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On limit points of spectra of first-order sentences with quantifier depth 4

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We study the asymptotic behavior of probabilities of first-order properties of sparse binomial random graphs. We consider properties with quantifier depth not more than 4. It is known that the only possible limit points of the spectrum (i.e., the set of all positive α such that $G(n, n^{-\alpha})$ does not obey the zero-one law with respect to the property) of such a property are 1/2 and 3/5. We prove that 1/2 is not a limit point of the spectrum.

1. Introduction

In this paper, we study the asymptotic behavior of probabilities of properties of Erdös-Rényi random graphs $G(N, p)$ that are expressible as sentences in the first-order logic. Here, p is typically a function $p = p(N)$. We consider first-order sentences of quantifier depth no more than k and sparse settings: $p = N^{-\alpha}$, where α is some positive number less than 1.

Recall the definition of the Erdös-Rényi random graph: $G(N, p)$ is a random element of the set of all simple graphs $\Omega_N = \{\mathcal{G} = (\mathcal{V}_N, \mathcal{E})\}$ with the set of vertices $\mathcal{V}_N = \{1, 2, \dots, N\}$, where N is a positive integer, with the distribution

$$\mathsf{P}_{N,p}(\mathcal{G}) = p^{|E(\mathcal{G})|}(1-p)^{\binom{N}{2}-|E(\mathcal{G})|},$$

where $0 \le p \le 1$.

We now define the class of graph first-order properties [Libkin 2004] as follows. A graph first-order property is a property defined by a first-order sentence in the vocabulary $\{=, \sim\}$ consisting of two predicate symbols, "=" expressing coincidence of vertices and "$\sim$" expressing adjacency.

Furthermore, we define the quantifier depth of a first-order property as the minimal quantifier depth of a first-order formula [Libkin 2004] expressing that property.

Definition 1. We say that for a function $p = p(N)$, *the zero-one law for first order logic holds* if for each first-order property the probability that the random graph $G(N, p)$ has that property tends to either 0 or 1 as $N \to \infty$.

Theorem 2 [Shelah and Spencer 1988]. *Let $p = N^{-\alpha}$ and α be a positive irrational number. Then the zero-one law for first-order logic holds.*

Note that for $p = N^{-\alpha}$ with α rational and $\alpha \le 1$, the zero-one law for first-order logic does not hold, as shown in the same paper.

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Definition 3. Let k be a positive integer. For $p = p(N)$, *the zero-one k-law holds* if for each first-order property with quantifier depth no more than k, the probability that the random graph $G(N, p)$ has that property tends to either 0 or 1 as $N \to \infty$.

The obvious corollary of Theorem 2 is that for functions $p = N^{-\alpha}$, the zero-one k-law also holds if α is irrational. For rational α, the situation gets complicated: the zero-one k-law may either hold or not hold, as demonstrated in the following theorem.

Theorem 4 [Zhukovskii 2012]. *Let $p(N) = N^{-\alpha}$ and $\alpha \in \big(0, \frac{1}{k-2}\big)$. Then the zero-one k-law holds. Moreover, if $\alpha = \frac{1}{k-2}$ then the zero-one k-law does not hold.*

Definition 5. *A spectrum of $k \in \mathbb{N}$* (or, simply, a k-spectrum) is the set of all $\alpha \in (0, 1)$ such that for $p(N) = N^{-\alpha}$ the zero-one k-law does not hold.

In other words, Theorem 4 states that the minimal element of the k-spectrum when $k \geq 3$ equals $\frac{1}{k-2}$.

In [Spencer 2001], it was proven that the spectrum of 14 is infinite. Further, it was proven in [Zhukovskii 2016] that $\frac{1}{2}$ is a limit point of the 5-spectrum. Moreover, it is known that the 3-spectrum is finite [Zhukovskii 2016]. In this paper we will study possible limit points of the 4-spectrum, the existence of which is neither proven nor disproven.

Theorem 6 [Matushkin and Zhukovskii 2018]. *The only limit points of the* 4*-spectrum may be $\frac{1}{2}$ and $\frac{3}{5}$.*

Our main result is stated below.

Theorem 7. *The point $\frac{1}{2}$ is not a limit point of the* 4*-spectrum.*

Thus, the only possible limit point of the 4-spectrum is $\frac{3}{5}$.

In the next section, we recall some known constructions needed to prove Theorem 7. In Sections 3 and 4, we introduce some new tools specific for α close to $\frac{1}{2}$; and finally, we prove the theorem in Section 5.

2. Known definitions and statements

We begin with recalling the Ehrenfeucht game $\mathrm{EHR}(A, B, k)$ on graphs A, B with two players: Spoiler and Duplicator, and a fixed number of rounds k [Ehrenfeucht 1961]. Each round of the game constitutes a move by Spoiler followed by a move by Duplicator: Spoiler selects a vertex from *either* of the two graphs A and B, whereas Duplicator selects a vertex from the *other* graph. Let x_ν denote the vertex selected from A in round ν, and y_ν that from B in round ν, for $1 \leq \nu \leq k$.

Duplicator wins the game if for each $s, t \in \{1, \ldots, k\}$, the following hold:

- $x_s = x_t$ if and only if $y_s = y_t$;
- $x_s \sim x_t$ if and only if $y_s \sim y_t$.

Otherwise, Spoiler wins.

The following theorem holds (see, for example, [Zhukovskii 2012]).

Theorem 8. *For $p = p(N)$ the zero-one k-law holds if and only if*

$$\lim_{N,M\to\infty} \mathsf{P}\big(\textit{Duplicator has a winning strategy in } \mathrm{EHR}(A, B, k)\big) = 1,$$

where A is a random graph distributed as $G(N, p(N))$, B is a random graph distributed as $G(M, p(M))$, and the two are independent of each other.

Definition 9. A graph property L is called *increasing (decreasing)* if for each two graphs G and G', with $V(G) = V(G')$ and $E(G) \subset (\supset) E(G')$, from $G \in L$ it follows that $G' \in L$.

Definition 10. A function $p_0(N)$ is called *the threshold function* for a property L if the following holds: If $p = o(p_0)$ (i.e., $\frac{p(N)}{p_0(N)} \to 0$, $N \to \infty$), then $\mathsf{P}(G(N, p) \in L) \to 0$ as $N \to \infty$; and if $p_0 = o(p)$, then $\mathsf{P}(G(N, p) \in L) \to 1$ as $N \to \infty$ (here, we restrict ourselves with increasing properties only; for decreasing properties, the threshold function is defined in the same way, but the o-conditions should be replaced; see, for example, [Janson et al. 2000]).

Define the *density of a graph* G as $\rho(G) = \frac{e(G)}{v(G)}$. Let us recall the result determining the threshold function for a property to contain an arbitrary given graph G as a subgraph. Define the *maximal density*: $\rho^{\max}(G) = \max_{H \subseteq G} \rho(H)$.

Theorem 11 [Ruciński and Vince 1985]. *The function $p = N^{-1/\rho^{\max}(G)}$ is the threshold function for the property of containing a copy of G.*

Consider graphs $G \supseteq H$, $\tilde{G} \supseteq \tilde{H}$ and positive integers k, l ($k \le l$) such that

$$V(H) = \{x_1, \dots, x_k\}, \quad V(G) = \{x_1, \dots, x_l\},$$
$$V(\tilde{H}) = \{\tilde{x}_1, \dots, \tilde{x}_k\}, \quad V(\tilde{G}) = \{\tilde{x}_1, \dots, \tilde{x}_l\}.$$

The graph $\tilde{G}$ is a (G, H)-*extension* of the graph $\tilde{H}$ if

$$(x_i, x_j) \in E(G) \setminus E(H) \Rightarrow (\tilde{x}_i, \tilde{x}_j) \in E(\tilde{G}) \setminus E(\tilde{H}).$$

If the relation

$$(x_i, x_j) \in E(G) \setminus E(H) \Leftrightarrow (\tilde{x}_i, \tilde{x}_j) \in E(\tilde{G}) \setminus E(\tilde{H})$$

holds then $\tilde{G}$ is called a *strict extension* of $\tilde{H}$, and the pairs (G, H) and $(\tilde{G}, \tilde{H})$ are called *isomorphic*.

Note that, throughout the paper, the labeling of vertices of graphs G, H, $\tilde{G}$, and $\tilde{H}$ will follow from the context. Thus, there will be no ambiguity in the definition of isomorphic pairs.

Fix a positive α. For graphs G and H with $H \subset G$, define

$$V(G, H) = V(G) \setminus V(H), \quad E(G, H) = E(G) \setminus E(H),$$
$$v(G, H) = |V(G, H)|, \quad e(G, H) = |E(G, H)|,$$
$$f_\alpha(G, H) = v(G, H) - \alpha e(G, H).$$

If for each graph S such that $H \subset S \subseteq G$, the inequality $f_\alpha(S, H) > 0$ holds, then the pair (G, H) is called α-*safe*. If for each graph S such that $H \subseteq S \subset G$ the inequality $f_\alpha(G, S) < 0$ holds, then the pair (G, H) is called α-*rigid*.

If for each graph S such that $H \subset S \subset G$ the inequality $f_\alpha(S, H) > 0$ holds, but also $f_\alpha(G, H) = 0$; then the pair (G, H) is called α-*neutral*. Note that if (G, H) is α-*neutral*, then for each graph S such that $H \subset S \subset G$, the relation $f_\alpha(G, S) < 0$ holds.

If the pair (G, H) where $H \subset G$ is α-safe, α-rigid or α-neutral, we call G, respectively, an *α-safe, α-rigid, or α-neutral extension* of H.

Let $\tilde{H} \subset \tilde{G} \subset \Gamma$ and $T \subset K$, with $v(T) \le v(\tilde{G})$. The pair $(\tilde{G}, \tilde{H})$ is called (K, T)-*maximal* in Γ, if each subgraph $\tilde{T} \subset \tilde{G}$, with $v(T) = v(\tilde{T})$ and $\tilde{T} \cap \tilde{H} \ne \tilde{T}$, does not have a (K, T)-extension $\tilde{K}$ in $\Gamma \setminus (\tilde{G} \setminus \tilde{T})$ having no edges between $V(\tilde{K}, \tilde{T})$ and $V(\tilde{G}, \tilde{T})$ [Zhukovskii and Raigorodskii 2015].

We will frequently use the following theorem (we write "a.a.s." meaning "asymptotically almost surely"):

Theorem 12 (J. Spencer, S. Shelah, 1988, [Alon and Spencer 1992]). *Let a pair (G, H) with $H \subset G$, be α-safe, and let t be a fixed positive integer. Let $r = v(H)$. Then a.a.s. (as $N \to \infty$) each induced subgraph of $G(N, N^{-\alpha})$ on r vertices has a strict (G, H)-extension that is (K, T)-maximal for each α-rigid pair (K, T) with $v(K, T) \le t$.*

3. New statements and constructions

Hereafter, for a graph G and a subset of its vertices $V \subseteq V(G)$, $G|_V$ denotes the subgraph of G induced on V.

To prove the main theorem, we need to build a finite set $\mathcal{G}$ of "bad" extensions for the following purpose. Let $x_1, \dots, x_k \in V(A)$ and $y_1, \dots, y_k \in V(B)$ be the vertices chosen in the game EHR$(A, B, 4)$ after some round k, for $1 \le k \le 4$. Consider two isomorphic extensions K and $\tilde{K}$ of $A|_{\{x_1,\dots,x_k\}}$ and $B|_{\{y_1,\dots,y_k\}}$, respectively, such that K and $\tilde{K}$ have a copy in $\mathcal{G}$. Playing as Duplicator, we have to ensure that K and $\tilde{K}$ have similar extensions of a special type; that is, constructed from the (K^*, T^*)-extension, where (K^*, T^*) is a pair of graphs with $v(T^*) = 2$, $e(T^*) = 0$, $v(K^*) = 3$, the only vertex in $V(K^*, T^*)$ being adjacent to each vertex from $V(T^*)$ (a "tick" extension).

To build the aforementioned set of "bad" extensions, we will repeatedly "extend" graphs in the following way. Let Ω be a graph, and let U be its induced subgraph (on an arbitrary subset of its vertices). Let also (A, B), $B \subset A$ be a pair of graphs such that $v(U) = v(B)$. We say that an algorithm *constructs* a strict (A, B)-extension W of U if it performs the following: to Ω it adds $v(A, B)$ vertices (those from the set $V(W) \setminus V(U)$) and $e(A, B)$ edges, so that the pairs (A, B) and (W, U) are isomorphic.

3.1. ***Building the set $\mathcal{G}$.*** Since we are only interested in isomorphism classes of graphs, it will be convenient to assume that all graphs in $\mathcal{G}$ contain a common singleton subgraph H with one vertex: $V(H) = \{z\}$.

Let T be some graph such that $H \subset T$. We call a pair of graphs (K, T), $T \subset K$, *bad* if the following conditions hold:

(1) (K, T) is $\frac{1}{2}$-rigid;

(2) $v(K, T) \le 6$;

(3) $v(T) \ge 2$;

(4) There is at least one edge between $V(K) \setminus V(T)$ and $V(T) \setminus \{z\}$ in the graph K.

Note that we bound $v(K, T)$ by 6, since we are only interested in relatively small extensions while playing the Ehrenfeucht game with 4 steps. The choice of the bound will be clarified in the proof of the main theorem.

Consider a set $\mathcal{G}$ of graphs with the common subgraph H and the following properties:

1. If a pair (G, H) is $\frac{1}{2}$-neutral or $\frac{1}{2}$-rigid and $v(G, H) \le 6$, then there is a graph isomorphic to G, in $\mathcal{G}$ (the image of z is z itself).

2. If $G \in \mathcal{G}$ and a graph $G' \supset G$ is such that the pair (G', G) is bad, then either G' has maximal density at least 2 or there is a graph isomorphic to G' in $\mathcal{G}$ (the image of z is z itself).

3. If $G \in \mathcal{G}$, then the graph obtained by adding some edge into G either has maximum density at least 2 or has an isomorphic graph in $\mathcal{G}$ (the image of z is z itself).

These properties are partially motivated by the fact that maximal density of each graph in $\mathcal{G}$ should be less than 2.

Statement 13. There exists a finite set $\mathcal{G}$ that satisfies 1–3.

Proof. We present an algorithm that constructs such a set $\mathcal{G}$.

(1) We begin with $\mathcal{G}$ that consists of the graphs isomorphic to all (nonempty) $\frac{1}{2}$-neutral extensions G of graph H for which $v(G, H) \leq 6$.

(2) While possible, we choose an arbitrary graph $G \in \mathcal{G}$ and construct $G' \supset G$ such that $\rho^{\max}(G') < 2$ and (G', G) is bad; if $\mathcal{G}$ does not contain a graph isomorphic to G', then G' is added to $\mathcal{G}$.

(3) While possible, we choose an arbitrary graph $G \in \mathcal{G}$ and construct G' such that $V(G) = V(G')$, $E(G) \subset E(G')$, $\rho^{\max}(G') < 2$; if $\mathcal{G}$ does not contain a graph isomorphic to G', then G' is added to $\mathcal{G}$.

We can assume that all the operations from Step 3 are performed after all the operations from Step 2, as the transposition of such operations does not change the resulting graph.

Call $G \in \mathcal{G}$ a 0-*stage graph* if G is added to $\mathcal{G}$ as a result of Step 1 of the algorithm. Call $G \in \mathcal{G}$ an s-*stage graph*, where $s \in \mathbb{N}$, if G is obtained as a result of applying the operation from Step 2 to an $(s-1)$-stage graph. Call a graph $G \in \mathcal{G}$ a *modified* s-*stage graph* if G is obtained from an s-stage graph by Step 3 of the algorithm.

