ERRATUM TO "DISCRETENESS AND COMPLETENESS FOR Θ_n-MODELS OF (∞, n)-CATEGORIES"

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In this erratum, we correct the statement of Proposition 5.4 of [1], which previously had a missing hypothesis, and clarify in the proof where this additional assumption is needed. We also correct some typos in the coskeleton computations in Example 7.9 of that paper.

We thank Miika Tuominen and Jack Romo for conversations about these mistakes and their clarification.

1. Correction to comparison of definitions of Segal objects

We briefly recall some definitions to begin.

Definition 1.1. A *Segal object in Θ_n-spaces* is a Reedy fibrant functor $W\colon \Delta^{\mathrm{op}} \to \mathcal{SS}ets^{\Theta_n^{\mathrm{op}}}$ such that, for every $m \geq 2$, the Segal map

$$W_m \to \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_{m}$$

is a weak equivalence in the model structure $\Theta_n\mathcal{CSS}$.

However, there is another definition that is more widely used in the literature, and that enables a cleaner description of the completeness condition. Here it is helpful to regard functors $W\colon \Delta^{\mathrm{op}} \to \mathcal{SS}ets^{\Theta_n^{\mathrm{op}}}$ instead as functors $W\colon \Delta^{\mathrm{op}} \times \Theta_n^{\mathrm{op}} \to \mathcal{SS}ets$.

Definition 1.2. Given a functor $W\colon \Delta^{\mathrm{op}} \times \Theta_n^{\mathrm{op}} \to \mathcal{SS}ets$ and any $x_0, x_1 \in W([0],[0])_0$, the *mapping object* $M_W^{\Delta}(x_0,x_1)\colon \Theta_n^{\mathrm{op}} \to \mathcal{SS}ets$ is defined levelwise by pullbacks

$$\begin{array}{ccc} M_W^{\Delta}(x_0,x_1)(c) & \longrightarrow & W([1],c) \\ \downarrow & & \downarrow \\ \{(x_0,x_1)\} & \longrightarrow & W([0],c) \times W([0],c). \end{array} \tag{1.3}$$

The following result is the corrected version of Proposition 5.4 of [1], adding an essential constancy condition. Recall that a Θ_n-space X is *essentially constant* if for any object $[m](c_1,\ldots,c_m)$ of Θ_n the unique map from it to $[0]$ induces a weak equivalence $X[0] \to X[m](c_1,\ldots,c_m)$.

Proposition 1.4. *A Reedy fibrant functor $W\colon \Delta^{\mathrm{op}} \times \Theta_n^{\mathrm{op}} \to \mathcal{SS}ets$ is a Segal object in Θ_n-spaces with W_0 essentially constant if and only if the following conditions hold:*

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(1) for any $m \geq 2$ and $c \in \mathrm{ob}(\Theta_n)$, the Segal map

$$W([m], c) \to W([1], c) \times_{W([0],c)} \cdots \times_{W([0],c)} W([1], c)$$

is a weak equivalence of simplicial sets; and

(2) for any $x_0, x_1 \in W([0], [0])_0$, the mapping object $M_W^\Delta(x_0, x_1)$ is a Θ_n-space.

Proof. Suppose that W is a Segal object in Θ_n-spaces, so for each $m \geq 2$ the map

$$W_m \to W_1 \times_{W_0} \cdots \times_{W_0} W_1$$

is a weak equivalence in $\Theta_n\mathcal{CSS}$. Since W is assumed to be Reedy fibrant, W_m is a Θ_n-space for each $m \geq 0$ [3, 15.3.12]. Since $\Theta_n\mathcal{CSS}$ is obtained as a localized model category, and local weak equivalences between fibrant objects are levelwise weak equivalences, each Segal map above is a levelwise weak equivalence of functors $\Theta_n^{\mathrm{op}} \to \mathcal{S}Sets$, i.e., the maps as in (1) are weak equivalences of simplicial sets.

To check (2), consider $M_W^\Delta(x, y)$ for fixed $x, y \in W([0], [0])_0$. Since W is assumed to be Reedy fibrant, the right vertical map in (1.3) is a fibration between Θ_n-spaces, which are the fibrant objects in $\Theta_n\mathcal{CSS}$. Since the discrete object $\{(x, y)\}$ is also a fibrant object in $\Theta_n\mathcal{CSS}$, the pullback must be as well. It follows that $M_W^\Delta(x, y)$ is fibrant, namely, a Θ_n-space.

Conversely, suppose W satisfies conditions (1) and (2). We first want to show that W is Reedy fibrant as a functor $W \colon \Delta^{\mathrm{op}} \to \Theta_n\mathcal{CSS}$. For any $m \geq 0$, let $M_m W$ denote the m-th matching object of W; using the definition of Reedy fibration [3, 15.3.3], we need to show that the map $W_m \to M_m W$ is a fibration in $\Theta_n\mathcal{CSS}$.

