 below.

For the second claim, recall first that $c'(u) = c'([X_1]) = [q_1]$. Second, the isomorphism $\sigma : \mathcal{M}^{\mathrm{gp}}_{X/S,y} \to H^1(\tilde{Y}_0,\mathbb{Z}(1))$ sends q_i to the pullback by ϕ_i of the canonical class $\theta \in H^1(\mathbb{S}^1,\mathbb{Z}(1))$. On the other hand, since x_2 is the coordinate on X_1, $v = \phi_2^*\theta$. Since $\phi_1\phi_2 = 1$ on $\tilde{X}_0$, $\phi_1^* + \phi_2^* = 0$, and hence $b^{-1}(u) = \sigma(c'(u)) = \phi_1^*\theta = -\phi_2^*\theta = -w$.

For the last claim, we note that the map

$$(r_1,\phi_1,r_2,\phi_2) \mapsto (r_1 - r_2, \phi_1) : \tilde{X}_0 \to \mathbb{R} \times \mathbb{S}^1$$

is an orientation-preserving homeomorphism (where the orientation sheaves of both source and target are identified with $\mathbb{Z}(1)$). Under this identification, w corresponds to the loop $0 \times \mathbb{S}^1$ (positively oriented), and v correspond to the "loop" $\mathbb{R} \times \{1\}$ oriented in the positive direction. These meet transversely at one point

(0, 1), and their tangent vectors form a negatively oriented basis at that point, thus $\langle w, v\rangle = -1$. □

Lemma 7.4.5. *Let $S = \mathbb{R} \cup \{\infty\}$ be the compactified real line, $Y = \{0\}$, $Z = \{\infty\}$, $X = \mathbb{R} = S \setminus Z$, $U = X \setminus Y$, $j : U \hookrightarrow X$. Let $e \in H^0(U, \mathbb{Z})$ equal 1 on $U_+ = (0, \infty)$ and 0 on $U_- = (-\infty, 0)$. As before, we have a short exact sequence*

$$0 \to \mathbb{Z}_X \to j_*\mathbb{Z}_U \to \mathcal{H}^1_Y(\mathbb{Z}_X) \to 0$$

and hence an identification $H^1_Y(X, \mathbb{Z}) \cong H^0(X, \mathcal{H}^1_Y(\mathbb{Z}_X)) \cong H^0(U, \mathbb{Z})/j^*H^0(X, \mathbb{Z})$. The element e thus gives a basis element u of $H^1_Y(X, \mathbb{Z})$. The orientation of the real axis gives a basis element of $\pi_1(S, \infty)$, and hence a basis element v of

$$\operatorname{Hom}(\pi_1(S, \infty), \mathbb{Z}) \cong H^1(S, \mathbb{Z}) \cong H^1_c(X, \mathbb{Z}).$$

Then the natural map $H^1_Y(X, \mathbb{Z}) \to H^1_c(X, \mathbb{Z})$ sends u to v.

Proof. By [SGA 4½ 1977, Cycle 1.1.5, p. 132], u corresponds to the partially trivialized $\mathbb{Z}_X$-torsor $(\mathbb{Z}_X, -e)$ (see Remark 7.2.4 and [SGA 4½ 1977, Cycle 1.1.4–5]). Let $(\mathcal{F}, f)$ be a $\mathbb{Z}_S$-torsor with a section $f \in H^0(\mathcal{F}, S \setminus Y)$ such that there exists an isomorphism $\iota : \mathcal{F}|_X \cong \mathbb{Z}_X$ identifying $f|_{X\setminus Y}$ with $-e$. Then the class $[\mathcal{F}]$ of $\mathcal{F}$ in $H^1(S, \mathbb{Z}) = H^1_c(X, \mathbb{Z})$ is the image of u. The image of 0 under the isomorphism ι yields a trivializing section g of $\mathcal{F}|_X$, and f is a trivializing section of $\mathcal{F}|_{S\setminus Y}$. On the intersection $X \cap (S \setminus Y) = U$, we have $f - g = 0 - e$; thus f is identified with g on U_-, and g is identified with $f + 1$ on U_+. So the positively oriented loop has monodromy $+1$ on $\mathcal{F}$, i.e., $[\mathcal{F}] = v$ as desired. □

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The Markov sequence problem for the Jacobi polynomials and on the simplex

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The Markov sequence problem aims at the description of possible eigenvalues of symmetric Markov operators with some given orthonormal basis as eigenvector decomposition. A fundamental tool for their description is the hypergroup property. We first present the general Markov sequence problem and provide the classical examples, most of them associated with the classical families of orthogonal polynomials. We then concentrate on the hypergroup property, and provide a general method to obtain it, inspired by a fundamental work of Carlen, Geronimo and Loss. Using this technique and a few properties of diffusion operators having polynomial eigenvectors, we then provide a simplified proof of the hypergroup property for the Jacobi polynomials (Gasper's theorem) on the unit interval. We finally investigate various generalizations of this property for the family of Dirichlet laws on the simplex.

1. Introduction

In this paper, we are interested in the Markov sequence problem and the related hypergroup property, and concentrate in particular on Beta measures on the interval and on Dirichlet measures on the simplex.

The general Markov sequence problem may be stated as follows: given a unit orthonormal $\mathcal{L}^2(\mu)$ basis $\{f_0 = \mathbf{1}, f_1, \dots, f_n, \dots\}$ on some probability space $(E, \mathcal{E}, \mu)$, one aims at the description of all sequences (λ_n), such that the linear operator K defined through $K(f_n) = \lambda_n f_n$ is a Markov operator, that is satisfies $K(\mathbf{1}) = \mathbf{1}$ and is positivity preserving. Since the first property amounts to $\lambda_0 = 1$, the problem is reduced to studying the positivity preserving property.

This problem arises in many areas, particularly in statistics, special function theory, orthogonal polynomials theory and so on (see, among many others, [Bakry et al. 2014; Bakry and Zribi 2017; Bochner 1954; Carlen et al. 2011; Connett and Schwartz 1990; Gasper 1971; 1972; Lasser 1983; Sarmanov and Bratoeva 1967]).

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The aim of this paper is to describe these Markov sequences for the family of Jacobi polynomials and their extension to some families of polynomials in many variables on the simplex $\{x_i \geq 0; \sum_{i=1}^n x_i \leq 1\} \subset \mathbb{R}^n$, orthogonal for the Dirichlet measures

$$C_{p_1 \cdots p_{n+1}} x_1^{p_1/2-1} \cdots x_n^{p_n/2-1} (1 - x_1 - \cdots x_n)^{p_{n+1}/2-1} \, dx_1 \cdots dx_n,$$

where $p_i > 0$, $i = 1, \ldots, n+1$. (The choice for this parametrization will be explained below).

These Dirichlet measures again play an important rôle in many areas (statistics, probability, mathematical biology, etc., see, for example, [Balakrishnan 2003; Gelman et al. 2004; Letac 2012; Letac and Massam 1998]), and are natural generalizations of Beta measures on $(-1, 1)$, associated with the Jacobi polynomials. For the Beta measure, we shall revisit the fundamental result of Gasper through a method introduced by Carlen, Geronimo and Loss [Carlen et al. 2011], and our aim is to use this technique to propose some extensions to the Dirichlet measures.

The Markov sequence set shares some basic generic properties, whatever the space E and the basis $\mathcal{F}$. We refer to [Bakry and Huet 2008] for further details.

As we already mentioned, since $f_0 = \mathbf{1}$, $\lambda_0 = 1$. Moreover, it is easily seen that for any n, $|\lambda_n| \leq 1$.

The set of Markov sequences is a convex set (a convex combination of sequences corresponds to the same convex combination of the associated Markov operators), and is closed under pointwise convergence on the sequences. Therefore, through Choquet's representation theorem, the description of all Markov sequences amounts to the description of the extremal ones.

Moreover, it is also stable under pointwise multiplication (which corresponds to the composition of the associated Markov operators).

Let us mention a few classical results concerning the Markov sequence problem.

(1) Hermite polynomials. The Hermite polynomials are the orthogonal polynomials for the Gaussian measure on $\mathbb{R}$, that is $\mu(dx) = (1/\sqrt{2\pi}) e^{-x^2/2} \, dx$. Sarmanov and Bratoeva [1967] proved that, for any Markov sequence, there exists a probability measure ν on $[-1, 1]$ such that $\lambda_n = \int_{-1}^1 x^n \, \nu(dx)$. In other words, the extremal Markov sequences are of the form $\lambda_n = e^{-nt}$ for some $t \geq 0$, or $(-1)^n e^{-nt}$, for some $t \geq 0$. The sequence (e^{-nt}) corresponds to a well known family of Markov operators K_t, namely the heat kernel associated with the Ornstein–Uhlenbeck operator. Indeed, $K_t = e^{tL}$, where $L(f)(x) = f'' - x f'$. This family of Markov kernels is known as the Ornstein–Uhlenbeck semigroup and there is a large literature devoted to it (see for example [Bakry et al. 2014; Gross 1975; 2006; Meyer 1982]). Moreover, the sequence $\lambda_n = (-1)^n$ corresponds to the symmetry $K(f)(x) = f(-x)$, so that those two operations generate all Markov sequences.

(2) Ultraspherical polynomials. The ultraspherical polynomials (P_n^α) form the family of orthogonal polynomials for $C_\alpha(1-x^2)^\alpha\,dx$, the ultraspherical probability measure on $(-1, 1)$, where $\alpha > -1$ is some real parameter and C_α the normalizing constant. Then, Bochner's theorem [1954] (see also [Bochner 1956; 1979; Lasser 1983]) asserts that a sequence (λ_n) is a Markov sequence for this basis if and only if there exists a probability measure ν on $(-1, 1)$ such that

$$\lambda_n = \int_{-1}^{1} \frac{P_n^\alpha(x)}{P_n^\alpha(1)}\,\nu(dx).$$

Indeed, at least formally, Sarmanov and Bratoeva's theorem may be deduced from Bochner's one, through a limiting procedure known as the Poincaré ansatz, that is considering the scaling of ultraspherical probability on $(-a, a)$ and letting a go to infinity. But the method followed in [Sarmanov and Bratoeva 1967] is completely different.

(3) Jacobi polynomials. Gasper's theorem [1970; 1971; 1972] concerns the Beta measures $C_{a,b}(1-x)^\alpha(1+x)^\beta\,dx$ on $(-1, 1)$, where $\alpha, \beta > -1$. As before, the basis is chosen to be the sequence of orthogonal polynomials for this measure, which are the Jacobi polynomials $P_n^{\alpha,\beta}$. Then, provided $\beta \geq \alpha \geq \frac{1}{2}$, a sequence (λ_n) is a Markov sequence for this family if and only if there exists a probability measure μ on $(-1, 1)$ such that, for any $n \in \mathbb{N}$,

$$\lambda_n = \int_{-1}^{1} \frac{P_n^{\alpha,\beta}(x)}{P_n^{\alpha,\beta}(1)}\,\nu(dx).$$

This example looks very close to the previous one, but is considerably more difficult. In Section 3 we shall come back to this result, which is central in our study.

(4) Eigenvectors of Sturm–Liouville operators. Another remarkable result in this direction is the Achour–Trimèche theorem, which may be stated as follows. Consider the interval $[-1, 1]$, and a probability measure μ on it, with a smooth density ρ, that we suppose bounded for simplicity ($0 < c \leq \rho \leq C < \infty$). Then, consider the diffusion operator $L(f) = f'' + \frac{\rho'}{\rho} f'$, which is symmetric in $\mathcal{L}^2(\mu)$. We choose as $\mathcal{L}^2(\mu)$ basis (f_n) the one formed by the eigenvectors of L with Neumann boundary condition, such that $f_0 = \mathbf{1}$. Then, provided that $\log\rho$ is concave and symmetric, for any Markov sequence (λ_n) associated with this family (f_n), there exists some probability measure ν on $(-1, 1)$ such that $\lambda_n = \int_{-1}^{1} f_n(x)/f_n(1)\,\nu(dx)$. Although not stated as presented here in [Achour and Trimèche 1979] or in the book [Bloom and Heyer 1995], one may find this result in [Bakry and Huet 2008].

This situation, where the extremal values for the Markov sequence problem are given by the values $f_n(x)/f_n(x_0)$ for some point x_0, appears in a number of situations. This property is described in [Bakry and Huet 2008], where it is called the hypergroup property at the point x_0, and is developed in Section 2. In particular, it is proven in [Bakry and Huet 2008] that, in the finite set case, the point x_0 must be of minimal mass for the measure μ. The sole exception in the above list is that of Hermite polynomials, which is in fact a degenerate case where the point x_0 is $+\infty$.

Although Gasper's result looks like a simple generalization of Bochner's one, which itself is a consequence of Achour and Trimèche's one, and contains as a limiting case the Hermite polynomial sequence, the proof of it is absolutely not straightforward. It has been considerably simplified by Carlen, Geronimo and Loss [Carlen et al. 2011] by a technique which we shall expose below in full generality, and is also used in [Bakry and Zribi 2017] for the corresponding question for the family of orthogonal polynomials associated to the A_2 root system. We provide here a further simplified proof of the proof of [Carlen et al. 2011]. It relies on the construction of some symmetric diffusion operator having polynomial eigenvectors in some 3 dimensional space.

Moreover, we study this Markov sequence problem for the most direct extensions of the Beta measures, which are the above mentioned Dirichlet measures on the simplex.

The paper is organized as follows. In Section 2, we introduce the hypergroup property, which is closely related to the Markov sequence problem. This is a property of some bases of $\mathcal{L}^2(\mu)$ which provides automatically the answer to the Markov sequence problem. In Section 3, we concentrate on the case of Jacobi polynomials, for which the hypergroup property holds true, thanks to Gasper's theorem. In particular, we present the Carlen–Geronimo–Loss method, which provides in the geometric case a simplified proof of Gasper's theorem. With the help of some basic results on diffusion processes with polynomial eigenvectors, we then provide a simplified proof of Gasper's theorem in the nongeometric situation, following the scheme of Carlen–Geronimo–Loss, and which avoids any tedious computation. Finally, in Section 4, we introduce the Dirichlet measure on the simplex, and the natural generalization of the Jacobi polynomials. Although the situation is much more complicated, and despite the fact that the hypergroup property is much harder to investigate, we provide some bases having the hypergroup property, and, for the generalized Jacobi polynomials, we provide a description of Markov sequences, but only for Markov operators which strongly commute with the operator for which these generalized Jacobi polynomials are eigenvectors.

2. The hypergroup property: general description

Hypergroups appear in the literature as a natural extension of the notion of locally compact groups, where the convolution of two Dirac masses is a probability measure and no longer a Dirac mass. For example, this happens naturally when one looks at the convolution of class functions in a group.

The hypergroup property (denoted HGP) as described in [Bakry and Huet 2008] is just a simplification of this theory, basically valid in the previous situation in the compact setting, and appears as a key tool in many subjects like probability, statistics, statistical mechanics, coding theory and algorithms, reversible Markov chain, etc., see [Bakry and Huet 2008].

The hypergroup property concerns some properties of a unit $\mathcal{L}^2(\mu)$ orthonormal basis on a probability space $(E, \mathcal{E}, \mu)$, which carries the answer to the Markov sequence problem, as in the above described examples. Consider indeed a probability space $(E, \mathcal{E}, \mu)$, where E is a topological space, $\mathcal{E}$ is the Borel σ-field, μ a probability measure. On this space is given an orthonormal basis $\mathcal{F} = (f_0, f_1, \dots, f_n, \dots)$ for $\mathcal{L}^2(\mu)$, where we suppose that $f_0 = \mathbf{1}$. For everything to make sense, we shall require that the functions f_n are continuous.

Then, as mentioned earlier, the Markov sequence problem aims at the description of all sequences (λ_n), with $\lambda_0 = 0$ such that the (unique) operator such $K(f_n) = \lambda_n f_n$ is a Markov operator, that is $K(\mathbf{1}) = \mathbf{1}$ and $f \geq 0 \Longrightarrow K(f) \geq 0$.

We already mentioned that the set of all Markov sequences is a compact set (under the pointwise convergence), and convex. Therefore, the description of all Markov sequences is reduced to the description of its extremal points.

Under very generic properties of the probability space, any Markov operator K may be represented as

$$K(f)(x) = \int f(y) K(x, dy),$$

where $K(x, dy)$ is a Markov transition kernel, that is, for each x, $K(x, \cdot)$ is a probability measure on E, and, for any $A \in \mathcal{E}$, $x \mapsto K(x, A)$ is measurable. Moreover, as soon as $\sum_n \lambda_n^2 < \infty$, then the operator is Hilbert–Schmidt, and the kernel $K(x, dy)$ has a density with respect to the measure μ, that is $K(x, dy) = k(x, y)\, \mu(dy)$, where

$$k(x, y) = \sum_n \lambda_n f_n(x) f_n(y),$$

where it is easily seen that the series converges in $\mathcal{L}^2(E^2, \mu \otimes \mu)$.

Then, as soon as $\lambda_0 = 1$ and $\sum_n \lambda_n^2 < \infty$, the Markov property amounts to checking that the function $k(x, y) = \sum_n \lambda_n f_n(x) f_n(y)$ is nonnegative. However, since every function f_n oscillates as soon as $n \geq 1$, since it satisfies $\int_E f_n(x)\mu(dx) = 0$, it is in general not at all easy to obtain this positivity property from the previous representation.

In [Bakry and Huet 2008], the semigroup property is introduced as follows:

Definition 2.1. The family $\mathcal{F}$ has the hypergroup property at the point x_0 if for any $x \in E$, the sequence $\lambda_n(x) = f_n(x)/f_n(x_0)$ is a Markov sequence.

The main consequence of [Bakry and Huet 2008], is that, when the hypergroup property holds at some point x_0, then the sequences $f_n(x)/f_n(x_0)$ form the set of extremal sequences, and therefore, in this situation, for any Markov operator K, there exists a probability measure ν_K on E such that

$$\lambda_n = \int_E \frac{f_n(x)}{f_n(x_0)} \, \nu_K(dx).$$

In the examples described in Section 1, this is the case for ultraspherical polynomials, for the Jacobi polynomials, and, for the basis of Neumann eigenvectors of Sturm–Liouville operators, as soon as the reference measure is log-concave and symmetric.

The hypergroup property may be restated (in some more or less formal way however) into the following: for any $(x, y, z) \in E^3$,

$$k(x, y, z) = \sum_i \frac{f_i(x) f_i(y) f_i(z)}{f_i(x_0)} \geq 0. \tag{2-1}$$

But it may happen that this series is not convergent in $\mathcal{L}^2(E^3, \mu \otimes \mu \otimes \mu)$, and that the formal measure $k(x, y, z)\, \mu(dz)$ is not even absolutely continuous with respect to the measure μ. Anyhow, one may describe, at least formally, the convolution $\mu_1 * \mu_2$ of two probability measures μ_1 and μ_2 as the measure μ_3 with density with respect to μ equal to $\int k(x, y, z)\, d\mu_1(x)\, d\mu_2(y)$, and then the measure $k(x, y, z)\, d\mu(z)$ appears as the convolutions of the Dirac masses in x and y. Then, again formally, one has

$$\int f_n(x)(\mu_1 * \mu_2)(dx) = \frac{1}{f_n(x_0)} \int f_n \, d\mu_1 \int f_n \, d\mu_2.$$

We can extend this convolution to all pairs of measures by bilinearity and from measures to functions by identifying f to the measure $f\, d\mu$. With this in mind, the link with the usual theory of hypergroups is easily done.

Another aspect of the 3 variable kernel $k(x, y, z)$ is that it allows some product formulas. Likewise, if we introduce the probability kernel

$$K(x, y, dz) = \sum_n \frac{f_n(x) f_n(y) f_n(z)}{f_n(x_0)} \mu(dz) = k(x, y, z)\mu(dz),$$

one may see that for each n, the function f_n satisfies the product formula

$$\frac{f_n(x) f_n(y)}{f_n(x_0)} = \int_E f_n(z) K(x, y, dz).$$

In practice, for all this to make sense, it is useful to have at disposal a family $\rho_n(t)$ of Markov sequences such that, for any $t > 0$, $\sum_n \rho_n^2(t) < \infty$, and which converges pointwise to 1 as $t \to 0$. Then, one applies all the previous formal computations to the Markov sequences $\rho_n(t) f_n(x)/f_n(x_0)$, and let t go to 0. In general, and in particular in the models studied below, this sequence $\rho_n(t)$ is provided by some adapted heat kernel.

An interesting aspect of the hypergroup property is its stability under tensorization. Namely,

Proposition 2.2. *Assume that* $(E_1, \mathcal{E}_1, \mu_1)$ *and* $(E_2, \mathcal{E}_2, \mu_2)$ *are two probability spaces on which there exist two unit orthonormal bases* $(f_0 = \mathbf{1}, f_1, \dots, f_n, \dots)$ *and* $(g_0 = \mathbf{1}, g_1, \dots, g_p, \dots)$, *satisfying the hypergroup property at points* $x_0 \in E_1$ *and* $y_0 \in E_2$, *respectively. Then, on the product space* $(E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2, \mu_1 \otimes \mu_2)$, *the unit orthonormal basis* $(f_n(x)g_p(y), n, p \geq 0)$ *satisfies the hypergroup property at the point* (x_0, y_0).

Proof. This is straightforward. If $K_1^x(x_1, dx_2)$ is a Markov kernel on E_1 with eigenvectors f_n associated with the eigenvalue $f_n(x)/f_n(x_0)$, and $K_2^y(y_1, dy_2)$ is a Markov kernel on E_2 with eigenvectors g_p associated with the eigenvalue $g_p(y)/g_p(y_0)$, then the product kernel $K_1^x \otimes K_2^y$ has eigenvectors $f_n(x_1)g_p(y_1)$ with associated eigenvalue $(f_n(x)/f_n(x_0))g_p(y)/g_p(y_0)$. □

Let us finally mention that this HGP property may be seen as the dual of the GKS property, named after Griffiths and Kelly and Sherman [1968], who described the so called GKS inequality in statistical mechanics, and assert that the product of two elements of the $\mathcal{L}^2(\mu)$ basis may be expressed as a linear combination of the elements of the basis with nonnegative coefficients (see [Bakry and Echerbault 1996]). However, we do not dispose at the moment of any efficient scheme similar to the one of [Carlen et al. 2011] to obtain this last property.

3. Gasper's theorem

3A. ***Jacobi Polynomials.*** As mentioned earlier, Gasper's theorem is the statement that the hypergroup property is valid for the family of Jacobi polynomials. One may find many proofs of it in the literature (see for example [Bakry and Huet 2008; Carlen et al. 2011; Connett and Schwartz 1990; Gasper 1970; 1971; 1972; Flensted-Jensen and Koornwinder 1979; Koornwinder 1974; 1977]). It plays an important role in many areas, even for example in the proof of Bieberbach conjecture, see [de Branges 1985].

As described in the introduction (and with a small change in the notation that will be justified later), the Beta measure $\beta_{p,q}(dx)$ on $(-1, 1)$ is defined as

$$\beta_{p,q}(dx) = C_{p,q}(1-x)^{\frac{1}{2}p-1}(1+x)^{\frac{1}{2}q-1}\,dx,$$

where p and q are positive and $C_{p,q}$ is the normalizing constant which makes $\beta_{p,q}$ a probability measure. In what follows, we find it convenient to move everything on (0, 1) through $x \mapsto \frac{1}{2}(1+x)$, so that the Beta measure is now, with another normalizing constant,

$$\beta_{p,q}(dx) = C_{p,q} x^{p/2-1}(1-x)^{q/2-1}\, dx.$$

The Jacobi polynomials are then defined as the unique family of orthogonal polynomials associated with $\beta_{p,q}$ and positive dominant coefficient. We shall denote by $P_n^{p,q}(x)$ the Jacobi polynomial of degree n.

The Jacobi polynomials are also the eigenvectors of the Jacobi operator on (0, 1)

$$J_{p,q} = x(1-x)\frac{d^2}{dx^2} + \Big[\frac{q}{2} - \Big(\frac{q+p}{2}\Big)x\Big]\frac{d}{dx} \tag{3-1}$$

with eigenvalue equal to $\lambda_n = -n\big(n + \frac{1}{2}(p+q) - 1\big)$, see [Bakry et al. 2014] for example. The specificity of these polynomials is that they represent the unique family of orthogonal polynomials in dimension 1 (together with their limiting cases, the Laguerre and Hermite polynomials) that are simultaneously the eigenvectors of diffusion operators, that is elliptic second order differential operators with no zero order terms (see [Bakry and Mazet 2003]).

Through a simple change of variables, $P_n^{p,q}(\cos^2(t))$ are the eigenvectors of the Sturm–Liouville operator

$$\frac{d^2}{dt^2} + ((q-1)\cot(t) - (p-1)\tan(t))\frac{d}{dt} \quad \text{on } [0, \pi],$$

with Neumann boundary condition, which is symmetric with respect of the measure $\sin^{q-1}(t)\cos^{p-1}(t)\, dt$.

Under this form, one may check that the density of the measure is log-concave as soon as $p, q > 1$, and is symmetric under the change $x \mapsto \pi - x$ whenever $p = q$. So that, after a translation of $-\pi/2$, the latter case enters in the scope of Achour–Trimèche theorem. However, this is not the case when $p \neq q$.

For this family, we have

Theorem 3.1 (Gasper). *Let $p, q > 0$. Then, the hypergroup property holds for the family of Jacobi polynomials at the point $x_0 = 1$ if and only if $q \geq p \geq 1$.*

As already mentioned in the introduction, Gasper's theorem is indeed an extension of a previous theorem due to Bochner [1954], which deals with the symmetric case $p = q$, that is the case of ultraspherical (or Gegenbauer) polynomials. However, although the arguments for the symmetric case are quite easy to follow, the proofs of Gasper's theorem remained quite complicated, up to the paper [Carlen et al. 2011], which provided an illuminating argument that we shall briefly recall below in Section 3B.

Moreover, in the case $p = q$, letting p go to ∞, scaling x to $x/\sqrt{p}$, then the measure $\mu_{p,p}$ converges to the Gaussian measure, the Jacobi polynomials converge to Hermite ones, and $\frac{2}{p} J_{p,p}$ converges to the Hermite operator. With this in mind, Sarmanov and Bratoeva's result may be seen again as a limiting case of Bochner's theorem.

In the Jacobi polynomials case, it is worth observing that the set of parameters for which the hypergroup property is valid is closed. Later on, Lemma 3.2 will allow us to restrict to cases where the auxiliary measures used in the proof have smooth densities.

Lemma 3.2. *If the hypergroup property for the Jacobi polynomials* $(P_n^{p_k,q_k})$ *holds true for a sequence* (p_k, q_k) *converging to* (p, q), *then it holds for* (p, q).

Proof. The family of orthogonal polynomials $P_n^{p,q}$ is obviously continuous in the parameters (p, q). The hypergroup property may be stated as the fact that the operator $K(x)$ with eigenvalues $P_n^{p,q}(x)/P_n^{p,q}(1)$ is positivity preserving. But this may be checked on polynomials, since any positive function may be approximated by positive polynomials, and any positive polynomial is a sum of squared polynomials. Therefore, it is enough to check that for any polynomial Q with degree K, one has $K(Q^2) \geq 0$.

But this translates into

$$K(Q^2)(y) = \int Q^2(z) \sum_{r=1}^{2k} \frac{P_r^{p,q}(x)}{P_r^{p,q}(1)} P_r^{p,q}(y) P_r^{p,q}(z)\, \mu_{p,q}(dz),$$

since Q^2 is orthogonal to $P_r^{p,q}$ for any $r > 2K$.

The polynomial Q being fixed, this property is obviously satisfied in the limit (p, q) as soon as it holds for a sequence (p_k, q_k). □

An important feature of the Jacobi operator is that, when p and q are integers, there is a natural interpretation of it through the unit sphere in dimension $p+q-1$. Then, the Jacobi operator (3-1) may be seen as an image of the spherical Laplace operator.

Indeed, if one considers the unit sphere $\mathbb{S}^{p+q-1} \subset \mathbb{R}^{p+q}$, there is a diffusion operator on it, namely the spherical Laplace operator $\Delta^{\mathbb{S}^{p+q-1}}$, which commutes to rotations and is unique up to scaling. If one considers the function

$$\mathbb{R}^{p+q} \to (0, 1), \quad \boldsymbol{x} = (x_1, \dots x_{p+q}) \mapsto y = \sum_{i=1}^{p} x_i^2,$$

one has, for any smooth function $f : (-1, 1) \to \mathbb{R}$,

$$\Delta^{\mathbb{S}^{p+q-1}}(f(y)) = 4J_{p,q}(f)(y). \tag{3-2}$$

As such, the Jacobi operator $J_{p,q}$ appears, as announced above, as an image of the spherical Laplace operator, and this remark is the key tool in the Carlen–Geronimo–Loss method to obtain the hypergroup property in this geometric case.

3B. ***The Carlen–Geronimo–Loss method.*** The Carlen–Geronimo–Loss scheme appears to be a quite general method to obtain the hypergroup property in various contexts (see for example [Bakry and Zribi 2017]).

Recall that we consider some probability space $(E, \mathcal{E}, \mu)$ on which we have a $\mathcal{L}^2(\mu)$ orthonormal basis $\mathcal{F} = (f_0 = 1, f_1, \dots, f_n, \dots)$. As before, in order for everything to make sense, we shall assume that E is a topological space, that $\mathcal{E}$ is the Borel sigma-algebra, and that all the functions f_i are continuous.

We assume that we have some dense linear subspace $\mathcal{A}$ in $\mathcal{L}^2(\mu)$, containing all the functions (f_n) of the basis $\mathcal{F}$, and a symmetric operator $\mathrm{L} : \mathcal{A} \to \mathcal{A}$. The basis $\mathcal{F}$ is formed of eigenvectors of L, that is $\mathrm{L}(f_n) = \rho_n f_n$, for some real sequence (ρ_n). In our example, $\mathcal{A}$ will be the space of polynomials.

We assume that there is an auxiliary topological space $(E_1, \mathcal{E}_1, \mu_1)$, endowed also with a dense subspace $\mathcal{A}_1 \subset \mathcal{L}^2(\mu_1)$, and another symmetric operator $\mathrm{L}_1 : \mathcal{A}_1 \to \mathcal{A}_1$. Moreover, there exists a continuous map $\pi : E_1 \to E$, and another continuous map $\phi : E_1 \to E_1$, with properties described in Theorem 3.3. We assume that the image of μ_1 under π is μ. For a function $f : E \to \mathbb{R}$, we denote by $\pi(f) : E_1 \to \mathbb{R}$ the function $\pi(f)(y) = f(\pi(y))$. Similarly, for a function $g : E_1 \to \mathbb{R}$, we denote $\phi(g)(y) = g(\phi(y))$. We also assume that $f \in \mathcal{A} \Longrightarrow \pi(f) \in \mathcal{A}_1$ and similarly $g \in \mathcal{A}_1 \Longrightarrow \phi(g) \in \mathcal{A}_1$.

Theorem 3.3. *Assume the following*:

(1) *For each n, the eigenspace of* L *associated with the eigenvalue ρ_n is one dimensional.*

(2) $\pi \mathrm{L} = \mathrm{L}_1 \pi$.

(3) $\phi \mathrm{L}_1 = \mathrm{L}_1 \phi$.

(4) *For two points x_0 and x in E, if Y is a random variable with values in E_1 with law μ_1, then the conditional law of $\pi(\phi(Y))$ given that $\pi(Y) = x_0$ is a Dirac mass at x.*

Then, the sequence $f_n(x)/f_n(x_0)$ is a Markov sequence for the basis (f_n). (If $f_n(x_0) = 0$, then the conclusion is that we also have $f_n(x) = 0$).

Remark 3.4. Point (4) requires a bit of explanation. Indeed, we assume that the probability measure μ_1 has a regular decomposition $\mu_1(dy) = \nu_x(dy)\, \mu(dx)$, where the measure $\nu_x(dy)$ has support the set $\pi(y) = x$, which means that, for any bounded measurable function $h : E_1 \to \mathbb{R}$,

$$\int_{E_1} h(y)\mu_1(dy) = \int_E \left(\int_{\{\mu(y)=x\}} h(\pi(y))\, \nu_x(dy) \right) \mu(dx),$$

and that the map $x \mapsto \nu_x$ is continuous. This allows us to make sense of ν_x for any $x \in E$ (since in general, those measures ν_x are just defined μ-almost everywhere). Then the hypothesis (4) asserts that the image measure through $\pi\phi$ of the measure ν_{x_0} is a Dirac mass δ_x.

Proof. Although the proof of this theorem is more or less implicit in [Carlen et al. 2011], and fully developed in [Bakry and Zribi 2017], we provide a sketch of it for completeness.

We denote $\langle f, g \rangle$ the scalar product in $\mathcal{L}^2(\mu)$ and $\langle f, g \rangle_1$ the scalar product in $\mathcal{L}^2(\mu_1)$.

We consider the correlation operator K defined on bounded Borel functions $f : E \to \mathbb{R}$ as

$$K(f)(x) = \mathbb{E}(\phi(\pi(f))(Y)/\pi(Y) = x),$$

where Y is a random variable with law μ_1. It is clearly a Markov operator. We shall see that $K(f_n) = \mu_n f_n$, where $\mu_n = f_n(x)/f_n(x_0)$.

The main remark is that the hypotheses imply that K commutes with L. Indeed, the operator K is entirely determined by the following property, which is just a rephrasing of the definition of a conditional expectation:

$$\text{for all } f, g \in \mathcal{A}, \quad \langle K(f), g \rangle = \langle \phi\pi(f), \pi g \rangle_1. \tag{3-3}$$

Indeed, using the measure decomposition introduced in Remark 3.4, one may introduces the operator π^*, such that

$$\pi^*(h)(x) = \mathbb{E}\big(h(Y)/\pi(Y) = x\big) = \int_{\{\pi(y)=x\}} h(y)\, \nu_x(dy),$$

the operator K may be written as $K = \pi^*\phi\pi$.

Then, for any pair $(f, g) \in \mathcal{A}$, we have

$$\begin{aligned}\langle \mathrm{L}K(f), g \rangle = \langle K(f), \mathrm{L}g \rangle &= \langle \phi\pi(f), \pi \mathrm{L}(g) \rangle_1 = \langle \phi\pi(f), \mathrm{L}_1\pi(g) \rangle_1 \\ &= \langle \mathrm{L}_1\phi\pi(f), \pi(g) \rangle_1 = \langle \phi \mathrm{L}_1\pi(f), \pi(g) \rangle_1 = \langle \phi\pi \mathrm{L}(f), \pi(g) \rangle_1 \\ &= \langle K\mathrm{L}(f), g \rangle,\end{aligned}$$

which proves the commutation property between K and L.

Therefore, if f_n is an eigenvector of L, with eigenvalue ρ_n, then $K(f_n)$ is again an eigenvector of L with the same eigenvalue. Since the eigenspaces of L are one dimensional, $K(f_n) = \mu_n f_n$ for some sequence (μ_n), which is therefore a Markov sequence.

Looking at the values at the point x_0, we get

$$f_n(x) = \mu_n f_n(x_0),$$

from which the conclusion follows. □

Corollary 3.5. *Under the hypothesis of Theorem 3.3, if, for any $x \in E$, there exists a map $\phi_x : E_1 \to E_1$ satisfying point (3) and such that the conditional law of $\pi\phi_x(Y)$ given $\pi(Y) = x_0$ is a Dirac mass at x, then the hypergroup property holds at x_0.*

Proof. It is an immediate consequence of Theorem 3.3. Indeed, if such happens, $f_n(x_0) \neq 0$, since otherwise one would get $f_n = 0$ everywhere, which may not be true for an element of a basis. □

With this in mind, Gasper's theorem in the geometric case follows easily. Of course, in this context, the auxiliary space E_1 is $\mathbb{S}^{p+q-1}$, L_1 is the spherical Laplace operator, and the map π is the map $\boldsymbol{x} \mapsto y = \sum_{i=1}^{p} x_i^2$ described in Section 3A.

The maps ϕ are as follows: since $p \le q$, for some point $\boldsymbol{x} = (x_1, \dots, x_{p+q}) \in \mathbb{R}^{p+q}$, we extract $\boldsymbol{x}_1 = (x_1, \dots, x_p)$, $\boldsymbol{x}_2 = (x_{p+1}, \dots, x_{2p})$ and $\boldsymbol{x}_3 = (x_{2p+1}, \dots, x_{p+q})$ (the last one may be empty). Then, for $\theta \in [0, 2\pi]$, $\phi_\theta(\boldsymbol{x}) = (\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{x}_3)$, where

$$\boldsymbol{y}_1 = \cos(\theta)\boldsymbol{x}_1 + \sin(\theta)\boldsymbol{x}_2, \qquad \boldsymbol{y}_2 = -\sin(\theta)\boldsymbol{x}_1 + \cos(\theta)\boldsymbol{x}_2. \tag{3-4}$$

Then, $\boldsymbol{x} \mapsto \phi_\theta(\boldsymbol{x})$ is a rotation in $\mathbb{R}^{p+q}$, and as such commutes with the spherical Laplace operator.

Then, it remains to observe that whenever $\pi(\boldsymbol{x}) = 1$, then $\boldsymbol{x}_2 = \boldsymbol{x}_3 = 0$, so that $\pi(\phi_\theta(\boldsymbol{x})) = \cos^2(\theta)$. Then, the conditional law property is satisfied (with $x = \cos^2(\theta)$ and $x_0 = 1$), and therefore we obtain the hypergroup property in this case.

To extend this proof to the general case, we shall require a few concepts from the general diffusion theory.

3C. *Symmetric diffusions and orthogonal polynomials.* Most of the material presented here is borrowed from [Bakry et al. 2014] for the general situation, and from [Bakry et al. 2013] for the particular case where orthogonal polynomials come into play.

A diffusion operator in an open set $\Omega \subset \mathbb{R}^d$ is a second order semielliptic differential operator with no zero order terms. As such, it may be written in a given system of coordinates as

$$\mathrm{L}(f)(x) = \sum_{ij} g^{ij}(x)\partial^2_{ij} f + \sum_i b^i(x)\partial_i f, \tag{3-5}$$

where, here and in what follows, the coefficients $g^{ij}(x)$ and $b^i(x)$ are assumed to be smooth (indeed, for our purpose, they always will be polynomials in the variables

(x_i) which are the coordinates of the point x). The matrix $g = (g^{ij})$ is always symmetric and, in this paper, positive definite in Ω (that is our operator L is indeed elliptic).

We are interested in the case where these operators are symmetric with respect to some measure $\mu(dx)$, which has a smooth positive density $\rho(x)$ with respect to the Lebesgue measure. That is, for any pair (f, g) of smooth functions $\Omega \to \mathbb{R}$, compactly supported in Ω, we require that

$$\int_\Omega \mathrm{L}(f)(x)g(x)\rho(x)\,dx = \int_\Omega f(x)\mathrm{L}(g)(x)\rho(x)\,dx. \tag{3-6}$$

For this to happen, a necessary and sufficient condition is that

$$\text{for all } i = 1, \dots, d, \quad b^i(x) = \sum_j \partial_j g^{ij}(x) + \sum_j g^{ij}(x)\partial_j \log(\rho)(x), \tag{3-7}$$

since, by integration by parts

$$\int_\Omega \mathrm{L}(f)(x)g(x)\rho(x)\,dx = -\int_\Omega g^{ij}\partial_i f\partial_j g\rho\,dx + \int_\Omega g\partial_i f[b_i - r_i]\rho\,dx, \tag{3-8}$$

where $r_i(x) = \sum_j \partial_j g^{ij}(x) + \sum_j g^{ij}(x)\partial_j \log(\rho)(x)$.

Such a measure is often called a reversible measure. It is unique in general, up to a multiplicative constant.

We then see that the coefficients b^i are entirely determined by the second order terms g^{ij} and by the density $\rho(x)$.

Moreover, let us introduce the carré du champ

$$\Gamma(f, g) = \tfrac{1}{2}(\mathrm{L}(fg) - f\mathrm{L}(g) - g\mathrm{L}(f)).$$

We have

$$\Gamma(f, g) = \sum_{ij} g^{ij}(x)\partial_i f\partial_j g,$$

and this bilinear operator characterizes the second order terms (g^{ij}) of the operator L. We have $g^{ij}(x) = \Gamma(x_i, x_j)$, and, when the operator L is symmetric, for any pair of smooth compactly supported functions (f, g), we have

$$\int_\Omega \mathrm{L}(f)g\rho(x)\,dx = -\int_\Omega \Gamma(f, g)\rho(x)\,dx. \tag{3-9}$$

This is the integration by parts formula.

Moreover, the operator Γ allows us to describe the so-called "change of variable formula," which is a way to describe in a general setting second order differential operators with no zero order terms. More precisely, when $f_1, \dots, f_q$ are smooth functions $\Omega \to \mathbb{R}$, then, for any smooth function $\Phi : \mathbb{R}^q \to \mathbb{R}$, one has

$$\mathrm{L}(\Phi(f_1, \dots, f_q)) = \sum_{ij} \Gamma(f_i, f_j)\partial^2_{ij}\Phi(f_1, \dots, f_q) + \sum_i \mathrm{L}(f_i)\partial_i\Phi(f_1, \dots, f_q). \quad (3\text{-}10)$$

It is also worth observing that Γ is a bilinear operator which is first order in each of its variables, which translates into

$$\Gamma\big(\Phi_1(f_1, \dots, f_q), \Phi_2(f_1, \dots, f_q)\big) = \sum_{ij} \Gamma(f_i, f_j)\partial_i\Phi_1(f_1, \dots, f_q)\partial_j\Phi_2(f_1, \dots, f_q). \quad (3\text{-}11)$$

From this, one sees that in order to describe locally a symmetric diffusion operator, it is enough to describe in some coordinate basis $(x_1, \dots, x_d)$ the quantities $\Gamma(x_i, x_j)$ and either ρ, or the functions $\mathrm{L}(x_i) = b^i(x)$ provided they satisfy Equation (3-7) for some ρ.

It is not necessary to restrict diffusion operators to open sets in $\mathbb{R}^d$. One may as well consider operators defined on smooth manifolds (and quite often compact manifolds such as spheres), or closed sets with boundaries. Then, the operator may be described through Equation (3-5) in any local system of coordinates, and formula (3-10) allows one to change coordinates to obtain a coherent system. However, when considering such operators on manifolds with boundaries, one has in general to describe to which functions one may apply the integration by parts formula (3-9). This is done in general through the prescription of the so called "boundary conditions" (such as Neumann or Dirichlet). In what follows, we shall require the possibility to apply this formula to any polynomial (and even any restriction to Ω of any smooth function defined in a neighborhood of Ω), and this requires some extra conditions concerning the behavior of the matrix (g^{ij}) at the boundary. Indeed, the fundamental property for that (assuming that the boundary is piecewise smooth) is that, for any regular point x_0 of the boundary, the normal unit vector belongs to the kernel of the matrix (g^{ij}): in this situation, the extra term in the integration by parts formula (3-9), coming from the boundary term in Stokes formula, vanishes (see [Bakry et al. 2013], for example). It is easily seen that this condition is also sufficient.

This is what is hidden indeed in the boundary equation (3-12) below, which is the translation of this property when the boundary is described through some algebraic equation (see [Bakry et al. 2013]).

A key feature is the notion of image of a diffusion operator L_1 on some set E_1. This is the basic tool to construct new diffusion operators L on a set E and maps $\pi : E_1 \to E$ such that $\pi\mathrm{L} = \mathrm{L}_1\pi$, as in Theorem 3.3.

Let E_1 be some space on which we have a diffusion operator L_1 and d applications $x_1, \dots, x_d : E_1 \to \mathbb{R}$. Consider the map $\pi : E_1 \to E \subset \mathbb{R}^d$, $\pi(y) = (x_1, \dots, x_d)$. Then, assume that for any i, $\mathrm{L}_1(x_i) = B^i(\pi)$, and for any $i, j = 1, \dots, d$, one has $\Gamma(x_i, x_j) = G^{ij}(\pi)$ for some functions B^i and G^{ij}, $E \to \mathbb{R}$. We say in this situation that we have a closed system. Then, the operator

$$\mathrm{L} = \sum_{ij} G^{ij} \partial^2_{ij} + \sum_i B^i \partial_i$$

defined on E is such that $\mathrm{L}_1 \pi = \pi \mathrm{L}$ (this is just the translation of Equation (3-10)). Moreover, L is a diffusion operator which is symmetric as soon as L_1 is, with reversible measure which is the image through π of the reversible measure μ_1 of L_1. In this situation, we say that L is the image of L_1 through π, or that L_1 projects onto L through π. An example of this is the case of the spherical Laplace operator $\Delta^{\mathbb{S}^{p+q-1}}$ which projects (up to the factor 4) onto the Jacobi operator through the map $\boldsymbol{y} = (x_1, \dots, x_{p+q}) \mapsto x = \sum_i x_i^2$ as described in Equation (3-2), so that the Beta measure $\beta_{p,q}$ is the image measure of the uniform measure on the sphere through this projection.

As mentioned above, the symmetry identity (3-6) is not enough for our purpose. We shall require it to be valid for pair of polynomials, when the symmetry property is only stated for compactly supported functions. In what follows, we shall be concerned with symmetric diffusion operators which may be diagonalized in a basis of orthogonal polynomials. That is, for every $n \geq 0$, there exists a basis of the space of polynomials in d variables with degree less than n, and which are at the same time eigenvectors for L. When this happens, we say that (Ω, Γ, ρ) is a polynomial model, and Ω is a polynomial domain.

When the set Ω is bounded with a piecewise $\mathcal{C}^1$ boundary, this requires the boundary of Ω to be an algebraic set and also some extra algebraic condition relating the boundary and the coefficients g^{ij}, called the boundary equation, see [Bakry et al. 2013].

More precisely, the boundary $\partial\Omega$ is included in an algebraic set $\{P_1 \cdots P_k = 0\}$, where P_i are real polynomials, which are irreducible in the complex field. Here, we assume that $P_1 \cdots P_k = 0$ is the reduced equation of the boundary, that is:

(1) For each regular point $x \in \partial\Omega$, there exists a neighborhood $\mathcal{V}(x)$ which contains x and a unique i such that $\mathcal{V}(x) \cap \partial\Omega = \mathcal{V}(x) \cap \{P_i = 0\}$.

(2) For $i = 1, \dots, k$, there exist a regular point $x \in \partial\Omega$ such that $P_i(x) = 0$.

Then, following [Bakry et al. 2013], bounded polynomial models are characterized by the following:

(1) For any $i, j = 1, \dots, d$, $g^{ij}(x)$ is a polynomial with degree at most 2.

(2) For any $i = 1, \dots, d$, $b^i(x)$ is a polynomial with degree at most 1.

(3) For any $i = 1, \dots, d$ and any $q = 1, \dots, k$, there exists a polynomial $L_{i,q}$ with degree at most 1 such that

$$\sum_j g^{ij} \partial_j \log P_q = L_{i,q}. \tag{3-12}$$

(We call this last Equation (3-12) the boundary equation).

As a consequence of the previous, each polynomial P_q is a factor of the polynomial $\det(g^{ij})$ (of degree at most $2d$). Moreover, every polynomial P_q has a constant sign on the open set Ω and we may decide that they are all positive on it. Beyond this, provided (g^{ij}) satisfies the boundary equation (3-12), for any choice of parameters $a_1, \dots, a_k$ such that $P_1^{a_1} \cdots P_k^{a_k}$ is integrable on Ω, the density measure

$$\rho(x) = C_{a_1 \cdots a_k} P_1^{a_1} \cdots P_k^{a_k}, \tag{3-13}$$

where $C_{a_1 \cdots a_k}$ is the normalizing constant, is such that (Ω, Γ, ρ) is a polynomial model.

Indeed, for the integration by parts formula to be true for a pair of polynomial functions, and thanks to the boundary equation (3-12), one may allow the parameters a_i in Equation (3-13) to be negative, as soon as $a_i > -1$, which is anyway a necessary condition for the measure $\rho(x)\,dx$ to be finite on Ω.

Sometimes one needs to extend those polynomial models using weighted degrees, that is deciding that the degree of a monomial $x_1^{p_1} \cdots x_d^{p_d}$ is $\sum_i n_i p_i$, where $n_1, \dots, n_d$ are some positive integers. All the picture remains valid, except that g^{ij} must have degree $n_i + n_j$ and b^i must have degree n_i. We call the sequence $(n_1, \dots, n_d)$ the weights of the polynomial model.

It is worth observing that whenever (Ω, Γ, ρ) is a polynomial model, and when we have a closed system $(y_1, \dots, y_q)$ where the functions y_i are polynomials, then the image model is again a polynomial model. But the degree may change. For example, if one starts from a polynomial model with the usual degree (that is $n_i = 1$ for any i), and if the degree of y_i is n_i, then we get a polynomial degree with weights $n_1, \dots, n_d$. Of course, one may always reduce to the case where the degrees have no common factor.

3D. ***A proof of Gasper's theorem in the general case.*** In this section, we extend the proof of Gasper's theorem provided in Section 3B which was valid only in the geometric case (that is when p and q are integers) to the general case. For this, we need to construct a model $(E_1, \mathrm{L}_1, \mu_1)$, with the adapted functions $\pi : E_1 \to E$ and $\phi_\theta : E_1 \to E_1$ with the properties required in Theorem 3.3. The key observation is that, in the geometric picture, one just requires the knowledge of $\|\boldsymbol{x}_1\|^2$, $\|\boldsymbol{x}_2\|^2$ and the scalar product $\boldsymbol{x}_1 \cdot \boldsymbol{x}_2$ to describe the action of the rotations ϕ_θ on $\|\boldsymbol{x}_1\|^2$.

For this, we first observe the action of the spherical Laplace operator on those variables. Following [Bakry et al. 2014], the spherical Laplace operator in dimension d may be described through its action on the coordinates, that is considering the restrictions of the various coordinates $x_1, \dots, x_{d+1}$ to the spheres as functions $\mathbb{S}^d \to \mathbb{R}$. Then, we get

$$\Gamma^{\mathbb{S}}(x_i, x_j) = \delta_{ij} - x_i x_j, \quad \Delta^{\mathbb{S}^d}(x_i) = -d x_i. \tag{3-14}$$

It is worth observing that $\Gamma^{\mathbb{S}}$ does not depend on the dimension d. The image through $\Delta^{\mathbb{S}^d}$ of a polynomial in the variables x_i with degree less than n is again a polynomial in the variables x_i with degree less than n. From this, it is easily seen that whenever we have a closed system made of polynomials, then the image of $\Delta^{\mathbb{S}^d}$ through this system is a polynomial model.

Now fix d large enough and, for $p \le [d/2]$, consider the 3 variables $\mathbb{S}^d \to \mathbb{R}$ defined as

$$X = \sum_{i=1}^{p} x_i^2, \quad Y = \sum_{i=p+1}^{2p} x_i^2, \quad U = \sum_{i=1}^{p} x_i x_{i+p}.$$

With the help of the change of variables formulas (3-10) and (3-11), we get

$$\begin{gathered}
\Gamma^{\mathbb{S}}(X, X) = 4X(1-X), \quad \Gamma^{\mathbb{S}}(Y, Y) = 4Y(1-Y), \\
\Gamma^{\mathbb{S}}(U, U) = X + Y - 4U^2, \\
\Gamma^{\mathbb{S}}(X, Y) = -4XY, \quad \Gamma^{\mathbb{S}}(X, U) = -4XU + 2U, \\
\Gamma^{\mathbb{S}}(Y, U) = -4YU + 2U, \\
\Delta^{\mathbb{S}^d}(X) = -2(d+1)X + 2p, \quad \Delta^{\mathbb{S}^d}(Y) = -2(d+1)Y + 2p, \\
\Delta^{\mathbb{S}^d}(U) = -2(d+1)U,
\end{gathered} \tag{3-15}$$

which shows that the triple (X, Y, U) forms a closed system for the spherical Laplace operator. (We omit the parameter d in $\Gamma^{\mathbb{S}}$ since it does not depend on the dimension d.)

It is worth observing that X itself is a closed subsystem of this closed system (and the image of the spherical Laplace operator is nothing other than the Jacobi operator, up to some affine transformation on the variable and scaling). Such is $\{X, Y\}$, but neither $\{U\}$ or $\{X, U\}$, for example.

Let us consider the image of the sphere under $x \mapsto (X, Y, U)$. It is a polynomial domain in $\mathbb{R}^3$ with boundary equation $\{(1 - X - Y)(XY - U^2) = 0\}$.

The image of $\mathbb{S}^d$ through the map (X, Y, U) is therefore a polynomial model, with domain E_1 being the bounded set which is the connected component in $\mathbb{R}^3$ of the complement of the set $\{(1-X-Y)(XY-U^2)=0\}$ which contains for example the point $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}\right)$. Observe that the boundary Equation (3-12) is automatically satisfied for this model. Indeed, since the spherical operator may be diagonalized in a basis of orthogonal polynomials in the variable (x_i) (the eigenvectors are the restrictions to the sphere of the harmonic homogeneous polynomials in dimension $d+1$), and one sees that the eigenvectors of this operator are nothing other than those polynomial eigenvectors which depend only on the variables X, Y, U.

The Γ operator is given in these coordinates by the matrix

$$G = (G^{ij}) := \begin{pmatrix} 4X(1-X) & -4XY & -4XU+2U \\ -4XY & 4Y(1-Y) & -4YU+2U \\ -4XU+2U & -4YU+2U & X+Y-4U^2 \end{pmatrix}, \tag{3-16}$$

and one may check (but, as already mentioned, this is automatic) that the two polynomials $1-X-Y$ and $XY-U^2$ satisfy the boundary equation (3-12). The reversible measure has density (up to a normalizing constant) $(1-X-Y)^a(XY-U^2)^b$, where the coefficients a and b may be computed through Equation (3-7). Then, we get

$$a = \frac{d-1}{2} - p, \quad b = \frac{p-3}{2},$$

Now, this diffusion operator again projects, up to a factor 4, on the Jacobi operator $J_{p,q}$ through the map $(X, Y, U) \mapsto X$, whenever $d = p+q-1$.

We may now consider this polynomial model (E_1, Γ) with a new measure with density $\rho(X, Y, U) = C(1-X-Y)^a(XY-U^2)^b$, where now a and b are real numbers.

It is easily seen that this measure is integrable on the domain E_1 as soon as $a > -1$ and $b > -1$. Setting $a = (q-p)/2 - 1$ and $b = (p-3)/2$, this requires $q > p > 1$, where now p and q are no longer integers but again real numbers.

As described in Section 3C, this provides a diffusion operator according to formula (3-5). The second order terms are provided by the matrix (3-16), and the first order coefficients may be computed explicitly through formula (3-7), with density $\rho = (1-X-Y)^a(XY-U^2)^b$ where, for given $q > p > 1$, we have $a = (q-p)/2-1$ and $b = (p-3)/2$.

More explicitly, one gets for the first order terms, exactly as in (3-15),

$$\begin{gathered} \mathrm{L}_1(X) = -2(p+q)X + 2p, \quad \mathrm{L}_1(Y) = -2(p+q)Y + 2p, \\ \mathrm{L}_1(U) = -2(p+q)U. \end{gathered} \tag{3-17}$$

The symmetry of the operators on a pair of polynomials is then insured by the fact that the first order coefficients b^i are chosen according to formula (3-7), and the fact that the boundary equation (3-12) is satisfied for the two factors $P_1(X, Y, U) = 1-X-Y$ and $P_2(X, Y, U) = XY-U^2$.

We get in such a way a model (E_1, Γ_1, μ_1) which projects through the map $\pi : (X, Y, U) \mapsto X$ on $4J_{pq}$, where J_{pq} is the Jacobi operator defined in Equation (3-1) (it is obvious: the variable X alone forms a closed system).

To complete the picture, it remains to describe the operators $\Phi_\theta : E_1 \to E_1$ which commute with L_1. From the geometric picture, when p and q are integers, one may describe the action of the rotations Φ_θ defined in Equation (3-4). We get $\Phi_\theta(X) = A(X, Y, U)$, where A is the linear operator with matrix

$$\begin{pmatrix} \cos^2(\theta) & \sin^2(\theta) & 2\sin(\theta)\cos(\theta) \\ \sin^2(\theta) & \cos^2(\theta) & -2\sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) & \cos(\theta)\sin(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{pmatrix}. \tag{3-18}$$

To check that it commutes with L_1, and following Section 3C, it is enough to check its action on the variables X, Y, U for L_1 and Γ. For example, writing $\Phi_\theta(X, Y, U) = (X_\theta, Y_\theta, U_\theta)$, and $\Gamma(X, Y) = G(X, Y) := -4XY$, one has to check that $\Gamma(X_\theta, Y_\theta) = -4X_\theta Y_\theta$ (there are 6 such formulas to check), and also, with $\mathrm{L}_1(X) = -2(p+q)X + 2p$, that $\mathrm{L}_1(X_\theta)) = -2(p+q)X_\theta + 2p$ (3 formulas to check).

The property for Γ comes from the geometric picture (the action of Γ on (X, Y, U) does not depend on the parameters p and q). As for the action of L_1, it may be checked directly, from $X_\theta = \cos^2(\theta)X + \sin^2(\theta)Y + 2\sin(\theta)\cos(\theta)U$, using (3-17).

As before, the point x_0 is 1. Whenever $\pi(X, Y, U) = 1$, then $(X, Y, U) = (1, 0, 0)$ and $\pi\Phi_\theta(1, 0, 0) = \cos^2(\theta)$.

This completes the proof of Gasper's theorem in the case $q > p > 1$. The general case $q \geq p \geq 1$ comes from Lemma 3.2.

Remark 3.6. If one considers the kernel $K_\theta(f)(\xi) = \mathrm{E}\big(f(\pi(R_\theta Z))/\pi(Z) = \xi\big)$, the previous representation allows one to compute it explicitly through some integral expression. However, the result is quite complicated, but one may check that the kernel $K_\theta(\xi, dy)$ has support $[0, (\sqrt{\xi}\cos\theta + \sqrt{1-\xi}\sin\theta)^2]$.

4. Dirichlet laws and diffusion processes on the simplex

4A. ***Dirichlet laws, and a first basis with the HGP property.*** The d-dimensional simplex $\mathbb{D}_d$ is the set of points $(x_1, \dots, x_d) \in \mathbb{R}^d$ such that, for all $i = 1, \dots, d$, $x_i \geq 0$ and such that $\sum_{i=1}^d x_i \leq 1$. In what follows, it will be convenient to set $x_{d+1} = 1 - \sum_{i=1}^d x_i$, so that $x_{d+1} \geq 0$ and $\sum_1^{d+1} x_i = 1$.

The Dirichlet laws $\mu_{d,\boldsymbol{p}}$ depend on a multi-index real parameter $\boldsymbol{p} = \{p_1, \dots, p_{d+1}\}$, where $p_i > 0$, $i = 1, \dots, d+1$, are probability measures on $\mathbb{D}_d$ with densities with respect to the Lebesgue measure $dx_1 \cdots dx_d$ of the form

$$C_{d,\boldsymbol{p}} x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d} x_{d+1}^{a_{d+1}},$$

where for $i = 1, \dots, d+1$, $a_i = \frac{p_i}{2} - 1$. The normalizing constant

$$C_{d,\boldsymbol{p}} = \frac{\Gamma\left(\sum_{i=1}^{d+1} a_i + d + 1\right)}{\prod_{i=1}^{d+1} \Gamma(a_i + 1)},$$

where Γ is the Euler function, which ensures that $\mu_{d,\boldsymbol{p}}$ is a probability. The choice of the parameters p_i instead of $a_i = \frac{p_i}{2} - 1$, similar to the choice made for Beta measures, comes from geometric considerations which will be described below.

Dirichlet measures appear as extensions the Beta measures on the interval. It turns out that the simplex is a polynomial domain as described in Section 3C, so that the Dirichlet laws are the natural measures associated to it, the boundary of the domain having reduced equation $x_1 \cdots x_d(1 - x_1 - \cdots - x_d) = 0$.

When the parameters p_i are integers, this Dirichlet law is the image measure of the uniform measure on the unit sphere in $\mathbb{R}^n$, with $n = \sum_1^{d+1} p_i$. Indeed, consider some partition of $\{1, \dots, n\}$ in sets $I_1, \dots, I_{d+1}$ with respective size $p_1, \dots, p_{d+1}$. Then, for $(y_1, \dots, y_n) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, consider the variables $x_i = \sum_{j \in I_i} y_j^2$. Then $(x_1, \dots, x_d) \in \mathbb{D}_d$, and the image measure of the uniform measure on the sphere through the map $y \mapsto (x_1, \dots, x_d)$ is $\mu_{d,\boldsymbol{p}}$. This will be obvious later on when we shall identify some diffusion operator on $\mathbb{D}_d$ with reversible measure $\mu_{d,\boldsymbol{p}}$ as the image of the spherical Laplace operator, as are the Beta measures on $[0, 1]$.

It is worth observing that the change of variables $x_i \mapsto 1 - x_{d+1}$ allows one to exchange the parameters p_i and p_{d+1}, so that one may order the parameters p_i, $i = 1, \dots, d+1$, in whichever order desired.

The change of variables $x_i = y_i(1 - x_1)$, for $i = 2, \dots, d$ transforms the measure $\mu_{d,\boldsymbol{p}}$ into a product measure $\beta_{p_1, n-p_1}(dx_1) \otimes \mu_{d-1,\boldsymbol{q}}(dy_2 \cdots dy_d)$, where $n = \sum_1^{d+1} p_i$, and $\boldsymbol{q} = \{p_2, \dots, p_{d+1}\}$. Iterating the procedure, one may transform the Dirichlet measure into a product of Beta measures on $[0, 1]^d$:

$$\beta_{p_1, n-p_1} \otimes \beta_{p_2, n-p_1-p_2} \otimes \cdots \otimes \beta_{p_d, n-n_1-\cdots-p_d}.$$

We may now choose a basis for $\mathcal{L}^2(\mathbb{D}_d, \mu_{d,\boldsymbol{p}})$ made of products of Jacobi polynomials associated to each of the factors (to be more precise, the image of these products under the inverse change of variables which maps $[0, 1]^d$ to $\mathbb{D}_d$). Now, provided that, for $i = 1, \dots, d+1$, $p_i \geq 1$, one may apply Gasper's theorem and the tensorization procedure of Proposition 2.2, and therefore get the hypergroup property for this basis.

Observe that this procedure depends on the choice of the ordering in the parameters $p_1, \dots, p_{d+1}$, so that one may construct in this way many different bases. But these bases are not the most natural direct extensions of the Jacobi polynomial bases on the simplex. In particular, in the coordinates $(x_1, \dots, x_d)$, they do not appear as polynomials, but as rational functions. On the other hand, on the simplex and for the Dirichlet measures, there are many choices of polynomial bases which are the natural extensions of the Jacobi polynomials, as we shall see in the next paragraph.

4B. ***Diffusion operators on the simplex having polynomial eigenvectors.*** To describe the diffusion processes which may be diagonalized in a system of orthogonal polynomials on the simplex, we have just to describe their carré du champ Γ, since the measure is given. It is a special feature of the simplex that there are many such Γ structures which answer the question, beyond the mere scaling factor, and this situation is very peculiar (in the dimension 2 classification of [Bakry et al. 2013], only the simplex, the circle, and a particular case of the double parabola have this property).

The various Γ operators on the simplex such that $(\mathbb{D}_d, \Gamma, \mu_{d,\boldsymbol{p}})$ are a polynomial model have been described for example in [Li 2019]. They depend on a symmetric parameter matrix $\boldsymbol{A}$ with entries A_{rs} as follows

$$g^{rs} := \Gamma_{\boldsymbol{A}}(x_r, x_s) = -A_{rs}x_r x_s + \delta_{rs}x_r \sum_{k=1}^{d+1} A_{rk}x_k, \quad 1 \le r \le s \le d, \tag{4-1}$$

where $A_{rs} = A_{sr},\ 1 \le r \le s \le d+1$ are nonnegative real parameters. The operator is elliptic on the simplex as soon as, for every $r \ne s$, $A_{rs} \ne 0$. One should check that the value of A_{ii} plays no role in the definition of $\Gamma_{\boldsymbol{A}}$, and we shall set $A_{ii} = 0$.

For this operator, and for the Dirichlet measure $\mu_{d,\boldsymbol{p}}$, one has

$$\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}(x_i) = \frac{1}{2}\sum_{k=1}^{d+1} A_{ik}(x_k p_i - x_i p_k).$$

One may check the validity of the boundary Equation (3-12), that is the fact that $\sum_{i=1}^{d} g^{ij}\partial_j \log P_p$ is an affine function for every boundary polynomial $P_p = x_1, \dots, x_{d+1}$.

Indeed, for $k = 1, \dots, d+1$, one has

$$\sum_{j=1}^{d} g^{ij}\partial_j \log x_k = -A_{ik}x_i + \sum_{q=1}^{d+1} A_{iq}x_q.$$

It is worth it to write $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ as

$$\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}} = \sum_{i<j} A_{ij}\mathrm{L}_{ij,\boldsymbol{p}},$$

where $\mathrm{L}_{ij,\boldsymbol{p}}$ has a carré du champ Γ_{ij} with

$$\Gamma_{ij}(x_r, x_s) = x_i x_j[\delta_{rs}(\delta_{ri} + \delta_{rj}) - (\delta_{ri}\delta_{sj} + \delta_{rj}\delta_{si})] \tag{4-2}$$

and

$$\mathrm{L}_{ij,\boldsymbol{p}}(x_r) = \tfrac{1}{2}(\delta_{ri} - \delta_{rj})(x_j p_i - x_i p_j). \tag{4-3}$$

In the case where all the A_{pq} are set to 1 (let us denote this matrix **1**), and when the parameters p_i are integers, there is a natural interpretation for this operator coming from the spherical Laplace operator in dimension $n = \sum_{i=1}^{d+1} p_i$, that is for the sphere imbedded in $\mathbb{R}^n$.

Indeed, let n be an integer and, as in the previous Section 4A, consider the $n-1$ dimensional spherical Laplace operator acting on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, defined through the equation $\sum_1^n y_i^2 = 1$. Let us look at a partition of the index set $\{1, \dots, n\}$ into $d+1$ disjoint sets $I_1, \dots, I_{d+1}$ with respective sizes $p_1, \dots, p_{d+1}$, and as before the variables $x_j = \sum_{i \in I_j} y_i^2$. As already observed, the map $y \in \mathbb{S}^{n-1} \mapsto (x_1, \dots, x_d)$ maps the sphere onto the simplex $\mathbb{D}_d$.

Moreover, following Equation (3-14), we see that

$$\Gamma^{\mathbb{S}}(x_i, x_j) = 4(\delta_{ij} x_i - x_i x_j), \quad \Delta^{\mathbb{S}^{n-1}}(x_i) = 2(p_i - n x_i). \tag{4-4}$$

The variables $(x_1, \dots, x_d)$ form a closed system, and we see that those formulas are the one obtained for $4\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$. This first shows that the Dirichlet measure $\mu_{d,\boldsymbol{p}}$ is the image of the uniform measure on the sphere through this map, as mentioned earlier. One may therefore address the question of the hypergroup property for the family of orthogonal polynomials which are the eigenvectors of this operator, following the same path. Unfortunately, it turns out that the eigenspaces for $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ are not one dimensional.

Indeed, consider a polynomial eigenvector of degree k, and look at the action of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ on its highest degree term $x_{\boldsymbol{k}} := x_1^{k_1} \cdots x_d^{k_d}$, where $k = \sum_1^d k_i$. The highest degree term of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}(x_{\boldsymbol{k}})$ is

$$-k\Big(k + \frac{n-2}{2}\Big) x_{\boldsymbol{k}},$$

so that the corresponding eigenvalue is $\nu_{\boldsymbol{k}} = -k\big(k + \frac{n-2}{2}\big)$, which depends only on $k = \sum_1^d k_i$. The corresponding eigenspace has then dimension $\binom{k+d-1}{k}$. However, for this operator, one may follow the scheme of [Carlen et al. 2011] and construct a new space E_1 (the sphere in the geometric case), with a symmetric diffusion operator L_1 on it, together with maps $\pi : E_1 \to \mathbb{D}_d$ and $\phi : E_1 \to E_1$ with the properties that $\pi \mathrm{L} = \mathrm{L}_1 \pi$, $\phi \mathrm{L}_1 = \mathrm{L}_1 \phi$, together with the conditional law property at the point $(1, 0, \dots, 0)$. But the fundamental property that the eigenspaces of L are one dimensional is missing, and the analysis of Markov sequences is therefore much more delicate.

Indeed, following the scheme of the proof of Gasper's theorem, one may first concentrate on the geometric case. To understand the difficulty, let us also concentrate on the case $d = 2$. In this situation, one has 3 integer parameters $p_1 \le p_2 \le p_3$, and, setting $n = p_1 + p_2 + p_3$, we look at the sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Then, one considers three subsets I_1, I_2, I_3 of $\{1, \dots, n\}$, with respective sizes p_1, p_2, p_3 and three vectors $\boldsymbol{x}_1 = (y_i, i \in I_1)$, $z_2 = (y_i, i \in I_2)$ and $z_3 = (y_i, i \in I_3)$. Moreover, we split I_2 and I_3 into disjoint sets $I_2 = J_2 \cup K_2$, $I_3 = J_3 \cup K_3$, with $|J_2| = |J_3| = p_1$. Then, we consider the vectors $\boldsymbol{x}_2 = (y_i, i \in J_2)$, $\boldsymbol{y}_3 = (y_i, i \in J_3)$, $\boldsymbol{y}_2 = (y_i, i \in K_2)$ and $\boldsymbol{y}_3 = (y_i, i \in K_3)$.

We consider now the variables $x_i = \|\boldsymbol{x}_i\|^2$, $i = 1, 2, 3$, and $y_i = \|z_i\|^2$, $i = 2, 3$. Moreover, we look at the variables $u_{ij} = \boldsymbol{x}_i \cdot \boldsymbol{x}_j$, $1 \le i < j \le 3$. For simplicity, we stick to the case where $p_1 < p_2 \le p_3$, and, observing that $y_3 = 1 - x_1 - x_2 - x_3 - y_2$, we are left to the 7 variables

$$(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23}).$$

It happens that these 7 variables form a closed system for the spherical Laplace operator, and we obtain some operator L_7 on some bounded polynomial domain $\Omega_7 \subset \mathbb{R}^7$. Moreover, the operator $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ is the image of L_7 under the map

$$\pi_1 : \Omega_7 \to \mathbb{D}_2, \quad (x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23}) \mapsto (x_1, x_2 + y_2).$$

Let us denote by π_2 the projection from the sphere onto Ω_7, and $\pi : \mathbb{S}^{n-1} \to \mathbb{D}_2$, $\pi = \pi_1 \pi_2$.

One then may consider the full $O(3)$ group acting in a horizontal way on the triple of vectors $(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$. For example the plane rotations R_θ^{ij}, $1 \le i < j \le 3$:

$$R_\theta^{ij}(\boldsymbol{x}_i, \boldsymbol{x}_j) = (\cos\theta \boldsymbol{x}_i + \sin\theta \boldsymbol{x}_j, -\sin\theta \boldsymbol{x}_i + \cos\theta \boldsymbol{x}_j). \tag{4-5}$$

For any of these horizontal rotations R, there exists some point x_R in the simplex such that whenever $\pi(Y) = (1, 0)$, then $\pi R(Y) = x_R$ (that is $x_R = \pi R(1, 0, \ldots, 0)$). One may see that for any point $x \in \mathbb{D}_d$, there exists such horizontal rotation $R \in \mathrm{SO}(3)$ such that $x_R = x$.

One may immediately see the action of these rotations on the variables

$$(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23}),$$

as we did in dimension 1.

In order to apply the one dimensional scheme, one may expect to find a common orthonormal base in the eigenspaces of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ in which the correlation operators $K_R(f)(x) = \mathbb{E}(\pi R f(Y)/\pi(Y) = x)$, where Y is uniformly distributed on the sphere, are jointly diagonalizable. (Observe that $R \mapsto K_R$ is not a representation of $O(3)$.) We shall see that it is impossible. Indeed, if such were the case, they would commute with each other. But this is not the case, as shown next in Proposition 4.1. For this, we just concentrate on the plane rotations R_θ^{ij} (4-5) and their conditional expectations $K_\theta^{ij}(f)(x) = \mathbb{E}(\pi R_\theta^{ij} f(Y)/\pi(Y) = x)$.

Proposition 4.1. *The operators K_θ^{12} and K_ϕ^{13} do not commute with each other.*

Proof. The operators K_θ^{ij} are not easy to describe. We may look at the easier operators $S_{ij} = \partial_\theta K^{ij}_{|\theta=0}$ But we shall see that those operators vanish identically. We may therefore compute $R_{ij} = \partial_\theta^2 K^{ij}_{|\theta=0}$.

To compute these operators S_{ij} and K_{ij} on the simplex, for the pairs $(1,2)$, $(1,3)$, $(2,3)$, we observe that for two bounded polynomial functions $f(x,y)$ and $g(x,y)$ on $\mathbb{D}_2$, up to a constant 2, we have

$$\langle S_{12}(f), g\rangle = 2\int_{\mathbb{S}^{n-1}} u_{12}(\partial_1 f - \partial_2 f)(\pi(\boldsymbol{y}))g(\pi\boldsymbol{y})\,d\boldsymbol{y},$$

where $\pi(x_1,x_2,x_3,y_2,u_{12},u_{13},u_{23}) = (x_1, x_2+y_2)$. Thus

$$S_{12}(f) = 2s_{12}(x)(\partial_i f - \partial_j f), \quad \text{where } s_{12}(x) = \mathbb{E}\big(u_{12}(\boldsymbol{y})/\pi(\boldsymbol{y}) = (x_1, x_2+y_2)\big),$$

which is 0 by symmetry, and

$$\langle K_{12}f, g\rangle = \int_{\mathbb{S}^{n-1}} \big(2(x_2-x_1)(\partial_1 f - \partial_2 f) + 4u_{12}^2(\partial_1-\partial_2)^2 f\big)\pi(\boldsymbol{y})g(\pi\boldsymbol{y})\,d\boldsymbol{y}.$$

Thus

$$K_{12}(f) = 2k_{12}(\partial_1-\partial_2)f + 4t_{12}(\partial_1-\partial_2)^2 f,$$

where

$$k_{12}(x,y) = \mathbb{E}\big(x_1 - x_2/(x_1, x_2+y_2) = (x,y)\big),$$
$$t_{12}(x,y) = \mathbb{E}\big(u_{12}^2(Y)/\pi(Y) = (x_1, x_2+y_2) = (x,y)\big).$$

For the operators S_{13} and K_{13}, we may perform a similar computation, and obtain a similar computation:

$$K_{13}(f) = 2k_{13}(\partial_1 f - \partial_2 f) + 4t_{13}(\partial_1-\partial_2)^2 f,$$

with

$$k_{13}(x,y) = \mathbb{E}\big(x_1 - x_3/(x_1, x_2+y_2) = (x,y)\big),$$
$$t_{13}(x,y) = \mathbb{E}\big(u_{13}^2(Y)/(x_1, x_2+y_2) = (x,y)\big),$$

and for K_{23}, we obtain

$$K_{23}(f) = 2k_{23}\partial_2 f + 4t_{23}\partial_2^2 f,$$

with

$$k_{23}(x,y) = \mathbb{E}\big(x_2 - x_3/(x_1, x_2+y_2) = (x,y)\big),$$
$$t_{23}(x,y) = \mathbb{E}\big(u_{23}^2/(x_1, x_2+y_2) = (x,y)\big),$$

It remains to compute these conditional laws.

Following the computations of Section 3D, we may compute the law of the set of variables $(x_1,x_2,x_3,y_2,u_{12},u_{13},u_{23})$ under the uniform measure on the sphere through the action of the spherical Laplace operator $\Delta_{\mathbb{S}^{n-1}}$ on these variables. The Gamma operator acts on the variables as

$$\Gamma(x_p,x_q) = 4x_p(\delta_{pq} - x_q), \quad \Gamma(x_i, y_2) = -4x_i y_2, \quad \Gamma(y_2,y_2) = 4y_2(1-y_2),$$

while

$$\Gamma(y_2, u_{ij}) = -4y_2u_{ij}, \quad \Gamma(x_i, u_{lk}) = -4x_iu_{kl} + 2\delta_{il}u_{ik} + 2\delta_{ik}u_{il},$$
$$\Gamma(u_{ij}, u_{kl}) = -4u_{ij}u_{kl} + \delta_{ik}u_{jl} + \delta_{il}u_{jk} + \delta_{jk}u_{il} + \delta_{jl}u_{ik}.$$

where, in the last formulas, u_{ii} stands for x_i. Moreover, with $n = p_1 + p_2 + p_3$, we have

$$\Delta_{\mathbb{S}^{n-1}}(x_i) = -2nx_i + 2p_1, \quad \Delta_{\mathbb{S}^{n-1}}(y_2) = -2ny_i + 2(p_2 - p_1),$$
$$\Delta_{\mathbb{S}^{n-1}}(u_{ij}) = -2nu_{ij}.$$

Then, the image measure of the sphere is the reversible measure for this operator, that we compute through Equation (3-7). Up to some normalizing constant, we may compute the density through formula (3-7). In order to compute this density with respect to the product measure $dx_1\, dx_2\, dx_3\, dy_1\, du_{12}\, du_{13}\, du_{23}$, we introduce

$$F_1 = x_1x_2x_3 + 2u_{12}u_{13}u_{23} - x_1u_{23}^2 - x_2u_{13}^2 - x_3u_{12}^2,$$
$$F_2 = 1 - x_1 - x_2 - x_3 - y_2$$

Observe that F_1 is the determinant of the Gram matrix associated with the vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$.