Let us prove that the algorithm terminates and constructs a finite set. It is sufficient to show that Step 2 of the algorithm terminates, as in Step 3 for each graph in $\mathcal{G}$ we only add a finite number of new graphs.

Consider the value $f_{1/2}(G) = v(G) - \frac{1}{2}e(G)$. Firstly, for each graph G this value is half-integer. Secondly, it is easy to show that for a 0-stage graph G the equality $f_{1/2}(G) = 1$ holds. Finally, if G' is an s-stage graph with $s \in \mathbb{N}$ then it is isomorphic to some $\frac{1}{2}$-rigid (K, T)-extension of some $(s-1)$-stage graph G. So,

$$f_{1/2}(G') = f_{1/2}(G) + f_{1/2}(K, T) \leq f_{1/2}(G) - \tfrac{1}{2}.$$

Thus, if $s \geq 2$ we have

$$f_{1/2}(G') \leq 0,$$

which implies $\rho^{\max}(G') \geq 2$. Consequently, in Step 2, only 1-stage graphs may be obtained. Thus, s-stage graphs with $s > 1$ are not in $\mathcal{G}$. Thus, $\mathcal{G}$ is finite.

If G' is a modified 1-stage graph we have $f_{1/2}(G') \leq \frac{1}{2} - \frac{1}{2} = 0$. Thus, there are also no modified s-stage graphs in $\mathcal{G}$, with $s > 0$.

Let us verify that the set $\mathcal{G}$ that we built satisfies all the three properties. Properties 1 and 3 are evident by construction. The second property may only be disrupted if in Step 3 of the algorithm there appears a graph G that has a $\frac{1}{2}$-rigid extension $G' \notin \mathcal{G}$, with $\rho^{\max}(G') < 2$. Let $\tilde{G} \in \mathcal{G}$ be the graph from which G is obtained in Step 3. Consider $\tilde{G}'$ that is a (G', G)-extension of $\tilde{G}$. Obviously, $E(\tilde{G}') \subset E(G')$. Therefore,

$\rho^{\max}(\tilde{G}') \leq \rho^{\max}(G') < 2$. Hence, $\tilde{G}'$ has been added in Step 2 of the algorithm. Thus, in Step 3 we add G', which leads to a contradiction.

The set $\mathcal{G}$ is successfully constructed. □

Remark 14. From the proof we obtain the equality

$$f_{1/2}(G) = 1 - \frac{s}{2},$$

for each s-stage $G \in \mathcal{G}$.

We now list some additional properties of $\mathcal{G}$ that will help prove the main theorem.

4. For each $G \in \mathcal{G}$ we have $\rho^{\max}(G) < 2$

Proof. Evident by construction. □

5. The degree of each vertex, except possibly for z, in any graph $G \in \mathcal{G}$ is greater than 2, and the degree of z is no less than 1.

Proof. Let us prove that the degree of each vertex except z is greater than 2. It is sufficient to verify the property for graphs of stages 0 and 1. Let some vertex $w \neq z$ from $V(G)$ have a degree no more than 2.

If G is a 0-stage graph then $f_{1/2}(G, G - w) \geq 0$, which contradicts (G, H) being $\frac{1}{2}$-neutral: indeed, it is evident that $G - w \neq H$, for each graph of $\mathcal{G}$ has at least 3 vertices.

If G is a 1-stage graph, let W be a subgraph of G from which G is obtained at Step 2 of the algorithm constructing $\mathcal{G}$. If $w \in V(W)$, a contradiction follows from the fact that W is a 0-stage graph. If $w \in V(G) \setminus V(W)$ the inequality $f_{1/2}(G, G - w) \geq 0$ contradicts the pair (G, W) being $\frac{1}{2}$-rigid.

Let us prove that the degree of z is no less than 1. It is sufficient to verify the property for 0-stage graphs. Assume the contrary. If the degree of z is less than 1 the $\frac{1}{2}$-neutrality of (G, H) implies that

$$2v\big(G|_{V(G)\setminus\{z\}}\big) = e(G, H) = e\big(G|_{V(G)\setminus\{z\}}\big).$$

Therefore, $\rho^{\max}(G) \geq 2$. We come to a contradiction, which proves the statement. □

6. Since there are only 0, 1 and modified 0-stage graphs in $\mathcal{G}$, each one of them has no more than 21 vertices.

7. If a graph $G \in \mathcal{G}$ is not a 0-stage graph, the pair (G, H) is $\frac{1}{2}$-rigid.

Proof. If $G \in \mathcal{G}$ is a modified 0-stage graph then the $\frac{1}{2}$-rigidness follows from the $\frac{1}{2}$-neutralness of the pair (G', H), where G' is a 0-stage graph from which G is obtained by adding edges.

If G is a 1-stage graph then it is a (K, T)-extension of a 0-stage graph G', where (K, T) is a $smash[b]\frac{1}{2}$-rigid pair. Consider an arbitrary graph S such that $H \subseteq S \subset G$. Because of the $\frac{1}{2}$-rigidness of the pair (G, G') and the $\frac{1}{2}$-neutralness of the pair (G', H), we have

$$\begin{aligned} e(G, S) &= e\big(G|_{V(S)\cup V(G')}, S\big) + e\big(G, G|_{V(S)\cup V(G')}\big) \\ &\geq e(G', S \cap G') + \big(G, G|_{V(S)\cup V(G')}\big) \\ &\geq 2\big(v(G') - v(S \cap G') + v(G) - v\big(G, G|_{V(S)\cup V(G')}\big)\big) \\ &= 2(v(G) - v(S)). \end{aligned}$$

Let us show that the last inequality is in fact strict. If it turns into equality then $e(G', S \cap G') = 2(v(G') - v(S \cap G'))$ and $e(G, G|_{V(S) \cup V(G')}) = 2(v(G) - v(G, G|_{V(S) \cup V(G')}))$. The first inequality implies that $S \cap G' = H$. The second one implies that $V(S) \cup V(G') = V(G)$. In other words, $V(S) = V(G) \setminus (V(G') \setminus V(H))$. But, in that case $e(G|_{V(S) \cup V(G')}, S) > e(G', S \cap G')$, because the pair (K, T) contains a bad pair and, therefore, between $V(S) \setminus V(H)$ and $V(G') \setminus V(S)$ there is at least one edge.

Thus, the last inequality is strict, which precisely means that (G, H) is $\frac{1}{2}$-rigid. □

8. Each graph $G \in \mathcal{G}$ is connected and remains connected after removing the vertex z.

Proof. Let us prove that $G - z$ is connected. The remaining part of the statement follows from the fact that the degree of z in G is at least 1.

We now verify the validity of this property for a 0-stage graph G. Assume the contrary: let $G - z$ be disconnected. Then there exist induced subgraphs $T_1, T_2 \subset G$, intersecting only in the vertex z, such that in G there are no edges between $V(T_1) \setminus \{z\}$ and $V(T_2) \setminus \{z\}$, with $v(T_1) > 1$, $v(T_2) > 1$, and $V(T_1) \cup V(T_2) = V(G)$. We then obtain the equality $0 = f_{1/2}(G, H) = f_{1/2}(T_1, H) + f_{1/2}(T_2, H)$. Therefore, there exists $h \in \{1, 2\}$ such that $f_{1/2}(T_h, H) \leq 0$, which contradicts the $\frac{1}{2}$-neutralness of (G, H). For a modified 0-stage graph the validity of the property follows from its validity for 0-stage graphs.

We now verify the validity of this property for a 1-stage graph G'. Let G be a 0-stage graph from which G' is obtained, for which (G', G) is a bad pair. When removing z, all vertices of G remain in a single connected component $S \subset G'$. Put $T_1 = G'|_{V(S) \cup \{z\}}$ and $T_2 = G'|_{V(G') \setminus V(S)}$. Assume the contrary: let T_2 be nonempty. Since the pair (G', G) is bad, the graph G' has got at least one edge between $V(G') \setminus V(G)$ and $V(G) \setminus \{z\}$. Therefore, $T_1 \neq G$ and $V(G) \cup V(T_2) \neq V(G')$. Then, because of the $\frac{1}{2}$-rigidness of the pair (G', G), we have $f_{1/2}(T_1, G) = f_{1/2}(G', G'|_{V(G) \cup V(T_2)}) < 0$. Besides, $f_{1/2}(G', T_1) < 0$. Then we have the inequality

$$\rho(G') = \frac{e(G) + e(T_1, G) + e(G', T_1)}{v(G) + v(T_1, G) + v(G', T_1)} \geq \frac{2(v(G) - 1) + 2v(T_1, G) + 1 + 2v(G', T_1) + 1}{v(G) + v(T_1, G) + v(G', T_1)} = 2,$$

which contradicts the inclusion $G' \in \mathcal{G}$. Thus, for 1-stage graphs the property is also proven. There are no more graphs in $\mathcal{G}$, which completes the proof. □

Remark 15. In order to avoid long notations and explanations, we will frequently write $G \in \mathcal{G}$ to mean that $\mathcal{G}$ contains a subgraph isomorphic to G.

3.2. *Occurrence of graphs of $\mathcal{G}$ in an arbitrary graph.*

Consider some graph Γ and a vertex $u \in V(\Gamma)$. We call a strict (G, H)-extension R of u *$\mathcal{G}$-maximal* if for each $G' \in \mathcal{G}$ there is no (G', H)-extension of $\Gamma|_u$ in Γ that strictly contains all vertices of R.

We call an induced subgraph $U \subset \Gamma$ *u-bad* if it is a $\mathcal{G}$-maximal (G, H)-extension of $\Gamma|_u$ for some $G \in \mathcal{G}$.

Statement 16. Let graph Γ not contain any subgraphs with maximal density no less than 2 with 41 or less vertices. Then for each vertex $u \in V(\Gamma)$ any two u-bad subgraphs of Γ only intersect in u.

Proof. Assume the contrary. Let $U, W \subseteq \Gamma$ be u-bad subgraphs with $v(U \cap W) > 1$. We can assume that either of U and W is a 0-stage or a 1-stage graph (neither 0′-stage nor 1′-stage).

Let U be an s-stage graph. Since $U \in \mathcal{G}$, there exists a sequence of nested graphs U_i, $0 \leq i \leq s + 1$, $U_0 = H$, $U_{s+1} = U$ such that (U_{i+1}, U_i) is bad for all $i \geq 1$, and the pair (U_1, U_0) is $\frac{1}{2}$-neutral or $\frac{1}{2}$-rigid with $v(U_1, U_0) \le 6$. But then the pair $(\Gamma|_{V(U_1)\cup V(W)}, W)$ is bad due to the $\frac{1}{2}$-neutralness of the pair (U_1, U_0). Hence, either the graph $\Gamma|_{V(U_1)\cup V(W)}$ is in $\mathcal{G}$, or $\rho^{\max}(\Gamma|_{V(U_1)\cup V(W)}) \ge 2$, which contradicts the hypothesis of Statement 16. Hence, $\Gamma|_{V(U_1)\cup V(W)} \in \mathcal{G}$.

Moreover, assume that $\Gamma|_{V(U_r)\cup V(W)}$ has a copy in $\mathcal{G}$ for some r. The pair $(\Gamma_{V(U_{r+1}\cup V(W))}, \Gamma_{V(U_r\cup V(W))})$ is bad. Thus, either $\Gamma|_{V(U_{r+1})\cup V(W)}$ is in $\mathcal{G}$, or $\rho^{\max}(\Gamma|_{V(U_{r+1})\cup V(W)}) \ge 2$, which contradicts the hypothesis of Statement 16. Hence, $\Gamma|_{V(U_{r+1})\cup V(W)}$ is in $\mathcal{G}$.

With $s + 1 = r$, we obtain that $\Gamma_{V(U\cup W)}$ has a copy in $\mathcal{G}$. Therefore, U is not H-maximal, which contradicts the condition. The statement is proven. □

Enumerate all graphs in $\mathcal{G}$: $\mathcal{G} = \{G_1, \dots, G_K\}$. Let

$$V(G_i) = \{z = z^i_1, z^i_2, \dots, z^i_{v(G_i)}\}.$$

Let $\mathcal{U}_i(u)$ be the set of all u-bad (G_i, H)-extensions of the vertex u in Γ.

Let the 0-*neighborhood of the vertex u* be the set

$$U_0(u) = V(\Gamma) \setminus \Big(\bigcup_{i=1}^{K} \bigcup_{U\in\mathcal{U}_i(u)} V(U) \Big).$$

3.3. ***(K*, T*)-neighborhood.*** Let U be an induced subgraph of Γ. We now define the (K^*, T^*)-*neighborhood* of U. Let

- $W_0 = V(U)$;
- $W_{i+1} = W_i \cup \big\{v \in V(\Gamma) : |\{w \in W_i : w \sim v\}| \ge 2\big\}, \quad i \in \mathbb{N}$;
- $W = \bigcup_{i=0}^{\infty} W_i$.

We now define $\Gamma|_W$ to be the (K^*, T^*)-neighborhood of U in Γ.

If for all i the sign "$\ge$" in item 2 is replaced by "$=$" and $W_{i+1} \setminus W_i$ is an independent set, then the set W is called the *strict* (K^*, T^*)-*neighborhood* of U in Γ.

We now prove an auxiliary statement in order to motivate the definition of (K^*, T^*)-neighborhood.

Statement 17. Let α be a positive rational. Consider graphs G, U, W with $W \subset U \subseteq G$ such that the pair (G, U) is α-neutral and (U, W) is α-safe. Let there also be at least one edge between $V(G) \setminus V(U)$ and $V(U) \setminus V(W)$ in G. Then the pair (G, W) is also α-safe.

Proof. Consider an arbitrary graph G' such that $W \subset G' \subseteq G$. Put $S = G' \cap U$. Due to the fact that (G, U) is α-neutral and (U, W) is α-safe, one can show a chain of inequalities

$$\begin{aligned} e(G', W) = e(S, W) + e(G', S) &\le \frac{v(S) - v(W)}{\alpha} + e\big(G|_{V(U)\cup V(G')}, U\big) \\ &\le \frac{v(S) - v(W)}{\alpha} + \frac{v(G') - v(S)}{\alpha} = \frac{v(G') - v(W)}{\alpha}. \end{aligned}$$

Note that $e(S, W) \le \dfrac{v(S) - v(W)}{\alpha}$ turns into an equality if and only if $S = W$. The inequality

$$e\big(G|_{V(U)\cup V(G')}, U\big) \le \frac{v(G') - v(S)}{\alpha}$$

turns into an equality if and only if $V(G') \subseteq V(U)$ or $V(U) \cup V(G') = V(G)$.

Assume both inequalities are equalities. Then we have $W = G' \cap U$. If $V(G') \subseteq V(U)$, then $G' = W$, which contradicts the fact that W is a proper subgraph of G'. If, on the other hand, $V(U) \cup V(G') = V(G)$, then $e(G', W) = \frac{v(G')-v(W)}{\alpha} = \frac{v(G)-v(U)}{\alpha} = e(G, U)$ (The first inequality in this chain easily follows from the fact that the previous inequalities turn into equalities). Therefore, there are no edges between the sets $V(G) \setminus V(U)$ and $V(U) \setminus V(W)$ in G, which contradicts the assumption. Hence, the inequality is in fact strict, which completes the proof. □

Corollary 18. *Let α be a positive rational and U be an induced subgraph of G with $\rho^{\max}(U) < \frac{1}{\alpha}$. Let the pair (G, U) be α-neutral. Let there also be at least one edge between the sets $V(U)$ and $V(G) \setminus V(U)$, in G. Then the inequality $\rho^{\max}(G) < \frac{1}{\alpha}$ holds.*

Proof. We obtain the corollary from Statement 17 by putting W to be the empty graph. □

We now formulate some corollaries from Statement 17 that motivate the definition of (K^*, T^*)-neighborhood.

