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Dedicated to the memory of Professor Naum Ilyitch Feldman

Let r, m be positive integers. Let $0 \leq x < 1$ be a rational number. We denote by $\Phi_s(x, z)$ the s -th Lerch function

$$\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+x+1)^s},$$

with $s = 1, 2, \dots, r$. When $x = 0$, this is the polylogarithmic function. Let $\alpha_1, \dots, \alpha_m$ be pairwise distinct algebraic numbers with $0 < |\alpha_j| < 1$ ($1 \leq j \leq m$). We state a linear independence criterion over algebraic number fields of all the $rm + 1$ numbers: $\Phi_1(x, \alpha_1), \Phi_2(x, \alpha_1), \dots, \Phi_r(x, \alpha_1), \Phi_1(x, \alpha_2), \Phi_2(x, \alpha_2), \dots, \Phi_r(x, \alpha_2), \dots, \Phi_1(x, \alpha_m), \Phi_2(x, \alpha_m), \dots, \Phi_r(x, \alpha_m)$ and 1. We obtain an explicit sufficient condition for the linear independence of values of the r Lerch functions $\Phi_1(x, z), \dots, \Phi_r(x, z)$ at m distinct points in an algebraic number field of arbitrary finite degree without any assumptions on r and m . When $x = 0$, our result implies the linear independence of polylogarithms of distinct algebraic numbers of arbitrary degree, subject to a metric condition. We give an outline of our proof together with concrete examples of linearly independent polylogarithms.

1. Introduction

Let s be a nonnegative integer and $0 \leq x < 1$ be a rational number. We study the linear independence of values of the s -th Lerch function defined by

$$\Phi_s(x, z) = \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+x+1)^s}, \quad z \in \mathbb{C}, |z| < 1.$$

The s -th Lerch function $\Phi_s(x, z)$ satisfies the inhomogeneous differential equation

$$\frac{d}{dz} \Phi_s(x, z) = \frac{1}{z} \Phi_{s-1}(x, z) - \frac{x}{z} \Phi_s(x, z), \quad s \geq 1. \quad (1)$$

Then the s -th Lerch function is a G -function in the sense of Siegel [1929]; see also [Feldman and Nesterenko 1998].

Note that in the case of $x = 0$, we have $\Phi_s(0, z) = \text{Li}_s(z)$, where

$$\text{Li}_s(z) = \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)^s}, \quad z \in \mathbb{C}, |z| < 1,$$

is the s -th polylogarithmic function.

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Let r, m be positive integers and K be an algebraic number field. Consider $\alpha_1, \dots, \alpha_m \in K \setminus \{0\}$ with $\alpha_{i_1} \neq \alpha_{i_2}$ for $1 \leq i_1 < i_2 \leq m$ and $0 \leq x \in \mathbb{Q}$.

We define the vector of formal power series $\vec{\Phi}$ by

$$\vec{\Phi} := {}^t(1, \Phi_1(x, \alpha_1 z), \dots, \Phi_r(x, \alpha_1 z), \dots, \Phi_1(x, \alpha_m z), \dots, \Phi_r(x, \alpha_m z)) \in K[[z]]^{rm+1},$$

the vector of rational functions $\vec{A}(\alpha_i) := {}^t(\alpha_i/(1 - \alpha_i z), 0, \dots, 0) \in K(z)^r$ and an $r \times r$ matrix $A(x)$ by

$$A(x) := \begin{pmatrix} -x/z & 0 & \cdots & 0 \\ 1/z & -x/z & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1/z & -x/z \end{pmatrix} \quad \text{if } r \geq 2, \quad A(x) = \begin{pmatrix} -x \\ z \end{pmatrix} \quad \text{if } r = 1.$$

Then, taking the differential equation (1) into account, the vector $\vec{\Phi}$ satisfies the following system of differential equations in \vec{y} :

$$\frac{d}{dz} \vec{y} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vec{A}(\alpha_1) & A(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vec{A}(\alpha_m) & 0 & \cdots & A(x) \end{pmatrix} \vec{y}. \quad (2)$$

We see that (2) is indeed a system of *homogenous* differential equations in \vec{y} .

We consider r Lerch functions $\Phi_s(x, z)$, $1 \leq s \leq r$. The linear independence of $\text{Li}_s(\alpha)$ at one rational number α , with $1 \leq s \leq r$, was studied by E. M. Nikishin [1979a]. It was generalized to the Lerch function by Kawashima [2014] and to algebraic cases by M. Hirose, M. Kawashima and N. Sato [Hirose et al. 2017]. See also [Hirata-Kohno et al. 2017] for examples. M. Hata [1990] adapted generalized Legendre polynomials modifying Padé-type constructions of G. V. Chudnovsky [1979; 1982; 1983], see also [Chudnovsky and Chudnovsky 1985], to obtain the linear independence of $\text{Li}_s(\alpha)$ (indeed of the Lerch transcendent function) for different s but at one rational number α . His result implies the irrationality of $\text{Li}_2(1/q)$ with q integer, $q \geq 12$, whereas it was announced in [Chudnovsky 1979] the irrationality of $\text{Li}_2(1/q)$ with $q \geq 14$. Later, Hata [1993] gave the irrationality of the value of $\text{Li}_2(1/q)$ with q integer, $q \geq 7$ or $q \leq -5$.

Rhin and C. Viola [2005] adapted their permutation group method, established in [Rhin and Viola 1996], to get the irrationality of $\text{Li}_2(\alpha)$ for certain $\alpha \in \mathbb{Q}$, involving the irrationality $\text{Li}_2(1/q)$, with $q \geq 6$, $q \in \mathbb{Z}$, in qualitative and quantitative forms. More recently, Viola and W. Zudilin [2018] extended the permutation group method with constructions to establish the linear independence of $1, \text{Li}_1(1/q), \text{Li}_2(1/q), \text{Li}_2(1/(1-q))$ over \mathbb{Q} with an integer $q \geq 9$ or $q \leq -8$ and more generally, that of $1, \text{Li}_1(\alpha), \text{Li}_2(\alpha), \text{Li}_2(\alpha/(\alpha-1))$ for certain $\alpha \in \mathbb{Q}$. See also important related works [Fischler et al. 2019; Marcovecchio 2006; Miladi 2001; Rivoal 2003; Zudilin 1996].

With respect to logarithms, G. Rhin and P. Toffin [1986] created a system of Padé approximants to show the linear independence of the natural logarithms of *distinct* $\alpha_1, \dots, \alpha_m$, either rational or quadratic imaginary numbers, under a *metric* condition requiring the points $\alpha_1, \dots, \alpha_m$ to be very close to the origin 0. This method provides a refinement of previous lower bounds for linear forms in logarithms, especially for effective bounds obtained by A. Baker [1975] and an essential improvement due to N. I. Feldman [1968], valid under the above-stated metric condition. This proof in [Rhin and Toffin 1986] opened a

new path, albeit unexplored systematically, during the next decades to show the linear independence of logarithms over \mathbb{Q} at *distinct* $\alpha \in \mathbb{Q}$, relying only on Padé approximations.

Since $\text{Li}_1(z)$ coincides with the usual natural logarithm, the Rhin–Toffin method suggests how to adapt Padé approximations to deal with the linear independence of polylogarithms at distinct points $\alpha_1, \dots, \alpha_m$.

We give a new criterion to show the linear independence of all the $rm + 1$ numbers: $\Phi_1(x, \alpha_1)$, $\Phi_2(x, \alpha_1)$, \dots , $\Phi_r(x, \alpha_1)$, $\Phi_1(x, \alpha_2)$, $\Phi_2(x, \alpha_2)$, \dots , $\Phi_r(x, \alpha_2)$, \dots , $\Phi_1(x, \alpha_m)$, $\Phi_2(x, \alpha_m)$, \dots , $\Phi_r(x, \alpha_m)$ and 1, over an algebraic number field K , supposing $\alpha_1, \dots, \alpha_m$ pairwise distinct in K , assumed to be sufficiently close to the origin, which we will make precise later. We also give an outline of our proof with basic ideas.