Rewriting the variables $(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23})$ as $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ in this order, (to have a more compact presentation of what follows), we get, with $G_{ij} = \frac{1}{4}\Gamma(x_i, x_j)$,

$$\sum_j \partial_j G_{ij} = 2 - 8x_i, \qquad i = 1, 2, 3,$$
$$\sum_j \partial_j G_{4j} = 1 - 8x_4,$$
$$\sum_j \partial_j G_{ij} = -8x_i, \qquad i = 5, 6, 7,$$
$$\sum_j G_{ij}\partial_j \log F_1 = 1 - 3x_i, \qquad i = 1, 2, 3,$$
$$\sum_j G_{ij}\partial_j \log F_1 = -3x_i, \qquad i = 4, 5, 6, 7,$$
$$\sum_i G_{ij}\partial_j \log F_2 = -x_i, \qquad i = 1, \dots, 7,$$
$$\sum_i G_{ij}\partial_j \log x_4 = -x_i + \delta_{i4}, \quad i = 1, \dots, 7.$$

In the end, through formula (3-7), we are able to compute the density of the measure, which is, up to some normalizing constant

$$\rho = F_1^{\alpha} F_2^{\beta} y_2^{\gamma},$$

with

$$\alpha = \frac{p_1}{2} - 2, \quad \beta = \frac{n - p_2}{2} - p_1 - 1, \quad \gamma = \frac{p_2 - p_1}{2} - 1.$$

Observe that the equation $F_1F_2y_2 = 0$ is indeed the reduced equation of the set Ω_7.

To compute the conditional law, it is worthwhile to change variables in order to transform the measure $\rho(x)\,dx$ into a product measure. For this, we set

$$u_{ij} = \sqrt{x_i x_j}\sigma_{ij}, \quad y_2 = z - x_2, x_2 = uz, \quad x_3 = v(1 - x_1 - z),$$

so that the measure becomes a product measure, of the form

$$\mu(dx_1, dz)\beta_1(du)\beta_2(dv)\gamma(d\sigma_{12}, d\sigma_{23}, d\sigma_{13}),$$

where μ is, as expected, the Dirichlet law in dimension 2, $\mu_{2,(p_1,p_2)}$.

With this in mind, it is easy to check that we have

$$\begin{aligned} k_{12} &= 2(x - a_1 y), & t_{12} &= b_1 xy,\\ k_{13} &= 2(x - a_2(1 - x - y)), & t_{13} &= b_2 x(1 - x - y),\\ k_{23} &= 2(a_3 y - a_4(1 - x - y)), & t_{23} &= b_3 y(1 - x - y), \end{aligned}$$

for some constants a_i, b_j that we are not going to identify directly, but where we may assert that $b_i > 0$, for example. (Indeed, knowing that those differential operators K_{ij} must commute with $\mathrm{L}_{2,\boldsymbol{p}}$ allows one to compute them up to some constant.)

Now, if one wants to see that these operators do not commute, we may look at $\bigl[\frac{1}{b_1}K_{12}, \frac{1}{b_3}K_{13}\bigr]$, for example. This is a third order operator whose leading term is $2(1 - x - y)(x - y)(\partial_1 - \partial_2)^3$, which clearly does not vanish. □

Remark 4.2. For any horizontal rotation R, the associated kernel

$$K_R(f)(x) = \mathbb{E}\bigl(f(\pi(R\boldsymbol{x}))/\pi(\boldsymbol{x}) = x\bigr)$$

leaves invariant all the eigenspaces of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$. But the question of their action on this space remains completely open. In particular, one may ask if any Markov operator which commutes with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ is a mixture of such conditional expectations of rotations K_R.

We now concentrate on the operators $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$. We shall show that in the generic case (that is for some dense set for the parameters A_{ij} and p_i), their eigenspaces are one dimensional.

There is still a geometric interpretation for them, in the geometric case $p_i \in \mathbb{N}$, as we shall see below. And this geometric interpretation allows us to use the same space E_1 with the projection $\pi : E_1 \to \mathbb{D}_d$, which may be extended to the general case $p_i \notin \mathbb{N}$ as we did in Section 3D. But the problem now is that the horizontal rotations do not commute with the lift of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ to the geometric model. Therefore, we may not apply the Carlen–Geronimo–Loss scheme to them.

The geometric interpretation of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ that we present now is inspired from [Li 2019], where a similar interpretation is carried out for the matrix simplex. In $\mathbb{R}^n$, consider the infinitesimal rotations in the coordinate plane (i, j), $D_{ij} = y_i\partial_j - y_j\partial_i$.

Consider now as before a partition $\{I_1, \dots, I_{d+1}\}$ of the set $\{1, \dots, n\}$, where $|I_i| = p_i$. For $i \neq j$ consider the following second order diffusion operator on the sphere $\mathbb{S}^{n-1}$:

$$\Delta_{ij} = \sum_{p \in I_i,\, q \in I_j} D^2_{pq}.$$

The action of Δ_{ij}, and its associated carré du champ Γ_{ij} on the variables $x_r = \sum_{p \in I_r} y_p^2$ and $x_s = \sum_{p \in I_s} y_p^2$ is as follows.

Proposition 4.3.
$$\Gamma_{ij}(x_r, x_s) = 4[\delta_{rs} x_i x_j (\delta_{ri} + \delta_{rj}) - (\delta_{ri}\delta_{sj} + \delta_{rj}\delta_{si}) x_r x_s],$$
$$\Delta_{ij}(x_r) = 2(\delta_{ir} - \delta_{jr})(x_j p_i - x_i p_j).$$

Proof. We start by the computation of this action on the variables $y_p, y_q : \mathbb{S}^{n-1} \to \mathbb{R}$.

$$\begin{aligned} \Delta_{i,j}(y_p) &= -y_p(\mathbf{1}_{p \in I_j} p_i + \mathbf{1}_{p \in I_i} p_j), \\ \Gamma_{i,j}(y_p, y_q) &= \delta_{pq}(\mathbf{1}_{p \in I_i} x_j + \mathbf{1}_{p \in I_j} x_i) - y_p y_q (\mathbf{1}_{p \in I_i}\mathbf{1}_{q \in I_j} + \mathbf{1}_{p \in I_j}\mathbf{1}_{q \in I_i}), \end{aligned} \tag{4-6}$$

where $\mathbf{1}_{p \in A}$ stands for $1_A(p)$, the indicator function of the set A. From this, using the change of variable formula (3-11), we get

$$\Gamma_{i,j}(x_p, x_q) = 4 x_i x_j [\delta_{pq}(\delta_{pi} + \delta_{pj}) - (\delta_{pi}\delta_{qj} + \delta_{pj}\delta_{qi})].$$

In the same way, we obtain the formula for $\Delta_{ij}(x_r)$ using formula (3-10). □

As a corollary, and comparing with formulae (4-2) and (4-3), we get:

Corollary 4.4. *The operator* $4\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ *is the image of the operator* $\sum_{i<j} A_{ij}\Delta_{ij}$ *through the map* $y \mapsto (x_1, \dots, x_d)$ *which maps* $\mathbb{S}^{n-1}$ *onto* $\mathbb{D}_d$, *where* $n = \sum_{i=1}^{d+1} p_i$.

Remark 4.5. In view of Equation (4-4), it is worth observing that the spherical Laplace operator may be written as $\sum_{i \le j} \Delta_{ij}$. Therefore, comparing with Corollary 4.4, we see that what is missing is the operator $\sum_i \Delta_{ii}$, where

$$\Delta_{ii} = \sum_{p<q,\, p \in I_i,\, q \in I_i} D^2_{pq}.$$

But it is easily seen that the action of Δ_{ii} on the variables x_p vanishes: $\Gamma_{ii}(x_p, x_q) = \Delta_{ii}(x_p) = 0$.

It is also worth observing that one may split some subset I_i into two subsets I_{i_1} and I_{i_2}. More precisely, suppose that we have a partition $\{I_1, \dots, I_{d+1})$ of $\{1, \dots, n\}$ and that we split say I_1 into two disjoint sets $I_{1a} \cup I_{1b}$. Then we may consider a new operator on $\mathbb{D}_{d+1}$ $\mathrm{L}_{\boldsymbol{A}_1,\boldsymbol{a}_1}$, for some matrix $\boldsymbol{A}_1$ and some vector $\boldsymbol{a}_1$. Then, provided that for any $j > 1$, $A_{1a,j} = A_{1b,j} = A_{1j}$, the image of $\mathrm{L}_{\boldsymbol{A}_1,\boldsymbol{a}_1}$ on $\mathbb{D}_d$ under the map $(x_{1a}, x_{1b}, x_2, \dots, x_d) \mapsto (x_{1a} + x_{1b}, x_2, \dots, x_d)$ is $\mathrm{L}_{\boldsymbol{A},\boldsymbol{a}}$, where $\boldsymbol{a} = (a_{1a} + a_{1b}, a_2, \dots, a_d)$.

Of course, the same reasoning applies for any parameter i instead of 1.

For the sake of completeness, we show below that the eigenspaces of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ have dimension 1 in the generic case.

Proposition 4.6. *For a dense set for the parameters A_{ij} and p_i, the eigenspaces of the operator $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ are one dimensional.*

Proof. Since the space $\mathcal{P}_n$ of polynomials with total degree n is preserved by $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$, one may concentrate on its action on $\mathcal{P}_n$. To understand the eigenvalues of this restriction, which do not come from the restriction to $\mathcal{H}_{n-1}$, it is enough to look at the restriction of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ to homogeneous polynomial of degree n, and consider for such polynomial P, the degree-n homogeneous part of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}(P)$.

Then, the eigenvalues of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ are the eigenvalues of this linear operator, represented by some matrix $M_{n,\boldsymbol{A},\boldsymbol{p}}$ in the natural basis of these homogeneous polynomials $e_{k_1,\dots,k_d} = \{x_1^{k_1}\cdots x_d^{k_d},\ \sum_i k_i = n\}$. We shall see that for each n, there exists a dense subset Ω_n of parameters (even with a complementary with Lebesgue measure 0) such that the eigenvalues of $M_{n\boldsymbol{A},\boldsymbol{p}}$ are all distinct for this parameters. Then, on $\bigcap_n \Omega_n$, which is dense by Baire's theorem, all the eigenvalues of $\mathrm{L}_{\boldsymbol{A},\boldsymbol{p}}$ are distinct.

To assert that the eigenvalues of $M_{n,\boldsymbol{A},\boldsymbol{p}}$ are distinct, it is enough to check that the characteristic polynomial has distinct roots, or in other words that its discriminant does not vanish. But the discriminant is a polynomial in the coefficients of the characteristic polynomial, which themselves are polynomials in the entries of the matrix, which themselves are polynomials in the variables A_{ij} and p_i. Therefore, there exists some polynomial Q in the variables A_{ij}, p_i, depending on the degree n, such that, if $Q \neq 0$, all the eigenvalues of $M_{n,\boldsymbol{A},\boldsymbol{p}}$ are distinct.

It remains to show that Q does not vanish identically, that is that there exists some choice of the parameters A_{ij} and p_i for which the eigenvalues are distinct.

Let us choose the matrix A_{ij} such that $A_{ij} = A_{i(d+1)}$ for $j > i$. Then, if we order the elements of the basis $\{e_{k_1,\dots,k_d},\ \sum_1^d k_i = n\}$ according to their inverse lexicographic order of $(k_1,\dots,k_{d-1})$ (so that $(n,\dots,0,0)$ is the lowest term), then one may check that all the elements of $M_{n,\boldsymbol{A},\boldsymbol{p}}$ which are above the diagonal vanish. Then, the eigenvalues of $M_{n,\boldsymbol{A},\boldsymbol{p}}$ are the diagonal elements. On the diagonal, the coefficient corresponding to $e_{k_1,\dots,k_d}$ is

$$-\sum_{i\neq j} k_i k_j A_{ij} - \sum_i k_i(k_i-1)A_{i,d+1} + \tfrac{1}{2}\sum_i k_i\Bigl(A_{i,d+1}p_i - \sum_{k=1}^{d+1} A_{ik}p_k\Bigr).$$

With the choice that we made, for $i \neq j$, $A_{i,j} = a_{\min(i,j)}$ for some sequence a_i, $i = 1,\dots,d$. Then, it is not hard to see that there exists a choice for the sequences a_i, $i = 1,\dots,d$ and p_i, $i = 1,\dots,d+1$ for which all these terms are different, for all the sequences of integers $(k_1,\dots,k_d)$ such that $\sum_1^d k_i = n$. □

4C. *Representations of Markov sequences.* In what follows, we restrict ourselves to the case where all the coefficients A_{ij}, $i \neq j$ are set to 1. Since the eigenspaces E_n are not one dimensional, we also restrict our attention to the study of Markov operators which have constant eigenvalues on the space E_n. That is, instead of looking at Markov operators which commute with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$, we look at Markov operators which are functions of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$. We say that such a Markov operator strongly commutes with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$

Observe first that, for any choice of a strict subset $I \subset \{1, \dots, d+1\}$, the projection $\pi : \mathbb{D}_d \to [0, 1]$, $\pi(x) = \sum_{i \in I} x_i$ maps the Dirichlet law $\mu_{d,\boldsymbol{p}}$ on the Beta measure $\beta_{q,n-q}$, where $q = \sum_{i \in I} p_i$ and $n = \sum_1^{d+1} p_i$. (We recall that by convention, $x_{d+1} = 1 - \sum_1^d x_i$). As usual, for any function $f : [0, 1] \to \mathbb{R}$, we denote $\pi f : \mathbb{D}_d \to \mathbb{R}$ the function $\pi f(y) = f(\pi(y))$. Then, with the Jacobi operator $J_{q,n-q} = \mathrm{L}_{1,q,n-q}$, one has

$$\pi J_{q,n-q} = \mathrm{L}_{\mathbf{1},\boldsymbol{p}}\pi,$$

as may be checked directly and easily, computing $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}\pi(x)$ and $\Gamma_{\mathbf{1},\boldsymbol{p}}(\pi(x), \pi(x))$.

Now, the eigenvalues of $J_{p,n-q}$ and $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ are the same (namely $-k\big(k + \frac{n-2}{2}\big)$, acting on polynomials of degree k). In other words, any eigenspace for $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$ contains an eigenvector of the form $P(\pi(x))$.

Now, let K be a Markov operator on $\mathbb{D}_d$ which strongly commutes with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$, with eigenvalue μ_k on E_k. For a Jacobi polynomial P_k, $K(\pi P_k) = \mu_k \pi P_k$. Therefore, for any polynomial P defined on $[0, 1]$, one sees that $K(\pi P) = \pi Q$, for some uniquely defined polynomial Q. This allows one to define a new Markov operator K_1 on $[0, 1]$ through its action on polynomials as $K(\pi P) = \pi K_1(P)$. It is clear that K_1 commutes with $J_{q,n-q}$.

If μ_k is the eigenvalue of K on the eigenspace E_k of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$, then, for any Jacobi polynomial with degree k, $K_1(P) = \mu_k P$. One may now apply Gasper's theorem and we have obtained:

Proposition 4.7. *Let K be a Markov operator on $\mathbb{D}_d$ which strongly commutes with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$, and let (μ_k) be the sequence of its eigenvalues on the eigenspace E_k of $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$. Choose $I \subset \{1, \dots, d+1\}$, $I \neq \{1, \dots, d+1\}$, and let $q = \sum_{i \in I} p_i$, and $n = \sum_1^{d+1} p_i$. Then, there exists a probability measure ν on $[0, 1]$ such that, for any $k \in \mathbb{N}$*

$$\mu_k = \int_0^1 \frac{P_k^{q,n-q}(x)}{P_n^{q,n-q}(x_0)}\, \nu(dx),$$

where $P_k^{p,n-q}$ is the Jacobi polynomial with degree k for the measure $\beta_{q,n-q}$, and $x_0 = 0$ or $x_0 = 1$ according to $p \leq n - q$ or not.

Remarks 4.8. (1) Contrary to the one dimensional case, it is not true in general that for any probability measure ν on $[0, 1]$, the associated sequence μ_n may be the sequence of eigenvalues of a Markov operator. Indeed, if such were the case, then for some value of $q = \sum_{i\in I} p_i$, one would have that the sequence $P_k^{q,n-q}(x)/P_k^{q,n-q}(1)$ is such a strong Markov sequence. Choosing another value of q, say q_1, associated to another subset I_1 of $\{1, \dots, d+1\}$, one would therefore get some measure $\nu(x, dy)$ on [0, 1] such that

$$\frac{P_k^{q,n-q}(x)}{P_k^{q,n-q}(1)} = \int_0^1 \frac{P_k^{q_1,n-q_1}(y)}{P_k^{q_1,n-q_1}(1)}\, \nu(x, dy).$$

Repeating the operation with $P_k^{q_1,n-q_1}(y)/P_k^{q_1,n-q_1}(1)$ and another measure $\nu_1(y, dz)$, one would get

$$\frac{P_k^{q,n-q}(x)}{P_k^{q,n-q}(1)} = \int \frac{P_k^{q,n-q}(z)}{P_k^{q,n-q}(1)}\, \nu_2(x, dz),$$

where $\nu_2(x, dz) = \int \nu(x, dy)\, \nu_1(y, dz)$.

Then, $\nu_2(x, dz)$ is the Dirac mass in x. As a consequence, for $\nu(x, dy)$ almost every y, $\nu_1(y, dz)$ is a Dirac mass in some point $h(y)$, and moreover this point is constant. This is clearly wrong, since the Jacobi polynomials for different values of the parameters do not coincide.

(2) In view of Theorem 3.3, in order to obtain the true hypergroup representation, that is the set of extremal points for Markov which strongly commutes with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$, it would be enough to produce the associated space E_1 and the corresponding operations π and ϕ such that the associated correlation operator $K(f) = \mathbb{E}\big(f(\phi\pi f(Y))/\pi(Y) = x\big)$ strongly commutes with $\mathrm{L}_{\mathbf{1},\boldsymbol{p}}$. Even in the geometric case, when the parameters p_i are integers, it does not seem to be the case for the horizontal rotations described in (3-4).

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The cohomology of C_2-equivariant $\mathcal{A}(1)$ and the homotopy of ko_{C_2}

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We compute the cohomology of the subalgebra $\mathcal{A}^{C_2}(1)$ of the C_2-equivariant Steenrod algebra $\mathcal{A}^{C_2}$. This serves as the input to the C_2-equivariant Adams spectral sequence converging to the completed $\mathrm{RO}(C_2)$-graded homotopy groups of an equivariant spectrum ko_{C_2}. Our approach is to use simpler $\mathbb{C}$-motivic and $\mathbb{R}$-motivic calculations as stepping stones.

1. Introduction

The $\mathrm{RO}(G)$-graded homotopy groups are among the most fundamental invariants of the stable G-equivariant homotopy category. This article is a first step towards systematic application of the equivariant Adams spectral sequence to calculate these groups.

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Araki and Iriye [1982; Iriye 1982] computed much information about the C_2-equivariant stable homotopy groups using EHP-style techniques in the spirit of Toda [1962]. Our approach is entirely independent from theirs.

We work only with the two-element group C_2 because it is the most elementary nontrivial case. In order to compute C_2-equivariant stable homotopy groups of the C_2-equivariant sphere spectrum using the Adams spectral sequence, one needs to work with the full C_2-equivariant Steenrod algebra $\mathcal{A}^{C_2}$ for the constant Mackey functor $\underline{\mathbb{F}}_2$. As the C_2-equivariant Eilenberg–Mac Lane spectrum for $\underline{\mathbb{F}}_2$ is flat [Hu and Kriz 2001, Corollary 6.45] the E_2-term of the Adams spectral sequence is given by the cohomology of the equivariant Steenrod algebra. In this article, we tackle a computationally simpler situation by working over the subalgebra $\mathcal{A}^{C_2}(1)$. This means that we are computing the C_2-equivariant stable homotopy groups not of the sphere but of the C_2-equivariant analogue of connective real K-theory ko. We will explicitly construct this C_2-equivariant spectrum ko_{C_2} in Section 10.

Our calculational program is carried out for $\mathcal{A}^{C_2}(1)$ in this article as a warmup for the full Steenrod algebra $\mathcal{A}$ to be studied in future work. Roughly speaking, $\mathcal{A}$ contains Steenrod squaring operations Sq^i with the expected properties, and $\mathcal{A}^{C_2}(1)$ is the subalgebra generated by Sq^1 and Sq^2. A key point is that our program works just as well in theory for $\mathcal{A}^{C_2}$ as for $\mathcal{A}^{C_2}(1)$, except that the details are even more complicated. It remains to be seen how far this can be carried out in practice.

Our strategy is to build up to the complexity of the C_2-equivariant situation by first studying the $\mathbb{C}$-motivic and $\mathbb{R}$-motivic situations. The relevant stable homotopy categories are related by functors as in the diagram

$$\begin{array}{ccc} \mathrm{Ho}(\mathbf{Sp}^{\mathbb{R}}) & \xrightarrow{-\otimes_{\mathbb{R}}\mathbb{C}} & \mathrm{Ho}(\mathbf{Sp}^{\mathbb{C}}) \\ \big\downarrow{\scriptstyle \mathbf{Re}} & & \big\downarrow{\scriptstyle \mathbf{Re}} \\ \mathrm{Ho}(\mathbf{Sp}^{C_2}) & \xrightarrow{\iota^*} & \mathrm{Ho}(\mathbf{Sp}) \end{array}$$

The vertical functors are Betti realization (see [Heller and Ormsby 2016, Section 4.4]). The functor ι^* restricts an equivariant spectrum to the trivial subgroup, yielding the underlying spectrum.

The $\mathbb{C}$-motivic cohomology of a point is equal to $\mathbb{F}_2[\tau]$ [Voevodsky 2003a] (see also [Dugger and Isaksen 2010, Section 2.1]). The $\mathbb{C}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{C}}$ is very similar to the classical Steenrod algebra, but there are some small complications related to τ. In particular, these complications allow the element h_1 in the cohomology of $\mathcal{A}^{\mathbb{C}}$ to be nonnilpotent, detecting the nonnilpotence of the motivic Hopf map $\eta_{\mathbb{C}}$ [Morel 2004, Corollary 6.4.5]. In the cohomology of $\mathcal{A}^{\mathbb{C}}(1)$, the nonnilpotence of h_1 is essentially the only difference to the classical case.

The $\mathbb{R}$-motivic cohomology of a point is equal to $\mathbb{F}_2[\tau, \rho]$ [Voevodsky 2003a] (again, see the discussion in [Dugger and Isaksen 2010, Section 2.1]). Now an additional complication enters because $\mathrm{Sq}^1(\tau) = \rho$. The computation of the cohomology of the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ becomes more difficult because the cohomology of a point is a nontrivial $\mathcal{A}^{\mathbb{R}}$-module. In addition, the $\mathbb{R}$-motivic Steenrod algebra $\mathcal{A}^{\mathbb{R}}$ has additional complications associated with terms involving higher powers of ρ [Voevodsky 2003b, Theorem 12.6].

A natural way to avoid this problem is to filter by powers of ρ. In the associated graded object, $\mathrm{Sq}^1(\tau)$ becomes zero and the associated graded Hopf algebroid is simply the $\mathbb{C}$-motivic Hopf algebra with an adjoined polynomial generator ρ. Therefore, the ρ-Bockstein spectral sequence starts from the cohomology of $\mathcal{A}^{\mathbb{C}}$ and converges to the cohomology of $\mathcal{A}^{\mathbb{R}}$.

This ρ-Bockstein spectral sequence has lots of differentials and hidden extensions. Nevertheless, a complete calculation for $\mathcal{A}^{\mathbb{R}}(1)$ is reasonable. A key point is to first carry out the ρ-inverted calculation. This turns out to be much simpler. With a priori knowledge of the ρ-inverted calculation in hand, there is just one possible pattern of ρ-Bockstein differentials.

Relying on our experience from the $\mathbb{R}$-motivic situation, we are now ready to tackle the C_2-equivariant situation. The C_2-equivariant cohomology of a point contains $\mathbb{F}_2[\tau, \rho]$, but there is an additional "negative cone" that is infinitely divisible by both τ and ρ [Hu and Kriz 2001, Proposition 6.2]. Except for the complications in the cohomology of a point, the C_2-equivariant Steenrod algebra $\mathcal{A}^{C_2}$ is no more complicated than the $\mathbb{R}$-motivic one [Hu and Kriz 2001, pp. 386–387].

Again, a ρ-Bockstein spectral sequence allows us to compute the cohomology of $\mathcal{A}^{C_2}(1)$. Because of infinite τ-divisibility, the starting point of the spectral sequence is more complicated than just the cohomology of $\mathcal{A}^{\mathbb{C}}(1)$. Once identified, this issue presents only a minor difficulty.

The ρ-inverted calculation determines the part of the cohomology of $\mathcal{A}^{C_2}(1)$ that supports infinitely many ρ multiplications. Dually, it is also helpful to determine in advance the part of the cohomology of $\mathcal{A}^{C_2}(1)$ that is infinitely ρ-divisible, i.e., the inverse limit of an infinite tower of ρ-multiplications. We anticipate that this approach via infinitely ρ-divisible classes will be essential in the more complicated calculation over the full Steenrod algebra $\mathcal{A}^{C_2}$, to be studied in future work.

As for the $\mathbb{R}$-motivic case, the ρ-Bockstein spectral sequence is manageable, even though it does have lots of differentials and hidden extensions.

All of these calculations lead to a thorough understanding of the cohomology of $\mathcal{A}^{C_2}(1)$. The charts in Section 12 display the calculation graphically.

The next step is to consider the C_2-equivariant Adams spectral sequence. For degree reasons, there are no nonzero Adams differentials. The same simple situation occurs in the classical, $\mathbb{C}$-motivic, and $\mathbb{R}$-motivic cases.

However, it turns out that there are many hidden extensions to be analyzed. The presence of so many hidden extensions suggests that the Adams filtration may not be optimal for equivariant purposes. Unfortunately, we do not have an alternative to propose.

The final description of the homotopy groups is complicated. Nevertheless, our computation establishes that the homotopy of ko_{C_2} is nearly periodic (see Theorem 11.15). We refer to Section 11 and the charts in Section 12 for details.

1A. *Organization.* In Section 2, we provide the basic algebraic input to our calculation by thoroughly describing the C_2-equivariant cohomology of a point and the C_2-equivariant Steenrod algebra $\mathcal{A}^{C_2}$. In Section 3, we set up the ρ-Bockstein spectral sequence, which is our main tool for computing the cohomology of $\mathcal{A}^{C_2}(1)$. In Sections 4 and 5, we carry out the ρ-inverted and the infinitely ρ-divisible calculations. In Section 6, we carry out the $\mathbb{R}$-motivic ρ-Bockstein spectral sequence as a warmup for the C_2-equivariant ρ-Bockstein spectral sequence in Section 7. Section 8 provides some information about Massey products in the C_2-equivariant cohomology of $\mathcal{A}(1)$, which is used in Section 9 to determine multiplicative structure that is hidden by the ρ-Bockstein spectral sequence. Section 10 gives the construction of the C_2-equivariant spectrum whose homotopy groups are computed by the cohomology of $\mathcal{A}^{C_2}(1)$, and Section 11 analyzes multiplicative structure in these homotopy groups that is hidden by the Adams spectral sequence. Finally, Section 12 includes a series of charts that graphically describe our calculation.

1B. *Notation.* We employ notation as follows:

(1) $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$ is the motivic cohomology of $\mathbb{C}$ with $\mathbb{F}_2$ coefficients, where τ has bidegree $(0, 1)$.

(2) $\mathbb{M}_2^{\mathbb{R}} = \mathbb{F}_2[\tau, \rho]$ is the motivic cohomology of $\mathbb{R}$ with $\mathbb{F}_2$ coefficients, where τ and ρ have bidegrees $(0, 1)$ and $(1, 1)$, respectively.

(3) $\mathbb{M}_2^{C_2}$ is the bigraded equivariant cohomology of a point with coefficients in the constant Mackey functor $\underline{\mathbb{F}}_2$. See Section 2A for a description of this algebra.

(4) NC is the "negative cone" part of $\mathbb{M}_2^{C_2}$. See Section 2A for a precise description.

(5) $H_{C_2}^{*,*}(X)$ is the C_2-equivariant cohomology of X, with coefficients in the constant Mackey functor $\underline{\mathbb{F}}_2$.

(6) $\mathcal{A}^{\mathrm{cl}}$, $\mathcal{A}^{\mathbb{C}}$, $\mathcal{A}^{\mathbb{R}}$, and $\mathcal{A}^{C_2}$ are the classical, $\mathbb{C}$-motivic, $\mathbb{R}$-motivic, and C_2-equivariant mod 2 Steenrod algebras.

(7) $\mathcal{A}^{\mathrm{cl}}(n)$, $\mathcal{A}^{\mathbb{C}}(n)$, $\mathcal{A}^{\mathbb{R}}(n)$, and $\mathcal{A}^{C_2}(n)$ are the classical, $\mathbb{C}$-motivic, $\mathbb{R}$-motivic, and C_2-equivariant subalgebras generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \ldots, \mathrm{Sq}^{2^n}$.

(8) $\mathcal{E}^{C_2}(1)$ is the subalgebra of $\mathcal{A}^{C_2}$ generated by

$$Q_0 = \mathrm{Sq}^1 \quad \text{and} \quad Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1.$$

(9) $\mathrm{Ext}_{\mathrm{cl}}$ is the bigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$, i.e., the cohomology of $\mathcal{A}^{\mathrm{cl}}$.

(10) $\mathrm{Ext}_{\mathbb{C}}$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$, i.e., the cohomology of $\mathcal{A}^{\mathbb{C}}$.

(11) $\mathrm{Ext}_{\mathbb{R}}$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$, i.e., the cohomology of $\mathcal{A}^{\mathbb{R}}$.

(12) Ext_{C_2} is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{C_2}}(\mathbb{M}_2^{C_2}, \mathbb{M}_2^{C_2})$, i.e., the cohomology of $\mathcal{A}^{C_2}$.

(13) $\mathrm{Ext}_{\mathrm{NC}}$ is the $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}$-module $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}}(\mathrm{NC}, \mathbb{M}_2^{\mathbb{R}})$.

(14) $\mathrm{Ext}_{\mathrm{cl}}(n)$ is the bigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}(n)}(\mathbb{F}_2, \mathbb{F}_2)$, i.e., the cohomology of $\mathcal{A}^{\mathrm{cl}}(n)$.

(15) $\mathrm{Ext}_{\mathbb{C}}(n)$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathbb{C}}(n)}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$, i.e., the cohomology of $\mathcal{A}^{\mathbb{C}}(n)$.

(16) $\mathrm{Ext}_{\mathbb{R}}(n)$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(n)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$, i.e., the cohomology of $\mathcal{A}^{\mathbb{R}}(n)$.

(17) $\mathrm{Ext}_{C_2}(n)$ is the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{C_2}(n)}(\mathbb{M}_2^{C_2}, \mathbb{M}_2^{C_2})$, i.e., the cohomology of $\mathcal{A}^{C_2}(n)$.

(18) $\mathrm{Ext}_{\mathrm{NC}}(n)$ is the $\mathrm{Ext}_{\mathbb{R}}(n)$-module $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(n)}(\mathrm{NC}, \mathbb{M}_2^{\mathbb{R}})$.

(19) E^+ is the ρ-Bockstein spectral sequence

$$\mathrm{Ext}_{\mathbb{C}}(1)[\rho] \Rightarrow \mathrm{Ext}_{\mathbb{R}}(1).$$

See Section 3.

(20) E^- is the ρ-Bockstein spectral sequence that converges to $\mathrm{Ext}_{\mathrm{NC}}(1)$. See Section 3.

(21) $\frac{\mathbb{F}_2[x]}{x^\infty}\{y\}$ is the infinitely x-divisible module $\mathrm{colim}_n\, \mathbb{F}_2[x]/x^n$, consisting of elements of the form $\frac{y}{x^k}$ for $k \geq 1$. See Remark 2.1.

(22) ko_{C_2} is a C_2-equivariant spectrum such that $H_{C_2}^{*,*}(\mathrm{ko}_{C_2}) \cong \mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1)$. See Section 10.

(23) $\pi_{*,*}(X)$ are the bigraded C_2-equivariant stable homotopy groups of X, completed at 2 so that the equivariant Adams spectral sequence converges.

(24) $\Pi_n(X)$ is the Milnor–Witt n-stem $\bigoplus_p \pi_{p+n,p}$.

We use grading conventions that are common in motivic homotopy theory but less common in equivariant homotopy theory. In equivariant homotopy theory, $\mathrm{RO}(C_2) \cong \mathbb{Z}[\sigma]/(\sigma^2 - 1)$ is the real representation ring of C_2, where σ is the 1-dimensional sign representation. The main points of translation are:

(1) Equivariant degree $p + q\sigma$ will be expressed, according to the motivic convention, as $(p+q, q)$, where $p+q$ is the total degree and q is the weight.

(2) The element τ in $\mathbb{M}_2^{\mathbb{R}}$ maps to u [Hill et al. 2016, Definition 3.12] under the realization map from $\mathbb{R}$-motivic to C_2-equivariant homotopy theory. We use the symbol τ in both cases.

(3) Similarly, realization takes the $\mathbb{R}$-motivic homotopy class $\rho : S^{-1,-1} \to S^{0,0}$ to a in $\pi_{-1,-1}$ [Hill et al. 2016, Definition 3.11]. We use the symbol ρ for both of these homotopy classes, and also for the corresponding elements of $\mathbb{M}_2^{\mathbb{R}}$ and $\mathbb{M}_2^{C_2}$.

We grade Ext groups in the form (s, f, w), where s is the stem, i.e., the total degree minus the homological degree; f is the Adams filtration, i.e., the homological degree; and w is the weight. We will also refer to the Milnor–Witt degree, which equals $s - w$.

2. Ext groups

2A. ***The equivariant cohomology of a point.*** The purpose of this section is to carefully describe the structure of the equivariant cohomology ring $\mathbb{M}_2^{C_2}$ of a point from a perspective that will be useful for our calculations. This section is a reinterpretation of results from [Hu and Kriz 2001, Proposition 6.2].

Additively, $\mathbb{M}_2^{C_2}$ equals

(1) $\mathbb{F}_2$ in degree (s, w) if $s \geq 0$ and $w \geq s$,

(2) $\mathbb{F}_2$ in degree (s, w) if $s \leq 0$ and $w \leq s - 2$,

(3) 0 otherwise.

This additive structure is represented by the dots in Figure 1. The nonzero element in degree (0, 1) is called τ, and the nonzero element in degree (1, 1) is called ρ. We remind the reader that we are here employing cohomological grading. Thus the class ρ has degree $(-1, -1)$ when considered as an element of the homology ring $\pi_{*,*} H\underline{\mathbb{F}}_2$.

The "positive cone" refers to the part of $\mathbb{M}_2^{C_2}$ in degrees (s, w) with $w \geq 0$. The positive cone is isomorphic to the $\mathbb{R}$-motivic cohomology ring $\mathbb{M}_2^{\mathbb{R}}$ of a point. Multiplicatively, the positive cone is just a polynomial ring on two variables, ρ and τ.

The "negative cone" NC refers to the part of $\mathbb{M}_2^{C_2}$ in degrees (s, w) with $w \leq -2$. Multiplicatively, the product of any two elements of NC is zero, so $\mathbb{M}_2^{C_2}$ is a square-zero extension of $\mathbb{M}_2^{\mathbb{R}}$. Also, multiplications by ρ and τ are nonzero in NC whenever they make sense. Thus, the elements of NC are infinitely divisible by both ρ and τ.

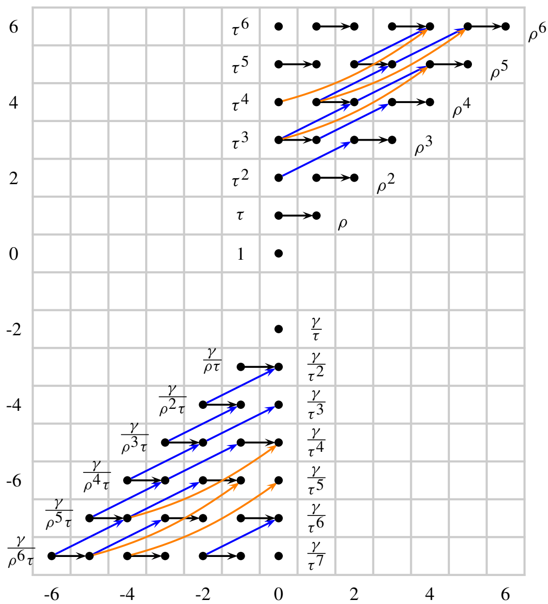


Figure 1. $\mathbb{M}_2^{C_2}$, with action by Sq^1, Sq^2, and Sq^4.

We use the notation $\frac{\gamma}{\rho^j\tau^k}$ for the nonzero element in degree $(-j, -1-j-k)$. This is consistent with the multiplicative properties described in the previous paragraph. So $\tau\cdot\frac{\gamma}{\rho^j\tau^k}$ equals $\frac{\gamma}{\rho^j\tau^{k-1}}$ when $k\geq 2$, and $\rho\cdot\frac{\gamma}{\rho^j\tau^k}$ equals $\frac{\gamma}{\rho^{j-1}\tau^k}$ when $j\geq 2$.

The symbol γ, which does not correspond to an actual element of $\mathbb{M}_2^{C_2}$, has degree $(0,-1)$.

The $\mathbb{F}_2[\tau]$-module structure on $\mathbb{M}_2^{C_2}$ is essential for later calculations, since we will filter by powers of ρ. Therefore, we explore further the $\mathbb{F}_2[\tau]$-module structure on NC.

Remark 2.1. Recall that $\mathbb{F}_2[\tau]/\tau^\infty$ is the $\mathbb{F}_2[\tau]$-module $\operatorname{colim}\mathbb{F}_2[\tau]/\tau^k$, which consists entirely of elements that are divisible by τ. We write $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{x\}$ for the infinitely divisible $\mathbb{F}_2[\tau]$-module consisting of elements of the form $\frac{x}{\tau^k}$ for $k\geq 1$. Note that x itself is not an element of $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{x\}$. The idea is that x represents the infinitely many relations $\tau^k\cdot\frac{x}{\tau^k}=0$ that define $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{x\}$.

With this notation in place, $\mathbb{M}_2^{C_2}$ is equal to

$$\mathbb{M}_2^{\mathbb{R}}\oplus \mathrm{NC} = \mathbb{M}_2^{\mathbb{R}}\oplus\bigoplus_{s\geq 0}\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\} \tag{2-1}$$

as an $\mathbb{F}_2[\tau]$-module.

2B. ***The equivariant Steenrod algebra.*** As a Hopf algebroid, the equivariant dual Steenrod algebra can be described [Ricka 2015, Proposition 6.10(2)] as

$$\mathcal{A}_*^{C_2} \cong \mathbb{M}_2^{C_2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}_*^{\mathbb{R}}. \tag{2-2}$$

Recall [Voevodsky 2003b] that

$$\mathcal{A}_*^{\mathbb{R}} \cong \mathbb{M}_2^{\mathbb{R}}[\tau_0, \tau_1, \ldots, \xi_1, \xi_2, \ldots]/(\tau_i^2 = \rho\tau_{i+1} + \tau\xi_{i+1} + \rho\tau_0\xi_{i+1}),$$

with $\eta_R(\rho) = \rho$ and $\eta_R(\tau) = \tau + \rho\tau_0$. The formula for the right unit η_R on the negative cone given in [Hu and Kriz 2001, Theorem 6.41] appears in our notation as

$$\eta_R\left(\frac{\gamma}{\rho^j\tau^k}\right) = \frac{\gamma}{\rho^j\tau^k}\left[\sum_{i\geq 0}\left(\frac{\rho}{\tau}\tau_0\right)^i\right]^k. \tag{2-3}$$

Note that the sum is finite because $\frac{\gamma}{\rho^j\tau^k} \cdot \rho^n = 0$ if $n > j$.

We have quotient Hopf algebroids

$$\mathcal{A}_*^{\mathbb{R}}(n) := \mathbb{M}_2^{\mathbb{R}}[\tau_0, \ldots, \tau_n, \xi_1, \ldots, \xi_n]/(\xi_i^{2^{n-i+1}}, \tau_i^2 = \rho\tau_{i+1} + \tau\xi_{i+1} + \rho\tau_0\xi_{i+1}).$$

and

$$\mathcal{E}_*^{\mathbb{R}}(n) := \mathbb{M}_2^{\mathbb{R}}[\tau_0, \ldots, \tau_n]/(\tau_i^2 = \rho\tau_{i+1}, \tau_n^2)$$

and their equivariant analogues

$$\mathcal{A}_*^{C_2}(n) := \mathbb{M}^{C_2} \otimes_{\mathbb{M}^{\mathbb{R}}} \mathcal{A}_*^{\mathbb{R}}(n), \qquad \mathcal{E}_*^{C_2}(n) := \mathbb{M}^{C_2} \otimes_{\mathbb{M}^{R}} E_*^{\mathbb{R}}(n) \tag{2-4}$$

Their duals are the subalgebras $\mathcal{A}^{C_2}(n) \subseteq \mathcal{A}^{C_2}$ and $\mathcal{E}^{C_2}(n) \subseteq \mathcal{A}^{C_2}$.

The relationship between the equivariant and $\mathbb{R}$-motivic Steenrod algebras leads to an analogous relationship between Ext groups.

Proposition 2.2. *Suppose that* Γ *is a Hopf algebroid over* A *and that* $B \cong A \oplus M$ *is a* Γ*-comodule which is a square-zero extension of* A, *meaning that the product of any two elements in* M *is zero. Then the* A*-module splitting of* B *induces a splitting*

$$\mathrm{Ext}_{B\otimes_A\Gamma}(B, B) \cong \mathrm{Ext}_\Gamma(A, A) \oplus \mathrm{Ext}_\Gamma(M, A)$$

of $\mathrm{Ext}_\Gamma(A, A)$*-modules. Furthermore, this is an isomorphism of* $\mathrm{Ext}_\Gamma(A, A)$*-algebras, if the right-hand side is taken to be a square-zero extension of* $\mathrm{Ext}_\Gamma(A, A)$.

Proof. We may express the cobar complex as:

$$\begin{aligned}\mathrm{coB}^s(B, B \otimes_A \Gamma) = B \otimes_B (\Gamma)^{\otimes s} &\cong B \otimes_B (B \otimes_A \Gamma)^{\otimes s} \\ &\cong B \otimes_A (\Gamma)^{\otimes s}.\end{aligned}$$

As the splitting of B is a splitting as Γ-comodules, there results a splitting

$$\mathrm{coB}^s(A, \Gamma) \oplus \mathrm{coB}^s(M, \Gamma)$$

of the cobar complex. This splitting is square-zero, in the sense that the product of any two elements in the second factor is equal to zero. This observation follows from the fact that the product of any two elements of M is zero.

In $\mathrm{Ext}_{B\otimes_A\Gamma}$, this yields

$$\mathrm{Ext}_{B\otimes_A\Gamma} \cong \mathrm{Ext}_\Gamma(A, A) \oplus \mathrm{Ext}_\Gamma(M, A).$$

The multiplication on $\mathrm{Ext}_\Gamma(M, A)$ is zero since this is already true in the cobar complex $\mathrm{coB}^s(M, \Gamma)$. □

Employing notation given in Section 1B, Proposition 2.2 applies to give isomorphisms

$$\mathrm{Ext}_{C_2} \cong \mathrm{Ext}_\mathbb{R} \oplus \mathrm{Ext}_{\mathrm{NC}}$$

and

$$\mathrm{Ext}_{C_2(n)} \cong \mathrm{Ext}_{\mathbb{R}(n)} \oplus \mathrm{Ext}_{\mathrm{NC}(n)}\,.$$

Thus from the point of view of $\mathbb{R}$-motivic homotopy theory, the cohomology of the negative cone is the only new feature in $\mathrm{Ext}_{\mathcal{A}^{C_2}}$ or $\mathrm{Ext}_{\mathcal{A}^{C_2(n)}}$.

3. The ρ-Bockstein spectral sequence

Our tool for computing $\mathbb{R}$-motivic or C_2-equivariant Ext is the ρ-Bockstein spectral sequence [Hill 2011; Dugger and Isaksen 2017a]. The ρ-Bockstein spectral sequence arises by filtering the cobar complex by powers of ρ. More precisely, we can define an $\mathcal{A}^\mathbb{R}$-module filtration on $\mathbb{M}_2^{C_2}$, where $F_p(\mathbb{M}_2^{C_2})$ is the part of $\mathbb{M}_2^{C_2}$ concentrated in degrees (s, w) with $s \geq p$. Dualizing, we get a filtration of comodules over the dual Steenrod algebra, which induces a filtration on the cobar complex that computes Ext_{C_2}.

Recall that the $\mathbb{C}$-motivic cohomology of a point is $\mathbb{M}_2^\mathbb{C} = \mathbb{F}_2[\tau]$, and the $\mathbb{C}$-motivic Steenrod algebra is $\mathcal{A}^\mathbb{C} = \mathcal{A}^\mathbb{R}/\rho$ [Voevodsky 2003a; 2003b]. For convenience, we write $\mathrm{Ext}_\mathbb{C}$ for $\mathrm{Ext}_{\mathcal{A}^\mathbb{C}}(\mathbb{M}_2^\mathbb{C}, \mathbb{M}_2^\mathbb{C})$.

Proposition 3.1. *There is a ρ-Bockstein spectral sequence*

$$E_1 = \mathrm{Ext}_{\mathrm{gr}_\rho \mathcal{A}^{C_2}}(\mathrm{gr}_\rho \mathbb{M}_2^{C_2}, \mathrm{gr}_\rho \mathbb{M}_2^{C_2}) \Rightarrow \mathrm{Ext}_{C_2}$$

such that a Bockstein differential d_r takes a class x of degree (s, f, w) to a class $d_r(x)$ of degree $(s-1, f+1, w)$. Under the splitting of Proposition 2.2, this decomposes as

$$E_1^+ = \mathrm{Ext}_\mathbb{C}[\rho] \Rightarrow \mathrm{Ext}_\mathbb{R}$$

and

$$E_1^- \Rightarrow \mathrm{Ext}_{\mathrm{NC}},$$

where E_1^- belongs to a split short exact sequence

$$\bigoplus_{s\geq 0} \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\} \otimes_{\mathbb{M}_2^{\mathbb{C}}} \mathrm{Ext}_{\mathbb{C}} \to E_1^- \to \bigoplus_{s\geq 0} \mathrm{Tor}_{\mathbb{M}_2^{\mathbb{C}}}\left(\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\}, \mathrm{Ext}_{\mathbb{C}}\right).$$

Remark 3.2. Beware that the short exact sequence for E_1^- does not split canonically.

Remark 3.3. The same spectral sequences occur in the same form when $\mathcal{A}^{C_2}$, $\mathcal{A}^{\mathbb{R}}$, and $\mathcal{A}^{\mathbb{C}}$ are replaced by $\mathcal{A}^{C_2}(n)$, $\mathcal{A}^{\mathbb{R}}(n)$, and $\mathcal{A}^{\mathbb{C}}(n)$.

Proof. See [Hill 2011, Proposition 2.3] (or [Dugger and Isaksen 2017a, Section 3]) for the description of E_1^+.

For E_1^-, the associated graded of NC is

$$\mathrm{gr}_\rho \, \mathrm{NC} \cong \bigoplus_{s\geq 0} \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\},$$

as described in Section 2A. It follows that the Bockstein spectral sequence begins with

$$E_0 \cong \bigoplus_{s\geq 0} \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\} \otimes_{\mathbb{M}_2^{\mathbb{C}}} \mathrm{coB}(\mathbb{M}_2^{\mathbb{C}}, \mathcal{A}_*^{\mathbb{C}}).$$

The ring $\mathbb{M}_2^{\mathbb{C}} \cong \mathbb{F}_2[\tau]$ is a graded principal ideal domain (in fact, it is a graded local ring with maximal ideal generated by τ). Therefore, the Künneth split exact sequence gives

$$\left(\bigoplus_{s\geq 0} \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\}\right) \otimes_{\mathbb{M}_2^{\mathbb{C}}} \mathrm{Ext}_{\mathbb{C}} \to E_1^- \to \mathrm{Tor}_{\mathbb{M}_2^{\mathbb{C}}}\left(\bigoplus_{s\geq 0} \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\}, \mathrm{Ext}_{\mathbb{C}}\right).$$

The first and third terms of the short exact sequence may be rewritten as in the statement of the proposition because the direct sum in each case is a splitting of $\mathbb{M}_2^{\mathbb{C}}$-modules. □

We shall completely analyze the spectral sequence

$$E_1^+ = \mathrm{Ext}_{\mathbb{C}}(1)[\rho] \Rightarrow \mathrm{Ext}_{\mathbb{R}}(1)$$

in Section 6. While nontrivial, this part of our calculation is comparatively straightforward.

On the other hand, analysis of the spectral sequence

$$E_1^- \Rightarrow \mathrm{Ext}_{\mathrm{NC}}(1)$$

requires significantly more work. The first step is to compute E_1^- more explicitly. In particular, we must describe the Tor groups that arise.

Lemma 3.4. (1) $\mathrm{Tor}^*_{\mathbb{M}_2^{\mathbb{C}}}\big(\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}, \mathbb{M}_2^{\mathbb{C}}\big)$ *equals* $\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}$, *concentrated in homological degree zero.*

(2) $\mathrm{Tor}^*_{\mathbb{M}_2^{\mathbb{C}}}\big(\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}, \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^k}\big)$ *equals* $\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^k}$, *concentrated in homological degree one.*

Proof. (1) This is a standard fact about the vanishing of higher Tor groups for free modules.

(2) This follows from direct computation, using the resolution

$$\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^k} \longleftarrow \mathbb{M}_2^{\mathbb{C}} \xleftarrow{\tau^k} \mathbb{M}_2^{\mathbb{C}} \longleftarrow 0.$$

After tensoring with $\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}$, this gives the map

$$\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\{x\} \longleftarrow \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\{y\}$$

that takes $\frac{y}{\tau^a}$ to $\frac{x}{\tau^{a-k}}$ if $a > k$, and takes $\frac{y}{\tau^a}$ to zero if $a \leq k$. This map is onto, and its kernel is isomorphic to $\mathbb{M}_2^{\mathbb{C}}/\tau^k$. □

Remark 3.5. Lemma 3.4 provides a practical method for identifying the E_1^- in Proposition 3.1. Copies of $\mathbb{M}_2^{\mathbb{C}}$ in $\mathrm{Ext}_{\mathbb{C}}(1)$ lead to copies of the negative cone in E_1^-. On the other hand, copies of $\mathbb{M}_2^{\mathbb{C}}/\tau$, such as the submodule generated by h_1^3, lead to copies of $\mathbb{M}_2^{\mathbb{C}}/\tau$ in E_1^- that are infinitely divisible by ρ. These copies of $\mathbb{M}_2^{\mathbb{C}}/\tau$ occur with a degree shift because they arise from Tor^1.

4. ρ-inverted $\mathrm{Ext}_{\mathbb{R}}(1)$

As a first step towards computing $\mathrm{Ext}_{C_2}(1)$, we will invert ρ in the $\mathbb{R}$-motivic setting and study $\mathrm{Ext}_{\mathbb{R}}(1)[\rho^{-1}]$. This gives partial information about $\mathrm{Ext}_{\mathbb{R}}(1)$ and also about $\mathrm{Ext}_{C_2}(1)$. Afterwards, it remains to compute ρ^k torsion, including infinitely ρ-divisible elements.

We write $\mathcal{A}^{\mathrm{cl}}$ for the classical Steenrod algebra. For convenience, we write $\mathrm{Ext}_{\mathrm{cl}}$ and $\mathrm{Ext}_{\mathrm{cl}}(n)$ for $\mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$ and $\mathrm{Ext}_{\mathcal{A}^{\mathrm{cl}}(n)}(\mathbb{F}_2, \mathbb{F}_2)$ respectively.

Proposition 4.1. *There is an injection* $\mathrm{Ext}_{\mathrm{cl}}(n-1)[\rho^{\pm 1}] \hookrightarrow \mathrm{Ext}_{\mathbb{R}}(n)[\rho^{-1}]$ *such that*:

(1) *The map is highly structured*, *i.e.*, *preserves products*, *Massey products*, *and algebraic squaring operations.*

(2) *The element* h_i *of* $\mathrm{Ext}_{\mathrm{cl}}(n-1)$ *corresponds to* h_{i+1} *of* $\mathrm{Ext}_{\mathbb{R}}(n)$.

(3) *The map induces an isomorphism*

$$\mathrm{Ext}_{\mathbb{R}}(n)[\rho^{-1}] \cong \mathrm{Ext}_{\mathrm{cl}}(n-1)[\rho^{\pm 1}] \otimes \mathbb{F}_2[\tau^{2^{n+1}}].$$

(4) *An element in* $\mathrm{Ext}_{\mathrm{cl}}(n-1)$ *of degree* (s, f) *corresponds to an element in* $\mathrm{Ext}_{\mathbb{R}}(n)$ *of degree* $(2s+f, f, s+f)$.

Proof. The proof is similar to the proof of [Dugger and Isaksen 2017a, Theorem 4.1]. Since localization is an exact functor, we may compute the cohomology of the Hopf algebroid $(\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}], \mathcal{A}^{\mathbb{R}}(n+1)_*[\rho^{-1}])$ to obtain $\mathrm{Ext}_{\mathbb{R}}(n+1)[\rho^{-1}]$. After inverting ρ, we have

$$\tau_{k+1} = \rho^{-1}\tau_k^2 + \rho^{-1}\tau\xi_{k+1} + \tau_0\xi_{k+1},$$

and it follows that

$$\mathcal{A}^{\mathbb{R}}(n)_*[\rho^{-1}] \cong \mathbb{M}_2^{\mathbb{R}}[\rho^{-1}][\tau_0, \xi_1, \dots, \xi_n]/(\tau_0^{2^{n+1}}, \xi_1^{2^n}, \dots, \xi_n^2).$$

This splits as

$$\big(\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}], \mathcal{A}(n)_*[\rho^{-1}]\big) \cong \big(\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}], \mathcal{A}'(n)\big) \otimes_{\mathbb{F}_2} (\mathbb{F}_2, \mathcal{A}''(n)),$$

where

$$\mathcal{A}'(n) = \mathbb{M}_2^{\mathbb{R}}[\rho^{-1}][\tau_0]/\tau_0^{2^{n+1}}$$

and

$$\mathcal{A}''(n) = \mathbb{F}_2[\xi_1, \dots, \xi_n]/(\xi_1^{2^n}, \dots, \xi_n^2).$$

Because it is isomorphic to the classical Hopf algebra $(\mathbb{F}_2, \mathcal{A}(n-1))$ with altered degrees, the Hopf algebra $(\mathbb{F}_2, \mathcal{A}''(n))$ has cohomology $\mathrm{Ext}_{\mathrm{cl}}(n-1)$.

For the Hopf algebroid $\big(\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}], \mathcal{A}'(n)\big)$, we have an isomorphism

$$(\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}], \mathcal{A}'(n)) \cong \mathbb{F}_2[\rho^{\pm 1}] \otimes_{\mathbb{F}_2} (\mathbb{F}_2[\tau], \mathbb{F}_2[\tau][x]/x^{2^{n+1}})$$

with

$$\eta_L(\tau) = \tau, \quad \eta_R(\tau) = \tau + x.$$

An argument like that of [Dugger and Isaksen 2017a, Lemma 4.2] shows that the cohomology of this Hopf algebroid is $\mathbb{F}_2[\tau^{2^{n+1}}]$. □

Corollary 4.2. $\mathrm{Ext}_{C_2}(1)[\rho^{-1}] \cong \mathrm{Ext}_{\mathbb{R}}(1)[\rho^{-1}] \cong \mathbb{F}_2[\rho^{\pm 1}, \tau^4, h_1]$.

Proof. The first isomorphism follows from Proposition 2.2, as $\mathrm{Ext}_{\mathrm{NC}}$ is ρ-torsion. The second isomorphism follows immediately from Proposition 4.1, given that $\mathrm{Ext}_{\mathrm{cl}}(0) \cong \mathbb{F}_2[h_0]$. □

Remark 4.3. Corollary 4.2 implies that the products $\tau^4 \cdot h_1^k$ are nonzero in $\mathrm{Ext}_{\mathbb{R}}(1)$. But $\tau^4 h_1^k = 0$ in $\mathrm{Ext}_{\mathbb{C}}(1)$ when $k \geq 3$, so the products $\tau^4 \cdot h_1^k$ are hidden in the ρ-Bockstein spectral sequence for $k \geq 3$. We will sort this out in detail in Section 6.

5. Infinitely ρ-divisible elements of $\mathrm{Ext}_{\mathcal{A}^{C_2}(1)}$

Having computed the effect of inverting ρ in Section 4, we now consider the dual question and study infinitely ρ-divisible elements. This gives additional partial information about $\mathrm{Ext}_{C_2}(1)$. Afterwards, it remains only to compute the ρ^k torsion classes that are not infinitely ρ-divisible.

In fact, this section is not strictly necessary to carry out the computation of $\mathrm{Ext}_{C_2}(1)$. Nevertheless, the infinitely ρ-divisible calculation works out rather nicely and provides some useful insight into the main computation. We also anticipate that this approach via infinitely ρ-divisible classes will be essential in the much more complicated calculation of Ext_{C_2}, to be studied in further work.

For a $\mathbb{F}_2[\rho]$-module M, the ρ-colocalization, or ρ-cellularization, is the limit $\lim_\rho M$ of the inverse system

$$\cdots \xrightarrow{\rho} M \xrightarrow{\rho} M.$$

While ρ-localization detects ρ-torsion-free elements, the ρ-colocalization detects infinitely ρ-divisible elements.

An alternative description is given by the isomorphism

$$\lim_\rho M \cong \mathrm{Hom}_{\mathbb{F}_2[\rho]}(\mathbb{F}_2[\rho^{\pm 1}], M)$$

because $\mathbb{F}_2[\rho^{\pm 1}]$ is isomorphic to $\mathrm{colim}_\rho\, \mathbb{F}_2[\rho]$. It follows that $\lim_\rho M$ is an $\mathbb{F}_2[\rho^{\pm 1}]$-module, and the functor $M \mapsto \lim_\rho M$ is right adjoint to the restriction

$$\mathrm{Mod}_{\mathbb{F}_2[\rho^{\pm 1}]} \to \mathrm{Mod}_{\mathbb{F}_2[\rho]}.$$

Lemma 5.1. (1) *Let M be a cyclic $\mathbb{F}_2[\rho]$-module $\mathbb{F}_2[\rho]$ or $\mathbb{F}_2[\rho]/\rho^k$. Then* $\lim_\rho M$ *is zero.*

(2) *Let M be the infinitely divisible $\mathbb{F}_2[\rho]$-module $\mathbb{F}_2[\rho]/\rho^\infty$. Then* $\lim_\rho M$ *is isomorphic to $\mathbb{F}_2[\rho^{\pm 1}]$.*

Proof. If M is cyclic, then no nonzero element is infinitely ρ-divisible, which implies the first statement. For the case $M = \mathbb{F}_2[\rho]/\rho^\infty$, a (homogeneous) element of the limit is either of the form

$$\left(\frac{1}{\rho^k}, \frac{1}{\rho^{k+1}}, \ldots\right)$$

or of the form

$$\left(0, \ldots, 0, 1, \frac{1}{\rho}, \frac{1}{\rho^2}, \ldots\right).$$

For $k \geq 0$, the isomorphism $\mathbb{F}_2[\rho^{\pm 1}] \to \lim_\rho M$ sends ρ^k to the tuple

$$\left(0, \ldots, 0, 1, \frac{1}{\rho}, \ldots\right)$$

having $k-1$ zeroes and sends $\frac{1}{\rho^k}$ to $\left(\frac{1}{\rho^k}, \frac{1}{\rho^{k+1}}, \ldots\right)$. □

We will now compute the ρ-colocalization of $\mathrm{Ext}_{C_2(1)}$.

Proposition 5.2.

$$\lim_\rho \mathrm{Ext}_{C_2}(1) \cong \bigoplus_{k\geq 1} \mathbb{F}_2[\rho^{\pm 1}, h_1]\left\{\frac{\gamma}{\tau^{4k}}\right\} \cong \mathbb{F}_2[\rho^{\pm 1}, h_1] \otimes \frac{\mathbb{F}_2[\tau^4]}{\tau^\infty}\{\gamma\}.$$

Recall that γ itself is not an element of $\lim_\rho \mathrm{Ext}_{C_2}(1)$, as described in Remark 2.1. The main point of Proposition 5.2 is that the elements $\frac{\gamma}{\tau^{4k}} h_1^j$ are infinitely ρ-divisible classes in $\mathrm{Ext}_{C_2}(1)$, and there are no other infinitely ρ-divisible families in $\mathrm{Ext}_{C_2}(1)$.

Proof. Since the cobar complex $\mathrm{coB}^*(\mathbb{M}_2^{C_2}, A^{C_2}(1))$ is finite-dimensional in each tridegree, the inverse systems

$$\cdots \xrightarrow{\rho} \mathrm{coB}^*(\mathbb{M}_2^{C_2}, A^{C_2}(1)) \xrightarrow{\rho} \mathrm{coB}^*(\mathbb{M}_2^{C_2}, A^{C_2}(1))$$

and

$$\cdots \xrightarrow{\rho} \mathrm{Ext}_{C_2}(1) \xrightarrow{\rho} \mathrm{Ext}_{C_2}(1)$$

satisfy the Mittag-Leffler condition, so that [Weibel 1994, Theorem 3.5.8]

$$\lim_\rho \mathrm{Ext}_{C_2}(1) \cong \mathrm{H}^*\big(\lim_\rho \mathrm{coB}^*(\mathbb{M}_2^{C_2}, A^{C_2}(1))\big).$$

Now we compute

$$\lim_\rho \mathrm{coB}^s(\mathbb{M}_2^{C_2}, A^{C_2}(1)) = \lim_\rho\big(\mathbb{M}_2^{C_2} \otimes_{\mathbb{M}_2^{C_2}} A^{C_2}(1)^{\otimes s}\big)$$
$$\cong \lim_\rho\big(\mathbb{M}_2^{C_2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} A^{\mathbb{R}}(1)^{\otimes s}\big).$$

The splitting $\mathbb{M}_2^{C_2} = \mathbb{M}_2^{\mathbb{R}} \oplus \mathrm{NC}$ yields a splitting

$$\big(\mathbb{M}_2^{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(1)^{\otimes s}\big) \oplus \big(\mathrm{NC} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(1)^{\otimes s}\big)$$

of $\mathbb{M}_2^{C_2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(1)^{\otimes s}$ as an $\mathbb{F}_2[\rho]$-module. The first piece of the splitting contributes nothing to the ρ-colocalization by Lemma 5.1(1) because $\mathbb{M}_2^{\mathbb{R}}$ is free as an $\mathbb{F}_2[\rho]$-module.

On the other hand, the $\mathbb{F}_2[\rho]$-module NC is a direct sum of copies of $\mathbb{F}_2[\rho]/\rho^\infty$. By Lemma 5.1(2), we have that $\lim_\rho\big(\mathrm{NC} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(1)^{\otimes s}\big)$ is isomorphic to

$$\left(\frac{\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}]}{\tau^\infty}\{\gamma\}\right) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}^{\mathbb{R}}(1)^{\otimes s}.$$

Now the argument of Proposition 4.1 provides a splitting

$$\mathrm{coB}^*_{\mathbb{M}_2^{\mathbb{R}}}\left(\frac{\mathbb{M}_2^{\mathbb{R}}[\rho^{-1}]}{\tau^\infty}\{\gamma\}, A^{\mathbb{R}}(1)\right)$$
$$\simeq \mathrm{coB}^*_{\mathbb{F}_2[\tau]}\left(\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\gamma\}, \frac{\mathbb{F}_2[\tau, x]}{x^4}\right)[\rho^{\pm 1}] \otimes_{\mathbb{F}_2} \mathrm{coB}^*_{\mathbb{F}_2}(\mathbb{F}_2, \mathbb{F}_2[\xi_1]/\xi_1^2),$$

where $x = \rho\tau_0$. The cohomology of the second factor is $\mathbb{F}_2[h_1]$.

It remains to show that the cohomology of

$$\mathrm{coB}^*_{\mathbb{F}_2[\tau]}\left(\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\gamma\}, \frac{\mathbb{F}_2[\tau,x]}{x^4}\right)$$

is equal to $\frac{\mathbb{F}_2[\tau^4]}{\tau^\infty}\{\gamma\}$. As in Formula (2-3), the comodule structure on $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\gamma\}$ is given by

$$\eta_R\left(\frac{\gamma}{\tau^k}\right) = \frac{\gamma}{\tau^k}\left(1+\frac{x}{\tau}+\frac{x^2}{\tau^2}+\frac{x^3}{\tau^3}\right)^k.$$

Now we filter $\mathrm{coB}^*_{\mathbb{F}_2[\tau]}\left(\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\gamma\}, \frac{\mathbb{F}_2[\tau,x]}{x^4}\right)$ by powers of x. We then have

$$E_1 \cong \frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{\gamma\} \otimes_{\mathbb{F}_2} \mathbb{F}_2[v_0, v_1],$$

where $v_0 = [x]$ and $v_1 = [x^2]$. The differential

$$d_1\left(\frac{\gamma}{\tau^{2k-1}}\right) = \frac{\gamma}{\tau^{2k}} v_0$$

gives

$$E_2 \cong \frac{\mathbb{F}_2[\tau^2]}{\tau^\infty}\{\gamma\} \otimes_{\mathbb{F}_2} \mathbb{F}_2[v_1].$$

Finally, the differential

$$d_2\left(\frac{\gamma}{\tau^{4k-2}}\right) = \frac{\gamma}{\tau^{4k}} v_1$$

gives

$$E_3 = E_\infty \cong \frac{\mathbb{F}_2[\tau^4]}{\tau^\infty}\{\gamma\}. \qquad \square$$

6. The cohomology of $\mathcal{A}^{\mathbb{R}}(1)$

Our next step in working towards the calculation of $\mathrm{Ext}_{C_2}(1)$ is to describe the simpler $\mathbb{R}$-motivic $\mathrm{Ext}_{\mathbb{R}}(1)$. The reader is encouraged to consult the charts on pages 616–619 to follow along with the calculations described in this section. This calculation was originally carried out in [Hill 2011]. We include the details of the $\mathbb{R}$-motivic ρ-Bockstein spectral sequence, but we take the approach of [Dugger and Isaksen 2017a], rather than [Hill 2011], in establishing ρ-Bockstein differentials. The point is that there is only one pattern of differentials that is consistent with the ρ-inverted calculation of Corollary 4.2. This observation avoids much technical work with Massey products that would otherwise be required to establish relations that then imply differentials.

For $\mathcal{A}^{\mathbb{R}}(1)$, the $\mathbb{R}$-motivic ρ-Bockstein spectral sequence takes the form

$$\mathrm{Ext}_{\mathbb{C}}(1)[\rho] \Rightarrow \mathrm{Ext}_{\mathbb{R}}(1),$$

where

$$\mathrm{Ext}_{\mathbb{C}}(1) \cong \mathbb{M}_2^{\mathbb{C}}[h_0, h_1, a, b]/\, h_0 h_1, \tau h_1^3, h_1 a, a^2 + h_0^2 b.$$

Proposition 6.1. *In the $\mathbb{R}$-motivic ρ-Bockstein spectral sequence, we have differentials*

(1) $$d_1(\tau) = \rho h_0,$$

(2) $$d_2(\tau^2) = \rho^2 \tau h_1,$$

(3) $$d_3(\tau^3 h_1^2) = \rho^3 a.$$

All other differentials on multiplicative generators are zero, and E_4 equals E_∞.

Proof. By Corollary 4.2, the infinite ρ-towers that survive the ρ-Bockstein spectral sequence occur in the Milnor–Witt $4k$-stem. All other infinite ρ-towers are either truncated by a differential or support a differential.

For example, the permanent cycle h_0 must be ρ-torsion in $\mathrm{Ext}_{\mathbb{R}}(1)$, which forces the Bockstein differential

$$d_1(\tau) = \rho h_0.$$

Next, the ρ-tower on τh_1 cannot survive, and the only possibility is that there is a differential

$$d_2(\tau^2) = \rho^2 \tau h_1.$$

Note that these differentials also follow easily from the right unit formula given in Section 2B. The ρ-tower on $\tau^3 h_1^2$ cannot survive, and we conclude that it must support a differential

$$d_3(\tau^3 h_1^2) = \rho^3 a.$$

There is no room for further nonzero differentials, so $E_4 = E_\infty$. □

Proposition 6.1 leads to an explicit description of the $\mathbb{R}$-motivic ρ-Bockstein E_∞-page. However, there are hidden multiplications in passing from E_∞ to $\mathrm{Ext}_{\mathbb{R}}(1)$.

Theorem 6.2. $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$ *is the $\mathbb{F}_2$-algebra on generators given in Table 1 with relations given in Table 2.*

The horizontal lines in Table 2 group the relations into families. The first family describes the ρ^k-torsion. The remaining families are associated to the classical products h_0^2, $h_0 h_1$, h_1^3, $h_0 a$, $h_1 a$, and $a^2 + h_0^2 b$ respectively.

Proof. The family of ρ^k-torsion relations follows from the ρ-Bockstein differentials of Proposition 6.1.

Many relations follow immediately from the ρ-Bockstein E_∞-page because there are no possible additional terms.

mw	(s, f, w)	generator
0	$(-1, 0, -1)$	ρ
0	$(0, 1, 0)$	h_0
0	$(1, 1, 1)$	h_1
1	$(1, 1, 0)$	τh_1
2	$(0, 1, -2)$	$\tau^2 h_0$
2	$(4, 3, 2)$	a
4	$(4, 3, 0)$	$\tau^2 a$
4	$(8, 4, 4)$	b
4	$(0, 0, -4)$	τ^4

Table 1. Generators for $\mathrm{Ext}_{\mathbb{R}}(1)$

mw	(s, f, w)	relation
0	$(-1, 1, -1)$	ρh_0
2	$(-1, 1, -3)$	$\rho \cdot \tau^2 h_0$
1	$(-1, 1, -2)$	$\rho^2 \cdot \tau h_1$
2	$(1, 3, -1)$	$\rho^3 a$
4	$(0, 2, -4)$	$(\tau^2 h_0)^2 + \tau^4 h_0^2$
0	$(1, 2, 1)$	$h_0 h_1$
1	$(1, 2, 0)$	$h_0 \cdot \tau h_1 + \rho h_1 \cdot \tau h_1$
2	$(1, 2, -1)$	$\tau^2 h_0 \cdot h_1 + \rho(\tau h_1)^2$
3	$(1, 2, -2)$	$\tau^2 h_0 \cdot \tau h_1$
1	$(3, 3, 2)$	$h_1^2 \cdot \tau h_1$
2	$(3, 3, 1)$	$h_1(\tau h_1)^2 + \rho a$
3	$(3, 3, 0)$	$(\tau h_1)^3$
4	$(3, 3, -1)$	$\tau^4 \cdot h_1^3 + \rho \cdot \tau^2 a$
4	$(4, 4, 0)$	$\tau^2 h_0 \cdot a + h_0 \cdot \tau^2 a$
6	$(4, 4, -2)$	$\tau^2 h_0 \cdot \tau^2 a + \tau^4 h_0 a$
2	$(5, 4, 3)$	$h_1 a$
3	$(5, 4, 2)$	$\tau h_1 \cdot a$
4	$(5, 4, 1)$	$h_1 \cdot \tau^2 a + \rho^3 b$
5	$(5, 4, 0)$	$\tau h_1 \cdot \tau^2 a$
4	$(8, 6, 4)$	$a^2 + h_0^2 b$
6	$(8, 6, 2)$	$a \cdot \tau^2 a + \tau^2 h_0 \cdot h_0 b$
8	$(8, 6, 0)$	$(\tau^2 a)^2 + \tau^4 h_0^2 b + \rho^2 \tau^4 h_1^2 b$

Table 2. Relations for $\mathrm{Ext}_{\mathbb{R}}(1)$.

mw	(s, f, w)	$x \in \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$	$q_* x \in \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}$
0	$(-1, 0, -1)$	ρ	ρ
0	$(0, 1, 0)$	h_0	h_0
0	$(1, 1, 1)$	h_1	0
1	$(1, 1, 0)$	τh_1	ρv_1
2	$(0, 1, -2)$	$\tau^2 h_0$	$\tau^2 h_0$
2	$(4, 3, 2)$	a	$h_0 v_1^2$
4	$(4, 3, 0)$	$\tau^2 a$	$\tau^2 h_0 v_1^2$
4	$(8, 4, 4)$	b	v_1^4
4	$(0, 0, -4)$	τ^4	τ^4

Table 3. The homomorphism $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)} \to \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}$.

Corollary 4.2 implies that $\tau^4 \cdot h_1^3$, is nonzero in $\mathrm{Ext}_{\mathbb{R}}(1)$. It follows that there must be a hidden relation

$$\tau^4 \cdot h_1^3 = \rho \cdot \tau^2 a.$$

Similarly, there is a hidden relation

$$h_1 \cdot \tau^2 a = \rho^3 b$$

because $\tau^4 \cdot h_1^4$ is nonzero in $\mathrm{Ext}_{\mathbb{R}}(1)$. This last relation then gives rise to the extra term $\rho^2 \tau^4 h_1^2 b$ in the relation for $(\tau^2 a)^2 + \tau^4 h_0^2 b$.

Shuffling relations for Massey products imply the remaining three relations, namely

$$h_0 \cdot \tau h_1 = h_0 \langle h_1, h_0, \rho \rangle = \langle h_0, h_1, h_0 \rangle \rho = \rho h_1 \cdot \tau h_1,$$

$$\tau^2 h_0 \cdot h_1 = \langle \rho \tau h_1, \rho, h_0 \rangle h_1 = \rho \tau h_1 \langle \rho, h_0, h_1 \rangle = \rho (\tau h_1)^2,$$

and

$$\rho a = \rho \langle h_0, h_1, \tau h_1 \cdot h_1 \rangle = \langle \rho, h_0, h_1 \rangle \tau h_1 \cdot h_1 = h_1 (\tau h_1)^2.$$

See Table 6 in Section 8 for more details on these Massey products, whose indeterminacies are all zero. □

Remark 6.3. For comparison purposes, we recall from [Hill 2011, Theorem 3.1] that

$$\mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)} \cong \mathbb{F}_2[\rho, \tau^4, h_0, \tau^2 h_0, v_1]/(\rho h_0, \rho^3 v_1, (\tau^2 h_0)^2 + \tau^4 h_0^2).$$

The ρ-torsion is created by the Bockstein differentials $d_1(\tau) = \rho h_0$ and $d_3(\tau^2) = \rho^3 v_1$. The class v_1 is in degree $(s, f, w) = (2, 1, 1)$.

Proposition 6.4. *The ring homomorphism* $q_* : \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)} \to \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}$ *induced by the quotient* $q : \mathcal{A}^R(1)_* \to \mathcal{E}^{\mathbb{R}}(1)_*$ *of Hopf algebroids is given as in Table 3.*

Proof. Many of the values $q_*(x)$ are already true over $\mathbb{C}$ and follow easily from their descriptions in the May spectral sequence. For instance, b is represented by $h_{2,1}^4$, and v_1 is represented by $h_{2,1}$. On the other hand, the value $q_*(\tau h_1)$ is most easily seen using the cobar complex. The class τh_1 in $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$ is represented by $\tau\xi_1 + \rho\tau_1$. This maps to $\rho\tau_1$ in the cobar complex for $\mathcal{E}^{\mathbb{R}}(1)$ and represents the class ρv_1 there. □

7. Bockstein differentials in the negative cone

We finally come to the key step in our calculation of $\mathrm{Ext}_{C_2}(1)$. We are now ready to analyze the ρ-Bockstein differentials associated to the negative cone, i.e., to the spectral sequence E^- of Proposition 3.1. We already analyzed the spectral sequence E^+ in Section 6.

7A. ***The structure of E_1^-.*** First, we need some additional information about the algebraic structure of E_1^-. Since $E_1 = E_1^+ \oplus E_1^-$ is defined in terms of Ext groups, it is a ring and has higher structure in the form of Massey products. The subobject E_1^- is a module over E_1^+, and it possesses Massey products of the form $\langle x_1, \dots, x_n, y\rangle$, where $x_1, \dots, x_n$ belong to E_1^+ and y belongs to E_1^-.

Definition 7.1. Suppose that x is a nonzero element of $\mathrm{Ext}_{\mathbb{C}}(1)$ such that τx is zero. According to Remark 3.5, for each $s \geq 0$, the element x gives rise to a copy of $\mathbb{M}_2^{\mathbb{C}}/\tau$ in $\mathrm{Tor}_{\mathbb{M}_2^{\mathbb{C}}}\big(\frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}, \mathrm{Ext}_{\mathbb{C}}(1)\big)\big\{\frac{\gamma}{\rho^s}\big\}$ that is infinitely divisible by ρ. In particular, it gives a nonzero element of the Tor group. Let $\frac{Q}{\rho^s}x$ be any lift to E_1^- of this nonzero element.

Remark 7.2. There is indeterminacy in the choice of Qx which arises from the first term of the short exact sequence for E_1^- in Proposition 3.1.

Lemma 7.3. *The element Qx of E_1^- is contained in the Massey product $\big\langle x, \tau, \frac{\gamma}{\tau}\big\rangle$.*

Proof. If $d(u) = \tau \cdot x$ in the cobar complex for $\mathrm{Ext}_{\mathbb{C}}(1)$, then $\frac{\gamma}{\tau}u$ is a cycle, since $\tau\frac{\gamma}{\tau} = 0$. This cycle $\frac{\gamma}{\tau}u$ represents both the Massey product as well as Qx. □

Remark 7.4. The most important example is the element Qh_1^3, which is defined because τh_1^3 equals zero in $\mathrm{Ext}_{\mathbb{C}}(1)$. Another possible name for Qh_1^3 is $\frac{\gamma}{\tau}v_1^2$, since v_1^2 is the element of the May spectral sequence that creates the relation τh_1^3.