Statement 19. Let Γ be an arbitrary graph. Let U, W be some induced subgraphs of Γ such that $W \subset U$ and (U, W) is $\frac{1}{2}$-safe. Let $G \subset \Gamma$ be the strict (K^*, T^*)-neighborhood of U in Ω, and let also each vertex from the set $V(G) \setminus V(U)$ be adjacent to not more than one vertex from $V(W)$ in Ω. Then the pair (G, W) is also $\frac{1}{2}$-safe.

Proof. Let $U_0 \subset U_1 \subset \ldots \subset U_s$ be a sequence of nested graphs such that $U_0 = U$, $U_s = G$, and for $1 \le i \le s$ the equalities $v(U_i, U_{i-1}) = 1$ hold and $e(U_i, U_{i-1}) = 2$. The pairs (U_i, U_{i-1}) are $\frac{1}{2}$-neutral for all i. Moreover, there are exactly two edges between $V(U_i) \setminus V(U_{i-1})$ and $V(U_{i-1})$ and no more than one edge between $V(U_i) \setminus V(U_{i-1})$ and $V(W)$. Hence, there is at least one edge between the sets $V(U_i) \setminus V(U_{i-1})$ and $V(U_{i-1}) \setminus V(W)$, in U_i.

Let us prove by induction on i that (U_i, W) is $\frac{1}{2}$-safe. The base case when $i = 0$ directly follows from the statement condition. The step case from $i-1$ to i follows from the fact that (U_{i-1}, W) is $\frac{1}{2}$-safe, (U_i, U_{i-1}) is $\frac{1}{2}$-neutral, and also Statement 17 for $\alpha = \frac{1}{2}$. Put $i = s$, and obtain the desired inequality. □

Corollary 20. *Let Γ be an arbitrary graph. Let U be an induced subgraph of Γ for which the inequality $\rho^{\max}(U) < 2$ holds. If G is the strict (K^*, T^*)-neighborhood of U in Γ, then the inequality $\rho^{\max}(G) < 2$ holds.*

Proof. We obtain the corollary from Statement 19 by putting W to be the empty graph. □

3.4. ***Some auxiliary notations.*** Let $u_1, \ldots, u_m, v_1, \ldots, v_r$ be some vertices in a graph Γ, and put $U = \Gamma|_{\{u_1,\ldots,u_m\}}$. For a tuple $(\alpha_1, \ldots, \alpha_r) \in \{0, 1\}^r$, put

$$\mathcal{N}\big(v_1^{\alpha_1}, \ldots, v_r^{\alpha_r}\big) = \big\{u \in V(\Gamma) \mid \forall k\, (v_k \sim u \leftrightarrow \alpha_k = 1)\big\} \setminus \{v_1, \ldots, v_r\}.$$

Also, in place of v_i^0 we write $\neg v_i$, and in place of v_i^1 we write just v_i. For example,

$$\mathcal{N}(\neg v_1, v_2, \neg v_3) = \big\{u \in V(\Gamma) \mid u \not\sim v_1, u \sim v_2, u \not\sim v_3, u \notin v_1, v_2, v_3\big\}.$$

Put

$$\delta\big(v_1^{\alpha_1}, \ldots, v_r^{\alpha_r}\big) = \big(\big|\mathcal{N}\big(v_1^{\alpha_1}, \ldots, v_r^{\alpha_r}\big)\big| \ge 1\big).$$

Finally, put

$$\mathcal{N}^U\big(v_1^{\alpha_1},\dots,v_m^{\alpha_m}\big)=\mathcal{N}\big(v_1^{\alpha_1},\dots,v_m^{\alpha_m}\big)\setminus V(U);$$
$$\delta^U\big(v_1^{\alpha_1},\dots,v_r^{\alpha_r}\big)=\big(\big|\mathcal{N}\big(v_1^{\alpha_1},\dots,v_r^{\alpha_r}\big)\setminus V(U)\big|\geq 1\big).$$

Given a set of vertices $V=\{v_1,v_2,\dots,v_r\}$, we write $\neg V$ in place of $\neg v_1,\dots,\neg v_r$.

3.5. ***Definition of the type of a bad subgraph.*** Fix a positive integer m. For a vertex $t\in V(\Gamma)\setminus V(U)$, define $j^U(t)\in\{0,1\}^{m+m2^m}$ as follows: For all k with $1\leq k\leq m$ and for all $S\subseteq V(U)$ put

$$j^U_{u_k}(t)=\delta^U(t,u_k,\neg(V(U)\setminus\{u_k\}));$$

$$j^U_{u_k,S}(t)=\Big(\exists s\in\mathcal{N}^U(t,u_k,\neg(V(U)\setminus\{u_k\})):\\ \big(\bigwedge_{l:\,u_l\in S}\delta^U(s,u_l,\neg t,\neg(V(U)\setminus\{u_l\}))\big)\wedge\big(\bigwedge_{l:\,u_l\in V(U)\setminus S}\neg\delta^U(s,u_l,\neg t,\neg(V(U)\setminus\{u_l\}))\big)\Big). \quad (1)$$

Hereafter, for predicate P we use the notation (P) that takes the value 1 if P is true and 0 otherwise. Finally,

$$j^U(t)=\big(\big(j^U_{u_k}(t)\big)_{k\in\{1,\dots,m\}},\big(j^U_{u_k,S}\big)_{k\in\{1,\dots,m\},S\subseteq V(U)}\big).$$

Note that $j^U_{u_k}(t)=\bigvee_{S\subseteq V(U)}j^U_{u_k,S}(t)$.

Let $u,w\in V(U)$. Define

$$J^U(u,w)=\big\{j^U(t)\ \mid\ t\in\mathcal{N}^U(u,w)\big\};$$
$$I^U(u,w)=\big(\big|\mathcal{N}^U(u,w)\big|>1\big).$$

Let a vertex $s\in V(\Gamma)\setminus V(U)$ satisfy $\delta^U(u,w,s)=1$. Define

$$\sigma^U_1(u,w,s)=\delta^U(u,s,\neg w);$$
$$\sigma^U_2(u,w,s)=\delta^U(w,s,\neg u);$$
$$\sigma^U(u,w,s)=\big(\sigma^U_1(u,w,s),\sigma^U_2(u,w,s)\big);$$

$$S^U(u,w)=\begin{cases}\big\{s\in\mathcal{N}^U(\neg u_1,\dots,\neg u_m)\mid\forall t\in\mathcal{N}^U(u,w):\,s\sim t\big\}, & I^U(u,w)=1;\\ \varnothing, & \text{else.}\end{cases}$$

Let

$$\Sigma^U(u,w)=\big\{\sigma^U(u,w,s)\mid s\in S^U(u,w)\big\}.$$

For a u-bad subgraph U isomorphic to G_i, define the *i-type* of U in the following manner (hereafter, u_k is the image of z^i_k under the isomorphism between G_i and U, for $1\leq k\leq m$; the image of $z\in V(G_i)$ is $u=u_1$):

$$T^i_1(U)=(J^U(u_k,u_n))_{k,n};$$
$$T^i_2(U)=(I^U(u_k,u_n))_{k,n};$$
$$T^i_3(U)=(\Sigma^U(u_k,u_n))_{k,n}.$$

The latter three definitions are tuples for which indices of vertices take all pairs of values $1 \le k < n \le m$ in lexicographical order. Let

$$T^i(U) = (T_1^i(U), T_2^i(U), T_3^i(U)).$$

Finally, put

$$\mathcal{J}^i(u) = \{T^i(U) \mid U \in \mathcal{U}_i(u)\}, \quad 1 \le i \le |\mathcal{G}|.$$

3.6. ***Construction of some graph pairs associated with the latter definitions.*** Now we are almost ready to move on to the lemmas. All we have left is to present a way of constructing graphs that serve as extensions for Duplicator to search when playing the Ehrenfeucht game. Our constructions will be closely related to the value $j^U(t)$.

Fix m vertices $\tilde{u}_1, \dots, \tilde{u}_m$ and an arbitrary value

$$j = (j_{k,S})_{k\in\{1,\dots,m\},\, S\subseteq\{\tilde{u}_1,\dots,\tilde{u}_m\}} \in \{0, 1\}^{m2^m}.$$

We build a pair of graphs (A, B) in the following manner:

a) Mark a vertex $\tilde{t}$.

b) For each nonzero $j_{k,S}$, mark a vertex $\tilde{s}_{k,S}$ and join it with $\tilde{u}_k$ and $\tilde{t}$.

c) For each vertex $\tilde{s}_{k,S}$ and for each l such that $\tilde{u}_l \in S$, mark a vertex and join it with $\tilde{u}_l$ and $\tilde{s}_{k,S}$.

d) The set of all the marked vertices, including $\tilde{u}_1, \dots, \tilde{u}_m$, and the set of all the drawn edges constitute graph A. The set of vertices $\{\tilde{u}_1, \dots, \tilde{u}_m, \tilde{t}\}$ and the empty set of edges constitute B.

Definition 21. The constructed pair (A, B) *corresponds* to the tuple $j \in \{0, 1\}^{m2^m}$.

Note that the inequality $v(A, B) \le m(m+1)2^m$ holds, since for each $j_{k,S}$ not more than $m+1$ vertices are added to $V(A, B)$. Also note that the construction of the pair (A, B) comes down to the sequential construction of several (K^*, T^*)-extensions. In other words, the set $V(A, B)$ is a subset of the strict (K^*, T^*)-neighborhood of B in A.

4. Three main lemmas

4.1. *The first lemma.*

Lemma 22. *There exists an* $\varepsilon > 0$ *such that for each* $\alpha \in \left(\frac{1}{2}, \frac{1}{2} + \varepsilon\right)$ *a.a.s.* (X, Y) *satisfies the following condition: for each vertex* $x_1 \in V(X)$ *there exists a vertex* $y_1 \in V(Y)$ *such that for all* $1 \le i \le |\mathcal{G}|$ *the equality*

$$\mathcal{J}^i(x_1) = \mathcal{J}^i(y_1)$$

holds. Here $X \sim G(N, N^{-\alpha})$ *and* $Y \sim G(M, M^{-\alpha})$ *are independent random graphs.*

Proof. The informal idea of the proof is as follows. For almost every graph X and an arbitrary vertex $x_1 \in V(X)$, we need to construct a graph Z with a vertex $z_1 \in V(Z)$ such that for all i from 1 to K the equality $\mathcal{J}^i(x_1) = \mathcal{J}^i(z_1)$ holds. We then find a “maximal” copy $\hat{Z}$ of Z (meaning that there are no $\frac{1}{2}$-neutral nor $\frac{1}{2}$-rigid extensions of bounded size of $\hat{Z}$ in Y) in almost every graph Y. Finally, we choose y_1 as the image of z_1 under the isomorphism between Z and $\hat{Z}$.

For each positive integer N let $\mathcal{X}_N \subseteq \Omega_N$ be a set of graphs on N vertices without any subgraphs G with $\rho^{\max}(G) \ge 2$ and $v(G) \le 41$. By Theorem 11, we have $\mathsf{P}(X \in \mathcal{X}_N) \to 1$ for each $\alpha > \frac{1}{2}$. Consider an arbitrary $X \in \mathcal{X}_N$ and a vertex $x_1 \in X$. By Statement 16, any two x_1-bad subgraphs of X only intersect in x_1. We now introduce an algorithm to build a graph $Z = Z(X, x_1)$ with $z_1 \in V(Z)$ such that $\rho^{\max}(Z) < 2$ and the equality $\mathcal{J}^i(x_1) = \mathcal{J}^i(z_1)$ holds for $1 \le i \le |\mathcal{G}|$.

(1) Mark a vertex z_1.

(2) For $1 \le i \le |\mathcal{G}|$, for all types $T \in \mathcal{J}^i(x_1)$:

   (a) Let $T = T^i(U)$ for some $U \in \mathcal{U}_i(x_1)$. Let $x_2, \dots, x_{v(G_i)}$ be the elements of $V(U) \setminus \{x_1\}$.

   (b) Mark $v(G_i) - 1$ vertices $z_2, \dots, z_{v(G_i)}$ and draw edges between them so that the subgraph of Z induced on $z_1, z_2, \dots, z_{v(G_i)}$ is isomorphic to U (with z_k being an image of x_k for all k).

   (c) For all pairs (k, n) such that $1 \le k < n \le v(G_i)$:

      (i) If $S^U(x_k, x_n)$ is nonempty and $|\mathcal{N}^U(x_k, x_n)| = 2$:

         (A) For each vertex $t \in \mathcal{N}^U(x_k, x_n)$ mark a vertex r_t, join it with z_k and z_n, and construct the strict (A, B)-extension of the subgraph induced on $z_1, z_2, \dots, z_{v(G_i)}, r_t$, where (A, B) is the pair of graphs that corresponds to $j^U(t)$ (see Section 3.6).

         (B) For each $\sigma = (\sigma_1, \sigma_2) \in \Sigma^U(x_k, x_n)$ (see Section 3.5) mark a vertex r_σ and join it with the vertices r_t for $t \in \mathcal{N}^U(x_k, x_n)$. If $\sigma_1 = 1$, mark a vertex r_1 and join it with r_σ and z_k. If $\sigma_2 = 1$, mark r_2 and join it with r_σ and z_n.

      (ii) Else:

         (A) For each $j \in J^U(x_k, x_n)$ mark r_j, join it with z_k and z_n, and construct a strict (A, B)-extension of the subgraph induced on $z_1, z_2, \dots, z_{v(G_i)}, r_j$, where (A, B) is the pair of graphs that corresponds to j.

         (B) If $|J^U(x_k, x_n)| = 1$, but $I^U(x_k, x_n) = 1$, repeat Step (A) once again.

(3) Marked vertices and drawn edges constitute the graph Z. □

Statement 23. The inequality $\rho^{\max}(Z) < 2$ holds.

Proof. Let R be an x_1-bad subgraph of X isomorphic to a modified 0-stage graph or a 1-stage graph. By Property 7 of $\mathcal{G}$, the inequality $e(R, R|_{\{x_1\}}) \ge 2v(R, R|_{\{x_1\}}) + 1$ holds. If there exist at least two x_1-bad subgraphs isomorphic to a modified 0-stage graph or a 1-stage graph (denote them by R_1, R_2), then for their union S we have

$$e(S) = e(R_1) + e(R_2) \ge 2v(R_1) + 2v(R_2) - 2 = 2v(S).$$

Thus, $\rho(S) \ge 2$, with $v(S) \le 41$, which contradicts the absence of small dense subgraphs in X.

Therefore, in Step (b) the algorithm constructs no more than one extension of $Z|_{\{z_1\}}$, isomorphic to a modified 0-stage or a 1-stage graph. Denote this subgraph (or the subgraph on a single vertex z_1 if there is no such extension) by U_0. Denote by Z_1 the subgraph induced on z_1 and all the other vertices added in Step (b). We call an induced subgraph $U \subseteq Z$ that contains z_1 *interesting* if $V(U) \setminus z_1$ is equal to the set of vertices built in Step (b) of the algorithm for some $T \in \mathcal{J}^i(x_1)$.

