SPECTRAL RADIUS OF RANDOM MATRICES WITH INDEPENDENT ENTRIES

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We consider random $n \times n$ matrices $X$ with independent and centered entries and a general variance profile. We show that the spectral radius of $X$ converges with very high probability to the square root of the spectral radius of the variance matrix of $X$ when $n$ tends to infinity. We also establish the optimal rate of convergence; that is a new result even for general i.i.d. matrices beyond the explicitly solvable Gaussian cases. The main ingredient is the proof of the local inhomogeneous circular law (Ann. Appl. Probab. 28:1 (2018), 148–203) at the spectral edge.

1. Introduction

Girko’s celebrated circular law [Girko 1984; Bai 1997]\(^1\) asserts that the spectrum of an $n \times n$ random matrix $X$ with centered, independent, identically distributed (i.i.d.) entries with variance $\mathbb{E}|x_{ij}|^2 = 1/n$ converges, as $n \to \infty$, to the unit disc with a uniform limiting density of eigenvalues. The cornerstone of the proof is the Hermitization formula (see (2-12)) that connects eigenvalues of $X$ to the eigenvalues of a family of Hermitian matrices $(X - z)^*(X - z)$ with a complex parameter $z$ [Girko 1984]. The circular law for i.i.d. entries with the minimal second moment condition was established by Tao and Vu [2010] after several partial results [Götze and Tikhomirov 2010; Pan and Zhou 2010; Tao and Vu 2008];

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\(^1\)The original proof in [Girko 1984] was not considered complete and Bai published a clean version under more restrictive conditions [Bai 1997]. An extended version of Girko’s original proof with explanations and corrections appeared in [Girko 1998, Chapter 6]; see also [Girko 2012].
see [Bordenave and Chafaï 2012] for the extensive history and literature. We also refer to the recent circular law for very sparse matrices [Rudelson and Tikhomirov 2019].

The circular law establishes the weak limit of the empirical density of eigenvalues and thus it accounts for most but not all of them. In particular, it does not give information on the spectral radius \( \varrho(X) \) of \( X \) since the largest (in absolute value) eigenvalue may behave very differently than the bulk spectrum. In fact, such outliers do not exist but this requires a separate proof. The convergence of the spectral radius of \( X \) to 1,

\[
\varrho(X) \to 1, \quad \text{almost surely as } n \to \infty, \tag{1-1}
\]

was proven by Bai and Yin [1986] under the fourth moment condition, \( \mathbb{E}[n^{1/2} x_{ij}]^4 \leq C \), using Wigner’s moment method. Under stronger conditions the upper bound in (1-1) was independently proven in [Geman 1986]; see also [Geman and Hwang 1982; Nemish 2018]. More recently in [Bordenave et al. 2018] the convergence \( \varrho(X) \to 1 \) in probability was shown assuming only finite \( 2 + \epsilon \) moment and a symmetric entry distribution.

Precise information on the spectral radius is available only for the Ginibre ensemble, i.e., when \( x_{ij} \) are Gaussian; in this case it is known [Rider 2003; Rider and Sinclair 2014] that

\[
\varrho(X) \approx 1 + \sqrt{\frac{\gamma_n}{4n}} + \frac{1}{\sqrt{4n\gamma_n}} \xi, \quad \gamma_n := \log \frac{n}{2\pi} - 2\log \log n, \tag{1-2}
\]

where \( \xi \) is a Gumbel distributed random variable.

In this paper we drop the condition that the matrix elements are identically distributed and we study the spectral radius of \( X \) when the variances \( \mathbb{E}|x_{ij}|^2 \) have a nontrivial profile given by the matrix \( \mathcal{S} = (\mathbb{E}|x_{ij}|^2)_{i,j=1}^n \). In our previous work [Alt et al. 2018] we showed that the spectral radius of \( X \) is arbitrarily close to the square root of the spectral radius of \( \mathcal{S} \). More precisely, for any fixed \( \epsilon > 0 \) we have

\[
\sqrt{\varrho(\mathcal{S})} - \epsilon \leq \varrho(X) \leq \sqrt{\varrho(\mathcal{S})} + \epsilon \tag{1-3}
\]

with very high probability for large \( n \). Motivated by (1-2) we expect that the precision of the approximation in (1-3) can be greatly improved and the difference between \( \varrho(X) \) and \( \sqrt{\varrho(\mathcal{S})} \) should not exceed \( n^{-1/2} \) by much. Indeed, our first main result proves that for any \( \epsilon > 0 \) we have

\[
\sqrt{\varrho(\mathcal{S})} - n^{-1/2+\epsilon} \leq \varrho(X) \leq \sqrt{\varrho(\mathcal{S})} + n^{-1/2+\epsilon} \tag{1-4}
\]

with very high probability for large \( n \). Apart from the \( n^{\epsilon} \) factor this result is optimal considering (1-2).

Note that (1-4) is new even for the i.i.d. case beyond Gaussian, i.e., there is no previous result on the speed of convergence in (1-1).

We remark that, compared with the spectral radius, much more is known about the largest singular value of \( X \) since it is equivalent to the (square root of the) largest eigenvalue of the sample covariance matrix \( XX^* \). For the top eigenvalues of \( XX^* \), precise limiting behavior (Tracy–Widom) is known if \( X \) has general i.i.d. matrix elements [Pillai and Yin 2014], and even general diagonal population matrices are allowed [Lee and Schnelli 2016]. Note, however, the largest singular value of \( X \) in the i.i.d. case converges to 2, i.e., it is very different from the spectral radius, indicating that \( X \) is far from being normal.
We stress that understanding the spectral radius is a genuinely non-Hermitian problem and hence in general it is much harder than studying the largest singular value.

While the largest singular value is very important for statistical applications, the spectral radius is relevant for time evolution of complex systems. More precisely, the spectral radius controls the eigenvalue with largest real part that plays an important role in understanding the long time behavior of large systems of linear ODEs with random coefficients of the form

$$\frac{d}{dt} u_t = -g u_t + X u_t$$

with a tunable coupling constant $g$. Such an ODE system was first introduced in an ecological model to study the interplay between complexity and stability in May’s seminal paper [1972]; see also the recent exposition [Allesina and Tang 2015]. It has since been applied to many situations when a transience phenomenon is modelled in dynamics of complex systems, especially for neural networks, e.g., [Sompolinsky et al. 1988; Hennequin et al. 2014; Grela 2017]. Structured neural networks require the generalization of May’s original i.i.d. model to nonconstant variance profile $\mathcal{S}$ [Aljadeff et al. 2015; Muir and Mrsic-Flogel 2015; Rajan and Abbott 2006; Gudowska-Nowak et al. 2020] which we study in full generality. The long time evolution of (1-5) at critical coupling $g_c := \sqrt{\varrho(\mathcal{S})}$ in the i.i.d. Gaussian case was computed in [Chalker and Mehlig 1998] after some nonrigorous steps; the full mathematical analysis even for general distribution and beyond the i.i.d. setup was given in [Erdős et al. 2018; 2019a]. The time-scale on which the solution of (1-5) at criticality can be computed depends on how precisely $\varrho(X)$ can be controlled by $\sqrt{\varrho(\mathcal{S})}$. In particular, the current improvement of this precision to (1-4) allows one to extend the result of [Erdős et al. 2018, Theorem 2.6] to very long time scales of order $n^{1/2-\varepsilon}$. These applications require a separate analysis; we will not pursue them in the present work.

We now explain the key novelties of this paper; more details will be given in Section 2A after presenting the precise results in Section 2. The spectral radius of $X$ is ultimately related to our second main result, the local law for $X$ near the spectral edges, i.e., a description of the eigenvalue density on local scales but still above the eigenvalue spacing; in this case $n^{-1/2}$. As a byproduct, we also prove the optimal $1/n$ speed of convergence in the inhomogeneous circular law [Alt et al. 2018; Cook et al. 2018]. Note that the limiting density has a discontinuity at the boundary of its support, the disk of radius $\sqrt{\varrho(\mathcal{S})}$ [Alt et al. 2018, Proposition 2.4], hence the typical eigenvalue spacing at the edge and in the bulk coincide, unlike for the Hermitian problems. The local law in the bulk for $X$ with a general variance profile has been established in [Alt et al. 2018, Theorem 2.5] on scale $n^{-1/2+\varepsilon}$ and with optimal error bounds. This entails an optimal local law near zero for the Wigner-type Hermitian matrix

$$H_z := \begin{pmatrix} 0 & X - z \\ (X - z)^* & 0 \end{pmatrix}$$

appearing in Girko’s formula. As long as $z$ is in the bulk spectrum of $X$, the relevant spectral parameter $0$ lies in the bulk spectrum of $H_z$. Still, the local law for Wigner-type matrices [Ajanki et al. 2017b] is not applicable since the flatness condition, which requires the variances of all matrix elements of $H_z$ be comparable, is violated by the large zero blocks in $H_z$. In fact, the corresponding Dyson equation has an
unstable direction due to the block symmetry of $H_z$. The main achievement of [Alt et al. 2018] was to handle this instability.

When $z$ is near the spectral edge of $X$, the density of $H_z$ develops a cusp singularity at 0. The optimal cusp local law for Wigner-type matrices with flatness condition was proven recently in [Erdős et al. 2020] relying on the improved fluctuation averaging mechanism and on the deterministic analysis of the corresponding Dyson equation in [Alt et al. 2020a]. Due to the cusp, the Dyson equation has a natural unstable direction and the corresponding non-Hermitian perturbation theory is governed by a cubic equation.

The Dyson equation corresponding to the matrix $H_z$ for $z$ near the spectral edge of $X$ exhibits both instabilities simultaneously. This leads to the main technical achievement of this paper: we prove an optimal local law in the cusp regime with the block instability. Most of the paper contains our refined analysis of the Dyson equation with two instabilities, a delicate synthesis of the methods developed in [Alt et al. 2018] and [Alt et al. 2020a]. The necessary fluctuation averaging argument, however, turns out to be simpler than in [Erdős et al. 2020]; the block symmetry here helps.

We remark that bulk and edge local laws for the i.i.d. case have been proven earlier [Bourgade et al. 2014a; 2014b] with the optimal scale at the edge in [Yin 2014] and later with improved moment assumptions in [Götze et al. 2017]; see also [Tao and Vu 2015] for similar results under the three moment matching condition. However, these works did not provide the improved local law outside of the spectrum that is necessary to identify the spectral radius. The main difference is that the i.i.d. case results in an explicitly solvable scalar-valued Dyson equation, so the entire stability analysis boils down to analysing explicit formulas. The inhomogeneous variance profile $\mathcal{S}$ leads to a vector-valued Dyson equation with no explicit solution at hand; all stability properties must be obtained inherently from the equation itself. Furthermore, even in the i.i.d. case the local law for $H_z$ in [Bourgade et al. 2014b; Yin 2014] was not optimal in the edge regime $|z| \approx 1$ and the authors directly estimated only the specific error terms in Girko’s formula. The optimality of our local law for $H_z$ at the edge is the main reason why the proof of the local circular law in Section 6 is very transparent. In fact, our current local law is formulated in the isotropic sense (see (5-3) later) which is more general than the result in [Bourgade et al. 2014a; 2014b] even in the i.i.d. case. This generalized version was an essential ingredient in the recent proof of edge universality for i.i.d. matrices [Cipolloni et al. 2021].

2. Main results

Let $X = (x_{ij})_{i,j=1}^n \in \mathbb{C}^{n \times n}$ be a matrix with independent, centered entries. Let $\mathcal{S} := (\mathbb{E}|x_{ij}|^2)_{i,j=1}^n$ be the matrix collecting the variances of the entries of $X$. Further, our main results will require a selection of the following assumptions (we remark that the last assumption (A3) can be substantially relaxed; see Remark 2.5).

**Assumptions.** (A1) The variance matrix $\mathcal{S}$ of $X$ is flat i.e., there are constants $s^* > s_* > 0$ such that

$$\frac{s_*}{n} \leq \mathbb{E}|x_{ij}|^2 \leq \frac{s^*}{n}$$

for all $i, j = 1, \ldots, n$.

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2The flatness condition in (2-1) agrees with the concept of flatness introduced for general matrices with independent entries in [Alt et al. 2018, Equation (2.1)]. For Hermitian matrices, flatness is defined slightly differently, see (3.6) in [Erdős et al. 2020] and the explanation thereafter.
The entries of $X$ have bounded moments in the sense that, for each $m \in \mathbb{N}$, there is $\mu_m > 0$ such that
\[
\mathbb{E}|x_{ij}|^m \leq \mu_m n^{-m/2}
\]
for all $i, j = 1, \ldots, n$.

Each entry of $\sqrt{n}X$ has a bounded density on $\mathbb{C}$ in the following sense. There are probability densities $\nu_{ij} : \mathbb{C} \to [0, \infty)$ such that
\[
\mathbb{P}(\sqrt{n} x_{ij} \in B) = \int_B \nu_{ij}(z) \, d^2 z
\]
for all $i, j = 1, \ldots, n$ and all Borel sets $B \subset \mathbb{C}$ and these densities are bounded in the sense that there are $\alpha, \beta > 0$ such that $\nu_{ij} \in L^{1+\alpha}(\mathbb{C})$ and
\[
\|\nu_{ij}\|_{1+\alpha} \leq n^\beta
\]
for all $i, j = 1, \ldots, n$.

In (A3) and in the following, $d^2 z$ denotes the Lebesgue measure on $\mathbb{C}$. The main results remain valid if $X$ has all real entries, i.e., the density $\nu_{ij}$ of $\sqrt{n} x_{ij}$ in (A3) is supported on $\mathbb{R}$ instead of $\mathbb{C}$ and we consider its $L^{1+\alpha}(\mathbb{R})$-norm. In fact, the proofs are completely analogous. Hence, for simplicity, we only present the proofs in the complex case.

The following theorem, our first main result, provides a convergence result for the spectral radius of the random matrix $X$. For any matrix $R \in \mathbb{C}^{n \times n}$, we write $\varrho(R)$ for its spectral radius, i.e.,
\[
\varrho(R) := \max_{\lambda \in \text{Spec}(R)} |\lambda|.
\]

**Theorem 2.1** (spectral radius of $X$). Let $X$ satisfy (A1)–(A3). Then, for each (small) $\varepsilon > 0$ and (large) $D > 0$, there is $C_{\varepsilon,D} > 0$ such that
\[
\mathbb{P}\left(\varrho(X) - \sqrt{\varrho(S)} \geq n^{-1/2+\varepsilon}\right) \leq \frac{C_{\varepsilon,D}}{n^D}
\]
for all $n \in \mathbb{N}$.

Here, the constant $C_{\varepsilon,D}$ depends only on $s_*, s^*$ from (A1), the sequence $(\mu_m)_{m \in \mathbb{N}}$ from (A2) and $\alpha, \beta$ from (A3) in addition to $\varepsilon$ and $D$.

**Remark 2.2** (upper bound on the spectral radius of $X$ without (A3)). Without Assumption (A3) our proof still implies the following upper bound on the spectral radius $\varrho(X)$ of $X$. That is, if (A1) and (A2) are satisfied then for each $\varepsilon > 0$ and $D > 0$, there is $C_{\varepsilon,D} > 0$ such that, for all $n \in \mathbb{N}$, we have
\[
\mathbb{P}(\varrho(X) \geq \sqrt{\varrho(S)} + n^{-1/2+\varepsilon}) \leq \frac{C_{\varepsilon,D}}{n^D}.
\]
In particular, $X$ does not have any eigenvalue of modulus bigger than $\sqrt{\varrho(S)} + n^{-1/2+\varepsilon}$ with very high probability. The assumption (A3) is only used to control the smallest singular value of $X - z$ when relating the eigenvalue density of $X$ and the one of the Hermitization $H_z$ of $X$ (see (2-5)) in the proof of Theorem 2.3. An eigenvalue of $X$ at $z$ can be excluded directly, without comparing the eigenvalue densities, if the kernel of $H_z$ is trivial. Therefore, (A3) is not needed for the upper bound on $\varrho(X)$.
The next main result, Theorem 2.3, shows that the eigenvalue density of \( X \) is close to a deterministic density on all scales slightly above the typical eigenvalue spacing when \( n \) is large. We now prepare the definition of this deterministic density. For each \( \eta > 0 \) and \( z \in \mathbb{C} \), we denote by \((v_1, v_2) \in (0, \infty)^n \times (0, \infty)^n\) the unique solution to the system of equations
\[
\frac{1}{v_1} = \eta + \mathcal{S} v_2 + \frac{|z|^2}{\eta + \mathcal{S}^t v_1}, \quad \frac{1}{v_2} = \eta + \mathcal{S}^t v_1 + \frac{|z|^2}{\eta + \mathcal{S} v_2}.
\] (2-3)

Here, any scalar is identified with the vector in \( \mathbb{C}^n \) whose components agree all with the scalar. E.g., \( \eta \) is identified with \((\eta, \ldots, \eta) \in \mathbb{C}^n \). Moreover, the ratio of two vectors in \( \mathbb{C}^n \) is defined componentwise. The existence and uniqueness of \((v_1, v_2)\) has been derived in [Alt et al. 2018, Lemma 2.2] from abstract existence and uniqueness results in [Helton et al. 2007].

In the following, we consider \( v_1 = v_1(z, \eta) \) and \( v_2 = v_2(z, \eta) \) as functions of \( \eta > 0 \) and \( z \in \mathbb{C} \). In Proposition 3.14, we will show that there is a probability density \( \sigma : \mathbb{C} \rightarrow [0, \infty) \) such that
\[
\sigma(z) = -\frac{1}{2\pi} \Delta \int_0^\infty \left( \langle v_1(z, \eta) \rangle - \frac{1}{1 + \eta} \right) d\eta,
\] (2-4)
where the equality and the Laplacian \( \Delta \) on \( \mathbb{C} \) are understood in the sense of distributions on \( \mathbb{C} \). Moreover, \( \langle v_1 \rangle \) denotes the mean of the vector \( v_1 \in \mathbb{C}^n \), i.e., \( (\mu) := \frac{1}{n} \sum_{i=1}^n u_i \) for any \( u = (u_i)_{i=1}^n \in \mathbb{C}^n \). In Lemma 3.15, we will show that the integral on the right-hand side of (2-4) exists for each \( z \in \mathbb{C} \). Proposition 3.14 also proves further properties of \( \sigma \), in particular, that the support of \( \sigma \) is a disk of radius \( \sqrt{\varrho(\mathcal{S})} \) around the origin.

In order to analyze the eigenvalue density of \( X \) on local scales, we consider shifted and rescaled test functions as follows. For any function \( f : \mathbb{C} \rightarrow \mathbb{C} \), \( z_0 \in \mathbb{C} \) and \( a > 0 \), we define
\[
f_{z_0,a} : \mathbb{C} \rightarrow \mathbb{C}, \quad f_{z_0,a}(z) := n^2 a f(n^a(z - z_0)).
\]
The eigenvalues of \( X \) are denoted by \( \xi_1, \ldots, \xi_n \). Now we are ready to state our second main result.

**Theorem 2.3** (local inhomogeneous circular law). Let \( X \) satisfy (A1)–(A3). Let \( a \in [0, 1/2] \) and \( \varphi > 0 \). Then, for every \( \varepsilon > 0 \) and \( D > 0 \), there is \( C_{\varepsilon,D} > 0 \) such that
\[
P \left( \left| \frac{1}{n} \sum_{i=1}^n f_{z_0,a}(\xi_i) - \int_{\mathbb{C}} f_{z_0,a}(z) \sigma(z) d\mathcal{S}^2 z \right| \geq \frac{\|\Delta f\|_{L^1}}{n^{1-2a-\varepsilon}} \right) \leq \frac{C_{\varepsilon,D}}{n^D}
\]
uniformly for all \( n \in \mathbb{N}, \ z_0 \in \mathbb{C} \) satisfying \( |z_0| \leq \varphi \) and \( f \in C^2_0(\mathbb{C}) \) satisfying \( \text{supp} f \subset \{ z \in \mathbb{C} : |z| \leq \varphi \} \). The point \( z_0 \) and the function \( f \) may depend on \( n \). In addition to \( \varepsilon \) and \( D \), the constant \( C_{\varepsilon,D} \) depends only on \( s_*, s^* \) from (A1), \( (\mu_m)_{m \in \mathbb{N}} \) from (A2), \( \alpha, \beta \) from (A3), and \( a \) and \( \varphi \).

The bulk regime, \( |z_0| < \sqrt{\varrho(\mathcal{S})} \), in Theorem 2.3 has already been proven in [Alt et al. 2018, Theorem 2.5]. Choosing \( a = 0 \) and \( z_0 = 0 \) in Theorem 2.3 amounts to the optimal \( 1/n \) speed of convergence in the inhomogeneous circular law.

Finally, we state a corollary of our result showing that all normalized eigenvectors \( u = (u_i)_{i=1}^n \in \mathbb{C}^n \) of \( X \) are completely delocalized in the sense that \( \max_{i=1}^n |u_i| \leq n^{-1/2+\varepsilon} \) with very high probability. Eigenvector
delocalization under somewhat different conditions and with very different methods has already been established in [Rudelson and Vershynin 2015] with recent refinements in [Rudelson and Vershynin 2016; Luh and O’Rourke 2020; Lytova and Tikhomirov 2020].

**Corollary 2.4 (eigenvector delocalization).** Let $X$ satisfy (A1) and (A2). Then, for each $\epsilon > 0$ and $D > 0$, there is $C_{\epsilon, D} > 0$ such that

$$
P(\exists u \neq 0 : Xu = \zeta u \text{ for some } \zeta \in \mathbb{C} \text{ and } \max_{i=1}^n |u_i| \geq n^{-1/2+\epsilon} \|u\|) \leq \frac{C_{\epsilon, D}}{n^D}
$$

for all $n \in \mathbb{N}$. Here, $\|u\|$ denotes the Euclidean norm of $u$.

**Remark 2.5 (alternative to Assumption (A3)).** Theorem 2.1, as well as Theorem 2.3 (with an additional condition $\|f\|_{L^{2+\epsilon}} \leq n^C \|f\|_{L^1}$, with some large constant $C$, on the test function $f$), holds if Assumption (A3) is replaced by the following anticoncentration condition. With the Lévy concentration function

$$
\mathcal{L}(Z, t) := \sup_{u \in \mathbb{R}} P(|Z - u| < t)
$$

we require that $\max_{i,j} \mathcal{L}(\sqrt{n} x_{ij}, t) \leq b$ for some constants $t \geq 0$ and $b \in (0, 1)$. In our main proofs, we use (A3) for pedagogical reasons and the necessary modifications will be explained in Remark 6.2 at the end of Section 6.

2A. **Outline of the proof.** In this subsection, we outline a few central ideas of the proofs of Theorem 2.1 and Theorem 2.3. The spectrum of the $n \times n$-matrix $X$ can conveniently be studied by analysing the kernel of the $2n \times 2n$ Hermitian matrices $H_z$ defined through

$$
H_z := \begin{pmatrix} 0 & X - z \\ (X - z)^* & 0 \end{pmatrix}
$$

for $z \in \mathbb{C}$. In fact, $z$ is an eigenvalue of $X$ if and only if the kernel of $H_z$ is nontrivial.

All spectral properties of a Hermitian matrix can be obtained from its resolvent. In fact, in many cases, the resolvent of a Hermitian random matrix becomes deterministic when its size tends to infinity and the limit is the solution to the associated matrix Dyson equation. In our setup, the matrix Dyson equation (MDE) for the deterministic counterpart $M = M(z, \eta)$ of the resolvent $G = G(z, \eta) = (H_z - i\eta)^{-1}$ of $H_z$ is given by

$$
-M^{-1}(z, \eta) = \begin{pmatrix} \eta & z \\ \bar{z} & \bar{\eta} \end{pmatrix} + S[M(z, \eta)]
$$

Here, $\eta > 0$ and $z \in \mathbb{C}$ are parameters and $i\eta$, $z$ and $\bar{z}$ are identified with the respective multiples of the $n \times n$ identity matrix. Moreover, we introduced the self-energy operator $S : \mathbb{C}^{2n \times 2n} \to \mathbb{C}^{2n \times 2n}$ given by

$$
S[R] = \begin{pmatrix} \mathcal{S} r_2 & 0 \\ 0 & \mathcal{S} r_1 \end{pmatrix}
$$

for $R = (r_{ij})_{i,j=1}^{2n} \in \mathbb{C}^{2n \times 2n}$, where $r_1 := (r_{ii})_{i=1}^n$, $r_2 := (r_{i,n+1})_{i=1}^{2n}$ and $\mathcal{S} := (\mathbb{E}|x_{ij}|^2)_{i,j=1}^{n+1}$. The matrix on the right-hand side of (2-7) denotes a $2n \times 2n$ diagonal matrix with the vector $(\mathcal{S} r_2, \mathcal{S} r_1) \in \mathbb{C}^{2n}$ on its diagonal.
Two remarks about (2-6) and (2-7) are in order. In this paper we are interested exclusively in the kernel of $H_z$. Otherwise $i\eta$ on the right-hand side of (2-6) had to be replaced by $E + i\eta$ for some $E \in \mathbb{R}$ (see [Ajanki et al. 2019; Erdős et al. 2019b; Alt et al. 2020b] for the general MDE in the random matrix setup). We also remark that the self-energy operator $S$ in (2-7) is chosen slightly differently compared to the choice of the self-energy operator for a Hermitian random matrix in [Ajanki et al. 2019; Erdős et al. 2019b; Alt et al. 2020b]. Instead, we follow here the convention from [Alt et al. 2018]. For further details, see Remark 5.4.

First, we discuss Theorem 2.1. Suppose we already know that $G$ is very well approximated by $M$. Owing to [Alt et al. 2018, Proposition 3.2] (see also Lemma 3.3), $\text{Im} M(z, \eta)$ vanishes sufficiently fast for $\eta \downarrow 0$ as long as $|z|^2 \geq \varrho(S) + n^{-1/2+\varepsilon}$. Then we can immediately conclude that the kernel of $H_z$ has to be trivial. Hence, any eigenvalue of $X$ has modulus smaller than $\sqrt{\varrho(S)} + n^{-1/2} + \varepsilon$. Similarly, under the condition $|z|^2 < \varrho(S) - n^{-1/2+\varepsilon}$, the imaginary part $\text{Im} M(z, \eta)$ is big enough as $\eta \downarrow 0$ due to [Alt et al. 2018, Proposition 3.2]. This will imply that $H_z$ has a nontrivial kernel and, hence, $X$ has an eigenvalue close to $z$, thus completing the proof of (2-2).

Therefore, what remains is to prove a local law for $H_z$, i.e., that $G$ is very well approximated by $M$. The resolvent $G$ satisfies a perturbed version of the MDE (2-6),

$$-G^{-1} = \left( \begin{array}{cc} i\eta & z \\ \bar{z} & i\eta \end{array} \right) + S[G] - DG^{-1}, \quad D := (H_z - \mathbb{E}H_z)G + S[G]G$$  \hspace{1cm} (2-8)

for all $\eta > 0$ and $z \in \mathbb{C}$. The error matrix $D$ will be shown to be small in Section 5. Consequently, we will consider (2-8) as a perturbed version of the MDE (2-6) and study its stability properties under small perturbations to conclude that $G$ is close to $M$.

A simple computation starting from (2-6) and (2-8) yields the stability equation associated to the MDE,\n
$$B[G - M] = MS[G - M](G - M) - MD.$$  \hspace{1cm} (2-9)

Here, $B : \mathbb{C}^{2n \times 2n} \to \mathbb{C}^{2n \times 2n}$ is the linear stability operator of the MDE, given explicitly by

$$B[R] := R - MS[R]M$$  \hspace{1cm} (2-10)

for any $R \in \mathbb{C}^{2n \times 2n}$.

The stability equation (2-9) is viewed as a general quadratic equation of the form

$$B[Y] - A[Y, Y] + Z = 0$$  \hspace{1cm} (2-11)

for the unknown matrix $Y (= G - M)$ in the regime where $Z (= MD)$ is small. Here, $B$ is a linear map and $A$ is a bilinear map on the space of matrices. This problem would be easily solved by a standard implicit function theorem if $B$ had a stable (i.e., bounded) inverse; this is the case in the bulk regime. When $B$ has unstable directions, i.e., eigenvectors corresponding to eigenvalues very close to zero, then these directions need to be handled separately.

The linear stability operator (2-10) for Wigner-type matrices with a flat variance matrix in the edge or cusp regime gives rise to one unstable direction $B$ with $B[B] \approx 0$. In this case, the solution is, to leading order, parallel to the unstable direction $B$, hence it can be written as $Y = \Theta B + \text{error}$ with some complex scalar coefficient $\Theta$, determining the leading behavior of $Y$. For such $Y$ the linear term in (2-11)
becomes lower order and the quadratic term and the error term in $Y$ play an important role. Systematically expanding $Y$ up to higher orders in the small parameter $\|Z\| \ll 1$, we arrive at an approximate cubic equation for $\Theta$ of the form $c_3 \Theta^3 + c_2 \Theta^2 + c_1 \Theta = \text{small}$, with very precisely computed coefficients. The full derivation of this cubic equation is given in [Erdős et al. 2020, Lemma A.1]. In the bulk regime $|c_1| \sim 1$, hence the equation is practically linear. In the regime where the density vanishes, we have $c_1 \approx 0$, hence higher order terms become relevant. At the edge we have $|c_2| \approx 1$, so we have a quadratic equation, while in the cusp regime $c_2 \approx 0$, but $|c_3| \sim 1$, so we have a cubic equation. It turns out that under the flatness condition no other cases are possible, i.e., $|c_1| + |c_2| + |c_3| \sim 1$. This trichotomic structural property of the underlying cubic equation was first discovered in [Ajanki et al. 2017a], developed further in [Alt et al. 2020a], and played an essential role in proving cusp local laws for Wigner-type matrices in [Ajanki et al. 2017b; Erdős et al. 2020].

In our current situation, lacking flatness for $H_\ast$, a second unstable direction of $B$ is present due to the specific block structure of the matrix $H_\ast$ which creates a major complication. We denote the unstable directions of $B$ by $B$ and $B_\ast$. One of them, $B$, is the relevant one and it behaves very similarly to the one present in [Alt et al. 2020a; 2020b; Erdős et al. 2020]. The novel unstable direction $B_\ast$ originates from the specific block structure of $H_\ast$ and $S$ in (2-5) and (2-7), respectively, and is related to the unstable direction in [Alt et al. 2018]. We need to treat both unstable directions separately. In a generic situation, the solution to (2-11) would be of the form $Y = \Theta B + \Theta_\ast B_\ast + \text{error}$, where the complex scalars $\Theta$ and $\Theta_\ast$ satisfy a system of coupled cubic equations that is hard to analyse. Fortunately, for our applications, we have an additional input, namely we know that there is a matrix, concretely $E_\ast$ and a careful use of the specific structure of $B$ in (2-9) requires an analysis of the small eigenvalues of $B$ in the regime, where $|z|^2$ is close to $\varphi(\mathcal{F})$ and $\eta$ is small. This analysis is based on viewing the non-normal operator $B$ as a perturbation around an operator of the form $1 - C \mathcal{F}$, where $C$ is unitary and $\mathcal{F}$ is Hermitian. The unperturbed operator, $1 - C \mathcal{F}$, is also non-normal but simpler to analyze compared to $B$. In fact, $1 - C \mathcal{F}$ has a single small eigenvalue and this eigenvalue has (algebraic and geometric) multiplicity two and we can construct appropriate eigendirections. A very fine perturbative argument reveals that after perturbation these two eigendirections will be associated to two different (small) eigenvalues $\beta$ and $\beta_\ast$. The distance between them is controlled from below which allows us to follow the perturbation of the eigendirections as well. Precise perturbative expansions of $B$ and $B_\ast$ around the corresponding eigenvectors of $1 - C \mathcal{F}$ and a careful use of the specific structure of $S$ in (2-7) reveal that, up to a small error term, $B$ is orthogonal to $E_\ast$ while $B_\ast$ is far from orthogonal to $E_\ast$.

Moreover, we have to show that $MD$ in (2-9) is sufficiently small in the unstable direction $B$ to compensate for the blow-up of $B^{-1}$ originating from the relevant small eigenvalue $\beta$. To that end, we need
to adjust the cusp fluctuation averaging mechanism discovered in [Erdős et al. 2020] to the current setup which will be done in Section 5B. This part also uses the specific block structure of $H_\varepsilon$ in (2-5). We can, thus, conclude that $G - M$ is small due to (2-9) which completes the sketch of the proof of Theorem 2.1.

The proof of Theorem 2.3 also follows from the local law for $H_\varepsilon$ since the observable of the eigenvalues of $X$ is related to the resolvent $G$ while the integral over $f_{z_0, a} \sigma$ is related to $M$. Indeed, [Alt et al. 2018, Equations (2.10), (2.13) and (2.14)] imply that

$$\frac{1}{N} \sum_{i=1}^{n} f_{z_0, a}(\xi_i) = \frac{1}{4\pi n} \int_{C} \Delta f_{z_0, a}(z) \log |\det H_\varepsilon| d^2z = -\frac{1}{4\pi n} \int_{C} \Delta f_{z_0, a}(z) \int_{0}^{\infty} \text{Im} \text{Tr} G(z, \eta) d\eta \, d^2z. \quad (2-12)$$

The first identity in (2-12) is known as Girko’s Hermitization formula, the second identity (after a regularization of the $\eta$-integral at infinity) was first used in [Tao and Vu 2015]. On the other hand, since the imaginary part of the diagonal of $M$ coincides with the solution $(v_1, v_2)$ of (2-3) (see (3-6)), the definition of $\sigma$ in (2-4) yields

$$\int_{C} f_{z_0, a}(z) \sigma(z) d^2z = -\frac{1}{4\pi n} \int_{C} \Delta f_{z_0, a}(z) \int_{0}^{\infty} \text{Im} \text{Tr} M(z, \eta) d\eta \, d^2z.$$

Therefore, Theorem 2.3 also follows once the closeness of $G$ and $M$ has been established as explained above.

**2B. Notation and conventions.** In this section, we collect some notation and conventions used throughout the paper. We set $[k] := \{1, \ldots, k\} \subset \mathbb{N}$ for any $k \in \mathbb{N}$. For $z \in \mathbb{C}$ and $r > 0$, we define the disk $D_r(z)$ in $\mathbb{C}$ of radius $r$ centered at $z$ through $D_r(z) := \{w \in \mathbb{C} : |z - w| < r\}$. We use $d^2z$ to denote integration with respect to the Lebesgue measure on $\mathbb{C}$.

We now introduce some notation used for vectors, matrices and linear maps on matrices. Vectors in $\mathbb{C}^{2n}$ are denoted by boldfaced small Latin letters like $\mathbf{x}$, $\mathbf{y}$ etc. For vectors $\mathbf{x} = (x_a)_{a \in [2n]}$, $\mathbf{y} = (y_a)_{a \in [2n]} \in \mathbb{C}^{2n}$, we consider the normalized Euclidean scalar product $\langle \mathbf{x}, \mathbf{y} \rangle$ and the induced normalized Euclidean norm $\|\mathbf{x}\|$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = (2n)^{-1} \sum_{a \in [2n]} x_a y_a, \quad \|\mathbf{x}\| := (\langle \mathbf{x}, \mathbf{x} \rangle)^{1/2}.$$

Functions of vectors such as roots, powers or inverse and operations such as products of vectors are understood entrywise.

Matrices in $\mathbb{C}^{2n \times 2n}$ are usually denoted by capitalized Latin letters. We especially use $G$, $H$, $J$, $M$, $R$, $S$ and $T$. For a matrix $R \in \mathbb{C}^{2n \times 2n}$, we introduce the real part $\text{Re} R$ and the imaginary part $\text{Im} R$ defined through

$$\text{Re} R := \frac{1}{2}(R + R^*), \quad \text{Im} R := \frac{1}{2i}(R - R^*).$$

We have $R = \text{Re} R + i \text{Im} R$ for all $R \in \mathbb{C}^{2n \times 2n}$. On $\mathbb{C}^{2n \times 2n}$, we consider the normalized trace $\langle \cdot \rangle$ and the normalized Hilbert–Schmidt scalar product $\langle \cdot, \cdot \rangle$ defined by

$$\langle V \rangle := \frac{1}{2n} \text{Tr}(V) = \frac{1}{2n} \sum_{i=1}^{2n} v_{ii}, \quad \langle V, W \rangle := \frac{1}{2n} \text{Tr}(V^* W) = \langle V^* W \rangle$$

for all $V = (v_{ij})_{i,j=1}^{2n}, W \in \mathbb{C}^{2n \times 2n}$. The norm on $\mathbb{C}^{2n \times 2n}$ induced by the normalized Hilbert–Schmidt
scalar product is denoted by $\| \cdot \|_2$, i.e., $\| V \|_2 := \langle V^* V \rangle^{1/2}$ for any $V \in \mathbb{C}^{2n \times 2n}$. Moreover, for $V \in \mathbb{C}^{2n \times 2n}$, we write $\| V \|$ for the operator norm of $V$ induced by the normalized Euclidean norm $\| \cdot \|$ on $\mathbb{C}^{2n}$.

We use capitalized calligraphic letters like $S$, $B$, and $T$ to denote linear maps on $\mathbb{C}^{2n \times 2n}$. In particular, for $A, B \in \mathbb{C}^{2n \times 2n}$, we define the linear map $C_{A,B} : \mathbb{C}^{2n \times 2n} \rightarrow \mathbb{C}^{2n \times 2n}$ through $C_{A,B}[R] := ARB$ for all $R \in \mathbb{C}^{2n \times 2n}$. This map satisfies the identities $C_{A,B}^* = C_{A^*,B^*}$ and $C_{A,B}^{-1} = C_{A^{-1},B^{-1}}$, where the second identity requires the matrices $A$ and $B$ to be invertible. We set $C_A := C_{A,A}$ for any matrix $A \in \mathbb{C}^{2n \times 2n}$. For a linear map $T$ on $\mathbb{C}^{2n \times 2n}$, we consider several norms. We denote by $\| T \|$ the operator norm of $T$ induced by $\| \cdot \|$ on $\mathbb{C}^{2n \times 2n}$. Moreover, $\| T \|_{2 \rightarrow 2}$ denotes the operator norm of $T$ induced by $\| \cdot \|_2$ on $\mathbb{C}^{2n \times 2n}$. We write $\| T \|_{2 \rightarrow \| \cdot \|}$ for the operator norm of $T$ when the domain is equipped with $\| \cdot \|_2$ and the target is equipped with $\| \cdot \|$.

In order to simplify the notation in numerous computations, we use the following conventions. In vector-valued relations, we identify a scalar with the vector whose components all agree with this scalar. Moreover, we use the block matrix notation

$$
\begin{pmatrix}
 a & b \\
 c & d
\end{pmatrix}
$$

exclusively for $2n \times 2n$-matrices. Here, each block is of size $n \times n$. If $a, b, c$ or $d$ are vectors (or scalars) then with a slight abuse of notations they are identified with the diagonal $n \times n$ matrices with $a, b, c$ or $d$, respectively, on the diagonal (or the respective multiple of the $n \times n$ identity matrix). Furthermore, we introduce the $2n \times 2n$ matrices $E_+$ and $E_-$ given in the block matrix notation of (2-13) by

$$
E_+ := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_- := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

We remark that $E_+$ coincides with the identity matrix in $\mathbb{C}^{2n \times 2n}$. In our argument, the following sets of $2n \times 2n$-matrices appear frequently. The diagonal matrices $M_d \subset \mathbb{C}^{2n \times 2n}$ and the off-diagonal matrices $M_o \subset \mathbb{C}^{2n \times 2n}$ are defined through

$$
M_d := \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{C}^n \right\}, \quad M_o := \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} : a, b \in \mathbb{C}^n \right\}.
$$

Note that the subspaces $M_d$ and $M_o$ are orthogonal with respect to the normalized Hilbert–Schmidt scalar product defined above.