Remark 7.5. Beware that the Massey product description for Qx holds in E_1^-, not in $\mathrm{Ext}_{C_2}(1)$. In fact, we have already seen in Section 6 that τ is not a permanent cycle in the ρ-Bockstein spectral sequence.

Nevertheless, minor variations on these Massey products may exist in $\mathrm{Ext}_{C_2}(1)$. For example, $\langle h_1^2, \tau h_1, \frac{\gamma}{\tau}\rangle$ equals Qh_1^3 in $\mathrm{Ext}_{C_2}(1)$.

We can now deduce a specific computational property of E_1^- that we will need later.

mw	(s, f, w)	element
0	$(-1, 0, -1)$	ρ
0	$(0, 1, 0)$	h_0
0	$(1, 1, 1)$	h_1
1	$(0, 0, -1)$	τ
2	$(4, 3, 2)$	a
4	$(8, 4, 4)$	b
0	$(4, 2, 4)$	Qh_1^3
$-k-1$	$(0, 0, k+1)$	$\frac{\gamma}{\tau^k}$

Table 4. Generators for the Bockstein E_1-page.

Lemma 7.6. *In E_1^-, there is a relation $h_0 \cdot Qh_1^3 = \frac{\gamma}{\tau} a$.*

Proof. Use Lemma 7.3 and the Massey product shuffle

$$h_0 \cdot Qh_1^3 = h_0 \left\langle h_1^3, \tau, \frac{\gamma}{\tau} \right\rangle = \langle h_0, h_1^3, \tau \rangle \frac{\gamma}{\tau} = \frac{\gamma}{\tau} a. \qquad \square$$

Table 4 gives multiplicative generators for the Bockstein E_1-page. The elements above the horizontal line are multiplicative generators for E_1^+. The elements below the horizontal generate E_1^- in the following sense. Every element of E_1^- can be formed by starting with one of the these listed elements, multiplying by elements of E_1^+, and then dividing by ρ. The elements in Table 7 are not multiplicative generators for $\mathrm{Ext}_{C_2}(1)$ in the usual sense, because we allow for division by ρ. The point of this notational approach is that the elements of E_1^- and of $\mathrm{Ext}_{\mathrm{NC}}$ are most easily understood as families of ρ-divisible elements.

7B. ***ρ-Bockstein differentials in E^-.*** Our next goal is to analyze the ρ-Bockstein differentials in E^-. We will rely heavily on the ρ-Bockstein differentials for E^+ established in Section 6, using that E^- is an E^+-module.

As an E_1^+-module, E_1^- is generated by the elements $\frac{\gamma}{\rho^j \tau^k}$ and $\frac{Q}{\rho^j} h_1^3$. This arises from the observation that the τ torsion in $\mathrm{Ext}_{\mathbb{C}}(1)$ is generated as an $\mathrm{Ext}_{\mathbb{C}}(1)$-module by h_1^3.

Proposition 7.7 gives the values of the ρ-Bockstein d_1 differential on these generators of E_1^-. All other d_1 differentials can then be deduced from the Leibniz rule and the E_1^+-module structure.

All of the differentials in E^- are infinitely divisible by ρ, in the following sense. When we claim that $d_r(x) = y$, we also have differentials $d_r\left(\frac{x}{\rho^j}\right) = \frac{y}{\rho^j}$ for all $j \geq 0$. For example, in Proposition 7.7, the formula $d_1\left(\frac{\gamma}{\rho\tau}\right) = \frac{\gamma}{\tau^2} h_0$ implies that

$$d_1\left(\frac{\gamma}{\rho^{j+1}\tau}\right) = \frac{\gamma}{\rho^j \tau^2} h_0 \quad \text{for all } j \geq 0.$$

Proposition 7.7. *For all $k \geq 0$,*

(1) $$d_1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right) = \frac{\gamma}{\tau^{2k+2}}h_0,$$

(2) $$d_1\left(\frac{Q}{\rho}h_1^3\right) = \frac{\gamma}{\tau^2}a.$$

These differentials are infinitely divisible by ρ.

Proof. We give three proofs for the first formula. First, it follows from

$$\mathrm{Sq}^1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right) = \frac{\gamma}{\tau^{2k+2}},$$

using the relationship between d_1 and the left and right units of the Hopf algebroid. Second, we have

$$\begin{aligned} 0 = d_1\left(\tau^{2k+1}\frac{\gamma}{\rho\tau^{2k+1}}\right) &= \tau^{2k+1}d_1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right) + \frac{\gamma}{\rho\tau^{2k+1}}\rho\tau^{2k}h_0 \\ &= \tau^{2k+1}d_1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right) + \frac{\gamma}{\tau}h_0. \end{aligned}$$

Third, we can use Proposition 5.2 to conclude that the infinitely ρ-divisible elements $\frac{\gamma}{\tau^{2k+1}}$ cannot survive the ρ-Bockstein spectral sequence. The only possibility is that they support a d_1 differential.

For the second formula, use the first formula to determine that $d_1\left(\frac{\gamma}{\rho\tau}a\right) = \frac{\gamma}{\tau^2}h_0a$. Then use the relation of Lemma 7.6. Alternatively, this differential is also forced by Proposition 5.2. □

It is now straightforward to compute E_2^-, since the ρ-Bockstein d_1 differential is completely known. The charts in Section 12 depict E_2^- graphically.

Next, Proposition 7.8 gives a ρ-Bockstein d_2 differential in E_2^-. This is the essential calculation, in the sense that the d_2 differential is zero on all other E_2^+-module generators of E_2^-.

Proposition 7.8. $d_2\left(\frac{\gamma}{\rho^2\tau^{4k+2}}\right) = \frac{\gamma}{\tau^{4k+3}}h_1$ *for all $k \geq 0$. This differential is infinitely divisible by ρ.*

Proof. As for Proposition 7.7, we give three proofs. First, $\mathrm{Sq}^2\left(\frac{\gamma}{\rho^2\tau^{4k+2}}\right) = \frac{\gamma}{\tau^{4k+3}}$. Second, we have

$$\begin{aligned} 0 = d_2\left(\tau^{4k+2}\frac{\gamma}{\rho^2\tau^{4k+2}}\right) &= \tau^{4k+2}d_2\left(\frac{\gamma}{\rho^2\tau^{4k+2}}\right) + \rho^2\tau^{4k+1}\frac{\gamma}{\rho^2\tau^{4k+2}}h_1 \\ &= \tau^{4k+2}d_2\left(\frac{\gamma}{\rho^2\tau^{4k+2}}\right) + \frac{\gamma}{\tau}h_1. \end{aligned}$$

Third, use Proposition 5.2 to conclude that the infinitely ρ-divisible elements $\frac{\gamma}{\tau^{4k+1}}$ cannot survive the ρ-Bockstein spectral sequence. The only possibility is that they support a d_2 differential. □

At this point, the behavior of E^- becomes qualitatively different from E^+. For E^+, there are nonzero d_3 differentials, and then the E_4^+-page equals the E_∞^+-page.

For E^-, it turns out that the d_r differential is nonzero for infinitely many values of r. This does not present a convergence problem, because there are only finitely many nonzero differentials in any given degree. One consequence is that the orders of the ρ-torsion in $\mathrm{Ext}_{C_2}(1)$ are unbounded. In other words, for every s, there exists an element x of such that $\rho^s x$ is nonzero but $\rho^{s+t} x$ is zero for some $t > 0$. This is fundamentally different from $\mathrm{Ext}_{\mathbb{R}}(1)$, where $\rho^3 x$ is zero if x is not ρ-torsion free.

Proposition 7.9 makes explicit these higher differentials.

Proposition 7.9. *For all $k \geq 1$,*

(1) $$d_{4k}\left(\frac{Q}{\rho^{4k}} h_1^{4k}\right) = \frac{\gamma}{\tau^{4k}} b^k,$$

(2) $$d_{4k+1}\left(\frac{Q}{\rho^{4k+1}} h_1^{4k+3}\right) = \frac{\gamma}{\tau^{4k+2}} ab^k.$$

These differentials are infinitely divisible by ρ.

Proof. We know that $\frac{\gamma}{\tau^{4k}}$ and b are permanent cycles. On the other hand, in $\mathrm{Ext}_{C_2}(1)$ the relation $\tau^4 h_1^4 = \rho^4 b$ gives

$$\frac{\gamma}{\tau^{4k}} b^k = \rho^4 \frac{\gamma}{\rho^4 \tau^{4k}} b^k = \tau^4 \frac{\gamma}{\rho^4 \tau^{4k}} h_1^4 b^{k-1}.$$

Thus $\frac{\gamma}{\tau^{4k}} b^k$ is h_1-divisible, which implies that it must be zero in $\mathrm{Ext}_{C_2}(1)$, as there is no surviving class in the appropriate degree to support the h_1-multiplication. The only Bockstein differential that could hit $\frac{\gamma}{\tau^{4k}} b^k$ is the claimed one.

For the second formula, the classes $\frac{\gamma}{\tau^{4k+2}} a$ and b are permanent cycles, yet

$$\frac{\gamma}{\tau^{4k+2}} ab^k = \rho^4 \frac{\gamma}{\rho^4 \tau^{4k+2}} ab^k = \tau^4 \frac{\gamma}{\rho^4 \tau^{4k+2}} a h_1^4 b^{k-1}$$

in $\mathrm{Ext}_{C_2}(1)$. But $h_1 a = 0$, so $\frac{\gamma}{\tau^{4k+2}} ab^k$ must be zero in $\mathrm{Ext}_{C_2}(1)$, forcing the claimed differential.

Alternatively, one can use Proposition 5.2 to obtain both differentials. □

Table 5 summarizes the Bockstein differentials that we computed in Sections 6 and 7B. The differentials above the horizontal line occur in E^+, while the differentials below the horizontal line occur in E^- and are infinitely divisible by ρ.

The ρ-Bockstein differentials of Sections 6 and 7 allow us to completely compute the E_∞-page of the ρ-Bockstein spectral sequence for $\mathrm{Ext}_{C_2}(1)$.

mw	(s, f, w)	element	r	d_r	proof
1	$(0, 0, -1)$	τ	1	ρh_0	Prop. 6.1
2	$(0, 0, -2)$	τ^2	2	$\rho^2 \tau h_1$	Prop. 6.1
3	$(2, 2, -1)$	$\tau^3 h_1^2$	3	$\rho^3 a$	Prop. 6.1
$-2k-2$	$(1, 0, 2k+3)$	$\frac{\gamma}{\rho\tau^{2k+1}}$	1	$\frac{\gamma}{\tau^{2k+2}} h_0$	Prop. 7.7
0	$(5, 2, 5)$	$\frac{Q}{\rho} h_1^3$	1	$\frac{\gamma}{\tau^2} a$	Prop. 7.7
$-4k-3$	$(2, 0, 4k+5)$	$\frac{\gamma}{\rho^2\tau^{4k+2}}$	2	$\frac{\gamma}{\tau^{4k+3}} h_1$	Prop. 7.8
0	$(8k+1, 4k-1, 8k+1)$	$\frac{Q}{\rho^{4k}} h_1^{4k}$	$4k$	$\frac{\gamma}{\tau^{4k}} b^k$	Prop. 7.9
0	$(8k+5, 4k+2, 8k+5)$	$\frac{Q}{\rho^{4k+1}} h_1^{4k+3}$	$4k+1$	$\frac{\gamma}{\tau^{4k+2}} ab^k$	Prop. 7.9

Table 5. Bockstein differentials.

7C. *ρ-Bockstein differentials in E^- for $\mathcal{E}^{C_2}(1)$.* For comparison, we also carry out the analogous but easier computation over $\mathcal{E}^{C_2}(1)$ rather than $\mathcal{A}^{C_2}(1)$. Besides $d_1\big(\frac{\gamma}{\rho\tau^{2k+1}}\big) = \frac{\gamma}{\tau^{2k+2}} h_0$, the only other Bockstein differential is given in the following result.

Proposition 7.10. *$d_3\big(\frac{\gamma}{\rho^3\tau^{4k+2}}\big) = \frac{\gamma}{\tau^{4k+4}} v_1$ for all $k \geq 0$. This differential is infinitely divisible by ρ.*

Proof. The differential $d_3(\tau^2) = \rho^3 v_1$ of Remark 6.3 gives

$$0 = d_3\left(\tau^{4k+2} \frac{\gamma}{\rho^3\tau^{4k+2}}\right) = \tau^{4k+2} d_3\left(\frac{\gamma}{\rho^3\tau^{4k+2}}\right) + \rho^3\tau^{4k} \frac{\gamma}{\rho^3\tau^{4k+2}} v_1$$
$$= \tau^{4k+2} d_3\left(\frac{\gamma}{\rho^3\tau^{4k+2}}\right) + \frac{\gamma}{\tau^2} v_1. \qquad \square$$

8. Some Massey products

The final step in the computation of $\mathrm{Ext}_{C_2}(1)$ is to determine multiplicative extensions that are hidden in the ρ-Bockstein E_∞-page. In order to do this, we will need some Massey products in $\mathrm{Ext}_{C_2}(1)$. Table 6 summarizes the information that we will need.

Theorem 8.1. *Some Massey products in* $\mathrm{Ext}_{C_2}(1)$ *are given in Table 6. All have zero indeterminacy.*

Proof. For some Massey products in Table 6, a ρ-Bockstein differential is displayed in the last column. In these cases, May's convergence theorem [May 1969; Isaksen 2014, Chapter 2.2] applies, and the Massey product can be computed with the given differential. Roughly speaking, May's convergence theorem says that Massey products in $\mathrm{Ext}_{C_2}(1)$ can be computed with any ρ-Bockstein differential. Beware that

mw	(s, f, w)	bracket	contains	proof
1	$(1, 1, 0)$	$\langle \rho, h_0, h_1 \rangle$	τh_1	$d_1(\tau) = \rho h_0$
1	$(2, 2, 1)$	$\langle h_0, h_1, h_0 \rangle$	τh_1^2	classical
2	$(4, 3, 2)$	$\langle \tau h_1 \cdot h_1, h_1, h_0 \rangle$	a	classical
2	$(0, 1, -2)$	$\langle \rho \tau h_1, \rho, h_0 \rangle$	$\tau^2 h_0$	$d_2(\tau^2) = \rho^2 \tau h_1$
4	$(8, 5, 4)$	$\langle a, h_1, \tau h_1^2 \rangle$	$h_0 b$	classical
-4	$(0, 0, 4)$	$\langle \tau^2 h_0, \rho, \frac{\gamma}{\tau^6} \rangle$	$\frac{\gamma}{\tau^3}$	$d_1(\tau^3) = \rho \tau^2 h_0$
-4	$(0, 0, 4)$	$\langle h_0, \rho, \frac{\gamma}{\tau^4} \rangle$	$\frac{\gamma}{\tau^3}$	$d_1(\tau) = \rho h_0$
-3	$(1, 0, 4)$	$\langle \rho, \frac{\gamma}{\tau^4}, \tau h_1 \rangle$	$\frac{\gamma}{\rho \tau^2}$	$d_2\left(\frac{\gamma}{\rho^2 \tau^2}\right) = \frac{\gamma}{\tau^3} h_1$
-3	$(0, 0, 3)$	$\langle \rho \tau h_1, \rho, \frac{\gamma}{\tau^4} \rangle$	$\frac{\gamma}{\tau^2}$	$d_2(\tau^2) = \rho^2 \tau h_1$
-2	$(4, 2, 6)$	$\langle \frac{\gamma}{\tau^3}, h_1, \tau h_1 \cdot h_1 \rangle$	$\frac{\gamma}{\rho^2 \tau} h_1^2$	$d_2\left(\frac{\gamma}{\rho^2 \tau^2}\right) = \frac{\gamma}{\tau^3} h_1$
-2	$(0, 0, 2)$	$\langle \tau^2 h_0, \rho, \frac{\gamma}{\tau^4} \rangle$	$\frac{\gamma}{\tau}$	$d_1(\tau^3) = \rho \tau^2 h_0$
-2	$(0, 0, 2)$	$\langle h_0, \rho, \frac{\gamma}{\tau^2} \rangle$	$\frac{\gamma}{\tau}$	$d_1(\tau) = \rho h_0$
-2	$(2, 1, 4)$	$\langle h_1, h_0, \frac{\gamma}{\tau^2} \rangle$	$\frac{\gamma}{\rho \tau} h_1$	$d_1\left(\frac{\gamma}{\rho \tau}\right) = \frac{\gamma}{\tau^2} h_0$
0	$(4,2,4) + (8k,4k,8k)$	$\langle \rho, \frac{\gamma}{\tau^{4k+2}}, ab^k \rangle$	$\frac{Q}{\rho^{4k}} h_1^{4k+3}$	$d_{4k+1}\left(\frac{Q}{\rho^{4k+1}} h_1^{4k+3}\right) = \frac{\gamma}{\tau^{4k+2}} ab^k$
0	$(8,3,8) + (8k,4k,8k)$	$\langle \rho, \frac{\gamma}{\tau^{4k+4}}, b^{k+1} \rangle$	$\frac{Q}{\rho^{4k+3}} h_1^{4k+4}$	$d_{4k+4}\left(\frac{Q}{\rho^{4k+4}} h_1^{4k+4}\right) = \frac{\gamma}{\tau^{4k+4}} b^{k+1}$

Table 6. Some Massey products in $\mathrm{Ext}_{C_2}(1)$.

May's Convergence Theorem requires technical hypotheses involving "crossing differentials" that are not always satisfied. Failure to check these conditions can lead to mistaken calculations.

The proofs for other Massey products in Table 6 are described as "classical". In these cases, the Massey product already occurs in $\mathrm{Ext}_{\mathrm{cl}}$. □

Remark 8.2. The eight Massey products in the middle Section of Table 6 are only the first examples of infinite families that are τ^4-periodic. For example, $\langle \tau^2 h_0, \rho, \frac{\gamma}{\tau^{4k+6}} \rangle$ equals $\frac{\gamma}{\tau^{4k+3}}$ for all $k \geq 0$, and $\langle \rho, \frac{\gamma}{\tau^{4k+4}}, \tau h_1 \rangle$ equals $\frac{\gamma}{\tau^{4k+3}}$ for all $k \geq 0$.

9. Hidden extensions

We now determine multiplicative extensions that are hidden in the ρ-Bockstein E_∞-page. We have already determined some of these hidden extensions in Section 6. In this section, we establish additional hidden relations on elements associated with the negative cone. We have not attempted a completely exhaustive analysis of the ring structure of $\mathrm{Ext}_{C_2}(1)$.

Recall that $\mathrm{Ext}_{C_2}(1)$ is a square-zero extension of $\mathrm{Ext}_{\mathbb{R}}(1)$. This eliminates many possible hidden extensions. For example, $(Qh_1^3)^2$ is zero in $\mathrm{Ext}_{C_2}(1)$.

Proposition 9.1. *For all $k \geq 0$,*

(1) $$h_0 \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \frac{\gamma}{\tau^{4k+1}} ab^k,$$

(2) $$a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \frac{\gamma}{\tau^{4k+1}} h_0 b^{k+1}.$$

Proof. (1) $h_0 \langle \rho, \frac{\gamma}{\tau^{4k+2}}, ab^k \rangle = \langle h_0, \rho, \frac{\gamma}{\tau^{4k+2}} \rangle ab^k$.

(2) Using part (1), we have that

$$h_0 a \cdot \frac{Q}{\rho^{4k} h_1^{4k+3}} = a \cdot \frac{\gamma}{\tau^{4k+1}} ab^k = \frac{\gamma}{\tau^{4k+1}} h_0^2 b^{k+1},$$

which is nonzero. Therefore, $a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3}$ must also be nonzero, and $\frac{\gamma}{\tau^{4k+1}} h_0 b^{k+1}$ is the only nonzero class in the appropriate tridegree. □

Proposition 9.2. *For all $k \geq 1$,*

(1) $$\tau^2 a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \frac{\gamma}{\tau^{4k-1}} h_0 b^{k+1} + \frac{Q}{\rho^{4k-3}} h_1^{4k+2} b,$$

(2) $$\tau^4 \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \frac{Q}{\rho^{4k-4}} h_1^{4k-1} b,$$

(3) $$\tau^2 h_0 \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \frac{\gamma}{\tau^{4k-1}} ab^k.$$

Proof. (1) Using Proposition 9.1(1), we have that

$$h_0 \cdot \tau^2 a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \tau^2 a \cdot \frac{\gamma}{\tau^{4k+1}} ab^k = \frac{\gamma}{\tau^{4k-1}} h_0^2 b^{k+1},$$

which is nonzero. Hence $\tau^2 a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3}$ is either $\frac{\gamma}{\tau^{4k-1}} h_0 b^{k+1}$ or $\frac{\gamma}{\tau^{4k-1}} h_0 b^{k+1} + \frac{Q}{\rho^{4k-3}} h_1^{4k+2} b$.

On the other hand,

$$h_1 \cdot \tau^2 a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \rho^3 b \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \frac{Q}{\rho^{4k-3}} h_1^{4k+3} b.$$

Therefore, $\tau^2 a \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3}$ must equal $\frac{\gamma}{\tau^{4k-1}} h_0 b^{k+1} + \frac{Q}{\rho^{4k-3}} h_1^{4k+2} b$.

(2) Using Proposition 9.1(1), we have that

$$h_0 \cdot \tau^4 \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3} = \tau^4 \frac{\gamma}{\tau^{4k+1}} ab^k = \frac{\gamma}{\tau^{4k-3}} ab^k,$$

which is nonzero. This shows that $\tau^4 \cdot \frac{Q}{\rho^{4k}} h_1^{4k+3}$ is also nonzero, and there is just one possible value.

(3) $$\tau^2 h_0 \left\langle \rho, \frac{\gamma}{\tau^{4k+2}}, ab^k \right\rangle = \left\langle \tau^2 h_0, \rho, \frac{\gamma}{\tau^{4k+2}} \right\rangle ab^k.$$ □

Proposition 9.3. *For all $k \geq 0$,*

(1) $$h_0 \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = \frac{\gamma}{\tau^{4k+3}} b^{k+1},$$

(2) $$\tau h_1 \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = \frac{\gamma}{\rho \tau^{4k+2}} b^{k+1},$$

(3) $$\tau^2 h_0 \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = \frac{\gamma}{\tau^{4k+1}} b^{k+1},$$

(4) $$a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2 b^{k+1},$$

(5) $$\tau^2 a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = \frac{Q}{\rho^{4k}} h_1^{4k+3} b.$$

Proof. (1) $h_0 \langle \rho, \frac{\gamma}{\tau^{4k+4}}, b^{k+1} \rangle = \langle h_0, \rho, \frac{\gamma}{\tau^{4k+4}} \rangle b^{k+1}$.

(2) $\rho \tau h_1 \langle \rho, \frac{\gamma}{\tau^{4k+4}}, b^{k+1} \rangle = \langle \rho \tau h_1, \rho, \frac{\gamma}{\tau^{4k+4}} \rangle b^{k+1}$.

(3) $\tau^2 h_0 \langle \rho, \frac{\gamma}{\tau^{4k+4}}, b^{k+1} \rangle = \langle \tau^2 h_0, \rho, \frac{\gamma}{\tau^{4k+4}} \rangle b^{k+1}$.

(4) Using part (1), we have that

$$h_0 a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = a \cdot \frac{\gamma}{\tau^{4k+3}} b^{k+1} = \frac{\gamma}{\tau^{4k+3}} ab^{k+1},$$

which is nonzero. Therefore, $a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4}$ must also be nonzero, and there is just one possibility.

(5) Using part (1), we have that

$$h_0 \cdot \tau^2 a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4} = \tau^2 a \cdot \frac{\gamma}{\tau^{4k+3}} b^{k+1} = \frac{\gamma}{\tau^{4k+1}} ab^{k+1},$$

which is nonzero. This shows that $\tau^2 a \cdot \frac{Q}{\rho^{4k+3}} h_1^{4k+4}$ is also nonzero, and there is just one possible value. □

Proposition 9.4. *For all $k \geq 0$,*

(1) $$h_0 \cdot \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2 = \frac{\gamma}{\tau^{4k+3}} a,$$

(2) $$a \cdot \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2 = \frac{\gamma}{\tau^{4k+3}} h_0 b,$$

(3) $$\tau^2 a \cdot \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2 = \frac{\gamma}{\tau^{4k+1}} h_0 b.$$

Proof. (1) $\left\langle \frac{\gamma}{\tau^{4k+3}}, h_1, \tau h_1 \cdot h_1 \right\rangle h_0 = \frac{\gamma}{\tau^{4k+3}} \langle h_1, \tau h_1 \cdot h_1, h_0 \rangle$.

(2) Using part (1), we have that

$$h_0 a \cdot \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2 = a \cdot \frac{\gamma}{\tau^{4k+3}} a,$$

which equals $\frac{\gamma}{\tau^{4k+3}} h_0^2 b$ modulo a possible error term involving higher powers of ρ. Using that $h_1 a = 0$, we conclude that the error term is zero.

(3) Using part (1), we have that

$$h_0 \cdot \tau^2 a \cdot \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2 = \tau^2 a \cdot \frac{\gamma}{\tau^{4k+3}} a = \frac{\gamma}{\tau^{4k+1}} h_0^2 b,$$

which is nonzero. This shows that $\tau^2 a \cdot \frac{\gamma}{\rho^2 \tau^{4k+1}} h_1^2$ is also nonzero, and there is just one possible value. □

Proposition 9.5. *For all $k \geq 0$,*

(1) $$h_0 \cdot \frac{\gamma}{\rho \tau^{4k+1}} h_1 = \frac{\gamma}{\tau^{4k+1}} h_1^2,$$

(2) $$h_0 \cdot \frac{\gamma}{\rho \tau^{4k+2}} = \frac{\gamma}{\tau^{4k+2}} h_1.$$

Proof. All of these extensions follow from Massey product shuffles:

(1) $h_0 \left\langle h_1, h_0, \frac{\gamma}{\tau^{4k+2}} \right\rangle = \langle h_0, h_1, h_0 \rangle \frac{\gamma}{\tau^{4k+2}}$.

(2) $h_0 \left\langle \rho, \frac{\gamma}{\tau^{4k+4}}, \tau h_1 \right\rangle = \left\langle h_0, \rho, \frac{\gamma}{\tau^{4k+4}} \right\rangle \tau h_1$. □

Proposition 9.6. *For all $k \geq 0$,*

(1) $$h_1 \cdot \frac{\gamma}{\rho \tau^{4k+4}} h_1^2 = \frac{\gamma}{\tau^{4k+6}} a,$$

(2) $$h_1 \cdot \frac{\gamma}{\rho^3 \tau^{4k+6}} a = \frac{\gamma}{\tau^{4k+8}} b.$$

Proof. (1) $\tau h_1 \cdot h_1 \left\langle h_1, h_0, \frac{\gamma}{\tau^{4k+6}} \right\rangle = \langle \tau h_1 \cdot h_1, h_1, h_0 \rangle \frac{\gamma}{\tau^{4k+6}}$. Alternatively, this h_1 extension is forced by Lemma 5.1.

(2) We have

$$h_1 \cdot \frac{\gamma}{\rho^3 \tau^{4k+6}} a = \frac{\gamma}{\rho^3 \tau^{4k+8}} h_1 \cdot \tau^2 a = \frac{\gamma}{\rho^3 \tau^{4k+8}} \rho^3 b = \frac{\gamma}{\tau^{4k+8}},$$

where the second equality follows from Table 2. □

Over $\mathcal{E}^{C_2}(1)$, the only hidden multiplication is

Proposition 9.7. *In* $\mathrm{Ext}_{\mathcal{E}^{C_2}(1)}$, *we have* $h_0 \cdot \frac{\gamma}{\rho^2 \tau^{4k+2}} v_1^n = \frac{\gamma}{\tau^{4k+3}} v_1^{n+1}$. *for all* $k, n \geq 0$.

Proof. $$h_0 \cdot \frac{\gamma}{\rho^2 \tau^2} = h_0 \left\langle \rho, \frac{\gamma}{\tau^4}, v_1 \right\rangle = \left\langle h_0, \rho, \frac{\gamma}{\tau^4} \right\rangle v_1 = \frac{\gamma}{\tau^3} v_1.$$ □

mw	(s, f, w)	element
0	$(-1, 0, -1)$	ρ
0	$(0, 1, 0)$	h_0
0	$(1, 1, 1)$	h_1
1	$(1, 1, 0)$	τh_1
2	$(0, 1, -2)$	$\tau^2 h_0$
2	$(4, 3, 2)$	a
4	$(0, 0, -4)$	τ^4
4	$(4, 3, 0)$	$\tau^2 a$
4	$(8, 4, 4)$	b
$-k-1$	$(0, 0, k+1)$	$\frac{\gamma}{\tau^k}$
0	$(4, 2, 4)$	Qh_1^3

Table 7. Generators for $\mathrm{Ext}_{C_2}(1)$.

9A. $\mathrm{Ext}_{C_2}(1)$. The charts in Section 12 depict $\mathrm{Ext}_{C_2}(1)$ graphically. Table 7 gives generators for $\mathrm{Ext}_{C_2}(1)$. The elements above the horizontal line are multiplicative generators for $\mathrm{Ext}_{\mathbb{R}}(1)$. The elements below the horizontal generate $\mathrm{Ext}_{\mathrm{NC}}$ in the following sense. Every element of $\mathrm{Ext}_{\mathrm{NC}}$ can be formed by starting with one of these listed elements, multiplying by elements of $\mathrm{Ext}_{\mathbb{R}}(1)$, and then dividing by ρ.

The elements in Table 7 are not multiplicative generators for $\mathrm{Ext}_{C_2}(1)$ in the usual sense, because we allow for division by ρ. For example, $\frac{\gamma}{\rho^2\tau}h_1^2$ is indecomposable in the usual sense, yet it does not appear in Table 7 because $\rho^2 \cdot \frac{\gamma}{\rho^2\tau}h_1^2 = \frac{\gamma}{\tau}h_1^2$ is decomposable.

The point of this notational approach is that the elements of $\mathrm{Ext}_{\mathrm{NC}}$ are most easily understood as families of ρ-divisible elements.

9B. *The ring homomorphism* $q_* : \mathrm{Ext}_{\mathcal{A}^{C_2}(1)} \to \mathrm{Ext}_{\mathcal{E}^{C_2}(1)}$. It is worthwhile to consider the comparison to $\mathrm{Ext}_{\mathcal{E}^{C_2}(1)}$. We already described the map on the summand arising from the positive cone in Proposition 6.4. The map on the summand for the negative cone is given as follows.

Proposition 9.8. *The homomorphism* $q_* : \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}(\mathrm{NC}, \mathbb{M}_2^{\mathbb{R}}) \to \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}(\mathrm{NC}, \mathbb{M}_2^{\mathbb{R}})$ *induced by the quotient* $q : \mathcal{A}^R(1)_* \to \mathcal{E}^{\mathbb{R}}(1)$ *of Hopf algebroids is given as in Table 8.*

Proof. For the classes of the form $\frac{\gamma}{\rho^j\tau^k}$, this is true on the cobar complex. For the classes of the form $\frac{Q}{\rho^j}h_1^n$, this follows from the h_0-extension given in Proposition 9.1 and the value $q_*(a) = h_0 v_1^2$. Similarly, the value on $\frac{\gamma}{\rho^2\tau}h_1^2$ is obtained by combining

mw	(s, f, w)	$x \in \mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}\,\mathrm{NC}$	$q_* x \in \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}\,\mathrm{NC}$
0	$(4, 2, 4) + k(8, 4, 4)$	$\frac{Q}{\rho^{4k}} h_1^{4k+3}$	$\frac{\gamma}{\tau} v_1^{4k+2}$
0	$(8, 3, 8) + k(8, 4, 4)$	$\frac{Q}{\rho^{4k+4}} h_1^{4k+3}$	$\frac{\gamma}{\rho^2\tau^2} v_1^{4k+3}$
−2	$(0, 0, 2)$	$\frac{\gamma}{\tau}$	$\frac{\gamma}{\tau}$
−2	$(2, 1, 4)$	$\frac{\gamma}{\rho\tau} h_1$	$\frac{\gamma}{\tau^2} v_1$
−2	$(4, 2, 6)$	$\frac{\gamma}{\rho^2\tau} h_1^2$	$\frac{\gamma}{\tau^3} v_1^2$
−3	$(1, 0, 4)$	$\frac{\gamma}{\rho\tau^2}$	$\frac{\gamma}{\rho\tau^2}$
−5	$(0, 0, 5)$	$\frac{\gamma}{\tau^4}$	$\frac{\gamma}{\tau^4}$

Table 8. The homomorphism $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}(\mathrm{NC}) \to \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}(\mathrm{NC})$.

Proposition 9.4 with the value of $q_*(a)$. Lastly, the value on $\frac{\gamma}{\rho\tau} h_1$ follows from $q_*(\tau h_1) = \rho v_1$. □

Remark 9.9. Note that, on the other hand, the hidden h_0-extensions on classes in $\mathrm{Ext}_{\mathcal{A}^{C_2}(1)}$, such as Qh_1^3, can also be deduced from the homomorphism q_* if its values are determined by other means.

10. The spectrum $\mathbf{ko}_{C_2}$

Let **Sp** denoted the category of spectra, and let $\mathbf{Sp}^{C_2}$ denote the category of "genuine" C_2-spectra [May 1996, Chapter XII], obtained from the category of based C_2-spaces by inverting suspension with respect to the one-point compactification $S^{2,1}$ of the regular representation $(\mathbb{C}, z \mapsto \bar{z})$. There are restriction and fixed-point functors

$$\iota^* : \mathrm{Ho}(\mathbf{Sp}^{C_2}) \to \mathrm{Ho}(\mathbf{Sp}), \quad (-)^{C_2} : \mathrm{Ho}(\mathbf{Sp}^{C_2}) \to \mathrm{Ho}(\mathbf{Sp})$$

which detect the homotopy theory of C_2-spectra, meaning that a map f in $\mathrm{Ho}(\mathbf{Sp}^{C_2})$ is an equivalence if and only if $\iota^*(f)$ and f^{C_2} are equivalences in $\mathrm{Ho}(\mathbf{Sp})$. Moreover, a sequence $X \to Y \to Z$ is a cofiber sequence in $\mathrm{Ho}(\mathbf{Sp}^{C_2})$ if and only if applying both functors ι^* and $(-)^{C_2}$ yield cofiber sequences. Both statements follow from the fact [Schwede and Shipley 2003, Example 3.4(i)] that the pair of C_2-spectra $\{\Sigma^{\infty}_{C_2} S^0, \Sigma^{\infty}_{C_2} C_{2+}\}$ give a compact generating set for $\mathrm{Ho}(\mathbf{Sp}^{C_2})$. Beware that we are discussing categorical fixed-point spectra here, not geometric fixed-point spectra.

Recall (see [Lewis 1995, Proposition 3.3]) that for a C_2-spectrum X, the equivariant connective cover $X\langle 0\rangle \xrightarrow{q} X$ is a C_2-spectrum such that:

(1) $\iota(q)$ is the connective cover of the underlying spectrum X, and

(2) q^{C_2} is the connective cover of X^{C_2}.

Recall that KO_{C_2} is the C_2-spectrum representing the $\mathbb{K}$-theory of C_2-equivariant real vector bundles [May 1996, Chapter XIV].

Definition 10.1. Let ko_{C_2} be the equivariant connective cover $\mathrm{KO}_{C_2}\langle 0\rangle$ of KO_{C_2}.

We also have a description from the point of view of equivariant infinite loop space theory.

Theorem 10.2 [Merling 2017, Theorem 7.1]. $\mathrm{ko}_{C_2} \simeq \mathbb{K}_{C_2}(\mathbb{R})$, *where* $\mathbb{R}$ *is considered as a topological ring with trivial* C_2*-action.*

The underlying spectrum of ko_{C_2} is ko.

Lemma 10.3. *The fixed-point spectrum of* ko_{C_2} *is* $(\mathrm{ko}_{C_2})^{C_2} \simeq \mathrm{ko} \vee \mathrm{ko}$.

Proof. This is a specialization of the statement that, if X is any space equipped with a trivial G-action, then $\mathrm{KO}_G(X)$ is isomorphic to $\mathrm{RO}(G) \otimes \mathrm{KO}(X)$ [May 1996, Section XIV.2]. Alternatively, from the point of view of algebraic $\mathbb{K}$-theory, we have $\mathbb{K}_{C_2}(\mathbb{R})^{C_2} \simeq \mathbb{K}(\mathbb{R}[C_2])$ [Merling 2017, Theorem 1.2], and $\mathbb{R}[C_2] \cong \mathbb{R} \times \mathbb{R}$. It follows that

$$(\mathrm{ko}_{C_2})^{C_2} \simeq \mathbb{K}_{C_2}(\mathbb{R})^{C_2} \simeq \mathbb{K}(\mathbb{R}) \times \mathbb{K}(\mathbb{R}) \simeq \mathrm{ko} \vee \mathrm{ko}. \qquad \square$$

We are working towards a description of the C_2-equivariant cohomology of ko_{C_2} as the quotient $\mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1)$. This will allow us to express the E_2-page of the Adams spectral sequence for ko_{C_2} in terms of the cohomology of $\mathcal{A}^{C_2}(1)$. The main step will be to establish the cofiber sequence of Proposition 10.13. In preparation, we first prove some auxiliary results.

Definition 10.4. Let ρ be the element of $\pi_{-1,-1}$ determined by the inclusion $S^{0,0} \hookrightarrow S^{1,1}$ of fixed points.

Note that the element $\rho \in \pi_{-1,-1}$ induces multiplication by ρ in cohomology under the Hurewicz homomorphism.

Recall that the real C_2-representation ring $\mathrm{RO}(C_2)$ is a rank two free abelian group. Generators are given by the trivial one-dimensional representation 1 and the sign representation σ. Let $A(C_2)$ denote the Burnside ring of C_2, defined as the Grothendieck group associated to the monoid of finite C_2-sets. This is also a rank two free abelian group, with generators the trivial one-point C_2-set 1 and the free C_2-set C_2. As a ring, $A(C_2)$ is isomorphic to $\mathbb{Z}[C_2]/(C_2^2 - 2C_2)$.

The linearization map $A(C_2) \to \mathrm{RO}(C_2)$ sending a C_2-set to the induced permutation representation is an isomorphism, sending the free orbit C_2 to the regular representation $1 \oplus \sigma$. Recall that the Euler characteristic moreover gives an isomorphism from $A(C_2)$ to $\pi_0(S^{0,0})$ [Segal 1971, Corollary to Proposition 1].

Lemma 10.5. *The* C_2*-fixed point spectrum of* $\Sigma^{1,1}\mathrm{ko}_{C_2}$ *is equivalent to* ko.

Proof. Recall the cofiber sequence $C_{2+} \xrightarrow{\pi} S^{0,0} \xrightarrow{\rho} S^{1,1}$ of C_2-spaces. This yields a cofiber sequence

$$C_{2+} \wedge \mathrm{ko}_{C_2} \xrightarrow{\pi} \mathrm{ko}_{C_2} \xrightarrow{\rho} \Sigma^{1,1}\mathrm{ko}_{C_2}$$

of equivariant spectra. Passing to fixed point spectra gives the cofiber sequence

$$\mathrm{ko} \xrightarrow{\pi^{C_2}} \mathrm{ko} \vee \mathrm{ko} \xrightarrow{\rho^{C_2}} (\Sigma^{1,1}\mathrm{ko}_{C_2})^{C_2}.$$

In the analogous sequence for the sphere $S^{0,0}$, the map π^{C_2} is induced by the split inclusion $\mathbb{Z} \to A(C_2)$ sending 1 to the free orbit C_2. It follows that the map π^{C_2} is induced by the split inclusion $\mathbb{Z} \to \mathrm{RO}(C_2)$ that takes 1 to the regular representation ρ_{C_2}, and this induces a splitting of the cofiber sequence. Therefore, $(\Sigma^{1,1}\mathrm{ko}_{C_2})^{C_2}$ is equivalent to ko. □

Recall that $k\mathbb{R}$ denotes the equivariant connective cover $K\mathbb{R}\langle 0\rangle$ of Atiyah's K-theory "with reality" spectrum $K\mathbb{R}$ [Atiyah 1966]. The latter theory classifies complex vector bundles equipped with a conjugate-linear action of C_2. The underlying spectrum of $k\mathbb{R}$ is ku, and its fixed-point spectrum is ko.

Theorem 10.6 [Merling 2017, Theorem 7.2]. $k\mathbb{R} \simeq \mathbb{K}_{C_2}(\mathbb{C})$, *where* $\mathbb{C}$ *is considered as a topological ring with* C_2*-action given by complex conjugation.*

Definition 10.7. The C_2-equivariant Hopf map η is

$$\mathbb{C}^2 - \{0\} \to \mathbb{CP}^1 : (x, y) \mapsto [x : y],$$

where both source and target are given the complex conjugation action.

As $\mathbb{C} \cong \mathbb{R}[C_2]$, the punctured representation $\mathbb{C}^2 - \{0\}$ is homotopy equivalent to $S^{3,2}$, and $\mathbb{CP}^1$ is homeomorphic to $S^{2,1}$. It follows that η gives rise to a stable homotopy class in $\pi_{1,1}$.

Remark 10.8. The element η only defines a specific element of $\pi_{1,1}$ after choosing isomorphisms $\mathbb{C}^2 - \{0\} \cong S^{3,2}$ and $\mathbb{CP}^1 \cong S^{2,1}$ in the homotopy category. We follow the choices of [Dugger and Isaksen 2013, Example 2.12]. By Proposition C.5 of [Dugger and Isaksen 2013], with these choices, the induced map $\eta^{C_2} : S^1 \to S^1$ on fixed points is a map of degree -2.

Lemma 10.9. *The element* $\rho\eta$ *in* $\pi_{0,0}$ *corresponds to the element* $C_2 - 2$ *of* $A(C_2)$.

Proof. In $\pi_{0,0}$, we have $(\eta\rho)^2 = -2\eta\rho$ [Morel 2004, Lemma 6.1.2]. The nonzero solutions to $x^2 = -2x$ in $A(C_2)$ are $x = -2$, $x = C_2 - 2$, and $x = -C_2$. The only such solution which restricts to zero at the trivial subgroup is $x = C_2 - 2$. □

Lemma 10.10. *The induced map* $\eta^{C^2} : (\Sigma^{1,1}\mathrm{ko}_{C_2})^{C_2} \to (\mathrm{ko}_{C_2})^{C_2}$ *is equivalent to* $\mathrm{ko} \xrightarrow{(-1,1)} \mathrm{ko} \vee \mathrm{ko}$.

Proof. To determine the fixed map η^{C_2}, we use that a map $X \xrightarrow{\varphi} Y$ of C_2-spectra induces a commutative diagram

$$\begin{array}{ccc} X^{C_2} & \xrightarrow{\varphi^{C_2}} & Y^{C_2} \\ \downarrow & & \downarrow \\ X^e & \xrightarrow[\varphi^e]{} & Y^e \end{array}$$

in which the vertical maps are the inclusions of fixed points. In the case of η on ko_{C_2}, this gives the diagram

$$\begin{array}{ccc} \mathrm{ko} \simeq (\Sigma^{1,1}\mathrm{ko}_{C_2})^{C_2} & \xrightarrow{\eta^{C_2}} & \mathrm{ko} \vee \mathrm{ko} \simeq (\mathrm{ko}_{C_2})^{C_2} \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \nabla} \\ \Sigma^1\mathrm{ko} & \xrightarrow[\iota^*\eta]{} & \mathrm{ko} \end{array}$$

where ∇ is the fold map, as both the sign representation σ and the trivial representation 1 of C_2 restrict to the 1-dimensional trivial representation of the trivial group. This shows that η^{C_2} factors through the fiber of ∇, so that η^{C_2} must be of the form $(k, -k)$ for some integer k. On the other hand, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{ko} \otimes \mathrm{RO}(C_2) & \longrightarrow & \mathrm{ko} & \longrightarrow & \mathrm{ko} \otimes \mathrm{RO}(C_2) \\ \cong\uparrow & & \cong\uparrow & & \cong\uparrow \\ (\mathrm{ko}_{C_2})^{C_2} & \xrightarrow{\rho^{C_2}} & (\Sigma^{1,1}\mathrm{ko}_{C_2})^{C_2} & \xrightarrow{\eta^{C_2}} & (\mathrm{ko}_{C_2})^{C_2} \\ \uparrow & & \uparrow & & \uparrow \\ (S^{0,0})^{C_2} & \xrightarrow{\rho^{C_2}} & (S^{1,1})^{C_2} & \xrightarrow{\eta^{C_2}} & (S^{0,0})^{C_2} \end{array}$$

According to Lemma 10.9, on the sphere $\eta\rho$ induces multiplication by $(C_2 - 2)$ under the isomorphism $\pi_{0,0} \cong A(C_2)$. The outer vertical compositions induce the linearization isomorphism $A(C_2) \cong \mathrm{RO}(C_2)$ on π_0. It follows that the top row induces multiplication by $(\sigma - 1)$ on homotopy. We conclude that η^{C_2} is $(-1, 1)$. □

Definition 10.11. The complexification map $\mathrm{KO}_{C_2} \xrightarrow{c} K\mathbb{R}$ assigns to an equivariant real bundle $E \to X$ the associated bundle $\mathbb{C} \otimes_{\mathbb{R}} E \to X$, where C_2 acts on $\mathbb{C}$ via complex conjugation. We denote by $\mathrm{ko}_{C_2} \xrightarrow{c} k\mathbb{R}$ the associated map on connective covers.

Remark 10.12. Alternatively, from the point of view algebraic $\mathbb{K}$-theory, the complexification map can be described as $\mathbb{K}_{C_2}(\iota)$, where $\mathbb{R} \xrightarrow{\iota} \mathbb{C}$ is the inclusion of C_2-equivariant topological rings.

Proposition 10.13. *The Hopf map η induces a cofiber sequence*

$$\Sigma^{1,1}\mathrm{ko}_{C_2} \xrightarrow{\eta} \mathrm{ko}_{C_2} \xrightarrow{c} k\mathbb{R}. \tag{10-1}$$

Proof. On underlying spectra, this is the classical cofiber sequence

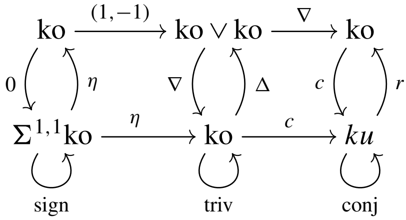
$$\Sigma\mathrm{ko} \xrightarrow{\eta} \mathrm{ko} \to ku.$$

On fixed points, according to Lemma 10.5 the sequence (10-1) induces a sequence

$$\mathrm{ko} \xrightarrow{\eta^{C_2}} \mathrm{ko} \vee \mathrm{ko} \xrightarrow{c^{C_2}} \mathrm{ko}.$$

By Lemma 10.10, the map η^{C_2} is of the form $(-1, 1)$. For any real C_2-representation V, the construction $\mathbb{C} \otimes_{\mathbb{R}} V$ only depends on the dimension of V, which implies that c^{C_2} is the fold map. So the fixed points sequence is also a cofiber sequence. □

Remark 10.14. From the point of view of spectral Mackey functors [Guillou and May 2011; Barwick 2017], the cofiber sequence (10-1) is the cofiber sequence of Mackey functors



where $ku \xrightarrow{r} \mathrm{ko}$ considers a rank n complex bundle as a rank $2n$ real bundle.

Theorem 10.15. *The C_2-equivariant cohomology of ko_{C_2}, as a module over $\mathcal{A}^{C_2}$, is*

$$H^{*,*}_{C_2}(\mathrm{ko}_{C_2}; \mathbb{F}_2) \cong \mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1).$$

Proof. According to [Ricka 2015, Corollary 6.19], we have $H^{*,*}_{C_2}(k\mathbb{R}) \cong \mathcal{A}^{C_2}/\!/\mathcal{E}^{C_2}(1)$. Since η induces the trivial map on equivariant cohomology, the sequence (10-1) induces a short exact sequence

$$0 \to H^{*-2,*-1}_{C_2}(\mathrm{ko}_{C_2}) \xrightarrow{i} \mathcal{A}^{C_2}/\!/\mathcal{E}^{C_2}(1) \xrightarrow{j} H^{*,*}_{C_2}(\mathrm{ko}_{C_2}) \to 0 \tag{10-2}$$

of A^{C_2}-modules.

The cofiber $C\eta$ is a 2-cell complex that supports a Sq^2 in cohomology. It follows that the composition

$$k\mathbb{R} \simeq \mathrm{ko}_{C_2} \wedge C(\eta) \to \Sigma^{2,1}\mathrm{ko}_{C_2} \hookrightarrow \Sigma^{2,1}\mathrm{ko}_{C_2} \wedge C(\eta)$$

induces the map

$$\mathcal{A}^{C_2}/\!/\mathcal{E}^{C_2}(1) \xrightarrow{ij} \mathcal{A}^{C_2}/\!/\mathcal{E}^{C_2}(1) : 1 \mapsto \mathrm{Sq}^2.$$

In particular, the composition $\mathcal{A}^{C_2} \to \mathcal{A}^{C_2}/\!/\mathcal{E}^{C_2}(1) \xrightarrow{j} H^{*,*}_{C_2}(\mathrm{ko}_{C_2})$ factors through $\mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1)$. Given the right $\mathcal{E}^{C_2}(1)$-module decomposition

$$\mathcal{A}^{C_2}(1) \cong \mathcal{E}^{C_2}(1) \oplus \Sigma^{2,1}\mathcal{E}^{C_2}(1),$$

it follows that the sequence (10-2) sits in a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{*-2,*-1}_{C_2}(\mathrm{ko}_{C_2}) & \longrightarrow & \mathcal{A}^{C_2}/\!/\mathcal{E}^{C_2}(1) & \longrightarrow & H^{*,*}_{C_2}(\mathrm{ko}_{C_2}) & \longrightarrow & 0 \\ & & \uparrow & & \| & & \uparrow & & \\ 0 & \longrightarrow & \Sigma^{2,1}\mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1) & \longrightarrow & \mathcal{A}^{C_2}/\!/\mathcal{E}^{C_2}(1) & \longrightarrow & \mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1) & \longrightarrow & 0 \end{array}$$

The outer two maps agree up to suspension, so they are both isomorphisms. □

Corollary 10.16. *The E_2-page of the Adams spectral sequence for* ko_{C_2} *is*

$$E_2 \cong \mathrm{Ext}_{\mathcal{A}^{C_2}}(H^{*,*}_{C_2}(\mathrm{ko}_{C_2}), \mathbb{M}^{C_2}_2) \cong \mathrm{Ext}_{C_2}(1).$$

Proof. This is a standard change of rings isomorphism [Ravenel 1986, Theorem A1.3.12], using that $H^{*,*}_{C_2}(\mathrm{ko}_{C_2})$ is isomorphic to $\mathcal{A}^{C_2}/\!/\mathcal{A}^{C_2}(1)$. Note that the change of rings theorem applies by [Ricka 2015, Corollary 6.15]. □

Remark 10.17. Working in the 2-complete category, it is also possible to build ko_{C_2} using the "Tate diagram" approach. See, for example, [Greenlees 2018] for a nice description of this approach. According to this approach, one specifies a C_2-spectrum X by giving three pieces of data:

(1) an underlying spectrum X^e with C_2-action,

(2) a geometric fixed points spectrum X^{gC_2}, and

(3) a map $X^{gC_2} \to (X^e)^{tC_2}$ from the geometric fixed points to the Tate construction.

In our case, the underlying spectrum is ko with trivial C_2-action. The rest of the Tate diagram information is given by the following result.

Proposition 10.18. *The geometric fixed points of* ko_{C_2} *is* $\bigvee_{k\geq 0} \Sigma^{4k} H\hat{\mathbb{Z}}_2$, *and the map* $(\mathrm{ko}_{C_2})^{gC_2} \to \mathrm{ko}^{tC_2}$ *is the connective cover.*

Proof. The Tate construction ko^{tC_2} was computed by Davis and Mahowald [1984, Theorem 1.4] to be $\bigvee_{n\in\mathbb{Z}} \Sigma^{4n} H\hat{\mathbb{Z}}_2$. For the interpretation of the Davis–Mahowald calculation in terms of the Tate construction, see [May 1996, Section XXI.3].

The geometric fixed points sit in a cofiber sequence

$$\mathrm{ko} \wedge \mathbb{RP}^\infty_+ \simeq \mathrm{ko}_{hC_2} \to (\mathrm{ko}_{C_2})^{C_2} \to (\mathrm{ko}_{C_2})^{gC_2},$$

which we can write as

$$\mathrm{ko} \vee (\mathrm{ko} \wedge \mathbb{RP}^\infty) \to \mathrm{ko} \vee \mathrm{ko} \to (\mathrm{ko}_{C_2})^{gC_2}.$$

The left map is a map of ko-modules, and we consider the simpler cofiber sequence

$$\mathrm{ko} \wedge \mathbb{RP}^\infty \xrightarrow{\mathrm{ko}\wedge t} \mathrm{ko} \to (\mathrm{ko}_{C_2})^{gC_2},$$

where $t : \mathbb{RP}^\infty \to S^0$ is the Kahn–Priddy transfer. As in [Ravenel 1986, Section 1.5], we write R for the cofiber of t, so that $(\mathrm{ko}_{C_2})^{gC_2} \simeq \mathrm{ko} \wedge R$. As Adams explained in [Adams 1974], the cohomology of R has a filtration as $\mathcal{A}^{\mathrm{cl}}(1)$-modules in which the associated graded object is $\bigoplus_{k\geq 0} \Sigma^{4k}\mathcal{A}^{\mathrm{cl}}(1)//\mathcal{A}^{\mathrm{cl}}(0)$. It follows that $\mathrm{ko} \wedge R \simeq \bigvee_{k\geq 0} \Sigma^{4k} H\hat{\mathbb{Z}}_2$.

Similarly, the associated graded for $\mathrm{colim}_n\, \mathrm{H}^*(\Sigma\mathbb{RP}^\infty_{-n})$ is

$$\bigoplus_{k\in\mathbb{Z}} \Sigma^{4k}\mathcal{A}^{\mathrm{cl}}(1)//\mathcal{A}^{\mathrm{cl}}(0).$$

The map $R \to \mathrm{holim}_n\, \Sigma\mathbb{RP}^\infty_{-n}$ is surjective on cohomology, and the same is true for the induced map $R \wedge \mathrm{ko} \to \mathrm{holim}_n(\mathbb{RP}^\infty_{-n} \wedge \Sigma\mathrm{ko})$. We conclude that the map

$$\bigvee_{k\geq 0} \Sigma^{4k} H\hat{\mathbb{Z}}_2 \simeq (\mathrm{ko}_{C_2})^{gC_2} \to \mathrm{ko}^{tC_2} \simeq \underset{n}{\mathrm{holim}}(\mathbb{RP}^\infty_{-n} \wedge \Sigma\mathrm{ko})$$

is a split inclusion in homotopy and therefore a connective cover. □

Remark 10.19. Note that the description of geometric fixed points given here is confirmed by Corollary 4.2. That is, the geometric fixed points of a C_2-spectrum X are given by the categorical fixed points of $S^{\infty,\infty} \wedge X$, where

$$S^{\infty,\infty} = \mathrm{colim}(S^{n,n} \xrightarrow{\rho} S^{n+1,n+1}).$$

Thus the geometric fixed points are computed by the ρ-inverted Adams spectral sequence. As we recall in the next section, the homotopy element 2 is detected by the element $h_0 + \rho h_1$ in Ext. Thus the element $\rho^k h_1^k \tau^{4j}$ of Corollary 4.2 detects 2^k in the $4j$-stem of the geometric fixed points.

11. The homotopy ring

In this section, we will describe the bigraded homotopy ring $\pi_{*,*}(\mathrm{ko}_{C_2})$ of ko_{C_2}. We are implicitly completing the homotopy groups at 2 so that the Adams spectral sequence converges [Hu and Kriz 2001, Corollary 6.47].

It turns out that the Adams spectral sequence collapses, so that $\mathrm{Ext}_{C_2}(1)$ is an associated graded object of $\pi_{*,*}(\mathrm{ko}_{C_2})$. Nevertheless, the Adams spectral sequence hides much of the multiplicative structure.

Recall that the Milnor–Witt stem of X is defined (see [Dugger and Isaksen 2017a]) as the direct sum

$$\Pi_n(X) \cong \bigoplus_i \pi_{n+i,i}(X).$$

Proposition 11.1. *There are no nonzero differentials in the Adams spectral sequence for* ko_{C_2}.

Proof. This follows by inspection of the E_2-page, shown in the charts in Section 12.

Adams d_r differentials decrease the stem by 1, increase the filtration by r, and preserve the weight. It follows that Adams differentials decrease the Milnor–Witt stem by 1. Every class in Milnor–Witt stem congruent to 3 modulo 4 is infinitely ρ-divisible. As there are no infinitely ρ-divisible classes in Milnor–Witt stem congruent to 2 modulo 4, it follows that there are no nonzero differentials supported in the Milnor–Witt $(4k+3)$-stem.

Every class in Milnor–Witt stem $4k$ supports an infinite tower of either h_0-multiples or h_1-multiples, while there are no such towers in Milnor–Witt stem $4k+1$. It follows that there cannot be any nonzero differentials emanating from the $(4k+1)$-Milnor–Witt-stem. Finally, direct inspection shows there cannot be any nonzero differentials starting in the Milnor–Witt $(4k+2)$ or $4k$-stems. □

The structure of the Milnor–Witt n-stem $\Pi_n(\mathrm{ko}_{C_2})$ of course depends on n. The description of these Milnor–Witt stems naturally breaks into cases, depending on the value of $n \pmod 4$.

The notation that we will use for specific elements of $\pi_{*,*}(\mathrm{ko}_{C_2})$ is summarized in Table 9. The definition of each element is discussed in detail in the following sections.

11A. ***The Milnor–Witt 0-stem.*** Our first task is to describe the Milnor–Witt 0-stem $\Pi_0(\mathrm{ko}_{C_2})$. The other Milnor–Witt stems are modules over $\Pi_0(\mathrm{ko}_{C_2})$, and we will use this module structure heavily in order to understand them.

Proposition 11.2. *Let X be a C_2-equivariant spectrum, and let α belong to $\pi_{n,k}(X)$. The element α is divisible by ρ if and only if its underlying class $\iota^*(\alpha)$ in $\pi_n(\iota^* X)$ is zero.*

Proof. The C_2-equivariant cofiber sequence

$$C_{2+} \to S^{0,0} \xrightarrow{\rho} S^{1,1}$$

induces a long exact sequence

$$\cdots \to \pi_{n+1,k+1}(X) \xrightarrow{\rho} \pi_{n,k}(X) \xrightarrow{\iota^*} \pi_n(\iota^* X) \to \pi_{n+2,k+1}(X) \xrightarrow{\rho} \cdots .$$ □

Corollary 11.3. *There is a hidden ρ extension from Qh_1^3 to h_1^3 in the Adams spectral sequence.*

mw	(s, w)	element	detected by	defining relation
0	$(-1, -1)$	ρ	ρ	
0	$(1, 1)$	η	h_1	
0	$(4, 4)$	α	Qh_1^3	$\rho\alpha = \eta^3$
0	$(0, 0)$	ω	h_0	$\omega = \eta\rho + 2$
4	$(0, -4)$	τ^4	τ^4	
0	$(8, 8)$	β	$\frac{Qh_1^4}{\rho^3}$	$4\beta = \alpha^2$
2	$(0, -2)$	$\tau^2\omega$	$\tau^2 h_0$	$(\tau^2\omega)^2 = 2\omega \cdot \tau^4$
-2	$(0, 2)$	$\tau^{-2}\omega$	$\frac{\gamma}{\tau}$	$\tau^4 \cdot \tau^{-2}\omega = \tau^2\omega$
-4	$(0, 4)$	$\tau^{-4}\omega$	$\frac{\gamma}{\tau^3}$	$\tau^4 \cdot \tau^{-4}\omega = \omega$
$-5-4k$	$(0, 5+4k)$	$\frac{\Gamma}{\tau^{4+4k}}$	$\frac{\gamma}{\tau^{4+4k}}$	$\tau^4 \cdot \frac{\Gamma}{\tau^{4+4k}} = \frac{\Gamma}{\tau^{4+4(k-1)}}$
1	$(1, 0)$	$\tau\eta$	τh_1	
2	$(4, 2)$	$\tau^2\alpha$	a	$2\tau^2\alpha = \alpha \cdot \tau^2\omega$

Table 9. Notation for $\pi_{*,*}(\mathrm{ko}_{C_2})$.

Proof. Recall that η^3 is zero in $\pi_3(\mathrm{ko})$. Proposition 11.2 implies that η^3 in $\pi_{3,3}(\mathrm{ko}_{C_2})$ is divisible by ρ. The only possibility is that there is a hidden extension from Qh_1^3 to h_1^3. □

Proposition 11.4. *The element η in $\pi_{1,1}(\mathrm{ko}_{C_2})$ is detected by h_1.*

Proof. The restriction $\iota^*(\eta)$ of η is the classical η, which is detected by the classical element h_1. As all other elements of $\mathrm{Ext}_{\mathcal{A}^{C_2}(1)}$ in the 1-stem and weight 1 all live in higher filtration, the result follows. □

Definition 11.5. Let α be an element in $\pi_{4,4}(\mathrm{ko}_{C_2})$ detected by Qh_1^3 such that $\rho\alpha = \eta^3$.

Corollary 11.3 guarantees that such an element α exists.

There are many elements of $\pi_{4,4}$ detected by Qh_1^3 because of the presence of elements in higher Adams filtration. The condition $\rho\alpha = \eta^3$ narrows the possibilities, but still does not determine a unique element because of the elements $\frac{\gamma}{\tau}h_0^k a$ in higher Adams filtration. For our purposes, this remaining choice makes no difference.

Definition 11.6. Let ω be the element $\eta\rho + 2$ of $\pi_{0,0}^{C_2}(\mathrm{ko}_{C_2})$.

As for ρ and η, the element ω comes from the homotopy groups of the equivariant sphere spectrum. Strictly speaking, there is no need for the notation ω since it can be written in terms of other elements. Nevertheless, it is convenient because ω plays a central role. According to Lemma 10.9, ω corresponds to the element C_2 of the Burnside ring $A(C_2)$.

Note that ω is detected by h_0, while 2 is detected by $h_0+\rho h_1$. For this reason, ω, rather than 2, plays the role of the zeroth Hopf map in the equivariant (and $\mathbb{R}$-motivic) context. Also note that ω equals $1-\epsilon$, where ϵ is the twist

$$S^{1,1}\wedge S^{1,1}\to S^{1,1}\wedge S^{1,1}.$$

Proposition 11.7. *The homotopy class η^5 is divisible by* 2.

Proof. The relation $\omega\eta=0$ was established by Morel [2004] in the $\mathbb{R}$-motivic stable stems, and the equivariant stems agree with the $\mathbb{R}$-motivic ones in the relevant degrees [Dugger and Isaksen 2017b, Theorem 4.1]. (See also [Dugger and Isaksen 2013] for a geometric argument for this relation given in the motivic context. This geometric argument works just as well equivariantly.)

Using the defining relation for α, it follows that

$$-2\eta\alpha=\rho\eta^2\alpha=\eta^5. \qquad \square$$

Proposition 11.7 was already known to be true in the homotopy of the C_2-equivariant sphere spectrum [Bredon 1968]. The divisibility of the elements η^k is very much related to work of Landweber [1969].

Definition 11.8. Let τ^4 be an element of $\pi_{0,-4}(\mathrm{ko}_{C_2})$ that is detected by τ^4.

The element τ^4 is not uniquely determined because of elements in higher Adams filtration. For our purposes, we may choose an arbitrary such element.

Proposition 11.9. (1) *There is a hidden τ^4 extension from Qh_1^3 to τ^2a.*

(2) *There is a hidden τ^4 extension from $\frac{Q}{\rho^3}h_1^4$ to b.*

Proof. (1) The product $\rho\alpha\cdot\tau^4$ equals $\tau^4\cdot\eta^3$, which is detected by $\tau^4\cdot h_1^3$. This last expression equals $\rho\cdot\tau^2a$ in Ext.

(2) Part (1) implies that there is a hidden τ^4 extension from Qh_1^4 to ρ^3b, since $h_1\cdot\tau^2a$ equals ρ^3b in Ext. This means that there is a hidden τ^4 extension from $\frac{Q}{\rho^3}h_1^4$ to b, since $\rho^3\cdot\frac{Q}{\rho^3}h_1^4$ equals Qh_1^4 in Ext. $\square$

Lemma 11.10. *The class α^2 in $\pi_{8,8}(\mathrm{ko}_{C_2})$ is divisible by* 4.

Proof. By Proposition 11.9, the multiplication map

$$\tau^4:\pi_{8,8}(\mathrm{ko}_{C_2})\xrightarrow{\cong}\pi_{8,4}(\mathrm{ko}_{C_2})$$

is an isomorphism. By considering the effect of multiplication by τ^4 in Ext, we see that

$$\tau^4:\pi_{8,4}(\mathrm{ko}_{C_2})\xrightarrow{\cong}\pi_{8,0}(\mathrm{ko}_{C_2})$$

is also an isomorphism. Thus it suffices to show that $(\tau^4)^2\alpha^2$ is 4-divisible in $\pi_{8,0}(\mathrm{ko}_{C_2})$. But $(\tau^4)^2\cdot\alpha^2$ is detected by $(\tau^2a)^2$ by Proposition 11.9 (1), which equals $(h_0+\rho h_1)^2\tau^4b$ in Ext. Finally, observe that $h_0+\rho h_1$ detects 2. $\square$

Definition 11.11. Let β be the element of $\pi_{8,8}(\mathrm{ko}_{C_2})$ detected by $\frac{Q}{\rho^3}h_1^4$ and satisfying $4\beta = \alpha^2$.

Note that β is uniquely determined by α, even though there are elements of higher Adams filtration, because there is no 2-torsion in $\pi_{8,8}(\mathrm{ko}_{C_2})$.

Proposition 11.12. $\rho^3\beta = \eta\alpha$.

Proof. The defining relation for β implies that $4\rho^3\beta$ equals $\rho^3\alpha^2$, which equals $\rho^2\eta^3\alpha$ by the defining relation for α. Using the relation $(\eta\rho+2)\eta = 0$, this element equals $4\eta\alpha$. Finally, there is no 2-torsion in $\pi_{5,5}(\mathrm{ko}_{C_2})$. □

Proposition 11.13. *The* (2*-completed*) *Milnor–Witt* 0*-stem of* ko_{C_2} *is*

$$\Pi_0(\mathrm{ko}_{C_2}) \cong \mathbb{Z}_2[\eta, \rho, \alpha, \beta]/(\rho(\eta\rho+2), \eta(\eta\rho+2), \rho\alpha-\eta^3, \rho^3\beta-\eta\alpha, \alpha^2-4\beta),$$

where the generators have degrees $(1, 1)$, $(-1, -1)$, $(4, 4)$, *and* $(8, 8)$ *respectively. These homotopy classes are detected by* h_1, ρ, Qh_1^3, *and* $\frac{Qh_1^4}{\rho^3}$ *in the Adams spectral sequence.*

Proof. The relations $\rho(\eta\rho+2)$ and $\eta(\eta\rho+2)$ are already true in the sphere [Morel 2004; Dugger and Isaksen 2013]. The third and fifth relations are part of the definitions of α and β, while the fourth relation is Proposition 11.12.

It remains to show that β^k is detected by $\frac{Q}{\rho^{4k-1}}h_1^{4k}$ and that $\alpha\beta^k$ is detected by $\frac{Q}{\rho^{4k-1}}h_1^{4k+4}$.

We assume for induction on k that β^k is detected by $\frac{Q}{\rho^{4k-1}}h_1^{4k}$. We have the relation $h_0 \cdot \frac{Q}{\rho^{4k-1}}h_1^{4k} = \frac{\gamma}{\tau^{4k-1}}b^k$ in Ext, so $\omega\beta^k$ is detected by $\frac{\gamma}{\tau^{4k-1}}b^k$ in Ext. Now b detects $\tau^4 \cdot \beta$ by Proposition 11.9 (2), so $\omega\beta^{k+1}$ is detected by $\frac{\gamma}{\tau^{4k-1}}b^{k+1}$. Finally, $\frac{\gamma}{\tau^{4k-1}}b^{k+1}$ equals $\tau^4 \cdot \frac{\gamma}{\tau^{4k+3}}b^{k+1}$ in Ext, which equals $\tau^4 \cdot h_0 \cdot \frac{Q}{\rho^{4k+3}}h_1^{4k+4}$.

We have now shown that $\tau^4 \cdot h_0 \cdot \frac{Q}{\rho^{4k+3}}h_1^{4k+4}$ detects $\tau^4 \cdot \omega\beta^{k+1}$. It follows that $\frac{Q}{\rho^{4k+3}}h_1^{4k+4}$ detects β^{k+1}.

A similar argument handles the case of $\alpha\beta^k$. □

11B. τ^4***-periodicity.*** Before analyzing the other Milnor–Witt stems of ko_{C_2}, we will explore a piece of the global structure involving the element τ^4 of $\pi_{0,-4}(\mathrm{ko}_{C_2})$.

Proposition 11.14. *There are hidden* τ^4 *extensions*

(1) *from* $\frac{\gamma}{\tau}$ *to* $\tau^2 h_0$,

(2) *from* $\frac{\gamma}{\rho^2\tau}h_1^2$ *to* a,

(3) *from* $\frac{\gamma}{\tau^3}$ *to* h_0,

(4) *from* $\frac{\gamma}{\rho\tau^2}$ *to* τh_1.

Proof. (1) Recall that $\frac{\gamma}{\tau} \cdot a$ equals $h_0 \cdot Qh_1^3$ in Ext, so the hidden τ^4 extension on Qh_1^3 from Proposition 11.9(1) implies that there is a hidden τ^4 extension from $\frac{\gamma}{\tau} \cdot a$ to $\tau^2 h_0 a$. It follows that there is a hidden τ^4 extension from $\frac{\gamma}{\tau}$ to $\tau^2 h_0$.

(2) Using that $h_1^2 \cdot \tau^2 h_0$ equals $\rho^2 a$ in Ext, part (1) implies that there is a hidden τ^4 extension from $\frac{\gamma}{\tau} h_1^2$ to $\rho^2 a$.

(3) Recall that $\frac{\gamma}{\tau^3} \cdot b$ equals $h_0 \cdot \frac{Q}{\rho^3} h_1^4$ in Ext, so the hidden τ^4 extension on $\frac{Q}{\rho^3} h_1^4$ from Proposition 11.9(2) implies that there is a hidden τ^4 extension from $\frac{\gamma}{\tau^3} \cdot b$ to $h_0 b$. It follows that there is a hidden τ^4 extension from $\frac{\gamma}{\tau^3}$ to h_0.

(4) Using that ρa equals $h_1(\tau h_1)^2$ in Ext, part (2) implies that there is a hidden τ^4 extension from $\frac{\gamma}{\rho\tau} h_1^2$ to $h_1(\tau h_1)^2$. Now $\frac{\gamma}{\rho\tau} h_1^2$ equals $\frac{\gamma}{\rho\tau^2} h_1 \cdot \tau h_1$, so there is also a hidden τ^4 extension on $\frac{\gamma}{\rho\tau^2}$. □

The homotopy of ko_{C_2} is nearly τ^4-periodic, in the following sense.

Theorem 11.15. *Multiplication by τ^4 gives a homomorphism on Milnor–Witt stems*

$$\Pi_n(\mathrm{ko}_{C_2}) \to \Pi_{n+4}(\mathrm{ko}_{C_2})$$

which is

(1) *injective if $n = -4$,*

(2) *surjective (and zero) if $n = -5$,*

(3) *bijective in all other cases.*

Proof. (1) This is already true in Ext, except in the 0-stem. But the 0-stem is handled by Proposition 11.14(3).

(2) There is nothing to prove here, given that $\Pi_{-1}(\mathrm{ko}_{C_2}) = 0$.

(3) We give arguments depending on the residue of n modulo 4.

- $n \equiv 0 \pmod 4$: If $n < -4$, this is already true in Ext. For $n \geq 0$, this follows from the relation $\rho\alpha = \eta^3$ and the hidden τ^4 extensions on α and β given in Proposition 11.9.
- $n \equiv 1 \pmod 4$: For $n < -3$, this is already true in Ext. For $n \geq -3$, this follows from Proposition 11.14(4).
- $n \equiv 2 \pmod 4$: For $n < -2$, this is already true in Ext. For $n \geq -2$, this follows from Proposition 11.14(1) and (2).
- $n \equiv 3 \pmod 4$: This is already true in Ext. □

Remark 11.16. Another way to view the τ^4-periodicity is via the Tate diagram (Proposition 10.18). We have a cofiber sequence

$$EC_{2+} \wedge \mathrm{ko} \to \mathrm{ko}_{C_2} \to S^{\infty,\infty} \wedge \mathrm{ko}_{C_2}.$$

The homotopy orbit spectrum therefore captures the ρ-torsion. If $x \in \pi_{*,*}\mathrm{ko}_{C_2}$ is ρ-torsion, then so is $\tau^4 \cdot x$. But multiplication by τ^4 is an equivalence on underlying spectra and therefore gives an equivalence on homotopy orbits. This implies the τ^4-periodicity in the ρ-torsion.

11C. ***The Milnor–Witt n-stem with $n \equiv 0 \pmod 4$.*** Theorem 11.15 indicates that τ^4 multiplications are useful in describing the structure of the homotopy groups of ko_{C_2}. Therefore, our next task is to build on our understanding of $\Pi_0(\mathrm{ko}_{C_2})$ and to describe the subring $\bigoplus_{k\in\mathbb{Z}} \Pi_{4k}(\mathrm{ko}_{C_2})$ of $\pi_{*,*}\mathrm{ko}_{C_2}$.

The Ext charts indicate that the behavior of these groups differs for $k \geq 0$ and for $k < 0$.

Proposition 11.17. $\bigoplus_{k\geq 0} \Pi_{4k}(\mathrm{ko}_{C_2})$ *is isomorphic to* $\Pi_0(\mathrm{ko}_{C_2})[\tau^4]$.

Proof. This follows immediately from Theorem 11.15. □

Definition 11.18. Define $\tau^2\omega$ to be an element in $\pi_{0,-2}(\mathrm{ko}_{C_2})$ that is detected by $\tau^2 h_0$ such that $(\tau^2\omega)^2 = 2\omega\cdot\tau^4$.

An equivalent way to specify a choice of $\tau^2\omega$ is to require that the underlying map $\iota^*(\tau^2\omega)$ equals 2 in $\pi_0(\mathrm{ko})$.

Definition 11.19. For $k \geq 1$, let $\frac{\Gamma}{\tau^k}$ be an element of $\pi_{0,k+1}$ detected by $\frac{\gamma}{\tau^k}$ such that

(1) $\tau^4 \cdot \frac{\Gamma}{\tau} = \tau^2\omega$,

(2) $\tau^4 \cdot \frac{\Gamma}{\tau^3} = \omega$,

(3) $\tau^4 \cdot \frac{\Gamma}{\tau^k} = \frac{\Gamma}{\tau^{k-4}}$ when $k \geq 5$.

According to Theorem 11.15, the elements $\frac{\Gamma}{\tau^k}$ are uniquely determined by the stated conditions. Proposition 11.14 (1) and (3) allow us to choose $\frac{\Gamma}{\tau}$ and $\frac{\Gamma}{\tau^3}$ with the desired properties. As suggested by the defining relations for these elements, we will often write $\tau^{-2-4k}\omega$ for $\frac{\Gamma}{\tau^{1+4k}}$ and $\tau^{-4-4k}\omega$ for $\frac{\Gamma}{\tau^{3+4k}}$.

Proposition 11.20. *As a* $\pi_0(\mathrm{ko}_{C_2})[\tau^4]$*-module,* $\bigoplus_{k\in\mathbb{Z}} \Pi_{4k}(\mathrm{ko}_{C_2})$ *is isomorphic to the* $\pi_0(\mathrm{ko}_{C_2})[\tau^4]$*-module generated by* 1 *and the elements* $\tau^{-4-4k}\omega$ *for* $k \geq 0$, *subject to the relations*

(1) $$\tau^4 \cdot \tau^{-4-4k}\omega = \tau^{-4k}\omega,$$

(2) $$\rho \cdot \tau^{-4-4k}\omega = 0,$$

(3) $$\eta \cdot \tau^{-4-4k}\omega = 0,$$

(4) $$\tau^4 \cdot \tau^{-4}\omega = \omega.$$

Proof. This follows by inspection of the Ext charts, together with the defining relations for $\tau^{-4-4k}\omega$. □

11D. ***The Milnor–Witt n-stem with $n \equiv 1 \pmod 4$.***

Definition 11.21. Denote by $\tau\eta$ an element of $\pi_{1,0}(\mathrm{ko}_{C_2})$ that is detected by τh_1.

Note that $\tau\eta$ is not uniquely determined because of elements in higher Adams filtration, but the choice makes no practical difference. One way to specify a choice of $\tau\eta$ is to use the composition

$$S^{1,0} \to S^{0,0} \to \mathrm{ko}_{C_2},$$

where the first map is the image of the classical Hopf map $\eta : S^1 \to S^0$, and the second map is the unit.

Proposition 11.22. *As a* $\Pi_0(\mathrm{ko}_{C_2})[\tau^4]$*-module, there is an isomorphism*

$$\bigoplus_{k\in\mathbb{Z}} \Pi_{1+4k}(\mathrm{ko}_{C_2}) \cong \big(\Pi_0(\mathrm{ko}_{C_2})[(\tau^4)^{\pm 1}]/(2, \rho^2, \eta^2, \alpha)\big)\{\tau\eta\}.$$

Proof. This follows from inspection of the Ext charts, together with Theorem 11.15. □

11E. *The Milnor–Witt n-stem with n ≡ 2* (mod 4). Recall from Definition 11.18 that $\tau^2\omega$ is an element of $\pi_{0,-2}(\mathrm{ko}_{C_2})$ that is detected by $\tau^2 h_0$.

