

Tunisian Journal of Mathematics

an international publication organized by the Tunisian Mathematical Society

A simple proof of the Hardy inequality on Carnot groups and for some hypoelliptic families of vector fields

François Vigneron

2020 vol. 2 no. 4



A simple proof of the Hardy inequality on Carnot groups and for some hypoelliptic families of vector fields

François Vigneron

We give an elementary proof of the classical Hardy inequality on any Carnot group, using only integration by parts and a fine analysis of the commutator structure, which was not deemed possible until now. We also discuss the conditions under which this technique can be generalized to deal with hypoelliptic families of vector fields, which, in this case, leads to an open problem regarding the symbol properties of the gauge norm.

The classical Hardy inequality [1934] on a smooth open domain $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) reads:

$$\text{for all } f \in H_0^1(\Omega), \quad \sup_{x_0 \in \Omega} \left(\int_{\Omega} \frac{|f(x)|^2}{|x - x_0|^2} dx \right) \leq \frac{4}{(n-2)^2} \int_{\Omega} |\nabla f(x)|^2 dx. \quad (1)$$

Since L. D'Ambrosio [2005] it has been well known that similar inequalities hold on nilpotent groups, but interest on this matter is still high; see, e.g., [Ruzhansky and Suragan 2017; Adimurthi and Mallick 2018; Ambrosio et al. 2019].

An important reference on this matter is a recent note by H. Bahouri, C. Fermanian-Kammerer, I. Gallagher [Bahouri et al. 2012]. It is dedicated to refined Hardy inequalities on graded Lie groups and relies on constructing a general Littlewood–Paley theory and, as such, involves the machinery of the Fourier transform on groups.

Expanding the generality towards hypoelliptic vector fields, G. Grillo's article [2003] contains an inequality that holds for L^p -norms, without an underlying group structure, and contains weights that allow positive powers of the Carnot–Carathéodory distance on the right-hand side. However, the proof of this generalization involves the whole power of the sub-Riemannian Calderon–Zygmund theory.

Another beautiful reference is the paper by P. Ciatti, M.G. Cowling, F. Ricci [Ciatti et al. 2015] that studies these matters on stratified Lie groups, but with the

MSC2010: primary 43A80; secondary 35R03, 42B37.

Keywords: Hardy inequality, Carnot group, stratified algebra, commutators, hypoelliptic vector fields, Hörmander bracket condition.

point of view of operator and interpolation theory (see also [Krugljak et al. 1999] to highlight some subtleties in this approach).

The main goal of the present paper is to prove a general result on Carnot groups, albeit slightly simpler than those of [Bahouri et al. 2012] or [Grillo 2003], by using only *elementary* techniques: most of the paper relies only on integrations by part and on a fine analysis of the commutator structure. We will occasionally use interpolation techniques, but it is only required here if fractional regularities are sought after.

A Carnot group is a connected, simply connected and nilpotent Lie group G whose Lie algebra \mathfrak{g} admits a stratification, i.e.,

$$\mathfrak{g} = \bigoplus_{j=1}^m V_j \quad \text{where } [V_1, V_j] = V_{j+1}, \quad (2)$$

with $V_m \neq \{0\}$ but $[V_1, V_m] = \{0\}$. The dimensions will be denoted by $q_j = \dim V_j$ and $q = \sum q_j = \dim \mathfrak{g}$. Given a basis $(Y_\ell)_{\ell=1, \dots, q}$ of \mathfrak{g} adapted to the stratification each index $i \in \{1, \dots, q\}$ can be associated to a unique *weight* $\omega_\ell \in \{1, \dots, m\}$ such that $Y_\ell \in V_{\omega_\ell}$, namely

$$\omega_\ell = j \quad \text{for } n_{j-1} < \ell \leq n_j, \quad (3)$$

where $n_0 = 0$ and $n_j = n_{j-1} + q_j$ for $j = 1, \dots, m$ is the sequence of cumulative dimensions. Note that $n_1 = q_1$ and $n_m = q$. The horizontal derivatives are the derivatives in the first layer (see Section 1.5 below) and they are collected together in the following notation:

$$\nabla_G f = (Y_1^L f, \dots, Y_{n_1}^L f). \quad (4)$$

The stratification hypothesis ensures that each derivative $Y_i f$ can be expressed as at most $\omega_i - 1$ commutators of horizontal derivatives. The *homogeneous dimension* is the integer

$$Q = \sum_{j=1}^m j q_j = \sum_{\ell=1}^q \omega_\ell. \quad (5)$$

For $k \in \mathbb{N}$, the Sobolev space $H^k(G)$ is the subspace of functions $\varphi \in L^2(G)$ such that $\nabla_G^\alpha \varphi \in L^2(G)$ for any multi-index α of length $|\alpha| \leq k$. Fractional spaces can, for example, be defined by interpolation. The main result that we intend to prove here is the following.

Theorem 1. *Let G be a Carnot group and $\|\cdot\|$ any homogeneous pseudonorm equivalent to the Carnot–Carathéodory distance to the origin. Then, for any real s*

with $0 \leq s < Q/2$, there exists a constant $C_s > 0$ such that

$$\int_G \frac{|f(g)|^2}{\|g\|_G^{2s}} dg \leq C_s \|f\|_{H^s(G)}^2 \quad (6)$$

for any function $f \in H^s(G)$.

A similar Hardy inequality was proved by the author in [Vigneron 2006] for families of vector fields that satisfy a Hörmander bracket condition of step 2; the proof was based on the ideas of [Bahouri et al. 2005a; Bahouri and Cohen 2011], but was never published independently. This result was part of a broader study [Bahouri et al. 2005b; 2009; Mustapha and Vigneron 2007; Vigneron 2007] aiming at characterizing the traces of Sobolev spaces on the Heisenberg group, along hypersurfaces with nondegenerate characteristic points. Here, instead, we concentrate (except in Section 3) on the case of stratified groups, but without restrictions on the step m of the stratification.

The mathematical literature already contains numerous Hardy-type inequalities either on the Heisenberg group, for the p -sub-Laplacian, for Grushin-type operators and H -type groups (see, e.g., [D'Ambrosio 2005; Kombe 2010]). Sometimes (e.g., in [Garofalo and Lanconelli 1990; Niu et al. 2001; Dou et al. 2007]), a weight is introduced in the left-hand side that vanishes along the center of the group, i.e., along the (most) subelliptic direction. For example, [Garofalo and Lanconelli 1990] contains the following inequality on the Heisenberg group $\mathbb{H}_n \simeq \mathbb{C}^n \times \mathbb{R}$:

$$\int_{\mathbb{H}_n} \frac{|f(x)|^2}{d_{\mathbb{H}_n}(x, 0)^2} \Phi(x) dx \leq A \sum_{j=1}^n (\|X_j f\|_{L^2}^2 + \|Y_j f\|_{L^2}^2) + B \|f\|_{L^2}^2, \quad (7)$$

where (X_j, Y_j) are a basis of the first layer of the stratification and $d_{\mathbb{H}_n}((z, t), 0) \simeq \sqrt[4]{|z|^4 + t^2}$ is the gauge distance and Φ is a cut-off function that vanishes along the center $z = 0$:

$$\Phi(z, t) = \frac{|z|^2}{\sqrt{|z|^4 + t^2}}.$$

A secondary goal of this article is to show that such a cut-off is usually not necessary.

The core of our proof of Theorem 1 (see Section 2.5) consists in an integration by part against the radial field (the infinitesimal generator of dilations). The radial field can be expressed in terms of the left-invariant vector fields but, as all strata are involved, this first step puts $m - s$ too many derivatives on the function. Next, one uses the commutator structure of the left-invariant fields to carefully backtrack all but one derivative and let them act instead on the coefficients of the radial field. This step requires that those coefficients have symbol-like properties. One

can then conclude by an iterative process that reduces the Hardy inequality with weight $\|g\|_G^{-s}$ to the one with weight $\|g\|_G^{-(s-1)}$ as long as $s < Q/2$.

Finally, it is worth mentioning that a byproduct of our elementary approach concerns the symbol properties of the Carnot–Carathéodory norm (or of any equivalent gauge). For general hypoelliptic families of vector fields, *the norm is not always a symbol of order 1* (see section Section 3). On the contrary, on Carnot groups, it happens to always be equivalent to such a symbol (Proposition 7). At the end of the article, we discuss sufficient conditions for this property to hold for families of hypoelliptic vector fields, based either on the order m of the Hörmander condition (Theorem 14), or on the way the commutators are structured (Theorems 15 and 16).

The structure of the article goes as follows. Section 1 is a brief survey of calculus on Carnot groups. It also sets the notations used subsequently. Section 2 contains the actual proof of Theorem 1 and concludes on Theorem 13, which is the homogeneous variant of the previous statement. Section 3 addresses an open question regarding families of vector fields that satisfy a Hörmander bracket condition, but lack an underlying group structure.

1. A brief survey of calculus on Carnot Groups

Let us first recall some classic definitions and facts about nilpotent Lie groups. We also introduce notations that will be needed in Section 2. For a more in-depth coverage of Lie groups, sub-Riemannian geometry and nilpotent groups, see, e.g., [Montgomery 2002; Folland and Stein 1982; Rossmann 2002] or the introduction of [Ambrosio and Rigot 2004].

1.1. Left-invariant vector fields and the exponential map. Let us consider a Lie group G and $\mathfrak{g} = T_e G$ its Lie algebra; e denotes the unit element of G . Left-translation is defined by $L_g(h) = gh$.

Definition. A vector field ξ is called *left-invariant* if $(L_g)_* \circ \xi = \xi \circ L_g$. Such a vector field is entirely determined by $v = \xi(e) \in \mathfrak{g}$. To signify that v generates ξ , one writes $\xi = v^L$ thus:

$$v^L(g) = d(L_g)|_e(v). \quad (8)$$

The tangent bundle TG identifies to $G \times \mathfrak{g}$ by the map $(g, v) \mapsto (g, v^L(g))$.

The flow Φ_t^v of a left-invariant vector field v^L exists for all time. Indeed, one has $\Phi_t^v(g) = L_g \circ \Phi_t^v(e)$, which implies that $\Phi_{t+s}^v(e) = L_{\Phi_s^v(e)} \circ \Phi_t^v(e)$, thus allowing the flow to be extended globally once it has been constructed locally.

Definition. The *exponential map* $\exp : \mathfrak{g} \rightarrow G$ is defined by $\exp(v) = \Phi_1^v(e)$ where $(\Phi_t^v)_{t \in \mathbb{R}}$ is the flow of the left-invariant vector field v^L .

One can check that the flow of v^L starting from $g \in G$ is $\Phi_t^v(g) = g \exp(tv)$. In particular,

$$\exp(sw) \exp(tv) = \Phi_t^v(\exp(sw)) \quad \text{and} \quad d\exp|_0 = \text{Id}_{\mathfrak{g}}.$$

1.2. The Baker–Campbell–Hausdorff formula. The commutator of two left-invariant vector fields is also a left-invariant field. Therefore, the commutator of $u, v \in \mathfrak{g}$ is defined by $[u, v] = [u^L, v^L](e) \in \mathfrak{g}$. The product law of G induces an extremely rigid relation between exponentials, known as the Baker–Campbell–Hausdorff formula:

$$\exp(u) \exp(v) = \exp(\mu(u, v)) \quad (9)$$

where $\mu(u, v) = u + v + \frac{1}{2}[u, v] + \frac{1}{12}([u, [u, v]] + [v, [v, u]]) + \dots$ is a universal Lie series in u, v i.e., an expression consisting of the iterated commutators of u and v . In general, this formula holds provided u and v are small enough for the series to converge (see, e.g., [Rossmann 2002, §1.3]). Subsequently, one will only use the linear part of (9) with respect to one variable:

$$d\mu(u, \cdot)|_0(w) = \frac{\text{ad}(u)}{1 - e^{-\text{ad}(u)}}(w) = w + \sum_{n=1}^{\infty} \frac{(-1)^n B_n}{n!} [u, \dots, [u, w]], \quad (10)$$

where B_n are the Bernoulli numbers (i.e., $(x)/e^x - 1 = \sum (B_n/n!)x^n$) and $\text{ad}(u) = [u, \cdot]$. This formula is classical and can be found, e.g., in [Rossmann 2002] or [Klarsfeld and Oteo 1989].

