Tunisian Journal of Mathematics

an international publication organized by the Tunisian Mathematical Society

2020 vol. 2 no. 4

msp

Tunisian Journal of Mathematics

msp.org/tunis

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Tunisian Journal of Mathematics (ISSN 2576-7666 electronic, 2576-7658 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

TJM peer review and production are managed by EditFlow® from MSP.

TUNISIAN JOURNAL OF MATHEMATICS
Vol. 2, No. 4, 2020
dx.doi.org/10.2140/tunis.2020.2.733

On log motives

Tetsushi Ito, Kazuya Kato, Chikara Nakayama and Sampei Usui

We define the categories of log motives and log mixed motives. The latter gives a new formulation for the category of mixed motives. We prove that the former is a semisimple abelian category if and only if the numerical equivalence and homological equivalence coincide, and that it is also equivalent to the latter being a Tannakian category. We discuss various realizations, formulate Tate and Hodge conjectures, and verify them in the curve case.

1. Introduction

1.1. In this paper, we define:

(1) the category of log motives over an fs log scheme, and

(2) the category of log mixed motives over an fs log scheme.

(1) is a generalization of the category of Grothendieck motives over a field with respect to the homological equivalence. The category (2) has $\oplus$, $\otimes$, dual, kernel and cokernel. We prove that the following (i), (ii), and (iii) are equivalent.

(i) The numerical equivalence and homological equivalence coincide in the category (1).

(ii) The category (1) is a semisimple abelian category.

(iii) The category (2) is a Tannakian category.

The equivalence of (i) and (ii) is the log version of the famous theorem of Jannsen [1992].

MSC2010: primary 14C15; secondary 14A20, 14F20.
Keywords: motive, mixed motive, log geometry.

1.2. We explain the organization briefly. In this paper, except in 2.1, an fs log scheme means an fs log scheme which has charts Zariski locally.

Let S be an fs log scheme. We fix a prime number ℓ and assume that ℓ is invertible over S.

After we give preparations in Section 2, we define in Section 3 the category of log motives over S, which is the log version of the category of motives of Grothendieck. In Section 4, we define the category of log mixed motives over S using the theory in Section 3.

Here we work modulo homological equivalence using ℓ-adic log étale cohomology theory.

In the case where the log structure of S is trivial, our construction gives a category of mixed motives over S modulo homological equivalence. This does not use the theory of Voevodsky [2000], though we hope our theory is connected to it. In the case $S = \mathrm{Spec}(k)$ for a field k of characteristic 0 with trivial log structure, our definition of the category of mixed motives over S is different from the definition of the category of mixed motives over k given by Jannsen [1990]. The difference lies in the definition of morphisms. We use K-theory whereas he uses absolute Hodge cycles.

Vologodsky [2015] and Park [2016] also defined log motives, respectively. They work with the formalism of triangulated categories à la Voevodsky. Our approach is more elementary to define the category of log mixed motives directly without defining its derived category. One can ask to compare our theory with theirs.

In Section 5, we introduce realizations that are not ℓ-adic. In Section 6, we discuss examples.

We explain each section of this paper more.

1.3. In Section 2, we give preparations on log geometry. We review results on log étale cohomology, log Betti cohomology, log de Rham cohomology, and log Hodge theory in 2.1, and then review or prove results on fans (2.2), on log modifications (2.3), and on the Grothendieck group of vector bundles on log schemes (2.4).

1.4. We explain more about Section 3.

Fix a prime number ℓ and let S be an fs log scheme on which ℓ is invertible. We define the category of log motives over S by imitating the definition of motive by Grothendieck modulo homological equivalence.

Recall that for a field k whose characteristic is not ℓ, the category of motives over k modulo (ℓ-adic) homological equivalence is defined as follows (see [Scholl 1994]). For a projective smooth scheme X over k and for $r \in \mathbb{Z}$, consider a symbol $h(X)(r)$. For projective smooth schemes X, Y over k and for $r, s \in \mathbb{Z}$, by a morphism $h(X)(r) \to h(Y)(s)$, we mean a homomorphism $\bigoplus_i H^i(X)_\ell(r) \to$

$\bigoplus_i H^i(Y)_\ell(s)$ which comes from $\mathrm{CH}(X \times Y)_\mathbb{Q}$. Here $H^i(X)_\ell$ is the étale cohomology group $H^i_{\text{ét}}(X \otimes_k \bar{k}, \mathbb{Q}_\ell)$ with $\bar{k}$ a fixed separable closure of k, (r) denotes the r-th Tate twist, the same for Y and s, and where $\mathrm{CH} = \bigoplus_i \mathrm{CH}^i$ is the Chow group and $(\cdot)_\mathbb{Q}$ means $\otimes\mathbb{Q}$. A motive over k is a pair $(h(X)(r), e)$, where X is a projective smooth scheme over k, $r \in \mathbb{Z}$, and e is an idempotent of the ring of endomorphisms of $h(X)(r)$.

Imitating this, we define the category of log motives over S is as follows. (See 3.1 for details.) For a projective vertical log smooth fs log scheme X over S and for $r \in \mathbb{Z}$, consider a symbol $h(X)(r)$. For projective vertical log smooth fs log schemes X, Y over S and for $r, s \in \mathbb{Z}$, by a morphism $h(X)(r) \to h(Y)(s)$, we mean a homomorphism $h : \bigoplus_i H^i(X)_\ell(r) \to \bigoplus_i H^i(Y)_\ell(s)$ satisfying the condition (C) below. Here $H^i(X)_\ell$ is the smooth $\mathbb{Q}_\ell$-sheaf on the log étale site on S defined to be the i-th relative log étale cohomology of X over S, (r) denotes the r-th Tate twist, and the same for Y and s.

(C) For any geometric standard log point p (2.1.11) over S, the pullback of h to p comes from an element of $\bigoplus_i \mathrm{gr}^i K(Z)_\mathbb{Q}$ for some log modification Z of $X_p \times_p Y_p$, where $K(Z)$ denotes the Grothendieck group of the category of vector bundles on Z and gr^i denotes the i-th graded quotient for the γ-filtration [SGA 6 1971].

A log motive over S is a pair $(h(X)(r), e)$, where X over S and r are as above and e is an idempotent of the ring of endomorphisms of $h(X)(r)$ (3.1.7).

The reason we need log modifications is explained in 3.1.5.

In the case where $S = \mathrm{Spec}(k)$ for a field k with the trivial log structure we have $\mathrm{gr}^i K(Z)_\mathbb{Q} = \mathrm{CH}^i(Z)_\mathbb{Q}$ for any smooth scheme Z over k and our category of log motives over S coincides with the category of motives over k modulo homological equivalence due to Grothendieck.

We will also define the category of log motives over S modulo numerical equivalence by taking the quotient of the set of morphisms by numerical equivalence. We prove the following log version of the theorem of Jannsen.

Theorem (Theorem 3.4.1). (1) *The category of log motives over S modulo numerical equivalence is a semisimple abelian category.*

(2) *The category of log motives over S* (defined in 3.1) *is a semisimple abelian category if and only if the numerical equivalence for morphisms of this category is trivial.*

1.5. We explain more about Section 4. Let S and ℓ be as in 1.4. Roughly speaking, we follow the method of Deligne [1971; 1974], who constructed mixed Hodge structures of geometric origin by using only projective smooth schemes over $\mathbb{C}$.

Our definition of log mixed motives is rather simple and is easily obtained by using the category of log (pure) motives in Section 3. This may seem strange because usually it is impossible to take care of mixed objects by using only pure objects. The reason why such a simple definition works is explained in 4.3.

We will prove the following result, which is a part of Theorem 4.4.2.

Theorem. *Assume that the category of log motives over S is semisimple*; *that is*, *the numerical equivalence coincides with the homological equivalence for this category* (see (2) of the previous theorem). *Then the category of log mixed motives over S is a Tannakian category. In particular*, *it is an abelian category.*

1.6. In Sections 2–4, our discussion only uses ℓ-adic étale realization. We consider in Section 5 more realizations, and formulate Tate conjecture and Hodge conjecture for log mixed motives. In the final section, Section 6, we prove that these conjectures are true in certain cases (Propositions 6.3.2, 6.3.4, 6.4.3). To prove the results on morphisms between H^1 of log curves (Propositions 6.3.4 and 6.4.3), we use the theory of log abelian varieties in [Kajiwara et al. 2008b] and the theory of log Jacobian varieties [Kajiwara 1993].

2. Preparations on log geometry

Basic references on log geometry are [Kato 1989; Illusie 1994]. Basic references on log étale cohomology are [Nakayama 1997; 1998; 2017b; Illusie 2002]. Basic references on algebraic cycles and K-groups are [SGA 6 1971; Fulton 1998].

In this paper, except in 2.1, for technical reasons, we consider only fs log schemes which have charts Zariski locally. (We hope that a generalization of our theory can be developed without such a restriction, but we guess that the resulting categories are not very different from the current ones.) A *monoid* means a commutative semigroup with a unit element which is usually denoted by 1.

Let X be an fs log scheme over an fs log scheme S. We say that X is *projective* if the underlying scheme of X is projective over the underlying scheme of S. We say that X is *vertical* if for any point x of X, whose image in S is denoted by s, the face of $M_{X,\bar{x}}$ spanned by the image of $M_{S,\bar{s}}$ is the whole $M_{X,\bar{x}}$. See [Nakayama 1997, Definition and Notation (7.3)].

A morphism $f : X \to Y$ of integral log schemes is *exact* if for any $x \in X$, an element of $M^{\mathrm{gp}}_{Y,\overline{f(x)}}$ whose image in $M^{\mathrm{gp}}_{X,\bar{x}}$ belongs to $M_{X,\bar{x}}$ belongs to $M_{Y,\overline{f(x)}}$. See [Kato 1989, Definition (4.6)].

2.1. ***Log cohomology theories.*** We review some theorems on log étale cohomology, log Betti cohomology, and log de Rham cohomology.

First we discuss the theorems on log étale cohomology. There are two versions of étale cohomology in log geometry. One is obtained using the Kummer étale (két) site, while the other is obtained using the full log étale (lét) site. In this paper we mainly use log étale cohomology defined using the full log étale site.

Let $f : X \to S$ be a morphism of fs log schemes. Let ℓ be a prime number which is invertible on S. Let $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ $(n \geq 1)$.

Proposition 2.1.1. *Assume that $f : X \to S$ is proper and log smooth. Then* $R^q f_{\text{lét}*}\Lambda$ (the higher direct image for the full log étale topology) *is locally constant and constructible* (see [Nakayama 2017b, 8.1] for the definition) *for all $q \in \mathbb{Z}$.*

Proof. This follows from [Nakayama 2017b, Theorem 13.1(1)]. □

2.1.2. As in the classical case, we define a *constructible $\mathbb{Z}_\ell$-sheaf* as an inverse system $(F_n)_n$, where F_n is a constructible sheaf of $\mathbb{Z}/\ell^{n+1}\mathbb{Z}$-modules such that $\mathbb{Z}/\ell^n\mathbb{Z} \otimes F_n \xrightarrow{\cong} F_{n-1}$. A *smooth $\mathbb{Z}_\ell$-sheaf* is a constructible $\mathbb{Z}_\ell$-sheaf $(F_n)_n$ with each F_n locally constant. The smooth $\mathbb{Z}_\ell$-sheaves form an abelian category. We define the category of *constructible $\mathbb{Q}_\ell$-sheaves* as the localization of this abelian category by torsion objects, that is, those killed by some power of ℓ. By the above proposition, we have, under the assumption there, a smooth $\mathbb{Q}_\ell$-sheaf on $S_{\text{lét}}$, which we denote by $R^q f_{\text{lét}*}\mathbb{Q}_\ell$.

Proposition 2.1.3 (Poincaré duality). *Let $d \geq 0$. Assume that $f : X \to S$ is proper, log smooth, vertical, and, full log étale locally on S, all fibers are of equi-d-dimensional. Then there is a natural isomorphism*

$$R^{2d-i} f_{\text{lét}*}\Lambda(d) \xrightarrow{\cong} \mathcal{H}om(R^i f_{\text{lét}*}\Lambda, \Lambda)$$

for any i.

Proof. This is by [Nakayama 2017b, Theorem 14.2(3)]. □

Corollary 2.1.4. *Under the same assumptions, suppose further that S is noetherian. Then, there is a natural isomorphism*

$$R^{2d-i} f_{\text{lét}*}\mathbb{Q}_\ell(d) \xrightarrow{\cong} \mathcal{H}om(R^i f_{\text{lét}*}\mathbb{Q}_\ell, \mathbb{Q}_\ell)$$

for any i.

Proposition 2.1.5 (Künneth formula). *Assume that S is quasicompact and that $f : X \to S$ is proper. Let $g : Y \to S$ be another proper morphism of fs log schemes. Let h be the induced morphism $X \times_S Y \to S$. Then there is a natural isomorphism*

$$Rf_{\text{lét}*}\Lambda \otimes^{\boldsymbol{L}}_\Lambda Rg_{\text{lét}*}\Lambda \xrightarrow{\cong} Rh_{\text{lét}*}\Lambda.$$

Proof. This is by [Nakayama 2017b, Theorem 9.1]. □

As a corollary, we have:

Corollary 2.1.6. *Assume that S is quasicompact and that $f : X \to S$ is proper and log smooth. Let $g : Y \to S$ be another proper and log smooth morphism of fs log* *schemes. Let h be the induced morphism $X \times_S Y \to S$. Then, for each $n \geq 0$, there is a natural isomorphism*

$$\bigoplus_{p+q=n} R^p f_{\mathrm{l\acute{e}t}*}\mathbb{Q}_\ell \otimes R^q g_{\mathrm{l\acute{e}t}*}\mathbb{Q}_\ell \xrightarrow{\cong} R^n h_{\mathrm{l\acute{e}t}*}\mathbb{Q}_\ell.$$

Proof. The natural homomorphism is seen to be bijective at stalks by the previous proposition. □

Next the theorems on log Betti cohomology are as follows. Let $f : X \to S$ be a morphism of fs log analytic spaces.

Proposition 2.1.7. *Assume that $f : X \to S$ is proper* (i.e., the underlying map is universally closed and separated) *and log smooth. Then $R^q f_*^{\log}\mathbb{Z}$ is a locally constant sheaf of finitely generated abelian groups for all $q \in \mathbb{Z}$.*

Proof. This is [Kajiwara and Nakayama 2008, Corollary 0.3]. □

Proposition 2.1.8 (Poincaré duality). *Let $d \geq 0$. Assume that $f : X \to S$ is proper, log smooth, vertical, and all fibers are of equi-d-dimensional. Then there is a natural isomorphism*

$$R^{2d-i} f_*^{\log}\mathbb{Q} \xrightarrow{\cong} \mathcal{H}om(R^i f_*^{\log}\mathbb{Q}, \mathbb{Q})$$

for any i.

Proof. The case where f is exact is by [Nakayama and Ogus 2010, Theorem 5.10(3)]. The general case is reduced to this case by exactification as follows. First, we assume that S has a chart by an fs monoid and fix such a chart. Then, by exactification [Illusie et al. 2005, Proposition (A.4.4)], there is a log blow-up [Illusie et al. 2005, Definition (6.1.1)] $p : S' \to S$ such that the base-changed morphism $f' : X' := X \times_S S' \to S'$ is exact. By the exact case, we have the natural isomorphism

$$R^{2d-i} f_*'^{\log}\mathbb{Q} \xrightarrow{\cong} \mathcal{H}om(R^i f_*'^{\log}\mathbb{Q}, \mathbb{Q}) \tag{$*$}$$

on $S'^{\log}$. Below we will prove that sending this by $p_*^{\log}$ gives us an isomorphism $R^{2d-i} f_*^{\log}\mathbb{Q} \xrightarrow{\cong} \mathcal{H}om(R^i f_*^{\log}\mathbb{Q}, \mathbb{Q})$ on $S^{\log}$. To see that the last isomorphism is independent of the choices of log blow-ups, we can argue as in [Nakayama 2017b, (14.10)], where the ℓ-adic analogue of the same problem is treated. Then, it implies that the isomorphism is independent also of the choices of charts, and is glued into the desired isomorphism.

Now we calculate $p_*^{\log}$ of each side of ($*$). Since $R^j f_*'^{\log}\mathbb{Q}$ is locally constant for any j (Proposition 2.1.7), by [Kajiwara and Nakayama 2008, Proposition 5.3(2)], we have

$$p_*^{\log} R f_*'^{\log}\mathbb{Q} = R p_*^{\log} R f_*'^{\log}\mathbb{Q} = R f_*^{\log} p_*^{\log}\mathbb{Q} = R f_*^{\log}\mathbb{Q},$$

where we denote the base-changed morphism of p by the same symbol and the last equality is by [Kajiwara and Nakayama 2008, Proposition 5.3(1)]. Hence,

$$p_*^{\log} R^{2d-i} f_*'^{\log}\mathbb{Q} = R^{2d-i} f_*^{\log}\mathbb{Q}.$$

On the other hand, as for the right-hand-side of $(*)$, again by [loc. cit., Proposition 5.3(2)], we have

$$R^i f_*'^{\log}\mathbb{Q} = p^{\log -1} p_*^{\log} R^i f_*'^{\log}\mathbb{Q},$$

and it is isomorphic to $p^{\log -1} R^i f_*^{\log}\mathbb{Q}$ by the same argument for the left-hand-side. Then,

$$\begin{aligned} p_*^{\log} \mathcal{H}om(R^i f_*'^{\log}\mathbb{Q}, \mathbb{Q}) &= p_*^{\log} \mathcal{H}om(p^{\log -1} R^i f_*^{\log}\mathbb{Q}, \mathbb{Q}) \\ &= \mathcal{H}om(R^i f_*^{\log}\mathbb{Q}, p_*^{\log}\mathbb{Q}) = \mathcal{H}om(R^i f_*^{\log}\mathbb{Q}, \mathbb{Q}), \end{aligned}$$

where the last equality is again by [loc. cit., Proposition 5.3(1)]. Thus we have an isomorphism

$$R^{2d-i} f_*^{\log}\mathbb{Q} \xrightarrow{\cong} \mathcal{H}om(R^i f_*^{\log}\mathbb{Q}, \mathbb{Q}). \qquad \square$$

Proposition 2.1.9. *Let $f : X \to S$ be a proper and log smooth morphism of fs log analytic spaces. Let $g : S' \to S$ be any morphism of fs log analytic spaces. Let $f' : X' := X \times_S S' \to S'$ and $g' : X' \to X$ be the base-changed morphisms. Let L be a locally constant sheaf of abelian groups on $X^{\log}$. Then the base change homomorphism*

$$g^{\log -1} R f_*^{\log} L \to R f_*'^{\log} g'^{\log -1} L$$

is an isomorphism.

Proof. We may assume that S has a chart. By exactification [Illusie et al. 2005, Proposition (A.4.4)], we take a log blow-up $p : S_1 \to S$ such that the base-changed morphism $f_1 : X_1 := X \times_S S_1 \to S_1$ is exact. Then, by proper log smooth base change theorem in log Betti cohomology [Kajiwara and Nakayama 2008, Theorem 0.1], the cohomologies of $R f_{1*}^{\log} p_X^{\log -1} L$ are locally constant, where p_X is the base-changed morphism $X_1 \to X$. Hence, by the invariance of cohomology under log blow-up [loc. cit., Proposition 5.3], to prove Proposition 2.1.9, we can replace f and g by the base-changed ones with respect to p, and L by its pullback $p_X^{\log -1} L$. Thus we may assume that f is exact. Then the conclusion follows from the log proper base change theorem [loc. cit., Proposition 5.1] (see [loc. cit., Remark 5.1.1]). $\square$

Proposition 2.1.10 (Künneth formula). *Let the notation and assumption be as in the previous proposition. Assume that g is proper. Let $h : X' \to S$ be the induced* *morphism. Then there is a natural isomorphism*

$$Rf^{\log}_{\mathrm{an}*}\mathbb{Q} \otimes^{L}_{\mathbb{Q}} Rg^{\log}_{\mathrm{an}*}\mathbb{Q} \xrightarrow{\cong} Rh^{\log}_{\mathrm{an}*}\mathbb{Q}.$$

Proof. This is by Proposition 2.1.9 and the usual projection formula. □

Next is a comparison between log Betti cohomology and log étale cohomology.

2.1.11. A *standard log point* means the fs log scheme $\mathrm{Spec}(k)$ for a field k endowed with the log structure associated to $\mathbb{N} \to k\ ;\ 1 \mapsto 0$. If we like to present k, we call it a *standard log point associated to* k. The standard log point associated to an algebraically closed field is called a *geometric standard log point*.

Proposition 2.1.12. *Let $f : X \to S$ be a proper, log smooth and vertical morphism of fs log schemes with S being of finite type over $\mathbb{C}$. Let*

$$X^{\log}_{\mathrm{an}} \xrightarrow{\eta} X_{\mathrm{k\acute{e}t}} \xleftarrow{\kappa} X_{\mathrm{l\acute{e}t}}$$

be natural morphisms of topoi (for η, see [Kato and Nakayama 1999, Remark (2.7)]). *Let $n \geq 1$ and $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$. Then we have*

$$\eta^* Rf_{\mathrm{k\acute{e}t}*}\Lambda = Rf^{\log}_{\mathrm{an}\,*}\Lambda, \quad \kappa^* Rf_{\mathrm{k\acute{e}t}*}\Lambda = Rf_{\mathrm{l\acute{e}t}*}\Lambda.$$

Proof. The second one is shown in 13.4 of [Nakayama 2017b]. We prove the first one. First, note that the cohomologies of the left-hand-side are locally constant and constructible by [loc. cit., Theorem 13.1(2)] and those of the right-hand-side are locally constant by Proposition 2.1.7.

We reduce to the case where f is exact. We may assume that S has a chart by an fs monoid and fix such a chart. Then, by [loc. cit., Lemma 3.10], there is a log blow-up $p : S' \to S$ such that the base-changed morphism $f' : X' := X \times_S S' \to S'$ is exact. By [loc. cit., Theorems 5.5(1) and 5.8(1)], we have

$$p^*_{\mathrm{k\acute{e}t}} Rf_{\mathrm{k\acute{e}t}*}\Lambda = p^*_{\mathrm{k\acute{e}t}} Rf_{\mathrm{k\acute{e}t}*} Rp_{\mathrm{k\acute{e}t}*}\Lambda = p^*_{\mathrm{k\acute{e}t}} Rp_{\mathrm{k\acute{e}t}*} Rf'_{\mathrm{k\acute{e}t}*}\Lambda = Rf'_{\mathrm{k\acute{e}t}*}\Lambda,$$

where we denote the base-changed morphism of p by the same symbol.

Similarly, by [Kajiwara and Nakayama 2008, Proposition 5.3], we have

$$p^{\log *} Rf^{\log}_{*}\Lambda = p^{\log *} Rf^{\log}_{*} Rp^{\log}_{*}\Lambda = p^{\log *} Rp^{\log}_{*} Rf'^{\log}_{*}\Lambda = Rf'^{\log}_{*}\Lambda.$$

Thus we may and will assume that f is exact.

Since the cohomologies of both sides are locally constant, we can work at stalks. Let s_0 be a point of S. By the following Proposition 2.1.13, there are a morphism $s \to S$ from the standard log point s over $\mathbb{C}$ whose image is s_0, and a log blow-up X' of $X_s := X \times_S s$ such that the composition $X' \to X_s \to s$ is strict semistable, i.e., a log deformation with smooth irreducible components. It is enough to show that the homomorphism at a stalk over a point of $s_0^{\log}$ is bijective. Then by the exact proper base change theorem ([Nakayama 1997, Theorem (5.1) and Remark (5.1.1)] for the log étale cohomology, [Kajiwara and Nakayama 2008, Proposition 5.1, Remark 5.1.1], and the usual proper base change theorem for topological spaces for the log Betti cohomology), we may assume that $S = s$, and further, by [Kajiwara and Nakayama 2008, Proposition 5.3(1); Nakayama 2017b, Theorem 5.5(1)], we may assume that $X = X'$, that is, in the original setting, we may assume that S is the standard log point over $\mathbb{C}$ and X is strict semistable over S.

Here we use the Steenbrink–Rapoport–Zink (SRZ, for short) spectral sequences as follows. In the proof of [Fujisawa and Nakayama 2003, Theorem 7.1], it is shown that there is a natural isomorphism between the ℓ-adic SRZ spectral sequence and the Betti SRZ spectral sequence. Since these converge to the stalk of ℓ-adic log étale cohomologies and that of log Betti cohomologies, respectively, we have the desired isomorphism. □

Proposition 2.1.13. *Let $s = (\operatorname{Spec} k, \mathbb{N})$ be a standard log point. Let $X \to s$ be a quasicompact, vertical, and log smooth morphism of fs log schemes. Then there are a positive integer n and a log blow-up* [Nakayama 2017b, 2.2] *$X' \to X \times_s s_n$, where $s_n := \left(\operatorname{Spec} k, \frac{1}{n}\mathbb{N}\right)$, such that the composition $X' \to s_n$ is strict semistable.*

This is a variant of the semistable reduction theorem of D. Mumford. The statement here is due to [Vidal 2004, Proposition 2.4.2.1]. (See [Kajiwara et al. 2008c, Remark after Assumption 8.1].) Another reference is [Saito 2004, Theorem 2.9]. Both papers based on the method of [Yoshioka 1995]. (Actually, [Yoshioka 1995; Saito 2004] treat the case of log smooth fs log schemes over a discrete valuation ring, but the proof is in the same way. [Saito 2004] treats the nonvertical case also.) See 2.3.14 for a variant of Proposition 2.1.13.

Finally, we discuss log de Rham cohomology and log Hodge theory.

Proposition 2.1.14. *Let k be a field of characteristic zero. Let $f : X \to S$ be a projective, log smooth and vertical morphism of fs log schemes with S being log smooth over k. Let $q \in \mathbb{Z}$. Then we have the following.*

(1) *$H^q_{\mathrm{dR}}(X/S) := R^q f_{\text{két}*}\omega^{\cdot,\text{két}}_{X/S}$ is a vector bundle endowed with a natural quasi-nilpotent integrable connection with log poles, and, for all p, the Hodge filters $R^q f_{\text{két}*}\omega^{\cdot\geq p,\text{két}}_{X/S}$ are subbundles of $H^q_{\mathrm{dR}}(X/S)$.*

(2) *When $k = \mathbb{C}$, we have a natural log Hodge structure on $S_{\text{két}}$ of weight q which is underlain by $H^q_{\mathrm{dR}}(X/S)$ with the Hodge filter.*

Proof. We may assume $k = \mathbb{C}$, and (1) is deduced from (2). We obtain (2) by [Kato et al. 2002, Theorem 8.1], the main theorem there. □

Lemma 2.1.15. *Let $f : X \to S$ be a proper, log smooth and vertical morphism of fs log analytic spaces with S being ideally log smooth over $\mathbb{C}$* [Illusie et al. 2005, Definition (1.5)]. *Assume that for any x, the cokernel of $(M_S/\mathcal{O}_S^\times)^{\mathrm{gp}}_{f(x)} \to (M_X/\mathcal{O}_X^\times)^{\mathrm{gp}}_x$*

is torsion-free. Assume also that either S is log smooth or f is exact. Then we have a canonical isomorphism

$$R^q f_* \omega_{X/S}^{\cdot,\text{két}} = \varepsilon^* R^q f_* \omega_{X/S}^{\cdot}$$

for any $q \in \mathbb{Z}$. Here ε is the forgetting-log morphism, i.e., the projection from the két site to the usual site.

Proof. By [Illusie et al. 2005, Theorems (6.2) and (6.3)], the local system $R^q f_*^{\log}\mathbb{C}$ corresponds to $R^q f_* \omega_{X/S}^{\cdot,\text{két}}$ by the két log Riemann–Hilbert correspondence, and it does to $R^q f_* \omega_{X/S}^{\cdot}$ by the nonkét log Riemann–Hilbert correspondence, respectively. Hence the desired isomorphism follows from the compatibility of the both Riemann–Hilbert correspondences [Illusie et al. 2005, Theorem (4.4)]. □

Lemma 2.1.16. *Let the notation and the assumption be as in the previous lemma. Let $X' \to X$ be a log blow-up and $f' : X' \to X \to S$ the composite. Then the canonical homomorphism*

$$R^q f_* \omega_{X/S}^{\cdot} \to R^q f'_* \omega_{X'/S}^{\cdot}$$

is an isomorphism.

Proof. By [Illusie et al. 2005, Theorem (6.3)], this homomorphism corresponds by the log Riemann–Hilbert correspondence to the homomorphism $R^q f_*^{\log}\mathbb{C} \to R^q f_*^{\prime\log}\mathbb{C}$ of local systems, which is an isomorphism by [Kajiwara and Nakayama 2008, Proposition 5.3(1)]. □

Proposition 2.1.17. *Let k be a field of characteristic zero. Let $f : X \to s$ be a projective, log smooth and vertical morphism of fs log schemes with s being the standard log point associated to k. Let $q \in \mathbb{Z}$. Then we have the following.*

(1) *$H^q_{\text{dR}}(X/s) := R^q f_{\text{két}*} \omega_{X/s}^{\cdot,\text{két}}$ is a vector bundle with a natural quasinilpotent integrable connection with log poles.*

(2) *When $k = \mathbb{C}$, $H^q_{\text{dR}}(X/s)$ carries a natural log Hodge structure on $s_{\text{két}}$ of weight q.*

Proof. We may assume $k = \mathbb{C}$, and (1) is deduced from (2). We prove (2). For this, we can use a general result in [Fujisawa and Nakayama 2018]. Here we give a direct proof, which is essentially a part of the arguments in [loc. cit.]. In [Fujisawa and Nakayama 2015], the nonkét version of the case of (2) where f is strict semistable is proved with the Hodge filter $R^q f_* \omega_{X/s}^{\cdot \geq p}$. We reduce (2) to this result as follows. To prove (2), we slightly generalized the statement to the case where s is the spectrum of a log Artin ring $\mathbb{C}[\mathbb{N}]/(x^n)$ for some $n \geq 1$, where x is the generator of log. In the rest of this proof, (2) means this generalized statement. We may assume that f satisfies the assumptions in Lemma 2.1.15 by két localization of the base s. By a variant of Proposition 2.1.13, we may assume further that there exists a log blow-up $X' \to X$ such that the special fiber of $X' \to s$ is strict semistable. By Lemma 2.1.15, we see that it is enough to show the nonkét version of (2). By the argument in [Illusie et al. 2007] and the strict semistable case in [Fujisawa and Nakayama 2015], $R^q f'_* \omega^{\cdot}_{X'/s}$ with the Hodge filters gives a log Hodge structure. The nonkét version of (2) is reduced to this by Lemma 2.1.16 and the induced Hodge filtration on $R^q f_* \omega^{\cdot}_{X/s}$ from $R^q f'_* \omega^{\cdot}_{X'/s}$ does not depend on the choice of X'. □

Proposition 2.1.18. *Let $f : X \to S$ be a projective, log smooth and vertical morphism of fs log schemes with S being log smooth over $\mathbb{C}$. Let $s \to S$ be a standard log point associated to $\mathbb{C}$ over S. Let $f_s : X_s \to s$ be the base-changed morphism. Let $q \in \mathbb{Z}$. Then the pullback of the log Hodge structure $H^q_{\mathrm{dR}}(X/S)$ is naturally isomorphic to the log Hodge structure $H^q_{\mathrm{dR}}(X_s/s)$.*

Proof. Since there is a natural base change map, it is enough to show that the local system can be base-changed, which is by Proposition 2.1.9. □

2.2. *Fans in log geometry.* Let (fs) be the category of fs log schemes which have charts Zariski locally. From now on, in the rest of this paper, an fs log scheme means an object of this (fs).

We review the formulation of fans in [Kato 1994] as unions of Spec of monoids. This is a variant of the theory of polyhedral cone decompositions in [Kempf et al. 1973; Oda 1988].

The material in Paragraphs 2.2.16 and 2.2.17 is new and was not discussed in [Kato 1994].

2.2.1. For a monoid P, an *ideal* of P means a subset I of P such that $ab \in I$ for any $a \in P$ and $b \in I$. A *prime ideal* of P means an ideal $\mathfrak{p}$ of P such that the complement $P \smallsetminus \mathfrak{p}$ is a submonoid of P. We denote the set of all prime ideals of P by $\mathrm{Spec}(P)$.

2.2.2. For a monoid P and for a submonoid S of P, we have the monoid $S^{-1}P = \{s^{-1}a \mid a \in P, s \in S\}$ obtained from P by inverting elements of S. Here $s_1^{-1}a_1 = s_2^{-1}a_2$ if and only if there is an $s_3 \in S$ such that $s_3 s_2 a_1 = s_3 s_1 a_2$.

In the case where $S = \{f^n \mid n \geq 0\}$ for $f \in P$, $S^{-1}P$ is denoted also by P_f.

2.2.3. By a *monoidal space*, we mean a topological space T endowed with a sheaf of monoids $\mathcal{P}$ such that $(\mathcal{P}_t)^\times = \{1\}$ for any $t \in T$. Here $\mathcal{P}_t$ denotes the stalk of $\mathcal{P}$ at t and $(\cdot)^\times$ means the subgroup consisting of all invertible elements.

2.2.4. For a monoid P, $\mathrm{Spec}(P)$ is regarded as a monoidal space in the following way.

We endow $\mathrm{Spec}(P)$ with the topology for which the sets

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}(P) \mid f \notin \mathfrak{p}\} \quad \text{with} \quad f \in P$$

form a basis of open sets.

The sheaf $\mathcal{P}$ of monoids on $\mathrm{Spec}(P)$ is characterized by the property that for $f \in P$, $\mathcal{P}(D(f)) = P_f/P_f^\times$.

The stalk of $\mathcal{P}$ at $\mathfrak{p} \in \mathrm{Spec}(P)$ is identified with $P_\mathfrak{p}/(P_\mathfrak{p})^\times$, where $P_\mathfrak{p} = (P \smallsetminus \mathfrak{p})^{-1}P$.

2.2.5. For a monoidal space Σ with the structure sheaf $\mathcal{P}$ of monoids and for a monoid P, the natural map $\mathrm{Mor}(\Sigma, \mathrm{Spec}(P)) \to \mathrm{Hom}(P, \mathcal{P}(\Sigma))$ is bijective.

2.2.6. A monoidal space is called a *fan* if it has an open covering $(U_\lambda)_\lambda$ such that each U_λ is isomorphic, as a monoidal space, to $\mathrm{Spec}(P_\lambda)$ for some monoid P_λ.

A fan which is isomorphic to $\mathrm{Spec}(P)$ for some monoid P is called an *affine fan*. The functor $P \mapsto \mathrm{Spec}(P)$ is an antiequivalence from the category of monoids P such that $P^\times = \{1\}$ to the category of affine fans. The converse functor is given by $\Sigma \mapsto \mathcal{P}(\Sigma)$, where $\mathcal{P}$ is the structure sheaf of Σ.

2.2.7. For a fan Σ, let

$$[\Sigma] : (\mathrm{fs}) \to (\mathrm{Sets})$$

be the contravariant functor which sends $X \in (\mathrm{fs})$ to the set of all morphisms $(X, M_X/\mathcal{O}_X^\times) \to \Sigma$ of monoidal spaces.

If $\Sigma = \mathrm{Spec}(P)$, we have $[\Sigma](X) = \mathrm{Hom}(P, \Gamma(X, M_X/\mathcal{O}_X^\times))$.

Lemma 2.2.8. *The functor* $\Sigma \mapsto [\Sigma]$ *from the category of fans to the category of contravariant functors* $(\mathrm{fs}) \to (\mathrm{Sets})$ *is fully faithful.*

Proof. Let Σ, Σ' be fans. We have to prove that

$$\mathrm{Mor}(\Sigma, \Sigma') \to \mathrm{Mor}([\Sigma], [\Sigma']) \tag{†}$$

is bijective.

First, we prove the case where both Σ and Σ' are affine, that is, we prove that the contravariant functor $P \mapsto [\mathrm{Spec}(P)]$ from the category of monoids P such that $P^\times = \{1\}$ to the category of contravariant functors $(\mathrm{fs}) \to (\mathrm{Sets})$ is fully faithful. For monoids P and Q such that $P^\times = \{1\}$ and $Q^\times = \{1\}$ and for $X = \mathrm{Spec}(\mathbb{Z}[Q])$, we have $[\mathrm{Spec}(P)](X) = \mathrm{Hom}(P, \Gamma(X, M_X/\mathcal{O}_X^\times)) = \mathrm{Hom}(P, Q)$. From this, we obtain easily that the map $\mathrm{Hom}(P, Q) \to \mathrm{Mor}([\mathrm{Spec}(Q)], [\mathrm{Spec}(P)])$ is bijective.

Next, we prove the case where $\Sigma = \mathrm{Spec}(Q)$ $(Q^\times = \{1\})$ is affine and Σ' is any. We prove that (†) is surjective. Let $f : [\Sigma] \to [\Sigma']$ be a morphism. Let x be an fs log point lying over $X = \mathrm{Spec}(\mathbb{Z}[Q])$ such that the homomorphism $Q \to (M_X/\mathcal{O}_X^\times)_{X,\bar{x}}$ is bijective. Let $((x, M_x/\mathcal{O}_x^\times) \to \Sigma') \in [\Sigma'](x)$ be the image by $f(x)$ of $((x, M_x/\mathcal{O}_x^\times) \to (X, M_X/\mathcal{O}_X^\times) \to \mathrm{Spec}(Q)) \in [\Sigma](x)$. Let U' be the smallest neighborhood in Σ' of the image s' of this morphism $(x, M_x/\mathcal{O}_x^\times) \to \Sigma'$. Then f factors through $[U']$, which is by the fact that any morphism $(X, M_X/\mathcal{O}_X^\times) \to \Sigma'$ sending x to s' factors through U'. Since U' is affine, the surjectivity of (†) is reduced to the previous case.

The injectivity of (†) is also reduced to the previous case as follows. Let a, b be two morphisms from Σ to Σ' and assume that the induced morphisms from $[\Sigma]$ to $[\Sigma']$ coincide. Considering an fs log point lying over each point of Σ, we see that the underlying maps of sets of a and b coincide. Then both a and b factor through the smallest neighborhood U' in Σ' of the image of the closed point. Since $[U'] \to [\Sigma']$ is injective, we reduce to the previous case. Alternatively, we use, instead of the previous case, the fact that $(X, M_X/\mathcal{O}_X^\times) \to \mathrm{Spec}(Q)$ is an epimorphism in the category of monoidal spaces.

Finally, the bijectivity of (†) for any Σ and any Σ' is reduced to the case where Σ is affine because Σ is the limit of an inductive system of affine fans and open immersions. □

2.2.9. According to Lemma 2.2.8, we will often identify a fan Σ with the functor $[\Sigma]$.

For an fs log scheme X and for a fan Σ, we will regard a morphism

$$(X, M_X/\mathcal{O}_X^\times) \to \Sigma$$

of monoidal spaces as a morphism $X \to [\Sigma]$ from the functor X on (fs) represented by X to the functor $[\Sigma]$. We sometimes also denote a morphism $X \to [\Sigma]$ simply by $X \to \Sigma$.

Lemma 2.2.10. *For an fs log scheme X, a fan Σ, and a morphism $X \to \Sigma$, the following conditions* (i) *and* (ii) *are equivalent.*

(i) *The corresponding morphism $(X, M_X/\mathcal{O}_X^\times) \to \Sigma$ of monoidal spaces is strict. Here we say that a morphism $f : (T, \mathcal{P}) \to (T', \mathcal{P}')$ of monoidal spaces is **strict** if $f^{-1}(\mathcal{P}') \to \mathcal{P}$ is an isomorphism.*

(ii) *Locally on X, there is an open set* $\mathrm{Spec}(P)$ *of Σ with P a monoid such that $X \to \Sigma$ factors as $X \to$* $\mathrm{Spec}(\mathbb{Z}[P]) \to \mathrm{Spec}(P) \subset \Sigma$, *where* $\mathrm{Spec}(\mathbb{Z}[P])$ *is endowed with the standard log structure and the homomorphism $P \to M_X$ corresponding to the first arrow is a chart of X* (that is, the first morphism is strict, where we say a morphism of log schemes $X \to Y$ is *strict* if the log structure of X coincides with the inverse image of the log structure of Y).

Proof. (ii) ⇒ (i). Since the projection $\mathrm{Spec}(\mathbb{Z}[P]) \to \mathrm{Spec}(P)$ satisfies the condition (i), (ii) implies (i).

(i) ⇒ (ii). Let $x \in X$ and we work around x. First, localizing X, we may assume that X has a chart P such that $P \to (M_X/\mathcal{O}_X^\times)_{\bar{x}}$ is bijective. Next, localizing Σ, we may assume $\Sigma = \mathrm{Spec}(Q)$ with $Q \to (M_X/\mathcal{O}_X^\times)_{\bar{x}}$ being bijective. Then P is isomorphic to Q and, after further localizing X, we may replace Q with P. □

2.2.11. We will say $X \to \Sigma$ is *strict* if the equivalent conditions in Lemma 2.2.10 are satisfied.

2.2.12. Polyhedral cone decompositions which appear in toric geometry [Kempf et al. 1973; Oda 1988] are related to the above notion of fan (2.2.6) as follows.

Let N be a free $\mathbb{Z}$-module of finite rank, and let $N_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} N$. A *rational polyhedral cone* in $N_{\mathbb{R}}$ is a subset of the form

$$\sigma = \left\{ \sum_{i=1}^{r} x_i N_i \;\middle|\; x_i \in \mathbb{R}_{\geq 0} \right\}$$

for some $N_1, \ldots, N_r \in N$. A rational polyhedral cone σ is called *strongly convex* if it does not contain a line, i.e., $\sigma \cap (-\sigma) = \{0\}$. A subset $\tau \subset \sigma$ is called a *face* of σ if there exists an element $h \in \mathrm{Hom}_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R})$ such that $\sigma \subset \{x \in N_{\mathbb{R}} \mid h(x) \geq 0\}$ and $\tau = \sigma \cap \{x \in N_{\mathbb{R}} \mid h(x) = 0\}$. A face of σ is also a rational polyhedral cone.

A rational polyhedral cone decomposition in $N_{\mathbb{R}}$ (or a rational fan in $N_{\mathbb{R}}$) is a nonempty set Σ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying the following two conditions: (i) If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$; (ii) If $\sigma, \tau \in \Sigma$, the intersection $\sigma \cap \tau$ is a face of σ.

We regard a rational fan Σ in $N_{\mathbb{R}}$ as a fan in the sense of 2.2.6 as follows.

We endow Σ with the topology for which the sets face(σ) of all faces of σ for $\sigma \in \Sigma$ form a basis of open sets.

We endow Σ with the sheaf $\mathcal{P}$ of monoids characterized by $\mathcal{P}(\mathrm{face}(\sigma)) = P_\sigma/(P_\sigma)^\times$, where

$$P_\sigma = \{h \in \mathrm{Hom}(N, \mathbb{Z}) \mid h(x) \geq 0 \text{ for all } x \in \sigma\}.$$

The open set face(σ) of Σ is identified with $\mathrm{Spec}(P_\sigma)$.

2.2.13. For a rational fan Σ in $N_{\mathbb{R}}$, we have the toric variety

$$\mathrm{Toric}_\Sigma = \textstyle\bigcup_{\sigma \in \Sigma} \mathrm{Spec}(\mathbb{Z}[P_\sigma])$$

over $\mathbb{Z}$ corresponding to Σ with the standard log structure, on which the torus $N \otimes \boldsymbol{G}_m$ acts naturally. We have

$$[\Sigma] = \mathrm{Toric}_\Sigma/(N \otimes \boldsymbol{G}_m)$$

as a sheaf on (fs), where Toric_Σ is identified with the sheaf on (fs) that it represents.

2.2.14. For an fs log scheme X, in the following Cases (i) and (ii), we can define a fan Σ_X associated to X and a strict morphism $X \to \Sigma_X$ in a canonical way.

Case (i). X is log regular ([Kato 1994]).

Case (ii). X is vertical and log smooth over a standard log point.

Case (i) was considered in [Kato 1994]. Case (ii) is explained below.

2.2.15. We first review Case (i). See [Kato 1994] for the definition of log regularity. As a set, Σ_X is the set of all points x of X such that the maximal ideal m_x of $\mathcal{O}_{X,x}$ is generated by the image of $M_{X,x} \setminus \mathcal{O}_{X,x}^\times$, where $M_{X,x}$ is the stalk at x of the direct image of M_X to the Zariski site. The topology of Σ_X is the restriction of the topology of X. The structural sheaf $\mathcal{P}$ of Σ_X is defined as the inverse image of the sheaf $M_X/\mathcal{O}_X^\times$ on X. The morphism $(X, M_X/\mathcal{O}_X^\times) \to \Sigma_X$ is defined as follows. As a map, it sends $x \in X$ to the point of X corresponding to the prime ideal of $\mathcal{O}_{X,x}$ generated by the image of $M_{X,x} \setminus \mathcal{O}_{X,x}^\times$. If $x \in X$ and if $y \in X$ is the image of x in Σ_X, there is a chart $P \to M_U$ for some open neighborhood U of x in X such that $P \to (M_X/\mathcal{O}_X^\times)_x$ is an isomorphism, and via the composite homomorphism $P \to (M_X/\mathcal{O}_X^\times)_x \to (M_X/\mathcal{O}_X^\times)_y$, $\mathrm{Spec}(P)$ is identified with an open neighborhood of y in Σ_X. The chart defines a morphism $(U, M_U/\mathcal{O}_U^\times) \to \mathrm{Spec}(P)$ and hence a morphism $(U, M_U/\mathcal{O}_U^\times) \to \Sigma_X$ and these local definitions are glued to a global definition of $(X, M_X/\mathcal{O}_X^\times) \to \Sigma_X$.

2.2.16. We consider Case (ii). As a set, Σ_X is the disjoint union $\Sigma'_X \coprod \{\eta\}$ of the set Σ'_X of all points x of X such that the maximal ideal m_x of $\mathcal{O}_{X,x}$ is generated by the image of $M_{X,x} \setminus \mathcal{O}_{X,x}^\times$ and the one-point-set $\{\eta\}$. The topology on Σ_X is as follows. First define the topology of Σ'_X to be the restriction of the topology of X. A closed subset of Σ_X is either a closed subset of Σ'_X or Σ_X. The structure sheaf $\mathcal{P}$ of monoids on Σ_X is defined as follows. First let the sheaf $\mathcal{P}'$ on Σ'_X be the inverse image of $M_X/\mathcal{O}_X^\times$. Let $\mathcal{P} = i_*\mathcal{P}'$, where $i : \Sigma'_X \to X$ is the inclusion map.

Then Σ_X is a fan. This is reduced to the log regular case as follows. Let $x \in X$ and let $P = M_{X,x}/\mathcal{O}_{X,x}^\times = M_{X,\bar{x}}/\mathcal{O}_{X,\bar{x}}^\times$. Since the problem is local on X, we can work around x. Since X is strict étale over some $\mathrm{Spec}\, k[Q]/(q)$, where Q is an fs monoid and q is an interior of Q, $\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})$ is locally isomorphic to the part $t = 0$ of a log regular scheme Y, where t is a section of log structure M_Y of Y such that the part of Y where t is invertible coincides with the part where M_Y is trivial. By Case (i), we have a fan Σ_Y, which is affine and naturally isomorphic to $\mathrm{Spec}(Q)$. Let $\Sigma'_{\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})}$ be the set of all points y of $\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})$ such that the maximal ideal at y is generated by the image of $M_{X,\bar{y}} \setminus \mathcal{O}_{X,\bar{y}}^\times$. We define a monoidal space $\Sigma_{\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})} = \Sigma'_{\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})} \coprod \{\eta\}$ similarly to Σ_X. Then this is isomorphic to Σ_Y.

On the other hand, since X has a chart Zariski locally, we may assume that X has a chart by P such that $P \to M_X \to M_{X,x}/\mathcal{O}_{X,x}^\times$ is the identity. Then, for any nonempty prime ideal $\mathfrak{p}$ of P, the ideal generated by the image of $\mathfrak{p}$ in $\mathcal{O}_{X,x}$ is a prime ideal because its image generates a prime ideal in the strict localization. Thus we have a map f from $\mathrm{Spec}(P) \setminus \{\varnothing\}$ to the set $\Sigma'_{\mathrm{Spec}(\mathcal{O}_{X,x})}$ of all points y of $\mathrm{Spec}(\mathcal{O}_{X,x})$ such that the maximal ideal at y is generated by the image of $M_{X,y} \setminus \mathcal{O}_{X,y}^\times$, and we also have a factorization of the above isomorphism $\mathrm{Spec}(P) \cong$

$\Sigma_Y \cong \Sigma_{\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})}$ as

$$\mathrm{Spec}(P) \to \Sigma'_{\mathrm{Spec}(\mathcal{O}_{X,x})} \coprod \{\eta\} \to \Sigma_{\mathrm{Spec}(\mathcal{O}_{X,\bar{x}})},$$

where the first morphism is induced from f, and the second is by the projection $\mathrm{Spec}(\mathcal{O}_{X,\bar{x}}) \to \mathrm{Spec}(\mathcal{O}_{X,x})$. We see that the second morphism is an isomorphism so that the first is also an isomorphism. Shrinking X if necessary, we may assume that $\Sigma'_{\mathrm{Spec}(\mathcal{O}_{X,x})} \cong \Sigma'_X$ so that $\mathrm{Spec}(P) \cong \Sigma_X$.

We define a map $X \to \Sigma_X$ in the similar way to Case (i) described above. The proof for the gluing also reduces to Case (i). The resulting map in fact factors through $X \to \Sigma'_X$.

2.2.17. Outside Cases (i) and (ii) in 2.2.14, it seems difficult to develop a general theory of fans canonically associated to fs log schemes (see [Abramovich et al. 2016]). We give an example of an fs log scheme X having the following nice property (1) but such that for any fan Σ, there is no strict morphism $(X, M_X/\mathcal{O}_X^\times) \to \Sigma$.

(1) X is locally isomorphic to a closed subscheme of a log regular scheme Y defined by an ideal of $\mathcal{O}_Y$ generated by the images of sections of the log structure M_Y of Y under $M_Y \to \mathcal{O}_Y$ endowed with the log structure induced by the log structure of Y. As a scheme, X is a union of two $\boldsymbol{P}^1_k$ obtained by identifying 0 of each $\boldsymbol{P}^1$ with ∞ of the other $\boldsymbol{P}^1$.

Let k be a field. Endow $\mathrm{Spec}(k[x_1, x_2, x_3, x_4])$ with the log structure associated to

$$\mathbb{N}^4 \to k[x_1, x_2, x_3, x_4], \quad n \mapsto \textstyle\prod_{i=1}^4 x_i^{n(i)}.$$

Let

$$Z = \mathrm{Spec}(k[x_1, x_2, x_3, x_4]/(x_1x_2, x_3, x_4))$$

with the induced log structure, and let Z' be a copy of Z. (Hence as schemes, Z and Z' are isomorphic to $\mathrm{Spec}(k[x, y]/(xy))$.) Denote the copy of x_i on Z' by x'_i. Let U be the part of Z on which x_1 is invertible and let V be the part of Z on which x_2 is invertible. Let U' and V' be the copies of U and V in Z', respectively. Let X be the union of Z and Z' which we glue by identifying the open set $U \coprod V$ of Z and the open set $U' \coprod V'$ of Z', as follows. We identify U and U' by identifying x'_1 with $1/x_1$, x'_2 with $x_1^2x_2$, x'_3 with x_3, and x'_4 with x_4 in the log structure. (Hence x_2 is identified with $(x'_1)^2x'_2$ in the log structure.) We identify V and V' by identifying x'_2 with $1/x_2$, x'_1 with $x_1x_2^2$, x'_3 with x_4, and x'_4 with x_3 in the log structure. (Hence x_1 is identified with $x'_1(x'_2)^2$ in the log structure.)

We show that there is no strict morphism $f : X \to \Sigma$ to any fan Σ.

Assume f exists. Let p be the point of Z at which all x_i have value 0, let $p' \in Z'$ be the copy of p, let u be the generic point of U, and let v be the generic point of V. Let $\mathcal{P}$ be the structure sheaf of monoids of Σ. Then $\mathcal{P}_{f(p)}$ is identified with

$(M_X/\mathcal{O}_X^\times)_p \cong \mathbb{N}^4$ which is generated by x_1, x_2, x_3, x_4. Σ has an open neighborhood which is identified with $\mathrm{Spec}(\mathcal{P}_{f(p)})$. Since p belongs to the closure of u in X, $f(u)$ belongs to $\mathrm{Spec}(\mathcal{P}_{f(p)})$. We have a commutative diagram

$$\begin{tikzcd} \mathcal{P}_{f(p)} \arrow[r] \arrow[d] & \mathcal{P}_{f(u)} \arrow[d] \\ (M_X/\mathcal{O}_X^\times)_p \arrow[r] & (M_X/\mathcal{O}_X^\times)_u \end{tikzcd}$$

in which vertical homomorphisms are isomorphisms, and hence $f(u)$ is the prime ideal of $\mathcal{P}_{f(p)}$ generated by x_2, x_3, x_4. The open neighborhood of u in Σ which is identified with $\mathrm{Spec}(\mathcal{P}_{f(u)})$ is regarded as an open set of $\mathrm{Spec}(\mathcal{P}_{f(p)})$. In this identification, the prime ideal of $\mathcal{P}_{f(p)}$ generated by x_3 is identified with the prime ideal of $\mathcal{P}_{f(u)}$ generated by x_3. Similarly, $\mathrm{Spec}(\mathcal{P}_{f(u)})$ is identified with an open set of $\mathrm{Spec}(\mathcal{P}_{f(p')})$ and the prime ideal of $\mathcal{P}_{f(u)}$ generated by x_3 is identified with the prime ideal of $\mathcal{P}_{f(p')}$ generated by x_3'.

Similarly $\mathrm{Spec}(\mathcal{P}_{f(v)})$ is identified with an open set of $\mathrm{Spec}(\mathcal{P}_{f(p)})$ and also with an open set of $\mathrm{Spec}(\mathcal{P}_{f(p')})$. The prime ideal of $\mathcal{P}_{f(v)}$ generated by x_4 is identified with the prime ideal of $\mathcal{P}_{f(p)}$ generated by x_4 and it is also identified with the prime ideal of $\mathcal{P}_{f(p')}$ generated by x_3'. This shows that the prime ideal of $\mathcal{P}_{f(p)}$ generated by x_3 is equal to the prime ideal generated by x_4. Contradiction.

2.3. *Subdivisions of fans and log modifications.*

2.3.1. We shall mainly consider fans Σ (2.2.6) satisfying the following condition (like in [Kato 1994]).

(S_{fan}) There exists an open covering $(U_\lambda)_\lambda$ such that for each λ, $U_\lambda \cong \mathrm{Spec}(P_\lambda)$ as a fan for some fs monoid P_λ.

2.3.2. Let N be as in 2.2.12, let σ be a strictly convex rational polyhedral cone in $N_{\mathbb{R}}$, and let Σ be the rational fan face(σ) in $N_{\mathbb{R}}$ consisting of all faces of σ. Then a *finite subdivision* of Σ means a finite rational fan Σ' in $N_{\mathbb{R}}$ such that $\sigma = \bigcup_{\tau \in \Sigma'} \tau$.

Lemma 2.3.3. *Let $\Sigma = (\Sigma, \mathcal{P})$ and $\Sigma' = (\Sigma', \mathcal{P}')$ be fans satisfying the condition S_{fan} and let $f : \Sigma' \to \Sigma$ be a morphism of fans. Then the following conditions* (i) *and* (ii) *are equivalent.*

- (i) *f satisfies*:
  - (i-1) *For any $t \in \Sigma$, the inverse image $f^{-1}(t)$ is finite.*
  - (i-2) *For any $t \in \Sigma'$, $\mathcal{P}_{f(t)}^{\mathrm{gp}} \to (\mathcal{P}')_t^{\mathrm{gp}}$ is surjective.*
  - (i-3) *The map* $\mathrm{Mor}(\mathrm{Spec}(\mathbb{N}), \Sigma') \to \mathrm{Mor}(\mathrm{Spec}(\mathbb{N}), \Sigma)$ *is bijective.*
- (ii) *There exists an open covering $(U_\lambda)_\lambda$ of Σ such that for each λ, there are a finitely generated free $\mathbb{Z}$-module N_λ, a strongly convex rational polyhedral*

cone σ_λ in $N_{\lambda,\mathbb{R}}$, a finite subdivision V_λ of face(σ_λ), *and a commutative diagram of fans*

$$\begin{tikzcd} U'_\lambda \arrow[r, "\cong"] \arrow[d] & V_\lambda \arrow[d] \\ U_\lambda \arrow[r, "\cong"] & \mathrm{face}(\sigma_\lambda) \end{tikzcd}$$

where U'_λ denotes the inverse image of U_λ in Σ'.

Proof. This is essentially proved in [Kato 1994, Section 9]. In fact, in (ii), each $V_\lambda \to \mathrm{face}(\sigma_\lambda)$ satisfies the condition (i) by [loc. cit., (9.5)]. Hence (ii) implies (i). Conversely, if f satisfies (i), then any base change of f by an open immersion from an affine fan U_λ to Σ also satisfies (i). Again by [loc. cit., (9.5)], we can find N_λ, σ_λ and so on. □

2.3.4. Let Σ be a fan satisfying S_{fan}. A *finite subdivision* of Σ (called a proper subdivision of Σ in [Kato 1994]) is a fan Σ' satisfying S_{fan} endowed with a morphism $\Sigma' \to \Sigma$ satisfying the equivalent conditions (i) and (ii) in Lemma 2.3.3.

Lemma 2.3.5. *Let Σ be a fan satisfying the condition S_{fan}, let X be an fs log scheme, let $X \to \Sigma$ be a morphism* (2.2.9), *and let Σ' be a finite subdivision of Σ. Then the functor $X \times_\Sigma \Sigma'$:* (fs) $\to$ (Sets) *is represented by an fs log scheme X' which is proper and log étale over X. Here $X \times_\Sigma \Sigma'$ denotes the fiber product of the functors $X =$* Mor$(\cdot, X)$ *and $\Sigma' = [\Sigma']$* (2.2.7) *on* (fs) *over the functor $\Sigma = [\Sigma]$ on* (fs) (it does not mean the set theoretic fiber product of X and Σ' over Σ).

Proof. We are reduced to the case $\Sigma = \mathrm{face}(\sigma)$ for a strongly convex rational polyhedral cone σ and Σ' is a finite subdivision of Σ. Locally on X, $X \to \Sigma$ is the composition $X \to \mathrm{Spec}(\mathbb{Z}[P_\sigma]) \to \Sigma$. Hence we are reduced to the case $X = \mathrm{Spec}(\mathbb{Z}[P_\sigma])$. Then $X \times_\Sigma \Sigma'$ is represented by the toric variety $\bigcup_{\tau\in\Sigma'} \mathrm{Spec}(\mathbb{Z}[P_\tau])$ over $\mathbb{Z}$ associated to Σ', which is proper and log étale over X. □

2.3.6. We call a morphism $X \to Y$ of fs log schemes a *log modification* if locally on Y, there exist a fan Σ satisfying S_{fan}, a morphism $Y \to \Sigma$, and a finite subdivision Σ' of Σ such that X represents $Y \times_\Sigma \Sigma'$.

Log modifications were studied in [Kato and Usui 2009] for fs log analytic spaces over $\mathbb{C}$.

The following lemma is easy to prove.

Lemma 2.3.7. (1) *A log modification is proper and log étale.*

(2) *If $X \to Y$ is a log modification, the induced morphism of functors* Mor$(\cdot, X) \to$ Mor$(\cdot, Y)$ *on* (fs) *is injective.*

(3) *If $X_i \to Y$ $(i = 1, 2)$ are log modifications, $X_1 \times_Y X_2 \to Y$ is a log modification. Here $X_1 \times_Y X_2$ denotes the fiber product in the category of fs log schemes.*

(4) *If $X \to Y$ and $Y \to Z$ are log modifications, the composition $X \to Z$ is a log modification.*

Proposition 2.3.8. *Let $f : X \to Y$ be a log modification of fs log schemes.*

(1) *Let F be a torsion sheaf of abelian groups on $Y_{\text{lét}}$. Then the natural homomorphism $F \to Rf_{\text{lét}*} f^*_{\text{lét}} F$ is an isomorphism.*

(2) *Let ℓ be a prime number which is invertible on Y. Then the natural homomorphism $\mathbb{Q}_\ell \to Rf_{\text{lét}*}\mathbb{Q}_\ell$ is an isomorphism.*

Proof. Assertion (2) is reduced to (1). Assertion (1) is a slight generalization of Theorem 5.5(2) of [Nakayama 2017b], and the proof is similar, which is reduced easily to Lemma 2.3.7(2). □

2.3.9. (1) Let Σ be a fan with the structure sheaf $\mathcal{P}$ of monoids. We say Σ is *free* if for any $t \in \Sigma$, the stalk $\mathcal{P}_t$ is isomorphic to $\mathbb{N}^{r(t)}$ for some $r(t) \geq 0$.

(2) Let X be an fs log scheme. We say $M_X/\mathcal{O}_X^\times$ is *free* if for any $x \in X$, $(M/\mathcal{O}_X^\times)_x \cong \mathbb{N}^{r(x)}$ for some $r(x) \geq 0$.

Proposition 2.3.10. *Let Σ be a finite fan satisfying the condition S_{fan}. Then there is a finite subdivision $\Sigma' \to \Sigma$ which is free* (2.3.9(1)).

This is already explained in [Kato 1994].

Lemma 2.3.11. *Let Σ be a finite fan satisfying the condition S_{fan} with the structural sheaf $\mathcal{P}$, let $t \in \Sigma$, and let P be an fs submonoid of $\mathcal{P}_t^{\text{gp}}$ containing $\mathcal{P}_t$. Then there is a finite subdivision Σ' of Σ such that there is an open immersion* $\operatorname{Spec}(P) \to \Sigma'$ *over Σ.*

Proof. Regard Σ as a conical polyhedral complex with an integral structure [Kempf et al. 1973, Chapter II, §1, Definitions 5 and 6, pp. 69–70]. Let σ be its cell corresponding to $\mathcal{P}_t$ and $\tau \subset \sigma$ be the subcone corresponding to P. Take a rational homomorphism $f : \sigma \to \mathbb{R}_{\geq 0}$ such that $f^{-1}(\{0\})$ is trivial, where $\mathbb{R}_{\geq 0}$ is the monoid of the nonnegative real numbers with addition. Let $f_0 : S := \bigcup_{\sigma' \in \Sigma} \operatorname{Sk}^1(\sigma') \cup \operatorname{Sk}^1(\tau) \to \mathbb{R}$ be the zero extension of the restriction of f to $\operatorname{Sk}^1(\tau)$, that is, for any $s \in S$, $f_0(s) = f(s)$ if $s \in \operatorname{Sk}^1(\tau)$ and $f_0(s) = 0$ otherwise. Here Sk^1 means the 1-skeleton [loc. cit., Chapter I, §2, p. 29]. Let $f_1 : |\Sigma| \to \mathbb{R}_{\geq 0}$ be the convex interpolation of f_0 [loc. cit., Chapter I, §2, p. 29 and Chapter II, §2, p. 92], where $|\Sigma|$ is the support of Σ. Then, f_1 coincides with f on τ, and the coarsest subdivision of the conical polyhedral complex Σ on any cell of which f_1 is linear owes τ as a cell. Hence the corresponding finite subdivision Σ' of the fan Σ satisfies the desired property. □

Proposition 2.3.12. *Let X be a quasicompact fs log scheme, let Σ be a finite fan satisfying the condition S_{fan} with the structure sheaf $\mathcal{P}$, and let $f : X \to \Sigma$ be a morphism* (2.2.9) *such that for any $x \in X$, the map $\mathcal{P}_{f(x)} \to (M_X/\mathcal{O}_X^\times)_x$ is surjective. Then for a sufficiently fine finite subdivision Σ' of Σ, $X \times_\Sigma \Sigma' \to \Sigma'$ is strict.*

Proof. First notice that the problem is local on X as the category of finite subdivisions of Σ is directed. Let $x \in X$, and let P be the fs submonoid of $(\mathcal{P}_{f(x)})^{\mathrm{gp}}$ consisting of all elements whose images in $(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x$ are contained in $(M_X/\mathcal{O}_X^\times)_x$. Then $P/P^\times \to (M_X/\mathcal{O}_X^\times)_x$ is an isomorphism. Since X is quasicompact and the problem is local on X, replacing X by an open neighborhood of x, we may assume that $X \to \Sigma$ factors as $X \to \mathrm{Spec}(P) \to \mathrm{Spec}(\mathcal{P}_{f(x)}) \to \Sigma$ and the first arrow is strict. Let Σ' be a finite subdivision of Σ such that there is an open immersion $\mathrm{Spec}(P) \to \Sigma'$ over Σ (Lemma 2.3.11). Then the morphism $X = X \times_\Sigma \Sigma' \to \Sigma'$ is strict because it is the composition of strict morphisms $X \to \mathrm{Spec}(P) \to \Sigma'$. □

Remark 2.3.13. This Proposition 2.3.12 will be used later in Proposition 3.1.4 to make the diagonal of a vertical log smooth fs log scheme over a standard log point a regular immersion, by log modification.

2.3.14. In the next section, we will use the following corollary of Proposition 2.1.13.

Let X be a projective vertical log smooth fs log scheme over a standard log point s. Then, for some morphism of standard log points $s' \to s$ whose underlying extension of the fields is an isomorphism, we have a projective strict semistable fs log scheme X' over s' which is a log blow-up of $X \times_s s'$.

2.4. *Grothendieck groups of vector bundles and log geometry.*

2.4.1. Recall the following theory in [SGA 6 1971] until 2.4.2.

For a scheme X, let $K(X)$ be the Grothendieck group of the category of locally free $\mathcal{O}_X$-modules on X of finite rank. It is a commutative ring in which the multiplication corresponds to tensor products.

The K-group $K(X)$ has a decreasing filtration $(F^r K(X))_{r\in\mathbb{Z}}$ called the *γ-filtration* (for details, see [SGA 6 1971; Fulton and Lang 1985, Chapter III, V]). It satisfies $F^0K(X) = K(X)$ and $F^r K(X) \cdot F^s K(X) \subset F^{r+s} K(X)$. We define

$$\mathrm{gr}^r K(X) := F^r K(X)/F^{r+1} K(X).$$

2.4.2. For a morphism $X \to Y$ of schemes, the pullback homomorphism $K(Y) \to K(X)$ is defined and it respects the γ-filtration.

On the other hand, for a morphism $f : X \to Y$ of schemes which is projective and locally of complete intersection (see [SGA 6 1971, Exposé VIII, définition 1.1]), the pushforward homomorphism $K(X) \to K(Y)$ is defined (see [SGA 6 1971, Exposé IV, 2.12]). It sends $F^i K(X)_\mathbb{Q}$ to $F^{i-d} K(Y)_\mathbb{Q}$. Here d is the relative dimension of f which is a locally constant function on X characterized as follows. Locally on X, f is a composition $X \xrightarrow{i} Z \xrightarrow{g} Y$, where i is a regular immersion and g is smooth. The relative dimension of f is $d_1 - d_2$, where d_1 is the relative dimension of g and d_2 is the codimension of i.

2.4.3. If X and Y are projective smooth schemes over a field k, any morphism $X \to Y$ over k is projective and locally of complete intersection and hence the pushforward homomorphism $K(X) \to K(Y)$ is defined. However, in log geometry, we have no such nice property if we replace the smoothness by log smoothness.

We give some preliminaries to treat log smooth situations which we encounter in later sections.

Proposition 2.4.4. *Let S be an fs log scheme of log rank ≤ 1* (this means that for any $s \in S$, $(M_S/\mathcal{O}_S^\times)_s$ is isomorphic to either $\mathbb{N}$ or $\{1\}$). *Let $f : X \to S$ be a log smooth morphism. Then the underlying morphism of schemes of f is flat.*

Proof. [Kato 1989, Corollaries (4.4), (4.5)]. □

Proposition 2.4.5. *Let S be an fs log scheme of log rank ≤ 1, and let $f : X \to Y$ be a morphism of fs log schemes over S. Assume that X, Y are log smooth over S, and assume that $M_X/\mathcal{O}_X^\times$ and $M_Y/\mathcal{O}_Y^\times$ are free* (2.3.9). *Then the underlying morphism of schemes of f is locally of complete intersection.*

Proof. Working étale locally on X and on Y, we may assume that f is the base change of $f' : X' \to Y'$ over $S' = \operatorname{Spec}(\mathbb{Z}[\mathbb{N}])$ by a strict morphism $S \to S'$, where S' is endowed with log by $\mathbb{N}$ and X' and Y' are log smooth over S'. By the assumption on the log of X and Y, we may assume that $M/\mathcal{O}^\times$ of X' and that of Y' are also free (2.3.9) and hence X' and Y' are smooth over $\mathbb{Z}$ as schemes. Hence f' is locally of complete intersection. Since X' and Y' are flat over S', f is also locally of complete intersection. Here we used the fact that any base change of a morphism $f' : X' \to Y'$ of locally complete intersection of schemes which are flat over a scheme is locally of complete intersection. A proof of this fact is as follows. Locally, f' is the composition of a regular immersion followed by a smooth morphism, and hence we may assume that f' is a regular immersion. But for a closed immersion defined by an ideal I being a regular immersion is equivalent to the condition that I/I^2 is locally free and $I^n/I^{n+1} = \operatorname{Sym}^n(I/I^2)$ for any n. The last property is stable under any base change. □

2.4.6. For an fs log scheme X, we define

$$K_{\lim}(X) := \varinjlim_{X'} K(X'),$$

where X' ranges over all log modifications (2.3.6) of X.

Lemma 2.4.7. *Let X be a quasicompact fs log scheme, let Σ be a finite fan satisfying the condition S_{fan} with the structure sheaf $\mathcal{P}$, and let $f : X \to \Sigma$ be a morphism* (2.2.9) *such that for any $x \in X$, the map $\mathcal{P}_{f(x)} \to (M_X/\mathcal{O}_X^\times)_x$ is surjective. Then we have an isomorphism*

$$\varinjlim_{\Sigma'} K(X \times_\Sigma \Sigma') \xrightarrow{\cong} K_{\lim}(X),$$

where Σ' ranges over all finite subdivisions of Σ.

Proof. Let $X' \to X$ be a log modification. Then the composition $f' : X' \to X \to \Sigma$ satisfies the condition that $\mathcal{P}_{f'(x)} \to (M_{X'}/\mathcal{O}_{X'}^{\times})_x$ is surjective for any $x \in X'$. Hence by Proposition 2.3.12, there is a finite subdivision Σ' of Σ such that the morphisms $X \times_\Sigma \Sigma' \to \Sigma$ and $X' \times_\Sigma \Sigma' \to \Sigma'$ are strict. This shows that the log modification $X' \times_\Sigma \Sigma' \to X \times_\Sigma \Sigma'$ is strict and hence $X' \times_\Sigma \Sigma' \xrightarrow{\cong} X \times_\Sigma \Sigma'$. □

2.4.8. Let s be a geometric standard log point (2.1.11), and let X be an fs log scheme over s. Let ℓ be a prime number which is different from the characteristic of s and let $H^m(X)_\ell := R^m f_* \mathbb{Q}_\ell$, where f is the morphism $X \to s$ and $R^m f_*$ is the m-th higher direct image for the log étale topology (2.1.2). We will identify $H^m(X)_\ell$ with its stalk.