In each section of this paper, we will specify a set of model parameters which are basic parameters of our model, e.g., $s_s$ and $s^*$ in (2-1). All of our estimates will hold uniformly for all models that satisfy our assumptions with the same model parameters. For $f, g \in [0, \infty)$, the comparison relation $f \lesssim g$ is true if $f \leq C g$ for some constant $C > 0$ that depends only on model parameters. We also write $f \gtrsim g$ if $g \lesssim f$ and $f \sim g$ if $f \lesssim g$ and $f \gtrsim g$. If $f(i)$ and $g(i)$ depend on a further parameter $i \in I$ and $f(i) \leq C g(i)$ for all $i \in I$ then we say $f \lesssim g$ uniformly for $i \in I$. We use the same notation for nonnegative vectors and positive semidefinite matrices. Here, for vectors $x = (x_a)_{a \in [2n]}$, $y = (y_a)_{a \in [2n]} \in [0, \infty)^{2n}$, the comparison relation $x \lesssim y$ means $x_a \lesssim y_a$ uniformly for all $a \in [2n]$, i.e., the implicit constant can be chosen independently of $a$. For positive semidefinite matrices $R_1, R_2 \in \mathbb{C}^{2n \times 2n}$, $R_1 \lesssim R_2$ if $\langle x, R_1 x \rangle \lesssim \langle x, R_2 x \rangle$ uniformly
for all $x \in \mathbb{C}^n$. For $\epsilon > 0$, scalars $f_1, f_2 \in \mathbb{C}$, matrices $R_1, R_2 \in \mathbb{C}^{2n \times 2n}$ and operators $T_1, T_2$ on $\mathbb{C}^{2n \times 2n}$, we write $f_1 = f_2 + O(\epsilon)$, $R_1 = R_2 + O(\epsilon)$ and $T_1 = T_2 + O(\epsilon)$ if $|f_1 - f_2| \lesssim \epsilon$, $\|R_1 - R_2\| \lesssim \epsilon$ and $\|T_1 - T_2\| \lesssim \epsilon$, respectively.

### 3. Analysis of the matrix Dyson equation

In this section, we study the linear stability of the MDE (2-6). According to the quadratic stability equation (2-9) associated to the MDE, the linear stability is governed by the behavior of the stability operator $B := 1 - C_M S$ (compare (2-10)). The main result of this section, Proposition 3.1, provides a complete understanding of the small, in absolute value, eigenvalues of $B$ in the regime when $\rho = \rho(z, \eta)$ is small. Here, $\rho = \rho(z, \eta)$ is defined through

$$\rho := \frac{1}{\pi} \langle \text{Im} \, M \rangle$$

(3-1)

for $\eta > 0$ and $z \in \mathbb{C}$, where $M$ is the solution to (2-6). For the small eigenvalues and their associated eigenvectors, very precise expansions in terms of $M$ are derived in Proposition 3.1.

We warn the reader that $\rho$ should not be confused with the spectral radii $\varrho(X)$ and $\varrho(S)$ used in Section 2. The function $\rho$ is the harmonic extension of the self-consistent density of states of $H_\epsilon$ (see, e.g., [Alt et al. 2020b, Equation (2)] for the definition of the self-consistent density of states).

In the remainder of the present section, we assume $\eta \in (0, 1]$ and $z \in D_\tau(0)$ for some fixed $\tau > 1$. In this section, the comparison relation $\lesssim$ introduced in Section 2B is understood with respect to the model parameters $\{s_\tau, s^*, \tau\}$. We recall that $s_\tau$ and $s^*$ constituted the bounds on the entries of $S$ in (2-1).

The following proposition is the main result of the present section.

**Proposition 3.1** (properties of the stability operator $B$). There are (small) $\rho_\eta \sim 1$ and $\epsilon \sim 1$ such that if $\rho + \eta/\rho \leq \rho_\eta$ then $B$ has two eigenvalues $\beta$ and $\beta_\eta$ in $D_\epsilon(0)$, i.e., $\text{Spec}(B) \cap D_\epsilon(0) = \{\beta, \beta_\eta\}$. Moreover, $\beta$ and $\beta_\eta$ have geometric and algebraic multiplicity one, $0 < |\beta_\eta| < |\beta|$ and

$$|\beta_\eta| \sim \eta/\rho, \quad |\beta| \sim \eta/\rho + \rho^2.$$  

(3-2)

Further, $B$ has left and right eigenvectors $\hat{B}_\eta, \hat{B}$ and $B_\eta, B$, corresponding to $\beta_\eta$ and $\beta$, respectively, i.e.,

$$B[B_\eta] = \beta_\eta B_\eta, \quad B[B] = \beta B, \quad B^*[\hat{B}_\eta] = \overline{\beta_\eta} \hat{B}_\eta, \quad B^*[\hat{B}] = \overline{\beta} \hat{B},$$

which satisfy

$$B = \rho^{-1} \text{Im} \, M - 2i \rho^{-1} (\text{Im} \, M)(\text{Im} \, M^{-1})(\text{Re} \, M) + O(\rho^2 + \eta/\rho),$$

(3-3a)

$$B_\eta = \rho^{-1} E_- \text{Im} \, M + O(\rho^2 + \eta/\rho),$$

(3-3b)

$$\hat{B} = -\rho^{-1} \text{Im}(M^{-1}) + O(\rho^2 + \eta/\rho),$$

(3-3c)

$$\hat{B}_\eta = -\rho^{-1} E_- \text{Im}(M^{-1}) + O(\rho^2 + \eta/\rho).$$

(3-3d)

For fixed $z$, the eigenvalues $\beta$ and $\beta_\eta$ and the eigenvectors $B, B_\eta, \hat{B}$ and $\hat{B}_\eta$ are continuous functions of $\eta$. 

as long as \( \rho + \eta/\rho \leq \rho_s \). We also have the expansions

\[
\beta \langle \hat{B}, B \rangle = \pi \eta \rho^{-1} + 2 \rho^2 \psi + \mathcal{O}(\rho^3 + \eta \rho + \eta^2/\rho^2),
\]

\[\beta_s \langle \hat{B}_s, B_s \rangle = \pi \eta \rho^{-1} + \mathcal{O}(\rho^3 + \eta \rho + \eta^2/\rho^2),\]

where \( \psi := \rho^{-4} \langle [\Im M] (\Im M^{-1})^2 \rangle \). We have \( \psi \sim 1 \), \( |\langle \hat{B}, B \rangle| \sim 1 \) and \( |\langle \hat{B}_s, B_s \rangle| \sim 1 \).

Moreover, the resolvent of \( B \) is bounded on the spectral subspace complementary to \( \beta \) and \( \beta_s \). That is, if \( Q \) is the spectral projection of \( B \) associated to \( \text{Spec}(B) \setminus \{ \beta, \beta_s \} \) then

\[
\|B^{-1}Q\| + \|(B^*)^{-1}Q^*\| \lesssim 1.
\]

We now make a few remarks about Proposition 3.1. First, owing to Lemma 3.3 (also note (3-9)), the condition \( \rho + \eta/\rho \leq \rho_s \) with \( \rho_s \sim 1 \) is satisfied if \( ||z||_2 - \rho(\mathcal{S})| \leq \delta \) and \( \eta \in (0, \delta] \) for some (small) \( \delta \sim 1 \).

Secondly, we note that \( B, B_s, \) etc. are called eigenvectors despite that they are in fact matrices in \( \mathbb{C}^{2n \times 2n} \).

Finally, the second term on the right-hand side of (3-3a) is of order \( \rho \), hence it is subleading compared to the first term \( \rho^{-1} \Im M \sim 1 \).

We now explain the relation between the solution \( M \) to the MDE (2-6), and the solution \((v_1, v_2)\) to (2-3). The \( 2n \times 2n \) matrix \( M \) satisfies

\[
M(z, \eta) = \begin{pmatrix}
iv_1 & -zu \\
-zu & iv_2
\end{pmatrix},
\]

where \((v_1, v_2)\) is the unique solution of (2-3) and \( u \) is defined through

\[
u_1 \eta + \mathcal{S} v_1 = \frac{v_2}{\eta + \mathcal{S} v_2}.
\]

Note that \( u \in (0, \infty)^n \). We remark that (3-6) is the unique solution to (2-6) with the side condition that \( \Im M \) is a positive definite matrix. The existence and uniqueness of such \( M \) follows from [Helton et al. 2007].

Throughout this section, the special structure of \( M \) as presented in (3-6) will play an important role. As a first instance, we see that the representation of \( M \) in (3-6) implies

\[
\Im M = \begin{pmatrix}
v_1 & 0 \\
0 & v_2
\end{pmatrix}, \quad \Re M = \begin{pmatrix}0 & -zu \\
-zu & 0
\end{pmatrix}.
\]

Therefore, \( \Im M \in \mathcal{M}_d \) and \( \Re M \in \mathcal{M}_o \). This is an important ingredient in the proof of the following corollary.

**Corollary 3.2.** There is \( \rho_s \sim 1 \) such that \( \rho + \eta/\rho \leq \rho_s \) implies

\[
|\langle E_-, B \rangle| \lesssim \rho^2 + \eta/\rho,
\]

where \( B \) is the right eigenvector of \( B \) from Proposition 3.1.

**Proof.** The expansion of \( B \) in (3-3a) yields

\[
\langle E_-, B \rangle = \rho^{-1} \langle E_-, \Im M \rangle - 2i\rho^{-1} \langle E_-, (\Im M)(\Im M^{-1})(\Re M) \rangle + \mathcal{O}(\rho^2 + \eta/\rho).
\]
We now conclude (3-8) by showing that the first two terms on the right-hand side vanish. The identity (3-12) implies \( \langle E_-, \text{Im} M \rangle = 0 \). Moreover, by (3-7), we have \( \text{Re} M \in \mathcal{M}_o \) and \( \text{Im} M \in \mathcal{M}_d \). Taking the imaginary part of (2-6) thus yields \( \text{Im} M^{-1} \in \mathcal{M}_d \). Therefore, \( \langle E_-, (\text{Im} M)(\text{Im} M^{-1})(\text{Re} M) \rangle = 0 \) since \( \text{Re} M \in \mathcal{M}_o \) while \( E_-(\text{Im} M)(\text{Im} M^{-1}) \in \mathcal{M}_d \). This completes the proof of (3-8).

\[ \square \]

3A. Preliminaries. The MDE (2-6) and its solution have a special scaling when \( S \) and, hence, \( \mathcal{S} \), are rescaled by \( \lambda > 0 \), i.e., \( S \) in (2-6) is replaced by \( \lambda S \). Indeed, if \( M = M(z, \eta) \) is the solution to (2-6) with positive definite imaginary part then \( M_{\lambda}(z, \eta) := \lambda^{-1/2}M(z \lambda^{-1/2}, \eta \lambda^{-1/2}) \) is the solution to

\[ -M_{\lambda}^{-1} = \begin{pmatrix} i\eta & z \\ \bar{z} & i\eta \end{pmatrix} + \lambda S[M_{\lambda}] \]

with positive imaginary part. The same rescaling yields the positive solution of (2-3) when \( \mathcal{S} \) is replaced by \( \lambda \mathcal{S} \) (see the explanations around (3.7) in [Alt et al. 2018]). Therefore, by a simple rescaling, we can assume that the spectral radius is one,

\[ \varrho(\mathcal{S}) = 1. \] (3-9)

We remark that the other assumptions on \( X \) are still satisfied since the flatness condition (A1) directly implies \( \varrho(\mathcal{S}) \sim 1 \). In the remainder of the paper, we will always assume (3-9).

Balanced polar decomposition of \( M \). We first introduce a polar decomposition of \( M \) that will yield a useful factorization of \( B \) which is the basis of its spectral analysis. To that end, we define

\[ U := \begin{pmatrix} i \sqrt{\frac{v_1 v_2}{u}} & -z \sqrt{u} \\ -\bar{z} \sqrt{u} & i \sqrt{\frac{v_1 v_2}{u}} \end{pmatrix}, \quad Q := \begin{pmatrix} \left( \frac{uv_1}{v_2} \right)^{1/4} & 0 \\ 0 & \left( \frac{uv_2}{v_1} \right)^{1/4} \end{pmatrix}, \] (3-10)

where roots and powers of vectors are taken entrywise. Starting from these definitions, an easy computation shows that \( M \) admits the following balanced polar decomposition

\[ M = QUQ. \] (3-11)

Such polar decomposition for the solution of the Dyson equation was introduced in [Ajanki et al. 2019].

The following lemma collects a few basic properties of \( M, \rho, U \) and \( Q \), mostly borrowed from [Alt et al. 2018].

Lemma 3.3 (basic properties of \( M, \rho, U \) and \( Q \)). (i) Let \( \eta \in (0, 1] \) and \( z \in D_\tau(0) \). We have

\[ \langle E_-, M \rangle = 0. \] (3-12)

Moreover, \( Q = Q^* \in \mathcal{M}_d \), \( \text{Im} U \in \mathcal{M}_d \), \( \text{Re} U \in \mathcal{M}_o \) and \( U \) is unitary.

(ii) Uniformly for all \( \eta \in (0, 1] \) and \( z \in D_\tau(0) \), \( \rho \) satisfies the scaling relations

\[ \rho \sim \begin{cases} \eta^{1/3} + (1 - |z|^2)^{1/2} & \text{if } |z| \leq 1, \\
\frac{\eta}{|z|^2 - 1 + \eta^{2/3}} & \text{if } 1 \leq |z| \leq \tau, \end{cases} \] (3-13)
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for the matrices $U$, $Q$ and $M$, we have the estimates

\[ \text{Im} U \sim \rho, \quad Q \sim 1, \quad \|M\| \lesssim 1, \]

and for the entries of $M$, we have

\[ v_1 \sim v_2 \sim \rho, \quad u \sim 1. \] (3-14)

(iii) For fixed $z \in \mathbb{C}$, $M$, $\rho$, $U$ and $Q$ are continuous functions of $\eta$.

Proof. First, we remark that $|z|^2$ was denoted by $\tau$ in [Alt et al. 2018].

The identity in (3-12) follows from [Alt et al. 2018, Equation (3.8)]. Obviously, (3-10) and $v_1, v_2, u > 0$ yield $Q = Q^* \in M_d$. $\text{Im} U \in M_d$ and $\text{Re} U \in M_0$. A simple computation reveals that $U$ is unitary as $u = v_1 v_2 + |z|^2 u^2$ due to [Alt et al. 2018, Equation (3.32)].

From [Alt et al. 2018, Equations (3.10), (3.11)], we conclude that $v_1, v_2$ and $\text{Im} M$ scale as the right-hand side of (3-13). Hence, (3-13) follows from the definition of $\rho$ in (3-1). Consequently, $v_1 \sim \rho \sim v_2$. Owing to [Alt et al. 2018, Equation (3.26)], we have $u \sim 1$ uniformly for all $z \in D_\tau(0)$ and $\eta \in (0, 1]$. Thus, $v_1 \sim \rho \sim v_2$ yields the first two scaling relations in (3-14). As $U$ is unitary we have $\|U\| = 1$. Thus, (3-11) and the first two scaling relations in (3-14) imply the last bound in (3-14).

For fixed $z \in \mathbb{C}$, the matrix $M$ is an analytic, hence, continuous function of $\eta$. Thus, $\rho, v_1$ and $v_2$ are continuous functions of $\eta$. Consequently, as $v_1, v_2, u > 0$, the matrices $U$ and $Q$ are also continuous in $\eta$. This completes the proof of Lemma 3.3. \qed

Factorization of $B$. We now present a factorization of $B$ which will be the basis of our spectral analysis of $B$ as a linear map on the Hilbert space $(\mathbb{C}^{2n \times 2n}, \langle \cdot, \cdot \rangle)$. From (3-11), we easily obtain

\[ B = 1 - C_M S = C_Q (1 - C_U F) C_Q^{-1}, \] (3-16)

where we introduced the positivity-preserving and Hermitian operator $F$ on $\mathbb{C}^{2n \times 2n}$ defined by

\[ F := C_Q S C_Q. \] (3-17)

Owing to (3-16) and $Q \sim 1$ by (3-14), the spectral properties of $B$ stated in Proposition 3.1 can be obtained by analysing $1 - C_U F$. If $\rho$ is small then $U$ is well approximated by $P$ defined through

\[ P := \text{sign}(\text{Re} U) = \frac{\text{Re} U}{|\text{Re} U|} = \begin{pmatrix} 0 & -z/|z| \\ -\bar{z}/|z| & 0 \end{pmatrix}. \] (3-18)

Indeed, $1 - |\text{Re} U| = 1 - \sqrt{1 - (\text{Im} U)^2} \lesssim (\text{Im} U)^2 \lesssim \rho^2$ implies that

\[ \|P - \text{Re} U\| \lesssim \rho^2, \quad \|P - U\| \lesssim \rho. \] (3-19)

Therefore, we will first analyse the operators $F$ and $C_P F$. The proof of Proposition 3.1 will then follow by perturbation theory since (3-19) implies

\[ \|1 - C_U F - (1 - C_P F)\| \lesssim \|P - U\| \lesssim \rho. \] (3-20)
Commutation relations.

**Lemma 3.4** (commutation relations of \(E_\) with \(M, Q, U\) and \(P\)). We have

\[
\begin{align*}
ME_- &= -E_- M^*, & M^* E_- &= -E_- M, \\
QE_- &= E_- Q, & Q^{-1} E_- &= E_- Q^{-1}, \\
UE_- &= -E_- U^*, & U^* E_- &= -E_- U, \\
PE_- &= -E_- P,
\end{align*}
\]

(3-21a) (3-21b) (3-21c) (3-21d)

*Proof.* The identities in (3-21a) follow by a simple computation starting from (3-6). Owing to \(Q \in M_d\) we immediately obtain (3-21b). The relations in (3-21c) are a direct consequence of (3-21a) and (3-21b). The matrix representation of \(P\) in (3-18) directly implies (3-21d). \(\square\)

Spectral properties of \(F\).

**Lemma 3.5** (spectral properties of \(F\)). For all \(\eta \in (0, 1]\) and \(z \in D_\tau(0)\), the following holds.

(i) The range of \(F\) is contained in the diagonal matrices, i.e., \(\text{ran } F \subset M_d\). Moreover, for all \(R \in \mathbb{C}^{{2n \times 2n}}\),

\[
\]

(3-22)

(ii) The top eigenvalue \(\|F\|_{2\to 2}\) of \(F\) is simple and satisfies

\[
1 - \|F\|_{2\to 2} \sim \eta / \rho.
\]

(3-23)

(iii) There is a unique positive definite eigenvector \(F\) with \(\|F\|_2 = 1\) associated to \(\|F\|_{2\to 2}\). It satisfies \(F \in M_d\).

(iv) The eigenvalue \(-\|F\|_{2\to 2}\) of \(F\) is also simple and \(E_- F\) is an eigenvector corresponding to it.

(v) There are \(\rho_* \sim 1\) and \(\vartheta \sim 1\) such that \(\eta / \rho \leq \rho_*\) implies

\[
\|F[R]\|_2 \leq \|F\|_{2\to 2}(1 - \vartheta) \|R\|_2
\]

for all \(R \in \mathbb{C}^{{2n \times 2n}}\) satisfying \(R \perp F\) and \(R \perp E_- F\).

Before the proof of Lemma 3.5, we introduce \(F_U\) defined through

\[
F_U := \rho^{-1} \text{Im } U = \rho^{-1} \begin{pmatrix}
\sqrt{v_1 v_2} / u & 0 \\
0 & \sqrt{v_1 v_2} / u
\end{pmatrix}.
\]

(3-24)

The importance of \(F_U\) originates from the approximate eigenvector relation

\[
(1 - F)[F_U] = \frac{\eta}{\rho} Q^2.
\]

(3-25)

which is a consequence of the MDE (2-6). Indeed, (2-6) and (3-11) imply

\[
-U^* = Q \begin{pmatrix}
\eta & z i \eta \\
\bar{z} & \eta i
\end{pmatrix} Q + F[U].
\]
Dividing the imaginary part of this identity by $\rho$ yields (3-25). Moreover, from (3-14), we directly deduce that

$$F_U \sim 1. \quad (3-26)$$

**Proof.** The definition of $S$ in (2-7) implies $\text{ran } S \subset \mathcal{M}_d$. Since $Q \in \mathcal{M}_d$ by (3-10) we deduce $\mathcal{F} \subset \mathcal{M}_d$. As $\text{ran } S \subset \mathcal{M}_d$, we also have $S[RE_-] = -S[R]E_- = -E_-S[R] = S[E_-R]$ for all $R \in \mathbb{C}^{2n \times 2n}$. This completes the proof of (i) due to (3-21b).

Since $\text{ran } \mathcal{F} \subset \mathcal{M}_d$, the restriction $\mathcal{F}|_{\mathcal{M}_d}$ contains all spectral properties of $\mathcal{F}$ (apart from information about the possible eigenvalue 0). The restriction $\mathcal{F}|_{\mathcal{M}_d}$ is given by

$$\mathcal{F}
\begin{pmatrix}
  r_1 & 0 \\
  0 & r_2
\end{pmatrix}
= 
\begin{pmatrix}
  F r_2 & 0 \\
  0 & F^t r_1
\end{pmatrix}$$

for $r_1, r_2 \in \mathbb{C}^n$, where we introduced the $n \times n$-matrix $\mathcal{F}$ defined by

$$\mathcal{F} r := \left(\frac{uv_1}{v_2}\right)^{1/2} \mathcal{F} \left(\frac{uv_2}{v_1}\right)^{1/2} \quad (3-27)$$

for $r \in \mathbb{C}^n$. Hence, in the standard basis of $\mathcal{M}_d \cong \mathbb{C}^{2n}$, the restriction $\mathcal{F}|_{\mathcal{M}_d}$ is represented by the $2n \times 2n$ matrix

$$F = 
\begin{pmatrix}
  0 & \mathcal{F} \\
  \mathcal{F}^t & 0
\end{pmatrix},$$

which was introduced in [Alt et al. 2018, Equation (3.27b)] and analyzed in [Alt et al. 2018, Lemma 3.4] using [Alt et al. 2017, Lemma 3.3]. Using the notation there, we have $L = 2$ due to (2-1) and $r_+ \sim r_- \sim 1$ due to (3-14). Thus, [Alt et al. 2017, Lemma 3.3] directly implies the simplicity of the top eigenvalue $\|F\|_{2 \rightarrow 2}$ and the existence of a unique positive definite eigenvector $F$ of $\mathcal{F}$ corresponding to $\|F\|_{2 \rightarrow 2}$ with $\|F\|_2 = 1$. Moreover, $F \in \mathcal{M}_d$. Owing to the second relation in (3-22), $E_- F$ is an eigenvector of $\mathcal{F}$ associated to $-\|F\|_{2 \rightarrow 2}$.

For the proof of (ii), we apply $\langle F, \cdot \rangle$ to (3-25) and obtain

$$1 - \|F\|_{2 \rightarrow 2} = \eta \frac{\langle F Q^2 \rangle}{\rho \langle F F_U \rangle} \sim \frac{\eta}{\rho} \sim \frac{\eta}{\rho}.$$

Here, we used the positive definiteness of $F$, $Q \sim 1$ by (3-14) and $F_U \sim 1$ by (3-26) in the second step. Since $\hat{\lambda} \sim 1$ due to [Alt et al. 2017, Equation (3.17)] and $r_+ \sim r_- \sim 1$ (see above), the bound in (v) for $R \in \mathcal{M}_d$ follows from [Alt et al. 2017, Lemma 3.3]. Since $\mathcal{F}$ vanishes on the orthogonal complement of $\mathcal{M}_d$, this completes the proof of Lemma 3.5.

\[ \square \]

**3B. Spectral properties of $\mathcal{C}_F \mathcal{F}$ and $\mathcal{C}_U \mathcal{F}$.** For brevity we introduce the following shorthand notation for the operators in the following lemma. We define

$$\mathcal{K} := 1 - \mathcal{C}_F \mathcal{F}, \quad \mathcal{L} := 1 - \mathcal{C}_U \mathcal{F}. \quad (3-28)$$

In the following lemma, we prove some resolvent bounds for these operators and show that they have at most two small eigenvalues.
Lemma 3.6 (resolvent bounds, number of small eigenvalues). There are (small) \( \rho_\ast \sim 1 \) and \( \varepsilon \sim 1 \) such that for all \( z \in D_\tau(0) \) and \( \eta > 0 \) satisfying \( \rho + \eta/\rho \leq \rho_\ast \) and, for all \( \mathcal{T} \in \{ \mathcal{K}, \mathcal{L} \} \), the following holds.

(i) For all \( \omega \in \mathbb{C} \) with \( \omega \notin D_\varepsilon(0) \cup D_{1-2\varepsilon}(1) \), we have
\[
\|(T - \omega)^{-1}\|_{2 \to 2} + \|(T - \omega)^{-1}\| + \|(T^* - \omega)^{-1}\| \lesssim 1.
\]

(ii) The spectral projection \( \mathcal{P}_\mathcal{T} \) of \( \mathcal{T} \), defined by
\[
\mathcal{P}_\mathcal{T} := -\frac{1}{2\pi i} \int_{\partial D_\varepsilon(0)} (T - \omega)^{-1} d\omega,
\]

satisfies \( \text{rank} \mathcal{P}_\mathcal{T} = 2 \). Moreover, for \( \mathcal{Q}_\mathcal{T} := 1 - \mathcal{P}_\mathcal{T} \), we have
\[
\|\mathcal{P}_\mathcal{T}\| + \|\mathcal{Q}_\mathcal{T}\| + \|\mathcal{P}_\mathcal{T}^*\| + \|\mathcal{Q}_\mathcal{T}^*\| + \|\mathcal{T}^{-1}\mathcal{Q}_\mathcal{T}\|_{2 \to 2} + \|\mathcal{T}^{-1}\mathcal{Q}_\mathcal{T}\| + \|\mathcal{T}^{-1}\mathcal{Q}_\mathcal{T}\| \lesssim 1.
\]

(iii) For fixed \( z \in D_\tau(0) \), the spectral projections \( \mathcal{P}_\mathcal{T} \) and \( \mathcal{Q}_\mathcal{T} \) are continuous in \( \eta \) as long as \( \rho + \eta/\rho \leq \rho_\ast \).

The proof of Lemma 3.6 is motivated by the proofs of [Alt et al. 2020b, Lemma 4.7] and [Alt et al. 2020a, Lemma 5.1]. However, the additional extremal eigendirection of \( \mathcal{F} \) requires a novel flow interpolating between \( 1 - \mathcal{F}^2 \) and \( 1 - (C_pF)^2 \) instead of \( 1 - \mathcal{F} \) and \( 1 - C_pF \).

**Proof.** From (2-1), we deduce that \( \mathcal{S}[R] \lesssim (R) \) for all positive semidefinite matrices \( R \in \mathbb{C}^{2n \times 2n} \). Thus, [Alt et al. 2020a, Lemma B.2(ii)] implies that \( \|\mathcal{S}\|_{2 \to 2 \to 2} \lesssim 1 \). Therefore, for all \( \mathcal{T} \in \{ \mathcal{K}, \mathcal{L} \} \), we have \( \|1 - \mathcal{T}\|_{2 \to 2 \to 2} \lesssim 1 \) due to \( Q \sim 1 \) and \( \|\mathcal{U}\| = 1 \) by Lemma 3.3. Hence, owing to [Alt et al. 2020a, Lemma B.2(ii)] and \( |\omega - 1| \gtrsim 1 \), it suffices to find \( \varepsilon \sim 1 \) such that, for \( \mathcal{T} \in \{ \mathcal{K}, \mathcal{L} \} \),

(i) uniformly for all \( \omega \notin D_\varepsilon(0) \cup D_{1-2\varepsilon}(1) \), we have
\[
\|(T - \omega)^{-1}\|_{2 \to 2} \lesssim 1,
\]

(ii) the rank of \( \mathcal{P}_\mathcal{T} \) equals 2, i.e., \( \text{rank} \mathcal{P}_\mathcal{T} = 2 \).

Both claims for \( \mathcal{T} = \mathcal{K} \) will follow from the corresponding statements for \( \mathcal{T} = 1 - (C_pF)^2 \) which we now establish by interpolating between \( 1 - \mathcal{F}^2 \) and \( 1 - (C_pF)^2 \). If \( \mathcal{T} = 1 - \mathcal{F}^2 \) then both assertions follow from Lemma 3.5. Moreover, a simple perturbation argument using Lemma 3.5 shows that
\[
\mathcal{F} = \|F_U\|_2^{-1} F_U + \mathcal{O}(\eta/\rho),
\]

where \( \mathcal{F} \) is the eigenvector of \( \mathcal{F} \) introduced in Lemma 3.5 (see the proof of [Alt et al. 2018, Lemma 3.5] for a similar argument).

In order to interpolate between \( 1 - \mathcal{F}^2 \) and \( 1 - (C_pF)^2 \) we use the following flow. For any \( t \in [0, 1] \), we define
\[
\mathcal{T}_t := 1 - \mathcal{V}_t \mathcal{F}, \quad \mathcal{V}_t := (1 - t) \mathcal{F} + t C_p F C_p.
\]

Then \( \mathcal{T}_0 = 1 - \mathcal{F}^2 \) and \( \mathcal{T}_1 = 1 - (C_pF)^2 \). We now show (3-31) for \( \mathcal{T} = \mathcal{T}_t \) uniformly for all \( t \in [0, 1] \). To that end, we verify that \( \|(\mathcal{T}_t - \omega)[R]\| \gtrsim 1 \) uniformly for \( R \in \mathbb{C}^{2n \times 2n} \) satisfying \( \|R\|_2 = 1 \). If \( |\omega| \geq 3 \) then this follows from \( \|\mathcal{V}_t\|_{2 \to 2} \leq \|\mathcal{F}\|_{2 \to 2} \leq 1 \) by (3-23). Let \( |\omega| \leq 3 \) and \( R \in \mathbb{C}^{2n \times 2n} \) satisfy \( \|R\|_2 = 1 \).
We have the orthogonal decomposition
\[ R = \alpha_+ F + \alpha_- E_- F + R_\perp, \]
where \( R_\perp \perp E_\pm F \) (recall \( E_\pm = 1 \) from (2-14)), and estimate
\[
\|(T_t - \omega)[R]\|_2^2 = |\omega|^2(\|\alpha_+\|^2 + |\alpha_-|^2) + \|(1 - \omega - \mathcal{V}_t \mathcal{F})[R_\perp]\|_2^2 + O(\eta/\rho)
\geq \varepsilon^2(|\alpha_+|^2 + |\alpha_-|^2) + (\vartheta - 2\varepsilon)^2 \|R_\perp\|_2^2 + O(\eta/\rho).
\]
(3-33)
We now explain how (3-33) is obtained. The identity in (3-33) follows from \( \mathcal{V}_t \mathcal{F}[E_\pm F] = E_\pm F + O(\eta/\rho) \) due to (3-32), (3-23), \( \mathcal{C}_p[F_U] = F_U \), (3-21d) and (3-22). The lower bound in (3-33) is a consequence of
\[
\|\mathcal{F}\|_2 \leq (1 - \omega - \mathcal{V}_t \mathcal{F})[R_\perp]\|_2 \geq (\vartheta - 2\varepsilon)\|R_\perp\|_2,
\]
where, in the first step, we used \( \|\mathcal{V}_t\|_2 \leq 1 \) and \( \|\mathcal{F}[R_\perp]\|_2 \leq \|\mathcal{F}\|_2(1 - \vartheta)\|R_\perp\|_2 \) due to part (v) of Lemma 3.5. In the second step, we employed \( \|\mathcal{F}\|_2 \leq 1 \) and \( |1 - \omega| \geq 1 - 2\varepsilon \). This shows (3-33) which implies (3-31) for \( T = T_t \) if \( \varepsilon \sim 1 \) and \( \rho_* \sim 1 \) are chosen sufficiently small.

A similar but simpler argument to the proof of (3-33) shows that \( \|(K - \omega)[R]\|_2 \geq 1 \) uniformly for all \( R \in \mathbb{C}^{2n \times 2n} \) satisfying \( \|R\|_2 = 1 \). This implies (3-31) for \( T = K \). In particular, by [Alt et al. 2020a, Lemma B.2(ii)] and \( |\omega - 1| \geq 1 \), the bound \( \|(K - \omega)^{-1}\|_2 \leq 1 \) from (3-31) implies the same bound in the norm \( \cdot \), i.e., Lemma 3.6(i) for \( T = K \). Hence, the contour integral representation for \( \mathcal{P}_T \) in (3-29) implies the bounds on the projections in (3-30). The remaining bounds in (3-30) follow similarly from \( \varepsilon \sim 1 \) and the contour integral representation
\[
K^{-1} Q_K = -\frac{1}{2\pi i} \int_{\partial D_{1-2\varepsilon}(1)} \omega^{-1}(K - \omega)^{-1} d\omega,
\]
which is a consequence of Lemma 3.6(i) for \( T = K \).

Owing to Lemma 3.3(iii) and \( \mathcal{F} = \mathcal{C}_Q SC_Q \) (see (3-17)), \( K \) and \( \mathcal{L} \) are continuous functions of \( \eta \). Hence, the contour integral representation of \( \mathcal{P}_T \) in (3-29) implies (iii).

What remains in order to complete the proof of Lemma 3.6 for \( T = K \) is showing rank \( \mathcal{P}_K = 2 \). The bound in (3-31) with \( T = T_t \) implies that \( \mathcal{P}_{T_t} \) is well defined for all \( t \in [0, 1] \). Moreover, the map \( t \mapsto \text{rank} \mathcal{P}_{T_t} \) is continuous and, hence, constant as a continuous, integer-valued map. Therefore,
\[
\text{rank} \mathcal{P}_{T_t} = \text{rank} \mathcal{P}_{T_0} = 2,
\]
(3-34)
where we used in the last step that \( T_0 = 1 - \mathcal{F}^2 \) and Lemma 3.5 (ii), (iv) and (v). Since the generalized eigenspace of \( 1 - (\mathcal{C}_p \mathcal{F})^2 \) corresponding to \( 1 - \mu^2 \) contains the generalized eigenspace of \( 1 - \mathcal{C}_p \mathcal{F} \) corresponding to \( 1 - \mu \) for any \( \mu \in \mathbb{C} \), the identity rank \( \mathcal{P}_{T_1} = 2 \) from (3-34) implies
\[
\text{rank} \mathcal{P}_K \leq 2.
\]
(3-35)
The following lemma provides the corresponding lower bound.

**Lemma 3.7** (eigenvalues of \( K \) in \( D_\varepsilon(0) \)). Let \( \varepsilon \) and \( \rho_* \) be chosen as in Lemma 3.6. If \( \rho + \eta/\rho \leq \rho_* \), then
\[ \text{Spec}(K) \cap D_\varepsilon(0) \]
consists of a unique eigenvalue \( \kappa \) of \( K \). This eigenvalue is positive, has algebraic and geometric multiplicity two and is a continuous function of \( \eta \) for fixed \( z \in D_\varepsilon(0) \).
Proof. Since rank $\mathcal{P}_K \leq 2$ by (3-35), the set Spec($K$) $\cap D(\rho)$ contains (counted with algebraic multiplicity) at most two eigenvalues of $K$. We will now show that it contains one eigenvalue of (algebraic and geometric) multiplicity two. As $\mathcal{C}_P \mathcal{F} \subset \mathcal{M}_d$ it suffices to study the corresponding eigenvalue problem on $\mathcal{M}_d$. Let $r_1, r_2 \in \mathbb{C}^n$ be vectors. We apply $\mathcal{C}_P \mathcal{F}$ to the diagonal matrix $R = \text{diag}(r_1, r_2) \in \mathcal{M}_d$ and obtain

$$
\mathcal{C}_P \mathcal{F}[R] = \mathcal{C}_P \mathcal{F}\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} = \begin{pmatrix} \mathcal{F'}r_1 & 0 \\ 0 & \mathcal{F'}r_2 \end{pmatrix},
$$

(3-36)

where $\mathcal{F}$ denotes the $n \times n$-matrix defined in (3-27) in the proof of Lemma 3.5. The spectral radii of the matrix $\mathcal{F}$ and its transpose $\mathcal{F}'$ agree. We denote this common spectral radius by $1 - \kappa$. Since $\|\mathcal{C}_P \mathcal{F}\|_2 \leq \|\mathcal{F}\|_2 < 1$ by Lemma 3.5 we have $\kappa > 0$. The entries of the matrices $\mathcal{F}$ and $\mathcal{F}'$ are strictly positive. Hence, by the Perron–Frobenius theorem, there are $r_1, r_2 \in (0, \infty)^n$ such that $\mathcal{F}'r_1 = (1 - \kappa)r_1$ and $\mathcal{F}r_2 = (1 - \kappa)r_2$. Thus, $K[R] = \kappa R$, where we used (3-36) and introduced $R := \text{diag}(r_1, r_2) \in \mathbb{C}^{2n \times 2n}$. Since $r_1, r_2 > 0$, $E_- R$ and $R$ are strictly independent. Moreover, $K[E_- R] = (1 - \mathcal{C}_P \mathcal{F})[E_- R] = E_- K[R] = \kappa E_- R$ due to (3-22) and (3-21d). Therefore, Spec($K$) $\cap D(\rho) = \{\kappa\}$ and $R$ and $E_- R$ span the eigenspace of $K$ associated to $\kappa$, i.e., rank $\mathcal{P}_K = 2$. Since $K$ and $\mathcal{P}_K$ are continuous functions of $\eta$, the eigenvalue $\kappa = \text{Tr}(K\mathcal{P}_K)/2$ is also continuous with respect to $\eta$. This completes the proof of Lemma 3.7.

Since rank $\mathcal{P}_K = 2$ by Lemma 3.7, this completes the proof of Lemma 3.6 for $T = K$.

Owing to (3-20), we have $\|L - K\|_2 \leq \rho$. Hence, possibly shrinking $\varepsilon \sim 1$ and $\rho_\ast \sim 1$ and a simple perturbation theory argument show the estimates in (i) and (3-30) for $T = L$. Moreover, viewing $L$ as perturbation of $K$ and using rank $\mathcal{P}_K = 2$ yield rank $\mathcal{P}_L = 2$ for sufficiently small $\rho_\ast \sim 1$. This completes the proof of Lemma 3.6.

Using the spectral properties of $K$ established in Lemma 3.6, we show in the following lemma that $E_\pm F_U$ are approximate eigenvectors of $K$ associated to its small eigenvalue $\kappa$ from Lemma 3.7.

**Lemma 3.8** (eigenvectors of $K$ associated to $\kappa$). Let $\varepsilon$ and $\rho_\ast$ be chosen as in Lemma 3.6 and $\mathcal{P}_K$ and $Q_K$ be as defined in Lemma 3.6 for $K = 1 - \mathcal{C}_P \mathcal{F}$. If $\rho + \eta/\rho \leq \rho_\ast$ then the following holds.