Our linear independence criterion for the values of the Lerch functions, including the case of polylogarithmic functions, at *distinct points in an algebraic number field of arbitrary finite degree*, is not covered by the previous criterion in [Galochkin 1974; 1975], as is explained below in Remarks 1.1, 1.3 and 1.4 and Example 6.3.

Remark 1.1. Let us describe here previous linear independence results concerning with values of the Lerch functions, at distinct rational or imaginary quadratic points, due to A. I. Galochkin [1974; 1975], Y. Z. Flicker [1977] K. Väänänen [1980], together with a result by K. Väänänen and G. Xu [Väänänen and Xu 1988]. First, we introduce the result of Galochkin [1974, Theorem 1]. All notation and conventions are those of the above-mentioned article, pages 385–388; see also [Nurmagomedov 1971].

Theorem 1.2 [Galochkin 1974, Theorem 1]. *Let I be \mathbb{Q} or an imaginary quadratic field and K be a finite extension of I with $[K : I] = \kappa < \infty$. For $1 \leq s \in \mathbb{Z}$, consider $f_1(z), \dots, f_s(z) \in K[[z]]$ which belong to the subclass $G(K, C_0, Q, \Lambda)$ with $C_0 Q \geq 2$, $C = \max(1, C_0)$ (see [Galochkin 1974, Definitions 1, 2]). Assume that the functions are not connected by any nonzero polynomial in s variables, of degree not exceeding N , with coefficients in $\mathbb{C}(z)$. Let $1 \leq d \in \mathbb{Z}$ and*

$$u := \binom{N+s}{s} + \kappa \binom{N-d+s}{s} - \kappa \binom{N+s}{s},$$

with $N \geq d$.

Suppose now

$$u > 0. \tag{3}$$

Then there exists an explicit constant $c_0 > 0$ which depends on N, d and $f_1(z), \dots, f_s(z)$, satisfying the following property: for any integer with $|q| > c_0$ and a nonzero polynomial $P(x_1, \dots, x_s) \in \mathbb{Z}[x_1, \dots, x_s]$ of degree $d \leq N$, we have

$$P(f_1(1/q), \dots, f_s(1/q)) \neq 0.$$

In particular, when $d = 1$, we have $su = \binom{N-1+s}{N} \{N + s(1 - \kappa)\}$. Thus, under the condition that $N > s(\kappa - 1)$ together with the assumption of the algebraic independence of the functions $f_1(z), \dots, f_s(z)$ over $\mathbb{C}(z)$, the linear independence of values of these s functions over K at the point $1/q$ follows.

It is worth noting that Flicker [1977] proved a p -adic analogue of Galochkin's theorem. Building on both Galochkin's and Flicker's work, Väänänen [1980] refined the above-mentioned results and generalized to a system of differential equations, both in the complex and the p -adic cases and also proved a Baker-type lower bound for linear combinations of classical logarithms and polylogarithms, also subject to a metric condition as above.

For these results to work, one needs that the G -functions belong to the subclass $G(K, C_0, Q, \Lambda)$ with $C_0 Q \geq 2$, that is, roughly speaking, a set of particular G -functions satisfying a system of linear differential equations, under a hypothesis called the Galochkin condition or (G, C) -condition, given in Definition 2 of [Galochkin 1974] (this is the same as (G, C) -function condition in [Chudnovsky 1984] and the (G, C) -assumption in [Chudnovsky and Chudnovsky 1985]).

More significant progress was made by Chudnovsky [1984], who proved that, for G -functions satisfying a differential equation system as in (2), Galochkin's condition automatically holds.

Summing up, thanks to the above-mentioned results, as soon as we can show that the considered G -functions satisfy a linear system of differential equations as in (2), as well as that the functions are *linearly independent* over $\mathbb{C}(z)$, we get the linear independence of the special values *provided* condition (3) is satisfied. Condition (3) comes from the use of Siegel's lemma to construct Padé approximants (whereas we avoid using Siegel's lemma in the present article).

We are now in a position to compare our results with the above-mentioned series of results. Restraining ourselves to the functions $1, \Phi_s(x, \alpha_i z)$, with $1 \leq i \leq m, 1 \leq s \leq r$, one can check they are *linearly independent* over $\mathbb{C}(z)$, in a similar way to [Väänänen 1980, pp. 292, 293]; see [David et al. ≥ 2020a] (it may be worth noting that Galochkin's condition can be checked by hand in this special case, and thus one can also proceed without using Chudnovsky's observation). Hence, we are in the case $N = 1$; thus necessarily $d = 1$.

However, for $N = d = 1$, condition (3) reads $u = s(1 - \kappa) + 1 \leq -s + 1 < 0$ if $\kappa \geq 2$; hence the assumption $u > 0$ of Galochkin's theorem never holds when $N = 1$ as soon as the base field considered is *not contained* in an imaginary quadratic field.

On the contrary, our criterion covers also such a case, since the base field can be an arbitrary number field. Namely our result gives the linear independence of values of the Lerch functions, when $N = 1$, applying our explicit construction of Padé approximations of $1, \Phi_s(x, \alpha_i/z)$ that is done around infinity, not around the origin (this is one of the reasons why our corresponding assumption is much weaker than that of Galochkin's theorem). Nevertheless, as we see in Example 6.3 below, our linear independence result for the values of Lerch functions is valid for algebraic points in K of arbitrary degree, to which neither Galochkin's [1974; 1975] nor Väänänen's [1980] results apply.

Remark 1.3. It is also worth noting that our result (see [David et al. ≥ 2020a] for details) is quantitative, with totally explicit constants, which is not the case of previous results.

Remark 1.4. A result of [Väänänen and Xu 1988] actually deals with general base fields as in our case. However, this is not applicable in our situation, because of the degenerate nature of the system (2).

The new ingredient in the article relies on a few points. First and foremost, we introduce a systematic construction of Padé approximants, which heavily relies on the computations made by past authors. Our modifications and generalizations of the method Nikishin [1979a; 1979b] developed, as well as of the Rhin–Toffin method [1986], supply a *formally regulated construction* of Padé approximants. Secondly an irrationality criterion, combined with the metric property provided for by Padé approximation, leads to the irrationality for the values of the Lerch functions at points sufficiently close to the origin (the precise sufficient condition, which we explain later, comes from the coupling of the criteria with Padé approximation). This strategy works only if one can ensure the *injectivity of evaluation maps* defined by systems of Padé approximation, which can be now interpreted as a nonvanishing property of a Hermite–

type determinant, which we succeed in proving. Our criterion also gives much more relaxed assumptions than the previous results in [Galochkin 1974; 1975], since we rely on our new *formal* construction of explicit Padé approximants, by avoiding the use of Siegel's lemma.

2. Notation and main results

We fix an algebraic closure of \mathbb{Q} and denote it by $\bar{\mathbb{Q}}$. For a finite subset $S \subset \bar{\mathbb{Q}}$, we define the denominator of S by

$$\text{den}(S) := \min\{0 < n \in \mathbb{Z} \mid n\alpha \text{ is an algebraic integer for any } \alpha \in S\}.$$

Let \mathbb{N} be the set of strictly positive integers. Let $m, r \in \mathbb{N}$ and K be an algebraic number field of finite degree over \mathbb{Q} . We denote the ring of integers of K by \mathcal{O}_K and the completion of K with respect to the fixed embedding $\iota_\infty : K \hookrightarrow \mathbb{C}$ by K_∞ . Then $[K_\infty : \mathbb{R}] = 1$ if $K_\infty \subset \mathbb{R}$, and $[K_\infty : \mathbb{R}] = 2$ otherwise.