We have a Chern class map $\mathrm{gr}^i K(X)_\mathbb{Q} \to H^{2i}_{\text{ét}}(X, \mathbb{Q}_\ell)(i)$ to the classical étale cohomology, which coincides with the Chern character map. By composing this with the canonical map $H^{2i}_{\text{ét}}(X, \mathbb{Q}_\ell)(i) \to H^{2i}(X)_\ell(i)$ and by going to the inductive limit for log modifications using the invariance Proposition 2.3.8 for the log étale cohomology, we obtain the Chern class map

$$\mathrm{gr}^i K_{\lim}(X)_\mathbb{Q} \to H^{2i}(X)_\ell(i).$$

Proposition 2.4.9. *Let X (resp. Y) be a projective and vertical log smooth fs log scheme over a geometric standard log point s (2.1.11) such that $M/\mathcal{O}^\times$ of X and that of Y are free (2.3.9). Let $f : X \to Y$ be a morphism over s of relative dimension d.* (d can be < 0. See 2.4.2.) *Let ℓ be a prime number which is different from the characteristic of s. Then for any $i \in \mathbb{Z}$, the following diagram is commutative.*

$$\begin{array}{ccc}
\mathrm{gr}^{i+d} K(X)_\mathbb{Q} & \longrightarrow & H^{2(i+d)}(X)_\ell(i+d) \\
\downarrow & & \downarrow \\
\mathrm{gr}^i K(Y)_\mathbb{Q} & \longrightarrow & H^{2i}(Y)_\ell(i)
\end{array}$$

Here the left vertical arrow is defined by Proposition 2.4.5 and 2.4.2 and the right vertical arrow is the pushforward map (the dual of $H^{2j}(Y)_\ell(j) \to H^{2j}(X)_\ell(j)$ for Poincaré duality (Corollary 2.1.4), where $j = \dim(Y) - i$).

Remark. In the above, d (resp. $\dim(Y)$) is considered as a locally constant function on X (resp. Y) (see 2.4.2). In general, if m is a locally constant function on X, $H^m(X)$ means $\bigoplus_i H^{m(i)}(X_i)$, where X_i are connected components of X and $m(i)$ is the value of m on X_i. The meaning of $\mathrm{gr}^m K(X)_\mathbb{Q}$ is similar.

Proof. Let X° (resp. Y°) be the underlying scheme of X (resp. Y). The morphism f is the composition of two morphisms $X \to \boldsymbol{P}^n \times Y \to Y$ in which the underlying morphism of schemes of the first arrow is a closed regular immersion and the second arrow is the projection. It is sufficient to prove Proposition 2.4.9 for each of these two morphisms. The proof for the latter morphism is standard. We consider the first morphism. It is sufficient to prove the commutativity of the two squares in the diagram

$$\begin{tikzcd} \mathrm{gr}^i K(X^\circ)_{\mathbb{Q}} \arrow[r] \arrow[d] & H^{2i}_{\text{ét}}(X^\circ, \mathbb{Q}_\ell)(i) \arrow[r] \arrow[d] & H^{2i}(X)_\ell(i) \arrow[d] \\ \mathrm{gr}^{i+c} K(Y^\circ)_{\mathbb{Q}} \arrow[r] & H^{2i+2c}_{\text{ét}}(Y^\circ, \mathbb{Q}_\ell)(i+c) \arrow[r] & H^{2i+2c}(Y)_\ell(i+c) \end{tikzcd}$$

assuming that the morphism $X^\circ \to Y^\circ$ is a closed regular immersion of codimension c. Here the central vertical arrow is the Gysin map which is defined as follows. Let $\xi \in H^{2c}_{X^\circ}(Y^\circ, \mathbb{Q}_\ell)(c)$ be the localized Chern class of the $\mathcal{O}_Y$-module $\mathcal{O}_X$ [Iversen 1976]. By using the cup product

$$H^i_{\text{ét}}(X^\circ, \mathbb{Q}_\ell) \times H^j_{X^\circ}(Y^\circ, \mathbb{Q}_\ell) \to H^{i+j}_{X^\circ}(Y^\circ, \mathbb{Q}_\ell),$$

let the Gysin map be the product with ξ. (See [Baum et al. 1975, Section 5.4].)

The left square is commutative by the Riemann–Roch theorem in Corollary 1 in Section 5.3 of [Baum et al. 1975] (see also [Fulton 1998]). We prove that the right square is commutative. By 2.3.14, we may assume that X and Y are strict semistable. Let X' be X° with the inverse image of the log structure of Y. Hence $X \to Y$ factors as $X \to X' \to Y$. Consider the diagram

$$\begin{tikzcd} H^i_{\text{ét}}(X^\circ, \mathbb{Q}_\ell) \arrow[r] \arrow[d] & H^i(X')_\ell \arrow[r] \arrow[d] & H^i(X)_\ell \arrow[d] \\ H^{i+2c}_{X^\circ}(Y^\circ, \mathbb{Q}_\ell)(c) \arrow[r] & H^{i+2c}_{X'}(Y)_\ell(c) \arrow[r] & H^{i+2c}(Y)_\ell(c) \end{tikzcd}$$

The left square is evidently commutative. The composition

$$H^i_{\text{ét}}(X^\circ, \mathbb{Q}_\ell) \to H^{i+2c}_{\text{ét}}(Y^\circ, \mathbb{Q}_\ell)(c) \to H^{i+2c}(Y)_\ell(c)$$

coincides with the composition

$$H^i_{\text{ét}}(X^\circ, \mathbb{Q}_\ell) \to H^{i+2c}_{X^\circ}(Y^\circ, \mathbb{Q}_\ell)(c) \to H^{i+2c}_{X'}(Y)_\ell(c) \to H^{i+2c}(Y)_\ell(c).$$

Hence it is sufficient to prove the commutativity of the right square. Let $p := \dim(X)$, so $\dim(Y) = p + c$. Let $j = 2p - i$. It is sufficient to prove that for $a \in H^i(X')_\ell$ and $b \in H^j(Y)_\ell(p)$, we have $(a \cup \xi \cup b)_Y = (a \cup b|_X)_X$ in $\mathbb{Q}_\ell$. Using $z = a \cup b|_{X'} \in H^{2p}(X')_\ell(p)$, we see that it is sufficient to prove that for $z \in H^{2p}(X')_\ell(p)$, the image of z under

$$H^{2p}(X')_\ell(p) \to H^{2p+2c}_{X'}(Y)_\ell(p+c) \to H^{2p+2c}(Y)_\ell(p+c) \to \mathbb{Q}_\ell$$

(the first arrow is the product with ξ) and the image of z under $H^{2p}(X')_\ell(p) \to H^{2p}(X)_\ell(p) \to \mathbb{Q}_\ell$ coincide. $H^{2p}(X')_\ell(p)$ is generated by the Chern classes of the

$\mathcal{O}_X$-modules $[\kappa(u)]$, where u ranges over all nonsingular closed points of X and $\kappa(u)$ is the residue field at u. For $z = [\kappa(u)]$, the image of z in $H^{2p+2c}(Y)_\ell(p+c)$ is the Chern class of the $\mathcal{O}_Y$-module $\kappa(u)$. Hence the image of this z in $\mathbb{Q}_\ell$ via $H^{2p+2c}(Y)_\ell(p+c)$ is 1. On the other hand, the image of this z in $\mathbb{Q}_\ell$ via $H^{2p}(X)_\ell(p)$ is 1. Thus both images coincide. □

Corollary 2.4.10. *Let X be a projective vertical log smooth fs log scheme over a geometric standard log point s. Let X' be a log blow-up of X such that $M_{X'}/\mathcal{O}_{X'}^\times$ is free* (2.3.9)*. Then the image of the Chern class map $\mathrm{gr}^i K_{\mathrm{lim}}(X) \to H^{2i}(X)_\ell(i)$ coincides with the image of the Chern class map $\mathrm{gr}^i K(X') \to H^{2i}(X)_\ell(i)$.*

Proof. Let Y be any log blow-up of X and let $a \in \mathrm{gr}^i K(Y)_\mathbb{Q}$. Take a log blow-up Y' of Y such that $M_{Y'}/\mathcal{O}_{Y'}^\times$ is free and such that Y' is also a log blow-up of X'. Let a' be the image of a in $\mathrm{gr}^i K(Y')$ by pullback, and let b be the image of a' in $\mathrm{gr}^i K(X')_\mathbb{Q}$ by pushforward. Then by Proposition 2.4.9, the image of a in $H^{2i}(X)_\ell(i)$ coincides with the image of b. □

2.4.11. The above Proposition 2.4.9 contains the following trace formula in [Kato and Saito 2004]. Let X be a projective vertical log smooth fs log scheme over a geometric standard log point s. Assume that X is purely of dimension d. Let $(X \times X)'$ be a log blow-up of $X \times X$, let $\alpha \in \mathrm{gr}^d K((X \times X)')_\mathbb{Q}$, and let f_α be the image of α under the composition

$$\mathrm{gr}^d K_{\mathrm{lim}}(X \times X)_\mathbb{Q} \to H^{2d}(X \times X)_\ell(d) \cong \bigoplus_i \mathrm{Hom}(H^i(X)_\ell, H^i(X)_\ell),$$

where the last isomorphism is by Poincaré duality (Corollary 2.1.4) and the Künneth formula (Corollary 2.1.6). We consider the trace $\mathrm{Tr}(f_\alpha)$. Let X' be the log blow-up $X \times_{X \times X} (X \times X)'$ of the diagonal, and let the intersection of α with the diagonal $\alpha \cdot \Delta_X \in \mathbb{Q}$ be the image of α under the composition

$$\mathrm{gr}^d K((X \times X)')_\mathbb{Q} \to \mathrm{gr}^d K(X')_\mathbb{Q} \to K(s)_\mathbb{Q} = \mathbb{Q},$$

where the first arrow is the pullback by $X' \to (X \times X)'$ and the second arrow is the pushforward. Then we have the trace formula

$$\mathrm{Tr}(f_\alpha) = \alpha \cdot \Delta_X \in \mathbb{Q}.$$

This follows from Proposition 2.4.9 as follows. Consider the diagram

$$\begin{tikzcd}
\mathrm{gr}^d K_{\mathrm{lim}}(X \times X)_\mathbb{Q} \arrow[r] \arrow[d] & \mathrm{gr}^d K_{\mathrm{lim}}(X)_\mathbb{Q} \arrow[r] \arrow[d] & \mathrm{gr}^0 K(s)_\mathbb{Q} = \mathbb{Q} \arrow[d] \\
H^{2d}(X \times X)_\ell(d) \arrow[r] & H^{2d}(X)_\ell(d) \arrow[r] & H^0(s)_\ell = \mathbb{Q}_\ell \\
\bigoplus_i \mathrm{Hom}(H^i(X)_\ell, H^i(X)_\ell) \arrow[u, "\cong"'] & &
\end{tikzcd}$$

where the first arrow in the lower row is the pullback by the diagonal. The left square is clearly commutative and, by Proposition 2.4.9, the right square is commutative. The image of $f_\alpha \in \bigoplus_i \operatorname{Hom}(H^i(X)_\ell, H^i(X)_\ell)$ in $\mathbb{Q}_\ell$ under the composition of the bottom isomorphism and the lower row is $\operatorname{Tr}(f_\alpha)$. This gives a proof of the trace formula.

3. Log motives

In this Section 3, let S be an fs log scheme and let ℓ be a prime number which is invertible on S. We define and study the category of log (pure) motives.

3.1. *The category of log motives.* We define the category of log motives over S.

3.1.1. For a projective vertical log smooth fs log scheme X over S and for $r \in \mathbb{Z}$, consider the symbol $h(X)(r)$.

Let

$$h(X)(r)_\ell := \bigoplus_m H^m(X)_\ell(r), \qquad \text{where } H^m(X)_\ell = R^m f_* \mathbb{Q}_\ell \quad \text{(see 2.4.8)}$$

with $f : X \to S$ and with $R^m f_*$ for the log étale topology. This is a smooth $\mathbb{Q}_\ell$-sheaf on the log étale site of S (see 2.1.2).

3.1.2. Let X and Y be projective vertical log smooth fs log schemes over a geometric standard log point (2.1.11). Let $r, s \in \mathbb{Z}$.

An element α of $\operatorname{gr}^i K_{\lim}(X \times Y)_{\mathbb{Q}}$ with $i = d+s-r$, where $d = \dim(X)$ induces a homomorphism $h(X)(r)_\ell \to h(Y)(s)_\ell$ as follows.

Let β be the image of α under the Chern class map

$$\operatorname{gr}^i K_{\lim}(X \times Y)_{\mathbb{Q}} \to H^{2i}(X \times Y)_\ell(i).$$

Then for $m, n \in \mathbb{Z}$ such that $m - 2r = n - 2s$, we have the composition

$$\begin{aligned} H^m(X)_\ell(r) \to H^m(X \times Y)_\ell(r) &\to H^{m+2i}(X \times Y)_\ell(r+i) \\ &\to H^{m+2i-2d}(Y)_\ell(r+i-d) = H^n(Y)_\ell(s). \end{aligned}$$

Here the first arrow is the pullback, the second arrow is the cup product with β, the third arrow is the pushforward by the projection $X \times Y \to Y$. This gives a map $h(X)(r)_\ell \to h(Y)(s)_\ell$.

3.1.3. Let X and Y be projective vertical log smooth fs log schemes over S and let $r, s \in \mathbb{Z}$.

By definition, a morphism $f : h(X)(r) \to h(Y)(s)$ is a homomorphism $f : h(X)(r)_\ell \to h(Y)(s)_\ell$ of $\mathbb{Q}_\ell$-sheaves such that for any geometric standard log point p over S, the pullback $h(X_p)(r)_\ell \to h(Y_p)(s)_\ell$ of f is induced by an element of $\operatorname{gr}^{d+s-r} K_{\lim}(X_p \times_p Y_p)_{\mathbb{Q}}$ with $d = \dim(X_p)$ in the above way.

Proposition 3.1.4. (1) *The identity morphism $h(X)(r)_\ell \to h(X)(r)_\ell$ is a morphism $h(X)(r) \to h(X)(r)$.*

(2) *More generally, for a morphism $Y \to X$ over S, the induced map $h(X)(r)_\ell \to h(Y)(r)_\ell$ is a morphism $h(X)(r) \to h(Y)(r)$.*

Proof. We may and do assume that S is a geometric standard log point s. Let d be the dimension of X.

We prove (1). Let $Z = X \times X$ (the fiber product over $S = s$) and consider the fan $\Sigma := \Sigma_Z$ associated to Z (2.2.16). By Proposition 2.3.12, there is a finite subdivision $\Sigma' \to \Sigma$ such that $X' := X \times_\Sigma \Sigma' \to \Sigma'$ and $Z' := Z \times_\Sigma \Sigma' \to \Sigma'$ are strict. Hence the morphism $X' \to Z'$ is a strict closed immersion. Since a strict closed immersion between log smooth schemes is a regular immersion as is seen as in the classical case (see [Kato 1989, Proposition (3.10)]), this morphism $X' \to Z'$ is a regular immersion. Consider the $\mathcal{O}_{Z'}$-module $\mathcal{O}_{X'}$ and its class $[\mathcal{O}_{X'}] \in \mathrm{gr}^d\, K(Z')_\mathbb{Q}$ with $d = \dim(X)$. By Poincaré duality (Corollary 2.1.4) and by the Künneth formula (Corollary 2.1.6), this class induces the identity map $h(X)_\ell(r) \to h(X)_\ell(r)$.

Assertion (2) follows from (1). The homomorphism $h(X)_\ell(r) \to h(Y)_\ell(r)$ associated to f is induced by an element of $\mathrm{gr}^d\, K_{\lim}(X \times Y)_\mathbb{Q}$ with $d = \dim(X)$ which is obtained from the above element of $\mathrm{gr}^d\, K_{\lim}(X \times X)_\mathbb{Q}$ giving the identity morphism, by pulling back by $1 \times f$. □

3.1.5. The above Proposition 3.1.4 explains the reason why we must use $K_{\lim}$ (not just K) in the definition of morphism of the category of log motives. For a projective vertical log smooth fs log scheme X over a geometric standard log point s, the diagonal $X \to X \times X$ is usually not a regular immersion and cannot define an element of $K(X \times X)$. We need a log modification $Z \to X \times X$ to have an element of $K(Z)$ corresponding to the diagonal, which gives the identity morphism $h(X) \to h(X)$.

Proposition 3.1.6. *For morphisms*

$$f : h(X_1)(r_1) \to h(X_2)(r_2) \quad \textit{and} \quad g : h(X_2)(r_2) \to h(X_3)(r_3),$$

the composition $g \circ f : h(X_1)(r_1) \to h(X_3)(r_3)$ is a morphism.

Proof. We may assume that S is a geometric standard log point. If f is induced by $\alpha \in \mathrm{gr}\, K_{\lim}(X_1 \times X_2)_\mathbb{Q}$ and g is induced by $\alpha' \in \mathrm{gr}\, K_{\lim}(X_2 \times X_3)_\mathbb{Q}$, $g \circ f$ is induced by the following element α'' of $\mathrm{gr}\, K_{\lim}(X_1 \times X_3)_\mathbb{Q}$. Let $u \in \mathrm{gr}\, K_{\lim}(X_1 \times X_2 \times X_3)_\mathbb{Q}$ be the product of the pullbacks of α and α'. Let $(X_1 \times X_3)'$ be a log blow-up of $X_1 \times X_3$ having free $M/\mathcal{O}^\times$ (2.3.9), and let $(X_1 \times X_2 \times X_3)'$ be a log blow-up of $X_1 \times X_2 \times X_3$ having free $M/\mathcal{O}^\times$ such that u comes from an element v of $\mathrm{gr}\, K((X_1 \times X_2 \times X_3)')_\mathbb{Q}$ and such that we have a morphism $(X_1 \times X_2 \times X_3)' \to$

$(X_1 \times X_3)'$ which is compatible with the projection $X_1 \times X_2 \times X_3 \to X_1 \times X_3$. Let α'' be the pushforward of v by the morphism $(X_1 \times X_2 \times X_3)' \to (X_1 \times X_3)'$. Then $g \circ f$ is induced by α'' by Proposition 2.4.9. $\square$

3.1.7. Imitating the definition of motives by Grothendieck, we define the *category* $\mathrm{LM}(S)$ *of log motives over* S as the category of the symbols $(h(X)(r), e)$, where e is an idempotent in the endomorphism ring of $h(X)(r)$. The set of morphisms is defined as

$$\mathrm{Hom}\big((h(X_1)(r_1), e_1), (h(X_2)(r_2), e_2)\big)$$
$$:= e_2 \circ \mathrm{Hom}(h(X_1), h(X_2)) \circ e_1 \subset \mathrm{Hom}(h(X_1), h(X_2)).$$

The identity morphism of $(h(X), e)$ is e.

The ℓ-adic realization M_ℓ of the log motive $M = (h(X), e)$ is defined to be $eh(X)_\ell$.

3.1.8. In the case where the underlying scheme of S is $\mathrm{Spec}(k)$ for a field k, there is a natural functor from the category of motives over k modulo homological equivalence defined by Grothendieck to our category $\mathrm{LM}(S)$ sending the motive defined by a projective smooth scheme X over k to the log motive defined by X endowed with the pullback log structure from S. This is because $\mathrm{CH}^r(X \times Y)_{\mathbb{Q}} = \mathrm{gr}^r K(X \times Y)_{\mathbb{Q}}$.

Further, when the log structure of S is trivial, this functor is an equivalence. This is because, in this case, we have $\mathrm{gr}^r K(X \times Y)_{\mathbb{Q}} = \mathrm{gr}^r K_{\lim}(X \times Y)_{\mathbb{Q}}$.

3.1.9. For a morphism $S' \to S$ of fs log schemes, we have the evident pullback functor $\mathrm{LM}(S) \to \mathrm{LM}(S')$.

3.1.10. For a két morphism $p' \to p$ of standard log points whose underlying extension of fields is Galois, we have

$$\mathrm{Hom}_{\mathrm{LM}(p)}(h(X)(r), h(Y))(s)) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{LM}(p')}(h(X')(r), h(Y')(s))^G,$$

where X' and Y' are the base-changed objects from X and Y, $r, s \in \mathbb{Z}$, and $(\cdot)^G$ denotes the G-invariant part for $G = \mathrm{Aut}_p(p')$.

3.2. *Basic things.*

3.2.1. Direct sums and direct products exist in $\mathrm{LM}(S)$, and they coincide.

In fact, we have $h(X) \oplus h(Y) := h\big(X \coprod Y\big)$, and if $r \le s$, $h(X)(r) \oplus h(Y)(s) = \big(h\big((X \times \boldsymbol{P}^n) \coprod Y\big)(s), e\big)$ for $n \ge s - r$ and for some e.

Conjecture 3.2.2. For a projective vertical log smooth fs log scheme X of relative dimension d over S, $h(X)$ has a decomposition

$$h(X) = h^0(X) \oplus h^1(X) \oplus \cdots \oplus h^{2d}(X)$$

in the category $\mathrm{LM}(S)$ of log motives such that $h^i(X)_\ell = H^i(X)_\ell$.

Note that such a decomposition is unique if it exists.

3.2.3. We have the following: $h(\boldsymbol{P}^n) = \bigoplus_{i=0}^n h^{2i}(\boldsymbol{P}^n)$. Canonically, $h^{2i}(\boldsymbol{P}^n) \cong \mathbb{Q}(-i)$ for $0 \le i \le n$. Here $\mathbb{Q} = h(S)$.

3.2.4. We define the category $\mathrm{LM}^{\mathrm{spl}}(S)$ as follows. For a projective vertical log smooth fs log scheme X over S and for $m, r \in \mathbb{Z}$, consider the symbol $h^m(X)(r)$.

For projective vertical log smooth fs log schemes X and Y over S and for $m, n, r, s \in \mathbb{Z}$, a morphism $h : h^m(X)(r) \to h^n(Y)(s)$ means a homomorphism $H^m(X)_\ell(r) \to H^n(Y)_\ell(s)$ of smooth $\mathbb{Q}_\ell$-sheaves on S satisfying the following condition. If $m - 2r \ne n - 2s$, then $h = 0$. If $m - 2r = n - 2s$, then for any geometric standard log point p over S, the pullback of h to p comes from an element of $\mathrm{gr}^{d+s-r} K_{\lim}(X_p \times_p Y_p)$, where $d = \dim(X_0)$.

An object of $\mathrm{LM}^{\mathrm{spl}}(S)$ is $(h^m(X)(r), e)$, where X is a projective vertical log smooth fs log scheme over S, $m, r \in \mathbb{Z}$, and e is an idempotent of the ring of endomorphism of $h^m(X)(r)$. Morphisms are defined like the case of $\mathrm{LM}(S)$.

3.2.5. Similarly to the case of $\mathrm{LM}(S)$ (3.2.1), direct sums exist in $\mathrm{LM}^{\mathrm{spl}}(S)$. We have a functor

$$\mathrm{LM}(S) \to \mathrm{LM}^{\mathrm{spl}}(S), \quad h(X)(r) \mapsto \bigoplus_m h^m(X)(r).$$

Conjecture 3.2.2 is that this functor is an equivalence of categories.

3.2.6. Tensor products are defined in $\mathrm{LM}(S)$ as follows:

$$(h(X)(r), e) \otimes (h(X')(s), e') := (h(X \times X')(r+s), e \otimes e').$$

For a log motive M over S, the Tate twist $M(-r)$ $(r \ge 0)$ is identified with $M \otimes h^{2r}(\boldsymbol{P}^n)$ with $n \ge r$.

3.2.7. Compared with $\mathrm{LM}(S)$, a disadvantage of the category $\mathrm{LM}^{\mathrm{spl}}(S)$ is that the tensor products cannot be defined.

3.2.8. Duals are defined in $\mathrm{LM}(S)$ as follows:

$$(h(X)(r), e)^* = (h(X)(d-r), e(d-2r)),$$

where d is the relative dimension of X over S.

Note that, by Poincaré duality (Corollary 2.1.4), any morphism $h(X)(r) \to h(Y)(s)$ induces a homomorphism $(h(Y)(s)^*)_\ell \to (h(X)(r)^*)_\ell$ of $\mathbb{Q}_\ell$-sheaves. We can easily check that this homomorphism gives a morphism

$$h(Y)(s)^* = h(Y)(d'-s) \to h(X)(d-r) = h(X)(r)^*$$

of motives, where d' is the relative dimension of Y over S by using the same elements of $\mathrm{gr}^i K_{\lim}(X_p \times_p Y_p)_{\mathbb{Q}}$, where p is a geometric standard log point over S and $i = d + s - r = d' + (d-r) - (d'-s)$.

3.2.9. Let X be a projective vertical log smooth fs log scheme over S. We conjecture that, for any morphism $s \to S$ from a standard log point associated to some finite field and for each $m \in \mathbb{Z}$, the filtration (the monodromy filtration) on the stalk over s of $H^m(X)_\ell$ determined by the monodromy operator coincides with the Frobenius weight filtration. We call this the *monodromy-weight conjecture* for X.

Proposition 3.2.10. *Let X and Y be projective vertical log smooth fs log schemes over S. Assuming the monodromy-weight conjecture for X and Y, we have the following*:

> *If $m - 2r > n - 2s$ and if S is of finite type over $\mathbb{Z}$, there is no nonzero homomorphism $H^m(X)_\ell(r) \to H^n(Y)_\ell(s)$.*

Proof. This is reduced to the case where S is a standard log point associated to a finite field k. Let $w = m - 2r$, $w' = n - 2s$. The monodromy-weight conjecture asserts that as a finite-dimensional $\mathbb{Q}_\ell$-vector space with actions of $\mathrm{Gal}(\bar{k}/k)$ and the monodromy operator $\mathcal{N}$, the stalk of $H^m(X)_\ell(r)$ (resp. $H^n(Y)_\ell(s)$) is isomorphic to a direct sum of subobjects Q (resp. R) being isomorphic to $\mathrm{Sym}^i H^1(E)_\ell \otimes V$, where the action of $\mathrm{Gal}(\bar{k}/k)$ on V is of weight $w - i$ (resp. $w' - i$) and the action of $\mathcal{N}$ on V is trivial. Hence, it is enough to show that there is no nonzero $\mathbb{Q}_\ell$-linear map $Q \to R$ which is compatible with the actions of $\mathrm{Gal}(\bar{k}/k)$ and $\mathcal{N}$. Let $Q \to R$ be such a map. For any nonzero element $x \in R$ of weight $u \geq w'$, we have $\mathcal{N}^{u-w'}(x) \neq 0$. But as a $\mathbb{Q}_\ell$-vector space with an action of $\mathcal{N}$, Q is generated by an element y of weight $u \geq w$ such that $\mathcal{N}^{u-w+1}(y) = 0$. The image x of this y in R is of weight $u \geq w'$ and $\mathcal{N}^{u-w'}(x) = 0$ because $u - w' \geq u - w + 1$. Hence $x = 0$. Therefore the map $Q \to R$ is the zero map. □

Remark 3.2.11. On the other hand, a nontrivial homomorphism $H^m(X)_\ell(r) \to H^n(Y)_\ell(s)$ can exist even if $m - 2r < n - 2s$ and even if S is of finite type over $\mathbb{Z}$. In fact, let S be a standard log point, $X = S$, and Y the log Tate curve. Then we have an exact sequence $0 \to \mathbb{Q}_\ell \to H^1(Y)_\ell \to \mathbb{Q}_\ell(-1) \to 0$. Hence a nontrivial homomorphism $H^0(X)_\ell \to H^1(Y)_\ell$ exists.

3.2.12. For an X strict semistable over a standard log point, $H^1(X_{\mathrm{Zar}}, M_X^{\mathrm{gp}}/\mathcal{O}_X^\times) = 0$ because $M_X^{\mathrm{gp}}/\mathcal{O}_X^\times \cong p_*\mathbb{Z}$, where $p : X' \to X$ is a normalization, is a flasque sheaf, which implies that $\mathrm{Pic}\,(X) = H^1(X_{\mathrm{Zar}}, \mathcal{O}_X^\times) \to H^1(X_{\mathrm{Zar}}, M_X^{\mathrm{gp}})$ is surjective. Hence by 2.3.14, we have:

> Let X, Y be projective vertical log smooth fs log schemes over an fs log scheme S. Then an element of $H^1((X \times Y)_{\mathrm{Zar}}, M_{X\times Y}^{\mathrm{gp}})$ gives a homomorphism $h(X)(r) \to h(Y)(r + 1 - d)$, where $r \in \mathbb{Z}$ and d is the relative dimension of X over S.

To see this, it is enough to show that the induced homomorphism $h(X)(r)_\ell \to h(Y)(r+1-d)_\ell$ comes from an element of the K-group after the base change to any geometric standard log point. We assume that the base S is a geometric standard log point over a field k. Apply 2.3.14 to $X \times Y$, and find a strict semistable X' over $X \times Y$ after the base change by the morphism $S' = (\operatorname{Spec} k, \mathbb{N}) \to S = (\operatorname{Spec} k, \mathbb{N})$ induced by the multiplication by n for some $n \geq 1$. If $n = 1$, since $\operatorname{Pic}(X') = \operatorname{gr}^1 K(X')$, we have a desired element of $\operatorname{gr}^1 K_{\lim}(X \times Y)_{\mathbb{Q}}$. For a general n, after the base change, take a desired element a of $\operatorname{gr}^1 K_{\lim}(X \times Y \times_S S')_{\mathbb{Q}}$. Then the $1/n$ times of $\operatorname{Tr}(a)$ is a desired element.

3.3. *Numerical equivalence.*

Proposition 3.3.1. *For any log motive M over S and for any morphism $f : M \to M$,* $\operatorname{Tr}(f) \in \mathbb{Q}_\ell$ *belongs to* $\mathbb{Q}$. (Precisely speaking, $\operatorname{Tr}(f)$ is a locally constant function $S \to \mathbb{Q}$. It is constant if S is connected.)

Proof. We are reduced to the case where S is a geometric standard log point. Then the result follows from the trace formula 2.4.11. □

Definition 3.3.2 (numerical equivalence). For objects M and M' of $\mathrm{LM}(S)$ and for a morphism $f : M \to M'$, we say that f is *numerically equivalent to* 0 if for any morphism $g : M' \to M$, we have $\operatorname{Tr}(gf) = 0$, that is, $\operatorname{Tr}(fg) = 0$. (Note that when S is the spectrum of a field endowed with the trivial log structure, it coincides with the usual definition; see [Jannsen 1992, Lemma 1].)

Morphisms $f, g : M \to M'$ are said to be *numerically equivalent* if $f - g$ is numerically equivalent to 0.

Lemma 3.3.3. *Let $\sim$ be the numerical equivalence. Let $f, g : M \to N$ be morphisms in* $\mathrm{LM}(S)$. *Assume $f \sim g$. Then*

(1) *$fh \sim gh$ for any morphism $h : L \to M$ from a log motive L over S.*

(2) *$hf \sim hg$ for any morphism $h : N \to L$ to a log motive L over S.*

Proof. We may assume that g is 0.

(1) Let $k : N \to L$ be any morphism. Then $\operatorname{Tr}(fhk) = \operatorname{Tr}(f(hk)) = 0$. Hence $fh \sim 0$.

(2) Let $k : L \to M$ be any morphism. Then $\operatorname{Tr}(khf) = \operatorname{Tr}((kh)f) = 0$. Hence $hf \sim 0$. □

3.3.4. By Lemma 3.3.3, we have the category $\mathrm{LM}_{\mathrm{num}}(S)$ of log motives over S modulo numerical equivalence.

Conjecture 3.3.5. In $\mathrm{LM}(S)$, $f \sim g$ implies $f = g$. That is, $\mathrm{LM}(S) = \mathrm{LM}_{\mathrm{num}}(S)$.

3.3.6. When S is a geometric standard log point, the category $\mathrm{LM}_{\mathrm{num}}(S)$ is independent of the choice of ℓ. This is a consequence of Proposition 3.3.1 since in this case, the group $\operatorname{Hom}(h(X)(r), h(Y)(s))$ is identified with a quotient of $\operatorname{gr}^{d+s-r} K_{\lim}(X \times_S Y)_{\mathbb{Q}}$ in the notation in 3.1.3.

3.4. *Semisimplicity.*

Theorem 3.4.1. (1) *The category* $\mathrm{LM}_{\mathrm{num}}(S)$ *is a semisimple abelian category.*

(2) *The category* $\mathrm{LM}(S)$ *is a semisimple abelian category if and only if the numerical equivalence for morphisms of this category is trivial.*

To prove this, we imitate the method of U. Jannsen [1992].

3.4.2. The following fact is known: A pseudoabelian category $\mathcal{C}$ is a semisimple abelian category if the following (i) and (ii) are satisfied for any objects X and Y.

(i) $\mathrm{Hom}\,(X, Y)$ is a $\mathbb{Q}$-vector space, the composition of morphisms is bilinear, and any idempotent of $\mathrm{End}\,(X)$ has a kernel.

(ii) $\mathrm{End}\,(X)$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra.

Lemma 3.4.3. *Let F be a field, A, B finite-dimensional F-vector spaces, $(\cdot, \cdot) : A \times B \to F$ an F-bilinear map, F_0 a subfield of F, A_0 an F_0-subspace of A, and B_0 an F_0-subspace of B. Assume that A is generated by A_0 over F, B is generated by B_0 over F, and $(a, b) \in F_0$ for any $a \in A_0$ and $b \in B_0$. Let*

$$K = \{a \in A \mid (a, b) = 0 \textit{ for any } b \in B\},$$
$$K_0 = \{a \in A_0 \mid (a, b) = 0 \textit{ for any } b \in B_0\}.$$

Then:

$$F \otimes_{F_0} A_0/K_0 \xrightarrow{\cong} A/K.$$

In particular, A_0/K_0 is finite-dimensional over F_0.

Proof. Take an F_0-subspace A'_0 of A_0 such that $F \otimes_{F_0} A'_0 \xrightarrow{\cong} A$ and an F_0-subspace B'_0 of B_0 such that $F \otimes_{F_0} B'_0 \xrightarrow{\cong} B$. Then A'_0 and B'_0 are finite-dimensional over F_0. Let $K'_0 = \{a \in A'_0 \mid (a, b) = 0 \text{ for any } b \in B\} = \{a \in A'_0 \mid (a, b) = 0 \text{ for any } b \in B'_0\}$. Let $L'_0 = \{b \in B'_0 \mid (a, b) = 0 \text{ for any } a \in A\} = \{b \in B'_0 \mid (a, b) = 0 \text{ for any } a \in A'_0\}$. The composition

$$A'_0/K'_0 \to A_0/K_0 \to \mathrm{Hom}(B'_0/L'_0, F_0)$$

is an isomorphism and the two arrows here are injective. Hence we have

$$A'_0/K'_0 \to A_0/K_0 \quad \text{is an isomorphism.} \tag{$\star$}$$

On the other hand, the paring $A \times B \to F$ is identified with $F \otimes_{F_0}$ of the pairing $A'_0 \times B'_0 \to F_0$. Hence we have

$$F \otimes_{F_0} A'_0/K'_0 \to A/K \quad \text{is an isomorphism.} \tag{$\star\star$}$$

By ($\star$) and ($\star\star$), we have that $F \otimes_{F_0} A_0/K_0 \to A/K$ is an isomorphism. □

Lemma 3.4.4. *Let F be a field of characteristic* 0, *V a finite-dimensional F-vector space, and A an F-subalgebra of* $\mathrm{End}_F(V)$. *Let J be the Jacobson radical of A, that is, J is the largest nilpotent two-sided ideal of A. Let*

$$I = \{a \in A \mid \mathrm{Tr}(ab) = 0 \textit{ for any } b \in A\}.$$

Here Tr *is the trace of an F-linear map $V \to V$. Then $I = J$.*

Proof. Let $a \in J$. Then for any $b \in A$, ab is nilpotent and hence $\mathrm{Tr}(ab) = 0$. Hence $a \in I$. Next we prove $I \subset J$. We may assume that F is algebraically closed. It is sufficient to prove that all elements of I are nilpotent. Let $a \in I$. Let $(\alpha_i)_{1 \le i \le n}$ $(n = \dim_F(V))$ be the eigenvalues of a counted with multiplicity. We have $0 = \mathrm{Tr}(a^n) = \sum_{i=1}^n \alpha_i^n$ for any $n \ge 1$. This proves that $\alpha_i = 0$ for all i. Hence a is nilpotent. □

Lemma 3.4.5. *Let F be a field of characteristic* 0, *V a finite-dimensional F-vector space, A an F-subalgebra of* $\mathrm{End}_F(V)$, *F_0 a subfield of F, and A_0 an F_0-subalgebra of A. Assume that A_0 generates the F-vector space A and assume that* $\mathrm{Tr}(a) \in F_0$ *for any $a \in A_0$. Let $I_0 = \{a \in A_0 \mid \mathrm{Tr}(ab) = 0$ for any $b \in A_0\}$. Then I_0 is a two-sided ideal of A_0, A_0/I_0 is a finite-dimensional semisimple F_0-algebra, and all elements of I_0 are nilpotent.*

Proof. The fact that I_0 is a two-sided ideal of A_0 is shown easily. Let

$$I = \{a \in A \mid \mathrm{Tr}(ab) = 0 \text{ for any } b \in A\}.$$

Then I is nilpotent and A/I is a semisimple algebra by Lemma 3.4.4. Hence all elements of I_0 are nilpotent. By Lemma 3.4.3, A_0/I_0 is finite-dimensional and $F \otimes_{F_0} A_0/I_0$ is isomorphic to A/I. Hence A_0/I_0 is semisimple. □

3.4.6. We prove Theorem 3.4.1(1). Let M be a log motive over S. In Lemma 3.4.5, take $F = \mathbb{Q}_\ell$, $F_0 = \mathbb{Q}$, and let A be the $\mathbb{Q}_\ell$-subalgebra of $\mathrm{End}_{\mathbb{Q}_\ell}(M_\ell)$ generated by $A_0 := \mathrm{End}_{\mathrm{LM}(S)}(M)$. Then the endomorphism ring of M in the category of log motives over S modulo numerical equivalence is A/I_0, where I_0 is as in Lemma 3.4.5. By Lemma 3.4.5, A/I_0 is a finite-dimensional semisimple $\mathbb{Q}$-algebra. This proves (1) of Theorem 3.4.1.

We prove Theorem 3.4.1(2). The if part follows from (1). We prove the only if part. Let $F = \mathbb{Q}_\ell$, $F_0 = \mathbb{Q}$, and A, A_0, I_0 be as in the proof of (1). By Lemma 3.4.5, all elements of I_0 are nilpotent. Assume that A_0 is semisimple. Since I_0 is a two-sided ideal of A_0 and all elements of I_0 are nilpotent, we have $I_0 = 0$. That is, the numerical equivalence is trivial.

4. Log mixed motives

We define the category of log mixed motives.

4.1. *The category $\mathcal{C}_S$.*

4.1.1. Let ℓ be a prime number. Let S be an fs log scheme over $\mathbb{Z}[1/\ell]$ of finite type.

Let $\mathcal{C}_S$ be the following category.

$$\text{Objects:}\quad \big(\mathcal{F}, W, (X_w)_{w\in\mathbb{Z}}, (V_{w,1})_{w\in\mathbb{Z}}, (V_{w,2})_{w\in\mathbb{Z}}, (\iota_w)_{w\in\mathbb{Z}}\big).$$

Here $\mathcal{F}$ is a smooth $\mathbb{Q}_\ell$-sheaf on the log étale site of S. W is an increasing filtration on $\mathcal{F}$ by smooth $\mathbb{Q}_\ell$-subsheaves. The X_w are projective vertical log smooth fs log schemes over S. For each $w \in \mathbb{Z}$, $V_{w,1}$ and $V_{w,2}$ are smooth $\mathbb{Q}_\ell$-subsheaves of $\bigoplus_{r\in\mathbb{Z}} H^{w+2r}(X_w)_\ell(r)$ such that $V_{w,1} \subset V_{w,2}$. The ι_w are isomorphisms $\mathrm{gr}^W_w \mathcal{F} \cong V_{w,2}/V_{w,1}$.

W is called the *weight filtration*.

A morphism

$$\big(\mathcal{F}, W, (X_w)_{w\in\mathbb{Z}}, (V_{w,1})_{w\in\mathbb{Z}}, (V_{w,2})_{w\in\mathbb{Z}}, (\iota_w)_{w\in\mathbb{Z}}\big) \\ \to \big(\mathcal{F}', W', (X'_w)_{w\in\mathbb{Z}}, (V'_{w,1})_{w\in\mathbb{Z}}, (V'_{w,2})_{w\in\mathbb{Z}}, (\iota'_w)_{w\in\mathbb{Z}}\big)$$

in $\mathcal{C}_S$ is a homomorphism of $\mathbb{Q}_\ell$-sheaves $\mathcal{F} \to \mathcal{F}'$ which respects the weight filtrations such that for each $w \in \mathbb{Z}$, the pullback of $\mathrm{gr}^W_w \mathcal{F} \to \mathrm{gr}^{W'}_w \mathcal{F}'$ to any geometric standard log point s over S is induced from the sum of morphisms $h(X_w \times_S s)(r) \to h(X'_w \times_S s)(r')$ for various $r, r' \in \mathbb{Z}$ which sends $V_{w,i}$ to $V'_{w,i}$ over s for $i = 1, 2$.

4.1.2. The category $\mathcal{C}_S$ has $\oplus$, kernels, and cokernels. Furthermore, $\otimes$, the dual, and Tate twists are defined in $\mathcal{C}_S$. These are explained in 4.1.3–4.1.7.

4.1.3. We have

$$\big(\mathcal{F}, W, (X_w)_{w\in\mathbb{Z}}, (V_{w,1})_{w\in\mathbb{Z}}, (V_{w,2})_{w\in\mathbb{Z}}, (\iota_w)_{w\in\mathbb{Z}}\big) \\ \oplus \big(\mathcal{F}', W', (X'_w)_{w\in\mathbb{Z}}, (V'_{w,1})_{w\in\mathbb{Z}}, (V'_{w,2})_{w\in\mathbb{Z}}, (\iota'_w)_{w\in\mathbb{Z}}\big) \\ = \Big(\mathcal{F} \oplus \mathcal{F}', W \oplus W', \big(X_w \textstyle\coprod X'_w\big)_{w\in\mathbb{Z}}, (V_{w,1} \oplus V'_{w,1})_{w\in\mathbb{Z}}, (V_{w,2} \oplus V'_{w,2})_{w\in\mathbb{Z}}, \\ (\iota_w \oplus \iota'_w)_{w\in\mathbb{Z}}\Big).$$

4.1.4. The kernel of a morphism

$$\big(\mathcal{F}, W, (X_w)_{w\in\mathbb{Z}}, (V_{w,1})_{w\in\mathbb{Z}}, (V_{w,2})_{w\in\mathbb{Z}}, (\iota_w)_{w\in\mathbb{Z}}\big) \\ \to \big(\mathcal{F}', W', (X'_w)_{w\in\mathbb{Z}}, (V'_{w,1})_{w\in\mathbb{Z}}, (V'_{w,2})_{w\in\mathbb{Z}}, (\iota'_w)_{w\in\mathbb{Z}}\big)$$

is $\big(\mathcal{F}'', W'', (X''_w)_{w\in\mathbb{Z}}, (V''_{w,1})_{w\in\mathbb{Z}}, (V''_{w,2})_{w\in\mathbb{Z}}, (\iota''_w)_{w\in\mathbb{Z}}\big)$, where $\mathcal{F}''$ is the kernel of $\mathcal{F} \to \mathcal{F}'$, W'' is induced from W, $X''_w = X_w$, $V''_{w,2}$ is the kernel of $V_{w,2} \to V'_{w,2}/V'_{w,1}$, $V''_{w,1} = V''_{w,2} \cap V_{w,1}$, and ι''_w is induced from ι_w.

4.1.5. The cokernel of the above morphism is

$$\big(\mathcal{F}'', W'', (X''_w)_{w\in\mathbb{Z}}, (V''_{w,1})_{w\in\mathbb{Z}}, (V''_{w,2})_{w\in\mathbb{Z}}, (\iota''_w)_{w\in\mathbb{Z}}\big),$$

where $\mathcal{F}''$ is the cokernel of $\mathcal{F}\to\mathcal{F}'$, W'' is induced from W', $X''_w = X'_w$,

$$V''_{w,2} = V'_{w,2} + \mathrm{Image}(V_{w,2}), \quad V''_{w,1} = V'_{w,1} + \mathrm{Image}(V_{w,2}),$$

and ι''_w is induced by ι'_w.

4.1.6.

$$\big(\mathcal{F}, W, (X_w)_{w\in\mathbb{Z}}, (V_{w,1})_{w\in\mathbb{Z}}, (V_{w,2})_{w\in\mathbb{Z}}, (\iota_w)_{w\in\mathbb{Z}}\big)$$
$$\otimes\big(\mathcal{F}', W', (X'_w)_{w\in\mathbb{Z}}, (V'_{w,1})_{w\in\mathbb{Z}}, (V'_{w,2})_{w\in\mathbb{Z}}, (\iota'_w)_{w\in\mathbb{Z}}\big)$$

is defined as

$$\big(\mathcal{F}'', W'', (X''_w)_{w\in\mathbb{Z}}, (V''_{w,1})_{w\in\mathbb{Z}}, (V''_{w,2})_{w\in\mathbb{Z}}, (\iota''_w)_{w\in\mathbb{Z}}\big),$$

where $\mathcal{F}'' = \mathcal{F}\otimes\mathcal{F}'$, W'' is the convolution of W and W', $X''_w = \coprod_{i+j=w} X_i \times X'_j$, $V''_{w,2} = \bigoplus_{i+j=w} V_{i,2}\otimes V'_{j,2}$, $V''_{w,1} = \bigoplus_{i+j=w}(V_{i,1}\otimes V'_{j,2} + V_{i,2}\otimes V'_{j,1})$, $\iota''_w = \bigoplus_{i+j=w} \iota_i\otimes\iota'_j$.

4.1.7. The definition of the dual and the Tate twists are the evident ones.

4.2. ***The category of log mixed motives.*** Deligne [1971; 1974], showed how we can obtain mixed Hodge structures of geometric origin basing on the theory of pure Hodge structures. We imitate his method to formulate objects of $\mathcal{C}_S$ of geometric origin.

In this 4.2, S denotes an fs log scheme and ℓ denotes a prime number which is invertible on S.

For an fs log scheme X over S, $H^m(X)(r)_\ell$ denotes $R^m f_*\mathbb{Q}_\ell(r)$, where f is the morphism $X\to S$.

4.2.1. Consider (U, X, D), where X is a projective vertical log smooth fs log scheme over S, $D = (D_\lambda)_{\lambda\in\Lambda}$ is a finite family of Cartier divisors on X, and U is the open subscheme of X defined as the complement of $\bigcup_{\lambda\in\Lambda} D_\lambda$ in X satisfying the following condition:

> For any subset Λ' of Λ, $D_{\Lambda'} := \bigcap_{\lambda\in\Lambda'} D_\lambda$ with the inverse image of the log structure of X is log smooth over S, and of codimension $\sharp(\Lambda')$ in X at each point of it.

To describe a typical example, let X be a projective and strict semistable family over a trait $S = \operatorname{Spec} A$ endowed with natural log structures. Let $D = (D_\lambda)_{\lambda\in\Lambda}$ be a finite family of Cartier divisors on X. Assume that strict étale locally on X, X is strict étale over $\operatorname{Spec}(A[T_1, \dots, T_n]/(T_1\cdots T_i - \pi))$, where $i\le n$, π is a prime element of A and the log of X is given by $T_1, \dots, T_i$, and that for some $i\le j\le n$, each of the $T_{i+1}, \dots, T_j$ gives some D_λ and the other D_λ are empty there. Then these satisfy the above condition.

4.2.2. Let the notation and the assumptions be as in 4.2.1. For $i \geq 0$, let $D^{(i)}$ be the disjoint union of $D_{\Lambda'}$ for all $\Lambda' \subset \Lambda$ such that $\sharp(\Lambda') = i$. In particular, $D^{(0)} = X$. For $i \geq 0$, we have a smooth $\mathbb{Q}_\ell$-sheaf $H^m(D^{(i)})_\ell$ on the log étale site of S (see 2.1.2).

4.2.3. Let the notation and the assumptions be as in 4.2.1. Endow U with the inverse image of the log structure of X.

Then $H^m(U)_\ell$ is a smooth $\mathbb{Q}_\ell$-sheaf on the log étale site of S and we have a spectral sequence

$$E_1^{i,j} = H^{2i+j}(D^{(-i)})_\ell(i) \Rightarrow E_\infty^m = H^m(U)_\ell$$

in the category of smooth $\mathbb{Q}_\ell$-sheaves. In fact, first, by relative purity in log étale cohomology [Higashiyama and Kamiya 2017], we have a spectral sequence with finite coefficients. By Proposition 2.1.1, the E_1-terms of this spectral sequence determines a smooth $\mathbb{Q}_\ell$-sheaves, which implies the above facts.

4.2.4. Consider a simplicial system $(U_\bullet, X_\bullet, D_\bullet)$ of objects (U, X, D) of 4.2.1 (here we follow [Deligne 1974]). Let $\boldsymbol{H}^m(U_\bullet)_\ell$ be the smooth $\mathbb{Q}_\ell$-sheaf on S defined to be the m-th hypercohomology (relative to S) of the simplicial system. The spectral sequence in 4.2.3 is generalized to the spectral sequence

$$E_1^{i,j} = \bigoplus_{s \geq 0} H^{j-2s}(D_{s+i}^{(s)})_\ell(-s) \Rightarrow E_\infty^m = \boldsymbol{H}^m(U_\bullet)_\ell.$$

4.2.5. Let the notation be as in 4.2.4. Let $m \in \mathbb{Z}$. We define an increasing filtration W on $\boldsymbol{H}^m(U_\bullet)_\ell$, which we call the *weight filtration*, as the filtration defined by the spectral sequence in 4.2.4.

4.2.6. If S is of finite type over $\mathbb{Z}[1/\ell]$, let $\mathcal{C}_S^{\mathrm{mot}}$ be the full subcategory of $\mathcal{C}_S$ consisting of objects which are obtained from the following standard objects in 4.2.7 below by taking $\oplus$, kernels, cokernels, $\otimes$, the duals, and Tate twists.

4.2.7. In the above, a standard object means:

> Consider $(U_\bullet, X_\bullet, D_\bullet, m)$, where $(U_\bullet, X_\bullet, D_\bullet)$ is as in 4.2.4 and $m \in \mathbb{Z}$. The associated standard object is as follows:

Let $\mathcal{F} = \boldsymbol{H}^m(U_\bullet)_\ell$ on S.

Let W be the filtration on $\boldsymbol{H}^m(U_\bullet)_\ell$ defined by the spectral sequence in 4.2.5. Then, for $w \in \mathbb{Z}$, $\mathrm{gr}_w^W \boldsymbol{H}^m(U_\bullet)_\ell = V'_{w,2}/V'_{w,1}$ for some $\mathbb{Q}_\ell$-subsheaves $V'_{w,1}$, $V'_{w,2}$ of $\bigoplus_{s \geq 0} H^{w-2s}(D_{s+m-w}^{(s)})_\ell(-s)$ such that $V'_{w,1} \subset V'_{w,2}$.

Let $X_w = \bigsqcup_{s \geq 0} D_{s+m-w}^{(s)}$. Consider the natural projection

$$\bigoplus_{r \in \mathbb{Z}} H^{w+2r}(X_w)_\ell(r) \to \bigoplus_{s \geq 0} H^{w-2s}(D_{s+m-w}^{(s)})_\ell(-s).$$

For $i = 1, 2$, let $V_{w,i}$ be the pullbacks of $V'_{w,i}$ by this natural projection. Then we have the isomorphism $\iota_w : \mathrm{gr}^W_w \mathcal{F} \cong V_{w,2}/V_{w,1}$.

4.2.8. If S is affine and is the inverse limit of the S_λ which are of finite type over $\mathbb{Z}[1/\ell]$, we define $\mathcal{C}_S^{\mathrm{mot}}$ as the inductive limit of the categories $\mathcal{C}_{S_\lambda}^{\mathrm{mot}}$. This does not depend on the choice of limits.

4.2.9. We define the category of log mixed motives $\mathrm{LMM}(S)$ over S as the Zariski sheafification of the categories $\mathcal{C}_S^{\mathrm{mot}}$ in 4.2.8. More precisely, to give a log mixed motive M over S is to give an affine covering $(S_i)_{i\in I}$ of S, objects M_i of $\mathcal{C}_{S_i}^{\mathrm{mot}}$, affine coverings $(S_{ij\lambda})_\lambda$ of $S_i \cap S_j$ for each $i, j \in I$, and isomorphisms between the restrictions of M_i and M_j to each $S_{ij\lambda}$ which are compatible to each other. The set of morphisms is similarly defined as the quotient of the set of compatible local morphisms over affine open sets under an appropriate equivalence.

4.2.10. For a morphism $S' \to S$ of fs log schemes, we have the pullback functor $\mathrm{LMM}(S) \to \mathrm{LMM}(S')$.

4.2.11. We have a fully faithful functor

$$\mathrm{LM}^{\mathrm{spl}}(S) \to \mathrm{LMM}(S)$$

which sends $H^m(X)(r)$ to the object associated to $(U_\bullet, X_\bullet, D_\bullet, m)(r)$ with $X_\bullet$ determined by X, $U_\bullet = X_\bullet$, $D_\bullet$ empty.

4.2.12. If the log structure of S is trivial, we define the category $\mathrm{MM}(S)$ of mixed motives to be the category of log mixed motives over S.

4.3. ***Justifications of our definition.*** Here we explain the reason why we think our definition of log mixed motives is reasonable.

4.3.1. The reader may feel strange that in our definition of a morphism of log mixed motives (4.1.1), we do not put much conditions other than the condition that its gr^W is motivic, though it is usually impossible to take care of mixed objects by using only pure objects.

We hope that the following Proposition 4.3.4 (resp. Proposition 4.3.5) justifies our definition of log mixed motive (resp. of morphism of log mixed motives) in 4.2 (resp. 4.1.1).

We hope that if S is of finite type over $\mathbb{Z}$ and if we take the category of log mixed motives over S and the category of smooth $\mathbb{Q}_\ell$-sheaves on the log étale site of S as $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, the conditions in 4.3.2 below are satisfied. (Especially we hope that the finiteness assumption on S assures that the condition (v) in 4.3.2 below is satisfied.)

4.3.2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be abelian categories. Assume that we have exact subfunctors $W_w : \mathcal{C}_1 \to \mathcal{C}_1$ ($w \in \mathbb{Z}$) of the identify functor $\mathcal{C}_1 \to \mathcal{C}_1$ such that $W_w \circ W_w = W_w$

and such that $W_{w'} \subset W_w$ if $w' \le w$. Assume that we have a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$. Assume that these satisfy the following six conditions.

(i) For each object M of $\mathcal{C}_1$, $W_w M = M$ if $w \gg 0$ and $W_w M = 0$ if $w \ll 0$.

(ii) The functor F is exact.

(iii) Let $w \in \mathbb{Z}$ and let M and N be objects of $\mathcal{C}_1$. Assume that M and N are pure of weight w (that is, $W_w M = M,\ W_{w-1} M = 0,\ W_w N = N,\ W_{w-1} N = 0$). Then the canonical map $\mathrm{Hom}_{\mathcal{C}_1}(M, N) \to \mathrm{Hom}_{\mathcal{C}_2}(F(M), F(N))$ is injective.

(iv) Let $w, w' \in \mathbb{Z}$ and assume $w > w'$. Let M and N be objects of $\mathcal{C}_1$ and assume that M is pure of weight w and N is pure of weight w'. Then $\mathrm{Hom}_{\mathcal{C}_1}(M, N) = 0$ and $\mathrm{Hom}_{\mathcal{C}_2}(F(M), F(N)) = 0$.

(v) Let $w, w' \in \mathbb{Z}$ and assume $w \ge w'$. Let M and N be objects of $\mathcal{C}_1$ and assume that M is pure of weight w and N is pure of weight w'. Then the canonical map $\mathrm{Ext}^1_{\mathcal{C}_1}(M, N) \to \mathrm{Ext}^1_{\mathcal{C}_2}(F(M), F(N))$ is injective.

(vi) Let $w \in \mathbb{Z}$. Then the full subcategory of $\mathcal{C}_1$ consisting of all objects which are pure of weight w is semisimple.

Remark. By Proposition 3.2.10, $\mathrm{Hom}(F(M), F(N)) = 0$ in the condition (iv) is reasonable. (This is clearly reasonable if the log structure of S is trivial, but not trivial otherwise.) Further, the condition (v) is related to Tate conjecture. In fact, it means that an extension of motives splits if the ℓ-adic realization splits; two extensions are isomorphic if their ℓ-adic realizations are isomorphic. These are analogues of Tate conjectures.

Lemma 4.3.3. *Let the notation and the assumptions be as in 4.3.2 and let M and N be objects of $\mathcal{C}_1$.*

(1) *The morphism* $\mathrm{Hom}_{\mathcal{C}_1}(M, N) \to \mathrm{Hom}_{\mathcal{C}_2}(F(M), F(N))$ *is injective.*

(2) *If there is a $w \in \mathbb{Z}$ such that $W_w M = 0$ and $W_w N = N$, then* $\mathrm{Hom}_{\mathcal{C}_1}(M, N) = 0$, $\mathrm{Hom}_{\mathcal{C}_2}(F(M), F(N)) = 0$, *and the map* $\mathrm{Ext}^1_{\mathcal{C}_1}(M, N) \to \mathrm{Ext}^1_{\mathcal{C}_2}(F(M), F(N))$ *is injective.*

Proof. By the induction on the lengths of the weight filtrations of M and N together with the assumptions (i) and (ii), both statements reduce to the case where M and N are pure. Let w (resp. w') be the weight of M (resp. N).

(1) If $w = w'$ (resp. $w > w'$), (1) is by (iii) (resp. (iv)). If $w < w'$, $\mathrm{Hom}_{\mathcal{C}_1}(M, N) = 0$, and (1) holds.

(2) Since $w > w'$, (iv) and (v) imply (2). □

Proposition 4.3.4. *Let the notation and the assumptions be as in 4.3.2. Let M be an object of $\mathcal{C}_1$ and let V be a subobject of $F(M)$ in $\mathcal{C}_2$ such that for any $w \in \mathbb{Z}$, the subobject $\mathrm{gr}^W_w V := (V \cap F(W_w M))/(V \cap F(W_{w-1} M))$ of $F(\mathrm{gr}^W_w M)$ is $F(N_w)$ for* *some subobject N_w of $\mathrm{gr}_w^W M$ in $\mathcal{C}_1$. Then there is a unique subobject N of M in $\mathcal{C}_1$ such that V coincides with $F(N)$.*

Proof. By downward induction on w, we may assume that $W_{w-1}M = 0$ and that if we denote $V \cap F(W_w M)$ by V', the subobject $V'' := V/V'$ of $F(M/W_w M)$ coincides with $F(N'')$ for some subobject N'' of $M/W_w M$. By the assumption, the subobject V' of $F(W_w M) = F(\mathrm{gr}_w^W M)$ coincides with $F(N')$ for some subobject N' of $W_w M$.

Let the exact sequence $0 \to W_w M \to U \to N'' \to 0$ be the pullback of the exact sequence $0 \to W_w M \to M \to M/W_w M \to 0$ by $N'' \to M/W_w M$. Then $\mathrm{class}(F(U)) \in \mathrm{Ext}^1_{\mathcal{C}_2}(F(N''), F(W_w M))$ coincides with the image of $\mathrm{class}(V) \in \mathrm{Ext}^1_{\mathcal{C}_2}(F(N''), F(N'))$ under the homomorphism

$$\mathrm{Ext}^1_{\mathcal{C}_2}(F(N''), F(N')) \to \mathrm{Ext}^1_{\mathcal{C}_2}(F(N''), F(W_w M))$$

induced by the morphism $N' \to W_w M$.

Claim 4.3.4.1. *There are an object N of $\mathcal{C}_1$ and an exact sequence*

$$0 \to N' \to N \to N'' \to 0$$

such that

$$\mathrm{class}(F(N)) \in \mathrm{Ext}^1_{\mathcal{C}_2}(F(N''), F(N')) \quad \textit{and} \quad \mathrm{class}(V) \in \mathrm{Ext}^1_{\mathcal{C}_2}(F(N'')), F(N'))$$

coincide.

We prove Claim 4.3.4.1. By the condition (vi) in 4.3.2 on semisimplicity, there is a morphism $W_w M = \mathrm{gr}_w^W M \to N'$ such that the composition $N' \to W_w M \to N'$ is the identity morphism. Let the exact sequence $0 \to N' \to N \to N'' \to 0$ be the pushforward of $0 \to W_w M \to U \to N'' \to 0$ under $W_w M \to N'$. Then this satisfies the condition in Claim 4.3.4.1.

Claim 4.3.4.2. $\mathrm{class}(U) \in \mathrm{Ext}^1_{\mathcal{C}_1}(N'', W_w M)$ *coincides with the image of* $\mathrm{class}(N) \in \mathrm{Ext}^1_{\mathcal{C}_1}(N'', N')$ *under the homomorphism induced by $N' \to W_w M$.*

This follows from the injectivity of

$$\mathrm{Ext}^1_{\mathcal{C}_1}(N'', W_w M) \to \mathrm{Ext}^1_{\mathcal{C}_2}(F(N''), F(W_w M))$$

(Lemma 4.3.3) and the fact that $\mathrm{class}(F(U)) \in \mathrm{Ext}^1_{\mathcal{C}_2}(F(N''), F(W_w M))$ coincides with the image of $\mathrm{class}(V) \in \mathrm{Ext}^1_{\mathcal{C}_2}(F(N''), F(N'))$ under the homomorphism induced by the morphism $N' \to W_w M$.

By Claim 4.3.4.2, there is a morphism $N \to M$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W_w M \arrow[r] & M \arrow[r] & M/W_w M \arrow[r] & 0
\end{tikzcd}$$

is commutative. This proves Proposition 4.3.4. □

Proposition 4.3.5. *Let the notation and the assumptions be as in* 4.3.2 (actually the condition (vi) is not used for this proposition). *Let M and N be objects of $\mathcal{C}_1$. Then we have a bijection from* $\mathrm{Hom}_{\mathcal{C}_1}(M, N)$ *to the set of pairs* $(h, (h_w)_{w\in\mathbb{Z}})$, *where h is a morphism $F(M) \to F(N)$ and h_w is a morphism $\mathrm{gr}^W_w M \to \mathrm{gr}^W_w N$ satisfying the following conditions* (i) *and* (ii).

(i) *h sends $F(W_w M)$ to $F(W_w N)$ for any $w \in \mathbb{Z}$.*

(ii) *For any $w \in \mathbb{Z}$, the morphism $F(\mathrm{gr}^W_w M) \to F(\mathrm{gr}^W_w N)$ induced by h coincides with $F(h_w)$.*

Proof. We first prove:

Claim 4.3.5.1. *Let M and N be objects of $\mathcal{C}_1$. Let $w \in \mathbb{Z}$. Then we have a bijection from* $\mathrm{Hom}_{\mathcal{C}_1}(M, N)$ *to the set of pairs (a, b), where a is a morphism $W_w M \to W_w N$ and b is a morphism $M/W_w M \to N/W_w N$ satisfying the following condition* $(*)$.

$(*)$ *The image of*

$$\mathrm{class}(M) \in \mathrm{Ext}^1_{\mathcal{C}_1}(M/W_w M, W_w M) \quad \textit{in} \;\; \mathrm{Ext}^1_{\mathcal{C}_1}(M/W_w M, W_w N)$$

under the map induced by a coincides with the image of

$$\mathrm{class}(N) \in \mathrm{Ext}^1_{\mathcal{C}_1}(N/W_w N, W_w N) \quad \textit{in} \;\; \mathrm{Ext}^1_{\mathcal{C}_1}(M/W_w M, W_w N)$$

under the map induced by b.

Proof of Claim 4.3.5.1. Let $a : W_w M \to W_w N$ and $b : M/W_w M \to N/W_w N$ be morphisms. Let the exact sequence $0 \to W_w N \to X \to M/W_w M \to 0$ be the pushforward of $0 \to W_w M \to M \to M/W_w M \to 0$ under a, and let the exact sequence $0 \to W_w N \to Y \to M/W_w M \to 0$ be the pullback of $0 \to W_w N \to N \to N/W_w N \to 0$ under b. Then the condition $(*)$ is that the extension classes of X and Y coincide. On the other hand, a morphism $M \to N$ which induces a and b corresponds bijectively to a morphism $X \to Y$ which induces identity morphisms of $W_w N$ and $M/W_w M$. By the first part of (2) of Lemma 4.3.3, we have $\mathrm{Hom}_{\mathcal{C}_1}(M/W_w M, W_w N) = 0$. Hence we have the bijection in Claim 4.3.5.1 □

We can prove similarly:

Claim 4.3.5.2. *Let M and N be objects of $\mathcal{C}_1$. Let $w \in \mathbb{Z}$. Then we have a bijection from* $\{h \in \mathrm{Hom}(F(M), F(N)) \mid h(F(W_w M)) \subset F(W_w N)\}$ *to the set of pairs (a, b), where a is a morphism $F(W_w M) \to F(W_w N)$ and b is a morphism $F(M/W_w M) \to F(N/W_w N)$ satisfying the following condition* $(**)$.

(∗∗) *The image of*

$$\mathrm{class}(F(M)) \in \mathrm{Ext}^1_{\mathcal{C}_2}(F(M/W_wM), F(W_wM))$$

in $\mathrm{Ext}^1_{\mathcal{C}_2} F((W_wM), F(W_wN))$ *under the map induced by a coincides with the image of*

$$\mathrm{class}(F(N)) \in \mathrm{Ext}^1_{\mathcal{C}_2}(F(N/W_wN), F(W_wN))$$

in $\mathrm{Ext}^1_{\mathcal{C}_2}(F(M/W_wM), F(W_wN))$ *under the map induced by b.*

Now we prove Proposition 4.3.5. By downward induction on w, we may assume that there is a $w \in \mathbb{Z}$ such that $W_{w-1}M = M$, $W_{w-1}N = N$ and such that Proposition 4.3.5 is true if we replace M and N by M/W_wM and N/W_wN, respectively. By Claims 4.3.5.1 and 4.3.5.2, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

where

$$A = \mathrm{Hom}_{\mathcal{C}_1}(M, N), \quad A' = \mathrm{Hom}_{\mathcal{C}_2,W}(F(M), F(N)),$$
$$B = \mathrm{Hom}_{\mathcal{C}_1}(W_wM, W_wN) \times \mathrm{Hom}_{\mathcal{C}_1}(M/W_wM, N/W_wN),$$
$$B' = \mathrm{Hom}_{\mathcal{C}_2}(F(W_wM), F(W_wN)) \times \mathrm{Hom}_{\mathcal{C}_2,W}(F(M/W_wM), F(N/W_wN)),$$
$$C = \mathrm{Ext}^1_{\mathcal{C}_1}(M/W_wM, W_wN), \quad C' = \mathrm{Ext}^1_{\mathcal{C}_2}(F(M/W_wM), F(W_wN)).$$

Here $\mathrm{Hom}_{\mathcal{C}_2,W}$ means the set of homomorphisms of $\mathcal{C}_2$ which respect the filtrations W. The vertical arrows are injective by Lemma 4.3.3. This proves

$$A \xrightarrow{\cong} \{x \in A' \mid \text{the image of } x \text{ in } B' \text{ comes from } B\},$$

which proves Proposition 4.3.5 by downward induction on w. □

4.4. *Main theorem.*

4.4.1. Recall that the following (i) and (ii) are equivalent (Theorem 3.4.1(2)).

(i) In the category of log motives, homological equivalence (i.e., the trivial equivalence) coincides with the numerical equivalence.

(ii) The category of log motives is a semisimple abelian category.

Theorem 4.4.2. (i) *and* (ii) *are equivalent to the following* (iii).

(iii) *The category of log mixed motives is a Tannakian category* [Saavedra Rivano 1972; Deligne 1990].

4.4.3. We prove (ii) $\Rightarrow$ (iii). It is sufficient to prove that a morphism f is an isomorphism if it induces an isomorphism $\mathcal{F} \to \mathcal{F}'$. By (ii), there is a morphism $h(X'_w) \to h(X_w)$ which induces the inverse map $V'_{w,2}/V'_{w,1} \to V_{w,2}/V_{w,1}$. Thus the inverse map $\mathcal{F}' \to \mathcal{F}$ is a morphism of log mixed motives.

We prove (iii) $\Rightarrow$ (i). Let X be a projective vertical log smooth fs log scheme over S. Consider a morphism $f : h(X) \to h(X)$ which is numerically equivalent to 0. We prove $f = 0$. Let V_1 be the kernel of $f : h(X)_\ell \to h(X)_\ell$ and let $V_2 = h(X)_\ell$. On the other hand, let $V'_1 = 0$ and V'_2 be the image of $f : h(X)_\ell \to h(X)_\ell$. Then f induces an isomorphism $f : V_2/V_1 \xrightarrow{\cong} V'_2/V'_1$. By (iii), there is a morphism $g : h(X) \to h(X)$ which induces the inverse map $V'_2/V'_1 \to V_2/V_1$. Then $fg : h(X)_\ell \to h(X)_\ell$ is a projection to V'_2. Hence $\mathrm{Tr}(fg) = \dim(V'_2)$. Hence $\mathrm{Tr}(fg) = 0$ implies $V'_2 = 0$ and hence $f = 0$.

4.4.4. One can consider the following unconditional variant of the above statement (iii).

Let $\mathrm{LMM}_{\mathrm{num}}(S)$ be the category of log mixed motives over S modulo numerical equivalence. Here morphisms $f, g : \mathcal{F} \to \mathcal{F}'$ of log mixed motives are said to be *numerically equivalent* if $\mathrm{gr}(f)$ and $\mathrm{gr}(g)$ are numerically equivalent. Then one can ask if $\mathrm{LMM}_{\mathrm{num}}(S)$ is a Tannakian category.

This is a mixed analogue of Theorem 3.4.1(1).

5. Formulation with various realizations

In Sections 3 and 4, we considered ℓ-adic realizations of log mixed motives fixing a prime number ℓ. Here we consider various realizations.

5.1. *Log motives and log mixed motives with many realizations.*

5.1.1. Let $\mathcal{R}$ be the union of the set of all prime numbers and the set of three letters $\{B, D, H\}$: B means Betti realization; D means de Rham realization; H means Hodge realization.

Let S be an fs log scheme. Let R be a nonempty subset of $\mathcal{R}$. If a prime number ℓ is contained in R, assume that S is over $\mathbb{Z}[1/\ell]$. If $B \in R$, assume that S is locally of finite type over $\mathbb{C}$. If $D \in R$, assume that S is log smooth over a field of characteristic 0 or S is a standard log point associated to a field of characteristic 0. If $H \in R$, assume that S is log smooth over $\mathbb{C}$ or S is the standard log point associated to $\mathbb{C}$.