1.3. Stratification. From now on, G is supposed to be *stratified*, i.e., it is a Carnot group as defined in the introduction of this paper. A stratified group is, in particular, nilpotent of step m . Moreover, elementary linear algebra gives restrictions on the possible dimensions $q_j = \dim V_j$ of the strata:

$$q_2 \leq \frac{q_1(q_1 - 1)}{2} \quad \text{and for } j \geq 2, \quad q_{j+1} < q_1 q_j.$$

The last inequality is strict because of the Jacobi identity

$$[u, [v, w]] - [v, [u, w]] = -[w, [u, v]].$$

For an exact count of the possible relations, see, e.g., [Reutenauer 1993].

Proposition 2. *Let G be a Carnot group. Then $\exp : \mathfrak{g} \rightarrow G$ is a global diffeomorphism that allows G to be identified with the set \mathfrak{g} equipped with the group law $u * v = \mu(u, v)$. The identity element is 0 and the inverse of u is $-u$.*

Proof. This claim is very standard so one only sketches the proof briefly. As $d\exp|_0 = \text{Id}_{\mathfrak{g}}$, there exists a neighborhood U_0 of e in G and V_0 of 0 in \mathfrak{g} such that $\exp : V_0 \rightarrow U_0$ is a diffeomorphism. As G is connected, it is generated by any

neighborhood of e and in particular by $U_0 = \exp(V_0)$. But, as \mathfrak{g} is nilpotent, the expression $\mu(u, v)$ is a Lie polynomial of order m , thus (9) holds for any $u, v \in \mathfrak{g}$. Combining these facts implies that the exponential map is surjective. Next, one can show that the pair (\mathfrak{g}, \exp) is a covering space of G . Indeed, given $g = \exp(v) \in G$, one gets a commutative diagram of diffeomorphisms

$$\begin{array}{ccc} V_0 & \xrightarrow{\exp} & U_0 \\ \mu(v', \cdot) \downarrow & & \downarrow L_g \\ V' & \xrightarrow{\exp} & g \cdot U_0 \end{array}$$

for each $v' \in \mathfrak{g}$ such that $\exp(v') = g$. Finally, by a standard covering space argument based on the fact that \mathfrak{g} is path connected (as a vector space) and G is simply connected (in the stratification assumption), one can claim that the exponential map is a global diffeomorphism. \square

Example. The Heisenberg group \mathbb{H} can be realized as a set of upper-triangular matrices with diagonal entries equal to 1. The group law in \mathbb{H} is the multiplication of matrices:

$$\begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & p' & r' \\ 0 & 1 & q' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & p+p' & r+r'+pq' \\ 0 & 1 & q+q' \\ 0 & 0 & 1 \end{pmatrix}.$$

The Lie algebra of \mathbb{H} is

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & p & r \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix} = pY_1 + qY_2 + rY_3; \ p, q, r \in \mathbb{R} \right\}.$$

Left-invariant vector fields on \mathbb{H} are linear combinations of

$$Y_1^L(g) = Y_1, \quad Y_2^L(g) = Y_2 + pY_3 \quad \text{and} \quad Y_3^L(g) = Y_3, \quad \text{where } g = \begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}.$$

\mathbb{H} is a stratified nilpotent group with $V_1 = \text{Span}(Y_1, Y_2)$ and $V_2 = \text{Span}(Y_3)$. The exponential map is the usual exponential of nilpotent matrices. It transfers the group structure to $\mathfrak{h} \simeq \mathbb{R}^3$ by (9):

$$(p, q, r)_{\mathfrak{h}} \simeq \begin{pmatrix} 0 & p & r \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}, \quad \exp \begin{pmatrix} 0 & p & r \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & p & r + \frac{1}{2}pq \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mu((p, q, r)_{\mathfrak{h}}, (p', q', r')_{\mathfrak{h}}) = (p + p', q + q', r + r' + \frac{1}{2}(pq' - qp'))_{\mathfrak{h}}.$$

In exponential coordinates on $\mathfrak{h} \simeq \mathbb{R}^3$, the left-invariant vector fields thus take the following form:

$$\tilde{Y}_1^L(p, q, r) = \partial_p - \frac{1}{2}q\partial_r, \quad \tilde{Y}_2^L(p, q, r) = \partial_q + \frac{1}{2}p\partial_r, \quad \tilde{Y}_3^L(p, q, r) = \partial_r.$$

The two different expressions of the fields correspond to the change of variables $(p, q, r) \mapsto (p, q, r - \frac{1}{2}pq)$.

Remarks. (1) In general (even if G is connected and nilpotent), only the Lie group action can be recovered from the exponential map but not G itself. For example, $G = \{z \in \mathbb{C}; |z| = 1\} \simeq \mathbb{S}^1$ with rotation law $(z, z') \mapsto zz'$ is a nilpotent group. One has $\mathfrak{g} = \mathbb{R}$ and $\mu(x, y) = x + y$ but the exponential map is $\exp(x) = e^{ix}$ and is obviously not a global diffeomorphism.

(2) Combined with (10), the commutative diagram of the proof of Proposition 2 provides a general formula for the differential of the exponential map, which we will need later on. For $v, w \in \mathfrak{g}$ and $g = \exp(v)$, one has

$$d\exp|_v(w) = \sum_{n=0}^{m-1} \frac{(-1)^n}{(n+1)!} [v, \dots, [v, w]]^L(g). \quad (11)$$

n times v

For example, on the Heisenberg group, one gets $d\exp|_v(w) = (w - \frac{1}{2}[v, w])^L(g)$.

Proposition 3. *For any indices $j, k \in \{1, \dots, m\}$, one has*

$$[V_j, V_k] \subset \begin{cases} V_{j+k} & \text{if } j+k \leq m, \\ \{0\} & \text{otherwise.} \end{cases} \quad (12)$$

Proof. By convention, let us write $V_n = \{0\}$ if $n > m$. For $k = 1$, the property holds by definition. For $k = 2$, as $V_2 = [V_1, V_1]$, any element can be written $[X, Y]$ with $X, Y \in V_1$. For $Z \in V_j$, one uses the identity $[A, BC] = [A, B]C + B[A, C]$ to get

$$[Z, [X, Y]] = [[Z, X], Y] - [[Z, Y], X] \in [[V_j, V_1], V_1] \subset [V_{j+1}, V_1] \subset V_{j+2}.$$

Next, one proceeds recursively. Assuming that for some $k \geq 2$, one has $[V_j, V_k] \subset V_{j+k}$ for any j , then given $Z \in V_{k+1} = [V_1, V_k]$, one writes $Z = [X, \zeta]$ with $X \in V_1$ and $\zeta \in V_k$. Then for any $W \in V_j$, the Jacobi identity gives

$$\begin{aligned} [W, Z] &= [W, [X, \zeta]] \\ &= -[X, [\zeta, W]] - [\zeta, [W, X]] \in [V_1, V_{j+k}] + [V_k, V_{j+1}] \subset V_{j+k+1} \end{aligned}$$

which makes the property hereditary in k . □

1.4. Stratified dilations. The next essential object in a Carnot group is the *dilation* of the Lie algebra:

$$\text{for all } r > 0, \quad \delta_r = \sum_{j=1}^m r^j \pi_j, \quad (13)$$

where $\pi_j : \mathfrak{g} \rightarrow V_j$ is the projection onto V_j with kernel $\bigoplus_{k \neq j} V_k$. Identifying G to $\exp(\mathfrak{g})$, one gets a one parameter family of group automorphisms that we will simply denote by

$$rg = \exp \circ \delta_r \circ \exp^{-1}(g) \quad (14)$$

for any $r > 0$ and $g \in G$.

The next result is an immediate consequence of the definition but should later be compared with the scaling property (41) of the radial vector field.

Proposition 4. *The dilation of a left-invariant vector field v^L is given by*

$$(\delta_r v)^L(rg) = ((L_{rg})_* \circ \delta_r \circ (L_g)_*^{-1})(v^L(g)). \quad (15)$$

Up to a constant factor, the Haar measure on G is given by the Lebesgue measure on $\mathfrak{g} \simeq \mathbb{R}^q$ and is commonly denoted dg .

Proposition 5. *One has, for all $\varphi \in L^1(G)$ and $r > 0$,*

$$\int_G \varphi(rg) dg = r^{-Q} \int_G \varphi(g) dg, \quad (16)$$

where Q is the homogeneous dimension (5) of G . Note that when $m \neq 1$, one has $Q > q$.

Given (arbitrary¹) Euclidean norms $\|\cdot\|_{V_j}$ on each V_j and $w = 2 \text{LCM}(1, \dots, m)$, the anisotropic *gauge-norm* of either $v \in \mathfrak{g}$ or of $g = \exp(v) \in G$ is defined by

$$\|v\|_{\mathfrak{g}} = \|g\|_G = \left(\sum_{j=1}^m \|\pi_j(v)\|_{V_j}^{w/j} \right)^{1/w}. \quad (17)$$

The gauge norm is homogeneous in the following sense:

$$\|rg\|_G = r\|g\|_G \quad \text{and} \quad |\square_{g,r}| = c_0 r^Q, \quad (18)$$

with a uniform constant c_0 and where the gauge-ball is defined by

$$\square_{g,r} = \{h \in G ; \|h^{-1}g\|_G < r\}.$$

¹For a given basis of \mathfrak{g} adapted to the stratification, one will chose here a Euclidean structure that renders this basis orthonormal. This is the natural choice when one proceeds to the identification $\mathfrak{g} \simeq \mathbb{R}^q$ through this basis.

Remark. The intrinsic metric objects of G are the so-called Carnot balls defined as the set of points that can be connected to a center $g_0 \in G$ by an absolutely continuous path γ whose velocity is subunitary for almost every time i.e., such that $\dot{\gamma}(t) \in (L_{\gamma(t)})_*(B_0)$ where $B_0 \subset V_1$ is a fixed Euclidean ball of the first layer of the stratification (up to some choice of a Euclidean metric on V_1). However, the ball-box theorem [Montgomery 2002] states that such intrinsic objects can be sandwiched between two gauge-balls of comparable radii. For our purpose (the analysis of Sobolev spaces), one can thus deal only with gauge-balls without impeding the generality.

1.5. Horizontal derivatives and Sobolev spaces on G . Vector fields ξ on G are identified with derivation operators on $C^\infty(G)$ by the Lie derivative formula:

$$(\xi\varphi)(g) = d\varphi|_g(\xi(g)). \quad (19)$$

Definition. The *horizontal derivatives* are the left-invariant vector-fields associated with V_1 .

Let us consider a basis $(Y_\ell)_{1 \leq \ell \leq q}$ of \mathfrak{g} that is adapted to the stratification, i.e.,

$$V_j = \text{Span}(Y_\ell)_{n_{j-1} < \ell \leq n_j}, \quad (20)$$

where n_j is defined by (3). An horizontal derivative is thus a vector field

$$\xi = \sum_{j \leq n_1} \alpha_j Y_j^L = \alpha \cdot \nabla_G,$$

where $\alpha = (\alpha_1, \dots, \alpha_{n_1}) \in \mathbb{R}^{n_1}$ and ∇_G is defined by (4). Noncommutative multi-indices are defined as follows: for $\gamma = (\gamma_1, \dots, \gamma_\ell) \in \{1, \dots, n_1\}^\ell$, one writes $l = |\gamma|$ and $\nabla_G^\gamma = Y_{\gamma_1}^L \circ \dots \circ Y_{\gamma_\ell}^L$.