(i) There are left and right eigenvectors $\widehat{K}_+$ and $K_+$ of $K$ corresponding to $\kappa$ such that

$$
K_+ = F_U - \frac{\eta}{\rho} K^{-1} Q_K C_P [Q^2], \quad \widehat{K}_+ = F_U - \frac{\eta}{\rho} (K^*)^{-1} Q_K [Q^2].
$$

(3-37)

They are elements of $\mathcal{M}_d$, continuous functions of $\eta$ for fixed $z \in D(\rho)$ and satisfy

$$
\langle \widehat{K}_+, K_+ \rangle = \langle F_U^2 \rangle + O(\eta^2/\rho^2).
$$

(3-38)

Moreover, we have

$$
\kappa = \frac{\eta}{\rho} \frac{\pi}{\langle F_U^2 \rangle} + O(\eta^2/\rho^2).
$$

(3-39)

(ii) Furthermore, $K_- := E_- K_+$ and $\widehat{K}_- := E_- \widehat{K}_+$ are also right and left eigenvectors of $K$ corresponding to $\kappa$ that are linearly independent of $K_+$ and $\widehat{K}_+$, respectively.
(iii) The projections $\mathcal{P}_K$ and $\mathcal{P}_K^*$ have the representation

$$
\mathcal{P}_K = \frac{\langle \hat{K}_+, \cdot \rangle}{\langle \hat{K}_+, \hat{K}_+ \rangle} K_+ + \frac{\langle \hat{K}_-, \cdot \rangle}{\langle \hat{K}_-, \hat{K}_- \rangle} K_-,
\mathcal{P}_K^* = \frac{\langle K_+ \cdot \rangle}{\langle K_+, K_+ \rangle} \hat{K}_+ + \frac{\langle K_- \cdot \rangle}{\langle K-, K_- \rangle} \hat{K}_-.
$$

In particular, $\text{ran} \mathcal{P}_K \subset \mathcal{M}_d$, $\text{ran} \mathcal{P}_K^* \subset \mathcal{M}_d$ and $\mathcal{P}_K \mathcal{M}_o = \mathcal{P}_K^* \mathcal{M}_o = \{0\}$.

For the proof, we note that the definition of $F_U$ in (3-24), (3-11) and the definition of $\rho$ in (3-1) imply

$$
\langle Q^2, F_U \rangle = \rho^{-1} \langle Q \text{ Im } Q \rangle = \pi.
$$

(3-40)

**Proof.** We start the proof of (i) by remarking that the eigenspace of $\mathcal{K}$ associated to $\kappa$ is contained in $\mathcal{M}_d$ since $\text{ran} \mathcal{C}_P \mathcal{F} \subset \mathcal{M}_d$. Next, we apply $\mathcal{Q}_K \mathcal{C}_P$ to (3-25), use $\mathcal{C}_P[F_U] = F_U$ and $\mathcal{K} = 1 - \mathcal{C}_P \mathcal{F}$ and obtain

$$
\mathcal{K} \mathcal{Q}_K[F_U] = \frac{\eta}{\rho} \mathcal{Q}_K \mathcal{C}_P[Q^2].
$$

Hence, setting $K_+ := \mathcal{P}_K[F_U]$ yields

$$
K_+ = \mathcal{P}_K[F_U] = F_U - \mathcal{Q}_K[F_U] = F_U - \frac{\eta}{\rho} \mathcal{K}^{-1} \mathcal{Q}_K \mathcal{C}_P[Q^2].
$$

This proves the expansion of $K_+$ in (3-37). For the proof of the expansion of $\hat{K}_+$, we use $\mathcal{C}_P[F_U] = F_U$ in (3-25), apply $\mathcal{Q}_K^*$ to the result and set $\hat{K}_+ := \mathcal{P}_K^*[F_U]$. Then the expansion of $\hat{K}_+$ follows in a similar way to that of $K_+$. The continuity of $F_U = \rho^{-1} \text{ Im } U$ due to Lemma 3.3(iii) and the continuity of $\mathcal{P}_K$ and $\mathcal{P}_K^*$ due to Lemma 3.6(iii) imply that $K_+$ and $\hat{K}_+$ are also continuous.

The relation in (3-38) follows directly from (3-37) since $\langle \hat{K}_+, K_+ - F_U \rangle = \langle \hat{K}_+ - F_U, K_+ \rangle = 0$ due to $\mathcal{Q}_K \mathcal{P}_K = 0$ and, thus,

$$
\langle \hat{K}_+, K_+ \rangle = \langle \hat{K}_+, F_U \rangle + \langle \hat{K}_+, K_+ - F_U \rangle = \langle F_U^2 \rangle + \langle \hat{K}_+ - F_U, K_+ \rangle + O(\eta^2/\rho^2).
$$

For the proof of (3-39), we deduce from (3-37) and $\mathcal{Q}_K[K_+] = 0$ that

$$
\kappa \langle \hat{K}_+, K_+ \rangle = \langle \mathcal{K}^*[\hat{K}_+], K_+ \rangle = \langle \mathcal{K}[F_U], K_+ \rangle = \frac{\eta}{\rho} \langle Q^2, F_U \rangle + O(\eta^2/\rho^2) = \frac{\eta \pi}{\rho} + O(\eta^2/\rho^2),
$$

where we used $\mathcal{C}_P[F_U] = F_U$, (3-25) and (3-40). Therefore, we obtain (3-39) due to (3-38).

We now prove (ii). From (3-22) and (3-21d), we deduce that $\mathcal{K}$ commutes with $\mathcal{C}_{E_-1}$. Hence, $E_- K_+$ and $E_- \hat{K}_+$ are right and left eigenvectors of $\mathcal{K}$ corresponding to $\kappa$ as well. For sufficiently small $\rho_+ \sim 1$ and $\eta/\rho \leq \rho_*$, $K_+$ and $\hat{K}_+$ are strictly positive definite. Hence, $K_-$ and $\hat{K}_-$ are linearly independent of $K_+$ and $\hat{K}_+$, respectively.

Part (iii) follows directly from Lemma 3.6, Lemma 3.7 and Lemma 3.8 (i) and (ii). This completes the proof of Lemma 3.8. \( \square \)

**3C. Eigenvalues of $L$ in $D_\rho(0)$.** In this section, we study the small eigenvalues of $L$ as perturbations of the small eigenvalue $\kappa$ of $\mathcal{K}$ (see Lemma 3.7).
Lemma 3.9 (eigenvalues of $\mathcal{L}$ in $D_\varepsilon(0)$). There are $\rho_\varepsilon \sim 1$ and $\varepsilon \sim 1$ such that if $\rho + \eta/\rho \leq \rho_\varepsilon$ then $\text{Spec}(\mathcal{L}) \cap D_\varepsilon(0)$ consists of two eigenvalues $\beta$ and $\beta_\varepsilon$. Each of these eigenvalues has algebraic and geometric multiplicity one. Moreover, they satisfy $|\beta_\varepsilon| < |\beta|$ and

$$\beta_\varepsilon = \kappa + \mathcal{O}(\rho^3 + \eta \rho), \quad \beta = \kappa + 2\rho^2 \frac{(F_\varepsilon^4)}{(F_\varepsilon^2)} + \mathcal{O}(\rho^3 + \eta \rho).$$  (3-41)

Furthermore, $\beta$ and $\beta_\varepsilon$ are continuous functions of $\eta$ for fixed $z \in D_\varepsilon(0)$.

We remark that the eigenvalues of $\mathcal{L}$ are denoted by $\beta$ and $\beta_\varepsilon$ since the spectra of $\mathcal{L}$ and $B$ agree. Indeed, $B$ and $\mathcal{L}$ are related through the similarity transform $B = CQ\mathcal{L}C^{-1}$ due to (3-16).

To lighten the notation in the following, we denote the difference between $\mathcal{L}$ and $\mathcal{K}$ by

$$\mathcal{D} := \mathcal{L} - \mathcal{K}.$$  (3-42)

Proof. We decompose $\mathcal{L}$ according to the splitting $\mathcal{P}_\mathcal{K} + \mathcal{Q}_\mathcal{K} = 1$, i.e., we write

$$\mathcal{L} = \begin{pmatrix} \mathcal{P}_\mathcal{K}\mathcal{L}\mathcal{P}_\mathcal{K} & \mathcal{P}_\mathcal{K}\mathcal{L}\mathcal{Q}_\mathcal{K} \\ \mathcal{Q}_\mathcal{K}\mathcal{L}\mathcal{P}_\mathcal{K} & \mathcal{Q}_\mathcal{K}\mathcal{L}\mathcal{Q}_\mathcal{K} \end{pmatrix}.$$  (3-43)

More precisely, by (3-43) we mean that we consider the (not necessarily orthogonal) decomposition $\mathbb{C}^{2n} = \text{ran} \mathcal{P}_\mathcal{K} + \text{ran} \mathcal{Q}_\mathcal{K}$ into two complementary subspaces and the operators in the right-hand side of (3-43) act among the appropriate subspaces in this decomposition, e.g., $\mathcal{Q}_\mathcal{K}\mathcal{L}\mathcal{P}_\mathcal{K}$ is a linear operator from $\text{ran} \mathcal{P}_\mathcal{K}$ to $\text{ran} \mathcal{Q}_\mathcal{K}$. Notice that $\mathcal{Q}_\mathcal{K}(\mathcal{L} - \omega)\mathcal{Q}_\mathcal{K}$ (viewed as a linear map on $\text{ran} \mathcal{Q}_\mathcal{K}$) is invertible if $|\omega| \leq \varepsilon$, where $\varepsilon$ was chosen as in Lemma 3.6. To see this, we use the identity

$$(A + B)(I + A^{-1}B)^{-1}A^{-1} = I$$  (3-44)

for $A = \mathcal{Q}_\mathcal{K}(\mathcal{L} - \omega)\mathcal{Q}_\mathcal{K}$, $B = \mathcal{Q}_\mathcal{K}(\mathcal{L} - \mathcal{K})\mathcal{Q}_\mathcal{K}$ and $I$ being the identity map on $\text{ran} \mathcal{Q}_\mathcal{K}$ and notice that $A$ (viewed as a map on $\text{ran} \mathcal{Q}_\mathcal{K}$) is invertible by Lemma 3.6; in fact $A^{-1} = \mathcal{Q}_\mathcal{K}(\mathcal{L} - \omega)^{-1}\mathcal{Q}_\mathcal{K}$ with $\|A^{-1}\| \leq 1$ by (3-30). Moreover $\|A^{-1}B\| \leq \|A^{-1}\|\|\mathcal{Q}_\mathcal{K}\|\|\mathcal{L} - \mathcal{K}\| \lesssim \rho$, where we used $\|\mathcal{L} - \mathcal{K}\| \lesssim \rho$ by (3-20) and $\|\mathcal{Q}_\mathcal{K}\| \leq 1 + \|\mathcal{P}_\mathcal{K}\| \lesssim 1$ by (3-30). Therefore $I + A^{-1}B$ is also invertible if $\rho$ is sufficiently small, yielding the invertibility of $A + B = \mathcal{Q}_\mathcal{K}(\mathcal{L} - \omega)\mathcal{Q}_\mathcal{K}$ from (3-44) and that

$$\|\mathcal{Q}_\mathcal{K}(\mathcal{L} - \omega)\mathcal{Q}_\mathcal{K}\|^{-1} \lesssim 1.$$  (3-45)

Moreover, we use (3-43) and Schur’s determinant identity to compute the determinant of $\mathcal{L} - \omega$ and obtain

$$\det(\mathcal{L} - \omega) = \det(\mathcal{Q}_\mathcal{K}(\mathcal{L} - \omega)\mathcal{Q}_\mathcal{K}) \det(\mathcal{P}_\mathcal{K}(\mathcal{L} - \omega)\mathcal{P}_\mathcal{K} - \mathcal{P}_\mathcal{K}\mathcal{L}\mathcal{Q}_\mathcal{K}(\mathcal{Q}_\mathcal{K}(\mathcal{L} - \omega)\mathcal{Q}_\mathcal{K})^{-1}\mathcal{Q}_\mathcal{K}\mathcal{L}\mathcal{P}_\mathcal{K}).$$  (3-46)

Since the first determinant on the right-hand side is not zero for $|\omega| \leq \varepsilon$, the small eigenvalues of $\mathcal{L}$ are exactly those $\omega$’s for which the second determinant vanishes. Note that this is a $2 \times 2$ determinant since $\text{ran} \mathcal{P}_\mathcal{K}$ is two dimensional. Now we write this determinant in a convenient basis. In the basis $(K_+, K_-)$ of $\text{ran} \mathcal{P}_\mathcal{K}$ (see Lemma 3.8), we have

$$\mathcal{P}_\mathcal{K}(\mathcal{L} - \omega)\mathcal{P}_\mathcal{K} = \mathcal{P}_\mathcal{K}(\mathcal{K} - \omega)\mathcal{P}_\mathcal{K} + \mathcal{P}_\mathcal{K}\mathcal{D}\mathcal{P}_\mathcal{K} = \begin{pmatrix} \kappa - \omega & 0 \\ 0 & \kappa - \omega \end{pmatrix} + \Lambda,$$  (3-47)
where we introduce the $2 \times 2$-matrix $\Lambda$ defined through

$$
\Lambda := \begin{pmatrix}
(\bar{K}_+, D[K_+]) & (\bar{K}_+, D[K_-]) \\
(\bar{K}_+, K_+) & (\bar{K}_+, K_-) \\
(\bar{K}_-, D[K_+]) & (\bar{K}_-, D[K_-]) \\
(\bar{K}_-, K_+) & (\bar{K}_-, K_-)
\end{pmatrix}. 
$$

(3-48)

The following lemma which will be shown in Section 3E provides a precise expansion of $\Lambda$ in the small $\rho$ regime.

**Lemma 3.10 (expansion of $\Lambda$).** For $\Lambda$ defined in (3-48), we have the expansion

$$
\Lambda = 2\rho^2 \begin{pmatrix}
(f_0^1) + (f_0^1) \\
0 & 0
\end{pmatrix} + O(\eta \rho).
$$

Lemma 3.10 and (3-47) imply

$$
P_\kappa (\mathcal{L} - \omega) P_\kappa = \begin{pmatrix}
\kappa + 2\rho^2 (f_0^1) - \omega & 0 \\
0 & \kappa - \omega
\end{pmatrix} + O(\eta \rho). 
$$

(3-49)

What remains in order to compute the $2 \times 2$-determinant in (3-46) is estimating $P_\kappa \mathcal{L} Q_\kappa$ and $Q_\kappa \mathcal{L} P_\kappa$. To that end we use the following lemma which will be proven in Section 3E below.

**Lemma 3.11 (expansion of $\mathcal{D}[K_\pm]$ and $\mathcal{D}^*[\hat{K}_\pm]$).** Let $\mathcal{D}$ be defined as in (3-42). Let $K_\pm$ and $\hat{K}_\pm$ be the eigenvectors of $K$ introduced in Lemma 3.8. Then we have

$$
\mathcal{D}[K_+] = O(\rho + \eta), \quad \mathcal{D}[K_-] = O(\eta), \quad \mathcal{D}^*[\hat{K}_+] = O(\rho^2 + \eta), \quad \mathcal{D}^*[\hat{K}_-] = O(\eta).
$$

As $Q_\kappa \mathcal{L} P_\kappa = Q_\kappa K P_\kappa + Q_\kappa D P_\kappa = Q_\kappa D P_\kappa$ and $P_\kappa \mathcal{L} Q_\kappa = P_\kappa K Q_\kappa + P_\kappa D Q_\kappa = P_\kappa D Q_\kappa$, the representation of $P_\kappa$ in Lemma 3.8(iii) yields

$$
\|Q_\kappa \mathcal{L} P_\kappa\| \lesssim \max\{\|Q_\kappa D[K_+]\|, \|Q_\kappa D[K_-]\|\} \lesssim \rho + \eta, \quad (3-50a)
$$

$$
\|P_\kappa \mathcal{L} Q_\kappa\| \lesssim \sup_{\|R\|=1} \max\{|(\hat{K}_+, D Q_\kappa[R])|, |(\hat{K}_-, D Q_\kappa[R])|\} \lesssim \rho^2 + \eta, \quad (3-50b)
$$

where the last steps follow from Lemma 3.11 and (3-30).

Therefore, we combine (3-45), (3-49), (3-50a) and (3-50b), and use $\eta \lesssim \rho$ to obtain

$$
P_\kappa (\mathcal{L} - \omega) P_\kappa - P_\kappa \mathcal{L} Q_\kappa (Q_\kappa (\mathcal{L} - \omega) Q_\kappa)^{-1} Q_\kappa \mathcal{L} P_\kappa = \begin{pmatrix}
\kappa + 2\rho^2 (f_0^1) - \omega & 0 \\
0 & \kappa - \omega
\end{pmatrix} + O(\rho^3 + \eta \rho) \quad (3-51)
$$

with respect to the basis vectors $K_+$ and $K_-$. We now analyse the small eigenvalues of $\mathcal{L}$. We have seen after (3-46) that, for any $|\omega| \leq \varepsilon$, we have

$$
\det(\mathcal{L} - \omega) = 0 \text{ if and only if }
$$

$$
\det(P_\kappa (\mathcal{L} - \omega) P_\kappa - P_\kappa \mathcal{L} Q_\kappa (Q_\kappa (\mathcal{L} - \omega) Q_\kappa)^{-1} Q_\kappa \mathcal{L} P_\kappa) = 0.
$$
Owing to (3-51), the latter relation is equivalent to

$$0 = (\kappa - \omega + \rho^2 \gamma + \delta_{11}) (\kappa - \omega + \delta_{22}) - \delta_{12} \delta_{21},$$

(3-52)

where $\gamma := 2 \langle F_U^4 \rangle / \langle F_U^2 \rangle$ and $\delta_{ij}$ are the entries of the error term on the right-hand side of (3-51). In particular, $\gamma \sim 1$ by (3-26) and $\delta_{ij} = O(\rho^3 + \eta \rho)$. The quadratic equation in (3-52) has the solutions

$$\omega_\pm = \kappa + \frac{\gamma}{2} \rho^2 (1 \pm 1) + O(\delta + \delta^2 / \rho^2),$$

where $\delta := \sup_{i,j} |\delta_{ij}|$. As $\kappa$, $\rho$, $\gamma$ and $\delta_{ij}$ are continuous in $\eta$ (the continuity of $\delta_{ij}$ follows from the continuity of $P_K$ and $L$), $\omega_\pm$ are continuous in $\eta$. Since $\delta = O(\rho^3 + \eta \rho)$ and $\rho \gtrsim \eta$, that is

$$\omega_+ = \kappa + 2 \rho^2 \frac{\langle F_U^4 \rangle}{\langle F_U^2 \rangle} + O(\rho^3 + \eta \rho), \quad \omega_- = \kappa + O(\rho^3 + \eta \rho).$$

Clearly, $\omega_+$ and $\omega_-$ are different from each other if $\rho + \eta / \rho \leq \rho_*$ and $\rho_* \sim 1$ is chosen small enough. Hence, $\omega_+$ and $\omega_-$ are two small eigenvalues of $L$. Lemma 3.6 implies that $L$ possesses at most two small eigenvalues, thus we have fully described the spectrum of $L$ close to zero.  

**3D. Eigenvectors of $L$ and proof of Proposition 3.1.** By Lemma 3.9, there are two eigenvalues of $L$ in $D_*(0)$. The following lemma relates the corresponding eigenvectors to $F_U$ via the eigenvectors of $K$ from Lemma 3.8. The eigenvectors of $L$ will be perturbations of those of $K$. The main mechanism is that the two small eigenvalues of $L$ are sufficiently separated, $|\beta - \beta_*| \sim \rho^2$ (see (3-41)). We will use that this separation is much larger than $\rho^3 + \eta \rho$, the effect of the perturbation $D$ between the unperturbed spectral subspaces $\text{ran} P_K$ and $\text{ran} Q_K$ (see (3-57)). Hence, regular perturbation theory applies.

Owing to $B = C_Q L C_Q^{-1}$ by (3-16), we will conclude Proposition 3.1 immediately from this lemma.

**Lemma 3.12** (eigenvectors of $L$). There is $\rho_* \sim 1$ such that if $\rho + \eta / \rho \leq \rho_*$ then there are right (left) eigenvectors $L$ and $L_*$ ($\hat{L}$ and $\hat{L}_*$) of $L$ associated to the eigenvalues $\beta$ and $\beta_*$ from Lemma 3.9, respectively, satisfying

$$L = F_U + 2 \rho i F_U^2 (\text{Re} U) + O(\rho^2 + \eta / \rho), \quad L_* = E_- F_U + O(\rho^2 + \eta / \rho),$$

(3-53a)

$$\hat{L} = F_U + O(\rho^2 + \eta / \rho), \quad \hat{L}_* = E_- F_U + O(\rho^2 + \eta / \rho).$$

(3-53b)

Moreover, $L$, $L_*$, $\hat{L}$ and $\hat{L}_*$ are continuous functions of $\eta$. For their scalar products, we have the expansions

$$\langle \hat{L}, L \rangle = \langle F_U^2 \rangle + O(\rho^2 + \eta / \rho), \quad \langle \hat{L}_*, L_* \rangle = \langle F_U^2 \rangle + O(\rho^2 + \eta / \rho).$$

(3-54)

Before the proof of Lemma 3.12, we first conclude Proposition 3.1 from Lemmas 3.6, 3.9 and 3.12.

**Proof of Proposition 3.1.** We choose $\varepsilon \sim 1$ as in Lemma 3.9 and $\rho_* \sim 1$ as in Lemma 3.12. Since $B = C_Q L C_Q^{-1}$ due to (3-16), the spectra of $B$ and $L$ agree. Hence, $\text{Spec}(B) \cap D_*(0) = \{\beta, \beta_*\}$, with $\beta$ and $\beta_*$ as introduced in Lemma 3.9. From Lemma 3.9, (3-39) and $\langle F_U^2 \rangle \sim 1$ by (3-26), we obtain the scaling relations in (3-2) by shrinking $\rho_* \sim 1$ if needed.
We now derive (3-4a) and (3-4b). From (3-11), (3-24) and \( \text{Im } U = - \text{Im } U^* = - \text{Im } U^{-1} \) for the unitary operator \( U \) (see Lemma 3.3(i)), we conclude \( \psi = \langle F_U^\perp \rangle \). Moreover, \( \psi \sim 1 \) due to (3-26). The identities \( B = C_Q \mathcal{L} C_Q^{-1} \) and \( Q = Q^* \) also imply \( B = C_Q[L], \ B_\pm = C_Q[L_\pm], \ \hat{B} = C_Q^{-1}[\hat{L}] \) and \( \hat{B}_\pm = C_Q^{-1}[\hat{L}_\pm] \). Hence, \( \langle \hat{B}, B \rangle = \langle \hat{L}, L \rangle \) and \( \langle \hat{B}_\pm, B_\pm \rangle = \langle \hat{L}_\pm, L_\pm \rangle \) as \( Q = Q^* \) (in particular, \( |\langle \hat{B}, B \rangle| \sim 1 \) and \( |\langle \hat{B}_\pm, B_\pm \rangle| \sim 1 \)). Therefore, the expansions of \( \beta \) and \( \beta_\mp \) in Lemma 3.9, the expansion of \( \kappa \) in (3-39), \( \langle F_U^2 \rangle \sim 1 \) due to (3-26) and (3-54) yield (3-4a) and (3-4b).

The balanced polar decomposition, (3-11), the definition of (3-24) and \( \text{Im } U = - \text{Im } U^* = - \text{Im } U^{-1} \) yield that

\[
Q E_\pm F_U Q = \rho^{-1} E_\pm \text{Im } M, \quad Q^{-1} E_\pm F_U Q^{-1} = -\rho^{-1} E_\pm \text{Im } M^{-1}, \quad Q F_U^2 (\text{Re } U) Q = Q F_U Q Q^{-1} F_U Q^{-1} (\text{Re } U) Q = -\rho^{-2} (\text{Im } M)(\text{Im } M^{-1})(\text{Re } M). \tag{3-55}
\]

Since \( B = C_Q[L], \ B_\pm = C_Q[L_\pm], \ \hat{B} = C_Q^{-1}[\hat{L}] \) and \( \hat{B}_\pm = C_Q^{-1}[\hat{L}_\pm] \), the expansions in (3-3), thus, follow from Lemma 3.12, (3-55) and \( Q \sim 1 \) in (3-14). Moreover, the continuity of \( Q, \ Q^{-1} \) and the eigenvectors of \( \mathcal{L} \) from Lemma 3.12 yield the continuity of the eigenvectors of \( B \).

The identity \( B = C_Q \mathcal{L} C_Q^{-1} \) also implies \( B^{-1} Q = C_Q \mathcal{L}^{-1} \mathcal{L} C_Q^{-1} \). Similarly, \( (B^*)^{-1} Q^* = C_Q^{-1} \mathcal{L} C_Q \). Hence, the bounds in (3-5) follow from (3-30) in Lemma 3.6. This completes the proof of Proposition 3.1. \( \square \)

The remainder of this subsection is devoted to the proof of Lemma 3.12.

**Proof of Lemma 3.12.** We fix \( \lambda \in \{\beta, \beta_\mp\} \). Since \( \beta \) and \( \beta_\mp \) have multiplicity one and together with \( \mathcal{L} \) they are continuous functions of \( \eta \) due to Lemma 3.9 and Lemma 3.3 (iii), respectively, we find an eigenvector \( L' \) of \( \mathcal{L} \) associated to \( \lambda \) such that \( L' \) is a continuous function of \( \eta \) and \( \|L'\| = 1 \).

We apply \( \mathcal{P}_\lambda \) to the eigenvector relation \( \lambda L' = \mathcal{L}[L'] \), use \( \mathcal{L} = \mathcal{K} + \mathcal{D} \) and \( \mathcal{P}_\lambda \mathcal{K} = \mathcal{K} \mathcal{P}_\lambda = \kappa \mathcal{P}_\lambda \) to obtain

\[
\lambda \mathcal{P}_\lambda[L'] = \mathcal{P}_\lambda(\mathcal{K} + \mathcal{D})[L'] = \kappa \mathcal{P}_\lambda[L'] + \mathcal{P}_\lambda \mathcal{D} \mathcal{P}_\lambda[L'] + \mathcal{P}_\lambda \mathcal{D} \mathcal{Q}_\lambda[K'][L']. \tag{3-56}
\]

We express (3-56) in the basis (\( K_+, \ K_- \)) of ran \( \mathcal{P}_\lambda \) (see Lemma 3.8 (iii)). We use that \( \mathcal{P}_\lambda \mathcal{D} \mathcal{P}_\lambda = \Lambda \) in this basis, where \( \Lambda \) is defined as in (3-48), and decompose \( \mathcal{P}_\lambda[L'] = \gamma_+ K_+ + \gamma_- K_- \) for some \( \gamma_+, \ \gamma_- \in \mathbb{C} \).

This yields

\[
(\Lambda - \delta) \begin{pmatrix} \gamma_+ \\ \gamma_- \end{pmatrix} = - \begin{pmatrix} \langle \hat{K}_+, \mathcal{D} \mathcal{Q}_\lambda[L'] \rangle \\ \langle \hat{K}_+, K_+ \rangle \end{pmatrix} = - \begin{pmatrix} \langle \mathcal{D}^* \hat{K}_+, \mathcal{Q}_\lambda[L'] \rangle \\ \langle \hat{K}_+, K_+ \rangle \end{pmatrix} = O((\rho^3 + \eta \rho)\|\mathcal{P}_\lambda[L']\|), \tag{3-57}
\]

where \( \delta := \lambda - \kappa \). Here, in the last step, we used Lemma 3.11 to estimate \( \mathcal{D}^* \hat{K}_\pm \). For the other factor, \( \mathcal{Q}_\lambda[L'] \), we use the general eigenvector perturbation result, Lemma B.1 in Appendix B. More precisely, applying \( \mathcal{Q}_\lambda \) to (B-2b) and using \( \mathcal{Q}_\lambda[K] = \mathcal{Q}_\lambda \mathcal{P}_\lambda[L'] = 0 \), we obtain \( \|\mathcal{Q}_\lambda[L']\| \lesssim \|\mathcal{D}\|\|\mathcal{P}_\lambda[L']\| \lesssim \rho \|\mathcal{P}_\lambda[L']\| \) since \( \|\mathcal{D}\| \lesssim \rho \) by (3-20). For the denominators in (3-57), we use that \( |\langle \hat{K}_s, K_s \rangle| \sim 1 \) for \( s \in \{\pm\} \) by Lemma 3.8 and (3-26).
From $\|L\| = 1$ and (3-30), we conclude $\| \mathcal{P}_K[L'] \| \lesssim 1$. Thus,

$$\gamma_\pm = \frac{(\hat{K}_\pm, L')}{(K_\pm, K_\pm)} = \mathcal{O}(1)$$

as $|(\hat{K}_\pm, K_\pm)| \sim 1$ by (3-38) and (3-26). Consequently, (3-57) and Lemma 3.10 imply

$$\left(2 \rho^2 \frac{(F^4_U)}{\langle F^2_U \rangle} - \delta \begin{bmatrix} 0 \\ -\delta \end{bmatrix} \begin{bmatrix} \gamma_+ \\ -\gamma_- \end{bmatrix} = \begin{bmatrix} \mathcal{O}(\rho^3 + \eta \rho) \\ \mathcal{O}(\eta \rho) \end{bmatrix}. \right. \quad (3-58)$$

In order to compute $\gamma_+$ and $\gamma_-$, we now distinguish the two cases $\lambda = \beta$ and $\lambda = \beta^*$ and apply Lemma 3.9 to estimate $\delta$. If $\lambda = \beta$ then $|\delta| \sim \rho^2$ by Lemma 3.9 and (3-26). Hence, (3-58) implies $|\gamma_-| \lesssim \eta / \rho$. Thus, $|\gamma_+| \sim 1$ as $|\gamma_+||K_+| \gtrsim ||L'|| - |\gamma_-||K_-| - ||Q_K[L']||$, $|\gamma_-| \lesssim \eta / \rho$, $||Q_K[L']|| \lesssim \rho$ and $||L'|| = 1 \sim ||K_\pm||$. In particular, $L := L'(\hat{K}_+, K_+)/\langle \hat{K}_+, L' \rangle = L'/\gamma_+$ is continuous in $\eta$ and

$$\mathcal{P}_K[L] = K_+ + \mathcal{O}(\eta / \rho) = F_U + \mathcal{O}(\eta / \rho),$$

where we used (3-37) in the last step. We now apply Lemma B.1 to compute $Q_K[L]$ with $K = \mathcal{P}_K[L] = F_U + \mathcal{O}(\eta / \rho)$. From (3-60a), we obtain $(K - \kappa)^{-1} Q_K \mathcal{D}[F_U] = -2\rho i F^2_U (\text{Re } U) + \mathcal{O}(\rho^2 + \eta)$ since $Q_K$ and $K$ agree with the identity map on $M_0$ and $F^2_U (\text{Re } U) \in M_0$. Hence, Lemma B.1 and $\|\mathcal{D}\| \lesssim \rho$ directly imply the expansion of $L$ in (3-53a).

We now consider the case $\lambda = \beta^*$. Lemma 3.9 with $\lambda = \beta^*$ implies $|\delta| \lesssim \rho^3 + \eta \rho$ and, thus,

$$2 \rho^2 \frac{(F^4_U)}{\langle F^2_U \rangle} - \delta \sim \rho^2.$$

Hence, $|\gamma_+| \lesssim \rho + \eta / \rho$ and, similarly to the other case, we set $L_* := L'/\gamma_-$ and conclude

$$\mathcal{P}_K[L_*] = E_- F_U + \mathcal{O}(\rho + \eta / \rho).$$

Owing to (B-2b) in Lemma B.1 with $K = \mathcal{P}_K[L_*]$, we have

$$Q_K[L_*] = -(K - \kappa)^{-1} Q_K \mathcal{D}[\mathcal{P}_K[L_*]] + \mathcal{O}(\rho^2) = -(K - \kappa)^{-1} Q_K \mathcal{D}[E_- F_U] + \mathcal{O}(\rho^2 + \eta) = \mathcal{O}(\rho^2 + \eta), \quad (3-59)$$

where the last step follows from (3-60b). Therefore, the second identity in (3-57), Lemma 3.10, Lemma 3.11 and $\rho \gtrsim \eta$ by (13-13) imply

$$\left(2 \rho^2 \frac{(F^4_U)}{\langle F^2_U \rangle} - \delta \begin{bmatrix} 0 \\ -\delta \end{bmatrix} \begin{bmatrix} \gamma_+ \\ -\gamma_- \end{bmatrix} = \begin{bmatrix} \mathcal{O}(\rho^4 + \eta \rho) \\ \mathcal{O}(\eta \rho) \end{bmatrix}. \right. \quad (3-59)$$

Since

$$2 \rho^2 \frac{(F^4_U)}{\langle F^2_U \rangle} - \delta \sim \rho^2,$$

we conclude $|\gamma_+| \lesssim \rho^2 + \eta / \rho$. Hence, $\mathcal{P}_K[L_*] = E_- F_U + \mathcal{O}(\rho^2 + \eta / \rho)$ and the expansion of $L_*$ in (3-53a) follows from (3-59).
A completely analogous argument starting from $\mathcal{L}^*[\hat{L}] = \hat{\lambda} \hat{L}$ yields the expansions of $\hat{L}$ and $\hat{L}_s$ in (3-53b). We leave the details to the reader. From (3-53a) and (3-53b), we directly obtain (3-54) since $F_U^3(\text{Re} U) \in M_0$ implies $\langle F_U^3(\text{Re} U) \rangle = 0$. This completes the proof of Lemma 3.12.

**3E. Proofs of the auxiliary Lemmas 3.10 and 3.11.** In this section, we show Lemmas 3.10 and 3.11 which were both stated in Section 3C.

**Proof of Lemma 3.11.** This lemma follows directly from Lemma 3.8 and the precise expansions of $D[E \pm F_U]$ and $D^*[E \pm F_U]$ established in (3-60).

In the following computations, we constantly use that $P$, $U$, $U^*$ and $F_U$ commute with each other. From (3-25), (3-19) and $1 - U^2 = -2i \text{Re} U \text{Im} U + 2(\text{Im} U)^2$, we obtain

$$D[F_U] = (C_P - C_U)F_U + O(\eta) = F_U(1 - U^2) + O(\eta) = -2i \rho F_U^3(\text{Re} U) + 2\rho^2 F_U^3 + O(\eta). \quad (3-60a)$$

Similarly, (3-25), (3-19), (3-21d) and $U E_\perp F_U U = -E_U F_U U = -E_- F_U$ by (3-21c) imply

$$D[E_- F_U] = -(C_P - C_U)E_- F_U + O(\eta) = E_- F_U - E_- F_U^3 + O(\eta) = O(\eta). \quad (3-60b)$$

Since $1 - (U^*)^2 = 2i \text{Re} U \text{Im} U + 2(\text{Im} U)^2$, $F_U^2(\text{Re} U) \in M_0$ by Lemma 3.3(i) and $F$ vanishes on $M_0$, $\mathcal{D}^*[F_U] = \mathcal{F}(C_P - C_U^*)[F_U] = \mathcal{F}[F_U(1 - (U^*)^2)] = 2i \rho \mathcal{F}[F_U^3(\text{Re} U)] + 2\rho^2 \mathcal{F}[F_U^3] = 2\rho^2 \mathcal{F}[F_U^3]. \quad (3-60c)$

From $P E_- F_U P = -E_- F_U = U^* E_- F_U U^*$ by (3-21d) and (3-21c), we deduce

$$\mathcal{D}^*[E_- F_U] = \mathcal{F}[-E_- F_U + E_- F_U] = 0. \quad (3-60d)$$

This completes the proof of Lemma 3.11.

**Proof of Lemma 3.10.** We first show that, for all $s_1, s_2 \in \{\pm\}$, we have

$$\langle \mathcal{K}_{s_1}, D[K_{s_2}] \rangle = \langle E_{s_1} F_U, D[E_{s_2} F_U] \rangle + O(\eta \rho + \eta^2). \quad (3-61)$$

In fact, it is easy to see that (3-61) follows from $K_{s_1}, \mathcal{K}_{s_2} \in M_d$ and (3-37) in Lemma 3.8 and that

$$\langle R_1, (C_P - C_U)[R_2] \rangle = O(\rho^2) \quad (3-62)$$

for all $R_1, R_2 \in M_d$ satisfying $\|R_1\|, \|R_2\| \lesssim 1$. For the proof of (3-62), we expand $U = \text{Re} U + i \text{Im} U$ and obtain

$$\langle R_1, (C_P - C_U)[R_2] \rangle = \langle R_1, (C_P - C_{Re U})[R_2] \rangle - i\langle R_1, \text{Im} U R_2 \text{Re} U + \text{Re} U R_2 \text{Im} U \rangle + O(\rho^2) = O(\rho^2).$$

Here, we used in the first step that $\text{Im} U = O(\rho)$. For the second step, we noted that the first term is $O(\rho^2)$ due to (3-19) and the second term vanishes as $\text{Re} U \in M_0$ while $R_1, R_2, \text{Im} U \in M_d$ due to Lemma 3.3(i).

We still need to compute $\langle E_{s_1} F_U, D[E_{s_2} F_U] \rangle = \langle D^*[E_{s_1} F_U], E_{s_2} F_U \rangle$. From (3-60c) and (3-25), we get

$$\langle D^*[F_U], F_U \rangle = 2\rho^2 \langle F_U^4, \rho(\eta) \rangle + O(\eta \rho), \quad \langle D^*[E_- F_U], F_U \rangle = -2\rho^2 \langle F_U^4 E_- \rangle + O(\eta \rho) = O(\eta \rho),$$

where we used in the very last step that $\langle F_U^4 E_- \rangle = 0$ since the diagonal $n$-vector components of $F_U$ are identical due to (3-24). Moreover, (3-60d) directly implies that $\langle D^*[E_- F_U], E_\pm F_U \rangle = 0.$
Hence, owing to (3-61), we deduce
\[
\Lambda = 2\rho^2 \left( \begin{pmatrix} \frac{(r_1^*)}{F_1^*} \\ \frac{(r_2^*)}{F_2^*} \end{pmatrix} 0 \\ 0 \right) + O(\eta\rho + \eta^2).
\]
Using that \(\rho \gtrsim \eta\) due to (3-13) completes the proof of Lemma 3.10. \(\square\)

3F. Derivatives of \(M\). As a first application of our analysis of the stability operator \(B\) we show the following bound on the derivatives of \(M\), the solution to the MDE (2-6), with respect to \(\eta\), \(z\) and \(\bar{z}\).

**Lemma 3.13** (bounds on derivatives of \(M\)). There is \(\rho_* \sim 1\) such that \(\rho + \eta/\rho \leq \rho_*\) implies
\[
\|\partial_\eta M\| + \|\partial_z M\| + \|\partial_\bar{z} M\| \lesssim \frac{1}{\rho^2 + \eta/\rho}.
\] (3-63)

**Proof.** We only show the bound on \(\partial_\eta M\). The estimates on \(\partial_z M\) and \(\partial_\bar{z} M\) are shown analogously.