5.1.2. We define the categories

$$\mathrm{LM}_R(S), \quad \mathrm{LMM}_R(S)$$

of log motives over S and of log mixed motives over S, respectively, with respect to realizations in R.

The definition of $\mathrm{LM}_R(S)$ is similar to Section 3. For a projective vertical log smooth fs log scheme $f : X \to S$ over S and for $r \in \mathbb{Z}$, consider the symbol $h_R(X)(r)$.

When a prime ℓ belongs to R, let

$$h_R(X)(r)_\ell := \bigoplus_m H^m(X)_\ell(r).$$

When $B \in R$, let

$$h_R(X)(r)_B := \bigoplus_m H^m(X)_B(r), \quad \text{where } H^m(X)_B = R^m f_*^{\log}\mathbb{Q}.$$

This is a locally constant sheaf of finite dimensional $\mathbb{Q}$-vector spaces on $S^{\log}$ (see Proposition 2.1.7).

When $D \in R$, let

$$h_R(X)(r)_D := \bigoplus_m H^m(X)_D(r), \quad \text{where } H^m(X)_D = R^m f_{\text{két}*}\omega^{\cdot,\text{két}}_{X/S}.$$

This is a locally free sheaf of $\mathcal{O}_{\text{két}}$-modules of finite rank with a quasinilpotent integrable connection with log poles on $S_{\text{két}}$ (see Propositions 2.1.14(1) and 2.1.17(1)).

When $H \in R$, let

$$h_R(X)(r)_H := \bigoplus_m H^m(X)_H(r), \quad \text{where } H^m(X)_H = R^m f_{\text{két}*}\omega^{\cdot,\text{két}}_{X/S}$$

endowed with the natural log Hodge structures. This is a log mixed Hodge structure on $S_{\text{két}}$ (see Propositions 2.1.14(2) and 2.1.17(2)).

A morphism $h_R(X)(r) \to h_R(Y)(s)$ is defined as a family of morphisms between realizations for each element of R satisfying, for any geometric standard log point p over S, the pull-backed morphism is induced by a common element of gr of the K-group. Note that we do not impose any comparison isomorphism between different realizations. The rest is the same as in 3.1, and we have the category $\mathrm{LM}_R(S)$. Here we use the Poincaré duality (Proposition 2.1.8) and the Künneth formula (Proposition 2.1.10) in log Betti cohomology, which implies the necessary corresponding theorems in log de Rham and log Hodge theory via log Riemann–Hilbert correspondence [Illusie et al. 2005, Theorem (6.2)]. We also use the Riemann–Roch theorems.

5.1.3. The definition of $\mathrm{LMM}_R(S)$ is also similar to the case where R consists of one prime. We first define $\mathcal{C}_{S,R}$ as follows.

First, for an R consisting of one prime ℓ, $\mathcal{C}_{S,R}$ is $\mathcal{C}_S$ in 4.1.1.

Second, for $R = \{B\}$, we define $\mathcal{C}_{S,R}$ as the following category.

$$\text{Objects:} \quad \big(\mathcal{F}, W, (X_w)_{w\in\mathbb{Z}}, (V_{w,1})_{w\in\mathbb{Z}}, (V_{w,2})_{w\in\mathbb{Z}}, (\iota_w)_{w\in\mathbb{Z}}\big).$$

Here $\mathcal{F}$ is a locally constant sheaf of finite dimensional $\mathbb{Q}$-vector spaces on $S_{\mathrm{an}}^{\log}$. The W is an increasing filtration on $\mathcal{F}$ by locally constant $\mathbb{Q}$-subsheaves. The X_w is a projective vertical log smooth fs log scheme over S. For each $w \in \mathbb{Z}$, $V_{w,1}$ and $V_{w,2}$ are locally constant $\mathbb{Q}$-subsheaves of $\bigoplus_{r\in\mathbb{Z}} H^{w+2r}(X_w)_B(r)$ such that $V_{w,1} \subset V_{w,2}$. The ι_w is an isomorphism $\mathrm{gr}_w^W \mathcal{F} \cong V_{w,2}/V_{w,1}$.

A morphism

$$\big(\mathcal{F}, W, (X_w)_{w\in\mathbb{Z}}, (V_{w,1})_{w\in\mathbb{Z}}, (V_{w,2})_{w\in\mathbb{Z}}, (\iota_w)_{w\in\mathbb{Z}}\big) \\ \to \big(\mathcal{F}', W', (X'_w)_{w\in\mathbb{Z}}, (V'_{w,1})_{w\in\mathbb{Z}}, (V'_{w,2})_{w\in\mathbb{Z}}, (\iota'_w)_{w\in\mathbb{Z}}\big)$$

is a homomorphism of $\mathbb{Q}$-sheaves $\mathcal{F} \to \mathcal{F}'$ which respects the weight filtrations such that for each $w \in \mathbb{Z}$, the pullback of $\mathrm{gr}_w^W \mathcal{F} \to \mathrm{gr}_w^{W'} \mathcal{F}'$ to any geometric standard log point s associated to $\mathbb{C}$ over S is induced from the sum of morphisms $h_{\{B\}}(X_w \times_S s)(r) \to h_{\{B\}}(X'_w \times_S s)(r')$ for various r, r' which sends $V_{w,i}$ to $V'_{w,i}$ over s for $i = 1, 2$.

Third, for $R = \{D\}$, we define $\mathcal{C}_{S,R}$ as the following category.

Objects: $\big(\mathcal{F}, W, (X_w)_{w\in\mathbb{Z}}, (V_{w,1})_{w\in\mathbb{Z}}, (V_{w,2})_{w\in\mathbb{Z}}, (\iota_w)_{w\in\mathbb{Z}}\big)$.

Here $\mathcal{F}$ is a locally free $\mathcal{O}_{S_{\text{két}}}$-modules of finite rank endowed with a quasinilpotent integrable connection with log poles. The W is an increasing filtration on $\mathcal{F}$ by locally free $\mathcal{O}_{S_{\text{két}}}$-submodules with the compatible connections such that the graded quotients are also locally free. The X_w is a projective vertical log smooth fs log scheme over S. For each $w \in \mathbb{Z}$, $V_{w,1}$ and $V_{w,2}$ are locally free $\mathcal{O}_{S_{\text{két}}}$-submodules with the compatible connections of $\bigoplus_{r\in\mathbb{Z}} H^{w+2r}(X_w)_D(r)$ such that $V_{w,1} \subset V_{w,2}$. The ι_w is an isomorphism $\mathrm{gr}_w^W \mathcal{F} \cong V_{w,2}/V_{w,1}$.

A morphism

$$\big(\mathcal{F}, W, (X_w)_{w\in\mathbb{Z}}, (V_{w,1})_{w\in\mathbb{Z}}, (V_{w,2})_{w\in\mathbb{Z}}, (\iota_w)_{w\in\mathbb{Z}}\big) \\ \to \big(\mathcal{F}', W', (X'_w)_{w\in\mathbb{Z}}, (V'_{w,1})_{w\in\mathbb{Z}}, (V'_{w,2})_{w\in\mathbb{Z}}, (\iota'_w)_{w\in\mathbb{Z}}\big)$$

is a homomorphism of $\mathcal{O}_{S_{\text{két}}}$-modules $\mathcal{F} \to \mathcal{F}'$ which respects the weight filtrations such that for each $w \in \mathbb{Z}$, the pullback of $\mathrm{gr}_w^W \mathcal{F} \to \mathrm{gr}_w^{W'} \mathcal{F}'$ to any geometric standard log point s over S is induced from the sum of morphisms

$$h_{\{D\}}(X_w \times_S s)(r) \to h_{\{D\}}(X'_w \times_S s)(r')$$

for various r, r' which sends $V_{w,i}$ to $V'_{w,i}$ over s for $i = 1, 2$.

Fourth, for $R = \{H\}$, we define $\mathcal{C}_{S,R}$ as the following category.

Objects: $\big(\mathcal{F}, W, (X_w)_{w\in\mathbb{Z}}, (V_{w,1})_{w\in\mathbb{Z}}, (V_{w,2})_{w\in\mathbb{Z}}, (\iota_w)_{w\in\mathbb{Z}}\big)$.

Here $(\mathcal{F}, W)$ is a log mixed Hodge structure on $S_{\text{két}}$. The X_w is a projective vertical log smooth fs log scheme over S. For each $w \in \mathbb{Z}$, $V_{w,1}$ and $V_{w,2}$ are sub-log

Hodge structures of $\bigoplus_{r\in\mathbb{Z}} H^{w+2r}(X_w)_H(r)$ such that $V_{w,1} \subset V_{w,2}$. The ι_w is an isomorphism $\mathrm{gr}_w^W \mathcal{F} \cong V_{w,2}/V_{w,1}$.

A morphism

$$\big(\mathcal{F}, W, (X_w)_{w\in\mathbb{Z}}, (V_{w,1})_{w\in\mathbb{Z}}, (V_{w,2})_{w\in\mathbb{Z}}, (\iota_w)_{w\in\mathbb{Z}}\big) \\ \to \big(\mathcal{F}', W', (X'_w)_{w\in\mathbb{Z}}, (V'_{w,1})_{w\in\mathbb{Z}}, (V'_{w,2})_{w\in\mathbb{Z}}, (\iota'_w)_{w\in\mathbb{Z}}\big)$$

is a homomorphism of log mixed Hodge structures $(\mathcal{F}, W) \to (\mathcal{F}', W')$ such that for each $w \in \mathbb{Z}$, the pullback of $\mathrm{gr}_w^W \mathcal{F} \to \mathrm{gr}_w^{W'} \mathcal{F}'$ to any standard log point s associated to $\mathbb{C}$ over S is induced from the sum of morphisms $h_{\{H\}}(X_w \times_S s)(r) \to h_{\{H\}}(X'_w \times_S s)(r')$ for various r, r' which sends $V_{w,i}$ to $V'_{w,i}$ over s for $i = 1, 2$.

Lastly, for any R, we define $\mathcal{C}_{S,R}$ as follows.

Objects: $(Y_\rho)_{\rho\in R}$, where Y_ρ is an object of $\mathcal{C}_{S,\{\rho\}}$, satisfying the condition that for any $w \in \mathbb{Z}$, the X_w of Y_ρ is common.

A morphism $(Y_\rho)_{\rho\in R} \to (Y'_\rho)_{\rho\in R}$ is $(f_\rho)_{\rho\in R}$, where $f_\rho : Y_\rho \to Y'_\rho$ is a morphism of $\mathcal{C}_{S,\{\rho\}}$, satisfying the condition that for any $w, r, r' \in \mathbb{Z}$ and any $s \to S$, the element of gr of the K-group inducing the morphism

$$h_\rho(X_w \times_S s)(r) \to h_\rho(X'_w \times_S s)(r')$$

is common.

Note that in this definition, we do not impose any comparison isomorphism between different realizations.

5.1.4. We define $\mathcal{C}_{S,R}^{\mathrm{mot}} \subset \mathcal{C}_{S,R}$ and $\mathrm{LMM}_R(S)$ imitating 4.2. Here the objects associated to standard objects for B, D, and H are defined by virtue of Propositions 2.1.7, 2.1.14, and 2.1.17.

5.2. ***Conjectures and results.*** We state the conjecture that our categories $\mathrm{LM}_R(S)$ and $\mathrm{LMM}_R(S)$ are independent of the choices of the family R of realizations. We also state Tate conjecture and Hodge conjecture. For the latter, we explain in Section 6 that they hold in a simple case. In there, we use the theories of log abelian varieties and log Jacobian varieties.

Conjecture 5.2.1. Let R' be a nonempty subset of R. Then the restriction of realizations give an equivalence of categories

$$\mathrm{LM}_R(S) \xrightarrow{\simeq} \mathrm{LM}_{R'}(S), \quad \mathrm{LMM}_R(S) \xrightarrow{\simeq} \mathrm{LMM}_{R'}(S).$$

Theorem 5.2.2. *The following* (i)–(iii) *are equivalent.*

(i) *In the category* $\mathrm{LM}_R(S)$, *homological equivalence* (i.e., the trivial equivalence) *coincides with the numerical equivalence.*

(ii) *The category* $\mathrm{LM}_R(S)$ *is a semisimple abelian category.*

(iii) *The category* $\mathrm{LMM}_R(S)$ *is a Tannakian category.*

Proof. Similar to Theorems 3.4.1(2) and 4.4.2. □

For $\rho \in R$, we denote the realization for ρ of $M \in \mathrm{LMM}(S)$ by M_ρ.

Conjecture 5.2.3 (Tate conjecture for log mixed motives). Assume that ℓ is invertible over S. Assume that either one of the following (i) and (ii) is satisfied.

(i) S is of finite type over some field which is finitely generated over the prime field.

(ii) S is of finite type over $\mathbb{Z}$.

Then for any objects M and N of $\mathrm{LMM}_{\{\ell\}}(S)$, we have

$$\mathbb{Q}_\ell \otimes \mathrm{Hom}(M, N) \xrightarrow{\cong} \mathrm{Hom}_W(M_\ell, N_\ell).$$

Here the right-hand-side denotes the set of homomorphisms of $\mathbb{Q}_\ell$-sheaves which respect the weight filtrations.

Remark. If either the log structure of S is trivial or M and N are pure, "W" on the right-hand-side in this conjecture can be eliminated (for the weight filtrations are automatically respected).

Conjecture 5.2.4 (the second Tate conjecture). Assume that S is of finite type over $\mathbb{Q}$ and let M, N be objects of $\mathrm{LMM}_{\{\ell,B\}}(S)$. Then we have a bijection from $\mathrm{Hom}(M, N)$ to the set of all pairs (a, b), where a is a morphism $M_\ell \to N_\ell$, and b is a homomorphism $M_B \to N_B$ defined on $(S \otimes \mathbb{C})^{\log}_{\mathrm{an}}$, such that the pullback of a on $(S \otimes \mathbb{C})^{\log}_{\mathrm{an}}$ is induced from b (see Proposition 2.1.12).

5.2.5. The above second Tate conjecture follows from Tate conjecture. In fact, in $\mathbb{Q}_\ell \otimes \mathrm{Hom}(M, N) \to \mathbb{Q}_\ell \otimes \{(a, b)\} \to \mathrm{Hom}(M_\ell, N_\ell)$, the composition is an isomorphism if Tate conjecture is true and the second map is an injection.

Conjecture 5.2.6 (Hodge conjecture for log mixed motives). Assume that S is log smooth over $\mathbb{C}$ or is the standard log point over $\mathbb{C}$. Let M and N be objects of $\mathrm{LMM}_{\{H\}}(S)$. Then we have

$$\mathrm{Hom}(M, N) \xrightarrow{\cong} \mathrm{Hom}(M_H, N_H).$$

By Proposition 2.1.18, the Conjecture 5.2.6 is reduced to the case where S is the standard log point associated to $\mathbb{C}$.

6. Examples

6.1. ***Log abelian varieties.*** This 6.1 and 6.2 are preparations for 6.3 and 6.4. In this 6.1, we review the theory of log abelian varieties [Kajiwara et al. 2008b] and supply some results. See [Nakayama 2017a] for a survey of the theory. We only consider log abelian varieties over a standard log point, for we need only this case in 6.3 and 6.4.

6.1.1. For an fs log scheme S, let (fs/S) be the category of fs log schemes over S, and let $(\mathrm{fs}/S)_{\text{ét}}$ be the site (fs/S) endowed with the classical étale topology. A log abelian variety over S is a sheaf of abelian groups on $(\mathrm{fs}/S)_{\text{ét}}$ satisfying certain conditions. If s is the standard log point associated to a field k, a log abelian variety over s is described as in 6.1.2–6.1.5 below.

6.1.2. Let $\boldsymbol{G}_{m,\log}$ be the sheaf $U \mapsto \Gamma(U, M_U^{\mathrm{gp}})$ on $(\mathrm{fs}/s)_{\text{ét}}$.

For a semiabelian variety G over k with the exact sequence $0 \to T \to G \to B \to 0$, where T is a torus over k and B is an abelian variety over k, let $G_{\log}$ be the pushout of $G \leftarrow T \to \mathcal{H}om(X(T), \boldsymbol{G}_{m,\log})$ in the category of sheaves of abelian groups on $(\mathrm{fs}/s)_{\text{ét}}$. Here $X(T) := \mathcal{H}om(T, \boldsymbol{G}_m)$ is the character group of T. We have $G \subset G_{\log}$.

Let $\mathcal{M}_1$ be the category of systems (Γ, G, h), where Γ is a locally constant sheaf of free $\mathbb{Z}$-modules of finite rank on $(\mathrm{fs}/s)_{\text{ét}}$, G is a semiabelian variety over k, and h is a homomorphism $\Gamma \to G_{\log}$.

An object of $\mathcal{M}_1$ was called a log 1-motif in [Kajiwara et al. 2008b].

6.1.3. For an object (Γ, G, h) of $\mathcal{M}_1$ with T the torus part of G, we have the $\mathbb{Z}$-bilinear paring

$$\langle\cdot,\cdot\rangle : X(T) \times \Gamma \to \mathbb{Z}$$

(called the *monodromy pairing*) defined as follows. The map h induces $\Gamma \to G_{\log} \to G_{\log}/G \cong T_{\log}/T$ and hence $X(T) \times \Gamma \to X(T) \times T_{\log}/T \to \boldsymbol{G}_{m,\log}/\boldsymbol{G}_m$. Since $\boldsymbol{G}_{m,\log}/\boldsymbol{G}_m$ restricted to the small étale site of the underlying scheme $\mathrm{Spec}(k)$ of s is $\mathbb{Z}$, we have the above monodromy pairing.

6.1.4. Let $E = (\Gamma, G, h)$ be an object of $\mathcal{M}_1$. The dual $E^* = (\Gamma^*, G^*, h^*)$ of E is an object of $\mathcal{M}_1$ defined as in [Kajiwara et al. 2008b]. We have $\Gamma^* = X(T)$, the torus part T^* of G^* is $\mathcal{H}om(\Gamma, \boldsymbol{G}_m)$, and the abelian variety G^*/T^* is the dual abelian variety B^* of $B = G/T$.

Let $E = (\Gamma, G, h)$ be an object of $\mathcal{M}_1$.

A *polarization* on E is a homomorphism $p : E \to E^*$ satisfying the following conditions (i)–(iv).

(i) The homomorphism $B \to B^*$ induced by p is a polarization of the abelian variety B.

(ii) The homomorphism $\Gamma \otimes \mathbb{Q} \to \Gamma^* \otimes \mathbb{Q}$ induced by p is an isomorphism.

(iii) The pairing $\Gamma \times \Gamma \to \mathbb{Z}$, $(a, b) \mapsto \langle p(a), b\rangle$ is a positive definite symmetric bilinear form, where $\langle\cdot,\cdot\rangle$ denotes the monodromy pairing (6.1.3) and p denotes the homomorphism $\Gamma \to \Gamma^* = X(T)$ induced by p.

(iv) The homomorphism $T_{\log} \to (T^*)_{\log}$ induced by p comes from

$$T \to T^* = \mathcal{H}om(\Gamma, \boldsymbol{G}_m)$$

which is dual to the homomorphism $\Gamma \to \Gamma^* = X(T)$ induced by p.

Let $\mathcal{M}_0$ be the full subcategory of $\mathcal{M}_1$ consisting of objects which have polarizations after base change to $\bar{k}$.

6.1.5. For an object (Γ, G, h) of $\mathcal{M}_1$, we have a subgroup sheaf $G_{\log}^{(\Gamma)}$ of $G_{\log}$ containing G and $h(\Gamma)$ defined as in [Kajiwara et al. 2008b].

A *log abelian variety over* s is a sheaf of abelian groups A on $(\mathrm{fs}/s)_{\text{ét}}$ such that $A = G_{\log}^{(\Gamma)}/h(\Gamma)$ for some object (Γ, G, h) of $\mathcal{M}_0$. Let $\mathrm{LAV}(s)$ be the category of log abelian varieties over s. We have an equivalence of categories

$$\mathcal{M}_0 \xrightarrow{\sim} \mathrm{LAV}(s), \quad (\Gamma, G, h) \mapsto G_{\log}^{(\Gamma)}/h(\Gamma)$$

by [Kajiwara et al. 2008b, Theorem 3.4] (see [loc. cit., Proposition 4.5 and Theorem 4.6(2)]).

6.1.6. Let E be an object of $\mathcal{M}_0$ and let A be the corresponding log abelian variety. Then the log abelian variety A^* corresponding to the dual E^* of E is called the *dual log abelian variety* of A. We have an embedding $A^* \subset \mathcal{E}xt^1(A, \mathbf{G}_{m,\log})$. A polarization of A gives a homomorphism $A \to A^*$.

6.1.7. For an additive category $\mathcal{C}$, let $\mathcal{C} \otimes \mathbb{Q}$ be the following category. Objects of $\mathcal{C} \otimes \mathbb{Q}$ are the same as those of $\mathcal{C}$. For objects E, E' of $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}\otimes\mathbb{Q}}(E, E') = \mathrm{Hom}_{\mathcal{C}}(E, E') \otimes \mathbb{Q}$.

6.1.8. The category $\mathcal{M}_1 \otimes \mathbb{Q}$ is an abelian category as is seen easily. $\mathcal{M}_0 \otimes \mathbb{Q}$ is stable in $\mathcal{M}_1 \otimes \mathbb{Q}$ under taking kernels, cokernels, and direct sums (see [Zhao 2017]), and hence, it is an abelian category. Hence $\mathrm{LAV}(s) \otimes \mathbb{Q}$ is an abelian category.

6.1.9. Let A be a log abelian variety over s corresponding to an object (Γ, G, h) of $\mathcal{M}_0$. Let ℓ be a prime number which is different from the characteristic of k. Then the ℓ-adic Tate module $T_\ell A$ is defined in the natural way as a smooth $\mathbb{Z}_\ell$-sheaf on the log étale site of s (see [Kajiwara et al. 2015, 18.9]). Let $V_\ell A = \mathbb{Q}_\ell \otimes T_\ell A$.

We have an exact sequence $0 \to T_\ell G \to T_\ell A \to \Gamma \otimes \mathbb{Z}_\ell \to 0$ (see [loc. cit., 18.10]).

We have $T_\ell(A^*) = \mathcal{H}om(T_\ell A, \mathbb{Z}_\ell(1))$.

If T is the torus part of G, the monodromy operator $\mathcal{N} : T_\ell A \to T_\ell A(-1)$ coincides with the composition $T_\ell A \to \Gamma \otimes \mathbb{Z}_\ell \to T_\ell T(-1) \to T_\ell A(-1)$, where the second arrow $\Gamma \otimes \mathbb{Z}_\ell \to T_\ell T(-1) = \mathrm{Hom}(X(T), \mathbb{Z}_\ell)$ is the map induced by the monodromy pairing $\langle\cdot,\cdot\rangle : X(T) \times \Gamma \to \mathbb{Z}$ (6.1.3).

6.1.10. Let A be a polarizable log abelian variety over s. Fix a polarization $p : A \to A^*$. Then p is an isomorphism in $\mathrm{LAV}(s) \otimes \mathbb{Q}$. For $f \in \mathrm{End}_{\mathrm{LAV}(s)\otimes\mathbb{Q}}(A)$, let $f^\sharp := p^{-1} f^* p \in \mathrm{End}_{\mathrm{LAV}(s)\otimes\mathbb{Q}}(A)$, where $f^* : A^* \to A^*$ is the dual of f.

Proposition 6.1.11. *Let A and p be as above and let $f \in \mathrm{End}_{\mathrm{LAV}(s)\otimes\mathbb{Q}}(A)$, $f \neq 0$. Then $\mathrm{Tr}(ff^\sharp) > 0$. Here Tr is the trace of the induced $\mathbb{Q}_\ell$-linear map $V_\ell A \to V_\ell A$.*

Proof. Let $E = (\Gamma, G, h)$ be an object of $\mathcal{M}_0$ corresponding to A, let T be the torus part of G, and let $B = G/T$ be the quotient abelian variety of G. Let $f_0, f_0^\sharp : \Gamma \otimes \mathbb{Q}_\ell \to \Gamma \otimes \mathbb{Q}_\ell$, $f_1, f_1^\sharp : V_\ell B \to V_\ell B$, and $f_2, f_2^\sharp : V_\ell T \to V_\ell T$ be the map induced by f, $f^\sharp$, respectively. Then

$$\mathrm{Tr}(ff^\sharp) = \sum_{i=0}^{2} \mathrm{Tr}(f_i f_i^\sharp).$$

By the usual theory of abelian varieties, $\mathrm{Tr}(f_1 f_1^\sharp) \geq 0$ and it is nonzero if $f_1 \neq 0$. $\mathrm{Tr}(f_0 f_0^\sharp) \geq 0$ and this is nonzero if $f_0 \neq 0$, for we have a positive definite symmetric form. We have $\mathrm{Tr}(f_2 f_2^\sharp) \geq 0$ and it is nonzero if $f_2 \neq 0$ by duality. Hence $\mathrm{Tr}(ff^\sharp) \geq 0$ and this is nonzero unless $f_0 = f_1 = f_2 = 0$. If $f_0 = f_1 = f_2 = 0$, $f = 0$ because any homomorphism $B \to T$ is zero. □

Corollary 6.1.12. *The category $\mathrm{LAV}(s) \otimes \mathbb{Q}$ is semisimple.*

Proof. This is deduced from the above proposition by the arguments in 3.4. □

6.1.13. Let A be a log abelian variety over s. Assume $k = \mathbb{C}$. Then we have the polarizable log Hodge structure over s of weight -1 corresponding to A [Kajiwara et al. 2008a], which we denote by $H_1(A)_H$. The underlying locally constant sheaf of finite-dimensional $\mathbb{Q}$-vector spaces on the topological space $s_{\mathrm{an}}^{\log}$ (which is homeomorphic to a circle S^1) will be denoted by $H_1(A)_B$. If (Γ, G, h) denotes the object of $\mathcal{M}_0$ corresponding to A, we have an exact sequence

$$0 \to \mathcal{H}_1(G, \mathbb{Z}) \to H_1(A)_B \to \Gamma \to 0.$$

Proposition 6.1.14. *Let A_1 and A_2 be log abelian varieties over s.*

(1) *If k is finitely generated over a prime field, we have*

$$\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{Hom}(A_1, A_2) \xrightarrow{\cong} \mathrm{Hom}(T_\ell A_1, T_\ell A_2).$$

(2) *If k is a subfield of $\mathbb{C}$ which is finitely generated over $\mathbb{Q}$, we have a bijection from $\mathrm{Hom}(A_1, A_2)$ to the set of pairs (a, b), where a is a homomorphism $T_\ell A_1 \to T_\ell A_2$ and b is a homomorphism $H_1(A_1)_B \to H_1(A_2)_B$ on $(s \otimes_k \mathbb{C})^{\log}$ such that the pullback of a on $(s \otimes_k \mathbb{C})^{\log}$ is induced by b.*

(3) *If $k = \mathbb{C}$, $\mathrm{Hom}(A_1, A_2) \xrightarrow{\cong} \mathrm{Hom}(H_1(A_1)_H, H_1(A_2)_H)$.*

Proof. For an object $E = (\Gamma, G, h)$ of $\mathcal{M}_1$, define the filtration W on E by $W_w E = E$ for $w \geq 0$, $W_{-1}E = (0, G, 0)$, $W_{-2}E = (0, T, 0)$ with T the torus part of G, and $W_w E = 0$ for $w \leq -3$. Then $\mathrm{gr}_0^W E = (\Gamma, 0, 0)$, $\mathrm{gr}_{-1}^W E = (0, B, 0)$, where B is the abelian variety G/T, $\mathrm{gr}_{-2}^W E = (0, T, 0)$, and $\mathrm{gr}_w^W E = 0$ for $w \neq 0, -1, -2$. Let

$\mathcal{C}_1 = \mathcal{M}_1 \otimes \mathbb{Q}$ and let $\mathcal{C}_2$ be the category of smooth $\mathbb{Q}_\ell$-sheaves on the log étale site of s. Then (1) and (2) follow from the Tate conjecture on homomorphisms of abelian varieties proved by Faltings [1983] and from the injectivity of $G(k) \otimes \mathbb{Q} \to H^1(k, V_\ell G)$ for a semiabelian variety G over k, by the method of 4.3.

Assertion (3) follows from [Kajiwara et al. 2008a]. $\square$

6.2. ***Log Jacobian varieties.*** We review the theory of log Jacobian varieties of log curves over a standard log point in [Kajiwara 1993], and supply some results. In this subsection and the next, we omit some details of proofs, which will be treated in a forthcoming paper.

6.2.1. Let s be the standard log point associated to a field k. Let X be a projective vertical log smooth connected curve over s which is strict semistable, whose double points are rational and whose components are geometrically irreducible.

Then we have a log abelian variety over s associated to X called the *log Jacobian variety* of X. We will denote it by J.

This J is essentially constructed by Kajiwara [1993]. We explain his construction below in 6.2.4.

This J has the following properties 6.2.2 and 6.2.3.

6.2.2. Let $\mathcal{H}^1(X, M^{\mathrm{gp}})$ be the sheafification of the presheaf $U \mapsto H^1(X \times_s U, M^{\mathrm{gp}})$ on $(\mathrm{fs}/s)_{\text{ét}}$. We have a degree map $\mathcal{H}^1(X, M^{\mathrm{gp}}) \to \mathbb{Z}$. Let $\mathcal{H}^1(X, M^{\mathrm{gp}})^0 \subset \mathcal{H}^1(X, M^{\mathrm{gp}})$ be the kernel of the degree map. Then J is a subgroup sheaf of $\mathcal{H}^1(X, M^{\mathrm{gp}})^0$.

6.2.3. Let $E = (\Gamma, G, h)$ be the object of $\mathcal{M}_0$ corresponding to J, let T be the torus part of G, and let $B = G/T$ be the quotient abelian variety of G. Then Γ, T, B are described as follows.

Let Γ be the first homology group of the graph of X as usual, that is, $\Gamma = \mathrm{Ker}\big(\bigoplus_{I_1} \mathbb{Z} \to \bigoplus_{I_0} \mathbb{Z}\big)$, where I_0 is the set of generic points of X, and I_1 is the set of singular points of X. $\mathcal{H}om\,(T, \boldsymbol{G}_m) = \mathrm{Hom}(\Gamma, \mathbb{Z})$. $B = \prod_{\nu \in I_0} J_{D(\nu)}$, where $D(\nu)$ is the closure of ν in X which is a projective smooth curve over k and $J_{D(\nu)}$ is the Jacobian variety of $D(\nu)$. We have a canonical isomorphism $J \cong J^*$ (see 6.2.7) which induces the evident isomorphisms $\Gamma \cong \Gamma^*$, $T \cong T^*$ and $B \cong B^*$.

6.2.4. We explain the construction of J, which is essentially due to Kajiwara. For simplicity, we assume that k is algebraically closed. By [Kajiwara 1993, (2.18)], we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \Gamma \arrow[d] & G_{\log} \arrow[d] & \mathcal{H}^1(X, M^{\mathrm{gp}}) \arrow[d, equal] & \\
0 \arrow[r] & \bigoplus_{I_1} \mathbb{Z} \arrow[r] \arrow[d] & P^{\log}_{X/s} \arrow[r] \arrow[d] & \mathcal{H}^1(X, M^{\mathrm{gp}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{I_0} \mathbb{Z} \arrow[r, equal] \arrow[d] & \bigoplus_{I_0} \mathbb{Z} \arrow[r] & 0 & \\
 & \mathbb{Z} & & &
\end{tikzcd}$$

with exact rows and columns, where $G = \mathrm{Ker}(\mathcal{H}^1(X, \boldsymbol{G}_m) \to \bigoplus_{I_0} \mathbb{Z})$, and $P^{\log}_{X/s}$ is defined in [Kajiwara 1993]. This diagram yields a log 1-motif $(\Gamma, G, h : \Gamma \to G_{\log})$ and the degree map $\mathcal{H}^1(X, M^{\mathrm{gp}}) \to \mathbb{Z}$ whose kernel $\mathcal{H}^1(X, M^{\mathrm{gp}})^0 \cong G_{\log}/h(\Gamma)$ contains $G^{(\Gamma)}_{\log}/h(\Gamma)$. The last sheaf is J.

6.2.5. Let $Y := X \times_s X$. We have a M_Y^{gp}-torsor on Y called the *Poincaré torsor*, defined as follows.

Let $U = Y \smallsetminus \bigcup_{x \in I_1}(\{x\} \times \{x\})$. Let M'_U be the pushout over the trivial log structure on U of the log structure $M_Y|_U$ and the log structure consisting of the sections of $\mathcal{O}_U$ which are invertible outside the diagonal X in Y. Let M'_Y be the unique fs log structure on Y whose restriction to U coincides with M'_U. (See the following local description for the existence of such an fs log structure.) We have $M_Y^{\mathrm{gp}} \subset (M'_Y)^{\mathrm{gp}}$. There is a unique global section t of $(M'_Y)^{\mathrm{gp}}/M_Y^{\mathrm{gp}}$ having the following property: Let π be a generator of the log structure of s. At any singular point x of X, let f_1, g_1 be generators of the log of the left X in $X \times_s X$ around x such that $f_1 g_1 = \pi$, and let f_2, g_2 be the copies of them for the right X in $X \times_s X$. Let $f_1 - f_2$ be the section of M'_Y around $\{x\} \times \{x\}$ which is $f_1 - f_2$ on the locus $\{g_1 = g_2 = 0\}$, which is f_1 on the locus $\{g_1 = f_2 = 0\}$, which is $-f_2$ on the locus $\{f_1 = g_2 = 0\}$, and which is $(-\pi g_1^{-1} g_2^{-1})(g_1 - g_2)$ on the locus $\{f_1 = f_2 = 0\}$. Define $g_1 - g_2$ similarly. Then, we have $g_1 - g_2 = (-\pi f_1^{-1} f_2^{-1})(f_1 - f_2)$ in $(M'_Y)^{\mathrm{gp}}$ and $-\pi f_1^{-1} f_2^{-1} \in M_Y^{\mathrm{gp}}$. The desired t coincides around $\{x\} \times \{x\}$ with the class of $f_1 - f_2$ which is also the class of $g_1 - g_2$. Note that the ideal of $\mathcal{O}_Y$ which defines the diagonal is generated around $\{x\} \times \{x\}$ by the image of $f_1 - f_2$ and by the image of $g_1 - g_2$.

Let the Poincaré torsor be the inverse image of t^{-1} in $(M'_Y)^{\mathrm{gp}}$ under $(M'_Y)^{\mathrm{gp}} \to (M'_Y)^{\mathrm{gp}}/M_Y^{\mathrm{gp}}$. This is an M_Y^{gp}-torsor.

If X is a projective smooth curve over k endowed with the pullback log structure from s, this Poincaré torsor comes from the usual Poincaré $\boldsymbol{G}_m$-torsor.

6.2.6. We have a morphism $\varphi : X \to \mathcal{H}^1(X, M^{\mathrm{gp}})$ which sends x to the pullback of the Poincaré torsor (6.2.5) with respect to $X \to X \times X,\ y \mapsto (x, y)$.

If b is a morphism $s \to X$ over s, we have a canonical morphism

$$\varphi_b : X \to J \subset \mathcal{H}^1(X, M^{\mathrm{gp}}), \quad x \mapsto \varphi(x) - \varphi(b)$$

called the *log Albanese mapping* associated to b.

6.2.7 (Self-duality of the log Jacobian). Let b and φ_b be as above. Then the pulling back via φ_b gives an isomorphism

$$\mathcal{E}xt^1(J, \boldsymbol{G}_{m,\log}) \xrightarrow{\cong} \mathcal{H}^1(X, M^{\mathrm{gp}})^0,$$

which is independent of the choice of b. Hence the subgroup sheaf J of $\mathcal{H}^1(X, M^{\mathrm{gp}})^0$ is regarded as a subgroup sheaf of $\mathcal{E}xt^1(J, \boldsymbol{G}_{m,\log})$. Via this, J is identified with the dual log abelian variety J^* of J. Since this isomorphism $J \cong J^*$ does not depend on b, it is defined canonically even if there is no b.

Proposition 6.2.8. *Let $b : s \to X$ be a morphism over s. Let A be any log abelian variety over s. Then the map*

$$\mathrm{Hom}(J, A) \to \mathrm{Mor}(X, A), \quad h \mapsto h \circ \varphi_b$$

is bijective.

Proof. The inverse map is given as follows. Let $f : X \to A$ be a morphism. Then we have $A^* \to \mathcal{E}xt^1(A, \boldsymbol{G}_{m,\log}) \to \mathcal{H}^1(X, M^{\mathrm{gp}})^0$, where the second arrow is the pullback by f. This induces $A^* \to J$. Taking the dual log abelian varieties, we have $J \to A$. □

6.2.9. Let ℓ be a prime number which is invertible in k. Then we have canonical isomorphisms

$$V_\ell J \cong H^1(X)_\ell(1) \cong \mathcal{H}om(H^1(X)_\ell, \mathbb{Q}_\ell).$$

6.3. *Examples I.* This subsection Examples I is for the pure case. Section 6.4, Examples II, is for the mixed case.

The following is a part of Conjecture 3.2.2.

Proposition 6.3.1. *Let X be a projective vertical log smooth curve over an fs log scheme S. Then $h(X) = h^0(X) \oplus h^1(X) \oplus h^2(X)$.*

Proof. It is enough to show that for $i = 0, 1, 2$, the composite of the i-th projection and the i-th inclusion

$$h(X)_\ell = \bigoplus_{j=0}^{2} H^j(X)_\ell \to H^i(X)_\ell \to \bigoplus_{j=0}^{2} H^j(X)_\ell = h(X)_\ell$$

comes from an element of K-group $\otimes\mathbb{Q}$ after pulling back to any geometric standard log point. So we may assume that S is a geometric standard log point over a field k. It is enough to show it for $i = 0, 2$. By the duality, the case $i = 2$ is reduced to $i = 0$. We prove the case $i = 0$. If there is a section $S \to X$, the composite for $i = 0$ coincides with $h(X)_\ell \to h(S)_\ell \to h(X)_\ell$ and induced by an element of K-group. In the general case, there is a section after Kummer log flat localization of the base [Nakayama 2009, Proposition 4.1], so we have the desired element a of K-group after the base change by $(\operatorname{Spec} k, n : \mathbb{N} \to \mathbb{N})$ for some $n \geq 1$. Then the $1/n$ times of $\mathrm{Tr}(a)$ is a desired element. □

Proposition 6.3.2. *Assume that S is the standard log point over $\mathbb{C}$. Let X be a connected projective strict semistable curve over S. Then the Hodge conjecture 5.2.6 for* $\mathrm{Hom}(\mathbb{Q}, h^2(X)(1))$ *is true.*

Proof. Assume that we are given a homomorphism $h : \mathbb{Q} \to H^2(X)_B(1)$. By invariant cycle theorem, this comes from the classical Betti cohomology $H^2(X_{\mathrm{an}}, \mathbb{Q}(1))$. Since $h(1)$ belongs to $\mathrm{Fil}^1 H^2(X)_H$, it vanishes in $H^2(X, \mathcal{O}_X)$. Hence it comes from the kernel of $H^2(X_{\mathrm{an}}, \mathbb{Q}(1)) \to H^2(X, \mathcal{O}_X)$. By the exponential sequence $0 \to \mathbb{Z}(1) \to \mathcal{O}_{X_{\mathrm{an}}} \to \mathcal{O}_{X_{\mathrm{an}}}^\times \to 0$, it comes from $\mathrm{Pic}(X) \otimes \mathbb{Q}$. □

The next proposition will be proved in a forthcoming paper.

Proposition 6.3.3. *Let s be a geometric standard log point of characteristic $\neq \ell$. For $i = 1, 2$, let X_i be a projective vertical log smooth curve over s which is strict semistable, and let J_i be the log Jacobian variety of X_i. For a homomorphism $h : H^1(X_1)_\ell \to H^1(X_2)_\ell$, the following two conditions* (i) *and* (ii) *are equivalent.*

(i) *h is a morphism $H^1(X_1) \to H^1(X_2)$ of log motives over s.*

(ii) *h comes from a morphism $J_1 \to J_2$ in* $\mathrm{LAV}(s)$ (via the isomorphisms in 6.2.9, $H^1(X_i)_\ell(1) \cong V_\ell J_i$).

Proposition 6.3.4. *Let X and Y be projective vertical log smooth curves over an fs log scheme S whose geometric fibers are connected.*

(1) *Assume that S is the standard log point over $\mathbb{C}$ and that X and Y are strict semistable over S. Then the Hodge conjecture 5.2.6 for* $\mathrm{Hom}(h(X), h(Y))$ *is true.*

(2) *Assume that S is of finite type over $\mathbb{Q}$. Then the second Tate conjecture 5.2.4 for* $\mathrm{Hom}(h(X), h(Y))$ *is true.*

(3) *Assume that S is a standard log point associated to a finitely generated field over a prime field whose characteristic is different from a prime number ℓ. Then the Tate conjecture 5.2.3 for* $\mathrm{Hom}(h(X), h(Y))$ *is true.*

(4) *For $f, g \in \mathrm{Hom}(h(X), h(Y))$, if f and g are numerically equivalent, then $f = g$.*

(5) *The endomorphism ring of $h(X)$ is a finite-dimensional semisimple algebra over $\mathbb{Q}$.*

Proof. By 2.3.14, Proposition 2.1.9 and ℓ-adic log proper base change theorem [Kajiwara and Nakayama 2008, Proposition 5.1] (see [loc. cit., Remark 5.1.1]), we may assume that S is a standard log point and X and Y are strict semistable and that their double points are rational and their components are geometrically irreducible. Let J and J' be the log Jacobian variety of X and Y, respectively. By Propositions 6.3.1, 6.3.3, and the method of 4.3, we can identify $\mathrm{Hom}(h(X), h(Y))$ with $\mathrm{Hom}_{\mathrm{LAV}(S)\otimes\mathbb{Q}}(J, J')$. Then we reduce to the results in 6.1. □

6.4. *Examples II.*

6.4.1. Let X be a projective vertical log smooth curve over an fs log scheme S. Let $n \geq 1$ and $s_1, \dots, s_n : S \to X$ be strict morphisms over S such that $s_i(S) \cap s_j(S) = \varnothing$ if $i \neq j$. Let $D := \bigcup_{i=1}^{n} s_i(S)$ and let $U := X \setminus D$.

We will denote the log mixed motive corresponding to the standard object associated to $(U, X, D, 1)$ over S by $H^1(U)$.

Let $\Gamma = \mathrm{Ker}(\mathrm{sum} : \mathbb{Z}^n \to \mathbb{Z})$. We have $W_0 M = M$, $W_{-2} M = 0$, $W_{-1} M = H^1(X)$, $\mathrm{gr}_0^W M = \Gamma \otimes \mathbb{Q}(-1)$, where $M = H^1(U)$.

The spectral sequence as in 4.2.4 for each realization degenerates at E_2.

6.4.2. Let the notation be as in 6.4.1.

If S is over $\mathbb{Z}[1/\ell]$, we have an exact sequence

$$0 \to H^1(X)_\ell \to H^1(U)_\ell \to \Gamma \otimes \mathbb{Q}_\ell(-1) \to 0 \tag{1}$$

of $\mathbb{Q}_\ell$-sheaves.

If S is either log smooth over $\mathbb{C}$ or the standard log point associated to $\mathbb{C}$, we have an exact sequence

$$0 \to H^1(X)_H \to H^1(U)_H \to \Gamma \otimes \mathbb{Q}(-1) \to 0 \tag{2}$$

of log mixed Hodge structures over S.

Assume that S is a standard log point associated to a field k, and assume that X is connected and strict semistable and that their double points are rational and their components are geometrically irreducible. Let J be the log Jacobian variety of X. Then $(s_i)_i$ induces a homomorphism $\psi := \varphi \circ (s_i)_i : \Gamma \to J$ by the log Albanese mapping φ (6.2.6).

Note that for any log abelian variety A over S, we have a canonical homomorphism

$$A(S) \otimes \mathbb{Q} \to \mathrm{Ext}^1(\mathbb{Q}_\ell, V_\ell A) \tag{3}$$

by Kummer theory, which is injective if k is finitely generated over a prime field. If $k = \mathbb{C}$, we have also a canonical injective map

$$A(S) \to \mathrm{Ext}^1(\mathbb{Z}, H_1(A)_H). \tag{4}$$

We have:

(5) Under the homomorphism $\mathrm{Hom}(\Gamma, J) \to \mathrm{Ext}^1(\Gamma \otimes \mathbb{Q}_\ell, V_\ell J)$ induced by (3) (applied to the log abelian variety $A = \mathcal{H}om(\Gamma, J)$), the extension class of (1) coincides with the image of $\psi : \Gamma \to J$.

(6) If $k = \mathbb{C}$, under the homomorphism $\mathrm{Hom}(\Gamma, J) \to \mathrm{Ext}^1(\Gamma, H_1(J)_H)$ induced by (4), the extension class of (2) coincides with the image of $\psi : \Gamma \to J$.

Proposition 6.4.3. *Let U_1, U_2 be objects as U in 6.4.1.*

(1) *Assume that S is the standard log point over $\mathbb{C}$ and that X_1 and X_2 are connected and strict semistable. Then for* $\mathrm{Hom}(H^1(U_1), H^1(U_2))$ *the Hodge conjecture 5.2.6 is true.*

(2) *Assume that S is of finite type over $\mathbb{Q}$. Then the second Tate conjecture 5.2.4 for* $\mathrm{Hom}(H^1(U_1), H^1(U_2))$ *is true.*

(3) *Assume that S is the standard log point associated to a finitely generated field over a prime field whose characteristic is different from a prime number ℓ. Then the Tate conjecture 5.2.3 for* $\mathrm{Hom}(H^1(U_1), H^1(U_2))$ *is true.*

Proof. Similarly as in Proposition 6.3.4, we may assume that S is a standard log point and X_i are connected and strict semistable and that their double points are rational and their components are geometrically irreducible. For $i = 1, 2$, let J_i be the log Jacobian variety of X_i. By (5) in 6.4.2, by the injectivity of the map (3) in 6.4.2, and by Proposition 6.3.3, the method of 4.3 shows that:

(∗) The set of morphisms $H^1(U_1) \to H^1(U_2)$ is identified with the set of pairs (a, b), where a is a homomorphism $\Gamma_1 \otimes \mathbb{Q} \to \Gamma_2 \otimes \mathbb{Q}$ and b is a morphism $J_1 \to J_2$ in $\mathrm{LAV}(s) \otimes \mathbb{Q}$ such that $\psi_2 \circ a = b \circ \psi_1$.

Hence by (5) in 6.4.2, by the injectivity of the map (3) in 6.4.2, and by this (∗), the method of 4.3 proves (2) and (3). Similarly, by (6) in 6.4.2, by the injectivity of the map (4) in 6.4.2, and by (∗), the method of 4.3 proves (1). □

Acknowledgments

A part of this paper was written while Ito was staying at the Institut des Hautes Études Scientifiques. He would like to thank the IHES for its hospitality. We thank Takeshi Saito and Takeshi Kajiwara for helpful discussions, the referee for careful reading and valuable comments, and Heer Zhao for pointing out an earlier error in 6.2.5. Kato is partially supported by NSF grants DMS 1303421 and DMS 1601861. Nakayama is partially supported by JSPS, KAKENHI (C) No. 22540011, (B) No. 23340008, and (C) No. 16K05093. Usui is partially supported by JSPS, KAKENHI (B) No. 23340008 and (C) No. 17K05200.

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Received 25 Dec 2017. Revised 12 Apr 2019.

TETSUSHI ITO:
tetsushi@math.kyoto-u.ac.jp
Department of Mathematics, Kyoto University, Kyoto, Japan

KAZUYA KATO:
kkato@math.uchicago.edu
Department of Mathematics, University of Chicago, Chicago, IL, United States

CHIKARA NAKAYAMA:
c.nakayama@r.hit-u.ac.jp
Department of Economics, Hitotsubashi University, Kunitachi, Tokyo, Japan

SAMPEI USUI:
usui@math.sci.osaka-u.ac.jp
Graduate School of Science, Osaka University, Toyonaka, Osaka, Japan

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TUNISIAN JOURNAL OF MATHEMATICS
Vol. 2, No. 4, 2020
dx.doi.org/10.2140/tunis.2020.2.791

Equidistribution and counting of orbit points for discrete rank one isometry groups of Hadamard spaces

Gabriele Link

Let X be a proper, geodesically complete Hadamard space, and $\Gamma < \mathrm{Is}(X)$ a discrete subgroup of isometries of X with the fixed point of a rank one isometry of X in its infinite limit set. In this paper we prove that if Γ has nonarithmetic length spectrum, then the Ricks–Bowen–Margulis measure — which generalizes the well-known Bowen–Margulis measure in the CAT(-1) setting — is mixing. If in addition the Ricks–Bowen–Margulis measure is finite, then we also have equidistribution of Γ-orbit points in X, which in particular yields an asymptotic estimate for the orbit counting function of Γ. This generalizes well-known facts for nonelementary discrete isometry groups of Hadamard manifolds with pinched negative curvature and proper CAT(-1)-spaces.

1. Introduction

Let (X, d) be a proper Hadamard space, $x, y \in X$ and $\Gamma < \mathrm{Is}(X)$ a discrete group. The *Poincaré series* of Γ with respect to x and y is defined by

$$P(s; x, y) := \sum_{\gamma\in\Gamma} \mathrm{e}^{-sd(x,\gamma y)};$$

its exponent of convergence

$$\delta_\Gamma := \inf\Big\{ s > 0 : \sum_{\gamma\in\Gamma} \mathrm{e}^{-sd(x,\gamma y)} \text{ converges} \Big\} \tag{1}$$

is called the *critical exponent* of Γ. By the triangle inequality the critical exponent is independent of $x, y \in X$. We will require that the critical exponent δ_Γ is *finite*, which is not a severe restriction as it is always the case when X admits a compact quotient or when Γ is finitely generated.

Obviously $P(s; x, y)$ converges for $s > \delta_\Gamma$ and diverges for $s < \delta_\Gamma$. The group Γ is said to be *divergent*, if $P(\delta_\Gamma; x, y)$ diverges, and *convergent* otherwise.

MSC2010: primary 20F69, 22D40; secondary 20F67, 37D25, 37D40.
Keywords: rank one space, Bowen–Margulis measure, mixing, equidistribution, orbit counting function.

Since X is proper, the *orbit counting function* with respect to x and y

$$N_\Gamma : [0,\infty) \to [0,\infty), \qquad R \mapsto \#\{\gamma \in \Gamma : d(x,\gamma y) \le R\} \tag{2}$$

satisfies $N_\Gamma(R) < \infty$ for all $R > 0$; moreover, it is related to the critical exponent via the formula

$$\delta_\Gamma = \limsup_{R\to+\infty} \frac{\ln\big(N_\Gamma(R)\big)}{R}.$$

One goal of this article is to give a precise asymptotic estimate for the orbit counting function for a discrete *rank one group* Γ as in [Link 2018] (that is a group with the fixed point of a so-called rank one isometry of X in its infinite limit set); for precise definitions we refer the reader to Section 3. Such a rank one group always contains a nonabelian free subgroup generated by two independent rank one elements, hence its critical exponent δ_Γ is strictly positive. Notice that our assumption on Γ obviously imposes severe restrictions on the Hadamard space X itself: it can neither be a higher rank symmetric space, a higher rank Euclidean building, nor a product of Hadamard spaces.

Using the Poincaré series from above, a remarkable Γ-equivariant family of measures $(\mu_x)_{x\in X}$ supported on the geometric boundary ∂X of X — a so-called conformal density — can be constructed in our very general setting (see [Patterson 1976; Sullivan 1979] for the original constructions in hyperbolic n-space).

Let $\mathcal{G}$ denote the set of parametrized geodesic lines in X endowed with the compact-open topology (which can be identified with the unit tangent bundle SX if X is a Riemannian manifold) and consider the action of $\mathbb{R}$ on $\mathcal{G}$ by reparametrization. This action induces a flow g_Γ on the quotient space $\Gamma\backslash\mathcal{G}$. If X is *geodesically complete*, then thanks to the construction due to R. Ricks [2017, Section 7] — which uses the conformal density $(\mu_x)_{x\in X}$ described above — we obtain a g_Γ-invariant Radon measure m_Γ on $\Gamma\backslash\mathcal{G}$. This possibly infinite measure will be called the *Ricks–Bowen–Margulis* measure, since it generalizes the classical Bowen–Margulis measure in the CAT(-1)-setting.

If Γ is divergent, then according to [Link 2018, Theorem 10.2] the dynamical system $(\Gamma\backslash\mathcal{G}, g_\Gamma, m_\Gamma)$ is conservative and ergodic. We also want to mention here that if X is a Hadamard *manifold*, then the Ricks–Bowen–Margulis measure m_Γ is equal to Knieper's measure first introduced in Section 2 of [Knieper 1998] for cocompact groups Γ (and which was used in [Link and Picaud 2016] for arbitrary rank one groups). In the cocompact case Knieper's work further implies that the Ricks–Bowen–Margulis measure is the unique measure of maximal entropy on the unit tangent bundle of the compact quotient $\Gamma\backslash X$ (see again [Knieper 1998, Section 2]).

In this article we will first address the question under which hypotheses the dynamical system $(\Gamma\backslash\mathcal{G}, g_\Gamma, m_\Gamma)$ is mixing. We remark that in our very general setting we cannot hope to get mixing without further restrictions on the group Γ: F. Dal'Bo [2000, Theorem A] showed that even in the special case of a CAT(-1)-Hadamard *manifold* X, the dynamical system $(\Gamma\backslash SX, g_\Gamma, m_\Gamma)$ with the classical Bowen–Margulis measure m_Γ is *not* mixing, if the length spectrum of Γ is arithmetic (that is if the set of lengths of closed geodesics in $\Gamma\backslash X$ is contained in a discrete subgroup of $\mathbb{R}$). However, we obtain the best possible result:

Theorem A. *Let X be a proper, geodesically complete Hadamard space and let $\Gamma < \mathrm{Is}(X)$ be a discrete, divergent rank one group. Then with respect to Ricks–Bowen–Margulis measure the geodesic flow on $\Gamma\backslash\mathcal{G}$ is mixing or the length spectrum of Γ is arithmetic.*

Notice that in the CAT(0)-setting Theorem A was already proved by M. Babillot [2002, Theorem 2] in the special case when X is a *manifold* and $\Gamma < \mathrm{Is}(X)$ is cocompact; moreover, in this case the second alternative cannot occur, that is the length spectrum of Γ *cannot* be arithmetic. It was then generalized by Ricks [2017, Theorem 4] to non-Riemannian proper Hadamard spaces X and discrete rank one groups $\Gamma < \mathrm{Is}(X)$ with *finite* Ricks–Bowen–Margulis measure. Under the additional hypothesis that the limit set of Γ is equal to the whole geometric boundary ∂X of X, Ricks also proved that the length spectrum of Γ can only be arithmetic if X is isometric to a tree with all edge lengths in $c\mathbb{N}$ for some $c > 0$. Here we allow both infinite Ricks–Bowen–Margulis measure and limit sets that are proper subsets of ∂X.

Let us mention that the restriction to divergent groups is quite reasonable: If the measure m_Γ is infinite, then the mixing property of $(\Gamma\backslash\mathcal{G}, g_\Gamma, m_\Gamma)$ only states that for all Borel sets $A, B \subset \Gamma\backslash\mathcal{G}$ with $m_\Gamma(A)$, $m_\Gamma(B)$ finite we have

$$\lim_{t\to\pm\infty} m_\Gamma(A \cap g_\Gamma^t B) = 0.$$

This condition is very weak and obviously neither implies conservativity nor ergodicity. Actually it is easily seen to hold true when $(\Gamma\backslash\mathcal{G}, g_\Gamma, m_\Gamma)$ is dissipative, which — according to [Link 2018, Theorem 10.2] — is equivalent to the fact that Γ is convergent.

In the second part of the article we use the mixing property in the case of finite Ricks–Bowen–Margulis measure to deduce an equidistribution result for Γ-orbit points in the vein of T. Roblin's results [2003, théorème 4.1.1] for CAT(-1)-spaces:

Theorem B. *Let X be a proper, geodesically complete Hadamard space and let $\Gamma < \mathrm{Is}(X)$ be a discrete rank one group with nonarithmetic length spectrum and finite Ricks–Bowen–Margulis measure m_Γ.*

Let f be a continuous function from $\bar{X}\times\bar{X}$ to $\mathbb{R}$, and $x, y\in X$. Then

$$\lim_{T\to\infty}\Bigg(\delta_\Gamma\cdot \mathrm{e}^{-\delta_\Gamma T}\sum_{\substack{\gamma\in\Gamma\\ d(x,\gamma y)\le T}} f(\gamma y,\gamma^{-1}x)\Bigg)=\frac{1}{\|m_\Gamma\|}\int_{\partial X\times\partial X} f(\xi,\eta)\,\mathrm{d}\mu_x(\xi)\,\mathrm{d}\mu_y(\eta).$$

Finally, from the equidistribution result Theorem B and its proof we get the following asymptotic estimates for the orbit counting function introduced in (2):

Theorem C. *Let X be a proper, geodesically complete Hadamard space, $x, y\in X$ and $\Gamma<\mathrm{Is}(X)$ a discrete rank one group.*

(a) *If Γ is divergent with nonarithmetic length spectrum and finite Ricks–Bowen–Margulis measure m_Γ, then*

$$\lim_{R\to\infty}\delta_\Gamma\cdot \mathrm{e}^{-\delta_\Gamma R}\#\{\gamma\in\Gamma : d(x,\gamma y)\le R\}=\mu_x(\partial X)\mu_y(\partial X)/\|m_\Gamma\|.$$

(b) *If Γ is divergent with nonarithmetic length spectrum and infinite Ricks–Bowen–Margulis measure, then*

$$\lim_{R\to\infty}\mathrm{e}^{-\delta_\Gamma R}\#\{\gamma\in\Gamma : d(x,\gamma y)\le R\}=0.$$

(c) *If Γ is convergent, then* $\lim_{R\to\infty}\mathrm{e}^{-\delta_\Gamma R}\#\{\gamma\in\Gamma : d(x,\gamma y)\le R\}=0.$

In work in progress with Jean-Claude Picaud we apply the equidistribution result Theorem B above to get asymptotic estimates for the number of closed geodesics modulo free homotopy in $\Gamma\backslash X$ which are much more general and much more precise than the ones given in [Link 2007].

The paper is organized as follows: Section 2 fixes some notation and recalls basic facts concerning Hadamard spaces and rank one geodesics. In Section 3 we introduce the notions of rank one isometry and $\mathrm{Is}(X)$-recurrence and state some important facts. We also recall the definition of a rank one group and give the weakest condition which ensures that a discrete group $\Gamma<\mathrm{Is}(X)$ is rank one. In Section 4 we introduce the notion of geodesic current and describe Ricks' construction of a geodesic flow invariant measure associated to such a geodesic current first on the quotient $\Gamma\backslash[\mathcal{G}]$ of parallel classes of parametrized geodesic lines and finally on the quotient $\Gamma\backslash\mathcal{G}$ of parametrized geodesic lines. Moreover, we recall from [Link 2018] a few results about the corresponding dynamical systems. Section 5 is devoted to the proof of Theorem A, which follows Babillot's strategy [2002, Section 2.2] and uses cross-ratios of quadrilaterals similar to the ones introduced by Ricks [2017, Section 10]. In Section 6 we introduce the notions of shadows, cones and corridors and state some important properties that are needed in the proof of Theorem B. Section 7 gives estimates for the so-called Ricks–Bowen–Margulis measure, which is the Ricks measure associated to the quasiproduct geodesic current coming from a conformal density. In Section 8 we prove Theorem B, and Section 9 finally deals with the orbit counting function and the proof of Theorem C.

2. Preliminaries on Hadamard spaces

The purpose of this section is to introduce terminology and notation and to summarize basic results about Hadamard spaces. Most of the material can be found in [Ballmann 1995; Bridson and Haefliger 1999] (see also [Ballmann 1982; Ballmann et al. 1985] in the special case of Hadamard manifolds and [Ricks 2017] for more recent results).

Let (X, d) be a metric space. For $y \in X$ and $r > 0$ we will denote $B_r(y) \subset X$ the open ball of radius r centered at $y \in X$. A *geodesic* is an isometric map σ from a closed interval $I \subset \mathbb{R}$ or $I = \mathbb{R}$ to X. For more precision we use the term *geodesic ray* if $I = [0, \infty)$ and *geodesic line* if $I = \mathbb{R}$.

We will deal here with *Hadamard spaces* (X, d), that is complete metric spaces in which for any two points $x, y \in X$ there exists a geodesic $\sigma_{x,y}$ joining x to y (that is a geodesic $\sigma = \sigma_{x,y} : [0, d(x, y)] \to X$ with $\sigma(0) = x$ and $\sigma(d(x, y)) = y$) and in which all geodesic triangles satisfy the CAT(0)-inequality. This implies in particular that X is simply connected and that the geodesic joining an arbitrary pair of points in X is unique. Notice however that in the non-Riemannian setting completeness of X does not imply that every geodesic can be extended to a geodesic line, so X need not be geodesically complete. The geometric boundary ∂X of X is the set of equivalence classes of asymptotic geodesic rays endowed with the cone topology (see for example [Ballmann 1995, Chapter II]). We remark that for all $x \in X$ and all $\xi \in \partial X$ there exists a unique geodesic ray $\sigma_{x,\xi}$ with origin $x = \sigma_{x,\xi}(0)$ representing ξ.

Given two geodesics $\sigma_1 : [0, T_1] \to X$, $\sigma_2 : [0, T_2] \to X$ with $\sigma_1(0) = \sigma_2(0) =: x$ the *Alexandrov angle* $\angle(\sigma_1, \sigma_2)$ is defined by

$$\angle(\sigma_1, \sigma_2) := \lim_{t_1, t_2 \to 0} \angle_{\bar{x}}\big(\overline{\sigma_1(t_1)}, \overline{\sigma_2(t_2)}\big),$$

where the angle on the right-hand side denotes the angle of a comparison triangle in the Euclidean plane of the triangle with vertices $\sigma_1(t_1)$, x and $\sigma_2(t_2)$ (compare [Bridson and Haefliger 1999, Proposition II.3.1]). By definition, every Alexandrov angle has values in $[0, \pi]$. For $x \in X$, $y, z \in \overline{X} \setminus \{x\}$ the angle $\angle_x(y, z)$ is then defined by

$$\angle_x(y, z) := \angle(\sigma_{x,y}, \sigma_{x,z}). \tag{3}$$

From here on we will require that X is proper; in this case the geometric boundary ∂X is compact and the space X is a dense and open subset of the compact space

$\bar{X} := X \cup \partial X$. Moreover, the action of the isometry group $\mathrm{Is}(X)$ on X naturally extends to an action by homeomorphisms on the geometric boundary.

If $x, y \in X$, $\xi \in \partial X$ and σ is a geodesic ray in the class of ξ, we set

$$\mathcal{B}_\xi(x, y) := \lim_{s\to\infty}\big(d(x, \sigma(s)) - d(y, \sigma(s))\big). \tag{4}$$

This number exists, is independent of the chosen ray σ, and the function

$$\mathcal{B}_\xi(\cdot, y) : X \to \mathbb{R}, \quad x \mapsto \mathcal{B}_\xi(x, y)$$

is called the *Busemann function* centered at ξ based at y (see also [Ballmann 1995, Chapter II]). Obviously we have

$$\mathcal{B}_{g\cdot\xi}(g\cdot x, g\cdot y) = \mathcal{B}_\xi(x, y) \quad \text{for all } x, y \in X \text{ and } g \in \mathrm{Is}(X),$$

and the cocycle identity

$$\mathcal{B}_\xi(x, z) = \mathcal{B}_\xi(x, y) + \mathcal{B}_\xi(y, z) \tag{5}$$

holds for all $x, y, z \in X$.

Since X is non-Riemannian in general, we consider (as a substitute of the unit tangent bundle SX) the set of parametrized geodesic lines in X which we will denote $\mathcal{G}$. We endow this set with the distance function d_1 given by

$$d_1(u, v) := \sup\{\mathrm{e}^{-|t|} d\big(v(t), u(t)\big) : t \in \mathbb{R}\} \quad \text{for } u, v \in \mathcal{G}; \tag{6}$$

this distance function induces the compact-open topology, and every isometry of X naturally extends to an isometry of the metric space $(\mathcal{G}, d_1)$.

Moreover, there is a natural map $p : \mathcal{G} \to X$ defined as follows: To a geodesic line $v : \mathbb{R} \to X$ in $\mathcal{G}$ we assign its origin $pv := v(0) \in X$. Notice that p is proper, 1-Lipschitz and $\mathrm{Is}(X)$-equivariant; if X is geodesically complete, then p is surjective.

For a geodesic line $v \in \mathcal{G}$ we denote its extremities $v^- := v(-\infty) \in \partial X$ and $v^+ := v(+\infty) \in \partial X$ the negative and positive end point of v; in particular, we can define the end point map

$$\partial_\infty : \mathcal{G} \to \partial X \times \partial X, \quad v \mapsto (v^-, v^+).$$

For $v \in \mathcal{G}$ we define the parametrized geodesic $-v \in \mathcal{G}$ by

$$(-v)(t) := v(-t) \quad \text{for all } t \in \mathbb{R}.$$

We say that a point $\xi \in \partial X$ can be joined to $\eta \in \partial X$ by a geodesic $v \in \mathcal{G}$ if $v^- = \xi$ and $v^+ = \eta$. Obviously the set of pairs $(\xi, \eta) \in \partial X \times \partial X$ such that ξ and η can be joined by a geodesic coincides with $\partial_\infty\mathcal{G}$, the image of $\mathcal{G}$ under the end point map ∂_∞. It is well-known that if X is CAT(-1), then any pair of distinct boundary points (ξ, η) belongs to $\partial_\infty\mathcal{G}$, and the geodesic joining ξ to η is unique up to reparametrization. In general however, the set $\partial_\infty\mathcal{G}$ is much smaller compared to $\partial X \times \partial X$ minus the diagonal due to the possible existence of flat subspaces in X. For $(\xi, \eta) \in \partial_\infty \mathcal{G}$ we denote by

$$(\xi\eta) := p\big(\{v \in \mathcal{G} : v^- = \xi,\ v^+ = \eta\}\big) = p \circ \partial_\infty^{-1}(\xi, \eta) \tag{7}$$

the subset of points in X which lie on a geodesic line joining ξ to η. It is well-known that $(\xi\eta) = (\eta\xi) \subset X$ is a closed and convex subset of X which is isometric to a product $C_{(\xi\eta)} \times \mathbb{R}$, where $C_{(\xi\eta)} = C_{(\eta\xi)}$ is again a closed and convex set.

For $x \in X$ and $(\xi, \eta) \in \partial_\infty \mathcal{G}$ we denote

$$v = v(x; \xi, \eta) \in \mathcal{G} \tag{8}$$

the unique parametrized geodesic line satisfying the conditions $v \in \partial_\infty^{-1}(\xi, \eta)$ and $d(x, v(0)) = d(x, (\xi\eta))$. Notice that its origin $pv = v(0)$ is precisely the orthogonal projection onto the closed and convex subset $C_{(\xi\eta)}$. Obviously we have

$$v(x; \eta, \xi) = -v(x; \xi, \eta) \quad \text{and} \quad \gamma v(x; \xi, \eta) = v(\gamma x; \gamma\xi, \gamma\eta) \quad \text{for all } \gamma \in \mathrm{Is}(X).$$

In order to describe the sets $(\xi\eta)$ and $C_{(\xi\eta)}$ more precisely and for later use we introduce, as in [Ricks 2017, Definition 5.4], for $x \in X$ the so-called *Hopf parametrization* map

$$H_x : \mathcal{G} \to \partial_\infty \mathcal{G} \times \mathbb{R}, \qquad v \mapsto \big(v^-, v^+, \mathcal{B}_{v^-}(v(0), x)\big) \tag{9}$$

of $\mathcal{G}$ with respect to x. We remark that compared to [Ricks 2017, Definition 5.4] and (5) in [Link 2018] we changed the sign in the last coordinate in order to make (13) below consistent. It is immediate that for a CAT(-1)-space X this map is a homeomorphism; in general it is only continuous and surjective. Moreover, it depends on the point $x \in X$ as follows: If $y \in X$, $v \in \mathcal{G}$ and $H_x(v) = (\xi, \eta, s)$, then

$$H_y(v) = \big(\xi, \eta, s + \mathcal{B}_\xi(x, y)\big)$$

by the cocycle identity (5) for the Busemann function (compare also [Coornaert and Papadopoulos 1994, Section 3]).

The Hopf parametrization map allows to define an equivalence relation $\sim$ on $\mathcal{G}$ as follows: if $u, v \in \mathcal{G}$, then $u \sim v$ if and only if $H_x(u) = H_x(v)$. Notice that this definition does not depend on the choice of $x \in X$ and that every point $(\xi, \eta, s) \in \partial_\infty \mathcal{G} \times \mathbb{R}$ uniquely determines an equivalence class $[v]$ with $v \in \mathcal{G}$. The *width* of $v \in \mathcal{G}$ is defined by

$$\mathrm{width}(v) := \sup\{d(u(0), w(0)) : u, w \in [v]\} = \mathrm{diam}(C_{(v^- v^+)}). \tag{10}$$

Notice that if X is CAT(-1) then for all $v \in \mathcal{G}$ we have $[v] = \{v\}$ and hence $\mathrm{width}(v) = 0$; in general, if $v(\mathbb{R})$ is contained in an isometric image of a Euclidean plane, then the width of v is infinite.

This motivates the following definitions: A geodesic line $v \in \mathcal{G}$ is called *rank one* if its width is finite; it is said to have zero width if $\mathrm{width}(v) = 0$. In the sequel we will use as in [Ricks 2017] the notation

$$\mathcal{R} := \{v \in \mathcal{G} : v \text{ is rank one}\}, \quad \text{respectively}$$
$$\mathcal{Z} := \{v \in \mathcal{G} : v \text{ is rank one of zero width}\}.$$

We remark that the existence of a rank one geodesic imposes severe restrictions on the Hadamard space X. For example, X can neither be a symmetric space or Euclidean building of higher rank nor a product of Hadamard spaces.

The following important lemma states that even though we cannot join any two distinct points in the geometric boundary ∂X of the Hadamard space X by a geodesic in X, given a rank one geodesic we can at least join all points in a neighborhood of its end points by a geodesic in X. More precisely, we have the following result:

Lemma 2.1 ([Ballmann 1995, Lemma III.3.1] reformulated). *Let $v \in \mathcal{R}$ be a rank one geodesic and $c >$ $\mathrm{width}(v)$. Then there exist open disjoint neighborhoods U^- of v^- and U^+ of v^+ in $\overline{X}$ with the following properties: If $\xi \in U^-$ and $\eta \in U^+$ then there exists a rank one geodesic joining ξ and η. For any such geodesic $w \in \mathcal{R}$ we have $d(w(t), v(0)) < c$ for some $t \in \mathbb{R}$ and* $\mathrm{width}(w) \le 2c$.