Let us unfold the commutator structure in \mathfrak{g} with the following notation:

$$[Y_{\ell_1}, \dots, [Y_{\ell_n}, Y_{\ell_{n+1}}]] = \sum_{\ell'} \kappa(\ell_1, \dots, \ell_n, \ell_{n+1}; \ell') Y_{\ell'}. \quad (21)$$

Note that according to (12), one can warrant that $\kappa(\ell_1, \dots, \ell_n, \ell_{n+1}; \ell') = 0$ if $\omega_{\ell'} \neq \omega_{\ell_1} + \dots + \omega_{\ell_{n+1}}$.

Remark. To simplify computations, one can always assume that the basis is chosen such that

$$\text{for all } \ell \in \{1, \dots, q\}, \quad Y_\ell = [Y_{\alpha_1(\ell)}, \dots, [Y_{\alpha_{k-1}(\ell)}, Y_{\alpha_k(\ell)}]], \quad (22)$$

where $k = \omega_\ell$ and $\alpha_i(\ell) \leq n_1$. Indeed, the Lie algebra is linearly generated by the commutators of the restricted family ∇_G and one just has to extract a basis from it.

Example. On the Heisenberg group \mathbb{H} , the horizontal derivatives are left-invariant vector fields of the form $\xi = (\alpha Y_1 + \beta Y_2)^L$ for $\alpha, \beta \in \mathbb{R}$.

Definition. For $s \in \mathbb{N}$, the Sobolev space $H^s(G)$ consists of the functions such that each composition of at most s horizontal derivatives belongs to $L^2(G)$. The norm is defined (up to the choice of the Y_ℓ) by

$$\|\varphi\|_{H^s(G)}^2 = \sum_{|\gamma| \leq s} \int_G |\nabla_G^\gamma \varphi(g)|^2 dg, \quad (23)$$

with γ a noncommutative multi-index.

Remark. The space $H^2(G)$ is the domain of the hypoelliptic Laplace operator

$$\mathcal{L}_G = - \sum_{\ell \leq n_1} (Y_\ell^L)^* Y_\ell^L. \quad (24)$$

A celebrated result of L. Hörmander [1967] states that $H^s(G) \subset H_{\text{loc}}^{s/m}(\mathbb{R}^q)$, where the last Sobolev space is the classical one (homogeneous and isotropic) on \mathbb{R}^q .

1.6. Exponential coordinates on a stratified group. Given a basis $(Y_\ell)_{1 \leq \ell \leq q}$ of \mathfrak{g} adapted to the stratification, one can define a natural coordinate system on G , called *exponential coordinates*. Given $g = \exp(v) \in G$, its coordinates $x(g) = (x_\ell(g))_{1 \leq \ell \leq q} \in \mathbb{R}^q$ are defined by

$$v = \sum_{\ell=1}^q x_\ell(g) Y_\ell. \quad (25)$$

The projections $(\pi_j)_{1 \leq j \leq m}$ introduced in (13) are

$$\text{for all } j \in \{1, \dots, m\}, \quad \pi_j(v) = \sum_{\ell=1+n_{j-1}}^{n_j} x_\ell(g) Y_\ell. \quad (26)$$

In exponential coordinates, the expression of stratified dilations (14) is

$$\text{for all } \ell \in \{1, \dots, q\}, \quad x_\ell(rg) = r^{\omega_\ell} x_\ell(g). \quad (27)$$

The gauge norm (17) is given (for some fixed large $w \in \mathbb{N}$) by

$$\|g\|_G = \|x(g)\|_{\mathfrak{g}} = \left(\sum_{j=1}^m \left(\sum_{\ell=1+n_{j-1}}^{n_j} |x_\ell(g)|^2 \right)^{w/j} \right)^{1/(2w)}. \quad (28)$$

One could however take any uniformly equivalent quantity as a gauge norm, which will be the case subsequently, after Proposition 7.

Example. With the previous notation, the exponential coordinates on the Heisenberg group \mathbb{H} are

$$x_1(g) = p, \quad x_2(g) = q, \quad x_3(g) = r - \frac{1}{2}pq \quad \text{for } g = \begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{H}.$$

1.7. Left-invariant basis of vector fields. When doing explicit computations, it is natural to identify \mathfrak{g} with \mathbb{R}^q through the previous coordinates. Given $v \in \mathfrak{g}$, the left-invariant vector field v^L on G defined by (8) matches a corresponding vector field on $\mathfrak{g} \simeq \mathbb{R}^q$ that we will denote by \tilde{v}^L . According to the Baker–Campbell–Hausdorff formula (10),

for all $x = (x_1, \dots, x_q) \in \mathfrak{g}$,

$$\tilde{v}^L(x) = v + \sum_{n=1}^{m-1} \sum_{\ell_1, \dots, \ell_n} \frac{(-1)^n B_n}{n!} x_{\ell_1} \dots x_{\ell_n} [Y_{\ell_1}, \dots, [Y_{\ell_n}, v]], \quad (29)$$

where each of the ℓ_i ranges over $\{1, \dots, q\}$.

After the identification $\mathfrak{g} \simeq \mathbb{R}^q$ and to avoid confusion, let us denote Y_ℓ by ∂_ℓ : the vectors $(\partial_\ell)_{1 \leq \ell \leq q}$ are the dual basis of the coordinates $(x_\ell)_{1 \leq \ell \leq q}$. The left-invariant basis then becomes explicit:

$$\text{for all } x = (x_1, \dots, x_q) \in \mathfrak{g}, \quad \tilde{Y}_\ell^L(x) = \partial_\ell + \sum_{\ell'=1}^q \zeta_{\ell, \ell'}(x_1, \dots, x_q) \partial_{\ell'}, \quad (30)$$

with, thanks to (29) and (21),

$$\zeta_{\ell, \ell'}(x) = \sum_{n=1}^{m-1} \frac{(-1)^n B_n}{n!} \sum_{\ell_1, \dots, \ell_n} \kappa(\ell_1, \dots, \ell_n, \ell; \ell') x_{\ell_1} \dots x_{\ell_n}. \quad (31)$$

Let us point out that $\zeta_{\ell, \ell'} = 0$ if $\omega_{\ell'} \leq \omega_\ell$ (because κ vanishes), thus the left-invariant correction to ∂_ℓ only involves derivatives of a strictly higher weight. In other words, the indices in (30) can be restricted to $\ell' > n_{\omega_\ell} = q_1 + q_2 + \dots + q_{\omega_\ell}$.

Note also that $\zeta \in C^\infty(\mathbb{R}^q, \mathcal{M}_{q,q}(\mathbb{R}))$ and $\zeta(0) = 0$. More precisely, this matrix represents the differential action of left-translations, expressed in exponential coordinates:

$$(dL_g)|_e \equiv \text{Id}_{\mathbb{R}^q} + \zeta(x(g)). \quad (32)$$

Moreover, as $|x_\ell(g)| \lesssim \|g\|_G^{\omega_\ell}$, one has $|\zeta_{\ell, \ell'}(x(g))| \lesssim \|g\|_G^{\omega_{\ell'} - \omega_\ell}$.

2. Proof of Theorem 1

This section is devoted to the proof of the main statement. The key idea is to prove the result for $s = 1$ and then “push” the result up to the maximal regularity using

only integrations by part. Adding an interpolation step once the result is known for $s = 1$, but before pushing it to a higher regularity, allows one to capture all eligible fractional derivatives. The actual proof is written in the last subsection Section 2.5 but some preliminary results are required.

2.1. Symbol classes $S_n^\alpha(G)$. Symbol classes are a convenient way to classify the coefficients involved in the computations in terms of how they vanish at the origin.

Definition. For $\alpha \in \mathbb{R}_+$ and $n \in \mathbb{N} \cup \{\infty\}$, the symbol class $S_n^\alpha(G)$ is defined as the set of functions $\varphi \in L_{\text{loc}}^\infty(G)$ such that for any multi-index γ of length $|\gamma| \leq n$, there exists a constant $C_\gamma > 0$ that ensures the following inequality:

$$\text{for all } g \in G, \quad \|g\|_G \leq 1 \implies |\nabla_G^\gamma \varphi(g)| \leq C_\gamma \|g\|_G^{(\alpha - |\gamma|)_+}. \quad (33)$$

For example, the symbols of class $S_0^0(G) = L_{\text{loc}}^\infty(G)$ are only required to be bounded near the origin. The symbol class $S_\infty^\alpha(G)$ is also denoted $S^\alpha(G)$.

The following properties hold.

(1) The Leibnitz formula gives

$$\varphi \in S_m^\alpha(G) \text{ and } \psi \in S_n^\beta(G) \implies \varphi\psi \in S_{\min(m,n)}^{\alpha+\beta}(G).$$

(2) As Y_ℓ^L is a linear combination of derivatives ∇_G^γ of length $|\gamma| = \omega_\ell$, one has (if $n \geq \omega_\ell$)

$$\varphi \in S_n^\alpha(G) \implies Y_\ell^L \varphi \in S_{n-\omega_\ell}^{(\alpha-\omega_\ell)_+}(G).$$

(3) As smooth functions are locally bounded, one has also

$$S_{\alpha-1}^\alpha(G) \cap C^\infty(G) \subset S^\alpha(G).$$

The coordinates and the coefficients of the left-invariant vector fields belong to the following classes.

Proposition 6. *One has*

$$\text{for all } \ell \in \{1, \dots, q\}, \quad x_\ell(g) \in S^{\omega_\ell}(G) \quad (34)$$

and

$$\text{for all } \ell, \ell' \in \{1, \dots, q\}, \quad \zeta_{\ell, \ell'}(x_1(g), \dots, x_q(g)) \in S^{\omega_{\ell'} - \omega_\ell}(G). \quad (35)$$

Proof. We already observed that $|x_\ell(g)| \lesssim \|g\|_G^{\omega_\ell}$ thus $x_\ell(g) \in S_0^{\omega_\ell}(G)$. Next, using (30), one gets

$$Y_{\ell_0}^L(x_\ell(g)) = \delta_{\ell_0, \ell} + \zeta_{\ell_0, \ell}(x(g)) = \begin{cases} \zeta_{\ell_0, \ell}(x(g)) & \text{if } \omega_{\ell_0} < \omega_\ell, \\ \delta_{\ell_0, \ell} & \text{if } \omega_{\ell_0} = \omega_\ell, \\ 0 & \text{if } \omega_{\ell_0} > \omega_\ell. \end{cases}$$

Assuming $\ell_0 \in \{1, \dots, n_1\}$, one gets $|\nabla_G x_\ell(g)| \leq C \|g\|_G^{\omega_\ell - 1}$ thus $x_\ell(g) \in S_1^{\omega_\ell}(G)$. One can now bootstrap this partial result in the expression (31), which gets us $\zeta_{\ell, \ell'}(x(g)) \in S_1^{\omega_{\ell'} - \omega_\ell}(G)$. The previous expression now reads $\nabla_G x_\ell(g) \in S_1^{\omega_\ell - 1}$ and thus $x_\ell(g) \in S_2^{\omega_\ell}(G)$. Iterating this process leads to $x_\ell(g) \in S_\infty^{\omega_\ell}(G)$ and $\zeta_{\ell, \ell'}(x(g)) \in S_\infty^{\omega_{\ell'} - \omega_\ell}(G)$. \square

The key result is that one can adjust the gauge norm to be a symbol of order 1 (see also Section 3).

Proposition 7. *There exists a symbol $\rho(g) \in S_1^1(G)$ that is uniformly equivalent to the gauge norm. For higher-order derivatives, it satisfies for any multi-index γ :*

$$\text{for all } g \in G, \quad \rho(g) \leq 1 \implies |\nabla_G^\gamma \rho| \leq \frac{C_\gamma}{\rho^{|\gamma|-1}}. \quad (36)$$

Moreover, there exists $w \in \mathbb{N}$ such that $\rho^w \in S^w(G)$.