Let $x \in \mathbb{Q} \cap [0, 1)$. Put

$$\mu(x) := \text{den}(x) \prod_{q: \text{prime}, q \mid \text{den}(x)} q^{1/(q-1)}.$$

Consider $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_m) \in (K \setminus \{0\})^m$, with $\alpha_i \neq \alpha_j$ for all $1 \leq i < j \leq m$. For $1 \leq g \leq [K : \mathbb{Q}]$, we denote by $\alpha^{(g)}$ the g -th conjugate of $\alpha \in K$ over \mathbb{Q} .

Let $\beta \in K \setminus \{0\}$ with $\max_{1 \leq i \leq m} (|\alpha_i|) < |\beta|$. We put

$$D(\boldsymbol{\alpha}, \beta) := \text{den}(\alpha_1, \dots, \alpha_m, \beta).$$

We also define

$$\begin{aligned} \mathbb{A}(\boldsymbol{\alpha}, \beta, x) &:= \log |\beta| - (rm + 1) \log \max_i (|\alpha_i|) \\ &\quad - \left\{ rm \left(\log D(\boldsymbol{\alpha}, \beta) + r [\text{den}(x) + \log \frac{5}{2}] \right) + r (\log 3 + \log \mu(x)) \right\}, \end{aligned}$$

$$\begin{aligned} \mathcal{A}^{(g)}(\boldsymbol{\alpha}, \beta, x) &:= rm (\log D(\boldsymbol{\alpha}, \beta) + \log \max(1, \min(|\alpha_i^{(g)}|)^{-1} \cdot |\beta^{(g)}|) + r [\text{den}(x) - \log 2]) \\ &\quad + r \left(\log \mu(x) + \sum_{i=1}^m \log (2^r |\alpha_i| + 3^r \max(|\alpha_i^{(g)}|, |\beta^{(g)}|)) \right) + \log 3 \quad \text{for } 1 \leq g \leq [K : \mathbb{Q}], \end{aligned}$$

$$V(\boldsymbol{\alpha}, \beta, x) := \mathbb{A}(\boldsymbol{\alpha}, \beta, x) + \mathcal{A}^{(1)}(\boldsymbol{\alpha}, \beta, x) - \frac{\sum_{g=1}^{[K:\mathbb{Q}]} \mathcal{A}^{(g)}(\boldsymbol{\alpha}, \beta, x)}{[K_\infty : \mathbb{R}]}.$$

We then obtain the following statement.

Theorem 2.1. *Assume $V(\boldsymbol{\alpha}, \beta, x) > 0$. Then the $rm + 1$ numbers*

$$1, \Phi_1(x, \alpha_1/\beta), \dots, \Phi_r(x, \alpha_1/\beta), \dots, \Phi_1(x, \alpha_m/\beta), \dots, \Phi_r(x, \alpha_m/\beta)$$

are linearly independent over K .

In the special case where K equals \mathbb{Q} or an imaginary quadratic field, Corollary 6 in [Väänänen 1980] gives an analogous quantitative result for polylogarithms, but the needed condition there is not so explicit as ours, $V(\alpha, \beta, 0) > 0$. For a general number field K , Theorem 2.1 is the first result to give the linear independence of the values of the Lerch function, even in the case of polylogarithms, at distinct algebraic numbers.

3. Construction of Padé approximants

We now explain how we construct Padé approximants of the Lerch functions. Since the full proof is long, it will be provided, with all relevant details, in the forthcoming articles [David et al. ≥ 2020a; ≥ 2020b], with a p -adic analogue as well as quantitative measures of linear independence.

First we recall the definition of Padé approximants of formal Laurent series. In the rest of this section, we denote by L a unique factorization domain of characteristic 0. We define the order function ord_∞ at $z = \infty$ by

$$\text{ord}_\infty : L[z][[1/z]] \rightarrow \mathbb{Z} \cup \{\infty\}, \quad \sum_k a_k \cdot \frac{1}{z^k} \mapsto \min\{k \in \mathbb{Z} \mid a_k \neq 0\}.$$

Lemma 3.1. *Let r be a positive integer, $f_1(z), \dots, f_r(z) \in 1/z \cdot L[[1/z]]$ and $\mathbf{n} := (n_1, \dots, n_r) \in \mathbb{N}^r$. Put $N := \sum_{i=1}^r n_i$. Let M be a positive integer with $M \geq N$. Then there exists a family of polynomials $(P_0(z), P_1(z), \dots, P_r(z)) \in L[z]^{r+1} \setminus \{\mathbf{0}\}$ satisfying the following conditions:*

- (i) $\deg P_0(z) \leq M$.
- (ii) $\text{ord}_\infty P_0(z) f_j(z) - P_j(z) \geq n_j + 1$ for $1 \leq j \leq r$.

Definition 3.2. Using the notation of Lemma 3.1, we call a family of polynomials $(P_0(z), P_1(z), \dots, P_r(z)) \in L[z]^{r+1}$ satisfying the properties (i) and (ii) *Padé-type approximants of (f_1, \dots, f_r) of weight \mathbf{n} and of degree M* .

For the Padé-type approximants $(P_0(z), P_1(z), \dots, P_r(z))$, of (f_1, \dots, f_r) of weight \mathbf{n} , we call the family of formal Laurent series $(P_0(z) f_j(z) - P_j(z))_{1 \leq j \leq r}$ *Padé-type approximation systems of (f_1, \dots, f_r) , of weight \mathbf{n} and of degree M* .

In the sequel, we take $x \in L \setminus \mathbb{Z}_{<0}$ and assume $x + k$ are invertible in L for any $k \in \mathbb{N}$.

We now introduce notation for formal primitive, derivation, and evaluation maps. Let I be a finite set; we assume that L contains $K[X_i, 1/X_i]_{X_i \in I}$, where K is a number field. In the sequel, it will be convenient to work formally and thus to treat as many quantities as variables as is useful, and we shall freely extend the set I as need arises.

Notation 3.3. (i) For $\alpha \in L$, we denote by Eval_α the linear evaluation map $L[T] \rightarrow L$, $P \mapsto P(\alpha)$.
(ii) For $P \in L[T]$, we denote by $[P]$ the multiplication by P ($Q \mapsto PQ$).
(iii) We also denote by Prim_x (formal primitive) the linear operator $L[T] \rightarrow L[T]$, defined by

$$P \mapsto \frac{1}{T^{1+x}} \int_0^T \xi^x P(\xi) d\xi.$$

(iv) We denote by Deri_x the derivative map

$$P \mapsto T^{-x} \frac{d}{dT} (T^{x+1} P(T)),$$

and by $S_{n,x}$ for $n \geq 1$ the map taking

$$T^k \mapsto \frac{(k+x+1)_n}{n!} T^k,$$

where $(k+x+1)_n := (k+x+1) \cdots (k+x+n)$, that is, the divided derivative mapping

$$P \mapsto \frac{1}{n!} T^{-x} \frac{d^n}{dT^n} (T^{n+x} P) = \frac{1}{n!} \left(\frac{d}{dT} + \frac{x}{T} \right)^n T^n (P),$$

so that $\text{Deri}_x = S_{1,x}$.

(v) If φ is an L -automorphism of an L -module M and k an integer, we define

$$\varphi^{(k)} := \begin{cases} \overbrace{\varphi \circ \cdots \circ \varphi}^{k \text{ times}} & \text{if } k > 0, \\ \text{id}_M & \text{if } k = 0, \\ \underbrace{\varphi^{-1} \circ \cdots \circ \varphi^{-1}}_{-k \text{ times}} & \text{if } k < 0. \end{cases}$$

For a given $l \in \mathbb{Z}$, we define the linear map $\varphi_{\alpha,x,l}$ as follows.