This lemma implies that the set $\mathcal{R}$ is open in $\mathcal{G}$; we emphasize that $\mathcal{Z}$ in general need not be an open subset of $\mathcal{G}$: In every open neighborhood of a *zero width* rank one geodesic there may exist a rank one geodesic of arbitrarily small but strictly positive width.

Let us now get back to the Hopf parametrization map defined in (9): As stated in [Ricks 2017, Proposition 5.10] the $\mathrm{Is}(X)$-action on $\mathcal{G}$ descends to an action on $\partial_\infty \mathcal{G} \times \mathbb{R} = H_x(\mathcal{G})$ by homeomorphisms via

$$\gamma(\xi, \eta, s) := \big(\gamma\xi, \gamma\eta, s + \mathcal{B}_{\gamma\xi}(\gamma x, x)\big) \quad \text{for } \gamma \in \mathrm{Is}(X).$$

Moreover, the action of $\mathrm{Is}(X)$ is well-defined on the set of equivalence classes $[\mathcal{G}]$ of elements in $\mathcal{G}$, and the (well-defined) map

$$[\mathcal{G}] \to \partial_\infty \mathcal{G} \times \mathbb{R}, \quad [v] \mapsto H_x(v) \tag{11}$$

is an $\mathrm{Is}(X)$-equivariant homeomorphism. For convenience we will frequently identify $\partial_\infty \mathcal{G} \times \mathbb{R}$ with $[\mathcal{G}]$. We also remark that the end point map $\partial_\infty : \mathcal{G} \to \partial X \times \partial X$ induces a well-defined map $[\mathcal{G}] \to \partial X \times \partial X$ which we will also denote ∂_∞.

As in Definition 5.4 of [Ricks 2017] we will say that a sequence $(v_n) \subset \mathcal{G}$ *converges weakly* to $v \in \mathcal{G}$ if and only if

$$v_n^- \to v^-, \quad v_n^+ \to v^+ \quad \text{and} \quad \mathcal{B}_{v_n^-}(v_n(0), x) \to \mathcal{B}_{v^-}(v(0), x); \tag{12}$$

notice that this definition is independent of the choice of $x \in X$. Obviously, weak convergence $v_n \to v$ is equivalent to the convergence $[v_n] \to [v]$ in $[\mathcal{G}]$, and $v_n \to v$ in $\mathcal{G}$ always implies $[v_n] \to [v]$ in $[\mathcal{G}]$.

We will also need the following result due to Ricks, which implies that the restriction of the Hopf parametrization map (9) to the subset $\mathcal{R}$ is proper:

Lemma 2.2 [Ricks 2017, Lemma 5.9]. *If a sequence $(v_n) \subset \mathcal{G}$ converges weakly to $\underline{v \in \mathcal{R}}$, then some subsequence of (v_n) converges to some $u \in \mathcal{G}$ with $u \sim v$.*

The topological space $\mathcal{G}$ can be endowed with the *geodesic flow* $(g^t)_{t\in\mathbb{R}}$ which is naturally defined by reparametrization of $v \in \mathcal{G}$. In particular we have

$$(g^t v)(0) = v(t) \quad \text{for all } v \in \mathcal{G} \text{ and } t \in \mathbb{R}.$$

The geodesic flow induces a flow on the set of equivalence classes $[\mathcal{G}]$ which we will also denote $(g^t)_{t\in\mathbb{R}}$; via the $\mathrm{Is}(X)$-equivariant homeomorphism $[\mathcal{G}] \to \partial_\infty\mathcal{G} \times \mathbb{R}$ the action of the geodesic flow $(g^t)_{t\in\mathbb{R}}$ on $[\mathcal{G}]$ is equivalent to the translation action on the last factor of $\partial_\infty\mathcal{G} \times \mathbb{R}$ given by

$$g^t(\xi, \eta, s) := (\xi, \eta, s + t). \tag{13}$$

3. Rank one isometries and rank one groups

As in the previous section we let (X, d) be a proper Hadamard space and denote $\mathrm{Is}(X)$ the isometry group of X.

Definition 3.1. An isometry $\gamma \in \mathrm{Is}(X)$ is called *axial* if there exists a constant $\ell = \ell(\gamma) > 0$ and a geodesic $v \in \mathcal{G}$ such that $\gamma v = g^\ell v$. We call $\ell(\gamma)$ the *translation length* of γ, and v an *invariant geodesic* of γ. The boundary point $\gamma^+ := v^+$ (which is independent of the chosen invariant geodesic v) is called the *attractive fixed point*, and $\gamma^- := v^-$ the *repulsive fixed point* of γ.

An axial isometry h is called *rank one* if one (and hence any) invariant geodesic of h belongs to $\mathcal{R}$; the *width* of h is then defined as the width of an arbitrary invariant geodesic of h.

Notice that if $\gamma \in \mathrm{Is}(X)$ is axial, then $\partial_\infty^{-1}(\gamma^-, \gamma^+) \subset \mathcal{G}$ is the set of parametrized invariant geodesics of γ, and every axial isometry $\widetilde{\gamma}$ commuting with γ satisfies

$$p \circ \partial_\infty^{-1}(\widetilde{\gamma}^-, \widetilde{\gamma}^+) = p \circ \partial_\infty^{-1}(\gamma^-, \gamma^+).$$

If h is rank one, then the fixed point set of h equals $\{h^-, h^+\}$; moreover, if g is an axial isometry commuting with h, then g and h clearly generate a virtually cyclic subgroup of $\mathrm{Is}(X)$.

The following important lemma describes the north-south dynamics of rank one isometries:

Lemma 3.2 [Ballmann 1995, Lemma III.3.3]. *Let h be a rank one isometry. Then*

(a) *every point $\xi \in \partial X \setminus \{h^+\}$ can be joined to h^+ by a geodesic, and all these geodesics are rank one,*

(b) *given neighborhoods U^- of h^- and U^+ of h^+ in $\bar{X}$ there exists $N \in \mathbb{N}$ such that $h^{-n}(\bar{X} \setminus U^+) \subset U^-$ and $h^n(\bar{X} \setminus U^-) \subset U^+$ for all $n \geq N$.*

We next prepare for an extension of part (a) of the lemma above, which replaces the fixed point h^+ of the rank one isometry h by the end point of a certain geodesic:

Definition 3.3 (compare Section 5 in [Ricks 2017]). Let $G < \mathrm{Is}(X)$ be any subgroup. An element $v \in \mathcal{G}$ is said to *(weakly) G-accumulate* on $u \in \mathcal{G}$ if there exist sequences $(g_n) \subset G$ and $(t_n) \nearrow \infty$ such that $g_n g^{t_n} v$ converges (weakly) to u as $n \to \infty$; v is said to be *(weakly) G-recurrent* if v (weakly) G-accumulates on v.

Notice that if v is an invariant geodesic of an axial isometry $\gamma \in \mathrm{Is}(X)$, then v is $\langle\gamma\rangle$-recurrent and hence in particular $\mathrm{Is}(X)$-recurrent. Moreover, if $v \in \mathcal{G}$ weakly G-accumulates on $\underline{u \in \mathcal{R}}$, then by Lemma 2.2 v G-accumulates on some element $w \sim u$. In particular, if $\underline{v \in \mathcal{Z}}$ is weakly G-recurrent, then it is already G-recurrent.

The following statements show the relevance of the previous notions.

Lemma 3.4 (see Section 6 in [Ricks 2017] or Lemma 3.11 in [Link 2018]). *If $v \in \mathcal{R}$ is weakly $\mathrm{Is}(X)$-recurrent then for every $\xi \in \partial X \setminus \{v^+\}$ there exists $w \in \mathcal{R}$ satisfying*

$$\mathrm{width}(w) \leq \mathrm{width}(v) \quad \textit{and} \quad (w^-, w^+) = (\xi, v^+).$$

We will also need the following generalization of a statement originally due to G. Knieper in the manifold setting; recall the definition of the distance function d_1 from (6).

Lemma 3.5 (Lemma 7.1 in [Link 2018] or Proposition 4.1 in [Knieper 1998]). *Let $u \in \mathcal{Z}$ be an $\mathrm{Is}(X)$-recurrent rank one geodesic of zero width. Then for all $v \in \mathcal{G}$ with $v^+ = u^+$ and $\mathcal{B}_{v^+}(v(0), u(0)) = 0$ we have*

$$\lim_{t\to\infty} d_1(g^t v, g^t u) = 0.$$

We will now deal with discrete subgroups Γ of the isometry group $\mathrm{Is}(X)$ of X. The *geometric limit set* L_Γ of Γ is defined by $L_\Gamma := \overline{\Gamma \cdot x} \cap \partial X$, where $x \in X$ is an arbitrary point.

If X is a CAT(-1)-space, then a discrete group $\Gamma < \mathrm{Is}(X)$ is called *nonelementary* if its limit set L_Γ is infinite. It is well-known that this implies that Γ contains two axial isometries with disjoint fixed point sets (which are actually rank one of zero width as $\mathcal{G} = \mathcal{Z}$ for any CAT(-1)-space). In the general setting this motivates the following

Definition 3.6. We say that two rank one isometries $g, h \in \mathrm{Is}(X)$ are *independent* if and only if $\{g^+, g^-\} \cap \{h^+, h^-\} \neq \emptyset$ (see for example [Link 2010, Section 2; Caprace and Fujiwara 2010, Section 2]). Moreover, a group $\Gamma < \mathrm{Is}(X)$ is called *rank one* if Γ contains a pair of independent rank one elements.

Obviously, if X is CAT(-1) then every nonelementary discrete isometry group is rank one. In general however, the notion of rank one group seems very restrictive at first sight. Nevertheless we have the following weak hypothesis which ensures that a discrete group $\Gamma < \mathrm{Is}(X)$ is rank one:

Lemma 3.7 [Link 2018, Lemma 4.4]. *If $\Gamma < \mathrm{Is}(X)$ is a discrete subgroup with infinite limit set L_Γ containing the positive end point v^+ of a weakly* $\mathrm{Is}(X)$*-recurrent element $v \in \mathcal{R}$, then Γ is a rank one group.*

Notice that the conclusion is obviously true when v^+ is a fixed point of a rank one isometry of X.

We will now define an important subset of the limit set L_Γ of Γ. For that we let $x, y \in X$ arbitrary. A point $\xi \in \partial X$ is called a *radial limit point* if there exists $c > 0$ and sequences $(\gamma_n) \subset \Gamma$ and $(t_n) \nearrow \infty$ such that

$$d\big(\gamma_n y, \sigma_{x,\xi}(t_n)\big) \le c \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

Notice that by the triangle inequality this condition is independent of the choice of $x, y \in X$. The *radial limit set* $L_\Gamma^{\mathrm{rad}} \subset L_\Gamma$ of Γ is defined as the set of radial limit points.

We will further denote

$$\mathcal{Z}_\Gamma^{\mathrm{rec}} := \{v \in \mathcal{Z} : v \text{ and } -v \text{ are } \Gamma\text{-recurrent}\} \tag{15}$$

the set of zero width parametrized geodesics which are Γ-recurrent in both directions. Notice that if $v \in \mathcal{Z}$ is weakly Γ-recurrent, then it is already Γ-recurrent according to the remark below Definition 3.3. We will also need the following:

Definition 3.8. An element $v \in \Gamma\backslash\mathcal{G}$ is called *positively and negatively recurrent*, if it possesses a lift $\tilde{v} \in \mathcal{G}$ such that both $\tilde{v}$ and $-\tilde{v}$ are Γ-recurrent.

4. Geodesic currents and the Ricks measure

In this section we want to describe the construction of the Ricks measure from an arbitrary geodesic current on $\partial_\infty\mathcal{R}$. We will also recall the properties of the Ricks measure which are relevant for our purposes. Our main references here are [Ricks 2017, Section 7; Link 2018, Section 5].

From here on X will always be a proper Hadamard space and $\Gamma < \mathrm{Is}(X)$ a discrete rank one group with

$$\mathcal{Z}_\Gamma := \{v \in \mathcal{Z} : v^-, v^+ \in L_\Gamma\} \neq \emptyset.$$

Notice that according to Proposition 1 in [Link 2018] the latter hypothesis is always satisfied when X is geodesically complete. For later use we further fix a point $o \in X$.

Recall that the support of a Borel measure ν on a topological space Y is defined as the set

$$\operatorname{supp}(\nu) = \big\{ y \in Y : \nu(U) > 0 \text{ for every open neighborhood } U \text{ of } y \big\}. \tag{16}$$

We also recall that a set $A \subset Y$ is said to have full ν-measure, if $\nu(Y \setminus A) = 0$.

We start with two finite Borel measures μ_-, μ_+ on ∂X with $\operatorname{supp}(\mu_\pm) = L_\Gamma$, and let $\bar\mu$ be a Γ-invariant Radon measure on $\partial_\infty\mathcal{R}$ which is absolutely continuous with respect to the product measure $(\mu_- \otimes \mu_+)|_{\partial_\infty\mathcal{R}}$. Such $\bar\mu$ is called a *quasiproduct geodesic current* on $\partial_\infty\mathcal{R}$ (see, for example, [Link 2018, Definition 5.2]).

Following Ricks' approach we can define a Radon measure $\overline{m} = \bar\mu \otimes \lambda$ on $[\mathcal{R}] \cong \partial_\infty\mathcal{R} \times \mathbb{R}$, where λ denotes Lebesgue measure on $\mathbb{R}$. Now according to Lemma 2.1 Γ acts properly on $[\mathcal{R}] \cong \partial_\infty\mathcal{R} \times \mathbb{R}$ which admits a proper metric. Since the action is by homeomorphisms and preserves the Borel measure $\overline{m} = \bar\mu \otimes \lambda$, there is (see, for instance, [Ricks 2015, Appendix A]) a unique Borel quotient measure $\overline{m}_\Gamma$ on $\Gamma\backslash[\mathcal{R}]$ satisfying the characterizing property

$$\int_{\bar A} \tilde h \,\mathrm{d}\overline{m} = \int_{\Gamma\backslash[\mathcal{R}]} (h \cdot f_{\bar A}) \,\mathrm{d}\overline{m}_\Gamma \tag{17}$$

for all Borel sets $\bar A \subset [\mathcal{R}]$ and Γ-invariant Borel maps $\tilde h : [\mathcal{R}] \to [0, \infty]$ and $\tilde f_{\bar A} : [\mathcal{R}] \to [0, \infty]$ defined by $\tilde f_{\bar A}([v]) := \#\{\gamma \in \Gamma : \gamma[v] \in \bar A\}$ for $[v] \in \mathcal{R}$, and with h and $f_{\bar A}$ the maps on $\Gamma\backslash[\mathcal{R}]$ induced from $\tilde h$ and $\tilde f_{\bar A}$.

Our final goal is to construct from a weak Ricks measure $\overline{m}_\Gamma$ a geodesic flow invariant measure on $\Gamma\backslash\mathcal{G}$. So let us first remark that $\mathcal{Z} \subset \mathcal{R}$ is a Borel subset by semicontinuity of the width function (10) (see [Ricks 2017, Lemma 8.4]), and that $H_o|_{\mathcal{Z}} : \mathcal{Z} \to \partial_\infty\mathcal{Z} \times \mathbb{R} \cong [\mathcal{Z}]$ is a homeomorphism onto its image; hence $[\mathcal{Z}] \subset [\mathcal{R}]$ is also a Borel subset. So if $\Gamma\backslash[\mathcal{Z}]$ has positive mass with respect to the weak Ricks measure $\overline{m}_\Gamma$ we may define (as in [Ricks 2017, Definition 8.12]) a geodesic flow and Γ-invariant measure m^0 on $\mathcal{G}$ by setting

$$m^0(E) := \overline{m}(H_o(E \cap \mathcal{Z})) \quad \text{for any Borel set } E \subset \mathcal{G}; \tag{18}$$

this measure m^0 then induces the *Ricks measure* m^0_Γ on $\Gamma\backslash\mathcal{G}$.

Notice that in general $\overline{m}_\Gamma(\Gamma\backslash[\mathcal{Z}]) = 0$ is possible; obviously this is always the case when $\mathcal{Z} = \emptyset$. However, Theorem 6.7 and Corollary 2 in [Link 2018] immediately imply:

Theorem 4.1. *Let X be a proper Hadamard space, and $\Gamma < \operatorname{Is}(X)$ a discrete rank one group with $\mathcal{Z}_\Gamma \neq \emptyset$ (which is always the case if X is geodesically complete). Let μ_-, μ_+ be nonatomic, finite Borel measures on ∂X with $\mu_\pm(L_\Gamma^{\mathrm{rad}}) = \mu_\pm(\partial X)$, and $\bar\mu \sim (\mu_- \otimes \mu_+)|_{\partial_\infty\mathcal{R}}$ a quasiproduct geodesic current.*

Then for the set $\mathcal{Z}_\Gamma^{\mathrm{rec}}$ *defined in* (15) *we have*

$$(\mu_-\otimes\mu_+)(\partial_\infty\mathcal{Z}_\Gamma^{\mathrm{rec}})=(\mu_-\otimes\mu_+)(\partial_\infty\mathcal{R})=\mu_-(\partial X)\cdot\mu_+(\partial X),$$

and in particular $\bar\mu\sim\mu_-\otimes\mu_+$.

So in this case the Ricks measure m_Γ^0 is actually equal to the weak Ricks measure $\overline{m}_\Gamma$ used for its construction. Moreover, for the measure m on $\mathcal{G}$, from which the Ricks measure descends, we have the formula

$$m(E)=\int_{\partial_\infty\mathcal{Z}}\lambda(p(E)\cap(\xi\eta))\,\mathrm{d}\bar\mu(\xi,\eta),\tag{19}$$

where λ again denotes Lebesgue measure, and $E\subset\mathcal{G}$ is an arbitrary Borel set. We further remark that if X is a manifold, then the Ricks measure is also equal to Knieper's measure m_Γ^{Kn} associated to $\bar\mu$ which descends from

$$m^{\mathrm{Kn}}(E):=\int_{\partial_\infty\mathcal{G}}\mathrm{vol}_{(\xi\eta)}(p(E)\cap(\xi\eta))\,\mathrm{d}\bar\mu(\xi,\eta)\quad\text{for any Borel set } E\subset\mathcal{G},$$

where $\mathrm{vol}_{(\xi\eta)}$ denotes the induced Riemannian volume element on the submanifold $(\xi\eta)\subset X$.

From here on we will therefore denote the Ricks measure m_Γ instead of m_Γ^0.

For later reference we want to summarize what we know from Theorem 7.4 and Lemma 7.5 in [Link 2018]. Before we can state the result we denote $\mathcal{B}(R)\subset\mathcal{G}$ the set of all parametrized geodesics $v\in\mathcal{G}$ with origin $pv=v(0)\in B_R(o)$ and define

$$\Delta:=\sup\left\{\frac{\ln\bar\mu(\partial_\infty\mathcal{B}(R))}{R}:R>0\right\}.\tag{20}$$

Theorem 4.2. *Let* X, $\Gamma<\mathrm{Is}(X)$, μ_-, μ_+ *and* $\bar\mu$ *as in Theorem 4.1. We further assume that the constant* Δ *defined via* (20) *is finite. Then the dynamical systems* $(\partial X\times\partial X,\Gamma,\mu_-\otimes\mu_+)$, $(\partial_\infty\mathcal{G},\Gamma,\bar\mu)$ *and* $(\Gamma\backslash\mathcal{G},g_\Gamma,m_\Gamma)$ *are ergodic.*

We will repeatedly make use of the following argument, which is immediate by Fubini's theorem:

Corollary 4.3. *Let* $X,\Gamma<\mathrm{Is}(X)$, μ_-, μ_+, $\bar\mu$ *and* $\Delta<\infty$ *as in Theorem 4.2. Then if* $\Omega\subset\Gamma\backslash\mathcal{Z}$ *is a subset of full* m_Γ*-measure, and* $\widetilde\Omega\subset\mathcal{Z}$ *the preimage of* Ω *under the projection map* $\mathcal{Z}\mapsto\Gamma\backslash\mathcal{Z}$, *the sets*

$$E^-:=\left\{\xi\in\partial X:(\xi,\eta')\in\partial_\infty\widetilde\Omega\ \textit{for } \mu^+\textit{-almost every } \eta'\in\partial X\right\}\quad\textit{and}$$
$$E^+:=\left\{\eta\in\partial X:(\xi',\eta)\in\partial_\infty\widetilde\Omega\ \textit{for } \mu^-\textit{-almost every } \xi'\in\partial X\right\}$$

satisfy $\mu_-(E^-)=\mu_-(\partial X)$ *and* $\mu_+(E^+)=\mu_+(\partial X)$.

Before proving the important Lemma 4.4 we want to recall a few notions from topology and geometric group theory: If Y is a topological space, then a collection of subsets of Y is said to be *locally finite* if every $y \in Y$ has an open neighborhood that intersects only finitely many sets in the collection. Notice that if the collection $\{U_\lambda : \lambda \in \Lambda\} \subset Y$ (with Λ a countable set) is locally finite, then the collection of the closures $\{\overline{U_\lambda} : \lambda \in \Lambda\} \subset Y$ is also locally finite. Moreover, for the closure of the countable union $\bigcup_{\lambda\in\Lambda} U_\lambda$ we have

$$\overline{\bigcup_{\lambda\in\Lambda} U_\lambda} = \bigcup_{\lambda\in\Lambda} \overline{U_\lambda}. \tag{21}$$

Indeed, if $(y_n) \subset \bigcup_{\lambda\in\Lambda} U_\lambda$ is a sequence converging to a point $y \in Y$, we let $U \subset Y$ be an open neighborhood of y such that $U \cap U_\lambda = \varnothing$ for all but finitely many $\lambda \in \Lambda$; denote the finite set of exceptions by

$$F := \{\lambda \in \Lambda : U \cap U_\lambda \neq \varnothing\}.$$

Then for n sufficiently large we have

$$y_n \in U \cap \bigcup_{\lambda\in\Lambda} U_\lambda \subset \bigcup_{\lambda\in F} U_\lambda, \quad \text{hence} \quad y \in \overline{\bigcup_{\lambda\in F} U_\lambda} = \bigcup_{\lambda\in F} \overline{U_\lambda} \subset \bigcup_{\lambda\in\Lambda} \overline{U_\lambda}.$$

The converse inclusion is trivial.

Assume now that a discrete group G acts by isometries on a proper metric space Y. An open set $D \subset Y$ is called a *fundamental domain* for the action of G on Y, if

$$Y = \bigcup_{g\in G} g \cdot \bar{D}, \quad \text{and} \quad gD \cap D = \varnothing \quad \text{for all } g \in G \setminus \{e\};$$

it is said to be locally finite (for the action of G on Y) if the collection of sets $\{g \cdot D : g \in G\}$ is locally finite; notice that this is equivalent to the fact that for any compact set $K \subset Y$ the number

$$\#\{g \in G : K \cap g \cdot \bar{D} \neq \varnothing\}$$

is finite.

For our purposes we will need a fundamental domain for the action of Γ on $\mathcal{G}$ whose boundary is negligible with respect to the measure m on $\mathcal{G}$ inducing the Ricks measure on $\Gamma\backslash\mathcal{G}$ (which is defined by (19)):

Lemma 4.4. *Let X, $\Gamma < \mathrm{Is}(X)$, μ_-, μ_+, $\bar{\mu}$ and $\Delta < \infty$ as in Theorem 4.2. Then there exists a Γ-invariant subset $\mathcal{G}' \subset \mathcal{G}$ of full m-measure and a locally finite fundamental domain $\mathcal{D} \subset \mathcal{G}'$ for the action of Γ on $\mathcal{G}'$ which satisfies $m(\partial\mathcal{D}) = 0$.*

Proof. We denote

$$\mathcal{F} := \left\{ v \in \mathcal{G} : \gamma v = v \text{ for some } \gamma \in \Gamma \setminus \{e\} \right\}$$

the set of parametrized geodesics in $\mathcal{G}$ which are fixed by a nontrivial element in Γ. Notice that this set is nonempty only if Γ contains elliptic elements.

Obviously $\mathcal{F}$ is closed, Γ-invariant and invariant by the geodesic flow. Moreover, $\mathcal{F} \cap \mathcal{Z}$ is a proper subset of the support of m. By ergodicity of m_Γ we conclude that $m(\mathcal{F}) = 0$.

Choose a point $x \in X$ with trivial stabilizer in Γ. Let $\mathcal{D}_\Gamma \subset \mathcal{G}$ denote the open *Dirichlet domain* for Γ with center x, that is the set of all parametrized geodesic lines with origin in

$$\big\{z \in X : d(z,x) < d(z,\gamma x) \text{ for all } \gamma \in \Gamma \setminus \{e\}\big\};$$

then by choice of x we have

$$\gamma \mathcal{D}_\Gamma \cap \mathcal{D}_\Gamma = \varnothing \quad \text{for all } \gamma \in \Gamma \setminus \{e\}.$$

Moreover, $\mathcal{D}_\Gamma$ is locally finite as X is proper and Γ is discrete. Notice that in general $\mathcal{D}_\Gamma$ need not be a fundamental domain for the action of Γ on $\mathcal{G}$, because

$$\textstyle\bigcup_{\gamma \in \Gamma} \gamma \overline{\mathcal{D}_\Gamma} \subsetneq \mathcal{G}$$

is possible as the following example provided by the anonymous referee shows: If X is the universal cover of a bouquet of circles of length 1 (that is a regular tree), $\Gamma < \mathrm{Is}(X)$ the group of deck transformations (which does not contain elliptic elements) and $x \in X$ the midpoint of an edge E of X, then the closure of the Dirichlet domain $\mathcal{D}_\Gamma \subset \mathcal{G}$ with center x consists of all parametrized geodesics v with origin $v(0) \in \bar{E}$ and $E \subset v(\mathbb{R})$. But if $w \in \mathcal{G}$ is a parametrized geodesic with $w(\mathbb{R}) \cap \bar{E} = \{w(0)\}$, then $w \notin \bigcup_{\gamma \in \Gamma} \gamma \overline{\mathcal{D}_\Gamma}$.

For this reason we consider the "enlarged boundary"

$$\begin{aligned}\tilde{\partial} \mathcal{D}_\Gamma = \big\{v \in \mathcal{G} : d(v(0), x) = d(v(0), \gamma x) &\text{ for some } \gamma \in \Gamma \setminus \{e\} \\ &\text{and } d(v(0), x) \le d(v(0), \gamma x) \text{ for all } \gamma \in \Gamma\big\} \supset \partial \mathcal{D}_\Gamma\end{aligned}$$

and use the set

$$\widehat{\mathcal{D}_\Gamma} := \mathcal{D}_\Gamma \cup \tilde{\partial} \mathcal{D}_\Gamma$$

instead of the closure $\overline{\mathcal{D}_\Gamma}$ of the Dirichlet domain. Then obviously

$$\textstyle\bigcup_{\gamma \in \Gamma} \gamma \widehat{\mathcal{D}_\Gamma} = \mathcal{G}, \quad \text{and} \quad \mathcal{F} \cap \widehat{\mathcal{D}_\Gamma} \subset \tilde{\partial} \mathcal{D}_\Gamma.$$

However, the problem is that in general the boundary $\partial \mathcal{D}_\Gamma$ of the Dirichlet domain (and also the enlarged boundary $\tilde{\partial} \mathcal{D}_\Gamma$) is very complicated, and in particular $m(\tilde{\partial} \mathcal{D}_\Gamma) \ge m(\partial \mathcal{D}_\Gamma) > 0$ is possible.

In order to get a fundamental domain with boundary of zero m-measure we will therefore modify the Dirichlet domain $\mathcal{D}_\Gamma$ in a neighborhood of the enlarged boundary $\tilde{\partial} \mathcal{D}_\Gamma$ as proposed by Roblin [2003, p. 13]: We first choose a covering of $\tilde{\partial} \mathcal{D}_\Gamma \setminus \mathcal{F}$ by a locally finite family of open sets $\{V_n : n \in \mathbb{N}\} \subset \mathcal{G} \setminus \mathcal{F}$ with a uniform upper bound on the diameter with respect to the distance function d_1 introduced in (6) such that for all $n \in \mathbb{N}$ we have

$$m(\partial V_n) = 0 \quad \text{and} \quad \overline{V_n} \cap \gamma \overline{V_n} = \varnothing \quad \text{for all} \;\; \gamma \in \Gamma \setminus \{e\}.$$

We first claim that the family of subsets $\{\Gamma \cdot V_n : n \in \mathbb{N}\} \subset \mathcal{G} \setminus \mathcal{F}$ is still locally finite. For the proof we choose for each $j \in \mathbb{N}$ a point $v_j \in V_j \cap \tilde{\partial}\mathcal{D}_\Gamma$; there exists $r > 0$ such that $V_j \subset B_r(v_j)$ for all $j \in \mathbb{N}$. Since the map $p : \mathcal{G} \to X,\ v \mapsto v(0)$ is 1-Lipschitz, we also have $pV_j \subset B_r(pv_j)$ for all $j \in \mathbb{N}$. Moreover, $v_j \in \tilde{\partial}\mathcal{D}_\Gamma$ implies that $d(pv_j, \gamma x) \geq d(pv_j, x)$ for all $\gamma \in \Gamma$.

Now assume that the family $\{\Gamma \cdot V_n : n \in \mathbb{N}\} \subset \mathcal{G} \setminus \mathcal{F}$ is not locally finite. Then there exists an open set $U \subset \mathcal{G} \setminus \mathcal{F}$ and infinite sets $\{\gamma_k : k \in \mathbb{N}\} \subset \Gamma,\ \{j_k : k \in \mathbb{N}\} \subset \mathbb{N}$ such that $U \cap \gamma_k V_{j_k} \neq \varnothing$ for all $k \in \mathbb{N}$. Let $R > 0$ such that $pU \subset B_R(x)$, where x is the center of the Dirichlet domain. For $k \in \mathbb{N}$ we pick $u_k \in U \cap \gamma_k V_{j_k}$; passing to a subsequence if necessary we can assume that (u_k) converges to a point $u \in \overline{U}$. Since Γ is discrete and $\{\gamma_k : k \in \mathbb{N}\}$ is infinite, we know that $d(x, \gamma_k^{-1} pu) \to \infty$ as $k \to \infty$, hence for all k sufficiently large we have $\gamma_k^{-1} pu_k \notin \overline{B_{R+2r}(x)}$.

Let $k \in \mathbb{N}$ such that $d(x, \gamma_k^{-1} pu_k) > R + 2r$. Notice that $u_k \in \gamma_k V_{j_k} \subset \gamma_k B_r(v_{j_k})$ implies $d(\gamma_k^{-1} pu_k, pv_{j_k}) < r$ and hence

$$d(x, pv_{j_k}) \geq d(x, \gamma_k^{-1} pu_k) - d(pv_{j_k}, \gamma_k^{-1} pu_k) > R + 2r - r = R + r.$$

By choice of $v_{j_k} \subset \tilde{\partial}\mathcal{D}_\Gamma$ we further know that $d(pv_{j_k}, \gamma x) \geq d(pv_{j_k}, x)$ for all $\gamma \in \Gamma$, hence in particular

$$d(x, \gamma_k pv_{j_k}) \geq d(x, pv_{j_k}) > R + r,$$

and therefore

$$d(x, pu_k) \geq d(x, \gamma_k pv_{j_k}) - d(pu_k, \gamma_k pv_{j_k}) > d(x, pv_{j_k}) - r > R;$$

this is an obvious contradiction to $pU \subset B_R(x)$.

We are now going to construct the desired fundamental domain. We start with the Dirichlet domain $\mathcal{D}_0 := \mathcal{D}_\Gamma$ from above and set $\mathcal{D}_1 := \big(\mathcal{D}_0 \setminus \Gamma \cdot \overline{V_1}\big) \cup V_1 \subset \mathcal{G} \setminus \mathcal{F}$. This set is open as a union of two open sets, and it is still locally finite for the action of Γ on $\mathcal{G} \setminus \mathcal{F}$; obviously we have $\gamma \cdot \mathcal{D}_1 \cap \mathcal{D}_1 = \varnothing$ for all $\gamma \in \Gamma \setminus \{e\}$. Hence defining $\mathcal{D}_n := (\mathcal{D}_{n-1} \setminus \Gamma \cdot \overline{V_n}) \cup V_n$ for $n \in \mathbb{N}$, we get a sequence of open subsets of $\mathcal{G} \setminus \mathcal{F}$ each of which is locally finite for the action of Γ on $\mathcal{G} \setminus \mathcal{F}$. The limit of this sequence exists and equals

$$\mathcal{D} = \big(\mathcal{D}_0 \setminus \cup_{i=1}^{\infty} \Gamma \cdot \overline{V_i}\big) \sqcup \bigcup_{j=1}^{\infty} \big(V_j \setminus \cup_{i>j} \Gamma \cdot \overline{V_i}\big).$$

We claim that $\mathcal{D}$ is a locally finite fundamental domain for the action of Γ on $\mathcal{G} \setminus \mathcal{F}$, but now with boundary $\partial\mathcal{D}$ of m-measure zero as it is contained in

$$\bigcup_{j=1}^{\infty} \Gamma \cdot \partial V_j \cup \mathcal{F}.$$

We first show that $\mathcal{G}\setminus\mathcal{F}\subset\Gamma\cdot\overline{\mathcal{D}}$: So let $v\in\mathcal{G}\setminus\mathcal{F}$ arbitrary. As $\bigcup_{\gamma\in\Gamma}\gamma\cdot\widehat{\mathcal{D}_0}=\mathcal{G}$ we may assume without loss of generality that $v\in\widehat{\mathcal{D}_0}$. For $v\in\mathcal{D}_0\setminus\bigcup_{i=1}^\infty\Gamma\cdot\overline{V_i}\subset\mathcal{D}$ we are done, so let

$$v\in\tilde\partial\mathcal{D}_0\cup\bigcup_{i=1}^\infty\Gamma\cdot\overline{V_i}\ \subset\ \bigcup_{i=1}^\infty\Gamma\cdot\overline{V_i}$$

(as $\{V_n : n\in\mathbb{N}\}$ is an open covering of $\tilde\partial\mathcal{D}_0\setminus\mathcal{F}$). Let $\ell\in\mathbb{N}$ be the largest integer such that $v\in\Gamma\cdot\overline{V_\ell}$; such ℓ exists by local finiteness of the family $\{\Gamma\cdot\overline{V_n} : n\in\mathbb{N}\}$. Hence for some $\gamma\in\Gamma$ we have $v\in\gamma\overline{V_\ell}\setminus\bigcup_{i>\ell}\Gamma\cdot\overline{V_i}\subset\gamma\overline{\mathcal{D}}$, which proves the claim.

We next show that $\mathcal{D}$ is locally finite for the action of Γ on $\mathcal{G}\setminus\mathcal{F}$. Notice that $\mathcal{D}\subset\mathcal{D}_0\cup\bigcup_{j=1}^\infty V_j$; as $\mathcal{D}_0$ is locally finite it suffices to prove that the collection of sets $\bigl\{\gamma\cdot\bigcup_{j=1}^\infty V_j : \gamma\in\Gamma\bigr\}\subset\mathcal{G}\setminus\mathcal{F}$ is locally finite. But this follows directly from the local finiteness of the family of sets $\{\Gamma\cdot V_n : n\in\mathbb{N}\}\subset\mathcal{G}\setminus\mathcal{F}$.

We finally show that $\partial\mathcal{D}\subset\bigcup_{j=1}^\infty\Gamma\cdot\partial V_j\cup\mathcal{F}$. As

$$\partial\mathcal{D}\subset\partial\bigl(\mathcal{D}_0\setminus\bigcup_{i=1}^\infty\Gamma\cdot\overline{V_i}\bigr)\cup\partial\Bigl(\bigcup_{j=1}^\infty\bigl(V_j\setminus\bigcup_{i>j}\Gamma\cdot\overline{V_i}\bigr)\Bigr)\cup\mathcal{F},$$

the claim will follow from the inclusion

$$\partial\bigl(\bigcup_{i=1}^\infty\Gamma\cdot V_i\bigr)\cap\mathcal{G}\setminus\mathcal{F}\ \subset\ \bigcup_{i=1}^\infty\Gamma\cdot\partial V_i.$$

But $v\in\partial\bigl(\bigcup_{i=1}^\infty\Gamma\cdot V_i\bigr)\cap\mathcal{G}\setminus\mathcal{F}$ implies $v\notin\bigcup_{i=1}^\infty\Gamma\cdot V_i$ and

$$v\in\overline{\bigcup_{i=1}^\infty\Gamma\cdot V_i}\cap\mathcal{G}\setminus\mathcal{F}\ \subset\ \bigcup_{i=1}^\infty\Gamma\cdot\overline{V_i}$$

according to (21), hence the assertion is true. □

5. Mixing of the Ricks measure

Let X be a proper Hadamard space as before, and $\Gamma<\mathrm{Is}(X)$ a discrete rank one group with $\mathcal{Z}_\Gamma\neq\varnothing$. Notice that if X is geodesically complete, then according to Proposition 1 in [Link 2018] the latter condition is automatically satisfied. We further fix a point $o\in X$.

From here on we will assume that μ_-, μ_+ are nonatomic, finite Borel measures on ∂X with $\mu_\pm(L_\Gamma^{\mathrm{rad}})=\mu_\pm(\partial X)$. We will further require that for the quasiproduct geodesic current $\bar\mu\sim(\mu_-\otimes\mu_+)|_{\partial_\infty\mathcal{R}}$ on $\partial_\infty\mathcal{R}$ the constant Δ defined in (20) is finite.

From Theorem 4.1 and Definition 3.8 we immediately get that the set

$$\bigl\{u\in\Gamma\backslash\mathcal{G} : u \text{ is positively and negatively recurrent}\bigr\}$$

has full m_Γ-measure (which is equivalent to conservativity of the dynamical system $(\partial_\infty\mathcal{G}, g_\Gamma, \bar\mu)$). Moreover, according to Theorem 4.2 the dynamical system $(\partial_\infty\mathcal{G}, g_\Gamma, \bar\mu)$ is ergodic and we can use its Corollary 4.3.

Our proof of mixing will closely follow Babillot's idea [2002]. However, as she only gives the proof for cocompact rank one isometry groups of Hadamard *manifolds*, for the convenience of the reader we want to give a detailed proof in our more general setting, which includes arbitrary discrete rank one isometry groups of non-Riemannian Hadamard spaces. We also emphasize that her set $\mathcal{R}$ in [Babillot 2002] is defined as the set of unit tangent vectors $v \in SX \cong \mathcal{G}$ which do not admit a parallel perpendicular Jacobi field; this is in general a proper open subset of our set $\mathcal{R}$ (which was defined as the set of parametrized geodesic lines with finite width) which is contained in $\mathcal{Z}$. In particular, her Proposition-Definition below Lemma 2 in [Babillot 2002] is not true when considering our set $\mathcal{R}$ instead of hers. We therefore have to work on the set $\mathcal{Z}$ (which is not open in $\mathcal{R}$) and use — up to a constant factor — the cross-ratio introduced by Ricks [2017, Definition 10.2] instead of Babillot's.

From the Busemann function introduced in (4) we first define for $(\xi, \eta) \in \partial_\infty \mathcal{G}$ the *Gromov product* of (ξ, η) with respect to $y \in X$ via

$$\mathrm{Gr}_y(\xi, \eta) = \tfrac{1}{2}\big(\mathcal{B}_\xi(y, z) + \mathcal{B}_\eta(y, z)\big), \tag{22}$$

where $z \in (\xi\eta)$ is an arbitrary point on a geodesic line joining ξ and η. It is related to Ricks' definition following [Ricks 2017, Lemma 5.1] via the formula $\mathrm{Gr}_y(\xi, \eta) = -2\beta_y(\xi, \eta)$ for all $(\xi, \eta) \in \partial_\infty \mathcal{G}$. We then make the following:

Definition 5.1 [Ricks 2017, Definition 10.1]. A quadruple of points $(\xi_1, \xi_2, \xi_3, \xi_4) \in \big(\partial X\big)^4$ is called a *quadrilateral*, if there exist $v_{13}, v_{14}, v_{23}, v_{24} \in \mathcal{R}$ such that

$$\partial_\infty v_{ij} = (\xi_i, \xi_j) \quad \text{for all } (i, j) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}.$$

The set of all quadrilaterals is denoted $\mathcal{Q}$, and we define

$$\mathcal{Q}_\Gamma = \mathcal{Q} \cap \big(L_\Gamma\big)^4.$$

Definition 5.2 (compare [Ricks 2017, Definition 10.2]). If $(\xi, \xi', \eta, \eta') \in \mathcal{Q}$ is a quadrilateral, its *cross-ratio* is defined by

$$\mathrm{CR}(\xi, \xi', \eta, \eta') = \mathrm{Gr}_o(\xi, \eta) + \mathrm{Gr}_o(\xi', \eta') - \mathrm{Gr}_o(\xi, \eta') - \mathrm{Gr}_o(\xi', \eta).$$

Notice that our definition corresponds to Ricks' via

$$\mathrm{CR}(\xi, \xi', \eta, \eta') = -2\mathrm{B}(\xi, \xi', \eta, \eta').$$

The properties of a cross-ratio listed in Proposition 10.5 of [Ricks 2017] are therefore satisfied for our cross-ratio CR. We further have:

Lemma 5.3 [Ricks 2017, Lemma 10.6]. *If $g \in \mathrm{Is}(X)$ is axial, then its translation length $\ell(g)$ is given by*

$$\ell(g) = \mathrm{CR}(g^-, g^+, \xi, g\xi).$$

From this we immediately get:

Proposition 5.4. *The length spectrum* $\{\ell(\gamma) : \gamma \in \Gamma$ *axial*$\}$ *of* Γ *is a subset of the cross-ratio spectrum* $\mathrm{CR}(\mathcal{Q}_\Gamma)$.

Theorem 5.5. *Let* $\Gamma < \mathrm{Is}(X)$ *be a discrete rank one group with nonarithmetic length spectrum and* $\mathcal{Z}_\Gamma \neq \varnothing$. *Let* μ_-, μ_+ *be nonatomic finite Borel measures on* ∂X *with* $\mu_\pm(L_\Gamma^{\mathrm{rad}}) = \mu_\pm(\partial X)$, *and*

$$\bar{\mu} \sim (\mu_- \otimes \mu_+)|_{\partial_\infty \mathcal{R}}$$

a quasiproduct geodesic current defined on $\partial_\infty \mathcal{R}$ *for which the constant* Δ *defined by* (20) *is finite. Let* m_Γ *be the associated Ricks measure on* $\Gamma\backslash\mathcal{G}$. *Then the dynamical system* $(\Gamma\backslash\mathcal{G}, g_\Gamma, m_\Gamma)$ *is mixing, that is for all Borel sets* $A, B \subset \Gamma\backslash\mathcal{G}$ *with* $m_\Gamma(A)$ *and* $m_\Gamma(B)$ *finite we have* (*with the abbreviation* $\|m_\Gamma\| = m_\Gamma(\Gamma\backslash\mathcal{G})$)

$$\lim_{t\to\pm\infty} m_\Gamma(A \cap g_\Gamma^{-t} B) = \begin{cases} \dfrac{m_\Gamma(A)\cdot m_\Gamma(B)}{\|m_\Gamma\|} & \text{if } m_\Gamma \text{ is finite,} \\ 0 & \text{if } m_\Gamma \text{ is infinite.} \end{cases}$$

Proof. We first remark that mixing is equivalent to the fact that for every square integrable function $\varphi \in \mathrm{L}^2(m_\Gamma)$ on $\Gamma\backslash\mathcal{G}$ the functions $\varphi \circ g_\Gamma^t$ converge weakly in $\mathrm{L}^2(m_\Gamma)$ to the constant

$$\frac{1}{\|m_\Gamma\|}\int \varphi \, \mathrm{d}m_\Gamma$$

as $t \to \pm\infty$. Moreover, since the continuous functions with compact support are dense in $\mathrm{L}^2(m_\Gamma)$ it suffices to show that for every $f \in \mathrm{C}_c(\Gamma\backslash\mathcal{G})$

$$f \circ g_\Gamma^t \to \frac{1}{\|m_\Gamma\|}\int f \, \mathrm{d}m_\Gamma$$

weakly in $\mathrm{L}^2(m_\Gamma)$ as $t \to \pm\infty$.

We argue by contradiction and assume that m_Γ is not mixing. Then there exists a function $f \in \mathrm{C}_c(\Gamma\backslash\mathcal{G})$ (without loss of generality we may assume $\int f \, \mathrm{d}m_\Gamma = 0$ if m_Γ is finite) and a sequence $(t_n) \nearrow \infty$ such that $f \circ g_\Gamma^{t_n}$ does not converge to 0 weakly in $\mathrm{L}^2(m_\Gamma)$ as $n \to \infty$. By [Babillot 2002, Lemma 1] there exists a sequence $(s_n) \nearrow \infty$ and a *nonconstant* function $\Psi \in \mathrm{L}^2(m_\Gamma)$ such that

$$f \circ g_\Gamma^{s_n} \to \Psi \quad \text{and} \quad f \circ g_\Gamma^{-s_n} \to \Psi$$

weakly in $\mathrm{L}^2(m_\Gamma)$ as $n \to \infty$. Without loss of generality we may assume that Ψ is defined on all of $\Gamma\backslash\mathcal{G}$. Let $\widetilde{\Psi} : \mathcal{G} \to \mathbb{R}$ denote the lift of Ψ to $\mathcal{G}$ and smooth it along the flow by considering for $\tau > 0$ the function

$$\widetilde{\Psi}_\tau : \widetilde{\Omega} \to \mathbb{R}, \qquad v \mapsto \int_0^\tau \widetilde{\Psi}(g^s v)\, \mathrm{d}s.$$

For fixed $\varepsilon > 0$ sufficiently small $\widetilde{\Psi}_\varepsilon$ is still nonconstant, and now there exists a set $E'' \subset \partial_\infty\mathcal{G}$ of full $\bar{\mu}$-measure such that for all $v \in \partial_\infty^{-1}E''$ the function

$$h_v : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \widetilde{\Psi}_\varepsilon(g^t v)$$

is continuous. Notice that according to Theorem 4.1 we can assume $E'' \subset \partial_\infty\mathcal{Z}_\Gamma^{\mathrm{rec}}$ as $\partial_\infty\mathcal{Z}_\Gamma^{\mathrm{rec}}$ has full $\bar{\mu}$-measure in $\partial_\infty\mathcal{G}$. To any such function we associate the set of its periods which is a closed subgroup of $\mathbb{R}$; it only depends on $(v^-, v^+) \in E''$. This gives a map from E'' into the set of closed subgroups of $\mathbb{R}$ which is Γ-invariant as $\widetilde{\Psi}_\varepsilon$ is. By ergodicity of $\bar{\mu}$ (Theorem 4.2) this map is constant $\bar{\mu}$-almost everywhere.

Assume that this constant image is the group $\mathbb{R}$. Hence for $\bar{\mu}$-almost every $(v^-, v^+) \in E''$ every real number is a period of h_v for some $v \in \partial_\infty^{-1}(v^-, v^+)$ which is only possible if h_v is independent of t. In this case $\widetilde{\Psi}_\varepsilon$ induces a Γ-invariant function on a subset $E' \subset E'' \subset \partial_\infty\mathcal{Z}_\Gamma^{\mathrm{rec}}$ of full $\bar{\mu}$-measure. Again by ergodicity of $\bar{\mu}$ this function is constant, which finally gives a contradiction to the fact that $\widetilde{\Psi}_\varepsilon$ is nonconstant. So we conclude that there exists a subset $E' \subset \partial_\infty\mathcal{Z}_\Gamma^{\mathrm{rec}}$ of full $\bar{\mu}$-measure and $a \geq 0$ such that the constant image of the map above restricted to E' is the closed subgroup $2a\mathbb{Z}$.

In order to get the desired contradiction, we will next show that the cross-ratio spectrum $\mathrm{CR}(\mathcal{Q}_\Gamma)$ is contained in the closed subgroup $a\mathbb{Z}$. We denote $\tilde{f} : \mathcal{G} \to \mathbb{R}$ the lift of f to $\mathcal{G}$, and define

$$\tilde{f}_\varepsilon : \mathcal{G} \to \mathbb{R}, \quad v \mapsto \int_0^\varepsilon \tilde{f}(g^s v)\,\mathrm{d}s.$$

Since $\tilde{f}$ is Γ-invariant, $\tilde{f}_\varepsilon$ is also Γ-invariant and therefore descends to a function f_ε on $\Gamma\backslash\mathcal{G}$. Moreover,

$$f_\varepsilon \circ g_\Gamma^{s_n} \to \Psi_\varepsilon \quad \text{and} \quad f_\varepsilon \circ g_\Gamma^{-s_n} \to \Psi_\varepsilon$$

weakly in $\mathrm{L}^2(m_\Gamma)$ as $n \to \infty$, where $\Psi_\varepsilon \in \mathrm{L}^2(m_\Gamma)$ is the function induced from the Γ-invariant function $\widetilde{\Psi}_\varepsilon$ above. According to the classical fact stated and proved in [Babillot 2002, Section 1] there exists a sequence $(n_k) \subset \mathbb{N}$ such that Ψ_ε is the almost sure limit of the Cesaro averages for positive and negative times

$$\frac{1}{K^2}\sum_{k=1}^{K^2} f_\varepsilon \circ g_\Gamma^{s_{n_k}} \quad \text{and} \quad \frac{1}{K^2}\sum_{k=1}^{K^2} f_\varepsilon \circ g_\Gamma^{-s_{n_k}}.$$

We denote $\widetilde{\Psi}_\varepsilon^+$, $\widetilde{\Psi}_\varepsilon^-$ the lifts of the almost sure limits of the Cesaro averages above and consider the set

$$\widetilde{\Omega} := \left\{u \in \mathcal{Z}_\Gamma^{\mathrm{rec}} : \widetilde{\Psi}_\varepsilon^+(u),\, \widetilde{\Psi}_\varepsilon^-(u) \text{ exist and } \widetilde{\Psi}_\varepsilon^+(u) = \widetilde{\Psi}_\varepsilon^-(u) = \widetilde{\Psi}_\varepsilon(u)\right\};$$

from the previous paragraph and the fact that $\partial_\infty \mathcal{Z}_\Gamma^{\mathrm{rec}}$ has full $\bar{\mu}$-measure we know that $\partial_\infty \widetilde{\Omega}$ has full $\bar{\mu}$-measure. The same is true for the set $E := E' \cap \partial_\infty \widetilde{\Omega}$, where $E' \subset \partial_\infty \mathcal{Z}_\Gamma^{\mathrm{rec}}$ is the set of full $\bar{\mu}$-measure from the first part of the proof. So in particular $v \in \partial_\infty^{-1} E$ implies that the periods of the continuous function $h_v \in \mathrm{C}(\mathbb{R})$ are contained in the closed subgroup $2a\mathbb{Z}$.

Since $\tilde{f}$ is the lift of a function $f \in \mathrm{C}_c(\Gamma\backslash\mathcal{G})$, both $\tilde{f}$ and $\tilde{f}_\varepsilon$ are uniformly continuous. So if $u, v \in \widetilde{\Omega} \subset \partial_\infty^{-1} E$ are arbitrary, then according to Lemma 3.5 we have the following statements:

(a) If $u^+ = v^+$ and $\mathcal{B}_{v^+}(u(0), v(0)) = 0$, then $\widetilde{\Psi}_\varepsilon^+(u) = \widetilde{\Psi}_\varepsilon^+(v)$.

(b) If $u^- = v^-$ and $\mathcal{B}_{v^-}(u(0), v(0)) = 0$, then $\widetilde{\Psi}_\varepsilon^-(u) = \widetilde{\Psi}_\varepsilon^-(v)$.

Now according to Corollary 4.3 the sets

$$E^- := \left\{\xi \in \partial X : (\xi, \eta') \in E \text{ for } \mu^+\text{-almost every } \eta' \in \partial X\right\} \quad \text{and}$$
$$E^+ := \left\{\eta \in \partial X : (\xi', \eta) \in E \text{ for } \mu^-\text{-almost every } \xi' \in \partial X\right\}$$

satisfy $\mu_-(E^-) = \mu_-(\partial X)$, $\mu_+(E^+) = \mu_+(\partial X)$, hence $E^- \times E^+$ has full $\bar{\mu}$-measure.

We first consider the set of special quadrilaterals

$$\mathcal{S} = \left\{(\xi, \xi', \eta, \eta') : (\xi, \eta) \in E \cap (E^- \times E^+),\ (\xi', \eta'), (\xi, \eta'), (\xi', \eta) \in E\right\} \subset \mathcal{Q}_\Gamma.$$

So let $(\xi, \eta) \in E \cap (E^- \times E^+)$ and choose $(\xi', \eta') \in E$ such that (ξ', η) and (ξ, η') also belong to E. In order to show that the cross-ratio $\mathrm{CR}(\xi, \xi', \eta, \eta')$ belongs to $a\mathbb{Z}$ we start with a geodesic $v \in \partial_\infty^{-1}(\xi, \eta)$.

Let $v_1 \in \partial_\infty^{-1}(\xi', \eta)$ such that $\mathcal{B}_\eta(v(0), v_1(0)) = 0$, $v_2 \in \partial_\infty^{-1}(\xi', \eta')$ such that $\mathcal{B}_{\xi'}(v_1(0), v_2(0)) = 0$, $v_3 \in \partial_\infty^{-1}(\xi, \eta')$ such that $\mathcal{B}_{\eta'}(v_2(0), v_3(0)) = 0$ and finally $v_4 \in \partial_\infty^{-1}(\xi, \eta)$ such that $\mathcal{B}_\xi(v_3(0), v_4(0)) = 0$. Then according to (a)

$$\widetilde{\Psi}_\varepsilon^+(v) = \widetilde{\Psi}_\varepsilon^+(v_1) = \widetilde{\Psi}_\varepsilon^-(v_1)$$

by choice of $\widetilde{\Omega}$. Moreover (b) gives

$$\widetilde{\Psi}_\varepsilon^-(v_1) = \widetilde{\Psi}_\varepsilon^-(v_2) = \widetilde{\Psi}_\varepsilon^+(v_2).$$

Again by (a) we get

$$\widetilde{\Psi}_\varepsilon^+(v_2) = \widetilde{\Psi}_\varepsilon^+(v_3) = \widetilde{\Psi}_\varepsilon^-(v_3)$$

and by (b)

$$\widetilde{\Psi}_\varepsilon^-(v_3) = \widetilde{\Psi}_\varepsilon^-(v_4) = \widetilde{\Psi}_\varepsilon^+(v_4).$$

Altogether this shows $\widetilde{\Psi}_\varepsilon(v_4) = \widetilde{\Psi}_\varepsilon(v)$, and since $\partial_\infty v_4 = \partial_\infty v$ we know that there exists $t \in \mathbb{R}$ such that $v = g^t v_4$. Hence t is a period of the function h_v and therefore

$t \in 2a\mathbb{Z}$ (as $\partial_\infty v \in E'$). On the other hand, we have

$$\begin{aligned}
2\,\mathrm{CR}(\xi,\xi',\eta,\eta') &= 2\big(\mathrm{Gr}_o(\xi,\eta)+\mathrm{Gr}_o(\xi',\eta')-\mathrm{Gr}_o(\xi,\eta')-\mathrm{Gr}_o(\xi',\eta)\big)\\
&= \mathcal{B}_\xi(o,v(0))+\mathcal{B}_\eta(o,v(0))+\mathcal{B}_{\xi'}(o,v_2(0))+\mathcal{B}_{\eta'}(o,v_2(0))\\
&\qquad -\mathcal{B}_\xi(o,v_3(0))-\mathcal{B}_{\eta'}(o,v_3(0))-\mathcal{B}_{\xi'}(o,v_1(0))-\mathcal{B}_\eta(o,v_1(0))\\
&= \underbrace{\mathcal{B}_\eta(v_1(0),v(0))}_{=0}+\underbrace{\mathcal{B}_{\xi'}(v_1(0),v_2(0))}_{=0}+\underbrace{\mathcal{B}_{\eta'}(v_3(0),v_2(0))}_{=0}\\
&\qquad +\underbrace{\mathcal{B}_\xi(v_4(0),v_3(0))}_{=0}+\mathcal{B}_\xi(v_3(0),v(0))\\
&= \mathcal{B}_\xi(v_4(0),v(0))=\mathcal{B}_\xi(v_4(0),v_4(t))=t\in 2a\mathbb{Z},
\end{aligned}$$

hence $\mathrm{CR}(\xi,\xi',\eta,\eta')\in a\mathbb{Z}$. This proves that $\mathrm{CR}(\mathcal{S})\subset a\mathbb{Z}$.

Finally, since the cross-ratio is continuous and the set of special quadrilaterals $\mathcal{S}$ is dense in $\mathcal{Q}_\Gamma$, the cross-ratio spectrum $\mathrm{CR}(\mathcal{Q}_\Gamma)$ is included in the discrete subgroup $a\mathbb{Z}$ of $\mathbb{R}$. So according to Proposition 5.4 the length spectrum is arithmetic in contradiction to the hypothesis of the theorem. □

We will often work in the universal cover X of $\Gamma\backslash X$ and therefore need the following

Corollary 5.6. *Let $\Gamma<\mathrm{Is}(X)$ be a discrete rank one group with nonarithmetic length spectrum and $\mathcal{Z}_\Gamma\neq\emptyset$. Let μ_-, μ_+ be nonatomic finite Borel measures on ∂X with $\mu_\pm(L_\Gamma^{\mathrm{rad}})=\mu_\pm(\partial X)$, and*

$$\bar\mu\sim(\mu_-\otimes\mu_+)|_{\partial_\infty\mathcal{R}}$$

a quasiproduct geodesic current defined on $\partial_\infty\mathcal{R}$ for which the constant Δ defined in (20) is finite. Let m_Γ be the associated Ricks measure on $\Gamma\backslash\mathcal{G}$, $A,B\subset\Gamma\backslash\mathcal{G}$ Borel sets with $m_\Gamma(A)$ and $m_\Gamma(B)$ finite, and $\tilde A,\tilde B\subset\mathcal{G}$ lifts of A and B. Then

$$\lim_{t\to\pm\infty}\Big(\sum_{\gamma\in\Gamma} m(\tilde A\cap g^{-t}\gamma\tilde B)\Big)=\begin{cases}\dfrac{m(\tilde A)\cdot m(\tilde B)}{\|m_\Gamma\|} & \text{if } m_\Gamma \text{ is finite},\\ 0 & \text{if } m_\Gamma \text{ is infinite}.\end{cases}$$

Proof. According to Lemma 4.4 there exists a Γ-invariant subset $\mathcal{G}'\subset\mathcal{G}$ of full m-measure and a locally finite fundamental domain $\mathcal{D}\subset\mathcal{G}'$ for the action of Γ on $\mathcal{G}'$ with $m(\partial\mathcal{D})=0$. Notice that for any measurable function $h\in\mathrm{L}^1(m_\Gamma)$ with lift $\tilde h:\mathcal{G}\to\mathbb{R}$ the integral $\int_{\mathcal{D}}\tilde h\,\mathrm{d}m$ is independent of the chosen fundamental domain $\mathcal{D}\subset\mathcal{G}$ as above. Moreover, we obviously get from (17) and (18)

$$\int_{\mathcal{D}}\tilde h\,\mathrm{d}m=\int_{\Gamma\backslash\mathcal{G}} h\,\mathrm{d}m_\Gamma.$$

Now let $A,B\subset\Gamma\backslash\mathcal{G}$ be Borel sets with $m_\Gamma(A)$ and $m_\Gamma(B)$ finite, and $\tilde A,\tilde B\subset\mathcal{G}$ lifts of A and B. Without loss of generality we may assume that $\tilde A,\tilde B\subset\overline{\mathcal{D}}$. For

$t \in \mathbb{R}$ consider the function $h_t \in \mathrm{L}^1(m_\Gamma)$ defined by

$$h_t = \mathbb{1}_{A \cap g_\Gamma^{-t} B}.$$

For its lift $\tilde{h}_t$ and $v \in \mathcal{G}$ we have

$$\tilde{h}_t(v) = 1 \quad \text{if } \gamma' v \in \tilde{A} \cap g^{-t} \gamma \tilde{B} \text{ for some } \gamma', \gamma \in \Gamma,$$

and $\tilde{h}_t(v) = 0$ otherwise. So

$$m_\Gamma(A \cap g_\Gamma^{-t} B) = \int_{\Gamma \backslash \mathcal{G}} h_t \, \mathrm{d}m_\Gamma = \int_{\mathcal{D}} \tilde{h}_t \, \mathrm{d}m = \sum_{\gamma \in \Gamma} m(\tilde{A} \cap g^{-t} \gamma \tilde{B}).$$

The claim now follows from Theorem 5.5, because

$$m_\Gamma(A) = \int_{\Gamma \backslash \mathcal{G}} \mathbb{1}_A \, \mathrm{d}m_\Gamma = \int_{\mathcal{D}} \mathbb{1}_{\Gamma \tilde{A}} \, \mathrm{d}m = m(\tilde{A}) \quad \text{and} \quad m_\Gamma(B) = m(\tilde{B}). \qquad \square$$

Notice that in general it is not so easy to determine whether a discrete rank one group has arithmetic length spectrum or not. As mentioned before, if $\Gamma < \mathrm{Is}(X)$ has finite Ricks–Bowen–Margulis measure and satisfies $L_\Gamma = \partial X$, then according to Theorem 4 in [Ricks 2017] the length spectrum of Γ is arithmetic if and only if X is a tree with all edge lengths in $c\mathbb{N}$ for some $c > 0$. This includes Babillot's observation that for cocompact discrete rank one groups of a Hadamard *manifold* the length spectrum is nonarithmetic. Moreover, we recall a few further results:

Proposition 5.7. *Let X be a proper CAT(-1) Hadamard space. A discrete rank one group $\Gamma < \mathrm{Is}(X)$ has nonarithmetic length spectrum if*

- Γ *contains a parabolic isometry* [Dal'bo and Peigné 1998],
- *the limit set L_Γ possesses a connected component which is not reduced to a point* [Bourdon 1995],
- *X is a manifold with constant sectional curvature* [Guivarc'h and Raugi 1986, Proposition 3],
- *X is a Riemannian surface* [Dal'bo 1999].

6. Shadows, cones and corridors

We keep the notation and conditions from the previous section. So in particular X is a proper Hadamard space and $\Gamma < \mathrm{Is}(X)$ a discrete rank one group. For our proof of the equidistribution theorem we will need a few definitions and preliminary statements. Recall that for $y \in X$ and $r > 0$ $B_r(y) \subset X$ denotes the open ball of radius r centered at $y \in X$. The *shadow* of $B_r(y) \subset X$ viewed from the source $x \in X$ is defined by

$$\mathcal{O}_r(x, y) := \{\eta \in \partial X : \sigma_{x,\eta}(\mathbb{R}_+) \cap B_r(y) \neq \varnothing\};$$

this is an open subset of the geometric boundary ∂X. If $\xi \in \partial X$ we define

$$\begin{aligned}\mathcal{O}_r(\xi, y) &:= \{\eta \in \partial X : \text{there exists } v \in \partial_\infty^{-1}(\xi, \eta) \text{ with } v(0) \in B_r(y)\}\\ &= \{\eta \in \partial X : (\xi, \eta) \in \partial_\infty \mathcal{G} \text{ and } d(y, (\xi\eta)) < r\}.\end{aligned}$$

Notice that due to the possible existence of flat subspaces in X a shadow $\mathcal{O}_r(\xi, y)$ with source $\xi \in \partial X$ need not be open: In a Euclidean plane such a shadow always consists of a single point in the boundary, no matter how large r is. In our context, the shadows with source ξ in the boundary ∂X will be larger, but still not necessarily open.

Remark. If ξ is the positive end point v^+ of a weakly $\mathrm{Is}(X)$-recurrent geodesic $v \in \mathcal{Z}$, then Lemma 3.4 and Lemma 2.1 imply that $\mathcal{O}_r(\xi, y)$ is open for any $y \in X$.

More generally, if there exists a geodesic $u \in \mathcal{Z}$ with $u^+ = \xi$ and $u(0) \in B_r(y)$, then according to Lemma 2.1 the shadow $\mathcal{O}_r(\xi, y)$ contains an open neighborhood of u^- in ∂X, but need not be open: If u is not $\mathrm{Is}(X)$-recurrent, then this open neighborhood of u^- can be much smaller than $\mathcal{O}_r(\xi, y)$, and there might exist a point $\eta \in \mathcal{O}_r(\xi, y)$ such that $(\xi\eta)$ is isometric to a Euclidean plane. But ξ cannot be joined to any point in the boundary of $(\xi\eta)$ different from η, no matter how close it is to η. In this case, every open neighborhood of η intersects the complement of the shadow $\mathcal{O}_r(\xi, y)$ in ∂X nontrivially (as this complement includes all the boundary points which cannot be joined to ξ by a geodesic), hence $\eta \in \partial\mathcal{O}_r(\xi, y)$.

We will now prove that this cannot happen if η is the end point of an $\mathrm{Is}(X)$-recurrent geodesic $v \in \mathcal{Z}$, that is if η belongs to the set

$$\partial X^{\mathrm{rec}} := \{\eta \in \partial X : \text{there exists } v \in \mathcal{Z} \text{ } \mathrm{Is}(X)\text{-recurrent with } \eta = v^+\}. \tag{23}$$

Lemma 6.1. *Let $\xi \in \partial X$, $x \in X$ and $r > 0$ arbitrary. Then for the closure $\overline{\mathcal{O}_r(\xi, x)}$ and the boundary $\partial\mathcal{O}_r(\xi, x)$ of the shadow $\mathcal{O}_r(\xi, x) \subset \partial X$ we have*

(a) $$\overline{\mathcal{O}_r(\xi, x)} \subset \{\zeta \in \partial X : (\xi, \zeta) \in \partial_\infty \mathcal{G} \text{ and } d\big(x, (\xi\zeta)\big) \le r\},$$

(b) $$\partial\mathcal{O}_r(\xi, x) \cap \partial X^{\mathrm{rec}} \subset \{\zeta \in \partial X^{\mathrm{rec}} \setminus \{\xi\} : d(x, (\xi\zeta)) = r\}.$$

Proof. In order to prove (a) we let $\zeta \in \overline{\mathcal{O}_r(\xi, x)}$ be arbitrary. Then there exists a sequence $(\zeta_n) \subset \mathcal{O}_r(\xi, x)$ with $\zeta_n \to \zeta$ as $n \to \infty$. For $n \in \mathbb{N}$ we let $v_n = v(x; \xi, \zeta_n) \in \mathcal{G}$ as defined in (8), hence in particular $v_n^- = \xi$, $v_n^+ = \zeta_n$ and $v_n(0) \in B_r(x)$. Passing to a subsequence if necessary we may assume that $v_n(0)$ converges to a point $z \in \overline{B_r(x)}$ (as $\overline{B_r(x)}$ is compact). Recall the definition of the Alexandrov angle from (3). According to Proposition II.9.2 in [Bridson and Haefliger 1999] we have

$$\angle_z(\xi, \zeta) \ge \limsup_{n\to\infty} \angle_{v_n(0)}(\xi, \zeta_n) = \pi,$$

since $v_n(0)$ is a point on the geodesic v_n joining ξ to ζ_n. From $\angle_z(\xi,\zeta)\in[0,\pi]$ we therefore get $\angle_z(\xi,\zeta)=\pi$, hence $z\in\overline{B_r(x)}$ is a point on a geodesic joining ξ to ζ, and in particular $(\xi,\zeta)\in\partial_\infty\mathcal{G}$. This proves (a).

For the proof of (b) we let $\zeta\in\partial\mathcal{O}_r(\xi,x)\cap\partial X^{\mathrm{rec}}$ be arbitrary. By definition of the boundary we know that $\zeta\in\overline{\mathcal{O}_r(\xi,x)}$ and that there exists a sequence $(\eta_n)\subset\partial X\setminus\mathcal{O}_r(\xi,x)$ with $\eta_n\to\zeta$ as $n\to\infty$. From (a) we know that $(\xi,\zeta)\in\partial_\infty\mathcal{G}$, hence in particular $\zeta\neq\xi$, and that $d(x,(\xi\zeta))\le r$. So it only remains to prove that $d(x,(\xi\zeta))\ge r$.

We will prove that every point $\eta\in(\partial X^{\mathrm{rec}}\setminus\{\xi\})\cap\mathcal{O}_r(\xi,x)$ is an interior point of $\mathcal{O}_r(\xi,x)$: Then $d(x,(\xi\zeta))<r$ would imply that ζ is an interior point of $\mathcal{O}_r(\xi,x)$ and therefore cannot be the limit of a sequence $(\eta_n)\subset\partial X\setminus\mathcal{O}_r(\xi,x)$, in contradiction to $\zeta\in\partial\mathcal{O}_r(\xi,x)$.

So let $\eta\in(\partial X^{\mathrm{rec}}\setminus\{\xi\})\cap\mathcal{O}_r(\xi,x)$ be arbitrary. From Lemma 3.4 we get that $(\xi,\eta)\in\partial_\infty\mathcal{Z}$, and with $v:=v(x;\xi,\eta)\in\mathcal{Z}$ we have $d(x,v(0))=d(x,(\xi\eta))<r$. Fix $\varepsilon=\frac{1}{2}\bigl(r-d(x,(\xi\eta))\bigr)>0$. According to Lemma 2.1 there exists an open neighborhood $U\subset\partial X$ of η such that any $u\in\mathcal{G}$ with $u^-=\xi$ and $u^+\in U$ satisfies $u\in\mathcal{R}$ and $d(v(0),u(\mathbb{R}))<\varepsilon$. Let $\eta'\in U$ arbitrary and $u\in\partial_\infty^{-1}(\xi,\eta')$ be parametrized such that $d(v(0),u(0))<\varepsilon$. Then

$$\begin{aligned} d(x,(\xi\eta'))\le d(x,u(0))\le d(x,v(0))+d(v(0),u(0))&<d(x,(\xi\eta))+\varepsilon\\ &<d(x,(\xi\eta))+\tfrac{1}{2}\bigl(r-d(x,(\xi\eta))\bigr)<r. \end{aligned}$$ □

Instead of using the boundary $\partial\mathcal{O}_r(\xi,x)$ we will work in the sequel with the set

$$\tilde\partial\mathcal{O}_r(\xi,x):=\{\eta\in\partial X:(\xi,\eta)\in\partial_\infty\mathcal{G}\text{ and }d(x,(\xi\eta))=r\}\tag{24}$$

whose intersection with $\partial X^{\mathrm{rec}}$ may be strictly larger than $\partial\mathcal{O}_r(\xi,x)\cap\partial X^{\mathrm{rec}}$. Notice that every point $\eta\in(\partial X^{\mathrm{rec}}\setminus\{\xi\})\cap(\partial X\setminus\tilde\partial\mathcal{O}_r(\xi,x))$ is an interior point of the complement $\partial X\setminus\tilde\partial\mathcal{O}_r(\xi,x)$ of $\tilde\partial\mathcal{O}_r(\xi,x)$ in ∂X.

