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Rigidity in equivariant algebraic K-theory

Niko Naumann and Charanya Ravi

If (R, I) is a henselian pair with an action of a finite group G and $n \geq 1$ is an integer coprime to $|G|$ such that $n \cdot |G| \in R^*$, then the reduction map of mod-n equivariant K-theory spectra

$$K^G(R)/n \xrightarrow{\simeq} K^G(R/I)/n$$

is an equivalence. We prove this by revisiting the recent proof of nonequivariant rigidity by Clausen, Mathew, and Morrow.

1. Introduction and statement of result

Rigidity is a fundamental feature of algebraic K-theory with finite coefficients which was established by Suslin [1983] for extensions of algebraically closed fields, and by Gabber and Gillet–Thomason [Gillet and Thomason 1984] for geometric henselian local rings. In [Gabber 1992], inspired by previous results of Suslin [1984] for henselian valuation rings of dimension one, Gabber proved a rigidity theorem for algebraic K-theory with finite coefficients for general henselian pairs:

Theorem 1.1 (Gabber). *If (R, I) is a henselian pair and $n \geq 1$ is an integer such that $n \in R^*$, then*

$$K(R)/n \xrightarrow{\simeq} K(R/I)/n$$

is an equivalence.

In all these results, the coefficients are assumed to be coprime to the characteristic. In [Clausen et al. 2018], the authors established the most comprehensive rigidity statement to date addressing the case of coefficients not necessarily coprime to the characteristic. To formulate it, we denote by K^{inv} the fiber of the cyclotomic trace $K \to \mathrm{TC}$. Then their result [Clausen et al. 2018, Theorem A] reads as follows:

Theorem 1.2 (Clausen, Mathew, Morrow). *If (R, I) is a henselian pair and $n \geq 1$ is an integer, then the reduction map*

$$K^{\mathrm{inv}}(R)/n \xrightarrow{\simeq} K^{\mathrm{inv}}(R/I)/n$$

is an equivalence.

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The purpose of the present note is to generalize this result to an equivariant situation for an action of a finite, abstract group G.

Given a commutative ring R with an action of G, there is associated the twisted group ring $R \wr G$; see Section 2 for a reminder. Our main theorem is below:

Theorem 1.3. *If the finite group G acts on the henselian pair (R, I), $|G| \in R^*$, and $n \geq 1$ is an integer coprime to $|G|$, then the reduction map*

$$K^{\mathrm{inv}}(R \wr G)/n \xrightarrow{\simeq} K^{\mathrm{inv}}((R/I) \wr G)/n$$

is an equivalence.

The more traditional invariant in equivariant algebraic K-theory is the spectrum $K^G(R)$, defined to be the connective K-theory of the exact category of finitely generated projective R-modules together with a semilinear G-action. We deduce the next result about this.

Corollary 1.4. *Assume in the situation of Theorem 1.3 that, in addition, $n \in R^*$ holds. Then the reduction map*

$$K^G(R)/n \xrightarrow{\simeq} K^G(R/I)/n$$

is an equivalence.

Proof. Since $n \in R^*$, the TC-term in the definition of $K^{\mathrm{inv}}(R \wr G)$ vanishes mod n, i.e., $K^{\mathrm{inv}}(R \wr G)/n \simeq K(R \wr G)/n$. Since $|G| \in R^*$, a finitely generated projective left $R \wr G$-module is the same thing as a finitely generated projective R-module with a semilinear G-action; hence $K(R \wr G) \simeq K^G(R)$, and similarly with R replaced with R/I. □

Remark 1.5.
- The appearance of $R \wr G$ might seem a bit spurious since all our results assume $|G| \in R^*$, which forces $K(R \wr G) \simeq K^G(R)$ (and similarly for TC). Since however several of the intermediate results work without assuming that $|G| \in R^*$, we decided to phrase things in terms of $R \wr G$.
- The referee points out that Theorem 1.3 might lend credibility to the following expectation about rigidity: If (A, I) is a henselian pair, $n \geq 1$ an integer and B a finite A-algebra, then $K^{\mathrm{inv}}(B)/n \to K^{\mathrm{inv}}(B/IB)/n$ should be an equivalence.

Corollary 1.4 is a generalization of Theorem 1.1 for equivariant algebraic K-theory. Rigidity results for equivariant algebraic K-theory have been previously studied for henselian local rings with trivial group actions (but for more general algebraic groups) in [Yagunov and Østvær 2009; Krishna 2010] and in [Yagunov and Østvær 2009] and [Tabuada 2018] for extensions of algebraically closed fields and extensions of separably closed fields, respectively. In [Heller et al. 2018], Corollary 1.4 was proved in the geometric case, assuming that G is abelian and that k contains $|G|$-th roots of unity.

The proofs of our results are direct generalizations of those of [Clausen et al. 2018]. We made an effort to make this paper reasonably self-contained, which results in repeating some arguments from [Clausen et al. 2018].

We conclude the introduction with an overview of the sections. In Section 2, we establish the equivariant generalization of the key finiteness property, called pseudocoherence, isolated in [Clausen et al. 2018]. This allows us to generalize equivariant rigidity from certain nice geometric situations to general henselian pairs. In Section 3 we establish a sufficient supply of equivariant rigidity in nice situations (see Proposition 3.1) by combining the nonequivariant result with decomposition results of Vistoli and Tabuada–Van den Bergh. Section 4 collects further technical results. Finally, Section 5 assembles the pieces into a proof of Theorem 1.3.

2. G-projective pseudocoherence

The aim of this section is to establish the equivariant generalizations of the finiteness properties [Clausen et al. 2018, Propositions 4.21, 4.25] of algebraic K-theory and of topological cyclic homology with finite coefficients. Fix a finite group G throughout.

Let R be a commutative ring and $I \subset R$ an ideal. Recall that the pair (R, I) is called a henselian pair if for every $f(t) \in R[t]$, $\bar{a} \in R/I$, such that $\bar{a}$ is a simple root of $\bar{f}(t) \in (R/I)[t]$, there exists $a \in R$ such that $a \mapsto \bar{a}$ and $f(a) = 0$. By a result of Gabber [1992, Corollary 1], the property of (R, I) being a henselian pair depends only on the ideal I, regarded as a nonunital ring, and not on R. We now briefly recall the definition of nonunital henselian algebras. For a detailed discussion see [Clausen et al. 2018, Section 3].