Notation 3.4.

$$\varphi_{\alpha,x,l} := [\alpha] \circ \text{Eval}_\alpha \circ \text{Prim}_x^{(l)}.$$

For any nonnegative integers k , note that $\varphi_{\alpha,x,s}(T^k)$ is a formal analogue of

$$\frac{1}{(s-1)!} \int_0^\alpha T^{k+x} \log^{s-1} \frac{1}{T} dT.$$

For convenience, we collect below the following elementary facts.

Facts 3.5. (i) The map Prim_x is an isomorphism and its inverse is Deri_x for $x \in L \setminus \mathbb{Z}_{<0}$. Hence $\varphi_{\alpha,x,s}$ is well-defined for $s \leq -1$.
(ii) For any integers $n_1 \geq 0, n_2 \geq 0$ and $x \in L \setminus \mathbb{Z}_{<0}$ with $x+k$ invertible in L for any $k \in \mathbb{N}$, the divided derivatives $S_{n_1,x}$ and $S_{n_2,x}$ commute; namely $S_{n_1,x} \circ S_{n_2,x} = S_{n_2,x} \circ S_{n_1,x}$.
(iii) For any integer $s \in \mathbb{Z}$ and any $\alpha \in L$, we have $\varphi_{\alpha,x,s} \circ \text{Deri}_x = \varphi_{\alpha,x,s-1}$.
(iv) By continuity, all the above-mentioned maps extend to $L[[T]]$ with respect to the natural valuation.
(v) The kernel of the map $\varphi_{\alpha,x,0}$ is the ideal $(T - \alpha)$ for any $x \in L \setminus \mathbb{Z}_{<0}$.

Using Facts 3.5(iv), the classical Lerch function is indeed expressed as a natural image by $\varphi_{\alpha,x,s}$, with $s \geq 1$, by

$$\varphi_{\alpha,x,s} \left(\frac{1}{z-T} \right) = \Phi_s \left(x, \frac{\alpha}{z} \right). \quad (4)$$

Consider $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_m) \in (L \setminus \{0\})^m$ with $\alpha_i \neq \alpha_j$ for $i \neq j$. We study Padé approximants of type II of the functions $(\Phi_s(x, \alpha_i/z))_{1 \leq i \leq m, 1 \leq s \leq r}$.

Let l be a nonnegative integer with $0 \leq l \leq rm$. For a positive integer n , we define the family of polynomials

$$P_{n,l}(\boldsymbol{\alpha}, x | z) := \text{Eval}_z \circ S_{n,x}^{(r)} \left(T^l \prod_{i=1}^m (T - \alpha_i)^{rn} \right), \quad (5)$$

$$P_{n,l,i,s}(\boldsymbol{\alpha}, x | z) := \varphi_{\alpha_i,x,s} \left(\frac{P_{n,l}(\boldsymbol{\alpha}, x | z) - P_{n,l}(\boldsymbol{\alpha}, x | T)}{z - T} \right) \quad \text{for } 1 \leq i \leq m, 1 \leq s \leq r. \quad (6)$$

Under the notation above, we obtain the following theorem.

Theorem 3.6. *For each $0 \leq l \leq rm$, the family of polynomials $(P_{n,l}(\boldsymbol{\alpha}, x | z), P_{n,l,i,s}(\boldsymbol{\alpha}, x | z))_{1 \leq i \leq m, 1 \leq s \leq r}$ forms a Padé-type approximants system of $(\Phi_s(x, \alpha_i/z))_{1 \leq i \leq m, 1 \leq s \leq r}$ of weight $(n, \dots, n) \in \mathbb{N}^{rm}$ and of degree $rmn + l$.*

Proof. By the definition of $P_{n,l}(\boldsymbol{\alpha}, x | z)$, we have

$$\deg P_{n,l}(\boldsymbol{\alpha}, x | z) = rmn + l.$$

Hence the condition on the degree is verified. We only need to check the condition on the valuation.

Put

$$R_{n,l,i,s}(\boldsymbol{\alpha}, x | z) := P_{n,l}(\boldsymbol{\alpha}, x | z) \Phi_s(x, \alpha_i/z) - P_{n,l,i,s}(\boldsymbol{\alpha}, x | z).$$

Then, by the definition of $R_{n,l,i,s}(\boldsymbol{\alpha}, x | z)$ with property (4), we obtain

$$\begin{aligned} R_{n,l,i,s}(\boldsymbol{\alpha}, x | z) &= P_{n,l}(\boldsymbol{\alpha}, x | z) \varphi_{\alpha_i, x, s} \left(\frac{1}{z - T} \right) - P_{n,l,i,s}(\boldsymbol{\alpha}, x | z) \\ &= \varphi_{\alpha_i, x, s} \left(\frac{P_{n,l}(\boldsymbol{\alpha}, x | T)}{z - T} \right) = \sum_{k=0}^{\infty} \varphi_{\alpha_i, x, s} (T^k P_{n,l}(\boldsymbol{\alpha}, x | T)) \frac{1}{z^{k+1}}. \end{aligned} \quad (7)$$

Note that in $\text{End}_K(K[T])$ we have the identities

$$\begin{aligned} S_{n,x} &= \frac{1}{n!} S_{1,x} \circ \dots \circ (S_{1,x} + n - 1) \quad \text{for } n \in \mathbb{N}, \\ [T^k] \circ S_{1,x} &= (S_{1,x} - k) \circ [T^k] \quad \text{for } k \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

By the definition of $P_{n,l}(\boldsymbol{\alpha}, x | T)$ and the identities above, for each $1 \leq s \leq r$, $0 \leq k \leq n - 1$, there exists a polynomial $U_{s,k}(X) \in \mathbb{Q}[X]$ of $\deg U_{s,k} = nr - s$, satisfying

$$T^k P_{n,l}(\boldsymbol{\alpha}, x | T) = S_{1,x}^{(s)} \circ U_{s,k}(S_{1,x}) \left(T^{k+l} \prod_{i=1}^m (T - \alpha_i)^{rn} \right).$$

By the Leibniz rule, we obtain that $U_{s,k}(S_{1,x}) \left(T^{k+l} \prod_{i=1}^m (T - \alpha_i)^{rn} \right)$ belongs to the ideal $(T - \alpha_i)$ for each $1 \leq i \leq m$. Hence we get

$$\varphi_{\alpha_i, x, s} (T^k P_{n,l}(\boldsymbol{\alpha}, x | T)) = \varphi_{\alpha_i, x, 0} \circ U_{s,k}(S_{1,x}) \left(T^{k+l} \prod_{i=1}^m (T - \alpha_i)^{rn} \right) = 0$$

for $1 \leq i \leq m$, $1 \leq s \leq r$ and $0 \leq k \leq n - 1$.

Consequently, by the expansion above of $R_{n,l,i,s}(\boldsymbol{\alpha}, x | z)$, we obtain

$$\text{ord}_{\infty} R_{n,l,i,s}(\boldsymbol{\alpha}, x | z) \geq n + 1 \quad \text{for } 1 \leq i \leq m, 1 \leq s \leq r.$$

Then Theorem 3.6 follows. \square

4. Metric approximations and linear independence criteria

We now give a few of the estimates associated with the Padé approximation we just constructed. They do not need involved arguments to be proven; however, due to the technical nature of the construction, computations are somewhat heavy and we skip them to keep in line with the spirit of this article.

The estimates in Lemma 4.1 can be combined with an appropriate linear independence criterion to provide for a measure.