Remark. The converse inclusions "$\supset$" in the above Lemma 6.1 are wrong in general: If X is a 4-regular tree with all edge lengths equal to 1, then

$$\overline{\mathcal{O}_r(\xi,x)}=\mathcal{O}_r(\xi,x)=\{\eta\in\partial X\setminus\{\xi\}:d(x,(\xi\eta))\le\lceil r\rceil-1\},$$

where $\lceil r\rceil\in\mathbb{N}$ is the smallest integer bigger than or equal to r. So for $n\in\mathbb{N}$ we have

$$\overline{\mathcal{O}_n(\xi,x)}\subsetneq\{\eta\in\partial X\setminus\{\xi\}:d(x,(\xi\eta))\le n\}.$$

Moreover,

$$\begin{aligned}\tilde\partial\mathcal{O}_n(\xi,x)&=\{\eta\in\partial X\setminus\{\xi\}:d(x,(\xi\eta))=n\}\\ &=\{\eta\in\partial X\setminus\{\xi\}:n\le d(x,(\xi\eta))<n+1\}\\ &=\mathcal{O}_{n+1}(\xi,x)\setminus\mathcal{O}_n(\xi,x)\neq\varnothing,\end{aligned}$$

whereas the boundary $\partial\mathcal{O}_r(\xi, x)$ is always empty. Since all points in ∂X are $\mathrm{Is}(X)$-recurrent, this shows that for all $n \in \mathbb{N}$

$$\varnothing = \partial\mathcal{O}_n(\xi, x) \cap \partial X^{\mathrm{rec}} \subsetneq \tilde{\partial}\mathcal{O}_n(\xi, x) = \{\zeta \in \partial X^{\mathrm{rec}} \setminus \{\xi\} : d\big(x, (\xi\zeta)\big) = n\}.$$

We will further need the following refined versions of the shadows above which were first introduced by Roblin [2003]: for $r > 0$ and $x, y \in X$ we set

$$\mathcal{O}_r^-(x, y) := \{\eta \in \partial X : \forall z \in B_r(x) \text{ we have } \sigma_{z,\eta}(\mathbb{R}_+) \cap B_r(y) \neq \varnothing\},$$
$$\mathcal{O}_r^+(x, y) := \{\eta \in \partial X : \exists z \in B_r(x) \text{ such that } \sigma_{z,\eta}(\mathbb{R}_+) \cap B_r(y) \neq \varnothing\}.$$

It is obvious from the definitions that for any $\rho > 0$ and for all $x', y' \in X$ we have

$$d(x, x') < \rho \text{ and } d(y, y') < \rho \implies \mathcal{O}_r^+(x, y) \subset \mathcal{O}_{r+\rho}^+(x', y'). \tag{25}$$

Notice also that $\mathcal{O}_r^-(x, y)$ need not be open as it is an uncountable intersection of open sets $\mathcal{O}_r(z, y)$ with $z \in B_r(x)$ (for a concrete example see the remark on page 820). If $\xi \in \partial X$, we set

$$\mathcal{O}_r^-(\xi, y) = \mathcal{O}_r^+(\xi, y) = \mathcal{O}_r(\xi, y).$$

Remark. In the middle of page 58 of [Roblin 2003] it is stated that in a CAT(-1)-space X every sequence $(z_n) \subset \bar{X}$ converging to a point $\xi \in \partial X$ satisfies

$$\lim_{n\to\infty} \mathcal{O}_r^{\pm}(z_n, x) = \mathcal{O}_r(\xi, x).$$

This is not true in a CAT(0)-space as the following example shows:

Let X be the Euclidean plane, $x \in X$ the origin $(0, 0)$, and identify ∂X with $\mathbb{S}^1 \cong [0, 2\pi)$. Let $\xi = \pi$ and $r > 0$. Then obviously $\mathcal{O}_r(\xi, x) = \{0\}$.

For $n \in \mathbb{N}$ we define $\varphi_n := 1/n$ and $z_n := \big(-rn\cos(\varphi_n), -rn\sin(\varphi_n)\big)$, hence

$$\sigma_{x,z_n}(-\infty) = \sigma_{z_n,x}(\infty) = \varphi_n \quad \text{and} \quad (z_n) \to \xi = \pi.$$

By elementary Euclidean geometry we further have $\mathcal{O}_r^-(z_n, x) = \{\varphi_n\}$, and thus

$$\lim_{n\to\infty} \mathcal{O}_r^-(z_n, x) = \varnothing \neq \{0\} = \mathcal{O}_r(\xi, x).$$

However, the following statement will be sufficient for our purposes.

Proposition 6.2. *Let $\xi \in \partial X$, $x \in X$, $r > 0$ and recall the definitions of $\tilde{\partial}\mathcal{O}_r(\xi, x)$ from (24) and of $\partial X^{\mathrm{rec}}$ from (23). Then for every sequence $(z_n) \subset \bar{X}$ converging to ξ the following inclusions hold*:

(a) $$\limsup_{n\to\infty}(\mathcal{O}_r^{\pm}(z_n, x) \cap \partial X^{\mathrm{rec}}) \subset \big(\mathcal{O}_r(\xi, x) \cup \tilde{\partial}\mathcal{O}_r(\xi, x)\big) \cap \partial X^{\mathrm{rec}},$$

(b) $$\liminf_{n\to\infty}(\mathcal{O}_r^{\pm}(z_n, x) \cap \partial X^{\mathrm{rec}}) \supset \mathcal{O}_r(\xi, x) \cap \partial X^{\mathrm{rec}}.$$

Proof. Let us first prove (a), which will follow from

$$\limsup_{n\to\infty}(\mathcal{O}_r^+(z_n,x)\cap\partial X^{\mathrm{rec}})\subset\big(\mathcal{O}_r(\xi,x)\cup\tilde{\partial}\mathcal{O}_r(\xi,x)\big)\cap\partial X^{\mathrm{rec}}.$$

If $\zeta\in\limsup_{n\to\infty}(\mathcal{O}_r^+(z_n,x)\cap\partial X^{\mathrm{rec}})$, then for all $n\in\mathbb{N}$ there exists $k_n\ge n$ such that $\zeta\in\mathcal{O}_r^+(z_{k_n},x)\cap\partial X^{\mathrm{rec}}$. Moreover, by definition of $\partial X^{\mathrm{rec}}$ and Lemma 3.4 there exists $w\in\mathcal{Z}$ with $w^-=\zeta$ and $w^+=\xi$. Reparametrizing w if necessary we may assume that its origin $w(0)$ satisfies $\mathcal{B}_\zeta(x,w(0))=0$.

Passing to a subsequence of (z_{k_n}) if necessary we may assume that either $(z_{k_n})\subset\partial X$ or $(z_{k_n})\subset X$.

Fix $n\in\mathbb{N}$. If $z_{k_n}\in\partial X$ we choose a geodesic line

$$u_n\in\partial_\infty^{-1}(\zeta,z_{k_n})\quad\text{with }\; u_n(\mathbb{R})\cap B_r(x)\ne\varnothing.$$

If $z_{k_n}\in X$ we first let σ_n be a geodesic ray in the class of ζ with $\sigma_n(0)\in B_r(z_{k_n})$ and $\sigma_n(\mathbb{R}_+)\cap B_r(x)\ne\varnothing$, and then $u_n\in\mathcal{G}$ a geodesic line with $u_n^-=\zeta$ whose image in X contains $\sigma_n(\mathbb{R}_+)$ (that is $-u_n\in\mathcal{G}$ extends the ray σ_n). From $\zeta\in\partial X^{\mathrm{rec}}$ and Lemma 3.4 we know that in both cases $u_n\in\mathcal{Z}$; up to reparametrization we can further assume that $\mathcal{B}_\zeta(x,u_n(0))=0$.

By choice of u_n we further know that $d(x,u_n(\mathbb{R}))<r$; we fix $s_n\in\mathbb{R}$ such that $d(x,u_n(s_n))=d(x,u_n(\mathbb{R}))$ (which is equivalent to $g^{s_n}u_n=v(x;\zeta,\xi)$). Then

$$|s_n|=|\mathcal{B}_\zeta(u_n(0),u_n(s_n))|=|\mathcal{B}_\zeta(x,u_n(s_n))|\le d(x,u_n(s_n))<r. \tag{26}$$

In the easy case that $(z_{k_n})\subset\partial X$ we have $(u_n^+)=(z_{k_n})\to\xi$, so (u_n) converges weakly to $w\in\mathcal{Z}$.

Otherwise, for $n\in\mathbb{N}$ we choose $t_n\in\mathbb{R}$ such that $u_n(t_n)=\sigma_n(0)\in B_r(z_{k_n})$. Since (z_{k_n}) converges to ξ we also have $u_n(t_n)\to\xi$, hence $(t_n)\nearrow\infty$. From the estimate (26) we get $d(x,u_n(0))<2r$, so $u_n(t_n)\to\xi$ implies $u_n^+\to\xi$, which proves that also in this case (u_n) converges weakly to $w\in\mathcal{Z}$.

Passing to a subsequence if necessary we may now assume that the sequence (s_n) from above converges to $s\in[-r,r]$ and that (u_n) converges to w in $\mathcal{G}$ (by Lemma 2.2). This finally gives

$$\begin{aligned}d(x,w(\mathbb{R}))&\le d(x,w(s))\\&\le\lim_{n\to\infty}\Big(\underbrace{d(x,u_n(s_n))}_{<r}+\underbrace{d(u_n(s_n),w(s_n))}_{\to 0}+\underbrace{d(w(s_n),w(s))}_{=|s_n-s|\to 0}\Big)\le r.\end{aligned}$$

For the proof of (b) we let $\zeta\in\mathcal{O}_r(\xi,x)\cap\partial X^{\mathrm{rec}}$ be arbitrary. By definition of $\partial X^{\mathrm{rec}}$ and Lemma 3.4 there exists $w\in\mathcal{Z}$ with $w^-=\xi$, $w^+=\zeta$. Reparametrizing w if necessary we may assume that $w=v(x;\xi,\zeta)$, hence $d(x,w(0))<r$.

Since $B_r(x)$ is open, there exists $\epsilon>0$ such that $B_\epsilon(w(0))\subset B_r(x)$. According to Lemma 2.1 there exist neighborhoods $U,V\subset\overline{X}$ of w^-, w^+ such that any two points in U, V can be joined by a rank one geodesic $u \in \mathcal{R}$ with $d(u(0), w(0)) < \epsilon$ and $\mathrm{width}(u) < 2\epsilon$. As $z_n \to \xi = w^-$ there exists $n \in \mathbb{N}$ such that for all $k \geq n$ we have $B_r(z_k) \subset U$ if $z_k \in X$ respectively $z_k \in U$ if $z_k \in \partial X$; for these k we immediately get $\zeta = w^+ \in \mathcal{O}_r^-(z_k, x) \subset \mathcal{O}_r^+(z_k, x)$ (since $B_\epsilon\big(w(0)\big) \subset B_r(x)$). □

We now fix nonatomic finite Borel measures μ_-, μ_+ on ∂X with $\mu_\pm(L_\Gamma^{\mathrm{rad}}) = \mu_\pm(\partial X)$ and such that $\bar\mu \sim (\mu_- \otimes \mu_+)|_{\partial_\infty \mathcal{R}}$ is a quasiproduct geodesic current on $\partial_\infty \mathcal{R}$ for which the constant Δ defined by (20) is finite. We will need the following:

Lemma 6.3. *Let $\xi \in \partial X$, $x \in X$ and recall definition (24). The set*

$$\{r > 0 : \mu_+(\tilde\partial \mathcal{O}_r(\xi, x)) > 0\}$$

is at most countable.

Proof. We first notice that the sets $\tilde\partial \mathcal{O}_r(\xi, x)$ are disjoint for different values of r. Hence by finiteness of μ_x we know that for $n \in \mathbb{N}$ arbitrary the set

$$A_n := \{r > 0 : \mu_+(\tilde\partial \mathcal{O}_r(\xi, x)) > 1/n\}$$

is finite. Therefore the set $\{r > 0 : \mu_+(\tilde\partial \mathcal{O}_r(\xi, x)) > 0\} = \bigcup_{n\in\mathbb{N}} A_n$ is at most countable. □

From Proposition 6.2 we get the following estimate on the μ_+-measure of the small and large shadows with source in the neighborhood of a given boundary point.

Corollary 6.4. *Let $\xi \in \partial X$, $x \in X$ and $r > 0$ such that*

$$\mu_+(\mathcal{O}_r(\xi, x)) > 0 \quad \text{and} \quad \mu_+(\tilde\partial \mathcal{O}_r(\xi, x)) = 0.$$

Then for all $\varepsilon > 0$ there exists a neighborhood $U \subset \bar X$ of ξ such that for all $z \in U$

$$\mathrm{e}^{-\varepsilon} \mu_+(\mathcal{O}_r(\xi, x)) < \mu_+(\mathcal{O}_r^\pm(z, x)) < \mathrm{e}^{\varepsilon} \mu_+(\mathcal{O}_r(\xi, x)).$$

Proof. We first recall the definition of $\mathcal{Z}_\Gamma^{\mathrm{rec}}$ from (15) and notice that $\Gamma\backslash\mathcal{Z}_\Gamma^{\mathrm{rec}}$ has full m_Γ-measure by Theorem 4.1. So according to Corollary 4.3 we have

$$\mu_+\big(\{\zeta \in \partial X : (\eta, \zeta) \in \partial_\infty \mathcal{Z}_\Gamma^{\mathrm{rec}} \text{ for } \mu_-\text{-almost every } \eta \in \partial X\}\big) = \mu_+(\partial X).$$

Hence from the obvious inclusion

$$\{\zeta \in \partial X : (\eta, \zeta) \in \partial_\infty \mathcal{Z}_\Gamma^{\mathrm{rec}} \text{ for } \mu_-\text{-almost every } \eta \in \partial X\} \subset \partial X^{\mathrm{rec}}$$

we obtain $\mu_+(\partial X^{\mathrm{rec}}) = \mu_+(\partial X)$.

Since μ_+ is a finite Borel measure, Proposition 6.2 implies

$$\begin{aligned}\mu_+(\mathcal{O}_r(\xi,x)) &= \mu_+(\mathcal{O}_r(\xi,x)\cap\partial X^{\mathrm{rec}}) \overset{\text{(b)}}{\le} \mu_+\Big(\liminf_{n\to\infty}(\mathcal{O}_r^{\pm}(z_n,x)\cap\partial X^{\mathrm{rec}})\Big)\\ &\le \liminf_{n\to\infty}\mu_+(\mathcal{O}_r^{\pm}(z_n,x)\cap\partial X^{\mathrm{rec}}) \le \limsup_{n\to\infty}\mu_+(\mathcal{O}_r^{\pm}(z_n,x)\cap\partial X^{\mathrm{rec}})\\ &\le \mu_+\Big(\limsup_{n\to\infty}(\mathcal{O}_r^{\pm}(z_n,x)\cap\partial X^{\mathrm{rec}})\Big)\\ &\overset{\text{(a)}}{\le} \mu_+\big((\mathcal{O}_r(\xi,x)\cup\tilde\partial\mathcal{O}_r(\xi,x))\cap\partial X^{\mathrm{rec}}\big) = \mu_+(\mathcal{O}_r(\xi,x)),\end{aligned}$$

because $\mu_+\big(\tilde\partial\mathcal{O}_r(\xi,x)\big)=0$. So we conclude

$$\lim_{n\to\infty}\mu_+(\mathcal{O}_r^{\pm}(z_n,x)) = \lim_{n\to\infty}\mu_+(\mathcal{O}_r^{\pm}(z_n,x)\cap\partial X^{\mathrm{rec}}) = \mu_+(\mathcal{O}_r(\xi,x)),$$

hence the claim. □

For a subset $A\subset\partial X$ we next define the small and large cones

$$\begin{aligned}\mathcal{C}_r^-(x,A) &:= \{z\in X : \mathcal{O}_r^+(x,z)\subset A\},\\ \mathcal{C}_r^+(x,A) &:= \{z\in X : \mathcal{O}_r^+(x,z)\cap A\neq\varnothing\}.\end{aligned}\tag{27}$$

Notice that our definition of the small cones $\mathcal{C}_r^-$ differs slightly from Roblin's in order to get Lemma 6.8. Moreover, they are related to our large cones via

$$\mathcal{C}_r^-(x,A)\subset\mathcal{C}_r^+(x,A)\quad\text{and}\quad \mathcal{C}_r^-(x,A)=\overline{X}\setminus\mathcal{C}_r^+(x,\partial X\setminus A).$$

From the latter equality and (25) we immediately get:

Lemma 6.5. *Let $\rho>0$, $x_0\in B_\rho(x)$ and $y_0\in B_\rho(y)$. Then*

(a) $$y\in\mathcal{C}_r^+(x,A)\Rightarrow y_0\in\mathcal{C}_{r+\rho}^+(x_0,A),$$

(b) $$y\in\mathcal{C}_{r+\rho}^-(x,A)\Rightarrow y_0\in\mathcal{C}_r^-(x_0,A).$$

This shows in particular that for $r<r'$ we have

$$\mathcal{C}_r^+(x,A)\subset\mathcal{C}_{r'}^+(x,A)\quad\text{and}\quad \mathcal{C}_r^-(x,A)\supset\mathcal{C}_{r'}^-(x,A).\tag{28}$$

In Sections 8 and 9 we will frequently need the following:

Lemma 6.6. *Let $x,y\in X$, $r>0$, and $\widehat{V}\subset\overline{X}$, $V\subset\partial X$ be arbitrary open sets.*

(a) *For $A\subset\partial X$ with $\overline{A}\subset\widehat{V}\cap\partial X$ only finitely many $\gamma\in\Gamma$ satisfy*

$$\gamma y\in\mathcal{C}_r^{\pm}(x,A)\setminus\widehat{V}.$$

(b) *For $\widehat{A}\subset\overline{X}$ with $\overline{\widehat{A}}\cap\partial X\subset V$ only finitely many $\gamma\in\Gamma$ satisfy*

$$\gamma y\in\widehat{A}\setminus\mathcal{C}_r^{\pm}(x,V).$$

Proof. We begin with the proof of (a) by contradiction. Assume that there exists a sequence $(\gamma_n) \subset \Gamma$ such that $\gamma_n y \in \mathcal{C}_r^+(x, A) \setminus \widehat{V}$ for all $n \in \mathbb{N}$. As Γ is discrete, every accumulation point of $(\gamma_n y) \subset X$ belongs to ∂X. Passing to a subsequence if necessary we will assume that $\gamma_n y \to \zeta \in L_\Gamma \subset \partial X$ as $n \to \infty$.

From $\gamma_n y \in \mathcal{C}_r^+(x, A)$ we know that $\mathcal{O}_r^+(x, \gamma_n y) \cap A \neq \varnothing$. We choose a geodesic line $v_n \in \mathcal{G}$ with $v_n^+ \in A$ whose image intersects $B_r(x)$ and then $B_r(\gamma_n y)$. Up to reparametrization we can assume that $\mathcal{B}_{v_n^+}(x, v_n(0)) = 0$ and $\mathcal{B}_{v_n^+}(\gamma_n y, v_n(t_n)) = 0$ for some $t_n > 0$. Then by an easy geometric estimate analogous to the one in the proof of Proposition 6.2 (a) we have $d(x, v_n(0)) < 2r$ and $d(\gamma_n y, v_n(t_n)) < 2r$. By convexity of the distance function and $\sigma_{x,v_n^+}(\infty) = v_n(\infty) = v_n^+$ we get

$$d(\sigma_{x,v_n^+}(T), v_n(T)) < 2r \quad \text{for all } T > 0.$$

Hence

$$d(\gamma_n y, \sigma_{x,v_n^+}(t_n)) \le d(\gamma_n y, v_n(t_n)) + d(v_n(t_n), \sigma_{x,v_n^+}(t_n)) < 4r$$

which implies $v_n^+ \to \zeta$ and therefore $\zeta \in \bar{A} \subset \widehat{V} \cap \partial X$.

On the other hand, as $\widehat{V}$ is open and $\gamma_n y \notin \widehat{V}$ for all $n \in \mathbb{N}$, we obviously have $\zeta \notin \widehat{V} \cap \partial X$, hence a contradiction. The claim for $\mathcal{C}_r^-(x, A) \setminus \widehat{V}$ follows from the obvious inclusion $\mathcal{C}_r^-(x, A) \subset \mathcal{C}_r^+(x, A)$.

For the proof of (b) we assume that there exists a sequence $(\gamma_n) \subset \Gamma$ such that $\gamma_n y \in \widehat{A} \setminus \mathcal{C}_r^-(x, V)$ for all $n \in \mathbb{N}$. Passing to a subsequence if necessary we will assume as above that $\gamma_n y \to \zeta \in L_\Gamma \subset \partial X$ as $n \to \infty$. Here $\gamma_n y \in \widehat{A}$ for all $n \in \mathbb{N}$ obviously implies $\zeta \in \overline{\widehat{A}} \cap \partial X \subset V$.

From $\gamma_n y \notin \mathcal{C}_r^-(x, V)$ we know that $\mathcal{O}_r^+(x, \gamma_n y) \not\subset V$. We choose a geodesic line $v_n \in \mathcal{G}$ with $v_n^+ \notin V$ whose image intersects $B_r(x)$ and then $B_r(\gamma_n y)$. As in the proof of (a) we get $v_n^+ \to \zeta$, and therefore $\zeta \in \overline{\partial X \setminus V} = \partial X \setminus V$ since V is open; this is an obvious contradiction to $\zeta \in V$. Again, the claim for $\widehat{A} \setminus \mathcal{C}_r^+(x, V)$ follows from the obvious inclusion $\mathcal{C}_r^+(x, V) \supset \mathcal{C}_r^-(x, V)$. □

Before we proceed we will state some results concerning the following corridors first introduced by Roblin [2003]: for $r > 0$ and $x, y \in X$ we set

$$\mathcal{L}_r(x, y) = \big\{(\xi, \eta) \in \partial_\infty \mathcal{G} : \text{there exists } v \in \partial_\infty^{-1}(\xi, \eta) \text{ and } t > 0 \\ \text{such that } v(0) \in B_r(x),\ v(t) \in B_r(y)\big\}. \quad (29)$$

Notice that if $(\xi, \eta) \notin \partial_\infty \mathcal{Z}$, then the element $v \in \partial_\infty^{-1}(\xi, \eta)$ satisfying the condition on the right-hand side is in general different from $v(x; \xi, \eta)$ (and from $g^{-t} v(y; \xi, \eta)$).

Remark. The inclusion $\mathcal{O}_r^-(y, x) \times \mathcal{O}_r^-(x, y) \subset \mathcal{L}_r(x, y)$ claimed in the middle of page 58 of [Roblin 2003] is wrong even in the hyperbolic plane $\mathbb{H}^2$ as the following counterexample provided by C. Pittet shows: Let $x = 1 + \mathrm{i}$, $y = \mathrm{e}^4 + \mathrm{i}\mathrm{e}^4$

and $r = d(x, \sqrt{2}\mathrm{i}) = d(y, \sqrt{2}\mathrm{e}^4\mathrm{i})$ (which is equal to the hyperbolic distance of x respectively y to the imaginary axis). Then elementary hyperbolic geometry shows that the geodesic line

$$\sigma : \mathbb{R} \to \mathbb{H}^2, \quad t \mapsto \mathrm{e}^t\mathrm{i}$$

satisfies $\sigma(-\infty) \in \mathcal{O}_r^-(y, x)$, $\sigma(\infty) \in \mathcal{O}_r^-(x, y)$, but $(\sigma(-\infty), \sigma(\infty)) \notin \mathcal{L}_r(x, y)$ (since $\sigma(\mathbb{R})$ is tangent to the open balls $B_r(x)$ and $B_r(y)$). Notice in particular that none of the sets $\mathcal{O}_r^-(y, x)$, $\mathcal{O}_r^-(x, y)$ is open.
As a replacement for the above inclusion we will prove Lemma 6.8 below.

From here on we fix $r > 0$, $\gamma \in \mathrm{Is}(X)$, points $x, y \in X$ and subsets $A, B \subset \partial X$. The following results relate corridors to cones and large shadows. The proof of the first one is straightforward.

Lemma 6.7. *If* $(\zeta, \xi) \in \mathcal{L}_r(x, \gamma y) \cap (\gamma B \times A)$, *then*

$$(\gamma y, \gamma^{-1}x) \in \mathcal{C}_r^+(x, A) \times \mathcal{C}_r^+(y, B) \quad \text{and} \quad (\zeta, \xi) \in \mathcal{O}_r^+(\gamma y, x) \times \mathcal{O}_r^+(x, \gamma y).$$

Lemma 6.8. *If* $(\gamma y, \gamma^{-1}x) \in \mathcal{C}_r^-(x, A) \times \mathcal{C}_r^-(y, B)$, *then*

$$\mathcal{L}_r(x, \gamma y) \cap (\gamma B \times A) \supset \{(\zeta, \xi) \in \partial X \times \partial X : \xi \in \mathcal{O}_r^-(x, \gamma y),\ \zeta \in \mathcal{O}_r(\xi, x)\}.$$

Proof. From $\zeta \in \mathcal{O}_r(\xi, x)$ we know that the geodesic line $w = v(x; \xi, \zeta) \in \mathcal{G}$ defined by (8) has origin $w(0) \in B_r(x)$. Then $v := -w \in \partial_\infty^{-1}(\zeta, \xi)$ satisfies $v(0) \in B_r(x)$, so $v^+ = \xi \in \mathcal{O}_r^-(x, \gamma y)$ implies $v(t) = \sigma_{v(0),\xi}(t) \in B_r(\gamma y)$ for some $t > 0$. We conclude $(\zeta, \xi) \in \mathcal{L}_r(x, \gamma y)$.

It remains to prove that $\zeta \in \gamma B$ and $\xi \in A$. By definition (27) $\gamma y \in \mathcal{C}_r^-(x, A)$ immediately gives $\mathcal{O}_r^-(x, \gamma y) \subset \mathcal{O}_r^+(x, \gamma y) \subset A$, hence $\xi \in A$. Moreover, from $(\zeta, \xi) \in \mathcal{L}_r(x, \gamma y)$ we directly get $\zeta \in \mathcal{O}_r^+(\gamma y, x)$. So $\gamma^{-1}\zeta \in \mathcal{O}_r^+(y, \gamma^{-1}x)$, and from $\gamma^{-1}x \in \mathcal{C}_r^-(y, B)$ we know that $\mathcal{O}_r^+(y, \gamma^{-1}x) \subset B$ according to definition (27). Hence $\gamma^{-1}\zeta \in B$ which is equivalent to $\zeta \in \gamma B$. □

We will further need the following Borel subsets of $\mathcal{G}$ which up to small details were already introduced by Roblin [2003]:

$$\begin{aligned} K_r(x) &= \big\{ g^s v(x; \xi, \eta) : (\xi, \eta) \in \partial_\infty \mathcal{Z} \text{ with } d(x, (\xi\eta)) < r,\ s \in (-r/2, r/2) \big\}, \\ K_r^+(x, A) &= \{ v \in K_r(x) : v^+ \in A \} =: K^+, \\ K_r^-(y, B) &= \{ w \in K_r(y) : w^- \in B \} =: K^-. \end{aligned} \tag{30}$$

Notice that by definition the orbit of an element $v \in \mathcal{Z}$ either never enters one of the sets above or spends precisely time r in them.

Moreover, we have the following relation to the corridors $\mathcal{L}_r(x, \gamma y)$ introduced in (29):

Lemma 6.9. *For all $\gamma \in \mathrm{Is}(X)$ with $d(x, \gamma y) \geq 3r$ we have*

$$\partial_\infty\big(\{K^+ \cap g^{-t}\gamma K^- : t > 0\}\big) = \mathcal{L}_r(x, \gamma y) \cap \partial_\infty \mathcal{Z} \cap (\gamma B \times A)$$

Proof. For the inclusion "$\subset$" we let $v \in K^+ \cap g^{-t}\gamma K^-$ for some $t > 0$. Then obviously $(\zeta, \xi) := (v^-, v^+) \in \partial_\infty \mathcal{Z}$, $\xi = v^+ \in A$ and $\zeta = v^- \in \gamma B$. Now consider $u := v(x; \zeta, \xi) \in \mathcal{Z}$ and let $\tau \in \mathbb{R}$ such that

$$v(\gamma y, \zeta, \xi) = g^\tau u;$$

such τ exists because $(\zeta, \xi) \in \partial_\infty \mathcal{Z}$. From the definition of $K_r(x)$ and $K_r(\gamma y)$ we further get $|d(x, \gamma y) - \tau| < 2r$; since $d(x, \gamma y) \geq 3r$ this implies $\tau > r > 0$. Hence $(\zeta, \xi) = (u^-, u^+) \subset \mathcal{L}_r(x, \gamma y)$.

For the converse inclusion "$\supset$" we let $(\zeta, \xi) \in \mathcal{L}_r(x, \gamma y) \cap \partial_\infty \mathcal{Z} \cap (\gamma B \times A)$ be arbitrary. Then by definition (29) there exists $v \in \mathcal{Z}$ and $t' > 0$ with

$$(v^-, v^+) = (\zeta, \xi), \quad d(x, v(0)) < r \quad \text{and} \quad d(\gamma y, v(t')) < r.$$

As above we set $u := v(x; \zeta, \xi)$ and let $\tau \in \mathbb{R}$ such that

$$v(\gamma y, \zeta, \xi) = g^\tau u.$$

Since $d(x, u(0)) \leq d(x, v(0)) < r$ and $u^+ = \xi \in A$ we have $u \in K^+$, and from $d(\gamma y, u(\tau)) \leq d(\gamma y, v(t')) < r$ and $u^- = \zeta \in \gamma B$ we further get $g^\tau u \in \gamma K^-$. Moreover we have $\tau > r > 0$ as above, so the claim is proved. □

7. The Ricks–Bowen–Margulis measure and some useful estimates

As before X will denote a proper Hadamard space and $\Gamma < \mathrm{Is}(X)$ a discrete rank one group with $\mathcal{Z}_\Gamma \neq \emptyset$. In order to get the equidistribution result Theorem B from the introduction we will have to work with the so-called Ricks–Bowen–Margulis measure: This is the Ricks measure from Section 5 associated to a particular quasiproduct geodesic current $\bar{\mu}$. We are going to describe this geodesic current now.

Definition 7.1. A δ-dimensional Γ-invariant *conformal density* is a continuous map μ of X into the cone of positive finite Borel measures on ∂X such that for all $x, y \in X$ and every $\gamma \in \Gamma$ we have

$$\begin{aligned} &\mathrm{supp}(\mu_x) \subset L_\Gamma, \\ &\gamma_* \mu_x = \mu_{\gamma x}, \quad \text{where } \gamma_*\mu_x(E) := \mu_x(\gamma^{-1}E) \text{ for all Borel sets } E \subset \partial X, \\ &\frac{\mathrm{d}\mu_x}{\mathrm{d}\mu_y}(\eta) = \mathrm{e}^{\delta \mathcal{B}_\eta(y,x)} \quad \text{for any } \eta \in \mathrm{supp}(\mu_x). \end{aligned} \tag{31}$$

Recall the definition of the critical exponent δ_Γ from (1) and notice that in our setting it is strictly positive, since Γ contains a nonabelian free subgroup generated by two independent rank one elements. For $\delta = \delta_\Gamma$ a conformal density as above can be explicitly constructed following the idea of S. J. Patterson [1976] originally used for Fuchsian groups (see for example [Knieper 1997, Lemma 2.2]). From here on we will therefore fix a δ_Γ-dimensional Γ-invariant conformal density $\mu = (\mu_x)_{x\in X}$.

With the Gromov product from (22) we will now consider as in Section 7 of [Ricks 2017] and in Section 8 of [Link 2018] for $x \in X$ the geodesic current $\bar{\mu}_x$ on $\partial_\infty \mathcal{G} \subset \partial X \times \partial X$ defined by

$$d\bar{\mu}_x(\xi,\eta) = \mathrm{e}^{2\delta_\Gamma \operatorname{Gr}_x(\xi,\eta)} \mathbb{1}_{\partial_\infty \mathcal{R}}(\xi,\eta)\, \mathrm{d}\mu_x(\xi)\, \mathrm{d}\mu_x(\eta).$$

As $\bar{\mu}_x$ does not depend on the choice of $x \in X$ we will write $\bar{\mu}$ in the sequel.

Since we want to apply Theorem 5.5 we will assume that $\mu_x(L_\Gamma^{\mathrm{rad}}) = \mu_x(\partial X)$; in view of Hopf–Tsuji–Sullivan dichotomy [Link 2018, Theorem 10.2] this is equivalent to the fact that Γ is divergent. Moreover, it is well-known that in this case the conformal density μ from above is nonatomic and unique up to scaling. So Theorem 4.1 implies that for all $x, y \in X$ we have

$$d\bar{\mu}(\xi,\eta) = \mathrm{e}^{2\delta_\Gamma \operatorname{Gr}_x(\xi,\eta)}\, \mathrm{d}\mu_x(\xi)\, \mathrm{d}\mu_x(\eta) = \mathrm{e}^{2\delta_\Gamma \operatorname{Gr}_y(\xi,\eta)}\, \mathrm{d}\mu_y(\xi)\, \mathrm{d}\mu_y(\eta) \tag{32}$$

and

$$(\mu_x \otimes \mu_x)(\partial_\infty \mathcal{Z}_\Gamma^{\mathrm{rec}}) = (\mu_x \otimes \mu_x)(\partial_\infty \mathcal{Z}) = \mu_x(\partial X)^2.$$

The Ricks measure m_Γ on $\Gamma\backslash\mathcal{G}$ associated to the geodesic current $\bar{\mu}$ from (32) is called the *Ricks–Bowen–Margulis measure*. It generalizes the well-known Bowen–Margulis measure in the CAT(-1)-setting. Moreover, for the measure m from which it descends we have the formula (19). Notice also that if X is a manifold and Γ is cocompact, then the Ricks–Bowen–Margulis measure is equal to the measure of maximal entropy m_Γ^{Kn} described in [Knieper 1998] (this is Knieper's measure associated to $\bar{\mu}$ from (32)). We further remark that the constant Δ defined in (20) is equal to $2\delta_\Gamma$ in this case (compare the last paragraph in Section 8 of [Link 2018]), hence in particular finite.

Fix $r > 0$, points $x, y \in X$ and subsets $A, B \subset \partial X$. We will first compute the measure of the sets introduced in (30). From (19), (32) and the remark below (30) we get

$$\begin{aligned} m(K^+) &= \int_{\partial_\infty \mathcal{Z}} \mathrm{d}\mu_x(\xi)\, \mathrm{d}\mu_x(\eta)\, \mathrm{e}^{2\delta_\Gamma \operatorname{Gr}_x(\xi,\eta)} \int \mathbb{1}_{K^+}\big(g^s v(x;\xi,\eta)\big)\, \mathrm{d}s \\ &= r \int_A \mathrm{d}\mu_x(\xi) \int_{\mathcal{O}_r(\xi,x)} \mathrm{d}\mu_x(\eta)\, \mathrm{e}^{2\delta_\Gamma \operatorname{Gr}_x(\xi,\eta)}, \end{aligned}$$

and similarly

$$m(K^-) = r \int_B \mathrm{d}\mu_y(\eta) \int_{\mathcal{O}_r(\eta,y)} \mathrm{d}\mu_y(\xi)\, \mathrm{e}^{2\delta_\Gamma \operatorname{Gr}_y(\xi,\eta)}.$$

From the nonnegativity of the Gromov-product (22) and the fact that

$$\mathrm{Gr}_x(\xi,\eta)\le r \quad \text{if } \eta\in\mathcal{O}_r(\xi,x)$$

we further get the useful estimates

$$r\int_A \mathrm{d}\mu_x(\xi)\,\mu_x(\mathcal{O}_r(\xi,x))\le m(K^+)\le r\mathrm{e}^{2\delta_\Gamma r}\int_A \mathrm{d}\mu_x(\xi)\,\mu_x(\mathcal{O}_r(\xi,x)), \tag{33}$$

$$r\int_B \mathrm{d}\mu_y(\eta)\,\mu_y(\mathcal{O}_r(\eta,y))\le m(K^-)\le r\mathrm{e}^{2\delta_\Gamma r}\int_B \mathrm{d}\mu_y(\eta)\,\mu_x(\mathcal{O}_r(\eta,y)).$$

We continue with the important:

Lemma 7.2. *Let* $T_0>6r,\ T>T_0+3r,\ \gamma\in\Gamma,\ (\xi,\eta)\in\mathcal{L}_r(x,\gamma y)\cap\partial_\infty\mathcal{Z}$ *and* $s\in(-r/2,r/2)$. *Then*

(a) $$\int_{T_0}^{T+3r}\mathrm{e}^{\delta_\Gamma t}\mathbb{1}_{K_r(\gamma y)}(g^{t+s}v(x;\xi,\eta))\,\mathrm{d}t\ge r\cdot\mathrm{e}^{-3\delta_\Gamma r}\mathrm{e}^{\delta_\Gamma d(x,\gamma y)}$$
if $T_0+3r<d(x,\gamma y)\le T$,

(b) $$\int_{T_0}^{T-3r}\mathrm{e}^{\delta_\Gamma t}\mathbb{1}_{K_r(\gamma y)}(g^{t+s}v(x;\xi,\eta))\,\mathrm{d}t\le r\cdot\mathrm{e}^{3\delta_\Gamma r}\mathrm{e}^{\delta_\Gamma d(x,\gamma y)},$$
and $$\int_{T_0}^{T-3r}\mathrm{e}^{\delta_\Gamma t}\mathbb{1}_{K_r(\gamma y)}(g^{t+s}v(x;\xi,\eta))\,\mathrm{d}t=0$$
if $d(x,\gamma y)\le T_0-3r$ *or* $d(x,\gamma y)>T$.

Proof. Denote $v=v(x;\xi,\eta)\in\mathcal{Z}$ and let $\tau>0$ such that $g^\tau v=v(\gamma y;\xi,\eta)$. Since $(\xi,\eta)\in\mathcal{L}_r(x,\gamma y)$, the triangle inequality yields

$$|d(x,\gamma y)-\tau|<2r.$$

By definition of $K_r(\gamma y)$ we have $g^{t+s}v\in K_r(\gamma y)$ if and only if $|t+s-\tau|<r/2$. Hence if $\tau-s-r/2\ge T_0$ and $\tau-s+r/2\le T+3r$, then

$$\int_{T_0}^{T+3r}\mathrm{e}^{\delta_\Gamma t}\mathbb{1}_{K_r(\gamma y)}(g^{t+s}v(x;\xi,\eta))\,\mathrm{d}t=\int_{\tau-s-r/2}^{\tau-s+r/2}\mathrm{e}^{\delta_\Gamma t}\,\mathrm{d}t$$
$$\ge r\cdot\mathrm{e}^{\delta_\Gamma(\tau-s-r/2)}\ge r\cdot e^{-3\delta_\Gamma r}\mathrm{e}^{\delta_\Gamma d(x,\gamma y)}.$$

Now $d(x,\gamma y)\in(T_0+3r,T]$ and $s\in(-r/2,r/2)$ imply

$$\tau-s-r/2\ge d(x,\gamma y)-2r-r/2-r/2\ge T_0\quad\text{and}$$
$$\tau-s+r/2\le d(x,\gamma y)+2r+r/2+r/2\le T+3r,$$

so (a) holds.

In order to prove (b) we first notice that

$$\int_{T_0}^{T-3r}\mathrm{e}^{\delta_\Gamma t}\mathbb{1}_{K_r(\gamma y)}(g^{t+s}v(x;\xi,\eta))\,\mathrm{d}t\le\int_{\tau-s-r/2}^{\tau-s+r/2}\mathrm{e}^{\delta_\Gamma t}\,\mathrm{d}t$$
$$\le r\cdot\mathrm{e}^{\delta_\Gamma(\tau-s+r/2)}\le r\cdot e^{3\delta_\Gamma r}\mathrm{e}^{\delta_\Gamma d(x,\gamma y)};$$

this proves the first assertion in (b).

Now if $d(x,\gamma y) \le T_0 - 3r$, then

$$\tau - s + r/2 \le d(x,\gamma y) + 2r + r \le T_0,$$

and if $d(x,\gamma y) \ge T$, then

$$\tau - s - r/2 \ge d(x,\gamma y) - 2r - r \ge T - 3r,$$

hence the integral in (b) equals zero in both cases. □

Moreover, from Lemma 6.9 we immediately get:

Corollary 7.3. *For all $\gamma \in \mathrm{Is}(X)$ with $d(x,\gamma y) > 3r$ and all $t > 0$ we have*

$$m(K^+ \cap g^{-t}\gamma K^-) = \int_{\mathcal{L}_r(x,\gamma y)\cap(\gamma B\times A)} \mathrm{d}\mu_x(\xi)\,\mathrm{d}\mu_x(\eta)\,\mathrm{e}^{2\delta_\Gamma \mathrm{Gr}_x(\xi,\eta)} \cdot \int_{-r/2}^{r/2} \mathbb{1}_{K_r(\gamma y)}(g^{t+s}v(x;\xi,\eta))\,\mathrm{d}s.$$

8. Equidistribution

We keep the notation and the setting from the previous section and will now address the question of equidistribution of Γ-orbit points in X. In order to get a reasonable statement we will have to require that the Ricks–Bowen–Margulis measure m_Γ is *finite*:

Theorem 8.1. *Let $\Gamma < \mathrm{Is}(X)$ be a discrete rank one group with nonarithmetic length spectrum, $\mathcal{Z}_\Gamma \ne \varnothing$ and finite Ricks–Bowen–Margulis measure m_Γ.*

Let f be a continuous function from $\bar{X}\times\bar{X}$ to $\mathbb{R}$, and $x, y \in X$. Then

$$\lim_{T\to\infty} \delta_\Gamma \mathrm{e}^{-\delta_\Gamma T} \sum_{\substack{\gamma\in\Gamma \\ d(x,\gamma y)\le T}} f(\gamma y, \gamma^{-1}x) = \frac{1}{\|m_\Gamma\|}\int_{\partial X\times\partial X} f(\xi,\eta)\,\mathrm{d}\mu_x(\xi)\,\mathrm{d}\mu_y(\eta).$$

Our proof will closely follow Roblin's strategy for his [2003, théorème 4.1.1]: Using mixing of the geodesic flow one proves that for all sufficiently small Borel sets $A, B \subset \partial X$ the limit inferior and the limit superior of the measures

$$\nu_{x,y}^T := \delta_\Gamma \cdot \mathrm{e}^{-\delta_\Gamma T} \sum_{\substack{\gamma\in\Gamma \\ d(x,\gamma y)\le T}} \mathcal{D}_{\gamma y} \otimes \mathcal{D}_{\gamma^{-1}x} \tag{34}$$

as T tends to infinity evaluated on products of "cones" of opening A, B is approximately $\mu_x(A)\cdot\mu_y(B)/\|m_\Gamma\|$.

In the first step we only deal with sufficiently small open neighborhoods of pairs of boundary points which are in a "nice" position with respect to x and y; then one shows that the estimates hold for all pairs of sufficiently small Borel sets A and B. The final step consists in the full proof by globalization with respect to A and B.

The following Proposition provides the first step in the proof of Theorem 8.1:

Proposition 8.2. *Let $\varepsilon > 0$, $(\xi_0, \eta_0) \in \partial X \times \partial X$ and $x, y \in X$ with trivial stabilizer in Γ and such that $x \in (\xi_0 v^+)$, $y \in (\eta_0 w^+)$ for some Γ-recurrent elements $v, w \in \mathcal{Z}$. Then there exist open neighborhoods $V, W \subset \partial X$ of ξ_0, η_0 such that for all Borel sets $A \subset V$, $B \subset W$*

$$\limsup_{T\to\infty} \nu^T_{x,y}\big(\mathcal{C}^-_1(x, A) \times \mathcal{C}^-_1(y, B)\big) \le \mathrm{e}^{\varepsilon} \mu_x(A)\mu_y(B)/\|m_\Gamma\|,$$
$$\liminf_{T\to\infty} \nu^T_{x,y}\big(\mathcal{C}^+_1(x, A) \times \mathcal{C}^+_1(y, B)\big) \ge \mathrm{e}^{-\varepsilon} \mu_x(A)\mu_y(B)/\|m_\Gamma\|.$$

Proof. Set $\rho := \min\{d(x, \gamma x),\ d(y, \gamma y) : \gamma \in \Gamma \setminus \{e\}\}$.

Let $\varepsilon > 0$ arbitrary. We first fix $r \in (0, \min\{1, \rho/3, \varepsilon/(30\delta_\Gamma)\})$ such that

$$\mu_x(\tilde{\partial}\mathcal{O}_r(\xi_0, x)) = 0 = \mu_y(\tilde{\partial}\mathcal{O}_r(\eta_0, y)).$$

Since $v^+ \in L_\Gamma \cap \mathcal{O}_r(\xi_0, x)$ and $w^+ \in L_\Gamma \cap \mathcal{O}_r(\eta_0, y)$, both shadows $\mathcal{O}_r(\xi_0, x)$ and $\mathcal{O}_r(\eta_0, y)$ contain an open neighborhood of a limit point of Γ by Lemma 2.1. So from $\mathrm{supp}(\mu_x) = \mathrm{supp}(\mu_y) = L_\Gamma$ and the definition (16) of the support of a measure we have

$$\mu_x(\mathcal{O}_r(\xi_0, x)) \cdot \mu_y(\mathcal{O}_r(\eta_0, y)) > 0.$$

Moreover, according to Lemma 2.1 and Corollary 6.4 there exist open neighborhoods $\widehat{V}, \widehat{W} \subset \overline{X}$ of ξ_0, η_0 such that if $(a, b) \in \widehat{V} \times \widehat{W}$, then a can be joined to v^+, b can be joined to w^+ by a rank one geodesic, and

$$\begin{aligned} \mathrm{e}^{-\varepsilon/30}\mu_x(\mathcal{O}_r(\xi_0, x)) &\le \mu_x(\mathcal{O}^\pm_r(a, x)) \le \mathrm{e}^{\varepsilon/30}\mu_x(\mathcal{O}_r(\xi_0, x)), \\ \mathrm{e}^{-\varepsilon/30}\mu_y(\mathcal{O}_r(\eta_0, y)) &\le \mu_y(\mathcal{O}^\pm_r(b, y)) \le \mathrm{e}^{\varepsilon/30}\mu_y(\mathcal{O}_r(\eta_0, y)). \end{aligned} \tag{35}$$

Let $V, W \subset \partial X$ be open neighborhoods of ξ_0, η_0 such that $\overline{V} \subset \widehat{V} \cap \partial X$ and $\overline{W} \subset \widehat{W} \cap \partial X$. Let $A \subset V$, $B \subset W$ be arbitrary Borel sets.

Roblin's method consists in giving upper and lower bounds for the asymptotics of the integrals

$$\int_{T_0}^{T\pm 3r} \mathrm{e}^{\delta_\Gamma t} \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-)\,\mathrm{d}t$$

as T tends to infinity: on the one hand one uses mixing to relate the integrals to $\mu_x(A) \cdot \mu_y(B)$; on the other hand one computes direct estimates for the integrals to get a relation to the measures $\nu^T_{x,y}\big(\mathcal{C}^\pm_1(x, A) \times \mathcal{C}^\pm_1(y, B)\big)$.

Let us start by exploiting the mixing property. Notice that by choice of $r < \rho/3$ and the definition of ρ we have

$$K_r(x) \cap \gamma K_r(x) = \varnothing \quad \text{and} \quad K_r(y) \cap \gamma K_r(y) = \varnothing \qquad \text{for all } \gamma \in \Gamma \setminus \{e\},$$

hence the projection map $\mathcal{G} \to \Gamma\backslash\mathcal{G}$ restricted to $K^{\pm}$ is injective. So we can apply Corollary 5.6 to get

$$\lim_{t\to\infty} \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-) = \frac{m(K^+)\cdot m(K^-)}{\|m_\Gamma\|}.$$

Hence there exists $T_0 > 6r$ such that for $t \geq T_0$ we have

$$\begin{aligned} e^{-\varepsilon/3} m(K^+)\cdot m(K^-) &\leq \|m_\Gamma\| \cdot \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-) \\ &\leq e^{\varepsilon/3} m(K^+)\cdot m(K^-). \end{aligned} \tag{36}$$

Combining (33) and the estimates (35) we obtain from $A \subset \widehat{V}$ and $B \subset \widehat{W}$

$$\begin{aligned} r e^{-\varepsilon/30} \mu_x(\mathcal{O}_r(\xi_0, x))\mu_x(A) \leq m(K^+) \leq r e^{2\delta_\Gamma r} e^{\varepsilon/30} \mu_x(\mathcal{O}_r(\xi_0, x))\mu_x(A), \\ r e^{-\varepsilon/30} \mu_y(\mathcal{O}_r(\eta_0, y))\mu_y(B) \leq m(K^-) \leq r e^{2\delta_\Gamma r} e^{\varepsilon/30} \mu_y(\mathcal{O}_r(\eta_0, y))\mu_y(B); \end{aligned}$$

using the abbreviation $M = r^2 \mu_x(\mathcal{O}_r(\xi_0, x))\mu_y(\mathcal{O}_r(\eta_0, y)) > 0$ and $\delta_\Gamma r \leq \varepsilon/30$, we get

$$e^{-\varepsilon/15} M \mu_x(A)\mu_y(B) \leq m(K^+)m(K^-) \leq e^{\varepsilon/5} M \mu_x(A)\mu_y(B). \tag{37}$$

Hence according to (36) we have for $t \geq T_0$

$$\begin{aligned} M\mu_x(A)\mu_y(B) &\leq e^{\varepsilon/15} e^{\varepsilon/3} \|m_\Gamma\| \cdot \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-), \\ M\mu_x(A)\mu_y(B) &\geq e^{-\varepsilon/5} e^{-\varepsilon/3} \|m_\Gamma\| \cdot \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-). \end{aligned}$$

We now integrate the inequalities to get

$$\begin{aligned} &\big(e^{\delta_\Gamma (T-3r)} - e^{\delta_\Gamma T_0}\big) M\mu_x(A)\mu_y(B) \\ &\qquad = \delta_\Gamma \int_{T_0}^{T-3r} e^{\delta_\Gamma t} M\mu_x(A)\mu_y(B)\,\mathrm{d}t \\ &\qquad \leq e^{2\varepsilon/5} \|m_\Gamma\| \cdot \delta_\Gamma \int_{T_0}^{T-3r} e^{\delta_\Gamma t} \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-), \end{aligned} \tag{38}$$

$$\begin{aligned}\left(\mathrm{e}^{\delta_\Gamma(T+3r)}-\mathrm{e}^{\delta_\Gamma T_0}\right)&M\mu_x(A)\mu_y(B)\\ &=\delta_\Gamma\int_{T_0}^{T+3r}\mathrm{e}^{\delta_\Gamma t}M\mu_x(A)\mu_y(B)\,\mathrm{d}t\\ &\ge e^{-8\varepsilon/15}\|m_\Gamma\|\cdot\delta_\Gamma\int_{T_0}^{T+3r}\mathrm{e}^{\delta_\Gamma t}\sum_{\gamma\in\Gamma}m(K^+\cap g^{-t}\gamma K^-).\end{aligned}\tag{39}$$

We will next give upper and lower bounds for the integrals on the right-hand side: For the upper bound we first remark that

$$(\xi,\eta)\in\mathcal{L}_r(x,\gamma y)\cap\partial_\infty\mathcal{Z}\quad\text{implies}\quad \mathrm{Gr}_x(\xi,\eta)<r.$$

Moreover, our choice of $T_0>6r$ guarantees that $K^+\cap g^{-t}\gamma K^-\neq\varnothing$ for some $t\ge T_0$ implies $d(x,\gamma y)>3r$. Applying Corollary 7.3 we therefore get

$$\begin{aligned}\int_{T_0}^{T-3r}&\mathrm{e}^{\delta_\Gamma t}\sum_{\gamma\in\Gamma}m(K^+\cap g^{-t}\gamma K^-)\,\mathrm{d}t\\ &\le\sum_{\gamma\in\Gamma}\int_{\mathcal{L}_r(x,\gamma y)\cap(\gamma B\times A)}\mathrm{d}\mu_x(\xi)\,\mathrm{d}\mu_x(\eta)\,\mathrm{e}^{2\delta_\Gamma r}\\ &\qquad\qquad\cdot\int_{-r/2}^{r/2}\left(\int_{T_0}^{T-3r}\mathbb{1}_{K_r(\gamma y)}(g^{t+s}v(x;\xi,\eta))\mathrm{e}^{\delta_\Gamma t}\,\mathrm{d}t\right)\mathrm{d}s\\ &\le\mathrm{e}^{2\delta_\Gamma r}\cdot r^2\cdot\mathrm{e}^{3\delta_\Gamma r}\sum_{\substack{\gamma\in\Gamma\\ d(x,\gamma y)\le T}}\int_{\mathcal{L}_r(x,\gamma y)\cap(\gamma B\times A)}\mathrm{d}\mu_x(\xi)\,\mathrm{d}\mu_x(\eta)\cdot\mathrm{e}^{\delta_\Gamma d(x,\gamma y)};\end{aligned}$$

here we used Lemma 7.2(b) in the last step. Lemma 6.7, $r\le1$ and the first estimate in (28) further imply

$$\begin{aligned}\int_{T_0}^{T-3r}&\mathrm{e}^{\delta_\Gamma t}\sum_{\gamma\in\Gamma}m\left(K^+\cap g^{-t}\gamma K^-\right)\mathrm{d}t\\ &\le r^2\mathrm{e}^{5\delta_\Gamma r}\sum_{\substack{\gamma\in\Gamma\\ d(x,\gamma y)\le T\\ (\gamma y,\gamma^{-1}x)\in\mathcal{C}_1^+(x,A)\times\mathcal{C}_1^+(y,B)}}\int_{\mathcal{O}_r^+(\gamma y,x)}\mathrm{d}\mu_x(\xi)\int_{\mathcal{O}_r^+(x,\gamma y)}\mathrm{d}\mu_x(\eta)\,\mathrm{e}^{\delta_\Gamma d(x,\gamma y)}.\end{aligned}$$

Using the fact that for all $\eta\in\mathcal{O}_r^+(x,\gamma y)$ we have $\mathcal{B}_\eta(x,\gamma y)\ge d(x,\gamma y)-4r$, Γ-equivariance and conformality (31) of μ imply

$$\int_{\mathcal{O}_r^+(x,\gamma y)}\mathrm{d}\mu_x(\eta)\,\mathrm{e}^{\delta_\Gamma d(x,\gamma y)}\le\mathrm{e}^{4\delta_\Gamma r}\mu_y(\mathcal{O}_r^+(\gamma^{-1}x,y)).$$

Moreover, since by Lemma 6.6(a) there are only finitely many $\gamma\in\Gamma$ such that

$$(\gamma y,\gamma^{-1}x)\in\left(\mathcal{C}_1^+(x,A)\times\mathcal{C}_1^+(y,B)\right)\setminus(\widehat{V}\times\widehat{W}),$$

restricting the summation to $\gamma \in \Gamma$ with

$$(\gamma y, \gamma^{-1}x) \in \big(\mathcal{C}_1^+(x, A) \times \mathcal{C}_1^+(y, B)\big) \cap (\widehat{V} \times \widehat{W})$$

only contributes a constant C to the upper bound. So with our choice of $r \le 1$ and $r \le \varepsilon/(30\delta_\Gamma)$ we conclude

$$\begin{aligned}
&\int_{T_0}^{T-3r} \mathrm{e}^{\delta_\Gamma t} \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-)\,\mathrm{d}t \\
&\quad\le r^2 \mathrm{e}^{9\varepsilon/30} \sum_{\substack{\gamma\in\Gamma \\ d(x,\gamma y)\le T \\ (\gamma y,\gamma^{-1}x)\in(\mathcal{C}_1^+(x,A)\times\mathcal{C}_1^+(y,B))\cap(\widehat{V}\times\widehat{W})}} \mu_x(\mathcal{O}_r^+(\gamma y, x))\mu_y(\mathcal{O}_r^+(\gamma^{-1}x, y)) + C \\
&\quad\overset{(35)}{\le} r^2 \mathrm{e}^{11\varepsilon/30} \sum_{\substack{\gamma\in\Gamma \\ d(x,\gamma y)\le T \\ (\gamma y,\gamma^{-1}x)\in(\mathcal{C}_1^+(x,A)\times\mathcal{C}_1^+(y,B))\cap(\widehat{V}\times\widehat{W})}} \mu_x(\mathcal{O}_r(\xi_0, x))\mu_y(\mathcal{O}_r(\eta_0, y)) + C, \\
&\quad\le \mathrm{e}^{11\varepsilon/30} M \frac{\mathrm{e}^{\delta_\Gamma T}}{\delta_\Gamma} \nu_{x,y}^T\big(\mathcal{C}_1^+(x, A) \times \mathcal{C}_1^+(y, B)\big) + C.
\end{aligned}$$

Plugging this in the inequality (38) divided by $M\mathrm{e}^{\delta_\Gamma(T-3r)} \cdot \|m_\Gamma\|$ we get (with a constant C' independent of T)

$$\begin{aligned}
&\frac{1-\mathrm{e}^{\delta_\Gamma(-T+3r+T_0)}}{\|m_\Gamma\|}\mu_x(A)\mu_y(B) \\
&\qquad\le \mathrm{e}^{2\varepsilon/5}\mathrm{e}^{11\varepsilon/30}\mathrm{e}^{3\delta_\Gamma r}\nu_{x,y}^T\big(\mathcal{C}_1^+(x, A) \times \mathcal{C}_1^+(y, B)\big) + C'\mathrm{e}^{-\delta_\Gamma T} \\
&\qquad\le \mathrm{e}^{13\varepsilon/15}\nu_{x,y}^T\big(\mathcal{C}_1^+(x, A) \times \mathcal{C}_1^+(y, B)\big) + C'\mathrm{e}^{-\delta_\Gamma T},
\end{aligned}$$

which proves

$$\liminf_{T\to\infty} \nu_{x,y}^T\big(\mathcal{C}_1^+(x, A) \times \mathcal{C}_1^+(y, B)\big) \ge \mathrm{e}^{-\varepsilon}\mu_x(A)\mu_y(B)/\|m_\Gamma\|.$$

We finally turn to the lower bound. Using again Corollary 7.3 and the nonnegativity of the Gromov product (22) we estimate

$$\begin{aligned}
&\int_{T_0}^{T+3r} \mathrm{e}^{\delta_\Gamma t} \sum_{\gamma\in\Gamma} m\big(K^+ \cap g^{-t}\gamma K^-\big)\,\mathrm{d}t \\
&\quad\ge \sum_{\gamma\in\Gamma} \int_{\mathcal{L}_r(x,\gamma y)\cap(\gamma B\times A)} \mathrm{d}\mu_x(\xi)\,\mathrm{d}\mu_x(\eta)\,\mathrm{e}^{2\delta_\Gamma\cdot 0} \\
&\qquad\qquad\cdot \int_{-r/2}^{r/2}\Big(\int_{T_0}^{T+3r} \mathbb{1}_{K_r(\gamma y)}(g^{t+s}v(x;\xi,\eta))\mathrm{e}^{\delta_\Gamma t}\,\mathrm{d}t\Big)\,\mathrm{d}s \\
&\quad\ge r^2\mathrm{e}^{-3\delta_\Gamma r} \sum_{\substack{\gamma\in\Gamma \\ T_0+3r<d(x,\gamma y)\le T}} \int_{\mathcal{L}_r(x,\gamma y)\cap(\gamma B\times A)} \mathrm{d}\mu_x(\xi)\,\mathrm{d}\mu_x(\eta)\cdot\mathrm{e}^{\delta_\Gamma d(x,\gamma y)},
\end{aligned}$$

where we used Lemma 7.2(a) in the last step.

By Lemma 6.8, $r \le 1$ and the second estimate in (28) we have for all $\gamma \in \Gamma$ with $(\gamma y, \gamma^{-1} x) \in \mathcal{C}_1^-(x, A) \times \mathcal{C}_1^-(y, B) \subset \mathcal{C}_r^-(x, A) \times \mathcal{C}_r^-(y, B)$

$$\mathcal{L}_r(x, \gamma y) \cap (\gamma B \times A) \supset \big\{(\zeta, \xi) \in \partial X \times \partial X : \xi \in \mathcal{O}_r^-(x, \gamma y),\ \zeta \in \mathcal{O}_r(\xi, x)\big\},$$

hence

$$\begin{aligned}&\int_{T_0}^{T+3r} e^{\delta_\Gamma t} \sum_{\gamma \in \Gamma} m(K^+ \cap g^{-t} \gamma K^-)\, dt \\ &\ge r^2 \cdot e^{-\varepsilon/10} \sum_{\substack{\gamma \in \Gamma \\ T_0 + 3r < d(x, \gamma y) \le T \\ (\gamma y, \gamma^{-1} x) \in (\mathcal{C}_1^-(x, A) \times \mathcal{C}_1^-(y, B)) \cap (\widehat{V} \times \widehat{W})}} \int_{\mathcal{O}_r^-(x, \gamma y)} d\mu_x(\xi)\, e^{\delta_\Gamma d(x, \gamma y)} \cdot \mu_x(\mathcal{O}_r(\xi, x)).\end{aligned}$$

Notice that

$$\gamma y \in \mathcal{C}_1^-(x, A) \subset \mathcal{C}_r^-(x, A) \quad \text{implies} \quad \mathcal{O}_r^-(x, \gamma y) \subset \mathcal{O}_r^+(x, \gamma y) \subset A \subset \widehat{V}$$

by definition of the small cones. Hence (35) shows that for all $\xi \in \mathcal{O}_r^-(x, \gamma y)$ we have

$$\mu_x(\mathcal{O}_r(\xi, x)) \ge e^{-\varepsilon/30} \mu_x(\mathcal{O}_r(\xi_0, x)).$$

By Γ-equivariance and conformality of μ we further have

$$\int_{\mathcal{O}_r^-(x, \gamma y)} d\mu_x(\xi)\, e^{\delta_\Gamma d(x, \gamma y)} \ge \mu_y(\mathcal{O}_r^-(\gamma^{-1} x, y)) \ge e^{-\varepsilon/30} \mu_y(\mathcal{O}_r(\eta_0, y)),$$

where the last inequality follows from $\gamma^{-1} x \in \widehat{W}$ and (35). Altogether this proves

$$\begin{aligned}&\int_{T_0}^{T+3r} e^{\delta_\Gamma t} \sum_{\gamma \in \Gamma} m(K^+ \cap g^{-t} \gamma K^-)\, dt \\ &\qquad \ge r^2 \cdot e^{-\varepsilon/6} \sum_{\substack{\gamma \in \Gamma \\ T_0 + 3r < d(x, \gamma y) \le T \\ (\gamma y, \gamma^{-1} x) \in \mathcal{C}_1^-(x, A) \times \mathcal{C}_1^-(y, B) \cap (\widehat{V} \times \widehat{W})}} \mu_x(\mathcal{O}_r(\xi_0, x)) \mu_y(\mathcal{O}_r(\eta_0, y)).\end{aligned}$$

Since the number of elements $\gamma \in \Gamma$ with $d(x, \gamma y) \le T_0 + 3r$ or with

$$(\gamma y, \gamma^{-1} x) \in \big(\mathcal{C}_1^-(x, A) \times \mathcal{C}_1^-(y, B)\big) \setminus (\widehat{V} \times \widehat{W})$$

is finite thanks to Lemma 6.6(a), there exists a constant $C > 0$ such that

$$\begin{aligned}\int_{T_0}^{T+3r} & \mathrm{e}^{\delta_\Gamma t} \sum_{\gamma\in\Gamma} m(K^+ \cap g^{-t}\gamma K^-)\,\mathrm{d}t \\ &\geq r^2 \cdot \mathrm{e}^{-\varepsilon/6} \sum_{\substack{\gamma\in\Gamma \\ d(x,\gamma y)\leq T \\ (\gamma y,\gamma^{-1}x)\in\mathcal{C}_1^-(x,A)\times\mathcal{C}_1^-(y,B)}} \mu_x(\mathcal{O}_r(\xi_0,x))\mu_y(\mathcal{O}_r(\eta_0,y)) - C \\ &\geq \mathrm{e}^{-\varepsilon/6} M \frac{\mathrm{e}^{\delta_\Gamma T}}{\delta_\Gamma} \nu_{x,y}^T\big(\mathcal{C}_1^-(x,A)\times\mathcal{C}_1^-(y,B)\big) - C. \end{aligned} \tag{40}$$

Plugging this in the inequality (39) divided by $M\mathrm{e}^{\delta_\Gamma(T+3r)}\cdot\|m_\Gamma\|$ we get (with a constant C' independent of T)

$$\begin{aligned}\frac{1-\mathrm{e}^{\delta_\Gamma(-T-3r+T_0)}}{\|m_\Gamma\|}&\mu_x(A)\mu_y(B) \\ &\geq \mathrm{e}^{-8\varepsilon/15}\mathrm{e}^{-\varepsilon/6}\mathrm{e}^{-3\delta_\Gamma r}\nu_{x,y}^T\big(\mathcal{C}_1^-(x,A)\times\mathcal{C}_1^-(y,B)\big) - C'\mathrm{e}^{-\delta_\Gamma T} \\ &= \mathrm{e}^{-12\varepsilon/15}\nu_{x,y}^T\big(\mathcal{C}_1^-(x,A)\times\mathcal{C}_1^-(y,B)\big) + C'\mathrm{e}^{-\delta_\Gamma T},\end{aligned}$$

which proves

$$\limsup_{T\to\infty} \nu_{x,y}^T\big(\mathcal{C}_1^-(x,A)\times\mathcal{C}_1^-(y,B)\big) \leq \mathrm{e}^{\varepsilon}\mu_x(A)\mu_y(B)/\|m_\Gamma\|. \qquad \square$$

The next Proposition is the second step in the proof of Theorem 8.1:

Proposition 8.3. *Let $\varepsilon > 0$ and $x, y \in X$ arbitrary. Then for all $(\xi_0,\eta_0)\in\partial X\times\partial X$ there exists $r > 0$ and open neighborhoods $V\subset\partial X$ of ξ_0, $W\subset\partial X$ of η_0 such that for all Borel sets $A\subset V,\ B\subset W$*

$$\begin{aligned}&\limsup_{T\to\infty} \nu_{x,y}^T\big(\mathcal{C}_r^-(x,A)\times\mathcal{C}_r^-(y,B)\big) \leq \mathrm{e}^{\varepsilon}\mu_x(A)\mu_y(B)/\|m_\Gamma\|, \\ &\liminf_{T\to\infty} \nu_{x,y}^T\big(\mathcal{C}_r^+(x,A)\times\mathcal{C}_r^+(y,B)\big) \geq \mathrm{e}^{-\varepsilon}\mu_x(A)\mu_y(B)/\|m_\Gamma\|.\end{aligned}$$

Proof. Let $(\xi_0,\eta_0)\in\partial X\times\partial X$ be arbitrary. Choose Γ-recurrent geodesics $v, w\in\mathcal{Z}$ and $x_0\in(\xi_0 v^+)$, $y_0\in(\eta_0 w^+)$ with trivial stabilizers in Γ. Let $V_0, W_0\subset\partial X$ be open neighborhoods of ξ_0 and η_0 such that the statement of Proposition 8.2 is true for x_0, y_0 instead of x, y, V_0, W_0 instead of V, W and $\varepsilon/3$ instead of ε.

Choose open neighborhoods $\widehat{V}_0, \widehat{W}_0$ of ξ_0, η_0 such that $\widehat{V}_0\cap\partial X\subset V_0$, $\widehat{W}_0\cap\partial X\subset W_0$ and

$$\begin{aligned}\big|d(x_0,a)-d(x,a)-\mathcal{B}_{\xi_0}(x_0,x)\big| &< \frac{\varepsilon}{6\delta_\Gamma}, \\ \big|d(y_0,b)-d(y,b)-\mathcal{B}_{\eta_0}(y_0,y)\big| &< \frac{\varepsilon}{6\delta_\Gamma}\end{aligned} \tag{41}$$

for all $(a,b)\in \widehat{V}_0\times\widehat{W}_0$. Notice that if $a=\xi\in\partial X$ we use the convention that $d(x_0,a)-d(x,a)=\mathcal{B}_a(x_0,x)$ and similarly for $b=\eta\in\partial X$.

Now let V, $W\subset\partial X$ be neighborhoods of ξ_0, η_0 such that for the closures we have $\overline{V}\subset\widehat{V}_0\cap\partial X$ and $\overline{W}\subset\widehat{W}_0\cap\partial X$. We further set

$$r=1+\max\{d(x,x_0),d(y,y_0)\},$$

and let $A\subset V$, $B\subset W$ be arbitrary Borel sets. From the choice of r above and Lemma 6.5(b) we immediately deduce that $(\gamma y,\gamma^{-1}x)\in\mathcal{C}_r^-(x,A)\times\mathcal{C}_r^-(y,B)$ implies

$$(\gamma y_0,\gamma^{-1}x_0)\in\mathcal{C}_1^-(x_0,A)\times\mathcal{C}_1^-(y_0,B).$$

For $r>0$ we set

$$\widehat{V}_{-r}:=\{z\in X:\overline{B_r(z)}\subset\widehat{V}_0\}\cup(\widehat{V}_0\cap\partial X).$$

If $d(x,\gamma y)\le T$ and $(\gamma y,\gamma^{-1}x)\in\widehat{V}_{-r}\times\widehat{W}_0$, then $(\gamma y_0,\gamma^{-1}x)\in\widehat{V}_0\times\widehat{W}_0$ and hence

$$\begin{aligned}
d(x_0,\gamma y_0)&\le d(x,\gamma y_0)+\mathcal{B}_{\xi_0}(x_0,x)+\frac{\varepsilon}{6\delta_\Gamma}=d(y_0,\gamma^{-1}x)+\mathcal{B}_{\xi_0}(x_0,x)+\frac{\varepsilon}{6\delta_\Gamma}\\
&\le d(y,\gamma^{-1}x)+\mathcal{B}_{\eta_0}(y_0,y)+\mathcal{B}_{\xi_0}(x_0,x)+\frac{\varepsilon}{3\delta_\Gamma}\\
&\le T+\mathcal{B}_{\eta_0}(y_0,y)+\mathcal{B}_{\xi_0}(x_0,x)+\frac{\varepsilon}{3\delta_\Gamma}.
\end{aligned}$$

So we conclude that for $T\gg 1$

$$\begin{aligned}
&\mathrm{e}^{-\delta_\Gamma T}\#\big\{\gamma\in\Gamma:d(x,\gamma y)\le T,\ (\gamma y,\gamma^{-1}x)\in\big(\mathcal{C}_r^-(x,A)\times\mathcal{C}_r^-(y,B)\big)\cap(\widehat{V}_{-r}\times\widehat{W}_0)\big\}\\
&\quad\le \mathrm{e}^{\varepsilon/3}\cdot\mathrm{e}^{\delta_\Gamma\left(\mathcal{B}_{\eta_0}(y_0,y)+\mathcal{B}_{\xi_0}(x_0,x)\right)}\cdot\mathrm{e}^{-\delta_\Gamma\left(T+\mathcal{B}_{\eta_0}(y_0,y)+\mathcal{B}_{\xi_0}(x_0,x)+\varepsilon/3\delta_\Gamma\right)}\\
&\qquad\cdot\#\Big\{\gamma\in\Gamma:d(x_0,\gamma y_0)\le T+\mathcal{B}_{\eta_0}(y_0,y)+\mathcal{B}_{\xi_0}(x_0,x)+\varepsilon/3\delta_\Gamma,\\
&\qquad\qquad(\gamma y_0,\gamma^{-1}x_0)\in\big(\mathcal{C}_1^-(x_0,A)\times\mathcal{C}_1^-(y_0,B)\big)\cap(\widehat{V}_0\times\widehat{W}_0)\Big\}.
\end{aligned}$$

Since the number of elements $\gamma\in\Gamma$ with

$$(\gamma y,\gamma^{-1}x)\in\big(\mathcal{C}_r^-(x,A)\times\mathcal{C}_r^-(y,B)\big)\setminus(\widehat{V}_{-r}\times\widehat{W}_0)$$

is finite by Lemma 6.6(a), we conclude that

$$\begin{aligned}
&\limsup_{T\to\infty}\nu_{x,y}^T\big(\mathcal{C}_r^-(x,A)\times\mathcal{C}_r^-(y,B)\big)\\
&\quad\le \mathrm{e}^{\varepsilon/3}\mathrm{e}^{\delta_\Gamma\left(\mathcal{B}_{\xi_0}(x_0,x)+\mathcal{B}_{\eta_0}(y_0,y)\right)}\\
&\qquad\cdot\limsup_{T\to\infty}\nu_{x_0,y_0}^{T+\mathcal{B}_{\xi_0}(x_0,x)+\mathcal{B}_{\eta_0}(y_0,y)+\varepsilon/3\delta_\Gamma}\big(\mathcal{C}_1^-(x_0,A)\times\mathcal{C}_1^-(y_0,B)\big)\\
&\quad\le \mathrm{e}^{2\varepsilon/3}\mathrm{e}^{\delta_\Gamma\left(\mathcal{B}_{\xi_0}(x_0,x)+\mathcal{B}_{\eta_0}(y_0,y)\right)}\mu_{x_0}(A)\mu_{y_0}(B)/\|m_\Gamma\|,
\end{aligned}$$

where we used Proposition 8.2 in the last estimate.