Proof. Let us now modify the gauge norm (28) into the uniformly equivalent gauge

$$\rho(g) = \left(\sum_{\ell=1}^q |x_\ell(g)|^{w/\omega_\ell} \right)^{1/w} \quad (37)$$

with $w = 2 \text{LCM}(1, \dots, m)$ to ensure that each $w/\omega_\ell \in 2\mathbb{N}$. In particular, $\rho(g)^w \in C^\infty(G)$. Next, one computes the first horizontal derivative of the norm, using (30):

$$\nabla_G \rho = \frac{\nabla_G(\rho^w)}{w\rho^{w-1}} = \left(\frac{1}{\rho^{w-1}} \left[x_\ell^{w-1} + \sum_{\ell'=1}^q \frac{\zeta_{\ell, \ell'}(x) x_{\ell'}^{w/\omega_{\ell'}-1}}{\omega_{\ell'}} \right] \right)_{\ell=1 \dots n_1}.$$

The expression in square brackets is a symbol of class $S^{w-1}(G)$ because of (34) and (35) and $\omega_\ell = 1$ for $\ell \leq n_1$. Thus $\nabla_G \rho$ is bounded near the origin which means that the modified gauge ρ belongs to $S_1^1(G)$. Next, one observes that for any $\alpha \geq 1$, if $\theta \in S^\alpha(G)$ then

$$\nabla_G \left(\frac{\theta}{\rho^\alpha} \right) = \left(\frac{\nabla_G \theta}{\rho^{\alpha-1}} - \alpha \frac{\theta}{\rho^\alpha} \nabla_G \rho \right) \frac{1}{\rho} = \left(\frac{\theta_1}{\rho^{\alpha-1}} + \frac{\theta_2}{\rho^{\alpha+w-1}} \right) \frac{1}{\rho}$$

with $\theta_1 \in S^{\alpha-1}(G)$ and $\theta_2 \in S^{\alpha+w-1}(G)$. One can thus claim by recurrence on the length of the multi-index γ that

$$\nabla_G^\gamma \rho = \left(\theta_{\gamma,0} + \sum_k \frac{\theta_{\gamma,k}}{\rho^{\alpha_k}} \right) \frac{1}{\rho^{|\gamma|-1}},$$

where $\theta_{\gamma,k} \in S^{\alpha_k}(G)$ is a polynomial in $x_\ell(g)$ with $\alpha_k \geq 1$ and $\theta_{\gamma,0}$ is a polynomial. Note that a polynomial in $S^0(G)$ is necessarily the sum of a constant and a polynomial in $S^1(G)$ and that, by (30), the horizontal derivatives of a polynomial

are also a polynomial. This ensures (36). The final assertion about $\|\cdot\|_G^w$ follows immediately from (34). \square

From now on, one will modify the gauge norm accordingly and assume that $\|\cdot\|_G = \rho \in S_1^1(G)$.

2.2. The radial vector field. The infinitesimal generator of dilations on \mathfrak{g} is the linear operator $\tilde{R} : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$\tilde{R} = \sum_{j=1}^m j\pi_j. \quad (38)$$

It is diagonalizable with positive eigenvalues; its trace $\text{Tr } \tilde{R} = Q$ is the homogeneous dimension. One checks immediately that $\delta_r = e^{(\log r)\tilde{R}}$, thus $\tilde{R}(x) = \frac{d}{dr}\delta_r(x)|_{r=1}$. The pair $(x, \tilde{R}(x))$ is a vector field on \mathfrak{g} whose expression in exponential coordinates follows from (26):

$$\text{for all } x = (x_1, \dots, x_q) \in \mathfrak{g}, \quad \tilde{R}(x) = \sum_{\ell=1}^q \omega_\ell x_\ell \partial_\ell. \quad (39)$$

Its exponential lift is called the *radial field* on G :

$$R(g) = d\exp_v(\tilde{R}(v)) = \frac{d}{dr}(rg)\Big|_{r=1}. \quad (40)$$

Proposition 8. *The radial vector field is scaling invariant:*

$$R(rg) = ((L_{rg})_* \circ \delta_r \circ (L_g)_*^{-1})(R(g)). \quad (41)$$

Moreover, it can be expressed in terms of left-invariant derivatives:

$$R(g) = \sum_{\ell=1}^q \sigma_\ell(x_1(g), \dots, x_q(g)) Y_\ell^L(g) \quad (42)$$

with $\sigma_\ell(x(g))$ equal to

$$\omega_\ell x_\ell(g) + \sum_{n=1}^{m-1} \frac{(-1)^n}{(n+1)!} \sum_{\ell_1, \dots, \ell_{n+1}} x_{\ell_1}(g) \cdots x_{\ell_{n+1}}(g) \cdot \omega_{\ell_{n+1}} \kappa(\ell_1, \dots, \ell_{n+1}; \ell) \in S^{\omega_\ell}(G).$$

Remarks. (1) Note that the variable x_{ℓ_i} that appears in the second term defining σ_ℓ must satisfy

$$\omega_\ell = \omega_{\ell_1} + \cdots + \omega_{\ell_{n+1}}$$

because if it is not the case, then $\kappa(\ell_1, \dots, \ell_{n+1}; \ell) = 0$. In particular, as there are at least $n+1 \geq 2$ factors, one has $\omega_{\ell_i} < \omega_\ell$ for each i .

- (2) Both expressions for $\tilde{R} = \sum \omega_\ell x_\ell \partial_\ell = \sum \sigma_\ell \tilde{Y}_\ell^L$ combined with (30) provide a remarkable identity embedded in the commutator structure. For any $\ell' \in \{1, \dots, q\}$,

$$\begin{aligned} \sum_{n=1}^{m-1} \frac{(-1)^n}{(n+1)!} \sum_{\ell_1, \dots, \ell_{n+1}} x_{\ell_1} \cdots x_{\ell_{n+1}} \omega_{\ell_{n+1}} \\ \cdot \left(\kappa(\ell_1, \dots, \ell_{n+1}; \ell') + \sum_{\ell} \kappa(\ell_1, \dots, \ell_{n+1}; \ell) \zeta_{\ell, \ell'}(x) \right) \\ = - \sum_{\ell} \omega_\ell x_\ell \zeta_{\ell, \ell'}(x). \end{aligned}$$

Note that when one substitutes $x = x(g)$, both sides are indeed symbols of class $S^{\omega_{\ell'}}(G)$.

Proof. Formula (41) follows, e.g., from the identities $\tilde{R} \circ \delta_r = \delta_r \circ \tilde{R}$ and $\text{ad} \circ \delta_r = \delta_r \circ \text{ad} \circ \delta_r^{-1}$:

$$\begin{aligned} R(rg) &= d\exp|_{\delta_r(v)} \circ \delta_r \circ \tilde{R}(v) \\ &= (L_{rg})_* \circ \left(\frac{1 - e^{-\text{ad}(\delta_r(v))}}{\text{ad}(\delta_r(v))} \right) \circ \delta_r \circ \tilde{R}(v) \\ &= (L_{rg})_* \circ \delta_r \circ \left(\frac{1 - e^{-\text{ad}(v)}}{\text{ad}(v)} \right) \circ \tilde{R}(v) \\ &= ((L_{rg})_* \circ \delta_r \circ (L_g)_*^{-1})(R(g)). \end{aligned}$$

The definition of $R(g)$ with $g = \exp(v)$ also reads

$$R(g) = \sum_{\ell=1}^q \omega_\ell x_\ell(g) (d\exp|_v Y_\ell).$$

Combining the expression for the differential of \exp given by (11), the identity $[u, v]^L = [u^L, v^L]$ and the fact that $v = \sum x_\ell(g) Y_\ell$ give

$$\begin{aligned} R(g) &= \sum_{\ell=1}^q \omega_\ell x_\ell(g) \left(\sum_{n=0}^{m-1} \frac{(-1)^n}{(n+1)!} [v, \dots, [v, Y_\ell]]_{n \text{ times } v} \right)^L(g) \\ &= \sum_{\ell=1}^q \omega_\ell x_\ell(g) \left(Y_\ell^L(g) + \sum_{n=1}^{m-1} \sum_{\ell_1, \dots, \ell_n} \frac{(-1)^n}{(n+1)!} x_{\ell_1}(g) \cdots x_{\ell_n}(g) [Y_{\ell_1}, \dots, [Y_{\ell_n}, Y_\ell]]^L(g) \right). \end{aligned}$$

This formula can be further simplified into (42) by using (21). The symbol property comes from (34) and the restriction on nonvanishing indices imposed by (21). \square

Examples. The previous computation can be simplified further by observing the antisymmetric role of ℓ_n and ℓ_{n+1} in $\omega_{\ell_{n+1}}[Y_{\ell_n}, Y_{\ell_{n+1}}]$ if $\omega_{\ell_n} = \omega_{\ell_{n+1}}$. For $m \leq 4$, one thus gets the following expressions for the radial field.

(1) For a group of step $m = 2$, the radial field is given by

$$R(g) = \sum_{\ell=1}^q \omega_{\ell} x_{\ell}(g) Y_{\ell}^L(g).$$

On the Heisenberg group \mathbb{H} with exponential coordinates introduced in Section 1.6, this formula boils down, as expected, to

$$R(g) = (pY_1 + qY_2 + 2(r - \frac{1}{2}pq)Y_3)^L(g) = p\partial_p + q\partial_q + 2r\partial_r.$$

(2) For a group of step $m = 3$, the radial field is “corrected” along V_3 :

$$R(g) = \sum_{\ell=1}^q \omega_{\ell} x_{\ell}(g) Y_{\ell}^L - \frac{1}{2} \sum_{\substack{1 \leq \ell_1 \leq n_1 \\ n_1 < \ell_2 \leq n_2}} x_{\ell_1}(g) x_{\ell_2}(g) [Y_{\ell_1}, Y_{\ell_2}]^L.$$

(3) For step $m = 4$, its expression involves a further “correction” along V_4 that is split among two types of commutators:

$$\begin{aligned} R(g) = \sum_{\ell=1}^q \omega_{\ell} x_{\ell}(g) Y_{\ell}^L - \frac{1}{2} \sum_{\substack{1 \leq \ell_1 \leq n_1 \\ n_1 < \ell_2 \leq n_3}} x_{\ell_1}(g) x_{\ell_2}(g) [Y_{\ell_1}, Y_{\ell_2}]^L \\ + \frac{1}{6} \sum_{\substack{1 \leq \ell_1, \ell_2 \leq n_1 \\ n_1 < \ell_3 \leq n_2}} x_{\ell_1}(g) x_{\ell_2}(g) x_{\ell_3}(g) [Y_{\ell_1}, [Y_{\ell_2}, Y_{\ell_3}]]^L. \end{aligned}$$

Proposition 9. *The gauge norm (37) and the radial field are related by the following formula:*

$$\text{for all } s > 0, \quad \frac{1}{\|\cdot\|_G^{2s}} = -\frac{1}{2s} R\left(\frac{1}{\|\cdot\|_G^{2s}}\right). \quad (43)$$

Proof. Applying the chain rule, one gets

$$\frac{\lambda(g)}{\|g\|_G^{2s}} = -\frac{1}{2s} R\left(\frac{1}{\|g\|_G^{2s}}\right)$$

with $\lambda(g) = R(\|g\|_G)/\|g\|_G$ and where the field R is obviously computed at the same point $g \in G$ as the function that is being differentiated. Let us also observe that

$$\lambda(g) = \frac{R(\|g\|_G^w)}{w\|g\|_G^w}$$

for any $w \in \mathbb{N}^*$ and in particular for $w = 2 \operatorname{LCM}(1, \dots, m)$ for which we know that $\|\cdot\|_G^w \in S^w(G)$ by Proposition 7. Using the formula (37) for the modified gauge norm and (39) for the expression of the radial field in exponential coordinates, one then gets (note that $w/\omega_\ell \in 2\mathbb{N}^*$)

$$\text{for all } x \in \mathfrak{g} \simeq \mathbb{R}^q, \quad \tilde{R}(\|x\|_{\mathfrak{g}}^w) = \sum_{\ell=1}^q \omega_\ell x_\ell \cdot \frac{w}{\omega_\ell} x_\ell^{w/\omega_\ell - 1} = w \|x\|_{\mathfrak{g}}^w$$

and thus $\lambda(g) = 1$ for any $g \in G$. □

2.3. Adjoints.

Proposition 10. *For the $L^2(G)$ scalar product, the adjoint vector field to R is*

$$R^*(g) = -Q - R(g).$$

Proof. The proof is simplest in exponential coordinates, using (39) and (5):

$$\text{for all } x \in \mathfrak{g} \simeq \mathbb{R}^q, \quad \tilde{R}(x) + \tilde{R}^*(x) = \operatorname{div} \tilde{R} = \sum_{\ell=1}^q \omega_\ell = Q.$$