Lemma 4.1. *Let n be a positive integer, x a rational number with $0 \leq x < 1$ and $\beta \in K \setminus \{0\}$. Then for any $1 \leq g \leq [K : \mathbb{Q}]$, we have*

$$\begin{aligned} \max_{\substack{0 \leq l \leq rm \\ 1 \leq i \leq m \\ 1 \leq s \leq r}} |P_{n,l,i,s}^{(g)}(\alpha, x | \beta)| &\leq \max(|\alpha_i^{(g)}|)^{rm} \left(\frac{3}{2}\right)^{r^2m+r} \left(\frac{3}{2^{rm}} \prod_{j=1}^m [2^r |\alpha_j^{(g)}| + 3^r \max(|\alpha_i^{(g)}|)]\right)^{rn} \\ &\times \begin{cases} (\min(|\alpha_i^{(g)}|)^{-1} |\beta^{(g)}|)^{rm(n+1)} & \text{if } \min(|\alpha_i^{(g)}|)^{-1} |\beta^{(g)}| > 1, \\ \min(|\alpha_i^{(g)}|)^{-1} |\beta^{(g)}| - 1 & \text{if } \min(|\alpha_i^{(g)}|)^{-1} |\beta^{(g)}| \leq 1 \\ rm(n+1) & \end{cases} \end{aligned}$$

for $1 \leq i \leq m$.

For the error term, we have

$$\max_{0 \leq l \leq rm} |R_{n,l,i,s}(\alpha, x | \beta)| \leq \frac{\max_{1 \leq i \leq m} (1, |\alpha_i|)^{rm+1}}{|\beta| - \max_j (|\alpha_j|)} \left(\frac{3}{2}\right)^{r^2m+r} \left(\frac{\max_j (|\alpha_j|)^{rm+1}}{|\beta|}\right)^n \left(3 \left(\frac{5}{2}\right)^{rm}\right)^{rn}.$$

We give here an outline of the proof. By (5) and (6), we have

$$\begin{aligned} P_{n,l}(\alpha, x | z) &= \sum_{k=0}^{rmn} \left[\sum_{\substack{1 \leq i \leq m \\ 0 \leq k_i \leq rn \\ \sum_i k_i = k}} \left(\prod_{i=1}^m \binom{rn}{k_i} (-\alpha_i)^{rn-k_i} \right) \right] \left(\frac{(k+l+x+1)_n}{n!} \right)^r z^{k+l}, \\ P_{n,l,i,s}(\alpha, x | z) &= \sum_{u=\max(l,1)-1}^{rmn+l-1} \left[\sum_{k=u+1}^{rmn+l} \left(\sum_{\substack{1 \leq i' \leq m \\ 0 \leq k_{i'} \leq rn \\ \sum_{i'} k_{i'} = k-l}} \prod_{i'=1}^m \binom{rn}{k_{i'}} (-\alpha_{i'})^{rn-k_{i'}} \right) \left(\frac{(1+k+x)_n}{n!} \right)^r \frac{\alpha_i^{k-u}}{(k-u)^s} \right] z^u. \end{aligned}$$

By the above equalities together with the triangle inequality, we obtain the upper bounds for $|P_{n,l,i,s}^{(g)}(\alpha, x | \beta)|$ and $|P_{n,l}(\alpha, x | \beta)|$.

For the term $|R_{n,l,i,s}(\alpha, x | \beta)|$, we use (7).

We then state a general linear independence criterion:

Proposition 4.2. *Let K be an algebraic number field of finite degree over \mathbb{Q} . We denote the completion of K with respect to the fixed embedding ι_∞ by K_∞ . Let $m \in \mathbb{N}$ and $\theta_0 := 1, \theta_1, \dots, \theta_m \in \mathbb{C} \setminus \{0\}$.*

Suppose that there exists a set of matrices

$$\{(A_{n,l,j})_{0 \leq l, j \leq m}\}_{n \in \mathbb{N}} \subset \mathbf{M}_{m+1}(\mathcal{O}_K) \cap \mathrm{GL}_{m+1}(K).$$

Assume further that there exist positive real numbers

$$\{\mathcal{A}^{(g)}\}_{1 \leq g \leq [K:\mathbb{Q}]}$$

and a positive real number \mathbb{A} satisfying the conditions

$$\max_{0 \leq l, j \leq m} |A_{n,l,j}^{(g)}| \leq e^{\mathcal{A}^{(g)} \cdot n + o(n)} \quad \text{for } 1 \leq g \leq [K:\mathbb{Q}] \ (n \rightarrow \infty), \quad (8)$$

$$\max_{\substack{0 \leq l \leq m \\ 1 \leq j \leq m}} |A_{n,l,0} \cdot \theta_j - A_{n,l,j}| \leq e^{-\mathbb{A} \cdot n + o(n)} \quad (n \rightarrow \infty). \quad (9)$$

We put

$$V := \mathbb{A} + \mathcal{A}^{(1)} - \frac{\sum_{g=1}^{[K:\mathbb{Q}]} \mathcal{A}^{(g)}}{[K_\infty : \mathbb{R}]}.$$

If $V > 0$, then the numbers $\theta_0, \dots, \theta_m$ are linearly independent over K .

Proof. Assume that there exists a vector $\beta := (\beta_0, \dots, \beta_m) \in \mathcal{O}_K \setminus \{0\}$ satisfying $\Lambda(\beta, \theta) := \sum_{i=0}^m \beta_i \theta_i = 0$. For $n \in \mathbb{N}$, as we have $\det(A_{n,l,j})_{0 \leq l, j \leq m} \neq 0$, there exists $0 \leq l_n \leq m$ satisfying

$$B_{l_n} := \sum_{j=0}^m A_{n,l_n,j} \beta_j \neq 0. \quad (10)$$

Put $R_{n,l,j} = A_{n,l,0} \theta_j - A_{n,l,j}$ for $1 \leq j \leq m$ and $0 \leq l \leq m$. Then by the definitions of $\Lambda(\beta, \theta)$, B_{l_n} , and $R_{n,l,j}$, we obtain

$$0 = A_{n,l_n,0} \Lambda(\beta, \theta) = B_{l_n} + \sum_{j=1}^m R_{n,l_n,j} \beta_j.$$

Using the product formula for $B_{l_n} \in \mathcal{O}_K \setminus \{0\}$, it follows that

$$1 \leq \prod_g' |B_{l_n}^{(g)}| \times |B_{l_n}|^{[K_\infty : \mathbb{R}]} = \prod_g' |B_{l_n}^{(g)}| \times \left| \sum_{j=1}^m R_{n,l_n,j} \beta_j \right|^{[K_\infty : \mathbb{R}]} \quad (11)$$

Here ' in \prod_g' means that g satisfies $2 \leq g \leq [K:\mathbb{Q}]$ if $K_\infty = \mathbb{R}$ and $3 \leq g \leq [K:\mathbb{Q}]$ if $K_\infty = \mathbb{C}$. Firstly, we look for an upper bound of $|B_{l_n}^{(g)}|$ for $g \neq 1$ if $K_\infty = \mathbb{R}$ and $g \neq 1, 2$ if $K_\infty = \mathbb{C}$. Using inequality (8), we have

$$|B_{l_n}^{(g)}| \leq e^{\mathcal{A}^{(g)} n + o(n)} \quad (n \rightarrow \infty). \quad (12)$$

Secondly, we give an upper bound for $|\sum_{j=1}^m R_{n,l_n,j} \beta_j|$. By (9), we get

$$\left| \sum_{j=1}^m R_{n,l_n,j} \beta_j \right| \leq e^{-\mathbb{A} n + o(n)} \quad (n \rightarrow \infty). \quad (13)$$

Substituting the inequalities (12) and (13) into inequality (11), taking the $1/[K_\infty : \mathbb{R}]$ -th power of the inequality, we obtain

$$1 \leq e^{-V n + o(n)} \quad (n \rightarrow \infty).$$

Since $V > 0$, we arrive at a contradiction for this inequality for all sufficiently large $n \in \mathbb{N}$. \square

Theorem 3.6 gives us the sequence of matrices. The growth control of the size of the matrices to carry out the approximations is provided for in Lemma 4.1. However, the matrices do not always have algebraic

integer entries. This is not a big deal. The defect of integrality comes from our operators Prim_x , Deri_x and it is corrected by multiplying by a suitable power of the least common multiple $d_n := \text{lcm}(1, \dots, n)$, which is standard in the theory.