If \(\rho_* \sim 1\) is chosen small enough and \(\rho + \eta/\rho \leq \rho_*\) then \(B\) is invertible due to Proposition 3.1. Thus, applying the implicit function theorem to (2-6) yields that \(M\) is differentiable with respect to \(\eta\) and \(\partial_\eta M = iB^{-1}[M^2]\). Hence, by Proposition 3.1, we have
\[
-i\partial_\eta M = \frac{\langle \hat{B}, M^2 \rangle}{\beta(\hat{B}, B)} B + \frac{\langle \hat{B}_*, M^2 \rangle}{\beta_* \langle \hat{B}_*, B_* \rangle} B_* + B^{-1} Q[M^2].
\] (3-64)

Moreover, differentiating \(\langle E_-, M \rangle = 0\), which holds due to (3-12), with respect to \(\eta\) yields
\[
\langle E_-, \partial_\eta M \rangle = 0.
\]

Hence, we apply \(\langle E_-, \cdot \rangle\) to (3-64) and obtain
\[
\frac{\langle \hat{B}_*, M^2 \rangle}{\beta_* \langle \hat{B}_*, B_* \rangle} \langle E_-, B_* \rangle = -\frac{\langle \hat{B}, M^2 \rangle}{\beta(\hat{B}, B)} \langle E_-, B \rangle + \langle E_-, B^{-1} Q[M^2] \rangle.
\] (3-65)

The right-hand side of (3-65) is \(O(1)\) since \(\hat{B}, M\) and \(B^{-1} Q[M^2]\) are bounded, \(|\langle \hat{B}, B \rangle| \sim 1\) and \(|\langle E_-, B \rangle| \lesssim |\beta|\) by (3-2) and (3-8). Since \(|\langle E_-, B_* \rangle| \sim 1\) and \(\|B_*\| \lesssim 1\) by (3-3b), we conclude that the second term on the right-hand side of (3-64) is \(O(1)\). Thus, the leading term to \(\partial_\eta M\) comes from the first term on the right-hand side of (3-64) which can be bounded by \((\rho^2 + \eta/\rho)^{-1}\) due to (3-2), \(|\langle \hat{B}, B \rangle| \sim 1\) and the boundedness of \(\hat{B}, M\) and \(B\). \(\square\)

3G. Existence and properties of \(\sigma\). The next proposition shows the existence of a probability density \(\sigma\) satisfying (2-4). In particular, the logarithmic potential of the probability measure \(\sigma(z)\) is given by the function \(-2\pi L\), where, for any \(z \in \mathbb{C}\), \(L(z)\) is defined through
\[
L(z) := -\frac{1}{2\pi} \int_0^\infty \left( \Im M(z, \eta) - \frac{1}{1 + \eta} \right) d\eta.
\] (3-66)

Throughout this subsection, we use the normalization \(\phi(\mathcal{S}) = 1\) (see (3-9)).

**Proposition 3.14** (properties of \(\sigma\)). There exists an integrable function \(\sigma : \mathbb{C} \to [0, \infty)\) such that \(\sigma = \Delta z L\).
in the sense of distributions. Moreover, \( \sigma \) can be chosen to satisfy the following properties:

(i) The function \( \sigma \) is a probability density on \( \mathbb{C} \) with respect to \( d^2z \).

(ii) For all \( z \in \mathbb{C} \) with \( |z| > 1 \), we have \( \sigma(z) = 0 \).

(iii) The restriction \( \sigma|_{D(0,1)} \) is infinitely often continuously differentiable and \( \sigma(z) \sim 1 \) uniformly for \( z \in D(0, 1) \).

The next lemma will directly imply Proposition 3.14 and be proved after the proof of this proposition.

**Lemma 3.15** (properties of \( L \)). The function \( L(z) \) from (3-66) has the following properties:

(i) The integrand on the right-hand side of (3-66) is Lebesgue-integrable for every \( z \in \mathbb{C} \).

(ii) \( L \) is a rotationally symmetric function and continuously differentiable with respect to \( z \) and \( \bar{z} \) on \( \mathbb{C} \).

(iii) \( \Delta_z L(z) \) exists for all \( z \in \mathbb{C} \) satisfying \( |z| \neq 1 \).

**Proof of Proposition 3.14.** By Lemma 3.15(iii), we know \( \Delta_z L \) defines a function on \( \mathbb{C} \setminus \{z \in \mathbb{C} : |z| = 1\} \). We set \( v(z, \eta) = (v_1(z, \eta), v_2(z, \eta)) \) and remark that \( (\text{Im} M)(\eta) = (v) \). Moreover, [Alt et al. 2018, Equation (4.2)] implies that \( v \) has a smooth extension to \( \mathbb{C} \setminus D(0, 1) \times [0, \infty) \) and \( \lim_{\eta \downarrow 0} v(z, \eta) = 0 \) for \( |z| > 1 \) due to (3-13). Therefore, we can follow the proof of [Alt et al. 2018, Equation (4.10)] and obtain

\[
\int_{\tau_0 \leq |z|^2 \leq \tau_1} \Delta_z L(z) d^2z = 0
\]

for all \( \tau_1 > \tau_0 > 1 \). As \( \tau_1 \) and \( \tau_0 \) are arbitrary, we conclude \( \sigma(z) = \Delta_z L(z) = 0 \) if \( |z| > 1 \).

Let \( f \in C_0^{\infty}(\mathbb{C}) \) be a smooth function with compact support. We compute

\[
\int_{\mathbb{C}} L(z) \Delta_z f(z) d^2z = \int_{D(0,1)} L(z) \Delta_z f(z) d^2z + \int_{\mathbb{C} \setminus D(0,1)} L(z) \Delta_z f(z) d^2z = \int_{\mathbb{C}} \sigma(z) f(z) d^2z. \tag{3-67}
\]

Here, we moved \( \Delta_z \) to \( L \) in the second step and used that \( \sigma(z) = \Delta_z L(z) = 0 \) for \( |z| > 1 \) and that the boundary terms cancel each other due to the continuity of \( L, \partial_z L \) and \( \partial_{\bar{z}} L \) from Lemma 3.15(ii). Since \( \sigma(z) = 0 \) for \( |z| > 1 \), setting \( \sigma(z) = 0 \) if \( |z| = 1 \) and using [Alt et al. 2018, Proposition 2.4] for the remaining properties completes the proof of Proposition 3.14.

In the proof of Lemma 3.15, we will make use of the following lemma whose proof we postpone until the end of this section.

**Lemma 3.16.** Uniformly for \( z \in \mathbb{C} \) and \( \eta > 0 \), we have

\[
\left\| M(z, \eta) - \frac{i}{1 + \eta} \right\| \lesssim \frac{1}{1 + \eta^2}, \tag{3-68}
\]

\[
\left\| \partial_z M(z, \eta) \right\| + \left\| \partial_{\bar{z}} M(z, \eta) \right\| \lesssim \frac{1}{\eta^{2/3} + \eta^2}. \tag{3-69}
\]

**Proof of Lemma 3.15.** The assertion in (i) follows immediately from (3-68). Moreover, since \( M(z, \eta) \) is continuously differentiable with respect to \( z \) and \( \bar{z} \), the bound (3-69) implies (ii). In [Alt et al. 2018, Proposition 2.4(i)], it was shown that \( \Delta_z (\text{Im} M(z, \eta)) = \Delta_z (v^T|_{r=|z|^2}) = 4\langle \partial^2_z v^T + \partial_{\bar{z}} v^T \rangle|_{r=|z|^2} \) (these equalities use the notation of [Alt et al. 2018]) is integrable in \( \eta > 0 \) on \([0, \infty)\) for all \( z \in \mathbb{C} \) with \( |z| < 1 \).
Completely analogously, the integrability can be shown if \(|z| > 1\). This shows that \(\Delta_z\) and the \(\eta\)-integral can be interchanged which proves (iii).

Proof of Lemma 3.16. From (2-6), it is easy to get \(|M| \leq \eta^{-1}\) for all \(z \in \mathbb{C}\) and \(\eta > 0\). As in the proof of Lemma 3.3, we see that, uniformly for \(|z| \geq 2\) or \(\eta \geq 1\), we have

\[
\rho \sim \frac{\eta}{|z|^2 + \eta^2}, \quad v_1 \sim v_2 \sim \rho, \quad u \sim \frac{1}{|z|^2 + \eta^2}, \quad \text{Im} \, U \sim \frac{\eta}{|z| + \eta}, \quad Q \sim \frac{1}{|z|^{1/2} + \eta^{1/2}}.
\]

(3-70)

In particular, \(|M| \lesssim (1 + \eta)^{-1}\) and (3-68) follows from multiplying (2-6) by \(i\eta^{-1}M\) and using \(|S| \lesssim 1\).

We remark that (3-69) follows from (3-63) and (3-13) using a similar proof to that of [Alt et al. 2018, Equation (4.2)] if \(|z| \leq 2\) and \(\eta \leq 1\). Hence, we assume \(|z| \geq 2\) or \(\eta \geq 1\) in the remainder of the proof. In this regime, we obtain \(1 - \|F\|_{2-2} \sim 1\) by following the proof of (3-23) and using (3-70). Therefore, (3-16), \(|S| \leq 1\) (see the proof of Lemma 3.6) and (3-70) imply \(|B^{-1}| \lesssim 1\). We differentiate (2-6) with respect to \(z\) and, thus, obtain

\[
\|\partial_z M(z, \eta)\| = \|B^{-1} \left[ M \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} M \right] \| \lesssim \eta^{-2}.
\]

Together with a similar argument for \(\partial_z M\) this completes the proof of Lemma 3.16.

4. Cubic equation associated to stability equation of the MDE

In this section, we study specific perturbations to the matrix Dyson equation (MDE). Throughout this section, \(M\) is the solution to the unperturbed MDE (2-6). We consider solutions \(G\) of the perturbed MDE (2-8), for some \(D \in \mathbb{C}^{2n \times 2n}\) with the additional constraint \((E_-, G) = 0\), keeping in mind that in our application the resolvent \(G = (H_- - i\eta)^{-1}\) (see (5-9)) satisfies this constraint. Since \(D\) is small, we need to study the stability of the MDE (2-6), under a small perturbation. The linear stability operator of this perturbation problem is \(B = 1 - C_M S\) (see (2-10)). When \(\rho\) is small, the inverse of \(B\) blows up, hence we need to expand to the next order, i.e., study the quadratic stability equation, (2-9). The following proposition describes the stability properties of (2-9) in this regime. In fact, the difference \(G - M\) is dominated by the contribution \(\Theta := (\tilde{B}, G - M)/\langle \tilde{B}, B \rangle\) of \(G - M\) in the unstable direction \(B\) of \(B\) (see Proposition 3.1). The scalar quantity \(\Theta\) satisfies a cubic equation. In order to control \(G - M\), we will control this cubic equation via a bootstrapping argument in \(\eta\) in Section 5.

In the following proposition and the rest of the paper, we use a special matrix norm to estimate the distance between \(G\) and \(M\). We denote this norm by \(\| \cdot \|_*\). It is slightly modified compared of those defined in [Alt et al. 2020b; Erdős et al. 2020] in order to account for the additional unstable direction of \(B\). Such norms are tailored to local law proofs by the cumulant method and have first appeared in [Erdős et al. 2019b]. We need some auxiliary notations in order to define \(\| \cdot \|_*\). For \(a, b, c, d \in [2n]\), we set \(\kappa_c(ab, cd) := \delta_{ac} \delta_{bd} s_{ab}\) and \(\kappa_d(ab, cd) := \delta_{ac} \delta_{bd} t_{ab}\), where \(s_{ab} := E|w_{ab}|^2\) and \(t_{ab} := E w_{ab}^2\) with \(W = (w_{ab})_{a,b \in [2n]} := H_e - E H_e\) (compare (2-5) for the definition of \(H_e\)). Moreover, for a vector \(x = (x_a)_{a \in [2n]}\), we write \(\kappa_c(xb, cd) = \sum_a x_a \kappa_c(ab, cd)\) for a weighted version of \(\kappa_c\). We use an analogous convention for \(\kappa_d\). If we replace an index of a scalar quantity by a dot (\(\cdot\)) then this
denotes the corresponding vector, where the omitted index runs through \([2n]\), e.g., \(R_a\) denotes the vector \((R_{ab})_{b\in[2n]}\). For an \(2n \times 2n\) matrix \(R\) and vectors \(x, y \in \mathbb{C}^{2n}\), we use the short-hand notation \(R_{xy}\) to denote the quadratic form \(\langle x, R y \rangle\) and \(R_{xa}\) to denote \(\langle x, R e_a \rangle\), where \(e_a\) is the \(a\)-th normalized standard basis vector. With these conventions, we define some sets of testvectors. For fixed vectors \(x, y \in \mathbb{C}^{2n}\), we define

\[
I_0 := \{x, y\} \cup \{e_a, (\hat{B}^{*})_{a}, ((\hat{B}^{*})_{a}) : a \in [2n]\},
I_{k+1} := I_k \cup \{Mu : u \in I_k\} \cup \{\kappa_e((Mu)a, b\cdot), \kappa_d((Mu)a, b\cdot) : u \in I_k, a, b \in [2n]\}.
\]

We now introduce the \(\|\cdot\|_a\)-norm defined by

\[
\|R\|_a := \|R\|^K_{x,y} := \sum_{0 \leq k < K} (2n)^{-k/2K} \|R\|_k + (2n)^{-1/2} \max_{u \in I_k} \|R_{a\cdot}\|_u, \quad \|R\|_I := \max_{u, v \in I} \frac{|R_{uv}|}{\|u\|_a \|v\|}.
\]

We remark that the norm \(\|\cdot\|_a\) depends on \(\eta\) and \(\tau\) via \(M = M(z, \eta)\). However, this will not play any important role in our arguments.

In this section, the model parameters for the comparison relation \(\lesssim\) are given by \(s_a\) and \(s^*\) from (2-1) as well as \(\tau\) from the upper bound on \(|z|\).

**Proposition 4.1** (cubic equation for \(\Theta\)). There is \(\rho_* \sim 1\) such that if \(\rho + \eta/\rho \leq \rho_*\) for some fixed \(z \in D_\tau(0)\) and \(\eta \in (0, 1]\) then the following holds. We fix \(K \in \mathbb{N}\), \(x, y \in \mathbb{C}^{2n}\) and set \(\|\cdot\|_a := \|\cdot\|_{x,y}^K\). If \(G\) and \(D\) satisfy (2-8), \(\langle E_-, G \rangle = 0\) and \(\|G - M\| + \|D\| \lesssim n^{-30/K}\) then

\[
G - M = \Theta B - B^{-1}Q[MD] + \Theta^2 B^{-1}Q[MS[B]B] + E, \tag{4.1}
\]

where \(\Theta := (\hat{B}, G - M) / (\hat{B}, B)\) and the error matrix \(E\) has the upper bound

\[
\|E\| \lesssim |\Theta|^2(\rho + \eta/\rho) + n^{16/K}(|\Theta|^3 + |\Theta|(\|D\|_a + \rho^2 + \eta/\rho) + \|D\|^2_a + |\langle R_1, D \rangle|) \tag{4.2}
\]

with \(R_1 := M^*(B^{-1}Q)^*[E_-]\), thus, the \(2n \times 2n\)-matrix \(R_1\) does not depend on \(G\) and \(D\) and satisfies \(\|R_1\| \lesssim 1\).

Moreover, \(\Theta\) fulfils the approximate cubic equation

\[
\Theta^3 + \xi_2 \Theta^2 + \xi_1 \Theta = \varepsilon_* \tag{4.3}
\]

whose coefficients \(\xi_2\) and \(\xi_1\) satisfy the scaling relations

\[
|\xi_2| \sim \rho, \quad |\xi_1| \sim \eta/\rho + \rho^2 \tag{4.4}
\]

and the error term \(\varepsilon_*\) is bounded by

\[
|\varepsilon_*| \lesssim n^{62/K}\left(\|D\|_a^3 + |\langle R_1, D \rangle|^3/2 + |\langle R_2, D \rangle|^3/2\right)
\]

\[
+ |\langle \hat{B}, MD \rangle| + |\langle \hat{B}, M(SB^{-1}Q[MD])(B^{-1}Q[MD]) \rangle| \tag{4.5}
\]

Here, the matrix \(R_2 \in \mathbb{C}^{2n \times 2n}\) does not depend on \(G\) and \(D\) and satisfies \(\|R_2\| \lesssim 1\).

We note that \(R_2\) has an explicit definition (see (4-8) below) but its exact form will not be important.
Proof. The proof follows from an application of Lemma A.1 to (2-9) with the choices $A[R, T] := \frac{1}{2} M(S[R]T + S[T]R)$, $B = 1 - C M S$ as in (2-10), $Y = G - M$ and $Z = MD$. Note that $\langle E_-, G - M \rangle = 0$ due to the assumption $\langle E_-, G \rangle = 0$ and (3-12). We first check the conditions of Lemma A.1 in (A-1) with $\| \cdot \| \equiv \| \cdot \|_*$ and $\lambda := n^{1/2K}$. Partly, they will be a consequence of the bounds

$$\| MS[R]T \|_* \lesssim n^{1/2K} \| R \|_* \| T \|_*, \quad \| MR \|_* \lesssim n^{1/2K} \| R \|_*, \quad \| Q \|_{* \rightarrow *} \lesssim 1, \quad \| B^{-1} Q \|_{* \rightarrow *} \lesssim 1,$$  

for all $R, T \in \mathbb{C}^{2n \times 2n}$. The proof of (4-6) is very similar to the one of [Alt et al. 2020b, Lemma 3.4]. Since (4-6b) does not have a counterpart therein, we provide the details at the end of this section.

Owing to (3-3a) and (3-3b), we have $\| B \|_* + \| B_* \|_* \lesssim \| B \| + \| B_* \| \lesssim 1$. The third, sixth and ninth terms in (A-1) are $\sim 1$ by Proposition 3.1. This completes the proof of (A-1) with $\lambda := n^{1/2K}$.

Therefore, Lemma A.1 with $\delta := \eta - \eta / \rho$, $\| \Theta \| \lesssim \| G - M \|_* \lesssim n^{-30/K}$ and $\| MD \|_* \lesssim n^{1/2K} \| D \|_*$ imply $\mu_3 \Theta^3 + \mu_2 \Theta^2 - \beta \langle \hat{B}, B \rangle \Theta$

$$= -\mu_0 + \langle R_2, D \rangle \Theta + \mathcal{O}(n^{-1/4K} |\Theta|^3 + n^{62/K} (\| D \|_*^3 + |\langle R_1, D \rangle|^{3/2}) + n^{20/K} |\Theta|^2 (|\langle E_-, B \rangle|^2 + |\langle \hat{B}, A [B, B_*] \rangle|^2)), \quad (4-7)$$

where $\mu_3, \mu_2$ and $\mu_0$ are defined as in (A-4), $R_1 = M^*(B^{-1} Q)^* [E_-]$ and we introduced

$$R_2 := M^*(B^{-1} Q)^* \left[ S[B^*] M^* \hat{B} + S[M^* \hat{B} B^*] - \frac{\langle B, E_- \rangle}{\langle B_*, E_- \rangle} \left( S[(B_*)^*] M^* \hat{B} + S[M^* \hat{B} (B_*^*)] \right) \right]. \quad (4-8)$$

Note that $R_1$ and $R_2$ are independent of $G$ and $D$ and satisfy $\| R_1 \| \lesssim 1$ and $\| R_2 \| \lesssim 1$ due to Proposition 3.1 and (3-14).

The remaining task is expanding $\mu_3, \mu_2, -\beta \langle \hat{B}, B \rangle$ and $\langle \hat{B}, A [B, B_*] \rangle$ on the right-hand side of (4-7) with the help of Proposition 3.1. The coefficient $-\beta \langle \hat{B}, B \rangle$ has already been identified in (3-4a). For the others, we will rewrite the expansions in Proposition 3.1 in terms of $U$, $Q$ and $F_U = \rho^{-1} \text{Im} U$ defined in (3-10) and (3-24) via $M = QUQ$ by (3-11). In particular,

$$\text{Im} M = Q(\text{Im} U) Q = \rho Q F_U Q,$$

$$- \text{Im} M^{-1} = Q^{-1} (\text{Im} U) Q^{-1} = \rho Q^{-1} F_U Q^{-1}, \quad (4-9)$$

$$\text{Re} M = Q(\text{Re} U) Q.$$  

Note $U, U^*$ and $F_U$ commute. Further, since $U$ is unitary (see Lemma 3.3(i)), the estimate (3-14) implies

$$(\text{Re} U)^2 = 1 - (\text{Im} U)^2 = 1 + \mathcal{O}(\rho^2). \quad (4-10)$$

We recall that $\psi$ defined in Proposition 3.1 satisfies $\psi = \langle F_U \rangle$ (see the proof of Proposition 3.1). In the following, we will frequently use that $\langle R \rangle = 0$ if $R \in \mathcal{M}_0$.

We now compute the coefficients from (A-4). Indeed, we now show that

$$\mu_3 = \psi + \mathcal{O}(\rho + \eta / \rho), \quad (4-11a)$$

$$\mu_2 = 3i \rho \psi + \mathcal{O}(\rho^2 + \eta / \rho). \quad (4-11b)$$
As a preparation of the proof of (4-11), we expand $A[B, B]$. Proposition 3.1, (4-9) and the definition $F = C_Q S C_Q$ from (3-17) yield

$$A[B, B] = M S[B]B = Q U F[F_U + 2i \rho F_U^2 \text{Re } U](F_U + 2i \rho F_U^2 \text{Re } U)Q + O(\rho^2 + \eta/\rho)$$

$$= Q U F_U(F_U + 2i \rho F_U^2 \text{Re } U)Q + O(\rho^2 + \eta/\rho)$$

$$= Q(F_U^2 \text{Re } U + 3i \rho F_U^3)Q + O(\rho^2 + \eta/\rho).$$

(4-12)

Here, we used that $F$ vanishes on $\mathcal{M}_o$ and $F_U^2 \text{Re } U \in \mathcal{M}_o$ as well as using (3-25) in the second step and $U = \text{Re } U + i \text{Im } U = \text{Re } U + i \rho F_U = \text{Re } U + O(\rho)$ by (3-14) in the last step.

We recall the definitions $L = 1 - C_U F$ and $K = 1 - C_U F$ from (3-28). Since $B^{-1} Q = C_Q L^{-1} Q K^{-1} = C_Q K^{-1} Q K^{-1} + O(\rho)$ by Lemma 3.6 and (3-20) we deduce from (4-12) that

$$B^{-1} Q A[B, B] = C_Q K^{-1} Q K^{-1} + O(\rho + \eta/\rho) = C_Q [F_U^2 \text{Re } U] + O(\rho + \eta/\rho).$$

(4-13)

The last step follows since $K^{-1} Q K$ acts as the identity map on $\mathcal{M}_o$ and $F_U^2 \text{Re } U \in \mathcal{M}_o$.

**Proof of (4-11a).** For the first term in the definition of $\mu_3$ of (A-4), we use Proposition 3.1, (4-9) and (4-13) to obtain

$$2 \langle \tilde{B}, A[B, B^{-1} Q A[B, B]] \rangle = \langle F_U U F(F_U)F_U^2 (\text{Re } U) \rangle + O(\rho + \eta/\rho) = \psi + O(\rho + \eta/\rho).$$

(4-14)

In the first step, we also employed that $S$ vanishes on $\mathcal{M}_o$ and $C_Q [F_U^2 \text{Re } U] \in \mathcal{M}_o$, which follows from $Q \in \mathcal{M}_d$ and $F_U^2 \text{Re } U \in \mathcal{M}_o$. The second step is a consequence of (3-25), (4-10) and $\psi = (F_U^4)$. To estimate the second term in the definition of $\mu_3$, we now estimate $\langle \tilde{B}, A[B, B_s] \rangle$. Proposition 3.1 and (4-9) imply

$$2 \langle \tilde{B}, A[B, B_s] \rangle$$

$$= \langle F_U^2 U F[F_U]E_- \rangle + \langle U F[E_\textrm{F-U}]F_U^2 (1 + 2i \rho F_U \text{Re } U) \rangle + O(\rho^2 + \eta/\rho)$$

$$= \langle F_U^3 \text{Re } U E_- \rangle + i \rho \langle F_U^4 E_- \rangle - \langle E_- \text{Re } U F_U^3 \rangle - 2i \rho \langle E_- \text{Re } U F_U^4 \rangle + O(\rho^2 + \eta/\rho)$$

$$= O(\rho^2 + \eta/\rho).$$

(4-15)

Here, we used that $F$ vanishes on $F_U^2 \text{Re } U \in \mathcal{M}_o$ in the first step. The second step follows from (3-25) and (3-22). In the last step, after cancelling the first and third terms, we employed (4-10) and $(F_U^3 E_-) = 0$ by (3-24).

The expansion in (4-13) and $E_- Q (\text{Re } U) F_U^2 Q \in \mathcal{M}_o$ imply

$$\langle E_-, B^{-1} Q A[B, B] \rangle = \langle E_- Q (\text{Re } U) F_U^2 Q \rangle + O(\rho + \eta/\rho) = O(\rho + \eta/\rho).$$

(4-16)

From (4-14), (4-15), $|\langle E_-, B_s \rangle| \sim 1$ by (3-3b) and (4-16) we conclude that $\mu_3$ defined in (A-4) satisfies (4-11a).

**Proof of (4-11b).** We now turn to the expansion of $\mu_2$. From Proposition 3.1, (4-9) and (4-12), we conclude

$$\mu_2 = \langle \tilde{B}, A[B, B] \rangle = \langle F_U^3 \text{Re } U + 3i \rho F_U^4 \rangle + O(\rho^2 + \eta/\rho) = 3i \rho \psi + O(\rho^2 + \eta/\rho).$$

(4-17)

Here, we used $F_U^3 (\text{Re } U) \in \mathcal{M}_o$ and $\psi = (F_U^4)$ in the last step. This completes the proof of (4-11b).
We now continue to estimate the right-hand side of (4-7). Young’s inequality implies that \(|\langle R_1, D \rangle_\Theta| \leq n^{-1/4K} |\Theta|^3 + n^{1/8K} |\langle R_1, D \rangle|^{3/2}\). Then, we incorporate the error terms on the right-hand side of (4-7) bounded by \(n^{-1/4K} |\Theta|^3\) and introduce \(\widetilde{\mu}_3\) such that \(\widetilde{\mu}_3 \Theta^3 = \mu_3 \Theta^3 + \mathcal{O}(n^{-1/4K} |\Theta|^3)\). Hence, \(|\widetilde{\mu}_3| \sim 1\) by (4-11a) and \(\psi \sim 1\) by Proposition 3.1. After this rearrangement, we divide (4-7) by \(\widetilde{\mu}_3\) and set
\[
\xi_2 := \mu_2 / \widetilde{\mu}_3, \quad \xi_1 := (-\beta \langle \hat{B}, B \rangle + \mathcal{O}(n^{20/K} |\Theta|(|\langle E_-, B \rangle|^2 + |\langle \hat{B}, A[B, B_\star] \rangle|^2))) / \widetilde{\mu}_3.
\]

Since \(|\widetilde{\mu}_3| \sim 1\), we conclude \(|\xi_2| \sim |\mu_2| \sim \rho\) due to (4-11b) and \(\psi \sim 1\). For the scaling relation of \(\xi_1\), we note that \(|\beta \langle \hat{B}, B \rangle| \sim \rho^2 + \eta / \rho\) by (3-2) and \(|\langle \hat{B}, B \rangle| \sim 1\) from Proposition 3.1. Moreover, from (4-15) and (3-8), we obtain \(n^{20/K} |\Theta|(|\langle E_-, B \rangle|^2 + |\langle \hat{B}, A[B, B_\star] \rangle|^2)) \lesssim n^{-10/K} (\rho^2 + \eta^2 / \rho^2)\). Hence, \(|\xi_1| \sim \rho^2 + \eta / \rho\). This completes the proof of (4-3), the scaling relations (4-4) and the bound on \(\|P[\pi]_\| \) in (4-5).

Finally, the expansion of \(G - M\) in (4-1) and the error estimate in (4-2) follow from (A-5) in Lemma A.1 together with (4-16), \(\|MD\| \lesssim n^{1/2K} \|D\|_\| \), (3-8) and \(R_1 = M^* (B^{-1})^* [E_-]\). This completes the proof of Proposition 4.1.

**Proof of (4-6b).** This proof is motivated by the proof of the bound on \(\|P[\pi]_\| \) in [Alt et al. 2020b, Lemma 3.4]. We compute
\[
\langle \hat{B}, R \rangle = \frac{1}{2n} \text{Tr}(\hat{B}^* R) = \frac{1}{2n} \sum_{a=1}^{2n} \langle \hat{B} e_a, R e_a \rangle = \frac{1}{2n} \sum_{a=1}^{2n} R \hat{B}^* e_a.
\]

Since \(\|\hat{B}\| \lesssim 1\) by Proposition 3.1, we have \(\|\hat{B}^* e_a\| \lesssim 1\). Thus, we obtain \(|R \hat{B}^* e_a| |e_a| \leq \|R\|_0 \|\hat{B}^* e_a\| \lesssim \|R\|_\|.\)

Using this in (4-18) completes the proof of the bound on \(\langle \hat{B}, R \rangle\) in (4-6b).

The bound for \(\langle \hat{B}^* e_a, R \rangle\) is proven in the same way. The proof of the estimate on \(\langle E_-, R \rangle\) is simpler. \(\square\)

5. **Local law for \(H_z\)**

The main result of this section, Theorem 5.2, is a precise expansion of the resolvent of \(H_z\) at \(i\eta\) when \(\eta > 0\) is sufficiently small and the modulus of \(z \in \mathbb{C}\) is close to 1. We recall that we assume \(\varrho(\mathcal{S}) = 1\) (see (3-9) and the associated explanations). For the formulation of Theorem 5.2 as well as the subsequent statements and arguments, we use the following notion for high probability estimates.

**Definition 5.1** (stochastic domination). Let \(\Phi = (\Phi^{(n)}), \Psi = (\Psi^{(n)})\) be two sequences of nonnegative random variables. We say \(\Phi\) is *stochastically dominated* by \(\Psi\) and write \(\Phi \prec \Psi\) if, for all (small) \(\varepsilon > 0\) and (large) \(D > 0\), there is a constant \(C_{\varepsilon, D} > 0\) such that
\[
\mathbb{P}(\Phi^{(n)} > n^{\varepsilon} \Psi^{(n)}) \leq \frac{C_{\varepsilon, D}}{n^{D}}
\]
for all \(n \in \mathbb{N}\). If \(\Phi^{(n)}\) and \(\Psi^{(n)}\) depend on some parameter family \(U^{(n)}\) and (5-1) holds for all \(u \in U^{(n)}\) then we say that \(\Phi \prec \Psi\) uniformly for all \(u \in U^{(n)}\).

In the following, let \(\rho = \langle \text{Im} M \rangle / \pi\) (see (3-1)), \(H_z\) be defined as in (2-5) and \(G := (H_z - i\eta)^{-1}\). Moreover, \(M\) denotes the solution of the MDE (2-6). For each \(z \in \mathbb{C}\), we now introduce the fluctuation...
scale \( \eta_t = \eta_t(z) \) of eigenvalues of \( H_z \) around zero: We set

\[
\eta_t(z) := \begin{cases} 
(1 - |z|^2)^{-1/2} n^{-1} & \text{if } |z|^2 \leq 1 - n^{-1/2}, \\
\frac{n^{-3/4}}{1 - n^{-1/2}} & \text{if } 1 - n^{-1/2} < |z|^2 \leq 1 + n^{-1/2}, \\
\frac{(|z|^2 - 1)^{1/6} n^{-2/3}}{1 + n^{-1/2}} & \text{if } 1 + n^{-1/2} < |z|^2 \leq 2, \\
n^{-2/3} & \text{if } |z|^2 > 2.
\end{cases}
\] (5-2)

The fluctuation scale describes the typical eigenvalue spacing of \( H_z \) at zero (first two cases) and at the spectral edges of the eigenvalue density of \( H_z \) close to zero (last two cases). The definition of \( \eta_t \) in (5-2) is motivated by the definition of the fluctuation scale in [Erdős et al. 2020] and the scaling relations of \( \rho \) from (3-13). For \( |z| > 1 \), the eigenvalue density of \( H_z \) has a gap of size \( \Delta \sim (|z|^2 - 1)^{3/2} \) around zero, hence, (5-2) is analogous to [Erdős et al. 2020, Equation (2.7)]. If \( |z| \leq 1 \) then the eigenvalue density of \( H_z \) has a small local minimum of height \( \rho_0 \sim (1 - |z|^2)^{1/2} \) at zero. So (5-2) should be compared to [Erdős et al. 2020, Equation (A.8a)].

**Theorem 5.2** (local law for \( H_z \)). Let \( X \) satisfy (A1) and (A2). Then there is \( \tau_* \sim 1 \) such that, for each \( \xi > 0 \), the estimates

\[
|\langle x, (G - M) y \rangle| < \|x\| \|y\| \left( \sqrt{\frac{\rho}{n\eta}} + \frac{1}{n\eta} \right), \quad |\langle R(G - M) \rangle| < \|R\| \frac{n^{-\gamma/3}}{n\eta}
\] (5-3)

hold uniformly for all \( z \in \mathbb{C} \) satisfying \( |z| - 1| \leq \tau_* \), for all \( \eta \in [n^{\xi} \eta_t(z), n^{100}] \), for any deterministic vectors \( x, y \in \mathbb{C}^{2n} \) and deterministic matrix \( R \in \mathbb{C}^{2n \times 2n} \).

Moreover, outside the spectrum, for each \( \xi > 0 \) and \( \gamma > 0 \), we have the improved bound

\[
|\langle R(G - M) \rangle| < \|R\| \frac{n^{-\gamma/3}}{n\eta}
\] (5-4)

uniformly for all \( z \in \mathbb{C} \) and \( \eta \in \mathbb{R} \) satisfying \( |z|^2 \geq 1 + (n^{\gamma} \eta)^{2/3} \), \( |z| \leq 1 + \tau_* \), \( n^{\xi} \eta_t(z) \leq \eta \leq \tau_* \) and \( R \in \mathbb{C}^{2n \times 2n} \).

We stress that the spectral parameter of the resolvent \( G \) in the previous theorem and throughout the entire paper lies on the imaginary axis and is given by \( i\eta \). With additional efforts our method can be extended to spectral parameters near the imaginary axis, but the Hermitization formula (2-12) requires to understand the resolvent of \( H_z \) only on the imaginary axis, so we restrict ourselves to this case. After the proof of Theorem 5.2, we will establish the following corollary that will directly imply Corollary 2.4.

**Corollary 5.3** (isotropic eigenvector delocalization). Let \( \tau_* \sim 1 \) be chosen as in Theorem 5.2. Let \( x \in \mathbb{C}^{2n} \) be a fixed deterministic vector. If \( u \in \mathbb{C}^{2n} \) is contained in the kernel of \( H_z \) for some \( z \in \mathbb{C} \) satisfying \( |z| - 1| \leq \tau_* \), i.e., \( H_z u = 0 \) then

\[
|\langle x, u \rangle| < n^{-1/2} \|x\| \|u\|.
\]

We remark that the conclusion of Corollary 5.3 is also true if \( |z| - 1| \leq \tau_* \). This can easily be shown following the proof of [Alt et al. 2018, Theorem 5.2], where certain steps of the proof of the local law
from [Ajanki et al. 2019] have been used except that now analogous inputs from [Erdős et al. 2019b] are needed instead of [Ajanki et al. 2019].

Proof of Corollary 2.4. Let \( u \in \mathbb{C}^n \) be an eigenvector of \( X \), i.e., \( Xu = \zeta u \) for some \( \zeta \in \mathbb{C} \). If \( |\zeta| \leq 1 - \tau_* \) then the claim follows from [Alt et al. 2018, Corollary 2.6]. Otherwise, we can assume that \( ||\zeta| - 1| \leq \tau_* \) by [Alt et al. 2018, Theorem 2.5 (ii)]. We set \( u := (0, u')^T \) and obtain \( H_u u = 0 \). Hence, we choose \( x = e_i \) in Corollary 5.3 and obtain a bound on \( |u_i| \). Finally, taking the maximum over \( i \in [n] \) and a simple union bound complete the proof of Corollary 2.4. \( \square \)

Remark 5.4 (choice of \( S \) in MDE). We warn the reader that in this paper the definition of the self-energy operator \( S \) given in (2-7) coincides with the definition in [Alt et al. 2018, Equation (3.3)]. However, this convention differs from the more canonical choice, \( R \mapsto E[(H - \mathbb{E}H)R(H - \mathbb{E}H)] \), typically used for a Hermitian random matrix \( H \) in several other works (e.g., [Ajanki et al. 2019; Erdős et al. 2019b; Alt et al. 2020b]). The present choice of \( S \) substantially simplifies the analysis of the associated MDE. As a price for this, we will need a simple adjustment when estimating the error term \( D \) in the perturbed Dyson equation, (2-8), since the convenient estimate on \( D \) builds upon the canonical choice of \( S \). Nevertheless, as Proposition 5.5 shows, the very same estimates on \( D \) as for the canonical choice [Erdős et al. 2020] can be obtained for the current choice.

We will establish Theorem 5.2 in Section 5A. The proof will consist of a bootstrapping argument using the stability properties of the MDE in the previous sections, in particular, Proposition 4.1, and the following bounds on the error term \( D \) in the perturbed MDE for \( G \), (2-8). To formulate these bounds, we now introduce some norms for random matrices and a spectral domain. For \( p \geq 1 \), a scalar-valued random variable \( Z \) and a random matrices \( Y \in \mathbb{C}^{2n \times 2n} \), we define the \( p \)-th-moment norms

\[
\|Z\|_p := (E|Z|^p)^{1/p}, \quad \|Y\|_p := \sup_{x,y} \frac{\|\langle x, y \rangle\|_p}{\|x\|\|y\|}.
\]

For \( \zeta > 0 \), we introduce the spectral domain

\[
\mathbb{D}_\zeta := \{(z, \eta) \in \mathbb{C} \times \mathbb{R} : n^{-1+\zeta} \leq \eta \leq \tau_* \}, \quad ||z| - 1| \leq \tau_* \},
\]

where \( \tau_* \sim 1 \) is chosen such that (3-13) implies \( \rho + \eta/\rho \leq \rho_* \) for all \( (z, \eta) \in \mathbb{D}_\zeta \) with \( \rho_* \) from Proposition 4.1.

In the following, we will work with several quantities that will depend on \( \epsilon \) and \( p \). In order to simplify the notation, we will use the following notation. Let \( f(\epsilon, p, a) \) and \( g(\epsilon, p, a) \) be two quantities that depend on \( \epsilon \) and \( p \) as well as a set \( a \) of other possible parameters. We write \( f \leq_{\epsilon, p} g \) if there is a constant \( C(\epsilon, p) \) such that \( f(\epsilon, p, a) \leq C(\epsilon, p)g(\epsilon, p, a) \) for all \( \epsilon, p \) and \( a \).

Proposition 5.5. Let \( D \) be the error matrix from (2-8). Under the assumptions of Theorem 5.2, there is a constant \( C > 0 \) such that for any \( p \geq 1, \ \epsilon > 0, \ (z, \eta) \in \mathbb{D}_0, \) and any deterministic \( x, y \in \mathbb{C}^n \), we have the moment bounds

\[
\|\langle x, Dy \rangle\|_p \leq_{\epsilon, p} \|x\|\|y\|n^\epsilon \psi'_q(1 + \|G\|_q)^C \left(1 + \frac{\|G\|_q}{\sqrt{n}}\right)^{Cp}, \quad (5-5a)
\]

\[
\|\langle RD \rangle\|_p \leq_{\epsilon, p} \|R\|n^\epsilon \psi'_q(1 + \|G\|_q)^C \left(1 + \frac{\|G\|_q}{\sqrt{n}}\right)^{Cp}. \quad (5-5b)
\]
Moreover, if $R \in \mathcal{M}_o$ then we have the improved estimate

$$
\| (RD)\|_{p} \leq \varepsilon, p \| R \| n^{\varepsilon} \sigma_q [\psi + \psi_q']^2 (1 + \| G \|_q)^{\varepsilon} \left(1 + \frac{\| G \|_q}{\sqrt{n}} \right)^{Cp}. 
$$

Here, we used the $z$-dependent control parameters

$$
\psi := \sqrt{\frac{p}{n\eta}}, \quad \psi' := \sqrt{\frac{\| \text{Im} \ G \|_q}{n\eta}}, \quad \psi'' := \| G - M \|_q, \quad \sigma_q := \rho + \psi + \sqrt{\frac{\eta}{\rho}} + \psi' + \psi''
$$

with $q := Cp^3/\varepsilon$.

**Remark 5.6.** This proposition is the exact counterpart of the cusp fluctuation averaging in [Erdős et al. 2020, Proposition 4.12] with $\sigma = 0$, hence the definition of $\sigma_q$ does not contain $\sigma$. Notice that $\sigma = 0$ in our case following from the fact that the spectral parameter $i\eta$ lies on the imaginary axis to which the spectrum is symmetric.