For a commutative ring R, a nonunital R-algebra is an R-module I endowed with a multiplication $I \otimes_R I \to I$ which is associative and commutative. A nonunital R-algebra I is said to be henselian if for every $n \geq 0$ and every $g(t) \in I[t]$ of degree at most n, the polynomial $f(t) = t(1+t)^n + g(t)$ has a (necessarily unique) root in I. Let $\mathrm{Ring}_R^{\mathrm{nu,h}}$ denote the category of nonunital, henselian R-algebras.

Definition 2.1. We denote by $\mathrm{Ring}_R^{\mathrm{nu,h},G}$ the category of G-objects in $\mathrm{Ring}_R^{\mathrm{nu,h}}$.

To ease the notation, we abbreviate $\mathrm{Ring}^{\mathrm{nu,h},G} := \mathrm{Ring}_{\mathbb{Z}}^{\mathrm{nu,h},G}$.

It is observed in [Clausen et al. 2018, Remark 3.10] that the category $\mathrm{Ring}_R^{\mathrm{nu,h}}$ is bicomplete, and that the forgetful functor

$$R : \mathrm{Ring}_R^{\mathrm{nu,h}} \to \mathrm{Sets}$$

to sets is a conservative right adjoint which commutes with sifted colimits.[1]

[1]Equivalently, as the categories are discrete, it commutes with filtered colimits and split coequalizers.

Denoting by

$$F_R : \mathrm{Sets} \to \mathrm{Ring}_R^{\mathrm{nu,h}}$$

its left-adjoint, this is remarked to imply that the subcategory $(\mathrm{Ring}_R^{\mathrm{nu,h}})_\Sigma \subseteq \mathrm{Ring}_R^{\mathrm{nu,h}}$ of compact projective objects is the idempotent completion of the full subcategory spanned by the free objects

$$F_R(n) := F_R(\{1, \dots, n\}) \quad (n \geq 0).$$

Moreover, $F_R(n)$ is identified in [Clausen et al. 2018, Example 3.9] as the ideal generated by the variables $X_1, \dots, X_n$ in the R-algebra given by the henselization of $R[X_1, \dots, X_n]$ along the ideal $(X_1, \dots, X_n)$.

This generalizes to the equivariant setting as follows: The category $\mathrm{Ring}^{\mathrm{nu,h},G}$ is bicomplete and the forgetful functor

$$R' : \mathrm{Ring}_R^{\mathrm{nu,h},G} \to \mathrm{Ring}_R^{\mathrm{nu,h}}$$

is a conservative right-adjoint which commutes with all colimits. This is clear by thinking of $\mathrm{Ring}^{\mathrm{nu,h},G}$ as the category of presheaves on G with values in $\mathrm{Ring}^{\mathrm{nu,h}}$. Consequently, denoting the left-adjoint of R' by

$$F'_R : \mathrm{Ring}_R^{\mathrm{nu,h}} \to \mathrm{Ring}_R^{\mathrm{nu,h},G},$$

and letting $F''_R := F'_R \circ F_R$, the subcategory $(\mathrm{Ring}_R^{\mathrm{nu,h},G})_\Sigma \subseteq \mathrm{Ring}_R^{\mathrm{nu,h},G}$ of compact projective objects is the idempotent completion of the full subcategory spanned by the free objects $F''_R(n) := F''_R(\{1, \dots, n\})$ $(n \geq 0)$. These can be identified explicitly:

Proposition 2.2. *For every $n \geq 0$, $F''_R(n)$ is the ideal generated by the variables $X_{\sigma,i}$ $(\sigma \in G, 1 \leq i \leq n)$ in the R-algebra given by the henselization of the polynomial R-algebra $R[X_{\sigma,i} \mid \sigma \in G, 1 \leq i \leq n]$ along the ideal $(X_{\sigma,i})$, and G-action determined by $\sigma(x_{\tau,i}) = x_{\sigma\tau,i}$.*

Said a bit more invariantly, $F''_R(n)$ is the henselization along the origin of the affine R-space afforded by the direct sum of n copies of the regular representation of G over R.

Proof of Proposition 2.2. Since henselization is a left-adjoint, it suffices to see the analogous statement before henselization. Then using the equivalence between nonunital R-algebras and augmented R-algebras, the claim follows because the augmented R-algebra with G-action $R[X_{\sigma,i} \mid \sigma \in G, 1 \leq i \leq n]$ has the required mapping property. □

For every $N \geq 1$, we denote by

$$[N] : F''_R(n) \to F''_R(n)$$

the "multiplication-by-N map", namely the unique map in $\mathrm{Ring}_R^{\mathrm{nu,h},G}$ which, under the identification of Proposition 2.2, maps every $X_{\sigma,i}$ to $NX_{\sigma,i}$.

Proposition 2.3. *For fixed $M \geq 1$ and $n \geq 0$, we have an isomorphism in* $\mathrm{Ring}_{\mathbb{Z}[1/M]}^{\mathrm{nu,h},G}$

$$\mathrm{colim}_{(N,M)=1} F''_{\mathbb{Z}[1/M]}(n) \simeq F''_{\mathbb{Q}}(n),$$

the (filtered) colimit being taken along the multiplication maps $[N]$ for all N coprime to M, partially ordered by divisibility.

Proof. This is proved exactly as in the special case $M = 1$, $G = \{e\}$, which is due to Gabber (see [Clausen et al. 2018, Corollary 3.20]). We leave the details to the reader. □

Recall the twisted group ring (e.g., [Curtis and Reiner 1981, §28]): If R is a commutative ring with a (left) G-action, then the twisted group ring $R \wr G$ is the finite free R-module on the set $\{e_\sigma : \sigma \in G\}$ with multiplication determined by $(re_\sigma)(r'e_\tau) = r\sigma(r')e_{\sigma\tau}$. This construction is functorial in R. It is rigged such that the datum of a left $R \wr G$-module is equivalent to the datum of an R-module together with a *semi*linear G-action. Observe that when the G-action on R is trivial, this construction gives the usual group ring, i.e., $R \wr G = R[G]$ in this case.

For an associative, unital ring A, we denote by $K(A)$ the connective K-theory spectrum of the category of finitely generated projective left A-modules; see [Quillen 1973]. Given any $I \in \mathrm{Ring}^{\mathrm{nu,h},G}$, we denote by $\mathbb{Z} \ltimes I$ the ring with G-action obtained from I by adjoining a unit (necessarily with trivial G-action). The augmentation $\mathbb{Z} \ltimes I \to \mathbb{Z}$ is G-equivariant and thus induces an augmentation

$$p : (\mathbb{Z} \ltimes I) \wr G \to \mathbb{Z}[G].$$

We need the following equivariant generalization of [Clausen et al. 2018, Lemma 4.20].