Plugging in these estimates in Proposition 4.2 leads us to the proof of the main theorem. The metric condition requiring the numbers to be sufficiently close to the origin is translated to the condition $V > 0$ in the linear independence criterion Proposition 4.2.

However, there is still a significant step to be performed. Now we need to prove that the matrices coming from the Padé approximation are indeed invertible. We describe this main step in the next section.

5. Nonvanishing of a determinant and the final step of the proof

In this section, we use the following notation. Let m, r be positive integers and K be a field with characteristic 0. We assume that $\alpha_1, \dots, \alpha_m, z, T$ all belong to the set of variables I , so our ring L contains $K[\alpha_i, z, T, 1/\alpha_i, 1/z, 1/T]$. Put $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_m)$.

For a positive integer l with $0 \leq l \leq rm$, and for $x \in K$, we put

$$\begin{aligned} P_{n,l}(z) &:= P_{n,l}(\boldsymbol{\alpha}, x \mid z), \\ P_{n,l,i,s}(z) &:= P_{n,l,i,s}(\boldsymbol{\alpha}, x \mid z) \quad \text{for } 1 \leq i \leq m, 1 \leq s \leq r. \end{aligned}$$

The polynomials in the right-hand sides above have been already defined in (5) and (6) respectively.

We define a column vector $\vec{p}_{n,l}(z) \in K[z]^{rm+1}$ by

$$\vec{p}_{n,l}(z) := {}^t(P_{n,l}(z), P_{n,l,1,1}(z), \dots, P_{n,l,1,r}(z), \dots, P_{n,l,m,1}(z), \dots, P_{n,l,m,r}(z)).$$

Proposition 5.1. *We use the same notation as above. For any positive integer n , we have*

$$\Delta_n(z) := \det(\vec{p}_{n,0}(z) \cdots \vec{p}_{n,rm}(z)) \in K(\alpha_1, \alpha_2, \dots, \alpha_m) \setminus \{0\}.$$

To prove this, we firstly prove that the determinant $\Delta_n = \Delta_n(z)$ is a constant, i.e., is independent of z . Secondly, we regard Δ_n as an element of $K(\alpha_1, \dots, \alpha_m)$ viewing $\alpha_1, \dots, \alpha_m$ as indeterminates, and factor it up to a constant depending only on n, m, r . We finally show that this absolute constant Δ_n is nonzero. For this last step, we identify this determinant with a certain real integral to show that it does not vanish.

We shall prove

$$\Delta_n(z) \in K(\alpha_1, \dots, \alpha_m) \quad \text{for all } n \in \mathbb{N}.$$

For this, denote $P_{n,l}(z)\Phi_s(x, \alpha_i/z) - P_{n,l,i,s}(z)$ by $R_{n,l,i,s}(z)$ as above ($0 \leq l \leq rm$, $1 \leq i \leq m$, $1 \leq s \leq r$).

In the matrix giving the determinant $\Delta_n(z)$, we add, the first row multiplied by the $\Phi_s(x, \alpha_i/z)$, to the $((i-1)r+s+1)$ -th row ($1 \leq i \leq m$, $1 \leq s \leq r$), to obtain

$$\Delta_n(z) = (-1)^{rm} \det \begin{pmatrix} P_{n,0}(z) & \dots & P_{n,rm}(z) \\ R_{n,0,1,1}(z) & \dots & R_{n,rm,1,1}(z) \\ \vdots & \ddots & \vdots \\ R_{n,0,1,r}(z) & \dots & R_{n,rm,1,r}(z) \\ \vdots & \ddots & \vdots \\ R_{n,0,m,1}(z) & \dots & R_{n,rm,m,1}(z) \\ \vdots & \ddots & \vdots \\ R_{n,0,m,r}(z) & \dots & R_{n,rm,m,r}(z) \end{pmatrix}.$$

We denote by $\Delta_{n,s,t}(z)$, the (s, t) -th cofactor of the matrix in the right-hand side of the identity above. Then we have, developing along the first row

$$\Delta_n(z) = (-1)^{rm} \left(\sum_{l=0}^{rm} P_{n,l}(z) \Delta_{n,1,l+1}(z) \right). \quad (14)$$

Since we have

$$\text{ord}_\infty R_{n,l,i,s}(z) \geq n+1 \quad \text{for } 0 \leq l \leq rm, 1 \leq i \leq m \text{ and } 1 \leq s \leq r,$$

we get

$$\text{ord}_\infty \Delta_{n,1,l+1}(z) \geq (n+1)rm.$$

Combining the fact $\deg P_{n,l}(z) = rmn + l$ with the lower bound of $\text{ord}_\infty \Delta_{n,1,l+1}(z)$ above, we obtain

$$\begin{aligned} P_{n,l}(z) \Delta_{n,1,l+1}(z) &\in 1/z \cdot K[[1/z]] \quad \text{for } 0 \leq l \leq rm-1, \\ P_{n,rm}(z) \Delta_{n,1,rm+1}(z) &\in K[[1/z]]. \end{aligned}$$

Note that in the relation above, the constant term of $P_{n,rm}(z) \Delta_{n,1,rm+1}(z)$ is

$$(\text{coefficient of } z^{rm(n+1)} \text{ of } P_{n,rm}(z)) \times (\text{coefficient of } 1/z^{rm(n+1)} \text{ of } \Delta_{n,1,rm+1}(z)).$$

Thus by (14), the determinant $\Delta_n(z)$ is a polynomial in z with nonpositive valuation with respect to ord_∞ ; consequently it turns to be a constant. Moreover, the terms of strictly negative valuation should be canceled out. Hence we have

$$\begin{aligned} \Delta_n = \Delta_n(z) &= (-1)^{rm} \left(\sum_{l=0}^{rm} P_{n,l}(z) \Delta_{n,1,l+1}(z) \right) \\ &= (-1)^{rm} (\text{constant term of } P_{n,rm}(z) \Delta_{n,1,rm+1}(z)) \in K. \end{aligned} \quad (15)$$

We now need to rewrite Δ_n as a rational function of $\alpha_1, \dots, \alpha_m$ in a workable way. We further extend the set of variables and assume that the set I contains the rm variables $t_{i,s}$, $1 \leq i \leq m$, $1 \leq s \leq r$, so that L contains

$$K[\alpha_1, \dots, \alpha_m, z, T, 1/\alpha_1, \dots, 1/\alpha_m, 1/z, 1/T][t_{i,s}].$$

For each variable $t_{i,s}$ and any integer l , we have a well-defined map for $\alpha \in L$,

$$\varphi_{\alpha, t_{i,s}, x, l} : L[t_{i,s}]_{1 \leq i \leq m, 1 \leq s \leq r} \rightarrow L[t_{i',s'}]_{(i',s') \neq (i,s)}, \quad t_{i,s}^k \mapsto \frac{\alpha^{k+1}}{(k+x+1)^l}$$

since $L[t_{i,s}]_{1 \leq i \leq m, 1 \leq s \leq r}$ can be regarded as a polynomial ring in one variable $L'[t_{i,s}]$ over $L' = L[t_{i',s'}]_{(i',s') \neq (i,s)}$.