We remark that $\psi$ in Proposition 5.5 is different from the $\psi$ defined in Proposition 3.1. This should not lead to any confusion since the latter notation is used only in Section 3 and 4 while the former is used exclusively in Proposition 5.5 and Section 5B. We prefer to stick to these notations for compatibility with the publications [Ajanki et al. 2017b; Alt et al. 2020a; Erdős et al. 2020]. We will show Proposition 5.5 in Section 5B.

**5A. Proof of Theorem 5.2.** This subsection is devoted to the proof of Theorem 5.2. To that end, we follow the arguments from [Erdős et al. 2020, Sections 3.2, 3.3], where the local law for a general Hermitian matrix close to a cusp regime was deduced from estimates on $D$ as provided in Proposition 5.5. We will present the main steps of the proof, focusing on the differences, but for arguments that require only simple (mostly notational) adjustments we will refer the reader to [Erdős et al. 2020]. When comparing to [Erdős et al. 2020], the reader should think of the following cases described in the notation of Equation (3.7b) there. For $|z| \leq 1$, the eigenvalue density of $H_z$ has a local minimum of size $\rho(\tau_0) \sim (1 - |z|^2)^{1/2}$ at $\tau_0 = 0$ and $\omega = 0$. For $|z| > 1$, the spectrum of $H_z$ has a symmetric gap of size $\Delta \sim (|z|^2 - 1)^{3/2}$ around zero and we study the resolvent of $H_z$ at the middle of this gap, $|\omega| = \Delta/2$. For a random matrix $Y \in \mathbb{C}^{2n \times 2n}$ and a deterministic control parameter $\Lambda = \Lambda(z)$, we define the notation $|Y| \prec \Lambda$ and $|Y|_{av} \prec \Lambda$ as follows:

$$
|Y| \prec \Lambda \iff |Y_{xy}| \prec \Lambda \|x\| \|y\| \quad \text{uniformly for all } x, y \in \mathbb{C}^{2n},
$$

$$
|Y|_{av} \prec \Lambda \iff \| (RY) \| \prec \Lambda \|R\| \quad \text{uniformly for all } R \in \mathbb{C}^{2n \times 2n}.
$$

We recall that by definition $Y_{xy} = \langle x, Y y \rangle$ for $x, y \in \mathbb{C}^{2n}$. The following lemma relates this notion of high probability bounds to the high moments estimates introduced above. We leave the simple adjustments of the proof of [Alt et al. 2020b, Lemma 3.7] to the reader.

**Lemma 5.7.** Let $Y$ be a random matrix in $\mathbb{C}^{2n \times 2n}$, $\Phi$ a deterministic control parameter such that $\Phi \geq n^{-C}$ and $\|Y\| \leq n^C$ for some $C > 0$. Let $K \in \mathbb{N}$ be fixed. Then we have

$$
\| Y \|_{*}^{K, x, y} \prec \Phi \quad \text{uniformly for } x, y \in \mathbb{C}^{2n} \iff |Y| \prec \Phi \iff \| Y \|_{p} \leq \varepsilon, p \| n^\varepsilon \Phi \quad \text{for all } \varepsilon > 0, p \geq 1.
$$
The next lemma adapts Proposition 4.1 to the random matrix setup with the help of Proposition 5.5. The lemma is the analog of [Erdős et al. 2020, Lemma 3.8] in our setup.

**Lemma 5.8.** Let \( \zeta, c > 0 \) be fixed and sufficiently small. We assume that \(|G - M| < \Lambda, |\text{Im}(G - M)| < \Xi \) and \(|\Theta| < \theta \) at some fixed \((z, \eta) \in \mathbb{D}_\zeta\) for some deterministic control parameters \(\Lambda, \Xi\) and \(\theta\) such that \(\Lambda + \Xi + \theta \lesssim n^{-c}\). Then, for any sufficiently small \(\delta > 0\), the estimates

\[
|\Theta^3 + \xi_2 \Theta^2 + \xi_1 \Theta| < n^{2\delta} \left( \rho + \frac{\eta^{1/2}}{\rho^{1/2}} + \left( \frac{\rho + \Xi}{n\eta} \right)^{1/2} \right) \frac{\rho + \Xi}{n\eta} + n^{-\delta} \theta^3
\]  

(5-6)

and

\[
|G - M| < \theta + \sqrt{\frac{\rho + \Xi}{n\eta}}, \quad |G - M|_{\text{av}} < \theta + \frac{\rho + \Xi}{n\eta}\]

(5-7)

hold, where \(\xi_2\) and \(\xi_1\) are chosen as in Proposition 4.1 and \(\Theta = \langle \hat{B}, G - M \rangle / \langle \hat{B}, B \rangle\).

Moreover, for fixed \(z\), \(\Theta\) is a continuous function of \(\eta\) as long as \((z, \eta) \in \mathbb{D}_\zeta\).

**Proof.** Owing to Lemma 5.7 and \(|G| < \|M\| + \Lambda \lesssim 1\), the high-moment bounds in (5-5a) and (5-5b) imply

\[
|D| < \sqrt{\frac{\rho + \Xi}{n\eta}}, \quad |D|_{\text{av}} < \frac{\rho + \Xi}{n\eta}.
\]

(5-8)

We conclude that the assumption on \(\|G - M\|_\ast + |D|_\ast\) in Proposition 4.1 is satisfied for sufficiently large \(K\) depending on \(c\) and \(\epsilon\) in the definition of \(<\) in Definition 5.1. What remains to ensure the applicability of Proposition 4.1 is checking \(\langle E_-, G \rangle = 0\). In fact, we now prove that, for each \(z \in \mathbb{C}\) and \(\eta > 0\), the resolvent \(G = (H_z - i\eta)^{-1}\) satisfies

\[
\langle E_-, G \rangle = 0.
\]

(5-9)

For the proof of (5-9), we denote by \(G_{11}, G_{22} \in \mathbb{C}^{n \times n}\) the upper-left and lower-right \(n \times n\)-minor of the resolvent \(G = (H_z - i\eta)^{-1} \in \mathbb{C}^{2n \times 2n}\). Then the block structure of \(H_z\) from (2-5) yields

\[
G_{11} = \frac{i\eta}{(X - z)(X - z)^\ast + \eta^2}, \quad G_{22} = \frac{i\eta}{(X - z)^\ast(X - z) + \eta^2}.
\]

Since \((X - z)(X - z)^\ast\) and \((X - z)^\ast(X - z)\) have the same eigenvalues we obtain \(2n\langle E_-, G \rangle = \text{Tr} \ G_{11} - \text{Tr} \ G_{22} = 0\). This shows (5-9) and, thus, ensures the applicability of Proposition 4.1.

The first bound in (5-8), the bounds on \(\mathcal{B}^{-1}Q\) and \(MR\) in (4-6) and Lemma 5.7 yield

\[
|\mathcal{B}^{-1}Q[MD]| < \frac{\rho + \Xi}{n\eta}.
\]

(5-10)

by choosing \(K\) sufficiently large to absorb various \(n^{1/K}\)-factors into \(<\). Similarly, we use (4-6), (5-8), the assumption \(|\Theta| < \theta\) and Lemma 5.7 to estimate the other terms in (4-1) and (4-2) and deduce (5-7).

What remains is estimating the right-hand side of (4-5) to obtain (5-6). Incorporating the \(n^{1/K}\) factors into \(<\), we see \(\|D\|_\ast^3, |\langle R_1, D \rangle|^{3/2}\) and \(|\langle R_2, D \rangle|^{3/2}\) are dominated by the right-hand side of (5-6) due to (5-5a) and Lemma 5.7. Recall \(M = \text{Re} \ M + \mathcal{O}(\rho)\) with \(\text{Re} \ M \in \mathcal{M}_0\) and \(\hat{B}^\ast = -\rho^{-1} \text{Im} \ M^{-1} + \mathcal{O}(\rho + \eta/\rho)\)
(see (3-3c)) with \(-\rho^{-1}\) Im \(M^{-1}\) \(\sim\) 1 and \(-\rho^{-1}\) Im \(M^{-1}\) \(\in\) \(\mathcal{M}_d\). Therefore, \(M\) is almost off-diagonal while \(\hat{B}^*\) is almost diagonal, so we find \(B_1 \in \mathcal{M}_e, B_2 \in \mathbb{C}^{2n \times 2n}\) such that \(\hat{B}^* M = B_1 + B_2\) and \(\|B_1\| \lesssim 1, \|B_2\| \lesssim \rho + \eta / \rho\). Hence, (5-5b) and (5-5c) imply

\[
|\langle \hat{B}, M D \rangle| \leq \left( \rho + \frac{n\eta}{\rho^{1/2}} + \delta \left( \frac{\rho + \eta}{n\eta} \right)^{1/2} \right) \frac{\rho + \eta}{n\eta} + n^{-\delta} \delta^3,
\]

where we used the bound on \(|G - M|\) from (5-7) in the second step and Young’s inequality in the last step.

We now conclude the proof of (5-6) by showing that

\[
|\langle \hat{B}, M (SB^{-1} Q[M D]) B^{-1} Q[M D] \rangle| \leq \left( \frac{\rho + \eta}{n\eta} \right)^{3/2}.
\]

Since \(\text{ran} S \subset \mathcal{M}_d\) and \(|\langle \hat{B}^* M \rangle| \leq \|\hat{B}^* M\| \lesssim 1\), we have

\[
|\langle \hat{B}, M (SB^{-1} Q[M D]) B^{-1} Q[M D] \rangle| = \frac{1}{2n} \left| \sum_{a \in [2n]} (\hat{B}^* M)_{aa} S[A]_{aa} A_{aa} \right| \lesssim \max_{a \in [2n]} |S[A]_{aa}| \max_{a \in [2n]} |A_{aa}|,
\]

where \(A := B^{-1} Q[M D]\). Owing to the second bound in (5-8), \(\|M^* (B^*)^{-1} Q^* S[e_a e_a^*]\| \lesssim 1\), and the definition of \(|\cdot|_{av}\) we obtain

\[
|S[A]_{aa}| = |\langle e_a, S[A] e_a \rangle| = |\langle S[e_a e_a^*], B^{-1} Q[M D] \rangle| = |\langle M^* (B^*)^{-1} Q^* S[e_a e_a^*], D \rangle| \lesssim \frac{\rho + \eta}{n\eta}.
\]

Therefore, (5-11) follows by using (5-10) and (5-13) in (5-12). This completes the proof of (5-6).

Finally, we note that \(\Theta\) is a continuous function of \(\eta\) as \(\hat{B}, B, G\) and \(M\) are continuous with respect to \(\eta\). This completes the proof of Lemma 5.8. \(\square\)

We now introduce \(\tilde{\xi}_2\) and \(\tilde{\xi}_1\) which will turn out to be comparable versions of the coefficients \(\xi_2\) and \(\xi_1\), respectively, (see (4-4) and Lemma 5.9(i)). Moreover, they depend explicitly and monotonically on \(\eta\) which will be important for our arguments. We define

\[
\tilde{\xi}_2 := |1 - |z|^2|^{1/2} + \eta^{1/3}, \quad \tilde{\xi}_1 := (\tilde{\xi}_2)^2.
\]

These definitions are chosen in analogy to [Erdős et al. 2020, Equation (3.7c)], where in the first case we chose \(|\omega| \sim \Delta \sim (|z|^2 - 1)^{3/2}\) and, in the second case, \(\rho(\tau_0) \sim (1 - |z|^2)^{1/2}\) and \(\omega = 0\).

**Lemma 5.9** (properties of \(\tilde{\xi}_2\) and \(\tilde{\xi}_1\)).

(i) For all \(z \in D_r(0)\) and \(\eta \in (0, 1]\), we have \(\rho^2 + \eta / \rho \sim \tilde{\xi}_1\). For any \(\eta \in (0, 1]\), we have \(\tilde{\xi}_2 \sim \rho\) if \(z \in \mathbb{C}\) satisfies \(|z| \leq 1\) and \(\tilde{\xi}_2 \gtrsim \rho\) if \(z \in D_r(0) \setminus D_1(0)\).

(ii) Uniformly for all \(z \in D_r(0)\) and \(\eta \geq \eta_\ell\), we have

\[
\tilde{\xi}_2 \gtrsim \frac{1}{n\eta} + \left( \frac{\rho}{n\eta} \right)^{1/2}, \quad \tilde{\xi}_1 \gtrsim \tilde{\xi}_2 \left( \rho + \frac{1}{n\eta} \right).
\]
Theorem 5.2, are identical to the proofs of [Erdős et al. 2020, Equations (3.28) and (3.30)]. Therefore, where \( \tilde{\eta} \) is proof. The scaling relations in (i) follow easily from the scaling relations for \( \rho \) in (3-13) by distinguishing the regimes \( |z| \leq 1 \) and \( |z| > 1 \).

The first bound in (ii) follows once \( \tilde{\xi}_2 \gtrsim 1/(n\eta) \) and \( (\tilde{\xi}_2)^2 \gtrsim \rho/(n\eta) \) are proven. For \( |z|^2 \leq 1 - n^{-1/2} \), we have \( (1 - |z|^2)^{1/2} \gtrsim 1/(n\eta) \) if \( \eta \geq \eta_l(z) \). If \( |z|^2 > 1 - n^{-1/2} \) then \( \eta_l(z) \gtrsim n^{-3/4} \) by definition and, hence, \( \eta^{1/3} \gtrsim 1/(n\eta) \). This shows \( \tilde{\xi}_2 \gtrsim 1/(n\eta) \) in all regimes. If \( |z| \leq 1 \) then \( \tilde{\xi}_2 \sim \rho \) by (i) and \( \rho^2 \gtrsim \rho/(n\eta) \) is easily verified due to (3-13). For \( |z| > 1 \), \( (\tilde{\xi}_2)^2 \gtrsim \rho/(n\eta) \) is equivalent to \( (|z|^2 - 1)^{1/2} + \eta^{1/3} \gtrsim n^{-1/4} \) which follows directly from \( \eta_l(z) \gtrsim n^{-3/4} \) in this regime. This shows the first bound in (ii).

We note that, owing to \( \tilde{\xi}_1 = (\tilde{\xi}_2)^2 \), the second bound in (ii) is equivalent to \( \tilde{\xi}_2 \gtrsim \rho + 1/(n\eta) \). But we know \( \tilde{\xi}_2 \gtrsim \rho \) from (i). This completes the proof of Lemma 5.9.

Proof of Theorem 5.2. We will only consider the bounds in (5-3) for \( \eta \leq \tau_* \) since the opposite regime is covered by [Erdős et al. 2019b, Theorem 2.1] due to \( \rho \sim \eta^{-1} \) for \( \eta \geq \tau_* \) by (3-13) and [Alt et al. 2018, Equation (3.9)]. The bounds (5-3) and (5-4) are the analogs of (3.28) and (3.30) in [Erdős et al. 2020], respectively. Given the preparations presented above, the proofs of (5-3) and (5-4), and thus, that of Theorem 5.2, are identical to the proofs of [Erdős et al. 2020, Equations (3.28) and (3.30)]. Therefore, we only describe the main strategy here and explain the applicability of certain inputs.

The proof of Theorem 5.2 starts with the following (isotropic) rough bound on \( G - M \).

Lemma 5.10 (rough bound). For any \( \zeta > 0 \), there exists a constant \( c > 0 \) such that the rough bounds

\[
|G - M| < n^{-c}
\]

holds on the spectral domain \( \mathbb{D}_\zeta \).

Proof. The proof of Lemma 5.10 is identical to the one of [Erdős et al. 2020, Lemma 3.9]. We explain the main idea. From [Erdős et al. 2019b, Theorem 2.1], an initial bound on \( |G - M| \) at \( \eta = \tau_* \) is deduced. We remark that that theorem is also applicable in our setup. Using the monotonicity of the map \( \eta \mapsto \eta\|G(z, \eta)\|_{\rho} \) (which is shown in the same way as in [Erdős et al. 2019b, Equation (5.11)]), the bootstrapping result [Erdős et al. 2020, Lemma 3.10] for cubic inequalities and Lemma 5.8, this initial bound is strengthened and propagated down to all (small) values of \( \eta \) in \( \mathbb{D}_\zeta \). Moreover, the assumptions in (ii) of [Erdős et al. 2020, Lemma 3.10] are easily checked by using the definitions of \( \tilde{\xi}_2 \) and \( \tilde{\xi}_1 \) in (5-14) and Lemma 5.9(i). This completes the proof of Lemma 5.10.

As in the proof of [Erdős et al. 2020, Theorem 2.5], we now record an intermediate local law in the next proposition. It is obtained by following the proof of [Erdős et al. 2020, Proposition 3.11] and employing Lemma 5.8 instead of [Erdős et al. 2020, Equation 3.8].

Proposition 5.11 (local law uniformly for \( \eta \geq n^{-1+\zeta} \)). Let \( \zeta > 0 \). On \( \mathbb{D}_\zeta \), we have the bounds

\[
|G - M| < \theta_* + \sqrt{\frac{\rho}{n\eta}} + \frac{1}{n\eta}, \quad |G - M|_{\text{av}} < \theta_* + \frac{\rho}{n\eta} + \frac{1}{(n\eta)^2},
\]

where \( \theta_* \) is defined through

\[
\theta_* := \min\{d_*^{1/3}, d_*^{1/2}/\tilde{\xi}_2^{1/2}, d_*/\tilde{\xi}_1\}, \quad d_* := \tilde{\xi}_2 \left( \frac{\tilde{\rho}}{n\eta} + \frac{1}{(n\eta)^2} \right) + \frac{1}{(n\eta)^3} + \left( \frac{\tilde{\rho}}{n\eta} \right)^{3/2}
\]

and \( \tilde{\rho} \) denotes the right-hand side of (3-13), i.e., \( \rho \sim \tilde{\rho} \).
Now, we follow the proof of [Erdős et al. 2020, Equation (3.28)] and use Lemma 5.9(ii) instead of [Erdős et al. 2020, Lemma 3.3] to obtain both bounds in (5-3).

We now strengthen (5-3) to (5-4) outside of the spectrum. If \(|z|^2 > 1 + (n^\gamma \eta)^{2/3}\) then \(\theta_\ast \leq d_\ast/\tilde{\xi}_1\), Lemma 5.9(ii), \((\tilde{\xi}_2)^2 = \tilde{\xi}_1\) and (3-13) imply

\[
\theta_\ast + \frac{\tilde{\rho}}{n^n} + \frac{1}{(n^n)^2} \lesssim \frac{\tilde{\xi}_2}{\xi_1} \left( \frac{\rho}{n^n} + \frac{1}{(n^n)^2} \right) \lesssim \frac{1}{(|z|^2 - 1)^{1/2}} \left( \frac{\eta}{|z|^2 - 1} + \frac{1}{n^n} \right) \frac{1}{n^n} \lesssim \frac{n^{-\gamma/3}}{n^n} \tag{5-16}
\]

for \(\eta \geq \eta_t(z) \gtrsim n^{-3/4}\) (see [Erdős et al. 2020, Equation (3.30)]). Applying (5-16) to the second bound in (5-15) yields the improved bound (5-4) and, thus, completes the proof of Theorem 5.2. \(\square\)

From Proposition 5.11, we now conclude Corollary 5.3.

**Proof of Corollary 5.3.** The bound on \(|G - M|\) in (5-15) and the argument from [Ajanki et al. 2017b, Corollary 1.14] directly imply Corollary 5.3. \(\square\)

We conclude this subsection by collecting two simple consequences of the previous results and [Alt et al. 2018]. For the remainder of Section 5A, \(\tau > 0\) will be a parameter bounding the spectral parameter \(z\) from above. The implicit constant in \langle-estimates is allowed to depend on \(\tau\).

**Corollary 5.12.** Let \(X\) satisfy (A1) and (A2). Let \(\zeta > 0\). Then we have

\[|\Im \langle G(z, \eta) - M(z, \eta) \rangle| \ll \frac{1}{n^n}\]

uniformly for all \(z \in D_\tau(0)\) and all \(\eta \in [n^\xi \eta_t(z), n^{100}]\).

**Proof.** The corollary is a direct consequence of the local law near the edge, (5-3), the local law away from the edge, [Alt et al. 2018, Equation (5.4)], and the definition of \(\eta_t\) in (5-2). \(\square\)

We denote the eigenvalues of \(H_z\) by \(\lambda_1(z), \ldots, \lambda_{2n}(z)\). The following lemma provides a simple bound on the number of eigenvalues of \(H_z\) in the interval \([-\eta, \eta]\). It is an extension of [Alt et al. 2018, Equation (5.22)] to the edge regime.

**Lemma 5.13** (eigenvalues of \(H_z\) near 0). Let \(X\) satisfy (A1) and (A2). Let \(\zeta > 0\). Then we have

\[|\{i \in [2n] : |\lambda_i(z)| \leq \eta\}| \ll n^n \rho + 1\]

uniformly for all \(z \in D_\tau(0)\) and all \(\eta \in [n^\xi \eta_t(z), n^{100}]\).

**Proof.** We define \(\Lambda_\eta := \{i \in [2n] : |\lambda_i(z)| \leq \eta\}\). For \(\eta \geq n^\xi \eta_t\), we obtain from Corollary 5.12 and \(\rho = \langle \Im M \rangle / \rho\) that

\[
\frac{|\Lambda_\eta|}{2\eta} \leq \sum_{i \in \Lambda_\eta} \frac{\eta}{\eta^2 + |\lambda_i(z)|^2} \leq 2n \Im \langle G(z, \eta) \rangle \ll \eta \left( \rho + \frac{1}{n^n} \right) \lesssim n^n \rho + \frac{1}{\eta}.
\]

This completes the proof of Lemma 5.13. \(\square\)
5B. Cusp fluctuation averaging — Proof of Proposition 5.5. In this subsection we will provide the proof of Proposition 5.5. Since the self-consistent density of states of the Hermitian matrix $H = H_z$ develops a cusp singularity at the origin in the regime $|z| - 1 \ll 1$, this result is analogous to [Erdős et al. 2020, Theorem 3.7], which provides an improved bound for specific averages of the random error matrix in the MDE. This improved bound is called cusp fluctuation averaging and takes the form (5-5c) in the current work. In [Erdős et al. 2020] the expectation $E H$ was diagonal and the self-energy operator was assumed to satisfy the flatness condition [Erdős et al. 2020, Equation (3.6)]. Both conditions are violated in our current setup and thus the result from [Erdős et al. 2020] is not directly applicable. However, with minor modifications the proof of [Erdős et al. 2020, Theorem 3.7] can be adjusted to yield Proposition 5.5.

In fact, the cancellation that underlies the cusp fluctuation averaging (5-5c) is simpler and more robust for $H$ with the bipartite structure (2-5). An indication of this fact is that (5-5c) holds for any $R \in M_o$ while the corresponding bound in [Erdős et al. 2020, Theorem 3.7] only holds when the error matrix is averaged against a specific vector that depends on $M$, the solution to the MDE.

For the purpose of being able to follow the strategy from [Erdős et al. 2020] very closely we define

$$W := H_z - E H_z, \quad \tilde{S}[R] := E W R W, \quad \tilde{D} := W G + \tilde{S}[G] G.$$  

The modified self-energy operator $\tilde{S}$ is introduced to match the convention of [Erdős et al. 2020] (see Remark 5.4). This differs from the self-energy operator $S$ defined in this paper (see (2-7) and Remark 5.4), which is the block diagonal part of $\tilde{S}$, consisting of the blocks $E X R_{22} X^*$ and $E X^* R_{11} X$, both themselves being diagonal since $X$ has independent entries. The difference between the two versions of the self-energy is

$$T[R] := (\tilde{S} - S)[R] = \begin{pmatrix} 0 & E X R_{21} X^* \\ E X^* R_{12} X^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & T \otimes R'_{21} \\ T \otimes R'_{12} & 0 \end{pmatrix} = T \otimes R'.$$  

where $\otimes$ indicates the entrywise (Hadamard) matrix product and we introduced the matrices $T = (t_{ij})_{i,j=1}^{2n} \in \mathbb{C}^{n \times n}$ and $T = (w_{ij})_{i,j=1}^{2n} \in \mathbb{C}^{2n \times 2n}$ with entries $t_{ij} = E w_{ij}^2$ being the second moments of the entries of $W = (w_{ij})_{i,j=1}^{2n}$ from (5-17). The modified error matrix $\tilde{D}$ was labelled $D$ in [Erdős et al. 2020] and is the natural error when considering the MDE with self-energy $\tilde{S}$ and corresponding solution $\tilde{M}$. In the current work we will stick to the convention from [Alt et al. 2018] with respect to the definition of $S$, $D$, $M$ in order to keep the MDE and its solution simple and thus we indicate the corresponding quantities $\tilde{S}$, $\tilde{D}$, $\tilde{M}$ from [Erdős et al. 2020] by a tilde. Another notational difference is that the dimension of $H$ was denoted by $N$ in [Erdős et al. 2020], that corresponds to $N = 2n$ in this paper.

We start the proof of Proposition 5.5 by showing that it suffices to establish its statement for $D$ replaced by $\tilde{D}$. Let us therefore assume the following proposition whose proof is the main content of this subsection.

**Proposition 5.14.** The statement of Proposition 5.5 holds with $D$ replaced by $\tilde{D}$, i.e., under the same assumptions and with the same constants we have the estimates

$$\| (x, \tilde{D} y) \|_p \leq \epsilon, \quad \| x \|_p \| x \|_{p'} \| y \|_{p'} (1 + \| G \|_q) C \left( 1 + \frac{\| G \|_q}{\sqrt{n}} \right)^{C p},$$  

$$\| (R \tilde{D}) \|_p \leq \epsilon, \quad \| R \| n^f [\psi^q]' \| n^f \| G \|_q \| C \left( 1 + \frac{\| G \|_q}{\sqrt{n}} \right)^{C p},$$  

where $C$ is the constant that appears in the definition of the self-energy $\tilde{S}$.


and for $R \in \mathcal{M}_o$ the improved estimate

$$
\| (R \tilde{D}) \|_p \leq_{\varepsilon, p} \| R \| n^\varepsilon \sigma_q [\psi + \psi_q']^2 (1 + \| G \|_q)^C \left( 1 + \frac{\| G \|_q}{\sqrt{n}} \right)^{C_p}. 
$$

(5-19c)

Furthermore, with $T$ from (5-18) and for an arbitrary deterministic matrix $R$,

$$
\| (RT[G]G) \|_p \leq_{\varepsilon, p} \| R \| n^\varepsilon \sigma_q [\psi + \psi_q']^2 (1 + \| G \|_q)^C \left( 1 + \frac{\| G \|_q}{\sqrt{n}} \right)^{C_p}. 
$$

(5-19d)

Given the bounds from Proposition 5.14 it suffices to estimate the difference $\tilde{D} - D = T[G]G$. First (5-5a) follows from (5-19a) because for normalized vectors $x, y \in \mathbb{C}^{2n}$ we have

$$
\| \langle x, T[G]G y \rangle \|_p = \| \sum_i G_{iv} G_{iy} \|_{1/p} \lesssim n^\varepsilon \| G \|_{1/\varepsilon} \left( \frac{\| \text{Im } G \|_p}{n \eta} \right)^{1/2} \leq n^\varepsilon \| G \|_q \psi_q',
$$

where in the equality we introduced the vectors $v_i = (t_i, \bar{x}_j)$ with $\| v_i \|_{\infty} \lesssim 1/n$ and in the first inequality we used the Ward identity in the second factor after applying the general bound $\| \sum_i X_i Y_i \|_p \leq n^\varepsilon \sup_i \| X_i \|_{1/\varepsilon} \| \sum_i Y_i \|_{2p}$ for any random variables $(X_i, Y_i)_{i=1}^{2n}$ and $\varepsilon \in (0, 1/2p)$. Then (5-19b) implies (5-5b) by taking the $\| \cdot \|_{p}$-norm on both sides of

$$
|\langle RT[G]G \rangle| \lesssim \frac{\| R \| \langle G^* G \rangle}{n} = \| R \| \frac{\langle \text{Im } G \rangle}{n \eta},
$$

where we used $\| T \|_{2 \rightarrow 2} \lesssim 1/n$. Finally, (5-5c) immediately follows from (5-19c) and (5-19d).

The remainder of this subsection is dedicated to proving Proposition 5.14. To avoid repetition we will only point out the necessary modifications to the proof of [Erdős et al. 2020, Theorem 3.7].

The proof of (5-19a) and (5-19b) is exactly the same as the proof of [Erdős et al. 2020, Equations (3.11a) and (3.11b)], which, in turn directly follow from [Erdős et al. 2019b, Theorem 4.1]. Note that this latter theorem does not assume flatness, i.e., a lower bound on $\tilde{S}$, hence it is directly applicable to our $H$ as well. We also remark that the proof of (5-19a) and (5-19b) requires only double index graphs (in the sense of [Erdős et al. 2019b]) and their estimates rely only on the power counting of Wardable edges. A self-contained summary of the necessary concepts can be found in [Erdős et al. 2020, Section 4.1–4.7], where the quite involved cumulant expansion from [Erdős et al. 2019b], originally designed to handle any correlation, is translated into the much simpler independent setup. This summary in [Erdős et al. 2020] has the advantage that it also introduces the single index graphs as a preparation for the more involved $\sigma$-cell estimates needed for the cusp fluctuation averaging.

In the rest of the proof we focus on (5-19c) and (5-19d) and we assume that the reader is familiar with [Erdős et al. 2020, Section 4], but no familiarity with [Erdős et al. 2019b] is assumed. We will exclusively work with single index graphs as defined in [Erdős et al. 2020, Section 4.2]. In the rest of this section we use $N = 2n$ for easier comparison with [Erdős et al. 2020].

We start with the proof of (5-19c). We write $R \in \mathcal{M}_o$ as $R = J \text{ diag}(r)$ for some $r \in \mathbb{R}^{2n}$, where we can without loss of generality assume that $R$ has real entries and the matrix $J$ that exchanges $\mathcal{M}_o$
and $M_4$ is

$$J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. $$

With this notation the left-hand side of (5-19c) takes the form $\langle \text{diag}(r) \tilde{D} J \rangle = \langle \text{diag}(r)(W + \tilde{S}[G])GJ \rangle$. This form exactly matches the left-hand side of [Erdős et al. 2020, Equation (3.11c)] with $r = pf$, except that the last factor inside the trace is $GJ$ instead of just $G$. To accommodate this change we slightly extend the set of single index graphs $\Gamma \in \mathcal{G}$ defined in [Erdős et al. 2020, Section 4.2] by allowing two additional types of $G$-edges in $\text{GE} = \text{GE}(\Gamma)$. We call the original $G$-edges from [Erdős et al. 2020] that encode the entries of $G$ and $G^*$, straight $G$-edges and add new twisted $G$-edges that represent the entries of $GJ$ and $(GJ)^* = JG^*$, respectively. Graphically $(GJ)_{ab}$ will be encoded by a solid directed line from vertex $a$ to vertex $b$ and with a superscript $J$ on the line. Similarly, $(GJ)^*_{ab}$ is a dashed line from $a$ to $b$ with a superscript $J$. Hence, the new twisted $G$-edges are represented by

$$GJ = \bullet \xrightarrow{J} \bullet, \quad (GJ)^* = \bullet \xleftarrow{J} \bullet.$$

The terminology $G$-edge will refer to all four types of edges. In particular, all of them are taken into account for the $G$-edge degree of vertices.

According to the single index graph expansion (see [Erdős et al. 2020, Equation (4.10)]) the $p$-th moment of $\langle R \tilde{D} \rangle$ can now be written as a sum over the values $\text{Val}(\Gamma)$ associated to the graphs $\Gamma$ within the subset $\mathcal{G}(p) \subset \mathcal{G}$ of single index graphs. This subset originates from the single index resolution (see [Erdős et al. 2020, Definition 4.2]) of double index graphs, i.e.,

$$\mathbb{E}[\langle \text{diag}(r) \tilde{D} J \rangle]^p = N^{-p} \sum_{\Gamma \in \mathcal{G}(p)} \text{Val}(\Gamma) + O(N^{-p}). \quad (5-20)$$

The twisted $G$-edges enter into the graphs $\mathcal{G}(p)$ through the following simple modification (iv)' of [Erdős et al. 2020, (iv) from Definition 4.2] that originates from the fact that a wiggled $G$-edge in double index graphs is now associated to the matrix $GJ \text{diag}(r)$ and its adjoint instead of $G \text{diag}(r)$ with $r = pf$ as in [Erdős et al. 2020]:

(iv)' If a wiggled $G$-edge is mapped to an edge $e$ from $u$ to $v$, then $v$ is equipped with a weight of $r$ and $e$ is twisted. If a wiggled $G^*$-edge is mapped to an edge $\tilde{e}$ from $u$ to $v$, then $u$ is equipped with weight $r$ and $\tilde{e}$ is twisted. All vertices with no weight specified in this way are equipped with constant weight $1$.

The above changes reveal a one-to-one correspondence between the set of graphs $\mathcal{G}(p)$ in [Erdős et al. 2020] and its modification in the current work. This correspondence shows that the single index graph expansion is entirely unaffected by the presence of the off-diagonal matrix $J$ apart from replacing each weight $pf$ in $\mathcal{G}(p)$ from [Erdős et al. 2020] by a weight $r$ and replacing $p$ straight $G$-edges by twisted ones. More precisely, if in a graph from [Erdős et al. 2020] a vertex $v$ had a weight $(pf)^k$ (see [Erdős et al. 2020, Fact 2 at the end of Section 4.3] for the introduction of the concept of a general weight and [Erdős et al. 2020, Equation (4.12)] for an example), then in its corresponding graph the vertex $v$ is adjacent to exactly $k_1$ twisted $G$-edges that end at $v$ and $k_2$ twisted $G^*$-edges that start from $v$ such that $k_1 + k_2 = k$. 
Since the graphs contained in the sets \( \mathcal{G} \) and \( \mathcal{G}(p) \) do not differ between the current work and [Erdős et al. 2020] once the distinction between straight and twisted edges is dropped and the exact form of the weight \( r \) is irrelevant, any result from [Erdős et al. 2020, Section 4] that is insensitive to these distinctions can be directly applied here. When determining whether a subset \( GE \subset GE \) is classified as Wardable (see [Erdős et al. 2020, Definition 4.6]), we take into account all \( G \)-edges, i.e., straight as well as twisted \( G \)-edges. This is justified since the Ward estimates (see [Erdős et al. 2020, Equation (4.14b)])

\[
\sum_a |(GJ)_{ab}|^2 = \frac{(J(\text{Im } G)J)_{bb}}{\eta} \lesssim N \psi^2, \quad \sum_b |(GJ)_{ab}|^2 = \frac{\text{Im } G_{aa}}{\eta} \lesssim N \psi^2, \quad \psi = \left(\frac{\rho}{N \eta}\right)^{1/2}
\]

are valid for twisted \( G \)-edges as well. As in [Erdős et al. 2020], the inequalities (5-21) are meant in a high moment sense. In particular, Lemmas 4.7, 4.8 and 4.11 there remain valid. Note that the Lemma 4.11 involves the concept of a \( \sigma \)-cell that we explain next.

The most relevant difference between our setup and [Erdős et al. 2020] concerns the specific mechanism behind the cusp fluctuation averaging. This mechanism is revealed by exploiting a local cancellation in the graphical representation of these three types of \( \sigma \)-cells (see (5-22)). Some and \( K \) and \( J \) here denote a straight- or a twisted-edge adjacent to each endpoint of \( e \), loops are not allowed, and to precisely one of the endpoints (say \( a \)) an additional weight \( r \) is attached. In a typical \( \sigma \)-cell there are four \( G \)-edges that connect external vertices \( x, y, u, v \) with \( (a, b) \) and the adjacent \( G \)-edges encode the expression

\[
\mathbb{E} \sum_r r_a (\tilde{G} J)_{xa} G^{(1)}_{ay} K_{ba} G^{(2)}_{ub} G^{(3)}_{vb} f_{xyuv}.
\]

Here \( f_{xyuv} \) represents the rest of the graph and is independent of \( a, b \); the sum runs over all vertex indices and \( K_{ba} \) is either \( E[w_{ba}]^2 \), \( E[w_{ba}^2] \) or \( E[w_{ab}^2] \). Furthermore, \( \tilde{G} \in \{G, \overline{G}\} \) and \( G^{(i)} \in \{G, G^t, G^*, \overline{G}\} \) (here \( G^t \) and \( \overline{G} \) just denote a \( G \)- or a \( G^2 \)-edge with opposite orientation to save us from writing out all possibilities in (5-22)). Some \( G \)'s in (5-22) may coincide giving rise to two additional options for a \( \sigma \)-cell; the first one with two external indices, and the second one with no external index:

\[
\mathbb{E} \sum_r r_a (\tilde{G} J)_{xa} G^{(1)}_{ab} K_{ba} G^{(2)}_{ub} f_{xu}, \quad \text{and} \quad \mathbb{E} \sum_r r_a (\tilde{G} J)_{xa} G^{(1)}_{ab} K_{ba} f.
\]

The graphical representation of these three types of \( \sigma \)-cells is the same as drawn in [Erdős et al. 2020, Definition 4.10] with weight \( r = pf \), except that one \( G \)-edge adjacent to \( a \) is twisted. For example, the \( \sigma \)-cell with four external indices (5-22) is represented by

where the solid lines are \( G \)-edges, exactly one of them twisted (indicated by \( J \)), and without indicating their orientation. The interaction edge \( K \) is depicted by the dotted line, while the weights \( r \) and \( 1 \) attached
to vertices are indicated by arrows pointing to these vertices. The weight 1 could be ignored; it plays no specific role in the current paper, we drew it only for consistency with the picture in [Erdős et al. 2020] where it was essential that exactly one edge of the \( \sigma \)-cell received a specific weight. The graphical picture of the other two types of \( \sigma \)-cells are analogous.

Exactly as in [Erdős et al. 2020] the cusp fluctuation mechanism will allow us to gain a factor \( \sigma_q \) as defined in Proposition 5.5 for every \( \sigma \)-cell inside each graph. This gain is stated in [Erdős et al. 2020] as Proposition 4.12. In our setup, its analog is the next proposition.

**Proposition 5.16.** Let \( c > 0 \) be any constant and \( \Gamma \in \mathcal{G} \) be a single index graph with at most \( cp \) vertices and \( cp^2 \) edges with a \( \sigma \)-cell \( (u, v) = e \in IE(\Gamma) \). Then there exists a finite collection of graphs \( \mathcal{G}_\Gamma \) with at most one additional vertex and at most \( 6p \) additional \( G \)-edges such that

\[
\text{Val}(\Gamma) = \sum_{\Gamma' \in \mathcal{G}_\Gamma} \text{Val}(\Gamma') + \mathcal{O}(N^{-p}), \quad \text{W-Est}(\Gamma') \leq \rho \sigma_q \text{W-Est}(\Gamma'), \quad \Gamma' \in \mathcal{G}_\Gamma
\]

and all graphs \( \Gamma' \in \mathcal{G}_\Gamma \) have exactly one \( \sigma \)-cell less than \( \Gamma \).

The statement of Proposition 5.16 coincides with [Erdős et al. 2020, Proposition 4.12] except for the missing term \( \sigma \text{Val}(\Gamma_a) \) in the expansion of \( \text{Val}(\Gamma) \) originating from \( \sigma = 0 \) (see Remark 5.6) and the modified definition of \( \sigma_q \) (see Proposition 5.5). Up to [Erdős et al. 2020, Proposition 4.12] (which is replaced by Proposition 5.16), we have now verified all ingredients in the proof of [Erdős et al. 2020, Theorem 3.7] and thus its analog Proposition 5.14. Therefore, we will finish this subsection by pointing out the necessary modifications to the proof of [Erdős et al. 2020, Proposition 4.12] in order to show Proposition 5.16.