Now for $\xi \in A \subset \widehat{V}_0 \cap \partial X$ and $\eta \in B \subset \widehat{W}_0 \cap \partial X$ we get from (41)

$$\mathcal{B}_{\xi_0}(x_0, x) < \mathcal{B}_{\xi}(x_0, x) + \frac{\varepsilon}{6\delta_\Gamma}, \quad \mathcal{B}_{\eta_0}(y_0, y) < \mathcal{B}_{\eta}(y_0, y) + \frac{\varepsilon}{6\delta_\Gamma},$$

hence

$$\begin{aligned} \mathrm{e}^{\delta_\Gamma \mathcal{B}_{\xi_0}(x_0,x)} \mu_{x_0}(A) &= \int_A \mathrm{e}^{\delta_\Gamma \mathcal{B}_{\xi_0}(x_0,x)} \,\mathrm{d}\mu_{x_0}(\xi) \\ &\le \mathrm{e}^{\varepsilon/6} \int_A \mathrm{e}^{\delta_\Gamma \mathcal{B}_{\xi}(x_0,x)} \frac{\mathrm{d}\mu_{x_0}}{\mathrm{d}\mu_x}(\xi)\,\mathrm{d}\mu_x(\xi) \overset{(31)}{=} \mathrm{e}^{\varepsilon/6} \mu_x(A), \end{aligned}$$

and similarly

$$\mathrm{e}^{\delta_\Gamma \mathcal{B}_{\eta_0}(y_0,y)} \mu_{y_0}(B) \le \mathrm{e}^{\varepsilon/6} \mu_y(B).$$

This finally proves

$$\limsup_{T\to\infty} \nu_{x,y}^T\big(\mathcal{C}_r^-(x, A) \times \mathcal{C}_r^-(y, B)\big) \le \mathrm{e}^{\varepsilon} \mu_x(A)\mu_y(B)/\|m_\Gamma\|.$$

The proof of the inequality for the limit inferior is analogous. □

Proof of Theorem 8.1. Let $x, y \in X$ and $\varepsilon > 0$ arbitrary. For $(\xi_0, \eta_0) \in \partial X \times \partial X$ we fix $r > 0$ and open neighborhoods $V, W \subset \partial X$ of ξ_0, η_0 such that the conclusion of Proposition 8.3 holds. Choose open sets $\widehat{V}, \widehat{W} \subset \overline{X}$ with $\widehat{V} \cap \partial X = V$ and $\widehat{W} \cap \partial X = W$, and let $\widehat{A}, \widehat{B} \subset \overline{X}$ be Borel sets with $\overline{\widehat{A}} \subset \widehat{V}$, $\overline{\widehat{B}} \subset \widehat{W}$ and

$$(\mu_x \otimes \mu_y)(\partial(\widehat{A} \times \widehat{B})) = 0. \tag{42}$$

Let $\alpha > 0$ be arbitrary, and choose open sets $A^+, B^+ \subset \partial X$ and compact sets $A^-, B^- \subset \partial X$ with the properties

$$\begin{gathered} A^- \subset \widehat{A}^\circ \cap \partial X \subset \overline{\widehat{A}} \cap \partial X \subset A^+ \subset V, \\ B^- \subset \widehat{B}^\circ \cap \partial X \subset \overline{\widehat{B}} \cap \partial X \subset B^+ \subset W, \\ \mu_x(\widehat{A}^\circ \setminus A^-) < \alpha, \quad \mu_x(A^+ \setminus \overline{\widehat{A}}) < \alpha, \\ \mu_y(\widehat{B}^\circ \setminus B^-) < \alpha, \quad \mu_y(B^+ \setminus \overline{\widehat{B}}) < \alpha. \end{gathered}$$

Notice that according to Lemma 6.6(b) the number of $\gamma \in \Gamma$ with

$$(\gamma y, \gamma^{-1} x) \in (\overline{\widehat{A}} \times \overline{\widehat{B}}) \setminus \big(\mathcal{C}_r^-(x, A^+) \times \mathcal{C}_r^-(y, B^+)\big)$$

is finite; the same is true for the number of $\gamma \in \Gamma$ with

$$(\gamma y, \gamma^{-1} x) \in \big(\mathcal{C}_r^+(x, A^-) \times \mathcal{C}_r^+(y, B^-)\big) \setminus (\widehat{A}^\circ \times \widehat{B}^\circ)$$

by Lemma 6.6(a). Hence

$$\|m_\Gamma\|\cdot\limsup_{T\to\infty}\nu^T_{x,y}(\widehat{A}\times\widehat{B})\le\|m_\Gamma\|\cdot\limsup_{T\to\infty}\nu^T_{x,y}\big(\mathcal{C}^-_r(x,A^+)\times\mathcal{C}^-_r(y,B^+)\big),$$
$$\|m_\Gamma\|\cdot\liminf_{T\to\infty}\nu^T_{x,y}(\widehat{A}\times\widehat{B})\ge\|m_\Gamma\|\cdot\liminf_{T\to\infty}\nu^T_{x,y}\big(\mathcal{C}^+_r(x,A^-)\times\mathcal{C}^+_r(y,B^-)\big).$$

Proposition 8.3 further implies

$$\begin{aligned}\|m_\Gamma\|\cdot\limsup_{T\to\infty}\nu^T_{x,y}(\widehat{A}\times\widehat{B})&\le \mathrm{e}^\varepsilon\mu_x(A^+)\mu_y(B^+)\\ &\le \mathrm{e}^\varepsilon\mu_x(\overline{\widehat{A}})\mu_y(\overline{\widehat{B}})+\alpha\mathrm{e}^\varepsilon(\mu_x(\partial X)+\mu_y(\partial X))\\ &\overset{(42)}{\le} \mathrm{e}^\varepsilon\mu_x(\widehat{A})\mu_y(\widehat{B})+\alpha\mathrm{e}^\varepsilon(\mu_x(\partial X)+\mu_y(\partial X))\end{aligned}$$

and

$$\begin{aligned}\|m_\Gamma\|\cdot\liminf_{T\to\infty}\nu^T_{x,y}(\widehat{A}\times\widehat{B})&\ge \mathrm{e}^{-\varepsilon}\mu_x(A^-)\mu_y(B^-)\\ &\ge \mathrm{e}^{-\varepsilon}\mu_x(\widehat{A}^\circ)\mu_y(\widehat{B}^\circ)-\alpha\mathrm{e}^{-\varepsilon}(\mu_x(\partial X)+\mu_y(\partial X))\\ &\overset{(42)}{\ge} \mathrm{e}^{-\varepsilon}\mu_x(\widehat{A})\mu_y(\widehat{B})-\alpha\mathrm{e}^{-\varepsilon}(\mu_x(\partial X)+\mu_y(\partial X))\end{aligned}$$

As α was arbitrarily small we get in the limit as α tends to zero

$$\limsup_{T\to\infty}\nu^T_{x,y}(\widehat{A}\times\widehat{B})\le \mathrm{e}^\varepsilon\mu_x(\widehat{A})\mu_y(\widehat{B})/\|m_\Gamma\|\quad\text{and}$$
$$\liminf_{T\to\infty}\nu^T_{x,y}(\widehat{A}\times\widehat{B})\ge \mathrm{e}^{-\varepsilon}\mu_x(\widehat{A})\mu_y(\widehat{B})\|m_\Gamma\|.$$

So for every continuous and positive function h with support in $\widehat{V}\times\widehat{W}$ we have

$$\begin{aligned}\frac{\mathrm{e}^{-\varepsilon}}{\|m_\Gamma\|}\int h\,(\mathrm{d}\mu_x\otimes\mathrm{d}\mu_y)&\le\liminf_{T\to\infty}\int h\,\mathrm{d}\nu^T_{x,y}\\ &\le\limsup_{T\to\infty}\int h\,\mathrm{d}\nu^T_{x,y}\le\frac{\mathrm{e}^\varepsilon}{\|m_\Gamma\|}\int h\,(\mathrm{d}\mu_x\otimes\mathrm{d}\mu_y).\end{aligned}$$

Now the compact set $\partial X\times\partial X$ can be covered by a finite number of open sets of type $V\times W$ with $V,W\subset\partial X$ as above, and similarly $\overline{X}\times\overline{X}$ by finitely many open sets $\widehat{V}\times\widehat{W}$ with $\widehat{V},\widehat{W}\subset\overline{X}$ as above. Using a partition of unity subordinate to such a finite cover we see that the inequalities above remain true for every continuous and positive function on $\overline{X}\times\overline{X}$. The claim now follows by taking the limit $\varepsilon\to 0$, and passing from positive continuous functions to arbitrary continuous functions via a standard argument. □

Corollary 8.4. *Let $\Gamma<\mathrm{Is}(X)$ be a discrete rank one group with nonarithmetic length spectrum, $\mathcal{Z}_\Gamma\neq\varnothing$ and finite Ricks–Bowen–Margulis measure m_Γ.*

Let $f:\overline{X}\to\mathbb{R}$ be a continuous function, and $x,y\in X$. Then

$$\lim_{T\to\infty}\delta_\Gamma\mathrm{e}^{-\delta_\Gamma T}\sum_{\substack{\gamma\in\Gamma\\ d(x,\gamma y)\le T}}f(\gamma y)=\frac{\mu_y(\partial X)}{\|m_\Gamma\|}\int_{\partial X}f(\xi)\,\mathrm{d}\mu_x(\xi).$$

9. Asymptotic estimates for the orbit counting function

In this section we let X be a proper Hadamard space and $\Gamma < \mathrm{Is}(X)$ a discrete rank one group with $\mathcal{Z}_\Gamma \neq \varnothing$. Recall that the orbit counting function with respect to x, $y \in X$ is defined by

$$N_\Gamma : [0,\infty) \to \mathbb{N}, \qquad R \mapsto \#\{\gamma \in \Gamma : d(x,\gamma y) \le R\}.$$

We first state a direct corollary of Theorem 8.1 (using $f = \mathbb{1}_{\bar{X}\times\bar{X}}$):

Proposition 9.1. *Let $\Gamma < \mathrm{Is}(X)$ be a discrete rank one group with nonarithmetic length spectrum, $\mathcal{Z}_\Gamma \neq \varnothing$ and finite Ricks–Bowen–Margulis measure m_Γ. Then for any x, $y \in X$ we have*

$$\lim_{R\to\infty} \delta_\Gamma \mathrm{e}^{-\delta_\Gamma R} N_\Gamma(R) = \frac{\mu_x(\partial X)\mu_y(\partial X)}{\|m_\Gamma\|}.$$

We next deal with the case that the Ricks–Bowen–Margulis measure is not finite:

Theorem 9.2. *Let $\Gamma < \mathrm{Is}(X)$ be a discrete rank one group with $\mathcal{Z}_\Gamma \neq \varnothing$ and infinite Ricks–Bowen–Margulis measure m_Γ. If Γ is divergent we further require that Γ has nonarithmetic length spectrum. Then for the orbit counting function with respect to arbitrary points x, $y \in X$ we have*

$$\lim_{t\to\infty} N_\Gamma(t)\mathrm{e}^{-\delta_\Gamma t} = 0.$$

As in the proof of Theorem 8.1 we define the measure

$$\nu^T_{x,y} := \delta_\Gamma \mathrm{e}^{-\delta_\Gamma T} \sum_{\substack{\gamma\in\Gamma \\ d(x,\gamma y)\le T}} \mathcal{D}_{\gamma y} \otimes \mathcal{D}_{\gamma^{-1}x};$$

here we only have to show that

$$\limsup_{T\to\infty} \nu^T_{x,y}(\bar{X}\times\bar{X}) = 0.$$

Again, the first step of the proof is provided by:

Lemma 9.3. *Let $(\xi_0,\eta_0) \in \partial X \times \partial X$ and x, $y \in X$ with trivial stabilizer in Γ and such that $x \in (\xi_0 v^+)$, $y \in (\eta_0 w^+)$ for some Γ-recurrent elements v, $w \in \mathcal{Z}$. Then there exist open neighborhoods V, $W \subset \partial X$ of ξ_0, η_0 such that for all Borel sets $A \subset V$, $B \subset W$*

$$\limsup_{T\to\infty} \nu^T_{x,y}\big(\mathcal{C}^-_1(x,A) \times \mathcal{C}^-_1(y,B)\big) = 0.$$

Proof. Let $\varepsilon > 0$ arbitrary and set $\rho := \min\{d(x,\gamma x), d(y,\gamma y) : \gamma \in \Gamma\}$.

As in the proof of Proposition 8.2 we fix $r \in (0, \min\{1, \rho/3, \varepsilon/(30\delta_\Gamma)\})$ such that

$$\mu_x(\tilde{\partial}\mathcal{O}_r(\xi_0,x)) = 0 = \mu_y(\tilde{\partial}\mathcal{O}_r(\eta_0,y))$$

and choose open neighborhoods $\widehat{V}, \widehat{W} \subset \bar{X}$ of ξ_0, η_0 such that if $(a,b) \in \widehat{V} \times \widehat{W}$, then a can be joined to v^+, b can be joined to w^+ by a rank one geodesic and (35) holds. Let $V \subset \widehat{V} \cap \partial X,\ W \subset \widehat{W} \cap \partial X$ be open neighborhoods of ξ_0, η_0, and $A \subset V,\ B \subset W$ arbitrary Borel sets; denote $K^+ = K_r^+(x, A),\ K^- = K_r^-(y, B)$, and $M = r^2 \mu_x(\mathcal{O}_r(\xi_0, x))\mu_y(\mathcal{O}_r(\eta_0, y)) > 0$. Then by mixing (or dissipativity in the case of a convergent group Γ) there exists $T_0 \gg 1$ such that

$$\sum_{\gamma \in \Gamma} m(K^+ \cap g^{-t}\gamma K^-) < M\varepsilon \cdot \mathrm{e}^{-\varepsilon/3}$$

for all $t \geq T_0$, which implies

$$(\mathrm{e}^{\delta_\Gamma(T+3r)} - \mathrm{e}^{\delta_\Gamma T_0}) M\varepsilon \cdot \mathrm{e}^{-\varepsilon/3} > \delta_\Gamma \int_{T_0}^{T+3r} \mathrm{e}^{\delta_\Gamma t} \sum_{\gamma \in \Gamma} m(K^+ \cap g^{-t}\gamma K^-)\,\mathrm{d}t.$$

We now use (40) to get

$$\delta_\Gamma \int_{T_0}^{T+3r} \mathrm{e}^{\delta_\Gamma t} \sum_{\gamma \in \Gamma} m(K^+ \cap g^{-t}\gamma K^-)\,\mathrm{d}t \\ \geq \mathrm{e}^{-\varepsilon/6} M \mathrm{e}^{\delta_\Gamma T} \nu_{x,y}^T\big(\mathcal{C}_1^-(x, A) \times \mathcal{C}_1^-(y, B)\big) - C$$

with a constant C independent of T. Dividing by $M\mathrm{e}^{\delta_\Gamma(T+3r)}$ then yields

$$\begin{aligned}(1 - \mathrm{e}^{\delta_\Gamma(-T-3r+T_0)})\varepsilon \cdot \mathrm{e}^{-\varepsilon/3} &> \mathrm{e}^{-\varepsilon/6}\mathrm{e}^{-3\delta_\Gamma r} \nu_{x,y}^T\big(\mathcal{C}_1^-(x, A) \times \mathcal{C}_1^-(y, B)\big) - C'\mathrm{e}^{-\delta_\Gamma T} \\ &= \mathrm{e}^{-4\varepsilon/15} \nu_{x,y}^T\big(\mathcal{C}_1^-(x, A) \times \mathcal{C}_1^-(y, B)\big) + C'\mathrm{e}^{-\delta_\Gamma T},\end{aligned}$$

where C' is again a constant independent of T. We conclude

$$\limsup_{T \to \infty} \nu_{x,y}^T\big(\mathcal{C}_1^-(x, A) \times \mathcal{C}_1^-(y, B)\big) < \varepsilon,$$

and the claim follows from the fact that $\varepsilon > 0$ was chosen arbitrarily small. □

The next statement shows that in fact we can omit the conditions on x and y in Lemma 9.3.

Lemma 9.4. *Let $x, y \in X$ arbitrary. Then for all $(\xi_0, \eta_0) \in \partial X \times \partial X$ there exists $r > 0$ and open neighborhoods $V \subset \partial X$ of ξ_0, $W \subset \partial X$ of η_0 such that for all Borel sets $A \subset V,\ B \subset W$*

$$\limsup_{T \to \infty} \nu_{x,y}^T\big(\mathcal{C}_r^-(x, A) \times \mathcal{C}_r^-(y, B)\big) = 0.$$

Proof. Let $(\xi_0, \eta_0) \in \partial X \times \partial X$ be arbitrary. Choose Γ-recurrent geodesics $v, w \in \mathcal{Z}$ and $x_0 \in (\xi_0 v^+),\ y_0 \in (\eta_0 w^+)$ with trivial stabilizers in Γ. Let $V, W \subset \partial X$ be open neighborhoods of ξ_0 and η_0 such that the statement of Lemma 9.3 holds for x_0, y_0 instead of x, y. Set

$$r = 1 + \max\{d(x, x_0), d(y, y_0)\}$$

and let $A \subset V$, $B \subset W$ be arbitrary Borel sets. From the choice of r above and Lemma 6.5(b) we know that $(\gamma y, \gamma^{-1}x) \in \mathcal{C}_r^-(x, A) \times \mathcal{C}_r^-(y, B)$ implies

$$(\gamma y_0, \gamma^{-1}x_0) \in \mathcal{C}_1^-(x_0, A) \times \mathcal{C}_1^-(y_0, B).$$

If $d(x, \gamma y) \le T$, then obviously

$$d(x_0, \gamma y_0) \le d(x_0, x) + d(x, \gamma y) + d(y, y_0) \le T + d(x_0, x) + d(y, y_0),$$

hence for $T \gg 1$

$$\begin{aligned}
&\mathrm{e}^{-\delta_\Gamma T}\#\bigl\{\gamma \in \Gamma : d(x, \gamma y) \le T,\ (\gamma y, \gamma^{-1}x) \in \bigl(\mathcal{C}_r^-(x, A) \times \mathcal{C}_r^-(y, B)\bigr)\bigr\}\\
&\quad\le \mathrm{e}^{\delta_\Gamma(d(x_0,x)+d(y,y_0))} \cdot \mathrm{e}^{-\delta_\Gamma(T+d(x_0,x)+d(y,y_0))}\\
&\qquad\qquad \cdot\#\Bigl\{\gamma \in \Gamma : d(x_0, \gamma y_0) \le T + d(x_0, x) + d(y, y_0),\\
&\qquad\qquad\qquad\qquad (\gamma y_0, \gamma^{-1}x_0) \in \bigl(\mathcal{C}_1^-(x_0, A) \times \mathcal{C}_1^-(y_0, B)\bigr)\Bigr\}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
&\limsup_{T\to\infty} \nu_{x,y}^T\bigl(\mathcal{C}_r^-(x, A) \times \mathcal{C}_r^-(y, B)\bigr)\\
&\qquad \le \mathrm{e}^{\delta_\Gamma(d(x_0,x)+d(y,y_0))} \limsup_{T\to\infty} \nu_{x_0,y_0}^{T+d(x_0,x)+d(y,y_0)}\bigl(\mathcal{C}_1^-(x_0, A) \times \mathcal{C}_1^-(y_0, B)\bigr) = 0,
\end{aligned}$$

where we used Lemma 9.3 in the last estimate. □

Proof of Theorem 9.2. Let $x, y \in X$ and $\varepsilon > 0$ arbitrary. For $(\xi_0, \eta_0) \in \partial X \times \partial X$ we fix $r > 0$ and open neighborhoods $V, W \subset \partial X$ of ξ_0, η_0 such that the conclusion of Lemma 9.4 holds. Choose open sets $\widehat{V}, \widehat{W} \subset \bar{X}$ with $\widehat{V} \cap \partial X = V$ and $\widehat{W} \cap \partial X = W$, and let $\widehat{A}, \widehat{B} \subset \bar{X}$ be Borel sets with

$$\overline{\widehat{A}} \subset \widehat{V} \quad \text{and} \quad \overline{\widehat{B}} \subset \widehat{W}.$$

Choose open sets $A, B \subset \partial X$ with the properties

$$\overline{\widehat{A}} \cap \partial X \subset A \subset V \quad \text{and} \quad \overline{\widehat{B}} \cap \partial X \subset B \subset W;$$

from Lemma 6.6(b) we know that the number of $\gamma \in \Gamma$ with

$$(\gamma y, \gamma^{-1}x) \in (\overline{\widehat{A}} \times \overline{\widehat{B}}) \setminus \bigl(\mathcal{C}_r^-(x, A) \times \mathcal{C}_r^-(y, B)\bigr)$$

is finite. Hence

$$\limsup_{T\to\infty} \nu_{x,y}^T(\widehat{A} \times \widehat{B}) \le \limsup_{T\to\infty} \nu_{x,y}^T\bigl(\mathcal{C}_r^-(x, A) \times \mathcal{C}_r^-(y, B)\bigr) = 0,$$

which implies that for every continuous and positive function with support in $\widehat{V} \times \widehat{W}$ we have

$$\limsup_{T\to\infty} \int h \, \mathrm{d}\nu_{x,y}^T = 0.$$

Now the compact set $\partial X \times \partial X$ can be covered by a finite number of open sets of type $V \times W$ with $V, W \subset \partial X$ as above, and similarly $\bar{X} \times \bar{X}$ by finitely many open sets $\widehat{V} \times \widehat{W}$ with $\widehat{V}, \widehat{W} \subset \bar{X}$ as above. Using a partition of unity subordinate to such a finite cover we see that the statement above remains true for every continuous and positive function on $\bar{X} \times \bar{X}$. $\square$

Acknowledgements

The author is extremely grateful to the referee for a thorough and critical reading of the article. She would like to give thanks for pointing out several inaccuracies and mistakes in previous versions of the paper, and in particular for providing the example and a suggestion for how to fix a former error in the proof of Lemma 4.4. She is also very grateful for the many valuable suggestions to improve the exposition.

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Received 25 Oct 2018. Revised 17 Jul 2019.

GABRIELE LINK:
gabriele.link@kit.edu
Institute for Algebra and Geometry, Karlsruhe Institute of Technology (KIT), Karlsruhe, Germany

TUNISIAN JOURNAL OF MATHEMATICS
Vol. 2, No. 4, 2020
dx.doi.org/10.2140/tunis.2020.2.841

A generalization of a power-conjugacy problem in torsion-free negatively curved groups

Rita Gitik

Let H and K be quasiconvex subgroups of a negatively curved torsion-free group G. We give an algorithm which decides whether an element of H is conjugate in G to an element of K.

1. Introduction

Max Dehn [1911] introduced three basic algorithmic problems in group theory: the word problem, the conjugacy problem, and the isomorphism problem. Let a group G be given by a presentation $G = \langle X | R \rangle$. The word problem asks if there exists an algorithm to decide if any word in the alphabet X represents the trivial element of G. The word problem was shown to be undecidable, in general, by Novikov [1955], and independently, by Boone [1958]. The conjugacy problem asks if there exists an algorithm which for any pair of words in the alphabet X decides whether they represent conjugate elements in G. A special case of the conjugacy problem, namely the existence of an algorithm deciding if a given word in the alphabet X represents an element of G conjugate to the identity of G, is the word problem. Hence the conjugacy problem is also undecidable, in general. The isomorphism problem asks if for any pair of presentations there exists an algorithm to decide if they define isomorphic groups. The isomorphism problem was shown to be undecidable, in general, by Adian [1957], and independently by Rabin [1958]. The membership problem for a subgroup H of a group G asks if there exists an algorithm deciding if any element of G belongs to H. As the word problem, in general, is undecidable, it follows that the membership problem is, in general, undecidable. The power-conjugacy problem for a group G asks if for any two elements of G there exists an algorithm to decide if one of them is conjugate to some power of the other. The generalized power-conjugacy problem for a group G asks if for any two elements of G there exists an algorithm to decide if some power of one of them is conjugate to some power of the other. A special case of the power-conjugacy problem, namely the existence of an algorithm deciding if

MSC2010: primary 20F10; secondary 20F65, 20F67.
Keywords: negatively curved group, conjugacy problem, quasiconvex subgroup.

any element of G is conjugate to some power of the identity element in G, is the word problem. Hence the power-conjugacy problem and the generalized power-conjugacy problem are undecidable, in general. For more detailed information about the aforementioned algorithmic problems see, for example, survey articles [Hurwitz 1984; Miller III 1992].

Even though the aforementioned algorithmic problems are undecidable in general, they are decidable in negatively curved groups. The solution of the word problem in negatively curved groups follows from the work of Greendlinger [1960]. The solution of the conjugacy problem for negatively curved groups was given by Gromov [1987, p. 199]. The solution of the isomorphism problem for negatively curved groups was given by Dahmani and Guirardel [2011]. The solution of the membership problem for quasiconvex subgroups of negatively curved groups was given by the author [Gitik 1996; 2017]. The power-conjugacy problem has been the subject of extensive research and was solved for several additional classes of groups, see for example, [Anshel and Stebe 1974; Bezverhnii 2016; Barker et al. 2016; Bezverkhniĭ and Kuznetsova 2008; Bogopolski et al. 2006; Comerford 1977; Lipschutz and Miller III 1971; Pride 2008]. In this paper we prove a generalized version of the power-conjugacy problem for torsion-free negatively curved groups.

Our solution of a generalized version of the power-conjugacy problem for torsion-free negatively curved groups implies that both the power-conjugacy problem and the generalized power-conjugacy problem are solvable in torsion-free negatively curved groups.

Theorem 1 (generalized power-conjugacy problem). *Let H and K be μ-quasiconvex subgroups of a δ-negatively curved torsion-free group G. There exists an algorithm to decide if an element of H is conjugate in G to an element of K.*

Corollary 2. *Let K be a quasiconvex subgroup of a torsion-free negatively curved group G, and let u be a nontrivial element of G. There exists an algorithm to decide whether some power of u is conjugate in G to an element of K.*

Proof. As a cyclic subgroup in a negatively curved group is quasiconvex [Gromov 1987, p. 210], we can apply Theorem 1 with H being the cyclic subgroup generated by u. □

Corollary 3. *Let K be a quasiconvex subgroup of a torsion-free negatively curved group G and let u be a nontrivial element of G. There exists an algorithm to decide whether u is conjugate in G to an element of K.*

Proof. Lemma 7, stated below, shows that if H is the cyclic subgroup generated by u and u is conjugate to an element of K, then there exists $g \in G$ with $|g| < C$ such that $gug^{-1} \in K$, (C is defined in the statement of Lemma 7). As G is finitely generated, there are only finitely many elements shorter than C in G. Hence we need to check if one of finitely many elements of the form gug^{-1} with $|g| < C$ is in K, which we can do because the membership problem for K in G is decidable. □

Corollary 4. *The power-conjugacy problem is decidable for torsion-free negatively curved groups.*

Proof. Let u be a nontrivial element of G and let v be any element of G. Corollary 3 implies that there is an algorithm to decide whether u is conjugate in G to an element of a cyclic group generated by v, which is the power-conjugacy problem. □

Corollary 5. *The generalized power-conjugacy problem is decidable for torsion-free negatively curved groups.*

Proof. Let u be a nontrivial element of G and let v be any element of G. Corollary 2 implies that there is an algorithm to decide whether some power of u is conjugate in G to an element of a cyclic group generated by v, which is the generalized power-conjugacy problem. □

Theorem 1 follows from three technical results stated below.

Remark 6. Note that if there exist $h \in H$ and $g \in G$ such that $ghg^{-1} \in K$, then for any $h_0 \in H$ and $k_0 \in K$, $(k_0 g h_0)(h_0^{-1} h h_0)(h_0^{-1} g^{-1} k_0^{-1}) \in K$. So if $g \in G$ conjugates an element of H to an element of K, then any g_0 in the double coset KgH has the same property.

Lemma 7. *Let H and K be μ-quasiconvex subgroups of a δ-negatively curved torsion-free group G, and let $g \in G$ be a shortest representative of the double coset KgH such that ghg^{-1} is in K for some nontrivial element h of H. Then g is shorter than $C = 4\delta + 2\mu + (m^2 + 1) \cdot L$, where L is the number of words in G with length less than $8\delta + \mu$, and m is the number of elements in G with length not greater than $42\delta + 12\mu$.*

Lemma 8. *Let H and K be μ-quasiconvex subgroups of a δ-negatively curved torsion-free group G and let h be a shortest nontrivial element of H such that ghg^{-1} is in K for some $g \in G$ with $|g| < C$. Then h is shorter than*

$$C' = (L' + 2)2\mu + 8\delta,$$

where L' is the number of words in G shorter than $(2\delta + 2\mu)$.

Proof of Theorem 1. Assume that there exists a nontrivial element $h \in H$ and an element $g \in G$ such that $ghg^{-1} \in K$. Let g_1 be a shortest element in the double coset KgH. Lemma 7 states that $|g_1| < C$. Remark 6 implies that there exists an element $h' \in H$ such that $g_1 h' g_1^{-1} \in K$. Let $h_1 \in H$ be a shortest nontrivial element such that $g_1 h_1 g_1^{-1} \in K$. Lemma 8 states that $|h_1| < C'$. As G is finitely generated, there are finitely many possible g_1 and h_1. Hence we need to form finitely many products $g_1 h_1 g_1^{-1}$ and to check if they belong to K, which we can verify because the generalized word problem is solvable for quasiconvex subgroups of negatively curved groups. □

2. Preliminaries

Let X be a set, let $X \cup X^{-1} = \{x, x^{-1} \mid x \in X\}$, and for $x \in X$ define $(x^{-1})^{-1} = x$. A word in $X \cup X^{-1}$ is any finite sequence of elements of $X \cup X^{-1}$. Denote the set of all words in $X \cup X^{-1}$ by $W(X)$, and denote the equality of two words by "$\equiv$".

Recall that the Cayley graph of $G = \langle X | R \rangle$, denoted Cayley($G$), is an oriented graph whose set of vertices is G and the set of edges is $G \times (X \cup X^{-1})$, such that an edge (g, x) begins at the vertex g and ends at the vertex gx. Since the Cayley graph depends on the generating set of the group, we work with a fixed generating set.

A geodesic in the Cayley graph is a shortest path joining two vertices. A geodesic triangle in the Cayley graph is a closed path $p = p_1 p_2 p_3$, where each p_i is a geodesic. A group $G = \langle X | R \rangle$ is δ-negatively curved if any side of any geodesic triangle in the Cayley graph of $G = \langle X | R \rangle$ belongs to the δ-neighborhood of the union of the other two sides.

A subgroup H of a group $G = \langle X | R \rangle$ is μ-quasiconvex in $G = \langle X | R \rangle$ if any geodesic in the Cayley graph of $G = \langle X | R \rangle$ with endpoints in H belongs to the μ-neighborhood of H. A subgroup is quasiconvex in $G = \langle X | R \rangle$ if it is μ-quasiconvex in $G = \langle X | R \rangle$ for some μ. As usual, we assume that all negatively curved groups are finitely generated.

The label of a path $p = (g, x_1)(g \cdot x_1, x_2) \cdots (g \cdot x_1 \cdots x_{n-1}, x_n)$ in Cayley(G) is the function $\mathrm{Lab}(p) \equiv x_1 x_2 \cdots x_n \in W(X)$. As usual, we identify the word $\mathrm{Lab}(p)$ with the corresponding element in G.

Theorem GMRS [Gitik et al. 1998]. *Let H be a μ-quasiconvex subgroup of a δ-negatively curved torsion-free group G, and let m be the number of elements in G with length not greater than $42\delta + 12\mu$. Let $S = \{g_i^{-1} H g_i \mid 1 \leq i \leq n\}$ be a collection of essentially distinct conjugates of H, where the conjugates $g_i^{-1} H g_i$ and $g_j^{-1} H g_j$ are called essentially distinct if $Hg_i \neq Hg_j$ for $i \neq j$. If $n > m^2$, then the intersection of some pair of elements of S is trivial.*

3. Proofs of the results

Let g be a shortest element in the double coset KgH such that $ghg^{-1} = k$ is in K for some nontrivial element h of H.

Let p, p_h and p' be geodesics in Cayley(G) such that $\mathrm{Lab}(p) \equiv \mathrm{Lab}(p') \equiv g$, $\mathrm{Lab}(p_h) = h$, p begins at 1 and ends at g, p' begins at ghg^{-1} and ends at gh, and p_h begins at g which is the endpoint of p and ends at gh which is the endpoint of p'.

Denote the vertices of p in their linear order by $1 = v_0, v_1, \ldots, v_n = g$ and denote the vertices of p' in their linear order by $ghg^{-1} = v'_0, v'_1, \ldots, v'_n = gh$. Note that $|g| = |p| = |p'| = n$.

Let p_k be a geodesic in Cayley(G) joining $v_0 = 1$ and $v'_0 = ghg^{-1}$. Then the paths p, p_k, p' and p_h form a geodesic 4-gon which is 2δ-thin in Cayley(G), because G is δ-negatively curved.

Lemma 9. *For any index i such that $2\delta + \mu \le i \le n - 2\delta - \mu$ the distance $d(v_i, v'_i)$ is less than $8\delta + \mu$.*

Proof. Let l be the biggest index such that the vertex v_l belongs to the 2δ-neighborhood of p_k, let w_l be a vertex in p_k closest to v_l, and let r be a geodesic joining w_l to v_l. By construction, $\mathrm{Lab}(p_k) = k \in K$. As K is μ-quasiconvex, p_k belongs to the μ-neighborhood of K in Cayley(G), hence there exists a vertex $u_l \in K$ such that $d(w_l, u_l) < \mu$. Let r' be a geodesic joining u_l to w_l. Let s_l be the subpath of p joining v_0 to v_l, let $\bar{s_l}$ be the inverse of the path s_l, and let t_l be the subpath of p joining v_l to v_n. Note that $\mathrm{Lab}(r'rt_l) = \mathrm{Lab}(r'r\bar{s_l})(s_lt_l) = \mathrm{Lab}(r'r\bar{s_l})g \in Kg$. As g is a shortest representative of KgH, it follows that $|g| = |p| = |s_l| + |t_l| \le |r'rt_l| < 2\delta + \mu + |t_l|$, so $|s_l| = d(v_0, v_l) = l < 2\delta + \mu$. Hence if $i \ge \mu + 2\delta$, then $d(v_i, p_k) > 2\delta$.

Let i be the smallest index such that the vertex v_i belongs to the 2δ-neighborhood of p_h. An argument, similar to the above, shows that for any $j \le n - 2\delta - \mu$, $d(v_j, p_h) > 2\delta$.

Therefore, for any index i such that $2\delta + \mu \le i \le n - 2\delta - \mu$, the vertex v_i belongs to the 2δ-neighborhood of p'. Similarly, for any index i such that $2\delta + \mu \le i \le n - 2\delta - \mu$ the vertex v'_i belongs to the 2δ-neighborhood of p.

Let $b = n - 2\delta - \mu$. We claim that $d(v_b, v'_b) < 4\delta + \mu$. Indeed, let $j(b) \le b$ be an index such that $d(v_b, v'_{j(b)}) < 2\delta$. Let t_b be the subpath of p joining v_b and v_n, let $t'_{j(b)}$ be the subpath of p' joining $v'_{j(b)}$ to v'_n, and let γ be a geodesic joining v_b and $v'_{j(b)}$. Consider the geodesic 4-gon formed by t_b, p_h, $t'_{j(b)}$ and γ.

As $b \le n - 2\delta - \mu$, it follows that $d(v'_b, p_h) > 2\delta$. If $d(v'_b, \gamma) < 2\delta$, then $d(v_b, v'_b) \le |\gamma| + d(v'_b, \gamma) < 4\delta$. If $d(v'_b, t_b) < 2\delta$, then $d(v_b, v'_b) \le |t_b| + d(v'_b, t_b) < 4\delta + \mu$.

Now consider $2\delta + \mu \le i \le n - 2\delta - \mu$. Let $j(i)$ be an index such that $d(v_i, v'_{j(i)}) < 2\delta$. By interchanging v_i and $v'_{j(i)}$, if needed, we can assume that $j(i) \ge i$. As p is a geodesic, $d(v_i, v_b) = b - i \le d(v_i, v'_{j(i)}) + d(v'_{j(i)}, v'_b) + d(v_b, v'_b) < 2\delta + (b - j(i)) + 4\delta + \mu$, hence $0 \le j(i) - i < 6\delta + \mu$. But then $d(v_i, v'_i) \le d(v_i, v'_{j(i)}) + d(v'_{j(i)}, v'_i) < 2\delta + (j(i) - i) < 8\delta + \mu$, proving Lemma 9. □

Proof of Lemma 7. Assume that $|g| = n \ge C$, where C is defined in the statement of Lemma 7. It follows that $(n - 2\delta - \mu) - (2\delta + \mu) \ge C - 4\delta - 2\mu = L \cdot (m^2 + 1)$.

Therefore Lemma 9 implies there exists a set of distinct indexes $\{i_j \mid 1 \le j \le m^2+1\}$ such that

(1) $n-2\delta-\mu \ge i_j \ge 2\delta+\mu$,

(2) the paths connecting v_{i_j} to v'_{i_j} have the same label, say a, for all i_j.

Recall that the paths p and p', defined at the beginning of the current section, have identical labels. Let s_{i_j} be the initial subpath of p connecting v_0 and v_{i_j} and let s'_{i_j} be the initial subpath of p' connecting v'_0 and v'_{i_j}. This definition implies that the paths s_{i_j} and s'_{i_j} have identical labels for all j.

Let k be the element of K and h be the element of H, defined at the beginning of the current section.

If $a=1$, then $v_{i_1}=v'_{i_1}$. It follows that $\mathrm{Lab}(s_{i_1})^{-1}k\,\mathrm{Lab}(s'_{i_1})=1$, hence $k=h=1$, contradicting the choice of h.

If $a \ne 1$, consider the set $S=\{\mathrm{Lab}(s_{i_j}^{-1})k\,\mathrm{Lab}(s_{i_j}) \mid 1 \le i_j \le m^2+1\}$. As $\mathrm{Lab}(s_{i_j})^{-1}k\,\mathrm{Lab}(s_{i_j})=a \ne 1$ for all $1 \le i_j \le m^2+1$, and as G is torsion-free, it follows that the intersection of any pair of elements of S is infinite.

However, the elements of S are essentially distinct. Indeed, assume that there exists $k_0 \in K$ such that $k_0\,\mathrm{Lab}(s_{i_j})=\mathrm{Lab}(s_{i_l})$.

Without loss of generality, $i_l > i_j$. Let t_{i_l} be the subpath of p joining v_{i_l} to v_n. Then $g=\mathrm{Lab}(s_{i_l})\mathrm{Lab}(t_{i_l})=k_0\mathrm{Lab}(s_{i_j})\mathrm{Lab}(t_{i_l})$. Hence the element $\mathrm{Lab}(s_{i_j})\mathrm{Lab}(t_{i_l})$ belongs to Kg and $|\,\mathrm{Lab}(s_{i_j})\,\mathrm{Lab}(t_{i_l})| \le |s_{i_j}|+|t_{i_j}| < |s_{i_l}|+|t_{i_l}|=|g|$, contradicting the choice of g as a shortest representative of the double coset KgH. So S is a collection of m^2+1 distinct conjugates of K such that any two elements of S have infinite intersection, contradicting Theorem GMRS.

Hence $|g| < C$, proving Lemma 7. □

Remark 10. By increasing the quasiconvexity constant μ if needed, we can assume that μ is a positive integer.

Lemma 11. *Let g be an element shorter than $4\delta+2\mu$ such that $ghg^{-1} \in K$ for a nontrivial $h \in H$. If h is longer than $(L'+2)2\mu+8\delta$, where L' is the number of words in G shorter than $2\delta+2\mu$, then there exist a nontrivial $h_0 \in H$ with $|h_0| \le 2\mu(L'+2)$ and $g_0 \in G$ with $|g_0| < 2\delta+2\mu$ such that $g_0h_0g_0^{-1} \in K$.*

Proof. Let p, p_k, p' and p_h be a geodesic 4-gon, as in the proof of Lemma 9. Denote the vertices of p_h in their linear order by $g=v_0^h, v_1^h, \ldots, v_f^h=gh$.

Let q' be the maximal initial subpath of p_h which belongs to the 2δ-neighborhood of p. Note that the length of q' is at most $4\delta+\mu$. Indeed, let $v_{q'}^h$ be the terminal vertex of q'. Let $v_{q'}$ be a vertex of p such that $d(v_{q'}^h, v_{q'}) \le 2\delta$. Let α be a geodesic in $\mathrm{Cayley}(G)$ which begins at $v_{q'}$ and ends at $v_{q'}^h$. Let $s_{q'}$ be the initial subpath of p joining v_0 to $v_{q'}$ and let $t_{q'}$ be the terminal subpath of p joining $v_{q'}$ to $v_n=g$. As H is μ-quasiconvex in G, there exists a vertex $x_{q'}$ in $\mathrm{Cayley}(G)$ and a geodesic α'

joining $v^h_{q'}$ to $x_{q'}$ such that $\mathrm{Lab}(q'\alpha') \in H$ and $|\alpha'| < \mu$. As g is a shortest element in the double coset KgH, it follows that

$$|g| = |s_{q'}| + |t_{q'}| \leq |s_{q'}| + |\alpha| + |\alpha'| \leq |s_{q'}| + 2\delta + \mu.$$

Hence $|t_{q'}| \leq 2\delta + \mu$. It follows that $|q'| \leq |t_{q'}| + |\alpha| \leq 4\delta + \mu$.

Similarly, the length of the maximal subpath of p_h which belongs to the 2δ-neighborhood of p' is at most $4\delta + \mu$.

Assume that h is longer than $(L'+2)2\mu + 8\delta$. Then there exists a subpath q of p_h of length at least $(L'+1)2\mu$ which belongs to the 2δ-neighborhood of p_k. By construction, q begins at the vertex $v^h_{q'}$. By definition of the path q, for any vertex v^h_i of q there exists a vertex $w(v^h_i)$ in p_k such that $d(v^h_i, w(v^h_i)) < 2\delta$. As H is μ-quasiconvex in G, for any vertex v^h_i of q there exists a vertex x_i such that $d(v^h_i, x_i) < \mu$, and the element x_i belongs to the coset gH. Similarly, there exists a vertex $k(v^h_i)$ such that $d(w(v^h_i), k(v^h_i)) < \mu$ and the element $k(v^h_i)$ belongs to K. Let β_i be a geodesic joining $k(v^h_i)$ and x_i. Then $|\beta_i| < 2\mu + 2\delta$.

Consider the subset of vertices of p_h with indexes

$$v^h_{q'}, v^h_{q'+2\mu}, \ldots, v^h_{q'+j\cdot 2\mu}, \ldots, v^h_{q'+L'\cdot 2\mu}.$$

The distance between two consecutive vertices in this subset is 2μ, hence $x_{q'+i\cdot 2\mu} \neq x_{q'+j\cdot 2\mu}$ for $i \neq j$.

By definition of the constant L', there exist indexes $i \neq j$ such that $\mathrm{Lab}(\beta_{q'+i\cdot 2\mu}) = \mathrm{Lab}(\beta_{q'+j\cdot 2\mu})$. By construction, $d(v_{q'+i\cdot 2\mu}, v_{q'+j\cdot 2\mu}) \leq 2\mu(L'+1)$, so

$$d(x_{q'+i\cdot 2\mu}, x_{q'+j\cdot 2\mu}) < 2\mu + 2\mu(L'+1) = 2\mu(L'+2).$$

By construction, if ν is a geodesic joining $x_{q'+i\cdot 2\mu}$ and $x_{q'+j\cdot 2\mu}$, then $\mathrm{Lab}(\nu) \in H$. Similarly, if ν' is a geodesic joining $k(v_{q'+i\cdot 2\mu})$ and $k(v_{q'+j\cdot 2\mu})$, then $\mathrm{Lab}(\nu') \in K$. So take $g_0 = \mathrm{Lab}(\beta(v_{q'+i\cdot 2\mu})$ and $h_0 = \mathrm{Lab}(\nu)$, proving Lemma 11. □

Proof of Lemma 8. Let h be a nontrivial element of H such that $ghg^{-1} \in K$ for some $g \in G$ with $|g| < C$, where C is defined in the statement of Lemma 7. We want to find $h_0 \in H$ with $|h_0| < C'$, where C' is defined in the statement of Lemma 8, and $g_0 \in G$, which might be different from g, with $|g_0| < C$ such that $g_0 h_0 g_0^{-1} \in K$.

Consider three cases.

(1) If $|g| < 4\delta + 2\mu$ and $|h| \leq (L'+2)2\mu + 8\delta$, take $g_0 = g$ and $h_0 = h$.

(2) If $|g| < 4\delta + 2\mu$ and $|h| > (L'+2)2\mu + 8\delta$, then Lemma 11 states that there exist a nontrivial $h_0 \in H$ with $|h_0| \leq 2\mu(L'+2)$ and $g_0 \in G$ with $|g_0| < 2\delta + 2\mu$ such that $g_0 h_0 g_0^{-1} \in K$.

(3) If $C > |g| \geq 4\delta + 2\mu$, let p, p', v_b, v'_b and p_h be as in the proof of Lemma 9. It is shown in Lemma 9 that $d(v_b, v'_b) < 4\delta + \mu$. Then

$$|h| = |p_h| \leq d(v_b, v_n) + d(v_b, v'_b) + d(v'_b, v'_n) < (\mu + 2\delta) + (\mu + 4\delta) + (\mu + 2\delta) < (3\mu + 8\delta).$$

Hence we can take $g_0 = g$ and $h_0 = h$, proving Lemma 8. □

Acknowledgment

The author would like to thank Dinakar Ramakrishnan and Eliyahu Rips for their support and the referee for the helpful comments.

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Received 4 Apr 2019. Revised 17 Aug 2019.

RITA GITIK:

ritagtk@umich.edu
Department of Mathematics, University of Michigan, Ann Arbor, MI, United States

TUNISIAN JOURNAL OF MATHEMATICS
Vol. 2, No. 4, 2020
dx.doi.org/10.2140/tunis.2020.2.851

A simple proof of the Hardy inequality on Carnot groups and for some hypoelliptic families of vector fields

François Vigneron

We give an elementary proof of the classical Hardy inequality on any Carnot group, using only integration by parts and a fine analysis of the commutator structure, which was not deemed possible until now. We also discuss the conditions under which this technique can be generalized to deal with hypoelliptic families of vector fields, which, in this case, leads to an open problem regarding the symbol properties of the gauge norm.

The classical Hardy inequality [1934] on a smooth open domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ reads:

$$\text{for all } f \in H_0^1(\Omega), \quad \sup_{x_0 \in \Omega} \left(\int_\Omega \frac{|f(x)|^2}{|x - x_0|^2} \, dx \right) \leq \frac{4}{(n-2)^2} \int_\Omega |\nabla f(x)|^2 \, dx. \tag{1}$$

Since L. D'Ambrosio [2005] it has been well known that similar inequalities hold on nilpotent groups, but interest on this matter is still high; see, e.g., [Ruzhansky and Suragan 2017; Adimurthi and Mallick 2018; Ambrosio et al. 2019].

An important reference on this matter is a recent note by H. Bahouri, C. Fermanian-Kammerer, I. Gallagher [Bahouri et al. 2012]. It is dedicated to refined Hardy inequalities on graded Lie groups and relies on constructing a general Littlewood–Paley theory and, as such, involves the machinery of the Fourier transform on groups.

Expanding the generality towards hypoelliptic vector fields, G. Grillo's article [2003] contains an inequality that holds for L^p-norms, without an underlying group structure, and contains weights that allow positive powers of the Carnot–Carathéodory distance on the right-hand side. However, the proof of this generalization involves the whole power of the sub-Riemannian Calderon–Zygmund theory.

Another beautiful reference is the paper by P. Ciatti, M.G. Cowling, F. Ricci [Ciatti et al. 2015] that studies these matters on stratified Lie groups, but with the

MSC2010: primary 43A80; secondary 35R03, 42B37.
Keywords: Hardy inequality, Carnot group, stratified algebra, commutators, hypoelliptic vector fields, Hörmander bracket condition.

point of view of operator and interpolation theory (see also [Krugljak et al. 1999] to highlight some subtleties in this approach).

The main goal of the present paper is to prove a general result on Carnot groups, albeit slightly simpler than those of [Bahouri et al. 2012] or [Grillo 2003], by using only *elementary* techniques: most of the paper relies only on integrations by part and on a fine analysis of the commutator structure. We will occasionally use interpolation techniques, but it is only required here if fractional regularities are sought after.

A Carnot group is a connected, simply connected and nilpotent Lie group G whose Lie algebra $\mathfrak{g}$ admits a stratification, i.e.,

$$\mathfrak{g} = \bigoplus_{j=1}^{m} V_j \quad \text{where } [V_1, V_j] = V_{j+1}, \tag{2}$$

with $V_m \neq \{0\}$ but $[V_1, V_m] = \{0\}$. The dimensions will be denoted by $q_j = \dim V_j$ and $q = \sum q_j = \dim \mathfrak{g}$. Given a basis $(Y_\ell)_{\ell=1,\dots,q}$ of $\mathfrak{g}$ adapted to the stratification each index $i \in \{1, \dots, q\}$ can be associated to a unique *weight* $\omega_\ell \in \{1, \dots, m\}$ such that $Y_\ell \in V_{\omega_\ell}$, namely

$$\omega_\ell = j \quad \text{for } n_{j-1} < \ell \leq n_j, \tag{3}$$

where $n_0 = 0$ and $n_j = n_{j-1} + q_j$ for $j = 1, \dots, m$ is the sequence of cumulative dimensions. Note that $n_1 = q_1$ and $n_m = q$. The horizontal derivatives are the derivatives in the first layer (see Section 1.5 below) and they are collected together in the following notation:

$$\nabla_G f = (Y_1^L f, \dots, Y_{n_1}^L f). \tag{4}$$

The stratification hypothesis ensures that each derivative $Y_i f$ can be expressed as at most $\omega_i - 1$ commutators of horizontal derivatives. The *homogeneous dimension* is the integer

$$Q = \sum_{j=1}^{m} j q_j = \sum_{\ell=1}^{q} \omega_\ell. \tag{5}$$

For $k \in \mathbb{N}$, the Sobolev space $H^k(G)$ is the subspace of functions $\varphi \in L^2(G)$ such that $\nabla_G^\alpha \varphi \in L^2(G)$ for any multi-index α of length $|\alpha| \leq k$. Fractional spaces can, for example, be defined by interpolation. The main result that we intend to prove here is the following.

Theorem 1. *Let G be a Carnot group and $\|\cdot\|$ any homogeneous pseudonorm equivalent to the Carnot–Carathéodory distance to the origin. Then, for any real s*

with $0 \leq s < Q/2$, *there exists a constant* $C_s > 0$ *such that*

$$\int_G \frac{|f(g)|^2}{\|g\|_G^{2s}}\, dg \leq C_s \|f\|^2_{H^s(G)} \tag{6}$$

for any function $f \in H^s(G)$.

A similar Hardy inequality was proved by the author in [Vigneron 2006] for families of vector fields that satisfy a Hörmander bracket condition of step 2; the proof was based on the ideas of [Bahouri et al. 2005a; Bahouri and Cohen 2011], but was never published independently. This result was part of a broader study [Bahouri et al. 2005b; 2009; Mustapha and Vigneron 2007; Vigneron 2007] aiming at characterizing the traces of Sobolev spaces on the Heisenberg group, along hypersurfaces with nondegenerate characteristic points. Here, instead, we concentrate (except in Section 3) on the case of stratified groups, but without restrictions on the step m of the stratification.

The mathematical literature already contains numerous Hardy-type inequalities either on the Heisenberg group, for the p-sub-Laplacian, for Grushin-type operators and H-type groups (see, e.g., [D'Ambrosio 2005; Kombe 2010]). Sometimes (e.g., in [Garofalo and Lanconelli 1990; Niu et al. 2001; Dou et al. 2007]), a weight is introduced in the left-hand side that vanishes along the center of the group, i.e., along the (most) subelliptic direction. For example, [Garofalo and Lanconelli 1990] contains the following inequality on the Heisenberg group $\mathbb{H}_n \simeq \mathbb{C}^n \times \mathbb{R}$:

$$\int_{\mathbb{H}_n} \frac{|f(x)|^2}{d_{\mathbb{H}_n}(x,0)^2} \Phi(x)\, dx \leq A \sum_{j=1}^{n} (\|X_j f\|^2_{L^2} + \|Y_j f\|^2_{L^2}) + B\|f\|^2_{L^2}, \tag{7}$$

where (X_j, Y_j) are a basis of the first layer of the stratification and $d_{\mathbb{H}_n}((z,t),0) \simeq \sqrt[4]{|z|^4 + t^2}$ is the gauge distance and Φ is a cut-off function that vanishes along the center $z = 0$:

$$\Phi(z,t) = \frac{|z|^2}{\sqrt{|z|^4 + t^2}}.$$

A secondary goal of this article is to show that such a cut-off is usually not necessary.

The core of our proof of Theorem 1 (see Section 2.5) consists in an integration by part against the radial field (the infinitesimal generator of dilations). The radial field can be expressed in terms of the left-invariant vector fields but, as all strata are involved, this first step puts $m - s$ too many derivatives on the function. Next, one uses the commutator structure of the left-invariant fields to carefully backtrack all but one derivative and let them act instead on the coefficients of the radial field. This step requires that those coefficients have symbol-like properties. One can then conclude by an iterative process that reduces the Hardy inequality with weight $\|g\|_G^{-s}$ to the one with weight $\|g\|_G^{-(s-1)}$ as long as $s < Q/2$.

Finally, it is worth mentioning that a byproduct of our elementary approach concerns the symbol properties of the Carnot–Carathéodory norm (or of any equivalent gauge). For general hypoelliptic families of vector fields, *the norm is not always a symbol of order 1* (see section Section 3). On the contrary, on Carnot groups, it happens to always be equivalent to such a symbol (Proposition 7). At the end of the article, we discuss sufficient conditions for this property to hold for families of hypoelliptic vector fields, based either on the order m of the Hörmander condition (Theorem 14), or on the way the commutators are structured (Theorems 15 and 16).

The structure of the article goes as follows. Section 1 is a brief survey of calculus on Carnot groups. It also sets the notations used subsequently. Section 2 contains the actual proof of Theorem 1 and concludes on Theorem 13, which is the homogeneous variant of the previous statement. Section 3 addresses an open question regarding families of vector fields that satisfy a Hörmander bracket condition, but lack an underlying group structure.

1. A brief survey of calculus on Carnot Groups

Let us first recall some classic definitions and facts about nilpotent Lie groups. We also introduce notations that will be needed in Section 2. For a more in-depth coverage of Lie groups, sub-Riemannian geometry and nilpotent groups, see, e.g., [Montgomery 2002; Folland and Stein 1982; Rossmann 2002] or the introduction of [Ambrosio and Rigot 2004].

1.1. ***Left-invariant vector fields and the exponential map.*** Let us consider a Lie group G and $\mathfrak{g} = T_eG$ its Lie algebra; e denotes the unit element of G. Left-translation is defined by $L_g(h) = gh$.

Definition. A vector field ξ is called *left-invariant* if $(L_g)_* \circ \xi = \xi \circ L_g$. Such a vector field is entirely determined by $v = \xi(e) \in \mathfrak{g}$. To signify that v generates ξ, one writes $\xi = v^L$ thus:

$$v^L(g) = d(L_g)_{|e}(v). \tag{8}$$

The tangent bundle TG identifies to $G \times \mathfrak{g}$ by the map $(g, v) \mapsto (g, v^L(g))$.

The flow Φ_t^v of a left-invariant vector field v^L exists for all time. Indeed, one has $\Phi_t^v(g) = L_g \circ \Phi_t^v(e)$, which implies that $\Phi_{t+s}^v(e) = L_{\Phi_s^v(e)} \circ \Phi_t^v(e)$, thus allowing the flow to be extended globally once it has been constructed locally.

Definition. The *exponential map* $\exp : \mathfrak{g} \to G$ is defined by $\exp(v) = \Phi_1^v(e)$ where $(\Phi_t^v)_{t\in\mathbb{R}}$ is the flow of the left-invariant vector field v^L.

One can check that the flow of v^L starting from $g \in G$ is $\Phi_t^v(g) = g\exp(tv)$. In particular,

$$\exp(sw)\exp(tv) = \Phi_t^v(\exp(sw)) \quad \text{and} \quad d\exp_{|0} = \mathrm{Id}_{\mathfrak{g}}\,.$$

1.2. ***The Baker–Campbell–Hausdorff formula.*** The commutator of two left-invariant vector fields is also a left-invariant field. Therefore, the commutator of $u, v \in \mathfrak{g}$ is defined by $[u, v] = [u^L, v^L](e) \in \mathfrak{g}$. The product law of G induces an extremely rigid relation between exponentials, known as the Baker–Campbell–Hausdorff formula:

$$\exp(u)\exp(v) = \exp(\mu(u, v)) \tag{9}$$

where $\mu(u, v) = u + v + \frac{1}{2}[u, v] + \frac{1}{12}([u, [u, v]] + [v, [v, u]]) + \cdots$ is a universal Lie series in u, v i.e., an expression consisting of the iterated commutators of u and v. In general, this formula holds provided u and v are small enough for the series to converge (see, e.g., [Rossmann 2002, §1.3]). Subsequently, one will only use the linear part of (9) with respect to one variable:

$$d\mu(u, \cdot)_{|0}(w) = \frac{\mathrm{ad}(u)}{1 - e^{-\mathrm{ad}(u)}}(w) = w + \sum_{n=1}^{\infty} \frac{(-1)^n B_n}{n!} \underbrace{[u, \ldots, [u, w]]}_{n \text{ times } u}, \tag{10}$$

where B_n are the Bernoulli numbers (i.e., $(x)/e^x - 1 = \sum (B_n/n!)x^n$) and $\mathrm{ad}(u) = [u, \cdot]$. This formula is classical and can be found, e.g., in [Rossmann 2002] or [Klarsfeld and Oteo 1989].

1.3. ***Stratification.*** From now on, G is supposed to be *stratified*, i.e., it is a Carnot group as defined in the introduction of this paper. A stratified group is, in particular, nilpotent of step m. Moreover, elementary linear algebra gives restrictions on the possible dimensions $q_j = \dim V_j$ of the strata:

$$q_2 \le \frac{q_1(q_1 - 1)}{2} \qquad \text{and for } j \ge 2, \quad q_{j+1} < q_1 q_j.$$

The last inequality is strict because of the Jacobi identity

$$[u, [v, w]] - [v, [u, w]] = -[w, [u, v]].$$

For an exact count of the possible relations, see, e.g., [Reutenauer 1993].

Proposition 2. *Let G be a Carnot group. Then* $\exp : \mathfrak{g} \to G$ *is a global diffeomorphism that allows G to be identified with the set* $\mathfrak{g}$ *equipped with the group law $u * v = \mu(u, v)$. The identity element is* 0 *and the inverse of u is $-u$.*

Proof. This claim is very standard so one only sketches the proof briefly. As $d\exp_{|0} = \mathrm{Id}_{\mathfrak{g}}$, there exists a neighborhood U_0 of e in G and V_0 of 0 in $\mathfrak{g}$ such that $\exp : V_0 \to U_0$ is a diffeomorphism. As G is connected, it is generated by any neighborhood of e and in particular by $U_0 = \exp(V_0)$. But, as $\mathfrak{g}$ is nilpotent, the expression $\mu(u, v)$ is a Lie polynomial of order m, thus (9) holds for any $u, v \in \mathfrak{g}$. Combining these facts implies that the exponential map is surjective. Next, one can show that the pair $(\mathfrak{g}, \exp)$ is a covering space of G. Indeed, given $g = \exp(v) \in G$, one gets a commutative diagram of diffeomorphisms

$$\begin{tikzcd} V_0 \arrow[r, "\exp"] \arrow[d, "\mu(v',\cdot)"'] & U_0 \arrow[d, "L_g"] \\ V' \arrow[r, "\exp"] & g \cdot U_0 \end{tikzcd}$$

for each $v' \in \mathfrak{g}$ such that $\exp(v') = g$. Finally, by a standard covering space argument based on the fact that $\mathfrak{g}$ is path connected (as a vector space) and G is simply connected (in the stratification assumption), one can claim that the exponential map is a global diffeomorphism. $\square$

Example. The Heisenberg group $\mathbb{H}$ can be realized as a set of upper-triangular matrices with diagonal entries equal to 1. The group law in $\mathbb{H}$ is the multiplication of matrices:

$$\begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & p' & r' \\ 0 & 1 & q' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & p+p' & r+r'+pq' \\ 0 & 1 & q+q' \\ 0 & 0 & 1 \end{pmatrix}.$$

The Lie algebra of $\mathbb{H}$ is

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & p & r \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix} = pY_1 + qY_2 + rY_3 \, ; \; p, q, r \in \mathbb{R} \right\}.$$

Left-invariant vector fields on $\mathbb{H}$ are linear combinations of

$$Y_1^L(g) = Y_1, \quad Y_2^L(g) = Y_2 + pY_3 \quad \text{and} \quad Y_3^L(g) = Y_3, \quad \text{where } g = \begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathbb{H}$ is a stratified nilpotent group with $V_1 = \mathrm{Span}(Y_1, Y_2)$ and $V_2 = \mathrm{Span}(Y_3)$. The exponential map is the usual exponential of nilpotent matrices. It transfers the group structure to $\mathfrak{h} \simeq \mathbb{R}^3$ by (9):

$$(p, q, r)_{\mathfrak{h}} \simeq \begin{pmatrix} 0 & p & r \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}, \quad \exp \begin{pmatrix} 0 & p & r \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & p & r + \frac{1}{2}pq \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mu\big((p, q, r)_{\mathfrak{h}}, (p', q', r')_{\mathfrak{h}}\big) = \big(p + p', q + q', r + r' + \tfrac{1}{2}(pq' - qp')\big)_{\mathfrak{h}}.$$

In exponential coordinates on $\mathfrak{h} \simeq \mathbb{R}^3$, the left-invariant vector fields thus take the following form:

$$\tilde{Y}_1^L(p,q,r) = \partial_p - \tfrac{1}{2} q \partial_r, \quad \tilde{Y}_2^L(p,q,r) = \partial_q + \tfrac{1}{2} p \partial_r, \quad \tilde{Y}_3^L(p,q,r) = \partial_r.$$

The two different expressions of the fields correspond to the change of variables $(p,q,r) \mapsto \big(p,q,r - \frac{1}{2}pq\big)$.

Remarks. (1) In general (even if G is connected and nilpotent), only the Lie group action can be recovered from the exponential map but not G itself. For example, $G = \{z \in \mathbb{C};\ |z| = 1\} \simeq \mathbb{S}^1$ with rotation law $(z, z') \mapsto zz'$ is a nilpotent group. One has $\mathfrak{g} = \mathbb{R}$ and $\mu(x, y) = x + y$ but the exponential map is $\exp(x) = e^{ix}$ and is obviously not a global diffeomorphism.

(2) Combined with (10), the commutative diagram of the proof of Proposition 2 provides a general formula for the differential of the exponential map, which we will need later on. For $v, w \in \mathfrak{g}$ and $g = \exp(v)$, one has

$$d\exp_{|v}(w) = \sum_{n=0}^{m-1} \frac{(-1)^n}{(n+1)!} [\underbrace{v, \dots, [v}_{n \text{ times} v}, w]]^L(g). \tag{11}$$

For example, on the Heisenberg group, one gets $d\exp_{|v}(w) = \big(w - \frac{1}{2}[v, w]\big)^L(g)$.

Proposition 3. *For any indices $j, k \in \{1, \dots, m\}$, one has*

$$[V_j, V_k] \subset \begin{cases} V_{j+k} & \text{if } j + k \le m, \\ \{0\} & \text{otherwise.} \end{cases} \tag{12}$$

Proof. By convention, let us write $V_n = \{0\}$ if $n > m$. For $k = 1$, the property holds by definition. For $k = 2$, as $V_2 = [V_1, V_1]$, any element can be written $[X, Y]$ with $X, Y \in V_1$. For $Z \in V_j$, one uses the identity $[A, BC] = [A, B]C + B[A, C]$ to get

$$[Z, [X, Y]] = [[Z, X], Y] - [[Z, Y], X] \in [[V_j, V_1], V_1] \subset [V_{j+1}, V_1] \subset V_{j+2}.$$

Next, one proceeds recursively. Assuming that for some $k \ge 2$, one has $[V_j, V_k] \subset V_{j+k}$ for any j, then given $Z \in V_{k+1} = [V_1, V_k]$, one writes $Z = [X, \zeta]$ with $X \in V_1$ and $\zeta \in V_k$. Then for any $W \in V_j$, the Jacobi identity gives

$$\begin{aligned} [W, Z] &= [W, [X, \zeta]] \\ &= -[X, [\zeta, W]] - [\zeta, [W, X]] \in [V_1, V_{j+k}] + [V_k, V_{j+1}] \subset V_{j+k+1} \end{aligned}$$

which makes the property hereditary in k. $\square$

1.4. ***Stratified dilations.*** The next essential object in a Carnot group is the *dilation* of the Lie algebra:

$$\text{for all } r>0, \qquad \delta_r=\sum_{j=1}^{m} r^j\pi_j, \tag{13}$$

where $\pi_j:\mathfrak{g}\to V_j$ is the projection onto V_j with kernel $\bigoplus_{k\neq j} V_k$. Identifying G to $\exp(\mathfrak{g})$, one gets a one parameter family of group automorphisms that we will simply denote by

$$rg=\exp\circ\,\delta_r\circ\exp^{-1}(g) \tag{14}$$

for any $r>0$ and $g\in G$.

The next result is an immediate consequence of the definition but should later be compared with the scaling property (41) of the radial vector field.

Proposition 4. *The dilation of a left-invariant vector field v^L is given by*

$$(\delta_r v)^L(rg)=\big((L_{rg})_*\circ\delta_r\circ(L_g)_*^{-1}\big)(v^L(g)). \tag{15}$$

Up to a constant factor, the Haar measure on G is given by the Lebesgue measure on $\mathfrak{g}\simeq\mathbb{R}^q$ and is commonly denoted dg.

Proposition 5. *One has, for all $\varphi\in L^1(G)$ and $r>0$,*

$$\int_G \varphi(rg)\,dg=r^{-Q}\int_G\varphi(g)\,dg, \tag{16}$$

where Q is the homogeneous dimension (5) *of G. Note that when $m\neq 1$, one has $Q>q$.*

Given (arbitrary[1]) Euclidean norms $\|\cdot\|_{V_j}$ on each V_j and $w=2\,\mathrm{LCM}(1,\ldots,m)$, the anisotropic *gauge-norm* of either $v\in\mathfrak{g}$ or of $g=\exp(v)\in G$ is defined by

$$\|v\|_{\mathfrak{g}}=\|g\|_G=\left(\sum_{j=1}^{m}\|\pi_j(v)\|_{V_j}^{w/j}\right)^{1/w}. \tag{17}$$

The gauge norm is homogeneous in the following sense:

$$\|rg\|_G=r\|g\|_G \quad\text{and}\quad |\Box_{g,r}|=c_0r^Q, \tag{18}$$

with a uniform constant c_0 and where the gauge-ball is defined by

$$\Box_{g,r}=\{h\in G\,;\ \|h^{-1}g\|_G<r\}.$$

[1]For a given basis of $\mathfrak{g}$ adapted to the stratification, one will chose here a Euclidean structure that renders this basis orthonormal. This is the natural choice when one proceeds to the identification $\mathfrak{g}\simeq\mathbb{R}^q$ through this basis.

Remark. The intrinsic metric objects of G are the so-called Carnot balls defined as the set of points that can be connected to a center $g_0 \in G$ by an absolutely continuous path γ whose velocity is subunitary for almost every time i.e., such that $\dot{\gamma}(t) \in (L_{\gamma(t)})_*(B_0)$ where $B_0 \subset V_1$ is a fixed Euclidean ball of the first layer of the stratification (up to some choice of a Euclidean metric on V_1). However, the ball-box theorem [Montgomery 2002] states that such intrinsic objects can be sandwiched between two gauge-balls of comparable radii. For our purpose (the analysis of Sobolev spaces), one can thus deal only with gauge-balls without impeding the generality.

1.5. ***Horizontal derivatives and Sobolev spaces on G.*** Vector fields ξ on G are identified with derivation operators on $C^\infty(G)$ by the Lie derivative formula:

$$(\xi\varphi)(g) = d\varphi_{|g}(\xi(g)). \tag{19}$$

Definition. The *horizontal derivatives* are the left-invariant vector-fields associated with V_1.