Let us show that

$$\rho^{\max}(Z_1) < 2.$$

Let $U_1, \dots, U_h$ be an enumeration of all interesting subgraphs of Z, but for U_0. For $1 \le k \le h$ denote

$$\Psi_k = Z|_{V(U_0) \cup V(U_1) \cup \ldots \cup V(U_k)}.$$

Put $\Psi_0 = U_0$. Since the algorithm does not introduce an edge between any pair of vertices belonging to two different U_k, for $1 \le k \le h$ the equalities

$$V(\Psi_k) \setminus V(\Psi_{k-1}) = V(U_k) \setminus \{z_1\} \quad \text{and} \quad E(\Psi_k) \setminus E(\Psi_{k-1}) = E(U_k) \setminus E\big(Z|_{\{z_1\}}\big)$$

hold. The pair $(U_k, Z|_{\{z_1\}})$ is $\frac{1}{2}$-neutral for $1 \le k \le h$. Moreover, from Property 5 of $\mathcal{G}$ it follows that for each k, there is at least one edge between z_1 and $V(U_k) \setminus \{z_1\}$. Thus, the pair (Ψ_k, Ψ_{k-1}) is $\frac{1}{2}$-neutral for $1 \le k \le h$, and there is at least one edge between $V(\Psi_{k-1})$ and $V(\Psi_k) \setminus V(\Psi_{k-1})$, in Ψ_k.

Let us prove by induction on k that $\rho^{\max}(\Psi_k) < 2$. The base case when $k = 0$ follows from the fact that the graph $\Psi_0 = U_0$ is isomorphic to a graph from set $\mathcal{G}$. Let us prove the step case from $k-1$ to k. We have $\rho^{\max}(\Psi_{k-1}) < 2$, pair (Ψ_k, Ψ_{k-1}) is $\frac{1}{2}$-neutral, and in Ψ_k there is at least one edge between $V(\Psi_{k-1})$ and $V(\Psi_k) \setminus V(\Psi_{k-1})$. From Corollary 18 with $\alpha = \frac{1}{2}$, we obtain $\rho^{\max}(\Psi_k) < 2$. With $k = h$, we have $\rho^{\max}(Z_1) < 2$.

The construction performed in the algorithm steps, but for Step (b), comes down to construction of strict (K^*, T^*)-extensions. Thus, the graph Z is a strict (K^*, T^*)-neighborhood of its subgraph Z_1. By Corollary 20, we have $\rho^{\max}(Z) < 2$. □

We now study the structure of z_1-bad subgraphs in Z.

Statement 24. A subgraph of Z is z_1-bad if and only if it is interesting.

Proof. The statement boils down to two parts. First, we show that each z_1-bad subgraph in Z is interesting.

Let us show that the vertices of each z_1-bad subgraph are contained in the set $V(U_0) \cup V(U_1) \cup \dots \cup V(U_h)$ (the graphs U_i are defined in the proof of Statement 23). Assume the contrary: let for some i the graph Z contain an induced subgraph $\tilde{G}$ isomorphic to G_i, with the image of $z \in V(G_i)$ under the isomorphism between $\tilde{G}$ and G_i being z_1; moreover, not all vertices of $\tilde{G}$ are marked in Step (b) of the algorithm. Of all such vertices we consider the last added vertex a. This vertex is adjacent to exactly two of the previous vertices. The vertices in $\tilde{G}$ marked later by the algorithm are in interesting subgraphs and, thus, cannot be adjacent to a. Therefore, the degree of a in $\tilde{G}$ is not more than 2, which contradicts Property 5 of $\mathcal{G}$.

Furthermore, let us show that every z_1-bad subgraph of Z has to be fully contained in a single interesting subgraph. Let this not be the case. Then a subgraph U intersects with two interesting subgraphs, in two vertices each (including z_1). But, different interesting subgraphs only intersect in z_1, and edges of Z do not join vertices from different interesting subgraphs. On removing z_1, however, U does not fall into components (by Property 8 of $\mathcal{G}$) and, consequently, cannot contain vertices from different interesting subgraphs, other than z_1. Finally, a z_1-bad subgraph cannot be strictly contained in an interesting subgraph, for this contradicts its $\mathcal{G}$-maximality. Thus, each z_1-bad subgraph in Z is interesting.

Now, consider an interesting subgraph $U \subseteq Z$. Let it not be z_1-bad. Since U is isomorphic to some $G \in \mathcal{G}$, there is a graph $G' \supset U$ that is z_1-bad. Then G' is also interesting (as proven above), which contradicts two interesting subgraphs intersecting only on z_1. The statement is proven. □

Statement 25. For all i the equality $\mathcal{J}^i(x_1) = \mathcal{J}^i(z_1)$ holds. Here x_1 is the vertex from the statement of Lemma 22.

Proof. Consider an arbitrary $T^i(U) \in \mathcal{J}^i(x_1)$ with some x_1-bad U being induced on vertices $x_1, x_2, \dots, x_m$. Let $\tilde{U}$ be a corresponding interesting subgraph of Z on vertices $z_1, z_2, \dots, z_m$. Let us show that $T^i(U) = T^i(\tilde{U})$. It is sufficient to check the equalities for T^i_1, T^i_2, and T^i_3.

Consider arbitrary $1 \le k, n \le m$. Note that if $S^U(x_k, x_n)$ is nonempty then $|\mathcal{N}^U(x_k, x_n)| = 2$. Indeed, first, by the definition of $S^U(x_k, x_n)$ there is $|\mathcal{N}^U(x_k, x_n)| \ge 2$. Moreover, if $|\mathcal{N}^U(x_k, x_n)| \ge 3$, denote by t_1, t_2, t_3 three arbitrary vertices of $\mathcal{N}^U(x_k, x_n)$ and by s an arbitrary vertex of $S^U(x_k, x_n)$. By definition, s is adjacent to each t_ℓ, $1 \le \ell \le 3$. Therefore, $(X|_{V(U)\cup\{t_1,t_2,t_3,s\}}, U)$ is bad, which contradicts the fact that U is x_1-bad.

Consider an arbitrary $t \in \mathcal{N}^U(x_k, x_n)$. Put $j = j^U(t)$. If $S^U(x_k, x_n)$ is nonempty, then $|\mathcal{N}^U(x_k, x_n)| = 2$; and in Step (A) the algorithm marks $r_t \in \mathcal{N}^{\tilde{U}}(z_k, z_n)$. If, on the other hand, $S^U(x_k, x_n)$ is empty, then in Step (A) the algorithm marks $r_j \in \mathcal{N}^{\tilde{U}}(z_k, z_n)$. Denote $\tilde{t} = r_t$ in the first case and $\tilde{t} = r_j$ in the second one.

Let us show that

$$j^U(t) = j^{\tilde{U}}(\tilde{t}).$$

Fix an arbitrary $S \subseteq V(U)$ and k, for some $1 \le k \le m$. Let $\varphi : V(U) \to V(\tilde{U})$ be the isomorphism between the graphs U and $\tilde{U}$ such that $\varphi(x_k) = z_k$ for each k. Put $\tilde{S} = \varphi(S)$. Let us show that $j^U_{k,S}(t) = j^{\tilde{U}}{}_{k,\tilde{S}}(\tilde{t})$. Consider the pair (A, B) that corresponds to $j^U(t)$ (see Section 3.6) and denote by $\tilde{A}$ the subgraph of Z that is the image of A constructed in Step (A) or (A) of the algorithm.

If $j^U_{k,S}(t) = 1$, then in $\tilde{A}$ there exists a vertex $s_{k,\tilde{S}} \in \mathcal{N}^{\tilde{U}}(z_k, \tilde{t})$ for which by construction of (A, B) the following holds:

$$\delta^{\tilde{U}}(s_{k,\tilde{S}}, z_l, \neg\tilde{t}, \neg(V(\tilde{U}) \setminus \{z_1\})) = \begin{cases} 1, & x_l \in S, \\ 0, & \text{else,} \end{cases}$$

which implies that $j^{\tilde{U}}_{k,\tilde{S}}(\tilde{t}) = 1$.

Let $j^U_{k,S}(t) = 0$. Assume that $j^{\tilde{U}}_{k,\tilde{S}}(\tilde{t}) = 1$. Then in $\mathcal{N}^{\tilde{U}}(\tilde{t}, z_k)$ there exists a vertex s such that

$$\delta^{\tilde{U}}(s, z_l, \neg\tilde{t}, \neg(V(\tilde{U}) \setminus \{z_1\})) = \begin{cases} 1, & x_l \in S, \\ 0, & \text{else.} \end{cases}$$

Note that s is not the image of any of the vertices $\tilde{s}_{k,\tilde{S}'} \in V(A)$. Indeed, otherwise we have

$$\delta^{\tilde{U}}\big(s, z_l, \neg\tilde{t}, \neg(V(\tilde{U}) \setminus \{z_1\})\big) = \begin{cases} 1, & x_l \in S' = \{l \mid z_l \in \tilde{S}'\}, \\ 0, & \text{else,} \end{cases}$$

but $S' \ne S$.

But the vertex $s \in \mathcal{N}^{\tilde{U}}(\tilde{t}, z_k)$ may only be marked as the image of some vertex $\tilde{s}_{k,\tilde{S}'} \in V(A)$ since $\tilde{t}$ and z_k have no other common neighbors. We obtain a contradiction. Thus, $j^{\tilde{U}}_{k,\tilde{S}}(\tilde{t}) = 0$, and the equality

$$j^U(t) = j^{\tilde{U}}(\tilde{t})$$

is proven.

We now verify that $T^i_1(U) = T^i_1(\tilde{U})$. The equality boils down to the componentwise equalities $J^U(x_k, x_n) = J^{\tilde{U}}(z_k, z_n)$ for all k, n.

Consider an arbitrary $j \in J^U(x_k, x_n)$ and a vertex $t \in \mathcal{N}^U(x_k, x_n)$ such that $j^U(t) = j$. As proven above, there exists $\tilde{t} \in \mathcal{N}^{\tilde{U}}(z_k, z_n)$ such that $j^U(t) = j^{\tilde{U}}(\tilde{t})$. Thus, the inclusion $J^U(x_k, x_n) \subseteq J^{\tilde{U}}(z_k, z_n)$ holds.

Moving on, an arbitrary vertex $\tilde{t} \in \mathcal{N}^{\tilde{U}}(z_k, z_n)$ may only be marked in Steps (A), (A), or (B) of the algorithm. This means that there exists a vertex $t \in \mathcal{N}^U(x_k, x_n)$ such that $j^U(t) = j^{\tilde{U}}(\tilde{t})$. We have shown the second inclusion, which completes the proof of $T_1^i(U) = T_1^i(\tilde{U})$.

We now verify that $T_2^i(U) = T_2^i(\tilde{U})$. The equality boils down to componentwise equalities $I^U(x_k, x_n) = I^{\tilde{U}}(z_k, z_n)$.

If $I^U(z_k, z_n) = 0$, then $S^U(z_k, z_n) = \varnothing$. Thus, vertices from $\mathcal{N}^{\tilde{U}}(z_k, z_n)$ may only be marked in Step (A) of the algorithm. Therefore, the algorithm marks not more than one vertex in $\mathcal{N}^{\tilde{U}}(z_k, z_n)$, which implies the equality $I^{\tilde{U}}(x_k, x_n) = 0$.

Let $I^U(x_k, x_n) = 1$. Consider the case $S^U(z_k, z_n) \neq \varnothing$. As proven above, $|\mathcal{N}^U(z_k, z_n)| = 2$, and in Step (A) at least two vertices from $\mathcal{N}^{\tilde{U}}(z_k, z_n)$ are marked, which implies the equality $I^U(x_k, x_n) = 1$. Consider the case $S^U(z_k, z_n) = \varnothing$. If $|J^U(x_k, x_n)| > 1$, then in Step (A) the algorithm marks at least two vertices of $\mathcal{N}^{\tilde{U}}(z_k, z_n)$, which implies the equality $I^{\tilde{U}}(x_k, x_n) = 1$. If $|J^U(x_k, x_n)| = 1$, then in Steps (A) and (B) the algorithm marks two vertices of $\mathcal{N}^{\tilde{U}}(z_k, z_n)$, which implies the equality $I^{\tilde{U}}(x_k, x_n) = 1$. The equality $T_2^i(U) = T_2^i(\tilde{U})$ is verified.

Finally, we verify that $T_3^i(U) = T_3^i(\tilde{U})$. The equality boils down to the componentwise equalities $\Sigma^U(x_k, x_n) = \Sigma^{\tilde{U}}(z_k, z_n)$.

If $\Sigma^U(x_k, x_n) = \varnothing$, then either $I^U(x_k, x_n) = 0$ or there exist $t_1, t_2 \in \mathcal{N}^U(x_k, x_n)$ such that $\delta^U(t_1, T_1) = 0$. In the first case, the algorithm marks not more than one vertex $r \in \mathcal{N}^{\tilde{U}}(z_k, z_n)$, which implies the equality $\Sigma^{\tilde{U}}(z_k, z_n) = \varnothing$. In the second case, the algorithm marks at least two vertices $r_1, r_2 \in \mathcal{N}^{\tilde{U}}(z_k, z_n)$, but at the same time the algorithm does not mark any vertices adjacent to r_1 and r_2 but for in $V(\tilde{U})$. Thus, $\Sigma^{\tilde{U}}(z_k, z_n) = \varnothing$.

Examine the case $\Sigma^U(x_k, x_n) \neq \varnothing$. As shown above, $|\mathcal{N}^U(x_k, x_n)| = 2$. Denote by t_1, t_2 the elements of the set $\mathcal{N}^U(x_k, x_n)$. In Step (A), the algorithm marks the vertices $r_{t_1}, r_{t_2} \in \mathcal{N}^{\tilde{U}}(z_k, z_n)$. By construction, the set $\mathcal{N}^{\tilde{U}}(z_k, z_n)$ consists exactly of r_{t_1} and r_{t_2}. Consider an arbitrary $\sigma \in \Sigma^U(x_k, x_n)$ defined by some $s \in S^U(x_k, x_n)$. In Step (A), the algorithm marks $r_\sigma \in \mathcal{N}^{\tilde{U}}(r_{t_1}, r_{t_2})$. Let us show the equality $\sigma^U(x_k, x_n, s) = \sigma^{\tilde{U}}(z_k, z_n, r_\sigma)$. Vertices from sets $\mathcal{N}^{\tilde{U}}(z_h, r_\sigma)$ with $h = k, n$ may only be marked by the algorithm in Step (B). In this step, the vertex that is adjacent to z_h and r_σ is constructed iff $\delta^U(x_h, s) = 1$, which implies the equality $\delta^U(x_h, s) = \delta^{\tilde{U}}(z_h, r_\sigma)$ with $h = k, n$. The equality $\sigma^U(x_k, x_n, s) = \sigma^{\tilde{U}}(z_k, z_n, r_\sigma)$ is verified. Moreover, $r_\sigma \in S^{\tilde{U}}(z_k, z_n)$, and thus the inclusion $\Sigma^U(x_k, x_n) \subseteq \Sigma^{\tilde{U}}(z_k, z_n)$ is verified.

We now verify the reverse inclusion. Consider an arbitrary $\sigma \in \Sigma^{\tilde{U}}(z_k, z_n)$ defined by some $\tilde{s} \in S^{\tilde{U}}(z_k, z_n)$. We have the inclusion $\tilde{s} \in \mathcal{N}^{\tilde{U}}(r_{t_1}, r_{t_2})$. Such a vertex may only be marked by the algorithm in Step (B). Therefore, $\tilde{s} = r_\sigma$ for some $\sigma \in \Sigma^U(x_k, x_n)$. Above we proved the equality $\sigma(\tilde{s}, z_k, z_n) = \sigma$. Thus, we have shown the inclusion $\Sigma^{\tilde{U}}(z_k, z_n) \subseteq \Sigma^U(x_k, x_n)$, which concludes the proof of the equality $T_3^i(U) = T_3^i(\tilde{U})$. Thus, for each i we have

$$\mathcal{J}^i(x_1) \subseteq \mathcal{J}^i(z_1).$$

Now consider an arbitrary z_1-bad $\tilde{U} \subseteq Z$ isomorphic to G_i. By Statement 24 $\tilde{U}$ is interesting; hence, in X there exists an x_1-bad subgraph U for which, by what we have proven above, the equality $T^i(U) = T^i(\tilde{U})$ holds. Therefore,

$$\mathcal{J}^i(z_1) \subseteq \mathcal{J}^i(x_1).$$

Thus, for all $1 \leq i \leq |\mathcal{G}|$ the equalities $\mathcal{J}^i(x_1) = \mathcal{J}^i(z_1)$ hold, and the statement is proven. □

Define $\mathcal{Z}_N = \{Z(X, x_1) \mid X \in \mathcal{X}_N, x_1 \in V(X)\}$.