**Proof of Proposition 5.16.** We follow the proof of [Erdős et al. 2020, Proposition 4.12] and explain the necessary changes. It has two ingredients. The first is an explicit computation that involves the projections on stable and unstable directions of the stability operator \( B \) of the MDE (see [Erdős et al. 2020, Equation (4.30)]). This computation is extremely delicate and involves the precise choice for \( K_{ba}, r_a, G^{(i)} \) and their relation in the \( \sigma \)-cell (5-22). Its outcome is that up to a sufficiently small error term it is possible to act with the stability operator on any vertex \( a \) of the \( \sigma \)-cell. This action of \( B \) on \( a \) leads to an improvement of the bound on the corresponding graph that is stated as [Erdős et al. 2020, Lemma 4.13].

In our current setup the gain \( \sigma_q \) for every \( \sigma \)-cell inside a graph \( \Gamma \) is much more robust than in [Erdős et al. 2020], it is basically a consequence of the almost off-diagonality of \( M \). There is no need to act with the stability operator and also the specific weights attached to the vertices of the \( \sigma \)-cells are not important. Instead, the value of any graph containing a \( \sigma \)-cell can be estimated directly by \( \sigma_q \) times the sum of values of graphs with one sigma cell locally resolved (removed). For this gain the concrete choice of \( K_{ab}, r_a \) and \( G^{(i)} \) does not matter as long as \( |K_{ba}| \lesssim \frac{1}{N} \) and \( r_a \lesssim 1 \). Furthermore, we will not make use of resolvents \( G^{(2)} \) and \( G^{(3)} \) in the corresponding calculations. Thus in the following, instead of (5-22), we will only consider the simplified expression

\[
\mathbb{E} \sum (\tilde{G}J K^{(b)} G^{(1)})_{xy} f_{xy} = \mathbb{E} \sum_a (\tilde{G}J)_{xa} k_a^{(b)} G_a^{(1)} f_{xy},
\]

where \( K^{(b)} := \text{diag}(k^{(b)}) \) is a diagonal matrix whose diagonal \( k^{(b)} \) has components \( k_a^{(b)} := r_a K_{ba} \), and
$f = f_{xyb}$ encodes the rest of the graph. This is exactly the reference graph $\Gamma$ at the beginning of the proof of [Erdős et al. 2020, Lemma 4.13], but now the left edge is twisted and a weight $k^{(b)}$ is attached to the vertex $a$. With the choice $\widetilde{G} = G$, $G^{(1)} = G^*$ we have

$$\Gamma := \begin{array}{c}
\circ \\
J \\
\circ
\end{array}$$

which corresponds to the case $GJK^{(b)}G^*$ in (5-24). Since for all possible choices $\widetilde{G} \in \{G, G^*\}$ and $G^{(1)} \in \{G, G^*, G^*, G^*\}$ the discussion is analogous, we will restrict ourselves to the case $GJK^{(b)}G^*$.

In complete analogy to [Erdős et al. 2020, Equation (4.24)], but using simply the identity operator instead of the stability operator $B$ and inserting the identity $G = M - GS[M]M - GWM$ for the first resolvent factor on the left into (5-24), we find by (5-18) that


Notice that the twisted $G$-edge corresponding to $GJ$ disappeared and $J$ now appears only together with $M$ in the form $MJK^{(b)}$.

We estimate the five summands inside the square brackets of (5-25). This means to show that their power counting estimate (defined as W-Est in [Erdős et al. 2020, Lemma 4.8]) is smaller than the W-Est of the left-hand side of (5-25), W-Est($\Gamma$), at least by a factor $\sigma q$, i.e., we have

$$\text{Val}(\Gamma') = \sum_{\Gamma' \in \mathcal{G}_{\Gamma}} \text{Val}(\Gamma') + O(N^{-p}) \quad \text{with W-Est}(\Gamma') \leq_p \sigma q \text{W-Est}(\Gamma), \quad (5-26)$$

where all graphs $\Gamma' \in \mathcal{G}_{\Gamma}$ have one $\sigma$-cell less than $\Gamma$. Note that in contrast to [Erdős et al. 2020, Lemma 4.13] no insertion of the stability operator $B$ is needed for (5-26) to hold and that in contrast to [Erdős et al. 2020, Proposition 4.12] the additional graph $\Gamma_\sigma$ is absent from the right-hand side. In this sense (5-26) combines these two statements in a simplified fashion.

We remark that here the notion of differential edge (defined around [Erdős et al. 2020, Equation (4.27)]) is understood to include twisted $G$-edges as well. The derivatives of the twisted edges follow the same rules as the derivatives of the untwisted edges with respect to the matrix elements of $W$, for example

$$\frac{\partial}{\partial w_{ab}}(GJ) = G\Delta^{ab}(GJ), \quad \text{with } (\Delta^{ab})_{ij} = \delta_{ia}\delta_{jb},$$

i.e., simply one of the resulting two $G$-edges remains twisted. In particular, the number of twisted edges remains unchanged.

The first, second and fourth summands in (5-25) correspond to the graphs treated in parts (a), (b) and (c) inside the proof of [Erdős et al. 2020, Lemma 4.13] and their estimates follow completely analogously. We illustrate this with the simplest first and the more complex fourth term. The first term
gives $\mathbb{E}(MJK^{(b)}G^*)_{xy}f$, which would exactly be case (a) in the proof of [Erdős et al. 2020, Lemma 4.13] if $MJK^{(b)}$ were diagonal. However, even if it has an off-diagonal part (in the sense of $M_a$), the same bound holds, i.e., we still have

$$W-Est\left(\sum_a (MJK^{(b)})_{xa}G^*_{ay}f\right) \leq \frac{1}{N\psi^2} W-Est\left(\sum_a (GJK^{(b)}G^*)_{xy}f\right) = \sigma_q W-Est(\Gamma), \quad (5-27)$$

where $1/N$ comes from the fact that the summation over $a$ collapses to two values, $a=x$ and $a=\hat{x}$ and $\psi^2$ accounts for the two Wardable edges in $\Gamma$. Here we defined $\hat{x} := x + n (\text{mod } 2n)$ to be the complementary index of $x$. In order to see (5-27) more systematically, set

$$m_d := \text{diag}(M) \quad \text{and} \quad m_o := \text{diag}(MJ)$$

to be the vectors representing the diagonal and off-diagonal parts of $M$. Then

$$MJK^{(b)} = \text{diag}(m_o k^{(b)}) + J \text{diag}(m_d k^{(b)}) = \text{diag}(m_o k^{(b)}) + \text{diag}(m_d \tilde{k}^{(b)}) J, \quad (5-28)$$

where for any $2n$ vector $v = (v_1, v_2)$ with $v_i \in \mathbb{C}^n$ we define $\tilde{v} := (v_2, v_1)$. Thus, graphically, the factor $MJK^{(b)}$ can be represented as a sum of two graphs with a weight assigned to one vertex and for one of them there is an additional twist operator $J$ which one may put on either side. Therefore, the graph on the left-hand side of (5-27) can be represented by

$$\begin{align*}
\begin{array}{ccc}
\bullet & m_o k^{(b)} & \bullet \\
x & a & y \\
\end{array}
+ \begin{array}{ccc}
\bullet & m_d \tilde{k}^{(b)} & \bullet \\
x & a & J \\
\end{array}
\end{align*} \quad (5-29)$$

Here the double solid line depicts the identity operator as in [Erdős et al. 2020]. Both graphs in (5-29) are exactly the same as the one in case (a) within the proof of [Erdős et al. 2020, Lemma 4.13] with the only changes being the modified vertex weights $m_o k^{(b)}$ and $m_d \tilde{k}^{(b)}$ instead of just $m$ and the twisted $G^*$-edge in the second graph of (5-29). To justify the Ward estimate (5-27) we use only the fact that $\|m_o\|_\infty \lesssim 1$ and $\|m_d\|_\infty \lesssim 1$, although the latter estimate can be improved to $\|m_d\|_\infty \lesssim \rho$. Here we denote by $\|x\|_\infty := \max_{a \in [2n]} |x_a|$ the maximum norm of a vector $x$.

The decomposition (5-28) of $MJK^{(b)}$ into a sum of two terms, each with usual weights $m$ and one of them with a twisted edge, can be reinterpreted in all other cases. Thus, the second and fourth term in (5-25) can be treated analogously to the cases (b) and (c) within the proof of [Erdős et al. 2020, Lemma 4.13]. In particular, the fourth term is split as

$$\mathbb{E}(GT(G-M)MJK^{(b)}G^*)_{xy}f = \mathbb{E} \sum (t_{ba}G_{xb}(G-M)_{ab}u_aG^*_{ay} + t_{ba}G_{xb}(G-M)_{ab}v_a(GJ^*)_{ay}) f, \quad (5-30)$$

for some bounded vectors $u, v$ with $\|u\|_\infty + \|v\|_\infty \lesssim 1$. Thus the corresponding two graphs exactly match the first graph depicted in (c) of the proof of [Erdős et al. 2020, Lemma 4.13] with locally modified weights and the second one having a twisted $G^*$-edge.
For the third term in (5-25) we use a cumulant expansion and find
\[
\mathbb{E}(G(\tilde{S}[G] + W)MJK^{(b)}G^{*})_{xy} f
\]
\[
= -\mathbb{E}(G\tilde{S}[MJK^{(b)}(G^{*})G]_{xy} f + \sum_{a,c}^{6\rho} \sum_{k=2}^{\beta} \sum_{\varepsilon \in I^{k}} \kappa(ac, \varepsilon) \mathbb{E} \partial_{\varepsilon}[G_{xa}(MJK^{(b)}G^{*})_{cy} f]
\]
\[
+ \sum_{a,c} \mathbb{E} G_{xa}(MJK^{(b)}G^{*})_{cy} (\mathbb{E} |w_{ac}|^{2} \partial_{ac} + \mathbb{E} w_{ac}^{2} \partial_{ca}) f + \mathcal{O}(N^{-p}). \tag{5-31}
\]
where \(\beta\) is a \(k\)-tuple of double indices from \(I = [N] \times [N]\), \(\varepsilon\) is the multiset formed out of the entries of \(\beta\) (multisets allow repetitions) and \(\partial_{\varepsilon} = \prod_{(ij) \in \varepsilon} \partial_{w_{ij}}\). The notation \(\kappa(ac, \varepsilon)\) denotes the higher order cumulants of \(w_{ab}\) and \(\{w_{\beta} : \beta \in \varepsilon\}\). The second and third summands on the right-hand side of (5-31) correspond to the graphs treated in (e) and (d) of the proof of [Erdős et al. 2020, Lemma 4.13], respectively, with the factor \(MJK^{(b)}\) reinterpreted as sum of two terms with some weight \(m\) as explained in (5-28).

Thus we focus our attention on the first summand which reflects the main difference between our setup and that in [Erdős et al. 2020].

This term is expanded further using
\[
\tilde{S}[MJK^{(b)}G^{*}] = S[MJK^{(b)}M^{*}] + S[MJK^{(b)}(G - M)^{*}] + \mathcal{T}[MJK^{(b)}G^{*}]. \tag{5-32}
\]
At this point the main mechanism behind the cusp fluctuation averaging is revealed by the fact that the leading term \(S[MJK^{(b)}M^{*}] = \text{diag}(x)\) for some vector \(x\) with \(|x_{i}| \lesssim \frac{\rho}{N}\), because the diagonal elements of \(M\) are of order \(\mathcal{O}(\rho)\). This is the only place where the smallness of the diagonal elements of \(M\) is used, in all other estimates we used only the block diagonal structure of \(M\) and the boundedness of its matrix elements. The other two terms in (5-32) are smaller order. In fact, the second term exactly corresponds to the term encoded by the fourth graph on the right-hand side of [Erdős et al. 2020, Equation (4.29)], taking into account that this graph now splits into two due to the decomposition (5-28) with an extra twisted edge on one of the resulting graphs similarly to (5-30). Therefore this term is estimated as explained in (b) of the proof of [Erdős et al. 2020, Lemma 4.13]. The term corresponding to the last summand in (5-32) is analogous to the sixth graph in [Erdős et al. 2020, Equation (4.29)] and thus treated as explained in (c) within the proof of [Erdős et al. 2020, Lemma 4.13].

Finally, we treat the last \(G\mathcal{T}[M]\) term in (5-25) which was absent in [Erdős et al. 2020] and stems from the difference between the two self-energy operators \(S\) and \(\tilde{S}\). This term leads to a contribution of the form \(\mathbb{E}(GKG^{*})_{xy} f\) with the block diagonal matrix \(L^{(b)} := \mathcal{T}[M]MJK^{(b)}\) that satisfies \(\|L^{(b)}\| \lesssim N^{-1}\) in (5-25) as \(\|\mathcal{T}[M]\| \lesssim N^{-1}\). Thus the ensuing two graphs \(G' \in \mathcal{G}_{\Gamma}\) have the same Ward estimates as \(\Gamma\) but an additional \(N^{-1}\)-edge weight. This completes the proof of Proposition 5.14. \(\square\)

6. Local law for \(X\)

In this section, we provide the proofs of Theorems 2.1 and 2.3. We start with the proof of Theorem 2.3 which, given Theorem 5.2, will follow a similar argument to the proof of [Alt et al. 2018, Theorem 2.5(i)].
In this section, the model parameters consist of \( s_\alpha, s^* \) from (A1), the sequence \((\mu_m)_m\) from (A2), \( \alpha, \beta \) from (A3), and \( \alpha \) and \( \varphi \) from Theorem 2.3. Therefore, the implicit constants in the comparison relation and the stochastic domination are allowed to depend on these parameters.

**Proof of Theorem 2.3.** Let \( T > 0 \). From [Alt et al. 2018, Equation (2.15)] and (3-67) in the proof of Proposition 3.14, we get that

\[
\frac{1}{n} \sum_{i=1}^{n} f_{z_0,a}(\xi_i) - \int_{\mathbb{C}} f_{z_0,a}(z) \sigma(z) d^2z = \frac{1}{4\pi n} \int_{\mathbb{C}} \Delta f_{z_0,a}(z) \log |\det(H_z - iT)| d^2z - \frac{1}{2\pi} \int_{\mathbb{C}} \Delta f_{z_0,a}(z) \int_{0}^{T} \text{Im}(G(z, \eta) - M(z, \eta)) d\eta d^2z + \frac{1}{2\pi} \int_{\mathbb{C}} \Delta f_{z_0,a}(z) \int_{T}^{\infty} \left( \text{Im}(M(z, \eta)) - \frac{1}{1 + \eta} \right) d\eta d^2z. \tag{6-1}
\]

Here, we used that \( (v^t_1(\eta)|_{t=|z|}) = \text{Im}(M(z, \eta)) \) and \( m^z(i\eta) = (G(z, \eta)) \), where \( m^z \) is the Stieltjes transform of the empirical spectral measure of \( H_z \) (see [Alt et al. 2018, Equation (2.12)]). We also employed \( f_{z_0,a} \subset z_0 + \text{supp} f \subset D_{2\varphi}(0) \). For the remainder of the proof, we choose \( T := n^{100} \).

The same arguments used in the proof of [Alt et al. 2018, Theorem 2.5] to control the first and third term on the right-hand side of (6-1) for \( |z_0| \leq 1 \) imply that those terms are stochastically dominated by \( n^{-1+2\alpha} \| \Delta f \|_1 \) for all \( z_0 \in \mathbb{C} \) such that \( |z_0| \leq \varphi \).

What remains is bounding the second term on right-hand side of (6-1). To that end, we fix \( z \) and estimate the \( d\eta \)-integral by

\[
I(z) := \int_{0}^{T} \text{Im}(G(z, \eta) - M(z, \eta)) d\eta
\]
for \( z \in D_{2\varphi}(0) \) via the following lemma which is an extension of [Alt et al. 2018, Lemma 5.8].

**Lemma 6.1.** For every \( \delta > 0 \) and \( p \in \mathbb{N} \), there is a positive constant \( C \), depending only on \( \delta \) and \( p \) in addition to the model parameters, such that

\[
\sup_{z \in D_{2\varphi}(0)} \mathbb{E}|I(z)|^p \leq C \frac{n^{\delta p}}{n^p}.
\]

We postpone the proof of Lemma 6.1 to the end of this section. Now, Lemma 6.1 implies Theorem 2.3 along the same steps used to conclude [Alt et al. 2018, Theorem 2.5(i)] from [Alt et al. 2018, Lemma 5.8]. This completes the proof of Theorem 2.3.

Before we prove Lemma 6.1, we first conclude Theorem 2.1 from Theorem 2.3 and the improved bound on \( G - M \) in (5-4) in Theorem 5.2.

**Proof of Theorem 2.1.** We first show the upper bound on \( g(X) \) as detailed in Remark 2.2. Throughout the proof, we say that an event \( \Xi = \Xi_n \) (in the probability space of \( n \times n \) random matrices \( X \)) holds with very high probability if for each \( D > 0 \), there is \( C > 0 \) such that \( \mathbb{P}(\Xi) \geq 1 - Cn^{-D} \) for all \( n \in \mathbb{N} \).
Proposition 5.7, which holds uniformly for $p$ from (A3) and on the following straightforward estimate. Let $\rho(\mathcal{M})$ defined in (5-2). Throughout the proof, we will omit the dependence of $\rho(\mathcal{M})$.

The proof proceeds analogously to the proof of [Alt et al. 2018, Lemma 5.8]. However, we have to replace the fluctuation scale in the bulk, $n^{-1}$, by the $z$-dependent fluctuation scale $\eta_z$ defined in (5-2). Throughout the proof, we will omit the dependence of $G(z, \eta)$ and $M(z, \eta)$ on $z$ and $\eta$ from our notation and write $G = G(z, \eta)$ and $M = M(z, \eta)$. Similarly, we denote the $2n$ eigenvalues of $H_z$ by $\lambda_1, \ldots, \lambda_{2n}$.

We will have to convert a few $\prec$-bounds into moment bounds. In order to do that, we will use the following straightforward estimate. Let $c > 0$. Then, for each $\delta > 0$ and $p \in \mathbb{N}$, there is $C$, depending on $c, \delta$ and $p$, such that any random variable $Y \geq 0$ satisfies

$$Y < n^{-1}, \quad Y \leq n^c \implies \mathbb{E}Y^p \leq Cn^{p(-1+\delta)}. \quad (6-2)$$

We fix $\varepsilon > 0$, choose $l > 0$ sufficiently large and decompose the integral in the definition of $I$ to obtain

$$I(z) = \frac{1}{n} \sum_{|\lambda_i| < n^{-l}} \log \left( 1 + \frac{\eta_i^2 n^{2\varepsilon}}{\lambda_i^2} \right) + \frac{1}{n} \sum_{|\lambda_i| \geq n^{-l}} \log \left( 1 + \frac{\eta_i^2 n^{2\varepsilon}}{\lambda_i^2} \right) - \int_0^{\eta_i n^{2\varepsilon}} \text{Im}(\mathcal{M})d\eta + \int_0^{\eta_i n^{2\varepsilon}} \text{Im}(G-M)d\eta. \quad (6-3)$$

We now estimate the $p$-th moment of each term on the right-hand side of (6-3) individually. Exactly as in the proof of [Alt et al. 2018, Lemma 5.8], we choose $l > 0$ sufficiently large, depending on $\alpha, \beta$ from (A3) and on $p$ such that the estimate on the smallest singular value of $X - z$ in [Alt et al. 2018, Proposition 5.7], which holds uniformly for $z \in D_{2\varepsilon}(0)$, implies

$$\mathbb{E} \left[ \frac{1}{n} \sum_{|\lambda_i| < n^{-l}} \log \left( 1 + \frac{\eta_i^2 n^{2\varepsilon}}{\lambda_i^2} \right) \right]^p \leq n^{-p}. $$
For the second term on the right-hand side of (6-3), we distinguish the three regimes, \(|\lambda_i| \in [n^{-\epsilon}, \eta n^\epsilon]\), \(|\lambda_i| \in [\eta n^\epsilon, \eta n^{1/2}]\) and \(|\lambda_i| > \eta n^{1/2}\). In the first regime, we use \(\eta_i^2 n^{2\epsilon} \lambda_i^{-2} \leq n^{2\epsilon+2}\) and Lemma 5.13 with \(\eta = \eta n^\epsilon\) and obtain
\[
\frac{1}{n} \sum_{|\lambda_i| \in [n^{-\epsilon}, \eta n^\epsilon]} \log \left(1 + \frac{\eta_i^2 n^{2\epsilon}}{\lambda_i^2}\right) \leq \frac{C \log n}{n} \left|\left\{ i : |\lambda_i| \leq \eta n^\epsilon \right\}\right| \lesssim \frac{n^{2\epsilon}}{n}.
\]
We decompose the second regime, \(|\lambda_i| \in [\eta n^\epsilon, \eta n^{1/2}]\), into the union of the intervals \([\eta_k, \eta_{k+1}]\), where \(\eta_k := \eta n^{\epsilon} 2^k\), \(k = 0, \ldots, N\) and \(N \lesssim \log n\). Hence, \(\log(1+x) \leq x\) for \(x > 0\) yields
\[
\frac{1}{n} \sum_{|\lambda_i| \in [\eta n^\epsilon, \eta n^{1/2}]} \log \left(1 + \frac{\eta_i^2 n^{2\epsilon}}{\lambda_i^2}\right) \leq \frac{2}{n} \sum_{k=0}^{N} \sum_{\lambda_i \in [\eta_k, \eta_{k+1}]} \eta_i^2 n^{2\epsilon} \lambda_i^{-2} \lesssim \frac{n^{3\epsilon}}{n}.
\]
In the third regime, \(|\lambda_i| > \eta n^{1/2}\), we conclude from \(|\lambda_i| > \eta n^{1/2}\) that
\[
\frac{1}{n} \sum_{|\lambda_i| > \eta n^{1/2}} \log \left(1 + \frac{\eta_i^2 n^{2\epsilon}}{\lambda_i^2}\right) \leq \frac{1}{n} \sum_{|\lambda_i| > \eta n^{1/2}} \log(1 + n^{-1} n^{2\epsilon}) \leq 2 \frac{n^{2\epsilon}}{n}.
\]
Therefore, (6-2) implies that the second term on the right-hand side of (6-3) satisfies the bound in Lemma 6.1.

For the third term in (6-3), we use (3-13) and distinguish the regimes in the definition of \(\eta_{t}\) in (5-2) and obtain
\[
\int_0^{\eta n^\epsilon} \langle \text{Im } M \rangle \, d\eta \sim \int_0^{\eta n^{\epsilon}} \rho \, d\eta \lesssim n^{-1+4\epsilon/3}.
\]
Before estimating the last term in (6-3), we conclude that \(\sup_{\eta \in [\eta n^\epsilon, T]} \eta |\text{Im}(G - M)| < n^{-1}\) from Corollary 5.12. Here, we also used a union bound and the Lipschitz-continuity of \(G\) and \(M\) as functions of \(\eta\) in the following sense: There is \(c \geq 2\) such that \(\|G(z, \eta_1) - G(z, \eta_2)\| + \|M(z, \eta_1) - M(z, \eta_2)\| \lesssim n^\epsilon |\eta_1 - \eta_2|\) for all \(\eta_1, \eta_2 \geq n^{-1}\) and \(z \in D_{2\varphi}(0)\). For \(G\) this follows from resolvent identities. For \(M\) this was shown in [Alt et al. 2019, Corollary 3.8]. The bound \(\sup_{\eta \in [\eta n^\epsilon, T]} \eta |\text{Im}(G - M)| < n^{-1}\) implies
\[
\int_{\eta n^\epsilon}^{T} |\text{Im}(G - M)| \, d\eta < n^{-1}.
\]
Owing to (6-2) this implies the desired estimate on the \(p\)-th moment of the last term on the right-hand side of (6-3). This completes the proof of Lemma 6.1. \(\square\)

**Remark 6.2** (alternative to Assumption (A3), modifications in the proof). Instead of Assumption (A3) we now assume the condition \(\max_{i,j} \mathcal{L}(\sqrt{n} x_{ij}, t) \leq b\) from Remark 2.5 and we explain the necessary modifications in the proof of Theorem 2.3. The only place where Assumption (A3) is used in our entire argument was in estimating the first term in (6-3). Under this new condition, it is a simple consequence of [Livshyts et al. 2019, Theorem 1.1] that
\[
|\log \lambda_1(z)| < 1,
\]
where we denoted by \(\lambda_1(z)\) the smallest, nonnegative eigenvalue of \(H_z\). With this bound at hand, the
\[\text{Note that the operator norm on } \mathbb{C}^{2n \times 2n} \text{ induced by the Euclidean norm on } \mathbb{C}^{2n} \text{ was denoted by } \| \cdot \|_2 \text{ in [Alt et al. 2019].}\]
The following general lemma determines the first few terms in the perturbative expansion of the solution \( H \) of the quadratic equation \( C \| \cdot \| \). Let \( C_2 \) be the Hilbert–Schmidt scalar product on \( \mathbb{C}^{2n \times 2n} \). Let \( A : \mathbb{C}^{2n \times 2n} \to \mathbb{C}^{2n \times 2n} \) be a bilinear map such that \( A[R, T] = A[T, R] \) for all \( R, T \in \mathbb{C}^{2n \times 2n} \). Let \( B : \mathbb{C}^{2n \times 2n} \to \mathbb{C}^{2n \times 2n} \) be a linear operator with two simple eigenvalues \( \beta \) and \( \beta_\ast \) with associated left and right eigenvectors \( \hat{B}, \hat{B}_\ast \) and \( B_\ast, B \), respectively.

For some \( \lambda \geq 1 \), we assume that
\[
\|A\| + \|B^{-1} Q\| + \frac{1}{|\langle \hat{B}, B \rangle|} + \|\langle \hat{B}, \cdot \rangle\| + \|B\| + \frac{1}{|\langle \hat{B}_\ast, B_\ast \rangle|} + \|\langle \hat{B}_\ast, \cdot \rangle\| + \|B_\ast\| + \frac{1}{|\langle E_-, B_\ast \rangle|} + \|E_-, \cdot \| \leq \lambda, \tag{A-1}
\]
where \( \| \cdot \| \) denotes the norm on bilinear maps, linear operators and linear forms induced by the norm on \( \mathbb{C}^{2n \times 2n} \).

Then there is a universal constant \( c > 0 \) such that for any \( Y, Z \in \mathbb{C}^{2n \times 2n} \) with \( \|Y\| + \|Z\| \leq c\lambda^{-12} \) that satisfy the quadratic equation
\[
B[Y] - A[Y, Y] + Z = 0 \tag{A-2}
\]
with the constraint \( \langle E_-, Y \rangle = 0 \) the following holds: For any \( \delta \in (0, 1) \), the coefficient
\[
\Theta := \frac{\langle \hat{B}, Y \rangle}{\langle \hat{B}, B \rangle}
\]
fulfills the cubic equation
\[\mu_3 \Theta^3 + \mu_2 \Theta^2 + \mu_1 \Theta + \mu_0 = 2\lambda_0 \Theta + \lambda_0^2 (|\Theta|^3 + |\Theta|^4 + \delta^{-2} (\|Z\|^3 + \langle E_-, B^{-1}Q[Z] \rangle^3/2) + |\Theta|^2 (\langle E_-, B \rangle^2 + (\langle \hat{B}, A[B, B] \rangle)^2)) \tag{A-3}\]
whose coefficients are given by
\[\mu_3 = 2 \langle \hat{B}, A[B, B^{-1}Q[A[B, B]]] \rangle \frac{2 \langle \hat{B}, A[B, B_0] \rangle \langle E_-, B^{-1}Q[A[B, B]] \rangle}{\langle E_-, B_0 \rangle},\]
\[\mu_2 = \langle \hat{B}, A[B, B] \rangle,\]
\[\mu_1 = -\beta \langle \hat{B}, B \rangle - 2 \langle \hat{B}, A[B, B^{-1}Q[Z]] \rangle + 2 \langle E_-, B \rangle \langle \hat{B}, A[B^{-1}Q[Z], B_0] \rangle,\]
\[\mu_0 = \langle \hat{B}, A[B^{-1}Q[Z], B^{-1}Q[Z]] - Z \rangle.\]

Moreover, \(Y\) can be expressed by \(\Theta\) and \(Z\) via
\[Y = \Theta B - B^{-1}Q[Z] + \Theta^2 B^{-1}Q[A[B, B]] - \Theta^2 \frac{\langle E_-, B^{-1}Q[A[B, B]] \rangle}{\langle E_-, B_0 \rangle} B_0 + \lambda_0^2 \Theta (|\Theta|^3 + |\Theta|(|\Theta|^3 + \|E_-, B\|) + \|Z\|^2 + \|E_-, B^{-1}Q[Z]\|). \tag{A-5}\]

**Proof.** We decompose \(Y\) according to the spectral subspaces of \(B\). This yields
\[Y = \Theta B + \Theta_0 B_0 + Q[Y], \quad \Theta := \frac{\langle \hat{B}, Y \rangle}{\langle \hat{B}, B \rangle}, \quad \Theta_0 := \frac{\langle B_0, Y \rangle}{\langle B_0, B_0 \rangle}. \tag{A-6}\]

We define
\[Y_1 := \Theta B - B^{-1}Q[Z],\]
\[Y_2 := Q[Y] + B^{-1}Q[Z] + \Theta_0 B_0 - Y_3,\]
\[Y_3 := \frac{\langle E_-, B^{-1}Q[Z] - \Theta B - B^{-1}Q[A[Y_1, Y_1]] \rangle}{\langle E_-, B_0 \rangle} B_0, \tag{A-7}\]

Obviously, \(Y = Y_1 + Y_2 + Y_3\), \(Y_1 = \lambda O_1\) and \(Y_3 = \lambda^7 O_2\), where we introduced the notation \(O_k = O(|\Theta|^k + \|Z\|^k + \langle E_-, B \rangle^k + \|E_-, B^{-1}Q[Z]\|^k/2)\)

and used the convention that \(R = O_k\) means \(\|R\| = O_k\). Here and in the following, the implicit constant in \(O\) will always be independent of \(\lambda\).

From \(\langle E_-, Y \rangle = 0\) and (A-6), we obtain
\[\Theta_0 \langle E_-, B_0 \rangle = -\Theta \langle E_-, B \rangle - \langle E_-, Q[Y] \rangle = \langle E_-, B^{-1}Q[Z] - \Theta B - B^{-1}Q[A[Y, Y]] \rangle.\]

Here, we used that \(Q[Y] = B^{-1}A[Y, Y] - B^{-1}Q[Z]\) by (A-2) in the second step. This shows that \(\Theta_0\) is the coefficient of \(B_0\) in the definition of \(Y_3\) up to replacing \(Y\) by \(Y_1\). Thus, we deduce that
\[\|\Theta_0 B_0 - Y_3\| = \lambda^5 O(\|Y_1\| (\|Y_2\| + \|Y_3\|) + \|Y_2\|^2 + \|Y_3\|^2). \tag{A-8}\]
We insert $Y = Y_1 + Y_2 + Y_3$ into (A-2) and obtain
\begin{equation}
\Theta \beta B + \Theta_\ast \beta_\ast B_\ast + B Q[Y_2] + (1 - Q)[Z] = A[Y, Y].
\end{equation}
(A-9)

Applying $B^{-1} Q$ to the previous relation implies
\begin{equation}
Q[Y_2] = B^{-1} Q A[Y, Y].
\end{equation}
(A-10)

Hence, $\| Q[Y_2] \| = \lambda^2 O(\| Y_1 \|^2 + \| Y_2 \|^2 + \| Y_3 \|^2)$. As $Y_2 = Q[Y_2] + \Theta_\ast B_\ast - Y_3$, we get from (A-8) that
\begin{equation}
\| Y_2 \| = \lambda^2 O(\| Y_1 \|^2 + \| Y_2 \|^2 + \| Y_3 \|^2).
\end{equation}
(A-11)

The definitions of $Y_1$, $Y_2$, $Y_3$ and the conditions (A-1) and $\| Z \| + \| Y \| \leq c \lambda^{-12}$ yield $\| Y_2 \| \leq c C \lambda^{-5}$ for some universal constant $C > 0$. Hence, (A-11) implies $Y_2 = \lambda^{19} O_2$ as $Y_1 = \lambda O_1$ and $Y_3 = \lambda^{7} O_2$ if $c$ is chosen sufficiently small independently of $\lambda$. Therefore, we conclude from (A-10), (A-8) and $|\Theta| + \| Z \| = O(\lambda^{-10})$ that
\begin{equation}
Y_2 = B^{-1} Q A[Y_1, Y_1] + \lambda^{30} O_3.
\end{equation}
(A-12)

In particular, this implies (A-5) as $Y = Y_1 + Y_2 + Y_3$.

Applying $\langle \hat{B}, \cdot \rangle$ to (A-9) yields
\begin{equation}
\Theta \beta \langle \hat{B}, B \rangle + \langle \hat{B}, Z \rangle = \langle \hat{B}, A[Y, Y] \rangle.
\end{equation}
(A-13)

Therefore, we now show (A-3) by computing $\langle \hat{B}, A[Y, Y] \rangle$. Using $Y_1 = \lambda O_1$, $Y_2 = \lambda^{19} O_2$, $Y_3 = \lambda^{7} O_2$ and (A-12), we deduce
\begin{equation}
\langle \hat{B}, A[Y, Y] \rangle = \langle \hat{B}, A[Y_1, Y_1] \rangle + 2 \langle \hat{B}, A[Y_1, Y_3] \rangle + 2 \langle \hat{B}, A[Y_1, B^{-1} Q A[Y_1, Y_1]] \rangle + \lambda^{40} O_4.
\end{equation}
(A-14)

For a linear operator $K_1$ and a bilinear operator $K_2$ with $\| K_1 \| + \| K_2 \| \leq 1$, we have
\begin{equation}
\| \Theta K_2 [R, R] \| \leq \delta |\Theta|^3 + \delta^{-1/2} \| R \|^3, \quad \| \Theta^2 K_1 [R] \| \leq \delta |\Theta|^3 + \delta^{-2} \| R \|^3
\end{equation}
for any matrix $R \in \mathbb{C}^{2n \times 2n}$ since $\delta > 0$. Therefore, as $\delta \in (0, 1)$, we obtain
\begin{equation}
\langle \hat{B}, A[Y_1, B^{-1} Q A[Y_1, Y_1]] \rangle = \Theta^3 \langle \hat{B}, A[B, B^{-1} Q A[B, B]] \rangle + \lambda^{7} O(\delta |\Theta|^3 + \delta^{-2} \| Z \|^3).
\end{equation}
(A-15)

Similarly, we conclude
\begin{equation}
\langle \hat{B}, A[Y_1, Y_3] \rangle = \\
= \Theta \langle \hat{B}, A[B^{-1} Q[Z], B_\ast] \rangle \langle E_-, B \rangle - \Theta \langle \hat{B}, A[B, B_\ast] \rangle \langle E_-, B^{-1} Q A[B, B] \rangle \\
+ \lambda^{10} O(\delta |\Theta|^3 + \delta^{-2} \| Z \|^3 + \delta^{-2} \| \langle E_-, B^{-1} Q[Z] \rangle \|^{3/2} + |\Theta|^2 \| \langle E_-, B \rangle \|^2 + |\langle \hat{B}, A[B, B_\ast] \rangle \|^2).
\end{equation}
(A-16)

Finally, we expand the first term on the right-hand side of (A-14) using the definition of $Y_1$ from (A-7) and insert (A-15) as well as (A-16) into (A-14) to compute the second and third term. We apply the result to (A-13) and obtain the cubic equation in (A-3) with the coefficients detailed in (A-4). □
Appendix B. Non-Hermitian perturbation theory

In this section, we present for the reader’s convenience the perturbation theory for a non-Hermitian operator $K$ on $\mathbb{C}^{2n \times 2n}$ with an isolated eigenvalue $\kappa$. We denote by $P_{K}$ the spectral projection of $K$ associated to $\kappa$ and set $Q_{K} := 1 - P_{K}$. We assume that the algebraic multiplicity of $\kappa$ coincides with its geometric multiplicity. In particular, this condition ensures that, for any $L \in \mathbb{C}^{2n \times 2n}$, we have

$$K[K] = \kappa K, \quad K^* [\hat{K}] = \overline{\kappa} \hat{K},$$

where $K := P_{K}[L]$ and $\hat{K} := P_{K^*}^{*}[L]$. That is, $K$ and $\hat{K}$ are right and left eigenvectors of $K$ corresponding to $\kappa$, respectively.

Throughout this section, we suppose that there is a constant $C > 0$ such that

$$\|K\| + \|(K - \kappa)^{-1} Q_{K}\| + \|P_{K}\| \leq C. \quad \text{(B-1)}$$

Here and in the following, $\| \cdot \|$ denotes the operator norm of operators on $\mathbb{C}^{2n \times 2n}$ induced by some norm $\| \cdot \|$ on the matrices in $\mathbb{C}^{2n \times 2n}$.

**Lemma B.1.** There is $\varepsilon > 0$, depending only on $C$ from (B-1), such that the following holds.

If $\mathcal{L}$ is a linear map on $\mathbb{C}^{2n \times 2n}$ satisfying $\|\mathcal{L} - \mathcal{K}\| \leq \varepsilon$ and $\lambda$ is an eigenvalue of $\mathcal{L}$ satisfying $|\kappa - \lambda| \leq \varepsilon$ then, for any right and left normalized eigenvectors $L$ and $\hat{L}$ of $\mathcal{L}$ associated to $\lambda$, we have

$$\lambda (\hat{L}, L) = \kappa (\hat{K}, K) + (\hat{K}, D[K]) + (\hat{K}, D Q_{K} (2\kappa - \kappa)(\kappa - \kappa)^{-2} Q_{K} D[K]) + O(\|D\|^{3}), \quad \text{(B-2a)}$$

$$L = K - (K - \kappa)^{-1} Q_{K} D[K] + L_{2} + O(\|D\|^{3}), \quad \text{(B-2b)}$$

$$\hat{L} = \hat{K} - (K^* - \kappa)^{-1} Q_{K}^{*} D^*[\hat{K}] + \hat{L}_{2} + O(\|D\|^{3}), \quad \text{(B-2c)}$$

where we used the definitions $D := \mathcal{L} - \mathcal{K}$, $K := P_{K}[L]$ and $\hat{K} := P_{K}^{*}[\hat{L}]$ as well as

$$L_{2} := (K - \kappa)^{-1} Q_{K} D(K - \kappa)^{-1} Q_{K} D[K] - (K - \kappa)^{-2} Q_{K} D P_{K} D[K],$$

$$\hat{L}_{2} := (K^* - \kappa)^{-1} Q_{K}^{*} D^*[K](K^* - \kappa)^{-1} Q_{K}^{*} D^*[\hat{K}] - (K^* - \kappa)^{-2} Q_{K}^{*} D^* P_{K}^{*} D^*[\hat{K}].$$

In the previous lemma and in the following, the implicit constants in the comparison relation $\lesssim$ and in $O$ depend only on $C$ from (B-1).

**Proof.** We first establish the relations (B-2b) and (B-2c) for the eigenvectors. The eigenvector relation $\mathcal{L}[L] = \lambda L$ together with the definition $\delta := \lambda - \kappa$ yields

$$K Q_{K}[L] + D[L] = \delta K + \beta Q_{K}[L]. \quad \text{(B-3)}$$

Here, we also employed $K[K] = \kappa K$ and $L = K + Q_{K}[L]$.