Let us consider a basis $(Y_\ell)_{1\le\ell\le q}$ of $\mathfrak{g}$ that is adapted to the stratification, i.e.,

$$V_j = \operatorname{Span}(Y_\ell)_{n_{j-1}<\ell\le n_j}, \tag{20}$$

where n_j is defined by (3). An horizontal derivative is thus a vector field

$$\xi = \sum_{j\le n_1} \alpha_j Y_j^L = \alpha \cdot \nabla_G,$$

where $\alpha = (\alpha_1, \dots, \alpha_{n_1}) \in \mathbb{R}^{n_1}$ and ∇_G is defined by (4). Noncommutative multi-indices are defined as follows: for $\gamma = (\gamma_1, \dots, \gamma_\ell) \in \{1, \dots, n_1\}^\ell$, one writes $l = |\gamma|$ and $\nabla_G^\gamma = Y_{\gamma_1}^L \circ \cdots \circ Y_{\gamma_l}^L$.

Let us unfold the commutator structure in $\mathfrak{g}$ with the following notation:

$$[Y_{\ell_1}, \dots, [Y_{\ell_n}, Y_{\ell_{n+1}}]] = \sum_{\ell'} \kappa(\ell_1, \dots, \ell_n, \ell_{n+1}; \ell') Y_{\ell'}. \tag{21}$$

Note that according to (12), one can warrant that $\kappa(\ell_1, \dots, \ell_n, \ell_{n+1}; \ell') = 0$ if $\omega_{\ell'} \neq \omega_{\ell_1} + \cdots + \omega_{\ell_{n+1}}$.

Remark. To simplify computations, one can always assume that the basis is chosen such that

$$\text{for all } \ell \in \{1, \dots, q\}, \quad Y_\ell = [Y_{\alpha_1(\ell)}, \dots, [Y_{\alpha_{k-1}(\ell)}, Y_{\alpha_k(\ell)}]], \tag{22}$$

where $k = \omega_\ell$ and $\alpha_i(\ell) \le n_1$. Indeed, the Lie algebra is linearly generated by the commutators of the restricted family ∇_G and one just has to extract a basis from it.

Example. On the Heisenberg group $\mathbb{H}$, the horizontal derivatives are left-invariant vector fields of the form $\xi = (\alpha Y_1 + \beta Y_2)^L$ for $\alpha, \beta \in \mathbb{R}$.

Definition. For $s \in \mathbb{N}$, the Sobolev space $H^s(G)$ consists of the functions such that each composition of at most s horizontal derivatives belongs to $L^2(G)$. The norm is defined (up to the choice of the Y_ℓ) by

$$\|\varphi\|^2_{H^s(G)} = \sum_{|\gamma|\leq s} \int_G |\nabla_G^\gamma \varphi(g)|^2 \, dg, \tag{23}$$

with γ a noncommutative multi-index.

Remark. The space $H^2(G)$ is the domain of the hypoelliptic Laplace operator

$$\mathcal{L}_G = -\sum_{\ell \leq n_1} (Y_\ell^L)^* \, Y_\ell^L. \tag{24}$$

A celebrated result of L. Hörmander [1967] states that $H^s(G) \subset H^{s/m}_{\mathrm{loc}}(\mathbb{R}^q)$, where the last Sobolev space is the classical one (homogeneous and isotropic) on $\mathbb{R}^q$.

1.6. ***Exponential coordinates on a stratified group.*** Given a basis $(Y_\ell)_{1\leq\ell\leq q}$ of $\mathfrak{g}$ adapted to the stratification, one can define a natural coordinate system on G, called *exponential coordinates*. Given $g = \exp(v) \in G$, its coordinates $x(g) = (x_\ell(g))_{1\leq\ell\leq q} \in \mathbb{R}^q$ are defined by

$$v = \sum_{\ell=1}^{q} x_\ell(g) Y_\ell. \tag{25}$$

The projections $(\pi_j)_{1\leq j\leq m}$ introduced in (13) are

$$\text{for all } j \in \{1, \ldots, m\}, \quad \pi_j(v) = \sum_{\ell=1+n_{j-1}}^{n_j} x_\ell(g) Y_\ell. \tag{26}$$

In exponential coordinates, the expression of stratified dilations (14) is

$$\text{for all } \ell \in \{1, \ldots, q\}, \qquad x_\ell(rg) = r^{\omega_\ell} x_\ell(g). \tag{27}$$

The gauge norm (17) is given (for some fixed large $w \in \mathbb{N}$) by

$$\|g\|_G = \|x(g)\|_{\mathfrak{g}} = \left(\sum_{j=1}^{m} \left(\sum_{\ell=1+n_{j-1}}^{n_j} |x_\ell(g)|^2 \right)^{w/j} \right)^{1/(2w)}. \tag{28}$$

One could however take any uniformly equivalent quantity as a gauge norm, which will be the case subsequently, after Proposition 7.

Example. With the previous notation, the exponential coordinates on the Heisenberg group $\mathbb{H}$ are

$$x_1(g) = p, \quad x_2(g) = q, \quad x_3(g) = r - \tfrac{1}{2}pq \qquad \text{for } g = \begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{H}.$$

1.7. ***Left-invariant basis of vector fields.*** When doing explicit computations, it is natural to identify $\mathfrak{g}$ with $\mathbb{R}^q$ through the previous coordinates. Given $v \in \mathfrak{g}$, the left-invariant vector field v^L on G defined by (8) matches a corresponding vector field on $\mathfrak{g} \simeq \mathbb{R}^q$ that we will denote by $\tilde{v}^L$. According to the Baker–Campbell–Hausdorff formula (10),

for all $x = (x_1, \dots, x_q) \in \mathfrak{g}$,

$$\tilde{v}^L(x) = v + \sum_{n=1}^{m-1} \sum_{\ell_1,\dots,\ell_n} \frac{(-1)^n B_n}{n!} x_{\ell_1} \dots x_{\ell_n} [Y_{\ell_1}, \dots, [Y_{\ell_n}, v]], \tag{29}$$

where each of the ℓ_i ranges over $\{1, \dots, q\}$.

After the identification $\mathfrak{g} \simeq \mathbb{R}^q$ and to avoid confusion, let us denote Y_ℓ by ∂_ℓ: the vectors $(\partial_\ell)_{1 \le \ell \le q}$ are the dual basis of the coordinates $(x_\ell)_{1 \le \ell \le q}$. The left-invariant basis then becomes explicit:

$$\text{for all } x = (x_1, \dots, x_q) \in \mathfrak{g}, \quad \tilde{Y}_\ell^L(x) = \partial_\ell + \sum_{\ell'=1}^{q} \zeta_{\ell,\ell'}(x_1, \dots, x_q) \partial_{\ell'}, \tag{30}$$

with, thanks to (29) and (21),

$$\zeta_{\ell,\ell'}(x) = \sum_{n=1}^{m-1} \frac{(-1)^n B_n}{n!} \sum_{\ell_1,\dots,\ell_n} \kappa(\ell_1, \dots, \ell_n, \ell; \ell') x_{\ell_1} \cdots x_{\ell_n}. \tag{31}$$

Let us point out that $\zeta_{\ell,\ell'} = 0$ if $\omega_{\ell'} \le \omega_\ell$ (because κ vanishes), thus the left-invariant correction to ∂_ℓ only involves derivatives of a strictly higher weight. In other words, the indices in (30) can be restricted to $\ell' > n_{\omega_\ell} = q_1 + q_2 + \cdots + q_{\omega_\ell}$.

Note also that $\zeta \in C^\infty(\mathbb{R}^q, \mathcal{M}_{q,q}(\mathbb{R}))$ and $\zeta(0) = 0$. More precisely, this matrix represents the differential action of left-translations, expressed in exponential coordinates:

$$(dL_g)_{|e} \equiv \mathrm{Id}_{\mathbb{R}^q} + \zeta(x(g)). \tag{32}$$

Moreover, as $|x_\ell(g)| \lesssim \|g\|_G^{\omega_\ell}$, one has $|\zeta_{\ell,\ell'}(x(g))| \lesssim \|g\|_G^{\omega_{\ell'} - \omega_\ell}$.

2. Proof of Theorem 1

This section is devoted to the proof of the main statement. The key idea is to prove the result for $s = 1$ and then "push" the result up to the maximal regularity using only integrations by part. Adding an interpolation step once the result is known for $s=1$, but before pushing it to a higher regularity, allows one to capture all eligible fractional derivatives. The actual proof is written in the last subsection Section 2.5 but some preliminary results are required.

2.1. *Symbol classes $S_n^\alpha(G)$.* Symbol classes are a convenient way to classify the coefficients involved in the computations in terms of how they vanish at the origin.

Definition. For $\alpha \in \mathbb{R}_+$ and $n \in \mathbb{N} \cup \{\infty\}$, the symbol class $S_n^\alpha(G)$ is defined as the set of functions $\varphi \in L^\infty_{\mathrm{loc}}(G)$ such that for any multi-index γ of length $|\gamma| \le n$, there exists a constant $C_\gamma > 0$ that ensures the following inequality:

$$\text{for all } g \in G, \quad \|g\|_G \le 1 \Longrightarrow |\nabla_G^\gamma \varphi(g)| \le C_\gamma \|g\|_G^{(\alpha-|\gamma|)_+}. \tag{33}$$

For example, the symbols of class $S_0^0(G) = L^\infty_{\mathrm{loc}}(G)$ are only required to be bounded near the origin. The symbol class $S_\infty^\alpha(G)$ is also denoted $S^\alpha(G)$.

The following properties hold.

(1) The Leibnitz formula gives

$$\varphi \in S_m^\alpha(G) \text{ and } \psi \in S_n^\beta(G) \Longrightarrow \varphi\psi \in S^{\alpha+\beta}_{\min(m,n)}(G).$$

(2) As Y_ℓ^L is a linear combination of derivatives ∇_G^γ of length $|\gamma| = \omega_\ell$, one has (if $n \ge \omega_\ell$)

$$\varphi \in S_n^\alpha(G) \Longrightarrow Y_\ell^L \varphi \in S^{(\alpha-\omega_\ell)_+}_{n-\omega_\ell}(G).$$

(3) As smooth functions are locally bounded, one has also

$$S^\alpha_{\alpha-1}(G) \cap C^\infty(G) \subset S^\alpha(G).$$

The coordinates and the coefficients of the left-invariant vector fields belong to the following classes.

Proposition 6. *One has*

$$\textit{for all } \ell \in \{1, \dots, q\}, \quad x_\ell(g) \in S^{\omega_\ell}(G) \tag{34}$$

and

$$\textit{for all } \ell, \ell' \in \{1, \dots, q\}, \quad \zeta_{\ell,\ell'}(x_1(g), \dots, x_q(g)) \in S^{\omega_{\ell'}-\omega_\ell}(G). \tag{35}$$

Proof. We already observed that $|x_\ell(g)| \lesssim \|g\|_G^{\omega_\ell}$ thus $x_\ell(g) \in S_0^{\omega_\ell}(G)$. Next, using (30), one gets

$$Y^L_{\ell_0}(x_\ell(g)) = \delta_{\ell_0,\ell} + \zeta_{\ell_0,\ell}(x(g)) = \begin{cases} \zeta_{\ell_0,\ell}(x(g)) & \text{if } \omega_{\ell_0} < \omega_\ell, \\ \delta_{\ell_0,\ell} & \text{if } \omega_{\ell_0} = \omega_\ell, \\ 0 & \text{if } \omega_{\ell_0} > \omega_\ell. \end{cases}$$

Assuming $\ell_0 \in \{1, \dots, n_1\}$, one gets $|\nabla_G x_\ell(g)| \le C\|g\|_G^{\omega_\ell - 1}$ thus $x_\ell(g) \in S_1^{\omega_\ell}(G)$. One can now bootstrap this partial result in the expression (31), which gets us $\zeta_{\ell,\ell'}(x(g)) \in S_1^{\omega_{\ell'}-\omega_\ell}(G)$. The previous expression now reads $\nabla_G x_\ell(g) \in S_1^{\omega_\ell - 1}$ and thus $x_\ell(g) \in S_2^{\omega_\ell}(G)$. Iterating this process leads to $x_\ell(g) \in S_\infty^{\omega_\ell}(G)$ and $\zeta_{\ell,\ell'}(x(g)) \in S_\infty^{\omega_{\ell'}-\omega_\ell}(G)$. □

The key result is that one can adjust the gauge norm to be a symbol of order 1 (see also Section 3).

Proposition 7. *There exists a symbol $\rho(g) \in S_1^1(G)$ that is uniformly equivalent to the gauge norm. For higher-order derivatives, it satisfies for any multi-index γ:*

$$\text{for all } g \in G, \quad \rho(g) \le 1 \Longrightarrow |\nabla_G^\gamma \rho| \le \frac{C_\gamma}{\rho^{|\gamma|-1}}. \tag{36}$$

Moreover, there exists $w \in \mathbb{N}$ such that $\rho^w \in S^w(G)$.

Proof. Let us now modify the gauge norm (28) into the uniformly equivalent gauge

$$\rho(g) = \left(\sum_{\ell=1}^{q} |x_\ell(g)|^{w/\omega_\ell} \right)^{1/w} \tag{37}$$

with $w = 2\,\mathrm{LCM}(1, \dots, m)$ to ensure that each $w/\omega_\ell \in 2\mathbb{N}$. In particular, $\rho(g)^w \in C^\infty(G)$. Next, one computes the first horizontal derivative of the norm, using (30):

$$\nabla_G \rho = \frac{\nabla_G(\rho^w)}{w\rho^{w-1}} = \left(\frac{1}{\rho^{w-1}} \left[x_\ell^{w-1} + \sum_{\ell'=1}^{q} \frac{\zeta_{\ell,\ell'}(x)\, x_{\ell'}^{w/\omega_{\ell'}-1}}{\omega_{\ell'}} \right] \right)_{\ell=1\dots n_1}.$$

The expression in square brackets is a symbol of class $S^{w-1}(G)$ because of (34) and (35) and $\omega_\ell = 1$ for $\ell \le n_1$. Thus $\nabla_G \rho$ is bounded near the origin which means that the modified gauge ρ belongs to $S_1^1(G)$. Next, one observes that for any $\alpha \ge 1$, if $\theta \in S^\alpha(G)$ then

$$\nabla_G\left(\frac{\theta}{\rho^\alpha}\right) = \left(\frac{\nabla_G \theta}{\rho^{\alpha-1}} - \alpha \frac{\theta}{\rho^\alpha} \nabla_G \rho \right) \frac{1}{\rho} = \left(\frac{\theta_1}{\rho^{\alpha-1}} + \frac{\theta_2}{\rho^{\alpha+w-1}} \right) \frac{1}{\rho}$$

with $\theta_1 \in S^{\alpha-1}(G)$ and $\theta_2 \in S^{\alpha+w-1}(G)$. One can thus claim by recurrence on the length of the multi-index γ that

$$\nabla_G^\gamma \rho = \left(\theta_{\gamma,0} + \sum_k \frac{\theta_{\gamma,k}}{\rho^{\alpha_k}} \right) \frac{1}{\rho^{|\gamma|-1}},$$

where $\theta_{\gamma,k} \in S^{\alpha_k}(G)$ is a polynomial in $x_\ell(g)$ with $\alpha_k \ge 1$ and $\theta_{\gamma,0}$ is a polynomial. Note that a polynomial in $S^0(G)$ is necessarily the sum of a constant and a polynomial in $S^1(G)$ and that, by (30), the horizontal derivatives of a polynomial are also a polynomial. This ensures (36). The final assertion about $\|\cdot\|_G^w$ follows immediately from (34). $\square$

From now one, one will modify the gauge norm accordingly and assume that $\|\cdot\|_G = \rho \in S_1^1(G)$.

2.2. ***The radial vector field.*** The *infinitesimal generator of dilations* on $\mathfrak{g}$ is the linear operator $\tilde{R} : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\tilde{R} = \sum_{j=1}^{m} j\pi_j. \tag{38}$$

It is diagonalizable with positive eigenvalues; its trace $\operatorname{Tr} \tilde{R} = Q$ is the homogeneous dimension. One checks immediately that $\delta_r = e^{(\log r)\tilde{R}}$, thus $\tilde{R}(x) = \frac{d}{dr}\delta_r(x)\big|_{r=1}$. The pair $(x, \tilde{R}(x))$ is a vector field on $\mathfrak{g}$ whose expression in exponential coordinates follows from (26):

$$\text{for all } x = (x_1, \dots, x_q) \in \mathfrak{g}, \quad \tilde{R}(x) = \sum_{\ell=1}^{q} \omega_\ell x_\ell \partial_\ell. \tag{39}$$

Its exponential lift is called the *radial field* on G:

$$R(g) = d\exp_{|v}(\tilde{R}(v)) = \frac{d}{dr}(rg)\Big|_{r=1}. \tag{40}$$

Proposition 8. *The radial vector field is scaling invariant*:

$$R(rg) = ((L_{rg})_* \circ \delta_r \circ (L_g)_*^{-1})(R(g)). \tag{41}$$

Moreover, *it can be expressed in terms of left-invariant derivatives*:

$$R(g) = \sum_{\ell=1}^{q} \sigma_\ell(x_1(g), \dots, x_q(g)) Y_\ell^L(g) \tag{42}$$

with $\sigma_\ell(x(g))$ equal to

$$\omega_\ell x_\ell(g) + \sum_{n=1}^{m-1} \frac{(-1)^n}{(n+1)!} \sum_{\ell_1,\dots,\ell_{n+1}} x_{\ell_1}(g) \cdots x_{\ell_{n+1}}(g) \cdot \omega_{\ell_{n+1}} \kappa(\ell_1, \dots, \ell_{n+1}; \ell) \in S^{\omega_\ell}(G).$$

Remarks. (1) Note that the variable x_{ℓ_i} that appears in the second term defining σ_ℓ must satisfy

$$\omega_\ell = \omega_{\ell_1} + \cdots + \omega_{\ell_{n+1}}$$

because if it is not the case, then $\kappa(\ell_1, \dots, \ell_{n+1}; \ell) = 0$. In particular, as there are at least $n + 1 \geq 2$ factors, one has $\omega_{\ell_i} < \omega_\ell$ for each i.

(2) Both expressions for $\tilde{R} = \sum \omega_\ell x_\ell \partial_\ell = \sum \sigma_\ell \tilde{Y}_\ell^L$ combined with (30) provide a remarkable identity embedded in the commutator structure. For any $\ell' \in \{1, \dots, q\}$,

$$\sum_{n=1}^{m-1} \frac{(-1)^n}{(n+1)!} \sum_{\ell_1,\dots,\ell_{n+1}} x_{\ell_1} \cdots x_{\ell_{n+1}} \omega_{\ell_{n+1}} \cdot \Big(\kappa(\ell_1, \dots, \ell_{n+1}; \ell') + \sum_\ell \kappa(\ell_1, \dots, \ell_{n+1}; \ell) \zeta_{\ell,\ell'}(x) \Big) = -\sum_\ell \omega_\ell x_\ell \zeta_{\ell,\ell'}(x).$$

Note that when one substitutes $x = x(g)$, both sides are indeed symbols of class $S^{\omega_{\ell'}}(G)$.

Proof. Formula (41) follows, e.g., from the identities $\tilde{R} \circ \delta_r = \delta_r \circ \tilde{R}$ and $\operatorname{ad} \circ \delta_r = \delta_r \circ \operatorname{ad} \circ \delta_r^{-1}$:

$$\begin{aligned} R(rg) &= d\exp_{|\delta_r(v)} \circ \delta_r \circ \tilde{R}(v) \\ &= (L_{rg})_* \circ \left(\frac{1 - e^{-\operatorname{ad}(\delta_r(v))}}{\operatorname{ad}(\delta_r(v))} \right) \circ \delta_r \circ \tilde{R}(v) \\ &= (L_{rg})_* \circ \delta_r \circ \left(\frac{1 - e^{-\operatorname{ad}(v)}}{\operatorname{ad}(v)} \right) \circ \tilde{R}(v) \\ &= ((L_{rg})_* \circ \delta_r \circ (L_g)_*^{-1})(R(g)). \end{aligned}$$

The definition of $R(g)$ with $g = \exp(v)$ also reads

$$R(g) = \sum_{\ell=1}^{q} \omega_\ell x_\ell(g) (d\exp_{|v} Y_\ell).$$

Combining the expression for the differential of exp given by (11), the identity $[u, v]^L = [u^L, v^L]$ and the fact that $v = \sum x_\ell(g) Y_\ell$ give

$$\begin{aligned} R(g) &= \sum_{\ell=1}^{q} \omega_\ell x_\ell(g) \Bigg(\sum_{n=0}^{m-1} \frac{(-1)^n}{(n+1)!} [\underbrace{v, \dots, [v}_{n \text{ times } v}, Y_\ell]] \Bigg)^L (g) \\ &= \sum_{\ell=1}^{q} \omega_\ell x_\ell(g) \Bigg(Y_\ell^L(g) + \sum_{n=1}^{m-1} \sum_{\ell_1,\dots,\ell_n} \frac{(-1)^n}{(n+1)!} x_{\ell_1}(g) \cdots x_{\ell_n}(g) [Y_{\ell_1}, \dots, [Y_{\ell_n}, Y_\ell]]^L(g) \Bigg). \end{aligned}$$

This formula can be further simplified into (42) by using (21). The symbol property comes from (34) and the restriction on nonvanishing indices imposed by (21). $\square$

Examples. The previous computation can be simplified further by observing the antisymmetric role of ℓ_n and ℓ_{n+1} in $\omega_{\ell_{n+1}}[Y_{\ell_n}, Y_{\ell_{n+1}}]$ if $\omega_{\ell_n} = \omega_{\ell_{n+1}}$. For $m \leq 4$, one thus gets the following expressions for the radial field.

(1) For a group of step $m = 2$, the radial field is given by

$$R(g) = \sum_{\ell=1}^{q} \omega_\ell x_\ell(g) Y_\ell^L(g).$$

On the Heisenberg group $\mathbb{H}$ with exponential coordinates introduced in Section 1.6, this formula boils down, as expected, to

$$R(g) = \big(pY_1 + qY_2 + 2(r - \tfrac{1}{2}pq)Y_3\big)^L(g) = p\partial_p + q\partial_q + 2r\partial_r.$$

(2) For a group of step $m = 3$, the radial field is "corrected" along V_3:

$$R(g) = \sum_{\ell=1}^{q} \omega_\ell x_\ell(g) Y_\ell^L - \tfrac{1}{2} \sum_{\substack{1 \leq \ell_1 \leq n_1 \\ n_1 < \ell_2 \leq n_2}} x_{\ell_1}(g) x_{\ell_2}(g) [Y_{\ell_1}, Y_{\ell_2}]^L.$$

(3) For step $m = 4$, its expression involves a further "correction" along V_4 that is split among two types of commutators:

$$\begin{aligned} R(g) = \sum_{\ell=1}^{q} \omega_\ell x_\ell(g) Y_\ell^L - \tfrac{1}{2} \sum_{\substack{1 \leq \ell_1 \leq n_1 \\ n_1 < \ell_2 \leq \boldsymbol{n}_3}} & x_{\ell_1}(g) x_{\ell_2}(g) [Y_{\ell_1}, Y_{\ell_2}]^L \\ & + \tfrac{1}{6} \sum_{\substack{1 \leq \ell_1, \ell_2 \leq n_1 \\ n_1 < \ell_3 \leq n_2}} x_{\ell_1}(g) x_{\ell_2}(g) x_{\ell_3}(g) [Y_{\ell_1}, [Y_{\ell_2}, Y_{\ell_3}]]^L. \end{aligned}$$

Proposition 9. *The gauge norm (37) and the radial field are related by the following formula*:

$$\text{for all } s > 0, \qquad \frac{1}{\|\cdot\|_G^{2s}} = -\frac{1}{2s} R\left(\frac{1}{\|\cdot\|_G^{2s}}\right). \tag{43}$$

Proof. Applying the chain rule, one gets

$$\frac{\lambda(g)}{\|g\|_G^{2s}} = -\frac{1}{2s} R\left(\frac{1}{\|g\|_G^{2s}}\right)$$

with $\lambda(g) = R(\|g\|_G)/\|g\|_G$ and where the field R is obviously computed at the same point $g \in G$ as the function that is being differentiated. Let us also observe that

$$\lambda(g) = \frac{R(\|g\|_G^w)}{w\|g\|_G^w}$$

for any $w \in \mathbb{N}^*$ and in particular for $w = 2\,\mathrm{LCM}(1, \dots, m)$ for which we know that $\|\cdot\|_G^w \in S^w(G)$ by Proposition 7. Using the formula (37) for the modified gauge norm and (39) for the expression of the radial field in exponential coordinates, one then gets (note that $w/\omega_\ell \in 2\mathbb{N}^*$)

$$\text{for all } x \in \mathfrak{g} \simeq \mathbb{R}^q, \quad \tilde{R}(\|x\|_{\mathfrak{g}}^w) = \sum_{\ell=1}^{q} \omega_\ell x_\ell \cdot \frac{w}{\omega_\ell} x_\ell^{w/\omega_\ell - 1} = w\|x\|_{\mathfrak{g}}^w$$

and thus $\lambda(g) = 1$ for any $g \in G$. □

2.3. ***Adjoints.***

Proposition 10. *For the $L^2(G)$ scalar product, the adjoint vector field to R is*

$$R^*(g) = -Q - R(g).$$

Proof. The proof is simplest in exponential coordinates, using (39) and (5):

$$\text{for all } x \in \mathfrak{g} \simeq \mathbb{R}^q, \quad \tilde{R}(x) + \tilde{R}^*(x) = \operatorname{div} \tilde{R} = \sum_{\ell=1}^{q} \omega_\ell = Q.$$

One can also prove this formula directly, using (42) and (30):

$$\text{for all } g \in \mathfrak{G}, \quad R(g) + R^*(g) = \sum_{\ell} \partial_\ell(\sigma_\ell) + \sum_{\ell,\ell'} \zeta_{\ell,\ell'} \cdot (\partial_{\ell'}\sigma_\ell) + \sigma_\ell \cdot (\partial_{\ell'}\zeta_{\ell,\ell'}).$$

In this sum, according to a remark that follows (30), the index ℓ' is restricted to $\ell' > n_{\omega_\ell}$ and, in particular, the definition (3) then implies $\omega_{\ell'} > \omega_\ell$. Now thanks to the remark that follows Proposition 8, one can claim that the variable x_ℓ does not appear in the second part of σ_ℓ, thus its derivative reads

$$\partial_\ell(\sigma_\ell) = \omega_\ell.$$

For a similar reason, $\partial_{\ell'}\sigma_\ell = 0$ for $\omega_{\ell'} > \omega_\ell$. One observes also that in (31), each ℓ_i involved in the expression of $\zeta_{\ell,\ell'}$ must satisfy $\omega_{\ell_i} < \omega_{\ell'}$. In particular, $\partial_{\ell'}\zeta_{\ell,\ell'} = 0$. One concludes using (5). □

The next property checks that left-invariant vector fields on a Carnot group are divergence-free.

Proposition 11. *For the $L^2(G)$ scalar product, the adjoint vector field to Y_ℓ^L is $-Y_\ell^L$. In particular, for any smooth function ψ on G and any $\ell_1, \dots, \ell_n \in \{1, \dots, q\}$,*

$$\int_G [Y_{\ell_1}^L, \dots, [Y_{\ell_{n-1}}^L, Y_{\ell_n}^L]]\psi(g) \cdot \psi(g)\, dg = 0. \tag{44}$$

Proof. The second "computational" proof of the previous proposition (the one based on (30)) also ensures that

$$\partial_{\ell'}\zeta_{\ell,\ell'} = 0$$

when $\omega_{\ell'} > \omega_\ell$ and therefore $(Y_\ell^L)^* = -Y_\ell^L$. As the commutator of two antisymmetric operators is also an antisymmetric one, the second statement follows immediately. □

2.4. ***A density result.*** The following density result can be proved by a scaling argument.

Proposition 12. *The space $\mathcal{D}(G\backslash\{e\})$ of C^∞ functions, compactly supported outside the origin, is dense in $H^s(G)$ for any $0 \le s < Q/2$.*

Proof. One can use a Hilbert space approach based on scaling and Schwartz's theorem for distributions. Let us assume additionally that $s \in \mathbb{N}$ and consider a function $u \in H^s(G)$ that is orthogonal to any $\varphi \in \mathcal{D}(G\backslash\{e\})$, e.g.,

$$(u,\varphi)_s = \sum_{|\gamma|\le s} \int_G \nabla_G^\gamma u(g) \cdot \nabla_G^\gamma \varphi(g)\, dg = 0.$$

Integrating by parts (using Proposition 11 and the notation γ^* for the multi-index γ in reverse order) yields

$$\sum_{|\gamma|\le s} (-1)^{|\gamma|} \int_G \nabla_G^{\gamma^*} \nabla_G^\gamma u(g) \cdot \varphi(g)\, dg = 0.$$

For fractional values of s, one would replace $\nabla_G^{\gamma^*} \nabla_G^\gamma$ by a fractional power of the sub-Laplacian (24) and what follows would go unchanged. Schwartz's theorem implies that the distributional support of

$$v = \sum_{|\gamma|\le s} (-1)^{|\gamma|} \nabla_G^{\gamma^*} \nabla_G^\gamma u$$

is reduced to the single point $\{e\}$ and thus $v = \sum (-1)^{|\alpha|} c_\alpha \partial^\alpha \delta$ where δ is the Dirac function at the origin. As v is at most a $2s$-th horizontal derivative of u, one has $v \in H^{-s}(G)$ and in particular for any test function $\psi \in \mathcal{D}(G)$,

$$\left| \int_G v(g)\psi(g)\, dg \right|^2 \le C \sum_{|\gamma|\le s} \int_G |\nabla_G^\gamma \psi(g)|^2\, dg.$$

The constant C does not depend on the support of ψ because $\operatorname{supp} v \subset \{e\}$. In particular, one can apply this inequality to the dilations $\psi(rg)$ for any $r > 1$:

$$\left|\int_G v(g)\psi(rg)\,dg\right|^2 \le C \sum_{|\gamma|\le s} r^{2|\gamma|} \int_G |\nabla_G^\gamma \psi(rg)|^2\,dg.$$

thus

$$\left|\int_G v(r^{-1}g)\psi(g)\,dg\right|^2 \le C \sum_{|\gamma|\le s} r^{2|\gamma|+Q} \int_G |\nabla_G^\gamma \psi(g)|^2\,dg.$$

Finally, one can compute the left-hand side using the homogeneity of the Dirac mass:

$$\int_G v(r^{-1}g)\psi(g)\,dg = \sum_\alpha c_\alpha r^{Q+\sum \alpha_j\omega_j}\partial^\alpha \psi(e).$$

Combining both formulas, one gets for any $r > 1$:

$$\left|\sum_\alpha c_\alpha r^{Q+\sum \alpha_j\omega_j}\partial^\alpha \psi(e)\right| \le C r^{Q/2+s}\|\psi\|_{H^s(G)}$$

and in particular with a suitable choice of ψ and $r \to \infty$,

$$c_\alpha \neq 0 \Longrightarrow s \ge \frac{Q}{2} + \sum \omega_j\alpha_j.$$

But as $s < Q/2$, each coefficient c_α vanishes, i.e., $v = 0$ in $H^{-s}(G)$ and thus using $u \in H^s(G)$ as a test function, one infers $u = 0$. □

Remark. When Q is even and $s = Q/2 \in \mathbb{N}$, the previous density result still holds. The only change in the proof is to observe that $\delta \notin H^{-Q/2}(G)$ by exhibiting an example of an unbounded function in $H^{Q/2}(G)$; the classical example $\log(-\log\|g\|)\psi(g)$ with a sooth cut-off ψ still works. However, when Q is odd, one still has $\delta \notin H^{-Q/2}(G)$ but the density result *fails* as it already does in $H^{n+1/2}(\mathbb{R}^{2n+1})$. For more details on this point, see [Vigneron 2006].

2.5. ***Hardy inequality.*** In this final section, let us combine the previous results into a proof of Theorem 1.

Given $f \in H^s(G)$ with $s < Q/2$ and the density result of the previous section, one can assume without restriction that f is compactly supported and that $0 \notin \operatorname{Supp} u$. Next, one will take a smooth cutoff function $\chi : \mathbb{R} \to [0, 1]$ such that $\chi(t) = 1$ if $|t| < \frac{1}{2}$. For any $\rho_0 > 0$, one has

$$\int_G \frac{|f(g)|^2}{\|g\|_G^{2s}} \le \int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} + \left(\frac{2}{\rho_0}\right)^{2s} \|f\|^2_{L^2(G)} \qquad \text{with } \varphi(g) = \chi\left(\frac{\|g\|_G}{\rho_0}\right) f(g). \tag{45}$$

Moreover, one has $\|\varphi\|_{H^s(G)} \leq C_s \rho_0^{-s} \|f\|_{H^s(G)}$. Without restriction, one can therefore assume that f (now denoted by φ) is compactly supported in a fixed but arbitrary small annular neighborhood around the origin.

The key of the computation is the following integration by part argument. Using (43), one has

$$\int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} = -\frac{1}{2s} \int_G R\left(\frac{1}{\|g\|^{2s}}\right) \cdot |\varphi(g)|^2.$$

Using Proposition 10 and the fact that $\operatorname{supp} \varphi$ is an annulus around the origin so that no boundary terms appear:

$$\left(\frac{Q}{2} - s\right) \int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} = -\int_G \frac{\varphi(g) R(\varphi(g))}{\|g\|^{2s}}.$$

According to (42), the radial field can be expressed with left-invariant vector fields:

$$\left(\frac{Q}{2} - s\right) \int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} = -\sum_{\ell=1}^{q} \int_G \frac{\sigma_\ell(x(g)) \varphi(g) Y_\ell^L(\varphi(g))}{\|g\|^{2s}}. \tag{46}$$

What we do next depends on the order of each derivative $Y_\ell^L \simeq \nabla_G^{\omega_\ell}$.

Case $m = 1$. In the Euclidean case, one uses Cauchy–Schwarz and Young's identity $|ab| \leq \varepsilon a^2 + \varepsilon^{-1} b^2$ with $\varepsilon > 0$ small enough so that $s + \varepsilon < Q/2$, which leads to

$$\left(\frac{Q}{2} - s - \varepsilon\right) \int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} \leq \varepsilon^{-1} C \int_G \frac{|\nabla_G \varphi|^2}{\|g\|_G^{2(s-1)}}. \tag{47}$$

This proves Hardy's inequality for $s = 1$. Interpolation with L^2 then ensures that the Hardy inequality holds for any $s \in [0, 1]$. Finally, the previous estimate provides a bootstrap argument from $s - 1$ to s for any $s < Q/2$.

Case $m = 2$. One uses the Euclidean technique to deal with the horizontal derivatives. For the stratum V_2, one uses the commutator structure to backtrack one "half" integration by part. More precisely, the right-hand side of (46) becomes, for $1 \leq \ell \leq n_1$,

$$\left| \int_G \frac{\sigma_\ell(x(g)) \varphi(g) Y_\ell^L(\varphi(g))}{\|g\|^{2s}} \right| \leq \varepsilon \int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} + C_\varepsilon \int_G \frac{|\nabla_G \varphi(g)|^2}{\|g\|_G^{2(s-1)}}$$

and using (22) for $n_1 < \ell \leq n_2 = q$ and Proposition 11 (notice the cancellation of the highest order term thanks to the commutator structure):

$$-\int_G \frac{\sigma_\ell(x(g))}{\|g\|^{2s}}\varphi(g)\cdot[Y^L_{\alpha_1(\ell)}, Y^L_{\alpha_2(\ell)}](\varphi(g))$$
$$= \int_G Y^L_{\alpha_1(\ell)}\Big(\frac{\sigma_\ell(x(g))}{\|g\|^{2s}}\Big)\varphi(g)\cdot Y^L_{\alpha_2(\ell)}(\varphi(g)) - \int_G Y^L_{\alpha_2(\ell)}\Big(\frac{\sigma_\ell(x(g))}{\|g\|^{2s}}\Big)\varphi(g)\cdot Y^L_{\alpha_1(\ell)}(\varphi(g)).$$

Using the symbol properties of $\|\cdot\|_G$ and $\sigma_\ell(x(g))$, both terms are bounded in the following way:

$$\int_G \frac{|\varphi(g)|\cdot|\nabla_G\varphi(g)|}{\|g\|_G^{2s-1}} \le \varepsilon\int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} + C_\varepsilon\int_G \frac{|\nabla_G\varphi(g)|^2}{\|g\|_G^{2(s-1)}}.$$

One thus gets (47) again and one can conclude the proof just as in the case $m = 1$.

Case $m \ge 3$. The additional terms on the right-hand side of (46) correspond to $n_2 < \ell \le q$. Thanks to (22), one can express each of them with commutators from the first stratum:

$$I_\ell(\varphi) = -\int_G \frac{\sigma_\ell(x(g))}{\|g\|^{2s}}\varphi(g)\cdot[Y^L_{\alpha_1(\ell)},\dots,[Y^L_{\alpha_{\omega_\ell-1}(\ell)}, Y^L_{\alpha_{\omega_\ell}(\ell)}]](\varphi(g)).$$

As in the case $m = 2$, the key is to use the commutator structure to put all the derivatives but one on the symbol. More precisely, using Proposition 11, one first gets

$$I_\ell(\varphi) = \tfrac{1}{2}\int_G [Y^L_{\alpha_1(\ell)},\dots,[Y^L_{\alpha_{\omega_\ell-1}(\ell)}, Y^L_{\alpha_{\omega_\ell}(\ell)}]]\Big(\frac{\sigma_\ell(x(g))}{\|g\|^{2s}}\Big)|\varphi(g)|^2.$$

Next, one puts the outermost derivative back out onto $\varphi(g)^2$:

$$I_\ell(\varphi) = -\sum_{i=1}^{\omega_\ell}\int_G W_{\ell,i}\Big(\frac{\sigma_\ell(x(g))}{\|g\|^{2s}}\Big)\varphi(g)\cdot Y^L_{\alpha_i(\ell)}\varphi(g),$$

where each $W_{\ell,i}$ is a derivative of order $\omega_\ell - 1$. The symbol property $\sigma_\ell(x(g)) \in S^{\omega_\ell}(G)$ given by Proposition 8 ensures that

$$\Big|W_{\ell,i}\Big(\frac{\sigma_\ell(x(g))}{\|g\|^{2s}}\Big)\Big| \le \frac{C_{\ell,i,s}}{\|g\|_G^{2s-1}}.$$

Again, one gets

$$|I_\ell(\varphi)| \le \varepsilon\int_G \frac{|\varphi(g)|^2}{\|g\|_G^{2s}} + C_\varepsilon\int_G \frac{|\nabla_G\varphi(g)|^2}{\|g\|_G^{2(s-1)}}$$

and (47) holds once more. As in the case $m = 1$, one thus gets the Hardy inequality for $s = 1$. Then, by interpolation with $L^2(G)$, one gets it for $s \in [0, 1]$. Finally, using (47) iteratively, one can collect it for any $s < Q/2$.

Remark. When Q is odd and $s \in \mathbb{N}$, one can always take ε small enough so that $Q/2 - s - \varepsilon \neq 0$ in (47). The previous iteration argument thus proves the Hardy inequality (6) for any $s \in \mathbb{N}$, but the result is then only valid for functions that belong to the $H^s(G)$-closure of smooth compactly supported functions whose support avoids the origin.

2.6. ***Homogeneous Hardy inequality.*** One can slightly improve (6) by using a simple scaling argument. For simplicity, we will only spell out the procedure for $s \in \mathbb{N}$ and $0 \leq s < Q/2$ though it would also work for fractional values of s if ∇_G^s was replaced by the corresponding power $\mathcal{L}_G^{s/2}$ of the subelliptic Laplace operator (24).

Theorem 13. *For* $0 \leq s < Q/2$, *the following homogeneous inequality holds*:

$$\text{for all } f \in H^s(G) \qquad \int_G \frac{|f(g)|^2}{\|g\|^{2s}}\, dg \leq 2C_s \|\nabla_G^s f\|_{L^2(G)}^2. \tag{48}$$

Proof. Let us indeed apply (6) to the function $f(r^{-1}g)$. After the change of variable $g = r\bar{g}$, one gets

$$r^{Q-2s} \int_G \frac{|f(\bar{g})|^2}{\|\bar{g}\|^{2s}}\, d\bar{g} \leq C_s \sum_{|\alpha| \leq s} r^{Q-2|\alpha|} \int_G |\nabla_G^\alpha f(\bar{g})|^2\, d\bar{g}$$

which, for $r < 1$, can be further simplified into

$$\int_G \frac{|f(\bar{g})|^2}{\|\bar{g}\|^{2s}}\, d\bar{g} \leq C_s \sum_{|\alpha|=s} \int_G |\nabla_G^\alpha f(\bar{g})|^2\, d\bar{g} + C_s r^2 \sum_{|\alpha| \leq s-1} \int_G |\nabla_G^\alpha f(\bar{g})|^2\, d\bar{g}.$$

Choosing

$$r^2 = \min\left\{1\,;\ \frac{\|\nabla_G^s f\|_{L^2(G)}^2}{\|f\|_{H^{s-1}(G)}^2}\right\}$$

instantly leads to (48). □

Remark. It would have been tempting to try using (45)–(46) without digging further in the commutator structure to get

$$\int_G \frac{|f(g)|^2}{\|g\|_G^{2s}}\, dg \leq \int_G |f(g)|^2\, dg + \left(\frac{Q}{2} - s - \varepsilon\right)^{-1} \sum_{\ell=1}^{q} \int_G \frac{|Y_\ell^L((\chi f)(g))|^2}{\|g\|^{2(s-\omega_\ell)}}\, dg.$$

For $s = 1$, it gives a Hardy inequality with $\|f\|_{H^m(G)}^2$ on the right-hand side. However, a scaling argument is then not sufficient to deduce the correct one, either (6) or (48). Indeed, one would simultaneously need to let $r \to \infty$ and $r \to 0$ to get rid of the superfluous derivatives without letting the lower-order $L^2(G)$ term get in the way, which is overall impossible.

3. A remark about the case of general hypoelliptic vector fields

For general families of vector fields that satisfy a Hörmander condition of step m, the technique of proving the Hardy inequality by integration by part works, but possibly with some restrictions.

3.1. ***A counterexample to the symbol property of the gauge.*** The main objection is the following one. When the group structure is discarded, the fact that one can chose a gauge pseudonorm in a symbol class of order 1 can fail.

For example, the family

$$Z_1 = \partial_1 + x_1\partial_3, \quad Z_2 = \partial_2 + x_4\partial_3 + x_5\partial_4 \quad \text{and} \quad Z_3 = \partial_5$$

is uniformly of rank 3 in $\mathbb{R}^5$ and satisfies a uniform Hörmander bracket condition of step 3:

$$\partial_4 = [Z_3, Z_2], \quad \partial_3 = [[Z_3, Z_2], Z_2].$$

However, the "natural" gauge,

$$\rho = (|x_1|^{12} + |x_2|^{12} + |x_3|^4 + |x_4|^6 + |x_5|^{12})^{1/12},$$

is *not* a symbol of order 1 because $|Z_1\rho| \geq c\rho^{-1}$ along $x_1^3 - x_3 = x_2 = x_4 = x_5 = 0$. Luckily, for this particular family, the change of variable $y_3 = x_3 - \frac{1}{2}x_1^2$ and $y_i = x_i$ $(i \neq 3)$ transforms the family into $Z_1' = \partial_{y_1}$, $Z_2' = \partial_{y_2} + y_4\partial_{y_3} + y_5\partial_{y_4}$ and $Z_3' = \partial_{y_5}$ and for this new family, the associated gauge is a symbol of order 1.

In [Vigneron 2006, Chapter 7], it was shown that up to a Hörmander condition of step 3, one can always modify the gauge by a local diffeomorphism to restore the symbol property. However, the same question for a family of vector fields that satisfy a Hörmander condition of step 4 or higher is still open. For the convenience of the reader, we will recall here briefly the key points of the discussion (and clarify the redaction), as this result was written in French and never published.

3.2. ***Regular hypoelliptic vector fields of step m.*** Let us consider a family $\mathfrak{X} = (X_\ell)_{1\leq\ell\leq n_1}$ of vector fields on some smooth open set $\Omega \subset \mathbb{R}^q$ and

$$\text{for all } x \in \Omega, \quad W_k(x) = \operatorname{Span}\big(X_i(x), \dots, [X_{j_1}, \dots, [X_{j_{k-1}}, X_{j_k}]](x)\big). \tag{49}$$

One assumes that $\underline{x} \in \Omega$ is a *regular Hörmander point*, i.e., that $n_k = \dim W_k$ is constant near $\underline{x}$ and that $n_m = q$ for some finite integer $m \geq 2$.

Remark. At the origin of a Carnot group (2), one would have $W_k(e) = \bigoplus_{j=1}^k V_j$.

Next, one introduces a local basis of vector fields $(Y_\ell(x))_{1\leq\ell\leq q}$, adapted to the stratification, i.e., $Y_\ell(x) \in W_{\omega_\ell}(x)$ where for each ℓ, the weight $\omega_\ell \in \{1, \dots, m\}$ is defined by (3). For simplicity, one will now restrict Ω to be a bounded and small enough neighborhood of $\underline{x}$ on which all those properties hold. The analog of horizontal derivatives is the family

$$\nabla_{\mathfrak{X}} = (Y_1, \dots, Y_{n_1}). \tag{50}$$

A local coordinate system $(x_\ell)_{1\le\ell\le q}$ is said to be *adapted* to the commutator structure of the vector fields $\mathfrak{X}$ near $\underline{x}$ if the dual basis $(\partial_\ell)_{1\le\ell\le q}$ satisfies $Y_\ell(\underline{x}) = \partial_\ell$.

Remark. Let us point out that *adapted* coordinates are not necessarily *privileged* in the sense of A. Bellaïche [1996] and M. Gromov [1996]: the point of coordinates $(x_\ell)_{1\le\ell\le q}$ does not necessarily match with the image of $\underline{x}$ under the composite action of the flows $e^{x_\ell Y_\ell}$ (for some predetermined order of composition).

In an adapted coordinate system, the *gauge* is defined by

$$\rho(x) = \left(\sum_{\ell=1}^{q} |x_\ell|^{w/\omega_\ell}\right)^{1/w}, \tag{51}$$

where $w = 2\,\mathrm{LCM}(1, \dots, m)$ and the basis of vector fields and their commutators satisfy

$$\text{for all } \ell \in \{1, \dots, q\}, \quad Y_\ell(x) = \partial_\ell + \sum_{\ell'=1}^{q} \zeta_{\ell,\ell'}(x)\partial_{\ell'}. \tag{52}$$

One obviously has $|x_\ell| \le \rho(x)^{\omega_\ell}$ and, using a Taylor expansion, $\zeta_{\ell,\ell'}(x_0) = 0$ implies $|\zeta_{\ell,\ell'}(x)| \le C\rho(x)$. However, for derivatives, one can only claim that $\nabla_{\mathfrak{X}}^{\gamma} x_\ell$ and $\nabla_{\mathfrak{X}}^{\gamma} \zeta_{\ell,\ell'}$ are bounded when $|\gamma| \ge 1$.

3.3. *A positive result for hypoelliptic fields of step 2.*

Theorem 14. *Let us consider a family of vector fields and $\underline{x} \in \Omega$ a regular Hörmander point of step $m = 2$. Then for any adapted coordinate system, the gauge ρ satisfies*

$$|\nabla_{\mathfrak{X}}^{\gamma} \rho| \le C_\gamma \rho^{1-|\gamma|} \tag{53}$$

in the neighborhood of $\underline{x}$, for any multi-index γ.

Proof. For $\gamma = 0$, the estimate (53) comes from the fact that ρ is smooth and vanishes at the origin and thus admits a Taylor expansion at the origin that is locally bounded by $\sum |x_\ell|$ and thus by ρ. For $|\gamma| = 1$, the computation is actually explicit:

$$\nabla_{\mathfrak{X}} \rho = \frac{1}{\rho^3}\left(x_\ell^3 + \sum_{\ell'\le n_1} \zeta_{\ell,\ell'} x_{\ell'}^3 + \tfrac{1}{2}\sum_{\ell'>n_1} \zeta_{\ell,\ell'} x_{\ell'}\right)_{1\le\ell\le n_1}.$$

In the parenthesis, the first term is locally bounded by ρ^3, the second by ρ^4 and the last one again by ρ^3, thus $\nabla_{\mathfrak{X}} \rho \in L^\infty(\Omega)$ provided Ω is small enough. To deal with the higher-order derivatives, let us introduce the class $\mathcal{P}_n$ of homogeneous polynomials of x_ℓ and $\zeta_{\ell,\ell'}$ with smooth coefficients, i.e.,

$$\sum_{\alpha,\beta} x_1^{\alpha_1} \cdots x_q^{\alpha_q} \zeta_{1,1}^{\beta_{1,1}} \zeta_{1,2}^{\beta_{1,2}} \cdots \zeta_{q,q}^{\beta_{q,q}} f_{\alpha,\beta}(x),$$

where $f_{\alpha,\beta} \in C^\infty(\Omega)$ and $\sum \alpha_i \omega_i + \sum \beta_{j,j'} = n$. For $n \leq 0$, one sets $\mathcal{P}_n = C^\infty(\Omega)$. With the Leibnitz formula, one checks immediately that

$$\partial_\ell(\mathcal{P}_n) \subset \begin{cases} \mathcal{P}_{n-1} + \mathcal{P}_n & \text{if } \ell \leq n_1, \\ \mathcal{P}_{n-2} + \mathcal{P}_{n-1} + \mathcal{P}_n & \text{if } \ell > n_1, \end{cases}$$

thus $\nabla_{\mathfrak{X}}(\mathcal{P}_n) \subset \mathcal{P}_{n-1} + \mathcal{P}_n + \mathcal{P}_{n+1}$. Moreover, for $m \geq n$, any expression in $\mathcal{P}_m$ is locally bounded by $C\rho^n$ for some constant C. We have shown above that $\nabla_{\mathfrak{X}}\rho \in \rho^{-3} \cdot (\mathcal{P}_3 + \mathcal{P}_4)$. One then gets recursively on $k = |\gamma|$ that $\nabla_{\mathfrak{X}}^\gamma \rho$ is a linear combination of expressions

$$\frac{\mathcal{P}_m}{\rho^{n+k-1}}$$

with $m \geq n$ and is thus locally bounded by $C\rho^{1-k}$. □

Remark. One has $\mathcal{P}_n \subset \mathcal{P}_{n-2}$. However, for $\ell > n_1$, one has $x_\ell^2 \in \mathcal{P}_4 \cap \mathcal{P}_2$ but $x_\ell^2 \notin \mathcal{P}_3$.

3.4. ***Two positive results for hypoelliptic fields of step $m \geq 3$.*** Let us now revert to the case of a general value for m. As pointed out at the beginning of this section, one can find a counterexample of a family of vector fields, a regular Hörmander point of step $m = 3$ and an adapted coordinate system for which (53) fails. If we tried to run the previous proof, the failure point would be that

$$\partial_{\ell'}(\mathcal{P}_n) \subset \mathcal{P}_{n-\omega_{\ell'}} + \cdots + \mathcal{P}_{n-1} + \mathcal{P}_n.$$

When computing $\nabla_{\mathfrak{X}}(\mathcal{P}_n)$, the multiplication by $\zeta_{\ell,\ell'} \in \mathcal{P}_1$ is then not able to compensate for the loss when $\omega_{\ell'} \geq 3$. The profound reason is that our knowledge about the way $\zeta_{\ell,\ell'}$ vanishes at the origin is too weak.

Definition. A coordinate system adapted to the commutator structure of the vector fields $\mathfrak{X}$ near a regular Hörmander point $\underline{x}$ of step m is called *well-adapted* if

$$\text{for all } \ell \in \{1, \dots, n_1\},\ \ell' \in \{1, \dots, q\}, \quad |\nabla_{\mathfrak{X}}^\gamma \zeta_{\ell,\ell'}| \leq C_\gamma \rho^{(\omega_{\ell'}-1-|\gamma|)_+} \tag{54}$$

in a neighborhood of $\underline{x}$. A family of vector fields that satisfies a regular Hörmander condition is called *well-structured* if it admits a well-adapted coordinate system.

One can check that in a well-adapted coordinate system, the gauge automatically satisfies (53).

Theorem 15 (If-theorem for arbitrary step m). *Let us consider a family of vector fields and $\underline{x} \in \Omega$ a regular Hörmander point of step m. Then for any* ***well*** *adapted coordinate system, the gauge ρ satisfies (53) in the neighborhood of $\underline{x}$.*

Proof. The key is to adapt the definition of $\mathcal{P}_n$ to capture the enhanced knowledge that we gained about $\zeta_{\ell,\ell'}$. Let us define $\widetilde{\mathcal{P}}_n$ as the subset of $C^\infty(\Omega)$ that consists of homogeneous polynomials with smooth coefficients of x_ℓ, $\zeta_{\ell,\ell'}$ and of the derivatives of $\zeta_{\ell,\ell'}$ for which we have estimates, i.e.,

$$\sum_{\alpha,\beta} x_1^{\alpha_1} \cdots x_q^{\alpha_q} \zeta_{1,1}^{\beta_{1,1}} \zeta_{1,2}^{\beta_{1,2}} \cdots \zeta_{q,q}^{\beta_{q,q}} \left(\prod_\gamma (\nabla_{\mathfrak{X}}^\gamma \zeta_{1,1})^{\delta_{\gamma;1,1}} \cdots (\nabla_{\mathfrak{X}}^\gamma \zeta_{n_1,q})^{\delta_{\gamma;n_1,q}} \right) f_{\alpha,\beta,\delta}(x),$$

where $f_{\alpha,\beta,\delta} \in C^\infty(\Omega)$, γ denotes multi-indices of length $|\gamma| \geq 1$ and

$$\sum_{1\leq i\leq q} \alpha_i \omega_i + \sum_{j=1}^q \sum_{j'=1}^q \beta_{j,j'}(\omega_{j'}-1) + \sum_{|\gamma|\geq 1} \sum_{j=1}^{n_1} \sum_{j'=1}^q \delta_{\gamma;j,j'}(\omega_{j'}-1-|\gamma|)_+ = n.$$

Note that only the factors for which $\omega_{j'} - 1 - |\gamma| > 0$ are significant; the others can simply be tossed into $f_{\alpha,\beta,\delta}$. For $n \leq 0$, one sets again $\widetilde{\mathcal{P}}_n = C^\infty(\Omega)$. We also introduce the linear span

$$\widetilde{\mathcal{P}}_n^+ = \sum_{m\geq n} \widetilde{\mathcal{P}}_m.$$

Using the Leibnitz formula, $\nabla_{\mathfrak{X}}(x_\ell) \in \widetilde{\mathcal{P}}_{\omega_\ell - 1}$ and $\nabla_{\mathfrak{X}}(\widetilde{\mathcal{P}}_n) \subset \widetilde{\mathcal{P}}_{n-1}^+$. One also has

$$\nabla_{\mathfrak{X}}\rho = \frac{\nabla_{\mathfrak{X}}(\rho^w)}{w\rho^{w-1}} \in \rho^{-(w-1)} \cdot \widetilde{\mathcal{P}}_{w-1}^+$$

and recursively (note that $\rho^w \in \widetilde{\mathcal{P}}_w$ allows one to convert $\widetilde{\mathcal{P}}_0$ into $\rho^{-w} \cdot \widetilde{\mathcal{P}}_w$)

$$\nabla_{\mathfrak{X}}^\gamma \rho \in \sum_{n\geq 1} \frac{\mathcal{P}_n^+}{\rho^{n+|\gamma|-1}}$$

from which (53) follows immediately. $\square$

The previous "abstract" theorem does not presume on the existence of a well-adapted coordinate system. However, when $m \leq 3$, it can actually be made to work.

Theorem 16. *Any family of vector fields that satisfies a regular Hörmander condition of step $m \leq 3$ admits at least one well-adapted coordinate system. It is therefore well-structured.*

Proof. For $m = 1$ and 2, any adapted coordinate system is well-adapted. Let us thus focus on $m = 3$ and use the previous notations. Writing down the Taylor expansion of the coefficients for $\ell \le n_1$,

$$\zeta_{\ell,\ell'}(x) = \sum_{i \le n_1} \left(\frac{\partial \zeta_{\ell,\ell'}}{\partial x_i}(\underline{x}) \right) x_i + O(\rho^2),$$

it appears that, for $m = 3$, a coordinate system is well-adapted if and only if

$$\text{for all } \ell_1, \ell_2 \in \{1, \dots, n_1\},\ \ell_3 \in \{n_2 + 1, \dots, q\}, \quad \frac{\partial \zeta_{\ell_1,\ell_3}}{\partial x_{\ell_2}}(\underline{x}) = 0. \tag{55}$$

Let us compute the following commutator:

$$[Y_{\ell_1}, Y_{\ell_2}] = \sum_{\ell=1}^{q} \left(\frac{\partial \zeta_{\ell_2,\ell}}{\partial x_{\ell_1}} - \frac{\partial \zeta_{\ell_1,\ell}}{\partial x_{\ell_2}} \right) \partial_\ell.$$

At the point $\underline{x}$, the terms corresponding to $\ell > n_2$ must belong to $W_2(\underline{x})$ and thus vanish, therefore

$$\text{for all } \ell_1, \ell_2 \in \{1, \dots, n_1\},\ \ell_3 \in \{n_2+1, \dots, q\}, \quad \frac{\partial \zeta_{\ell_1,\ell_3}}{\partial x_{\ell_2}}(\underline{x}) = \frac{\partial \zeta_{\ell_2,\ell_3}}{\partial x_{\ell_1}}(\underline{x}). \tag{56}$$

One can now define a new coordinate system $(y_\ell)_{1 \le \ell \le q}$ whose dual basis satisfies

$$\frac{\partial}{\partial y_\ell} = \frac{\partial}{\partial x_\ell} + \mathbf{1}_{\ell \le n_1} \sum_{\ell' > n_2} \sum_{i \le n_1} \left(\frac{\partial \zeta_{\ell,\ell'}}{\partial x_i}(\underline{x}) \right) x_i \cdot \frac{\partial}{\partial x_{\ell'}}.$$

This coordinate system is (locally) well defined because the fields $\frac{\partial}{\partial y_\ell}$ commute with each other thanks to (56). By construction, this coordinate system satisfies (55) and is therefore a well-adapted one. □

Remark. The generalization of Theorem 16 for $m \ge 4$ is an *open* question. One can check that a coordinate system is well-adapted if and only if

$$\frac{\partial^\alpha \zeta_{\ell,\ell'}}{\partial x_1^{\alpha_1} \cdots \partial x_q^{\alpha_q}}(\underline{x}) = 0 \tag{57}$$

for any indices such that $\omega_\ell = 1$, $\omega_{\ell'} \ge 3$ and $\sum_{i=1}^{q} \omega_i \alpha_i \le \omega_{\ell'} - 2$. However, for $m \ge 4$, it is not clear whether the regular Hörmander assumption of step m is enough to ensure that the vector fields

$$\frac{\partial}{\partial y_\ell} = \frac{\partial}{\partial x_\ell} + \mathbf{1}_{\ell \le n_1} \sum_{\ell' > n_2} \left(\sum_{\sum \alpha_i \omega_i \le \omega_{\ell'} - 2} \left(\frac{\partial^\alpha \zeta_{\ell,\ell'}}{\partial x_1^{\alpha_1} \cdots \partial x_q^{\alpha_q}}(\underline{x}) \right) x_1^{\alpha_1} \cdots x_q^{\alpha_q} \cdot \frac{\partial}{\partial x_{\ell'}} \right)$$

commute with each other.

3.5. ***From the symbol property of the gauge to Hardy inequality.*** For well-structured families of vector fields, symbols of class $S^k(\mathfrak{X};\rho)$ are functions f such that

$$|\nabla_{\mathfrak{X}}^{\gamma} f(x)| \leq C_\gamma \rho(x)^{(k-|\gamma|)_+} \tag{58}$$

in a neighborhood of $\underline{x}$, for any multi-index γ. Once the symbol property is established for the gauge, the path that leads to the Hardy inequality is open. The key (see [Vigneron 2006, Chapter 7]) is to define a "radial" vector field that admits both expressions:

$$R(x) = \sum_{\ell=1}^{q} \sigma_\ell(x) Y_\ell(x) = \sum_{k=1}^{q} (\omega_k x_k + \tilde{\sigma}_k(x))\partial_k \tag{59}$$

in well-adapted coordinates, with $\sigma_\ell \in S^{\omega_\ell}(\mathfrak{X};\rho)$ and $\tilde{\sigma}_k \in S^{\omega_k+1}(\mathfrak{X};\rho)$. One can then check that

$$\operatorname{div} R = Q + O(\rho) \quad \text{and} \quad \lambda = \frac{R\rho}{\rho} \qquad \text{satisfies} \quad \begin{cases} \lambda(x) = 1 + O(\rho), \\ R\lambda = O(\rho). \end{cases} \tag{60}$$

The computations of Section 2.5 can then be carried out in a small enough neighborhood of $\underline{x}$.

Acknowledgements

I would like to express my gratitude to J.-Y. Chemin, who brought this problem to my attention a long time ago, and to wish him a happy 60th birthday.

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Received 14 Aug 2019.

FRANÇOIS VIGNERON:

francois.vigneron@u-pec.fr
Université Paris-Est, LAMA (UMR 8050), UPEC, UPEM, CNRS, 61, avenue du Général de Gaulle, F94010 Créteil, France

TUNISIAN JOURNAL OF MATHEMATICS
Vol. 2, No. 4, 2020
dx.doi.org/10.2140/tunis.2020.2.881

Trigonometric series with a given spectrum

Yves Meyer

To the memory of Salah Baouendi

Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set. The vector space consisting of all trigonometric sums whose frequencies belong to Λ is denoted by $\mathcal{T}_\Lambda$. Given an exponent $p \in [1, \infty]$ we say that Λ is p-coherent if there exist a compact set $K \subset \mathbb{R}^n$ and a continuous function ω defined on $\mathbb{R}^n$ with values in $[1, \infty)$ such that for every $P \in \mathcal{T}_\Lambda$ and every $y \in \mathbb{R}^n$ one has $\left(\int_{|x-y|\le 1} |P(x)|^p \, dx\right)^{1/p} \le \omega(y)\left(\int_K |P(x)|^p \, dx\right)^{1/p}$. Several properties of p-coherent sets are proved in this essay.

1. Four problems on trigonometric sums

1A. *Summary.* Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set. The vector space consisting of all trigonometric sums whose frequencies belong to Λ is denoted by $\mathcal{T}_\Lambda$. Given an exponent $p \in [1, \infty]$ we say that Λ is p-coherent if there exist a compact set $K \subset \mathbb{R}^n$ and a continuous function ω defined on $\mathbb{R}^n$ with values in $[1, \infty)$ such that for every $P \in \mathcal{T}_\Lambda$ and every $y \in \mathbb{R}^n$ one has $\left(\int_{|x-y|\le 1} |P(x)|^p \, dx\right)^{1/p} \le \omega(y)\left(\int_K |P(x)|^p \, dx\right)^{1/p}$.

A survey of the still incomplete L^2 theory is given in Section 2. A remarkable theorem by S. Jaffard, M. Tucsnak, and E. Zuazua on weighted L^2 estimates is stated and proved in Section 3. Examples of sets which are not p-coherent sets are given in the five following sections. A p-coherent set Λ has a finite Beurling and Malliavin density, as is proved in Section 4. The role of Section 4 is to bridge the gap between the problems raised in Section 1 and growth estimates satisfied by mean periodic functions with a given spectrum. When this growth cannot be controlled by a weight ω we say that Λ is a wild set. In other words a wild set is a set which is not ∞-coherent. We prove (Theorem 5.4) that the digital cone $\Lambda \subset \mathbb{R}^3$ is not p-coherent if $2 < p \le \infty$. However the digital cone is 2-coherent. A more involved example is the Pisot set. The Pisot set Λ_θ is 2-coherent. If θ is a Pisot number, the Pisot set is contained in a quasicrystal. Therefore it is p-coherent for $1 \le p \le \infty$. When θ is not a Pisot number the Pisot set is a wild set. The proof uses

MSC2010: primary 42A32; secondary 42B10.
Keywords: mean periodic functions, almost periodic functions, trigonometric sums.

a famous theorem by Charles Pisot. Unfortunately we do not know whether the Pisot set is p-coherent or not when θ is not a Pisot number and when $2 < p < \infty$. A partial answer is given in Section 8. A third example of a "wild set" is given in Section 6, Theorem 6.1. This wild set has a finite Beurling and Malliavin density but is not p-coherent when $1 \leq p \leq \infty$. Theorem 6.1 is given another proof in Section 7, where we show in full generality that a p-coherent set has a finite upper uniform density. Finally in Section 8 we provide the reader with sufficient conditions implying that a set Λ is p-coherent and relate these L^p estimates to the spectral properties or to the additive properties of Λ.

1B. ***The wave equation.*** One of the motivations of this essay is control theory [Avdonin 1974; Avdonin and Ivanov 1995; Lions 1984]. To control the vibrations of a surface, one is led to study the wave equation on a bounded domain. Solutions of the wave equation on a compact Riemannian manifold or on a bounded domain are nonperiodic trigonometric series. That is why precise estimates on nonperiodic trigonometric sums are so important. Here are some details of this discussion. Let M be a compact Riemannian manifold and $\Delta : \mathcal{C}^\infty(M) \mapsto \mathcal{C}^\infty(M)$ be the corresponding Laplace–Beltrami operator. The wave equation on M is

$$\partial_t^2 u - \Delta_x u = 0. \tag{1}$$

A solution of (1) is a series

$$u(x,t) = \sum_0^\infty [a_k(x)\exp(i\lambda_k t) + b_k(x)\exp(-i\lambda_k t)], \tag{2}$$

where $-\lambda_k^2 \leq 0$ are the eigenvalues of the Laplace–Beltrami operator and the functions a_k and b_k belong to the corresponding eigenspaces. We have $\Delta a_k = -\lambda_k^2 a_k$, $\Delta b_k = -\lambda_k^2 b_k$.

The series (2) is not a periodic function of the time variable in general. Therefore even if $u(x,t)$ is a global continuous solution of (1), its large time behavior can be quite unexpected and surprising. More generally if $T > 0$, the growth as $t \to \infty$ of $I_p(t) = \bigl(\int_M \int_t^{t+T} |u(x,s)|^p \, dx \, ds\bigr)^{1/p}$ can strongly differ if $p \neq 2$ from what happens if $p = 2$. This essay focuses on such problems.

1C. ***Notation.*** Let us fix some notation. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$. The Fourier transform $\mathcal{F}(f) = \hat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} \exp(-2\pi i x \cdot \xi) \, dx. \tag{3}$$

Throughout, assume $\Lambda \subset \mathbb{R}^n$ is a closed and discrete set. Then Λ can always be ordered as a sequence λ_j, $j \in \mathbb{N}$, with $|\lambda_j|$ tending to infinity. Such a Λ is uniformly discrete if there exists a $\beta > 0$ such that,

$$\text{for all } \lambda \in \Lambda, \text{ for all } \lambda' \in \Lambda, \quad \lambda' \neq \lambda \implies |\lambda' - \lambda| \geq \beta. \tag{4}$$

One writes $P \in \mathcal{T}_\Lambda$ if P is a trigonometric sum whose frequencies belong to Λ:

$$P(x) = \sum_{\lambda \in \Lambda} c(\lambda) \exp(2\pi i \lambda \cdot x). \tag{5}$$

In the case of the wave equation on a compact manifold, x is replaced by the time variable and (5) takes the form

$$u(x_0, t) = \sum_0^\infty [a_k(x_0) \exp(i\lambda_k t) + b_k(x_0) \exp(-i\lambda_k t)]. \tag{6}$$

1D. *Four properties.* We now return to the general case. We are given a closed and discrete set $\Lambda \subset \mathbb{R}^n$. Four properties of Λ are discussed in this essay. The first one is the L^2 theory.

Property 1.1. There exist a compact set $K \subset \mathbb{R}^n$ of positive Lebesgue measure and a constant C such that for every $P \in \mathcal{T}_\Lambda$ one has

$$\left(\sum_{\lambda \in \Lambda} |c(\lambda)|^2 \right)^{1/2} \leq C \left(\int_K |P(x)|^2 \, dx \right)^{1/2}. \tag{7}$$

On the one hand (7) implies that Λ is uniformly discrete. Conversely if Λ is uniformly discrete there exists a positive number $R(\Lambda)$ such that (7) is satisfied when K is a ball of radius larger than $R(\Lambda)$. In dimension $n \geq 2$ we do not know how to compute $R(\Lambda)$ [Kahane 1962].

Definition 1.1. The compact set K in the right-hand side of (7) is minimal if (7) does not hold any more when K is replaced by a compact set $L \subset K$, $|L| < |K|$, the constant C being possibly replaced by a larger constant C'.

In an equivalent formulation of (7) the roles of Λ and K are exchanged. One starts with the Paley–Wiener space $\mathrm{PW}(K) \subset L^2(\mathbb{R}^n)$. It is the Hilbert space consisting of all $f \in L^2(\mathbb{R}^n)$ whose Fourier transform $\hat{f}$ is supported by K.

Definition 1.2. Let $K \subset \mathbb{R}^n$ be a compact set with a positive measure. A uniformly discrete $\Lambda \subset \mathbb{R}^n$ is a set of stable interpolation for the Paley–Wiener space $\mathrm{PW}(K)$ if for every square summable sequence $c(\lambda)$, $\lambda \in \Lambda$, there exists a function $f \in \mathrm{PW}(K)$ such that

$$f(\lambda) = c(\lambda) \quad \text{for all } \lambda \in \Lambda. \tag{8}$$

We then have:

Lemma 1.1. *Let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set and $K \subset \mathbb{R}^n$ be a compact set with a positive measure. Then Λ is a set of stable interpolation for* $\mathrm{PW}(K)$ *if and only if* (7) *is satisfied.*

References are [Landau 1967; Meyer 2018b; Olevskii and Ulanovskii 2008].

Let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set. If (7) is satisfied for a compact set K it is also satisfied for every compact set L containing K. Given Λ one tries to find a compact set K as small as possible for which (7) is valid. As was already said, K is minimal if there does not exist a compact subset L of K with $L \neq K$ for which (7) is valid. Two examples of minimal sets are given by Theorems 2.1 and 2.2. But K cannot be too small. The Lebesgue measure $|K|$ cannot be smaller than the upper uniform density of Λ. This was proved by H. J. Landau [1967]. We will return to Landau's theorem in Section 2.

Our second problem has the same structure but L^2 norms are replaced by L^∞ norms. This second problem was raised by J.-P. Kahane [1957].

Property 1.2. A uniformly discrete set Λ is a coherent set of frequencies if there exist a compact set K and a constant C such that for every $P \in \mathcal{T}_\Lambda$ one has

$$\|P\|_\infty \leq C \sup_{x \in K} |P(x)|. \tag{9}$$

Coherent sets of frequencies are studied in [Kahane 1957; Meyer 1972]. Property (9) is labeled $Q(\Lambda)$ in Kahane's seminal work. Property $Q(\Lambda)$ also implies that Λ is uniformly discrete but the converse is not true whatever be the size of K. This was observed in [Kahane 1957]. Given a coherent set of frequencies Λ one is interested in finding K as small as possible in (9). Theorem 8.1 gives an answer to this problem. The definition of a minimal compact set K for (9) is the same as the one given for (7). If $\Lambda = \mathbb{Z}$, then $K = [0, 1]$ is minimal for (7) and (9). If $\Lambda = \mathbb{Z} \cup \left\{\frac{1}{2}\right\}$, then $K = [0, 1]$ is still minimal for (9) but (7) does not hold.