Lemma 11.23. *The product* $\alpha \cdot \tau^2\omega$ *in* $\pi_{4,2}(\mathrm{ko}_{C_2})$ *is detected by* $h_0 a$.

Proof. The product $\tau^4 \cdot \alpha \cdot \tau^2\omega$ is detected by $\tau^4 h_0 a$ by Proposition 11.9(1). □

Definition 11.24. Define $\tau^2\alpha$ to be an element of $\pi_{4,2}(\mathrm{ko}_{C_2})$ that is detected by a such that $2 \cdot \tau^2\alpha$ equals $\alpha \cdot \tau^2\omega$.

Proposition 11.25. *As a* $\Pi_0(\mathrm{ko}_{C_2})[\tau^4]$*-module,* $\bigoplus_{k\in\mathbb{Z}} \Pi_{2+4k}(\mathrm{ko}_{C_2})$ *is isomorphic to the free* $\Pi_0(\mathrm{ko}_{C_2})[(\tau^4)^{\pm 1}]$*-module generated by* $\tau^2\omega$, $(\tau\eta)^2$, *and* $\tau^2\alpha$, *subject to the relations*

(1) $$\rho \cdot \tau^2\omega = 0,$$

(2) $$\alpha \cdot \tau^2\omega = 2 \cdot \tau^2\alpha,$$

(3) $$\rho(\tau\eta)^2 = \eta \cdot \tau^2\omega,$$

(4) $$2(\tau\eta)^2 = 0,$$

(5) $$\eta(\tau\eta)^2 = \rho \cdot \tau^2\alpha,$$

(6) $$\alpha(\tau\eta)^2 = 0,$$

(7) $$\eta \cdot \tau^2\alpha = 0,$$

(8) $$\alpha \cdot \tau^2\alpha = 2\beta \cdot \tau^2\omega.$$

Proof. Except for the last relation, this follows from inspection of the Ext charts, together with Theorem 11.15.

For the last relation, use that $2\alpha \cdot \tau^2\alpha$ equals $\tau^2\omega \cdot \alpha^2$ by the definition of $\tau^2\alpha$, and that $\tau^2\omega \cdot \alpha^2$ equals $4\beta \cdot \tau^2\omega$ by the defining relation for β. As there is no 2-torsion in this degree, relation (8) follows. □

11F. ***The Milnor–Witt n-stem with $n \equiv 3 \pmod 4$.*** The structure of

$$\bigoplus_{k\in\mathbb{Z}} \Pi_{4k+3}(\mathrm{ko}_{C_2})$$

is qualitatively different than the other cases because it contains elements that are infinitely divisible by ρ. The Ext charts show that $\bigoplus_{k\in\mathbb{Z}} \Pi_{4k+3}(\mathrm{ko}_{C_2})$ is concentrated in the range $k \le -2$.

The elements $\frac{\Gamma}{\tau^{4k}}$ are infinitely divisible by both ρ and τ^4. We write $\frac{\Gamma}{\rho^j\tau^{4k}}$ for an element such that $\rho^j \cdot \frac{\Gamma}{\rho^j\tau^{4k}}$ equals $\frac{\Gamma}{\tau^{4k}}$.

By inspection of the Ext charts, we see that $\bigoplus_{k\le 0} \Pi_{4k-5}(\mathrm{ko}_{C_2})$ is generated as an abelian group by the elements $\frac{\Gamma}{\rho^j\tau^{4k}}$. The $\Pi_0(\mathrm{ko}_{C_2})[\tau^4]$-module structure on $\bigoplus_{k\le 0} \Pi_{4k-5}(\mathrm{ko}_{C_2})$ is then governed by the orders of these elements, together with the relations

$$\alpha \cdot \frac{\Gamma}{\tau^{4k}} = -8\frac{\Gamma}{\rho^4\tau^{4k}}$$

and

$$\beta \cdot \frac{\Gamma}{\tau^{4k}} = 16\frac{\Gamma}{\rho^8\tau^{4k}}.$$

The first relation follows from the calculation

$$\alpha \cdot \frac{\Gamma}{\tau^{4k}} = \rho\alpha \cdot \frac{\Gamma}{\rho\tau^{4k}} = \eta^3 \cdot \frac{\Gamma}{\rho\tau^{4k}} = (\eta\rho)^3 \cdot \frac{\Gamma}{\rho^4\tau^{4k}} = (-2)^3 \cdot \frac{\Gamma}{\rho^4\tau^{4k}} = -8\frac{\Gamma}{\rho^4\tau^{4k}}.$$

The second relation follows from a similar argument, using that $\rho^3\beta = \eta\alpha$.

Proposition 11.26. *The order of* $\frac{\Gamma}{\tau^{4k}\rho^j}$ *is* $2^{\varphi(j)+1}$, *where* $\varphi(j)$ *is the number of positive integers* $0 < i \le j$ *such that* $i \equiv 0, 1, 2$ *or* $4 \pmod 8$.

Proof. Since $h_0 + \rho h_1$ detects the element 2, the result is represented by the chart on page 625, in stems zero to sixteen. As the top edge of the region is $(8, 4)$-periodic, this gives the result in higher stems as well. □

Remark 11.27. Proposition 11.26 is an independent verification of a well-known calculation. We follow the argument given in [Dugger 2005, Appendix B].

Let $\mathbb{R}^{q,q}$ be the antipodal C_2-representation on $\mathbb{R}^q$. Consider the cofiber sequence

$$S(q,q) \to D(q,q) \to S^{q,q},$$

where $S(q,q) \subset D(q,q) \subset \mathbb{R}^{q,q}$ are the unit sphere and unit disk respectively. Since $D(q,q)$ is equivariantly contractible, this gives the exact sequence

$$\pi_{m,0}(\mathrm{ko}_{C_2}) \leftarrow \pi_{m+q,q}(\mathrm{ko}_{C_2}) \leftarrow \mathrm{ko}_{C_2}^{-m-1,0}(S(q,q)) \leftarrow \pi_{m+1,0}(\mathrm{ko}_{C_2}).$$

If $m \le -2$, the outer groups vanish. Moreover, C_2 acts freely on $S(q,q)$, and the orbit space is $S(q,q)/C_2 \cong \mathbb{RP}^{q-1}$. It follows [May 1996, Section XIV.1] that

$$\mathrm{ko}_{C_2}^{-m-1}(S(q,q)) \cong \mathrm{ko}^{-m-1}(\mathbb{RP}^{q-1})$$

when $m \leq -2$ and $q \geq 1$. In particular,

$$\pi_{j,j+5}(\mathrm{ko}_{C_2}) \cong \mathrm{ko}^4(\mathbb{RP}^{j+4}),$$

and the latter groups are known (see [Davis and Mahowald 1979, Section 2]) to be cyclic of order $\varphi(j)$.

Having described all of the Milnor–Witt stems as $\Pi_0(\mathrm{ko}_{C_2})[\tau^4]$-modules, it remains only to understand products of the various $\Pi_0(\mathrm{ko}_{C_2})[\tau^4]$-module generators.

Proposition 11.28. *In the homotopy groups of* ko_{C_2}, *we have the relations*

$$(\tau^2\omega)^2 = 2\omega \cdot \tau^4, \tag{1}$$

$$\tau^2\omega \cdot \tau^2\alpha = \tau^4 \cdot \omega\alpha, \tag{2}$$

$$(\tau^2\alpha)^2 = 2\tau^4 \cdot \omega\beta. \tag{3}$$

Proof. The first relation is part of the definition of $\tau^2\omega$.

For the second relation, use the definitions of $\tau^2\alpha$ and of $\tau^2\omega$ to see that

$$2 \cdot \tau^2\omega \cdot \tau^2\alpha = (\tau^2\omega)^2\alpha = 2\tau^4 \cdot \omega\alpha.$$

The group $\pi_{4,0}(\mathrm{ko}_{C_2})$ has no 2-torsion, so it follows that $\tau^2\omega \cdot \tau^2\alpha$ equals $\tau^4 \cdot \omega\alpha$.

The proof of the third relation is similar. Use the definitions of $\tau^2\alpha$ and β and part (2) to see that

$$2(\tau^2\alpha)^2 = \tau^2\omega \cdot \tau^2\alpha \cdot \alpha = \tau^4 \cdot \omega\alpha^2 = 4\tau^4 \cdot \omega\beta.$$

The group $\pi_{8,4}(\mathrm{ko}_{C_2})$ has no 2-torsion. □

11G. *The homotopy ring of $k\mathbb{R}$.* We may similarly describe the homotopy of $k\mathbb{R}$. Since this has already appeared in the literature (see [Greenlees and Meier 2017, Section 11]), we do not give complete details.

We use the forgetful exact sequence of Proposition 11.2 to define the homotopy classes listed in Table 10. In each case, the forgetful map is injective, and we stipulate that τ^4 restricts to 1, that v_1 and $\tau^{-4}v_1$ restrict to the Bott element, and that $\tau^2\omega$, $\tau^{-2}\omega$, and $\tau^{-4}\omega$ all restrict to 2.

Proposition 11.29. *There are τ^4-extensions*

$$\tau^4 \cdot \tau^{-2}\omega = \tau^2\omega, \quad \tau^4 \cdot \tau^{-4}\omega = 2, \quad \tau^4 \cdot \tau^{-4}v_1 = v_1.$$

Proof. These all follow from the definition of these classes using the forgetful exact sequence of Proposition 11.2. Since the forgetful map is a ring homomorphism, we get that

$$\iota^*(\tau^4 \cdot \tau^{-2}\omega) = \iota^*(\tau^4) \cdot \iota^*(\tau^{-2}\omega) = 1 \cdot 2 = 2.$$

Since the forgetful map is injective in this degree, we conclude that $\tau^4 \cdot \tau^{-2}\omega = \tau^2\omega$. The same argument handles the other relations just as well. □

In order to describe the Milnor–Witt 0-stem of $k\mathbb{R}$, it is convenient to write $\alpha = \tau^{-2}\omega v_1^2$ and $\beta = \tau^{-4} v_1 \cdot v_1^3$.

Proposition 11.30. *The* (2*-completed*) *Milnor–Witt* 0*-stem of* $k\mathbb{R}$ *is*

$$\Pi_0(k\mathbb{R}) \cong \mathbb{Z}_2[\rho, \alpha, \beta]/(2\rho, \rho\alpha, \rho^3\beta, \alpha^2 - 4\beta),$$

where the generators have degrees $(-1, -1)$, $(4, 4)$, *and* $(8, 8)$ *respectively. These homotopy classes are detected by* ρ, $\frac{\gamma}{\tau} v_1^2$, *and* $\frac{\gamma}{\rho^2\tau^2} v_1^3$ *in the Adams spectral sequence.*

The other Milnor–Witt stems, aside from those in degree $-5-4k$, can all be described cleanly as ideals in $\Pi_0(k\mathbb{R})$. The τ^4-periodicities asserted in the following results all hold already on the level of Ext, except for the τ^4-multiplications from $\mathrm{Ext}_{\mathrm{NC}}$ to $\mathrm{Ext}_{\mathcal{E}(1)}$. Those are handled by Proposition 11.29. We recommend the reader to consult the diagram on page 630 in order to visualize the following results.

Proposition 11.31. *The map* $\Pi_{-4}(k\mathbb{R}) \xrightarrow{\tau^4} \Pi_0(k\mathbb{R})$ *is a monomorphism and identifies* $\Pi_{-4}(k\mathbb{R})$ *with the ideal generated by* 2, α, *and* β. *If* $k \neq -1$, *then multiplication by* τ^4 *is an isomorphism* $\Pi_{4k}(k\mathbb{R}) \cong \Pi_{4(k+1)}(k\mathbb{R})$.

Thus the Milnor–Witt stems of degree $4k$ break up into two families, which are displayed as the first two rows of the diagram on page 630.

Proposition 11.32. *The map* $\Pi_{-1}(k\mathbb{R}) \xrightarrow{v_1} \Pi_0(k\mathbb{R})$ *is a monomorphism and identifies* $\Pi_{-1}(k\mathbb{R})$ *with the ideal generated by* α *and* β. *Multiplication by* τ^4 *is a split epimorphism*

$$\frac{\mathbb{F}_2[\rho]}{\rho^\infty} \to \Pi_{-5}(k\mathbb{R}) \xrightarrow{\tau^4} \Pi_{-1}(k\mathbb{R}).$$

If $k \neq -1$, *then multiplication by* τ^4 *is an isomorphism* $\Pi_{-1+4k}(k\mathbb{R}) \cong \Pi_{3+4k}(k\mathbb{R})$.

Proposition 11.33. *The map* $\Pi_{-2}(k\mathbb{R}) \xrightarrow{v_1} \Pi_{-1}(k\mathbb{R})$ *is an isomorphism. Multiplication by* τ^4 *is an isomorphism* $\Pi_{4k-2}(k\mathbb{R}) \cong \Pi_{4k+2}(k\mathbb{R})$ *for all* $k \in \mathbb{Z}$.

Proposition 11.34. *The map* $\Pi_{-3}(k\mathbb{R}) \xrightarrow{v_1^3} \Pi_0(k\mathbb{R})$ *is a monomorphism and identifies* $\Pi_{-3}(k\mathbb{R})$ *with the ideal generated by* β. *Multiplication by* τ^4 *is an isomorphism* $\Pi_{4k-3}(k\mathbb{R}) \cong \Pi_{4k+1}(k\mathbb{R})$ *for all* $k \in \mathbb{Z}$.

Combining the information from Table 3 and Table 8 yields the induced homomorphism on homotopy groups as described in Table 11. Note that all values $c_*(x)$ are to be interpreted as correct modulo higher powers of 2.

Remark 11.35. Note that the results of this section provide another means of demonstrating the τ^4-periodicity in ko_{C_2} established in Section 11B. More specifically, the τ^4-extensions given in Proposition 11.29, together with the homomorphism c_* as described in Table 11, imply the τ^4-extensions given in Proposition 11.14.

mw	(s, w)	element	detected by	definition
0	$(-1, -1)$	ρ	ρ	
1	$(2, 1)$	v_1	v_1	$\iota^*(v_1) = v_1$
4	$(0, -4)$	τ^4	τ^4	$\iota^*(\tau^4) = 1$
2	$(0, -2)$	$\tau^2\omega$	$\tau^2 h_0$	$\iota^*(\tau^2\omega) = 2$
-2	$(0, 2)$	$\tau^{-2}\omega$	$\frac{\gamma}{\tau}$	$\iota^*(\tau^{-2}\omega) = 2$
-4	$(0, 4)$	$\tau^{-4}\omega$	$\frac{\gamma}{\tau^3}$	$\iota^*(\tau^{-4}\omega) = 2$
-3	$(2, 5)$	$\tau^{-4}v_1$	$\frac{\gamma}{\rho^2\tau^2}$	$\iota^*(\tau^{-4}v_1) = v_1$
-5	$(0, 5)$	$\frac{\Gamma}{\tau^4}$	$\frac{\gamma}{\tau^4}$	

Table 10. Notation for $\pi_{*,*}(k\mathbb{R})$.

mw	(s, w)	$x \in \pi_{*,*}(\mathrm{ko}_{C_2})$	$c_*x \in \pi_{*,*}(k\mathbb{R})$
0	$(-1, -1)$	ρ	ρ
0	$(1, 1)$	η	0
0	$(4, 4)$	α	$\tau^{-2}\omega \cdot v_1^2$
0	$(0, 0)$	ω	2
4	$(0, -4)$	τ^4	τ^4
0	$(8, 8)$	β	$\tau^{-4}v_1 \cdot v_1^3$
2	$(0, -2)$	$\tau^2\omega$	$\tau^2\omega$
-2	$(0, 2)$	$\tau^{-2}\omega$	$\tau^{-2}\omega$
-4	$(0, 4)$	$\tau^{-4}\omega$	$\tau^{-4}\omega$
-5	$(j, j+5)$	$\frac{\Gamma}{\rho^j\tau^4}$	$\frac{\Gamma}{\rho^j\tau^4}$
1	$(1, 0)$	$\tau\eta$	ρv_1
2	$(4, 2)$	$\tau^2\alpha$	$2v_1^2$

Table 11. The homomorphism $\pi_{*,*}(\mathrm{ko}_{C_2}) \xrightarrow{c_*} \pi_{*,*}(k\mathbb{R})$, modulo higher powers of 2.

12. Charts

12A. ***Bockstein E^+ and $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$ charts.*** The charts on pages 616–619 depict the Bockstein E^+ spectral sequence that converges to $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$. The details of this calculation are described in Section 6.

The E_2^+-page is too complicated to present conveniently in one chart, so this page is separated into two parts by Milnor–Witt stem modulo 2. Similarly, the E_3^+-page is separated into four parts by Milnor–Witt stem modulo 4. The E_4^+-page in Milnor–Witt stems 0 or 1 modulo 4 is not shown, since it is identical to the E_3^+-page in those Milnor–Witt stems. The E_4^+-page in Milnor–Witt stems 3 modulo 4 is not shown because it is zero.

Here is a key for reading the Bockstein charts:

(1) Gray dots and green dots indicate groups as displayed on the charts.

(2) Horizontal lines indicate multiplications by ρ.

(3) Vertical lines indicate multiplications by h_0.

(4) Diagonal lines indicate multiplications by h_1.

(5) Horizontal arrows indicate infinite sequences of multiplications by ρ.

(6) Vertical arrows indicate infinite sequences of multiplications by h_0.

(7) Diagonal arrows indicate infinite sequences of multiplications by h_1.

Here is a key for the charts of $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$:

(1) Gray dots indicate copies of $\mathbb{F}_2[\tau^4]$ that arise from a copy of $\mathbb{F}_2[\tau^4]$ in the E_∞^+-page.

(2) Green dots indicate copies of $\mathbb{F}_2[\tau^4]$ that arise from a copy of $\mathbb{F}_2$ and a copy of $\mathbb{F}_2[\tau^4]$ in the E_∞^+-page, connected by a τ^4 extension that is hidden in the Bockstein spectral sequence. For example, the green dot at (3, 3) arises from a hidden τ^4 extension from h_1^3 to $\rho \cdot \tau^2 a$.

(3) Blue dots indicate copies of $\mathbb{F}_2[\tau^4]$ that arise from two copies of $\mathbb{F}_2$ and one copy of $\mathbb{F}_2[\tau^4]$ in the E_∞^+-page, connected by τ^4 extensions that are hidden in the Bockstein spectral sequence. For example, the blue dot at (7, 7) arises from hidden τ^4 extensions from h_1^7 to $\rho^4 h_1^3 b$, and from $\rho^4 h_1^3 b$ to $\rho^5 \cdot \tau^2 a \cdot b$.

(4) Horizontal lines indicate multiplications by ρ.

(5) Vertical lines indicate multiplications by h_0.

(6) Diagonal lines indicate multiplications by h_1.

(7) Dashed lines indicate extensions that are hidden in the Bockstein spectral sequence.

(8) Orange horizontal lines indicate ρ multiplications that equal τ^4 times a generator. For example, $\rho \cdot \tau^2 a$ equals $\tau^4 \cdot h_1^3$.

(9) Horizontal arrows indicate infinite sequences of multiplications by ρ.

(10) Vertical arrows indicate infinite sequences of multiplications by h_0.

(11) Diagonal arrows indicate infinite sequences of multiplications by h_1.

12B. ***Bockstein E^- and*** $\mathbf{Ext_{NC}}$ ***charts for $\mathcal{A}^{C_2}(1)$.*** The charts on pages 620–624 depict the Bockstein E^- spectral sequence that converges to $\mathrm{Ext}_{\mathrm{NC}}$. The details of this calculation are described in Section 7.

The E_2^--page is too complicated to present conveniently in one chart, so this page is separated into two parts by Milnor–Witt stem modulo 2. Similarly, the E_3^--page is separated into four parts by Milnor–Witt stem modulo 4. The E_4^--page in Milnor–Witt stems 0 or 3 modulo 4 is not shown, since it is identical to the E_3^--page in those Milnor–Witt stems. The E_5^--page and E_6^--page in Milnor–Witt stems 1 or 2 modulo 4 is not shown, since it is identical to the E_4^--page in those Milnor–Witt stems.

Here is a key for reading the Bockstein charts:

(1) Gray dots and green dots indicate groups as displayed on the charts.

(2) Horizontal lines indicate multiplications by ρ.

(3) Vertical lines indicate multiplications by h_0.

(4) Diagonal lines indicate multiplications by h_1.

(5) Horizontal rightward arrows indicate infinite sequences of divisions by ρ, i.e., infinitely ρ-divisible elements.

(6) Vertical arrows indicate infinite sequences of multiplications by h_0.

(7) Diagonal arrows indicate infinite sequences of multiplications by h_1.

The structure of $\mathrm{Ext}_{\mathrm{NC}}$ is too complicated to present conveniently in one chart, so it is separated into parts by Milnor–Witt stem modulo 4. Unfortunately, the part in positive Milnor–Witt stems 0 modulo 4 alone is still too complicated to present conveniently in one chart. Instead, we display Ext_{C_2}, including both $\mathrm{Ext}_{\mathcal{A}^{\mathbb{R}}(1)}$ and $\mathrm{Ext}_{\mathrm{NC}}$, for the Milnor–Witt 0-stem and the Milnor–Witt 4-stem.

Here is a key for the charts of $\mathrm{Ext}_{\mathrm{NC}}$:

(1) Gray dots indicate copies of $\mathbb{F}_2[\tau^4]/\tau^\infty$.

(2) Horizontal lines indicate multiplications by ρ.

(3) Vertical lines indicate multiplications by h_0.

(4) Diagonal lines indicate multiplications by h_1.

(5) Dashed lines indicate extensions that are hidden in the Bockstein spectral sequence.

(6) Dashed lines of slope -1 indicate ρ extensions that are hidden in the Adams spectral sequence.

(7) Horizontal rightward arrows indicate infinite sequences of divisions by ρ, i.e., infinitely ρ-divisible elements.

(8) Vertical arrows indicate infinite sequences of multiplications by h_0.

(9) Diagonal arrows indicate infinite sequences of multiplications by h_1.

12C. ***Bockstein and* Ext *charts for* $\mathcal{E}^{C_2}$(1).** The Bockstein E^+ and E^- spectral sequences that converge to $\mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}$ and $\mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}(\mathrm{NC}, \mathbb{M}_2^{\mathbb{R}})$, respectively, are shown in the charts on page 627. The details of this calculation are described in Remark 6.3 and Section 7C. For legibility, we have split each of the E_∞^+, E_4^-, and $\mathrm{Ext}_{\mathrm{NC}}$ pages into a pair of charts, organized by families of v_1-multiples rather than by Milnor–Witt stems.

Here is a key for reading the Bockstein and $\mathrm{Ext}_{\mathrm{NC}}$ charts:

(1) Gray dots indicate groups as displayed on the charts.

(2) Horizontal lines indicate multiplications by ρ.

(3) Vertical lines indicate multiplications by h_0. Dashed vertical lines denote h_0-multiplications that are hidden in the Bockstein spectral sequence

(4) Horizontal rightward arrows indicate infinite sequences of divisions by ρ, i.e., infinitely ρ-divisible elements.

(5) Vertical arrows indicate infinite sequences of multiplications by h_0.

12D. ***Milnor–Witt stems.*** The diagrams on pages 629 and 630 depict the Milnor–Witt stems for ko_{C_2} and $k\mathbb{R}$ in families as described in Section 11.

The top figure on page 629 represents the Milnor–Witt $4k$-stem, where $k \geq 0$. The middle three figures represent the τ^4-periodic classes, as in Theorem 11.15. The bottom figure represents the Milnor–Witt stem Π_n, where $n \equiv 3 \pmod 4$ and $n \leq -5$.

Here is a key for reading the Milnor–Witt charts:

(1) Black dots indicate copies of $\mathbb{F}_2$.

(2) Hollow circles indicate copies of $\mathbb{Z}_2^2$.

(3) Circled numbers indicate cyclic groups of given order. For instance, the 1-stem of Π_{-5} is $\mathbb{Z}/4$.

(4) Blue lines indicate multiplications by η.

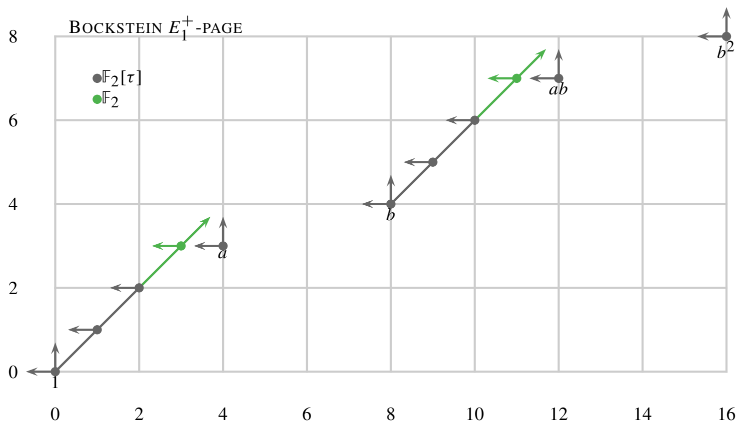
(5) Red lines indicate multiplications by ρ.

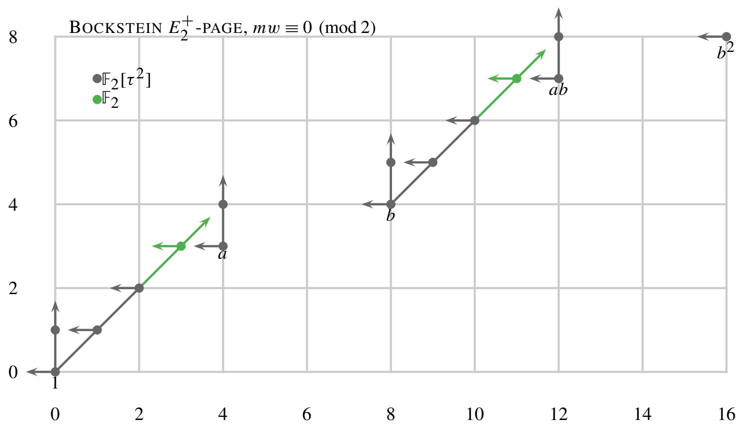
(6) Curved green lines denote multiplications by α.

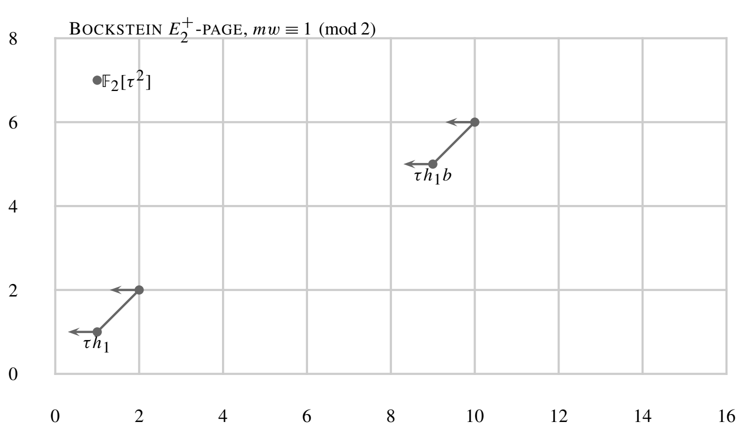
(7) Lines labeled with numbers indicate that a multiplication equals a multiple of an additive generator. For example, $\alpha \cdot \eta^4$ equals $4\eta\rho\beta$ in Π_0.

For clarity, some α multiplications are not shown in the first and last diagrams of page 629. For example, the α multiplication on η is not shown in the first diagram.

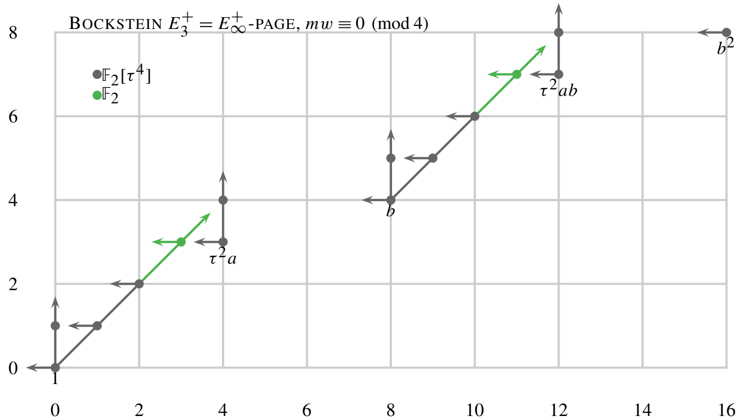
Bockstein charts for $\mathcal{A}^{\mathbb{R}}(1)$

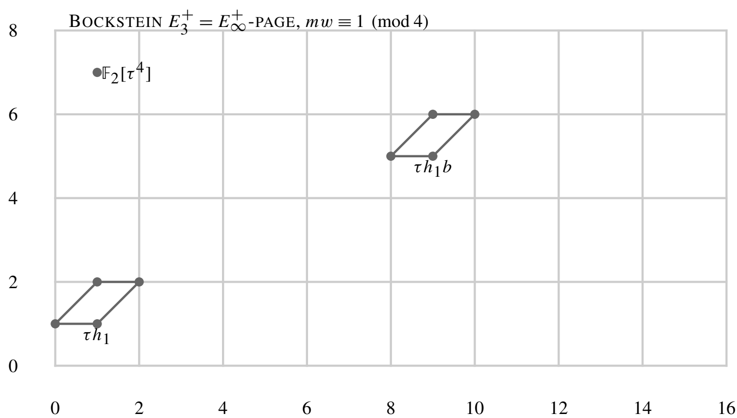


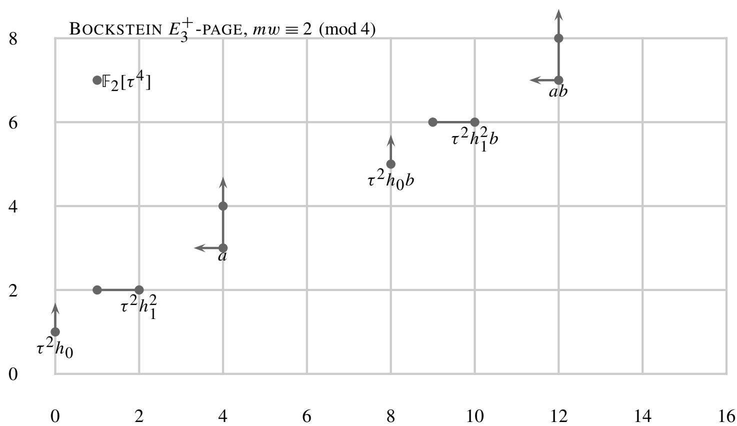




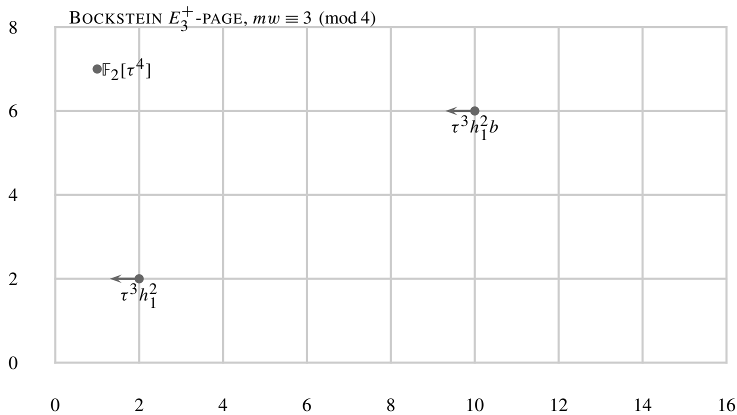
Bockstein charts for $\mathcal{A}^{\mathbb{R}}(1)$

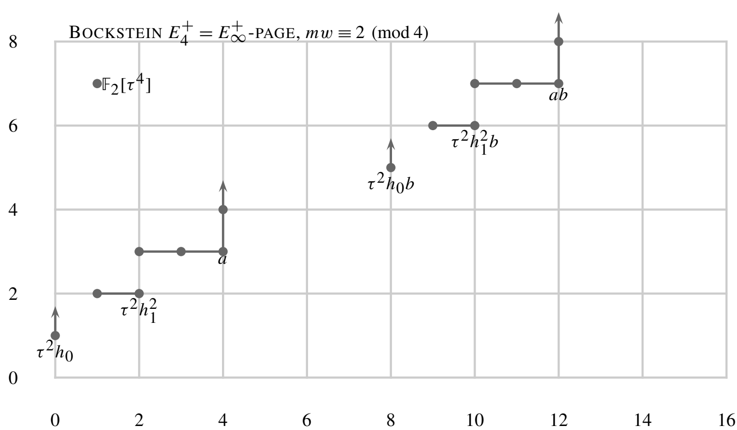




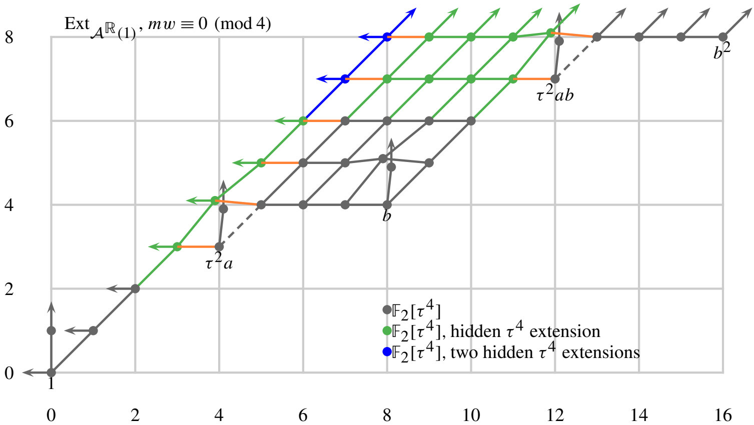


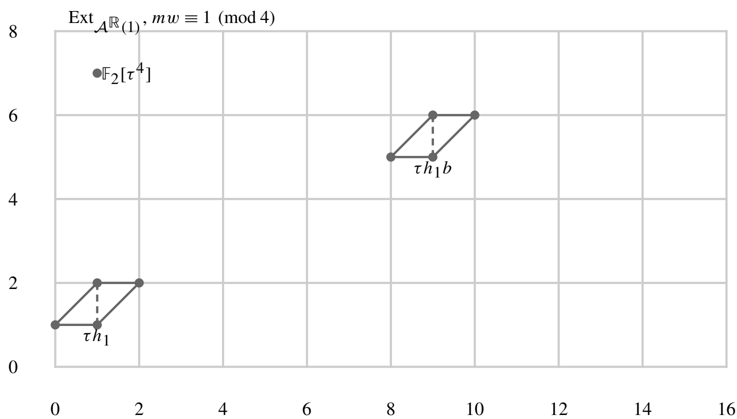
Bockstein charts for $\mathcal{A}^{\mathbb{R}}(1)$

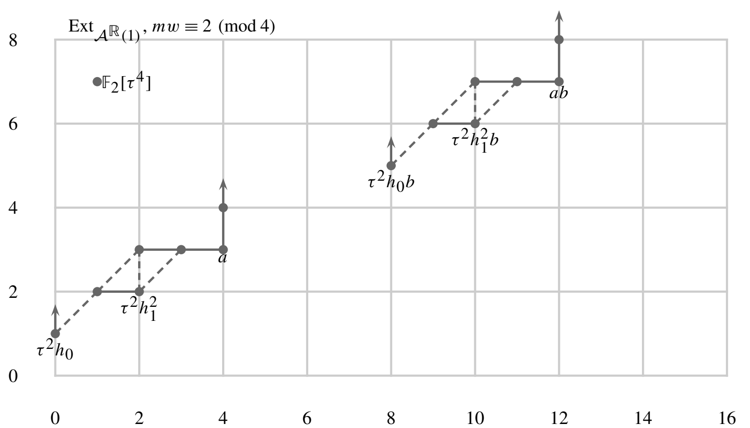




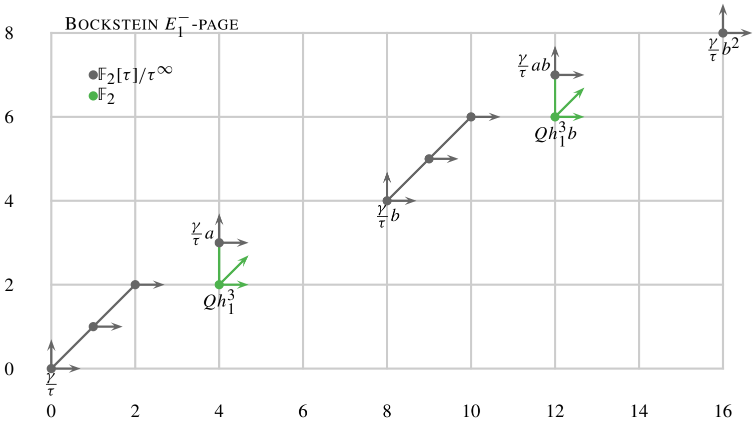
Ext charts for $\mathcal{A}^{\mathbb{R}}(1)$

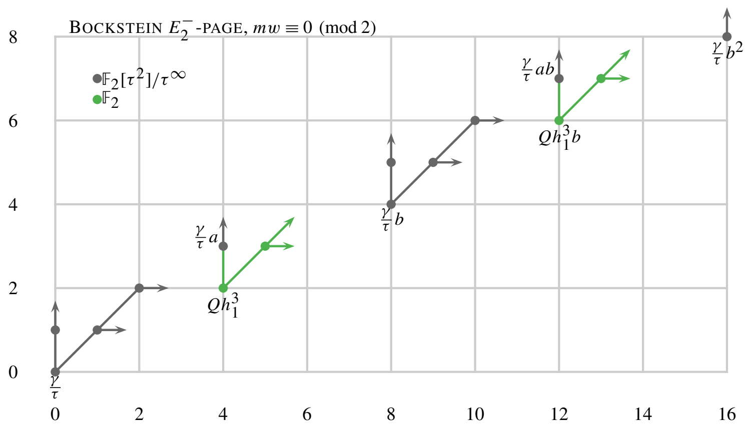


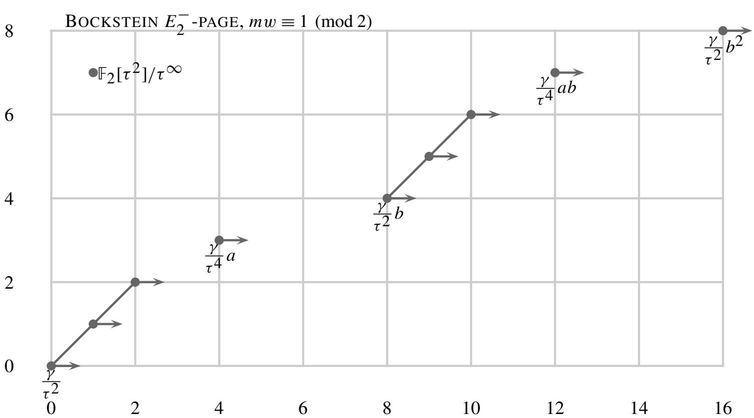




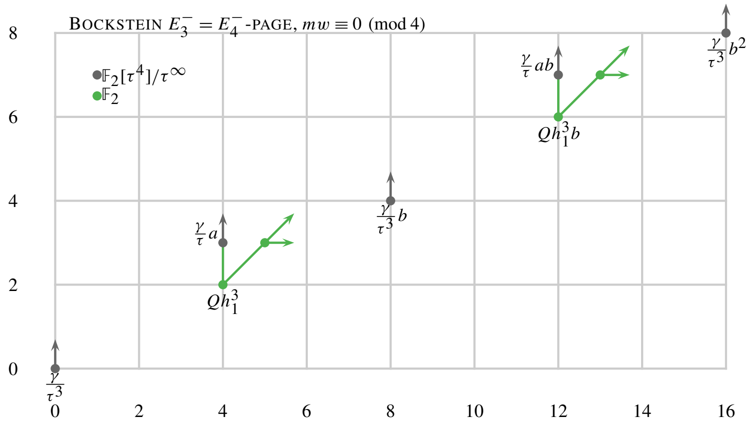
Bockstein E^- charts for $\mathcal{A}^{C_2}(1)$

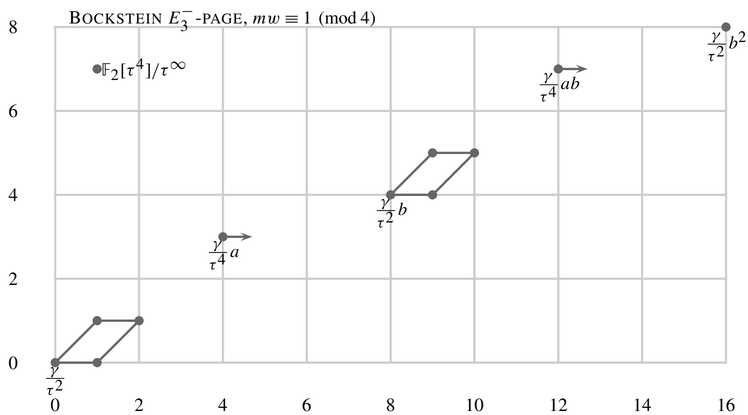


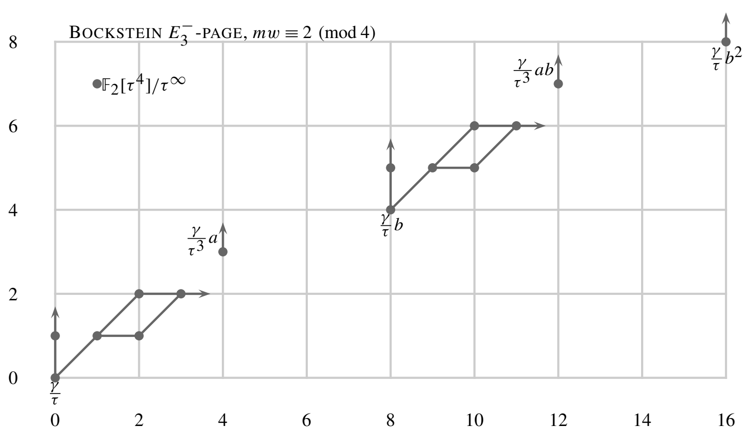




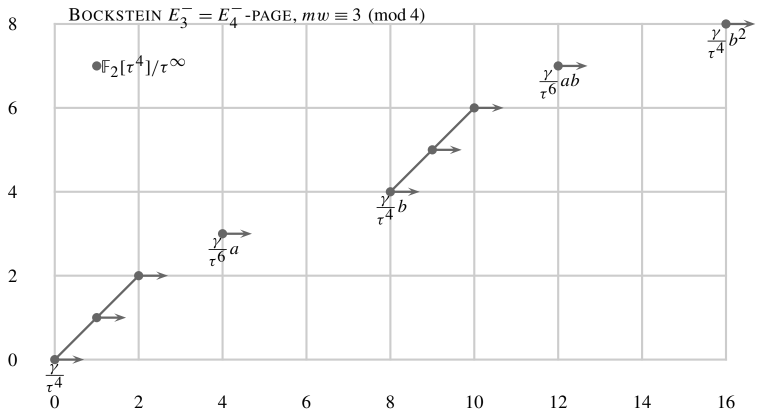
Bockstein E^- charts for $\mathcal{A}^{C_2}(1)$

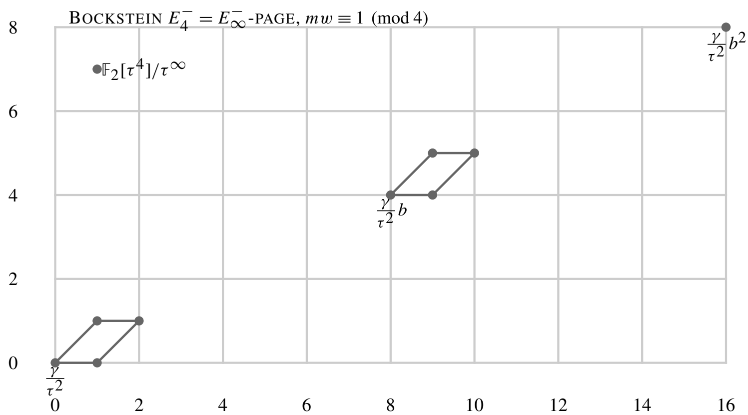


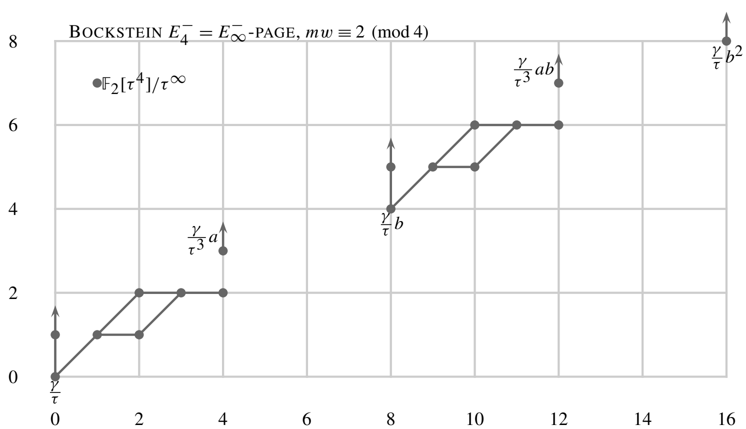




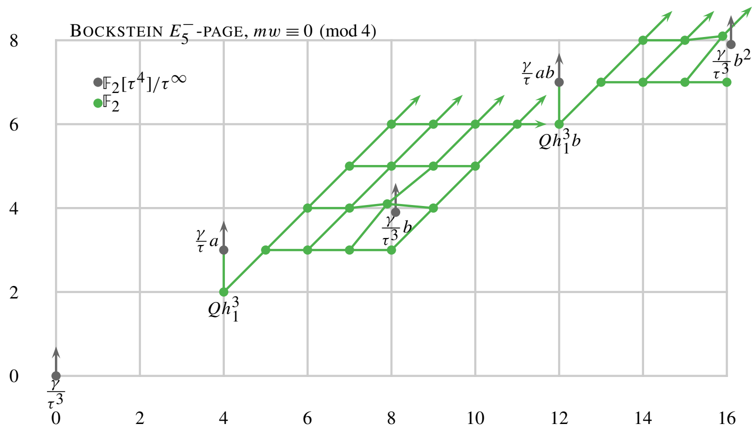
Bockstein E^- charts for $\mathcal{A}^{C_2}(1)$

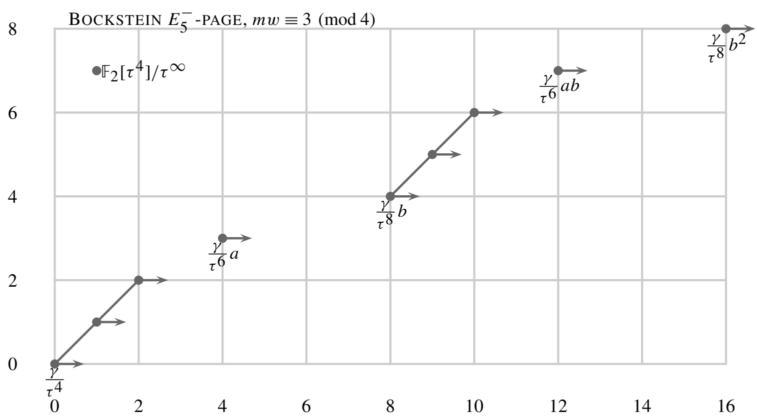






Bockstein E^- charts for $\mathcal{A}^{C_2}(1)$





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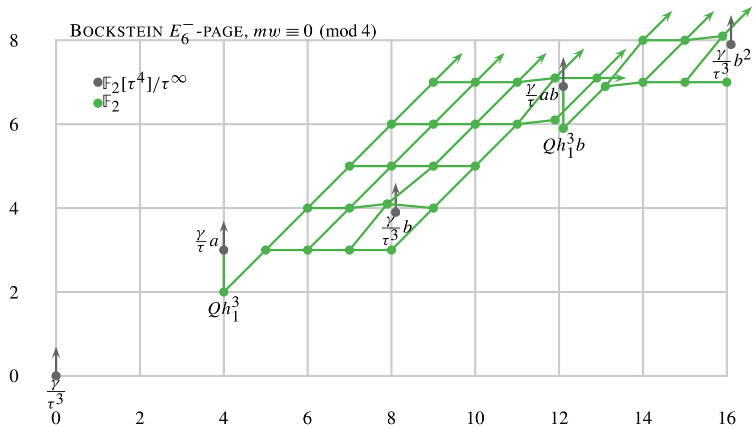
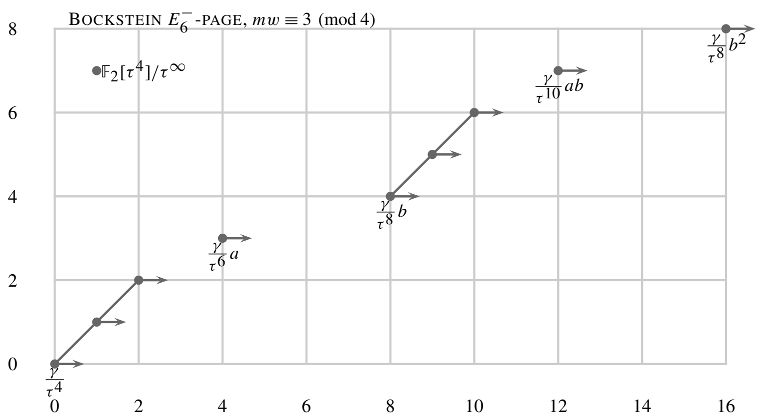
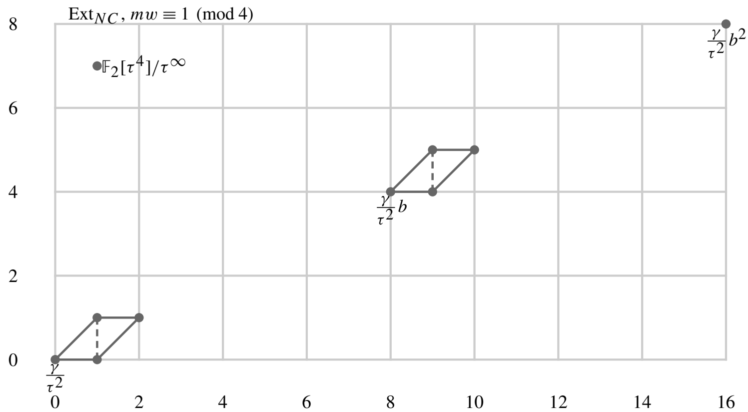
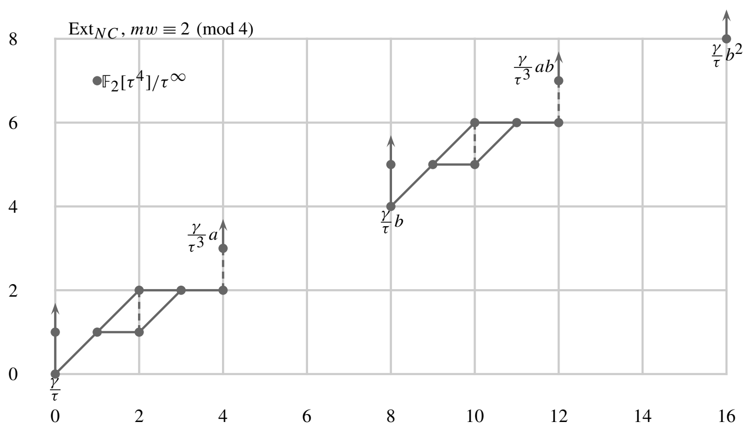
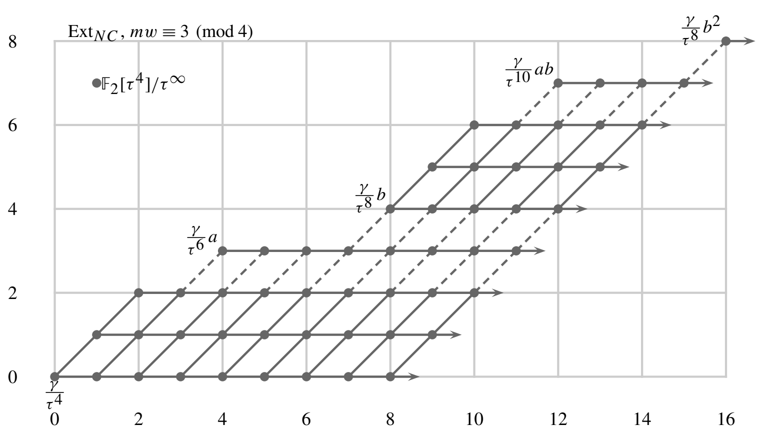




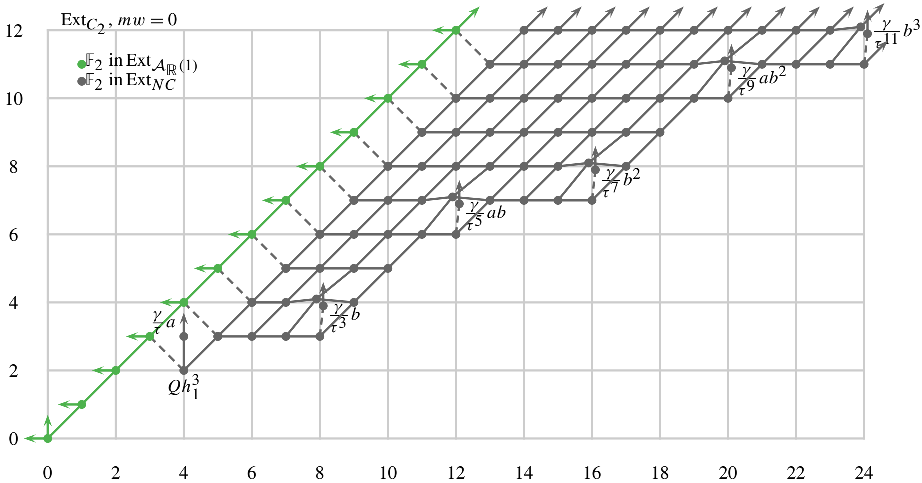
$\mathrm{Ext}_{\mathrm{NC}}$ charts for $\mathcal{A}^{C_2}(1)$

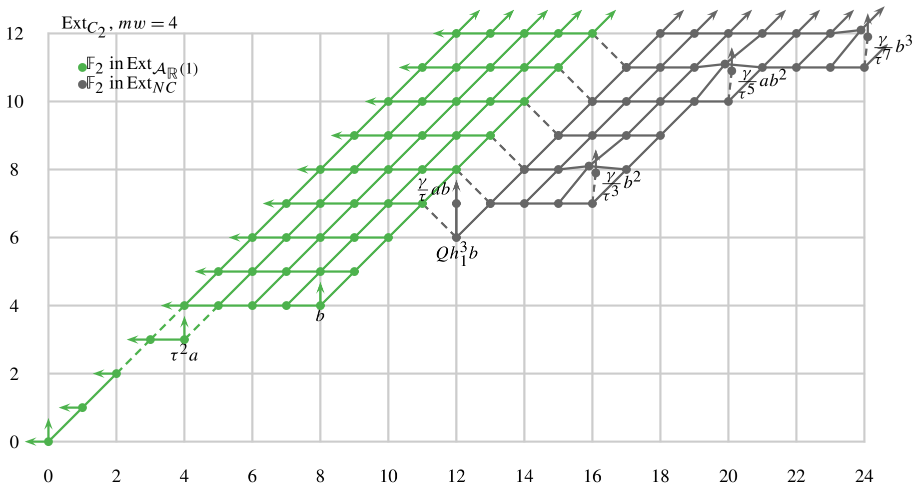




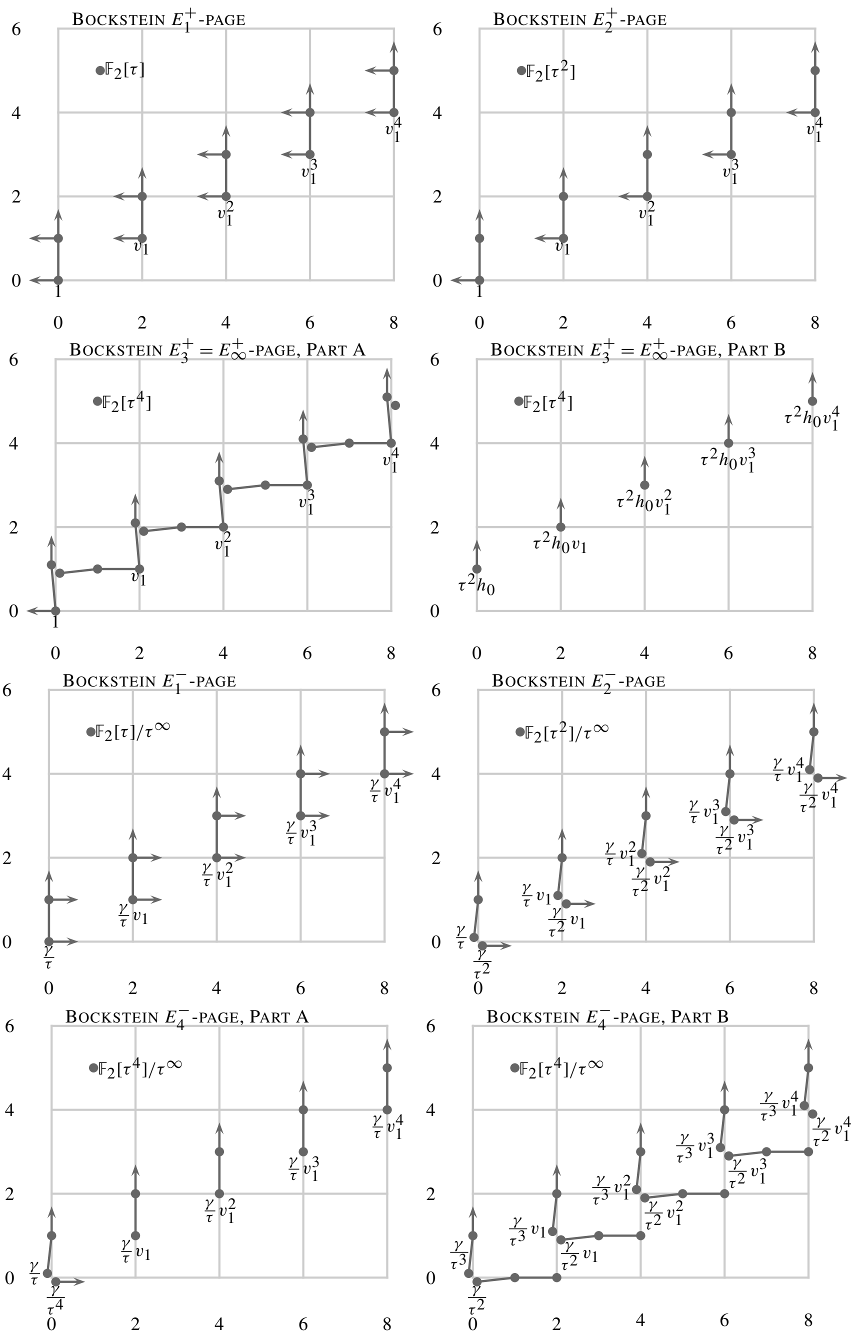


Ext charts for $\mathcal{A}^{C_2}(1)$ in $mw = 0$ and $mw = 4$

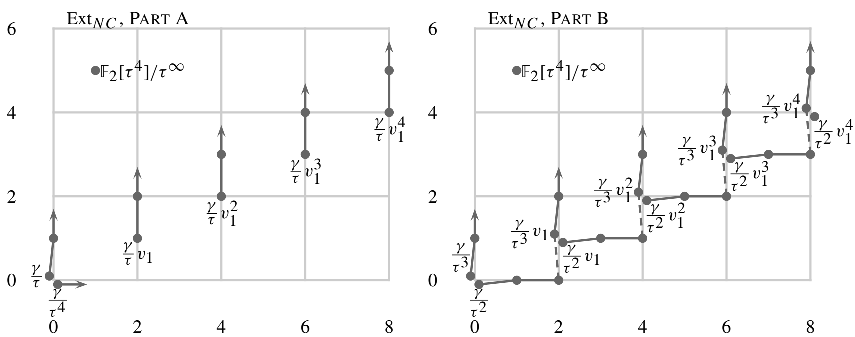




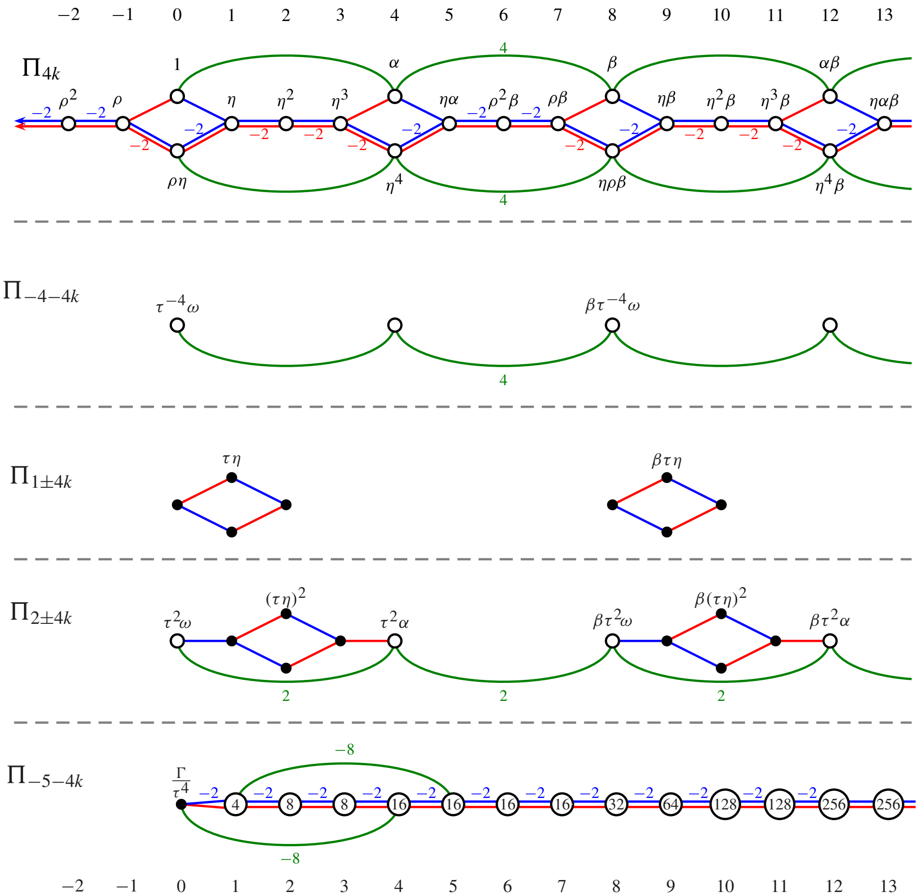
Bockstein charts for $\mathcal{E}^{C_2}(1)$



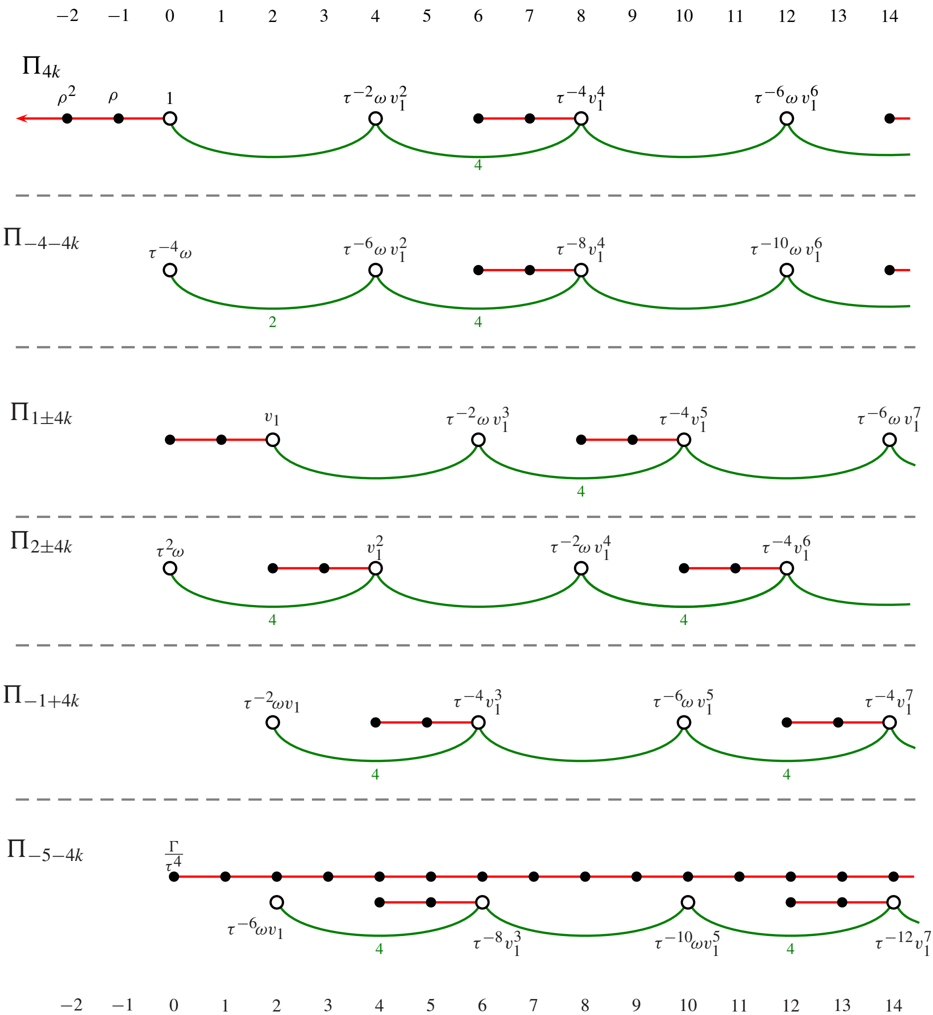
$\mathrm{Ext}_{\mathrm{NC}}$ charts for $\mathcal{E}^{C_2}(1)$



Milnor–Witt modules for ko_{C_2}



Milnor–Witt modules for $k\mathbb{R}$



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Degeneracy loci, virtual cycles and nested Hilbert schemes, I

Amin Gholampour and Richard P. Thomas

Given a map of vector bundles on a smooth variety, consider the deepest degeneracy locus where its rank is smallest. We show it carries a natural perfect obstruction theory whose virtual cycle can be calculated by the Thom–Porteous formula.

We show nested Hilbert schemes of points on surfaces can be expressed as degeneracy loci. We show how to modify the resulting obstruction theories to recover the virtual cycles of Vafa–Witten and reduced local DT theories. The result computes some Vafa–Witten invariants in terms of Carlsson–Okounkov operators. This proves and extends a conjecture of Gholampour, Sheshmani, and Yau and generalises a vanishing result of Carlsson and Okounkov.

1. Introduction

The prototype of a scheme Z with *perfect obstruction theory* [Behrend and Fantechi 1997] is the zero locus of a section of a vector bundle E on a smooth ambient variety A. We recall the construction in the next Section.

All perfect obstruction theories are *locally* of this form. In the rare situations where this is also true *globally*, the natural virtual cycle [ibid.] pushes forward to

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what we might expect, namely the Euler class of the bundle:

$$\iota_*[Z]^{\mathrm{vir}} = c_r(E) \in A_{\mathrm{vd}}(A). \tag{1.1}$$

Here $\iota : Z \hookrightarrow A$ is the inclusion, $r = \operatorname{rank} E$, $\mathrm{vd} = \dim A - r$ is the virtual dimension of the problem, and $[Z]^{\mathrm{vir}}$ lies in $A_{\mathrm{vd}}(Z)$ or $H_{2\mathrm{vd}}(Z)$.

Equation (1.1) can help in computing integrals over the virtual cycle. Examples include the computation of the number 27 of lines on a cubic surface, numbers of lines and conics on quintic threefolds, and the quantum hyperplane principle. A more relevant example to us is the reduced stable pair computations in [Kool and Thomas 2014], carried out by writing the moduli space of stable pairs (and its reduced perfect obstruction theory) as the zero locus of a section of a tautological bundle over a certain Hilbert scheme.

In this paper we study a generalisation of zero loci, namely degeneracy loci. We show these give another prototype of a perfect obstruction theory.[1] Again, when this can be done *globally*, it allows us to express integrals over the virtual cycle in terms of integrals over the ambient space, via the Thom–Porteous formula.

So fix a two term complex of vector bundles $E_\bullet = \{E_0 \xrightarrow{\sigma} E_1\}$ on a smooth ambient space A. Set $n = \dim A$, $r_i = \operatorname{rank}(E_i)$, and denote the r-th degeneracy locus by

$$Z_r := \big\{x \in A : \operatorname{rank}(\sigma|_x) \le r\big\}.$$

We work with the smallest r for which $Z := Z_r$ is nonempty. Our first result is the following, made more precise in Theorem 3.6.

Theorem. *Assume $Z_{r-1} = \varnothing$. Then both*

$$h^0(E_\bullet|_Z) = \ker(\sigma|_Z) \quad \textit{and} \quad h^1(E_\bullet|_Z) = \operatorname{coker}(\sigma|_Z)$$

are locally free on $Z := Z_r$, which inherits a perfect obstruction theory

$$\big\{h^1(E_\bullet|_Z)^* \otimes h^0(E_\bullet|_Z) \to \Omega_A|_Z\big\} \to \mathbb{L}_Z.$$

The push-forward of the resulting virtual cycle $[Z]^{\mathrm{vir}} \in A_{n-k}(Z)$ to A is given by the Thom–Porteous formula,

$$\Delta^{r_0-r}_{r_1-r}\big(c(E_1 - E_0)\big) \in A_{n-k}(A),$$

where $k = (r_0 - r)(r_1 - r)$ and $\Delta^a_b(c) := \det(c_{b+j-i})_{1\le i,j\le a}$.

[1] In fact we prove this by reducing to the model (2.1) in a bigger ambient space.

Nested Hilbert schemes. Our main application is to the punctual Hilbert schemes of *nested* subschemes of a fixed projective surface S. Full details and notation will be described later; for now for simplicity we restrict attention to the simplest case of the 2-step nested punctual Hilbert scheme

$$S^{[n_1,n_2]} := \big\{ I_1 \subseteq I_2 \subseteq \mathcal{O}_S : \operatorname{length}(\mathcal{O}_S/I_i) = n_i \big\}.$$

Now $S^{[n_1,n_2]}$ lies in the ambient space $S^{[n_1]} \times S^{[n_2]}$ as the locus of points (I_1, I_2) for which there is a nonzero map $\operatorname{Hom}_S(I_1, I_2) \neq 0$. Thus it can be seen as the degeneracy locus of the complex of vector bundles

$$R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2) \quad \text{over } S^{[n_1]} \times S^{[n_2]} \tag{1.2}$$

which, when restricted to the point (I_1, I_2), computes $\operatorname{Ext}^*_S(I_1, I_2)$. When $H^{0,2}(S) = 0$ this complex is 2-term, so we can apply the above theory. The resulting perfect obstruction theory on $S^{[n_1,n_2]}$ agrees with that of [Gholampour et al. 2017b]. In turn this arises in local DT theory [Gholampour et al. 2017a], so we can express DT integrals in terms of Chern classes of tautological sheaves over $S^{[n_1]} \times S^{[n_2]}$.

When $H^{0,1}(S) \neq 0$ the result is zero; when $H^{0,2}(S) \neq 0$ the theory does not apply. So for a general projective surface S we modify the complex $\operatorname{Ext}^*_S(I_1, I_2)$ with $H^1(\mathcal{O}_S)$ and $H^2(\mathcal{O}_S)$ terms. The modification is canonical over $S^{[n_1,n_2]}$, recovering the *reduced* version of the local DT deformation theory that arises in the $\mathrm{SU}(r)$ Vafa–Witten theory of S [Tanaka and Thomas 2017].

Splitting trick. We would like to extend this modification over the rest of $S^{[n_1]} \times S^{[n_2]}$, so we can apply the Thom–Porteous formula. Such modifications exist locally but *not* globally, so in Section 6A we use a trick reminiscent of the splitting principle in topology, pulling back to a certain bundle over $S^{[n_1]} \times S^{[n_2]}$ where there *is* a canonical modification. This allows us to prove the following (whose notation will be explained more fully in Sections 5–7, in particular (6.31)).

Theorem. *Let S be any smooth projective surface. The k-step nested Hilbert scheme $S^{[n_1,\dots,n_k]}$ can be seen as an intersection of degeneracy loci after pulling back to an affine bundle over $S^{[n_1]} \times \cdots \times S^{[n_k]}$. The resulting perfect obstruction theory $F^\bullet \to \mathbb{L}_{S^{[n_1,\dots,n_k]}}$ has virtual tangent bundle*

$$(F^\bullet)^\vee \cong \big\{ T_{S^{[n_1]}} \oplus \cdots \oplus T_{S^{[n_k]}} \to \mathscr{E}xt^1_p(\mathcal{I}_1, \mathcal{I}_2)_0 \oplus \cdots \oplus \mathscr{E}xt^1_p(\mathcal{I}_{k-1}, \mathcal{I}_k)_0 \big\},$$

the same as the one in Vafa–Witten theory [Tanaka and Thomas 2017] *or "reduced local DT theory"* [Gholampour et al. 2017b; 2017a]. *The virtual cycle*

$$[S^{[n_1,\dots,n_k]}]^{\mathrm{vir}} \in A_{n_1+n_k}(S^{[n_1,\dots,n_k]})$$

pushes forward to

$$c_{n_1+n_2}\big(R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)[1]\big) \cup \cdots \cup c_{n_{k-1}+n_k}\big(R\mathscr{H}om_\pi(\mathcal{I}_{k-1}, \mathcal{I}_k)[1]\big) \tag{1.3}$$

in $A_{n_1+n_k}(S^{[n_1]} \times \cdots \times S^{[n_k]})$.

The formula (1.3) for the push-forward of the virtual class was conjectured in [Gholampour et al. 2017b] for $k = 2$ and proved for toric surfaces. It was also shown to be true for more general surfaces when integrated against some natural classes. The classes $c_{n_{i-1}+n_i}\big(R\mathscr{H}om_\pi(\mathcal{I}_{i-1}, \mathcal{I}_i)[1]\big)$, considered as maps

$$H^*\big(S^{[n_{i-1}]}\big) \to H^{*+2n_i-2n_{i-1}}\big(S^{[n_i]}\big),$$

are called Carlsson–Okounkov operators. Carlsson and Okounkov [2012] calculate them in terms of Grojnowski–Nakajima operators, and prove vanishing of the higher Chern classes:

$$c_{n_1+n_2+i}\big(R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)[1]\big) = 0, \quad i > 0, \tag{1.4}$$

by showing the left side is a universal expression in Chern numbers of S, and that this universal expression vanishes for toric surfaces by a localisation computation. This gives enough relations to prove the universal expression is in fact zero. In Section 8 we reprove the vanishing (1.4) rather easily and geometrically using the Thom–Porteous formula, as well as the following generalisation.

Theorem. *Let S be any smooth projective surface. For any curve class $\beta \in H_2(S, \mathbb{Z})$, any Poincaré line bundle $\mathcal{L} \to S \times \mathrm{Pic}_\beta(S)$, and any $i > 0$,*

$$c_{n_1+n_2+i}\big(R\pi_*\mathcal{L} - R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2 \otimes \mathcal{L})\big) = 0 \quad \textit{on } S^{[n_1]} \times S^{[n_2]} \times \mathrm{Pic}_\beta(S).$$

The other degeneracy loci. In the companion paper [Gholampour and Thomas 2019] we work with *all* the degeneracy loci Z_k. These do not generally admit perfect obstruction theories when $k > r$. However there are natural spaces $\widetilde{Z}_k \to Z_k$ dominating them which are actually resolutions of their singularities in the transverse case (when all the Z_k have the correct codimension). For this reason we call the $\widetilde{Z}_k$ "*virtual resolutions*". Though they are singular in general, we show they admit natural perfect obstruction theories and virtual cycles whose push-forwards we can again describe by Chern class formulae.[2]

In this paper the natural application was to nested *punctual* Hilbert schemes of a smooth surface S. In [Gholampour and Thomas 2019] the natural application is to nested Hilbert schemes of both points *and curves* in S. Fundamentally the difference is the following. Letting $I_1, I_2 \subset \mathcal{O}_S$ be ideal sheaves of 0-dimensional subschemes of S, then

$$\mathrm{Hom}(I_1, I_2) \tag{1.5}$$

either vanishes, or — for $I_1 \subset I_2$ in the nested Hilbert scheme — is at most $\mathbb{C}$. Hence $S^{[n_1,n_2]}$ is the degeneracy locus of the complex (1.2). Conversely, when I_1 or I_2 have divisorial components, (1.5) can become arbitrarily big, and different elements

[2]Since $\widetilde{Z}_r \cong Z_r$ the constructions in [Gholampour and Thomas 2019] and this paper coincide when $k = r$.

correspond to different subschemes of S. (In the case $I_1 = \mathcal{O}_S(-D)$ and $I_2 = \mathcal{O}_S$, elements correspond — up to scale — to divisors in the same linear system as the divisor $D \subset S$.) Therefore the corresponding nested Hilbert scheme *dominates* the degeneracy locus of the complex (1.2) but need not equal it. In [Gholampour and Thomas 2019] we show it is naturally a virtual resolution of the type $\widetilde{Z}_k$.