By applying $(\kappa - \kappa)^{-1} Q_{K}$ in (B-3), we get

$$Q_{K}[L] = -(\kappa - \kappa)^{-1} Q_{K} D[L] + \delta (\kappa - \kappa)^{-1} Q_{K}[L]. \quad \text{(B-4)}$$

This relation immediately implies

$$\|Q_{K}[L]\| \lesssim \|D\| + \|\delta\| Q_{K}[L]\|. $$
Thus, we obtain $\|Q_K[L]\| \lesssim \|D\|$ by choosing $\epsilon$ sufficiently small. Hence, (B-3) implies

$$|\delta|\|K\| \leq |\beta|\|Q_K[L]\| + \|KQ_K[L]\| + \|D[L]\| = O(\|D\|).$$

Since $\|K\| \geq \|L\| - \|Q_K[L]\| \geq 1/2$ for sufficiently small $\epsilon$, we conclude

$$|\delta| \lesssim \|D\|.$$

We start from $L = K + Q_K[L]$ and iteratively replace $Q_K[L]$ by using (B-4) to obtain

$$L = K - (K - \kappa)^{-1} Q_K D[L] + \delta (K - \kappa)^{-1} Q_K[L] \tag{B-5}$$

$$= K - (K - \kappa)^{-1} Q_K D[L] - \delta (K - \kappa)^{-2} Q_K D[L] + O(\|D\|^3)$$

$$= K - (K - \kappa)^{-1} Q_K D[L] + (K - \kappa)^{-1} Q_K D(K - \kappa)^{-1} Q_K D[L] - \delta (K - \kappa)^{-2} Q_K D[K] + O(\|D\|^3)$$

$$= K - (K - \kappa)^{-1} Q_K D[L] + (K - \kappa)^{-1} Q_K D(K - \kappa)^{-1} Q_K D[L] - (K - \kappa)^{-2} Q_K D[Q_K D[L] + O(\|D\|^3).$$

Here, we also used that $|\delta| + \|Q_K[L]\| = O(\|D\|)$. The last step in (B-5) follows from $\delta K = D[L] + \kappa Q_K[L] - \beta Q_K[L] = D[K] + (K - \kappa) Q_K[L] + O(\|D\|^2) = D[K] - Q_K D[K] + O(\|D\|^2)$, which is a consequence of (B-3) and (B-4). This completes the proof of (B-2b). A completely analogous argument yields (B-2c).

For the proof of (B-2a), we first define $L_1 := - (K - \kappa)^{-1} Q_K D[K]$ and $\hat{L}_1 := -(K^* - \kappa)\hat{Q}_K D\hat{Q}_K[L]$. Using (B-2b) and (B-2c) in the relation $\beta \langle \hat{L}, L \rangle = \langle \hat{L}, (K + D)[L]\rangle$ yields

$$\beta \langle \hat{L}, L \rangle = \kappa \langle \hat{K}, K \rangle + \kappa \langle \hat{L}_1, K \rangle + \kappa \langle \hat{L}_2, K \rangle + \langle \hat{K}, \kappa L_1 \rangle + \langle \hat{K}, \kappa L_2 \rangle + \langle \hat{L}_1, K \rangle$$

$$+ \langle \hat{K}, D[K] \rangle + \langle \hat{L}_1, D[K] \rangle + \langle \hat{K}, D[L_1] \rangle + O(\|D\|^3)$$

since $L_1, \hat{L}_1 = O(\|D\|)$ and $L_2, \hat{L}_2 = O(\|D\|^2)$. We remark that $\langle \hat{L}_1, K \rangle = \langle \hat{K}, \kappa L_1 \rangle = \langle \hat{L}_2, K \rangle = \langle \hat{K}, \kappa L_2 \rangle = 0$ since $Q_K[K] = 0$ and $Q_K[\hat{K}] = 0$. For the remaining terms, we get $\langle \hat{L}_1, K \rangle = \langle \hat{K}, DK(K - \kappa)^{-1} Q_K D[K] \rangle$, $\langle \hat{L}_1, D[K] \rangle = \langle \hat{K}, D[L_1] \rangle = - \langle \hat{K}, D(K - \kappa)^{-1} Q_K D[K] \rangle$.

Therefore, a simple computation yields (B-2a). This completes the proof of Lemma B.1. \hfill \square

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NONCONVEX INTERACTIONS IN MEAN-FIELD SPIN GLASSES

JEAN-CHRISTOPHE MOURRAT

We propose a conjecture for the limit free energy of mean-field spin glasses with a bipartite structure, and show that the conjectured limit is an upper bound. The conjectured limit is described in terms of the solution to an infinite-dimensional Hamilton–Jacobi equation. A fundamental difficulty of the problem is that the nonlinearity in this equation is not convex. We also question the possibility to characterize this conjectured limit in terms of a saddle-point problem.

1. Introduction

Let \((J_{ij})_{i,j \geq 1}\) be independent standard Gaussian random variables, and, for every \(\sigma = (\sigma_{1,1}, \ldots, \sigma_{1,N}, \sigma_{2,1}, \ldots, \sigma_{2,N}) \in \mathbb{R}^{2N}\), let

\[
H_N(\sigma) := N^{-\frac{1}{2}} \sum_{i,j=1}^{N} J_{ij} \sigma_{1,i} \sigma_{2,j}.
\]

(1-1)

The main goal of this paper is to study the large-\(N\) behavior of the free energy

\[
\frac{1}{N} \mathbb{E} \log \int_{\mathbb{R}^{2N}} \exp(\beta H_N(\sigma)) \, dP_N(\sigma),
\]

(1-2)

where \(\beta \geq 0\) and \(P_N\) is a “simple” probability measure over \(\mathbb{R}^{2N}\). For convenience, we assume that there exist two probability measures \(\pi_1\) and \(\pi_2\) on \(\mathbb{R}\) with compact support such that, for every \(N \geq 1\),

\[
P_N = \pi_1^\otimes N \otimes \pi_2^\otimes N.
\]

(1-3)

Without loss of generality, we assume that the supports of \(\pi_1\) and \(\pi_2\) are subsets of \([-1, 1]\). For every metric space \(E\), we denote by \(\mathcal{P}(E)\) the space of Borel probability measures on \(E\), and, for every \(p \in [1, \infty]\), by \(\mathcal{P}_p(E)\) the subspace of \(\mathcal{P}(E)\) of probability measures with finite \(p\)-th moment. We write \(\delta_x\) for the Dirac probability measure at \(x \in E\). For every \(\nu \in \mathcal{P}(\mathbb{R}_+)\) and \(r \in [0, 1]\), we define

\[
F_\nu^{-1}(r) := \inf\{s \geq 0 : \nu([0, s]) \geq r\},
\]

(1-4)

and, for \(U\) a uniform random variable over \([0, 1]\), we write

\[
X_\nu := F_\nu^{-1}(U).
\]

(1-5)

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\textit{Keywords:} spin glass, Hamilton–Jacobi equation, Wasserstein space.
Recall that the law of $X_t$ is $\nu$, and that this construction provides us with a joint coupling of all probability measures over $\mathbb{R}_+$. For every $\mu = (\mu_1, \mu_2) \in (P(\mathbb{R}_+))^2$, we denote by $\widehat{\mu} \in P(\mathbb{R}_+^2)$ the law of the pair $(X_{\mu_1}, X_{\mu_2})$. Here is the main result of this paper.

**Theorem 1.1.** For every $t \geq 0$, we have

$$\liminf_{N \to \infty} -\frac{1}{N} \mathbb{E} \log \int \exp(\sqrt{2t}H_N(\sigma) - N^{-1}t|\sigma_1|^2|\sigma_2|^2) \, dP_N(\sigma) \geq f(t, (\delta_0, \delta_0)), \quad (1-6)$$

where $f = f(t, \mu) : \mathbb{R}_+ \times (P_2(\mathbb{R}_+))^2 \to \mathbb{R}$ is the solution to

$$\begin{align*}
\partial_t f - \int \partial_{\mu_1} f \partial_{\mu_2} f \, d\widehat{\mu} = 0 & \quad \text{on } \mathbb{R}_+ \times (P_2(\mathbb{R}_+))^2, \\
f(0, \cdot) = \psi & \quad \text{on } (P_2(\mathbb{R}_+))^2, \quad (1-7)
\end{align*}$$

and the initial condition $\psi$ is defined in (2-19).

We start by clarifying the meaning of the Hamilton–Jacobi equation in (1-7). Alternative expressions for the integral in (1-7) read

$$\int \partial_{\mu_1} f \partial_{\mu_2} f \, d\widehat{\mu} = \int_{\mathbb{R}_+^2} \partial_{\mu_1} f(t, \mu, x_1) \partial_{\mu_2} f(t, \mu, x_2) \, d\widehat{\mu}(x_1, x_2) = \mathbb{E}[\partial_{\mu_1} f(t, \mu, X_{\mu_1}) \partial_{\mu_2} f(t, \mu, X_{\mu_2})].$$

The notion of derivative at play here is not of Fréchet type (which would express the linear response to the addition of a small signed measure of zero total mass), but rather of transport type. Informally, for a “smooth” function $g = g(\nu) : P_2(\mathbb{R}_+) \to \mathbb{R}$, the derivative $\partial_\nu g(\nu, \cdot) \in L^2(\mathbb{R}_+, \nu)$ is characterized by the first-order expansion

$$g(\nu') = g(\nu) + \mathbb{E}[\partial_\nu g(\nu, X_\nu)(X_{\nu'} - X_\nu)] + o(\mathbb{E}[(X_{\nu'} - X_\nu)^2]^{1/2}).$$

More concretely, given some integer $k \geq 1$, and setting, for every $x = (x_1, \ldots, x_k) \in \mathbb{R}_+^k$ such that $x_1 \leq \cdots \leq x_k$,

$$g^{(k)}(x_1, \ldots, x_k) := g\left(\frac{1}{k} \sum_{\ell=1}^k \delta_{x_\ell}\right),$$

we have, for every $n \in \{1, \ldots, k\}$,

$$\partial_\nu g\left(\frac{1}{k} \sum_{\ell=1}^k \delta_{x_\ell}, x_n\right) = k \partial_{x_n} g^{(k)}(x).$$

This suggests natural finite-dimensional approximations of (1-7). Denoting

$$\overline{U}_k := \{q = (q_{1,1}, q_{1,2}, \ldots, q_{1,k}, q_{2,1}, \ldots, q_{2,k}) \in \mathbb{R}_+^{2k} : \text{ for all } a \in \{1, 2\}, \ q_{a,1} \leq \cdots \leq q_{a,k}\},$$

these approximations take the form

$$\partial_t f^{(k)} - k \sum_{\ell=1}^k \partial_{q_{1,\ell}} f^{(k)} \partial_{q_{2,\ell}} f^{(k)} = 0 \quad \text{on } \mathbb{R}_+ \times \overline{U}_k. \quad (1-8)$$
We will define the solution to (1-7) as the limit of such finite-dimensional approximations.\footnote{It may seem somewhat contrived to impose the ordering of the variables \( q_{a, 1} \leq \cdots \leq q_{a, k} \). However, in the proof of Theorem 1.1, this formulation will allow for a clearer treatment of the boundary condition on the “diagonal part”, i.e., whenever \( q_{a, \ell} = q_{a, \ell + 1} \) for some \( \ell \in \{1, \ldots, k - 1\} \). (This point was overlooked in a preliminary version of the paper.) Moreover, in more general models, the relevant variables are matrix-valued, and there is no simple “symmetrization” of an ordered tuple of symmetric matrices, so there is no way around working with a set of the form of \( \mathcal{U}_k \) in this more general setting.}

That there exists a connection between the free energy of spin glass models and certain infinite-dimensional Hamilton–Jacobi equations was first observed in the context of mixed \( p \)-spin models [Mourrat 2019]. In these models, the energy function \( H_N \) is a centered Gaussian field such that the covariance between \( H_N(\sigma) \) and \( H_N(\tau) \) is proportional to \( \xi(\sigma \cdot \tau / N) \), where the function \( \xi \) is fixed and can be written in the form \( \xi(r) = \sum_{p \geq 2} \beta_p r^p \), for some family of coefficients \( \beta_p \geq 0 \) that decays sufficiently fast. (The constraint \( \beta_p \geq 0 \) is necessary and sufficient in order for \( \xi \) to define a covariance kernel for every \( N \) [Schoenberg 1942].) For these models, the corresponding Hamilton–Jacobi equation takes the form

\[
\partial_t f - \int \xi(\partial_\mu f) \, d\mu = 0 \quad \text{on } \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}_+). \tag{1-9}
\]

With this in mind, it is natural to distinguish between three increasingly large classes of models. The first is the class of models for which the mapping \( \xi \) is convex over \( \mathbb{R} \); roughly speaking, these are the models whose limit free energy can be identified using the methods of [Guerra 2003; Talagrand 2006b; 2011a; 2011b] (in fact, the precise condition is slightly more restrictive; see [Talagrand 2011b, (14.101)]). An extension of this approach, developed in [2013a; Panchenko 2013b], allows us to cover all mixed \( p \)-spin models. The convexity property, once properly understood, is still fundamental in this setting. More precisely, one can check that the relevant solution to (1-9) satisfies \( \partial_\mu f \geq 0 \). On the other hand, in view of the form of \( \xi \), this function is convex over \( \mathbb{R}_+ \). In other words, we can redefine the function \( \xi \) to be \(+\infty\) over \((-\infty, 0)\); with this new definition, the relevant Hamilton–Jacobi equation is still (1-9), and now the convexity of the nonlinearity has been restored. This convexity is crucial to the validity of a Hopf–Lax formula for the solution, and this variational formula forms the basis of the arguments for identifying the limit free energy in these approaches.

The third class of models corresponds to situations in which the nonlinearity in the Hamilton–Jacobi equation may be genuinely nonconvex; a representative example in this class is the focus of the present paper. In this case, it is unclear whether the limit free energy can be described as a (reasonable) variational problem. The classical Hopf–Lax variational formula requires that the nonlinearity in the equation be convex (or concave), which it is clearly not in our setting. Alternatively, irrespective of the structure of the nonlinearity, the solution to a Hamilton–Jacobi equation can always be written as a saddle-point problem, provided that the initial condition is concave (or convex) [Hopf 1965; Bardi and Evans 1984; Lions and Rochet 1986]. This motivates us to study the concavity of the initial condition in (1-7), that is, the function \( \psi \) in (1-7). In the context of mixed \( p \)-spin models, the main result of [Auffinger and Chen 2015] implies the concavity of this function. I do not know whether this argument can be generalized to cover the bipartite model investigated here. But in any case, this does not seem to be the appropriate notion of concavity to guarantee the validity of a saddle-point formulation for the solution to (1-7).
order for this to work, we would need instead that the function $\psi$ be transport-concave (one may also say “displacement-concave”); but we will see that this is not so in general. At present, my impression is that it is not possible to express the limit free energy as a saddle-point problem in general, and that it would be very difficult to circumvent a description of this limit involving Hamilton–Jacobi equations.

We now discuss the intuition behind Theorem 1.1. The simplest setting in which to explain the idea is that of the Curie–Weiss model; see for instance [Mourrat 2017]. The main point is to enrich the model to include “non-interacting” terms in the energy function, with the hope that, if these simpler terms are sufficiently “expressive”, then certain asymptotic relations between the derivatives of the free energy will have to be satisfied. In our context, a first attempt is to try to compare $\sum J_{ij}\sigma_i\sigma_j$ with a linear combination of $z_1\cdot\sigma_1$ and $z_2\cdot\sigma_2$, where $z = (z_1, z_2) = (z_{1,1}, \ldots, z_{1,N}, z_{2,1}, \ldots, z_{2,N})$ is a vector of independent standard Gaussians. In other words, we consider, for every $t, p_1, p_2 \geq 0$, the free energy $G_N(t, p_1, p_2) :=$

$$-\frac{1}{N} \mathbb{E} \log \int \exp \left( \sqrt{2t} H_N(\sigma) - N^{-1} t |\sigma|^2 + \sum_{a=1}^2 \left( \sqrt{2p_a} z_a \cdot \sigma_a - p_a |\sigma_a|^2 \right) \right) \, dP_N(\sigma). \quad (1-10)$$

(Parametrizations of the form $\sqrt{t} X$ where $X$ is a Gaussian random variable are of course natural: think of Brownian motion. Each random variable in the exponential comes with a compensating term, so that the expectation of the exponential is equal to 1.) Denoting by $\langle \cdot \rangle$ the expectation with respect to the Gibbs measure proportional to $\exp(\cdot \cdot \cdot) \, dP_N(\sigma)$, one can check that

$$\partial_t G_N = N^{-2} \mathbb{E} \langle (\sigma_1 \cdot \sigma_1') (\sigma_2 \cdot \sigma_2') \rangle,$$

where $\sigma'$ denotes an independent copy of $\sigma$ under $\langle \cdot \rangle$. On the other hand,

$$\partial_{P_a} G_N = N^{-1} \mathbb{E} (\sigma_a \cdot \sigma_a') \quad (a \in \{1, 2\}),$$

so that

$$\partial_t G_N - \partial_{P_1} G_N \partial_{P_2} G_N = N^{-2} \mathbb{E} \left[ (\sigma_1 \cdot \sigma_1' - \mathbb{E} (\sigma_1 \cdot \sigma_1')) (\sigma_2 \cdot \sigma_2' - \mathbb{E} (\sigma_2 \cdot \sigma_2')) \right]. \quad (1-11)$$

Hence, if the overlaps $\sigma_a \cdot \sigma_a'$ were concentrated, we would then infer that $G_N$ converges to $g = g(t, p_1, p_2) : \mathbb{R}_+^3 \to \mathbb{R}$ solution to

$$\partial_t g - \partial_{P_1} g \partial_{P_2} g = 0. \quad (1-12)$$

However, as is well-known, the concentration of the overlaps is only valid in a high-temperature (that is, small $t$) region; a more refined enriched system is necessary to “close the equation” in general. The formal manipulation allowing us to obtain the true equation from the “naive” (or replica-symmetric) one given in (1-12) consists simply in replacing the variables $(p_1, p_2)$ encoding the strength of the extraneous random magnetic field by probability measures on $\mathbb{R}_+$, thus leading to the equation in (1-7). Intuitively, the reason why this makes sense is as follows. In the term $\sqrt{2p_a z_a \cdot \sigma_a}$, the magnetic field acting on $\sigma_a$ has a “trivial” structure. However, we need to have access to a richer term that allows to represent extraneous magnetic fields with an ultrametric structure, and this structure is described by its overlap distribution, a probability measure on $\mathbb{R}_+$. This construction, explained precisely below, defines an enriched free energy $\bar{F}_N = \bar{F}_N(t, \mu_1, \mu_2) : \mathbb{R}_+ \times (\mathcal{P}(\mathbb{R}_+))^2 \to \mathbb{R}$, and we will show that this enriched free energy is
asymptotically bounded from below by the solution to (1-7); see Theorem 2.7 for a precise statement. As will be seen in the next section, the corresponding enriched Gibbs measure features extraneous variables, denoted $\alpha$, which are in correspondence with the overlap structure of the random magnetic fields. A crucial step of the argument consists in showing that “typically”, the overlaps $\sigma_a \cdot \sigma'_a$ can be inferred from the knowledge of the overlap between $\alpha$ and $\alpha'$.

We now discuss related works. Fundamental insights on spin glasses, most notably the ultrametricity property, were first identified in the physics literature [Parisi 1979; 1980; Mézard et al. 1987], where variational formulas for limit free energies were predicted. These predictions were then proved rigorously in [Guerra 2003; Talagrand 2006b; 2011a; 2011b] in the setting of mixed $p$-spin models discussed above, under the assumption that the function $\xi$ is convex over $\mathbb{R}$. The extension to the case of general $\xi$ was achieved in [Panchenko 2013a; 2013b], and relies in particular on the justification that typical Gibbs measures are indeed organized along an asymptotically ultrametric structure. Further studies of particular relevance to the current paper concern the synchronization property, for models with multiple types of spins, or vector-valued spins [Panchenko 2015; 2018b; 2018a]. Earlier works on spin-glass models with spins of multiple types include [Talagrand 2009; Bovier and Klimovsky 2009; Barra et al. 2011; 2015; Auffinger and Chen 2014; Baik and Lee 2020].

Heuristic connections between limit free energies and partial differential equations were first pointed out in [Guerra 2001; Barra et al. 2010; 2013; Agliari et al. 2012], under a replica-symmetric or one-step replica symmetry breaking assumption. A rigorous identification of limit free energies of disordered systems in terms of Hamilton–Jacobi equations was obtained in [Mourrat 2017; 2020; Chen 2020; Chen and Xia 2020], in the context of problems of statistical inference. In this latter context, particular properties of the models allow us to “close the equation” using only a finite number of additional variables; in other words, the Hamilton–Jacobi equations appearing there are finite-dimensional. The relevant partial differential equation for mixed $p$-spin models, namely (1-9), was then identified in [Mourrat 2019]; an extension of this convergence, valid for the relevant enriched free energy, was conjectured there, and then proved in [Mourrat and Panchenko 2020]. This last reference also describes how to “remove” compensating terms such as the term $N^{-1} t |\sigma_1|^2 |\sigma_2|^2$ appearing in (1-6), so that we can indeed end up with an upper bound on the limit of (1-2).

The rest of the paper is organized as follows. In Section 2, we define the enriched free energy, record some of its basic properties, and state a generalized version of Theorem 1.1; see Theorem 2.7. In Section 3, we define the precise notion of viscosity solution for (1-8), and define the solution to (1-7) as the limit of such finite-dimensional solutions. In Section 4, we show that if we restrict the free energy to measures that are sums of $k$ Dirac masses with equal weights, then the function we obtain is a supersolution to (1-8), up to an error that goes to 0 as $k$ goes to infinity; this allows us to conclude the proof of Theorem 2.7 (and thus also of Theorem 1.1). A crucial ingredient used in Section 4 is the fact that overlaps synchronize, and the justification of this is deferred to Section 5. In this section, we revisit the synchronization results of [Panchenko 2015], emphasizing the notion of monotone couplings, and giving a “finitary” version of the statement of asymptotic synchronization. Finally, in Section 6, we discuss possible attempts at writing the solution to (1-7) as a saddle-point problem, and show that these tentative formulas are invalid. The Appendix collects a handful of basic results on Gaussian integrals.
2. Definitions and basic properties

We write \(\mathbb{N} = \{0, 1, \ldots\}\) to denote the set of natural numbers, \(\mathbb{N}_\ast := \mathbb{N} \setminus \{0\}\), and \(\mathbb{R}_+ := [0, \infty)\). For every \(x, y \in \mathbb{R}^N\), we write
\[
x \cdot y := \sum_{i=1}^{N} x_i y_i, \quad |x|^2 = x \cdot x.
\]
We always understand a vector \(\sigma \in \mathbb{R}^{2N}\) to be indexed according to \(\sigma = (\sigma_{1,1}, \ldots, \sigma_{1,N}, \sigma_{2,1}, \ldots, \sigma_{2,N})\). We recall that \(H_\sigma(\sigma)\) was defined in (1-1), and notice that, for every \(\sigma, \sigma' \in \mathbb{R}^{2N}\),
\[
\mathbb{E}[H_{\sigma}(\sigma)H_{\sigma'}(\sigma')] = N^{-1} \sum_{i,j=1}^{N} \sigma_{1,i} \sigma_{1',i} \sigma_{2,j} \sigma_{2',j}.
\]
For every \(t \geq 0\), we define
\[
H_N^t(\sigma) := \sqrt{2t} H_N(\sigma) - N^{-1} t |\sigma_1|^2 |\sigma_2|^2.
\]
We are now going to introduce another energy function, parametrized by \(\mu = (\mu_1, \mu_2) \in (\mathcal{P}(\mathbb{R}_+))^2\). It is much more convenient to describe and to work with this object in the case when the measures are discrete, and then simply argue by continuity. We therefore give ourselves an integer \(k \geq 0\), and parameters
\[
0 = \zeta_0 < \zeta_1 \leq \zeta_2 \leq \cdots \leq \zeta_{k-1} \leq \zeta_k < \zeta_{k+1} = 1, \quad (a \in \{1, 2\}),
\]
and we set, for every \(a \in \{1, 2\}\),
\[
\mu_a = \sum_{\ell=0}^{k} (\zeta_{\ell+1} - \zeta_\ell) \delta_{q_a,\ell}.
\]
These measures will serve to parametrize certain ultrametric structures with a prescribed overlap distribution. We instantiate the rooted tree with (countably) infinite degree and depth \(k\) as
\[
\mathcal{A} := \mathbb{N}^0 \cup \mathbb{N} \cup \mathbb{N}^2 \cup \cdots \cup \mathbb{N}^k,
\]
where \(\mathbb{N}^0 = \{\emptyset\}\), and \(\emptyset\) represents the root of the tree. For every \(\alpha \in \mathbb{N}^\ell\), we write \(|\alpha| := \ell\) to denote the depth of the vertex \(\alpha\) in the tree \(\mathcal{A}\). For every leaf \(\alpha = (n_1, \ldots, n_k) \in \mathbb{N}^k\) and \(\ell \in \{0, \ldots, k\}\), we write
\[
\alpha|_\ell := (n_1, \ldots, n_\ell),
\]
with the understanding that \(\alpha|_0 = \emptyset\). We also give ourselves a family \((\zeta_{a,\alpha})_{a \in \mathcal{A}, \alpha \in \mathbb{N}}\) of independent standard Gaussians, independent of \(H_N\), and we let \((v_a)_{a \in \mathbb{N}}\) be a Poisson–Dirichlet cascade with weights given by the family \((\zeta_\ell)_{1 \leq \ell \leq k}\). We refer to [Panchenko 2013b, (2.46)] for a precise definition, and only mention here a few important points. First, in the case \(k = 0\), we simply set \(v_\emptyset = 1\). Second, in the case \(k = 1\), the weights \((v_a)_{a \in \mathbb{N}}\) are obtained by normalizing a Poisson point process on \((0, \infty)\) with intensity measure \(\zeta_1 x^{-1-\zeta_1} \, dx\) so that \(\sum_{a} v_a = 1\). Third, for general \(k \geq 1\), the progeny of each non-leaf
We also allow ourselves to consider multiple independent copies, or “replicas”, of the random variables \((z_{\sigma, i})_{\sigma \in A, i \in \{1, 2\}, 1 \leq i \leq N}\). For every \(\sigma \in \mathbb{R}^{2N}\) and \(\alpha \in \mathbb{N}^k\), we set

\[
H_N^\mu(\sigma, \alpha) := \sum_{a=1}^{2} \left( \sum_{\ell=0}^{k} \left( 2q_a, \ell - 2q_a, \ell - 1 \right) \frac{1}{2} z_{a,\ell, a} \cdot \sigma_a - q_a, k | \sigma_a^2 \right),
\]

where we write \(z_{a, \ell, a} \sigma_a = \sum_{i=1}^{N} z_{a, i, a} \sigma_a, i\). The random variables \((H_N^\mu(\sigma, \alpha))_{\sigma \in \mathbb{R}^{2N}, \alpha \in \mathbb{N}^k}\) form a Gaussian family which is independent of \((H_N(\sigma))_{\sigma \in \mathbb{R}^{2N}}\). We understand that the symbol \(\mathbb{E}\) stands for the expectation with respect to \((J_{ij}), (z_a)_{a \in A}\) and \((v_a)_{a \in \mathbb{N}^k}\). Notice that, for each fixed choice of \(\alpha, \alpha' \in \mathbb{N}^k\), we have

\[
\frac{1}{N} \mathbb{E} \left[ \left( \sum_{\ell=0}^{k} \left( 2q_a, \ell - 2q_a, \ell - 1 \right) \frac{1}{2} z_{a,\ell, a} \right) \cdot \left( \sum_{\ell=0}^{k} \left( 2q_a, \ell - 2q_a, \ell - 1 \right) \frac{1}{2} z_{a', \ell, a} \right) \right] = 2q_a, \alpha \wedge \alpha',
\]

where we write

\[
\alpha \wedge \alpha' := \sup \{ \ell \leq k : \alpha_{\ell} = \alpha'_{\ell} \}.
\]

The point of the construction in (2-7) is to provide with a more refined “external field” than that introduced in (1-10). Indeed, if we sample two independent copies \(\alpha, \alpha' \in \mathbb{N}^k\) according to the weights \((v_a)_{a \in \mathbb{N}^k}\), then the law of overlap

\[
\frac{1}{N} \left( \sum_{\ell=0}^{k} \left( 2q_a, \ell - 2q_a, \ell - 1 \right) \frac{1}{2} z_{a, \ell, a} \right) \cdot \left( \sum_{\ell=0}^{k} \left( 2q_a, \ell - 2q_a, \ell - 1 \right) \frac{1}{2} z_{a', \ell, a} \right)
\]

under the measure in which we average over \((z_a)\) and \((v_a)\) is \(\mu_a\) (this can be inferred from Lemma 2.3 or, more directly, from [Panchenko 2013b, (2.34)]). We define

\[
F_N(t, \mu) := -\frac{1}{N} \log \int \sum_{\alpha \in \mathbb{N}^k} \exp(H_N^t(\sigma) + H_N^\mu(\sigma, \alpha)) \, v_a \, dP_N(\sigma).
\]

We also define the Gibbs measure \(\langle \cdot \rangle\), with canonical random variable \((\sigma, \alpha)\) taking values in \(\mathbb{R}^{2N} \times \mathbb{N}^k\), in such a way that, for any bounded measurable function \(f\),

\[
\langle f(\sigma, \alpha) \rangle := \exp(N F_N(t, \mu)) \int \sum_{\alpha \in \mathbb{N}^k} f(\sigma, \alpha) \exp(H_N^t(\sigma) + H_N^\mu(\sigma, \alpha)) \, v_a \, dP_N(\sigma).
\]

We also allow ourselves to consider multiple independent copies, or “replicas”, of the random variable \((\sigma, \alpha)\), which we may denote by \((\sigma', \alpha'), (\sigma'', \alpha'')\), and so on. Alternatively, in situations where many independent replicas need to be considered, we also denote these replicas by \((\sigma^\ell, \alpha^\ell)_{\ell \geq 1}\). Recall that the measure \(\langle \cdot \rangle\) is itself random; while the replicas are independent under \(\langle \cdot \rangle\), conditionally on the randomness “extraneous” to the measure, they are no longer independent after we average further.
We denote by $\widetilde{F}_N$ the average of $F_N$ with respect to the random variables $(z_a)$ and $(v_a)$. Since the only additional source of randomness in the problem comes from the $J$’s in the definition of $H_N$, and since these are independent random variables, we can write

$$\widetilde{F}_N(t, \mu) = \mathbb{E}[F_N(t, \mu) \mid (H_N(\sigma))_{\sigma \in \mathbb{R}^{2N}}].$$

(2-12)

We also define the fully averaged free energy

$$\overline{F}_N(t, \mu) := \mathbb{E}[F_N(t, \mu)].$$

(2-13)

The notation just introduced suggests that these quantities depend on the parameters $\zeta$ and $q$ in (2-3) and (2-4) only insofar as they affect the measures $\mu_1$ and $\mu_2$. The next proposition states that this is indeed the case, at least as far as the quantities $\widetilde{F}_N(t, \mu)$ and $\overline{F}_N(t, \mu)$ are concerned. (It would make more sense to speak of distributional identities for $H^\mu_N$ and $F_N(t, \mu)$; since such considerations will not play any role in this paper, we simply accept a slightly abusive notation for these latter two quantities.) It also states that $\overline{F}_N(t, \mu)$, and therefore also $\widetilde{F}_N(t, \mu)$, satisfy a Lipschitz estimate in their dependence in $\mu$. Recall that the random variables of $X_\nu$ appearing in the statement were defined in (1-5).

**Proposition 2.1** (Lipschitz continuity of $\overline{F}_N$). The functions $\widetilde{F}_N(t, \mu)$ and $\overline{F}_N(t, \mu)$ depend in the parameters $\zeta$ and $q$ in (2-3) and (2-4) only through their effect on the measures $(\mu_1, \mu_2)$ in (2-5). Moreover, for every $t \geq 0$ and any two pairs $\mu, \mu' \in (\mathcal{P}(\mathbb{R}_+))^2$ of measures of finite support, we have

$$|\overline{F}_N(t, \mu) - \overline{F}_N(t, \mu')| \leq \sum_{a=1}^{2} \mathbb{E}[|X_{\mu_a} - X_{\mu'_a}|],$$

(2-14)

and the same inequality also holds with $\widetilde{F}_N$ replaced by $\overline{F}_N$. In particular, $\widetilde{F}_N$ and $\overline{F}_N$ can be extended by continuity to $\mathbb{R}_+ \times (\mathcal{P}(\mathbb{R}_+))^2$.

One possible way to prove Proposition 2.1 is to rely on the following two results. The first one describes a relatively concrete procedure for computing averages over Poisson–Dirichlet cascades; see [Panchenko 2013b, Theorem 2.9] for a proof.

**Proposition 2.2** (integration of Poisson–Dirichlet cascades). Let $(\omega_a)_{a \in \mathbb{A}}$ be independent and identically distributed random variables taking values in some measurable space $E$, independent of the Poisson–Dirichlet cascade $(v_a)_{a \in \mathbb{A}}$. Let $X_k : E^k \to \mathbb{R}$ be a measurable function, and denote

$$X_{-1} := \mathbb{E} \log \sum_{a \in \mathbb{A}} \exp(X_k(\omega_{a_0}, \ldots, \omega_{a_k})) v_a.$$

In the expression above, the expectation $\mathbb{E}$ is with respect to the law of $(\omega_a)_{a \in \mathbb{A}}$ and $(v_a)_{a \in \mathbb{A}}$. Define recursively, for every $\ell \in \{1, \ldots, k\}$, the measurable function $X_{\ell-1} : E^{k-1} \to \mathbb{R} \cup [+\infty]$ given by

$$X_{\ell-1}(\omega_0, \ldots, \omega_{\ell-1}) := \zeta_{\ell}^{-1} \log \mathbb{E}_{\omega_0} \exp(\zeta_{\ell} X_{\ell}(\omega_0, \ldots, \omega_{\ell})).$$

where, for every $\ell \in \{0, \ldots, k\}$, we write $\mathbb{E}_{\omega_0}$ to denote the integration of the variable $\omega_{\ell}$ along the law of any of the variables $(\omega_a)_{a \in \mathbb{A}}$. We have

$$X_{-1} = \mathbb{E}_{\omega_0}[X_0(\omega_0)].$$
In the statement above, the random variables under each expectation are implicitly assumed to be integrable. In our context, we can apply this lemma in the following way: we set \( \omega_a := \frac{a}{\alpha} \) and, for every \( y_0 = (\omega_a, a \in [1,2], 1 \leq i \leq N), \ldots, y_k = (\omega_a, a \in [1,2], 1 \leq i \leq N) \in \mathbb{R}^{2N} \),

\[
X_k(y_0, \ldots, y_k) := \log \int \exp \left( H_N(\sigma) + \sum_{a=1}^{2N} \left( \sum_{\ell=0}^{k} (2 q_{a,\ell} - 2 q_{a,\ell-1}) \frac{1}{2} y_{\ell,a} \cdot \sigma_{a} - q_{a,k} |\sigma_a|^2 \right) \right) dP_N(\sigma). \quad (2-15)
\]

We then define recursively, for every \( 1 \leq i \leq k \),

\[
X_{\ell-1}(y_0, \ldots, y_{\ell-1}) := \xi_{\ell}^{-1} \log \mathbb{E}_{y_\ell} \exp(\xi_{\ell} X_{\ell}(y_0, \ldots, y_{\ell})), \quad (2-16)
\]

where, for every \( 0 \leq \ell, \ldots, k \), we write \( \mathbb{E}_{y_\ell} \) to denote the integration of the variable \( y_\ell \in \mathbb{R}^{2N} \) along the standard Gaussian measure. Proposition 2.2 then ensures that

\[
-N \tilde{F}_N(t, \mu) = \mathbb{E}_{y_0}[X_0(y_0)].
\]

(A more careful argument would start by using Proposition 2.2 to verify that \( |F_N(t, \mu)| \) is indeed integrable.)

The next lemma identifies the law of the overlap \( \alpha \wedge \alpha' \) under the averaged measure \( \mathbb{E}(\cdot) \). The proof can be found for instance in [Panchenko 2013b, (2.82)] or [Mourrat 2019, Lemma 2.3].

**Lemma 2.3** (overlapping of Poisson–Dirichlet cascades). For every \( \ell \in \{0, \ldots, k\} \), we have

\[
\mathbb{E}(\mathbb{1}_{\{\alpha \wedge \alpha' = \ell\}}) = \xi_{\ell+1} - \xi_{\ell}.
\]

The combination of Proposition 2.2 and Lemma 2.3 allows us to prove Proposition 2.1; see for instance [Mourrat 2019, Proposition 2.1].

While we usually think of \( \tilde{F}_N \) and \( \bar{F}_N \) as functions of the pair of measures \( \mu \), we also allow ourselves to speak of \( \partial_{q_{a,\ell}} F_N \) and \( \partial_{q_{a,\ell}} \tilde{F}_N \); this is meant to refer to the point of view in which these are seen as functions of the families of parameters \( q \) and \( \xi \) in (2-4) and (2-3). Another consequence of Proposition 2.2, which can be found for instance in [Talagrand 2011b, Proposition 14.3.2] or [Mourrat 2019, Lemma 2.4], is that the derivatives of \( \tilde{F}_N \) with respect to each of the parameters \( q_{a,\ell} \) in (2-4) are nonnegative, and they increase with \( \ell \) after suitable normalization. The precise statement is as follows.

**Lemma 2.4.** For every \( a \in \{1, 2\} \) and \( \ell \leq \ell' \in \{0, \ldots, k\} \), we have

\[
\partial_{q_{a,\ell}} \tilde{F}_N \geq 0, \quad (2-17)
\]

and

\[
(\xi_{\ell+1} - \xi_{\ell})^{-1} \partial_{q_{a,\ell}} \tilde{F}_N \leq (\xi_{\ell' + 1} - \xi_{\ell'})^{-1} \partial_{q_{a,\ell}} \tilde{F}_N. \quad (2-18)
\]

**Remark 2.5.** Clearly, the statement of Lemma 2.4 is also valid with \( \tilde{F}_N \) replaced by \( \bar{F}_N \). It is part of the statement of this lemma that the quantity \( (\xi_{\ell+1} - \xi_{\ell})^{-1} \partial_{q_{a,\ell}} \tilde{F}_N \) can be defined even when \( \xi_{\ell+1} = \xi_{\ell} \), by continuity.

Yet another consequence of Proposition 2.2 concerns the “initial condition” for \( \tilde{F}_N \). Under the assumption of (1-3), the verification that \( \tilde{F}_N(0, \mu) \) converges as \( N \) tends to infinity is particularly simple.
Lemma 2.6 (initial condition for product measures). Recall that we assume (1-3). For every $N \geq 1$ and $\mu \in (\mathcal{P}_1(\mathbb{R}_+))^2$, we have

$$F_N(0, \mu) = F_1(0, \mu).$$

Proof. The argument can be found for instance in [Panchenko 2013b, (2.60)]; we present it briefly here for the reader’s convenience. When $t = 0$, and under the assumption of (1-3), the definition of $X_k$ given in (2-15) can be rewritten as

$$X_k(y_0, \ldots, y_k) = \sum_{i=1}^N \log \int \exp \left( \sum_{a=1}^2 \left( \sum_{\ell=0}^{k-1} (2q_{a,\ell} - 2q_{a,\ell-1}) \frac{1}{2} y_{\ell,a,i} - q_{a,k} \sigma_a^2 \right) \right) \, dP_1(\sigma).$$

Recall that $P_1$ is a probability measure over $\mathbb{R}^2$, so in the integral above, the variable $\sigma$ takes the form $\sigma = (\sigma_a)_{a \in \{1, 2\}} \in \mathbb{R}^2$. In particular, we have written $X_k$ as a sum of independent and identically distributed random variables. Moreover, the law of each of these random variables does not depend on $N$. These properties are preserved as we go along the recursive procedure described in (2-16). As we reach $X_{-1}$, all randomness has been integrated out, and the result is thus $N$ times some constant, as desired. □

With an eye towards the initial condition in (1-7), we set, for every $\mu \in (\mathcal{P}_1(\mathbb{R}_+))^2$,

$$\psi(\mu) := F_1(0, \mu).$$

It is worth keeping in mind that the relatively simple definition of the initial condition in (2-19) is possible only because we made the assumption in (1-3) that the underlying measure has a product structure. In general, we only want to ascertain that for every $\mu \in (\mathcal{P}_1(\mathbb{R}_+))^2$,

$$\lim_{N \to \infty} F_N(0, \mu)$$

exists; and in this case, we call the limit $\psi(\mu)$. (We also use in the course of the proof that the support of $P_N$ lies in a ball of fixed radius.) Other choices of reference measure are thus possible: for instance, one may replace $P_N$ by the uniform measure on the product of two $N$-dimensional spheres of radius $\sqrt{N}$. See, for instance, [Mourrat 2019, part (2) of Proposition 3.1] for a similar calculation in this case (which itself borrows from [Talagrand 2006a]).