Statement 26. The number of vertices in an arbitrary graph $Z \in \mathcal{Z}_N$ is bounded above by an absolute constant independent of N.

Proof. The number of iterations of each step of the algorithm is clearly bounded above by an absolute constant. In each step the algorithm marks a bounded number of vertices. Thus, the number of vertices in each $Z \in \mathcal{Z}_N$ is also bounded. □

Define $\mathcal{Z} = \bigcup_{N \in \mathbb{N}} \mathcal{Z}_N$. Put

$$\varepsilon = \frac{1}{\sup_{Z \in \mathcal{Z}} \rho^{\max}(Z)} - \frac{1}{2}.$$

Since $\mathcal{Z}$ is finite and all graphs in $\mathcal{Z}$ have maximal density less than 2, we have $\varepsilon > 0$.

Now, consider the set $\mathcal{Y}_M \subseteq \Omega_M$ of graphs Y such that for each $Z \in \mathcal{Z}$ the following holds: in Y there exists a strict copy of Z that is (K, T)-maximal for each $\frac{1}{2}$-rigid or $\frac{1}{2}$-neutral pair (K, T) with $v(K, T) \le 20$ and, in particular, (K^*, T^*)-maximal. By Theorem 12, for each $\alpha \in \left(\frac{1}{2}, \frac{1}{2} + \varepsilon\right)$ we have $\mathsf{P}(Y \in \mathcal{Y}_M) \to 1$.

Consider arbitrary graphs $Z \in \mathcal{Z}$ and $Y \in \mathcal{Y}_M$. In Y, let us find a copy $\hat{Z}$ of Z that is (K, T)-maximal for each $\frac{1}{2}$-rigid or $\frac{1}{2}$-neutral pair (K, T) with $v(K, T) \le 20$. Let $\psi : V(Z) \to V(\hat{Z})$ be an isomorphism between the graphs. Put $y_1 = \psi(z_1)$.

Statement 27. For all i the equalities $\mathcal{J}^i(z_1) = \mathcal{J}^i(y_1)$ hold.

Proof. We first show that y_1-bad subgraphs of Y are exactly images of z_1-bad subgraphs.

First, we verify that for each z_1-bad subgraph $\tilde{U}$, the subgraph $W = \psi(\tilde{U})$ is y_1-bad. Indeed, otherwise in graph Y there is a subgraph $G' \supset W$ that is y_1-bad. If $G' \subset \hat{Z}$ then $\psi^{-1}(G') \subset Z$ and $\tilde{U} \subset \psi^{-1}(G')$, which contradicts the $\mathcal{G}$-maximality of U. Otherwise, denote by W' the graph $Y|_{V(G') \cap V(\hat{Z})}$. The pair (G', W') is either $\frac{1}{2}$-rigid or $\frac{1}{2}$-neutral by Property 7 of $\mathcal{G}$, with $v(G', W') \le v(G') - 1 \le 20$ (by Property 6 of $\mathcal{G}$). This contradicts (K, T)-maximality of $\hat{Z}$ in Y for each $\frac{1}{2}$-rigid or $\frac{1}{2}$-neutral pair (K, T) with $v(K, T) \le 20$.

Second, we verify that each y_1-bad subgraph W is an image of a z_1-bad subgraph in Z. Assume the contrary. Examine two cases. If $W \not\subseteq \hat{Z}$, then the pair $(W, W|_{V(\hat{Z}) \cap V(W)})$ is either $\frac{1}{2}$-rigid or $\frac{1}{2}$-neutral by Property 7 of $\mathcal{G}$, with $v(W, W|_{V(\hat{Z}) \cap V(W)}) \le 20$. This contradicts (K, T)-maximality of $\hat{Z}$ in Y for each $\frac{1}{2}$-rigid or $\frac{1}{2}$-neutral pair (K, T) with $v(K, T) \le 20$. If, on the other hand, $W \subseteq \hat{Z}$, then the prototype $\tilde{U}$ of W under ψ is defined. If $\tilde{U}$ is not z_1-bad, then there exists a subgraph G' of Z that contains $\tilde{U}$. Then $\psi(G')$ contains W, which contradicts the fact that W is z_1-bad.

Thus, the set of all y_1-bad subgraphs of Y is equal to the set of images of z_1-bad subgraphs of Z under the isomorphism ψ.

Once again, consider an arbitrary z_1-bad subgraph $\tilde{U}$ of type i and its image W. Let us show that $T^i(\tilde{U}) = T^i(W)$. It is evident from the definition of $\mathcal{J}^i(u)$ that the only vertices that influence the i-type of an y_1-bad subgraph are the ones from the (K^*, T^*)-neighborhood of the subgraph. In other words, upon removing from Y any vertex not in the (K^*, T^*)-neighborhood of $\hat{Z}$, the type $T^i(W)$ does not change. None of the vertices of $V(Y) \setminus V(\hat{Z})$ is in the (K^*, T^*)-neighborhood of subgraph $\hat{Z}$, due to the (K^*, T^*)-maximality of $\hat{Z}$ in Y. Upon removing all vertices of the set $V(Y) \setminus V(\hat{Z})$ from Y, we have the graph $\hat{Z}$ isomorphic to Z. Thus, we have the equality $T^i(\tilde{U}) = T^i(W)$. Similarly, for each y_1-bad subgraph W of Y with type i, there is the equality $T^i(W) = T^i(\psi^{-1}(W))$. Thus, the equalities

$$\mathcal{J}^i(x_1) = \mathcal{J}^i(y_1)$$

are verified for all i.

To sum up, for each $X \in \mathcal{X}_N$ and an arbitrary $x_1 \in V(X)$, we have found a graph $Z = Z(X, x_1)$ and a vertex $z_1 \in V(Z)$ such that $\mathcal{J}^i(x_1) = \mathcal{J}^i(z_1)$ for each i. For every graph $Y \in \mathcal{Y}_M$, we have found a vertex $y_1 \in V(Y)$ such that

$$\mathcal{J}^i(z_1) = \mathcal{J}^i(y_1)$$

for each i. Thus, for each $X \in \mathcal{X}_N$ and a vertex $x_1 \in V(X)$ and for each $Y \in \mathcal{Y}_M$, there exists $y_1 \in V(Y)$ such that

$$\mathcal{J}^i(z_1) = \mathcal{J}^i(y_1)$$

for each i. Finally, as stated at the beginning of the proof, we have $\mathsf{P}(X \in \mathcal{X}_N)$ and $\mathsf{P}(Y \in \mathcal{Y}_M)$ tend to 1, which concludes the proof. □

4.2. *The second lemma.* Let, as previously, $u, w \in V(\Gamma)$. Define $W = \Gamma|_{\{u,w\}}$. Let

$$\begin{aligned} j(u, w, t) &= j^W(t); \\ I(u, w) &= I^W(u, w); \\ J(u, w) &= J^W(u, w) = \{ j(u, w, t) \mid t \in \mathcal{N}(u, w) \}. \end{aligned}$$

Let a vertex $s \in V(\Gamma) \setminus V(W)$ satisfy $\delta^W(u, w, s) = 1$. Define

$$\begin{aligned} \sigma_1(u, w, s) &= \sigma_1^W(u, w, s); \\ \sigma_2(u, w, s) &= \sigma_2^W(u, w, s); \\ \sigma(u, w, s) &= (\sigma_1(u, w, s), \sigma_2(u, w, s)); \end{aligned}$$

$$S(u, w) = S^W(u, w) = \begin{cases} \{ s \in \mathcal{N}(\neg u, \neg w) \mid \forall t \in \mathcal{N}(u, w) : s \sim t \}, & I(u, w) = 1; \\ \varnothing, & \text{else}; \end{cases}$$

$$\Sigma(u, w) = \{ \sigma(u, w, s) \mid s \in S(u, w) \}.$$

Lemma 28. *There exists an $\varepsilon > 0$ such that for each $\alpha \in \left(\frac{1}{2}, \frac{1}{2} + \varepsilon\right)$ a.a.s. (X, Y) satisfies the following condition: For each pair of vertices $x_1, x_2 \in V(X)$ such that $x_2 \sim x_1$ and x_2 is in 0-neighborhood $U^0(x_1)$, and for any vertex $y_1 \in V(Y)$, there exists $y_2 \in V(Y)$ such that*

$$\begin{aligned} &y_2 \in U^0(y_1); \\ &y_2 \sim y_1; \\ &J(x_1, x_2) = J(y_1, y_2); \\ &I(x_1, x_2) = I(y_1, y_2); \\ &\Sigma(x_1, x_2) = \Sigma(y_1, y_2). \end{aligned}$$

Here X, Y are independent random graphs $G(N, N^{-\alpha})$, $G(M, M^{-\alpha})$.

Proof. Unlike the previous proof, we construct a pair of graphs, not a single graph. We then find a strict extension isomorphic to the constructed pair in Y.

For each N, put $\mathcal{X}_N = \Omega_N$. Consider an arbitrary $X \in \mathcal{X}_N$ and two vertices $x_1 \in V(X)$, $x_2 \in U^0(x_1)$ such that $x_1 \sim x_2$.

Let $F = (\{z_1\}, \varnothing)$. We propose the algorithm of constructing a graph $Z = Z(X, x_1, x_2)$, $F \subset Z$ such that the pair (Z, F) is $\frac{1}{2}$-safe and

$$z_2 \in U^0(z_1);$$
$$z_2 \sim z_1;$$
$$J(x_1, x_2) = J(z_1, z_2);$$
$$I(x_1, x_2) = I(z_1, z_2);$$
$$\Sigma(x_1, x_2) = \Sigma(z_1, z_2).$$

(1) Mark vertices z_1, z_2, and join them with an edge.
  (a) If $S(x_1, x_2)$ is nonempty and $|\mathcal{N}(x_1, x_2)| = 2$:
    (i) For each $t \in \mathcal{N}(x_1, x_2)$, mark a vertex r_t, join it with z_1 and z_2, and construct a strict (A, B)-extension of the subgraph induced on z_1, z_2, r_t, where (A, B) is the pair of graphs that corresponds to $j(x_1, x_2, t)$.
    (ii) For each $\sigma \in \Sigma(x_1, x_2)$ mark a vertex r_σ and join it with r_t for all $t \in \mathcal{N}(x_1, x_2)$. For $k = 1, 2$, if $\sigma_k = 1$, mark r_k and join it with r_σ and z_k.
  (b) Else:
    (i) For each $j \in J(x_1, x_2)$, mark r_j, join it with z_1 and z_2, and construct a strict (A, B)-extension of the subgraph induced on z_1, z_2, r_j, where (A, B) is the pair of graphs corresponding to j.
    (ii) If $|J(x_1, x_2)| = 1$, but $I(x_1, x_2) = 1$, repeat Step (i) once again.

(2) Marked vertices and drawn edges constitute the graph Z.

Let us show that if $\Sigma(x_1, x_2)$ is nonempty, then the equality $|\mathcal{N}(x_1, x_2)| = 2$ holds. First, we have

$$|\mathcal{N}(x_1, x_2)| \geq 2,$$

by definition of $S(x_1, x_2)$. Let $|\mathcal{N}(x_1, x_2)| > 2$. Then in the graph X, there exist vertices t_1, t_2, $t_3 \in \mathcal{N}(x_1, x_2)$ and $s \in \mathcal{N}(t_1, t_2, t_3)$ different from x_1 and x_2. Then $X|_{\{x_1,x_2,t_1,t_2,t_3,s\}}$ is a $\frac{1}{2}$-neutral extension of $X|_{\{x_1\}}$, with $v(X_{\{x_1,x_2,t_1,t_2,t_3,s\}}, X|_{\{x_1\}}) = 5$, which contradicts the inclusion $x_2 \in U^0(x_1)$. Thus, $|\mathcal{N}(x_1, x_2)| = 2$.

Statement 29. The constructed pair (Z, F) is $\frac{1}{2}$-safe.

Proof. Construction performed in the algorithm steps other then (1) boils down to sequential construction of strict (K^*, T^*)-extensions of some subgraphs of the graph being constructed. Therefore, Z is a strict (K^*, T^*)-neighborhood of its subgraph $Z|_{\{z_1,z_2\}}$. Moreover, the pair $(Z|_{\{z_1,z_2\}}, F)$ is $\frac{1}{2}$-safe. By Statement 19 we obtain the desired conclusion. □

Statement 30. The following equalities hold:

$$z_2 \in U^0(z_1);$$
$$z_2 \sim z_1;$$
$$J(x_1, x_2) = J(z_1, z_2);$$
$$I(x_1, x_2) = I(z_1, z_2);$$
$$\Sigma(x_1, x_2) = \Sigma(z_1, z_2).$$

Proof. We first prove that $z_2 \in U^0(z_1)$. Assume the contrary, i.e., there exists an induced subgraph $\tilde{U} \subseteq Z$ isomorphic to some $G \in \mathcal{G}$ containing z_1 and z_2, with z_1 being the image of $z \in V(G)$ under the isomorphism between G and $\tilde{U}$. Then $(\tilde{U}, F)$ is either $\frac{1}{2}$-rigid or $\frac{1}{2}$-neutral by Property 7 of $\mathcal{G}$. We obtain the contradiction with the $\frac{1}{2}$-safeness of (Z, F).

The relation $z_2 \sim z_1$ holds by construction of Z.

The other equalities can be verified similar to the corresponding ones in the proof of Statement 25, with $U = X|_{\{x_1,x_2\}}$. □

Define $\mathcal{Z}_N = \{Z(X, x_1, x_2) \mid X \in \mathcal{X}_N;\ x_1, x_2 \in V(X)\}$. As in Lemma 22, the total number of vertices in any $Z \in \mathcal{Z}_N$ is bounded above by an absolute constant.

Put $\mathcal{Z} = \bigcup_{N \in \mathbb{N}} \mathcal{Z}_N$. Let

$$\varepsilon = \inf_{Z \in \mathcal{Z}} \frac{v(Z, F)}{e(Z, F)} - \frac{1}{2}.$$

Obviously, $\varepsilon > 0$.

Let $\mathcal{Y}_M \subseteq \Omega_M$ be a set of graphs on M vertices such that for each $Z \in \mathcal{Z}$ and $y_1 \in V(Y)$ there exists a strict (Z, F)-extension of $Y|_{\{y_1\}}$ in Y that is (K, T)-maximal for each $\frac{1}{2}$-rigid or $\frac{1}{2}$-neutral (K, T) with $v(K, T) \leq 19$. By Theorem 12, for each $\alpha \in \left(\frac{1}{2}, \frac{1}{2} + \varepsilon\right)$ we have $\mathsf{P}(Y \in \mathcal{Y}_M) \to 1$.