Now for a positive integer n and an integer l with $0 \leq l \leq rm$, we put

$$A_{n,l}(T) := T^l \prod_{i=1}^m (T - \alpha_i)^{rn}.$$

By the definition of $A_{n,l}(T)$, we have $P_{n,l}(z) = \text{Eval}_z \circ S_{n,x}^{(r)}(A_{n,l}(T))$.

Let us define a column vector $\vec{r}_{n,l} \in L^{rm}$ by

$$\vec{r}_{n,l} := {}^t(\varphi_{\alpha_1, t_{1,1}, x, 1}(t_{1,1}^n A_{n,l}(t_{1,1})), \dots, \varphi_{\alpha_1, t_{1,r}, x, r}(t_{1,r}^n A_{n,l}(t_{1,r})), \dots, \varphi_{\alpha_m, t_{m,1}, x, 1}(t_{m,1}^n A_{n,l}(t_{m,1})), \dots, \varphi_{\alpha_m, t_{m,r}, x, r}(t_{m,r}^n A_{n,l}(t_{m,r}))).$$

Lemma 5.2. *Under the notation above, we obtain the identity*

$$\Delta_n = (-1)^{rmn} \left(\frac{(1 + rmn + rm + x)_n}{n!} \right)^r \det(\vec{r}_{n,0} \cdots \vec{r}_{n,rm-1}).$$

Proof. Using (15), we calculate constant term of $P_{n,rm}(z) \Delta_{n,1,rm+1}(z) \in K[[1/z]]$.

We need to deal with the noncommutativity of the multiplication by $[T]$ and the morphisms $S_{n,x}^{(k)}$. The defect of the commutativity is given by the following identity: there exists a set of rational numbers $\{e_{n,k}\}_{0 \leq k \leq rn} \subset \mathbb{Q}$ with $e_{n,0} = (-1)^{rn}$ and

$$[T^n] \circ S_{n,x}^{(r)} = \sum_{k=0}^{rn} e_{n,k} S_{1,x}^{(k)} \circ [T^n].$$

Then we obtain

$$\begin{aligned} \varphi_{\alpha_i, x, s}(T^n P_{n,l}(T)) &= \sum_{k=0}^{rn} e_{n,k} \varphi_{\alpha_i, x, s} \circ S_{1,x}^{(k)} \circ [T^n](A_{n,l}(T)) \\ &= \sum_{k=0}^{s-1} e_{n,k} \varphi_{\alpha_i, x, s-k} \circ [T^n](A_{n,l}(T)) + \sum_{k=s}^{rn} e_{n,k} \varphi_{\alpha_i, x, 0} \circ S_{1,x}^{(k-s)} \circ [T^n](A_{n,l}(T)) \\ &= \sum_{k=0}^{s-1} e_{n,k} \varphi_{\alpha_i, x, s-k}(T^n A_{n,l}(T)) \end{aligned}$$

for $1 \leq i \leq m$ and $1 \leq s \leq r$; the conclusion follows, interpreting the above relations as linear manipulations of lines and columns leaving the determinant unchanged. \square

Now, for nonnegative integers u, n , we put

$$P_{u,n}(t_{i,s}) = \prod_{i=1}^m \prod_{s=1}^r \left[t_{i,s}^u \prod_{j=1}^m (t_{i,s} - \alpha_j)^{rn} \right] \prod_{(i_1, s_1) < (i_2, s_2)} (t_{i_2, s_2} - t_{i_1, s_1}),$$

where the order $(i_1, s_1) < (i_2, s_2)$ follows lexicographically.

By \bigcirc , we denote the composite of morphisms. When no confusion is deemed to occur, we omit the subscripts $\underline{\alpha} = (\alpha_1, \dots, \alpha_m)$ and write

$$\psi = \psi_{\underline{\alpha}} := \bigcirc_{i=1}^m \bigcirc_{s=1}^r \varphi_{\alpha_i, t_{i,s}, x, s}.$$

Note that, by definition of $\det(\vec{r}_{n,0} \cdots \vec{r}_{n,rm-1})$, we have

$$\det(\vec{r}_{n,0} \cdots \vec{r}_{n,rm-1}) = \psi(P_{n,n}).$$

Let u be a nonnegative integer. We are going to study the value

$$C_{n,u,m} := \psi(P_{u,n}).$$

By induction, we obtain the following proposition.

Proposition 5.3. *There exists a nonzero constant $c_{n,u,m} \in K$ satisfying*

$$C_{n,u,m} = c_{n,u,m} \prod_{i=1}^m \alpha_i^{r(u+1)+r^2n+\binom{r}{2}} \prod_{1 \leq i_1 < i_2 \leq m} (\alpha_{i_2} - \alpha_{i_1})^{(2n+1)r^2},$$

with $\binom{r}{2} = 0$ if $r = 1$.

We write the details of the proof of the proposition in the forthcoming articles [David et al. \geq 2020a; \geq 2020b]; however, we describe here our basic idea. Indeed, we prove the proposition by reducing to the case $m = 2$ and showing:

- (i) $C_{n,u,2}$ is homogeneous of degree $2r(u+1) + 2r^2n + 2\binom{r}{2} + (2n+1)r^2$.
- (ii) $(\alpha_1\alpha_2)^{r(u+1)+r^2n+\binom{r}{2}}$ divides $C_{n,u,2}$.
- (iii) $(\alpha_1 - \alpha_2)^{(2n+1)r^2}$ divides $C_{n,u,2}$.

Here, we explain how the constant $c_{n,u,m}$ in Proposition 5.3 becomes nonzero. Whenever it is shown, then the determinant does not vanish.

We use the same notation as in Proposition 5.3. Define

$$D_{n,u,m} := \frac{C_{n,u,m}}{\prod_{i=1}^m \alpha_i^{r(u+1)+r^2n+\binom{r}{2}}} = c_{n,u,m} \times \prod_{1 \leq i_1 < i_2 \leq m} (\alpha_{i_2} - \alpha_{i_1})^{(2n+1)r^2}.$$

A straightforward calculation of an integral gives us

$$\begin{aligned} D_{n,u,m} = & \bigcirc_{i'=1}^m \bigcirc_{s'=1}^r \varphi_{1,t_{i',s'},x,s'} \left(\prod_{i=1}^m \prod_{s=1}^r \left[t_{i,s}^u \cdot (t_{i,s} - 1)^{rn} \cdot \prod_{\substack{1 \leq \tilde{i} \leq m \\ \tilde{i} \neq i}} (\alpha_i t_{i,s} - \alpha_{\tilde{i}})^{rn} \right] \right. \\ & \times \prod_{i=1}^m \left(\prod_{s_1 < s_2} (t_{i,s_2} - t_{i,s_1}) \right) \prod_{i_1 < i_2} \prod_{1 \leq s_1, s_2 \leq r} (\alpha_{i_2} t_{i_2,s_2} - \alpha_{i_1} t_{i_1,s_1}) \left. \right). \end{aligned}$$