Observe that $W_m = \underline{\mathrm{Map}}(\Delta[m], W)$, the functor $\Theta_n^{\mathrm{op}} \to \mathcal{S}Sets$ defined by

$$[p](c_1, \ldots, c_p) \mapsto W([m], [p](c_1, \ldots, c_p)).$$

Similarly, $M_m W = \underline{\mathrm{Map}}(\partial\Delta[m], W)$. Using the inclusion $\partial\Delta[m] \to \Delta[m]$, one can check that the map $W_m \to M_m W$ is indeed a Reedy fibration. It remains to show it is a fibration in $\Theta_n\mathcal{CSS}$, for which it suffices by [3, 15.3.13] to show that W_n is fibrant, i.e, a Θ_n-space.

We apply the right adjoint R to the inclusion functor of Segal precategory objects, or functors $\Delta^{\mathrm{op}} \to \Theta_n Sp$ with discrete space in degree zero, into all simplicial objects in $\Theta_n Sp$, where RW is the pullback

$$\begin{array}{ccc} RW & \longrightarrow & \mathrm{cosk}_0(W([0], [0])) \\ \downarrow & & \downarrow \\ W & \longrightarrow & \mathrm{cosk}_0(W_0). \end{array}$$

The essential constancy of W_0 implies that $\mathrm{cosk}_0(W([0], [0])$ is levelwise weakly equivalent to $\mathrm{cosk}_0(W_0)$, and hence $RW \to W$ is also a levelwise weak equivalence. But

$$(RW)_1 = \coprod_{(x,y)} \mathrm{map}_W(x, y),$$

which is a Θ_n-space by assumption, thus W_1 must be as well; a similar argument can be used for $n \geq 1$.

Finally, we need to check that for any $m \geq 2$ the Segal map

$$W_m \to W_1 \times_{W_0} \cdots \times_{W_0} W_1$$

is a weak equivalence in $\Theta_n \mathcal{CSS}$. We know by assumption that for any $m \geq 2$ and any object c of Θ_n^{op}, the map

$$W([m], c) \to W([1], c) \times_{W([0],c)} \cdots \times_{(W([0],c)} W([1], c)$$

is a weak equivalence of simplicial sets. It follows that the Segal map above is a levelwise weak equivalence of simplicial sets, hence also a weak equivalence in $\Theta_n \mathcal{CSS}$. □

Observe that the proof of the forward direction did not use the essential constancy condition, but it was necessary to prove the converse.

2. Correction to coskeleton computations

Here, we revise Example 7.9 of [1], correcting some mistakes in the original version.

Example 2.1. Suppose that $\mathcal{C} = \Theta_2^{\mathrm{op}}$, and let us consider the coskeleta associated to subsets T of

$$S = \{[0], [1]([0])\} \subseteq \mathrm{ob}(\Theta_2^{\mathrm{op}}).$$

We start with the case when T is the subset consisting of the object $[0]$; we denote the associated coskeleton functor by $\mathrm{cosk}_{[0]}$. Given a functor $X\colon \Theta_2^{\mathrm{op}} \to \mathcal{SS}ets$, we can use the fact that Θ_2 is built from Δ in particular ways to describe $\mathrm{cosk}_{[0]}(X)$.

First, when we evaluate at any object of the form $[q]([0], \ldots, [0])$, we can use the description of the 0-coskeleton of a simplicial space to see that

$$(\mathrm{cosk}_{[0]} X)[q]([0], \ldots, [0]) \cong X[0]^{q+1}.$$

In particular, we have

$$(\mathrm{cosk}_{[0]} X)[1]([0]) \cong X[0]^2.$$

Now, we can make use of the simplicial structure built into the objects $[1]([c])$ to observe that

$$(\mathrm{cosk}_{[0]} X)[1]([c]) \cong X[0]^2,$$

and indeed one can check that

$$(\mathrm{cosk}_{[0]} X)[q]([c_1], \ldots, [c_q]) \cong \left(X[0]^{q+1}\right)$$

for any q.

Now, let us consider instead the case when T is the subset containing only the object $[1]([0])$. In this situation, the simplicial 0-coskeleton appears in the objects $[1]([c])$, for any $c \geq 0$, in that

$$(\mathrm{cosk}_{[1]([0])} X)[1]([c]) \cong X[1]([0])^{c+1}.$$

At the object $[0]$, we must have

$$(\mathrm{cosk}_{[1]([0])} X)[0] \cong \Delta[0].$$

For the object $[1]([1])$, we must get

$$(\mathrm{cosk}_{[1]([0])} X)[1]([1]) \cong X[1]([0]) \times X[1][0];$$

a general formula for evaluating at objects $[q]([c_1], \ldots, [c_q])$ quickly becomes more complicated.

Finally, we consider the coskeleton associated to S itself. Here, we get

$$(\operatorname{cosk}_S X)[0] = X[0]$$

and

$$(\operatorname{cosk}_S X)[1]([0]) = X[1]([0]).$$

It is not hard to check that

$$(\operatorname{cosk}_S X)[q]([0], \ldots, [0]) \cong X[1]([0]) \times_{X[0]} \cdots \times_{X[0]} X[1]([0]).$$

We leave the descriptions upon evaluating at a general $[q]([c_1], \ldots, [c_q])$ to the reader.

References

[1] Julia E. Bergner, Discreteness and completeness for Θ_n-models of (∞, n)-categories, *Tunis. J. Math.* 6 (2024) no. 1, 49–96.

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[3] Philip S. Hirschhorn, *Model Categories and Their Localizations, Mathematical Surveys and Monographs 99*, AMS, 2003.

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