Consider an arbitrary $Z \in \mathcal{Z}$, $Y \in \mathcal{Y}_M$ and an arbitrary vertex $y_1 \in V(Y)$. Find a strict (Z, F)-extension $\hat{Z}$ of $Y|_{\{y_1\}}$ in Y such that $\hat{Z}$ is (K, T)-maximal for each $\frac{1}{2}$-rigid or $\frac{1}{2}$-neutral pair (K, T) with $v(K, T) \leq 19$. Let y_2 be an image of z_2 under the isomorphism between Z and $\hat{Z}$.

Statement 31. The following equalities hold:

$$y_2 \in U^0(y_1);$$
$$y_2 \sim y_1;$$
$$J(z_1, z_2) = J(y_1, y_2);$$
$$I(z_1, z_2) = I(y_1, y_2);$$
$$\Sigma(z_1, z_2) = \Sigma(y_1, y_2).$$

Proof. We first verify that $y_2 \in U^0(y_1)$. It is sufficient to show that there exist no induced subgraphs $W \subseteq Y$ isomorphic to some $G \in \mathcal{G}$ such that $y_2 \in V(W)$ and y_1 is the image of z under the isomorphism between G and W. Assume the contrary. Note that $W \not\subseteq \hat{Z}$, because $z_2 \in U^0(z_1)$. Then $(W, W|_{V(W) \cap V(\hat{Z})})$ is $\frac{1}{2}$-rigid by Property 7 of $\mathcal{G}$, with $v(W, W|_{V(W) \cap V(\hat{Z})}) \leq 19$, which contradicts (K, T)-maximality of $\hat{Z}$ in Y for each $\frac{1}{2}$-rigid (K, T) with $v(K, T) \leq 19$.

Moreover, obviously, $y_1 \sim y_2$. The equalities $J(z_1, z_2) = J(y_1, y_2)$, $I(z_1, z_2) = I(y_1, y_2)$, $\Sigma(z_1, z_2) = \Sigma(y_1, y_2)$ can be verified similar to the corresponding ones in Lemma 22. □

Thus, for each $X \in \mathcal{X}_N$ and vertices $x_1, x_2 \in V(X)$ such that $x_1 \sim x_2$, we constructed $Z = Z(X, x_1, x_2)$ and vertices $z_1, z_2 \in V(Z)$ such that the equalities for x_1, x_2, z_1, z_2 from Lemma 28 hold. For each $Y \in \mathcal{Y}_M$ and $y_1 \in V(Y)$ we found $y_2 \in V(Y)$ such that the equalities for x_1, x_2, z_1, z_2 from Lemma 28 hold. Since $\mathsf{P}(X \in \mathcal{X}_N)$ and $\mathsf{P}(Y \in \mathcal{Y}_M)$ tend to 1 (as stated at the beginning of the proof), Lemma 28 is proved. □

4.3. ***The third lemma.*** We finally introduce our last definitions. Let

$$J'(u, w) = \big\{ j(u, w, t) \mid (t \in \mathcal{N}(u, w)) \wedge (\delta(u, w, t) = 0) \big\};$$
$$I'(u, w) = (\exists t \in \mathcal{N}(u, w)\ \delta(u, w, \neg t) = 1);$$
$$T(u, w) = \big\{ \{t_1, t_2\} \mid t_1, t_2 \in \mathcal{N}(u, w), t_1 \sim t_2 \big\};$$
$$\tau(u, w) = \big\{ \{j(u, w, t_1), j(u, w, t_2)\} \mid t_1, t_2 \in \mathcal{N}(u, w), t_1 \sim t_2 \big\}.$$

Let us formulate and prove the next lemma.

Lemma 32. *There exists an $\varepsilon > 0$, such that for each $\alpha \in \left(\frac{1}{2}, \frac{1}{2} + \varepsilon\right)$ a.a.s. (X, Y) satisfies the following condition: For each pair of vertices $x_1, x_2 \in V(X)$ such that $x_2 \not\sim x_1$ and x_2 is in 0-neighborhood $U^0(x_1)$, and for any vertex $y_1 \in V(Y)$, there exists $y_2 \in V(Y)$ such that*

$$y_2 \in U^0(y_1);$$
$$y_2 \not\sim y_1;$$
$$J'(x_1, x_2) = J'(y_1, y_2);$$
$$I'(x_1, x_2) = I'(y_1, y_2);$$
$$\Sigma(x_1, x_2) = \Sigma(y_1, y_2);$$
$$\tau(x_1, x_2) = \tau(y_1, y_2);$$
$$|T(x_1, x_2)| \leq 1.$$

Here X, Y are independent random graphs $G(N, N^{-\alpha})$, $G(M, M^{-\alpha})$.

Proof. For each N, put $\mathcal{X}_N = \Omega_N$. Consider an arbitrary $X \in \mathcal{X}_N$ and two vertices $x_1 \in V(X)$, $x_2 \in U^0(x_1)$, $x_1 \not\sim x_2$.

Let us show that $|T(x_1, x_2)| \leq 1$. Assume the contrary. Then in X there are at least two pairs of vertices t_1, t_2 and t_3, t_4 that are adjacent to each other as well as to each of x_1 and x_2. If these pairs do not intersect, i.e., $\{t_1, t_2\} \cap \{t_3, t_4\} = \varnothing$, then $X|_{\{x_1,x_2,t_1,t_2,t_3,t_4\}}$ is a $\frac{1}{2}$-neutral extension of $X|_{\{x_1\}}$ with $v(X|_{\{x_1,x_2,t_1,t_2,t_3,t_4\}}, X|_{\{x_1\}}) = 5$. If these pairs do intersect, for instance, $t_1 = t_3$, then $X|_{\{x_1,x_2,t_1,t_2,t_4\}}$ is a $\frac{1}{2}$-neutral extension of $X|_{\{x_1\}}$, with $v(X|_{\{x_1,x_2,t_1,t_2,t_4\}}, X|_{\{x_1\}}) = 4$. Both alternatives contradict $x_2 \in U^0(x_1)$.

Further, let $|T(x_1, x_2)| = 1$ and $S(x_1, x_2)$ be nonempty. Let us prove that $|\mathcal{N}(x_1, x_2)| = 2$. If $|\mathcal{N}(x_1, x_2)| \geq 3$, let t_1, t_2, t_3 be some vertices from $\mathcal{N}(x_1, x_2)$ such that $t_1 \sim t_2$. Since $S(x_1, x_2)$ (see Section 3.5) is nonempty, in the graph X there is a vertex s adjacent to each t_l, with $l = 1, 2, 3$. Then $X|_{\{x_1,x_2,t_1,t_2,t_3,s\}}$ is a $\frac{1}{2}$-neutral extension of $X|_{\{x_1\}}$, with $v(X|_{\{x_1,x_2,t_1,t_2,t_3,s\}}, X|_{\{x_1\}}) = 5$, which contradicts $x_2 \in U^0(x_1)$.

So, put $F = (\{z_1\}, \varnothing)$. We now introduce an algorithm for constructing a graph $Z = Z(X, x_1, x_2)$, $F \subset Z$, such that (Z, F) is $\frac{1}{2}$-safe and

$$z_2 \in U^0(z_1);$$
$$z_2 \not\sim z_1;$$
$$J'(x_1, x_2) = J'(z_1, z_2);$$
$$I'(x_1, x_2) = I'(z_1, z_2);$$
$$\Sigma(x_1, x_2) = \Sigma(z_1, z_2);$$
$$\tau(x_1, x_2) = \tau(z_1, z_2).$$

(1) Mark z_1, z_2.

(2) If $T(x_1, x_2)$ is nonempty:
   (a) Let t_1, t_2 be some adjacent vertices of $\mathcal{N}(x_1, x_2)$.
   (b) Mark r_{t_1} and r_{t_2}, join each with z_1 and z_2 and to each other. With $k = 1, 2$ construct a strict (A, B)-extension of the subgraph induced on z_1, z_2, r_{t_k}, where (A, B) is the pair of graphs that corresponds to $j(x_1, x_2, t_k)$;
   (c) If $S(x_1, x_2)$ is nonempty:
      (i) For each $\sigma \in \Sigma(x_1, x_2)$ mark a vertex r_σ and join it with r_{t_ℓ} for $\ell = 1, 2$. For $k = 1, 2$, if $\sigma_k = 1$, mark r_k and join it with r_σ and z_k.
      (ii) For each $j \in J'(x_1, x_2)$ mark r_j, join it with z_1, z_2, and construct a strict (A, B)-extension of the subgraph induced on z_1, z_2, r_j, where (A, B) is a pair of graphs corresponding to j.

(3) Else:
   (a) If $S(x_1, x_2)$ is nonempty:
      (i) For each $t \in \mathcal{N}(x_1, x_2)$ mark r_t, join it with z_1, z_2, and construct a strict (A, B)-extension of the subgraph induced on z_1, z_2, r_t, where (A, B) is the pair of graphs that corresponds to $j(x_1, x_2, t)$.
      (ii) For each $\sigma \in \Sigma(x_1, x_2)$ mark r_σ and join it with r_t for all $t \in \mathcal{N}(x_1, x_2)$. For $k = 1, 2$, if $\sigma_k = 1$, mark r_k and join it with r_σ and z_k.
   (b) Else:
      (i) For each $j \in J(x_1, x_2)$ mark r_j, join it with z_1, z_2 and construct a strict (A, B)-extension of the subgraph induced on z_1, z_2, r_j, where (A, B) is the pair of graphs corresponding to j.
      (ii) If $|J(x_1, x_2)| = 1$ but $I(x_1, x_2) = 1$, repeat Step (i) once again.

(4) Marked vertices and constructed edges constitute Z.

The rest of the proof of Lemma 32 is similar to the proof of Lemma 28. We conclude the proof of Lemma 32 by proving two statements similar to Statements 29 and 30. □

Statement 33. The pair (Z, F) is $\frac{1}{2}$-safe.

Proof. Let us examine the case $|T(x_1, x_2)| = 0$.

If $S(x_1, x_2) = \varnothing$ or $|\mathcal{N}(x_1, x_2)| = 2$, then construction in all steps of the algorithm but for Step (1) are similar to ones in Lemma 28. Thus, the proof boils down to the proof of the similar statement in Lemma 28.

Let $S(x_1, x_2) \neq \varnothing$ and $|\mathcal{N}(x_1, x_2)| > 2$. If $|\mathcal{N}(x_1, x_2)| \geq 4$, denote by t_1, t_2, t_3, t_4 some vertices of $\mathcal{N}(x_1, x_2)$. Since $S(x_1, x_2)$ is nonempty, in graph X there exists a vertex s joined with each of t_ℓ for $1 \leq \ell \leq 4$, different from x_1 and x_2. Therefore $X|_{\{x_1,x_2,t_1,t_2,t_3,t_4,s\}}$ is a $\frac{1}{2}$-neutral extension of $X|_{\{x_1\}}$, with $v(X_{\{x_1,x_2,t_1,t_2,t_3,t_4,s\}}, X|_{\{x_1\}}) = 6$, which contradicts $x_2 \in U^0(x_1)$. Thus, $|\mathcal{N}(x_1, x_2)| = 3$. Denote by t_1, t_2, t_3 the vertices of $\mathcal{N}(x_1, x_2)$.

Finally, assume $|S(x_1, x_2)| \geq 2$. Let s_1, s_2 be some elements of $S(x_1, x_2)$. The graph $X|_{\{x_1,x_2,t_1,t_2,t_3,s_1,s_2\}}$ is a $\frac{1}{2}$-neutral extension of $X|_{\{x_1\}}$, with $v(X|_{\{x_1,x_2,t_1,t_2,t_3,s_1,s_2\}}, X|_{\{x_1\}}) = 6$, which contradicts $x_2 \in U^0(x_1)$. Thus, $|S(x_1, x_2)| = 1$. By definition of Σ (see Section 3.5), we obtain that $|\Sigma(x_1, x_2)| = 1$.

Denote by σ the single element of $\Sigma(x_1, x_2)$. The algorithm marks the vertices r_{t_l} for $1 \leq l \leq 3$ that are joined with z_1 and z_2 and also a vertex $r_\sigma \in \mathcal{N}(r_{t_1}, r_{t_2}, r_{t_3})$.

Note that $(Z|_{\{z_1,z_2,r_{t_1},r_{t_2},r_{t_3},r_\sigma\}}, F)$ is $\frac{1}{2}$-safe. Moreover, Z is obviously a strict (K^*, T^*)-neighborhood of $Z|_{\{z_1,z_2,r_{t_1},r_{t_2},r_{t_3},r_\sigma\}}$. By Statement 19, we obtain the desired statement for $|T(x_1, x_2)| = 0$.

If $|T(x_1, x_2)| = 1$, then $(Z|_{\{z_1,z_2,r_{t_1},r_{t_2}\}}, F)$ is $\frac{1}{2}$-safe and Z is the strict (K^*, T^*)-neighborhood of $Z|_{\{z_1,z_2,r_{t_1},r_{T_1}\}}$. By Statement 19, the pair (Z, F) is $\frac{1}{2}$-safe, which completes the proof. □

Statement 34. The equalities

$$z_2 \in U^0(z_1);$$
$$z_2 \not\sim z_1;$$
$$J'(x_1, x_2) = J'(z_1, z_2);$$
$$I'(x_1, x_2) = I'(z_1, z_2);$$
$$\Sigma(x_1, x_2) = \Sigma(z_1, z_2);$$
$$\tau(x_1, x_2) = \tau(z_1, z_2)$$

hold.

Proof. We verify that $z_2 \in U^0(z_1)$. Assume the contrary, i.e., let there exist an induced subgraph $\tilde{U} \subseteq Z$ isomorphic to some $G \in \mathcal{G}$ that contains z_1 and z_2, such that z_1 is the image of $z \in V(G)$ under the isomorphism between G and $\tilde{U}$. Then $(\tilde{U}, F)$ is either $\frac{1}{2}$-rigid or $\frac{1}{2}$-neutral due to Property 7 of $\mathcal{G}$. We obtain a contradiction with the $\frac{1}{2}$-safeness of (Z, F).

The relation $z_2 \not\sim z_1$ holds by construction of Z.

Consider an arbitrary $t \in \mathcal{N}^U(x_1, x_2)$. Let $j = j(x_1, x_2, t)$. If $\delta(x_1, x_2, t) = 1$, then $|T(x_1, x_2)| = 1$ and in Step (b) the algorithm marks $r_t \in \mathcal{N}(z_1, z_2)$. If $\delta(x_1, x_2, t) = 0$ and $S(x_1, x_2)$ is nonempty, then in Step (i) the algorithm marks $r_t \in \mathcal{N}(z_1, z_2)$. If $S(x_1, x_2)$ is empty, then in Step (i) the algorithm marks $r_j \in \mathcal{N}(z_1, z_2)$. Let $\tilde{t} = r_t$ in the first or the second case and $\tilde{t} = r_j$ in the third one.

We verify the equality

$$j(x_1, x_2, t) = j(z_1, z_2, \tilde{t}).$$

Let $U = X|_{\{x_1,x_2\}}$. If $\delta(x_1, x_2, t) = 0$, then the proof is similar to the corresponding statement in Lemma 22. Let $\delta(x_1, x_2, t) = 1$. Denote by t' the only vertex of $\mathcal{N}(x_1, x_2, t)$. Note that in the definition of $j^U(t)$, all formulas only contain the existence of vertices adjacent to exactly one vertex of U. Thus, t' does not influence $j^U(t)$ in the sense that upon removing t' from X, the value $j^U(t)$ does not change. With that note, the equality $j(x_1, x_2, t) = j(z_1, z_2, \tilde{t})$ can be proven analogously to the corresponding statement in Lemma 22.