We substitute $\alpha_m = 0$ in $D_{n,u,m}$; then we have

$$\begin{aligned} D_{n,u,m}|_{\alpha_m=0} = & c_{n,u,m} \prod_{i=1}^{m-1} (-\alpha_i)^{(2n+1)r^2} \prod_{1 \leq i_1 < i_2 \leq m-1} (\alpha_{i_2} - \alpha_{i_1})^{(2n+1)r^2} \\ = & \pm \prod_{i=1}^{m-1} \alpha_i^{(2n+1)r^2} \bigcirc_{s'=1}^r \varphi_{1,t_{m,s'},x,s'} \left(\prod_{s=1}^r [t_{m,s}^u \cdot (t_{m,s} - 1)^{rn}] \times \prod_{1 \leq s_1 < s_2 \leq r} (t_{m,s_2} - t_{m,s_1}) \right) \\ & \times \bigcirc_{i'=1}^{m-1} \bigcirc_{s'=1}^r \varphi_{1,t_{i',s'},x,s'} \left(\prod_{i=1}^{m-1} \prod_{s=1}^r \left[t_{i,s}^{u+r(n+1)} \cdot (t_{i,s} - 1)^{rn} \cdot \prod_{\substack{1 \leq \tilde{i} \leq m-1 \\ \tilde{i} \neq i}} (\alpha_i t_{i,s} - \alpha_{\tilde{i}})^{rn} \right] \right. \\ & \times \prod_{i=1}^{m-1} \left(\prod_{1 \leq s_1 < s_2 \leq r} (t_{i,s_2} - t_{i,s_1}) \right) \times \prod_{1 \leq i_1 < i_2 \leq m-1} \prod_{1 \leq s_1, s_2 \leq r} (\alpha_{i_2} t_{i_2,s_2} - \alpha_{i_1} t_{i_1,s_1}) \left. \right) \\ = & \pm \prod_{i=1}^{m-1} \alpha_i^{(2n+1)r^2} \bigcirc_{s'=1}^r \varphi_{1,t_{m,s'},x,s'} \left(\prod_{s=1}^r [t_{m,s}^u \cdot (t_{m,s} - 1)^{rn}] \times \prod_{1 \leq s_1 < s_2 \leq r} (t_{m,s_2} - t_{m,s_1}) \right) \\ & \times c_{n,u+r(n+1),m-1} \prod_{1 \leq i_1 < i_2 \leq m-1} (\alpha_{i_2} - \alpha_{i_1})^{(2n+1)r^2}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} c_{n,u,m} &= \pm \bigcirc_{s'=1}^r \varphi_{1,t_{m,s'},x,s'} \left(\prod_{s=1}^r [t_{m,s}^u \cdot (t_{m,s} - 1)^{rn}] \times \prod_{1 \leq s_1 < s_2 \leq r} (t_{m,s_2} - t_{m,s_1}) \right) c_{n,u+r(n+1),m-1} \\ &= \pm \prod_{i=1}^m \left(\bigcirc_{s'=1}^r \varphi_{1,t_{s'},x,s'} \left(\prod_{s=1}^r [t_s^{u+(i-1)r(n+1)} \cdot (t_s - 1)^{rn}] \times \prod_{1 \leq s_1 < s_2 \leq r} (t_{s_2} - t_{s_1}) \right) \right). \end{aligned}$$

We are then in a position to conclude. Indeed, using the definition of the operators $\varphi_{1,t_s,x,s}$, the composition of these operators is nothing but an integral over $[0, 1]^r$. More precisely,

$$\begin{aligned} \bigcirc_{s'=1}^r \varphi_{1,t_{s'},x,s'} &\left(\prod_{s=1}^r [t_s^u \cdot (t_s - 1)^{rn}] \cdot \prod_{1 \leq s_1 < s_2 \leq r} (t_{s_2} - t_{s_1}) \right) \\ &= \prod_{s'=1}^r \frac{1}{(s'-1)!} \int_0^1 \cdots \int_0^1 \prod_{s=1}^r \left[t_s^{u+x} (t_s - 1)^{rn} \log^{s-1} \frac{1}{t_s} \right] \cdot \prod_{1 \leq s_1 < s_2 \leq r} (t_{s_2} - t_{s_1}) \prod_{s=1}^r dt_s; \end{aligned}$$

then a direct computation enables us to show this last integral does not vanish, which yields Proposition 5.1.

The statement of Theorem 2.1 now follows from Proposition 4.2, since the determinant is a nonvanishing algebraic constant.

6. Examples

We show here three examples of linearly independent polylogarithms, which are shown by our criterion.

Example 6.1. Put $r = m = 10$ and $x = 0$. Take $\alpha := (1, \frac{1}{2}, \dots, \frac{1}{10})$ and $\beta = b$ with $|b| \geq e^{2715}$. Then we have $D(\alpha, b) = d_{10} = 2520$. Since we have the inequalities

$$\log 2520 < 7.84, \quad \log 3 < 1.10, \quad \log \frac{5}{2} < 0.92,$$

we have

$$\log |b| > 100(10 + \log 2520 + 10 \log \frac{5}{2}) + 10 \log 3.$$

Then the $10^2 + 1$ numbers

$$1, \text{Li}_1(1/b), \dots, \text{Li}_{10}(1/b), \dots, \text{Li}_1(1/(10b)), \dots, \text{Li}_{10}(1/(10b))$$

are linearly independent over \mathbb{Q} .

Example 6.2. Let $k \geq 2$ be an integer, set $r = m = 10^k$, $x = 0$. Take $\alpha := (j)_{1 \leq j \leq 10^k}$ and $\beta = b \in \mathbb{Z}$ with $|b| \geq \exp(2 \cdot 10^{3k})$. Since $k \geq 2$, we can easily verify

$$\begin{aligned} \log |b| &> (rm + 1) \log 10^k + (r^2 m (1 + \log \frac{5}{2}) + r \log 3) \\ &= k(10^{2k} + 1) \log 10 + (10^{3k} (1 + \log \frac{5}{2}) + 10^k \log 3). \end{aligned}$$

Then the $10^{2k} + 1$ numbers

$$1, \text{Li}_1(1/b), \dots, \text{Li}_{10^k}(1/b), \dots, \text{Li}_1(10^k/b), \dots, \text{Li}_{10^k}(10^k/b)$$

are linearly independent over \mathbb{Q} . For instance, we take $r = m = 10^4$ and $b = 3^{2 \cdot 10^{12}}$ then the $10^8 + 1$ numbers

$$1, \text{Li}_1(1/3^{2 \cdot 10^{12}}), \dots, \text{Li}_{10^4}(1/3^{2 \cdot 10^{12}}), \dots, \text{Li}_1(10^4/3^{2 \cdot 10^{12}}), \dots, \text{Li}_{10^4}(10^4/3^{2 \cdot 10^{12}}),$$

are all linearly independent over \mathbb{Q} .

Example 6.3. Let $M \geq 5$ be a natural number. Define the polynomial

$$f_M(X) := \left(2 + \frac{1}{M}\right)X^2 - 2X + \frac{2}{M}.$$

Then $X = (M \pm \sqrt{M^2 - 4M - 2})/(2M + 1)$ are roots of $f_M(X)$.

Put

$$\beta := \frac{2M + 1}{M - \sqrt{M^2 - 4M - 2}},$$

$K := \mathbb{Q}(\beta)$ and $\delta := e^{7908}$. We take $r = m = 10$, $\alpha := (1, \frac{1}{2}, \dots, \frac{1}{10})$ and

$$M > \frac{2\delta^2 - \delta + 1 + \sqrt{4\delta^4 + 4\delta^3 - 3\delta^2 - 6\delta + 5}}{4\delta - 4}.$$

Then we have

$$V(\alpha, \beta, 0) = \mathbb{A}(\alpha, \beta, 0) - \mathcal{A}^{(2)}(\alpha, \beta, 0) > \log |\beta| - 7908 > 0.$$

Thus by Theorem 2.1, the $10^2 + 1$ numbers

$$1, \text{Li}_1(1/\beta), \dots, \text{Li}_{10^k}(1/\beta), \dots, \text{Li}_1(1/10\beta), \dots, \text{Li}_{10^k}(1/10\beta),$$

are linearly independent over K . For example, we take $M \geq e^{15817}$, the $10^2 + 1$ numbers

$$1, \text{Li}_1(1/\beta), \dots, \text{Li}_{10^k}(1/\beta), \dots, \text{Li}_1(1/10\beta), \dots, \text{Li}_{10^k}(1/10\beta),$$

are linearly independent over K .

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