One can also prove this formula directly, using (42) and (30):

$$\text{for all } g \in \mathfrak{G}, \quad R(g) + R^*(g) = \sum_{\ell} \partial_{\ell}(\sigma_{\ell}) + \sum_{\ell, \ell'} \zeta_{\ell, \ell'} \cdot (\partial_{\ell'} \sigma_{\ell}) + \sigma_{\ell} \cdot (\partial_{\ell'} \zeta_{\ell, \ell'}).$$

In this sum, according to a remark that follows (30), the index ℓ' is restricted to $\ell' > n_{\omega_{\ell}}$ and, in particular, the definition (3) then implies $\omega_{\ell'} > \omega_{\ell}$. Now thanks to the remark that follows Proposition 8, one can claim that the variable x_{ℓ} does not appear in the second part of σ_{ℓ} , thus its derivative reads

$$\partial_{\ell}(\sigma_{\ell}) = \omega_{\ell}.$$

For a similar reason, $\partial_{\ell'} \sigma_{\ell} = 0$ for $\omega_{\ell'} > \omega_{\ell}$. One observes also that in (31), each ℓ_i involved in the expression of $\zeta_{\ell, \ell'}$ must satisfy $\omega_{\ell_i} < \omega_{\ell'}$. In particular, $\partial_{\ell'} \zeta_{\ell, \ell'} = 0$. One concludes using (5). □

The next property checks that left-invariant vector fields on a Carnot group are divergence-free.

Proposition 11. *For the $L^2(G)$ scalar product, the adjoint vector field to Y_{ℓ}^L is $-Y_{\ell}^L$. In particular, for any smooth function ψ on G and any $\ell_1, \dots, \ell_n \in \{1, \dots, q\}$,*

$$\int_G [Y_{\ell_1}^L, \dots, [Y_{\ell_{n-1}}^L, Y_{\ell_n}^L]] \psi(g) \cdot \psi(g) dg = 0. \quad (44)$$

Proof. The second “computational” proof of the previous proposition (the one based on (30)) also ensures that

$$\partial_{\ell'} \zeta_{\ell, \ell'} = 0$$

when $\omega_{\ell'} > \omega_{\ell}$ and therefore $(Y_{\ell}^L)^* = -Y_{\ell}^L$. As the commutator of two anti-symmetric operators is also an antisymmetric one, the second statement follows immediately. \square

2.4. A density result. The following density result can be proved by a scaling argument.

Proposition 12. *The space $\mathcal{D}(G \setminus \{e\})$ of C^{∞} functions, compactly supported outside the origin, is dense in $H^s(G)$ for any $0 \leq s < Q/2$.*

Proof. One can use a Hilbert space approach based on scaling and Schwartz’s theorem for distributions. Let us assume additionally that $s \in \mathbb{N}$ and consider a function $u \in H^s(G)$ that is orthogonal to any $\varphi \in \mathcal{D}(G \setminus \{e\})$, e.g.,

$$(u, \varphi)_s = \sum_{|\gamma| \leq s} \int_G \nabla_G^{\gamma} u(g) \cdot \nabla_G^{\gamma} \varphi(g) dg = 0.$$

Integrating by parts (using Proposition 11 and the notation γ^* for the multi-index γ in reverse order) yields

$$\sum_{|\gamma| \leq s} (-1)^{|\gamma|} \int_G \nabla_G^{\gamma^*} \nabla_G^{\gamma} u(g) \cdot \varphi(g) dg = 0.$$

For fractional values of s , one would replace $\nabla_G^{\gamma^*} \nabla_G^{\gamma}$ by a fractional power of the sub-Laplacian (24) and what follows would go unchanged. Schwartz’s theorem implies that the distributional support of

$$v = \sum_{|\gamma| \leq s} (-1)^{|\gamma|} \nabla_G^{\gamma^*} \nabla_G^{\gamma} u$$

is reduced to the single point $\{e\}$ and thus $v = \sum (-1)^{|\alpha|} c_{\alpha} \partial^{\alpha} \delta$ where δ is the Dirac function at the origin. As v is at most a $2s$ -th horizontal derivative of u , one has $v \in H^{-s}(G)$ and in particular for any test function $\psi \in \mathcal{D}(G)$,

$$\left| \int_G v(g) \psi(g) dg \right|^2 \leq C \sum_{|\gamma| \leq s} \int_G |\nabla_G^{\gamma} \psi(g)|^2 dg.$$

The constant C does not depend on the support of ψ because $\text{supp } v \subset \{e\}$. In particular, one can apply this inequality to the dilations $\psi(rg)$ for any $r > 1$:

$$\left| \int_G v(g) \psi(rg) dg \right|^2 \leq C \sum_{|\gamma| \leq s} r^{2|\gamma|} \int_G |\nabla_G^\gamma \psi(rg)|^2 dg.$$

thus

$$\left| \int_G v(r^{-1}g) \psi(g) dg \right|^2 \leq C \sum_{|\gamma| \leq s} r^{2|\gamma|+Q} \int_G |\nabla_G^\gamma \psi(g)|^2 dg.$$

Finally, one can compute the left-hand side using the homogeneity of the Dirac mass:

$$\int_G v(r^{-1}g) \psi(g) dg = \sum_{\alpha} c_{\alpha} r^{Q+\sum \alpha_j \omega_j} \partial^{\alpha} \psi(e).$$

Combining both formulas, one gets for any $r > 1$:

$$\left| \sum_{\alpha} c_{\alpha} r^{Q+\sum \alpha_j \omega_j} \partial^{\alpha} \psi(e) \right| \leq C r^{Q/2+s} \|\psi\|_{H^s(G)}$$

and in particular with a suitable choice of ψ and $r \rightarrow \infty$,

$$c_{\alpha} \neq 0 \implies s \geq \frac{Q}{2} + \sum \alpha_j \omega_j.$$

But as $s < Q/2$, each coefficient c_{α} vanishes, i.e., $v = 0$ in $H^{-s}(G)$ and thus using $u \in H^s(G)$ as a test function, one infers $u = 0$. \square

Remark. When Q is even and $s = Q/2 \in \mathbb{N}$, the previous density result still holds. The only change in the proof is to observe that $\delta \notin H^{-Q/2}(G)$ by exhibiting an example of an unbounded function in $H^{Q/2}(G)$; the classical example $\log(-\log\|g\|)\psi(g)$ with a smooth cut-off ψ still works. However, when Q is odd, one still has $\delta \notin H^{-Q/2}(G)$ but the density result *fails* as it already does in $H^{n+1/2}(\mathbb{R}^{2n+1})$. For more details on this point, see [Vigneron 2006].

2.5. Hardy inequality. In this final section, let us combine the previous results into a proof of Theorem 1.

Given $f \in H^s(G)$ with $s < Q/2$ and the density result of the previous section, one can assume without restriction that f is compactly supported and that $0 \notin \text{Supp } u$. Next, one will take a smooth cutoff function $\chi : \mathbb{R} \rightarrow [0, 1]$ such that $\chi(t) = 1$ if $|t| < \frac{1}{2}$. For any $\rho_0 > 0$, one has

$$\int_G \frac{|f(g)|^2}{\|g\|_G^{2s}} \leq \int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} + \left(\frac{2}{\rho_0}\right)^{2s} \|f\|_{L^2(G)}^2 \quad (45)$$

with $\varphi(g) = \chi\left(\frac{\|g\|_G}{\rho_0}\right) f(g)$.

Moreover, one has $\|\varphi\|_{H^s(G)} \leq C_s \rho_0^{-s} \|f\|_{H^s(G)}$. Without restriction, one can therefore assume that f (now denoted by φ) is compactly supported in a fixed but arbitrary small annular neighborhood around the origin.

The key of the computation is the following integration by part argument. Using (43), one has

$$\int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} = -\frac{1}{2s} \int_G R\left(\frac{1}{\|g\|_G^{2s}}\right) \cdot |\varphi(g)|^2.$$

Using Proposition 10 and the fact that $\text{supp } \varphi$ is an annulus around the origin so that no boundary terms appear:

$$\left(\frac{Q}{2} - s\right) \int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} = - \int_G \frac{\varphi(g) R(\varphi(g))}{\|g\|_G^{2s}}.$$

According to (42), the radial field can be expressed with left-invariant vector fields:

$$\left(\frac{Q}{2} - s\right) \int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} = - \sum_{\ell=1}^q \int_G \frac{\sigma_\ell(x(g)) \varphi(g) Y_\ell^L(\varphi(g))}{\|g\|_G^{2s}}. \quad (46)$$

What we do next depends on the order of each derivative $Y_\ell^L \simeq \nabla_G^{\omega_\ell}$.

Case $m = 1$. In the Euclidean case, one uses Cauchy–Schwarz and Young’s identity $|ab| \leq \varepsilon a^2 + \varepsilon^{-1} b^2$ with $\varepsilon > 0$ small enough so that $s + \varepsilon < Q/2$, which leads to

$$\left(\frac{Q}{2} - s - \varepsilon\right) \int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} \leq \varepsilon^{-1} C \int_G \frac{|\nabla_G \varphi|^2}{\|g\|_G^{2(s-1)}}. \quad (47)$$

This proves Hardy’s inequality for $s = 1$. Interpolation with L^2 then ensures that the Hardy inequality holds for any $s \in [0, 1]$. Finally, the previous estimate provides a bootstrap argument from $s - 1$ to s for any $s < Q/2$.

Case $m = 2$. One uses the Euclidean technique to deal with the horizontal derivatives. For the stratum V_2 , one uses the commutator structure to backtrack one “half” integration by part. More precisely, the right-hand side of (46) becomes, for $1 \leq \ell \leq n_1$,

$$\left| \int_G \frac{\sigma_\ell(x(g)) \varphi(g) Y_\ell^L(\varphi(g))}{\|g\|_G^{2s}} \right| \leq \varepsilon \int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} + C_\varepsilon \int_G \frac{|\nabla_G \varphi(g)|^2}{\|g\|_G^{2(s-1)}}$$

and using (22) for $n_1 < \ell \leq n_2 = q$ and Proposition 11 (notice the cancellation of the highest order term thanks to the commutator structure):

$$\begin{aligned}
& - \int_G \frac{\sigma_\ell(x(g))}{\|g\|^{2s}} \varphi(g) \cdot [Y_{\alpha_1(\ell)}^L, Y_{\alpha_2(\ell)}^L](\varphi(g)) \\
& = \int_G Y_{\alpha_1(\ell)}^L \left(\frac{\sigma_\ell(x(g))}{\|g\|^{2s}} \right) \varphi(g) \cdot Y_{\alpha_2(\ell)}^L(\varphi(g)) \\
& \quad - \int_G Y_{\alpha_2(\ell)}^L \left(\frac{\sigma_\ell(x(g))}{\|g\|^{2s}} \right) \varphi(g) \cdot Y_{\alpha_1(\ell)}^L(\varphi(g)).
\end{aligned}$$

Using the symbol properties of $\|\cdot\|_G$ and $\sigma_\ell(x(g))$, both terms are bounded in the following way:

$$\int_G \frac{|\varphi(g)| \cdot |\nabla_G \varphi(g)|}{\|g\|_G^{2s-1}} \leq \varepsilon \int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} + C_\varepsilon \int_G \frac{|\nabla_G \varphi(g)|^2}{\|g\|_G^{2(s-1)}}.$$

One thus gets (47) again and one can conclude the proof just as in the case $m = 1$.