We verify that

$$J'(x_1, x_2) = J'(z_1, z_2).$$

Consider an arbitrary $j \in J'(x_1, x_2)$ given by some $t \in \mathcal{N}(x_1, x_2)$, $\delta(x_1, x_2, t) = 0$. In one of the Steps (i), (i), (ii) a vertex $\tilde{t}$ is marked such that $j(z_1, z_2, \tilde{t}) = j$. Note that the algorithm does not mark any vertices adjacent to $\tilde{t}$, z_1, z_2 at the same time; therefore, $\delta(z_1, z_2, \tilde{t}) = 0$ and $j \in J'(z_1, z_2)$. Thus, the inclusion $J'(x_1, x_2) \subseteq J'(z_1, z_2)$ is shown. Consider now an arbitrary $j \in J'(z_1, z_2)$ given by a vertex $\tilde{t} \in \mathcal{N}(z_1, z_2)$, $\delta(z_1, z_2, \tilde{t}) = 0$. Vertex $\tilde{t}$ may only be marked in one of the aforementioned steps of the algorithm. Thus, there exists a vertex $t \in \mathcal{N}(x_1, x_2)$, $\delta(t, x_1, x_2) = 0$ such that $j(x_1, x_2, t) = j(z_1, z_2, \tilde{t})$. The reverse inclusion is shown, as well as the equality $J'(x_1, x_2) = J'(z_1, z_2)$.

We verify that

$$I'(x_1, x_2) = I'(z_1, z_2).$$

If $T(x_1, x_2)$ is empty, then $T(z_1, z_2)$ is also empty and the equality boils down to $I(x_1, x_2) = I(z_1, z_2)$, which can be proven analogously to the similar one in Lemma 22. Let $T(x_1, x_2)$ be nonempty. If $I'(x_1, x_2) = 1$, then $J'(x_1, x_2)$ is nonempty, and in Step (ii) the algorithm marks the vertex $r_j \in \mathcal{N}(z_1, z_2)$, with $\delta(z_1, z_2, r_j) = 0$, which guarantees $I'(z_1, z_2) = 1$. If, on the other hand, $I'(x_1, x_2) = 0$, then $J'(x_1, x_2)$ is empty and the algorithm does not mark any vertices joined with both z_1 and z_2, but for two adjacent vertices. Thus, $I'(z_1, z_2) = 0$, and the equality $I'(x_1, x_2) = I'(z_1, z_2)$ is shown.

We verify that

$$\Sigma(x_1, x_2) = \Sigma(z_1, z_2).$$

If $S(x_1, x_2)$ is empty, then either $|\mathcal{N}(x_1, x_2)| \leq 1$ and consequently $|\mathcal{N}(z_1, z_2)| \leq 1$ and $S(z_1, z_2)$ is also empty, or $|\mathcal{N}(x_1, x_2)| > 1$; but in this case the algorithm does not mark any vertices adjacent to each vertex of $\mathcal{N}(z_1, z_2)$, but for z_1 and z_2. Thus, $S(z_1, z_2)$ is empty. To sum up, if $S(x_1, x_2)$ is empty, then $S(z_1, z_2)$ is also empty and $\Sigma(x_1, x_2) = \varnothing = \Sigma(z_1, z_2)$.

Let $S(x_1, x_2)$ be nonempty. If $T(x_1, x_2)$ is empty, then the proof of the equality $\Sigma(x_1, x_2) = \Sigma(z_1, z_2)$ is similar to the corresponding equality in Lemma 22. Let $T(x_1, x_2)$ be nonempty. Denote the elements of $\mathcal{N}(x_1, x_2)$ by t_1, t_2. Consider an arbitrary $\sigma \in \Sigma(x_1, x_2)$ given by a vertex $s \in \mathcal{N}^U(t_1, t_2)$. In Step (i), the vertex $r_\sigma \in \mathcal{N}^U(t_1, t_2)$ is marked. Let us show that $\sigma(x_1, x_2, s) = \sigma(z_1, z_2, r_\sigma)$. Fix $k \in \{0, 1\}$. If $\sigma_k(x_1, x_2, s) = 1$, then in Step (i) the algorithm marks a vertex $r_k \in \mathcal{N}(z_k, r_\sigma)$ that is not adjacent to z_{2-k}. Thus, $\sigma_k(z_1, z_2, r_\sigma) = 1$. If $\sigma_k(x_1, x_2, s) = 0$, then each vertex of Z adjacent to z_k and r_σ is a vertex of $\mathcal{N}(z_1, z_2)$. Consequently, $\sigma_k(z_1, z_2, r_\sigma) = 0$. Thus, $\sigma(x_1, x_2, s) = \sigma(z_1, z_2, r_\sigma)$. Finally, a vertex from $S(z_1, z_2)$ may only be marked in Step (i) as r_σ for some $\sigma \in \Sigma(x_1, x_2)$. The equality $\Sigma(x_1, x_2) = \Sigma(z_1, z_2)$ is verified.

Let us show that

$$\tau(x_1, x_2) = \tau(z_1, z_2).$$

If $\tau(x_1, x_2)$ is empty, then the algorithm does not mark any pair of $\mathcal{N}(z_1, z_2)$ adjacent to one another, and $\tau(z_1, z_2)$ is empty. Let $\tau(x_1, x_2)$ be nonempty. Denote by t_1, t_2 the vertices of $\mathcal{N}(x_1, x_2)$ adjacent to one another. The algorithm marks $\tilde{t}_1, \tilde{t}_2 \in \mathcal{N}(z_1, z_2)$ adjacent to one another, and by what we have proved above, we have $j(x_1, x_2, t_k) = j(z_1, z_2, \tilde{t}_k)$ for $k = 1, 2$. There are no other pairs of vertices in Z adjacent to one another and to z_1 and z_2. The equality $\tau(x_1, x_2) = \tau(z_1, z_2)$ is verified. $\square$

5. Proof of the main theorem

Proof of Theorem 7. By Theorem 8, it is sufficient to show that there exists an $\varepsilon > 0$ such that for each $\alpha \in \left(\frac{1}{2}, \frac{1}{2}+\varepsilon\right)$ Duplicator a.a.s. has a winning strategy in the game $\mathrm{EHR}(A, B, 4)$, where $A \sim G(N, N^{-\alpha})$ and $B \sim G(M, M^{-\alpha})$ are independent random graphs.

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be the values obtained from Lemmas 22, 28, 32. Let

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}.$$

Consider an $\alpha \in \left(\frac{1}{2}, \frac{1}{2}+\varepsilon\right)$.

Consider a set $\mathcal{P}$ of all graph pairs (X, Y) satisfying the following conditions:

(1) For each $x_1 \in V(X)$ there exists $y_1 \in V(Y)$ such that $\mathcal{J}^i(x_1) = \mathcal{J}^i(y_1)$ for all i; and, vice versa, for each $y_1 \in V(Y)$ there exists $x_1 \in V(X)$ such that $\mathcal{J}^i(x_1) = \mathcal{J}^i(y_1)$ for all i.

(2) For each pair of vertices $x_1, x_2 \in V(X)$ such that $x_1 \sim x_2$, $x_2 \in U^0(x_1)$ and $y_1 \in V(Y)$, there exists $y_2 \in V(Y)$, $y_2 \sim y_1$ such that the conditions of Lemma 28 hold, and vice versa.

(3) For each pair of vertices $x_1, x_2 \in V(X)$ such that $x_1 \not\sim x_2$, $x_2 \in U^0(x_1)$ and $y_1 \in V(Y)$, there exists $y_2 \in V(Y)$, $y_2 \not\sim y_1$ such that the conditions of Lemma 32 hold, and vice versa.

(4) For each subgraph $S \subseteq X$ with $v(X) \leq 3$ and each $\frac{1}{2}$-safe pair (G, H) such that $v(G, H) \leq 22 \cdot 23 \cdot 2^{22}$ in X, there exists a strict (G, H)-extension of S that is (K, T)-maximal for each $\frac{1}{2}$-rigid or $\frac{1}{2}$-neutral (K, T) with $v(K, T) \leq 6$, and vice versa.

From Lemmas 22, 28, 32, and Theorem 12 it follows that $\mathsf{P}((A, B) \in \mathcal{P}) \to 1$. Let us prove that in such conditions Duplicator has a winning strategy.

For simplicity, we will often use the following denotation: Let A be an induced subgraph of some graph Γ (for example, $\Gamma = X, Y$) and $a_1, \ldots, a_l$ are some vertices from $V(\Gamma) \setminus V(A)$. By $A + a_1 + \ldots + a_l$ we denote the graph $\Gamma|_{V(A) \cup \{a_1, \ldots, a_l\}}$.

Without loss of generality we may assume that in the first round Spoiler chooses a vertex $x_1 \in V(X)$. Duplicator chooses a vertex $y_1 \in (Y)$ such that

$$\mathcal{J}^i(x_1) = \mathcal{J}^i(y_1)$$

for all i.

From now on during the proof we will only use the equality $\mathcal{J}^i(x_1) = \mathcal{J}^i(y_1)$ and Properties 1–4 of the pair (X, Y). We will not use the explicit construction from Lemma 22. Thus, X and Y are symmetric.

Without loss of generality we may assume that in the second round Spoiler chooses a vertex $x_2 \in V(X)$. We examine several cases.

Case 1. Let x_2 be in a subgraph $U_X \in \mathcal{U}_i(x_1)$ for some $i \geq 1$. Due to the equality $\mathcal{J}^i(x_1) = \mathcal{J}^i(y_1)$, the graph Y contains a y_1-bad subgraph U_Y of type i for which we have

$$T^i(U_X) = T^i(U_Y).$$

Let $\varphi : V(U_X) \to V(U_Y)$ be an isomorphism between U_X and U_Y. Duplicator answers by choosing $y_2 = \varphi(x_2) \in V(Y)$. Note that X and Y are still symmetric.

Without loss of generality we may assume that in the third round Spoiler chooses a vertex of $x_3 \in V(X)$. We now examine several subcases of Case 1.

Subcase 1a. Assume that $x_3 \in V(U_X)$.

Duplicator answers by choosing

$$y_3 = \varphi(x_3) \in V(Y).$$

In round 4, without loss of generality, Spoiler chooses $x_4 \in V(X)$. If $x_4 \in V(U_X)$, then Duplicator chooses $y_4 = \varphi(x_4) \in V(Y)$ and wins due to isomorphism of U_X and U_Y.

Otherwise, if x_4 is joined with not more than one vertex from those chosen before, then the pair $(X|_{\{x_1,x_2,x_3,x_4\}}, X|_{\{x_1,x_2,x_3\}})$ is $\frac{1}{2}$-safe. By Property 4 of the pair (X, Y), in Y there exists a vertex y_4 such that $Y|_{\{y_1,y_2,y_3,y_4\}}$ is a strict $(X|_{\{x_1,x_2,x_3,x_4\}}, X|_{\{x_1,x_2,x_3\}})$-extension of $Y|_{\{y_1,y_2,y_3\}}$. Duplicator chooses y_4 and wins.

If x_4 is adjacent to each of the vertices x_1, x_2, x_3, then $(X|_{\{x_1,x_2,x_3,x_4\}}, X|_{\{x_1,x_2,x_3\}})$ is $\frac{1}{2}$-rigid with $v(X|_{\{x_1,x_2,x_3,x_4\}}, X|_{\{x_1,x_2,x_3\}}) = 1$. Therefore, the subgraph U_X is not x_1-bad in X, which contradicts the definition of U_X.

Let x_4 be adjacent to exactly two of the vertices x_1, x_2, x_3, say x_k and x_n. Then we have $\delta^{U_X}(x_k, x_n) = 1 = \delta^{U_Y}(y_k, y_n)$. By definition, in $V(Y) \setminus V(U_Y)$ there exists a vertex y_4 adjacent to y_k and y_n. Moreover, y_4 is not adjacent to the remaining vertex, same as x_4 cannot be adjacent to each of x_ℓ for $1 \le \ell \le 3$ at the same time. Thus, Duplicator chooses y_4 and wins due to isomorphism of U_X and U_Y and properties of x_4, y_4.

Subcase 1a has been examined.

If $x_3 \notin V(U_X)$, then $|\mathcal{N}(x_3) \cap V(U_X)| \le 2$. Otherwise, the pair $(U_X + x_3, U_X)$ is $\frac{1}{2}$-rigid, with $v(U_X + x_3, U_X) = 1$, which contradicts the fact that U_X is x_1-bad.

Subcase 1b. Assume that $x_3 \notin V(U_X)$ and $|\mathcal{N}(x_3) \cap V(U_X)| = 2$.

Let z_{1X} and z_{2X} be the vertices from U_X adjacent to x_3. Put $z_{1Y} = \varphi(z_{1X})$, $z_{2Y} = \varphi(z_{2X})$. From the equality $J^{U_X}(z_{1X}, z_{2X}) = J^{U_Y}(z_{1Y}, z_{2Y})$, it follows that in Y there exists a vertex $y_3 \in \mathcal{N}^{U_Y}(z_{1Y}, z_{2Y})$ such that

$$j^{U_X}(x_3) = j^{U_Y}(y_3).$$

Duplicator chooses y_3. Note that for all $z \in V(U_X)$, we have $x_3 \sim z$ if and only if $y_3 \sim \varphi(z)$.

In round 4, without loss of generality, Spoiler chooses $x_4 \in V(X)$. If $x_4 \in V(U_X)$, then Duplicator chooses $y_4 = \varphi(x_4)$ and wins.

Otherwise, if x_4 is adjacent to not more than one vertex from x_1, x_2, x_3, then Duplicator wins due to α-safeness of $(X|_{\{x_1,x_2,x_3,x_4\}}, X|_{\{x_1,x_2,x_3\}})$.

If x_4 is adjacent to each of the vertices x_1, x_2, x_3, then the pair $(U_X + x_3 + x_4, U_X)$ is $\frac{1}{2}$-rigid, with $v(U_X + x_3 + x_4, U_X) = 2$, which contradicts the fact that U_X is x_1-bad.

If x_4 is adjacent to exactly two of x_1, x_2, x_3, then two possibilities may occur.

If x_4 is adjacent to x_1, x_2, then $x_4 \in \mathcal{N}^{U_X}(x_1, x_2)$. Let us show that there is a vertex in $V(Y) \setminus V(U_Y)$ adjacent to y_1 and y_2, different from y_3. If $x_3 \in \mathcal{N}^{U_X}(x_1, x_2)$, then $I^{U_X}(x_1, x_2) = 1 = I^{U_Y}(y_1, y_2)$. If, on the other hand, $x_3 \notin \mathcal{N}^{U_X}(x_1, x_2)$, then $y_3 \notin \mathcal{N}^{U_Y}(y_1, y_2)$ and $\delta^{U_X}(x_1, x_2) = 1 = \delta^{U_Y}(y_1, y_2)$. In both cases there exists a vertex $y_4 \in V(Y) \setminus V(U_Y)$ adjacent to y_1 and y_2. Note that y_4 cannot be adjacent to y_3, otherwise U_Y is not y_1-bad in Y. Thus, Duplicator chooses y_4 and wins.

Finally, let x_4 be adjacent to x_3 and some x_k for $k \in \{1, 2\}$, but not adjacent to x_l, $l \in \{1, 2\} \setminus \{k\}$. Consider the equality $j^{U_X}(x_3) = j^{U_Y}(y_3)$. We have

$$j^{U_X}_{x_k}(x_3) = 1 = j^{U_Y}_{y_k}(y_3),$$

which implies the existence of a vertex $y_4 \in \mathcal{N}^{U_Y}(y_k, y_3, \neg y_l)$ in Y. Duplicator chooses $y_4 \in V(Y)$ and wins.