Proposition 2.4. *Given $I \in \mathrm{Ring}^{\mathrm{nu,h},G}$, the map $p^* : K_0((\mathbb{Z} \ltimes I) \wr G) \xrightarrow{\simeq} K_0(\mathbb{Z}[G])$ is an isomorphism.*

Proof. This is a special case of Proposition 4.6. □

We denote by Sp the ∞-category of spectra. We recall from [Clausen et al. 2018, Definition 4.4] the notions of perfectness and pseudocoherence of spectrum-valued functors on a category relative to a subcategory: Given a small full subcategory $\mathcal{D}$ of a locally small category $\mathcal{C}$, a functor $F : \mathcal{C} \to \mathrm{Sp}$ is called $\mathcal{D}$-perfect if F belongs to the thick subcategory generated by the functors $\{\Sigma^\infty_+ \mathrm{Hom}_{\mathcal{C}}(D, -) \mid D \in \mathcal{D}\}$ in the presentable, stable ∞-category $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp})$. A functor $F \in \mathrm{Fun}(\mathcal{C}, \mathrm{Sp})$ is said to be $\mathcal{D}$-pseudocoherent if for each $n \in \mathbb{Z}$, there exists a $\mathcal{D}$-perfect functor F_n and a map $F_n \to F$ such that $\tau_{\leq n} F_n(C) \to \tau_{\leq n} F(C)$ is an equivalence for all $C \in \mathcal{C}$. In the particular case when $\mathcal{D} = (\mathrm{Ring}^{\mathrm{nu,h}})_\Sigma \subseteq \mathcal{C} = \mathrm{Ring}^{\mathrm{nu,h}}$, F is called projectively pseudocoherent; see [Clausen et al. 2018, Definition 4.12, (2)]. We pose the immediate equivariant generalization of this as a definition.

Definition 2.5. A functor $F : \mathrm{Ring}_R^{\mathrm{nu,h},G} \to \mathrm{Sp}$ is called G-projectively pseudocoherent (G-pscoh for short), if it is $(\mathrm{Ring}_R^{\mathrm{nu,h},G})_\Sigma$-pseudocoherent.

Our first aim then is to establish the following generalization of [Clausen et al. 2018, Proposition 4.21].

Proposition 2.6. *The functor* $\mathrm{Ring}^{\mathrm{nu,h},G} \to \mathrm{Sp}$, $I \mapsto K((\mathbb{Z} \ltimes I) \wr G)$ *is* G*-pscoh.*

Proof. Using the fiber sequence of functors

$$\tau_{\geq 1} K((\mathbb{Z} \ltimes (-)) \wr G) \to K((\mathbb{Z} \ltimes (-)) \wr G) \to \tau_{\leq 0} K((\mathbb{Z} \ltimes (-)) \wr G) = K_0((\mathbb{Z} \ltimes (-)) \wr G)$$

and the fact that G-pscoh functors form a thick subcategory [Clausen et al. 2018, Proposition 4.8, (1)], it suffices to see separately the G-projective pseudocoherence of $\tau_{\geq 1} K((\mathbb{Z} \ltimes (-)) \wr G)$ and of $K_0((\mathbb{Z} \ltimes (-)) \wr G)$.

For the latter, Proposition 2.4 yields an isomorphism $K_0((\mathbb{Z} \ltimes (-)) \wr G) \simeq K_0(\mathbb{Z}[G])$ to the constant functor with value the finitely generated abelian group $K_0(\mathbb{Z}[G])$; see [Kuku 2007, Theorem 2.2.1]. This settles the claim for this term.

To see that the other term is G-pscoh, we use the criterion [Clausen et al. 2018, Propositions 4.10 and 4.11] to reduce to seeing that the functor

$$H\mathbb{Z} \otimes \Sigma_+^\infty \Omega^\infty \tau_{\geq 1} K((\mathbb{Z} \ltimes (-)) \wr G)$$

is G-pscoh. It is well known (see [Weibel 2013, Chapter IV, §1]) that this functor is equivalent to $C_*(\mathrm{BGL}((\mathbb{Z} \ltimes (-)) \wr G); \mathbb{Z})$, the complex of integral chains on the classifying space of the infinite general linear group. We now use homology stability as given by [van der Kallen 1980, Theorem in Section 4.11] for the associative ring $A(-) := (\mathbb{Z} \ltimes (-)) \wr G$. To do so, we need to see that the stable range of $A(-)$ is bounded independently of the argument $- \in \mathrm{Ring}^{\mathrm{nu,h},G}$. Firstly, it is easy to see that dividing out a radical ideal does not change the stable range (see [Lam 1999, p. 32] and [Weibel 2013, Chapter I, Exercise 1.12(v)]), and at the beginning of the proof of Proposition 4.6 we will see that $(-) \wr G$ is a radical ideal in $A(-)$ with quotient ring $\mathbb{Z}[G]$. This already gives the independence of the stable range of $A(-)$ of the argument $(-)$, and since $\mathbb{Z}[G]$ is finite over its central subring $\mathbb{Z}$, this is bounded by (in fact, equal to) the stable range of $\mathbb{Z}$ (according to Bass's stable range theorem [1968, Chapter V, Theorem 3.5]). We conclude that for every $n \geq 1$ the obvious map on truncations

$$\tau_{\leq n} C_*(\mathrm{BGL}_{2n+1}((\mathbb{Z} \ltimes (-)) \wr G); \mathbb{Z}) \to \tau_{\leq n} C_*(\mathrm{BGL}((\mathbb{Z} \ltimes (-)) \wr G); \mathbb{Z})$$

is an equivalence. Renaming indices, this reduces us to seeing that for a fixed $n \geq 1$, the functor

$$C_*(\mathrm{BGL}_n((\mathbb{Z} \ltimes (-)) \wr G); \mathbb{Z})$$

is G-pscoh. There is a short exact sequence of groups

$$1 \to X(-) \to \mathrm{GL}_n((\mathbb{Z} \ltimes (-)) \wr G) \xrightarrow{\pi} \mathrm{GL}_n(\mathbb{Z}[G]) \to 1,$$

defining $X(-)$.[2] This gives an equivalence

$$C_*(\mathrm{BGL}_n((\mathbb{Z} \ltimes (-)) \wr G); \mathbb{Z}) \simeq (C_*(BX(-); \mathbb{Z}))_{h(\mathrm{GL}_n(\mathbb{Z}[G]))}.$$