Notation. Given a map $f : X \to Y$, we often use the same letter f to denote its basechange by any map $Z \to Y$, i.e., $f : X \times_Y Z \to Z$. We also sometimes suppress pullback maps f^* on sheaves.

2. Zero loci

We start by recalling the standard construction of a perfect obstruction theory, on the zero scheme Z of a section σ of a vector bundle E over a smooth ambient space A:

$$\begin{array}{ccc} & & E \\ & & \downarrow \big)\, \sigma \\ Z = \sigma^{-1}(0) & \subset & A \end{array} \tag{2.1}$$

On Z the derivative of this diagram gives

$$\begin{array}{ccc} E^*|_Z & \xrightarrow{d\sigma|_Z} & \Omega_A|_Z \\ \sigma \big\downarrow & & \big\| \\ I/I^2 & \xrightarrow{d} & \Omega_A|_Z \end{array} \tag{2.2}$$

where $I \subset \mathcal{O}_A$ is the ideal sheaf of Z generated by σ. The bottom row is a representative of the truncated cotangent complex $\mathbb{L}_Z$ of Z; denoting the two-term locally free complex on the top row by $F^\bullet$ we get a morphism[3]

$$F^\bullet \to \mathbb{L}_Z \tag{2.3}$$

in $D(\mathrm{Coh}\, Z)$ which induces an isomorphism on 0th cohomology sheaves h^0 and a surjection on h^{-1}. This data is called a *perfect obstruction theory* [Behrend and Fantechi 1997] on Z, and induces a virtual cycle

$$[Z]^{\mathrm{vir}} \in A_{\mathrm{vd}}(Z) \to H_{2\mathrm{vd}}(Z)$$

satisfying natural properties. Here H denotes Borel–Moore homology, and $\mathrm{vd} := \dim A - \mathrm{rank}\, E$ is the *virtual dimension* of the perfect obstruction theory.

[3]Diagram (2.1) also induces a natural map from $F^\bullet$ to the *full* cotangent complex of Z [Behrend and Fantechi 1997, Section 6], but we shall not need this.

3. Degeneracy loci

We work on a smooth complex quasiprojective variety A with a map

$$E_0 \xrightarrow{\sigma} E_1$$

between vector bundles of ranks r_0 and r_1. We denote by

$$Z_k \subset A \tag{3.1}$$

the degeneracy locus where $\operatorname{rank}(\sigma)$ drops to $\leq k$. This has a scheme structure defined by the vanishing of the $(k+1)\times(k+1)$ minors of σ, i.e., of

$$\bigwedge^{k+1}\sigma : \bigwedge^{k+1} E_0 \to \bigwedge^{k+1} E_1. \tag{3.2}$$

The Z_k can be characterised by the rank of the cokernel of σ over them [Eisenbud 1995, Section 20.2]. In Section 6 we will need a characterisation in terms of the kernel. Though this does not basechange well, it works for the smallest Z_k.

That is, let r denote the minimal rank of σ, so that $Z_{r-1} = \varnothing$, and set $Z := Z_r$. This is the largest subscheme of A on which $\ker \sigma|_Z$ is locally free of rank $r_0 - r$:

Lemma 3.3. *For a map of schemes $f : T \to A$, the following are equivalent*:

(1) *f factors through $Z = Z_r \subset A$.*

(2) $\ker(f^*\sigma : f^*E_0 \to f^*E_1)$ *is a rank $r_0 - r$ subbundle of f^*E_0.*

(3) $\ker(f^*\sigma : f^*E_0 \to f^*E_1)$ *has a locally free subsheaf of rank $r_0 - r$.*

Proof. If f factors through Z then $\bigwedge^{r+1} f^*\sigma \cong f^*\bigwedge^{r+1}\sigma|_Z \equiv 0$. Since $Z_{r-1} = \varnothing$ it follows from [Eisenbud 1995, Proposition 20.8] that $\operatorname{coker} f^*\sigma$ is locally free of rank $r_1 - r$. Thus $\ker f^*\sigma$ is a rank $r_0 - r$ subbundle of f^*E_0. This proves (1)$\Rightarrow$(2)$\Rightarrow$(3).

For (3)$\Rightarrow$(1), we suppose the kernel K of $f^*E_0 \to f^*E_1$ contains a locally free subsheaf of rank $r_0 - r$. Therefore the rank of $f^*\sigma$ on the generic point of T is $\leq r$, and thus in fact equal to r since we are assuming it drops no lower. In particular, $\operatorname{coker}(f^*\sigma)$ is a rank $r_1 - r$ sheaf.

By lower semicontinuity of rank, $f^*\sigma|_t$ is of rank $\leq r$ for any closed point $t \in T$, so, by our assumption on r again, it is equal to r. Combined with the exact sequence

$$f^*E_0|_t \xrightarrow{\sigma|_t} f^*E_1|_t \to (\operatorname{coker} f^*\sigma)|_t \to 0, \tag{3.4}$$

i.e., the fact that $\operatorname{coker}(f^*\sigma|_t) = (\operatorname{coker} f^*\sigma)|_t$, this shows that $(\operatorname{coker} f^*\sigma)|_t$ has dimension $r_1 - r$ for every closed point t. Therefore $\operatorname{coker} f^*\sigma$ is locally free of rank $r_1 - r$ by the Nakayama lemma. This implies that $\ker f^*\sigma$ is a rank $r_0 - r$ subbundle (rather than just a locally free subsheaf) of f^*E_0.

In particular f^*E_0/K is locally free of rank r, so $\bigwedge^{r+1}(f^*E_0/K)=0$. But

$$f^*\textstyle\bigwedge^{r+1}\sigma=\bigwedge^{r+1}f^*\sigma:\bigwedge^{r+1}f^*E_0\to\bigwedge^{r+1}f^*E_1$$

factors through $\bigwedge^{r+1}(f^*E_0/K)$, so it is also zero. That is, f factors through the zero scheme $Z\big(\bigwedge^{r+1}\sigma\big)=Z_r$ of $\bigwedge^{r+1}\sigma$. □

So $\sigma|_Z$ has rank precisely r, and its kernel $h^0:=h^0(E_\bullet|_Z)$ and cokernel $h^1:=h^1(E_\bullet|_Z)$ are *vector bundles* on Z of rank r_0-r and r_1-r respectively,

$$0\to h^0\to E_0|_Z\xrightarrow{\sigma|_Z}E_1|_Z\to h^1\to 0. \tag{3.5}$$

For instance if $r=r_0-1$ then σ is generically injective (and globally injective as a map of coherent sheaves) and Z is the locus where it fails to be injective as a map of bundles. Its kernel is a line bundle over Z. If $E_0=\mathcal{O}_A$ then Z is the zero locus of σ and we are back in the setting of Section 2.

Theorem 3.6. *The degeneracy locus $Z=Z_r$ inherits a 2-term perfect obstruction theory*

$$\big\{(h^1)^*\otimes h^0\to\Omega_A|_Z\big\}\to\mathbb{L}_Z.$$

The push-forward of the resulting virtual cycle $[Z]^{\mathrm{vir}}\in A_{n-k}(Z)$ to A is given by the Thom–Porteous formula

$$\Delta_{r_1-r}^{r_0-r}\big(c(E_1-E_0)\big)\in A_{n-k}(A).$$

Here $n=\dim A,\ k=(r_0-r)(r_1-r)$ and $\Delta_b^a(c):=\det(c_{b+j-i})_{1\le i,j\le a}$.

Proof. We work on the relative Grassmannian of $(r_0{-}r)$-dimensional subspaces of E_0,

$$\mathrm{Gr}:=\mathrm{Gr}(r_0-r,\,E_0)\xrightarrow{q}A$$

with universal subbundle $U\hookrightarrow q^*E_0$. Composing with $q^*\sigma$ gives a section

$$\tilde\sigma\in\Gamma(U^*\otimes q^*E_1). \tag{3.7}$$

Claim. *The zero locus $Z(\tilde\sigma)\subset\mathrm{Gr}$ is isomorphic to $Z\subset A$ under the restriction $\bar q:Z(\tilde\sigma)\to A$ of the projection $q:\mathrm{Gr}\to A$.*

At the level of closed points this is obvious: for $x\in A$

$$\begin{aligned}x\in Z&\iff\operatorname{rank}(\sigma|_x)=r\\&\iff\operatorname{rank}(\ker(\sigma_x))=r_0-r\\&\iff(E_0)|_x\text{ has a unique }(r_0{-}r)\text{-dimensional subspace}\\&\qquad\qquad U_x=\ker(\sigma_x)\text{ on which }\sigma|_x\text{ vanishes}\\&\iff U_x\text{ is the unique point of }Z(\tilde\sigma)\cap q^{-1}\{x\}.\end{aligned}$$

So $\bar{q}$ maps $Z(\tilde{\sigma})$ bijectively to $Z \subset A$. To see it maps scheme theoretically, note that, by construction, the composition

$$U \hookrightarrow q^*E_0 \xrightarrow{q^*\sigma} q^*E_1$$

is zero over $Z(\tilde{\sigma})$, so $\ker(\bar{q}^*\sigma)$ contains a locally free sheaf $U|_{Z(\tilde{\sigma})}$ of rank $r_0 - r$. Thus $\bar{q}$ factors through $Z \subset A$ by Lemma 3.3.

By Lemma 3.3 again, $\ker(\sigma|_Z)$ is a rank $r_0 - r$ subbundle of E_0. Its classifying map $Z \to \mathrm{Gr}(r_0 - r, E_0)$ has image in $Z(\tilde{\sigma})$ and clearly defines a right inverse to $\bar{q} : Z(\tilde{\sigma}) \to Z$. So to prove that $\bar{q}$ is an isomorphism to Z we need only show that the inverse image $\bar{q}^{-1}\{x\}$ of any closed point $x \in Z$ is a closed point of $Z(\tilde{\sigma})$.

Given a rank r linear map $\Sigma : V \to W$ between vector space of dimensions r_0, r_1, an elementary calculation show that the composition

$$U \hookrightarrow V \otimes \mathcal{O} \xrightarrow{\Sigma} W \otimes \mathcal{O}$$

on the Grassmannian $\mathrm{Gr}(r_0 - r, V)$ cuts out the *reduced* point $[\ker \Sigma \subset V] \in \mathrm{Gr}(r_0 - r, V)$. Applying this to $\Sigma = \sigma|_x$ proves the claim.

Perfect obstruction theory. Since $Z \cong Z(\tilde{\sigma})$ is cut out of Gr by $\tilde{\sigma} \in \Gamma(U^* \otimes q^*E_1)$, it inherits the standard perfect obstruction theory (2.2), i.e.,

$$U \otimes q^*E_1^*|_{Z(\tilde{\sigma})} \xrightarrow{d\tilde{\sigma}|_{Z(\tilde{\sigma})}} \Omega_{\mathrm{Gr}}|_{Z(\tilde{\sigma})} \tag{3.8}$$

mapping to $\mathbb{L}_{Z(\tilde{\sigma})} = \mathbb{L}_Z$. Now (3.8) fits into a diagram

$$\begin{array}{ccc} U|_Z \otimes (h^1)^* & \longrightarrow & q^*\Omega_A|_{Z(\tilde{\sigma})} \\ \downarrow & & \downarrow \\ U \otimes E_1^*|_Z & \xrightarrow{d\tilde{\sigma}|_{Z(\tilde{\sigma})}} & \Omega_{\mathrm{Gr}}|_{Z(\tilde{\sigma})} \\ {\scriptstyle \mathrm{id}_U \otimes} \downarrow {\scriptstyle \sigma^*} & & \downarrow \\ U|_Z \otimes \big(E_0|_Z \big/ \ker \sigma\big)^* & = & \Omega_{\mathrm{Gr}/A}|_{Z(\tilde{\sigma})} \end{array} \tag{3.9}$$

with left-hand column the short exact sequence $U|_Z \otimes$ (3.5)*, and right-hand column the natural short exact sequence of the fibration $\mathrm{Gr} \to A$. The bottom equality is dual to the standard identification $T_{\mathrm{Gr}/A} \cong \mathcal{H}om(U, E_0/U)$.

Assuming (3.9) is commutative for now, we can consider it as providing a quasi-isomorphism between the top row and the middle row (which is (3.8)). Hence the perfect obstruction theory (3.8) is

$$h^0 \otimes (h^1)^* \to \Omega_A|_Z,$$

as claimed. Just as in (1.1), the push-forward of the resulting virtual cycle to Gr is the Euler class $c_{(r_0-r)r_1}(U^* \otimes q^*E_1)$. Pushing this down to A gives the push-forward of $[Z]^{\mathrm{vir}}$ to A, by the commutativity of the diagram

$$\begin{array}{ccc} Z(\tilde{\sigma}) & \hookrightarrow & \mathrm{Gr} \\ \| & & \downarrow q \\ Z & \hookrightarrow & A \end{array}$$

But pushing forward $c_{(r_0-r)r_1}(U^* \otimes q^*E_1)$ to A gives $\Delta_{r_1-r}^{r_0-r}\big(c(E_1-E_2)\big)$ by [Fulton 1984, Theorem 14.4]. So we are left to prove:

Claim. *The diagram (3.9) is commutative.*

We need only show that the lower square of (3.9) commutes; the upper one is then induced from it. Let $\mathrm{Gr}_Z := \mathrm{Gr} \times_A Z$ and observe $Z(\tilde{\sigma}) \subset \mathrm{Gr}_Z$, with ideal sheaf I, say. We let

$$2Z \hookrightarrow \mathrm{Gr}_Z$$

be its scheme-theoretic doubling with ideal sheaf I^2. Let $p := q|_{2Z}$ be the induced projection $2Z \to Z$ and consider the maps

$$U|_{2Z} \hookrightarrow (q^*E_0)|_{2Z} \cong p^*(E_0|_Z) \to p^*(E_0/U|_Z) \xrightarrow{\sigma|_Z} p^*(E_1|_Z). \tag{3.10}$$

The final arrow is constructed from $\sigma|_Z : E_0|_Z \to E_1|_Z$ by recalling that $U|_Z \cong \ker(\sigma|_Z)$.

The composition of the first two arrows of (3.10) is a section of

$$U^*|_{2Z} \otimes p^*(E_0/U|_Z) \quad \text{on } 2Z$$

which vanishes on Z. Since the ideal of $Z \subset 2Z$ is $\Omega_{\mathrm{Gr}_Z/Z}$ it is a section of

$$(U|_Z)^* \otimes (E_0/U|_Z) \otimes \Omega_{\mathrm{Gr}_Z/Z}.$$

This is precisely the (adjoint of) the standard description of the isomorphism

$$U|_Z \otimes (E_0/U)|_Z^* \cong \Omega_{\mathrm{Gr}_Z/Z},$$

i.e., the bottom row of (3.9).

Since $p^*(E_1|_Z) = (q^*E_1)|_{2Z}$, the composition of all the arrows in (3.10) is just $\tilde{\sigma}|_{2Z}$. It vanishes on Z, defining the section $[d\tilde{\sigma}|_Z]$ of

$$(U|_Z)^* \otimes E_1|_Z \otimes I/I^2 \cong \mathscr{H}om\big(U \otimes E_1^*|_Z, \Omega_{\mathrm{Gr}/A}|_Z\big)$$

which defines the central arrow of (3.9). Thus (3.9) commutes. □

3A. ***Higher Thom–Porteous formula.*** When $r_0 - r = 1$, so the sheaf h^0 is a line bundle on the degeneracy locus Z, the following “higher” Thom–Porteous formula will be useful later. Let $\iota : Z \hookrightarrow A$ denote the inclusion.

Proposition 3.11. *If* $r_0 - r = 1$ *then the Thom–Porteous formula becomes*

$$\iota_*[Z]^{\mathrm{vir}} = c_{r_1 - r_0 + 1}(E_1 - E_0)$$

in $A_{n+r-r_1}(A)$, *and for any* $i \geq 0$ *we have the following extension to higher Chern classes*:

$$\iota_*\big(c_1\big((h^0)^*\big)^i \cap [Z]^{\mathrm{vir}}\big) = c_{r_1 - r_0 + 1 + i}(E_1 - E_0). \tag{3.12}$$

Proof. The first part follows from the simplification

$$\Delta_b^a(c(\,\cdot\,)) = c_b(\,\cdot\,)$$

when $a = r_0 - r = 1$.

For the second part, recall from (3.7) that Z is cut out of $\mathbb{P}(E_0) \xrightarrow{q} A$ by the vanishing of the composition

$$\mathcal{O}_{\mathbb{P}(E_0)}(-1) \hookrightarrow q^*E_0 \xrightarrow{q^*\sigma} q^*E_1.$$

Moreover, over this copy of Z, we see that the kernel h^0 of $E_0 \to E_1$ is $\mathcal{O}_{\mathbb{P}(E_0)}(-1)$. Therefore, denoting Segre classes by s_i, we have

$$\begin{aligned}
\iota_*\big(c_1((h^0)^*)^i \cap \big[Z\big]^{\mathrm{vir}}\big) &= q_*\big(c_1(\mathcal{O}_{\mathbb{P}(E_0)}(1))^i \cup c_{r_1}(q^*E_1(1))\big) \\
&= q_*\Big(c_1(\mathcal{O}_{\mathbb{P}(E_0)}(1))^i \cup \sum_{j=0}^{r_1} c_j(q^*E_1) \cup c_1(\mathcal{O}_{\mathbb{P}(E_0)}(1))^{r_1 - j}\Big) \\
&= \sum_{j=0}^{r_1} q_*\big(c_1(\mathcal{O}_{\mathbb{P}(E_0)}(1))^{i + r_1 - j} \cup q^*c_j(E_1)\big) \\
&= \sum_{j=0}^{r_1} s_{i + r_1 - j - r_0 + 1}(E_0) \cap c_j(E_1) \\
&= c_{r_1 - r_0 + i + 1}(E_1 - E_0).
\end{aligned}$$

□

Working throughout this Section with $\sigma^* : E_1^* \to E_0^*$ instead of $\sigma : E_0 \to E_1$ gives the same results, up to some reindexing of notation.

4. Jumping loci of direct image sheaves

Suppose $f : X \to Y$ is a morphism of projective schemes, with Y smooth. Fix either a coherent sheaf $\mathcal{F}$ on X which is *flat over* Y, or a perfect complex $\mathcal{F}$ on X and assume that X *is flat over* Y.

We assume that the cohomologies of $\mathcal{F}$ on any closed fibre X_y, $y \in Y$, are concentrated in only two adjacent degrees i, $i+1$. Let a denote the maximal dimension of $h^i(X_y, \mathcal{F}_y)$ as y varies throughout Y. That is, we assume there exists $i \in \mathbb{Z}$ such that

$$h^j(X_y, \mathcal{F}_y) = 0 \quad \text{for all } j \notin \{i, i+1\}, \ y \in Y,$$
$$h^i(X_y, \mathcal{F}_y) \le a \quad \text{for all } y \in Y.$$

It follows that $h^{i+1}(X_y, \mathcal{F}_y)$ has maximal dimension $b := a - (-1)^i \chi(\mathcal{F}_y)$.

Now $Rf_*\mathcal{F}$ is a perfect complex on Y which, by basechange and the Nakayama lemma, can be trimmed to be a 2-term complex of locally free sheaves

$$Rf_*\mathcal{F} \simeq \{E_i \to E_{i+1}\}$$

in degrees i and $i+1$. On restriction to the maximal degeneracy locus

$$Z_a := \left\{ y \in Y : h^i(X_y, \mathcal{F}_y) = a \right\} \subset Y$$

it has kernel of rank a. (Note this labelling convention differs slightly from (3.1).) Let $X_Z := X \times_Y Z$ and $\bar{f} := f|_{X_Z}$. By (3.2) and Theorem 3.6 we deduce the following.

Proposition 4.1. *The maximal jumping locus $Z = Z_a$ has a natural scheme structure and perfect obstruction theory*

$$\left\{ (R^{i+1}\bar{f}_*\mathcal{F})^* \otimes R^i \bar{f}_*\mathcal{F} \to \Omega_Y|_Z \right\} \to \mathbb{L}_Z,$$

with the $R^j\bar{f}_*\mathcal{F}$ locally free. The resulting virtual cycle

$$[Z]^{\mathrm{vir}} \in A_d(Z), \quad d := \dim Y - ab,$$

when pushed forward to Y, is given by

$$\Delta_b^a\big(c(Rf_*\mathcal{F}[i+1])\big) \in A_d(Y).$$

The result can also be applied to jump loci of relative Ext sheaves (the cohomology sheaves of $R\mathscr{H}om_f(A, B) := Rf_*R\mathscr{H}om(A, B)$) by setting $\mathcal{F} := R\mathscr{H}om(A, B)$. We shall use this on punctual Hilbert schemes next.

5. Nested Hilbert schemes on surfaces with $b_1 = 0 = p_g$

Given positive integers $n_1 \ge n_2 \ge \cdots \ge n_k$, the k-step nested punctual Hilbert scheme of S is, as a set,

$$\begin{aligned} S^{[n_1,n_2,\ldots,n_k]} &:= \left\{ S \supseteq Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_k : \text{length}(Z_i) = n_i \right\} \\ &= \left\{ I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subset \mathcal{O}_S : \text{length}(\mathcal{O}_S/I_i) = n_i \right\}. \end{aligned}$$

As a scheme it represents the functor which takes any base scheme B to the set of ideals $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_k \subset \mathcal{O}_{S\times B}$, flat over B, such that the restriction of $\mathcal{I}_i$ to any closed fibre $S \times \{b\}$ has colength n_i.

For simplicity we restrict to $k = 2$ for now; we will return to general k in Section 7.

Let S be a smooth complex projective surface with (for now) $h^{0,1}(S)=0=h^{0,2}(S)$, and fix integers $n_1 \geq n_2$. Over

$$S^{[n_1]} \times S^{[n_2]} \times S \xrightarrow{\pi} S^{[n_1]} \times S^{[n_2]}$$

we have the two universal subschemes $\mathcal{Z}_1$, $\mathcal{Z}_2$ and their ideal sheaves $\mathcal{I}_1, \mathcal{I}_2$. We will apply Proposition 4.1 to the perfect complex

$$R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2) := R\pi_* R\mathscr{H}om(\mathcal{I}_1, \mathcal{I}_2)$$

on $S^{[n_1]} \times S^{[n_2]}$. Over the closed point $(I_1, I_2) \in S^{[n_1]} \times S^{[n_2]}$ we have

$$\operatorname{Ext}^i(I_1, I_2) = 0, \quad i \neq 0, 1, \tag{5.1}$$

by Serre duality. Moreover

$$\operatorname{Hom}(I_1, I_2) = \begin{cases} 0, & Z_1 \not\supseteq Z_2, \\ \mathbb{C}, & Z_1 \supseteq Z_2, \end{cases} \tag{5.2}$$

is generically zero and jumps by 1 (but never more) over the nested Hilbert scheme

$$S^{[n_1,n_2]} := \big\{ Z_2 \subseteq Z_1 \subset S,\ \operatorname{length}(Z_i) = n_i \big\}, \tag{5.3}$$

at least set-theoretically. Despite our usual notational conventions (to denote π basechanged by $S^{[n_1,n_2]} \hookrightarrow S^{[n_1]} \times S^{[n_2]}$ also by π) we reserve

$$p : S^{[n_1,n_2]} \times S \to S^{[n_1,n_2]}$$

for the obvious projection. Since $\mathcal{I}_1, \mathcal{I}_2$ are flat over $S^{[n_1]} \times S^{[n_2]}$ they restrict to ideal sheaves over $S^{[n_1,n_2]}$; we denote them by the same letters.

Proposition 5.4. *If $h^{0,1}(S) = 0 = h^{0,2}(S)$ then the 2-step nested Hilbert scheme $S^{[n_1,n_2]}$ carries a perfect obstruction theory*

$$\Big((\mathscr{E}xt^1_p(\mathcal{I}_1, \mathcal{I}_2))^* \to \Omega_{S^{[n_1]}\times S^{[n_2]}}|_{S^{[n_1,n_2]}} \Big) \to \mathbb{L}_{S^{[n_1,n_2]}} \tag{5.5}$$

and virtual cycle

$$[S^{[n_1,n_2]}]^{\mathrm{vir}} \in A_{n_1+n_2}(S^{[n_1,n_2]}).$$

Its push-forward to $S^{[n_1]} \times S^{[n_2]}$ is given by

$$c_{n_1+n_2}\big(R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)[1]\big) \in A_{n_1+n_2}(S^{[n_1]} \times S^{[n_2]}). \tag{5.6}$$

Proof. By (5.2) we may apply Proposition 4.1 to the degeneracy locus Z of $R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2)$ by setting $\mathcal{F}=R\mathscr{H}om(\mathcal{I}_1,\mathcal{I}_2)$. By (5.1) and the Nakayama lemma $\mathcal{F}$ is quasi-isomorphic to a 2-term complex of vector bundles.

As sets $Z\cong S^{[n_1,n_2]}$ by (5.2). Over the degeneracy locus Z we have the exact sequence (3.5) with h^0 a rank one locally free sheaf, i.e., a line bundle L. Thus over $Z\times S$ we obtain a map

$$\mathcal{I}_1\otimes p^*L\to\mathcal{I}_2$$

which is nonzero on any fibre of p. Taking determinants or double duals shows that L is trivial, $h^0\cong\mathcal{O}_{S^{[n_1,n_2]}}$, and we get a map $\mathcal{I}_1\to\mathcal{I}_2$ whose classifying map gives a morphism $Z\to S^{[n_1,n_2]}$.

Conversely, since $p_*\mathscr{H}om(\mathcal{I}_1,\mathcal{I}_2)=\mathcal{O}$ over $S^{[n_1,n_2]}$, the latter lies in the degeneracy locus of $R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2)$, i.e., $S^{[n_1,n_2]}\subset Z$. It is clear these two maps are inverses.

The rest follows from Proposition 4.1, simplified as in Proposition 3.11, and the fact that $h^0\cong\mathcal{O}_{S^{[n_1,n_2]}}$. □

Remarks. In Theorem 7.1 we will identify our virtual cycle with that of [Gholampour et al. 2017b]. The formula (5.6) for the push-forward of this cycle was conjectured in [Gholampour et al. 2017b], proved there for toric surfaces, and shown to be true for more general surfaces when integrated against some natural classes.

From (3.9) one can work out that the dual of the first arrow in (5.5) is

$$\mathscr{E}xt^1_p(\mathcal{I}_1,\mathcal{I}_1)\oplus\mathscr{E}xt^1_p(\mathcal{I}_2,\mathcal{I}_2)\xrightarrow{(\iota,\,-\iota^*)}\mathscr{E}xt^1_p(\mathcal{I}_1,\mathcal{I}_2),$$

where $\iota:\mathcal{I}_1\to\mathcal{I}_2$ is the natural inclusion. This complex is therefore the virtual tangent bundle of our perfect obstruction theory on $S^{[n_1,n_2]}$.

6. Removing $H^1(\mathcal{O}_S)$ and $H^2(\mathcal{O}_S)$ on arbitrary surfaces

When $h^{0,1}(S)>0$ the virtual cycle constructed in the last section becomes zero due to a trivial $H^1(\mathcal{O}_S)$ piece in its obstruction sheaf. And when $h^{0,2}(S)>0$ the perfect complex $R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2)$ over $S^{[n_1]}\times S^{[n_2]}$ becomes *3-term*, as it has nonzero $h^2=\mathscr{E}xt^2_\pi(\mathcal{I}_1,\mathcal{I}_2)$.

So we want to modify $R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2)$ with $H^1(\mathcal{O}_S)$ and $H^2(\mathcal{O}_S)$ terms. The correct geometric way to do this is to take the product of our ambient space $S^{[n_1]}\times S^{[n_2]}$ with $\mathrm{Jac}(S)$—we do this in Section 9 when $h^{0,2}(S)=0$.[4] In this Section we use a more *ad hoc* fix which is less geometric but appears to give stronger results.

[4]When $h^{0,2}(S)>0$ one should do the same with the *derived scheme* $\mathrm{Jac}(S)$ with nonzero obstruction bundle $H^2(\mathcal{O}_S)\otimes\mathcal{O}$. We don't go this far.

To describe it, consider the natural composition

$$\begin{array}{ccccccc} H^2(\mathcal{O}_S)\otimes_{\mathbb{C}}\mathcal{O}_{S^{[n_1]}\times S^{[n_2]}} & \cong & R^2\pi_*\mathcal{O} & \cong & R^2\pi_*\mathcal{I}_2 & = & \mathscr{E}xt^2_\pi(\mathcal{O},\mathcal{I}_2) \\ & \searrow & & & & & \downarrow \iota_1^* \\ & & & & & & \mathscr{E}xt^2_\pi(\mathcal{I}_1,\mathcal{I}_2) \end{array} \tag{6.1}$$

induced by $\iota_1 : \mathcal{I}_1 \to \mathcal{O}_{S^{[n_1]}\times S^{[n_2]}\times S}$. Since $\mathscr{E}xt^3_\pi(\mathcal{O}/\mathcal{I}_1,\mathcal{I}_2)=0$ (because π has relative dimension 2) the composition (6.1) is *surjective*. Therefore, if there were a lifting

$$\begin{array}{ccc} H^2(\mathcal{O}_S)\otimes\mathcal{O}[-2] & & \\ \vdots & \searrow\!\!\!\!\searrow & \\ R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2) & \longrightarrow & \mathscr{E}xt^2_\pi(\mathcal{I}_1,\mathcal{I}_2)[-2], \end{array} \tag{6.2}$$

then the cone on the dotted arrow in (6.2) would have no h^2 and so would be quasi-isomorphic to a 2-term complex of vector bundles. So we could replace $R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2)$ by this cone: they have the same h^0 jumping locus $S^{[n_1,n_2]}$ (this is proved in Lemma 6.17; it is not true for the $h^{\geq 1}$ jumping loci, however) and the same Chern classes. Assuming we could find a similar lift for $H^1(\mathcal{O}_S)\otimes\mathcal{O}[-1]$ as well, applying Theorem 3.6 to the cone would give the following.

Theorem 6.3. *Let S be any smooth projective surface. The 2-step nested Hilbert scheme $S^{[n_1,n_2]}$ carries a natural*[5] *perfect obstruction theory and virtual cycle*

$$[S^{[n_1,n_2]}]^{\mathrm{vir}} \in A_{n_1+n_2}(S^{[n_1,n_2]})$$

whose push-forward to $S^{[n_1]}\times S^{[n_2]}$ is $c_{n_1+n_2}\big(R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2)[1]\big)$.

Unfortunately the lifting (6.2) does not exist in general, so to prove the Theorem we will use a trick borrowed from the splitting principle in topology: we pull back to a bigger space $\mathcal{A}\to S^{[n_1]}\times S^{[n_2]}$ where there is such a splitting, then show the passage does not destroy any information.

For the rest of this section we carry this out, dealing similarly with $H^1(\mathcal{O}_S)$ at the same time.

We denote by $R^{\geq 1}\pi_*\mathcal{O}$ the truncation $\tau^{\geq 1}R\pi_*\mathcal{O}$. Choosing once and for all a splitting of $R\Gamma(\mathcal{O}_S)$ into its cohomologies induces a splitting

$$R^{\geq 1}\pi_*\mathcal{O} \cong H^1[-1] \,\oplus\, H^2[-2], \tag{6.4}$$

where

$$H^i := H^i(\mathcal{O}_S)\otimes\mathcal{O}_{S^{[n_1]}\times S^{[n_2]}}$$

[5]Naturality will follow from the fact that the lift (6.2) is canonical on restriction to $S^{[n_1,n_2]}\subset S^{[n_1]}\times S^{[n_2]}$; see (6.10).

is the trivial vector bundle of rank $h^{0,i}(S)$ over $S^{[n_1]}\times S^{[n_2]}$. As described above, we wish to map this to $R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2)$ in an appropriate way, which we will do by factoring through the map

$$\iota_1^*: R\pi_*\mathcal{I}_2 \to R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2) \tag{6.5}$$

induced by $\iota_1:\mathcal{I}_1\to\mathcal{O}$. We relate $R\pi_*\mathcal{I}_2$ and $R^{\geq 1}\pi_*\mathcal{O}$ by the commutative diagram of exact triangles

$$\begin{array}{ccccc}
 & & \mathcal{O} & = & \mathcal{O} \\
 & & {\scriptstyle h^0}\downarrow & & \downarrow \\
R\pi_*\mathcal{I}_2 & \longrightarrow & R\pi_*\mathcal{O} & \longrightarrow & \pi_*(\mathcal{O}/\mathcal{I}_2) \\
\| & & \downarrow & & \downarrow \\
R\pi_*\mathcal{I}_2 & \longrightarrow & R^{\geq 1}\pi_*\mathcal{O} & \longrightarrow & \mathcal{O}^{[n_2]}/\mathcal{O}
\end{array}$$

Here $\mathcal{O}^{[n_2]}:=\pi_*(\mathcal{O}/\mathcal{I}_2)$ is the tautological vector bundle, and the top two rows induce the bottom one. This gives the exact triangle

$$\mathcal{O}^{[n_2]}/\mathcal{O}\,[-1] \longrightarrow R\pi_*\mathcal{I}_2 \rightleftarrows R^{\geq 1}\pi_*\mathcal{O} \tag{6.6}$$

which we want to split (to then compose with (6.5)). To write this more explicitly, we split $R^{\geq 1}\pi_*\mathcal{O}$ by (6.4) and fix a 2-term locally free resolution $F_1\to F_2$ of $R\pi_*\mathcal{I}_2$, with F_i in degree i. Then (6.6) gives

$$\begin{array}{ccccccccccc}
 & & \mathcal{O}^{[n_2]}/\mathcal{O} & & & & & & & & \\
 & & \downarrow & & & & & & & & \\
0 & \longrightarrow & R^1\pi_*\mathcal{I}_2 & \xrightarrow{h^1} & F_1 & \longrightarrow & F_2 & \xrightarrow{h^2} & R^2\pi_*\mathcal{I}_2 & \longrightarrow & 0 \\
 & & {\scriptstyle \iota_2}\downarrow\ \uparrow{\scriptstyle \phi_1} & & & & & \nwarrow{\scriptstyle \phi_2} & \| & & \\
 & & H^1 & & & & & & H^2 & &
\end{array} \tag{6.7}$$

where $\iota_2:\mathcal{I}_2\to\mathcal{O}$ and the left hand column is a short exact sequence. Choices of splittings ϕ_1,ϕ_2 would induce a splitting of (6.6).

Since the H^i are *free*, splittings (ϕ_1,ϕ_2) of (6.7) exist *locally*. But unfortunately we can show they do *not* exist globally in general. So we use a trick, pulling back to a bigger space $\mathcal{A}\to S^{[n_1]}\times S^{[n_2]}$ where there is a tautological such splitting.

6A. ***A splitting trick.*** Inside the total space of the bundle

$$\mathcal{E} := (H^1)^*\otimes R^1\pi_*\mathcal{I}_2 \oplus (H^2)^*\otimes F_2$$

over $S^{[n_1]} \times S^{[n_2]}$ there is a natural affine bundle[6] $\mathcal{A} \subset \mathcal{E}$ of pointwise splittings (ϕ_1, ϕ_2) of (6.7). That is, the surjective map of locally free sheaves

$$(1 \otimes \iota_2, 1 \otimes h^2) : \mathcal{E} \to \operatorname{End} H^1 \oplus \operatorname{End} H^2$$

induces a map on the total spaces of the associated vector bundles. Taking the inverse image of the section $(\mathrm{id}_{H^1}, \mathrm{id}_{H^2})$ defines the affine bundle

$$\rho : \mathcal{A} \to S^{[n_1]} \times S^{[n_2]}.$$

Pulling (6.7) back to $\mathcal{A}$, it now has a *canonical* tautological splitting $\Phi = (\phi_1, \phi_2)$, giving

$$\Phi : \rho^* H^1[-1] \oplus \rho^* H^2[-2] \to \rho^* R\pi_* \mathcal{I}_2 \tag{6.8}$$

as sought in (6.6). That is, composing Φ with (the pullback by ρ^* of)

$$\iota_2 : R\pi_* \mathcal{I}_2 \to R^{\geq 1}\pi_* \mathcal{O}$$

gives the identity: $\iota_2 \circ \Phi = \mathrm{id}$.

So finally we may compose (6.8) with (the pullback by ρ^* of) ι_1^* (6.5) to give a map

$$\iota_1^* \circ \Phi : \rho^* R^{\geq 1}\pi_* \mathcal{O} \to \rho^* R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2). \tag{6.9}$$

By construction, on taking h^2 it induces (the pullback by ρ^* of) the surjection (6.1). Therefore the cone $C(\iota_1^* \circ \Phi)$ on (6.9) has no h^2 and is quasi-isomorphic to a 2-term complex of locally free sheaves.

We next give a more explicit description of $C(\iota_1^* \circ \Phi)$. It is nicest over $\rho^{-1}(S^{[n_1,n_2]})$, since on $S^{[n_1,n_2]}$ the natural inclusion $\iota : \mathcal{I}_1 \to \mathcal{I}_2$ induces a *canonical* lift given by the composition

$$R^{\geq 1}\pi_* \mathcal{O} \to R\pi_* \mathcal{O} \xrightarrow{\mathrm{id}} R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_1) \xrightarrow{\iota} R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2). \tag{6.10}$$

Lemma 6.11. *The cone $C(\iota_1^* \circ \Phi)$ can be represented by a 3-term complex of vector bundles*[7]

$$\begin{array}{ccccc} \rho^* E_0 & \xrightarrow{\rho^*\sigma_1} & \rho^* E_1 & \xrightarrow{\rho^*\sigma_2} & \rho^* E_2 \\ \oplus & \nearrow_{\psi_1} & \oplus & \nearrow_{\psi_2} & \\ \rho^* H^1 & & \rho^* H^2 & & \end{array} \tag{6.12}$$

where $E_0 \to E_1 \to E_2$ represents $R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)$.

---

[6]Modelled on the vector bundle $(H^1)^* \otimes (\mathcal{O}^{[n_2]}/\mathcal{O}) \oplus (H^2)^* \otimes \ker(h^2)$. Bhargav Bhatt pointed out that we could have used the Jouanolou trick here to find an affine bundle whose total space is an affine variety on which therefore there exist (noncanonical) splittings.

[7]This can be truncated to a 2-term complex of vector bundles by removing the third term and replacing the second term by the kernel of the surjection $(\rho^*\sigma_2, \psi_2)$.

Moreover the maps may be chosen so that, on restriction to $\rho^{-1}(S^{[n_1,n_2]})$, they are the pullbacks by ρ^* of maps on $S^{[n_1,n_2]}$, and $C(\iota_1^* \circ \Phi)$ is the pullback $\rho^* C$ of the cone C on the composition (6.10).

Remark. Recall that by our notation convention, we are using the same notation ρ for the restriction of ρ to $\rho^{-1}(S^{[n_1,n_2]})$.

The lemma tells us that on $\rho^{-1}(S^{[n_1,n_2]})$, the explicit resolution (6.12) can be taken to be constant on the fibres of ρ — i.e., independent on the choice of lifts (ϕ_1, ϕ_2) of (6.7) — since, after composition with ι_1^*, all lifts become quasi-isomorphic to the canonical one (6.10) on $\rho^{-1}(S^{[n_1,n_2]})$.

Proof. First we show that $C(\iota_1^* \circ \Phi)$ restricted to $\rho^{-1}(S^{[n_1,n_2]})$ is quasi-isomorphic to $\rho^* C$. Consider the diagram

$$\begin{array}{ccccccc}
\rho^* R^{\geq 1}\pi_*\mathcal{O} & \longrightarrow & \rho^* R\pi_*\mathcal{O} & \xrightarrow{\text{id}} & \rho^* R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_1) & \xrightarrow{\iota} & \rho^* R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2) \\
\Phi \big\downarrow \big\uparrow & \nearrow_{\iota_2} & & & & \nearrow_{\iota_1^*} & \\
\rho^* R\pi_*\mathcal{I}_2 & & & & & &
\end{array}$$

on $\rho^{-1}(S^{[n_1,n_2]})$, where we have the canonical map $\iota : \rho^*\mathcal{I}_1 \hookrightarrow \rho^*\mathcal{I}_2$. Here the curved arrow is from (6.6) and makes the first triangle commute. Since by construction Φ is a right inverse to this map, the first triangle also commutes if we start at the top left corner. Since the second triangle also commutes, everything does, which means that $\iota_1^*\Phi$ equals the composition of the arrows along the top row.

Next we resolve $R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)^\vee$ by a complex of *very negative* vector bundles $G^\bullet$. This means that they behave like projectives in the abelian category of coherent sheaves. In particular, by making them sufficiently negative, we can arrange that the map $(\iota_1^*\Phi)^\vee$ can be represented by a genuine map of complexes

$$\rho^* G^\bullet \to \rho^*(H^1)^*[1] \oplus \rho^*(H^2)^*[2], \tag{6.13}$$

and, on $S^{[n_1,n_2]}$, the dual of the composition (6.10) is represented by a genuine map of complexes

$$G^\bullet \to (H^1)^*[1] \oplus (H^2)^*[2]. \tag{6.14}$$

On restriction to $\rho^{-1}(S^{[n_1,n_2]}) \subset \mathcal{A}$, we have shown that the first map (6.13) is quasi-isomorphic to the pullback by ρ^* of the second (6.14). Again we may assume we took the G^i sufficiently negative that — by the usual proof that quasi-isomorphic maps of complexes of projectives are homotopic — there is a homotopy between (6.13) and ρ^*(6.14). This homotopy is a pair of maps

$$\rho^* G^0 \to \rho^*(H^1)^*, \quad \rho^* G^{-1} \to \rho^*(H^2)^*,$$

over $\rho^{-1}(S^{[n_1,n_2]})$. By the sufficient negativity of the G^i they can be extended[8] to maps on all of $\mathcal{A}$. Modifying (6.13) by this homotopy, dualising and then truncating $(G^\bullet)^\vee$ to a 3-term complex now gives (6.12). □

So $C(\iota_1^* \circ \Phi)$ is quasi-isomorphic to the 2-term complex of vector bundles

$$\rho^*(E_0 \oplus H^1) \xrightarrow{\sigma} F, \tag{6.15}$$

where F is defined to be the kernel

$$0 \to F \to \rho^*(E_1 \oplus H^2) \to \rho^* E_2 \to 0. \tag{6.16}$$

And over $\rho^{-1}(S^{[n_1,n_2]})$, the complex (6.15) can be seen as a pull back by ρ^*.

Lemma 6.17. *The h^0 jumping locus of $C(\iota_1^* \circ \Phi)$ is $\rho^{-1}(S^{[n_1,n_2]})$ — the same as that of $\rho^* R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)$.*

Proof. Given any map $T \xrightarrow{f} \mathcal{A} \to S^{[n_1]} \times S^{[n_2]}$, we denote the basechange of π by

$$\pi_T : T \times S \to T.$$

We denote the pull backs of $\mathcal{I}_1, \mathcal{I}_2$ to $T \times S$ by the same notation. Pulling $C(\iota_1^* \circ \Phi)$ back to T, the long exact sequence associated to the cone becomes

$$0 \to \mathscr{H}om_{\pi_T}(\mathcal{I}_1, \mathcal{I}_2) \to h^0\big(f^* C(\iota_1^* \circ \Phi)\big) \to R^1\pi_{T*}\mathcal{O} \xrightarrow{\iota_1^*\Phi} \mathscr{E}xt^1_{\pi_T}(\mathcal{I}_1, \mathcal{I}_2).$$

It remains to prove that the last arrow is an injection, since that implies

$$\mathscr{H}om_{\pi_T}(\mathcal{I}_1, \mathcal{I}_2) \cong h^0\big(f^* C(\iota_1^* \circ \Phi)\big)$$

on any T, to which we can apply Lemma 3.3 to conclude.

The last arrow is the composition $\iota_1^* \circ \Phi$ in the diagram

$$\begin{array}{ccc} R^1\pi_{T*}\mathcal{O} & \overset{\Phi}{\underset{\iota_2}{\longleftarrow}} & R^1\pi_{T*}\mathcal{I}_2 \\ \downarrow \iota_1^* & & \downarrow \iota_1^* \\ \mathscr{E}xt^1_{\pi_T}(\mathcal{I}_1, \mathcal{O}) & \xleftarrow{\iota_2} & \mathscr{E}xt^1_{\pi_T}(\mathcal{I}_1, \mathcal{I}_2) \end{array}$$

To prove it is an injection it is sufficient to do so after composing with ι_2 along the bottom. Since the diagram commutes and Φ is a right inverse of the ι_2 along the top, this is equivalent to the left hand ι_1^* being injective. But this follows from the vanishing of $\mathscr{E}xt^1_{\pi_T}(\mathcal{O}/\mathcal{I}_1, \mathcal{O})$. □

[8] For $N \gg 0$ the restriction $\mathrm{Hom}_{\mathcal{A}}(G(-N), F) \to \mathrm{Hom}_{\rho^{-1}(S^{[n_1,n_2]})}(G(-N), F)$ is *onto* for locally free F and G.

For brevity we set $Z := S^{[n_1,n_2]}$. By Lemmas 6.17 and 6.11 we can see $\rho^{-1}(Z)$ as the degeneracy locus of any of the four maps

$$\rho^*\sigma_1 : \rho^* E_0 \to \rho^* E_1, \tag{6.18}$$

$$(\rho^*\sigma_1, \psi_1) : \rho^*(E_0 \oplus H^1) \to \rho^* E_1, \tag{6.19}$$

$$\begin{pmatrix} \rho^*\sigma_1 & \psi_1 \\ 0 & 0 \end{pmatrix} : \rho^*(E_0 \oplus H^1) \to \rho^*(E_1 \oplus H^2), \tag{6.20}$$

$$\sigma : \rho^*(E_0 \oplus H^1) \to K, \tag{6.21}$$

where

$$K := \ker\big(\rho^*(E_1 \oplus H^2) \to \rho^* E_2\big).$$

These give rise to four different perfect obstruction theories for $\rho^{-1}(Z)$. The one we are interested in is the fourth (6.21), but we will use the third (6.20) and the second (6.19) to relate this to the first (6.18) which has the desirable property that it is ρ-invariant: it is pulled back from a perfect obstruction theory on Z.

By Lemma 6.11 we can write each of (6.18)–(6.21) as the degeneracy locus of a map

$$s : \rho^* A \to B,$$

which on restriction to $\rho^{-1}(Z)$ becomes a pullback from Z—i.e., there exists a bundle B' on Z and $s' : A|_Z \to B'$ such that

$$B|_{\rho^{-1}(Z)} \cong \rho^* B' \quad \text{and} \quad s|_{\rho^{-1}(Z)} \cong \rho^* s'. \tag{6.22}$$

Now apply Section 3 with $r_0 - r = 1$ to this. We see $\rho^{-1}(Z)$ as being cut out of

$$\rho^*\mathbb{P}(A) \cong \mathbb{P}(\rho^* A) \xrightarrow{q} \mathcal{A}$$

by the induced section $\tilde{s}$ (3.7) of $q^* B(1)$, inducing the perfect obstruction theory (3.8)

$$\begin{array}{ccc} q^* B^*(-1)|_{\rho^{-1}(Z)} & \xrightarrow{d\tilde{s}} & \Omega_{\rho^*\mathbb{P}(A)}|_{\rho^{-1}(Z)} \\ \Big\downarrow{\scriptstyle \tilde{s}} & & \Big\| \\ \rho^*(I/I^2) & \xrightarrow{d} & \Omega_{\rho^*\mathbb{P}(A)}|_{\rho^{-1}(Z)} \end{array} \tag{6.23}$$

Here I is the ideal of $Z \subset \mathbb{P}(A)$, so the bottom row is the truncated cotangent complex $\mathbb{L}_{\rho^{-1}(Z)}$.

The bottom arrow factors through $\rho^*\Omega_{\mathbb{P}(A)}|_{\rho^{-1}(Z)}$, so using (6.22) the diagram factors through

$$\begin{array}{ccc} q^*\rho^*(B')^*(-1)|_{\rho^{-1}(Z)} & \xrightarrow{d\tilde{s}} & \rho^*\Omega_{\mathbb{P}(A)}|_{\rho^{-1}(Z)} \\ \tilde{s}\downarrow & & \| \\ \rho^*(I/I^2) & \xrightarrow{d} & \rho^*\Omega_{\mathbb{P}(A)}|_{\rho^{-1}(Z)} \end{array} \tag{6.24}$$

All of the sheaves here are pullbacks by ρ^*. Although on $\rho^{-1}(Z)$ the map s is also a pullback (6.22), that does *not* immediately mean that the maps in the above diagram are pulled back — they use the restriction of s not just to $\rho^{-1}(Z)$ but to its scheme theoretic doubling defined by the ideal $\rho^* I^2$.

However, in the first set-up (6.18) the maps clearly are pulled back. Using the second (6.19) and third (6.20) we will prove the same is true for the fourth (6.21), so that it descends to give a perfect obstruction theory for Z independent of the (ϕ_1, ϕ_2) choices built into $\mathcal{A}$.

Proposition 6.25. *Using the description* (6.21) *of* $\rho^{-1}(Z)$, *the resulting diagram* (6.24) *is* ρ-*invariant: it is the pullback by* ρ^* *of a perfect obstruction theory* $F^\bullet \to \mathbb{L}_Z$ *for* $Z = S^{[n_1,n_2]}$.

Proof. Applying (6.24) to the first set-up (6.18) gives

$$\begin{array}{ccc} \rho^*q^*E_1^*(-1)|_{\rho^{-1}(Z)} & \xrightarrow{\rho^*d(\tilde{\sigma}_1)} & \rho^*\Omega_{\mathbb{P}(E_0)}|_{\rho^{-1}(Z)} \\ \rho^*\tilde{\sigma}_1\downarrow & & \| \\ \rho^*(I/I^2) & \xrightarrow{d} & \rho^*\Omega_{\mathbb{P}(E_0)}|_{\rho^{-1}(Z)} \end{array}$$

where I is the ideal of $Z \subset \mathbb{P}(E_0)$.

Applied instead to the second (6.19), we get the diagram

$$\begin{array}{ccc} \rho^*q^*E_1^*(-1)|_{\rho^{-1}(Z)} & \xrightarrow{d(\widetilde{\rho^*\sigma_1, \psi_1})} & \rho^*\Omega_{\mathbb{P}(E_0\oplus H^1)}|_{\rho^{-1}(Z)} \\ (\widetilde{\rho^*\sigma_1, \psi_1})\downarrow & & \| \\ J/J^2 & \xrightarrow{d} & \rho^*\Omega_{\mathbb{P}(E_0\oplus H^1)}|_{\rho^{-1}(Z)} \end{array} \tag{6.26}$$

where J is the ideal of $\rho^{-1}(Z) \subset \mathbb{P}(\rho^*E_0 \oplus H^1)$. (Throughout this proof we denote q^*H^i, ρ^*H^i and $q^*\rho^*H^i$ simply by H^i.) This inclusion factors

$$\rho^{-1}(Z) \subset \mathbb{P}(\rho^*E_0) \subset \mathbb{P}(\rho^*E_0 \oplus H^1).$$

The first has conormal sheaf ρ^*I/I^2, while the second has conormal bundle $(H^1)^*(-1)$. The splitting of $\rho^*E_0 \oplus H^1$ induces a splitting

$$\Omega_{\mathbb{P}(\rho^*E_0\oplus H^1)}|_{\rho^{-1}(Z)} \cong \Omega_{\mathbb{P}(\rho^*E_0)}|_{\rho^{-1}(Z)} \oplus H(-1)|_{\rho^{-1}(Z)}$$

and so

$$J/J^2 = \rho^*(I/I^2) \oplus (H^1)^*(-1).$$

When substituted into (6.26) it becomes

$$\begin{array}{ccc} \rho^* q^* E_1^*(-1)|_{\rho^{-1}(Z)} & \xrightarrow{(\rho^* d\tilde{\sigma}_1,\, \psi_1^*)} & \rho^*\Omega_{\mathbb{P}(E_0)}|_{\rho^{-1}(Z)} \oplus (H^1)^*(-1) \\ {\scriptstyle (\rho^*\tilde{\sigma}_1,\, \psi_1^*)}\downarrow & & \| \\ \rho^*(I/I^2) \oplus (H^1)^*(-1) & \xrightarrow{(d,\mathrm{id})} & \rho^*\Omega_{\mathbb{P}(E_0)}|_{\rho^{-1}(Z)} \oplus (H^1)^*(-1) \end{array} \tag{6.27}$$

The *key point* of this proof is that the above diagram is pulled back by ρ^* from a similar diagram on Z. This is clear of all the bundles involved, and also clear of the first summand of the upper and left hand arrows. But these are the only parts of the arrows which depend on the thickening of $\rho^{-1}(Z)$. The other summands ψ_1^* depend only on their restriction to $\rho^{-1}(Z)$, where they are also pull backs by Lemma 6.11.

So the second degeneracy locus description of $\rho^{-1}(Z)$ (6.19) gives rise to a diagram which descends to (a perfect obstruction theory on) Z. For the third description (6.20) we add an extra $(H^2)^*(-1)$ summand to the diagram (6.27) with all maps from it zero:

$$\begin{array}{ccc} \rho^* q^* (E_1 \oplus H^2)^*(-1)|_{\rho^{-1}(Z)} & \xrightarrow[\oplus (0,0)]{(\rho^* d\tilde{\sigma}_1,\, \psi_1^*)} & \rho^*\Omega_{\mathbb{P}(E_0)}|_{\rho^{-1}(Z)} \oplus (H^1)^*(-1) \\ {\scriptstyle (\rho^*\tilde{\sigma}_1,\, \psi_1^*)}\downarrow{\scriptstyle \oplus (0,0)} & & \| \\ \rho^*(I/I^2) \oplus (H^1)^*(-1) & \xrightarrow{(d,\mathrm{id})} & \rho^*\Omega_{\mathbb{P}(E_0)}|_{\rho^{-1}(Z)} \oplus (H^1)^*(-1) \end{array} \tag{6.28}$$

This is therefore also a pullback by ρ^*. Finally, since (6.12) is a complex, the map (6.20) takes values in $K \subset \rho^*(E_1 \oplus H^2)$. Thus the equation cutting out $\rho^{-1}(Z)$ takes values in $q^*K(1) \subset q^*\rho^*(E_1 \oplus H^2)(1)$. Therefore the upper horizontal and left-hand vertical arrows of (6.28) factor through $q^*K^*(-1)$, giving

$$\begin{array}{ccc} q^*K^*(-1)|_{\rho^{-1}(Z)} & \longrightarrow & \rho^*\Omega_{\mathbb{P}(E_0)}|_{\rho^{-1}(Z)} \oplus (H^1)^*(-1) \\ \downarrow & & \| \\ \rho^*(I/I^2) \oplus (H^1)^*(-1) & \xrightarrow{(d,\mathrm{id})} & \rho^*\Omega_{\mathbb{P}(E_0)}|_{\rho^{-1}(Z)} \oplus (H^1)^*(-1) \end{array} \tag{6.29}$$

which is the diagram (6.24) applied to the fourth degeneracy locus (6.21).

By Lemma 6.11, both K and its inclusion into $\rho^*E_1 \oplus H^2$ are ρ-invariant. Thus the quotient diagram (6.29) of the diagram (6.28) is also a pull back by ρ^*. □

Proof of Theorem 6.3. Applying (6.23) (with $A = E_0 \oplus H^1$ and $B = K$) to the fourth description (6.21) induces a perfect obstruction theory on $\rho^{-1}(S^{[n_1,n_2]})$. And diagram (6.24) applied to (6.21) gives (6.29), which descends — by Proposition 6.25 — to give a compatible perfect obstruction theory on $S^{[n_1,n_2]}$. This compatibility means they satisfy

$$\rho^*[S^{[n_1,n_2]}]^{\mathrm{vir}} = [\rho^{-1}(S^{[n_1,n_2]})]^{\mathrm{vir}} \in A_{\dim \mathcal{A}-k}(\mathcal{A}).$$

By Theorem 3.6 the second term is $\Delta^{r_0-r}_{r_1-r}\big(c(K - (\rho^* E_0 \oplus H^1))\big)$. But the Chern classes of $K - (\rho^* E_0 \oplus H^1)$ are the same as those of $\rho^*(-E_0 + E_1 - E_2)$ and so those of $\rho^* R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)[1]$. Thus

$$\rho^*[S^{[n_1,n_2]}]^{\mathrm{vir}} = \rho^* \Delta^{r_0-r}_{r_1-r}\big(c(R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)[1])\big) \in A_{\dim \mathcal{A}-k}(\mathcal{A}).$$

Here $r_0 - r = 1$ is the rank of $\ker(\rho^* E_0 \to \rho^* E_1)$ over the degeneracy locus, and

$$\begin{aligned} r_1 - r_0 &= \operatorname{rank} K - \operatorname{rank} E_0 - h^1(\mathcal{O}_S) \\ &= \operatorname{rank} E_1 + h^2(\mathcal{O}_S) - \operatorname{rank} E_2 - \operatorname{rank} E_0 - h^1(\mathcal{O}_S) \\ &= -\chi(I_1, I_2) + \chi(\mathcal{O}_S) - 1 \\ &= n_1 + n_2 - 1, \end{aligned}$$

so $r_1 - r = n_1 + n_2$ and $k = (r_0 - r)(r_1 - r) = n_1 + n_2$. Therefore the above becomes

$$\rho^*[S^{[n_1,n_2]}]^{\mathrm{vir}} = \rho^* c_{n_1+n_2}\big(R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)[1]\big) \in A_{\dim \mathcal{A}-n_1-n_2}(\mathcal{A}).$$

But since ρ is an affine bundle,

$$\rho^* : A_{n_1+n_2}(S^{[n_1]} \times S^{[n_2]}) \to A_{\dim \mathcal{A}-n_1-n_2}(\mathcal{A}) \tag{6.30}$$

is an isomorphism [Kresch 1999, Corollary 2.5.7], so the result follows. □

Over the degeneracy locus $\rho^{-1}(S^{[n_1,n_2]})$, our complex $C(\iota_1^*\Phi)$ has

$$h^0 = \mathcal{O},$$

trivialised by the inclusion $\iota : \mathcal{I}_1 \hookrightarrow \mathcal{I}_2$. And $h^1[-1]$ is the cone on

$$h^0(C(\iota_1^*\Phi)) \cong \mathcal{O}_{\rho^{-1} S^{[n_1,n_2]}} \xrightarrow{h^0} C(\iota_1^*\Phi)|_{\rho^{-1} S^{[n_1,n_2]}}.$$

By Lemma 6.11 and the description (6.10), this is

$$R\mathscr{H}om_p(\mathcal{I}_1, \mathcal{I}_2)_0 := \operatorname{Cone}\big(Rp_*\mathcal{O} \xrightarrow{\iota\cdot\mathrm{id}} R\mathscr{H}om_p(\mathcal{I}_1, \mathcal{I}_2)\big), \tag{6.31}$$

where we recall that p is the basechange of π to $S^{[n_1,n_2]} \subset S^{[n_1]} \times S^{[n_2]}$. Thus

$$h^1 = \mathscr{E}xt^1_p(\mathcal{I}_1, \mathcal{I}_2)_0. \tag{6.32}$$

Theorem 3.6 shows the perfect obstruction theory of a degeneracy locus has virtual tangent bundle

$$T_{\mathcal{A}}|_{\rho^{-1}(Z)} \to (h^0)^* \otimes h^1.$$

As in the proof of Theorem 6.3 this descends to give our perfect obstruction theory on $Z = S^{[n_1,n_2]}$, yielding the following.

Corollary 6.33. *The perfect obstruction theory on $S^{[n_1,n_2]}$ of Theorem 6.3 can be written, in the notation of (6.31), as*

$$\left\{T_{S^{[n_1]}\times S^{[n_2]}}|_{S^{[n_1,n_2]}} \to \mathscr{E}xt^1_p(\mathcal{I}_1,\mathcal{I}_2)_0\right\}^\vee \to \mathbb{L}_{S^{[n_1,n_2]}}. \tag{6.34}$$

7. k-step nested Hilbert schemes

For $n_1 \geq n_2 \geq \cdots \geq n_k$, the k-step Hilbert scheme

$$S^{[n_1,n_2,\ldots,n_k]} := \left\{I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \subseteq \mathcal{O}_S,\ \operatorname{length}(\mathcal{O}_S/I_i) = n_i\right\}$$

can be seen inside $S^{[n_1]} \times \cdots \times S^{[n_k]}$ as the intersection of the $(k-1)$-degeneracy loci

$$\left\{\operatorname{Hom}(I_i, I_{i+1}) = \mathbb{C}\right\}, \quad i = 1, 2, \ldots, k-1,$$

where the maps in the complexes $R\mathscr{H}om_\pi(\mathcal{I}_i,\mathcal{I}_{i+1})$ drop rank.

So when $H^{\geq 1}(\mathcal{O}_S) = 0$ we can employ the exact same method as in Proposition 5.4, using $k-1$ sections of tautological bundles on a $(k-1)$-fold fibre product of relative Grassmannians, to describe a perfect obstruction theory, virtual cycle, and product of Thom–Porteous terms to compute its push-forward.

For general S, possibly with $H^{\geq 1}(\mathcal{O}_S) \neq 0$, we can replace the complexes $R\mathscr{H}om_\pi(\mathcal{I}_i,\mathcal{I}_{i+1})$ with their modifications $C(\iota_i^* \circ \Phi_i)$ of (6.9) after pulling back to an affine bundle of splittings. Then we use the same method as in Theorem 6.3 to produce the following result. We use the projections

$$\pi : S^{[n_1]} \times \cdots \times S^{[n_k]} \times S \to S^{[n_1]} \times \cdots \times S^{[n_k]},$$
$$p : S^{[n_1,\ldots,n_k]} \times S \to S^{[n_1,\ldots,n_k]},$$

and, when $I \subset J$, the same $\operatorname{Ext}(I, J)_0$ notation as in (6.31) and (6.32).

Theorem 7.1. *Fix a smooth complex projective surface S. Via degeneracy loci the k-step nested Hilbert scheme $S^{[n_1,\ldots,n_k]}$ inherits a perfect obstruction theory $F^\bullet \to \mathbb{L}_{S^{[n_1,\ldots,n_k]}}$ with virtual tangent bundle*

$$(F^\bullet)^\vee \cong \left\{T_{S^{[n_1]}} \oplus \cdots \oplus T_{S^{[n_k]}} \to \mathscr{E}xt^1_p(\mathcal{I}_1,\mathcal{I}_2)_0 \oplus \cdots \oplus \mathscr{E}xt^1_p(\mathcal{I}_{k-1},\mathcal{I}_k)_0\right\},$$

where the arrow is the obvious direct sum of the maps (6.34). This is isomorphic to the virtual tangent bundle

$$\mathrm{Cone}\left\{\left(\bigoplus_{i=1}^{k} R\mathscr{H}om_p(\mathcal{I}_i,\mathcal{I}_i)\right)_0 \to \bigoplus_{i=1}^{k-1} R\mathscr{H}om_p(\mathcal{I}_i,\mathcal{I}_{i+1})\right\}$$

of the perfect obstruction theory of [Gholampour et al. 2017b] *or Vafa–Witten theory* [Tanaka and Thomas 2017] *when the latter are defined. The push-forward of the resulting virtual cycle*

$$[S^{[n_1,\dots,n_k]}]^{\mathrm{vir}} \in A_{n_1+n_k}(S^{[n_1,\dots,n_k]})$$

to $S^{[n_1]}\times\cdots\times S^{[n_k]}$ *is given by the product*

$$c_{n_1+n_2}\big(R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2)[1]\big)\cup\cdots\cup c_{n_{k-1}+n_k}\big(R\mathscr{H}om_\pi(\mathcal{I}_{k-1},\mathcal{I}_k)[1]\big).$$

Remark. Note that we are not claiming the two perfect obstruction theories are the same, although they undoubtedly are. Proving this would involve identifying the map $F^\bullet \to \mathbb{L}$ produced by our degeneracy locus construction with the one induced by Atiyah classes in [Gholampour et al. 2017a; Tanaka and Thomas 2017]. We do not need this because the virtual cycles depend only on the scheme structure of $S^{[n_1,\dots,n_k]}$ and the K-theory class of $F^\bullet$.

Proof. All that is left to do is relate the two virtual tangent bundles. The virtual tangent bundle of [Gholampour et al. 2017b] is the cone on the bottom row of the diagram

$$\begin{array}{ccc} Rp_*\mathcal{O} & & \\ \big\downarrow{\scriptstyle \bigoplus_{i=1}^k \mathrm{id}} & & \\ \bigoplus_{i=1}^k R\mathscr{H}om_p(\mathcal{I}_i,\mathcal{I}_i) & \longrightarrow & \bigoplus_{i=1}^{k-1} R\mathscr{H}om_p(\mathcal{I}_i,\mathcal{I}_{i+1}) \\ \big\downarrow & & \big\| \\ \big(\bigoplus_{i=1}^k R\mathscr{H}om_p(\mathcal{I}_i,\mathcal{I}_i)\big)_0 & \longrightarrow & \bigoplus_{i=1}^{k-1} R\mathscr{H}om_p(\mathcal{I}_i,\mathcal{I}_{i+1}) \end{array} \tag{7.2}$$

Here the left hand column is an exact triangle which defines the term in the lower left corner. The central horizontal arrow acts on the j-th summand ($1\le j\le k$) of the left-hand side by taking it to $(0,\dots,0,-i_{j-1}^*,i_j,0,\dots,0)$ on the right-hand side, where i_j appears in the j-th position and is the canonical map $\mathcal{I}_j \hookrightarrow \mathcal{I}_{j+1}$. (For $j=1$ we ignore the $-i_{j-1}^*$ term to get $(i_1,0,\dots,0)$; for $j=k$ we ignore the i_j term to get $(0,\dots,0,-i_{k-1}^*)$.) This has zero composition with $\bigoplus_{i=1}^k \mathrm{id}$, so induces the lower horizontal arrow.

The identity map from $(Rp_*\mathcal{O})^{\oplus k} = Rp_*\mathcal{O}\otimes\mathbb{C}^k$ to the central left-hand term of (7.2) induces a map from $Rp_*\mathcal{O}\otimes(\mathbb{C}^k/\mathbb{C})$ to the bottom left-hand term, where

$\mathbb{C}$ sits in $\mathbb{C}^k$ via $(1, 1, \dots, 1)$. Projecting the elements

$$(1, 0, \dots, 0), (1, 1, 0, \dots, 0), \dots, (1, 1, \dots, 1, 0)$$

of $\mathbb{C}^k$ defines a basis in $\mathbb{C}^k/\mathbb{C}$ and so identifies $Rp_*\mathcal{O} \otimes (\mathbb{C}^k/\mathbb{C}) \cong (Rp_*\mathcal{O})^{\oplus(k-1)}$. Using our description of the central arrow, this identifies the induced map

$$Rp_*\mathcal{O} \otimes (\mathbb{C}^k/\mathbb{C}) \to \bigoplus_{i=1}^{k-1} R\mathscr{H}om_p(\mathcal{I}_i, \mathcal{I}_{i+1})$$

with

$$(Rp_*\mathcal{O})^{\oplus(k-1)} \xrightarrow{\text{diag}(i_1, i_2, \cdots, i_{k-1})} \bigoplus_{i=1}^{k-1} R\mathscr{H}om_p(\mathcal{I}_i, \mathcal{I}_{i+1}).$$

Taking the cone on these two maps from $(Rp_*\mathcal{O})^{\oplus(k-1)}$ to the two entries on the bottom row of (7.2) shows the bottom row is quasi-isomorphic to

$$\bigoplus_{i=1}^{k} R\mathscr{H}om_p(\mathcal{I}_i, \mathcal{I}_i)_0 \to \bigoplus_{i=1}^{k-1} R\mathscr{H}om_p(\mathcal{I}_i, \mathcal{I}_{i+1})_0$$

in the notation of (6.31). Each of these complexes has cohomology only in degree 1, so the virtual tangent bundle of [Gholampour et al. 2017b] is the cone on

$$\bigoplus_{i=1}^{k} \mathscr{E}xt_p^1(\mathcal{I}_i, \mathcal{I}_i)_0 \to \bigoplus_{i=1}^{k-1} \mathscr{E}xt_p^1(\mathcal{I}_i, \mathcal{I}_{i+1})_0$$

in the notation of (6.32). On the j-th summand on the left the arrow is

$$(0, \dots, 0, -i_{j-1}^*, i_j, 0, \dots, 0).$$

But this is $(F^\bullet)^\vee$, as required.

In [Gholampour et al. 2017a] it is shown that the perfect obstruction theory of [Gholampour et al. 2017b] is a summand of the obstruction theory one gets from localised local DT theory. The piece one has to remove is explained in terms of a more global perfect obstruction theory arising in Vafa–Witten theory in [Tanaka and Thomas 2017]. □

8. Generalised Carlsson–Okounkov vanishing

Theorem 6.3 expresses $[S^{[n_1,n_2]}]^{\text{vir}}$ as a degeneracy class. This allows us to give a topological proof of the following result of Carlsson and Okounkov [2012], which we will then generalise below.

Corollary 8.1. *Let S be any smooth projective surface. Over $S^{[n_1]} \times S^{[n_2]}$ we have the vanishing*

$$c_{n_1+n_2+i}\big(R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)[1]\big) = 0, \quad i > 0. \tag{8.2}$$

Proof. We apply the higher Thom–Porteous formula (3.12) to our modified complex $C(\iota_1^*\circ\Phi)$ (6.9) on $\mathcal{A}$. It has degeneracy locus $\rho^{-1}(S^{[n_1,n_2]})$, over which h^0 is just $\mathcal{O}$, trivialised by the tautological inclusion $\mathcal{I}_1 \hookrightarrow \mathcal{I}_2$ over the nested Hilbert scheme. Hence (3.12) gives

$$c_{r_1-r_0+i+1}\big(C(\iota_1^*\circ\Phi)[1]\big)=0$$

for $i>0$, where $r_1-r_0=n_1+n_2-1$.

Since $C(\iota_1^*\circ\Phi)[1]$ only differs from $\rho^* R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2)[1]$ by some trivial bundles H^1, H^2, this gives

$$\rho^* c_{n_1+n_2+i}\big(R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2)[1]\big)=0.$$

But $\rho^*: A_{n_1+n_2-i}(S^{[n_1]}\times S^{[n_2]})\to A_{\dim\mathcal{A}-n_1-n_2-i}(\mathcal{A})$ is an isomorphism [Kresch 1999, Corollary 2.5.7], which gives the result. □

The rest of this section is devoted to proving the following generalisation.

Theorem 8.3. *Let S be any smooth projective surface. For any curve class $\beta\in H_2(S,\mathbb{Z})$, any Poincaré line bundle $\mathcal{L}\to S\times \mathrm{Pic}_\beta(S)$, and any $i>0$,*

$$c_{n_1+n_2+i}\big(R\pi_*\mathcal{L}-R\mathscr{H}om_\pi(\mathcal{I}_1,\mathcal{I}_2\otimes\mathcal{L})\big)=0 \tag{8.4}$$

on $S^{[n_1]}\times S^{[n_2]}\times \mathrm{Pic}_\beta(S)$.

To prove this we will work with more general nested Hilbert schemes of subschemes $S\supset Z_1\supseteq Z_2$, by allowing Z_1 to have dimension ≤ 1 instead of just 0. Separating out its divisorial and 0-dimensional parts, we are then led, for $\beta\in H_2(S,\mathbb{Z})$, to the nested Hilbert scheme $S_\beta^{[n_1,n_2]}$. As a set it is

$$S_\beta^{[n_1,n_2]} := \big\{I_1(-D)\subset I_2\subset\mathcal{O}_S : \mathrm{length}(\mathcal{O}_S/I_i)=n_i,\ D \text{ Cartier with } [D]=\beta\big\}. \tag{8.5}$$

As a scheme it represents the functor taking schemes B to families of nested ideals $\mathcal{I}_1(-\mathcal{D})\hookrightarrow\mathcal{I}_2\hookrightarrow\mathcal{O}_{S\times B}$, flat over B. Here $\mathcal{D}$ is a Cartier divisor, the $\mathcal{O}_S/\mathcal{I}_i$ are finite over B of length n_i, and — on restriction to any closed fibre S_b — $\mathcal{D}_b$ has class β and the maps are still injections.

Setting $\beta=0$ and $n_1\ge n_2$ recovers the punctual nested Hilbert scheme (5.3). Instead setting $n_1=0=n_2$ gives the Hilbert scheme of curves S_β, which fibres over $\mathrm{Pic}_\beta(S)\ni L$ with fibres $\mathbb{P}(H^0(L))$.

In the sequel [Gholampour and Thomas 2019] we will construct a natural perfect obstruction theory and virtual cycle on $S_\beta^{[n_1,n_2]}$ for any β. Here we only sketch a less general construction for classes $\beta\gg 0$ since we do not actually need the virtual class, only the degeneracy locus expression, in order to prove Theorem 8.3.

8A. ***Another degeneracy locus construction.*** So fix $\beta \gg 0$ sufficiently positive that $H^{\geq 1}(L) = 0$ for all $L \in \mathrm{Pic}_\beta(S)$. The Abel–Jacobi map $\mathrm{AJ} : S_\beta \to \mathrm{Pic}_\beta(S)$ is then a projective bundle. Let $\mathcal{D}$ be the universal curve in $S_\beta \times S$ (or any basechange thereof) and as usual let π denote any projection down S. Then

$$R\mathscr{H}om_\pi(\mathcal{I}_1(-\mathcal{D}), \mathcal{O}) \quad \text{over } S^{[n_1]} \times S^{[n_2]} \times S_\beta$$

has $h^2 = 0$. Also $h^0 = \pi_* \mathcal{O}(\mathcal{D})$ and

$$h^1 = \mathscr{E}xt^1_\pi(\mathcal{I}_1(-\mathcal{D}), \mathcal{O}) \cong \mathscr{E}xt^2_\pi(\mathcal{O}_{\mathcal{Z}_1}(-\mathcal{D}), \mathcal{O}) \cong \big[(K_S(-\mathcal{D}))^{[n_1]}\big]^*,$$

with the last isomorphism[9] given by Serre duality down the fibres of π.

Thus $R\mathscr{H}om_\pi(\mathcal{I}_1(-\mathcal{D}), \mathcal{O})$ can be trimmed to a 2-term complex of vector bundles $E_0 \to E_1$ sitting in an exact sequence

$$0 \to \pi_* \mathcal{O}(\mathcal{D}) \to E_0 \to E_1 \to \big[(K_S(-\mathcal{D}))^{[n_1]}\big]^* \to 0,$$

all of whose terms are locally free.

So just as in Section 6A we may work on an affine bundle $\rho : \mathcal{A} \to S^{[n_1]} \times S^{[n_2]} \times S_\beta$ over which this splits canonically, giving an isomorphism

$$\rho^* R\mathscr{H}om_\pi(\mathcal{I}_1(-\mathcal{D}), \mathcal{O}) \cong \rho^* \pi_* \mathcal{O}(\mathcal{D}) \oplus \rho^* \big[(K_S(-\mathcal{D}))^{[n_1]}\big]^* [-1]$$

which induces the identity on cohomology sheaves. From now on we shall omit ρ^* from our notation and work as if this splitting holds on $S^{[n_1]} \times S^{[n_2]} \times S_\beta$ since we know that ρ^* induces an isomorphism on Chow groups (6.30).

In particular we get an induced composition

$$\begin{array}{ccccc} R\mathscr{H}om_\pi(\mathcal{I}_1(-\mathcal{D}), \mathcal{I}_2) & \longrightarrow & R\mathscr{H}om_\pi(\mathcal{I}_1(-\mathcal{D}), \mathcal{O}) & \longrightarrow & \pi_* \mathcal{O}(\mathcal{D}) \\ & \searrow^{\Psi} & & & \downarrow \\ & & & & \dfrac{\pi_* \mathcal{O}(\mathcal{D})}{s_{\mathcal{D}} \cdot \mathcal{O}} \end{array} \tag{8.6}$$

where $s_{\mathcal{D}} : \mathcal{O} \to \pi_* \mathcal{O}(\mathcal{D})$ is induced by adjunction from the section $s_{\mathcal{D}} : \pi^* \mathcal{O} \to \mathcal{O}(\mathcal{D})$ cutting out $\mathcal{D}$. At a closed point (I_1, I_2, D) of $S^{[n_1]} \times S^{[n_2]} \times S_\beta$, the horizontal composition along the top of (8.6) acts on h^0 as follows. It takes a nonzero element of $\mathrm{Hom}(I_1(-D), I_2)$ — i.e., a point of the nested Hilbert scheme up to scale — to its divisorial part in $H^0(\mathcal{O}(D))$; this is injective. The vertical map then compares

[9] Given any line bundle L on S, there is a tautological rank n_1 vector bundle

$$L^{[n_1]} := \pi_* \big[(\mathcal{O}_{S^{[n_1]}} \boxtimes L) \otimes \mathcal{O}_{\mathcal{Z}_1}\big]$$

over $S^{[n_1]}$ whose fibre over $Z_1 \in S^{[n_1]}$ is $\Gamma(L|_{Z_1})$. Here we are using the obvious family generalisation applied to the line bundle $K_S(-\mathcal{D})$ over $S \times S_\beta$.

this to the divisor D. Thus $h^0(\Psi)$ has one dimensional kernel $\mathcal{O}$ (canonically trivialised by $s_{\mathcal{D}}$) at precisely the points of the nested Hilbert scheme

$$S_\beta^{[n_1,n_2]} \xhookrightarrow{\iota} S^{[n_1]} \times S^{[n_2]} \times S_\beta, \tag{8.7}$$

and the kernel is never any bigger. Said differently, the 2-term complex of vector bundles

$$\mathrm{Cone}(\Psi)[-1]$$

drops rank by 1 on the subset (8.7), and no further. By working very similar to that in Proposition 5.4 one can easily show that (8.7) also describes the degeneracy locus scheme-theoretically, inducing a perfect obstruction theory on $S_\beta^{[n_1,n_2]}$. By the Thom–Porteous formula of Proposition 3.11 the resulting virtual cycle therefore satisfies

$$\iota_*[S_\beta^{[n_1,n_2]}]^{\mathrm{vir}} = c_b(\mathrm{Cone}(\Psi)),$$

where $b = \chi(\mathrm{Cone}(\Psi)) + 1 = n_1 + n_2$. More generally, by (3.12),

$$\iota_*\big(c_1((h^0)^*)^i \cap [S_\beta^{[n_1,n_2]}]^{\mathrm{vir}}\big) = c_{n_1+n_2+i}(\mathrm{Cone}(\Psi)).$$

Since we have already observed that $h^0(\mathrm{Cone}(\Psi)[-1]) \cong \mathcal{O}$ is trivialised by the restriction of $s_{\mathcal{D}}$ to (8.7), this gives

$$c_{n_1+n_2+i}\big(R\pi_*\mathcal{O}(\mathcal{D}) - R\mathcal{H}om_\pi(\mathcal{I}_1(-\mathcal{D}), \mathcal{I}_2)\big) = 0 \quad \text{on } S^{[n_1]} \times S^{[n_2]} \times S_\beta \tag{8.8}$$

for $\beta \gg 0$ and all $i > 0$. Notice how close this is to the result claimed in Theorem 8.3.