Case $m \geq 3$. The additional terms on the right-hand side of (46) correspond to $n_2 < \ell \leq q$. Thanks to (22), one can express each of them with commutators from the first stratum:

$$I_\ell(\varphi) = - \int_G \frac{\sigma_\ell(x(g))}{\|g\|^{2s}} \varphi(g) \cdot [Y_{\alpha_1(\ell)}^L, \dots, [Y_{\alpha_{\omega_\ell-1}(\ell)}^L, Y_{\alpha_{\omega_\ell}(\ell)}^L]](\varphi(g)).$$

As in the case $m = 2$, the key is to use the commutator structure to put all the derivatives but one on the symbol. More precisely, using Proposition 11, one first gets

$$I_\ell(\varphi) = \frac{1}{2} \int_G [Y_{\alpha_1(\ell)}^L, \dots, [Y_{\alpha_{\omega_\ell-1}(\ell)}^L, Y_{\alpha_{\omega_\ell}(\ell)}^L]] \left(\frac{\sigma_\ell(x(g))}{\|g\|^{2s}} \right) |\varphi(g)|^2.$$

Next, one puts the outermost derivative back out onto $\varphi(g)^2$:

$$I_\ell(\varphi) = - \sum_{i=1}^{\omega_\ell} \int_G W_{\ell,i} \left(\frac{\sigma_\ell(x(g))}{\|g\|^{2s}} \right) \varphi(g) \cdot Y_{\alpha_i(\ell)}^L \varphi(g),$$

where each $W_{\ell,i}$ is a derivative of order $\omega_\ell - 1$. The symbol property $\sigma_\ell(x(g)) \in S^{\omega_\ell}(G)$ given by Proposition 8 ensures that

$$\left| W_{\ell,i} \left(\frac{\sigma_\ell(x(g))}{\|g\|^{2s}} \right) \right| \leq \frac{C_{\ell,i,s}}{\|g\|_G^{2s-1}}.$$

Again, one gets

$$|I_\ell(\varphi)| \leq \varepsilon \int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} + C_\varepsilon \int_G \frac{|\nabla_G \varphi(g)|^2}{\|g\|_G^{2(s-1)}}$$

and (47) holds once more. As in the case $m = 1$, one thus gets the Hardy inequality for $s = 1$. Then, by interpolation with $L^2(G)$, one gets it for $s \in [0, 1]$. Finally, using (47) iteratively, one can collect it for any $s < Q/2$.

Remark. When Q is odd and $s \in \mathbb{N}$, one can always take ε small enough so that $Q/2 - s - \varepsilon \neq 0$ in (47). The previous iteration argument thus proves the Hardy inequality (6) for any $s \in \mathbb{N}$, but the result is then only valid for functions that belong to the $H^s(G)$ -closure of smooth compactly supported functions whose support avoids the origin.

2.6. Homogeneous Hardy inequality. One can slightly improve (6) by using a simple scaling argument. For simplicity, we will only spell out the procedure for $s \in \mathbb{N}$ and $0 \leq s < Q/2$ though it would also work for fractional values of s if ∇_G^s was replaced by the corresponding power $\mathcal{L}_G^{s/2}$ of the subelliptic Laplace operator (24).

Theorem 13. *For $0 \leq s < Q/2$, the following homogeneous inequality holds:*

$$\text{for all } f \in H^s(G) \quad \int_G \frac{|f(g)|^2}{\|g\|^{2s}} dg \leq 2C_s \|\nabla_G^s f\|_{L^2(G)}^2. \quad (48)$$

Proof. Let us indeed apply (6) to the function $f(r^{-1}g)$. After the change of variable $g = r\bar{g}$, one gets

$$r^{Q-2s} \int_G \frac{|f(\bar{g})|^2}{\|\bar{g}\|^{2s}} d\bar{g} \leq C_s \sum_{|\alpha| \leq s} r^{Q-2|\alpha|} \int_G |\nabla_G^\alpha f(\bar{g})|^2 d\bar{g}$$

which, for $r < 1$, can be further simplified into

$$\int_G \frac{|f(\bar{g})|^2}{\|\bar{g}\|^{2s}} d\bar{g} \leq C_s \sum_{|\alpha|=s} \int_G |\nabla_G^\alpha f(\bar{g})|^2 d\bar{g} + C_s r^2 \sum_{|\alpha| \leq s-1} \int_G |\nabla_G^\alpha f(\bar{g})|^2 d\bar{g}.$$

Choosing

$$r^2 = \min \left\{ 1; \frac{\|\nabla_G^s f\|_{L^2(G)}^2}{\|f\|_{H^{s-1}(G)}^2} \right\}$$

instantly leads to (48). □

Remark. It would have been tempting to try using (45)–(46) without digging further in the commutator structure to get

$$\int_G \frac{|f(g)|^2}{\|g\|^{2s}} dg \leq \int_G |f(g)|^2 dg + \left(\frac{Q}{2} - s - \varepsilon \right)^{-1} \sum_{\ell=1}^q \int_G \frac{|Y_\ell^L((\chi f)(g))|^2}{\|g\|^{2(s-\omega_\ell)}} dg.$$

For $s = 1$, it gives a Hardy inequality with $\|f\|_{H^m(G)}^2$ on the right-hand side. However, a scaling argument is then not sufficient to deduce the correct one, either (6) or (48). Indeed, one would simultaneously need to let $r \rightarrow \infty$ and $r \rightarrow 0$ to get rid of the superfluous derivatives without letting the lower-order $L^2(G)$ term get in the way, which is overall impossible.

3. A remark about the case of general hypoelliptic vector fields

For general families of vector fields that satisfy a Hörmander condition of step m , the technique of proving the Hardy inequality by integration by part works, but possibly with some restrictions.

3.1. A counterexample to the symbol property of the gauge. The main objection is the following one. When the group structure is discarded, the fact that one can chose a gauge pseudonorm in a symbol class of order 1 can fail.

For example, the family

$$Z_1 = \partial_1 + x_1 \partial_3, \quad Z_2 = \partial_2 + x_4 \partial_3 + x_5 \partial_4 \quad \text{and} \quad Z_3 = \partial_5$$

is uniformly of rank 3 in \mathbb{R}^5 and satisfies a uniform Hörmander bracket condition of step 3:

$$\partial_4 = [Z_3, Z_2], \quad \partial_3 = [[Z_3, Z_2], Z_2].$$

However, the “natural” gauge,

$$\rho = (|x_1|^{12} + |x_2|^{12} + |x_3|^4 + |x_4|^6 + |x_5|^{12})^{1/12},$$

is *not* a symbol of order 1 because $|Z_1 \rho| \geq c \rho^{-1}$ along $x_1^3 - x_3 = x_2 = x_4 = x_5 = 0$. Luckily, for this particular family, the change of variable $y_3 = x_3 - \frac{1}{2}x_1^2$ and $y_i = x_i$ ($i \neq 3$) transforms the family into $Z'_1 = \partial_{y_1}$, $Z'_2 = \partial_{y_2} + y_4 \partial_{y_3} + y_5 \partial_{y_4}$ and $Z'_3 = \partial_{y_5}$ and for this new family, the associated gauge is a symbol of order 1.

In [Vigneron 2006, Chapter 7], it was shown that up to a Hörmander condition of step 3, one can always modify the gauge by a local diffeomorphism to restore the symbol property. However, the same question for a family of vector fields that satisfy a Hörmander condition of step 4 or higher is still open. For the convenience of the reader, we will recall here briefly the key points of the discussion (and clarify the redaction), as this result was written in French and never published.

3.2. Regular hypoelliptic vector fields of step m . Let us consider a family $\mathfrak{X} = (X_\ell)_{1 \leq \ell \leq n_1}$ of vector fields on some smooth open set $\Omega \subset \mathbb{R}^q$ and

$$\text{for all } x \in \Omega, \quad W_k(x) = \text{Span}(X_i(x), \dots, [X_{j_1}, \dots, [X_{j_{k-1}}, X_{j_k}]](x)). \quad (49)$$

One assumes that $\underline{x} \in \Omega$ is a *regular Hörmander point*, i.e., that $n_k = \dim W_k$ is constant near \underline{x} and that $n_m = q$ for some finite integer $m \geq 2$.

Remark. At the origin of a Carnot group (2), one would have $W_k(e) = \bigoplus_{j=1}^k V_j$.

Next, one introduces a local basis of vector fields $(Y_\ell(x))_{1 \leq \ell \leq q}$, adapted to the stratification, i.e., $Y_\ell(x) \in W_{\omega_\ell}(x)$ where for each ℓ , the weight $\omega_\ell \in \{1, \dots, m\}$ is defined by (3). For simplicity, one will now restrict Ω to be a bounded and

small enough neighborhood of \underline{x} on which all those properties hold. The analog of horizontal derivatives is the family

$$\nabla_{\mathfrak{X}} = (Y_1, \dots, Y_{n_1}). \quad (50)$$

A local coordinate system $(x_\ell)_{1 \leq \ell \leq q}$ is said to be *adapted* to the commutator structure of the vector fields \mathfrak{X} near \underline{x} if the dual basis $(\partial_\ell)_{1 \leq \ell \leq q}$ satisfies $Y_\ell(\underline{x}) = \partial_\ell$.

Remark. Let us point out that *adapted* coordinates are not necessarily *privileged* in the sense of A. Bellaïche [1996] and M. Gromov [1996]: the point of coordinates $(x_\ell)_{1 \leq \ell \leq q}$ does not necessarily match with the image of \underline{x} under the composite action of the flows $e^{x_\ell Y_\ell}$ (for some predetermined order of composition).

In an adapted coordinate system, the *gauge* is defined by

$$\rho(x) = \left(\sum_{\ell=1}^q |x_\ell|^{w/\omega_\ell} \right)^{1/w}, \quad (51)$$

where $w = 2 \text{LCM}(1, \dots, m)$ and the basis of vector fields and their commutators satisfy

$$\text{for all } \ell \in \{1, \dots, q\}, \quad Y_\ell(x) = \partial_\ell + \sum_{\ell'=1}^q \zeta_{\ell, \ell'}(x) \partial_{\ell'}. \quad (52)$$

One obviously has $|x_\ell| \leq \rho(x)^{\omega_\ell}$ and, using a Taylor expansion, $\zeta_{\ell, \ell'}(x_0) = 0$ implies $|\zeta_{\ell, \ell'}(x)| \leq C\rho(x)$. However, for derivatives, one can only claim that $\nabla_{\mathfrak{X}}^\gamma x_\ell$ and $\nabla_{\mathfrak{X}}^\gamma \zeta_{\ell, \ell'}$ are bounded when $|\gamma| \geq 1$.

3.3. A positive result for hypoelliptic fields of step 2.

Theorem 14. *Let us consider a family of vector fields and $\underline{x} \in \Omega$ a regular Hörmander point of step $m = 2$. Then for any adapted coordinate system, the gauge ρ satisfies*

$$|\nabla_{\mathfrak{X}}^\gamma \rho| \leq C_\gamma \rho^{1-|\gamma|} \quad (53)$$

in the neighborhood of \underline{x} , for any multi-index γ .