To conclude the argument exactly as in the proof of [Clausen et al. 2018, Proposition 4.19], it remains to establish that, firstly, the functor $C_*(BX(-); \mathbb{Z})$ is G-pscoh and that, secondly, there is a finite index normal subgroup $N \subseteq \mathrm{GL}_n(\mathbb{Z}[G])$ such that its classifying space BN is equivalent to a finite CW-complex. The first claim follows as in [loc. cit.], because $X(I) \simeq I^{|G|\cdot n^2}$ (as sets), and the second claim follows from work of Borel and Serre, specifically [Serre 1971, Section 2.4, Théorème 4 and Section 1.5, Proposition 10], if we can show that $\mathrm{GL}_n(\mathbb{Z}[G])$ is an arithmetic subgroup of a suitable reductive group $\mathcal{G}$ over $\mathbb{Q}$. Indeed, one can take for $\mathcal{G}$ the group of units of the $\mathbb{Q}$-algebra $M_n(\mathbb{Q}[G])$: it is clear that

$$\mathrm{GL}_n(\mathbb{Z}[G]) \subseteq \mathcal{G}(\mathbb{Q}) = \mathrm{GL}_n(\mathbb{Q}[G])$$

is an arithmetic subgroup, and since $\mathbb{Q}[G] \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}[G]$ is a product of full matrix rings over $\mathbb{C}$, the group $\mathcal{G} \otimes_{\mathbb{Q}} \mathbb{C}$ is a finite product of $\mathrm{GL}_{i,\mathbb{C}}$ for various i, and hence is (connected and) reductive. □

The following generalization of [Clausen et al. 2018, Proposition 4.25] is even more immediate.

Proposition 2.7. *For every prime p, the functor*

$$\mathrm{Ring}^{\mathrm{nu,h},G} \to \mathrm{Sp}, \quad I \mapsto \mathrm{TC}((\mathbb{Z} \ltimes I) \wr G)/p$$

is G-pscoh.

Proof. This is identical to [loc. cit.], and we leave the details to the reader. Recall at least that the core part of the argument, namely [Clausen et al. 2018, Proposition 2.19], is a result about $\mathrm{TC}(-)/p$ considered on the category of cyclotomic spectra, which applies equally well to the case at hand. □

Recall that we write K^{inv} for the fiber of the cyclotomic trace $K \to \mathrm{TC}$. We introduce a relative term $K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G)$ to sit in a fiber sequence

$$K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G) \to K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G) \to K^{\mathrm{inv}}(\mathbb{Z} \wr G) = K^{\mathrm{inv}}(\mathbb{Z}[G]).$$

Combining Propositions 2.6 and 2.7 yields the following, which is the finiteness result to be used in the proof of Theorem 1.3.

[2] To see that π is onto, recall that the augmentation $(\mathbb{Z} \ltimes (-)) \wr G \to \mathbb{Z}[G]$ is *split* surjective.

Proposition 2.8. *For every prime p, the functor*

$$\mathrm{Ring}^{\mathrm{nu,h},G} \to \mathrm{Sp}, \quad I \mapsto K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G)/p$$

is G-pscoh.

3. A geometric special case

The purpose of this section is to establish a geometric special case of our main result, Theorem 1.3. This equivariant rigidity result will follow from its nonequivariant special case [Clausen et al. 2018, Theorem A] together with decomposition results of Vistoli and Tabuada–Van den Bergh [Tabuada and Van den Bergh 2018]. To formulate it, fix for the rest of this section a finite group G, a field k of characteristic not dividing $|G|$,[3] and a prime p not dividing $|G|$ (but possibly equal to the characteristic of k). Let X be an affine, smooth k-algebra with a G-action and assume given a rational point $x \in X(k)$ fixed by G. Then G acts canonically on the henselization $\mathcal{O}^h_{X,x}$ of the local ring $\mathcal{O}_{X,x}$, and the canonical map $\pi : \mathcal{O}^h_{X,x} \to k$ to the residue field is G-equivariant (for k endowed with the trivial G-action). Hence it induces a map on twisted group rings $\mathcal{O}^h_{X,x} \wr G \to k \wr G = k[G]$. The result then is the following.

Proposition 3.1. *In the above situation, the map*

$$K^{\mathrm{inv}}(\mathcal{O}^h_{X,x} \wr G)/p \xrightarrow{\simeq} K^{\mathrm{inv}}(k[G])/p \tag{3.2}$$

induced by π is an equivalence.

Proof. We start by setting the stage to apply [Tabuada and Van den Bergh 2018]. We let $E := \pi_*(K^{\mathrm{inv}}(-)/p)$, and observe that this is an additive invariant taking values in $\mathbb{Z}[1/|G|]$-modules and commuting with filtered colimits; for algebraic K-theory, this is classical and for $\mathrm{TC}(-)/p$ it follows from [Clausen et al. 2018, Theorem 2.7]. Now, [Tabuada and Van den Bergh 2018, Remark 1.3(ii) and (iii)] implies that

$$E([X/G]) \xrightarrow{\simeq} \Big(\bigoplus_{\sigma \subseteq G \text{ cyclic}} \widetilde{E}(X^\sigma \times \mathrm{Spec}(k[\sigma])) \Big)^G, \tag{3.3}$$

where $X^\sigma \subseteq X$ is the subscheme fixed by σ, and $\widetilde{E}$ refers to a certain functorially defined direct summand of E (depending on σ). Since we do not require knowledge of the exact shape of that summand, we do not review its definition here.

We observe that the G-fixed point $x \in X(k)$ determines a map

$$\bar{x} : [\mathrm{Spec}(k)/G] \to [X/G]$$

[3] By convention, this condition is satisfied if k is of characteristic zero.

such that $E(\bar{x})$ participates in a commutative diagram

$$\begin{array}{ccc} E([X/G]) & \xrightarrow{\simeq\ (3.3)} & \Big(\bigoplus_{\sigma\subseteq G \text{ cyclic}} \widetilde{E}(X^\sigma \times \operatorname{Spec}(k[\sigma]))\Big)^G \\ \Big\downarrow{\scriptstyle E(\bar{x})} & & \Big\downarrow{\scriptstyle \bigoplus_\sigma \widetilde{E}(x_\sigma \times \mathrm{id})} \\ E([\operatorname{Spec}(k)/G]) & \xrightarrow{\simeq} & \Big(b \bigoplus_{\sigma\subseteq G \text{ cyclic}} \widetilde{E}(\operatorname{Spec}(k[\sigma]))\Big)^G \end{array} \tag{3.4}$$

where x_σ denotes the unique factorization of x through $X^\sigma \subseteq X$.