Subcase 1c. Assume that $x_3 \notin V(U_X)$, $|\mathcal{N}(x_3) \cap V(U_X)| = 1$, and in $V(X) \setminus V(U_X)$ there exists a vertex t_X adjacent to x_3 such that $|\mathcal{N}(t_X) \cap V(U_X)| = 2$.

Denote by z_{1X} a vertex of U_X adjacent to x_3, and put $z_{1Y} = \varphi(z_{1X})$.

Let z_{2X} and z_{3X} be the vertices of U_X adjacent to t_X. Put $z_{2Y} = \varphi(z_{2X})$, $z_{3Y} = \varphi(z_{3X})$. Due to $J^{U_X}(z_{2X}, z_{3X}) = J^{U_Y}(z_{2Y}, z_{3Y})$, there exists a vertex $t_Y \in \mathcal{N}^{U_Y}(z_{2Y}, z_{3Y})$ such that the equality $j^{U_X}(t_X) = j^{U_Y}(t_Y)$ holds. Note that for all $z \in V(U_X)$, we have $t_X \sim z$ if and only if $t_Y \sim \varphi(z))$. Define

$$S_X = \big\{z \in V(U_X) \mid \delta^U(x_3, z, \neg(V(U_X) \setminus z))\big\};$$
$$S_Y = \big\{\varphi(z) \mid z \in S_X\big\}.$$

From the equality

$$j^{U_X}_{z_{1X}, S_X}(t_X) = j^{U_Y}_{z_{1Y}, S_Y}(t_Y),$$

it follows that in $\mathcal{N}^{U_Y}(t_Y, z_{1Y}, \neg(V(U_Y) \setminus \{z_{1Y}\}))$ there exists a vertex y_3 such that

$$\Big(\bigwedge_{z \in S_Y} \delta^{U_Y}\big(y_3, z, \neg t_Y, \neg(V(U_Y) \setminus \{z\})\big)\Big) \wedge \Big(\bigwedge_{z \in V(U_Y) \setminus S_Y} \neg\delta^{U_Y}\big(y_3, z, \neg t_Y, \neg(V(U_Y) \setminus \{z\})\big)\Big).$$

Duplicator chooses such a vertex $y_3 \in V(Y)$.

Without loss of generality, in round 4 Spoiler chooses $x_4 \in V(X)$.

If $x_4 \in V(U_X)$, then Duplicator chooses a vertex $\varphi(y_4) \in V(Y)$ and, obviously, wins. Otherwise, if the vertex x_4 is adjacent to each of the previous vertices, then $(U_X + x_3 + x_4 + t_X, U_X)$ is $\frac{1}{2}$-rigid, with $v(U_X + x_3 + x_4 + t_X, U_X) = 3$; and, consequently, U_X is not x_1-bad, which leads to a contradiction.

If y_4 is adjacent to precisely two previously chosen vertices, then two options are possible. If y_4 is adjacent to x_1 and x_2, then Duplicator's winning strategy is analogous to the one in Subcase 1b.

Finally, let x_4 be adjacent to x_3 and some x_k with $k \in \{1, 2\}$, but not adjacent to x_ℓ, $\ell \in \{1, 2\} \setminus \{k\}$. Then, due to the choice of y_3 in Y there is a vertex

$$y_4 \in \mathcal{N}^{U_Y}\big(y_3, y_k, \neg t_Y, \neg(V(U_Y) \setminus \{y_k\})\big).$$

Duplicator chooses such a vertex y_4 and wins.

If y_4 is adjacent to one or none of the previous vertices, then Duplicator wins due to $\frac{1}{2}$-safeness of the corresponding pair of graphs.

Subcase 1d. Assume that $x_3 \notin V(U_X)$, $|\mathcal{N}(x_3) \cap V(U_X)| = 1$, and in $V(U_X) \setminus V(U)$ there is no vertex adjacent to x_3 for which $|\mathcal{N}(t_X) \cap V(U_X)| = 2$. Denote by z_{1X} a vertex from $V(U_X)$ adjacent to x_3, and put $z_{1Y} = \varphi(z_{1X})$.

Consider the pair of graphs $(\tilde{A}_1, \tilde{B}_1)$, where $V(\tilde{B}_1) = \{\tilde{z}_1, \dots, \tilde{z}_m, \tilde{t}\}$, that corresponds to $j^{U_X}(x_3)$. Let

$$\tilde{B}_1' = \tilde{B}_1|_{\{\tilde{z}_1, \dots, \tilde{z}_v\}} \quad \text{and} \quad \tilde{A}_1' = \tilde{A}_1 + \{\tilde{z}_1, t\},$$

which indicates the graph $\tilde{A}_1$ along with the edge $\{\tilde{z}_1, \tilde{t}\}$. Here, $\tilde{z}_1$ is the image of z_{1X} in $\tilde{B}_1'$.

Statement 35. The pair $(\tilde{A}'_1, \tilde{B}'_1)$ is $\frac{1}{2}$-safe.

Proof. The statement follows from the $\frac{1}{2}$-safeness of $(\tilde{B}_1, \tilde{B}'_1)$, the fact that $\tilde{A}'$ is a strict (K^*, T^*)-neighborhood of $\tilde{B}_1$, and also Statement 19. □

Finally, note that $v(\tilde{A}'_1, \tilde{B}'_1) \le m(m+1)2^m$. By Property 4 of pair (X, Y), in Y there is a strict (K^*, T^*)-maximal $(\tilde{A}'_1, \tilde{B}'_1)$-extension of subgraph U_Y. Thus, Duplicator chooses $y_3 \in V(Y)$ that corresponds to $\tilde{t} \in V(\tilde{A}_1)$. It is easy to verify that

$$j^{U_X}(x_3) = j^{U_Y}(y_3).$$

Indeed, the only vertices that may influence the value $j^{U_Y}(y_3)$ are vertices from (K^*, T^*)-neighborhood of $U_Y + y_3$. But, from (K^*, T^*)-maximality of the strict $(\tilde{A}'_1, \tilde{B}'_1)$-extension, it follows that the (K^*, T^*)-neighborhood coincides with an already found extension.

Without loss of generality assume Spoiler chooses $x_4 \in V(X)$ in round 4.

If $x_4 \in V(U_X)$, then Duplicator chooses a vertex $y_4 = \varphi(x_4) \in V(Y)$ and wins. Otherwise, if x_4 is adjacent to exactly two vertices chosen before, then Duplicator wins due to the equalities

$$j^{U_X}(x_1, x_2, x_3) = j^{U_Y}(y_1, y_2, y_3) \quad \text{and} \quad \delta^{U_X}(x_1, x_2) = \delta^{U_Y}(y_1, y_2)$$

and the fact that in $V(Y) \setminus V(U_Y)$ there is no vertex adjacent to all three chosen before. If Spoiler chooses a vertex that is adjacent to not more than one of the previously chosen, then Duplicator wins due to the $\frac{1}{2}$-safeness of a corresponding extension. Spoiler also cannot choose a vertex that is adjacent to all three of the previous ones and not in U_X.

Subcase 1e. Assume that $x_3 \notin V(U_X)$, x_3 does not have neighbors in U_X and $\delta^{U_X}(x_1, x_2, x_3) = 0$. This case can be verified analogously to Subcase 1d. In Subcase 1e, we need to remove the edge $\{\tilde{z}_1, \tilde{t}\}$ from the pair $(\tilde{A}'_1, \tilde{B}'_1)$.

Subcase 1f. Assume that $x_3 \notin V(U_X)$, x_3 does not have neighbors in U_X and $\delta^{U_X}(x_1, x_2, x_3) = 1$.

Let $x_3 \notin S^{U_X}(x_1, x_2)$. Let t_X be a vertex of $\mathcal{N}^{U_X}(x_1, x_2, x_3)$.

Consider the pair of graphs $(\tilde{A}_2, \tilde{B}_2)$, where $V(\tilde{B}_2) = \{\tilde{z}_1, \dots, \tilde{z}_v, \tilde{t}_X, \tilde{t}\}$, that corresponds to $j^{U_X + t_X}(x_3)$. Put

$$\tilde{B}'_2 = \tilde{B}_2|_{\{\tilde{z}_1, \dots, \tilde{z}_v, \tilde{t}_X\}}, \tilde{A}'_2 = \tilde{A}_2 + \{\tilde{t}_X, \tilde{t}\},$$

where $\tilde{t}_X$ is an image of t_X in B'. As in Statement 35, the pair $(\tilde{A}', \tilde{B}')$ is $\frac{1}{2}$-safe. Finally, note that $v(\tilde{A}'_1, \tilde{B}'_1) \le (m+1)(m+2)2^{m+1}$.

Since $\delta^{U_Y}(y_1, y_2) = 1$, in $\mathcal{N}^{U_Y}(y_1, y_2)$ there is at least one vertex t_Y. By Property 4 of the pair (X, Y), in Y there is a strict (K^*, T^*)-maximal $(\tilde{A}'_2, \tilde{B}'_2)$-extension of $U_Y + t_Y$. Thus, Duplicator chooses a vertex $y_3 \in V(Y)$ that corresponds to $\tilde{t} \in V(\tilde{A}_2)$. It is not hard to verify that

$$j^{U_X + t_X}(x_3) = j^{U_Y + t_Y}(y_3).$$

Moreover, not a single vertex of U_X may be adjacent to y_3, due to strictness of the found $(\tilde{A}'_2, \tilde{B}'_2)$-extension.

Without loss of generality, in round 4 Spoiler chooses a vertex $x_4 \in V(X)$. If $x_4 \in V(U_X)$, then Duplicator chooses $y_4 = \varphi(x_4)$ and wins. If $x_4 \in \mathcal{N}^{U_X}(x_1, x_2, x_3)$, then Duplicator chooses t_Y and wins. If x_4 is adjacent to no more than two of the previous vertices, then two options are possible. If x_4 is adjacent to x_1 and x_2, then $I^{U_X}(x_1, x_2) = 1 = I^{U_Y}(y_1, y_2)$. Spoiler chooses an arbitrary element of

$\mathcal{N}^{U_Y}(y_1, y_2)$, different from t_Y, as y_4. Due to strictness of the found $(\tilde{A}'_2, \tilde{B}'_2)$-extension, y_4 is not adjacent to y_3, and Duplicator wins. If x_4 is adjacent to x_3 and x_k for some $k \in \{1, 2\}$, then the existence of the winning strategy for Duplicator can be verified as in Subcase 1d.

Let $x_3 \in S^{U_X}(x_1, x_2)$. Due to the equality $\Sigma^{U_X}(x_1, x_2) = \Sigma^{U_Y}(y_1, y_2)$, in Y there is a vertex $y_3 \in S^{U_Y}(y_1, y_2)$ such that

$$\sigma^{U_X}(x_1, x_2, x_3) = \sigma^{U_Y}(y_1, y_2, y_3).$$

Duplicator chooses this vertex y_3.

Without loss of generality, in round 4 Spoiler chooses a vertex $x_4 \in V(X)$. If $x_4 \in V(U_X)$, then Duplicator chooses a vertex $y_4 = \varphi(x_4)$ and wins, since x_4 is not adjacent to x_3 and y_4 is not adjacent to y_3. Otherwise, if x_4 is adjacent to each of the previously chosen, then it is enough to verify that there is a vertex $y_4 \in V(Y)$ adjacent to each vertex chosen before in Y. First, due to inclusion $y_3 \in S^{U_Y}(y_1, y_2)$, the set $\mathcal{N}^{U_Y}(y_1, y_2)$ is nonempty; and, second, each element of this set is adjacent to y_3. Hence, such a vertex y_4 exists.

If x_4 is adjacent to exactly two of the previously chosen vertices, then one of them is x_3, since $x_3 \in S^{U_X}(x_1, x_2)$, which means that x_3 is adjacent to each common neighbor of x_1 and x_2. Let also $x_k \sim x_4$ for some $k \in \{1, 2\}$. Then we have $\sigma_k^{U_X}(x_1, x_2, x_3) = 1 = \sigma_k^{U_Y}(y_1, y_2, y_3)$, which implies the existence of $y_4 \in V(Y)$ adjacent to y_3 and y_k, but not adjacent to x_l, $l \in \{1, 2\} \setminus \{k\}$. Thus, Duplicator chooses x_4 and wins. If x_4 is adjacent to not more than one of the previous vertices, then Duplicator wins due to $\frac{1}{2}$-safeness of the corresponding pair of graphs.

Case 1 has been examined.

Case 2. Let x_2 be an element of the 0-neighborhood of x_1, with $x_2 \sim x_1$. By Property 2 of the pair (X, Y), in Y there is a vertex y_2 adjacent to y_1 for which the conclusion of Lemma 28 holds.

Duplicator chooses this vertex y_2. All properties of the graphs X and Y are symmetric. Let $U_X = X|_{\{x_1,x_2\}}$ and $U_Y = Y|_{\{y_1,y_2\}}$. Further analysis of Case 2 is analogous to that of Case 1.

Case 3. Let x_2 be an element of the 0-neighborhood of x_1, with $x_2 \not\sim x_1$. By definition of (X, Y), in Y there is a vertex y_2 not adjacent to y_1 such that Lemma 32 conditions hold. Duplicator chooses a vertex y_2. Note that X and Y are now symmetric.

Without loss of generality, in round 3 Spoiler chooses a vertex $x_3 \in V(X)$.

Subcase 3a. Assume that x_3 is adjacent to both x_1, x_2, and in X there is a vertex t_X adjacent to each x_k for $1 \le k \le 3$. Since $\tau(x_1, x_2) = \tau(y_1, y_2)$, in Y there are vertices $y_3, t_Y \in \mathcal{N}(y_1, y_2)$ such that $j(x_1, x_2, x_3) = j(y_1, y_2, y_3)$ and $t_Y \sim y_3$. Duplicator chooses the vertex y_3.

Further, without loss of generality, in round 4 Spoiler chooses a vertex $x_4 \in V(X)$. If x_4 is adjacent to each of the previous vertices, then Duplicator chooses $t_Y \in V(Y)$ and wins. If x_4 is adjacent to each of x_1, x_2, then

$$I'(x_1, x_2) = 1 = I'(y_1, y_2).$$

Thus, in $\mathcal{N}(y_1, y_2)$ there is at least one vertex besides y_3 and t_Y, with $y_4 \not\sim y_3$. Therefore Duplicator wins. If x_4 is adjacent to x_3 and some x_k for $1 \le k \le 2$, then Duplicator wins due to the equality $j(x_1, x_2, x_3) = j(y_1, y_2, y_3)$. If, finally, x_4 is adjacent to not more than one of the previous vertices, then Duplicator wins due to the $\frac{1}{2}$-safeness of a corresponding extension.

Subcase 3b. Assume that x_3 is adjacent to both x_1, x_2 and in X there is no vertex adjacent to each of x_k for $1 \le k \le 3$. Since $J'(x_1, x_2) = J'(y_1, y_2)$, in Y there exists a vertex y_3 such that firstly,

$$j(x_1, x_2, x_3) = j(y_1, y_2, y_3);$$

and, second, in Y there is no vertex adjacent to each of y_k for $1 \le k \le 3$. With this note, Subcase 3b is similar to Subcase 1a.

Subcase 3c. Assume that x_3 is adjacent to exactly one of x_1, x_2. This subcase is similar to Subcases 1c and 1d.

Subcase 3d. Assume that x_3 is not adjacent to any of the previous vertices. The item is similar to Subcases 1e and 1f.

Case 3 has been examined, and the theorem is fully proven. □

6. Conclusion

In this paper, we achieved progress in studying limit points of 4-spectrum within the context of the zero-one k-law. Future research on this topic can be aimed at studying the last possible limit point of 4-spectrum, namely $\frac{3}{5}$.

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