Proof. For $\gamma = 0$, the estimate (53) comes from the fact that ρ is smooth and vanishes at the origin and thus admits a Taylor expansion at the origin that is locally bounded by $\sum |x_\ell|$ and thus by ρ . For $|\gamma| = 1$, the computation is actually explicit:

$$\nabla_{\mathfrak{X}} \rho = \frac{1}{\rho^3} \left(x_\ell^3 + \sum_{\ell' \leq n_1} \zeta_{\ell, \ell'} x_{\ell'}^3 + \frac{1}{2} \sum_{\ell' > n_1} \zeta_{\ell, \ell'} x_{\ell'} \right)_{1 \leq \ell \leq n_1}.$$

In the parenthesis, the first term is locally bounded by ρ^3 , the second by ρ^4 and the last one again by ρ^3 , thus $\nabla_{\mathfrak{X}} \rho \in L^\infty(\Omega)$ provided Ω is small enough. To deal

with the higher-order derivatives, let us introduce the class \mathcal{P}_n of homogeneous polynomials of x_ℓ and $\zeta_{\ell,\ell'}$ with smooth coefficients, i.e.,

$$\sum_{\alpha,\beta} x_1^{\alpha_1} \cdots x_q^{\alpha_q} \zeta_{1,1}^{\beta_{1,1}} \zeta_{1,2}^{\beta_{1,2}} \cdots \zeta_{q,q}^{\beta_{q,q}} f_{\alpha,\beta}(x),$$

where $f_{\alpha,\beta} \in C^\infty(\Omega)$ and $\sum \alpha_i \omega_i + \sum \beta_{j,j'} = n$. For $n \leq 0$, one sets $\mathcal{P}_n = C^\infty(\Omega)$. With the Leibnitz formula, one checks immediately that

$$\partial_\ell(\mathcal{P}_n) \subset \begin{cases} \mathcal{P}_{n-1} + \mathcal{P}_n & \text{if } \ell \leq n_1, \\ \mathcal{P}_{n-2} + \mathcal{P}_{n-1} + \mathcal{P}_n & \text{if } \ell > n_1, \end{cases}$$

thus $\nabla_{\mathfrak{X}}(\mathcal{P}_n) \subset \mathcal{P}_{n-1} + \mathcal{P}_n + \mathcal{P}_{n+1}$. Moreover, for $m \geq n$, any expression in \mathcal{P}_m is locally bounded by $C\rho^n$ for some constant C . We have shown above that $\nabla_{\mathfrak{X}}\rho \in \rho^{-3} \cdot (\mathcal{P}_3 + \mathcal{P}_4)$. One then gets recursively on $k = |\gamma|$ that $\nabla_{\mathfrak{X}}^\gamma \rho$ is a linear combination of expressions

$$\frac{\mathcal{P}_m}{\rho^{n+k-1}}$$

with $m \geq n$ and is thus locally bounded by $C\rho^{1-k}$. \square

Remark. One has $\mathcal{P}_n \subset \mathcal{P}_{n-2}$. However, for $\ell > n_1$, one has $x_\ell^2 \in \mathcal{P}_4 \cap \mathcal{P}_2$ but $x_\ell^2 \notin \mathcal{P}_3$.

3.4. Two positive results for hypoelliptic fields of step $m \geq 3$. Let us now revert to the case of a general value for m . As pointed out at the beginning of this section, one can find a counterexample of a family of vector fields, a regular Hörmander point of step $m = 3$ and an adapted coordinate system for which (53) fails. If we tried to run the previous proof, the failure point would be that

$$\partial_{\ell'}(\mathcal{P}_n) \subset \mathcal{P}_{n-\omega_{\ell'}} + \cdots + \mathcal{P}_{n-1} + \mathcal{P}_n.$$

When computing $\nabla_{\mathfrak{X}}(\mathcal{P}_n)$, the multiplication by $\zeta_{\ell,\ell'} \in \mathcal{P}_1$ is then not able to compensate for the loss when $\omega_{\ell'} \geq 3$. The profound reason is that our knowledge about the way $\zeta_{\ell,\ell'}$ vanishes at the origin is too weak.

Definition. A coordinate system adapted to the commutator structure of the vector fields \mathfrak{X} near a regular Hörmander point \underline{x} of step m is called *well-adapted* if

$$\text{for all } \ell \in \{1, \dots, n_1\}, \ell' \in \{1, \dots, q\}, \quad |\nabla_{\mathfrak{X}}^\gamma \zeta_{\ell,\ell'}| \leq C_\gamma \rho^{(\omega_{\ell'}-1-|\gamma|)_+} \quad (54)$$

in a neighborhood of \underline{x} . A family of vector fields that satisfies a regular Hörmander condition is called *well-structured* if it admits a well-adapted coordinate system.

One can check that in a well-adapted coordinate system, the gauge automatically satisfies (53).

Theorem 15 (If-theorem for arbitrary step m). *Let us consider a family of vector fields and $\underline{x} \in \Omega$ a regular Hörmander point of step m . Then for any **well** adapted coordinate system, the gauge ρ satisfies (53) in the neighborhood of \underline{x} .*

Proof. The key is to adapt the definition of \mathcal{P}_n to capture the enhanced knowledge that we gained about $\zeta_{\ell, \ell'}$. Let us define $\tilde{\mathcal{P}}_n$ as the subset of $C^\infty(\Omega)$ that consists of homogeneous polynomials with smooth coefficients of x_ℓ , $\zeta_{\ell, \ell'}$ and of the derivatives of $\zeta_{\ell, \ell'}$ for which we have estimates, i.e.,

$$\sum_{\alpha, \beta} x_1^{\alpha_1} \cdots x_q^{\alpha_q} \zeta_{1,1}^{\beta_{1,1}} \zeta_{1,2}^{\beta_{1,2}} \cdots \zeta_{q,q}^{\beta_{q,q}} \left(\prod_{\gamma} (\nabla_{\mathfrak{X}}^{\gamma} \zeta_{1,1})^{\delta_{\gamma;1,1}} \cdots (\nabla_{\mathfrak{X}}^{\gamma} \zeta_{n_1,q})^{\delta_{\gamma;n_1,q}} \right) f_{\alpha, \beta, \delta}(x),$$

where $f_{\alpha, \beta, \delta} \in C^\infty(\Omega)$, γ denotes multi-indices of length $|\gamma| \geq 1$ and

$$\sum_{1 \leq i \leq q} \alpha_i \omega_i + \sum_{j=1}^q \sum_{j'=1}^q \beta_{j,j'} (\omega_{j'} - 1) + \sum_{|\gamma| \geq 1} \sum_{j=1}^{n_1} \sum_{j'=1}^q \delta_{\gamma; j, j'} (\omega_{j'} - 1 - |\gamma|)_+ = n.$$

Note that only the factors for which $\omega_{j'} - 1 - |\gamma| > 0$ are significant; the others can simply be tossed into $f_{\alpha, \beta, \delta}$. For $n \leq 0$, one sets again $\tilde{\mathcal{P}}_n = C^\infty(\Omega)$. We also introduce the linear span

$$\tilde{\mathcal{P}}_n^+ = \sum_{m \geq n} \tilde{\mathcal{P}}_m.$$

Using the Leibnitz formula, $\nabla_{\mathfrak{X}}(x_\ell) \in \tilde{\mathcal{P}}_{\omega_\ell - 1}$ and $\nabla_{\mathfrak{X}}(\tilde{\mathcal{P}}_n) \subset \tilde{\mathcal{P}}_{n-1}^+$. One also has

$$\nabla_{\mathfrak{X}} \rho = \frac{\nabla_{\mathfrak{X}}(\rho^w)}{w \rho^{w-1}} \in \rho^{-(w-1)} \cdot \tilde{\mathcal{P}}_{w-1}^+$$

and recursively (note that $\rho^w \in \tilde{\mathcal{P}}_w$ allows one to convert $\tilde{\mathcal{P}}_0$ into $\rho^{-w} \cdot \tilde{\mathcal{P}}_w$)

$$\nabla_{\mathfrak{X}}^{\gamma} \rho \in \sum_{n \geq 1} \frac{\mathcal{P}_n^+}{\rho^{n+|\gamma|-1}}$$

from which (53) follows immediately. \square

The previous “abstract” theorem does not presume on the existence of a well-adapted coordinate system. However, when $m \leq 3$, it can actually be made to work.

Theorem 16. *Any family of vector fields that satisfies a regular Hörmander condition of step $m \leq 3$ admits at least one well-adapted coordinate system. It is therefore well-structured.*

Proof. For $m = 1$ and 2 , any adapted coordinate system is well-adapted. Let us thus focus on $m = 3$ and use the previous notations. Writing down the Taylor expansion

of the coefficients for $\ell \leq n_1$,

$$\zeta_{\ell, \ell'}(\underline{x}) = \sum_{i \leq n_1} \left(\frac{\partial \zeta_{\ell, \ell'}}{\partial x_i}(\underline{x}) \right) x_i + O(\rho^2),$$

it appears that, for $m = 3$, a coordinate system is well-adapted if and only if

$$\text{for all } \ell_1, \ell_2 \in \{1, \dots, n_1\}, \ell_3 \in \{n_2 + 1, \dots, q\}, \quad \frac{\partial \zeta_{\ell_1, \ell_3}}{\partial x_{\ell_2}}(\underline{x}) = 0. \quad (55)$$

Let us compute the following commutator:

$$[Y_{\ell_1}, Y_{\ell_2}] = \sum_{\ell=1}^q \left(\frac{\partial \zeta_{\ell_2, \ell}}{\partial x_{\ell_1}} - \frac{\partial \zeta_{\ell_1, \ell}}{\partial x_{\ell_2}} \right) \partial_{\ell}.$$

At the point \underline{x} , the terms corresponding to $\ell > n_2$ must belong to $W_2(\underline{x})$ and thus vanish, therefore

$$\text{for all } \ell_1, \ell_2 \in \{1, \dots, n_1\}, \ell_3 \in \{n_2 + 1, \dots, q\}, \quad \frac{\partial \zeta_{\ell_1, \ell_3}}{\partial x_{\ell_2}}(\underline{x}) = \frac{\partial \zeta_{\ell_2, \ell_3}}{\partial x_{\ell_1}}(\underline{x}). \quad (56)$$

One can now define a new coordinate system $(y_{\ell})_{1 \leq \ell \leq q}$ whose dual basis satisfies

$$\frac{\partial}{\partial y_{\ell}} = \frac{\partial}{\partial x_{\ell}} + \mathbf{1}_{\ell \leq n_1} \sum_{\ell' > n_2} \sum_{i \leq n_1} \left(\frac{\partial \zeta_{\ell, \ell'}}{\partial x_i}(\underline{x}) \right) x_i \cdot \frac{\partial}{\partial x_{\ell'}}.$$

This coordinate system is (locally) well defined because the fields $\frac{\partial}{\partial y_{\ell}}$ commute with each other thanks to (56). By construction, this coordinate system satisfies (55) and is therefore a well-adapted one. \square

Remark. The generalization of Theorem 16 for $m \geq 4$ is an *open* question. One can check that a coordinate system is well-adapted if and only if

$$\frac{\partial^{\alpha} \zeta_{\ell, \ell'}}{\partial x_1^{\alpha_1} \dots \partial x_q^{\alpha_q}}(\underline{x}) = 0 \quad (57)$$

for any indices such that $\omega_{\ell} = 1$, $\omega_{\ell'} \geq 3$ and $\sum_{i=1}^q \omega_i \alpha_i \leq \omega_{\ell'} - 2$. However, for $m \geq 4$, it is not clear whether the regular Hörmander assumption of step m is enough to ensure that the vector fields

$$\frac{\partial}{\partial y_{\ell}} = \frac{\partial}{\partial x_{\ell}} + \mathbf{1}_{\ell \leq n_1} \sum_{\ell' > n_2} \left(\sum_{\sum \alpha_i \omega_i \leq \omega_{\ell'} - 2} \left(\frac{\partial^{\alpha} \zeta_{\ell, \ell'}}{\partial x_1^{\alpha_1} \dots \partial x_q^{\alpha_q}}(\underline{x}) \right) x_1^{\alpha_1} \dots x_q^{\alpha_q} \cdot \frac{\partial}{\partial x_{\ell'}} \right)$$

commute with each other.