Next we want to pass to henselizations. To do this, we observe that everywhere in the above argument, one can replace $([X/G], x)$ with a pointed étale neighborhood (Y, y) such that $\kappa(x) \xrightarrow{\simeq} \kappa(y)$ is an isomorphism on residue fields. We obtain a commutative diagram generalizing (3.4):

$$\begin{array}{ccc} E([Y/G]) & \xrightarrow{\simeq} & \Big(\bigoplus_{\sigma\subseteq G \text{ cyclic}} \widetilde{E}(Y^\sigma \times \operatorname{Spec}(k[\sigma]))\Big)^G \\ \Big\downarrow{\scriptstyle E(\bar{y})} & & \Big\downarrow{\scriptstyle \bigoplus_\sigma \widetilde{E}(y_\sigma \times \mathrm{id})} \\ E([\operatorname{Spec}(k)/G]) & \xrightarrow{\simeq} & \Big(\bigoplus_{\sigma\subseteq G \text{ cyclic}} \widetilde{E}(\operatorname{Spec}(k[\sigma]))\Big)^G \end{array} \tag{3.5}$$

Passing to the filtered colimit of all such (Y, y) and recalling that henselization commutes with the closed immersions $X^\sigma \subseteq X$ (and more generally with integral extensions [Stacks 2005–, tag 0DYE]), we obtain

$$\begin{array}{ccc} E([\operatorname{Spec}(\mathcal{O}^h_{X,x})/G]) & \xrightarrow{\simeq} & \Big(\bigoplus_{\sigma\subseteq G \text{ cyclic}} \widetilde{E}(\operatorname{Spec}(\mathcal{O}^h_{X^\sigma,x}) \times \operatorname{Spec}(k[\sigma]))\Big)^G \\ \Big\downarrow{\scriptstyle E(\iota)} & & \Big\downarrow{\scriptstyle \bigoplus_\sigma \widetilde{E}(\iota_\sigma \times \mathrm{id})} \\ E([\operatorname{Spec}(k)/G]) & \xrightarrow{\simeq} & \Big(\bigoplus_{\sigma\subseteq G \text{ cyclic}} \widetilde{E}(\operatorname{Spec}(k[\sigma]))\Big)^G \end{array} \tag{3.6}$$

Here, $\iota_\sigma : \operatorname{Spec}(k) \hookrightarrow \operatorname{Spec}(\mathcal{O}^h_{X^\sigma,x})$ and $\iota : [\operatorname{Spec}(k)/G] \hookrightarrow [\operatorname{Spec}(\mathcal{O}^h_{X,x})/G]$ are (induced by) the canonical projection to the residue fields. Since each ι_σ is a closed immersion with henselian defining ideal, so is each $\iota_\sigma \times \mathrm{id}_{\operatorname{Spec}(k[\sigma])}$, and by [Clausen et al. 2018, Theorem A], every map $\widetilde{E}(\iota_\sigma \times \mathrm{id})$ is an isomorphism, and hence so is $E(\iota)$.

To equate $E(\iota)$ with $\pi_*((3.2))$, and thus to conclude the proof, it remains to recall that $E(-) = \pi_*(K^{\mathrm{inv}}(-)/p)$ and that since the order $|G|$ is invertible, a finitely generated projective module with a semilinear G-action is the same thing as a finitely generated projective left module over the twisted group ring, so that we have an equivalence of ∞-categories of perfect modules

$$\mathrm{Perf}([\mathrm{Spec}(\mathcal{O}^h_{X,x})/G]) \simeq \mathrm{Perf}(\mathcal{O}^h_{X,x} \wr G),$$

and similarly with $\mathcal{O}^h_{X,x}$ replaced by k. □

4. Nil-invariance, excision and exactness

Nil-invariance.

Proposition 4.1. *Let G be a finite group and $\pi : R \to R'$ a surjective homomorphism of commutative rings with a G-action such that* $\ker(\pi)$ *is nilpotent. Then* $K^{\mathrm{inv}}(R \wr G) \xrightarrow{\simeq} K^{\mathrm{inv}}(R' \wr G)$ *is an equivalence.*

Proof. This will follow from [Dundas et al. 2013, Chapter VII, Theorem 0.0.2] if we can show that the kernel of (the obviously surjective) ring homomorphism $\pi \wr G : R \wr G \to R' \wr G$ is nilpotent. However, an immediate computation shows that for every $n \geq 0$ we have

$$(\ker(\pi \wr G))^n \subseteq (\ker(\pi))^n \wr G. \qquad \square$$

Excision. Assume that

$$\begin{array}{ccc} R & \longrightarrow & S \\ \big\downarrow & & \big\downarrow g \\ R' & \longrightarrow & S' \end{array} \tag{4.2}$$

is a Milnor square of commutative rings, i.e., a pull-back diagram of rings with g surjective; see [Bass 1968, Chapter IX, §5] for an early account and [Land and Tamme 2019] for a current development. If, in addition, a finite group G acts on (4.2), then the induced square of twisted group rings

$$\begin{array}{ccc} R \wr G & \longrightarrow & S \wr G \\ \big\downarrow & & \big\downarrow g\wr G \\ R' \wr G & \longrightarrow & S' \wr G \end{array} \tag{4.3}$$

is clearly still a Milnor square. Denoting by $\mathbb{K}$ nonconnective algebraic K-theory and by $\mathbb{K}^{\mathrm{inv}}$ the fiber of the cyclotomic trace $\mathbb{K} \to \mathrm{TC}$, we then deduce the following from [Land and Tamme 2019, Theorem 3.3].

Proposition 4.4. *In the above situation,*

$$\begin{array}{ccc} \mathbb{K}^{\mathrm{inv}}(R \wr G) & \longrightarrow & \mathbb{K}^{\mathrm{inv}}(S \wr G) \\ \big\downarrow & & \big\downarrow \\ \mathbb{K}^{\mathrm{inv}}(R' \wr G) & \longrightarrow & \mathbb{K}^{\mathrm{inv}}(S' \wr G) \end{array} \tag{4.5}$$

is a pull-back square.

To pass to connective K-theory here, we need the equivariant generalization of [Clausen et al. 2018, Corollary 4.34], namely Proposition 4.6 below.

For an associative, unital ring A, we denote by

$$\mathbb{P}\mathrm{roj}(A)$$

the set of isomorphism classes of finitely generated projective left A-modules.

We start by establishing the equivariant generalization of [Clausen et al. 2018, Lemma 4.20]:

Proposition 4.6. *Let (R, I) be a henselian pair and G a finite group acting on (R, I). Then the obvious homomorphism*

$$K_0(R \wr G) \xrightarrow{\simeq} K_0((R/I) \wr G)$$

is an isomorphism.