Proof of Theorem 8.3.. We want to descend (8.8) from S_β to $\mathrm{Pic}_\beta(S)$ and then extend from $\beta \gg 0$ to all $\beta \in H_2(S, \mathbb{Z})$. We will use the formula of [Manivel 2016, Proposition 1],

$$c_{n+i}(F \otimes M) = \sum_{j=0}^{n+i} \binom{\mathrm{rank}\, F - j}{n+i-j} c_j(F) c_1(M)^{n+i-j},$$

for any perfect complex F and line bundle M, using the usual conventions for negative binomial coefficients. Applying this to $F = R\pi_*\mathcal{O}(\mathcal{D}) - R\mathcal{H}om_\pi(\mathcal{I}_1(-\mathcal{D}), \mathcal{I}_2)$ of rank $n := n_1 + n_2$ gives

$$c_{n_1+n_2+i}(F \otimes M) = \sum_{j=n_1+n_2+1}^{n_1+n_2+i} \binom{n_1+n_2-j}{n_1+n_2+i-j} c_j(F) c_1(M)^{n_1+n_2+i-j}, \tag{8.9}$$

because for smaller j the inequalities $n_1 + n_2 + i - j > n_1 + n_2 - j \geq 0$ force the binomial coefficient to vanish. By the vanishing (8.8) this gives

$$c_{n_1+n_2+i}(F \otimes M) = 0 \tag{8.10}$$

for $i > 0$ and any line bundle M on $S^{[n_1]} \times S^{[n_2]} \times S_\beta$. For any Poincaré line bundle $\mathcal{L}$ pulled back from $S \times \mathrm{Pic}_\beta(S)$, the line bundle $\mathcal{L}(-\mathcal{D})$ is trivial on each S fibre and is the pullback $\pi^* M$ of a line bundle M on $S^{[n_1]} \times S^{[n_2]} \times S_\beta$. (In fact $M = \mathcal{O}(-1)$ is the tautological bundle if we consider $S_\beta \to \mathrm{Pic}_\beta(S)$ to be the projectivisation of the vector bundle $\pi_* \mathcal{L}$.) Substituting into (8.10) gives

$$c_{n_1+n_2+i}\big(R\pi_* \mathcal{L} - R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2 \otimes \mathcal{L})\big) = 0$$

on $S^{[n_1]} \times S^{[n_2]} \times S_\beta$. Since this is pulled back from $S^{[n_1]} \times S^{[n_2]} \times \mathrm{Pic}_\beta(S)$ the Leray–Hirsch theorem shows we have the same vanishing there.

So we have proved the vanishing (8.4) for $\beta \gg 0$, and we need to generalise it to all $\beta \in H_2(S, \mathbb{Z})$. We write the left-hand side of (8.4) on $S^{[n_1]} \times S^{[n_2]} \times \mathrm{Pic}_\beta(S)$ in terms of characteristic classes using the Grothendieck–Riemann–Roch theorem applied to π. The result is an $H^{2(n_1+n_2+i)}\big(S^{[n_1]} \times S^{[n_2]} \times \mathrm{Pic}_\beta(S)\big)$-valued polynomial expression in the variables

$$\begin{array}{ccc} (\beta, \mathrm{id}, \gamma) & \in & H^2(S) \oplus H^1(S) \otimes H^1(S)^* \oplus H^2(\mathrm{Pic}_\beta(S)) \\ \| & & \| \\ c_1(\mathcal{L}) & \in & H^2(\mathrm{Pic}_\beta(S) \times S). \end{array}$$

We have shown that this polynomial vanishes on an open cone of classes $\beta \gg 0$ (for any γ). It therefore vanishes for all β. □

Corollary 8.11. *For any curve class β, let $\mathcal{D} \subset S \times S_\beta$ be the universal divisor. Then for $i > 0$*

$$c_{n_1+n_2+i}\big(R\pi_* \mathcal{O}(\mathcal{D}) - R\mathscr{H}om_\pi(\mathcal{I}_1(-\mathcal{D}), \mathcal{I}_2)\big) = 0 \quad \text{on } S^{[n_1]} \times S^{[n_2]} \times S_\beta.$$

Proof. By [Dürr et al. 2007, Lemma 2.15] we can identify the Hilbert scheme S_β with the projective cone $\mathbb{P}^*(R^2\pi_*\mathcal{L}^*(K_S))$ of quotient line bundles of $R^2\pi_*\mathcal{L}^*(K_S)$, in such a way that its natural projection to $\mathrm{Pic}_\beta(S)$ is given by the Abel–Jacobi morphism, and $\mathcal{O}(\mathcal{D}) \cong \mathrm{AJ}^* \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^*}(1)$ over $S \times S_\beta$. Now substitute

$$F := R\pi_* \mathcal{L} - R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2 \otimes \mathcal{L}), \qquad M := \mathcal{O}_{\mathbb{P}^*}(1)$$

over $S^{[n_1]} \times S^{[n_2]} \times S_\beta$ into (8.9). Each of the terms on the right-hand side vanishes for any β by Theorem 8.3. □

Remark. This result suggests that $R\pi_* \mathcal{O}(\mathcal{D}) - R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2(\mathcal{D}))$ has the same K-theory class as an honest vector bundle of rank $n_1 + n_2$ on $S^{[n_1]} \times S^{[n_2]} \times S_\beta$. We show in [Gholampour and Thomas 2019, Equation 4.27] that this is actually true after we pull back an affine bundle over $S^{[n_1]} \times S^{[n_2]} \times S_\beta$. Therefore its higher Chern classes are zero after pulling back to this affine bundle. Since this pullback is an isomorphism on Chow groups [Kresch 1999, Corollary 2.5.7], this gives another explanation for the vanishing of Corollary 8.11.

Aravind Asok kindly pointed out that it is possible that any bundle on the affine bundle is pulled back from the base; this would prove

$$R\pi_*\mathcal{O}(\mathcal{D}) - R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2(\mathcal{D}))$$

is represented by a bundle on $S^{[n_1]} \times S^{[n_2]} \times S_\beta$.

9. Alternative approach to the virtual cycle using Jac(S)

Instead of removing $H^1(\mathcal{O}_S)$ by hand, as we did in Section 6, we can do it geometrically by replacing the moduli space $S^{[n]}$ of ideal sheaves by the moduli space $S^{[n]} \times \mathrm{Jac}(S)$ of rank-1 torsion free sheaves.

Let $\mathcal{L}$ be a Poincaré line bundle over $S \times \mathrm{Jac}(S)$, and let

$$\mathcal{L}_1, \mathcal{L}_2 \to [S^{[n_1]} \times \mathrm{Jac}(S)] \times [S^{[n_2]} \times \mathrm{Jac}(S)] \times S$$

be $\pi_{25}^*\mathcal{L}$ and $\pi_{45}^*\mathcal{L}$ respectively, where π_{ij} is projection to the product of the i-th and j-th factors.

Then the degeneracy locus of the 2-term complex[10]

$$R\mathscr{H}om_\pi(\mathcal{I}_1 \otimes \mathcal{L}_1, \mathcal{I}_2 \otimes \mathcal{L}_2) \tag{9.1}$$

is

$$S^{[n_1,n_2]} \times \mathrm{Jac}(S) \subset [S^{[n_1]} \times \mathrm{Jac}(S)] \times [S^{[n_2]} \times \mathrm{Jac}(S)],$$

where the map is the product of the usual inclusion $S^{[n_1,n_2]} \subset S^{[n_1]} \times S^{[n_2]}$ with the diagonal map $\mathrm{Jac}(S) \subset \mathrm{Jac}(S) \times \mathrm{Jac}(S)$.

Therefore, just as in Sections 3 and 5, $S^{[n_1,n_2]} \times \mathrm{Jac}(S)$ inherits a perfect obstruction theory

$$(\mathscr{E}xt^1_p(\mathcal{I}_1, \mathcal{I}_2))^* \to \Omega_{S^{[n_1]} \times \mathrm{Jac}(S) \times S^{[n_2]} \times \mathrm{Jac}(S)}\big|_{S^{[n_1,n_2]} \times \mathrm{Jac}(S)}$$

(note the $\mathcal{L}_i$ cancel over the diagonal $\mathrm{Jac}(S)$). And the resulting virtual cycle, pushed forward to $S^{[n_1]} \times \mathrm{Jac}(S) \times S^{[n_2]} \times \mathrm{Jac}(S)$, is

$$c_{n_1+n_2+g}\big(R\mathscr{H}om_\pi(\mathcal{I}_1 \otimes \mathcal{L}_1, \mathcal{I}_2 \otimes \mathcal{L}_2)\big), \quad g := h^{0,1}(S).$$

Everything so far has been invariant under the obvious diagonal action of $\mathrm{Jac}(S)$. Taking a slice by pulling back to $\{\mathcal{O}_S\} \times \mathrm{Jac}(S) \subset \mathrm{Jac}(S) \times \mathrm{Jac}(S)$ gives the following.

Proposition 9.2. *There is a perfect obstruction theory*

$$\big(\mathscr{E}xt^1_p(\mathcal{I}_1, \mathcal{I}_2)\big)^* \to \Omega_{S^{[n_1]} \times S^{[n_2]} \times \mathrm{Jac}(S)}\big|_{S^{[n_1,n_2]} \times \{\mathcal{O}_S\}} \tag{9.3}$$

[10] It is only 2-term if $p_g(S) = 0$. If $p_g(S) > 0$ then we can pull back to an affine bundle where $H^2(\mathcal{O}_S)$ splits off, as in Section 6A.

on $S^{[n_1,n_2]}$. *The push-forward of the resulting virtual cycle*

$$[S^{[n_1,n_2]}]^{\mathrm{vir}} \in A_{n_1+n_2}(S^{[n_1,n_2]})$$

to $S^{[n_1]} \times S^{[n_2]} \times \mathrm{Jac}(S)$ *is*

$$c_{n_1+n_2+g}\big(R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2 \otimes \mathcal{L})[1]\big). \tag{9.4}$$

Remark. The canonical section

$$\mathcal{O} \to \mathscr{H}om(\mathcal{I}_1, \mathcal{I}_2) \to R\mathscr{H}om(\mathcal{I}_1, \mathcal{I}_2)$$

over $S^{[n_1,n_2]} \times S$ gives

$$R^1 p_* \mathcal{O} \to \mathscr{E}xt^1_p(\mathcal{I}_1, \mathcal{I}_2). \tag{9.5}$$

Dualising gives

$$\big(\mathscr{E}xt^1_p(\mathcal{I}_1, \mathcal{I}_2)\big)^* \to H^1(\mathcal{O}_S)^* \otimes \mathcal{O}_{S^{[n_1,n_2]}} \cong \Omega_{\mathrm{Jac}(S)}.$$

One can show that this map is the projection of (9.3) to $\Omega_{\mathrm{Jac}(S)}$.

So letting $\mathscr{E}xt^1_p(\mathcal{I}_1, \mathcal{I}_2)_0$ denote the cokernel of the injection (9.5), we can simplify the perfect obstruction theory (9.3) to

$$\big(\mathscr{E}xt^1_p(\mathcal{I}_1, \mathcal{I}_2)_0\big)^* \to \Omega_{S^{[n_1]} \times S^{[n_2]}}|_{S^{[n_1,n_2]}},$$

recovering the one of Section 6 by Corollary 6.33.

Remark. The degeneracy locus $S^{[n_1,n_2]}$ of Proposition 9.2 lies in

$$S^{[n_1]} \times S^{[n_2]} \times \{\mathcal{O}_S\} \overset{j}{\hookrightarrow} S^{[n_1]} \times S^{[n_2]} \times \mathrm{Jac}(S), \tag{9.6}$$

and (9.4) gives an expression for the push-forward of the virtual cycle to the right-hand side of (9.6). It would be nice to deduce a similar expression for the push-forward of the virtual cycle to the left-hand side of (9.6) (as we managed in Theorem 6.3 using the *ad hoc* method of Section 6A to remove $H^1(\mathcal{O}_S)$). The more geometric method of this section does not seem to give such an expression directly. But we *can* deduce it from (9.4) if we use the generalised Carlsson–Okounkov vanishing result of Theorem 8.3. This allows us to write

$$\begin{aligned} &c_{n_1+n_2+g}\big(R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2 \otimes \mathcal{L})[1]\big) \\ &\qquad = c_g(R\pi_*\mathcal{L}[1]) \cdot c_{n_1+n_2}\big(R\pi_*\mathcal{L} - R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2 \otimes \mathcal{L})\big) \end{aligned} \tag{9.7}$$

on $S^{[n_1]} \times S^{[n_2]} \times \mathrm{Jac}(S)$, because the higher Chern classes of

$$R\pi_*\mathcal{L} - R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2 \otimes \mathcal{L})$$

vanish. (The lower Chern classes do not feature because they are multiplied by $c_{>g}(R\pi_*(\mathcal{L}))$ which are pulled back from $\mathrm{Jac}(S)$ of dimension g and so are zero.)

Setting $n_1 = 0 = n_2$ in (9.4) shows $c_g(R\pi_*\mathcal{L}[1])$ is Poincaré dual to the origin $\mathcal{O}_S \in \mathrm{Jac}(S)$ (all multiplied by $S^{[n_1]} \times S^{[n_2]}$). Since $\mathcal{L}$ and $R\pi_*\mathcal{L}$ become trivial on this locus, the right hand side of (9.7) becomes

$$j_* c_{n_1+n_2}\big(R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)[1]\big),$$

using the push-forward map (9.6). Combined again with (9.4) this recovers the result of Theorem 6.3, that the virtual cycle's push-forward to $S^{[n_1]} \times S^{[n_2]}$ is $c_{n_1+n_2}\big(R\mathscr{H}om_\pi(\mathcal{I}_1, \mathcal{I}_2)[1]\big)$. This argument would only not be circular, however, if we could prove the generalised Carlsson–Okounkov vanishing of Theorem 8.3 without using Theorem 6.3.

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There are some constructions in the literature which are closely related to ours; see for instance [Neguț 2015, Equation 6.2] for $S = \mathbb{P}^2$, [Neguț 2019, Section 2.3] for some special types of nested sheaves, and [Neguț 2019, Equation (2.23)] for their perfect obstruction theory. Just before posting this paper we became aware of the old notes [Maulik and Okounkov], which describe the local DT obstruction theory [Gholampour et al. 2017a] on the nested Hilbert scheme, and a K-theoretic version of the Carlsson–Okounkov operator on toric surfaces. With hindsight it seems that Okounkov et al. probably knew of some form of relationship between the virtual class and the Thom–Porteous formula for toric surfaces with $H^{\geq 1}(\mathcal{O}_S) = 0$, even if they're too modest to admit it now.

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Almost $\mathbb{C}_p$ Galois representations and vector bundles

Jean-Marc Fontaine

Let K be a finite extension of $\mathbb{Q}_p$ and G_K the absolute Galois group. Then G_K acts on the fundamental curve X of p-adic Hodge theory and we may consider the abelian category $\mathcal{M}(G_K)$ of coherent $\mathcal{O}_X$-modules equipped with a continuous and semilinear action of G_K.

An *almost* $\mathbb{C}_p$*-representation of* G_K is a p-adic Banach space V equipped with a linear and continuous action of G_K such that there exists $d \in \mathbb{N}$, two G_K-stable finite dimensional sub-$\mathbb{Q}_p$-vector spaces U_+ of V, U_- of $\mathbb{C}_p^d$, and a G_K-equivariant isomorphism

$$V/U_+ \to \mathbb{C}_p^d/U_-.$$

These representations form an abelian category $\mathcal{C}(G_K)$. The main purpose of this paper is to prove that $\mathcal{C}(G_K)$ can be recovered from $\mathcal{M}(G_K)$ by a simple construction (and vice-versa) inducing, in particular, an equivalence of triangulated categories

$$D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K)).$$

1. Introduction

1A. We fix a prime number p, an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and a finite extension K of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$. We set $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ and $\mathbb{C}_p$ the p-adic completion of $\overline{\mathbb{Q}}_p$ on which G_K acts by continuity.

The fundamental curve $X_{\mathbb{Q}_p,\mathbb{C}_p^\flat}$ of p-adic Hodge theory, denoted by X below, was introduced in [Fargues and Fontaine 2018]. It is a separated noetherian regular scheme of dimension 1 defined over $\mathbb{Q}_p$; i.e., $H^0(X, \mathcal{O}_X) = \mathbb{Q}_p$. The structural sheaf is naturally equipped with a topology (Section 3D): if U is any open subset of X, then $\mathcal{O}_X(U)$ is a locally convex $\mathbb{Q}_p$-algebra. There is a natural action of

Jean-Marc Fontaine passed away on 29 January 2019. I saw him last in late November 2018, when he mentioned to me that he wanted to submit this paper to Tunisian Journal of Mathematics after making some small changes, and asked me if I could take care of the paper in case he could not do it himself; to which I, of course, agreed. Contributing to Fontaine's program has been one of the joys of my mathematical career and this paper puts the final touch to the geometrization of this program via the Fargues–Fontaine curve. – Pierre Colmez, 4 August 2019.

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G_K on X which is continuous. We may consider the abelian category $\mathcal{M}(G_K)$ of *G_K-equivariant coherent $\mathcal{O}_X$-modules*, that is of coherent $\mathcal{O}_X$-modules equipped with a semilinear and continuous action of G_K.

Any nonzero $\mathcal{F} \in \mathrm{Ob}(\mathcal{M}(G_K))$ has a degree $\deg(\mathcal{F}) \in \mathbb{Z}$ and a rank $\mathrm{rk}(\mathcal{F}) \in \mathbb{N}$, hence also a slope $s(\mathcal{F}) = \deg(\mathcal{F})/\mathrm{rk}(\mathcal{F}) \in \mathbb{Q} \cup \{+\infty\}$ (with the convention that $s(\mathcal{F}) = +\infty$ if $\mathcal{F}$ is a torsion $\mathcal{O}_X$-module). As in the classical case, one says that a coherent $\mathcal{O}_X[G_K]$-module $\mathcal{F}$ is *semistable* if $\mathcal{F} \neq 0$ and if $s(\mathcal{F}') \leq s(\mathcal{F})$ for any nonzero subobject $\mathcal{F}'$ of $\mathcal{F}$.

We may consider the full subcategory $\mathcal{M}^0(G_K)$ of $\mathcal{M}(G_K)$ whose objects are semistable of slope 0. One of the main results of [Fargues and Fontaine 2018] is that, if $\mathcal{F}$ is any object of $\mathcal{M}^0(G_K)$, then $\mathcal{F}(X) = H^0(X, \mathcal{F})$ is a finite-dimensional $\mathbb{Q}_p$-vector space, hence is an object of the abelian category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of p-adic representations of G_K (that is of finite-dimensional $\mathbb{Q}_p$-vector spaces equipped with a linear and continuous action of G_K) and that the functor

$$\mathcal{M}^0(G_K) \to \mathrm{Rep}_{\mathbb{Q}_p}(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

is an equivalence of categories (with $V \mapsto \mathcal{O}_X \otimes_{\mathbb{Q}_p} V$ as a quasi-inverse).

The main purpose of this paper is to discuss the following question: Is there an extension of this result enabling us to give an analogous Galois description of all objects of $\mathcal{M}(G_K)$?

1B. In [Fontaine 2003], I introduced the category of *almost $\mathbb{C}_p$-representations of G_K*: A *Banach representation of G_K* is a p-adic Banach space (i.e., a topological $\mathbb{Q}_p$-vector space whose topology can be defined by a norm and which is complete) equipped with a linear and continuous action of G_K. With an obvious definition of morphisms, Banach representations of G_K form an additive category $\mathcal{B}(G_K)$ containing the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ as a full subcategory. By continuity, G_K acts on the p-adic completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$ has a natural structure of a Banach representation. The category $\mathcal{C}(G_K)$ of *almost $\mathbb{C}_p$-representations of G_K* is the full subcategory of $\mathcal{B}(G_K)$ whose objects are those V's for which one can find $d \in \mathbb{N}$, two G_K-stable finite-dimensional sub-$\mathbb{Q}_p$-vector spaces U_+ of V and U_- of $\mathbb{C}_p^d$ and an isomorphism $V/U_+ \to \mathbb{C}_p^d/U_-$ in $\mathcal{B}(G_K)$. This category turns out to be abelian (*loc. cit.*).

The curve X has only one closed point ∞ which is G_K-stable and the orbit under G_K of any other closed point is infinite. This implies that a torsion object of $\mathcal{M}(G_K)$ is supported at ∞. As the completion of $\mathcal{O}_{X,\infty}$ is the ring B_{dR}^+ of p-adic periods, the category $\mathcal{M}^\infty(G_K)$ of torsion objects of $\mathcal{M}(G_K)$ ($\Longleftrightarrow$ semistable objects of slope ∞) can be identified with the category $\mathrm{Rep}_{B_{dR}^+}^{\mathrm{tor}}(G_K)$ of B_{dR}^+-modules of finite length equipped with a semilinear and continuous action of G_K. The topology of any B_{dR}^+-module of finite length is the topology of a p-adic Banach space and we may consider the forgetful functor

$$\mathrm{Rep}^{\mathrm{tor}}_{B^+_{dR}}(G_K) \to \mathcal{B}(G_K).$$

We proved in *loc. cit.* that this functor is fully faithful and that the essential image $\mathcal{C}^\infty(G_K)$ is contained in $\mathcal{C}(G_K)$. Hence, setting $\mathcal{C}^0(G_K) = \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, we see that for $s \in \{0, \infty\}$, the functor $\mathcal{F} \mapsto \mathcal{F}(X)$ induces an equivalence of categories

$$\mathcal{M}^s(G_K) \to \mathcal{C}^s(G_K).$$

Similarly as for a smooth projective curve over a field, we defined in [Fargues and Fontaine 2018] the Harder–Narasimhan filtration of any $\mathcal{F} \in \mathcal{M}(G_K)$: this is the unique filtration

$$0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^{r-1} \subset \mathcal{F}^r = \mathcal{F}$$

such that all the $\mathcal{F}^i/\mathcal{F}^{i-1}$ are semistable and that $s(\mathcal{F}^i/\mathcal{F}^{i-1}) > s(\mathcal{F}^{i+1}/\mathcal{F}^i)$ for $0 < i < r$. We call the $s(\mathcal{F}^i/\mathcal{F}^{i-1})$, for $1 \leq i \leq r$, the *HN-slopes of* $\mathcal{F}$.

Let $\mathcal{M}^{\geq 0}(G_K)$ the full subcategory of $\mathcal{M}(G_K)$ whose objects are *effective*, i.e., such that all their HN-slopes are ≥ 0.

Similarly let $\mathcal{C}^{\geq 0}(G_K)$ the full subcategory of $\mathcal{C}(G_K)$ whose objects are *effective*, i.e., those V's which are isomorphic to a subobject (in $\mathcal{C}(G_K)$) of an object of $\mathcal{C}^\infty(G_K)$.

If $\mathcal{F}$ is any coherent $\mathcal{O}_X[G_K]$-module, then $\mathcal{F}(X)$ is a topological $\mathbb{Q}_p$-vector space equipped with a linear and continuous action of G_K. Our main result is this:

Theorem 5.9. *If $\mathcal{F}$ is any coherent $\mathcal{O}_X[G_K]$-module, $\mathcal{F}(X)$ is an effective almost $\mathbb{C}_p$-representation of G_K. By restriction to $\mathcal{M}^{\geq 0}(G_K)$ the functor $\mathcal{F} \mapsto \mathcal{F}(X)$ induces an equivalence of categories*

$$\mathcal{M}^{\geq 0}(G_K) \to \mathcal{C}^{\geq 0}(G_K).$$

This equivalence doesn't extend to an equivalence between $\mathcal{M}(G_K)$ and $\mathcal{C}(G_K)$. Nevertheless each of these two categories can be reconstructed from the other: The above functor induces an equivalence of triangulated categories

$$D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K))$$

and each of them can be reconstructed as the heart of a t-structure. More precisely:

- Denote by $\mathcal{M}^{<0}(G_K)$ the full subcategory of $\mathcal{M}(G_K)$ whose objects are those for which all HN-slopes are < 0. Then $t = (\mathcal{M}^{\geq 0}(G_K), \mathcal{M}^{<0}(G_K))$ is what is called a torsion pair on $\mathcal{M}(G_K)$. From this torsion pair, we can construct an other abelian category $\mathrm{f}(\mathcal{M}(G_K))^t$ which is the full subcategory of $D^b(\mathcal{M}(G_K))$ whose objects are those $\mathcal{F}^\bullet$ such that $\mathcal{F}^i = 0$ for $i \notin \{0, 1\}$, while

$$H^0(\mathcal{F}^\bullet) \text{ is an object of } \mathcal{M}^{<0}(G_K) \quad \text{and} \quad H^1(\mathcal{F}^\bullet) \text{ is an object of } \mathcal{M}^{\geq 0}(G_K).$$

There is a natural equivalence $(\mathcal{M}(G_K))^t \to \mathcal{C}(G_K)$.

• Similarly, denote by $\mathcal{C}^{<0}(G_K)$ the full subcategory of $\mathcal{C}(G_K)$ whose objects are those V's for which $\mathrm{Hom}(V, W) = 0$ for all W in $\mathcal{C}^{\infty}(G_K)$. Then

$$t' = (\mathcal{C}^{<0}(G_K), \mathcal{C}^{\geq 0}(G_K))$$

is a torsion pair on $\mathcal{C}(G_K)$ which can be used to define the abelian subcategory $(\mathcal{C}(G_K))^{t'}$ which is the full subcategory of $D^b(\mathcal{C}(G_K)$ whose objects are those $V^\bullet$ such that $V^i = 0$ for $i \notin \{0, 1\}$, while

$$H^0(V^\bullet) \text{ is an object of } \mathcal{C}^{\geq 0}(G_K) \quad \text{and} \quad H^1(V^\bullet) \text{ is an object of } \mathcal{C}^{<0}(G_K).$$

There is a natural equivalence $(\mathcal{C}(G_K))^{t'} \to \mathcal{M}(G_K)$.

A description à la Beauville–Lazlo of vector bundles on X gives an equivalence of categories between G_K-equivariant vector bundles on X and Berger's B-pairs [Berger 2008]. Specializing the above results to the subcategory $\mathrm{Bund}_X(G_K)$ of $\mathcal{M}(G_K)$ of vector bundles recovers (via this equivalence of categories) some results of Berger [2009].

1C. ***Contents.*** In Section 2, we recall and slightly extend the results of [Fontaine 2003] on almost $\mathbb{C}_p$-representations. We first recall (Section 2A) some basic facts about locally convex spaces over a nonarchimedean field. We introduce (Section 2B) the category of (p-adic) ind-Fréchet representations (of G_K). Then (Section 2C), we recall some basic facts about the ring of periods B^+_{dR} and B_{dR} that we equip with a locally convex topology. In Section 2D, we discuss some properties of B^+_{dR}-representations and B_{dR}-representations (of G_K).

We describe (Section 2E) the main properties of the category $\mathcal{C}(G_K)$ of almost $\mathbb{C}_p$-representations and of its full subcategories $\mathcal{C}^0(G_K)$ of finite-dimensional p-adic representations and $\mathcal{C}^{\infty}(G_K)$ of B^+_{dR}-representations of finite length. In Section 2E, we also introduce the category $\widehat{\mathcal{C}}(G_K)$ of representations of G_K which are suitable limits (in the category of locally convex p-adic representations of G_K) of almost $\mathbb{C}_p$-representations. In Section 2F, we recall the notion of almost split exact sequence of $\mathcal{B}(G_K)$ and the fact that an extension in $\mathcal{B}(G_K)$ of two almost $\mathbb{C}_p$-representations is an almost $\mathbb{C}_p$-representation if and only if the associated short exact sequence almost splits.

In Section 3, we study the category $\mathrm{Rep}_{B_e}(G_K)$ of *B_e-representations of G_K* (several of the results we obtain are already in [Berger 2008; 2009]). We also recall and make more precise some of the results of [Fargues and Fontaine 2018] on *coherent $\mathcal{O}_X[G_K]$-modules*. We first recall (Section 3A) some basic facts about the sub-$\mathbb{Q}_p$-algebras B^+_{cris} and B_e of B_{dR} which are stable under the action of G_K and equipped with a natural topology of locally convex algebras. Then we introduce (Section 3B) $\mathrm{Rep}_{B_e}(G_K)$ and show that this is a $\mathbb{Q}_p$-linear abelian category.

We recall (Section 3C) the definition of the fundamental curve $X = X_{\mathbb{Q}_p,\mathbb{C}_p^\flat}$ of p-adic Hodge theory introduced in [Fargues and Fontaine 2018] on which G_K acts and give a description of the category $\mathrm{Coh}(\mathcal{O}_X)$ of coherent $\mathcal{O}_X$-modules. We discuss (Section 3D) the topology on the structural sheaf $\mathcal{O}_X$ and give a description of the category $\mathcal{M}(G_K)$ of coherent $\mathcal{O}_X[G_K]$-modules (Section 3E). We describe (Section 3F) the Harder–Narasimhan filtration on any $\mathcal{F} \in \mathcal{M}(G_K)$.

We consider two full subcategories of $\mathcal{M}(G_K)$:

– the category $\mathcal{M}^0(G_K)$ of the semistable objects of slope 0,

– the category $\mathcal{M}^\infty(G_K)$ of objects whose underlying $\mathcal{O}_X$-module is torsion.

We show (Section 3G) that the global sections functor induces equivalence of categories

$$\mathcal{M}^0(G_K) \to \mathcal{C}^0(G_K) \quad \text{and} \quad \mathcal{M}^\infty(G_K) \to \mathcal{C}^\infty(G_K).$$

In Section 3H, we introduce two kinds of twists of the objects of $\mathcal{M}(G_K)$, the *Tate twists* and the *Harder–Narasimhan twists*.

Say that a B_e-representation Λ is *trivialisable* if there exists $U \in \mathcal{C}^0(G_K)$ and an isomorphism $B_e \otimes_{\mathbb{Q}_p} U \to \Lambda$. In Section 3I, we show that $\mathrm{Rep}_{B_e}(G_K)$ is the smallest subcategory of itself containing trivialisable B_e-representations and stable under taking extensions and direct summands.

In Section 3A0, we show that, if Λ is a B_e-representation of G_K, then the underlying topological $\mathbb{Q}_p$-vector space equipped with its action of G_K is an object of $\widehat{\mathcal{C}}(G_K)$ and that the forgetful functor

$$\mathrm{Rep}_{B_e}(G_K) \to \widehat{\mathcal{C}}(G_K)$$

is exact and fully faithful. (This was already known to Berger [2009, théorème B].)

We conclude this section by discussing the cohomology of coherent $\mathcal{O}_X$-modules (Section 3A1) and of coherent $\mathcal{O}_X[G_K]$-modules (Section 3A2). We show that, taking the global sections, we get a functor

$$\mathcal{M}(G_K) \to \mathcal{C}(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X) = H^0(X, \mathcal{O}_X)$$

whose essential image is contained in $\mathcal{C}^{\geq 0}(G_K)$.

The aim of Section 4 is to construct a left adjoint

$$\mathcal{C}(G_K) \to \mathcal{M}(G_K), \quad V \mapsto \mathcal{F}_V$$

of the functor $\mathcal{F} \mapsto \mathcal{F}(X)$.

We show (Section 4C) that any almost $\mathbb{C}_p$-representation V has a *B_e-hull*, i.e., there is a pair $V_e = (V_e, \iota_e^V)$ with V_e a B_e-representation (of G_K) and $\iota_e^V : V \to V_e$ a morphism in $\widehat{\mathcal{C}}(G_K)$ such that, for all $\Lambda \in \mathrm{Rep}_{B_e}(G_K)$, the map

$$\mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_K)}(V_e, \Lambda) \to \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, \Lambda)$$

induced by ι_e^V is bijective.

Similarly with obvious definitions, we show that V has a B_{dR}^{+}-hull V_{dR}^{+} and a B_{dR}-hull V_{dR}.

Using the existence of these hulls and the relations between them and knowing the description of $\mathcal{M}(G_K)$ given in Section 3E, the construction of the functor $V \mapsto \mathcal{F}_V$ is quite simple.

The proof of the existence of these hulls relies heavily on the description of all extensions in $\mathcal{C}(G_K)$ of an object of $\mathcal{C}^{\infty}(G_K)$ by an object of $\mathcal{C}^0(G_K)$, which is given in Section 4B.

The aim of Section 5 is to prove our main result (Theorem 5.9).

We show in Section 5A (resp. 5B) that $\mathcal{M}^{\geq 0}(G_K)$ (resp. $\mathcal{C}^{\geq 0}(G_K)$) is the smallest full subcategory of $\mathcal{M}(G_K)$ (resp. $\mathcal{C}(G_K)$) containing $\mathcal{M}^0(G_K)$ and $\mathcal{M}^{\infty}(G_K)$ (resp. $\mathcal{C}^0(G_K)$ and $\mathcal{C}^{\infty}(G_K)$) and stable under extensions and direct summands.

In Section 5C we prove by dévissage that the functor

$$\mathcal{M}^{\geq 0}(G_K) \to \mathcal{C}^{\geq 0}(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

is an equivalence of exact categories (see Section 1E), the functor $V \mapsto \mathcal{F}_V$ being a quasi-inverse.

The purpose of Section 6 is to extend the main result to the categories $\mathcal{M}(G_K)$ and $\mathcal{C}(G_K)$.

After some general nonsense on derived categories of exact subcategories of abelian categories (Section 6A), we first extend the main result to an equivalence of of triangulated categories (Section 6B),

$$D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K)).$$

To go further, we need to introduce the full subcategories $\mathcal{M}^{<0}(G_K)$ of $\mathcal{M}(G_K)$ and $\mathcal{C}^{<0}(G_K)$ of $\mathcal{C}(G_K)$ of coeffective objects. The main theorem said that, if $\mathcal{F} \in \mathcal{M}^{\geq 0}(G_K)$, then $H^0(X, \mathcal{F})$ has a natural structure of an object of $\mathcal{C}^{\geq 0}(G_K)$ and this structure determines $\mathcal{F}$. We prove in Section 6C that, if $\mathcal{F} \in \mathcal{M}^{<0}(G_K)$, then $H^1(X, \mathcal{F})$ has a natural structure of an object of $\mathcal{C}^{<0}(G_K)$ and this structure determines $\mathcal{F}$.

Using this result, we can build $\mathcal{C}(G_K)$ from $\mathcal{M}(G_K)$ and conversely. We give two different recipes (with independent proofs) for that. In Section 6D we describe explicitly the heart of the t-structure on $D^b(\mathcal{M}(G_K))$ corresponding to $\mathcal{C}(G_K)$ and of the t-structure on $D^b(\mathcal{C}(G_K))$ corresponding to $\mathcal{M}(G_K))$. In Section 6E, we explain that $(\mathcal{M}^{\geq 0}(G_K), \mathcal{M}^{<0}(G_K))$ is a torsion pair on $\mathcal{M}(G_K)$. One can use it to construct a new abelian category equipped with a torsion pair. Up to equivalence, it is $\mathcal{C}(G_K)$ equipped with the torsion pair $(\mathcal{C}^{<0}(G_K), \mathcal{C}^{\geq 0}(G_K))$.

1D. ***A remark on possible generalisations.*** The results of this paper are obviously a special case of a much more general result where K is replaced by any reasonable rigid analytic, Berkovich or adic space. Let's sketch a description of the case where K is now any field complete with respect to a nonarchimedean nontrivial absolute value with perfect residue field of characteristic p.

• We can define the abelian category $\mathrm{Coh}(\mathcal{O}_{X_K})$ of *coherent modules on the curve* X_K. When K is a perfectoid field, X_K is the curve $X_{\mathbb{Q}_p,K^\flat}$ constructed in [Fargues and Fontaine 2018]. If K is not perfectoid, then X_K doesn't exist but one can define the category of coherent modules over this virtual curve. When K is a finite extension of $\mathbb{Q}_p$, there is a natural equivalence of categories

$$\mathrm{Coh}(\mathcal{O}_{X_K}) \to \mathcal{M}(G_K).$$

• We still have the Harder–Narasimhan filtration on $\mathrm{Coh}(\mathcal{O}_{X_K})$ and may consider its exact subcategories $\mathrm{Coh}^{\geq 0}(\mathcal{O}_{X_K})$ and $\mathrm{Coh}^{<0}(\mathcal{O}_{X_K})$ which form a torsion pair t on $\mathrm{Coh}(\mathcal{O}_{X_K})$.

• The construction of the curve X_K is functorial in K. If C is the completion of a separable closure K^s of K, for any coherent $\mathcal{O}_{X_K}$-module $\mathcal{F}$, we may consider the pull-back $f^*\mathcal{F}$ of $\mathcal{F}$ via $f : X_C \to X_K$.

If $G_K = \mathrm{Gal}(K^s/K)$, we may consider the exact category $\mathcal{B}(G_K)$ of p-adic Banach representations of G_K and we have exact and faithful functors

$$\mathrm{Coh}^{\geq 0}(\mathcal{O}_{X_K}) \to \mathcal{B}(G_K), \quad \mathcal{F} \mapsto H^0(X_C, f^*\mathcal{F}),$$
$$\mathrm{Coh}^{<0}(\mathcal{O}_{X_K}) \to \mathcal{B}(G_K), \quad \mathcal{F} \mapsto H^1(X_C, f^*\mathcal{F}).$$

But, in general, these functors are not fully faithful. Working with $\mathcal{B}(G_K)$ amounts to work over the small pro-étale site of K and we need to work with a bigger site. A possibility is to use the big pro-étale site $K_{\text{proét}}$ of K as defined in [Scholze 2017, §8][1] and to replace $\mathcal{B}(G_K)$ with the category $\mathrm{Vect}_{\mathbb{Q}_p}(K)$ of $\mathbb{Q}_p$-sheaves over $K_{\text{proét}}$, and $\mathcal{C}(G_K)$ with the category of *pseudo-geometric* $\mathbb{Q}_p$*-sheaves*, an abelian full subcategory of $\mathrm{Vect}_{\mathbb{Q}_p}(K)$ defined by imitating the definition of $\mathcal{C}(G_K)$ as a full subcategory of $\mathcal{B}(G_K)$.

The correspondence $K \mapsto X_K$ can be extended to a functor

$$U \mapsto X_U$$

[1] More precisely, we fix an uncountable cardinal κ satisfying the properties of [Scholze 2017, Lemma 4.1]. The underlying category is the category of perfectoid spaces over K which are κ-small [loc. cit., Definition 4.3] and coverings are as defined in [loc. cit., Definition 8.1] (the only difference with the big pro-étale site of Scholze is that we restrict ourself to perfectoid spaces lying over the given nonarchimedean field K).

from the category of perfectoid spaces to the category of $\mathbb{Q}_p$-schemes. We also have exact and faithful functors

$$\mathrm{Coh}^{\geq 0}(\mathcal{O}_{X_K}) \to \mathrm{Vect}_{\mathbb{Q}_p}(K), \quad \mathcal{F} \mapsto \big(U \mapsto H^0(X_U, f_U^*\mathcal{F})\big),$$
$$\mathrm{Coh}^{<0}(\mathcal{O}_{X_K}) \to \mathrm{Vect}_{\mathbb{Q}_p}(K), \quad \mathcal{F} \mapsto \big(U \mapsto H^1(X_U, f_U^*\mathcal{F})\big),$$

where $f_U : X_U \to X_K$ is the structural morphism.

It seems likely (and not so hard to prove) that these functors are fully faithful and that one can describe their essential images $\mathrm{Vect}_{\mathbb{Q}_p}^{pg,\geq 0}(K)$ and $\mathrm{Vect}_{\mathbb{Q}_p}^{pg,<0}(K)$. These two functors seem to induce an equivalence of categories

$$(\mathrm{Coh}(\mathcal{O}_{X_K}))^t \to \mathrm{Vect}_{\mathbb{Q}_p}^{pg}(K)$$

the induced torsion pair on $\mathrm{Vect}_{\mathbb{Q}_p}^{pg}(K)$ being $t' = (\mathrm{Vect}_{\mathbb{Q}_p}^{<0}(K), \mathrm{Vect}_{\mathbb{Q}_p}^{\geq 0}(K))$.

In the case where K is the p-adic completion of an algebraic closure of $\mathbb{Q}_p$, this result has been proved by Le Bras [2018]. We hope to come back soon to this generalisation.

1E. ***Conventions and notations.*** If $\mathcal{C}$ is a category, we often write $C \in \mathcal{C}$ for $C \in \mathrm{Ob}(\mathcal{C})$.

An *exact subcategory of an abelian category* $\mathcal{A}$ is a strictly full subcategory of $\mathcal{A}$ containing 0 and stable under extensions.

If $\mathcal{B}$ is an exact subcategory of $\mathcal{A}$, we say that a sequence of morphisms of $\mathcal{A}$ is *exact* if it is exact as a sequence of morphisms in $\mathcal{A}$. In particular, we have the obvious notion of a *short exact sequence*. It is easy to see that, equipped with this class of short exact sequences, $\mathcal{B}$ is an exact category in the sense of Quillen (cf. [Quillen 1973], see also [Laumon 1983]). Actually, any exact category $\mathcal{B}$ in the sense of Quillen can be viewed as an exact subcategory of an abelian category (cf. [Quillen 1973, §2]).

As usual $\mathbb{Z}_p(1)$ is the Tate module of the multiplicative group, and, for all $n \in \mathbb{N}$,

$$\mathbb{Z}_p(n) = \mathrm{Sym}^n_{\mathbb{Z}_p} \mathbb{Z}_p(1), \quad \mathbb{Z}_p(-n) = \mathcal{L}_{\mathbb{Z}_p}(\mathbb{Z}_p(n), \mathbb{Z}_p).$$

If M is any $\mathbb{Z}_p$-module equipped with a linear action of G_K, for all $n \in \mathbb{Z}$,

$$M(n) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n).$$

2. Representations of G_K

In this paper, each time we say “representation", we mean “representation of G_K". In this section, we introduce a few categories of such representations and describe some of their properties. Most of them are already known (see in particular [Fontaine 2003]) or easy consequences of known properties.

2A. ***Banach, Fréchet, ind-Banach and ind-Fréchet.*** We refer to [Emerton 2017] and [Schneider 2002] for basic facts about p-adic functional analysis. All results of this paragraph are either contained or easy consequences of results contained in at least one of these two memoirs.

We fix a nonarchimedean field E, i.e., a field complete with respect to a non-trivial nonarchimedean absolute value, and denote by $\mathcal{O}_E$ its valuation ring. In the applications in this paper, E will be $\mathbb{Q}_p$.

- A *locally convex E-vector space* is a topological E vector space V such that the open sub-$\mathcal{O}_E$-modules of V form a fundamental system of neighbourhood of 0.
- A *Fréchet E-vector space* or an *E-Fréchet* is a locally convex E-vector space which is metrisable and complete.
- A *Banach E-vector space* or an *E-Banach* is a Fréchet vector space whose topology can be defined by a norm.
- An *ind-Fréchet* (resp. *ind-Banach*) E-vector space or an *ind-E-Fréchet* (resp. *ind-E-Banach*) is a locally convex E-vector space V, such that one can find an increasing sequence $(V_n)_{n\in\mathbb{N}}$ of closed sub-E-vector spaces such that
  (i) $V=\bigcup_{n\in\mathbb{N}} V_n$,
  (ii) each V_n, with the induced topology, is an E-Fréchet (resp. an E-Banach),
  (iii) the topology of V is the coarsest locally convex topology with these properties.

Condition (iii) is equivalent to the fact that a sub-$\mathcal{O}_E$-module L of V is open if and only if $L\cap V_n$ is open in V_n for all $n\in\mathbb{N}$.

If V is a topological E-vector space, V is an E-Fréchet if and only if V is complete and its topology can be defined by a countable family $(q_n)_{n\in\mathbb{N}}$ of seminorms. In this situation, replacing each q_n by $q_n'=\sup_{0\le i\le n} q_i$, we may assume that $q_n\le q_{n+1}$ for all n. Then, if $\overline{V}_n$ is the Hausdorff completion of V, with respect to q_n, this is an E-Banach and we have an homeomorphism

$$V\mapsto \varprojlim_{n\in\mathbb{N}} \overline{V}_n$$

(with the inverse limit topology on the RHS). Conversely, any inverse limit, indexed by $\mathbb{N}$, of E-Banach is an E-Fréchet.

Let V be a topological E-vector space. We say that a decreasing filtration $(F^nV)_{n\in\mathbb{Z}}$ by closed sub-E-vector spaces of V is *admissible* if

(i) $\bigcup_{n\in\mathbb{Z}} F^nV=V$ and $\bigcap_{n\in\mathbb{Z}} F^nV=0$,
(ii) if $m\in\mathbb{Z}$ and $r\in\mathbb{N}$, then $F^mV/F^{m+r}V$, equipped with the induced topology, is an E-Banach,

(iii) if $m \in \mathbb{Z}$, the natural map

$$F^m V \to \varprojlim_{r \in \mathbb{N}} F^m V / F^{m+r} V$$

is an homeomorphism (with the inverse limit topology on the RHS),

(iv) a sub-$\mathcal{O}_E$-module L of V is open if and only if $L \cap F^n V$ is open in $F^n V$ for all n.

The following result is obvious:

Proposition 2.1. *Let V be a topological E-vector space.*

(i) *V is an ind-E-Fréchet if and only if it has an admissible filtration.*

(ii) *V is an E-Banach* (*resp. an E-Fréchet*, *resp. an ind-E-Banach*) *if and only if has an admissible filtration $(F^n V)_{n \in \mathbb{Z}}$ such that $F^0 V = V$ and $F^1 V = 0$* (*resp. $F^0 V = V$*, *resp. $F^1 V = 0$*).

Proposition 2.2. *Let V_1 and V_2 two ind-E-Fréchet*, *$(F^n V_1)_{n \in \mathbb{Z}}$ an admissible filtration of V_1 and $(F^n V_2)_{n \in \mathbb{Z}}$ an admissible filtration of V_2. Let $u : V_1 \to V_2$ an E-linear map. The following are equivalent*:

(i) *The map u is continuous. For all $m \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $u(F^m V_1) \subset F^n V_2$ and the induced map*

$$F^m V_1 \to F^n V_2$$

is continuous.

Proof. (ii)$\Longrightarrow$(i): It is enough to show that, if L is an open lattice in V_2, then $f^{-1}(L)$ is open in V_1 which means that if $m \in \mathbb{Z}$, then $f^{-1}(L) \cap F^m V_1$ is open in $F^m V_1$ which is indeed true as, if n is such that $f(F^m V_1) \subset F^n V_2$, this is the inverse image of the continuous map $F^m V_1 \to F^n V_2$ which is induced by f.

(i)$\Longrightarrow$(ii): All the $F^n V_2$ are E-Fréchet. For each fixed m, so is $F^m V_1$ and the existence of such an n is explained in [Schneider 2002, Corollary 8.9]. □

Corollary 2.3. *Let V be an ind-E-Fréchet and $(F^n V)_{n \in \mathbb{Z}}$ an admissible filtration. Then V is an E- Banach* (*resp. an E-Fréchet*, *resp. an ind-E-Banach*) *if and only if there exists $m \leq n$ such that $F^m V = V$ and $F^n V = 0$* (*resp. m such that $F^m V = V$*, *resp. n such that $F^n V = 0$*).

Corollary 2.4. *Let V be an ind-E-Fréchet and $(F_1^n V)_{n \in \mathbb{Z}}$ and $(F_2^n V)_{n \in \mathbb{Z}}$ two admissible filtrations. For all $m \in \mathbb{Z}$, there exists $n \in \mathbb{Z}$ such that $F_1^m V \subset F_2^n V$.*

An *ind Fréchet E-algebra* is a topological E-algebra B which has a *multiplicative admissible filtration*, i.e., an admissible filtration $(F^n B)_{n \in \mathbb{Z}}$ of the underlying topological E-vector space such that, if $m, n \in \mathbb{Z}$, and, if $b \in F^m B$, $b' \in F^n B$, then $bb' \in F^{m+n} B$.

A *Banach* (resp. *Fréchet*, resp. *ind-Banach*) E-algebra is an ind Fréchet E-algebra B which has a multiplicative admissible filtration $(F^n B)_{n\in\mathbb{Z}}$ such that $F^0 B = B$ and $F^1 B = 0$ (resp. $F^0 B = B$, resp. $F^1 B = 0$).

2B. ***Ind-Fréchet representations.*** From now on E will be $\mathbb{Q}_p$. We will say *Banach*, *Fréchet*, *ind-Banach*, *ind-Fréchet* instead of $\mathbb{Q}_p$-Banach, $\mathbb{Q}_p$-Fréchet, ind-$\mathbb{Q}_p$-Banach, ind-$\mathbb{Q}_p$-Fréchet. We will say *Banach algebra*, *Fréchet algebra*, and so on, instead of $\mathbb{Q}_p$-Banach algebra, $\mathbb{Q}_p$-Fréchet algebra.

The category $\mathcal{IF}(G_K)$ of *ind-Fréchet representations* (*of* G_K) is the category whose objects are ind-Fréchet equipped with a $\mathbb{Q}_p$-linear and continuous action of G_K, and whose morphisms are G_K-equivariant continuous $\mathbb{Q}_p$-linear map.

The category $\mathcal{IF}(G_K)$ is an additive $\mathbb{Q}_p$-linear category and any morphism

$$f : V_1 \to V_2$$

has a kernel and a cokernel: the kernel is the G_K-stable closed sub-$\mathbb{Q}_p$-vector space which is the kernel of the underlying $\mathbb{Q}_p$-linear map. The cokernel is the quotient of V_2 by the G_K-stable closed sub-$\mathbb{Q}_p$-vector space which is the closure of $f(V_1)$.

We say that a morphism f is *strict* if the map

$$\mathrm{Coim}(f) \to \mathrm{Im}(f)$$

is an homeomorphism.

Similarly one can define in an obvious way the categories $\mathcal{B}(G_K)$, $\mathcal{IB}(G_K)$ and $\mathcal{F}(G_K)$ of *Banach, ind-Banach, Fréchet representations (of* G_K). This is consistent with the definition of $\mathcal{B}(G_K)$ already given in the introduction.

2C. ***The rings B_{dR}^+ and B_{dR} and their topologies.*** We denote by B_{dR} the usual field of p-adic periods. Recall (from [Fontaine 1994, §1.5], for instance) that this is the fraction field of a discrete valuation ring B_{dR}^+, that G_K acts naturally on these two $\mathbb{Q}_p$-algebras and that $\mathbb{Z}_p(1)$ is naturally a G_K-stable sub-$\mathbb{Z}_p$-module of B_{dR}^+. We choose a generator t of $\mathbb{Z}_p(1)$. This is also a generator of the maximal ideal of B_{dR}^+. Therefore, for all $d \in \mathbb{Z}$, the d-th power of this ideal is

$$\mathrm{Fil}^d B_{dR} = B_{dR}^+ . t^d = B_{dR}^+(d)$$

and is stable under G_K. For each $d \geq 0$, we set

$$B_d = B_{dR}^+ / \mathrm{Fil}^d B_{dR}.$$

Recall [Fontaine 1994, §1.5.3] that B_d has a natural structure of a Banach algebra on which the action of G_K is continuous, that, in particular, $B_1 = \mathbb{C}_p$, and that, for each $d \in \mathbb{N}$, the projection $B_{d+1} \to B_d$ is also continuous. Equipped with the topology of the inverse limit, B_{dR}^+ becomes a Fréchet algebra on which G_K acts continuously.

For all $n \in \mathbb{Z}$, multiplication by t^n defines a bijection $B_{dR}^+ \to \mathrm{Fil}^n B_{dR}$ and we equip $\mathrm{Fil}^n B_{dR}$ with the induced topology (for which the action of G_K is continuous); note that multiplication by t^n does not commute with the action of G_K.

If $n \in \mathbb{Z}$, then $\mathrm{Fil}^{n+1} B_{dR}$ is closed in $\mathrm{Fil}^n B_{dR}$ and we equip B_{dR} with its natural locally convex topology. (A sub-$\mathbb{Z}_p$-module L of B_{dR} is open if and only if, for all $n \in \mathbb{Z}$, the $\mathbb{Z}_p$-module $L \cap \mathrm{Fil}^n B_{dR}$ is open in $\mathrm{Fil}^n B_{dR}$.)

We see that B_{dR} is an ind-Fréchet K-algebra, with $(\mathrm{Fil}^n B_{dR})_{n\in\mathbb{Z}}$ as a G_K-equivariant multiplicative admissible filtration. In particular B_{dR} has a natural structure of an ind-Fréchet K-representation of G_K.

2D. *B_{dR}^+ and B_{dR}-representations.* Any B_{dR}^+-module of finite type has a natural structure of a K-Fréchet and any finite-dimensional B_{dR}-vector space has a natural structure of an ind-Fréchet K-vector space.

A B_{dR}^+-représentation (resp. a *B_{dR}-representation) (of G_K)* is a B_{dR}^+-module of finite type (resp. a finite -dimensional B_{dR}-vector space) equipped with a semilinear and continuous action of G_K. With the G_K-equivariant B_{dR}^+-linear maps as morphisms, these representations form a category that we denote by $\mathrm{Rep}_{B_{dR}^+}(G_K)$ (resp. $\mathrm{Rep}_{B_{dR}}(G_K)$).

The category $\mathrm{Rep}^{\mathrm{tor}}_{B_{dR}^+}(G_K) = \mathcal{C}^\infty(G_K)$ of *torsion B_{dR}^+-representations (of G_K)* defined in the introduction (Section 1B) is the full subcategory of $\mathrm{Rep}_{B_{dR}^+}(G_K)$ whose objects are such that the underlying B_{dR}^+-module is torsion ($\Longleftrightarrow$ of finite length).

Recall (from [Stacks, 02MN], for instance) that a *Serre subcategory* $\mathcal{C}$ of an abelian category $\mathcal{A}$ is a strictly full subcategory of $\mathcal{A}$ containing 0 which is stable under subobjects, quotients and extensions. In particular, this is an abelian category. Given $\mathcal{A}$ and $\mathcal{C}$, one can define the quotient category $\mathcal{A}/\mathcal{C}$ which is an abelian category, solution of the obvious universal problem.

Proposition 2.5. *The category $\mathcal{C}^\infty(G_K)$ is a Serre subcategory of* $\mathrm{Rep}_{B_{dR}^+}(G_K)$. *The functor*

$$\mathrm{Rep}_{B_{dR}^+}(G_K) \to \mathrm{Rep}_{B_{dR}}(G_K), \quad W \mapsto B_{dR} \otimes_{B_{dR}^+} W$$

is essentially surjective and induces an equivalence

$$\mathrm{Rep}_{B_{dR}^+}(G_K)/\mathcal{C}^\infty(G_K) \xrightarrow{\simeq} \mathrm{Rep}_{B_{dR}}(G_K).$$

Proof. The essential surjectivity comes from the fact that, for any B_{dR}-representation W, there is a G_K-stable lattice B_{dR}^+-lattice W^+. This result itself comes from the fact that if W_0^+ is a B_{dR}^+-lattice of W, then W_0 is an ind-Fréchet K-vector space with $(t^n W_0^+)_{n\in\mathbb{Z}}$ forming an admissible filtration. For each $w \in W$, the $g(w)$'s for $g \in G_K$ form a compact subset of W, hence it is bounded which implies (by [Schneider 2002, Proposition 5.6]) that it is contained in $t^{-n}W_0^+$ for $n \gg 0$. Hence, if $e_1, e_2, \ldots, e_d$ is a basis of W over B_{dR}, there exists $n \in \mathbb{N}$ such that $g(e_i) \in t^{-n}W_0^+$ for $1 \le i \le d$ and $g \in G_K$. Therefore the sub-B_{dR}^+-module W^+ of W generated by all these $g(e_i)$'s is also contained in $t^{-n}W_0^+$ and is a G_K-stable B_{dR}^+-lattice of W. The continuity of the action of G_K on W implies the continuity of the action on W^+ which is an object of $\mathrm{Rep}_{B_{dR}^+}(G_K)$. We have an obvious identification of $B_{dR} \otimes_{B_{dR}^+} W^+$ to W and the functor is essentially surjective.

The rest of the proof is straightforward. □

If W is any object of $\mathcal{C}^\infty(G_K)$, there is an integer d such that the underlying B_{dR}^+-module is a B_d-module of finite type. As B_d is a Banach $\mathbb{Q}_p$-algebra, the underlying topological $\mathbb{Q}_p$-vector space is a Banach and W has a natural structure of a p-adic Banach representation.

Proposition 2.6 [Fontaine 2003, théorème 3.1]. *The forgetful functor*

$$\mathcal{C}^\infty(G_K) \to \mathcal{B}(G_K)$$

is fully faithful.

In other words, given a p-adic Banach representation W of G_K, there is at most one structure of B_{dR}^+-module of finite length on W extending the action of $\mathbb{Q}_p$ such that W becomes a torsion B_{dR}^+-representation.

We use this result to identify $\mathcal{C}^\infty(G_K)$ to a full subcategory of $\mathcal{B}(G_K)$.

We denote by

$$\widehat{\mathcal{C}}^\infty(G_K)$$

the full subcategory of $\mathcal{IF}(G_K)$ whose objects are those W's which admit a G_K-equivariant admissible filtration $(F^nW)_{n\in\mathbb{Z}}$ such that $F^mW/F^nW \in \mathcal{C}^\infty(G_K)$ for all $m \le n$ in $\mathbb{Z}$. By passing to the limit, the previous proposition implies that, on such a W, there is a unique structure of B_{dR}^+-module such that the action of G_K is semilinear and each F^mW is a sub-B_{dR}^+-module (and this structure is independent of the choice of $(F^nW)_{n\in\mathbb{Z}}$). We also see that $\widehat{\mathcal{C}}^\infty(G_K)$ is an abelian category and that any morphism of $\widehat{\mathcal{C}}^\infty(G_K)$ is B_{dR}^+-linear.

Moreover $\mathrm{Rep}_{B_{dR}^+}(G_K)$ can be identified with a full subcategory of $\widehat{\mathcal{C}}^\infty(G_K)$. Proposition 2.5 implies that this is also true for $\mathrm{Rep}_{B_{dR}}(G_K)$.

Proposition 2.7. *Let $d \in \mathbb{N}$.*

(i) *Let W_1 be an object of $\mathcal{C}^\infty(G_K)$ such that* $\mathrm{length}_{B_{dR}^+} W_1 \ge d$. *There exists a finite extension K' of K contained in $\overline{\mathbb{Q}}_p$ and a $G_{K'}$-stable sub-B_{dR}^+-module W_1' of W_1 of length d.*

(ii) *Let W_2 be an object of* $\mathrm{Rep}_{B^+_{dR}}(G_K)$ *with* $\mathrm{length}_{B^+_{dR}} W_2 \geq d$. *There exists a finite extension K' of K contained in $\overline{\mathbb{Q}}_p$ and a $G_{K'}$-stable sub-B^+_{dR}-module W'_2 of W_2 such that* $\mathrm{length}_{B^+_{dR}} W_2/W'_2 = d$.

Proof. (i) Via an obvious induction, we see that it is enough to check it for $d = 1$. Replacing W_1 by the kernel of the multiplication by t in W_1, we may assume that W_1 is a $\mathbb{C}_p$-representation.

Recall some basic facts of Sen's theory [1980/81]:

Let $\chi : G_K \to \mathbb{Z}_p^*$ be the cyclotomic character, H_K the kernel of χ and $L = (\mathbb{C}_p)^{H_K}$ which is also the completion of $K_\infty = \overline{\mathbb{Q}}_p^{H_K}$. We set $\Gamma_K = G_K/H_K = \mathrm{Gal}(K_\infty/K)$. The character χ factors through a character $\Gamma_K \to \mathbb{Z}_p^*$ that we still denote by χ.

For any $\mathbb{C}_p$-representation W (of G_K), denote by W_K^f the union of the finite-dimensional sub-K-vector spaces of W^{H_K} stable under the action of G_K (acting through Γ_K). This is a finite dimensional K_∞-vector space equipped with a semi-linear action of Γ_K. With obvious notations, we have:

- The functor
$$\mathrm{Rep}_{\mathbb{C}_p}(G_K) \to \mathrm{Rep}_{K_\infty}(\Gamma_K), \quad W \mapsto W_K^f$$
is exact and fully faithful.
- For any $W \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$, the obvious map
$$\mathbb{C}_p \otimes_{K_\infty} W_K^f \to W$$
is an isomorphism.
- For all $W \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$, there exists a unique endomorphism $\alpha_{W,K}$ of the K_∞-vector space W_K^f such that

  for all $w \in W_K^f$, there is an open subgroup Γ_w of Γ_K such that, if $\gamma \in \Gamma_w$, then
$$\gamma(w) = \exp(\log(\chi(\gamma)).\alpha_{W,K})(w).$$
  (The series $\exp(\lambda\alpha_{W,K})$ converges to an endomorphism of W_K^f for all small enough $\lambda \in \mathbb{Z}_p$.)

It is easy to see that, if K_1 is a finite extension of K contained in $\overline{\mathbb{Q}}_p$, then $W_{K_1}^f$ can be identified with $(K_1)_\infty \otimes_{K_\infty} W_K^f$ and that α_{W,K_1} is the $(K_1)_\infty$-endomorphism of $W_{K_1}^f$ deduced from $\alpha_{W,K}$ by scalar extension.

Choose such a K_1 containing an eigenvalue λ of $\alpha_{W,K}$, hence also of α_{W,K_1} and choose a nonzero eigenvector $\omega_0 \in W_{K_1}^f$ for α_{W,K_1}^f. There is a finite extension K' of K_1 contained in $\overline{\mathbb{Q}}_p$ such that, for all $\gamma \in \Gamma_{K'}$, we have
$$\gamma(w_0) = \exp(\log(\chi(\gamma)).\lambda).w.$$

We can view w_0 as a nonzero element of $W^f_{K'}$ and we see that for all $b \in K'$ and all $\gamma \in \Gamma_{K'}$, we have

$$\gamma(bw_0) = \gamma(b).\exp(\log(\chi(\gamma)).\lambda).w,$$

hence the K'-line of $W^f_{K'}$ generated by w_0 is stable under the action of $\Gamma_{K'}$. Therefore the $\mathbb{C}_p$-line W_1' of W_1 generated by w_0 is stable under the action of $G_{K'}$.

(ii) Replacing W_2 by $W_2/t^r W_2$ with r big enough, we may assume that W_2 is an object of $\mathcal{C}^\infty(G_K)$. The result follows by duality from the assertion (i) applied to the Pontryagin dual $W = \mathcal{L}_{B^+_{dR}}(W_2, B_{dR}/B^+_{dR})$ of W_2. □

2E. *Almost $\mathbb{C}_p$-representations.* If V_1 and V_2 are two objects of $\mathcal{IF}(G_K)$, an *almost isomorphism*

$$f : V_1 \rightsquigarrow V_2, \quad \text{also denoted by} \quad \tilde{f} : V_1/U_1 \to V_2/U_2,$$

is a triple $f = (U_1, U_2, \tilde{f})$ where U_1 is a finite-dimensional G_K-stable sub-$\mathbb{Q}_p$-vector space of V_1, U_2 is a finite dimensional G_K-stable sub-$\mathbb{Q}_p$-vector space of V_2 and

$$\tilde{f} : V_1/U_1 \to V_2/U_2$$

is an isomorphism of ind-Fréchet representations.

We say that two objects V_1 and V_2 of $\mathcal{IF}(G_K)$ are *almost isomorphic* if there exists an almost isomorphism

$$f : V_1 \rightsquigarrow V_2.$$

Proposition 2.8 [Fontaine 2003, théorème 5.3]. *Let V be an object of $\mathcal{B}(G_K)$. The following are equivalent*:

(i) *V is almost isomorphic to a torsion B^+_{dR}-representation.*

(ii) *V is almost isomorphic to a $\mathbb{C}_p$-representation.*

(iii) *There is $d \in \mathbb{N}$ such that V is almost isomorphic to $\mathbb{C}_p^d$ (equipped with the natural action of G_K).*

We denote by $\mathcal{C}(G_K)$ the category of *almost $\mathbb{C}_p$-representations (of G_K)*, that is the full subcategory of $\mathcal{B}(G_K)$ whose objects satisfy the equivalent conditions of the previous proposition. This is coherent with the definition given in the introduction (Section 1B).

The category $\mathcal{C}(G_K)$ contains $\mathcal{C}^\infty(G_K) = \mathrm{Rep}^{\mathrm{tor}}_{B^+_{dR}}(G_K)$ and $\mathcal{C}^0(G_K) = \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ as full subcategories.

A *weak Serre subcategory* $\mathcal{B}$ of an abelian category $\mathcal{A}$ is a strictly full subcategory which is abelian, such that the inclusion functor is exact and which is closed under taking extensions.

The following results are essentially contained in [Fontaine 2003]:

Theorem 2.9. *The category $\mathcal{C}(G_K)$ is abelian and any morphism of $\mathcal{C}(G_K)$ is strict as a morphism of $\mathcal{B}(G_K)$. A sequence of morphisms of $\mathcal{C}(G_K)$ is exact if and only if the underlying sequence of $\mathbb{Q}_p$-vector spaces is exact. The category $\mathcal{C}^0(G_K)$ is a Serre subcategory of $\mathcal{C}(G_K)$and $\mathcal{C}^\infty(G_K)$ is a weak Serre subcategory of $\mathcal{C}(G_K)$.*

Furthermore:

(i) *If $U \in \mathcal{C}^0(G_K)$ and $W \in \mathcal{C}^\infty(G_K)$, then $\mathrm{Hom}_{\mathcal{C}(G_K)}(W, U) = 0$.*

(ii) *There exists additive functions*

$$d : \mathrm{Ob}\,\mathcal{C}(G_K) \to \mathbb{N} \quad \textit{and} \quad h : \mathrm{Ob}\,\mathcal{C}(G_K) \to \mathbb{Z},$$

uniquely determined respectively by $d(U) = 0$ if $U \in \mathcal{C}^0(G_K)$ and $d(\mathbb{C}_p) = 1$ (resp. $h(U) = \dim_{\mathbb{Q}_p}(U)$ if $U \in \mathcal{C}^0(G_K)$ and $h(\mathbb{C}_p) = 0$); moreover, if $W \in \mathcal{C}^\infty(G_K)$, then $d(W) = \mathrm{length}_{B^+_{dR}}(W)$ and $h(W) = 0$.

Proof. This is [Fontaine 2003, théorème 5.1] with some extras:

- The fact that $\mathcal{C}^0(G_K)$ is a Serre subcategory of $\mathcal{C}(G_K)$, which is a triviality.
- The fact that $\mathcal{C}^\infty(G_K)$ is a weak Serre subcategory of $\mathcal{C}(G_K)$. The only thing which is not obvious is the stability under extensions of $\mathcal{C}^\infty(G_K)$ inside of $\mathcal{C}(G_K)$, which is contained in [loc. cit., proposition 6.3].
- The fact that if $U \in \mathcal{C}^0(G_K)$ and $W \in \mathcal{C}^\infty(G_K)$, then $\mathrm{Hom}_{\mathcal{C}(G_K)}(W, U) = 0$, which is the corollary [loc. cit., théorème 5.1]. □

For instance, we see that, if U is a G_K-stable finite dimensional sub-$\mathbb{Q}_p$-vector space of $\mathbb{C}_p$, then $d(\mathbb{C}_p/U) = 1$ and $h(\mathbb{C}_p/U) = -\dim_{\mathbb{Q}_p} U$.

If $V \in \mathcal{C}(G_K)$, $W \in \mathcal{C}^\infty(G_K)$ and $\tilde{f} : V/U_+ \to W/U_-$ is an almost isomorphism, from the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_+ & \longrightarrow & V & \longrightarrow & V/U_+ & \longrightarrow & 0 \\ & & & & & & \downarrow \simeq & & \\ 0 & \longrightarrow & U_- & \longrightarrow & W & \longrightarrow & W/U_- & \longrightarrow & 0 \end{array}$$

whose lines are exact, we deduce that

$$d(V) = d(W), \quad h(V) = h(U_+) - h(U_-) = \dim_{\mathbb{Q}_p}(U_+) - \dim_{\mathbb{Q}_p}(U_-).$$

Corollary 2.10. (i) *For any $V \in \mathcal{C}(G_K)$, we have $V \in \mathcal{C}^0(G_K) \iff d(V) = 0$ (in which case $h(V) = \dim_{\mathbb{Q}_p} V \geq 0$).*

(ii) *If $g : V \to W$ is a monomorphism of $\mathcal{C}(G_K)$ with $W \in \mathcal{C}^\infty(G_K)$ such that $d(V) = d(W)$, then g is an isomorphism.*

Proof. Looking at an almost isomorphism as above, the first assertion is immediate.

For the second, let U be the cokernel of g. We have $d(U) = 0$, hence $U \in \mathcal{C}^0(G_K)$, hence $U = 0$, as there is no nontrivial morphism from W to an object of $\mathcal{C}^0(G_K)$. □

Remark 2.11. As $\mathcal{C}^0(G_K)$ is a Serre subcategory of $\mathcal{C}(G_K)$, we may consider the quotient

$$\widetilde{\mathcal{C}}(G_K) = \mathcal{C}(G_K)/\mathcal{C}^0(G_K)$$

It is known [Fontaine 2003, proposition 7.1] that this abelian category is semisimple with exactly one isomorphism class of simple objects which is the class of $\mathbb{C}_p$ viewed as an object of this category. Hence $\widetilde{\mathcal{C}}(G_K)$ is completely determined, up to equivalence, by the somewhat mysterious huge skew field $\mathcal{D}_K$ of the endomorphisms of $\mathbb{C}_p$ in this category [loc. cit., proposition 7.2].

We denote by

$$\widehat{\mathcal{C}}(G_K)$$

the full subcategory of $\mathcal{IF}(G_K)$ whose objects are those V's which admit a G_K-equivariant admissible filtration $(F^nV)_{n\in\mathbb{Z}}$ such that $F^mV/F^nV \in \mathcal{C}(G_K)$ for all $m \le n$ in $\mathbb{Z}$.

By passing to the limit, we see that the previous theorem implies:

Proposition 2.12. *Any morphism of $\widehat{\mathcal{C}}(G_K)$ is strict (as a morphism of $\mathcal{IF}(G_K)$) and this category is abelian. A sequence of morphisms of $\widehat{\mathcal{C}}(G_K)$ is exact if and only if the underlying sequence of $\mathbb{Q}_p$-vector spaces is exact. The category $\mathcal{C}(G_K)$ is a Serre subcategory of $\widehat{\mathcal{C}}(G_K)$ of which $\widehat{\mathcal{C}}^\infty(G_K)$ is a weak Serre subcategory.*

Remark 2.13. As $\mathrm{Rep}_{B_{dR}^+}(G_K)$ and $\mathrm{Rep}_{B_{dR}}(G_K)$ are Serre subcategories of $\widehat{\mathcal{C}}^\infty(G_K)$, these two categories are also weak Serre subcategories of $\widehat{\mathcal{C}}(G_K)$.

2F. *Almost split exact sequences.* We say that a sequence of morphisms of $\mathcal{IF}(G_K)$ is *exact* if the underlying sequence of $\mathbb{Q}_p$-vector spaces is exact.

An *almost splitting* of a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

in $\mathcal{IF}(G_K)$ is a G_K-stable closed sub-$\mathbb{Q}_p$-vector space S of V such that

(i) the compositum $S \subset V \to V''$ is onto,

(ii) the $\mathbb{Q}_p$-vector space $S \cap V'$ is finite-dimensional.

We say that such an exact sequence *almost splits* if there exists such an almost splitting. This is equivalent to saying that there exists a G_K-stable finite-dimensional sub-$\mathbb{Q}_p$-vector space U of V' such that the sequence

$$0 \to V'/U \to V/U \to V'' \to 0$$

splits.

We observe that any almost splitting S of a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

defines, in an obvious way, almost isomorphisms

$$V \rightsquigarrow V' \oplus V'' \rightsquigarrow S \oplus V''.$$

Proposition 2.14 [Fontaine 2003, théorème 5.2]. *Let*

$$0 \to W' \to W \to W'' \to 0$$

be a short exact sequence in $\mathcal{IF}(G_K)$ with W' and W'' in $\mathcal{C}^\infty(G_K)$. Then W is in $\mathcal{C}^\infty(G_K)$ if and only if the sequence almost splits.

Proposition 2.15 [Fontaine 2003, proposition 5.2]. *Let*

$$0 \to V' \to V \to V'' \to 0$$

be a short exact sequence in $\mathcal{IF}(G_K)$ with V' and V'' in $\mathcal{C}(G_K)$. Then V is in $\mathcal{C}(G_K)$ if and only if the sequence almost splits.

Corollary 2.16. *Among the strictly full subcategories of $\mathcal{B}(G_K)$ which are abelian, containing $\mathbb{C}_p$ and $\mathcal{C}^0(G_K)$ and stable under almost split extensions, there is a smallest one. This is $\mathcal{C}(G_K)$.*

Proof. Clear! □

3. B_e-representations and coherent $\mathcal{O}_X[G_K]$-modules

3A. ***The topological $\mathbb{Q}_p$-algebras B^+_{cris} and B_e.*** Recall (from, e.g., [Fontaine 1994, §2.3 and §4.1]) that B^+_{cris} is a Banach algebra equipped with a continuous endomorphism φ and a continuous action of G_K commuting with φ. There is a natural G_K-equivariant continuous injective homomorphism of topological $\mathbb{Q}_p$-algebras

$$B^+_{\mathrm{cris}} \to B^+_{dR}$$

that we use to identify B^+_{cris} to a subring of B^+_{dR} containing t.

For each $d \in \mathbb{N}$, we set

$$P^d = \{b \in B^+_{\mathrm{cris}} \mid \varphi(b) = p^d b\}.$$

This is a G_K-stable closed sub-$\mathbb{Q}_p$-vector space of B^+_{cris} as well as of B^+_{dR} (e.g. [Kisin 2003, Lemma 3.3]). Moreover B^+_{cris} and B^+_{dR} induce the same topology on P^d which can be viewed as a Banach representation of G_K. We have a canonical short exact sequence (see [Colmez and Fontaine 2000, proposition 1.3], for instance)

$$0 \to \mathbb{Q}_p(d) \to P^d \to B_d \to 0$$

where $\mathbb{Q}_p(d) = \mathbb{Q}_p t^d$ and $P^d \to B_d$ is the compositum $P^d \subset B^+_{\mathrm{cris}} \subset B^+_{dR} \overset{\mathrm{proj}}{\to} B_d$. In particular we see that P^d is an almost $\mathbb{C}_p$-representation with $d(P^d) = d$ and $h(P^d) = 1$.

As usual, we set $B_{\mathrm{cris}} = B_{\mathrm{cris}}^{+}[1/t]$, which we can view as a G_K-stable subring of B_{dR}.

We have $\varphi(t) = pt$ and φ extends uniquely to B_{cris}. Moreover the natural map $B_{\mathrm{cris}} \to B_{dR} = B_{dR}^{+}[1/t]$ is still injective and we use it to identify B_{cris} to a G_K-stable sub-$\mathbb{Q}_p$-algebra of B_{dR}.

Recall that

$$B_e = \{b \in B_{\mathrm{cris}} \mid \varphi(b) = b\}$$

is also a G_K-stable sub-$\mathbb{Q}_p$-algebra of B_{dR}. We endow it with the topology induced by the (locally convex) topology of B_{dR}.

Then, we have

$$B_e = \varinjlim_{d\in\mathbb{N}} \mathrm{Fil}^{-d} B_e = \bigcup_{d\in\mathbb{N}} \mathrm{Fil}^{-d} B_e$$

where, for all $d \in \mathbb{N}$,

$$\mathrm{Fil}^{-d} B_e = B_e \cap B_{dR}^{+} t^{-d} = P^d.t^{-d} = P^d(-d)$$

is an almost $\mathbb{C}_p$-representation (with $d(P^d(-d)) = d$ and $h(P^d(-d)) = 1$) homeomorphic to P^d as a Banach. Setting $P^d = P^d(-d) = 0$ for $d > 0$, we see that B_e is an ind-Banach algebra with $(P^{-n}(n))_{n\in\mathbb{Z}}$ a G_K-stable multiplicative admissible filtration.

3B. *B_e-representations.* The topology of B_e induces on each B_e-module of finite type a natural topology for which it is an ind-Fréchet (actually an ind-Banach). A *B_e-representation (of G_K)* is a B_e-module of finite type equipped with a semilinear and continuous action of G_K. With the G_K-equivariant B_e-linear maps as morphisms, B_e-representations form a category that we denote by $\mathrm{Rep}_{B_e}(G_K)$.