In a weaker version of (9), L^∞ norms are replaced by weighted L^∞ norms.

Definition 1.3. A weight is a continuous function ω defined on $\mathbb{R}^n$ with values in $[1, \infty)$. A weight is submultiplicative if $\omega(x+y) \leq \omega(x)\omega(y)$ for all $x, y \in \mathbb{R}^n$.

Property 1.3. There exist a compact set K and a weight ω such that for every $P \in \mathcal{T}_\Lambda$ and every $y \in \mathbb{R}^n$ one has

$$|P(y)| \leq \omega(y) \sup_{x \in K} |P(x)|. \tag{10}$$

This no longer implies that Λ is uniformly discrete. For instance (10) is valid if $\Lambda = \mathbb{Z} \cup \sqrt{2}\,\mathbb{Z}$. We then have $\omega(x) = 1 + |x|$. This easy observation is proved in [Meyer 2018a].

More generally given $p \in [1, \infty]$ the following problem will be studied:

Property 1.4. A closed and discrete set Λ is p-coherent if there exist a compact set K and a weight ω such that for every $P \in \mathcal{T}_\Lambda$ and for every $y \in \mathbb{R}^n$ one has

$$\left(\int_{|x-y|\leq 1} |P(x)|^p \, dx\right)^{1/p} \leq \omega(y)\left(\int_K |P(x)|^p \, dx\right)^{1/p}. \tag{11}$$

If $p = \infty$, this is Property 1.3. If $p = 2$ and ω is a constant we are back to Property 1.1. This leads to the following definition:

Definition 1.4. If $1 \le p \le \infty$, if $K \subset \mathbb{R}^n$ is a compact set, and if ω is a weight, $\mathcal{L}(K, \omega, p)$ is the collection of all closed and discrete sets Λ fulfilling (11). Let $\mathcal{L}(p)$ be the union $\bigcup_{K,\omega} \mathcal{L}(K, \omega, p)$. This union is taken over all compact sets K and all weights ω. Finally if $1 \le p \le \infty$, we say that a closed and discrete set Λ is p-wild if it does not belong to $\mathcal{L}(p)$.

Our first task is to find a criterion on Λ implying $\Lambda \in \mathcal{L}(p)$. Our second task is to try to replace (11) by a sharper estimate. This estimate is sharper if the pair (K, ω) is replaced by (K', ω') where K' is "smaller" than K and similarly ω' is smaller than ω. We do not know if $\mathcal{L}(p) \subset \mathcal{L}(q)$ for $2 \le q \le p$. We do not know if $\mathcal{L}(p)$ is stable by finite unions. Lemma 1.2 is the only fact we know.

Lemma 1.2. *Let $1 \le p \le \infty$. We have $\mathcal{L}(\infty) \subset \mathcal{L}(p) \subset \mathcal{L}(2)$.*

The proof of the inclusion $\mathcal{L}(p) \subset \mathcal{L}(2),\ 1 \le p \le \infty$, will be given in a forthcoming paper. Let us prove the first assertion of Lemma 1.2. Property 1.3 is equivalent to the following assertion: there exist a compact set K and a constant C such that for every $y \in \mathbb{R}^n$ one can find a Radon measure μ_y with the following properties:

(a) μ_y is supported by K.

(b) $\|\mu_y\| \le \omega(y)$.

(c) $\hat{\mu}_y(\lambda) = \exp(2\pi i \lambda \cdot y)$ for all $\lambda \in \Lambda$.

To prove this remark we consider the linear form L_y on $\mathcal{T}_\Lambda$ defined by $L_y(P) = P(y)$. We consider the Banach space $\mathcal{C}(K)$ of continuous functions on K equipped with the sup-norm. Then (10) implies that the norm of L_y does not exceed $\omega(y)$. Using the Hahn–Banach theorem one extends L_y to $\mathcal{C}(K)$ with the same norm. This provides us with a Radon measure μ_y on K such that $\int_K P\, d\mu_y = P(y)$. Then (a), (b), and (c) are proved.

We now return to the proof of Property 1.4. We observe that (c) implies $P * \mu_y(x) = P(x + y)$ for every $P \in \mathcal{T}_\Lambda$. Therefore

$$\left(\int_{K+y} |P(u)|^p\, du\right)^{1/p} = \left(\int_K |P(x+y)|^p\, dx\right)^{1/p} = \left(\int_K |P * \mu_y|^p(x)\, dx\right)^{1/p}.$$

We now define $Q = P\chi_{(K-K)}$, where χ_E is the indicator function of E. We then have $\left(\int_K |P * \mu_y|(x)^p\, dx\right)^{1/p} \le \left(\int_{\mathbb{R}^n} |Q * \mu_y|(x)^p\, dx\right)^{1/p} \le \|\mu_y\|\, \|Q\|_p$, which ends the proof of Lemma 1.2.

Property 1.4 was studied in [Meyer 1974] on a simpler example. In the one-dimensional case, it was assumed that $\Lambda = \{k + r_k \mid k \in \mathbb{Z}\}$, where $r_k \to 0$ as $|k| \to \infty$, and finally it was assumed that ω has a polynomial growth at infinity. These assumptions are satisfied in the case of a vibrating sphere.

1E. *The vibrating sphere.* These problems become trivial when Λ is a lattice and when K is a fundamental domain for the dual lattice Λ^*. The dual lattice Λ^* is defined by $\Lambda^* = \{x \in \mathbb{R}^n \mid \exp(2\pi i x \cdot y) = 1 \text{ for all } y \in \Lambda\}$. A fundamental domain K for a lattice Γ is defined by the following condition: if sets of zero measure are ignored, the translated compacts $\gamma + K$, $\gamma \in \Gamma$, are an exact paving of $\mathbb{R}^n$.

The wave equation on the sphere $\mathbb{S}^2$ which is discussed in [Meyer 1973] provides us with a natural example where property (9) is not satisfied but where (10) holds true.

Theorem 1.1. *There exists a constant C such that for every continuous solution $u(x,t)$ of the wave equation on the sphere $\mathbb{S}^2$, for every $t \geq 2\pi$, and every $x \in \mathbb{S}^2$, one has*

$$|u(x,t)| \leq C\sqrt{t} \sup_{s \in [0,2\pi]} |u(x,s)| \tag{12}$$

and this estimate is optimal.

This follows from [Meyer 1973; 2018a] and explains why there exist continuous solutions of the wave equation on the sphere which are not almost periodic. If the sphere is replaced by the torus, this is no longer true, as will be proved in Theorem 5.4.

2. L^2 estimates

2A. *Landau's theorem.* The L^2 theory of nonperiodic trigonometric sums (in the sense given by Property 1.1) was born in the thirties [Ingham 1936; Paley and Wiener 1934]. In the sixties this theory was revitalized by some important applications to control theory [Avdonin 1974; Avdonin and Ivanov 1995; Lions 1984] and to signal processing [Landau 1967]. A main breakthrough was achieved in [Landau 1967]. While he was working at the Bell Labs in Murray Hill, Landau proved that (7) implies $|K| \geq \overline{\text{dens}}\,\Lambda$. The upper uniform density of Λ will be defined below and $|K|$ denotes the Lebesgue measure of K. Can the converse implication be true? Does $|K| \geq \overline{\text{dens}}\,\Lambda$ imply (7)? The simplest counterexample is given by $\Lambda = \mathbb{Z}$ and $K = \left[0, \frac{1}{3}\right] \cup \left[1, 1+\frac{1}{3}\right] \cup \left[2, 2+\frac{1}{3}\right] \cup \left[3, 3+\frac{1}{3}\right]$. The measure of K is $\frac{4}{3}$, which exceeds $\text{dens}\,\Lambda$, but (7) is not true since $P \in \mathcal{T}_\Lambda$ is one-periodic and each of the four intervals of K gives the same information on P. We return to the definition of the upper uniform density of Λ. First for every $R > 0$ one computes $N(R) = \sup_{x \in \mathbb{R}^n} \#(\Lambda \cap B(x,R))$, where $B(x,R)$ denotes the ball of radius R centered at x. Then the upper uniform density of Λ is $\limsup_{R \to \infty} N(R)/(c_n R^n)$, where c_n denotes the volume of the unit sphere.

Property 1.1 is well understood if $n = 1$ and if K is an interval: $|K| \geq \overline{\text{dens}}\,\Lambda$ is necessary and $|K| > \overline{\text{dens}}\,\Lambda$ is sufficient. But Property 1.1 is mostly open when $n = 1$ and K is a finite union of intervals, or when $n \geq 2$. Then the arithmetical structure of Λ plays a seminal role, as will be illustrated by Theorem 2.1. For example if $n = 2$, if Λ is a lattice, and if K is a disk, Landau's bound $|K| = \operatorname{dens}\Lambda$ cannot be approached. Indeed we have (7) $\Rightarrow |K| \geq 2\pi/(3\sqrt{3}) \operatorname{dens}\Lambda$ and $2\pi/(3\sqrt{3}) > 1$. This gap comes from the fact that the plane cannot be paved with translated copies of a disk. We conclude that in dimension $n \geq 2$ sharp results are not related to Landau's theorem but depend on a deeper analysis of the structure of Λ.

2B. ***The converse implication in Landau's theorem.*** The fundamental question raised by Landau's theorem is the following: given a discrete and closed set Λ, is it possible that (7) holds for every compact Riemann integrable set K such that $|K| > \overline{\operatorname{dens}}\,\Lambda$? This natural question was only recently solved. As was observed, such a Λ cannot be a lattice. A first solution was given in [Olevskii and Ulanovskii 2008] and then a second one in [Matei and Meyer 2010]. In the latter, we proved (7) when Λ is a simple quasicrystal and K is a compact Riemannian integrable set K such that $|K| > \operatorname{dens}\Lambda$. S. Grepstad and N. Lev [2014] settled the limiting case $|K| = \operatorname{dens}\Lambda$. For the sake of simplicity their result will be stated on an example. Let $\lfloor x \rfloor$ be the integral part of a real number x. Then $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. Let us assume $\alpha > 0, \beta > 0,\ \alpha \notin \mathbb{Q},\ \alpha + \beta^{-1} \notin \mathbb{Q}$. Let $\lambda_k = k + \beta\{\alpha k\},\ k \in \mathbb{Z}$, and $\Lambda_\alpha = \{\lambda_k \mid k \in \mathbb{Z}\}$. Then Sigrid Grepstad and Nir Lev proved the following theorem.

Theorem 2.1. *Let K be a finite union of disjoint intervals with endpoints in $\alpha\mathbb{Z}+\mathbb{Z}$. Then the exponential functions $\exp(2\pi i\lambda\cdot x)$, $\lambda \in \Lambda_\alpha$, are a Riesz basis of $L^2(K)$ if and only if $|K| = 1$.*

Definition 2.1. If H is a Hilbert space, a Riesz basis of H is the image of an orthonormal basis of H by an isomorphism $T : H \mapsto H$.

Let us observe that $|K| = 1$ is Landau's bound. Theorem 2.1 implies that such a K is minimal for Λ_α. Is there an L^p analogue of Theorem 2.1 when $p \neq 2$? We do not know since the proof of Theorem 2.1 given in [Grepstad and Lev 2014] is using Plancherel formula.

Corollary 2.1. *Let K be a finite union of disjoint intervals with endpoints in $\alpha\mathbb{Z}+\mathbb{Z}$. If $|K| = 1$ every $f \in L^2(K)$ is the sum of the Fourier series*

$$f(x) = \sum_{\lambda\in\Lambda_\alpha} c(\lambda)\exp(2\pi i\lambda\cdot x), \tag{13}$$

which converges to f in $L^2(K)$.

Moreover we have $C_1\|f\|_{L^2(K)} \leq \left(\sum_{\lambda\in\Lambda_\alpha}|c(\lambda)|^2\right)^{1/2} \leq C_2\|f\|_{L^2(K)}$ and there exists a dual family $g_\lambda(x) \in L^2(K)$ such that $c(\lambda) = \int_K f(x)\bar{g}_\lambda(x)\,dx$.

2C. ***A second example of a minimal set.*** Another example of a minimal set is given by the following construction. Let $\alpha > 0,\ \beta > 0\ \alpha \notin \mathbb{Q},\ \beta|\sin(\pi\alpha)| \in \left(0, \frac{1}{2}\right)$, and $\lambda_k^{(\alpha,\beta)} = k + \beta\sin(2\pi\alpha k),\ k \in \mathbb{Z}$. Let $\Lambda_{\alpha,\beta} = \{\lambda_k^{(\alpha,\beta)} \mid k \in \mathbb{Z}\}$.

Theorem 2.2. *The functions* $\exp(2\pi i\lambda x),\ \lambda \in \Lambda_{\alpha,\beta}$, *are a Riesz basis of* $L^2([0,1])$.

The proof of Theorem 2.2 mimics what was achieved in [Grepstad and Lev 2014]. The condition $0 < \beta|\sin(\pi\alpha)| < \frac{1}{2}$ implies that $\Lambda_{\alpha,\beta}$ is uniformly discrete. Moreover there exists an integer N such that uniformly in k we have

$$\beta\left|\frac{1}{N}\sum_{k}^{k+N-1}\sin(2\pi\alpha j)\right| \le \theta < \tfrac{1}{5}. \tag{14}$$

Then S. A. Avdonin's theorem [1974], see also [Avdonin and Ivanov 1995], yields the result. (Does an analogue of Grepstad and Lev's theorem hold?) Therefore $[0,1]$ is a minimal set for $\Lambda_{\alpha,\beta}$. Finally if $\beta|\sin(\pi\alpha)| \ge \frac{1}{2}$, then $\Lambda_{\alpha,\beta}$ is not uniformly discrete and $\exp(2\pi i\lambda x),\ \lambda \in \Lambda_{\alpha,\beta}$, cannot be a basis.

3. Weighted L^2 estimates

If Λ is uniformly discrete then Property 1.1 is satisfied. Given such a Λ, the main issue is to find "small" compact set K for which (7) holds. If $\overline{\operatorname{dens}}\,\Lambda < \infty$ then Λ is the union of at most N uniformly discrete sets $\Lambda_j,\ 1 \le j \le N$. We conjecture that (11) holds with $p = 2$ and $\omega(x) \le C(1+|x|)^{N-1}$. This was proved by Jaffard, Tucsnak and Zuazua [Jaffard et al. 1997] in a slightly narrower setting. In their theorem Λ is the union of two uniformly discrete sets of real numbers. Their theorem is proved here in a slightly simplified version. In this section 2π is omitted in the definition of the Fourier transform.

Theorem 3.1. *Let* $\lambda_k,\ k \in \mathbb{Z}$, *be an increasing sequence of real numbers such that for a constant* $M \ge 2$ *we have* $1 \le \lambda_{k+2} - \lambda_k \le M,\ k \in \mathbb{Z}$. *Then there exist a* $T > 0$ *and two constants* C_0 *and* C_1 *such that for every sequence* $a_k,\ k \in \mathbb{Z}$,

$$\begin{aligned} C_0\int_{-T}^{T}\left|\sum_{k\in\mathbb{Z}} a_k\exp(i\lambda_k t)\right|^2 dt &\le \sum_{k\in\mathbb{Z}}[|a_{k+1}+a_k|^2 + |\lambda_{k+1}-\lambda_k|^2(|a_{k+1}|^2+|a_k|^2)] \\ &\le C_1\int_{-T}^{T}\left|\sum_{k\in\mathbb{Z}} a_k\exp(i\lambda_k t)\right|^2 dt. \end{aligned}$$

If moreover we have $\lambda_{k+1} - \lambda_k \ge \beta > 0$ for all $k \in \mathbb{Z}$, then Λ is uniformly discrete and Theorem 3.1 coincides with Property 1.1 since $\sum_{k\in\mathbb{Z}}|a_{k+1}+a_k|^2 \le 2\sum_{k\in\mathbb{Z}}(|a_{k+1}|^2 + |a_k|^2)$. Let us prove Theorem 3.1. Let ϕ be a real-valued even function in the Schwartz class $\mathcal{S}(\mathbb{R})$ such that $\phi(t) > 0$ for all $t \in \mathbb{R}$, $\|\phi\|_2 = 1$ and $\hat{\phi} = 0$ outside $\left[-\frac{1}{10}, \frac{1}{10}\right]$. We then have:

Lemma 3.1. *There exist four positive constants C, C', c and c' (depending on the choice of ϕ) such that for every $\epsilon \in [-1/(2\sqrt{C}), 1/(2\sqrt{C})]$ and any coefficients a and b we have*

$$C'\epsilon^2(|a|^2+|b|^2)+(1-C\epsilon^2)|a+b|^2 \le \int_{-\infty}^{+\infty} |(a\exp(i\epsilon t)+b)\phi(t)|^2\,dt \le C\epsilon^2(|a|^2+|b|^2)+(1-C'\epsilon^2)|a+b|^2.$$

If $1/(2\sqrt{C}) \le |\epsilon| \le 1$ we have

$$c(|a|^2+|b|^2) \le \int_{-\infty}^{+\infty} |(a\exp(i\epsilon t)+b)\phi(t)|^2\,dt \le c'(|a|^2+|b|^2).$$

We set $\Phi=\phi^2$; we obviously have $\int_{-\infty}^{+\infty}\Phi(t)t^2\,dt>0$ and Φ is even. Therefore there exist $C>C'>0$ such that

$$1-C\epsilon^2 \le \widehat{\Phi}(\epsilon) \le 1-C'\epsilon^2 \quad \text{for all } \epsilon\in[-1,1].$$

On the other hand we have

$$\int_{-\infty}^{+\infty} |(a\exp(i\epsilon t)+b)\phi(t)|^2\,dt = |a|^2+|b|^2+\widehat{\Phi}(\epsilon)(a\bar{b}+b\bar{a}) = \widehat{\Phi}(\epsilon)|a+b|^2+(1-\widehat{\Phi}(\epsilon))(|a|^2+|b|^2).$$

This and the estimates on $\widehat{\Phi}(\epsilon)$ end the proof of the first assertion in Lemma 3.1. When $|\epsilon|\ge 1/(2\sqrt{C})$, we simply use the uniform bound $0\le\widehat{\Phi}(\epsilon)\le\theta<1$ and conclude as above.

We return to the proof of Theorem 3.1 and estimate $\|P\phi\|_2$ when $P(t)=\sum_{k\in\mathbb{Z}} a_k\exp(i\lambda_k t)$. As in [Jaffard et al. 1997] we denote by A the set of all integers k such that either $\lambda_{k+1}-\lambda_k\le\frac{1}{5}$ or $\lambda_k-\lambda_{k-1}\le\frac{1}{5}$. Then we set $B=\mathbb{Z}\setminus A$. If $k\in B$ and $k+1\in A$ we have $\lambda_{k+1}-\lambda_k>\frac{4}{5}$. Similarly if $k\in B$ and $k-1\in A$ we have $\lambda_k-\lambda_{k-1}>\frac{4}{5}$. These estimates hold since $\lambda_{k+2}-\lambda_k\ge 1$. If both k and $k+1$ belong to B we have $\lambda_{k+1}-\lambda_k>\frac{1}{5}$ and similarly if k and $k-1$ belong to B. In all cases we have $\lambda_{k+1}-\lambda_k>\frac{1}{5}$ if $k\in B$. Then if $P(t)=\sum_{k\in\mathbb{Z}}a_k\exp(i\lambda_k t)$, we have $P\phi=f_1+f_2$, where $f_1=\sum_{k\in A}a_k\exp(i\lambda_k t)\phi(t)$ and $f_2=\sum_{k\in B}a_k\exp(i\lambda_k t)\phi(t)$. By the definition of B, the terms in the sum f_2 are pairwise orthogonal since they have disjoint supports in the Fourier domain. Similarly the terms in the sum f_1 appear as pairs $g_k(t)=(a_k\exp(i\lambda_k t)+a_{k+1}\exp(i\lambda_{k+1}t))\phi(t)$, where $\lambda_{k+1}-\lambda_k\le\frac{1}{5}$. These g_k are pairwise orthogonal and are orthogonal to f_2. Then

$$\|P\phi\|_2^2=\|f_1\|_2^2+\|f_2\|_2^2=\sum_{k\in A}\|g_k\|_2^2+\sum_{k\in B}|a_k|^2\,\|\phi\|_2^2.$$

Lemma 3.1 is applied to each term $\|g_k\|_2$. We obtain the right-hand side of Theorem 3.1. Let us observe that in this right-hand side we ignored the sets A and B, which is harmless since $\lambda_{k+1}-\lambda_k>\frac{1}{5}$ if $k\in B$. Then Theorem 3.1 would be proved if the left-hand side was $\|P\phi\|_2^2$. This is not the case but this will be repaired using the following corollary of our calculation.

Corollary 3.1. *We have for every real number* τ

$$I(\tau)=\int_{-\infty}^{+\infty}|P(t+\tau)|^2\phi(t)\,dt\le C(1+\tau^2)\int_{-\infty}^{+\infty}|P(t)|^2\phi(t)\,dt.$$

Indeed we estimate $I(\tau)$ and $I(0)$ by a sum of coefficients given by Theorem 3.1. The only terms that differ in the two sums are

$$S(\tau)=\sum_{k\in\mathbb{Z}}|a_{k+1}\exp(i\tau\lambda_{k+1})+a_k\exp(i\tau\lambda_k)|^2$$

compared to $S(0)=\sum_{k\in\mathbb{Z}}|a_{k+1}+a_k|^2$. Then it suffices to observe that

$$\begin{aligned}|a_{k+1}\exp(i\tau\lambda_{k+1})+a_k\exp(i\tau\lambda_k)|&=|a_{k+1}\exp(i\tau(\lambda_{k+1}-\lambda_k))+a_k|\\&\le|a_{k+1}\exp(i\tau(\lambda_{k+1}-\lambda_k))-a_{k+1}|+|a_{k+1}+a_k|\\&\le|\tau|\,|a_{k+1}(\lambda_{k+1}-\lambda_k)|+|a_{k+1}+a_k|.\end{aligned}$$

Finally

$$\begin{aligned}S(\tau)&\le 2S(0)+2|\tau|^2\sum_{k\in\mathbb{Z}}|a_{k+1}(\lambda_{k+1}-\lambda_k)|^2\\&\le 2(1+\tau^2)\sum_{k\in\mathbb{Z}}[|a_{k+1}+a_k|^2+|\lambda_{k+1}-\lambda_k|^2(|a_{k+1}|^2+|a_k|^2)],\end{aligned}$$

which ends the proof.

To end the proof of Theorem 3.1 we use an obvious trick and prove the equivalence between $\|P\phi\|_2$ and $\left(\int_{-T}^{T}|P(t)|^2\right)^{1/2}$. On the one hand we have

$$\int_{-\infty}^{+\infty}|P(t)\phi(t)|^2\,dt\ge c\int_{-T}^{+T}|P(t)|^2\,dt$$

for a positive constant c. On the other hand

$$\int_{-\infty}^{+\infty}|P(t)\phi(t)|^2\,dt=\sum_{k\in\mathbb{Z}}\int_{(k-1)T}^{(k+1)T}|P(t)\phi(t)|^2\,dt=\sum_{k\in\mathbb{Z}}I_k.$$

Each I_k is estimated from above by Corollary 3.1 when $k\neq 0$. The total contribution does not exceed $\eta(T)\|P\phi\|_2$, while I_0 is estimated from below by $c\|P\phi\|_2$. We chose T large enough to have $c>\eta(T)$. This ends the proof of Theorem 3.1.

4. Mean periodic functions

The goal of this section is to prove that a p-coherent set $\Lambda\subset\mathbb{R}$ has a finite Beurling and Malliavin density. A sharper theorem will be proved in Section 8. Let us begin with the n-dimensional case. Let $\mathcal{C}(\mathbb{R}^n)$ denote the vector space of all continuous functions on $\mathbb{R}^n$, equipped with the topology of *uniform convergence on compact sets*.

Lemma 4.1. *If Λ is a discrete and closed set and if $\mathcal{T}_\Lambda$ is dense in $\mathcal{C}(\mathbb{R}^n)$ for the topology of uniform convergence on compact sets then Λ cannot be a p-coherent set.*

If $\mathcal{T}_\Lambda$ is dense in $\mathcal{C}(\mathbb{R}^n)$ then Property 1.4 cannot be true. Otherwise (11) would still hold for every continuous function, which is clearly impossible.

Lemma 4.2. *The following two properties are equivalent for a closed and discrete set $\Lambda \subset \mathbb{R}^n$:*

(a) *$\mathcal{T}_\Lambda$ is not dense in $\mathcal{C}(\mathbb{R}^n)$ for the topology of uniform convergence on compact sets.*

(b) *There exists a compactly supported Radon measure $\mu \neq 0$ whose Fourier transform vanishes on Λ.*

This is provided by the Hahn–Banach theorem. In the one-dimensional case Beurling and Malliavin [1967] proved that (a) and (b) are equivalent to a remarkable density condition on Λ. It will be proved in Section 6 that there exists a closed and discrete set Λ satisfying conditions (a) and (b) with an infinite upper uniform density. We conclude these remarks by Lemma 4.3.

Lemma 4.3. *If $\Lambda \subset \mathbb{R}$ is a p-coherent set, the Beurling and Malliavin density of Λ is finite.*

As it was observed in [Kahane 1957], Properties 1.2 and 1.3 are strongly motivated by the theory of mean periodic functions.

Definition 4.1. A mean periodic function is a function $f \in \mathcal{C}(\mathbb{R}^n)$ for which there exists a compactly supported Radon measure $\mu \neq 0$ such that $f * \mu = 0$.

Definition 4.2. Let us assume that $\mathcal{T}_\Lambda$ is not dense in $\mathcal{C}(\mathbb{R}^n)$ for the topology of uniform convergence on compact sets. Then the closure of $\mathcal{T}_\Lambda$ in $\mathcal{C}(\mathbb{R}^n)$ will be denoted by $\mathcal{C}_\Lambda$.

These definitions and Lemma 4.2 imply that every function $f \in \mathcal{C}_\Lambda$ is a mean periodic function.

Definition 4.3. The spectrum of a mean periodic function f is the set $S \subset \mathbb{C}^n$ of all $\lambda \in \mathbb{C}^n$ such that $\exp(2\pi i x \cdot \lambda)$ is a limit, for the topology of uniform convergence on compact sets, of linear combinations of translates of f.

If μ is a compactly supported Radon measure such that $\mu * f = 0$ the spectrum of f is contained in the set of zeros of the Fourier–Laplace transform of μ. If $f \in \mathcal{C}_\Lambda$ its spectrum is contained in Λ.

Here is a more constructive definition of a mean periodic function.

Lemma 4.4. *Let us assume that* (a) $f \in \mathcal{C}(\mathbb{R}^n)$ *has polynomial growth at infinity and* (b) $\hat{f} = \sum_{\lambda \in \Lambda} c(\lambda)\delta_\lambda$, *where* δ_a *is the Dirac measure at* a. *Then* $f \in \mathcal{C}_\Lambda$.

Is the converse implication true? What are the sets Λ enjoying the property that every $f \in \mathcal{C}_\Lambda$ has polynomial growth at infinity? This is Problem 1.3 and our essay partially answers this natural question. Problem 1.2 has a simple formulation as the following theorem shows.

Theorem 4.1. *Let* $\Lambda \subset \mathbb{R}^n$ *be a closed and discrete set. Then the following two conditions are equivalent*:

(i) *Every* $f \in \mathcal{C}_\Lambda$ *is an almost periodic function in the sense of H. Bohr.*

(ii) *Property* 1.2 *is satisfied.*

This is proved in [Kahane 1957].

5. Wild sets of frequencies

5A. *Equivalent definitions.*

Definition 5.1. A closed and discrete set $\Lambda \subset \mathbb{R}^n$ is wild if $\Lambda \notin \mathcal{L}(\infty)$.

A wild set Λ is an ∞-wild set. We treat the one-dimensional case and investigate the structure of wild sets. Property 1.3 is given an equivalent version in the following lemma:

Lemma 5.1. *Let* Λ *be a closed and discrete set of real numbers. Then Property* 1.3 *is equivalent to the following*: *there exists a* $T > 0$ *and for each* $R \geq T$ *a finite constant* $C(R)$ *such that for every* $P \in \mathcal{T}_\Lambda$ *one has*

$$\sup_{|x| \leq R} |P(x)| \leq C(R) \sup_{|x| \leq T} |P(x)|. \tag{15}$$

An L^p version of Lemma 5.1 will be given below. One direction is obvious and we can choose $C(R) = \sup_{|x| \leq R} \omega(x)$. The other direction would be obvious if continuity was not imposed on the weight. Here is the argument. We first replace $C(R)$ by

$$\widetilde{C}(R) = \sup_{P \in \mathcal{T}_\Lambda} \frac{\sup_{|x| \leq R} |P(x)|}{\sup_{|x| \leq T} |P(x)|}.$$

Then $\widetilde{C}(R) \leq C(R)$ and $\widetilde{C}(R)$ is obviously a nondecreasing function of R. Then (15) remains true if $C(R)$ is replaced by $\widetilde{C}(R)$. Finally there exists a continuous function $\omega(R)$ such that $\widetilde{C}(R) \leq \omega(R)$. This implies (10).

In summary if Λ is wild, then for every $R > 0$ there exists a $T > R$ and a sequence $P_j \in \mathcal{T}_\Lambda$ such that $\sup_{|x| \leq R} |P_j(x)|$ tends to 0, while $\sup_{|x| \leq T} |P_j(x)| = 1$. We now prove that if Λ is wild, this property is true for every $T > R$, which is a stronger statement.

Theorem 5.1. *Let Λ be a closed and discrete set of real numbers. Then Λ is wild if and only if one of the two following conditions is satisfied*:

(a) *There exists a sequence $P_j \in \mathcal{T}_\Lambda$ such that $P_j(0) = 1$ and such that for every compact set K contained in $(0, \infty)$ we have*

$$\sup_{y \in K} |P_j(y)| \to 0, \quad j \to \infty. \tag{16}$$

(b) *There exists a sequence $Q_j \in \mathcal{T}_\Lambda$ such that $Q_j(0) = 1$ and such that for every compact set K contained in $(-\infty, 0)$ we have $\sup_{y \in K} |Q_j(y)| \to 0,\ j \to \infty$.*

If Property 1.3 holds with $K = [a, b]$, every trigonometric sum $P \in \mathcal{T}_\Lambda$ can be extrapolated outside $[a, b]$ from its knowledge on $[a, b]$. In fact it suffices to extrapolate P on two points, one less than a, the other one larger than b, as the following lemma tells us.

Lemma 5.2. *Let Λ be a closed and discrete set of real numbers. If there exist an interval $I = [a, b]$, a real number $x_0 < a$, and a constant C such that for every $P \in \mathcal{T}_\Lambda$ one has*

$$|P(x_0)| \le C \sup_{y \in I} |P(y)| \tag{17}$$

then there exist a weight $\omega(x)$ and a compact set K such that (10) *holds true for every $x < a$.*

Let $\eta = a - x_0$ and $J = [a, b + \eta]$. Let us fix $u \in [x_0, a]$. Since the space $\mathcal{T}_\Lambda$ is translation invariant, (17) can be applied to $Q(x) = P(x + u - x_0)$. Therefore (17) remains valid when x_0 is replaced by u and I by J. We then proceed inductively from the interval $E_m = [x_0 - m\eta, a - m\eta]$ to $E_{m+1},\ m \in \mathbb{N}$. This inductive procedure yields (10) with an exponential weight.

Needless to say Lemma 5.2 is also true if $x_0 < a$ is replaced by $x_1 > b$. The conclusion is the validity of (10) for $x > b$. Finally Lemma 5.2 implies Theorem 5.1.

Here is the argument. Let us assume that Λ is wild. Then one of the two following conditions is satisfied: (1) either for any interval $I = [a, b]$ and any $x_0 < a$ there exists a sequence $P_j \in \mathcal{T}_\Lambda$ such that $P_j(x_0) = 1$ and P_j converges to 0 uniformly on I or (2) for any interval $I = [a, b]$ and any $x_1 > b$ there exists a sequence $P_j \in \mathcal{T}_\Lambda$ such that $P_j(x_1) = 1$ and P_j converges to 0 uniformly on I. Everything being translation invariant, we can assume $x_0 = 0$ if the first case occurs. We set $I_m = [m^{-1}, m]$. For every integer $j \ge 1$ there exists an integer j_m such that $P_{j_m} \in \mathcal{T}_\Lambda,\ P_{j_m}(0) = 1$, and $\sup_{y \in I_m} |P_{j_m}(y)| \le 2^{-j}$. This sequence $P_{j_m},\ m \in \mathbb{N}$, is the sequence announced in Theorem 5.1. The second alternative is similar.

Let $\Lambda \subset \mathbb{R}^n$ be a closed and discrete set and let $K \subset \mathbb{R}^n$ be a compact set. We denote by $\mathcal{C}(K)$ the Banach space of all continuous functions on K and by $\mathcal{C}_\Lambda(K)$ the closure of $\mathcal{T}_\Lambda$ in $\mathcal{C}(K)$.

Corollary 5.1. *Property* 1.3 *is satisfied by a closed and discrete set* Λ *of real numbers if and only if there exist two intervals* $[a, b]$ *and* $[c, d]$ *with* $c < a < b < d$ *such that the restriction operator* $R : \mathcal{C}_\Lambda([c, d]) \mapsto \mathcal{C}_\Lambda([a, b])$ *is an isomorphism.*

It would be interesting to know whether or not this property is valid in a more general setting. Given a closed and discrete set $\Lambda \subset \mathbb{R}^n$ and two compact sets $K, L \subset \mathbb{R}^n$ such that K is contained in the interior of L, we assume that the restriction operator $R : \mathcal{C}_\Lambda(L) \mapsto \mathcal{C}_\Lambda(K)$ is an isomorphism. Does it imply Property 1.3?

5B. *The Pisot set.* Here is an illustration of Theorem 5.1. Let $\theta \geq 2$ be a real number and let Λ_θ be the set of all finite sums $\sum_{k\geq 0} \epsilon_k \theta^k$, $\epsilon_k \in \{0, 1\}$. This set Λ_θ is uniformly discrete and will be named the Pisot set.

Theorem 5.2. *Let us assume that* θ *is not a Pisot–Thue–Vijayaraghavan number. Then* Λ_θ *is wild.*

We consider the sequence $P_m(x)$ of finite products

$$\prod_0^{m-1} \left(\frac{1 + \exp(2\pi i \theta^k x)}{2} \right).$$

The spectrum of P_m is contained in Λ. By Pisot's theorem we know that $|P_m(x)| = \prod_0^{m-1} |\cos(\pi\theta^k x)|$ converge uniformly to 0 on every compact set not containing the origin. We have $P_m(0) = 1$, which concludes the proof.

The converse is true. If θ is a Pisot–Thue–Vijayaraghavan number then Λ_θ satisfies Property 1.2, as is proved in [Meyer 1972]. If the sequence θ^k, $k \in \mathbb{N}$, is replaced by a lacunary sequence θ_k such that $\sum_0^\infty (\theta_k/\theta_{k+1})^2 < \infty$, the arithmetical properties of θ_k do not play any role. The set of all finite sums $\sum_{k\geq 0} \epsilon_k \theta_k$, $\epsilon_k \in \{0, 1\}$, satisfies Property 1.2 [Meyer 1972, Theorem IV, Chapter VIII].

5C. *The wave equation.* Here is a second example illustrating the definition of wild sets of frequencies. We consider the wave equation on the three-dimensional torus $\mathbb{T}^3$.

Theorem 5.3. *For every* $T_1 > T_0 > 0$ *and every* $\epsilon > 0$ *there exists a solution* $v(x, t)$ *of the wave equation on* $\mathbb{T}^3$ *such that* $v(0, 0) = 1$ *and* $|v(x, t)| \leq \epsilon$ *for all* $t \in [T_0, T_1]$, *for all* $x \in \mathbb{T}^3$.

Corollary 5.2. *The digital cone* $\Lambda \subset \mathbb{R}^4$, *defined by* $\Lambda = \{(k, \pm|k|) \mid k \in \mathbb{Z}^3\}$, *is wild.*

The proof of this simple observation depends on the following remarks. Let $w(x, t)$ be defined on $\mathbb{T}^3 \times \mathbb{R}$ by

$$w(x, t) = t + \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{\sin(2\pi t|k|)}{2\pi|k|} \exp(2\pi i k \cdot x).$$

Then $w(x, t)$ is the solution to the following Cauchy problem (named C-1) for the wave equation on $\mathbb{T}^3 \times \mathbb{R}$:

$$\frac{\partial^2}{\partial t^2}u(x,t) = \Delta u(x,t),$$
$$u(x,0) = 0, \quad \frac{\partial}{\partial t}u(x,0) = \delta_0(x).$$

But $w(x, t)$ can also be computed by periodizing the solution of a similar Cauchy problem (named C-2) on $\mathbb{R}^3 \times \mathbb{R}$. This scheme is detailed now. Let $\sigma_t,\ t \in \mathbb{R}$, be the normalized surface measure on the sphere $B_t \subset \mathbb{R}^3$ centered at 0 with radius $|t|$ (the total mass of σ_t is 1). Then $u(x,t) = t\sigma_t(x)$ belongs to $\mathcal{C}^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^3))$ and is the solution of the Cauchy problem C-2:

$$\frac{\partial^2}{\partial t^2}u(x,t) = \Delta u(x,t),$$
$$u(x,0) = 0, \quad \frac{\partial}{\partial t}u(x,0) = \delta_0(x).$$

Therefore

$$w(x,t) = \sum_{k \in \mathbb{Z}^3} t\sigma_t(x-k) \tag{18}$$

is the solution of the Cauchy problem C-1. Let us consider the distribution $\tau(x,t) = \frac{\partial}{\partial t}w(x,t)$. Let ϕ be a compactly supported smooth function defined on $\mathbb{R}^3$ and for $\epsilon > 0$ let $\phi_\epsilon(x) = g(x/\epsilon)$. If ϵ is small enough, ϕ_ϵ can be viewed as a function defined on $\mathbb{T}^3$. Let $g_\epsilon(x,t)$ be the solution of the wave equation defined by $g_\epsilon = \tau * \phi_\epsilon$, where the convolution product takes place on $\mathbb{T}^3$. Then $g_\epsilon(0,0) = 1$, while $|g_\epsilon(x,t)| \le C\epsilon$ if $t \in [T_0, T_1]$, as simple estimates show.

The same construction is performed on $\mathbb{T}^2$ instead of $\mathbb{T}^3$. It gives a natural example of a uniformly discrete set Λ for which Property 1.4 fails if $p > 2$. Indeed if $0 < T_0 < T_1$ we have $\int_{T_0}^{T_1} \int_{\mathbb{T}^2} |g_\epsilon|^p \, dt \, dx \le C\epsilon^{p/2+1}$, while for $\eta > 0$ we have $\int_\eta^\eta \int_{\mathbb{T}^2} |g_\epsilon|^p \, dt \, dx \simeq \epsilon^2$. We can conclude:

Theorem 5.4. *The digital cone $\Lambda \subset \mathbb{R}^3$, defined by $\Lambda = \{(k, \pm|k|) \mid k \in \mathbb{Z}^2\}$, is not p-coherent if $2 < p \le \infty$.*

Let us observe that Λ is uniformly discrete. Therefore Λ is 2-coherent.

6. Unions of lattices

An interesting example of a wild set of frequencies is detailed in this section. It will now be assumed that

$$\Lambda = \bigcup_1^\infty \omega_j \mathbb{Z}, \tag{19}$$

where $1 = \omega_1 < \cdots < \omega_j < \cdots$ and $\sum_1^\infty 1/\omega_j < \infty$. Then we have:

Lemma 6.1. *If Λ is defined by* (19) *then $\mathcal{T}_\Lambda$ is not dense in $\mathcal{C}(\mathbb{R}^n)$ for the topology of uniform convergence on compact sets.*

For proving Lemma 6.1 it suffices to construct a function h with the following properties: h is not identically 0, it is compactly supported, and its Fourier transform vanishes on Λ. Let us begin with $g_j(x)$, $j \in \mathbb{N}$, which is defined by $g_j(x) = \pi\omega_j$ on $[-1/(2\pi\omega_j), 1/(2\pi\omega_j)]$ and $g_j(x) = 0$ outside this interval. The convolution products $g_1 * g_2 * \cdots * g_j$ converge to a $\mathcal{C}^\infty$ function g. We have $g \geq 0$, $\int g = 1$ and $\hat{g} = 0$ on $\Lambda \setminus \{0\}$. The function $h = \frac{d}{dx} g$ has the required properties. This ends the proof.

Corollary 6.1. *The Beurling and Malliavin density of the set Λ defined by* (19) *is finite.*

Theorem 6.1. *Let us furthermore assume that $1, 1/\omega_1, 1/\omega_2, \ldots, 1/\omega_j, \ldots$ are linearly independent over $\mathbb{Q}$. Then the upper uniform density of $\Lambda = \bigcup_1^\infty \omega_j \mathbb{Z}$ is infinite and Λ is wild.*

In the next section it will be proved that the property "infinite upper uniform density" implies "p-wild". Here the two properties will be proved by the same argument. We argue by contradiction and assume that Λ is not wild. Then (10) holds true. The proof of Theorem 6.1 begins with the following definition.

Definition 6.1. Let $\Lambda \subset \mathbb{R}$ be a closed and discrete set and let F be a finite set. We write $F \in \mathcal{F}(\Lambda)$ if there exists a sequence x_j, $j \in \mathbb{N}$, of real numbers such that $F + x_j \subset \Lambda + [-1/j, 1/j]$.

If $|x_j| \leq C$, then $F \in \mathcal{F}(\Lambda)$ simply means $F \subset \Lambda + a$ for some a. If $\Lambda = \{k + 2^{-k} \mid k \in \mathbb{N}\}$, then $F \subset \mathbb{Z}$ implies $F \in \mathcal{F}(\Lambda)$. The proof of Theorem 6.1 depends on the following lemma:

Lemma 6.2. *Let $\Lambda \subset \mathbb{R}$ be a closed and discrete set. Let us assume that* (10) *holds true for a pair (Λ, K) and for a weight ω. If $F \in \mathcal{F}(\Lambda)$* (10) *also holds true for the pair (F, K) and the same weight ω.*

Let $P(x) = \sum_{y \in F} c(y) \exp(2\pi i \lambda y)$ be an arbitrary trigonometric sum with frequencies in F. We want to prove that

$$|P(x)| \leq \omega(x) \sup_{y \in K} |P(y)|. \tag{20}$$

For $j \geq 1$ every $y \in F$ can be written as $y = \lambda_{j,y} - x_j + \epsilon_j$, where $|\epsilon_j| \leq 1/j$. We approach $P(x)$ by $P_j(x) = \exp(-2\pi i x_j x) Q_j(x)$, where

$$Q_j(x) = \sum_{y \in F} c(y) \exp(2\pi i \lambda_{y,j} x).$$

But (10) is true for Q_j by assumption. Since $|P_j| = |Q_j|$, (10) is also true for P_j and it suffices to let j tend to infinity to conclude the proof of Lemma 6.2.

Lemma 6.3. *If Λ is defined by* (19) *and if the real numbers $1/\omega_j,\ j \in \mathbb{N}$, are linearly independent over $\mathbb{Q}$ then the finite set $F_{\epsilon,n} = \{0, \epsilon, 2\epsilon, \dots, (n-1)\epsilon\}$ belongs to $\mathcal{F}(\Lambda)$ for every $\epsilon > 0$ and every integer n.*

Lemma 6.3 obviously implies that the upper uniform density of Λ is infinite. This upper uniform density is defined as

$$\limsup_{T\to\infty} T^{-1} \sup_{x\in\mathbb{R}} \#([x, x+T] \cap \Lambda). \tag{21}$$

We return to the proof of Lemma 6.3. The linear independence over $\mathbb{Q}$ of $1, 1/\omega_1, 1/\omega_2, \dots, 1/\omega_n$ implies that the subgroup $(\exp(2\pi i k/\omega_m))_{1\le m\le n},\ k \in \mathbb{Z}$, is dense in $\mathbb{T}^n$. Therefore there exists a sequence k_j of integers and n sequences $l_{j,m}$, $j \in \mathbb{Z},\ 1 \le m \le n$, of integers such that, for $1 \le m \le n$, we have $k_j/\omega_m - l_{j,m} + m\epsilon/\omega_m \to 0,\ j \to \infty$. This convergence takes place on the real line. It yields $k_j - \omega_m l_{j,m} + m\epsilon \to 0,\ j \to \infty$. Returning to Definition 6.1 we have $x_j = k_j$ and $m\epsilon = \lim_{j\to\infty} \omega_m l_{j,m} - x_j$, as announced.

We now disprove the uniform validity of (10) when Λ is replaced by $F_{\epsilon,m}$ and $\epsilon \to 0,\ m \to \infty$. To this end we form $P_{\epsilon,m} = \epsilon^{-m} \sum_0^{m-1} c_k \exp(2\pi i \epsilon k x)$, where $\sum_0^{m-1} c_k k^q = 0,\ 0 \le q \le m-1$. Lemma 6.2 implies that (10) is satisfied by $P_{\epsilon,m}$ uniformly with respect to ϵ and m. But $\lim_{\epsilon\to 0} P_{\epsilon,m} = cx^m$. Therefore (10) is satisfied by x^m uniformly in $m \in \mathbb{N}$, which is impossible. The same argument can be used to disprove (11).

If Λ is replaced by a finite union $\Lambda_N = \bigcup_1^N \omega_j \mathbb{Z}$ then (10) is satisfied with $\omega(x) = (1+|x|)^{N-1}$, as is proved in [Meyer 2018a].

7. Upper uniform densities

Theorem 6.1 is a special instance of a more general fact which is valid for every $p \in [1, \infty]$.

Theorem 7.1. *The upper uniform density of a p-coherent set of real numbers is finite.*

The proof begins with the following lemma:

Lemma 7.1. *A closed and discrete set Λ is p-coherent if and only if there exist a $T > 0$ and for every $R \ge T$ a constant $C(R)$ such that for every $P \in \mathcal{T}_\Lambda$ one has*

$$\left(\int_{|x|\le R} |P(x)|^p\,dx\right)^{1/p} \le C(R)\left(\int_{|x|\le T} |P(x)|^p\,dx\right)^{1/p}. \tag{22}$$

The proof is immediate. On one hand if (11) is satisfied, it suffices to set $C(R) = \sup_{|x|\le R} \omega(x)$. On the other hand if (22) is satisfied, we first optimize this estimate. We replace $C(R)$ by the lower bound $\gamma(R) \le C(R)$ of all possible constants for which (22) holds. Then $\gamma(R)$ is an increasing function of R and there exists a continuous increasing $\tilde{\gamma}(R) \geq \gamma(R)$. Finally it suffices to set $\omega(y) = \tilde{\gamma}(|y|)$.

We return to the proof of Theorem 7.1 and fix the notation used in Lemma 7.2. If Λ is p-coherent then $\Lambda + y$ is also p-coherent with the same constants $C(R)$ in (22) and this holds true for every $y \in \mathbb{R}$. Without loss of generality it can be assumed that $K = [-a, a]$. Indeed K can be replaced by a larger compact. We fix $R = a + 1$ in (22). We then write $\gamma = C(R)$.

Lemma 7.2. *Let Λ be a p-coherent set of real numbers. Then for every interval J with length $|J| \geq 1$ we have*

$$\#(J \cap \Lambda) \leq C(a, \gamma)|J|, \tag{23}$$

where $C(a, \gamma)$ only depends on a and γ.

Lemma 7.2 obviously implies Theorem 7.1. To prove Lemma 7.2 for an arbitrary J it suffices to do it when $|J| = [0, 1]$. Indeed the translation invariance of (22) in the Fourier domain will imply Lemma 7.2 for every J with length 1. It suffices to add these estimates to obtain (23) for $|J| \geq 1$. We now prove (23) when $|J| = [0, 1]$. Let q be the conjugate exponent defined by $1/p + 1/q = 1$. We have $K = [-a, a]$ and without loss of generality it can be assumed that $a \geq 1$. Let $I = [a, a + 1]$ and let χ_I be the indicator function of I. By (22) and a duality argument there exists a function $g \in L^q(K)$ carried by K such that $\|g\|_q \leq \gamma$ and $\hat{g}(\lambda) = \hat{\chi}_I(\lambda)$ for every $\lambda \in \Lambda$. We now consider the entire function

$$F(z) = \int_{\mathbb{R}} \exp(-i2\pi zt)(\chi_I(t) - g(t))\, dt. \tag{24}$$

Then F vanishes on Λ. Let us compute $F(iy)$ for $y \geq 1$. We have $F(iy) = \int_a^{a+1} \exp(2\pi yt)\, dt - \int_K \exp(2\pi yt) g(t)\, dt$. Therefore

$$\begin{aligned} |F(iy)| &\geq \frac{\exp(2\pi y(a+1)) - \exp(2\pi ya)}{2\pi y} - \|g\|_q \left(\int_K \exp(2\pi pyt)\, dt \right)^{1/p} \\ &\geq C_1 \frac{\exp(2\pi y(a+1))}{y} - C_2 \gamma \frac{\exp(2\pi ya)}{y}. \end{aligned} \tag{25}$$

Here C_1 and C_2 are two absolute constants. Finally there exists a $y_0 \geq 1$, depending only on a and γ, such that $|F(iy_0)| \geq 1$. This y_0 is now fixed. We consider a disc D centered at $z_0 = iy_0$ with radius $R = 2\sqrt{1 + y_0^2}$. Then the disc centered at z_0 with radius $r = R/2$ contains J. The following corollary of Jensen's formula is applied to F and D:

Lemma 7.3. *If F is holomorphic on a neighborhood of a disc D centered at z_0 with radius R, if $|F|$ is bounded by M on D, and if $F(z_0) \neq 0$ then for every $r \in (0, R)$ the number of zeros of F inside the disc $|z - z_0| \leq r$ does not exceed $(\log(R/r))^{-1} \log(M/|F(z_0)|)$.*

We have $z_0 = iy_0$ and $R = 2r = 2\sqrt{1+y_0^2}$. We already know that $|F(z_0)| \geq 1$. Estimating $|F(z)|$ on D is trivial. Indeed if $y \geq 0$ and $x+iy = z \in D$, we have

$$|F(z)| \leq \frac{\exp(2\pi y(a+1)) - \exp(2\pi ya)}{2\pi y} + \gamma\left(\int_K \exp(2\pi pyt)\,dt\right)^{1/p} \leq M(a,\gamma)$$

since $y \leq y_0 + 2\sqrt{1+y_0^2}$ on D. The case $y \leq 0$ is similar. Finally Lemma 7.2 implies (23).

This proof raises a few questions. It seems that we are not using the full strength of the hypothesis since the proof is based on the value $R = a+1$ when $K = [-a, a]$. But this special instance of (22) implies (22) in full generality, as is proved in Lemma 5.1. The second issue is the generalization of Theorem 7.1 to the n-dimensional case. The third problem is the converse statement. If $p = 2$ and if Λ has a finite upper uniform density, does it imply that Λ is 2-coherent? A partial answer is given in [Jaffard et al. 1997].

8. L^p-estimates

8A. ***A sufficient condition.*** In a particular case the problem raised by Property 1.4 of Section 1 can be answered. The aim of this section is to show how much the L^p theory differs from the L^2 theory when $p \neq 2$. From now on $\Lambda \subset \mathbb{R}^n$ will be a uniformly discrete set and we set $\sigma_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$.

Definition 8.1. We say that Λ is a gentle set if the distributional Fourier transform $\hat{\sigma}_\Lambda$ of σ_Λ is a Radon measure.

A lattice is a gentle set. A finite union of gentle sets is a gentle set. But the set Λ defined by (19) is not a gentle set. The calculation of the Fourier transform of σ_Λ is an amusing exercise. The set $\Lambda_{\alpha,\beta}$, which was defined in Section 2, is a gentle set, as was proved in [Meyer 2018a]. In this case $\hat{\sigma}_\Lambda$ is an atomic measure. This fact will be used later on.

A gentle set has a finite upper uniform density, which explains why the set defined by (19) is not a gentle set. Indeed let ϕ be a compactly supported continuous function whose Fourier transform $\hat{\phi}$ is nonnegative. Then for every $y \in \mathbb{R}^n$ we have $\int \hat{\phi}(y-x)\,d\sigma_\Lambda(x) = \int \exp(2\pi i y\cdot u)\phi(u)\,d\hat{\sigma}_\Lambda(u) = I(y)$ and $|I(y)| \leq C$ since $\hat{\sigma}_\Lambda$ is a Radon measure.

Let Λ be a gentle set. Then $\mu = \hat{\sigma}_\Lambda$ is a Radon measure. We set $w(y) = \int_{B_y} d|\mu|$, $y \in \mathbb{R}^n$, where B is the ball centered at 0 with radius 1 and $B_y = B + y$. The following theorem was proved in [Meyer 2018a]:

Theorem 8.1. *Let Λ be a gentle set. Let us assume that w has a polynomial growth at infinity and let $\omega \geq 1$ be a continuous and submultiplicative function which is a majorant of w. Then there exists a compact set K such that for every $f \in \mathcal{C}_\Lambda$ we* *have*,

$$\text{for all } y \in \mathbb{R}^n, \quad |f(y)| \le \omega(y) \sup_{u \in K} |f(u)|. \tag{29}$$

This estimate is optimal if $w \simeq \omega$. Indeed we let g be a continuous function supported by the unit ball and normalized by $\|g\|_\infty = 1$. We consider the convolution product $f = \mu * g$. This function belongs to $\mathcal{C}_\Lambda$ and (29) is satisfied. For a given x we have

$$\left| \int g(y-x)\, d\mu(y) \right| \le C' \omega(x), \tag{30}$$

where C' is the total mass of μ on K. We now take the supremum of the left-hand side with respect to g and obtain $w(x) = |\mu|(B_x) \le C'\omega(x)$.

The L^p version of this theorem is given now.

Theorem 8.2. *Let Λ be a gentle set. Let us assume that w has polynomial growth at infinity and let $\omega \ge 1$ be a continuous submultiplicative function such that $\omega \ge w$. Let $1 \le p \le \infty$. Then there exists a compact set K such that for every $f \in \mathcal{C}_\Lambda$ and every $y \in \mathbb{R}^n$ we have*

$$\left(\int_{K+y} |f(x)|^p \, dx \right)^{1/p} \le C\omega(y)^{|2-p|/p} \left(\int_K |f(x)|^p \, dx \right)^{1/p}. \tag{31}$$

Theorem 8.2 is sharp. Indeed the estimate given by (31) is optimal in many instances, as will be proved below. Therefore the L^p theory strongly differs from the L^2 theory. If $p = 2$, Theorem 8.2 is a trivial statement. Indeed Λ is assumed to be uniformly discrete and the L^2 theory is given for free. If $p = \infty$, Theorem 8.2 is identical to Theorem 8.1. The proof of Theorem 8.2 is obtained by interpolating between these two cases. But the Riesz–Thorin interpolation theorem is not true if the operator to which it is applied is restricted to a subspace $V \subset L^2 \cap L^\infty$. That is why we need to build the interpolation scheme on the whole of $L^2 \cap L^\infty$. To prepare the notation for the proof we define $\beta > 0$ by

$$\inf_{\{\lambda \ne \lambda' | \lambda, \lambda' \in \Lambda\}} |\lambda - \lambda'| = \beta > 0.$$

Let $0 < r < r' < \beta/2$, let B_r and B'_r be the balls centered at 0 with radii r and r' respectively. Let ϕ be a function in the Schwartz class $\mathcal{S}$ such that $\hat{\phi} = 1$ on B_r and $\hat{\phi} = 0$ outside B'_r. Let μ_y be the Radon measure μ translated by $-y$ and let $\chi_y(x) = \exp(2\pi i x y)$. Then the Fourier transform of the product $\nu_y = \phi\mu_y$ is the convolution product $\hat{\phi} * \chi_y \sigma_\Lambda$. The following lemma resumes this discussion:

Lemma 8.1. *We have*

$$\widehat{\phi\mu_y}(\xi) = \sum_{\lambda \in \Lambda} \exp(2\pi i \lambda \cdot y) \hat{\phi}(\xi - \lambda). \tag{32}$$

We now estimate the norm of the measure $\phi\mu_y$. We have

$$\|\nu_y\| \leq C\omega(y). \tag{33}$$

This estimate results from the definition of w, the rapid decay of ϕ, and the slow growth of w.

The operator norm of the convolution with the measure $\nu_y = \phi\mu_y$ acting on $L^\infty(\mathbb{R}^n)$ does not exceed $\|\nu_y\| \leq C\omega(y)$. The same bound is valid on $L^1(\mathbb{R}^n)$. On the other hand this convolution operator acts on $L^2(\mathbb{R}^n)$ with a norm not exceeding C. Indeed (34) shows that $\|\hat{\nu}_y\|_\infty \leq C$ uniformly in y. An interpolation between L^2 and L^∞ or L^1 yields the following:

Lemma 8.2. *Let $p \in [1, \infty]$. Then we have, for every $y \in \mathbb{R}^n$ and every $f \in L^p$,*

$$\|\nu_y * f\|_p \leq C\omega(y)^{|2/p-1|}\|f\|_p. \tag{34}$$

We now return to the proof of Theorem 8.2. Let g be a positive function in the Schwartz class whose Fourier transform is supported by the ball centered at 0 with radius r. This function g will be used to localize $P \in \mathcal{T}_\Lambda$. Let us set $P_y(x) = P(x+y)$. Then the product gP_y gives access to P around y.

Lemma 8.3. *For every $y \in \mathbb{R}^n$ we have*

$$gP_y = \nu_y * (Pg). \tag{35}$$

It suffices to prove Lemma 9.3 when $P(x) = \exp(2\pi i\lambda \cdot x)$, $\lambda \in \Lambda$. Then the Fourier transform of the left-hand side of (35) is $\exp(2\pi i\lambda \cdot y)\hat{g}(\xi - \lambda)$, while the Fourier transform of the right-hand side is $\exp(2\pi i\lambda \cdot y)\hat{\phi}(\xi - \lambda)\hat{g}(\xi - \lambda)$. But $\hat{\phi} = 1$ on the support of $\hat{g}$, which ends the proof of Lemma 9.3.

We now return to the proof of (31). For simplifying the notation let us set $\omega_p(y) = \omega(y)^{|2/p-1|}$. Then

Lemma 8.4. *For every $P \in \mathcal{T}_\Lambda$, every $y \in \mathbb{R}^n$, and every $R \geq 1$ we have*

$$\left(\int_{|x-y|\leq R} |P(x)|^p\,dx\right)^{1/p} \leq C_R\omega_p(y)\|Pg\|_p. \tag{36}$$

We have by (34) and (35)

$$\|P_y g\|_p = \|\nu_y * (Pg)\|_p \leq C\omega_p(y)\|Pg\|_p. \tag{37}$$

Then (37) implies (36). Indeed it suffices to observe that $g(x) \geq c_R > 0$ on the ball centered at 0 with radius R.

If g were compactly supported, (36) would end the proof of Theorem 8.2. This is not the case but the problem can be easily fixed since g has a rapid decay at infinity. We now give the details of this argument.

Lemma 8.5. *Let Q_T be the cube defined by $|x_1| \le T, \dots, |x_n| \le T$, and let $\mathcal{R}_T = \mathbb{R}^n \setminus Q_T$. For every $\epsilon > 0$ there exists an integer $T \ge 1$ such that for every $P \in \mathcal{T}_\Lambda$ we have*

$$\left(\int_{\mathcal{R}_T} |Pg|^p \, dx\right)^{1/p} \le \epsilon \|Pg\|_p. \tag{38}$$

For proving this estimate we pave $\mathcal{R}_T$ by a disjoint union of cubes Q^j, $j \in \mathbb{N}$, of size 1. Then (38) implies $\int_{Q^j} |P|^p \, dx \le C\omega_p^p(x_j)\|Pg\|_p^p$, where x_j is the center of Q^j. Therefore

$$\int_{Q^j} |Pg|^p \, dx \le C\omega_p^p(x_j) \sup_{x \in Q_j} |g(x)|^p \, \|Pg\|_p^p. \tag{39}$$

Adding these estimates yields $\int_{\mathcal{R}_T} |Pg|^p \, dx \le C \sum_j \omega_p^p(x_j) \sup_{Q_j} |g|^p \, \|Pg\|_p^p \le \epsilon^p \|Pg\|_p^p$, which proves Lemma 9.5.

Corollary 8.1. *The three norms $\left(\int_{Q_T} |Pg|^p \, dx\right)^{1/p}$, $\left(\int_{Q_T} |P|^p \, dx\right)^{1/p}$, and $\|Pg\|_p$ are equivalent on $\mathcal{T}_\Lambda$ if T is large enough.*

Finally Corollary 8.1 and (3) imply Theorem 8.2.

8B. ***Optimality.*** In a special case which is detailed below, these estimates are optimal and this follows from a general result which is given now. Let us assume that $\hat{\sigma}_\Lambda$ is the atomic measure $\mu = \sum_0^\infty a_j \delta_{x_j}$ and, if $1 \le p \le \infty$, let $\omega_{p,R}(x) = \left(\sum_{|x-x_j| \le R} |a_j|^p\right)^{1/p}$. We further assume that there exists a constant C such that $\omega_{p,2R}(x) \le C\omega_{p,R}(x)$ holds true for $R \ge 1$. We write $\omega_{p,1} = \omega_p$.

Theorem 8.3. *Then for every compact set K and for every $y \in \mathbb{R}^n$ there exists a nontrivial $f \in \mathcal{C}_\Lambda$ such that*

$$\left(\int_{|x-y| \le 1} |f(x)|^p \, dx\right)^{1/p} \ge C_R \omega_p(y) \left(\int_K |f(x)|^p \, dx\right)^{1/p}. \tag{40}$$

Let us observe that $\omega_2 \simeq 1$ since Λ is uniformly discrete. It implies $\omega_p(y) \le C$ if $p \ge 2$. Moreover if $1 \le p \le 2$, Hölder's inequality yields

$$\omega_{p,R}(x) = \left(\sum_{|x-x_j| \le R} |a_j|^p\right)^{1/p} \le \left(\sum_{|x-x_j| \le R} |a_j|\right)^{(2-p)/p} \left(\sum_{|x-x_j| \le R} |a_j|^2\right)^{(p-1)/p}.$$

It shows that $\omega_{p,R}(x) \le C\omega(x)^{(2-p)/p}$. Therefore (31) and (40) are compatible. In general there is a gap between the upper bound given by (31) and the lower bound given by (40). But in some exceptional cases $\omega_{p,R}(x) \simeq \omega(x)^{(2-p)/p}$. An example is given below (Theorem 9.4). The proof of (40) mimics the argument used in the proof of Lemma 6.5 in [Matei and Meyer 2010]. Let $\epsilon > 0$ and let ϕ be an even compactly supported smooth function. We define $\phi_{\epsilon,p}(x) = \epsilon^{-n/p}\phi(x/\epsilon)$ and we have $\|\phi_{\epsilon,p}\|_p = \|\phi\|_p$. We consider the convolution product $f_{\epsilon,p} = \mu * \phi_{\epsilon,p}$. This function belongs to $\mathcal{C}_\Lambda$ and is our candidate to prove (40). Local L^p norms of $f_{\epsilon,p}$ are computed as follows:

Lemma 8.6. *Let a_j, $j \in \mathbb{N}$, be a sequence in l^1 and let $x_j \in \mathbb{R}^n$ be a sequence of pairwise disjoint points. Let K be a Riemann integrable compact set whose boundary ∂K does not contain any x_j. Then for $1 \le p \le \infty$ we have*

$$\lim_{\epsilon\to 0}\biggl(\int_K \Bigl|\sum_j a_j \phi_{\epsilon,p}(x-x_j)\Bigr|^p\,dx\biggr)^{1/p} = \biggl(\sum_{x_j\in K} |a_j|^p\biggr)^{1/p}. \tag{41}$$

Given $\eta > 0$ one fixes N such that $\sum_{N+1}^{\infty} |a_j| \le \eta$. The triangle inequality implies

$$\biggl(\int_K \Bigl|\sum_{N+1}^{\infty} a_j \phi_{\epsilon,p}(x-x_j)\Bigr|^p\,dx\biggr)^{1/p} \le \eta.$$

Next $\epsilon_N > 0$ is fixed such that the supports of $\phi_{\epsilon,p}(x-x_j)$, $x_j \in K$, $0 \le j \le N$, $0 < \epsilon \le \epsilon_N$, are pairwise disjoint. Then

$$\biggl(\int_K \Bigl|\sum_{0}^{N} a_j \phi_{\epsilon,p}(x-x_j)\Bigr|^p\,dx\biggr)^{1/p} = \biggl(\sum_{x_j\in K,\,0\le j\le N} |a_j|^p\biggr)^{1/p}. \tag{42}$$

This ends the proof of Lemma 9.6.

If one the condition $x_j \notin \partial K$ is dropped, (41) shall be replaced by

$$\lim_{\epsilon\to 0}\biggl(\int_K \Bigl|\sum_j a_j \phi_{\epsilon,p}(x-x_j)\Bigr|^p\,dx\biggr)^{1/p} \ge \biggl(\sum_{x_j\in L} |a_j|^p\biggr)^{1/p}, \tag{43}$$

where L is any compact set contained in the interior of K. Finally (41) and (43) imply Theorem 8.3.

We illustrate these theorems by the one-dimensional example of the set $\Lambda_{\alpha,\beta}$, which was defined in Section 2. In this case the Fourier transform of the measure $\sigma_{\Lambda_{\alpha,\beta}}$ is an explicit atomic measure [Meyer 2018a, Proposition 6.1]. Then Theorems 8.2 and 8.3 and the explicit calculation in [Meyer 2018a] imply the following:

Theorem 8.4. *Let $1 \le p \le \infty$ and $\omega_p(x) = C(1+|x|)^{|1/p-1/2|}$. Then for every $f \in \mathcal{C}_{\Lambda_{\alpha,\beta}}$ and for every y we have*

$$\biggl(\int_{y-1}^{y+1} |f(x)|^p\,dx\biggr)^{1/p} \le \omega_p(y)\biggl(\int_{-1}^{1} |f(x)|^p\,dx\biggr)^{1/p} \tag{44}$$

and this estimate is optimal if $1 \le p \le 2$.

8C. L^4 *estimates.* In the preceding examples L^p estimates were provided by interpolation between L^∞ and L^2. Here is an example where we do not have L^∞ estimates but a direct access to L^4.

Theorem 8.5. *Let $\Lambda \subset \mathbb{R}^n$ be a uniformly discrete set such that $\Lambda + \Lambda$ is also a uniformly discrete set. Then there exists a compact set $K \subset \mathbb{R}^n$ and a constant C such that for every $P \in \mathcal{T}_\Lambda$ and every $y \in \mathbb{R}^n$ we have*

$$\left(\int_{K+y} |P(x)|^4\,dx\right)^{1/4} \le C\left(\int_K |P(x)|^4\,dx\right)^{1/4}. \tag{45}$$

The proof of Theorem 9.5 will be given after a few remarks. Let us observe that $\Lambda + \Lambda$ is uniformly discrete if and only if $\Lambda - \Lambda$ is uniformly discrete.

But this condition is not necessary for obtaining (45). A one-dimensional counterexample is given by the union $\Lambda = \Lambda_1 \cup \Lambda_2$, where $\Lambda_1 = \{2^k \mid k \in \mathbb{N}\}$ and $\Lambda_2 = \{2^k + r_k \mid k \in \mathbb{N}\}$, with $0 < r_1 < \cdots < r_k < r_{k+1} < \cdots < \frac{1}{2}$. The proof of this remark is based on the following lemma:

Lemma 8.7. *Using this notation, there exists a $T > 0$ such that the norms $\left(\int_0^T |P(x)|^2\,dx\right)^{1/2}$ and $\left(\int_0^T |P(x)|^4\,dx\right)^{1/4}$ are equivalent on $\mathcal{T}_\Lambda$.*

If Lemma 9.7 is admitted, (45) follows. Indeed Λ is uniformly discrete. Therefore Property 1.1 is satisfied if $K = [0, T]$ and if T is large enough. Then Lemma 9.7 implies (45).

We now prove Lemma 9.7. Every $P \in \mathcal{T}_\Lambda$ is a sum $P_1 + P_2$, where the spectrum of P_1 is contained in Λ_1 and the spectrum of P_2 in Λ_2. As was already observed, Property 1.1 is satisfied by Λ. Therefore there exists a $T \ge 1$ such that

$$\int_0^T |P(x)|^2\,dx \simeq \int_0^T |P_1(x)|^2\,dx + \int_0^T |P_2(x)|^2\,dx.$$

But

$$\left(\int_0^T |P_1(x)|^2\,dx\right)^{1/2} \simeq \left(\int_0^T |P_1(x)|^4\,dx\right)^{1/4}$$

since Λ_1 is a Sidon set. The same holds true for P_2. Finally

$$\left(\int_0^T |P(x)|^2\,dx\right)^{1/2} \simeq \left(\int_0^T |P(x)|^4\,dx\right)^{1/4}.$$

A Delone set is, by definition, a uniformly discrete set $\Lambda \subset \mathbb{R}^n$ which is relatively dense: for a compact ball B we have $\Lambda + B = \mathbb{R}^n$. We now return to the hypothesis in Theorem 9.5. If Λ and $\Lambda - \Lambda$ are Delone sets, then Λ is a Meyer set and Theorem 9.5 follows from known results on quasicrystals. Finally Theorem 9.5 is not a new fact when Λ is a Delone set.

For proving Theorem 9.5 one observes that the L^4 norm of P is the square root of the L^2 norm of $Q = P^2$. The spectrum of Q is contained in $M = \Lambda + \Lambda$. But M is uniformly discrete. Therefore Property 1.1 is satisfied for the pair (M, K) if K is a large enough ball. More precisely there exists a constant C such that for every $Q \in \mathcal{T}_M$ and every $y \in \mathbb{R}^n$ we have

$$\left(\int_{K+y} |Q(x)|^2\, dx\right)^{1/2} \le C\left(\int_K |Q(x)|^2\, dx\right)^{1/2}, \tag{46}$$

which is exactly (45).

Theorem 9.5 can be applied to the set Λ_θ defined in Section 5. If $\theta \ge 3$ then $\Lambda_\theta + \Lambda_\theta$ is uniformly discrete. Therefore (45) holds true and the arithmetical properties of θ do not play any role. It would be interesting to know if this conclusion remains valid when the assumption $\theta \ge 3$ is replaced by $\theta \ge 2$ and when the exponent 4 is replaced by any $p \in (1, \infty)$. Finally beyond quasicrystals there are many other examples of Delone sets Λ satisfying Property 1.4. These examples cannot be constructed by Theorem 9.5.

Acknowledgements

The anonymous referee was supportive and inspiring. This work was supported by a grant from the Simons Foundation (601950, YM).

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Received 15 Jan 2019. Revised 24 Jul 2019.

YVES MEYER:

yves.meyer@cmla.ens-cachan.fr
CMLA, ENS-Cachan, CNRS, Université Paris-Saclay, France

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Tunisian Journal of Mathematics

2020 vol. 2 no. 4