Proof. To see that the map is injective, according to [Bass 1968, Chapter IX, Proposition 1.3] it suffices to check that the kernel of the projection $R \wr G \to (R/I) \wr G$, namely

$$I \wr G := \left\{ \sum_{\sigma \in G} a(\sigma) e_\sigma \;\middle|\; a(\sigma) \in I \right\} \subseteq R \wr G$$

is contained in the radical of $R \wr G$. Otherwise, $I \wr G$ was not contained in some maximal left ideal $\mathfrak{n} \subseteq R \wr G$. Then the subset

$$I \wr G + \mathfrak{n} := \{x + y \mid x \in I \wr G,\ y \in \mathfrak{n}\} \subseteq R \wr G$$

was a left-ideal properly containing $\mathfrak{n}$, and hence

$$I \wr G + \mathfrak{n} = R \wr G. \tag{4.7}$$

We consider $R = Re_e \subseteq R \wr G$ as a (noncentral!) subring. Then (4.7) holds as an equality of R-modules, and since $I \wr G = I(R \wr G)$ and $R \wr G$ is a finite (and free) R-module, Nakayama's lemma[4] implies that $\mathfrak{n} = R \wr G$, a contradiction which completes the proof of injectivity.

To see the surjectivity, we establish the stronger claim that the reduction map

$$\mathbb{P}\mathrm{roj}(R \wr G) \twoheadrightarrow \mathbb{P}\mathrm{roj}((R/I) \wr G) \tag{4.8}$$

is surjective. Write $\bar{R} := R/I$ and fix some $\overline{M} \in \mathbb{P}\mathrm{roj}(\bar{R} \wr G)$. We first descend everything to a situation of finite type over the integers. The ring with G-action $R = \bigcup_\alpha R_\alpha$ is the union of its finitely generated, G-stable subrings $R_\alpha \subseteq R$. Accordingly, we also have $\bar{R} = \bigcup_\alpha (R_\alpha/I_\alpha) =: \bigcup_\alpha \bar{R}_\alpha$ for $I_\alpha := R_\alpha \cap I$. Since then also $\bar{R} \wr G = \bigcup_\alpha (\bar{R}_\alpha \wr G)$, the given $\overline{M}$ descends to some $\overline{M}_\alpha \in \mathbb{P}\mathrm{roj}(\bar{R}_\alpha \wr G)$ for suitably large indices α.

[4]Recall that I is contained in the radical of R.

Write $(R_\alpha, I_\alpha)^h$ for the henselization of R_α along $I_\alpha \subseteq R_\alpha$, and note that G naturally acts on $(R_\alpha, I_\alpha)^h$, because henselization is functorial. It suffices to lift the given $\overline{M}_\alpha \in \mathbb{P}\mathrm{roj}(\overline{R}_\alpha \wr G)$ to some element of $\mathbb{P}\mathrm{roj}((R_\alpha, I_\alpha)^h \wr G)$, because the inclusion $R_\alpha \subseteq R$ factors through $(R_\alpha, I_\alpha)^h$ G-equivariantly.

We next claim an inclusion of (G-invariant) ideals for all sufficiently large $M \gg 0$, namely

$$I_\alpha^M \subseteq I_\alpha^G \cdot R_\alpha \subseteq I_\alpha \subseteq R_\alpha. \tag{4.9}$$

Indeed, the second inclusion is obvious, and the first one follows from the fact that I_α is finitely generated together with the below relation, valid for every $x \in I_\alpha$:

$$0 = \prod_{g\in G}(x - g(x)) =: x^{|G|} + \sum_{i=0}^{|G|-1} a_i x^i \quad \text{with } a_i \in I_\alpha \cap R_\alpha^G = I_\alpha^G,$$

which implies that $x^{|G|} \in I_\alpha^G \cdot R_\alpha$.

By (4.9), the kernel of the projection

$$R_\alpha / I_\alpha^G \cdot R_\alpha \simeq R_\alpha^G / I_\alpha^G \otimes_{R_\alpha^G} R_\alpha \twoheadrightarrow R_\alpha / I_\alpha = \overline{R}_\alpha$$

is nilpotent, and an easy calculation then shows that so is the kernel of the projection

$$(R_\alpha / I_\alpha^G \cdot R_\alpha) \wr G \twoheadrightarrow \overline{R}_\alpha \wr G$$

(see the proof of Proposition 4.1).

By [Bass 1968, Chapter III, Corollary 2.4 and Proposition 2.12] then, we can lift the given $\overline{M}_\alpha \in \mathbb{P}\mathrm{roj}(\overline{R}_\alpha \wr G)$ to some $\overline{M}'_\alpha \in \mathbb{P}\mathrm{roj}((R_\alpha / I_\alpha^G \cdot R_\alpha) \wr G)$.

As a final piece of preparation, we need to see what happens to the G-invariants under henselization. Since $R_\alpha^G \subseteq R_\alpha$ is integral, and (4.9) shows that

$$\sqrt{I_\alpha^G \cdot R_\alpha} = \sqrt{I_\alpha},$$

[Stacks 2005–, tag 0DYE] implies that the canonical map

$$(R_\alpha^G, I_\alpha^G)^h \otimes_{R_\alpha^G} R_\alpha \simeq (R_\alpha, I_\alpha)^h \tag{4.10}$$

is an isomorphism.

We are now in a position to lift the given $\overline{M}'_\alpha \in \mathbb{P}\mathrm{roj}((R_\alpha / I_\alpha^G \cdot R_\alpha) \wr G)$ using [Greco 1969, Theorem 4.1], as follows:[5]

As our henselian pair, we take $(R_\alpha^G, I_\alpha^G)^h$, and as our algebra, $A := (R_\alpha, I_\alpha)^h \wr G$. The algebra A is a finite $(R_\alpha^G)^h$-module, because it is clearly finite over R_α^h, and (4.10) shows that R_α^h is finite over $(R_\alpha^G)^h$, because $R_\alpha^G \subseteq R_\alpha$ is finite, being both integral and of finite type.

[5]The application of this theorem here is a bit involved because in general neither is $R \wr G$ an R-algebra in any obvious way (but only an R^G-algebra), nor is $R^G \subseteq R$ finite.