Proposition 3.1. *The B_e-module underlying any B_e-representation is free of finite rank. The category $\mathrm{Rep}_{B_e}(G_K)$ is a $\mathbb{Q}_p$-linear abelian category.*

Proof. Recall that B_e is a principal ideal domain [Fargues and Fontaine 2018, théorème 6.5.2]. In particular it is a noetherian ring and the fact that $\mathrm{Rep}_{B_e}(G_K)$ is a $\mathbb{Q}_p$-linear abelian category is obvious.

Moreover [loc. cit., proposition 10.1.1], for any maximal ideal $\mathfrak{p}$ of B_e, the orbit of $\mathfrak{p}$ under the action of G_K is infinite. This implies that there is no nontrivial G_K-equivariant ideal of B_e. If Λ is any nonzero B_e-representation of G_K, the annihilator of its torsion sub-module is a proper G_K-equivariant ideal and must be 0. Therefore the B_e-module underlying Λ is torsion free, hence free of finite rank. □

Remark 3.2. Let C_e be the fraction field of B_e. This is the union of the fractional ideals of B_e. For each such ideal $\mathfrak{a}$, the choice of a generator a defines a bijection

$$B_e \to \mathfrak{a}, \quad b \mapsto ba,$$

and we put on $\mathfrak{a}$ the topology defined by transport de structure, which is independent of the choice of the generator. Hence each $\mathfrak{a}$ is naturally an ind-Banach ($\mathbb{Q}_p$-vector space). If $\mathfrak{a} \subset \mathfrak{b}$ are two fractional ideals, this inclusion is continuous and $\mathfrak{a}$ is a closed sub-$\mathbb{Q}_p$-vector space of $\mathfrak{b}$. Hence we may endow C_e with the coarsest locally convex topology such that, for all fractional ideal $\mathfrak{a}$, the map $\mathfrak{a} \to C_e$ is continuous (a lattice $\mathcal{L}$ in C_e is open if and only if $\mathcal{L} \cap \mathfrak{a}$ is open in $\mathfrak{a}$ for all $\mathfrak{a}$).

The action of G_K on C_e is continuous for this topology (but C_e doesn't seem to be an object of $\mathcal{IF}(G_K)$) and we may consider the category $\mathrm{Rep}_{C_e}(G_K)$ of C_e-representations (of G_K), that is of finite-dimensional C_e-vector spaces equipped with a semilinear and continuous action of G_K. This is obviously a $\mathbb{Q}_p$-linear abelian category.

We have an obvious exact $\mathbb{Q}_p$-linear functor

$$\mathrm{Rep}_{B_e}(G_K) \to \mathrm{Rep}_{C_e}(G_K), \quad \Lambda \mapsto C_e \otimes_{B_e} \Lambda.$$

This functor is fully faithful: if $M \in \mathrm{Rep}_{C_e}(G_K)$ is a C_e-representation of dimension d, there is at most one G_K-equivariant sub-B_e-module of rank d because if Λ_1 and Λ_2 are two of them, so are $\Lambda_1 + \Lambda_2$ and $(\Lambda_1 + \Lambda_2)/\Lambda_1$ is torsion, hence 0.

Remark 3.3. If Λ is any B_e-representation of G_K, the underlying $\mathbb{Q}_p$-vector space is locally convex and Λ inherits a natural structure of an object of $\mathcal{IF}(G_K)$. We will see later that the forgetful functor

$$\mathrm{Rep}_{B_e}(G_K) \to \mathcal{IF}(G_K)$$

is fully faithful (Proposition 3.11) and that its essential image is contained in $\widehat{\mathcal{C}}(G_K)$ (Proposition 3.12).

Proposition 3.4. *Let $W \in \mathcal{C}^\infty(G_K)$ and $\Lambda \in \mathrm{Rep}_{B_e}(G_K)$. Then*

$$\mathrm{Hom}_{\mathcal{IF}(G_K)}(W, \Lambda) = 0.$$

Proof. Let $f : W \to \Lambda$ such a morphism. We see that $B_{dR} \otimes_{B_e} \Lambda$ is a B_{dR}-representation of G_K and that

$$g : \Lambda \to B_{dR} \otimes_{B_e} \Lambda, \quad \lambda \mapsto 1 \otimes \lambda$$

is a morphism of $\mathcal{IF}(G_K)$. But $gf : W \to B_{dR} \otimes_{B_e} \Lambda$ must be B_{dR}^+-linear (Section 2D). As the B_{dR}^+-module W is torsion, and $B_{dR} \otimes \Lambda$ is torsion free, we have $gf = 0$, hence also $f = 0$ as g is injective. □

3C. *Coherent $\mathcal{O}_X$-modules.* We know that B_e is a PID and we may consider the "open curve"

$$X_e = \mathrm{Spec}\; B_e,$$

a noetherian regular affine scheme of dimension 1 whose function field is the fraction field C_e of B_e that we can see as a subfield of B_{dR}. For each closed point x of X, the local ring $\mathcal{O}_{X,x}$ is a DVR and we denote by v_x the corresponding valuation on C_e normalised by $v_x(C_e^*) = \mathbb{Z}$.

Recall (cf. [Fargues and Fontaine 2018, §6.5.1]) that the curve $X = X_{\mathbb{Q}_p, \mathbb{C}_p^\flat}$ can be defined as the compactification at ∞ of X_e. More precisely, as B_{dR} is the fraction field of the discrete valuation ring B_{dR}^+, it is naturally equipped with a valuation v_{dR}: if $b \in B_{dR}$ is $\neq 0$, then $v_{dR}(b)$ is the largest $n \in \mathbb{Z}$ such that $b \in \mathrm{Fil}^n B_{dR}$. We denote by v_∞ the restriction of v_{dR} to C_e. The topological space underlying X is obtained from the topological space underlying X_e by adding the closed point ∞ defined by v_∞. Hence, the function field of X is C_e and, if U is any nonempty open subspace of X, we have

$$\mathcal{O}_X(U) = \{b \in C_e \mid v_x(b) \geq 0,\ \forall x \in U\}.$$

We have $X \setminus \{\infty\} = X_e$, the ring B_{dR}^+ is the completion of $\mathcal{O}_{X,\infty}$ and B_{dR} is the completion of C_e for the topology defined by v_∞.

Consider the following category $\mathrm{Coh}(\mathcal{O}_X)$:

- An object of $\mathrm{Coh}(\mathcal{O}_X)$ is a triple $(\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_\mathcal{F})$ with $\mathcal{F}_e$ a B_e-module of finite type, $\mathcal{F}_{dR}^+$ a B_{dR}^+-module of finite type and
$$\iota_\mathcal{F} : \mathcal{F}_{dR}^+ \to B_{dR} \otimes_{B_e} \mathcal{F}_e$$
a B_{dR}^+-linear map inducing an isomorphism of B_{dR}-vector spaces
$$B_{dR} \otimes_{B_{dR}^+} \mathcal{F}_{dR}^+ \to B_{dR} \otimes_{B_e} \mathcal{F}_e.$$
- A morphism $(\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_\mathcal{F}) \to (\mathcal{G}_e, \mathcal{G}_{dR}^+, \iota_\mathcal{G})$ is a pair (f_e, f_{dR}^+) with $f_e : \mathcal{F}_e \to \mathcal{G}_e$ a B_e-linear map and $f_{dR}^+ : \mathcal{F}_{dR}^+ \to \mathcal{G}_{dR}^+$ a B_{dR}^+-linear map such that the obvious diagram commutes.

To any coherent $\mathcal{O}_X$-module $\mathcal{F}$, we can associate an object $(\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_\mathcal{F})$ of this category:

- $\mathcal{F}_e = \mathcal{F}(X_e)$,
- $\mathcal{F}_{dR}^+ = B_{dR}^+ \otimes_{\mathcal{O}_{X,\infty}} \mathcal{F}_\infty$, the completion of the fiber of $\mathcal{F}$ at ∞,
- the completion at ∞ of the general fiber is $B_{dR} \otimes_{B_{dR}^+} \mathcal{F}_{dR}^+$ as well as $B_{dR} \otimes_{B_e} \mathcal{F}_e$ and $\iota_\mathcal{F} : \mathcal{F}_{dR}^+ \to B_{dR} \otimes_{B_e} \mathcal{F}_e$ is the natural map.

This correspondence is obviously functorial and it is immediate to see that it gives an equivalence of categories. We use it to identify the category of coherent $\mathcal{O}_X$-modules to $\mathrm{Coh}(\mathcal{O}_X)$. In this equivalence we see that the category $\mathrm{Bund}(X)$ of vector bundles over X, i.e., of torsion free coherent $\mathcal{O}_X$-modules, can be identified with the full subcategory of $\mathrm{Coh}(\mathcal{O}_X)$ whose objects are triples $(\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_{\mathcal{F}})$ such that the B_e-module $\mathcal{F}_e$ and the B_{dR}^+-module $\mathcal{F}_{dR}^+$ are torsion free ($\Longleftrightarrow$ free).

3D. *The topology on $\mathcal{O}_X$.* The curve X can be also described ([FF], §6.5.1) as

$$X = \mathrm{Proj} \bigoplus_{d \in \mathbb{N}} P^d$$

and there is (*loc. cit.*, théorème 6.5.2) a one to one correspondence between the closed points of X and the $\mathbb{Q}_p$-lines in P^1 (the map associating to such a line the prime ideal of $P = \bigoplus_{d\in\mathbb{N}} P^d$ that it generates is a bijection between the set of these lines and the set of nonzero homogeneous prime ideals of P different from $\bigoplus_{d>0} P^d$). In this correspondence ∞ corresponds to the line generated by t.

Moreover, if $x_1, x_2, \dots, x_r$ are closed points of X and if, for $1 \le i \le r$, we choose a generator t_i of the $\mathbb{Q}_p$-line associated to x_i, we see that the $\mathbb{Q}_p$-algebra $\mathcal{O}_X(X \setminus \{x_1, x_2, \dots, x_r\})$ has a natural topology: If we set $u = t_1 t_2 \dots t_r$, we have

$$\mathcal{O}_X(X \setminus \{x_1, x_2, \dots, x_r\}) = \bigcup_{n \in \mathbb{N}} P^{nr} u^{-n}$$

and we see that it is an ind-Banach algebra with $(P^{nr}u^{-n})_{n\in\mathbb{N}}$ a multiplicative admissible Banach filtration. Thus we may consider $\mathcal{O}_X$ as a sheaf of ind-Banach algebras (the restriction maps are obviously continuous).

3E. *The category $\mathcal{M}(G_K)$.* The group G_K acts continuously on X and it makes sense to speak of the category $\mathcal{M}(G_K)$ of coherent $\mathcal{O}_X[G_K]$-modules, that is of coherent $\mathcal{O}_X$-modules equipped with a semilinear and continuous action of G_K.

We see that:

– the open subset $X_e = \mathrm{Spec}\, B_e$ is stable under G_K and G_K acts continuously on the ind-Banach algebra B_e,

– the point ∞ is fixed by G_K and the action of G_K on the Fréchet algebra B_{dR}^+ (resp. on the ind-Fréchet algebra B_{dR}), completion at ∞ of $\mathcal{O}_{X,\infty}$ (resp. of the function field C_e of X) is continuous.

From the description of coherent $\mathcal{O}_X$-modules of the previous paragraph, we see that we can identify $\mathcal{M}(G_K)$ to the following category:

- An object is a triple $\mathcal{F} = (\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_{\mathcal{F}})$, where $\mathcal{F}_e$ is a B_e-representation, $\mathcal{F}_{dR}^+$ is a B_{dR}^+-representation and

$$\iota_{\mathcal{F}} : \mathcal{F}_{dR}^+ \to B_{dR} \otimes_{B_e} \mathcal{F}_e$$

is a G_K-equivariant homorphism of B_{dR}^+-modules such that the induced B_{dR}-linear map

$$B_{dR} \otimes_{B_{dR}^+} \mathcal{F}_{dR}^+ \to B_{dR} \otimes_{B_e} \mathcal{F}_e$$

is bijective.

- A morphism

$$f : (\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_{\mathcal{F}}) \to (\mathcal{G}_e, \mathcal{G}_{dR}^+, \iota_{\mathcal{G}})$$

is a pair (f_e, f_{dR}^+) with $f_e : \mathcal{F}_e \to \mathcal{G}_e$ (resp. $f_{dR}^+ : \mathcal{F}_{dR}^+ \to \mathcal{G}_{dR}^+$) a morphism of B_e-representations (resp. B_{dR}^+-representations) such that the obvious diagram commutes.

When there is no ambiguity about the map $\iota_{\mathcal{F}}$, we write abusively

$$\mathcal{F} = (\mathcal{F}_e, \mathcal{F}_{dR}^+)$$

We also denote by

$$\mathcal{F}_{dR} = B_{dR} \otimes_{B_e} \mathcal{F}_e = B_{dR} \otimes_{C_e} (C_e \otimes_{B_e} \mathcal{F}_e)$$

the completion at ∞ of the generic fiber $\mathcal{F}_\eta = C_e \otimes_{B_e} \mathcal{F}_e$ of $\mathcal{F}$.

The category $\mathrm{Bund}_X(G_K)$ of *G_K-equivariant vector bundles over X* is the full subcategory of $\mathcal{M}(G_K)$ whose objects are those for which the underlying $\mathcal{O}_X$-module is torsion free. From the fact that any B_e-representation is torsion free, we see that, if $\mathcal{F}$ is any coherent $\mathcal{O}_X[G_K]$-module, there is no torsion away from ∞. Therefore $\mathrm{Bund}_X(G_K)$ is the full subcategory of $\mathcal{M}(G_K)$ whose objects are those $\mathcal{F}$ such that the B_{dR}^+-module $\mathcal{F}_{dR}^+$ is free ($\iff$ torsion free), i.e., the B-pairs of [Berger 2008].

3F. *The Harder–Narasimhan filtration.* The abelian category $\mathrm{Coh}(\mathcal{O}_X)$ is equipped with two additive functions, the *rank* and the *degree* [Fargues and Fontaine 2018, chapitre 5]:

$$\mathrm{rk} : \mathrm{Coh}(\mathcal{O}_X) \to \mathbb{N}, \quad \deg : \mathrm{Coh}(\mathcal{O}_X) \to \mathbb{Z}$$

The rank of $\mathcal{F} = (\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_{\mathcal{F}})$ is the rank of the B_e-module $\mathcal{F}_e$. It is 0 if and only if $\mathcal{F}$ is torsion. It is more difficult to compute the degree. But this additive function is characterised by the following facts:

- if D is a divisor, then

$$\deg(\mathcal{L}(D)) = \deg(D) = \sum_{\substack{\text{closed} \\ \text{points of } X}} n_x \quad \text{if } D = \sum n_x[x],$$

- if $\mathcal{F}$ is a vector bundle of rank r, then

$$\deg(\mathcal{F}) = \deg(\wedge^r \mathcal{F}),$$

- if $\mathcal{F}$ is a torsion $\mathcal{O}_X$-module, then

$$\deg(\mathcal{F}) = \sum_{\substack{\text{closed} \\ \text{points of } X}} \mathrm{length}_{\mathcal{O}_{X,x}} \mathcal{F}_x.$$

The slope of a nonzero coherent O_X-module $\mathcal{F}$ is

$$\mathrm{slope}(\mathcal{F}) = \deg(\mathcal{F})/\mathrm{rank}(\mathcal{F}) \in \mathbb{Q} \cup \{+\infty\}$$

(with the convention that the slope of a nonzero torsion coherent $\mathcal{O}_X$-module is $+\infty$).

The following statements are similar to the classical case:

- A coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *semistable* if it is nonzero and if $\mathrm{slope}(\mathcal{F}') \le \mathrm{slope}(\mathcal{F})$ for any nonzero coherent sub-$\mathcal{O}_X$-module of $\mathcal{F}$.
- The Harder–Narasimhan filtration of a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is the unique increasing filtration

  $$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_{m-1} \subset \mathcal{F}_m = \mathcal{F}$$

  by coherent sub-$\mathcal{O}_X$-modules such that each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable with

  $$\mathrm{slope}(\mathcal{F}_1/\mathcal{F}_0) > \mathrm{slope}(\mathcal{F}_2/\mathcal{F}_1) > \cdots > \mathrm{slope}(\mathcal{F}_{m-1}/\mathcal{F}_{m-2}) > \mathrm{slope}(\mathcal{F}_m/\mathcal{F}_{m-1}).$$

The Harder–Narasimhan filtration splits continuously but not canonically. The slopes of the $\mathcal{F}_i/\mathcal{F}_{i-1}$ for $1 \le i \le m$ are called *the HN-slopes of* $\mathcal{F}$.

If $\mathcal{F}$ is an object of $\mathcal{M}(G_K)$, the unicity of the Harder–Narasimhan filtration implies that this filtration is by subobjects in $\mathcal{M}(G_K)$. In general, there is no G_K-equivariant splitting of this filtration.

3G. ***The equivalences $\mathcal{M}^0(G_K) \to \mathcal{C}^0(G_K)$ and $\mathcal{M}^\infty(G_K) \to \mathcal{C}^\infty(G_K)$.*** For all $s \in \mathbb{Q} \cup \{+\infty\}$, we denote by $\mathcal{M}^s(G_K)$ the full subcategory of $\mathcal{M}(G_K)$ whose objects are semistable of slope s. We also write $\mathcal{M}^\infty(G_K) = \mathcal{M}^{+\infty}(G_K)$.

We have $H^0(X, \mathcal{O}_X) = \mathbb{Q}_p$. A central result of [Fargues and Fontaine 2018] (théorème 8.2.10) is that a coherent $\mathcal{O}_X$-module $\mathcal{F}$ is semistable of slope 0 if and only if it is isomorphic to $\mathcal{O}_X^r$ for some positive integer r. From that we deduce:

Proposition 3.5. *If $\mathcal{F} \in \mathcal{M}^0(G_K)$, then $\mathcal{F}(X) \in \mathcal{C}^0(G_K)$ and $\mathrm{rank}(\mathcal{F}) = \dim_{\mathbb{Q}_p} \mathcal{F}(X)$. The functor*

$$\mathcal{M}^0(G_K) \to \mathcal{C}^0(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

is an equivalence of categories. The functor

$$\mathcal{C}^0(G_K) \to \mathcal{M}^0(G_K), \quad U \mapsto \mathcal{O}_X \otimes U = (B_e \otimes_{\mathbb{Q}_p} U, B_{dR}^+ \otimes_{\mathbb{Q}_p} U)$$

is a quasi-inverse.

If $\mathcal{F} \in \mathcal{M}(G_K)$, as there is no torsion away from ∞, we have $\mathcal{F} \in \mathcal{M}^\infty(G_K)$ if and only if $\mathcal{F}_e = 0$. From that, we deduce:

Proposition 3.6. *If $\mathcal{F} \in \mathcal{M}^\infty(G_K)$, then $\mathcal{F}(X) = \mathcal{F}^+_{dR}$ and belongs to $\mathcal{C}^\infty(G_K)$. Moreover*

$$\deg(\mathcal{F}) = \mathrm{length}_{B^+_{dR}} \mathcal{F}(X).$$

The functor

$$\mathcal{M}^\infty(G_K) \to \mathcal{C}^\infty(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

is an equivalence of categories. The functor

$$\mathcal{C}^\infty(G_K) \to \mathcal{M}^\infty(G_K), \quad W \mapsto \underline{W} = (0, W)$$

is a quasi-inverse.

For any $s \in \mathbb{Q}$, we denote by $\mathcal{M}^{\geq s}(G_K)$ (resp. $\mathcal{M}^{<s}(G_K)$) the full subcategory of $\mathcal{M}(G_K)$ whose objects are those which have all their HN-slopes $\geq s$ (resp. $< s$).

For any $\mathcal{F} \in \mathcal{M}(G_K)$, we denote by $\mathcal{F}^{\geq 0}$ the largest term of the Harder–Narasimhan filtration which belongs to $\mathcal{M}^{\geq 0}(G_K)$ and $\mathcal{F}^{<0} = \mathcal{F}/\mathcal{F}^{\geq 0}$. We have a short exact sequence

$$0 \to \mathcal{F}^{\geq 0} \to \mathcal{F} \to \mathcal{F}^{<0} \to 0$$

with $\mathcal{F}^{\geq 0} \in \mathcal{M}^{\geq 0}(G_K)$ and $\mathcal{F}^{<0} \in \mathcal{M}^{<0}(G_K)$.

The category $\mathcal{M}(G_K)$ is equipped with a tensor product. From the classification of vector bundles over X [Fargues and Fontaine 2018, théorème 8.2.10], we get the fact that if $s, t \in \mathbb{Q} \cup \{+\infty\}$, if $\mathcal{F} \in \mathcal{M}^s(G_K)$ and if $\mathcal{G} \in \mathcal{M}^t(G_K)$, then $\mathcal{F} \otimes \mathcal{G} \in \mathcal{M}^{s+t}(G_K)$ (with the convention that $s + t = +\infty$ if s or t is $+\infty$).

The additive category $\mathrm{Bund}_X(G_K)$ has an internal hom

$$(\mathcal{F}, \mathcal{G}) \mapsto \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$$

We see that $(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_e = \mathcal{L}_{B_e}(\mathcal{F}_e, \mathcal{G}_e)$ is the B_e-module of the B_e-linear maps $\mathcal{F}_e \to \mathcal{G}_e$, and $(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))^+_{dR} = \mathcal{L}_{B^+_{dR}}(\mathcal{F}^+_{dR}, \mathcal{G}^+_{dR})$ is the B^+_{dR}-module of the B^+_{dR}-linear maps $\mathcal{F}^+_{dR} \to \mathcal{G}^+_{dR}$.

In $\mathrm{Bund}_X(G_K)$, there is also a duality: The dual of $\mathcal{F}$ is $\mathcal{F}^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If $\mathcal{F}, \mathcal{G} \in \mathrm{Bund}_X(G_K)$, then $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathcal{F}^\vee \otimes \mathcal{G}$. If $\mathcal{F}$ is semistable of slope s, then $\mathcal{F}^\vee$ is semistable of slope $-s$.

3H. ***Tate and Harder–Narasimhan twists.*** Recall that, for any p-adic vector space V equipped with a linear action of G_K and $n \in \mathbb{Z}$, the n-th Tate's twist of V is

$$V(n) = V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$$

where $\mathbb{Q}_p(n) = \mathbb{Q}_p t^n \subset B_{dR}$. This construction is functorial.

For any $n \in \mathbb{Z}$, we denote by

$$\mathcal{O}_X(n)_T = \mathcal{O}_X \otimes \mathbb{Q}_p(n) = (B_e(n), B^+_{dR}(n)) = (B_e.t^n, B^+_{dR}.t^n)$$

(where $B_e.t^n$ (resp. $B_{dR}^+.t^n$) is the sub-B_e-module (resp. B_{dR}^+-module) of B_{dR} generated by t^n) the G_K-equivariant line bundle of slope 0 associated to $\mathbb{Q}_p(n)$.

For $\mathcal{F} \in \mathcal{M}(G_K)$ and $n \in \mathbb{Z}$, the *n-th Tate twist* of $\mathcal{F}$ is

$$\mathcal{F}(n)_T = \mathcal{F} \otimes \mathcal{O}_X(n)_T = (\mathcal{F}_e(n), \mathcal{F}_{dR}^+(n), \iota_{\mathcal{F}}(n)).$$

It has the same degree, the same rank and the same slope as $\mathcal{F}$.

For any $n \in \mathbb{Z}$, we consider the G_K-equivariant line bundle

$$\mathcal{O}_X(n)_{HN} = (B_e, B_{dR}^+(-n)) = (B_e, B_{dR}^+.t^{-n}).$$

There is an obvious short exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_X \to \mathcal{O}_X(n)_{HN} \to (0, B_n(-n)) \to 0 \quad & \text{if } n \geq 0, \\ 0 \to \mathcal{O}_X(n)_{HN} \to \mathcal{O}_X \to (0, B_{-n}) \to 0 \quad & \text{if } n < 0, \end{aligned}$$

In particular, $\mathcal{O}_X(n)_{HN}$ is a modification of $\mathcal{O}_X$ and is of degree n. It is semistable of slope n.

For $\mathcal{F} \in \mathcal{M}(G_K)$ and $n \in \mathbb{Z}$, we define the *n-th Harder–Narasimhan twist* of $\mathcal{F}$ as

$$\mathcal{F}(n)_{HN} = \mathcal{F} \otimes \mathcal{O}_X(n)_{HN} = (\mathcal{F}_e, \mathcal{F}_{dR}^+(-n), \iota_{\mathcal{F}}(-n)) = (\mathcal{F}_e, t^{-n}.\mathcal{F}_{dR}^+, \iota_{\mathcal{F}}(-n)).$$

It has the same rank as $\mathcal{F}$. If $\mathcal{F}$ is semistable of slope s, then $\mathcal{F}(n)_{HN}$ is semistable of slope $s+n$.

These two constructions are obviously functorial and commute with Harder–Narasimhan filtration. In particular:

- If $\mathcal{F}$ is semistable of slope s, then $\mathcal{F}(n)_T$ is semistable of slope s, and $\mathcal{F}(n)_{HN}$ is semistable of slope $s+n$.
- The HN-slopes of $\mathcal{F}(n)_T$ are the same as the HN-slopes of $\mathcal{F}$, and the HN-slopes of $\mathcal{F}(n)_{HN}$ are the $s+n$ for s running through the HN-slopes of $\mathcal{F}$.

These constructions commute: for $m, n \in \mathbb{Z}$, we have

$$\mathcal{F}(m)_T(n)_{HN} = \mathcal{F}(n)_{HN}(m)_T.$$

Remark 3.7. In [Fargues and Fontaine 2018, définition 8.2.1] the G_K-equivariant line bundle $\mathcal{O}_X(n)_{HN}(n)_T$ is denoted $\mathcal{O}_X(n)$. We have to avoid confusion between the three G_K-equivariant line bundles $\mathcal{O}_X(n)_T$, $\mathcal{O}_X(n)_{HN}$ and

$$\mathcal{O}_X(n) = (B_e(n), B_{dR}^+) = (B_e.t^n, B_{dR}^+).$$

3I. *Potentially trivialisable B_e-representations.* Let Λ be a B_e-representation of G_K and K' a finite extension of K contained in $\overline{\mathbb{Q}}_p$. We say that Λ is $G_{K'}$-*trivialisable* if there is $U \in \mathcal{C}^0(G_{K'})$ and a $G_{K'}$-equivariant isomorphism of B_e-modules

$$B_e \otimes_{\mathbb{Q}_p} U \simeq \Lambda.$$

We say that Λ is *trivialisable* if it is G_K-trivialisable and *potentially trivialisable* if there is a finite extension K' of K contained in $\overline{\mathbb{Q}}_p$ such that Λ is $G_{K'}$-trivialisable.

Proposition 3.8. *Any absolutely irreducible B_e-representation of G_K is potentially trivialisable.*

Proof. Let Λ be such a B_e-representation. Then $\Lambda_{dR} = B_{dR} \otimes_{B_e} \Lambda$ is a B_{dR}-representation. Let $\mathcal{L}$ be the set of G_K-stable B_{dR}^+-lattices of Λ_{dR}. We know (Proposition 2.5) that $\mathcal{L}$ is not empty. For each $L \in \mathcal{L}$, we may consider the G_K-equivariant vector bundle over X

$$\mathcal{F}_L = (\Lambda, L).$$

Such an $\mathcal{F}_L$ is semistable (otherwise the Harder–Narasimhan filtration would be nontrivial and would induce a nontrivial filtration of the B_e-representation $(\mathcal{F}_L)_e = \Lambda$ which is not possible as Λ is irreducible).

Chose such an $\mathcal{F}_L$. Replacing $\mathcal{F}_L$ with $\mathcal{F}_L(n)_{HN}$ with $n \in \mathbb{N}$ big enough, we may assume that the degree d of $\mathcal{F}_L$ is ≥ 0. By Proposition 2.7, we can find a finite extension K' of K contained in $\overline{\mathbb{Q}}_p$ and a $G_{K'}$-stable sub-B_{dR}^+-lattice $L_0 \subset L$ such that $\text{length}_{B_{dR}^+}(L/L_0) = d$. Then $\mathcal{F}_{L_0} = (\Lambda, L_0)$ is a $G_{K'}$-equivariant vector bundle over X of degree $d - d = 0$. As the B_e-representation Λ is absolutely irreducible, it is irreducible as a B_e-representation of $G_{K'}$. Hence, $\mathcal{F}_{L_0}$ is semistable of slope 0. By Proposition 3.5, there is a $\mathbb{Q}_p$-representation U of $G_{K'}$ such that

$$\mathcal{F}_{L_0} \simeq \mathcal{O}_X \otimes U.$$

Therefore Λ, as a B_e-representation of $G_{K'}$, is isomorphic to $B_e \otimes_{\mathbb{Q}_p} U$. □

Corollary 3.9. *The category* $\text{Rep}_{B_e}(G_K)$ *is the smallest full subcategory of itself containing potentially trivialisable B_e-representations and stable under taking extensions. This is also the smallest full subcategory of itself containing trivialisable B_e-representations and stable under taking extensions and direct summands.*

Proof. For any B_e-representation Λ of G_K, one can find a finite extension K_1 of K contained in $\overline{\mathbb{Q}}_p$ such that Λ, viewed as a B_e-representation of G_{K_1}, can be viewed as a successive extension of absolutely irreducible B_e-representations of G_{K_1} and the first assumption results from the previous proposition. Hence we may find a finite extension K' of K contained in $\overline{\mathbb{Q}}_p$ such that Λ, as a B_e-representation of $G_{K'}$, is a successive extension of $G_{K'}$-trivialisable B_e-representations. Therefore the induced B_e-representation of G_K

$$\Lambda' = B_e[G_K] \otimes_{B_e[G_{K'}]} \Lambda = \mathbb{Q}[G_K] \otimes_{\mathbb{Q}[G_{K'}]} \Lambda$$

is a successive extension of trivialisable B_e-representations of G_K. But the obvious G_K-equivariant projection $\Lambda' \to \Lambda$ splits (as, if $\Lambda^\vee$ denotes the B_e-dual of Λ and if $H = \mathrm{Gal}(K'/K)$, we have a short exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_K)}(\Lambda, \Lambda') \to \mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_{K'})}(\Lambda, \Lambda) \to H^1(H, \Lambda^\vee \otimes_{B_e} \Lambda')$$

and, as B_e is of characteristic 0, we have $H^1(H, \Lambda^\vee \otimes_{B_e} \Lambda') = 0$). Therefore, Λ is a direct summand of a successive extension of trivialisable B_e-representations. □

Remark 3.10. The results of this paragraph can also be deduced from the work of Berger ([Berger 2008] and [Berger 2009]) relating (φ, Γ)-modules on the Robba ring and B_e-pairs.

3J. *The forgetful functor* $\mathbf{Rep}_{B_e}(G_K) \to \widehat{\mathcal{C}}(G_K)$.

Proposition 3.11. *The forgetful functor*

$$\mathrm{Rep}_{B_e}(G_K) \to \mathcal{IF}(G_K)$$

is fully faithful.

Proof. Let Λ and Λ' two B_e-representations. We want to prove that any G_K-equivariant continuous map

$$\Lambda \xrightarrow{\alpha} \Lambda'$$

is B_e-linear.

Let K' be a finite Galois extension of K contained in $\overline{\mathbb{Q}}_p$ such that Λ and Λ' are successive extensions of trivialisable B_e-representations of $G_{K'}$. If $H = \mathrm{Gal}(K'/K)$, we have

$$\mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_K)}(\Lambda, \Lambda') = (\mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_{K'})}(\Lambda, \Lambda'))^H,$$
$$\mathrm{Hom}_{\mathcal{IF}(G_K)}(\Lambda, \Lambda') = (\mathrm{Hom}_{\mathcal{IF}(G_{K'})}(\Lambda, \Lambda'))^H.$$

Therefore, replacing K by K' we may assume again that there is $r \in \mathbb{N}$ and a filtration of Λ by sub-B_e-representations

$$0 = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{r-1} \subset \Lambda_r = \Lambda$$

such that each Λ_i/Λ_{i-1} is trivialisable.

We proceed by induction on r, the case $r = 0$ being trivial. Assume $r \geq 1$ and that $\Lambda_r/\Lambda_{r-1} = B_e \otimes_{\mathbb{Q}_p} U$ for some $U \in \mathcal{C}^0(G_K)$. Chose a B_e-linear section $s : B_e \otimes U \to \Lambda$ of the projection $\Lambda \to B_e \otimes U$. We have a decomposition of Λ as a B_e-module into a direct sum

$$\Lambda = \Lambda_{r-1} \oplus s(B_e \otimes U) = \Lambda_{r-1} \oplus (B_e \otimes s(U)).$$

By induction, the restriction of α to Λ_{r-1} is B_e-linear. Hence there is a unique B_e-linear map

$$\alpha_0 : \Lambda \to \Lambda'$$

such that $\alpha_0(\lambda) = \alpha(\lambda)$ if $\lambda \in \Lambda_{r-1}$ and $\alpha_0(s(u)) = \alpha(s(u))$ for all $u \in U$. It is easy to check that α_0 is continuous and G_K-equivariant. The maps

$$\alpha, \alpha_0 : \Lambda \to \Lambda'$$

coincide on $\Lambda_{r-1} \oplus s(U)$ and the map $\alpha - \alpha_0$ induces, by going to the quotient, a morphism in $\mathcal{IF}(G_K)$

$$\beta : \Lambda/(\Lambda_{r-1} \oplus s(U)) \to \Lambda'.$$

Recall (cf. eg [Colmez and Fontaine 2000], proposition 1.3) that $B_{dR} = B_e + B_{dR}^+$, and $B_e \cap B_{dR}^+ = \mathbb{Q}_p$. Hence, if we set $\widetilde{B}_{dR} = B_{dR}/B_{dR}^+$, we can identify $B_e/\mathbb{Q}_p$ to $\widetilde{B}_{dR}$.

Therefore we have

$$\Lambda/(\Lambda_{r-1} \oplus s(U)) = (\Lambda_r/\Lambda_{r-1})/U = B_e \otimes U/U = \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U.$$

and $\beta \in \mathrm{Hom}_{\mathcal{IF}(G_K)}(\widetilde{B}_{dR} \otimes U, \Lambda')$.

We see that $\widetilde{B}_{dR}$ is the direct limit of the $B_d(-d)$, for $d \in \mathbb{N}$, hence

$$\widetilde{B}_{dR} \otimes U = \varinjlim_{d\in\mathbb{N}} B_d(-d) \otimes_{\mathbb{Q}_p} U.$$

Each $B_d(-d) \otimes U$ is an object of $\mathcal{C}^\infty(G_K)$. Hence, Proposition 3.4, implies that

$$\mathrm{Hom}_{\mathcal{IF}(G_K)}(B_d(-d) \otimes U, \Lambda') = 0.$$

Therefore $\beta = 0$ and $\alpha = \alpha_0$ is B_e-linear. □

We use this result to identify $\mathrm{Rep}_{B_e}(G_K)$ to a full subcategory of $\mathcal{IF}(G_K)$.

Proposition 3.12. *We have*

$$\mathrm{Rep}_{B_e}(G_K) \subset \widehat{\mathcal{C}}(G_K).$$

More precisely, for any B_e-representation Λ of G_K, there is a G_K-equivariant admissible filtration $(F^n\Lambda)_{n\in\mathbb{Z}}$ with $F^1\Lambda = 0$ and $F^n\Lambda \in \mathcal{C}(G_K)$ for all n. Moreover, we may choose this filtration so that, if $b \in \mathrm{Fil}^{-d} B_e$ and $\lambda \in F^n\Lambda$ (with $d \in \mathbb{N}$, $n \in \mathbb{Z}$), then $b\lambda \in F^{n-d}\Lambda$.

Proof. Assume first that Λ is a successive extension of trivialisable B_e-representations, i.e., that there is $r \in \mathbb{N}$ and a filtration by sub-B_e-representations

$$0 = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_{r-1} \subset \Lambda_r = \Lambda$$

such that each Λ_i/Λ_{i-1} is trivialisable. We proceed by induction on r, the case $r = 0$ being trivial. Assume $r \geq 1$. Setting $\Lambda_{r-1} = \Lambda'$ and choosing $U \in \mathcal{C}^0(G_K)$

such that $\Lambda_r/\Lambda_{r-1} \simeq B_e \otimes_{\mathbb{Q}_p} U$, we may assume that we have a short exact sequence of B_e-representations

$$0 \to \Lambda' \to \Lambda \to B_e \otimes U \to 0$$

and, using induction hypothesis, that we have an admissible filtration $(F^n\Lambda')_{n\in\mathbb{Z}}$ of Λ' satisfying the required properties. Let $s : B_e \otimes U \to \Lambda$ a B_e-linear section of the projection $\Lambda \to B_e \otimes U$, so that we have a decomposition of the B_e-module Λ into a direct sum

$$\Lambda = \Lambda' \oplus s(B_e \otimes U) = \Lambda' \oplus (B_e \otimes s(U)).$$

The map

$$\rho : G_K \times U \to \Lambda', \quad (g, u) \mapsto g(s(u)) - s(g(u))$$

is continuous. Therefore, if T is a G_K-stable lattice of U, then $\rho(G_K \times T)$ is compact, hence bounded which implies (by [Schneider 2002, proposition 5.6]) that there exists $m \in \mathbb{Z}$ such that $\rho(G_K \times T)$, hence also $\rho(G_K \times U)$ is contained in $F^m\Lambda'$.

If, for $n \in \mathbb{Z}$, we set

$$F^n\Lambda = \begin{cases} F^n\Lambda' \oplus (F^{n-m}B_e \otimes U) & \text{if } n \leq m, \\ 0 & \text{if } n > m, \end{cases}$$

we see that $(F^n\Lambda)_{n\in\mathbb{N}}$ is an admissible filtration satisfying the required properties.

– In the general case, we choose a finite extension K' of K such that Λ is a successive extension of trivialisable B_e-representation of $G_{K'}$. Therefore we can find a $G_{K'}$-equivariant decreasing admissible filtration

$$(F^n_{K'}\Lambda)_{n\in\mathbb{Z}}$$

such that, if $n \in \mathbb{Z}$, then $F^n_{K'}\Lambda \in \mathcal{C}(G_{K'})$ and that, if $b \in \mathrm{Fil}^{-d}B_e$, for some $d \in \mathbb{N}$ and $\lambda \in F^n_{K'}\Lambda$, then $b\lambda \in F^{n-d}_{K'}\Lambda$.

For each $n \in \mathbb{Z}$, denote by $F^n\Lambda$ the smallest sub-$\mathbb{Q}_p$-vector space of Λ containing $F^n_{K'}\Lambda$ and stable under G_K. This is also the image of the obvious map

$$\mathbb{Q}_p[G_K] \otimes_{\mathbb{Q}_p[G_{K'}]} F^n_{K'}\Lambda \to \Lambda.$$

If $h_1, h_2, \dots, h_m$ is a system of representatives of $G_K/G_{K'}$ in G_K, this is also $\sum_{i=1}^m h_i(F^n\Lambda_{K'}) \subset \Lambda$ which is still bounded and it is clear that the $(F^n\Lambda)_{n\in\mathbb{Z}}$ satisfy the required properties. □

Remark 3.13. We see immediately that $\mathrm{Rep}_{B_e}(G_K)$ is a weak Serre subcategory of $\widehat{\mathcal{C}}(G_K)$.

3K. ***Cohomology of coherent $\mathcal{O}_X$-modules.*** We denote by $\overline{B}_{dR}$ the B_e-module B_{dR}/B_e. It is not of finite type but, as the cokernel of the inclusion $B_e \to B_{dR}$ which is a morphism of $\widehat{\mathcal{C}}(G_K)$, it can be viewed as an object of this category. The equalities $B_{dR} = B_e + B_{dR}^+$ and $\mathbb{Q}_p = B_e \cap B_{dR}^+$ imply that $\overline{B}_{dR}$, as an object of $\widehat{\mathcal{C}}(G_K)$, can also be identified with $B_{dR}^+/\mathbb{Q}_p$.

If $\mathcal{F} = (\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_{\mathcal{F}}) \in \mathrm{Coh}(\mathcal{O}_X)$. The map

$$\mathcal{F}_e \to \mathcal{F}_{dR} = B_{dR} \otimes_{B_e} \mathcal{F}_e, \quad x \mapsto 1 \otimes x$$

is injective, we use it to identify $\mathcal{F}_e$ to a sub-B_e-module of $\mathcal{F}_{dR}$ and we denote by $\overline{\mathcal{F}}_{dR}$ the quotient $\mathcal{F}_{dR}/\mathcal{F}_e$.

Proposition 3.14 [Fargues and Fontaine 2018, proposition 8.2.3]. *For any $\mathcal{F} \in \mathrm{Coh}(\mathcal{O}_X)$, we have $H^i(X, \mathcal{F}) = 0$ for $i \notin \{0, 1\}$ and*

$$\begin{aligned} \mathcal{F}(X) = H^0(X, \mathcal{F}) \neq 0 &\iff \mathcal{F}^{\geq 0} \neq 0, \\ H^1(X, \mathcal{F}) \neq 0 \qquad &\iff \mathcal{F}^{<0} \neq 0. \end{aligned}$$

Moreover, there is a canonical exact sequence of $\mathbb{Q}_p$-vector spaces

$$0 \to H^0(X, \mathcal{F}) \to \mathcal{F}_e \oplus \mathcal{F}_{dR}^+ \xrightarrow{d_{\mathcal{F}}} \mathcal{F}_{dR} \to H^1(X, \mathcal{F}) \to 0 \tag{1}$$

(where $d_{\mathcal{F}}(x, y) = \iota_{\mathcal{F}}(y) - x$) which is functorial in $\mathcal{F}$.

We have a commutative diagram of $\mathbb{Q}_p$-vector spaces

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{F}_e & = & \mathcal{F}_e & & \\
 & & \downarrow & & \downarrow & & \\
0 \longrightarrow & H^0(X, \mathcal{F}) \longrightarrow & \mathcal{F}_e \oplus \mathcal{F}_{dR}^+ & \longrightarrow & \mathcal{F}_{dR} H^1(X, \mathcal{F}) & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{F}_{dR}^+ & \longrightarrow & \overline{\mathcal{F}}_{dR} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

whose columns and the two first lines are exact. Hence we have also an exact sequence

$$0 \to H^0(X, \mathcal{F}) \to \mathcal{F}_{dR}^+ \xrightarrow{\overline{d}_{\mathcal{F}}} \overline{\mathcal{F}}_{dR} \to H^1(X, \mathcal{F}) \to 0 \tag{2}$$

where $\overline{d}_{\mathcal{F}}(y)$ is the image of $\iota_{\mathcal{F}}(y)$ in $\overline{\mathcal{F}}_{dR}$.

3L. ***Cohomology of coherent $\mathcal{O}_X[G_K]$-modules.*** We say that an almost $\mathbb{C}_p$-representation is *effective* if this object of $\mathcal{C}(G_K)$ is isomorphic to a sub-object of $\mathcal{C}^{\infty}(G_K)$. We denote by $\mathcal{C}^{\geq 0}(G_K)$ the full subcategory of $\mathcal{C}(G_K)$ whose objects are those which are effective.

Proposition 3.15. *Let $f : W \to V$ a morphism of $\mathcal{C}(G_K)$ with $W \in \mathcal{C}^{\infty}(G_K)$ and $V \in \mathcal{C}^{\geq 0}(G_K)$. Then the kernel of f belongs to $\mathcal{C}^{\infty}(G_K)$.*

Proof. By assumption, there exists a monomorphism $g : V \to W'$ in $\mathcal{C}(G_K)$ with $W' \in \mathcal{C}^{\infty}(G_K)$. The kernel of f is the same as the kernel of $gf : W \to W'$. As W and W' are in $\mathcal{C}^{\infty}(G_K)$, so is this kernel. □

Proposition 3.16. *Let $\mathcal{F} \in \mathcal{M}(G_K)$. Then $H^0(X, \mathcal{F}) \in \mathcal{C}^{\geq 0}(G_K)$.*

Proof. We see that $\mathcal{F}_e$, $\mathcal{F}_{dR}^+$ and $\mathcal{F}_{dR}$ can be viewed as objects of the abelian category $\widehat{\mathcal{C}}(G_K)$. The inclusion $\mathcal{F}_e \hookrightarrow \mathcal{F}_{dR}$ is a morphism of this category, hence $\overline{\mathcal{F}}_{dR}$ can also viewed as an object of $\widehat{\mathcal{C}}(G_K)$. The map $\overline{d}_{\mathcal{F}}$ of the exact sequence (2) is obviously a morphism of this category, hence

$$H^0(X, \mathcal{F}) = \ker \overline{d}_{\mathcal{F}} \quad \text{and} \quad H^1(X, \mathcal{F}) = \operatorname{coker} \overline{d}_{\mathcal{F}}$$

are objects of $\widehat{\mathcal{C}}(G_K)$.

For $m \in \mathbb{N}$, big enough, $\mathcal{F}(-m)_{HN}$ has all its HN-slopes strictly negative and $H^0(X, \mathcal{F}(-m)_{HN}) = 0$. But this is the kernel of the map

$$\mathcal{F}_{dR}^+(m) \to \overline{\mathcal{F}}_{dR}, \quad b \otimes t^m \mapsto t^m b \ (\text{mod } \mathcal{F}_e).$$

Hence we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathcal{F}_{dR}^+(m) & \longrightarrow & \overline{\mathcal{F}}_{dR} \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & H^0(X, \mathcal{F}) & \longrightarrow & \mathcal{F}_{dR}^+ & \longrightarrow & \overline{\mathcal{F}}_{dR}
\end{array}$$

(the first nonzero vertical arrow sends $b \otimes t^m$ to $t^m b$) whose lines are exact. Therefore, the compositum $H^0(X, \mathcal{F}) \to \mathcal{F}_{dR}^+ \to \mathcal{F}_{dR}^+ / t^m \mathcal{F}_{dR}^+$ is injective and $H^0(X, \mathcal{F})$, subobject in $\widehat{\mathcal{C}}(G_K)$ of $\mathcal{F}_{dR}^+ / t^m . \mathcal{F}_{dR}^+ \in \mathcal{C}^{\infty}(G_K)$ is in $\mathcal{C}^{\geq 0}(G_K)$. □

4. Hulls and construction of the functor $V \mapsto \mathcal{F}_V$

4A. ***Generalities.*** In what follows, $B_?$ is either B_e, B_{dR}^+ or B_{dR}.

We know (Remarks 3.13 and 2.13) that $\mathrm{Rep}_{B_?}(G_K)$ can be identified with a weak Serre subcategory of $\widehat{\mathcal{C}}(G_K)$. We have "inclusions" of weak Serre subcategories

$$\begin{array}{ccc} \mathrm{Rep}_{B_{dR}^+}(G_K) & & \\ & \searrow & \\ & & \mathrm{Rep}_{B_{dR}}(G_K) \longrightarrow \widehat{\mathcal{C}}(G_K) \\ & \nearrow & \\ \mathrm{Rep}_{B_e}(G_K) & & \end{array}$$

Let V be an almost $\mathbb{C}_p$-representation. We say that V *has a $B_?$-hull* if the functor

$$\mathrm{Rep}_{B_?}(G_K) \to \mathbb{Q}_p\text{-vector spaces}, \quad W \mapsto \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, W)$$

is representable, i.e., if there is a (necessarily unique up to unique isomorphism) pair $(V_?, \iota_?^V)$, with $V_?$ a $B_?$-representation and $\iota_?^V : V \to V_?$ a G_K-equivariant continuous $\mathbb{Q}_p$-linear map, such that, for all $B_?$-representation W, the map

$$\mathrm{Hom}_{\mathrm{Rep}_{B_?}(G_K)}(V_?, W) \to \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, W),$$

induced by $\iota_?^V$, is bijective.

When it is the case, we call $(V_?, \iota_?^V)$, or abusively $V_?$, the *$B_?$-hull of V*.

Our purpose is to show that such an hull always exists and to use these hulls to construct a functor

$$\mathcal{C}(G_K) \to \mathcal{M}(G_K), \quad V \mapsto \mathcal{F}_V.$$

Remark 4.1. Let V be an almost $\mathbb{C}_p$-representation and let I_V the class of morphisms

$$\iota : V \to W_\iota$$

of $\widehat{\mathcal{C}}(G_K)$ whose source is V and target a $B_?$-representation. With suitable conventions and abuses, to say that V has a $B_?$-hull means that the directed inverse limit

$$V_? = \varprojlim_{\iota \in I_V} W_\iota$$

exists and that the $B_?$-module underlying this "pro-$B_?$-representation of G_K" is of finite type.

Restricted to the full subcategory of $\mathcal{C}(G_K)$ of almost $\mathbb{C}_p$-representations admitting a $B_?$-hull, the correspondence $V \mapsto V_?$ is obviously functorial.

Let $V \in \mathcal{C}(G_K)$ such that, with obvious notations, $(V_{dR}^+, \iota_{dR}^{V,+})$ exists, let $M \in \mathrm{Rep}_{B_{dR}}(G_K)$ and $f : V \to M$ a morphism in $\widehat{\mathcal{C}}(G_K)$. We see that the sub B_{dR}^+-module W of M generated by $f(V)$ is an object of $\mathcal{C}^\infty(G_K)$, hence there is a unique morphism (in $\widehat{\mathcal{C}}(G_K)$ or, in this case, equivalently in $\mathcal{C}^\infty(G_K)$)

$$g : V_{dR}^+ \to W \subset M$$

such that $f = g \circ \iota_{dR}^{V,+}$ and we have

$$\mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, M) = \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V_{dR}^+, M) =$$
$$\mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(B_{dR} \otimes_{B_{dR}^+} V_{dR}^+, M) = \mathrm{Hom}_{\mathrm{Rep}_{B_{dR}(G_K)}}(B_{dR} \otimes_{B_{dR}^+} V_{dR}^+, M).$$

Therefore V_{dR} exists and can be identified with $B_{dR} \otimes_{B_{dR}^+} V_{dR}^+$.

The same argument applies to the case where (V_e, ι_e^V) exists. Hence we have:

Proposition 4.2. *Let $V \in \mathcal{C}(G_K)$.*

(i) *If V_{dR}^+ exists, V_{dR} exists and is, canonically and functorially, $B_{dR} \otimes_{B_{dR}^+} V_{dR}^+$.*

(ii) *If V_e exists, V_{dR} exists and is, canonically and functorially, $B_{dR} \otimes_{B_e} V_e$.*

Proposition 4.3. *Let $B_?$ as above and let V be an almost $\mathbb{C}_p$-representation of G_K which has a $B_?$-hull $(V_?, \iota_?^V)$.*

(i) *The image of $\iota_?^V$ generates $V_?$ as a $B_?$-module.*

(ii) *If moreover*

$$0 \to V' \to V \to V'' \to 0$$

is a short exact sequence in $\mathcal{C}(G_K)$, then V'' has a $B_?$-hull which is the quotient of $V_?$ by the sub-$B_?$-module of $V_?$ generated by the image of V'.

(iii) *In this situation, if V' has a $B_?$-hull, then the sequence*

$$V'_? \to V_? \to V''_? \to 0$$

is exact.

Proof. (i) Let W_0 be the sub-$B_?$-module of $V_?$ generated by the image of V. As $B_?$ is noetherian, this is a $B_?$-module of finite type. By the universal property of $V_?$, there is a unique morphism $\nu : V_? \to W_0$ such that the map $V \to W_0$ is $\nu \circ \iota_?^V$ and we see that $V_? = W_0 \oplus \ker \nu$. The fact that $\mathrm{id}_{V_?}$ is the unique endomorphism of $V_?$ such that $\nu \circ \iota_?^V = \iota_?^V$ forces $\ker \nu$ to be 0.

(ii) If W is any $B_?$-representation, we have

$$\begin{aligned}\mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V'', W) &= \left\{ f \in \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, W) \mid f(V') = 0 \right\} \\ &= \left\{ f \in \mathrm{Hom}_{\mathrm{Rep}_{B_?}(G_K)}(V_?, W) \mid f(\iota_?^V(V')) = 0 \right\} \\ &= \mathrm{Hom}_{\mathrm{Rep}_{B_?}(G_K)}(V_?/B_?\iota_?^V(V'), W).\end{aligned}$$

(iii) Let N be the kernel of the projection $V_? \to V''_?$. The image of $V'_?$ in $V_?$ is clearly contained in N. As N is the sub-$B_?$-module generated by the image of V', the map $V'_? \to N$ is surjective and

$$V'_? \to V_? \to V''_? \to 0$$

is exact. □

4B. *Construction of trivialisable almost $\mathbb{C}_p$-representations.* A *trivialisation* of an almost $\mathbb{C}_p$-representation V is a short exact sequence

$$0 \to U \to V \to W \to 0$$

in $\mathcal{C}(G_K)$ with $U \in \mathcal{C}^0(G_K)$ and $W \in \mathcal{C}^\infty(G_K)$.

An almost $\mathbb{C}_p$-representation is *trivialisable* if it admits a trivialisation.

If $V \in \mathcal{C}(G_K)$, if $\tilde{f} : V/U_+ \to W/U_-$ is an almost isomorphism with $W \in \mathcal{C}(G_K)$ and if $\widehat{V} = W \times_{W/U_-} V$, we have, in $\mathcal{C}(G_K)$, a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U_+ & = & U_+ & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_- & \longrightarrow & \widehat{V} & \longrightarrow & V & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_- & \longrightarrow & W & \longrightarrow & V/U_+ & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and V is a quotient of $\widehat{V}$ which is trivialisable as it is an extension of W by $U_+ \in \mathcal{C}^0(G_K)$.

Given $U \in \mathcal{C}^0(G_K)$ and $W \in \mathcal{C}^\infty(G_K)$, it is easy to construct all almost $\mathbb{C}_p$-representations which are extensions of W by U:

Recall that

$$B_{dR} = B_e + B_{dR}^+ \quad \text{and} \quad B_e \cap B_{dR}^+ = \mathbb{Q}_p$$

and that we set

$$\widetilde{B}_{dR} = B_{dR}/B_{dR}^+ = B_e/\mathbb{Q}_p.$$

Let U be an object of $\mathcal{C}^0(G_K)$ and W an object of $\mathcal{C}^\infty(G_K)$. Tensoring the exact sequence

$$0 \to \mathbb{Q}_p \to B_e \to \widetilde{B}_{dR} \to 0$$

by U we get a short exact sequence in $\widehat{\mathcal{C}}(G_K)$

$$0 \to U \to B_e \otimes_{\mathbb{Q}_p} U \to \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U \to 0$$

inducing a map

$$\begin{array}{ccc}
\delta_{U,W} : \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(W, \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U) & \longrightarrow & \mathrm{Ext}^1_{\widehat{\mathcal{C}}(G_K)}(W, U) \\
\| & & \| \\
\mathrm{Hom}_{\widehat{\mathcal{C}}^\infty(G_K)}(W, \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U) & & \mathrm{Ext}^1_{\mathcal{C}(G_K)}(W, U)
\end{array}$$

Proposition 4.4 [Fontaine 2003, proposition 3.7]. *Let* $U \in \mathcal{C}^0(G_K)$ *and* $W \in \mathcal{C}^\infty(G_K)$. *The map*

$$\delta_{U,W} : \mathrm{Hom}_{\widehat{\mathcal{C}}^\infty(G_K)}(W, \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U) \to \mathrm{Ext}^1_{\mathcal{C}(G_K)}(W, U)$$

is an isomorphism.

Hence if V is a trivialisable almost $\mathbb{C}_p$-representation and if

$$(T) \qquad 0 \to U \to V \to W_0 \to 0$$

is a trivialisation of V, there is a unique

$$\rho_T \in \mathrm{Hom}_{\widehat{\mathcal{C}}^\infty(G_K)}(W_0, \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U)$$

such that the square

$$\begin{array}{ccc} V & \longrightarrow & W_0 \\ \downarrow & & \downarrow \rho_T \\ B_e \otimes_{\mathbb{Q}_p} U & \longrightarrow & \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U \end{array}$$

is cartesian.

4C. *Construction of the hulls.*

Proposition 4.5. *Any almost* $\mathbb{C}_p$*-representation* V *has a* B_e*-hull* V_e, *a* B^+_{dR}*-hull* V^+_{dR} *and a* B_{dR}*-hull* V_{dR}. *We have*

$$V_{dR} = B_{dR} \otimes_{B_e} V_e = B_{dR} \otimes_{B^+_{dR}} V^+_{dR},$$
$$\mathrm{rank}_{B^+_{dR}} V^+_{dR} = \mathrm{rank}_{B_e} V_e = \dim_{B_{dR}} V_{dR} \geq h(V)$$

and equality holds when V *is trivialisable.*

Moreover:

(i) *If* $U \in \mathcal{C}^0(G_K)$, *then* $U_e = B_e \otimes_{\mathbb{Q}_p} U$ *and* $U^+_{dR} = B^+_{dR} \otimes_{\mathbb{Q}_p} U$,

(ii) *If* $W \in \mathcal{C}^\infty(G_K)$, *then* $W_e = 0$ *and* $W^+_{dR} = W$,

(iii) *If*

$$(T) \qquad 0 \to U \to V \to W_0 \to 0$$

is a trivialisation of an almost $\mathbb{C}_p$*-representation* V, *then*

(a) *the map* $U_e = B_e \otimes_{\mathbb{Q}_p} U \to V_e$ *is an isomorphism, and*

(b) *we have a short exact sequence*

$$0 \to B^+_{dR} \otimes_{\mathbb{Q}_p} U \to V^+_{dR} \to W_0 \to 0$$

More precisely, V^+_{dR} *is the fiber product* $(B_{dR} \otimes_{\mathbb{Q}_p} U) \times_{\widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U} W_0$ (*where* $W_0 \to \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U$ *is the map* ρ_T).

Proof. From Proposition 4.2, we see that the existence of V_e and V_{dR}^+ implies the existence of V_{dR} and the equalities:

$$V_{dR} = B_{dR} \otimes_{B_e} V_e = B_{dR} \otimes_{B_{dR}^+} V_{dR}^+,$$
$$\mathrm{rank}_{\mathrm{B_{dR}^+}} \mathrm{V_{dR}^+} = \mathrm{rank}_{\mathrm{B_e}} \mathrm{V_e} = \dim_{\mathrm{B_{dR}}} \mathrm{V_{dR}}.$$

(i) Let $U \in \mathcal{C}^0(G_K)$. By adjunction, for any B_e-representation Λ, we have

$$\mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(U, \Lambda) = \mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_K)}(B_e \otimes_{\mathbb{Q}_p} U, \Lambda)$$

hence U_e exists and is $B_e \otimes_{\mathbb{Q}_p} U$. Similarly, for any object $W_0 \in \mathcal{C}^\infty(G_K)$, we have

$$\mathrm{Hom}_{\mathcal{C}(G_K)}(U, W_0) = \mathrm{Hom}_{\widehat{\mathcal{C}}^\infty(G_K)}(B_{dR}^+ \otimes_{\mathbb{Q}_p} U, W_0)$$

hence U_{dR}^+ exists and is $B_{dR}^+ \otimes_{\mathbb{Q}_p} U$. In particular, $\dim_{B_{dR}} U_{dR} = h(U)$.

(ii) Let $W \in \mathcal{C}^\infty(G_K)$. For all B_e-representation Λ, we have $\mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(W, \Lambda) = 0$ (Proposition 3.4). Therefore W_e exists and is $= 0$. For any $W_0 \in \mathcal{C}^\infty(G_K)$, we have $\mathrm{Hom}_{\mathcal{C}(G_K)}(W, W_0) = \mathrm{Hom}_{\mathcal{C}^\infty(G_K)}(W, W_0)$ (Proposition 2.6) hence W_{dR}^+ exists and is W. In particular $\dim_{B_{dR}} W_{dR} = 0 = h(W)$.

(iii) Let V a trivialisable almost $\mathbb{C}_p$-representation and

$$(T) \qquad 0 \to U \to V \to W_0 \to 0$$

a trivialisation.

(a) Let Λ be a B_e-representation. The inclusion $U \to V$ induces a map

$$\alpha : \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, \Lambda) \to \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(U, \Lambda) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Rep}_{B_e}(G_K)}(B_e \otimes_{\mathbb{Q}_p} U, \Lambda) = \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(B_e \otimes_{\mathbb{Q}_p} U, \Lambda)$$

(Propositions 3.11 and 3.12). We have a cartesian square (Section 4B)

$$(S) \qquad \begin{array}{ccc} V & \longrightarrow & W_0 \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho_T} \\ B_e \otimes_{\mathbb{Q}_p} U & \longrightarrow & \widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U \end{array}$$

and we may use ρ to get a map

$$\beta : \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(B_e \otimes_{\mathbb{Q}_p} U, \Lambda) \to \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, \Lambda)$$

Let $f \in \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(B_e \otimes_{\mathbb{Q}_p} U, \Lambda)$ and $f' = \alpha(\beta(f))$. If $\sum b_i \otimes u_i \in B_e \otimes_{\mathbb{Q}_p} U$, we have

$$f'\big(\sum b_i \otimes u_i\big) = \sum b_i(\beta(f)(u_i)) = \sum b_i f(u_i) = f\big(\sum b_i \otimes u_i\big)$$

as f is B_e-linear, hence $f' = f$.

Let $g \in \mathrm{Hom}_{\widehat{\mathcal{C}}(G_K)}(V, \Lambda)$ and $g' = \alpha(\beta(g))$. If $u \in U$, as $\rho(u) = u$, we have

$$g'(u) = \beta(\alpha(g))(u) = \alpha(g)(u) = g(u)$$

Hence $g' - g$ factors through a morphism in $\widehat{\mathcal{C}}(G_K)$

$$W_0 \to \Lambda$$

which is necessarily 0 (Theorem 2.9), hence $g' = g$. Therefore we see that α is an isomorphism. It is implies that V_e exists and is equal to $U_e = B_e \otimes_{\mathbb{Q}_p} U$.

(b) We want to show that V_{dR}^+ exists and is equal to

$$W_1 = (B_{dR} \otimes_{\mathbb{Q}_p} U) \times_{\widetilde{B}_{dR} \otimes_{\mathbb{Q}_p} U} W_0.$$

Using the cartesian square (S) and the inclusion $B_e \otimes_{\mathbb{Q}_p} U \subset B_{dR} \otimes_{\mathbb{Q}_p} U$, we get a morphism of $\widehat{\mathcal{C}}(G_K)$

$$V \to W_1$$

and we have a commutative diagram in $\widehat{\mathcal{C}}(G_K)$

$$(*) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & U_{dR}^+ & \longrightarrow & W_1 & \longrightarrow & W_0 & \longrightarrow & 0 \end{array}$$

whose lines are exact.

If W is any B_{dR}^+-representation, we have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \mathrm{Hom}(W_0, W) & \longrightarrow & \mathrm{Hom}(V, W) & \longrightarrow & \mathrm{Hom}(U, W) & \longrightarrow & \mathrm{Ext}^1(W_0, W) \\ \| & & \downarrow & & \downarrow\simeq & & \| \\ 0 \longrightarrow \mathrm{Hom}(W_0, W) & \longrightarrow & \mathrm{Hom}(W_1, W) & \longrightarrow & \mathrm{Hom}(U_{dR}^+, W) & \longrightarrow & \mathrm{Ext}^1(W_0, W) \end{array}$$

(where all the Hom and Ext^1 are computed in $\widehat{\mathcal{C}}(G_K)$) which implies that

$$\mathrm{Hom}_{\mathcal{C}(G_K)}(V, W) \to \mathrm{Hom}_{\mathcal{C}(G_K)}(W_1, W) = \mathrm{Hom}_{\mathcal{C}^\infty(G_K)}(W_1, W)$$

is an isomorphism. Hence V_{dR}^+ exists and is equal to W_1.

Finally, let V be any object of $\mathcal{C}(G_K)$. We can find an exact sequence

$$0 \to U \to \widehat{V} \to V \to 0$$

with $\widehat{V}$ trivialisable. The existence of $\widehat{V}_e$ and $\widehat{V}_{dR}^+$ implies (Proposition 4.3) the existence of V_e and V_{dR}^+. The exactness of the sequence

$$U_{dR} \to \widehat{V}_{dR} \to V_{dR} \to 0$$

implies that

$$\dim_{B_{dR}} V_{dR} \geq \dim_{B_{dR}} \widehat{V}_{dR} - \dim_{B_{dR}} U_{dR} = h(\widehat{V}) - h(U) = h(V). \qquad \square$$

4D. ***The functor $V \mapsto \mathcal{F}_V$.*** For any almost $\mathbb{C}_p$-representation V, denote

$$\iota_V : V_{dR}^+ \to V_{dR} = B_{dR} \otimes_{B_e} V_e$$

the natural map. It induces an isomorphism $B_{dR} \otimes_{B_{dR}^+} V_{dR}^+ \to V_{dR}$. Therefore

$$\mathcal{F}_V = (V_{dR}^+, V_e, \iota_V)$$

is a coherent $\mathcal{O}_X[G_K]$-module. This construction is clearly functorial and we get an additive functor

$$\mathcal{C}(G_K) \to \mathcal{M}(G_K), \quad V \mapsto \mathcal{F}_V.$$

From the universal properties of the functor $V \mapsto V_{dR}^+$ and $V \mapsto V_e$, we deduce the fact that $V \mapsto \mathcal{F}_V$ is left adjoint to $\mathcal{F} \mapsto \mathcal{F}(X)$.

5. The equivalence $\mathcal{M}^{\geq 0}(G_K) \to \mathcal{C}^{\geq 0}(G_K)$

5A. ***A characterisation of effective coherent $\mathcal{O}_X[G_K]$-modules.***

Theorem 5.1. *The category $\mathcal{M}^{\geq 0}(G_K)$ is the smallest strictly full subcategory of $\mathcal{M}(G_K)$ containing $\mathcal{M}^0(G_K)$ and $\mathcal{M}^\infty(G_K)$ and stable under taking extensions and direct summands.*

Lemma 5.2. *Let s be a positive rational number. There exists $\mathcal{G}_s \in \mathcal{M}^s(G_K)$ which is an extension of an object of $\mathcal{M}^\infty(G_K)$ by an object of $\mathcal{M}^0(G_K)$.*

Proof of the theorem given the lemma. As a subcategory of $\mathcal{M}(G_K)$, the category $\mathcal{M}^{\geq 0}(G_K)$ is obviously stable under taking extensions and direct summands. Hence, it suffices to show that any $\mathcal{F} \in \mathcal{M}^{\geq 0}(G_K)$ can be written as a direct summand of successive extensions of direct summands of objects which are extensions of an object of $\mathcal{M}^\infty(G_K)$ by an object of $\mathcal{M}^0(G_K)$. Using the Harder–Narasimhan filtration, it is enough to show that, if $\mathcal{F}$ is semistable of slope $s \geq 0$, then $\mathcal{F}$ is such a direct summand.

If $s = 0$, then $\mathcal{F} \in \mathcal{M}^0(G_K)$ and, if $s = +\infty$, then $\mathcal{F} \in \mathcal{M}^\infty(G_K)$ and we may assume that s is a positive rational number.

Let $\mathcal{G}_s$ as in the lemma, so that we have a short exact sequence

$$0 \to \mathcal{G}_s^0 \to \mathcal{G}_s \to \mathcal{G}_s^\infty \to 0$$

with $\mathcal{G}_s^0 \in \mathcal{M}^0(G_K)$ ans $\mathcal{G}_s^\infty \in \mathcal{M}^\infty(G_K)$. As $\mathcal{G}_s$ is a vector bundle (it has no torsion), its dual $\mathcal{G}_s^\vee$ is well defined and semistable of slope $-s$. Therefore

$$\mathcal{F}_0 = \mathcal{F} \otimes \mathcal{G}_s^\vee$$

is semistable of slope 0. We have a short exact sequence

$$0 \to \mathcal{F}_0 \otimes \mathcal{G}_s^0 \to \mathcal{F}_0 \otimes \mathcal{G}_s \to \mathcal{F}_0 \otimes \mathcal{G}_s^\infty \to 0$$

and $\mathcal{F}_0 \otimes \mathcal{G}_s$ is an extension of $\mathcal{F}^0 \otimes \mathcal{G}_s^\infty \in \mathcal{M}^\infty(G_K)$ by $\mathcal{F}^0 \otimes \mathcal{G}_s^0 \in \mathcal{M}^0(G_K)$.

But, with obvious notations,

$$\mathcal{F}_0 \otimes \mathcal{G}_s = \mathcal{F} \otimes \mathcal{G}_s^\vee \otimes \mathcal{G}_s = \mathcal{F} \otimes \mathrm{End}(\mathcal{G}_s).$$

If $\mathrm{End}^0(\mathcal{G}_s)$ is the subsheaf of elements of trace 0 in $\mathrm{End}(\mathcal{G}_s)$, we have

$$\mathrm{End}(\mathcal{G}_s) = \mathcal{O}_X \oplus \mathrm{End}^0(\mathcal{G}_s)$$

hence

$$\mathcal{F}_0 \otimes \mathcal{G}_s = \mathcal{F} \otimes (\mathcal{O}_X \oplus \mathrm{End}^0(\mathcal{G}_s)) = \mathcal{F} \oplus (\mathcal{F} \otimes \mathrm{End}^0(G_s))$$

and $\mathcal{F}$ is a direct summand of $\mathcal{F}_0 \otimes \mathcal{G}_s$. □

Proof of the lemma. We may assume $K = \mathbb{Q}_p$. Recall the following facts ([Fargues and Fontaine 2018, proposition 10.5.3]; see also [Colmez and Fontaine 2000, §5]):

- A *filtered φ-module over* $\mathbb{Q}_p$ is a pair (D, Fil) consisting of
  (a) a *φ-module over* $\mathbb{Q}_p$, i.e., a finite-dimensional $\mathbb{Q}_p$-vector space D equipped with an automorphism $\varphi : D \to D$,
  (b) an exhausted and separated decreasing filtration $(\mathrm{Fil}^n D)_{n\in\mathbb{Z}}$.
  (i) There is a fully faithful additive functor

$$(D, \mathrm{Fil}) \mapsto \mathcal{F}_{D,\mathrm{Fil}}$$

from the category of filtered φ-modules over $\mathbb{Q}_p$ to the category of $G_{\mathbb{Q}_p}$-equivariant vector bundles over X (the essential image consists of those equivariant vector bundles which are *crystalline*, i.e., those $\mathcal{F}$'s such that the natural map

$$B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \left(B_{\mathrm{cris}} \otimes_{B_e} \mathcal{F}_e\right)^{G_K} \to B_{\mathrm{cris}} \otimes_{B_e} \mathcal{F}_e$$

is bijective): we have $\mathcal{F}_{D,\mathrm{Fil}} = (\mathcal{F}_{D,\mathrm{Fil},e}, \mathcal{F}^+_{D,\mathrm{Fil},dR})$ where
– $\mathcal{F}_{D,\mathrm{Fil},e}$ is the B_e-module $(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} D)_{\varphi=1}$ which implies that

$$\mathcal{F}_{D,\mathrm{Fil},dR} = B_{dR} \otimes_{B_e} \mathcal{F}_{D,e} = B_{dR} \otimes_{\mathbb{Q}_p} D,$$

– $\mathcal{F}^+_{D,\mathrm{Fil},dR} = \mathrm{Fil}^0(B_{dR} \otimes_{\mathbb{Q}_p} D) = \sum_{n\in\mathbb{Z}} \mathrm{Fil}^{-n} B_{dR} \otimes \mathrm{Fil}^n D.$

Set $s = d/h$ with d, h positive integers, prime together.

Consider the φ-module D over $\mathbb{Q}_p$ whose underlying $\mathbb{Q}_p$-vector space is of dimension h, with $(e_r)_{r\in\mathbb{Z}/h\mathbb{Z}}$ as a basis and

$$\varphi(e_r) = \begin{cases} e_{r+1} & \text{if } r+1 \neq 0, \\ p^{-d} e_0 & \text{if } r+1 = 0. \end{cases}$$

We equip D with two distinct filtrations Fil and Fil_0:

$$\mathrm{Fil}^n D = \begin{cases} D & \text{if } n \le 0, \\ 0 & \text{if } n > 0, \end{cases} \qquad \mathrm{Fil}_0^n D = \begin{cases} D & \text{if } n \le -d, \\ \bigoplus_{r \neq 0} \mathbb{Q}_p e_r & \text{if } -d < n \le 0, \\ 0 & \text{if } n > d. \end{cases}$$

Set $\mathcal{G}_s = \mathcal{F}_{D,\mathrm{Fil}}$ and $\mathcal{G}_s^0 = \mathcal{F}_{D,\mathrm{Fil}_0}$. Both are coherent $\mathcal{O}_X[G_K]$-module of rank h. As the polynomial $X^h - p^{-d}$ is irreducible over $\mathbb{Q}_p$, the $\mathbb{Q}_p[\varphi]$-module D is irreducible which implies that $\mathcal{G}_s$ and $\mathcal{G}_s^0$ are stable, hence semistable. An easy computation shows that $\deg(\mathcal{G}_s) = d$ and $\deg(\mathcal{G}_s^0) = 0$, hence $\mathcal{G}_s$ is semistable of slope $d/h = s$ and $\mathcal{G}_s^0$ is semistable of slope 0, hence belongs to $\mathcal{M}^0(G_K)$. We see that $\mathcal{G}_{s,e}^0 = \mathcal{G}_{s,e}$ and that $(\mathcal{G}_s^0)_{dR}^+ \subset (\mathcal{G}_s)_{dR}$. Therefore $\mathcal{G}_s^0$ is a subobject of $\mathcal{G}_s$ and the cokernel $\mathcal{G}_s^\infty$ is torsion, and so belongs to $\mathcal{M}^\infty(G_K)$. □

5B. ***Some properties of effective almost $\mathbb{C}_p$-representations.*** Recall (Section 1E) that an exact subcategory of an abelian category is a strictly full subcategory containing 0 and stable under extensions. For instance the previous theorem shows that $\mathcal{M}^{\ge 0}(G_K)$ is an exact subcategory of $\mathcal{M}(G_K)$.

Theorem 5.3. *Let $V \in \mathcal{C}(G_K)$. The following conditions are equivalent*:

(i) *V is effective (i.e., $V \in \mathcal{C}^{\ge 0}(G_K)$).*

(ii) *There is a finite extension K' of K contained in $\overline{Q}_p$ such that V, as an object of $\mathcal{C}(G_{K'})$ is a successive extension of objects belonging either to $\mathcal{C}^0(G_{K'})$ or to $\mathcal{C}^\infty(G_{K'})$.*

(iii) *V belongs to the smallest strictly full subcategory of $\mathcal{C}(G_K)$ containing $\mathcal{C}^0(G_K)$ and $\mathcal{C}^\infty(G_K)$ and stable under taking extensions and direct summands.*

Moreover $\mathcal{C}^{\ge 0}(G_K)$ is an exact subcategory of $\mathcal{C}(G_K)$.

Before proving this theorem, let's state an other result. Recall (Section 4D) that, to any $V \in \mathcal{C}(G_K)$, we associated the coherent $\mathcal{O}_X[G_K]$-module

$$\mathcal{F}_V = (V_{dR}^+, V_e, \iota_V).$$

We have

$$(\mathcal{F}_V)_{dR}^+ = V_{dR}^+, \quad (\mathcal{F}_V)_e = V_e, \quad \iota_{\mathcal{F}_V} = \iota_V.$$

Therefore, if we set $\overline{V}_{dR} = \overline{\mathcal{F}_V}_{dR} = V_{dR}/V_e$, we have (cf. Section 3L) an exact sequence

$$(C) \qquad 0 \to H^0(X, \mathcal{F}_V) \to V_{dR}^+ \xrightarrow{\bar{\iota}_V} \overline{V}_{dR} \to H^1(X, \mathcal{F}_V) \to 0$$

(where $\bar{\iota}_V = \bar{\iota}_{\mathcal{F}_V}$ is the compositum of ι_V with the projection $V_{dR} \to V_{dR}/V_e$) and, as $V \subset V_e$ is injective, the image of V in V_{dR}^+ is contained in $\mathcal{F}_V(X) = H^0(X, \mathcal{F}_V)$.

Proposition 5.4. *Let* $V \in \mathcal{C}^{\geq 0}(G_K)$.

(i) *We have* $h(V) \geq 0$ *and* $\dim_{B_{dR}} V_{dR} = h(V)$.

(ii) *We have* $V \in \mathcal{C}^{\infty}(G_K) \Longleftrightarrow h(V) = 0$,

(iii) *The sequence*

$$0 \to V \to V_{dR}^{+} \xrightarrow{\bar{\iota}_V} \overline{V}_{dR} \to 0$$

is exact.

(iv) *the map* $V \to H^0(X, \mathcal{F}_V)$ *is bijective and* $\mathcal{F}_V \in \mathcal{M}^{\geq 0}(G_K)$.

Moreover, the restriction to $\mathcal{C}^{\geq 0}(G_K)$ *of the four functors* $\mathcal{C}(G_K) \to \widehat{\mathcal{C}}(G_K)$

$$V \mapsto V_{dR}^{+}, \quad V \mapsto V_e, \quad V \mapsto V_{dR} \quad V \mapsto \overline{V}_{dR}$$

and of the functor

$$\mathcal{C}(G_K) \to \mathcal{M}(G_K), \quad V \mapsto \mathcal{F}_V$$

are exact.

Proof of the theorem and beginning of the proof of the proposition. For any $V \in \mathcal{C}^{\geq 0}(G_K)$, we denote by d_V the infimum of the $d(W)$'s for all $W \in \mathcal{C}^{\infty}(G_K)$ such that V is isomorphic to a subobject of W (note that $d(V) \leq d_V$).