3.5. From the symbol property of the gauge to Hardy inequality. For well-structured families of vector fields, symbols of class $S^k(\mathfrak{X}; \rho)$ are functions f such that

$$|\nabla_{\mathfrak{X}}^{\gamma} f(x)| \leq C_{\gamma} \rho(x)^{(k-|\gamma|)_{+}} \quad (58)$$

in a neighborhood of \underline{x} , for any multi-index γ . Once the symbol property is established for the gauge, the path that leads to the Hardy inequality is open. The key (see [Vigneron 2006, Chapter 7]) is to define a “radial” vector field that admits both expressions:

$$R(x) = \sum_{\ell=1}^q \sigma_{\ell}(x) Y_{\ell}(x) = \sum_{k=1}^q (\omega_k x_k + \tilde{\sigma}_k(x)) \partial_k \quad (59)$$

in well-adapted coordinates, with $\sigma_{\ell} \in S^{\omega_{\ell}}(\mathfrak{X}; \rho)$ and $\tilde{\sigma}_k \in S^{\omega_k+1}(\mathfrak{X}; \rho)$. One can then check that

$$\operatorname{div} R = Q + O(\rho) \quad \text{and} \quad \lambda = \frac{R\rho}{\rho} \quad \text{satisfies} \quad \begin{cases} \lambda(x) = 1 + O(\rho), \\ R\lambda = O(\rho). \end{cases} \quad (60)$$

The computations of Section 2.5 can then be carried out in a small enough neighborhood of \underline{x} .

Acknowledgements

I would like to express my gratitude to J.-Y. Chemin, who brought this problem to my attention a long time ago, and to wish him a happy 60th birthday.

References

- [Adimurthi and Mallick 2018] Adimurthi and A. Mallick, “A Hardy type inequality on fractional order Sobolev spaces on the Heisenberg group”, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **18**:3 (2018), 917–949. MR Zbl
- [Ambrosio and Rigot 2004] L. Ambrosio and S. Rigot, “Optimal mass transportation in the Heisenberg group”, *J. Funct. Anal.* **208**:2 (2004), 261–301. MR Zbl
- [Ambrosio et al. 2019] L. Ambrosio, A. Pinamonti, and G. Speight, “Weighted Sobolev spaces on metric measure spaces”, *J. Reine Angew. Math.* **746** (2019), 39–65. MR Zbl
- [Bahouri and Cohen 2011] H. Bahouri and A. Cohen, “Refined Sobolev inequalities in Lorentz spaces”, *J. Fourier Anal. Appl.* **17**:4 (2011), 662–673. MR Zbl
- [Bahouri et al. 2005a] H. Bahouri, J.-Y. Chemin, and I. Gallagher, “Precised Hardy inequalities on \mathbb{R}^d and on the Heisenberg group \mathbb{H}^d ”, exposé, XIX in *Séminaire: Équations aux Dérivées Partielles, 2004–2005*, École Polytech., Palaiseau, 2005. MR Zbl
- [Bahouri et al. 2005b] H. Bahouri, J.-Y. Chemin, and C.-J. Xu, “Trace and trace lifting theorems in weighted Sobolev spaces”, *J. Inst. Math. Jussieu* **4**:4 (2005), 509–552. MR Zbl
- [Bahouri et al. 2009] H. Bahouri, J.-Y. Chemin, and C.-J. Xu, “Trace theorem on the Heisenberg group”, *Ann. Inst. Fourier (Grenoble)* **59**:2 (2009), 491–514. MR Zbl

- [Bahouri et al. 2012] H. Bahouri, C. Fermanian-Kammerer, and I. Gallagher, “Refined inequalities on graded Lie groups”, *C. R. Math. Acad. Sci. Paris* **350**:7-8 (2012), 393–397. MR Zbl
- [Bellaïche 1996] A. Bellaïche, “The tangent space in sub-Riemannian geometry”, pp. 1–78 in *Sub-Riemannian geometry*, edited by A. Bellaïche and J.-J. Risler, Progr. Math. **144**, Birkhäuser, Basel, 1996. MR Zbl
- [Ciatti et al. 2015] P. Ciatti, M. G. Cowling, and F. Ricci, “Hardy and uncertainty inequalities on stratified Lie groups”, *Adv. Math.* **277** (2015), 365–387. MR Zbl
- [D’Ambrosio 2005] L. D’Ambrosio, “Hardy-type inequalities related to degenerate elliptic differential operators”, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **4**:3 (2005), 451–486. MR Zbl
- [Dou et al. 2007] J. Dou, P. Niu, and Z. Yuan, “A Hardy inequality with remainder terms in the Heisenberg group and the weighted eigenvalue problem”, *J. Inequal. Appl.* (2007), Art. ID 32585. MR Zbl
- [Folland and Stein 1982] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes **28**, Princeton University Press, 1982. MR
- [Garofalo and Lanconelli 1990] N. Garofalo and E. Lanconelli, “Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation”, *Ann. Inst. Fourier (Grenoble)* **40**:2 (1990), 313–356. MR Zbl
- [Grillo 2003] G. Grillo, “Hardy and Rellich-type inequalities for metrics defined by vector fields”, *Potential Anal.* **18**:3 (2003), 187–217. MR Zbl
- [Gromov 1996] M. Gromov, “Carnot–Carathéodory spaces seen from within”, pp. 79–323 in *Sub-Riemannian geometry*, edited by A. Bellaïche and J.-J. Risler, Progr. Math. **144**, Birkhäuser, Basel, 1996. MR Zbl
- [Hardy et al. 1934] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, Univ. Press, 1934. Zbl
- [Hörmander 1967] L. Hörmander, “Hypoelliptic second order differential equations”, *Acta Math.* **119** (1967), 147–171. MR
- [Klarsfeld and Oteo 1989] S. Klarsfeld and J. A. Oteo, “The Baker–Campbell–Hausdorff formula and the convergence of the Magnus expansion”, *J. Phys. A* **22**:21 (1989), 4565–4572. MR Zbl
- [Kombe 2010] I. Kombe, “Sharp weighted Rellich and uncertainty principle inequalities on Carnot groups”, *Commun. Appl. Anal.* **14**:2 (2010), 251–271. MR Zbl
- [Krugljak et al. 1999] N. Krugljak, L. Maligranda, and L.-E. Persson, “The failure of the Hardy inequality and interpolation of intersections”, *Ark. Mat.* **37**:2 (1999), 323–344. MR
- [Montgomery 2002] R. Montgomery, *A tour of sub-Riemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs **91**, Amer. Math. Soc., Providence, RI, 2002. MR Zbl
- [Mustapha and Vigneron 2007] S. Mustapha and F. Vigneron, “Construction of Sobolev spaces of fractional order with sub-Riemannian vector fields”, *Ann. Inst. Fourier (Grenoble)* **57**:4 (2007), 1023–1049. MR
- [Niu et al. 2001] P. Niu, H. Zhang, and Y. Wang, “Hardy type and Rellich type inequalities on the Heisenberg group”, *Proc. Amer. Math. Soc.* **129**:12 (2001), 3623–3630. MR Zbl
- [Reutenauer 1993] C. Reutenauer, *Free Lie algebras*, London Mathematical Society Monographs. New Series **7**, Oxford University Press, 1993. MR Zbl
- [Rossmann 2002] W. Rossmann, *Lie groups: an introduction through linear groups*, Oxford Graduate Texts in Mathematics **5**, Oxford University Press, 2002. MR Zbl

- [Ruzhansky and Suragan 2017] M. Ruzhansky and D. Suragan, “On horizontal Hardy, Rellich, Caffarelli–Kohn–Nirenberg and p -sub-Laplacian inequalities on stratified groups”, *J. Differential Equations* **262**:3 (2017), 1799–1821. MR Zbl
- [Vigneron 2006] F. Vigneron, *Function spaces associated with a family of vector fields*, Ph.D. thesis, Ecole Polytechnique X, 2006, Available at <https://pastel.archives-ouvertes.fr/tel-00136144>.
- [Vigneron 2007] F. Vigneron, “The trace problem for Sobolev spaces over the Heisenberg group”, *J. Anal. Math.* **103** (2007), 279–306. MR Zbl

Received 14 Aug 2019.

FRANÇOIS VIGNERON:

francois.vigneron@u-pec.fr

Université Paris-Est, LAMA (UMR 8050), UPEC, UPEM, CNRS, 61, avenue du Général de Gaulle, F94010 Créteil, France

Tunisian Journal of Mathematics

msp.org/tunis

EDITORS-IN-CHIEF

- Ahmed Abbes CNRS & IHES, France
abbes@ihes.fr
- Ali Baklouti Faculté des Sciences de Sfax, Tunisia
ali.baklouti@fss.usf.tn

EDITORIAL BOARD

- Hajer Bahouri CNRS & LAMA, Université Paris-Est Créteil, France
hajer.bahouri@u-pec.fr
- Arnaud Beauville Laboratoire J. A. Dieudonné, Université Côte d'Azur, France
beauville@unice.fr
- Philippe Biane CNRS & Université Paris-Est, France
biane@univ-mlv.fr
- Ewa Damek University of Wrocław, Poland
edamek@math.uni.wroc.pl
- Bassam Fayad CNRS & Institut de Mathématiques de Jussieu - Paris Rive Gauche, France
bassam.fayad@imj-prg.fr
- Benoit Fresse Université Lille 1, France
benoit.fresse@math.univ-lille1.fr
- Dennis Gaitsgory Harvard University, United States
gaitsgde@gmail.com
- Paul Goerss Northwestern University, United States
pgoerss@math.northwestern.edu
- Emmanuel Hebey Université de Cergy-Pontoise, France
emmanuel.hebey@math.u-cergy.fr
- Mohamed Ali Jendoubi Université de Carthage, Tunisia
ma.jendoubi@gmail.com
- Sadok Kallel Université de Lille 1, France & American University of Sharjah, UAE
sadok.kallel@math.univ-lille1.fr
- Minhyong Kim Oxford University, UK & Korea Institute for Advanced Study, Seoul, Korea
minhyong.kim@maths.ox.ac.uk
- Toshiyuki Kobayashi The University of Tokyo & Kavli IPMU, Japan
toshi@kurims.kyoto-u.ac.jp
- Patrice Le Calvez Institut de Mathématiques de Jussieu - Paris Rive Gauche & Sorbonne Université, France
patrice.le-calvez@imj-prg.fr
- Yanyan Li Rutgers University, United States
yyli@math.rutgers.edu
- Nader Masmoudi Courant Institute, New York University, United States
masmoudi@cims.nyu.edu
- Haynes R. Miller Massachusetts Institute of Technology, United States
hrm@math.mit.edu
- Nordine Mir Texas A&M University at Qatar & Université de Rouen Normandie, France
nordine.mir@qatar.tamu.edu
- Enrique Pujals City University of New York, United States
epujals@gc.cuny.edu
- Mohamed Sifi Université Tunis El Manar, Tunisia
mohamed.sifi@fst.utm.tn
- Daniel Tataru University of California, Berkeley, United States
tataru@math.berkeley.edu
- Sundaram Thangavelu Indian Institute of Science, Bangalore, India
veluma@math.iisc.ernet.in
- Nizar Touzi Centre de mathématiques appliquées, Institut Polytechnique de Paris, France
nizar.touzi@polytechnique.edu

PRODUCTION

- Silvio Levy (Scientific Editor)
production@msp.org

The Tunisian Journal of Mathematics is an international publication organized by the Tunisian Mathematical Society (<http://www.tms.rnu.tn>) and published in electronic and print formats by MSP in Berkeley.

See inside back cover or msp.org/tunis for submission instructions.

The subscription price for 2020 is US \$320/year for the electronic version, and \$380/year (+\$20, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Tunisian Journal of Mathematics (ISSN 2576-7666 electronic, 2576-7658 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

TJM peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY



mathematical sciences publishers
nonprofit scientific publishing

Tunisian Journal of Mathematics

2020 vol. 2 no. 4

On log motives	733
TETSUSHI ITO, KAZUYA KATO, CHIKARA NAKAYAMA and SAMPEI USUI	
Equidistribution and counting of orbit points for discrete rank one isometry groups of Hadamard spaces	791
GABRIELE LINK	
A generalization of a power-conjugacy problem in torsion-free negatively curved groups	841
RITA GITIK	
A simple proof of the Hardy inequality on Carnot groups and for some hypoelliptic families of vector fields	851
FRANÇOIS VIGNERON	
Trigonometric series with a given spectrum	881
YVES MEYER	