We then compute the reduction of $A = (R_\alpha, I_\alpha)^h \wr G$ to be

$$\bar{A} := A/I_\alpha^G \cdot A \simeq (R_\alpha/I_\alpha^G \cdot R_\alpha) \wr G.$$

Now, [Greco 1969, Theorem 4.1] shows that the given $\bar{M}'_\alpha \in \mathbb{P}\mathrm{roj}((R_\alpha/I_\alpha^G \cdot R_\alpha) \wr G) = \mathbb{P}\mathrm{roj}(\bar{A})$ lifts to some element of $\mathbb{P}\mathrm{roj}(A) = \mathbb{P}\mathrm{roj}((R_\alpha, I_\alpha)^h \wr G)$, as desired. □

We can now start to work on the version of Proposition 4.4 for connective algebraic K-theory, at least for those diagrams (4.2) coming from suitable maps of henselian pairs. In the following, fix henselian pairs (R, I) and (S, J) with an action of the finite group G, and assume that $(R, I) \to (S, J)$ is a map of pairs which respects the G-action and maps I isomorphically to J. Then

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ S & \longrightarrow & S/J \end{array} \tag{4.11}$$

is a diagram as in (4.2), i.e., a Milnor-square with a G-action.

We then define $K(R \wr G, I \wr G)$ by the fiber sequence

$$K(R \wr G, I \wr G) \to K(R \wr G) \to K((R/I) \wr G),$$

and analogously for $K(S \wr G, J \wr G)$ and with K replaced by $\mathbb{K}$. The map of pairs $(R, I) \to (S, J)$ induces a map

$$K(R \wr G, I \wr G) \to K(S \wr G, J \wr G),$$

and similarly for $\mathbb{K}$. Recall that there is a canonical transformation $K \to \mathbb{K}$.

Proposition 4.12. *In the above situation, the diagrams*

(i)
$$\begin{array}{ccc} K(R \wr G, I \wr G) & \longrightarrow & K(S \wr G, J \wr G) \\ \downarrow & & \downarrow \\ \mathbb{K}(R \wr G, I \wr G) & \longrightarrow & \mathbb{K}(S \wr G, J \wr G) \end{array} \tag{4.13}$$

and

(ii)
$$\begin{array}{ccc} K^{\mathrm{inv}}(R \wr G, I \wr G) & \longrightarrow & K^{\mathrm{inv}}(S \wr G, J \wr G) \\ \downarrow & & \downarrow \\ \mathbb{K}^{\mathrm{inv}}(R \wr G, I \wr G) & \longrightarrow & \mathbb{K}^{\mathrm{inv}}(S \wr G, J \wr G) \end{array} \tag{4.14}$$

are pull-back squares.

Proof. To prove part (i), let $F \to \mathbb{F}$ denote the map induced by (4.13) on horizontal fibers. The claim is that this map is an equivalence. Since $K \to \mathbb{K}$ induces an isomorphism on π_k for $k \geq 0$, we have $\pi_k(\mathbb{F}) \simeq \pi_k(F)$ for $k \geq 0$. The excision theorem of Milnor–Bass–Murthy [Bass 1968, Chapter XII, Theorem 8.3] applied to the diagram obtained from (4.11) by passing to twisted group rings shows that $\pi_k(\mathbb{F}) = 0$ for all $k \leq -1$. It remains to see that $\pi_k(F) = 0$ in this range, too. Since $K_0(R \wr G) \xrightarrow{\simeq} K_0((R/I) \wr G)$ is an isomorphism (Proposition 4.6) and the maps

$$K_1(R \wr G) \to K_1((R/I) \wr G) \quad \text{and} \quad K_1(S \wr G) \to K_1((S/J) \wr G)$$

are surjections (this is true more generally for any surjective ring homomorphism with kernel contained in the radical [Bass 1968, Chapter IX, Proposition 1.3(1)]), we conclude that the fibers $K(R \wr G, I \wr G)$ and $K(S \wr G, J \wr G)$ are concentrated in degrees ≥ 1, and thus F is concentrated in degrees ≥ 0, as claimed.

Part (ii) follows from part (i) by passage to fibers over TC, because the canonical transformation $K \to \mathrm{TC}$ factors as $K \to \mathbb{K} \to \mathrm{TC}$. □

Corollary 4.15. *If (R, I) is a henselian pair with a G-action and $R \to S$ is a map of commutative rings with G-action mapping I isomorphically to an ideal $J \subseteq S$, then*

$$K^{\mathrm{inv}}(R \wr G, I \wr G) \xrightarrow{\simeq} K^{\mathrm{inv}}(S \wr G, J \wr G)$$

is an equivalence.

Proof. The diagram

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ S & \longrightarrow & S/J \end{array}$$

is a Milnor-square with G-action. Note that the pair (S, J) is also henselian by [Clausen et al. 2018, Lemma 3.18]. Therefore, an application of Proposition 4.12(ii) reduces our claim to the analogous statement with K^{inv} replaced with $\mathbb{K}^{\mathrm{inv}}$. This is then a special case of Proposition 4.4. □

These results will be used to reduce rigidity of arbitrary pairs to rigidity of those pairs of the form $(\mathbb{Z} \ltimes I, I)$ already encountered in Section 2.

Corollary 4.16. *For a fixed finite group G, there is an equivalence of spectra, functorial in the henselian pair (R, I) with G-action*

$$K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G) \xrightarrow{\simeq} K^{\mathrm{inv}}(R \wr G, I \wr G).$$

Exactness. We call a sequence $I' \to I \to \bar{I}$ in $\mathrm{Ring}^{\mathrm{nu,h},G}$ *short exact* if it is so when considered nonequivariantly, i.e., in $\mathrm{Ring}^{\mathrm{nu,h}}$, i.e., if the underlying sequence of abelian groups is short exact; see [Clausen et al. 2018, Definition 3.4]. We consider the functor

$$F : \mathrm{Ring}^{\mathrm{nu,h},G} \to \mathrm{Sp}, \quad F(I) := K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G),$$

and claim that it is exact:

Proposition 4.17. *Given a short exact sequence $I' \to I \to \bar{I}$ in $\mathrm{Ring}^{\mathrm{nu,h},G}$, then $F(I') \to F(I) \to F(\bar{I})$ is a fiber sequence.*

Proof. We consider the following commutative diagram:

$$\begin{array}{ccccc}
F(I') & \longrightarrow & F(I) & \longrightarrow & F(\bar{I}) \\
\downarrow & & \downarrow & & \downarrow \\
F(I') & \longrightarrow & K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G) & \longrightarrow & K^{\mathrm{inv}}((\mathbb{Z} \ltimes \bar{I}) \wr G) \\
 & & \downarrow & & \downarrow \\
 & & K^{\mathrm{inv}}(\mathbb{Z}[G]) & \xrightarrow{=} & K^{\mathrm{inv}}(\mathbb{Z}[G])
\end{array}$$