Denote by $\mathcal{K}$ the set of finite extensions L of K contained in $\overline{\mathbb{Q}}_p$. For any $L \in \mathcal{K}$, let $\mathcal{C}^{?}(G_L)$ the full subcategory of $\mathcal{C}(G_L)$ whose objects can be written as a successive extension of objects belonging either to $\mathcal{C}^{0}(G_L)$ or to $\mathcal{C}^{\infty}(G_L)$.

We now show assertion (i) of the proposition and the implication (i)$\Longrightarrow$(ii) of the theorem, i.e., that, if $V \in \mathcal{C}^{\geq 0}(G_K)$, then

$$\dim_{B_{dR}} V_{dR} = h(V) \ \text{ (so } h(V) \geq 0\text{) and there exists } K' \in \mathcal{K} \text{ such that } V \in \mathcal{C}^{?}(G_{K'}).$$

We proceed by induction on d_V, the case $d_V = 0$ being trivial.

Let $V \subset W$ an embedding of V into an object $W \in \mathcal{C}^{\infty}(G_K)$ satisfying $d(W) = d_V > 0$. We can find (cf. Proposition 2.7) $K_1 \in \mathcal{K}$ and a G_{K_1}-stable sub-B_{dR}^{+}-module W' of W of length 1. Setting $W'' = W/W'$, $V' = V \cap W'$ and denoting V'' the image of V in W'', we get a commutative diagram in $\mathcal{C}(G_{K_1})$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & W'' & \longrightarrow & 0 \end{array}$$

whose rows are exact and vertical arrows are injective which implies that V' and V'' belong to $\mathcal{C}^{\geq 0}(G_{K_1})$. We have $d(V') \leq d(W') = 1$. From Corollary 2.10, we get that either $d(V') = 1$ in which case $V' = W'$ or $d(V') = 0$ which implies that $V' \in \mathcal{C}^{0}(G_{K_1})$.

– If $V' = W'$, we have $h(V') = 0$ and $(V')_{dR}^{+} = W'$ hence $V'_{dR} = 0$.

– If $V' \in \mathcal{C}^{0}(G_{K_1})$, we have $h(V') = \dim_{\mathbb{Q}_p} V'$ and $V'_{dR} = B_{dR} \otimes_{\mathbb{Q}_p} V'$.

In both cases, we have $\dim_{B_{dR}} V'_{dR} = h(V')$. By induction, we have $\dim_{B_{dR}} V''_{dR} = h(V'')$. The exactness of the sequence

$$V'_{dR} \to V_{dR} \to V''_{dR} \to 0$$

implies that

$$\dim_{B_{dR}} V_{dR} \le \dim_{B_{dR}} V'_{dR} + \dim_{B_{dR}} V''_{dR} = h(V') + h(V'') = h(V),$$

hence, as $\dim_{B_{dR}} V_{dR} \ge h(V)$ (Proposition 4.5), we get $\dim_{B_{dR}} V_{dR} = h(V)$, i.e the assertion (i) of the proposition.

Also by induction, as V'' belongs to $\mathcal{C}^{\ge 0}(G_{K_1})$, there is $K' \in \mathcal{K}$ containing K_1 such that $V'' \in \mathcal{C}^?(G_{K'})$. Then V, as a representation of $G_{K'}$, is an extension of V'' by either an object of $\mathcal{C}^\infty(G_{K'})$ (if $d(V') = 1$) or by an object of $\mathcal{C}^0(G_{K'})$ (if $d(V') = 0$). In both cases, V belongs to $\mathcal{C}^?(G_{K'})$.

Therefore, given $V \in \mathcal{C}^{\ge 0}(G_K)$, there is $K' \in \mathcal{K}$ and a filtration of V by subobjects in $\mathcal{C}^+(G_{K'})$

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{r-1} \subset V_r = V$$

such that, if $i = 1, 2, \ldots, r$, then V_i / V_{i-1} belongs either to $\mathcal{C}^0(G_{K'})$ or to $\mathcal{C}^\infty(G_{K'})$. This proves the implication (i)$\Longrightarrow$(ii) of the theorem.

In particular, we have $h(V) = \sum_{i=1}^r h(V_i / V_{i-1})$ which is > 0 unless $h(V_i / V_{i-1})$ vanishes for all i, which means that V_i / V_{i-1} belongs to $\mathcal{C}^\infty(G_K)$. As $\mathcal{C}^\infty(G_K)$ is stable under taking extensions, we get the equivalence

$$h(V) = 0 \Longleftrightarrow V \in \mathcal{C}^\infty(G_K)$$

which is the assertion (ii) of the proposition.

The implication (ii)$\Longrightarrow$(iii) of the theorem is obvious: If V satisfies (ii), the induced representation $\mathbb{Q}_p[G_K] \otimes_{\mathbb{Q}_p[G_{K'}]} V$ belongs to $\mathcal{C}^?(G_K)$ and V is a direct summand of this representation.

As a full subcategory of $\mathcal{C}(G_K)$, the category $\mathcal{C}^{\ge 0}(G_K)$ is obviously stable under taking direct summands. Hence, we see that the implication (iii)$\Longrightarrow$(i) of the theorem and the fact that $\mathcal{C}^{\ge 0}(G_K)$ is an exact subcategory of $\mathcal{C}(G_K)$ result from the following:

Lemma 5.5. *Assume we have a short exact sequence in $\mathcal{C}(G_K)$*

$$0 \to V_0 \to V_1 \to V_2 \to 0 \tag{1}$$

with $V_2 \in \mathcal{C}^{\ge 0}(G_K)$ and V_0 belonging either to $\mathcal{C}^0(G_K)$ or to $\mathcal{C}^\infty(G_K)$. Then $V_1 \in \mathcal{C}^{\ge 0}(G_K)$ and the sequence

$$0 \to V^+_{0,dR} \to V^+_{1,dR} \to V^+_{2,dR} \to 0$$

is exact.

Proof of the lemma. Assume first that V_0 belongs to $\mathcal{C}^0(G_K)$: we have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & V_0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & V_{0,dR}^+ & \longrightarrow & V_{1,dR}^+ & \longrightarrow & V_{2,dR}^+ & \longrightarrow 0 \end{array}$$

whose rows are exact, the maps $V_0 \to V_{0,dR}^+$ and $V_2 \to V_{2,dR}^+$ being injective. I will show that the map $V_{0,dR}^+ \to V_{1,dR}^+$ is injective. As $V_{0,dR}^+ = B_{dR}^+ \otimes_{\mathbb{Q}_p} V_0$ is a torsion free B_{dR}^+-module, it is enough to check that $V_{0,dR} \to V_{1,dR}$ is injective. If it were not true, we would have

$$\dim_{B_{dR}} V_{1,dR} < \dim_{B_{dR}} V_{0,dR} + \dim_{B_{dR}} V_{2,dR} = h(V_0) + h(V_2) = h(V_1).$$

As we have (Proposition 4.5) $\dim_{B_{dR}} V_{1,dR} \geq h(V_1)$, this can't happen. This forces $V_1 \to V_{1,dR}^+$ to be also injective, hence $V_1 \in \mathcal{C}^{\geq 0}(G_K)$.

Now assume instead that V_0 belongs to $\mathcal{C}^\infty(G_K)$. As the sequence (1) almost splits (Proposition 2.15), we can find an extension S in $\mathcal{C}^0(G_K)$ of V_2 by some $U \in \mathcal{C}^0(G_K)$ such that $V_1 = V_0 \oplus_U S$. By what we just saw, $S \in \mathcal{C}^{\geq 0}(G_K)$ and we have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & U & \longrightarrow & S & \longrightarrow & V_2 & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & U_{dR}^+ & \longrightarrow & S_{dR}^+ & \longrightarrow & V_{2,dR}^+ & \longrightarrow 0 \end{array}$$

whose line are exacts and vertical arrows are injective.

We also have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & U & \longrightarrow & W \oplus S & \longrightarrow & V_1 & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & U_{dR}^+ & \longrightarrow & W \oplus S_{dR}^+ & \longrightarrow & V_{1,dR}^+ & \longrightarrow 0 \end{array}$$

(the map $U \to W \oplus S$ send u to $(u, -u)$) whose rows are exact and the two first vertical arrows are injective.

The injectivity of $U_{dR}^+ \to S_{dR}^+$ implies the injectivity of $U_{dR}^+ \to W \oplus S_{dR}^+$. To finish the proof we only need to show that the map $V_1 \to V_{1,dR}^+$ is injective or, with obvious identifications, that inside of $W \oplus S_{dR}^+$, we have

$$U_{dR}^+ \cap (W \oplus S) = U.$$

Assume $(w, s) \in W \oplus S$ belongs to U_{dR}^+. This implies that $s \in S \cap U_{dR}^+$ which is U as the map $V_2 \to V_{2,dR}^+$ is injective. We then need $w = -s$ and (w, s) is the image of $-s \in U$. □

Proof of the exactness of the functors $V \mapsto V_{dR}$, $V \mapsto V_e$ and $V \mapsto \overline{V}_{dR}$. If

$$0 \to V' \to V \to V'' \to 0$$

is a short exact sequence in $\mathcal{C}^{\geq 0}(G_K)$, we know that the sequences

$$V'_e \to V_e \to V''_e \to 0, \quad V'_{dR} \to V_{dR} \to V''_{dR} \to 0$$

are exact. As

$$\dim_{B_{dR}} V_{dR} = h(V) = h(V') + h(V'') = \dim_{B_{dR}} V'_{dR} = \dim_{B_{dR}} V''_{dR}$$

the map $V'_{dR} \to V_{dR}$ must be injective and the functor $V \mapsto V_{dR}$ is exact.

As the B_e-modules V'_e, V_e and V''_e are torsion free and as

$$\begin{aligned}\operatorname{rank}_{B_e}(V'_e) &= \dim_{B_{dR}} V'_{dR},\\ \operatorname{rank}_{B_e}(V_e) &= \dim_{B_{dR}} V_{dR},\\ \operatorname{rank}_{B_e}(V''_e) &= \dim_{B_{dR}} V''_{dR}.\end{aligned}$$

the same argument shows the exactness of $V \mapsto V_e$.

We then have a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V'_e & \longrightarrow & V_e & \longrightarrow & V''_e & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V'_{dR} & \longrightarrow & V_{dR} & \longrightarrow & V''_{dR} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{V}'_{dR} & \longrightarrow & \overline{V}_{dR} & \longrightarrow & \overline{V}''_{dR} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

whose three columns and the two first rows are exact. This implies the exactness of the last row.

Lemma 5.6. *Let*

$$0 \to V' \to V \to V'' \to 0$$

a short exact sequence in $\mathcal{C}^{\geq 0}(G_K)$. Assume the sequences

$$0 \to V' \to (V')^+_{dR} \to \overline{V}'_{dR} \to 0$$

and

$$0 \to V'' \to (V'')^+_{dR} \to \overline{V}''_{dR} \to 0$$

are exact. Then the sequences

$$0 \to V \to V_{dR}^{+} \to \overline{V}_{dR} \to 0$$

and

$$0 \to (V')_{dR}^{+} \to V_{dR}^{+} \to (V'')_{dR}^{+} \to 0$$

are exact.

Proof of the lemma: We have a commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (V')_{dR}^{+} & \longrightarrow & V_{dR}^{+} & \longrightarrow & (V'')_{dR}^{+} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{V}'_{dR} & \longrightarrow & \overline{V}_{dR} & \longrightarrow & \overline{V}''_{dR} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whose first and third rows are exact. By assumption, the first and the third columns are also exact. We also know that, except may be in $(V')_{dR}^{+}$, the second line is exact and, as $V \in \mathcal{C}^{\geq 0}(G_K)$, that the map $V \to V_{dR}^{+}$ is injective. By diagram chasing, we get the fact that the second line and the second column are also exact. □

We resume the proof of the proposition.

We first prove (iii), i.e., for all $V \in \mathcal{C}^{\geq 0}(G_K)$, the exactness of the sequence

$$0 \to V \to V_{dR}^{+} \xrightarrow{\bar{\iota}_V} \overline{V}_{dR} \to 0.$$

(a) If $V \in \mathcal{C}^{\infty}(G_K)$, as $V_{dR}^{+} = V$ and $V_{dR} = \overline{V}_{dR} = 0$, exactness is obvious.

(b) If $V \in \mathcal{C}^{0}(G_K)$, this sequence can be rewritten

$$0 \to V \to B_{dR}^{+} \otimes_{\mathbb{Q}_p} V \to \overline{B}_{dR} \otimes_{\mathbb{Q}_p} V \to 0$$

and exactness is deduced by tensoring with V from the exactness of

$$0 \to \mathbb{Q}_p \to B_{dR}^{+} \to \overline{B}_{dR} \to 0$$

(recall that $B_{dR} = B_e + B_{dR}^{+}$, that $\overline{B}_{dR} = B_{dR}/B_e$ and that $B_e \cap B_{dR}^{+} = \mathbb{Q}_p$).

(c) In general, we proceed by induction on the smallest integer r_V such that there is $K' \in \mathcal{K}$ with the property that V is a successive extension of r_V objects belonging either to $\mathcal{C}^{0}(G_{K'})$ or to $\mathcal{C}^{\infty}(G_{K'})$. Replacing K by K' if necessary, we may assume

$K' = K$. We just proved it is OK if $r_V = 1$. Assume $r_V \geq 2$, so that we can find a short exact sequence in $\mathcal{C}^{\geq 0}(G_K)$

$$0 \to V' \to V \to V'' \to 0$$

with $r_{V'}$ and $r_{V''} < r_V$. Then, by induction, the sequences

$$0 \to V' \to (V')^+_{dR} \to \overline{V'}_{dR} \to 0$$
$$0 \to V'' \to (V'')^+_{dR} \to (\overline{V''})_{dR} \to 0$$

are exact and the result follows from the two assertions of the previous lemma.

From the exact sequence (C), we see that $V = H^0(X, \mathcal{F}_V)$ and that $H^1(X, \mathcal{F}_V) = 0$ hence that $\mathcal{F} \in \mathcal{M}^{\geq 0}(G_K)$, which proves (iv).

We are left to prove the exactness of the functors $V \mapsto V^+_{dR}$ and $V \mapsto \mathcal{F}_V$, i.e., that, if

$$0 \to V' \to V \to V'' \to 0$$

is a short exact sequence in $\mathcal{C}^{\geq 0}(G_K)$, then the sequences

$$0 \to (V')^+_{dR} \to V^+_{dR} \to (V'')^+_{dR} \to 0$$

and

$$0 \to \mathcal{F}_{V'} \to \mathcal{F}_V \to \mathcal{F}_{V''} \to 0$$

are exact. As we now know the assertion (iii) of the proposition, the exactness of the first sequence is a consequence of the previous lemma. Finally, we see that exactness of the second is equivalent to the exactness of

$$0 \to (V')^+_{dR} \to V^+_{dR} \to (V'')^+_{dR} \to 0$$

and of

$$0 \to V'_e \to V_e \to V''_e \to 0$$

and we are done. □

Proposition 5.7. *Let $V \in \mathcal{C}(G_K)$. Any decreasing sequence of subobjects of V*

$$V_1 \supset V_2 \supset \cdots \supset V_n \supset V_{n+1} \supset \cdots$$

is stationary.

Proof. Chose $\widehat{V} \in \mathcal{C}^{\geq 0}(G_K)$ such that V is a quotient of $\widehat{V}$. For all $n \in \mathbb{N}$, set

$$\widehat{V}_n = \widehat{V} \times_V V_n.$$

The $\widehat{V}_n$ form a decreasing sequence of subobject of $\widehat{V}$ and, for all $n \in \mathbb{N}$, we have a canonical isomorphism $\widehat{V}_n / \widehat{V}_{n+1} \simeq V_n / V_{n+1}$. In particular

$$V_{n+1} = V_n \iff \widehat{V}_{n+1} = \widehat{V}_n.$$

Replacing V by $\widehat{V}$ and the V_n's by the $\widehat{V}_n$'s if necessary we ay assume that V, therefore also the V_n's are in $\mathcal{C}^{\geq 0}$.

As $d(V_{n+1}) \leq d(V_n)$ and $d(V_n) \geq 0$, there is an integer m such that $d(V_n) = d(V_{n+1})$ for $n \geq m$.

For $n \geq m$, we have $d(V_n/V_{n+1}) = 0$, hence $V_n/V_{n+1} \in \mathcal{C}^0(G_K)$ and, if we set $h_n = \dim_{\mathbb{Q}_p}(V_n/V_{n+1})$ $(\in \mathbb{N})$, we have $h(V_{n+1}) = h(V_n) - h_n$. As $V_{n+1} \in \mathcal{C}^{\geq 0}(G_K)$, we have $h(V_{n+1}) \geq 0$. Therefore, there is an integer $m' \geq m$ such that $h_n = 0$ if $n \geq m'$. This implies that $V_{n+1} = V_n$. □

Remark 5.8. On the other hand, there are objects of $\mathcal{C}(G_K)$ which admit nonstationary increasing sequences of subobjects. For instance, it is easy to see that $\mathbb{C}_p$ contains infinitely many subobjects belonging to $\mathcal{C}^0(G_K)$. From that, one can constructs nonstationary increasing sequences

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset V_{n+1} \subset \cdots$$

of subobjects of $\mathbb{C}_p$ belonging to $\mathcal{C}^0(G_K)$.

5C. ***The main result.*** We may consider the functors

$$\mathcal{M}^{\geq 0}(G_K) \to \mathcal{C}^{\geq 0}(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

and

$$\mathcal{C}^{\geq 0}(G_K) \to \mathcal{M}^{\geq 0}(G_K), \quad V \mapsto \mathcal{F}_V.$$

Theorem 5.9. *The functor*

$$\mathcal{M}^{\geq 0}(G_K) \to \mathcal{C}^{\geq 0}(G_K), \quad \mathcal{F} \mapsto \mathcal{F}(X)$$

is an equivalence of exact categories and

$$\mathcal{C}^{\geq 0}(G_K) \to \mathcal{M}^{\geq 0}(G_K), \quad V \mapsto \mathcal{F}_V$$

is a quasi-inverse.

Proof. As the functor $V \mapsto \mathcal{F}_V$ is left adjoint to $\mathcal{F} \mapsto \mathcal{F}(X)$ (Section 4D), we are reduced to checking the following claims:

(i) If $V \in \mathcal{C}^{\geq 0}(G_K)$, the map $V \to \mathcal{F}_V(X)$ coming from adjunction is an isomorphism,

(ii) If $\mathcal{F} \in \mathcal{M}^{\geq 0}(G_K)$, the map $\mathcal{F}_{\mathcal{F}_V(X)} \to \mathcal{F}$ coming from adjunction is an isomorphism.

(iii) If

$$0 \to V' \to V \to V'' \to 0$$

is a short exact sequence of $\mathcal{C}^{\geq 0}(G_K)$, the sequence

$$0 \to \mathcal{F}_{V'} \to \mathcal{F}_V \to \mathcal{F}_{V''} \to 0$$

is exact.

(iv) If

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence of $\mathcal{M}^{\geq 0}(G_K)$, the sequence

$$0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X) \to 0$$

is exact.

(1) and (3) have already been proved (Proposition 5.4) and (4) results from the fact that, if $\mathcal{F}' \in \mathcal{M}^{\geq 0}(G_K)$, then $H^1(X, \mathcal{F}') = 0$ (Proposition 3.14).

Let's prove (2): Let $\mathcal{M}$ the full subcategory of $\mathcal{M}^{\geq 0}(G_K)$ whose objects are those $\mathcal{F}$'s for which $\mathcal{F}_{\mathcal{F}_V(X)} \to \mathcal{F}$ is an isomorphism. It is obviously stable under taking direct summands. By exactness of the functors $\mathcal{F} \to \mathcal{F}(X)$ and $V \mapsto \mathcal{F}_V$, it is stable under extensions. It contains $\mathcal{M}^0(G_K)$ and $\mathcal{M}^\infty(G_K)$. Then Theorem 5.1 implies that $\mathcal{M} = \mathcal{M}^{\geq 0}(G_K)$. □

6. From $\mathcal{M}(G_K)$ to $\mathcal{C}(G_K)$ and conversely

6A. ***Some general nonsense.*** Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ be an exact subcategory of $\mathcal{A}$. Recall (cf., e.g., [Laumon 1983, §1.1]) that one can define the derived category of bounded complexes of $\mathcal{B}$ that we denote $D^b_{\mathcal{A}}(\mathcal{B})$: in the triangulated category $\mathcal{K}^b(\mathcal{B})$ of bounded complexes of $\mathcal{B}$ up to homotopies, the full subcategory $\mathcal{N}$ of bounded acyclic complexes (in $\mathcal{B}$) form a null system and we set

$$\mathcal{D}^b_{\mathcal{A}}(\mathcal{B}) = \mathcal{K}^b(\mathcal{B})/\mathcal{N}.$$

Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ an exact subcategory of $\mathcal{A}$ and $\mathcal{D}$ a strictly full subcategory of $\mathcal{B}$ which is a Serre subcategory of $\mathcal{A}$ (hence $\mathcal{D}$ is abelian).

- We say that *the exact embedding $\mathcal{B} \hookrightarrow \mathcal{A}$ is left big with respect to $\mathcal{D}$* if,

(i) any quotient in $\mathcal{A}$ of an object of $\mathcal{B}$ belongs to $\mathcal{B}$,

(ii) for any object A of $\mathcal{A}$, one can find a short exact sequence

$$0 \to A \to B \to D \to 0$$

of $\mathcal{A}$ with B an object of $\mathcal{B}$ and D an object of $\mathcal{D}$.

- We say that *the exact embedding $\mathcal{B} \hookrightarrow \mathcal{A}$ is right big with respect to $\mathcal{D}$* if $\mathcal{B}^{\mathrm{op}} \hookrightarrow \mathcal{A}^{\mathrm{op}}$ is left big with respect to $\mathcal{D}^{\mathrm{op}}$ which amounts to requiring that

(i) any subobject in $\mathcal{A}$ of an object of $\mathcal{B}$ belongs to $\mathcal{B}$,

(ii) for any object A of $\mathcal{A}$, one can find a short exact sequence

$$0 \to D \to B \to A \to 0$$

of $\mathcal{A}$ with B an object of $\mathcal{B}$ and D an object of $\mathcal{D}$.

We say that an exact embedding $\mathcal{B} \hookrightarrow \mathcal{A}$ is *left big* (resp. *right big*) if one can find a Serre subcategory $\mathcal{D}$ of $\mathcal{A}$ contained in $\mathcal{B}$ such that $\mathcal{B} \hookrightarrow \mathcal{A}$ is left big (resp. right big) with respect to $\mathcal{D}$.

Proposition 6.1. *Let $\mathcal{B} \hookrightarrow \mathcal{A}$ an exact embedding which is either left big or right big. Then the natural functor*

$$D^b_{\mathcal{A}}(\mathcal{B}) \to D^b(\mathcal{A})$$

is an equivalence of triangulated categories.

It is a formal consequence of the more precise following statement:

Proposition 6.2. *Let $\mathcal{B} \hookrightarrow \mathcal{A}$ be an exact embedding and $\mathcal{D}$ a Serre subcategory of $\mathcal{A}$ contained in $\mathcal{B}$ such that $\mathcal{B} \hookrightarrow \mathcal{A}$ is left big* (*resp. right big*) *with respect to $\mathcal{D}$ and let $A^\bullet$ a bounded complex of $\mathcal{A}$.*

(i) *There exists a short exact sequence of bounded complexes of $\mathcal{A}$*

$$0 \to A^\bullet \to B^\bullet \to D^\bullet \to 0 \ \ (\textit{resp. } \ 0 \to D^\bullet \to B^\bullet \to A^\bullet \to 0\,)$$

with $B^\bullet$ a bounded complex of $\mathcal{B}$ and $D^\bullet$ an acyclic complex of $\mathcal{D}$.

(ii) *If*

$$0 \to A^\bullet \to B'^\bullet \to D'^\bullet \to 0 \ (\textit{resp. } \ 0 \to D'^\bullet \to B'^\bullet \to A^\bullet \to 0\,)$$

isan other short exact sequence of the same kind, *there exists a a third short exact sequence of the same kind*

$$0 \to A^\bullet \to B''^\bullet \to D''^\bullet \to 0 \ (\textit{resp. } \ 0 \to D''^\bullet \to B''^\bullet \to A^\bullet \to 0\,)$$

together with morphisms of complexes

$$B^\bullet \to B''^\bullet \textit{ and } B'^\bullet \to B''^\bullet \ (\textit{resp. } B''^\bullet \to B^\bullet \textit{ and } B''^\bullet \to B'^\bullet)$$

such that the diagram

$$\begin{array}{ccccc} & & A^\bullet & & \\ & \swarrow & \downarrow & \searrow & \\ B'^\bullet & \longrightarrow & B''^\bullet & \longleftarrow & B^\bullet \end{array} \quad \left(\textit{resp.} \quad \begin{array}{ccccc} B'^\bullet & \longrightarrow & B''^\bullet & \longleftarrow & B^\bullet \\ & \searrow & \downarrow & \swarrow & \\ & & A^\bullet & & \end{array} \right)$$

is commutative.

Proof. It is enough to treat the case were the strict embedding is right big. Assume this is the case. To prove (i), by induction, we are reduced to proving this:

Lemma 6.3. *Let $r \in \mathbb{Z}$ and let*

$$0 \to D_r^\bullet \to B_r^\bullet \to A^\bullet \to 0$$

a short exact sequence of bounded complexes of $\mathcal{A}$. Assume that $D_r^\bullet$ is an acyclic complex of $\mathcal{D}$, that $D_r^n = 0$ for $n \geq r$ and that B_r^n is an object of $\mathcal{B}$ for all $n < r$. Then, there exists a short exact sequence of bounded complexes of $\mathcal{A}$

$$0 \to D_{r+1}^\bullet \to B_{r+1}^\bullet \to A^\bullet \to 0$$

where $D_{r+1}^\bullet$ is an acyclic complex of $\mathcal{D}$ with $D_{r+1}^n = 0$ for $n \geq r+1$ and B_{r+1}^n an object of $\mathcal{B}$ for all $n < r+1$.

Proof of the lemma. We can identify B_r^n to A^n for $n \geq r$. Granted to right bigness of $\mathcal{B} \hookrightarrow \mathcal{A}$, we can find a short exact sequence

$$0 \to D \to B \to A^r \to 0$$

with B an object of $\mathcal{B}$ and D an object of $\mathcal{D}$. Set

$$B_{r+1}^n = \begin{cases} B_r^n & \text{for } n < r-1, \\ B_r^{r-1} \times_{A_r} B & \text{for } n = r-1, \\ B & \text{for } n = r, \\ B_r^n = A^n & \text{for } n > r. \end{cases}$$

We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & D & = & D & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\cdots \longrightarrow & B_{r+1}^{r-2} \longrightarrow & B_{r+1}^{r-1} & \longrightarrow & B_{r+1}^{r} & \longrightarrow & B_{r+1}^{r+1} & \longrightarrow \cdots \\
 & \| & \downarrow & & \downarrow & & \| & \\
\cdots \longrightarrow & B_{r}^{r-2} \longrightarrow & B_{r}^{r-1} & \longrightarrow & B_{r}^{r} & \longrightarrow & B_{r}^{r+1} & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

whose rows are complexes and columns are exact. Hence we have a quasi-isomorphism $B_{r+1}^\bullet \to B_r^\bullet$. Moreover B_{r+1}^n is an object of $\mathcal{B}$ for all $n < r+1$ (for $n = r-1$, this is due to the fact that B_{r+1}^{r-1} is a subobject of $B_r^{r-1} \oplus B$ which belongs to $\mathcal{B}$).

The compositum

$$B_{r+1}^\bullet \to B_r^\bullet \to A^\bullet$$

is a surjective morphism of complexes which is a quasi-isomorphism. Then the kernel $D^\bullet_{r+1}$ is acyclic. As it is the complex

$$\cdots \to D_r^{r-3} \to D_r^{r-2} \to D_{r+1}^{r-1} \to D \to 0 \to \cdots \to 0 \to \cdots$$

we see that $D^n_{r+1} = 0$ for $n \ge r+1$ and that all the D^n_{r+1} belong to $\mathcal{D}$ (for $n = r-1$, this is due to the fact that we have a short exact sequence

$$0 \to D'' \to D_{r+1}^{r-1} \to D \to 0$$

with $D'' = \operatorname{coker}(D_r^{r-3} \to D_r^{r-2}) \in \mathcal{D}$, hence, as D_{r+1}^{r-1} is an extension in $\mathcal{A}$ of $D \in \mathcal{D}$ by $D'' \in \mathcal{D}$, it belongs to $\mathcal{D}$). □

To prove part (ii) of the proposition we take, for each $n \in \mathbb{Z}$, the fiber product

$$B''^n = B'^n \times_{A^n} B^n.$$

For each n, we have an exact sequence

$$0 \to D''^n \to B''^n \to A^n \to 0$$

with $D''^n = D'^n \oplus D^n$ and all the required properties are obviously fulfilled. □

6B. *The equivalence of triangulated categories.*

Theorem 6.4. *The equivalence of categories of Theorem 5.9 extends uniquely to an equivalence of triangulated categories*

$$D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K)).$$

Proof. Uniqueness is obvious.

Recall (Section 5C) that $\mathcal{M}^{\ge 0}(G_K)$ is an exact subcategory of $\mathcal{M}(G_K)$ and $\mathcal{C}^{\ge 0}(G_K)$ is an exact subcategory of $\mathcal{C}(G_K)$.

• The category $\mathcal{M}^{\infty}(G_K)$ is a Serre subcategory of $\mathcal{M}(G_K)$ contained in $\mathcal{M}^{\ge 0}(G_K)$ and any quotient $\mathcal{F}''$ in $\mathcal{M}(G_K)$ of an object $\mathcal{F}$ of $\mathcal{M}^{\ge 0}(G_K)$ is in $\mathcal{M}^{\ge 0}(G_K)$

$$(\text{as } \mathcal{F} \in \mathcal{M}^{\ge 0}(G_K) \iff H^1(X, \mathcal{F}) = 0 \Longrightarrow H^1(X, \mathcal{F}'') = 0 \iff \mathcal{F}'' \in \mathcal{M}^{\ge 0}(G_K)).$$

If $\mathcal{F} \in \mathcal{M}(G_K)$, for all $n \in \mathbb{N}$, as, for all $n \in \mathbb{N}$, the HN-slopes of $\mathcal{F}(n)_{HN}$ are the $s+n$ for s describing the HN-slopes of $\mathcal{F}$ (cf. Section 3H), for $n \gg 0$, we have $\mathcal{F}(n)_{HN} \in \mathcal{M}^{\ge 0}(G_K)$.

Tensoring with $\mathcal{F}$ the short exact sequence (Section 3H)

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(n)_{HN} \to (0, B_n(-n)) \to 0$$

we get a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}(n)_{HN} \to (0, \mathcal{F}^+_{dR} \otimes_{B^+_{dR}} B_n(-n)) \to 0.$$

As $\mathcal{F}(n)_{HN}$ belongs to $\mathcal{M}^{\geq 0}(G_K)$ and $(0, \mathcal{F}^+_{dR} \otimes_{B^+_{dR}} B_n(-n))$ belongs to $\mathcal{M}^{\infty}(G_K)$, it shows that the exact embedding $\mathcal{M}^{\geq 0}(G_K) \to \mathcal{M}(G_K)$ is left big with respect to $\mathcal{M}^{\infty}(G_K)$. Therefore (Proposition 6.1) the natural functor

$$D^b_{\mathcal{M}(G_K)}(\mathcal{M}^{\geq 0}(G_K)) \to D^b(\mathcal{M}(G_K))$$

is an equivalence of triangulated categories.

- Similarly, the category $\mathcal{C}^0(G_K)$ is a Serre subcategory of $\mathcal{C}(G_K)$ contained in $\mathcal{C}^{\geq 0}(G_K)$ and any subobject in $\mathcal{C}(G_K)$ of an object of $\mathcal{C}^{\geq 0}(G_K)$ belongs to $\mathcal{C}^{\geq 0}(G_K)$.

Let $V \in \mathcal{C}(G_K)$ and choose an almost isomorphism $V/U_+ \simeq W/U_-$ with $W \in \mathcal{C}^{\infty}(G_K)$ (cf. Section 2E). Set

$$\hat{V} = V \times_{W/U_-} W$$

(where the map $V \to W/U_-$ is the compositum of the projection $V \to V/U_+$ with the isomorphism $V/U_+ \to W/U_-$).

We have a diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & U_+ & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & U_- & \longrightarrow & \widehat{V} & \longrightarrow & V & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & W & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

whose line and column are exacts. The column shows that $\widehat{V} \in \mathcal{C}^{\geq 0}(G_K)$ and, therefore, the line shows that V is a quotient of an object of $\mathcal{C}^{\geq 0}(G_K)$ by an object of $\mathcal{C}^0(G_K)$. In other words, the exact embedding $\mathcal{C}^{\geq 0}(G_K) \to \mathcal{C}(G_K)$ is right big with respect to $\mathcal{C}^0(G_K)$. Hence (Proposition 6.1) the natural functor

$$D^b_{\mathcal{C}(G_K)}(\mathcal{C}^{\geq 0}(G_K)) \to D^b(\mathcal{C}(G_K))$$

is an equivalence of triangulated categories.

As the equivalence $\mathcal{M}^{\geq 0}(G_K) \xrightarrow{\simeq} \mathcal{C}^{\geq 0}(G_K)$ is an equivalence of exact categories, it extends uniquely to an equivalence of triangulated categories

$$D^b_{\mathcal{M}(G_K)}(\mathcal{M}^{\geq 0}(G_K)) \to D^b_{\mathcal{C}(G_K)}(\mathcal{C}^{\geq 0}(G_K)).$$

- It is now clear that there is a unique equivalence of triangulated categories

$$D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K))$$

such that the square

$$\begin{array}{ccc} D^b_{\mathcal{M}(G_K)}(\mathcal{M}^{\geq 0}(G_K)) & \longrightarrow & D^b_{\mathcal{C}(G_K)}(\mathcal{C}^{\geq 0}(G_K)) \\ \downarrow & & \downarrow \\ D^b(\mathcal{M}(G_K)) & \longrightarrow & D^b(\mathcal{C}(G_K)) \end{array}$$

is commutative and that this equivalence extends that of Theorem 5.9. □

6C. *The equivalence $\mathcal{M}^{<0}(G_K) \to \mathcal{C}^{<0}(G_K)$.* We say that a coherent $\mathcal{O}_X[G_K]$-module is *co-effective* if all its HN slopes are < 0. We saw (Proposition 3.14) that $\mathcal{F} \in \mathcal{M}(G_K)$ is co-effective if and only if $H^0(X, \mathcal{F}) = 0$. The full subcategory of $\mathcal{M}(G_K)$ whose objects are co-effective is $\mathcal{M}^{<0}(G_K)$ and is stable under taking subobjects and extensions.

Any $\mathcal{F} \in \mathcal{M}(G_K)$ as a biggest quotient $\mathcal{F}^{<0}$ belonging to $\mathcal{M}^{<0}(G_K)$and the sequence

$$0 \to \mathcal{F}^{\geq 0} \to \mathcal{F} \to \mathcal{F}^{<0} \to 0$$

is exact.

We say that an almost $\mathbb{C}_p$-representation V is *co-effective* if, for all $W \in \mathcal{C}^{\infty}(G_K)$, we have $\mathcal{H}om_{\mathcal{C}(G_K)}(V, W) = 0$. We denote $\mathcal{C}^{<0}(G_K)$ the full subcategory of $\mathcal{C}(G_K)$ whose objects are co-effective. It is obviously stable undertaking quotients and extensions.

Proposition 6.5. *Let V be an almost $\mathbb{C}_p$-representation. The following conditions are equivalent*:

(i) *V is co-effective.*

(ii) $V^+_{dR} = 0$.

(iii) $\mathcal{F}_V = 0$.

These conditions also imply

$$V_e = V_{dR} = \overline{V}_{dR} = 0.$$

Proof. The equivalence (i)⟺(ii) results from the universal property of V^+_{dR} and (ii)⟺(iii) is trivial. If $V^+_{dR} = 0$, we have $V_{dR} = B_{dR} \otimes_{B^+_{dR}} V^+_{dR} = 0$, hence also $V_e = 0$ as the map $V_e \to V_{dR}$ is injective and therefore $\overline{V}_{dR} = V_{dR}/V_e = 0$. □

Proposition 6.6. *Let $V \in \mathcal{C}(G_K)$. The set of subobjects of V in $\mathcal{C}(G_K)$ belonging to $\mathcal{C}^{<0}(G_K)$ has a biggest element $V^{<0}$ and the set of quotients of V in $\mathcal{C}(G_K)$ belonging to $\mathcal{C}^{\geq 0}(G_K)$ as a biggest element $V^{\geq 0}$. Moreover $V^{<0}$* (*resp. $V^{\geq 0}$*) *is the kernel* (*resp. the image*) *of the natural map $V \to V^+_{dR}$. The sequence*

$$0 \to V^{<0} \to V \to V^{\geq 0} \to 0$$

is exact.

Proof. If V' and V'' are subobjects of V belonging to $\mathcal{C}^{<0}(G_K)$, we see that $V'+V''$ also. Hence to show the existence of $V^{<0}$ it is enough to show that any increasing sequence

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset V_{n+1} \subset \cdots$$

of subobjects of V belonging to $\mathcal{C}^{<0}(G_K)$ is stationary. As the sequence of the integers $d(V_n)$ is increasing and bounded by $d(V)$, there exists $m \in \mathbb{N}$ such that $d(V_n) = d(V_m)$ for all $n \geq m$. For such an n, we have $d(V_{n+1}/V_n) = 0$, hence $V_{n+1}/V_n \in \mathcal{C}^0(G_K)$. This implies $V_{n+1} = V_n$ as, otherwise, the compositum of the projection of V_{n+1} onto V_n with the injective map

$$V_{n+1}/V_n \to B_{dR}^+ \otimes_{\mathbb{Q}_p} (V_{n+1}/V_n), \quad v \mapsto 1 \otimes v$$

would be a nonzero morphism from V_{n+1} to an object of $\mathcal{C}^\infty(G_K)$.

If $\overline{V}'$ and $\overline{V}''$ are quotients of V belonging to $\mathcal{C}^{\geq 0}(G_K)$, then the image of $V \to \overline{V}' \oplus \overline{V}''$ also (as it is a subobject of $\overline{V}' \oplus \overline{V}'' \in \mathcal{C}^{\geq 0}(G_K)$). Hence to show the existence of $V^{\geq 0}$ it suffices to show that any sequence

$$\cdots \to \underline{V}_{n+1} \to \underline{V}_n \to \cdots \to \underline{V}_1 \subset \underline{V}_0$$

of quotients of V (belonging to $\mathcal{C}^{<0}(G_K)$) is stationary. If $\widetilde{V}_n$ is the kernel of the projection $V \to \overline{V}_n$, the sequence $(\widetilde{V}_n)_{n\in\mathbb{N}}$ is a decreasing sequence of objects of $\mathcal{C}(G_K)$, hence is stationary (Proposition 5.7), therefore the sequence of the $\overline{V}_n$'s also.

Set $V_0 = \ker(V \to V_{dR}^+)$. We obviously have $V^{<0} \subset V_0$ and to show the equality it is enough to show that $V_0 \in \mathcal{C}^{<0}(G_K)$. Otherwise, we could find a nonzero morphism $f : V^0 \to W$ with $W \in \mathcal{C}^\infty(G_K)$. Let $V_1 = \ker f$ and consider the short exact sequence

$$0 \to V_0/V_1 \to V/V_1 \to V/V_0 \to 0.$$

As V_0/V_1 injects into W, it belongs to $\mathcal{C}^{\geq 0}(G_K)$. As V/V_0 injects into V_{dR}^+, it also belongs to $\mathcal{C}^{\geq 0}(G_K)$. Therefore, as $\mathcal{C}^{\geq 0}(G_K)$ is stable under extensions, $V/V_1 \in \mathcal{C}^{\geq 0}(G_K)$. Hence the sequence

$$0 \to (V_0/V_1)_{dR}^+ \to (V/V_1)_{dR}^+ \to (V/V_0)_{dR}^+ \to 0$$

is exact. As obviously $(V/V_1)_{dR}^+ = (V/V_0)_{dR}^+ = V_{dR}^+$, it contradicts the fact that, as V_0/V_1 is a nonzero object of $\mathcal{C}^{\geq 0}(G_K)$, we have $(V_0/V_1)_{dR}^+ \neq 0$.

Let $V_2 = \mathrm{im}(V \to V_{dR}^+)$. As the map $V_2 \to V_{dR}^+$ is injective, V_2 belongs to $\mathcal{C}^{\geq 0}(G_K)$ and is, therefore a quotient of $V^{\geq 0}$. The kernel V_3 of the projection

$V^{\geq 0} \to V_2$ belongs also to $\mathcal{C}^{\geq 0}(G_K)$ (as this category is stable under taking subobjects) and we have an exact sequence in $\mathcal{C}^{\geq 0}(G_K)$

$$0 \to V_3 \to V^{\geq 0} \to V_2 \to 0$$

Therefore the sequence

$$0 \to V_{3,dR}^+ \to V_{dR}^{\geq 0,+} \to V_{2,dR}^+ \to 0$$

is also exact.

As $V^{\geq 0}$ is a quotient of V, we see that $V_{dR}^{\geq 0,+}$ is a quotient of V_{dR}^+. But clearly $V_{2,dR}^+ = V_{dR}^+$. Therefore $V_{dR}^{\geq 0,+} = V_{dR}^+$ and $V_{3,dR}^+ = 0$. As $V_3 \in \mathcal{C}^{\geq 0}(G_K)$, this implies $V_3 = 0$, hence $V^{\geq 0} = V_2$.

The exactness of

$$0 \to V^{<0} \to V \to V^{\geq 0} \to 0$$

is now clear. □

Remarks 6.7. (i) To any $V \in \mathcal{C}(G_K)$, we just associated the canonical short exact sequence

$$0 \to V^{<0} \to V \to V^{\geq 0} \to 0$$

It is worth comparing with the canonical short exact sequence

$$0 \to \mathcal{F}^{\geq 0} \to \mathcal{F} \to \mathcal{F}^{<0} \to 0$$

associated to any $\mathcal{F} \in \mathcal{M}(G_K)$.

(ii) We know that, for any $\mathcal{F} \in \mathcal{M}(G_K)$, the natural map $\mathcal{F}^{\geq 0}(X) \to \mathcal{F}(X)$ is an isomorphism.The two previous propositions together imply that, for any $V \in \mathcal{C}(G_K)$, the natural map $\mathcal{F}_V \mapsto \mathcal{F}_{V^{\geq 0}}$ is an isomorphism. In particular, $\mathcal{F}_V$ always belongs to $\mathcal{M}^{\geq 0}(G_K)$.

It is clear that $\mathcal{M}^{<0}(G_K)$ is an exact subcategory of $\mathcal{M}(G_K)$, and $\mathcal{C}^{<0}(G_K)$ is an exact subcategory of $\mathcal{C}(G_K)$.

Proposition 6.8. *If $\mathcal{F} \in \mathcal{M}(G_K)$, then $H^1(X, \mathcal{F}) \in \mathcal{C}^{<0}(G_K)$ and the map*

$$H^1(X, \mathcal{F}) \mapsto H^1(X, \mathcal{F}^{<0})$$

is an isomorphism.

Moreover, the functor

$$\mathcal{M}^{<0}(G_K) \to \mathcal{C}^{<0}(G_K), \quad \mathcal{F} \mapsto H^1(X, \mathcal{F})$$

is an equivalence of exact categories.

Proof. If $\mathcal{F} \in \mathcal{M}(G_K)$, we may find a short exact sequence in $\mathcal{M}(G_K)$

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to 0$$

with $\mathcal{F}^0 \in \mathcal{M}^{\geq 0}(G_K)$. As $H^1(X, \mathcal{F}^0) = 0$, we see that $H^1(X, \mathcal{F})$ is the cokernel of $H^0(X, \mathcal{F}^0) \to H^0(X, \mathcal{F}^1)$, hence belongs to $\mathcal{C}(G_K)$.

We know that $H^1(X, \mathcal{F})$ is a quotient of $\overline{\mathcal{F}}_{dR}$ therefore also of $\mathcal{F}_{dR}$. If $f : H^1(X, \mathcal{F}) \to W$ were a nonzero morphism of $\mathcal{C}(G_K)$ with $W \in \mathcal{C}^\infty(G_K)$, the compositum $\mathcal{F}_{dR} \to H^1(X, \mathcal{F}) \to W$ would be a nonzero morphism in $\widehat{\mathcal{C}}^\infty(G_K)$ and, therefore, would be B_{dR}^+-linear. As multiplication by t is invertible in $\mathcal{F}_{dR}$ and nilpotent in W, the map must be 0 which implies that $H^1(X, \mathcal{F}) \in \mathcal{C}^{<0}(G_K)$.

If A is an object of an abelian category and $d \in \mathbb{Z}$, we denote $A[d]$ the bounded complex in $\mathcal{A}$ which is A in degree $-d$ and 0 elsewhere.

Denote by

$$\Gamma : D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K)) \ \text{(resp. } \Delta : D^b(\mathcal{C}(G_K)) \to D^b(\mathcal{M}(G_K))\text{)}$$

the functor extending $\mathcal{F} \mapsto \mathcal{F}(X)$ (resp. $V \mapsto \mathcal{F}_V$). If $\mathcal{F} \in \mathcal{M}^{<0}(G_K)$ and if

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to 0$$

is as above (observe that $\mathcal{F}^0 \in \mathcal{M}^{\geq 0}(G_K) \Longrightarrow \mathcal{F}^1 \in \mathcal{M}^{\geq 0}(G_K)$), we see that (with obvious conventions)

$$\Gamma(\mathcal{F}[0]) = \Gamma(\mathcal{F}^0 \to \mathcal{F}^1) = (H^0(X, \mathcal{F}^0) \to H^0(X, \mathcal{F}^1)) = H^1(X, \mathcal{F})[-1]$$

(as $\mathcal{F} \in \mathcal{M}^{<0}(G_K)$ and $\mathcal{F}^0 \in \mathcal{M}^{\geq 0}(G_K)$, the sequence

$$0 \to H^0(X, \mathcal{F}^0) \to H^0(X, \mathcal{F}^1) \to H^1(X, \mathcal{F}) \to 0$$

is exact).

Let $V \in \mathcal{C}^{<0}(G_K)$. We can find a short exact sequence in $\mathcal{C}(G_K)$

$$0 \to V^0 \to V^1 \to V \to 0$$

with $V^1 \in \mathcal{C}^{\geq 0}(G_K)$ which implies $V^0 \in \mathcal{C}^{\geq 0}(G_K)$. With obvious conventions, we have

$$\Delta(V[-1]) = \Delta(V^0 \to V^1) = (\mathcal{F}_{V^0} \to \mathcal{F}_{V^1}) = \mathcal{F}[0]$$

with $\mathcal{F}$ the kernel of $\mathcal{F}_{V^0} \to \mathcal{F}_{V^1}$ (as $V \in \mathcal{C}^{<0}(G_K)$, we have $V_{dR}^+ = V_e = 0$ which implies that

$$\mathcal{F}_{V^0} = (V_{dR}^{0+}, V_e^0, \iota_{V^0}) \to \mathcal{F}_{V^1} = (V_{dR}^{1+}, V_e^1, \iota_{V^1})$$

is an epimorphism).

We have a commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H^0(X,\mathcal{F}) & \longrightarrow & V^0 & \longrightarrow & V^1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}^+_{dR} & \longrightarrow & V^{0+}_{dR} & \to & V^{1+}_{dR} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}_e & \longrightarrow & V^0_e & \longrightarrow & V^1_e \end{array}$$

whose rows and columns are exact. The injectivity of $V^0 \to V^1$ implies that $H^0(X,\mathcal{F}) = 0$, i.e., that $\mathcal{F} \in \mathcal{M}^{<0}(G_K)$.

Finally, we see that, if we view

- $\mathcal{M}^{<0}(G_K)$ as the full subcategory of $D^b(\mathcal{M}(G_K))$ whose objects are of the form $\mathcal{F}[0]$ with $\mathcal{F} \in \mathcal{M}^{<0}(G_K)$,
- $\mathcal{C}^{<0}(G_K)$ as the full subcategory of $D^b(\mathcal{C}(G_K))$ whose objects are of the form $V[-1]$ with $V \in \mathcal{C}^{<0}(G_K)$,

then Γ induces the required equivalence of categories. □

6D. ***t*-Structures and hearts.** The functors

$$\Gamma : D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K)) \ \text{ and } \ \Delta : D^b(\mathcal{C}(G_K)) \to D^b(\mathcal{M}(G_K))$$

are as in the proof of the previous proposition.

Let $(D^{\leq 0}_{\mathcal{M}}, D^{\geq 0}_{\mathcal{M}})$ be the canonical t-structure on $D^b(\mathcal{M}(G_K))$: we see that $D^{\leq 0}_{\mathcal{M}}$ (resp. $D^{\geq 0}_{\mathcal{M}}$) is the full subcategory of $D^b(\mathcal{M}(G_K))$ whose objects are those $\mathcal{F}^\bullet$ such that $H^i(\mathcal{F}^\bullet) = 0$ for $i > 0$ (resp. $i < 0$). Therefore if we denote by $\Gamma(D^{\leq 0}_{\mathcal{M}})$ (resp. $\Gamma(D^{\geq 0}_{\mathcal{M}})$) the essential image under Γ of $D^{\leq 0}_{\mathcal{M}}$ (resp. $D^{\geq 0}_{\mathcal{M}}$), we see that $(\Gamma(D^{\leq 0}_{\mathcal{M}}), \Gamma(D^{\geq 0}_{\mathcal{M}}))$ is a t-structure on $D^b(\mathcal{C}(G_K))$ whose heart $\Gamma(D^{\leq 0}_{\mathcal{M}}) \cap \Gamma(D^{\geq 0}_{\mathcal{M}})$ is an abelian category equivalent via Δ to $\mathcal{M}(G_K)$.

Similarly, let $(D^{\leq 0}_{\mathcal{C}}, D^{\geq 0}_{\mathcal{C}})$ the canonical t-structure on $D^b(\mathcal{C}(G_K))$: hence $D^{\leq 0}_{\mathcal{C}}$ (resp . $D^{\geq 0}_{\mathcal{C}}$) is the full subcategory of $D^b(\mathcal{C}G_K)$ whose objects are those $V^\bullet$ such that $H^i(V^\bullet) = 0$ for $i > 0$ (resp. $i < 0$). Therefore if we denote by $\Delta(D^{\leq 0}_{\mathcal{C}})$ (resp. $\Delta(D^{\geq 0}_{\mathcal{C}})$) the essential image under Δ of $D^{\leq 0}_{\mathcal{C}}$ (resp. $D^{\geq 0}_{\mathcal{C}}$), we see that $(\Delta(D^{\leq 0}_{\mathcal{C}}), \Delta(D^{\geq 0}_{\mathcal{C}}))$ is a t-structure on $D^b(\mathcal{M}(G_K))$ whose heart $\Delta(D^{\leq 0}_{\mathcal{C}}) \cap \Delta(D^{\geq 0}_{\mathcal{C}})$ is an abelian category equivalent via Γ to $\mathcal{C}(G_K)$.

Proposition 6.9. (i) $\Gamma(D^{\geq 0}_{\mathcal{M}})$ *(resp.* $\Gamma(D^{\leq 0}_{\mathcal{M}})$*) is the full subcategory of* $D^b(\mathcal{C}(G_K))$ *whose objects are those* $V^\bullet$*'s such that* $H^r(V^\bullet) = 0$ *for* $r < 0$ *and* $H^0(V^\bullet) \in \mathcal{C}^{\geq 0}(G_K)$ *(resp.* $H^r(V^\bullet) = 0$ *for* $r > 1$ *and* $H^1(V^\bullet) \in \mathcal{C}^{<0}(G_K)$*).*

(ii) $\Delta(D_{\mathcal{C}}^{\geq 0})$ (*resp.* $\Delta(D_{\mathcal{C}}^{\leq 0})$) *is the full subcategory of* $D^b(\mathcal{M}(G_K))$ *whose objects are those* $\mathcal{F}^\bullet$*'s such that* $H^r(\mathcal{F}^\bullet) = 0$ *for* $r < -1$ *and* $H^{-1}(\mathcal{F}^\bullet) \in \mathcal{M}^{<0}(G_K)$ (*resp.* $H^r(\mathcal{F}^\bullet) = 0$ *for* $r > 0$ *and* $H^0(\mathcal{F}^\bullet) \in \mathcal{M}^{\geq 0}(G_K)$).

Proof. Let's prove that the description of $\Gamma(D_{\mathcal{M}}^{\geq 0})$ is correct (the proof of the three other statements are similar):

Any object $\underline{\mathcal{F}}$ of $D_{\mathcal{M}}^{\geq 0}$ can be represented by a bounded complex $\mathcal{F}^\bullet$ such that $\mathcal{F}^i = 0$ for $i < 0$. From the fact that, for any $\mathcal{F} \in \mathcal{M}(G_K)$, one can find a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}_0 \to \mathcal{F}_1 \to 0$$

with $\mathcal{F}_0, \mathcal{F}_1 \in \mathcal{M}^{\geq 0}(G_K)$ and the fact that any quotient, in $\mathcal{M}(G_K)$, of an object of $\mathcal{M}^{\geq 0}(G_K)$ still belongs to $\mathcal{M}^{\geq 0}(G_K)$, one easily deduces that the complex $\mathcal{F}^\bullet$ is quasi-isomorphic to a bounded complex $\mathcal{F}_0^\bullet$ with $\mathcal{F}_0^r = 0$ for $r < 0$ and $\mathcal{F}_0^r \in \mathcal{M}^{\geq 0}(G_K)$ for all $r \in \mathbb{N}$. Therefore $\Gamma(\underline{\mathcal{F}})$ is represented by the bounded complex

$$\cdots \to 0 \to \cdots \to 0 \to \mathcal{F}_0^0(X) \to \mathcal{F}_0^1(X) \to \cdots \to \mathcal{F}_0^r(X) \to \mathcal{F}_0^{r+1}(X) \to \cdots$$

all of whose terms belong to $\mathcal{C}^{\geq 0}(G_K)$. In particular, as $\mathcal{C}^{\geq 0}(G_K)$ is stable under taking subobjects in $\mathcal{C}(G_K)$, we see that $\Gamma(\underline{\mathcal{F}})$ belongs to the full subcategory $D_{\mathcal{C},\mathcal{M}}^{\geq 0}$ of $D^b(\mathcal{C}(G_K))$ whose objects are those $\underline{V}$'s such that $H^r(\underline{V}) = 0$ for $r < 0$ and $H^0(\underline{V}) \in \mathcal{C}^{\geq 0}(G_K)$.

Conversely, any object $\underline{V}$ of $D_{\mathcal{C},\mathcal{M}}^{\geq 0}(G_K)$ can be represented by a complex $V_0^\bullet$ such that $V_0^r = 0$ for $r < 0$ and that the kernel of $V^0 \to V^1$ belongs to $\mathcal{C}^{\geq 0}(G_K)$. Using the fact that, for any $V \in \mathcal{C}(G_K)$ one can find a short exact sequence in $\mathcal{C}(G_K)$

$$0 \to V_1 \to V_0 \to V \to 0$$

with $V_1, V_0 \in \mathcal{C}^{\geq 0}(G_K)$, one easily deduces that the complex $V_0^\bullet$ is quasi-isomorphic to a bounded complex $V^\bullet$ with $V^r = 0$ for $r < 0$ and $V^r \in \mathcal{C}^{\geq 0}(G_K)$ for $r > 0$.

We have a short exact sequence (with $d : V^0 \to V^1$ the differential in the complex $V^\bullet$)

$$0 \to (V_{d=0}^0) \to V^0 \to dV^0 \to 0$$

The inclusion $dV^0 \subset V^1$ implies that $dV^0 \in \mathcal{C}^{\geq 0}(G_K)$. As $V_{d=0}^0 = H^0(V^\bullet)$, we have $(V^0)_{d=0} \in \mathcal{C}^{\geq 0}(G_K)$. We know that $\mathcal{C}^{\geq 0}(G_K)$, as a full subcategory of $\mathcal{C}(G_K)$, is stable under extension. Therefore $V^0 \in \mathcal{C}^{\geq 0}(G_K)$.

As all the V^r's belong to $\mathcal{C}^{\geq 0}(G_K)$, we see that $\Delta(\underline{V})$ is represented by the bounded complex

$$\cdots \to 0 \to \cdots \to 0 \to \mathcal{F}_{V^0} \to \mathcal{F}_{V^1} \to \cdots \to \mathcal{F}_{V^r} \to \mathcal{F}_{V^{r+1}} \to \cdots$$

hence belong to $D_{\mathcal{M}}^{\geq 0}$. □

6E. ***Torsion pairs in $\mathcal{M}(G_K)$ and in $\mathcal{C}(G_K)$.*** The langage of torsion pairs (see [Happel et al. 1996, Chapter 1]) is very convenient to give an explicite description of the way to go from $\mathcal{M}(G_K)$ to $\mathcal{C}(G_K)$ and conversely. The results of this subsection and of the next one are independent of those of the previous one and give an other proof of the description of the heart of the t-structures we considered (Proposition 6.9).

Recall (*loc. cit.*) that a *torsion pair* in an abelian category $\mathcal{A}$ is a pair $t = (\mathcal{A}^+, \mathcal{A}^-)$ of full subcategories of $\mathcal{A}$ containing 0 such that:

(i) If B is an object of $\mathcal{A}^+$ and C is an object of $\mathcal{A}^-$, then $\mathrm{Hom}_{\mathcal{A}}(B, C) = 0$,

(ii) for any object A of $\mathcal{A}$, there is a short exact sequence in $\mathcal{A}$

$$0 \to A^+ \to A \to A^- \to 0$$

with $A^+ \in \mathrm{Ob}(\mathcal{A}^+)$ and $A^- \in \mathrm{Ob}(\mathcal{A}^-)$.

Condition (1) implies that the exact sequence of (2) is unique up to a unique isomorphism and that the correspondences $A \mapsto A^+$ and $A \mapsto A^-$ are functorial.

We define the *heart* $\mathcal{A}^t$ of t as the full subcateogry of the derived category $D^b(\mathcal{A})$ whose objects are those $A^\bullet$ such that

$$H^{-1}(A^\bullet) \in \mathrm{Ob}(\mathcal{A}^-), \quad H^0(A^\bullet) \in \mathrm{Ob}(\mathcal{A}^+), \quad H^n(A^\bullet) = 0 \text{ if } n \notin \{-1, 0\}.$$

Proposition 6.10. *Let $t = (\mathcal{A}^+, \mathcal{A}^-)$ be a torsion pair in an abelian category $\mathcal{A}$. Consider the full subcategories $D^{\le 0} = D_t^{\le 0}(\mathcal{A})$ and $D^{\ge 0} = D_t^{\ge 0}(\mathcal{A})$ of $D = D^b(\mathcal{A})$ defined by*

(i) $\mathrm{Ob}(D^{\le 0}) = \{A^\bullet \in \mathrm{Ob}(D^b(\mathcal{A})) \mid H^1(A^\bullet) \in \mathrm{Ob}(\mathcal{A}^+)$ *and* $H^n(A^\bullet) = 0, \forall n > 1\}$,

(ii) $\mathrm{Ob}(D^{\ge 0}) = \{A^\bullet \in \mathrm{Ob}(D^b(\mathcal{A})) \mid H^0(A^\bullet) \in \mathrm{Ob}(\mathcal{A}^-)$ *and* $H^n(A^\bullet) = 0, \forall n < 0\}$.

Then $(D^{\le 0}, D^{\ge 0})$ is a t-structure on D whose heart is $\mathcal{A}^t$.

Proof. To show that $(D^{\ge 0}, D^{\le 0})$ is a t-structure, we have to check (cf. [Kashiwara and Schapira 1990, Definition 10.1.1]) that (with standard notations)

(i) $D^{\le -1} \subset D^{\le 0}$ and $D^{\ge 1} \subset D^{\ge 0}$,

(ii) $\mathrm{Hom}_D(X, Y) = 0$ for $X \in \mathrm{Ob}(D^{\le 0})$ and $Y \in \mathrm{Ob}(D^{\ge 1})$,

(iii) For any $X \in \mathrm{Ob}(D)$, there exists a distinguished triangle $X_0 \to X \to X_1 \xrightarrow[+1]{}$ in D with $X_0 \in \mathrm{Ob}(D^{\ge 0})$ and $X_1 \in \mathrm{Ob}(D^{\ge 1})$.

(1) is obvious. (2) is clear as, if $f : X \to Y$ with $X \in \mathrm{Ob}(D^{\le 0})$ and $Y \in \mathrm{Ob}(D^{\ge 1})$, we have $H^n(f) = 0$ for $n \le 0$ (as $H^n(Y) = 0$), for $n > 1$ (as $H^n(X) = 0$) and for $n = 1$ (as $H^1(X) \in \mathrm{Ob}(\mathcal{A}^+)$ and $H^1(Y) \in \mathrm{Ob}(\mathcal{A}^-)$). Let's check (3): we have

$H^1(X) = X^1_{d=0}/dX^0$. Let $U = (\widehat{H^1(X)})^+$ where $\widehat{H^1(X)}$ is the inverse image of $H^1(X)$ in $X^1_{d=0}$. We have a short exact sequence of complexes

$$0 \to X_0 \to X \to X_1 \to 0$$

where

$$X_0^n = \begin{cases} X^n & \text{if } n < 1, \\ U & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \qquad X_1^n = \begin{cases} 0 & \text{if } n < 1, \\ X^1/U & \text{if } n = 1, \\ X^n & \text{if } n > 1, \end{cases}$$

which gives the desired distinguished triangle.

We have $\mathcal{A}^t = D^{\leq 0} \cap D^{\geq 0}$ and the last assertion is obvious. □

In particular, $\mathcal{A}^t$ is an abelian category [Kashiwara and Schapira 1990, proposition 10.1.11].

Denote by $\mathcal{A}_0^t$ the full subcategory of $\mathcal{A}^t$ whose objects are those $A^\bullet$ such that $A^n = 0$ for $n \notin \{0, 1\}$. To give an object $A^\bullet$ of $\mathcal{A}_0^t$ amounts to give a morphism

$$d_A = d^0_{A^\bullet} : A^0 \to A^1$$

of $\mathcal{A}$ such that $\ker(d_A)$ is an object of $\mathcal{A}^-$ and $\operatorname{coker}(d_A)$ an object of $\mathcal{A}^+$.

The inclusion functor $\mathcal{A}_0^t \to \mathcal{A}^t$ is obviously an equivalence of categories: there is even a canonical quasi-inverse

$$\mathcal{A}^t \to \mathcal{A}_0^t,$$

which sends $A^\bullet$ to $A^{-1}/dA^{-2} \to (A^0)_{d=0}$.

We have an obvious functor

$$\iota_t^+ : \mathcal{A}^+ \to \mathcal{A}_0^t, \quad A \mapsto (0 \to A).$$

It is easy to check that this functor is fully faithful and we denote $\mathcal{A}_0^{t,-}$ its essential image.

Similarly, it is easy to check that the functor

$$\iota_t^- : \mathcal{A}^- \to \mathcal{A}_0^t \ : \ A \mapsto (A \to 0)$$

is fully faithful and we denote by $\mathcal{A}_0^{t,+}$ its essential image.

It is also easy to check that $\widetilde{t} = (\mathcal{A}_0^{t,+}, \mathcal{A}_0^{t,-})$ is a torsion pair in $\mathcal{A}_0^t$.

Proposition 6.11. (i) $t = (\mathcal{M}^{\geq 0}(G_K), \mathcal{M}^{<0}(G_K))$ *is a torsion pair in* $\mathcal{M}(G_K)$.

(ii) $t' = (\mathcal{C}^{<0}(G_K), \mathcal{C}^{\geq 0}(G_K))$ *is a torsion pair in* $\mathcal{C}(G_K)$.

Proof. (i) We already know (Section 6C) that, for any object $\mathcal{F}$ of $\mathcal{M}(G_K)$, we have a canonical exact sequence

$$0 \to \mathcal{F}^{\geq 0} \to \mathcal{F} \to \mathcal{F}^{<0} \to 0$$

with $\mathcal{F}^{\geq 0} \in \mathcal{M}^{\geq 0}(G_K)$ and $\mathcal{F}^{<0} \in \mathcal{M}^{<0}(G_K)$.

If $f : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{M}(G_K)$, it sends $\mathcal{F}^{\geq 0}$ to $\mathcal{G}^{\geq 0}$. Therefore if $\mathcal{F} \in \mathcal{M}^{\geq 0}(G_K)$ ($\iff \mathcal{F}^{\geq 0} = \mathcal{F}$) and if $\mathcal{G} \in \mathcal{M}^{<0}(G_K)$ ($\iff \mathcal{G}^{\geq 0} = 0$), we have $f = 0$.

(ii) We already know (Proposition 6.6) that, for any object V of $\mathcal{C}(G_K)$, we have a canonical exact sequence

$$0 \to V^{<0} \to V \to V^{\geq 0} \to 0$$

with $V^{<0} \in \mathcal{C}^{<0}(G_K)$ and $V^{\geq 0} \in \mathcal{C}^{\geq 0}(G_K)$. Let $f : V_1 \to V_2$ be a morphism of $\mathcal{C}(G_K)$ with $V_1 \in \mathcal{C}^{<0}(G_K)$ and $V_2 \in \mathcal{C}^{\geq 0}(G_K)$. We can find a monomorphism $V_2 \to W$ with $W \in \mathcal{C}^{\infty}(G_K)$. As any morphism from V_1 to W is 0, the compositum $V_1 \to V_2 \to W$ is 0, hence $f = 0$. □

Denote by $\mathrm{Ar}^t(\mathcal{M}(G_K))$ the full subcategory of the categories of arrows of $\mathcal{M}^{\geq 0}(G_K)$ whose objects are those $d_{\mathcal{F}} : \mathcal{F}^0 \to \mathcal{F}^1$ such that $\ker d_{\mathcal{F}} \in \mathcal{M}^{<0}(G_K)$. Denote $(\mathcal{M}(G_K))^t_{00}$ the full subcategory of $(\mathcal{M}(G_K))^t_0$ whose objects are of the form

$$d_{\mathcal{F}} : \mathcal{F}^0 \to \mathcal{F}^1$$

with $\mathcal{F}^0$ and $\mathcal{F}^1$ objects of $\mathcal{M}^{\geq 0}(G_K)$.

As $\mathcal{M}^{\geq 0}(G_K)$ is stable by taking quotients, $(\mathcal{M}(G_K))^t_{00}$ and $\mathrm{Ar}^t(\mathcal{M}(G_K))$ have the same objects. With obvious conventions, $(\mathcal{M}(G_K))^t_{00}$ is the category deduced from $\mathrm{Ar}^t(\mathcal{M}(G_K))$ by working up to homotopies and inverting quasi-isomorphisms.

Proposition 6.12. *The inclusion functor*

$$(\mathcal{M}(G_K))^t_{00} \to (\mathcal{M}(G_K))^t_0$$

is an equivalence of categories.

Proof. It means that any object $d_{\mathcal{F}} : \mathcal{F}^0 \to \mathcal{F}^1$ of $(\mathcal{M}(G_K))^t_0$ is quasi-isomorphic to an object of $(\mathcal{M}(G_K))^t_{00}$. Indeed, we may find a monomorphism $\mathcal{F}^0 \to \mathcal{G}^0$ of $\mathcal{M}(G_K)$ with $\mathcal{G}^0 \in \mathcal{M}^{\geq 0}(G_K)$. Set

$$\mathcal{G}^1 = \mathcal{G}^0 \oplus_{\mathcal{F}^0} \mathcal{F}^1.$$

We have a short exact sequence

$$0 \to \overline{\mathcal{G}}^0 \to \mathcal{G}^1 \to \operatorname{coker} d_{\mathcal{F}} \to 0$$

where $\overline{\mathcal{G}}^0$ is a quotient of $\mathcal{G}^0$. Then $\operatorname{coker} d_{\mathcal{F}} \in \mathcal{M}^{\geq 0}(G_K)$ by assumption and $\overline{\mathcal{G}}^0$ also because $\mathcal{M}^{\geq 0}(G_K)$ is stable under taking quotients. As it is also stable under extensions, $\mathcal{G}^1$ also belongs to $\mathcal{M}^{\geq 0}(G_K)$. Hence, $\mathcal{G}^0 \to \mathcal{G}^1$ is an object of $(\mathcal{M}(G_K))^t_{00}$ which is quasi-isomorphic to $\mathcal{F}^0 \to \mathcal{F}^1$. □

Similarly, denote by $\mathrm{Ar}^{t'}(\mathcal{C}(G_K))$ the full subcategory of the categories of arrows of $\mathcal{C}^{\geq 0}(G_K)$ whose objects are those $d_V : V^0 \to V^1$ such that $\operatorname{coker} d_V \in \mathcal{C}^{<0}(G_K)$. Denote $(\mathcal{C}(G_K))^{t'}_{00}$ the full subcategory of $(\mathcal{C}(G_K))^{t'}_0$ whose objects are of the form

$$d_V : V^0 \to V^1$$

with V^0 and V^1 objects of $\mathcal{C}^{\geq 0}(G_K)$.

As $\mathcal{C}^{\geq 0}(G_K)$ is stable by taking subobjects, $(\mathcal{C}(G_K))^{t'}_{00}$ and $\mathrm{Ar}^{t'}(\mathcal{C}(G_K))$ have the same objects. With obvious conventions, $(\mathcal{C}(G_K))^{t'}_{00}$ is the category deduced from $\mathrm{Ar}^{t'}(\mathcal{C}(G_K))$ by working up to homotopies and inverting quasi-isomorphisms.

Proposition 6.13. *The inclusion functor*

$$(\mathcal{C}(G_K))^{t'}_{00} \to (\mathcal{C}(G_K))^{t'}_0$$

is an equivalence of categories.

Proof. The proof is entirely similar to the proof of the previous proposition: It means that any object $d_V : V^0 \to V^1$ of $\mathcal{C}(G_K)^{t'}_0$ is quasi-isomorphic to an object of $(\mathcal{C}(G_K))^{t'}_{00}$. Indeed, we may find an epimorphism $W^1 \to V^1$ of $\mathcal{C}(G^K)$ with $V^1 \in \mathcal{C}^{\geq 0}(G_K)$. Set

$$W^0 = V_0 \times_{V^1} W^1$$

We have a short exact sequence

$$0 \to \ker d_V \to W^0 \to W' \to 0$$

where W' is a subobject of $\mathcal{G}^0$. Then $\ker d_V \in \mathcal{C}^{\geq 0}(G_K)$ by assumption and W' also because $\mathcal{C}^{\geq 0}(G_K)$ is stable under taking subobjects. As it is also stable under extensions, W^0 also belongs to $\mathcal{C}^{\geq 0}(G_K)$. Hence, $V^0 \to V^1$ is an object of $(\mathcal{C}(G_K))^{t'}_{00}$ which is quasi-isomorphic to $V^0 \to V^1$. ☐

Theorem 6.14. (i) *The functor*

$$\widehat{\Gamma} : \mathrm{Ar}^t(\mathcal{M}(G_K)) \to \mathcal{C}(G_K), \quad (d_{\mathcal{F}} : \mathcal{F}^0 \to \mathcal{F}^1) \mapsto \operatorname{coker}(\mathcal{F}^0(X) \to \mathcal{F}^1(X))$$

factors uniquely through a functor

$$\Gamma : \mathcal{M}(G_K)^t_{00} \to \mathcal{C}(G_K)$$

and Γ is an equivalence of categories.

(ii) *The functor*

$$\widehat{\Delta} : \mathrm{Ar}^{t'}(\mathcal{C}(G_K)) \to \mathcal{M}(G_K), \quad (d_V : V^0 \to V^1) \mapsto \ker(\mathcal{F}_{V^0} \to \mathcal{F}_{V^1})$$

factors uniquely through a functor

$$\Delta : \mathcal{C}(G_K)^{t'}_{00} \to \mathcal{M}(G_K)$$

and Δ *is an equivalence of categories.*

Proof. Let's prove (i). Set $\widehat{\mathcal{M}} = \mathrm{Ar}^t(\mathcal{M}(G_K))$ and $\mathcal{M} = \mathcal{M}(G_K)^t_{00}$. If $d_{\mathcal{F}} = \mathcal{F}^0 \to \mathcal{F}^1$ is an object of one of these categories we denote it also $d_{\mathcal{F}}$ or $\mathcal{F}^0 \to \mathcal{F}^1$.

We see that $\mathcal{M}$ has an obvious structure of an exact category and that the natural functor $\widehat{\mathcal{M}} \to \mathcal{M}$ is exact.

• Let $\widehat{\mathcal{M}}^+$ (resp. $\mathcal{M}^+$) the full subcategory of $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) whose objects are those $d_{\mathcal{F}}$'s such that $\operatorname{coker} d_{\mathcal{F}} = 0$. For such an object, as $\ker d_{\mathcal{F}} \in \mathcal{M}^{<0}(G_K)$, and $\mathcal{F}^0$ and $\mathcal{F}^1$ belong to $\mathcal{M}^{\geq 0}(G_K)$, the long exact sequence of coherent cohomology associated to the exact sequence of $\mathcal{M}(G_K)$

$$0 \to \ker d_{\mathcal{F}} \to \mathcal{F}^0 \to \mathcal{F}^1 \to 0$$

is reduced to

$$0 \to \mathcal{F}^0(X) \to \mathcal{F}^1(X) \to H^1(X, \ker d_{\mathcal{F}}) \to 0.$$

Granted Proposition 6.8, this shows that the restriction of $\widehat{\Gamma}$ to $\widehat{\mathcal{M}}^+$ factors through a functor

$$\Gamma^+ : \mathcal{M}^+ \to \mathcal{C}^{<0}(G_K)$$

which is an equivalence of categories.

• Let $\widehat{\mathcal{M}}^-$ (resp. $\mathcal{M}^-$) the full subcategory of $\widehat{\mathcal{M}}$ (resp. $\mathcal{M}$) whose objects are those $d_{\mathcal{F}}$ such that $\mathcal{F}^0 = 0$. The natural functor $\widehat{\mathcal{M}}^- \to \mathcal{M}^-$ is an equivalence of categories and, granted Theorem 5.9, the restriction of $\widehat{\Gamma}$ to $\widehat{\mathcal{M}}^+$ factors through an equivalence of categories

$$\Gamma^- : \mathcal{M}^- \to \mathcal{C}^{\geq 0}(G_K).$$

• For any $d_{\mathcal{F}} \in \widehat{M}$, we have a canonical short exact sequence

$$0 \to d_{\mathcal{F}_+} \to d_{\mathcal{F}} \to d_{\mathcal{F}_-} \to 0$$

with $d_{\mathcal{F}_+} = (\mathcal{F}^0 \to \mathrm{im}\, d_{\mathcal{F}}) \in \widehat{\mathcal{M}}^+$ and $d_{\mathcal{F}_-} = (0 \to \mathcal{F}^1) \in \widehat{\mathcal{M}}^-$ and this construction is functorial. Moreover, we see that the sequence

$$0 \to \widehat{\Gamma}(d_{\mathcal{F}_+}) \to \widehat{\Gamma}(d_{\mathcal{F}}) \to \widehat{\Gamma}(d_{\mathcal{F}_-}) \to 0$$

is exact.

From these facts, we see that $\widehat{\Gamma}$ factors through a functor $\Gamma : \mathcal{M} \to \mathcal{C}(G_K)$ and that this functor is faithful. It is also straightforward to check that it is exact.

We are left to check the essential surjectivity: Let $V \in \mathcal{C}(G_K)$. We can find a short exact sequence in $\mathcal{C}(G_K)$

$$0 \to U \to \widehat{V} \to V \to 0$$

with $U \in \mathcal{C}^0(G_K)$ and $\widehat{V} \in \mathcal{C}^{\geq 0}(G_K)$. Let $\mathcal{F}^-$ be the kernel of the morphism $\mathcal{F}_U \to \mathcal{F}_{\widehat{V}}$ of $\mathcal{M}(G_K)$. As the functor global section is left exact, we have an exact sequence

$$0 \to \mathcal{F}^-(X) \to \mathcal{F}_U(X) \to \mathcal{F}_{\widehat{V}}(X).$$

But $\mathcal{F}_U(X) = U$, $\mathcal{F}_{\widehat{V}}(X) = \widehat{V}$ and the map $U \to \widehat{V}$ is the given map which is injective. Therefore $\mathcal{F}^-(X) = 0$ which means that $\mathcal{F}^- \in \mathcal{M}^{<0}(G_K)$ and

$$d_{\mathcal{F}} = (\mathcal{F}_U \to \mathcal{F}_{\widehat{V}})$$

is an object of $\mathcal{M}$. Clearly $\Gamma(d_{\mathcal{F}}) = V$, i.e., Γ is essentially surjective.

The proof of (ii) is entirely similar and we leave it to the reader. □

Remark 6.15. The category $\mathcal{M}(G_K)^t_{00}$ is a full subcategory of $D^b(\mathcal{M}(G_K))$ and $\mathcal{C}(G_K)$ is a full subcategory of $D^b(\mathcal{C}(G_K))$. The functor Γ of the previous theorem is the restriction to $\mathcal{M}(G_K)^t_{00}$ of the functor $\Gamma : D^b(\mathcal{M}(G_K)) \to D^b(\mathcal{C}(G_K))$ considered in Section 6D. Similarly, the functor Δ of the previous theorem is the restriction to $\mathcal{C}(G_K)^{t'}_{00}$ of the functor $\Delta : D^b(\mathcal{C}(G_K)) \to D^b(\mathcal{M}(G_K))$ considered in Section 6D.

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