The top row is the one we want to recognize as a fiber sequence. The two right columns are the fiber sequences defining $F(I)$ and $F(\bar{I})$. The indicated equality implies that the upper right square is a pull-back. Hence the top row is a fiber sequence if and only if so is the second row. We verify this by observing that using Corollary 4.15 for the obvious map of pairs with G-action $(\mathbb{Z} \ltimes I', I') \to (\mathbb{Z} \ltimes I, I')$ gives

$$\begin{aligned}
F(I') &\stackrel{\mathrm{def}}{=} K^{\mathrm{inv}}((\mathbb{Z} \ltimes I') \wr G, I' \wr G) \xrightarrow{\simeq} K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I' \wr G) \\
&\stackrel{\mathrm{def}}{=} \mathrm{fiber}\big(K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G) \to K^{\mathrm{inv}}(((\mathbb{Z} \ltimes I)/I') \wr G)\big).
\end{aligned}$$

Since $(\mathbb{Z} \ltimes I)/I' \simeq \mathbb{Z} \ltimes \bar{I}$, this concludes the proof. □

5. The proof of the main result

In this section, we give the proof of our main result, Theorem 1.3, which we restate for convenience.

Theorem 5.1. *If the finite group G acts on the henselian pair (R, I), $|G| \in R^*$, and $n \geq 1$ is an integer coprime to $|G|$, then the reduction map*

$$K^{\mathrm{inv}}(R \wr G)/n \xrightarrow{\simeq} K^{\mathrm{inv}}((R/I) \wr G)/n$$

is an equivalence.

Proof. We can assume that $n = p$ is a prime (not dividing $|G|$). Since

$$K^{\mathrm{inv}}(R \wr G, I \wr G)/p \simeq K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G)/p$$

(see Corollary 4.16), our claim is that the functor

$$F : \mathrm{Ring}^{\mathrm{nu,h},G}_{\mathbb{Z}[1/|G|]} \to \mathrm{Sp}, \qquad F(I) := K^{\mathrm{inv}}((\mathbb{Z} \ltimes I) \wr G, I \wr G)/p$$

is trivial. We start by collecting properties of F that were established previously. By Proposition 2.8,

$$F \text{ is } G\text{-pscoh}. \tag{5.2}$$

By Proposition 4.17,

$$F \text{ sends short exact sequences to fiber sequences} \tag{5.3}$$

and by Proposition 4.1,

$$F \text{ vanishes on nilpotent arguments.} \tag{5.4}$$

For every prime field Ω of characteristic not dividing $|G|$,[6] recall the compact projective generators $F''_\Omega(n) \in \mathrm{Ring}_\Omega^{\mathrm{nu,h},G}$ ($n \geq 0$) from Proposition 2.2. We deduce from Proposition 3.1 that $F(F''_\Omega(n)) = 0$ for all $n \geq 0$.[7] Since by (5.2), the restriction of F to $\mathrm{Ring}_\Omega^{\mathrm{nu,h},G}$ is G-pscoh, and in particular left Kan extended from its subcategory of compact projective objects (see [Clausen et al. 2018, Lemma 4.6]), which in turn is the idempotent completion of all the $F''_\Omega(n)$, we see that for every prime field Ω of characteristic not dividing $|G|$, we have

$$F(\mathrm{Ring}_\Omega^{\mathrm{nu,h},G}) = 0. \tag{5.5}$$

We now boot-strap to see that for every $N \geq 1$, $I \in \mathrm{Ring}_{\mathbb{Z}[1/|G|]}^{\mathrm{nu,h},G}$:

$$\text{if } (N, |G|) = 1 \text{ and } NI = 0, \text{ then } F(I) = 0. \tag{5.6}$$

Since F preserves finite products, we can assume that $N = q^r$ is a prime-power (with the prime q not dividing $|G|$) and then consideration of the short exact sequence $qI \to I \to I/qI$ together with (5.3), (5.4) and (5.5) (for $\Omega = \mathbb{F}_q$) proves (5.6).

Since F is bounded below, there is an integer $d \in \mathbb{Z}$ such that

$$\pi_k F = 0 \quad \text{for every } k < d. \tag{5.7}$$

We will be done if we can show that the functor to abelian groups

$$F_0 := \pi_d F : \mathrm{Ring}_{\mathbb{Z}[1/|G|]}^{\mathrm{nu,h},G} \to \mathrm{Ab}$$

vanishes, because d being arbitrary, this will imply that $F = 0$. To see this, we will establish that

$$\begin{aligned}&\text{there is some } N \text{ coprime to } |G| \text{ such that for all } I \in \mathrm{Ring}_{\mathbb{Z}[1/|G|]}^{\mathrm{nu,h},G},\\ &\qquad\qquad F_0(NI) \to F_0(I) \text{ is the zero map.}\end{aligned} \tag{5.8}$$

Given this, using (5.3) and (5.7), we obtain an exact sequence

$$F_0(NI) \xrightarrow{0} F_0(I) \to F_0(I/NI) \to 0,$$

where $F_0(I/NI) = 0$ by (5.6), and hence $F_0(I) = 0$.

[6] By convention, this is fulfilled for characteristic zero.

[7] This is the step which forces us to assume that p does not divide $|G|$, and that the characteristic of Ω does not divide $|G|$.

To prove (5.8), we recall (Proposition 2.3) that we have a relation between free objects

$$F''_{\mathbb{Q}}(n) = \mathrm{colim}_{(N,|G|)=1} F''_{\mathbb{Z}[1/|G|]}(n).$$

Since $F_0(F''_{\mathbb{Q}}(n)) = 0$ by (5.5) for $\Omega = \mathbb{Q}$ and F_0 commutes with filtered colimits, we deduce that for every $x \in F_0(F''_{\mathbb{Z}[1/|G|]}(n))$ there is some N coprime to $|G|$ (depending on x and n) such that $[N](x) = 0$. To deduce from this the more uniform statement (5.8) one uses that F_0 is finitely generated and takes the product of all N for the generators. Since the details of this step are literally the same as in the proof of [Clausen et al. 2018, Lemma 4.16], we omit them here. □

The above proof establishes the following axiomatic rigidity result, which generalizes [Clausen et al. 2018, Proposition 4.15].

Proposition 5.9. *Assume G is a finite group and $F : \mathrm{Ring}^{\mathrm{nu,h},G}_{\mathbb{Z}[1/|G|]} \to \mathrm{Sp}$ is a G-pscoh functor such that*

(1) *for each prime field Ω of characteristic not dividing $|G|$ and $n \geq 0$, we have $F(F''_{\Omega}(n)) = 0$;*

(2) *F sends short exact sequences to fiber sequences*;

(3) *F vanishes on nilpotent arguments.*

Then $F = 0$.

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