We now state the extended version of Theorem 1.1 that will be the main focus of the rest of the paper.

Theorem 2.7. For every $t \geq 0$ and $\mu \in (\mathcal{P}_2(\mathbb{R}_+))^2$, we have

$$\liminf_{N \to \infty} F_N(t, \mu) \geq f(t, \mu),$$

where $f : \mathbb{R}_+ \times (\mathcal{P}_2(\mathbb{R}_+))^2 \to \mathbb{R}$ is the solution to (1-7).

The statement of Theorem 1.1 corresponds to the case $\mu = (\delta_0, \delta_0)$ in Theorem 2.7. We now discuss why one should expect that $F_N$ indeed converges to the solution to (1-7). We first observe that

$$\partial_t F_N = \frac{1}{N} \left\{ \frac{1}{\sqrt{2t}} H_N(\sigma) - \frac{1}{N} |\sigma_1|^2 |\sigma_2|^2 \right\}.$$  (2-20)
Taking the expectation, recalling (2-1), and using a Gaussian integration by parts, see (A-2), we obtain that
\[
\frac{\partial_t F_N}{N} = \frac{1}{N^2} \mathbb{E}\langle (\sigma_1 \cdot \sigma_1')(\sigma_2 \cdot \sigma_2') \rangle. \tag{2-21}
\]
By the same reasoning (or see, for instance, [Mourrat 2019, (2.17)]), we have
\[
\frac{\partial_{q_{a,\ell}} F_N}{N} = \frac{1}{N} \mathbb{E}\langle \sigma_a \cdot \sigma_a' | \alpha \land \alpha' = \ell \rangle. \tag{2-22}
\]
Using Lemma 2.3, we can rewrite this identity as
\[
(\zeta_{\ell+1} - \zeta_{\ell})^{-1} \frac{\partial_{q_{a,\ell}} F_N}{N} = \frac{1}{N} \mathbb{E}\langle \sigma_a \cdot \sigma_a' | \alpha \land \alpha' = \ell \rangle,
\]
where the conditional expectation is understood with respect to the measure \(\mathbb{E}\langle \cdot \rangle\). We deduce that
\[
\int \partial_{\mu_1} F_N \partial_{\mu_2} F_N \, d\hat{\mu} = \sum_{\ell=1}^k (\zeta_{\ell+1} - \zeta_{\ell})^{-1} \partial_{q_{1,\ell}} F_N \partial_{q_{2,\ell}} F_N
\]
\[
= \frac{1}{N^2} \sum_{\ell=1}^k \mathbb{E}\langle \mathbb{1}_{\{\alpha \land \alpha' = \ell\}} \mathbb{E}\langle \sigma_1 \cdot \sigma_1' | \alpha \land \alpha' = \ell \rangle \mathbb{E}\langle \sigma_2 \cdot \sigma_2' | \alpha \land \alpha' = \ell \rangle \rangle
\]
\[
= \frac{1}{N^2} \mathbb{E}\langle \mathbb{E}\langle \sigma_1 \cdot \sigma_1' | \alpha \land \alpha' \rangle \mathbb{E}\langle \sigma_2 \cdot \sigma_2' | \alpha \land \alpha' \rangle \rangle.
\]
We can now compare this expression with (2-21), and also with the situation encountered in the more naive attempt leading to (1-11). In the naive attempt, we could only hope to close the equation in situations for which the overlaps \(\sigma_a \cdot \sigma_a'\) are concentrated. In our current more refined attempt, we have instead
\[
\frac{\partial_t F_N}{N} - \int \partial_{\mu_1} F_N \partial_{\mu_2} F_N \, d\hat{\mu} = \frac{1}{N^2} \mathbb{E}\langle (\sigma_1 \cdot \sigma_1' - \mathbb{E}\langle \sigma_1 \cdot \sigma_1' | \alpha \land \alpha' \rangle)(\sigma_2 \cdot \sigma_2' - \mathbb{E}\langle \sigma_2 \cdot \sigma_2' | \alpha \land \alpha' \rangle) \rangle,
\]
and in particular,
\[
|\frac{\partial_t F_N}{N} - \int \partial_{\mu_1} F_N \partial_{\mu_2} F_N \, d\hat{\mu}| \leq \frac{1}{N^2} \sum_{a \in \{1,2\}} \mathbb{E}\langle (\sigma_a \cdot \sigma_a' - \mathbb{E}\langle \sigma_a \cdot \sigma_a' | \alpha \land \alpha' \rangle)^2 \rangle. \tag{2-23}
\]
In other words, we need to argue that the conditional variance of the overlaps \(\sigma_a \cdot \sigma_a'\), given the overlap \(\alpha \land \alpha'\), is small. This is precisely what the synchronization property should give us. (Moreover, there is some flexibility in that we do not need that this conditional variance be small for any single choice of the parameters.) From this point of view, the synchronization property becomes central even for models with a single type, since the point is to monitor synchronization with the extraneous random variables provided by the Poisson–Dirichlet cascade.

3. Viscosity solutions

The first goal of this section is to clarify the exact notion of solution for finite-dimensional approximations to (1-7), and show comparison principles for these finite-dimensional problems. The second goal is to
show that as we increase the dimension, the sequence of finite-dimensional solutions converges to some limit. We then interpret the limit as the solution to (1-7).

In this context, the convex cone $\mathbb{R}^2_+$ plays a fundamental role. In more general models of mean-field spin glasses, this convex cone would have to be replaced by the set of positive semidefinite matrices. In the setting of the bipartite model, matrices are not obviously showing up because, in some sense, we are only looking at the diagonal entries of a 2-by-2 matrix: observe that there is no term of the form $\sigma_1 \cdot \sigma'_2$ on the right side of (2-1). In order to avoid future repetitions, I found it useful to write this section so that it covers the two settings at once. Throughout this section, we keep the integer $D \geq 1$ fixed, and denote by $S_D^+$, $S_D^{++}$, and $S_D^{+++}$ the set of symmetric $D$-by-$D$ matrices, and the subsets of positive semidefinite and positive definite matrices respectively. We define

$$C := (0, \infty)^D \quad \text{or} \quad C := S_D^{++}, \quad (3-1)$$

its closure, given respectively by

$$\overline{C} := \mathbb{R}_+^D \quad \text{or} \quad \overline{C} := S_D^+, \quad (3-2)$$

and observe that $\overline{C}$ is a convex cone within its natural ambient vector space, namely

$$\mathcal{E} := \mathbb{R}^D \quad \text{or} \quad \mathcal{E} := S^D. \quad (3-3)$$

When $\mathcal{E} = S^D$, we interpret the scalar product between two matrices $a = (a_{dd'})_{1 \leq d,d' \leq D}$ and $b = (b_{dd'})_{1 \leq d,d' \leq D}$ in $S^D$ according to

$$a \cdot b := \sum_{d,d' = 1}^D a_{dd'}b_{dd'} = \text{tr}(a^*b),$$

with $a^*$ denoting the transpose of the matrix $a$. We also write $|a| := (a \cdot a)^{1/2}$. In both settings, the convex cone $\overline{C}$ defines a partial order: for every $x, y \in \mathcal{E}$, we write $x \leq y$ whenever $y-x \in \overline{C}$. We could also use the notation $x < y$ whenever $y-x \in \overline{C}$, however I will refrain from doing so, in order to avoid possible confusions that would arise from the fact that the conjunction of $x \leq y$ and $x \neq y$ does not imply $x < y$.

Let $K \geq 1$ be an integer. We define the open set

$$U_K := \{x = (x_1, \ldots, x_K) \in C^K : \text{ for all } k \in \{1, \ldots, K-1\}, \ x_{k+1} - x_k \in \mathcal{C}\}, \quad (3-4)$$

and its closure

$$\overline{U}_K := \{x = (x_1, \ldots, x_K) \in \overline{C}^K : x_1 \leq \cdots \leq x_K\}. \quad (3-5)$$

The first goal of this section is to study the existence and uniqueness of solutions to the equation

$$\partial_t f - H(\nabla f) = 0 \quad \text{in } (0, T) \times U_K, \quad (3-6)$$

for a given locally Lipschitz function $H : \mathcal{E}^K \to \mathbb{R}, \ T \in (0, \infty]$, and with a prescribed initial condition at $t = 0$. In the expression above, we use the notation, with the understanding that $f = f(t, x)$ with $x = (x_1, \ldots, x_K)$,

$$\nabla f := (\partial_{x_1} f, \ldots, \partial_{x_K} f),$$

where in this expression, each $\partial_{x_i} f$ takes values in the set $\mathcal{E}$. We will also impose a Neumann boundary
condition on \( \partial U_K \) for solutions to (3-6). Since the domain \( \overline{U}_K \) has corners, we define the outer normal to a point \( x \in \partial U_K \) as the set
\[
\mathbf{n}(x) := \{ \nu \in \mathcal{E}^K : |\nu| = 1 \text{ and for all } y \in \overline{U}_K, \ (y - x) \cdot \nu \leq 0 \}.
\]
(This definition would have to be modified for non-convex domains.) To display the Neumann boundary condition, we write the equation formally as
\[
\begin{align*}
\partial_t f - H(\nabla f) &= 0 \quad \text{in } (0, T) \times U_K, \\
n \cdot \nabla f &= 0 \quad \text{on } (0, T) \times \partial U_K.
\end{align*}
\] (3-7)

In order to study (3-7), we rely on the notion of viscosity solutions. Although the techniques used here to handle (3-7) do not differ much from classical arguments, I could not find results in the literature that would prove the well-posedness of viscosity solutions in non-smooth domains such as \( \overline{U}_K \). The best result I could find is [Dupuis and Ishii 1991], where the authors consider the case where the domain is the intersection of a finite number of open sets with a smooth boundary that satisfy certain conditions.

Our main interest for studying solutions to (3-7) resides in the fact that we will then define the solution to (1-7) as the limit of solutions to such finite-dimensional problems. Several other works have also considered Hamilton–Jacobi equations posed on spaces of probability measures or other infinite-dimensional spaces [Crandall and Lions 1985; 1986a; 1986b; Feng and Katsoulakis 2009; Feng and Kurtz 2006; Cardaliaguet and Quincampoix 2008; Cardaliaguet and Souquière 2012; Gangbo et al. 2008; Gangbo and Święch 2014; Ambrosio and Feng 2014; Cardaliaguet 2012]. However, I am not aware of results that show the well-posedness of equations of the type of (1-7); or that include the handling of a boundary condition; or that discuss the convergence of finite-dimensional approximations. These aspects will be covered here.

The remainder of this section is made of two parts. We first study finite-dimensional equations of the form (3-7); and then show how to pass to the limit and identify the solution to (1-7).

3A. Analysis of finite-dimensional equations. The precise definition of solutions to (3-7) reads as follows.

**Definition 3.1.** We say that a function \( f \in C([0, T) \times \overline{U}_K) \) is a *viscosity subsolution* to (3-7) if for every \( (t, x) \in (0, T) \times \overline{U}_K \) and \( \phi \in C^\infty((0, T) \times \overline{U}_K) \) such that \( (t, x) \) is a local maximum of \( f - \phi \), we have
\[
(\partial_t \phi - H(\nabla \phi))(t, x) \leq 0 \quad \text{if } x \in U_K,
\]
while, if \( x \in \partial U_K \),
\[
\min(\inf_{\nu \in \mathbf{n}(x)} \nabla \phi \cdot \nu, \partial_t \phi - H(\nabla \phi))(t, x) \leq 0.
\]
(3-8)

We say that a function \( f \in C([0, T) \times \overline{U}_K) \) is a *viscosity supersolution* to (3-7) if for every \( (t, x) \in (0, T) \times \overline{U}_K \) and \( \phi \in C^\infty((0, T) \times \overline{U}_K) \) such that \( (t, x) \) is a local minimum of \( f - \phi \), we have
\[
(\partial_t \phi - H(\nabla \phi))(t, x) \geq 0 \quad \text{if } x \in U_K.
\]
(3-9)
while, if \( x \in \partial U_K \),
\[
\max\left( \sup_{v \in n(x)} \nabla \phi \cdot v, \partial_t \phi - H(\nabla \phi) \right)(t, x) \geq 0. 
\] (3-10)
We say that a function \( f \in C([0, T) \times \overline{U}_K) \) is a \textit{viscosity solution} to (3-7) if it is both a viscosity subsolution and a viscosity supersolution to (3-7).

We may drop the qualifier \textit{viscosity} and simply talk about subsolutions, supersolutions, and solutions to (3-7). We say that a function \( f \in C([0, T) \times U_K) \) is a \textit{solution} to
\[
\begin{cases}
\partial_t f - H(\nabla f) \leq 0 & \text{in } (0, T) \times U_K, \\
n \cdot \nabla f \leq 0 & \text{on } (0, T) \times \partial U_K,
\end{cases}
\] (3-11)
whenever it is a subsolution to (3-7); and similarly with the inequalities reversed for supersolutions.

Historically, the notion of viscosity solutions emerged from the following construction of solutions: for a small parameter \( \varepsilon > 0 \), one considers the solution of the partial differential equation
\[
\begin{cases}
\partial_t f_\varepsilon - H(\nabla f_\varepsilon) = \varepsilon f_\varepsilon & \text{in } (0, T) \times U_K, \\
n \cdot \nabla f_\varepsilon = 0 & \text{on } (0, T) \times \partial U_K,
\end{cases}
\] and then one identifies the viscosity solution to (3-7) as the limit of \( f_\varepsilon \) as \( \varepsilon \) tends to zero. As will be seen below, the limit satisfies a form of maximum principle, as each of these approximations do. One can also consult [Evans 2010, Section III.10.1] for more intuition concerning the definition of viscosity solutions.

The most useful result concerning solutions to (3-7) for our purposes is a comparison principle. For every \( x \in E^K \), we write
\[
|x| := \left( \sum_{k=1}^K |x_k|^2 \right)^{\frac{1}{2}},
\] where \( |x_k| \) stands for the standard Euclidean norm in \( E \), and, for every \( r \in \mathbb{R} \), we write \( r_+ := \max(r, 0) \).

**Proposition 3.2** (comparison principle). Let \( T \in (0, \infty) \), and let \( u \) and \( v \) be respectively a sub- and a super-solution to (3-7) that are both uniformly Lipschitz continuous in the \( x \) variable. We have
\[
\sup_{(0,T) \times \overline{U}_K} (u - v) = \sup_{[0] \times \overline{U}_K} (u - v). 
\] (3-12)
More precisely, let
\[
L := \max(\| \nabla u \|_{L^\infty([0,T) \times U_K)}, \| \nabla v \|_{L^\infty([0,T) \times U_K)}),
\] (3-13)
and, for some arbitrary \( \delta > 0 \), let
\[
V := \sup\left\{ \frac{|H(p') - H(p)|}{|p' - p|} : |p|, |p'| \leq L + \delta \right\}.
\] (3-14)
For every \( R, M \in \mathbb{R} \) such that
\[
M > \sup_{0 \leq t < T, x \in \overline{U}_K} \frac{u(t, x) - v(t, x)}{1 + |x|},
\] (3-15)
the mapping
\[(t, x) \mapsto u(t, x) - v(t, x) - M(|x| + Vt - R)_{+}\]  
achieves its supremum at a point in \([0] \times \overline{U}_K\).

Before turning to the proof of this proposition, it will be useful to identify the cone dual to the convex cone \(\overline{U}_K\).

**Lemma 3.3 (dual cone to \(\overline{U}_K\)).** Let \(\overline{U}_K^*\) denote the cone dual to \(\overline{U}_K\), that is,
\[\overline{U}_K^* := \{x \in \mathcal{E}^K : \text{for all } v \in \overline{U}_K, \ x \cdot v \geq 0\}.\]

We have
\[\overline{U}_K^* = \left\{ x \in \mathcal{E}^K : \text{for all } k \in \{1, \ldots, K\}, \sum_{\ell=k}^{K} x_\ell \geq 0 \right\},\]  
and
\[\overline{U}_K = \{ v \in \mathcal{E}^K : \text{for all } x \in \overline{U}_K^*, \ x \cdot v \geq 0 \}.\]

**Proof.** For concreteness, we write the proof in the case when \(C = S^D_{++}\). Let \(x, v \in (S^D)^K\). Setting \(v_0 := 0\), we have
\[x \cdot v = \sum_{k=1}^{K} x_k \cdot v_k = \sum_{k=1}^{K} \sum_{\ell=k}^{K} x_\ell \cdot (v_k - v_{k-1}).\]  
It is therefore clear that the set on the right side of (3-18) is contained in \(\overline{U}_K^*\) (recall that if \(a, b \in S^D_+\), then \(a \cdot b = |\sqrt{a} \cdot \sqrt{b}|^2 \geq 0\)). Conversely, if \(x \in (S^D)^K\) does not belong to the set on the right side of (3-18), then there exists \(k \in \{1, \ldots, K\}\) such that
\[\sum_{\ell=k}^{K} x_\ell \notin S^D_+\].

Letting \(p\) denote the orthogonal projection onto the eigenspaces with negative eigenvalues of the matrix on the left side of the display above, and setting
\[v = (0, \ldots, 0, p, \ldots, p) \in \overline{U}_K,\]
where there are \(k - 1\) zero terms, we find that \(x \cdot v < 0\), so \(x \notin \overline{U}_K^*\). This shows (3-18).

The proof of (3-19) could be derived from a general statement concerning the bidual of closed convex cones; see Step 1 of the proof of Proposition 3.6 below. We rather provide with a more elementary and explicit argument. It is clear from (3-20) that \(\overline{U}_K\) is contained in the set on the right side of (3-19). Conversely, recall that a matrix \(a \in S^D\) belongs to \(S^D_+\) if and only if, for every \(b \in S^D_+\), we have \(a \cdot b \geq 0\) (see also [Mourrat 2020, Lemma 2.2]). Let \(v\) belong to the set on the right side of (3-19). To see that \(v \in \overline{U}_K\), it thus suffices to show that for every \(b_1, \ldots, b_K \in S^D_+\), one can find \(x_1, \ldots, x_K \in S^D\) such that, for every \(k \in \{1, \ldots, K\}\), we have
\[\sum_{\ell=k}^{K} x_\ell = b_k.\]

It suffices to set, for every \(k \in \{1, \ldots, K\}\), \(x_k := b_k - b_{k+1}\), with the notation \(b_{K+1} := 0\). \(\square\)
Proof of Proposition 3.2. For concreteness, we write the proof in the case when \( C = S^D_{++} \), and leave to the reader the simpler proof in the case when \( C = (0, \infty)^D \).

Since the second part of Proposition 3.2 implies the first part, we focus on the former. Without loss of generality, we assume that the functions \( u \) and \( v \) are continuous on \([0, T] \times \overline{U}_K\) (once the result is proved in this case, we can obtain the general case by approximating \( T \) with a sequence that converges to \( T \) increasingly). We argue by contradiction, and assume that the mapping

\[(t, x) \mapsto u(t, x) - v(t, x) - M(|x| + Vt - R) \]

does not achieve its supremum on \([0] \times \overline{U}_K\). Let \( \varepsilon_0 > 0 \), let \( \theta \in C^\infty(\mathbb{R}) \) be an increasing smooth function such that

\[ \text{for all } r \in \mathbb{R}, \quad (r - \varepsilon_0)_+ \leq \theta(r) \leq r_+, \]

and consider

\[ \Phi(t, x) := M\theta \left( \varepsilon_0 + \sum_{k=1}^K |x_k|^2 \right)^{1/2} \left( \varepsilon_0 + \sum_{\ell=1}^K |x_\ell|^2 \right)^{1/2} + Vt - R \].

(3-21)

We fix \( \varepsilon_0 \in (0, 1] \) sufficiently close to 0 so that

\[ \sup_{[0, T] \times \overline{U}_K} (u - v - \Phi) > \sup_{[0] \times \overline{U}_K} (u - v - \Phi). \]

(3-22)

For every \( k \in \{1, \ldots, K\} \), we have

\[ \partial_{x_k} \Phi(t, x) = \frac{Mx_k}{(\varepsilon_0 + \sum_{\ell=1}^K |x_\ell|^2)^{1/2}} \theta' \left( \varepsilon_0 + \sum_{\ell=1}^K |x_\ell|^2 \right)^{1/2} + Vt - R \]

(3-23)

In particular, we see that

\[ \partial_t \Phi \geq V|\nabla \Phi|. \]

(3-24)

We also record for future use that for every \( t \geq 0 \) and \( x \in \overline{U}_K \),

\[ \Phi(t, x) \geq M(|x| + Vt - R - 1)_. \]

(3-25)

We set \( C_0 := 2V|\vec{t}| + 1 \), where the vector \( \vec{t} \) is explicitly defined in (3-32). For some constant \( \varepsilon > 0 \) to be determined, we define the function

\[ \chi(t, x, x') := \Phi(t, x) + C_0\varepsilon t + \frac{\varepsilon}{T - t} - \varepsilon \sum_{k=1}^K \text{Id} \cdot (x_k + x'_k), \]

and set \( \tilde{\chi}(t, x) := \chi(t, x, x) \). In view of (3-22), we can choose \( \varepsilon > 0 \) sufficiently small that

\[ \sup_{[0, T] \times \overline{U}_K} (u - v - \tilde{\chi}) > \sup_{[0] \times \overline{U}_K} (u - v - \tilde{\chi}). \]

(3-26)

For later purposes, we also impose that \( 2\varepsilon|\vec{t}| \leq \delta \). We now introduce, for every \( \alpha \geq 1, \ t \in [0, T) \),
By (3-25), the definitions of \( M \) and \( \chi \), and the fact that the functions \( u \) and \( v \) are uniformly Lipschitz, we see that the supremum of \( \Psi_\alpha \) is achieved, at a point which we denote by \( (t_\alpha, x_\alpha, t'_\alpha, x'_\alpha) \). We also see that this maximizing quadruple stays in a bounded region as \( \alpha \) tends to infinity, and thus that the quantity

\[
\alpha(|t_\alpha - t'_\alpha|^2 + |x_\alpha - x'_\alpha|^2)
\]

must remain bounded as \( \alpha \) tends to infinity. We infer that, up to the extraction of a subsequence, there exist \( t_0 \in [0, T) \) and \( x_0 \in \overline{U}_K \) such that, as \( \alpha \) tends to infinity, we have \( t_\alpha \to t_0 \), \( t'_\alpha \to t_0 \), \( x_\alpha \to x_0 \), \( x'_\alpha \to x_0 \). Since

\[
\Psi_\alpha(t_\alpha, x_\alpha, t'_\alpha, x'_\alpha) \leq u(t_\alpha, x_\alpha) - v(t'_\alpha, x'_\alpha) - \chi(t, x_\alpha, x'_\alpha),
\]

and

\[
\Psi_\alpha(t_\alpha, x_\alpha, t'_\alpha, x'_\alpha) \geq \sup_{[0, T) \times \overline{U}_K} (u - v - \tilde{\chi})(t_0, x_0),
\]

we deduce, by continuity of \( u, v \), and \( \tilde{\chi} \), that

\[
(u - v - \tilde{\chi})(t_0, x_0) = \sup_{[0, T) \times \overline{U}_K} (u - v - \tilde{\chi}).
\]

In particular, by (3-26), we must have \( t_0 > 0 \), and thus \( t_\alpha > 0 \) and \( t'_\alpha > 0 \) for every \( \alpha \) sufficiently large. By construction, the function

\[
(t, x) \mapsto u(t, x) - v(t'_\alpha, x'_\alpha) - \frac{\alpha}{2}(|t - t'_\alpha|^2 + |x - x'_\alpha|^2) - \chi(t, x, x'_\alpha)
\]

reaches its maximum at \( (t_\alpha, x_\alpha) \). Since \( u \) is a subsolution, at least one of the following two statements hold:

\[
\alpha(t_\alpha - t'_\alpha) + C_0 \varepsilon + \partial_t \Phi(t_\alpha, x_\alpha) - H(\alpha(x_\alpha - x'_\alpha) + \nabla x \chi(t_\alpha, x_\alpha, x'_\alpha)) \leq 0,
\]

\[
x_\alpha \in \partial U_K \quad \text{and} \quad \inf_{\nu \in n(x_\alpha)} (\alpha(x_\alpha - x'_\alpha) + \nabla_x \chi(t_\alpha, x_\alpha, x'_\alpha)) \cdot \nu \leq 0.
\]

In (3-29), we dropped an additional term of \( \varepsilon/(T - t)^2 \) on the left side; this is possible since this term is nonnegative. Notice also that we somewhat reorganized the set of two conditions in (3-8)–(3-9), so that we also allow for the possibility that \( x_\alpha \in \partial U_K \) in (3-29). We now argue that (3-30) cannot hold. By definition of \( n(x_\alpha) \), for every \( \nu \in n(x_\alpha) \), we have \( \nu \cdot (x_\alpha - x'_\alpha) \geq 0 \). We observe that

\[
\nabla_x \chi = \nabla \Phi - \varepsilon \tilde{I},
\]

where we have set

\[
\tilde{I} := \begin{pmatrix}
\text{Id} \\
2 \text{Id} \\
\vdots \\
K \text{Id}
\end{pmatrix} \in (S^D)^K.
\]
Moreover, \( \nabla \Phi(t, x) \) is a vector proportional to \( x \). We now see that, for every \( x \in \partial U_K \) and \( v \in n(x) \), we have \( x \cdot v = 0 \). Indeed, since \( U_K \) is a cone, we have that \( \lambda x \in U_K \) for every \( \lambda \geq 0 \). In particular, we must have that \( (\lambda x - x) \cdot v \leq 0 \) for every \( \lambda \geq 0 \). This can only happen if \( x \cdot v = 0 \). Finally, we show that there exists a constant \( c > 0 \) such that for every \( x \in \partial U_K \) and \( v \in n(x) \), we have
\[
-x \cdot v \geq c.
\] (3-33)

By (3-20), we have
\[
i \cdot v = \sum_{k=1}^{K} \text{Id} \cdot \sum_{\ell=k}^{K} v_{\ell}.
\]

Notice also that \(-v \in U_K^\circ\). By Lemma 3.3, each matrix \( -\sum_{\ell=k}^{K} v_{\ell} \) therefore belongs to \( S_{+}^{D} \). Moreover, for every \( a \in S_{+}^{D} \), we have
\[
\text{Id} \cdot a \geq |a|,
\]
(the left side is the \( \ell^1 \) norm of the eigenvalues of \( a \), the right side the \( \ell^2 \) norm), and thus
\[
-x \cdot v \geq \sum_{k=1}^{K} \left| \sum_{\ell=k}^{K} v_{\ell} \right|.
\]

The right side of this inequality, as a function of \( v \), defines a norm on \((S_{+}^{D})^K\). Using the equivalence of norms and that \( |v| = 1 \), we conclude that (3-33) holds. Combining the preceding observations, we conclude that (3-30) cannot be valid, so (3-29) holds instead.

Similarly, since the function
\[
(i', x') \mapsto v(i', x') - u(i_{\alpha}, x_{\alpha}) + \frac{\alpha}{2} (|i' - i_{\alpha}|^2 + |x' - x_{\alpha}|^2) + \chi (i_{\alpha}, x_{\alpha}, x')
\]
has a local minimum at \((i'_{\alpha}, x'_{\alpha})\), and since \( v \) is a supersolution, at least one of the following two statements must be valid:
\[
\alpha (i_{\alpha} - i'_{\alpha}) - H(\alpha (x_{\alpha} - x'_{\alpha}) + \epsilon i) \geq 0, \quad (3-34)
\]
\[
x'_{\alpha} \in \partial U_K \quad \text{and} \quad \sup_{v \in n(x'_{\alpha})} (\alpha (x_{\alpha} - x'_{\alpha}) + \epsilon i) \cdot v \geq 0. \quad (3-35)
\]

In view of (3-33), we see that (3-35) cannot hold, and therefore (3-34) is valid.

We now show that (3-29) and (3-34) cannot hold simultaneously, thereby reaching the desired contradiction. We temporarily admit that the vectors
\[
\alpha (x_{\alpha} - x'_{\alpha}) + \nabla x \chi (i_{\alpha}, x_{\alpha}, x'_{\alpha}) \quad \text{and} \quad \alpha (x_{\alpha} - x'_{\alpha}) + \epsilon i \quad (3-36)
\]
are both of norm smaller than \( L + \delta \). Admitting this, we use (3-31), the Lipschitz property of \( H \), the fact that \( C_0 = 2V |i| + 1 \), and (3-24), to deduce that (3-29) implies
\[
\alpha (i_{\alpha} - i'_{\alpha}) + \epsilon - H(\alpha (x_{\alpha} - x'_{\alpha}) + \epsilon i) \leq 0,
\]
in contradiction with (3-34).
There remains to verify that the vectors in (3-36) are bounded by $L + \delta$. For convenience, we rewrite the mapping in (3-28) as

$$(t, x) \mapsto u(t, x) - \psi(t, x).$$

We recall that this mapping achieves its maximum at $(t_\alpha, x_\alpha)$, and we aim to show that $|\nabla \psi(t_\alpha, x_\alpha)| \leq L + \delta$. Since $u$ is $L$-Lipschitz, we have, for every $y \in \overline{U}_K$,

$$\psi(t_\alpha, y) - \psi(t_\alpha, x_\alpha) \geq u(t_\alpha, y) - u(t_\alpha, x_\alpha) \geq -L|y - x_\alpha|.$$ 

If $x_\alpha \in U_K$, the desired conclusion follows. Otherwise, we can only infer that, for every $v$ in the set

$$C := \{ \lambda (y - x_\alpha) : \lambda \in [0, \infty), y \in \overline{U}_K \},$$

we have

$$v \cdot \nabla \psi(t_\alpha, x_\alpha) \geq -L|v|.$$ 

We now recall that

$$\nabla \psi(t_\alpha, x_\alpha) = \alpha(x_\alpha - x'_\alpha) + \nabla \Phi(t_\alpha, x_\alpha) - \varepsilon \vec{t}.$$ 

Moreover, we have that $\nabla \Phi(t_\alpha, x_\alpha)$ is proportional to $x_\alpha$, say $\nabla \Phi(t_\alpha, x_\alpha) = \beta x_\alpha$, for some $\beta \geq 0$. Since $\alpha(x'_\alpha - x_\alpha) - \beta x_\alpha = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta} x'_\alpha - x_\alpha\right) \in C$,

we deduce that

$$(\nabla \psi(t_\alpha, x_\alpha) + \varepsilon \vec{t}) \cdot \nabla \psi(t_\alpha, x_\alpha) \leq L|\nabla \psi(t_\alpha, x_\alpha) + \varepsilon \vec{t}|.$$ 

This yields

$$|\nabla \psi(t_\alpha, x_\alpha) + \varepsilon \vec{t}| \leq L + \varepsilon|\vec{t}|,$$

and thus

$$|\nabla \psi(t_\alpha, x_\alpha)| \leq L + 2\varepsilon|\vec{t}|.$$ 

This is the desired result, since we have chosen $\varepsilon > 0$ sufficiently small that $2\varepsilon|\vec{t}| \leq \delta$. The argument for the second vector in (3-36) is similar.

We next provide with the following result on existence of solutions.

**Proposition 3.4** (existence of solutions). *For every uniformly Lipschitz initial condition $f_0 : U_K \to \mathbb{R}$, there exists a viscosity solution $f$ to (3-7) that satisfies $f(0, \cdot) = f_0$. Moreover, the function $f$ is Lipschitz continuous, and we have

$$\| \nabla f \|_{L^\infty(\mathbb{R}_+ \times U_K)} = \| \nabla f_0 \|_{L^\infty(U_K)}.$$ (3-38)

*Proof of Proposition 3.4.* We will prove below that the proposition is valid if we assume furthermore that $H$ is uniformly Lipschitz and that the initial condition is bounded. We first explain why this is sufficient. Denote the right side of (3-38) by $L$. The proof of Proposition 3.2 makes it clear that, if $u$ and $v$ are solutions to (3-7) with the same $L$-Lipschitz initial condition and with the nonlinearity $H$ replaced by $H_1$ and $H_2$ respectively, and if $H_1$ and $H_2$ coincide on a ball of radius $L + 1$, then $u = v$. It follows that, in order...
to build a solution to (3-7), we may as well replace \( H \) by a globally Lipschitz nonlinearity that coincides with \( H \) on the ball of radius \( L + 1 \). Finally, once this is done, we can use the property of finite speed of propagation proved in Proposition 3.2 to remove the constraint that the initial condition is bounded.

The argument for the existence of a solution is as in [Crandall et al. 1992] or [Barles 2013, Theorem 7.1] (in the latter, the initial condition is not assumed continuous, but this additional assumption allows us to conclude that the solution is continuous as well). For bounded initial conditions, this construction provides with bounded solutions.

We now turn to the proof of the fact that the solution thus constructed, which we denote by \( f \), is Lipschitz, and that the identity (3-38) holds. Again we fix \( C = S_{++}^D \) for concreteness, the case \( C = (0, \infty)^D \) being only easier. We argue by contradiction, assuming instead that

\[
\| |\nabla f| |_{L^\infty}(R_+ \times U_K) > \| |\nabla f_0| |_{L^\infty(U_K)} = L.
\]

We recall that we assume here that \( H \) is uniformly Lipschitz; we denote its Lipschitz constant by \( V \). For a constant \( R \in \mathbb{R} \) to be chosen, and \( M = 1 \), we define the function \( \Phi \) as in (3-21). We then set, for constants \( \varepsilon > 0 \), and \( T < \infty \) to be chosen, and every \( t \in [0, T) \),

\[
u(t, x) := f(t, x) + \Phi(t, x).
\]

as well as, for every \( t \geq 0 \) and \( x \in \overline{U}_K \),

\[
u(t, x) := f(t, x) + \Phi(t, x).
\]

Proceeding as in the proof of Proposition 3.2, we verify that \( u_\varepsilon \) and \( v \) are a sub- and a supersolution to (3-7) respectively. We have, for every \( t \in [0, T) \) and \( x \in \overline{U}_K \),

\[
u(t, x) := f(t, x) + \Phi(t, x).
\]

We then choose \( \varepsilon > 0 \) sufficiently small and \( T \) and \( R \in \mathbb{R} \) sufficiently large that

\[
\sup_{0 \leq t < T, x, x' \in \overline{U}_K} (u_\varepsilon(t, x) - v(t, x')) - L|x - x'| > \sup_{x, x' \in \overline{U}_K} (u_\varepsilon(0, x) - v(0, x') - L|x - x'|). \quad (3-40)
\]

We denote by \( \eta > 0 \) the difference between the left side and the right side of this inequality. We also remark that, by (3-39),

\[
\sup_{0 \leq t < T, x, x' \in \overline{U}_K} (u_\varepsilon(t, x) - v(t, x)) = \sup_{x, x' \in \overline{U}_K} (u_\varepsilon(0, x) - v(0, x)). \quad (3-41)
\]

We now let \( \delta' \leq \delta \in (0, 1] \) to be chosen (we will first fix \( \delta \) in terms of \( \eta \), and then \( \delta' \) in terms of \( \delta, \eta \) and moduli of continuity of \( u_\varepsilon, v, \) and \( H \)), and for every \( \alpha \in [1, \infty) \), \( t < T \), \( t' \geq 0 \), and \( x, x' \in \overline{U}_K \), we consider

\[
\Psi_\alpha(t, x, t', x') := u_\varepsilon(t, x) - v(t', x') - (L + \delta t)|x - x'| - \frac{\alpha}{2}|t_\alpha - t'_\alpha|^2 + \delta' \sum_{k=1}^{K} k \text{Id} \cdot (x_k + x'_k).
\]

Since we assume \( f \) to be bounded, the maximum of \( \Psi_\alpha \) is achieved at a point, which we denote
by \((t_0, x_0, t'_0, x'_0)\), and this point remains in a bounded region as \(\alpha\) tends to infinity (this bounded region can be chosen irrespectively of our choice of \(\delta\) and \(\delta'\) sufficiently small). Extracting a subsequence if necessary, we can further assume that \(t_0 \to t_0\), \(t'_0 \to t_0\), \(x_0 \to x_0\), and \(x'_0 \to x'_0\) (the limits of \(t_0\) and \(t'_0\) must be the same, since \(|t_0 - t'_0|^2 = O(\alpha^{-1})\)).

We also have that, for some constant \(C < \infty\),

\[
\Psi_\alpha(t_0, x_0, t_0, x'_0) \geq -C \delta + \sup_{0 \leq t < T} \sup_{x, x' \in \overline{U}_K} (u_v(t, x) - v(t, x') - L|x - x'|),
\]

while

\[
\Psi_\alpha(0, x_0, 0, x'_0) \leq C \delta + \sup_{x, x' \in \overline{U}_K} (u_v(0, x) - v(0, x') - L|x - x'|),
\]

and, using (3-41),

\[
\Psi_\alpha(t_0, x_0, t_0, x_0) \leq C \delta + \sup_{x, x' \in \overline{U}_K} (u_v(0, x) - v(0, x') - L|x - x'|).
\]

Choosing \(\delta > 0\) such that \(4C\delta \leq \eta\), we can thus guarantee that \(t_0 \neq 0\) and \(x_0 \neq x'_0\). More precisely, with this choice of \(\delta > 0\), and using the continuity of \(u_v\) and \(v\), we can ensure that there exists \(\gamma > 0\), not depending on \(\delta'\), such that \(|x_0 - x'_0| \geq \gamma\). As a consequence, we have \(t_0 > 0\), and \(|x_0 - x'_0| \geq \gamma/2\) for every \(\alpha\) sufficiently large. We use again the notation \(i\) from (3-32). Since \(u_v\) is a subsolution, at least one of the following statements holds:

\[
\alpha(t_0 - t'_0) + \delta |x_0 - x'_0| - H \left( (L + \delta t_0) \frac{x_0 - x'_0}{|x_0 - x'_0|} - \delta i \right) \leq 0, \tag{3-42}
\]

\[
x_0 \in \partial \overline{U}_K \quad \text{and} \quad \inf_{\nu \in \mathbf{n}(x_0)} \nu \cdot \left( (L + \delta t_0) \frac{x_0 - x'_0}{|x_0 - x'_0|} - \delta i \right) \leq 0. \tag{3-43}
\]

By (3-33) and the definition of \(\mathbf{n}(x_0)\), the statement in (3-43) cannot hold, and therefore (3-42) is valid. Conversely, since \(v\) is a supersolution, at least one of the following statements holds:

\[
\alpha(t_0 - t'_0) - \delta |x_0 - x'_0| - H \left( (L + \delta t_0) \frac{x_0 - x'_0}{|x_0 - x'_0|} + \delta i \right) \geq 0, \tag{3-44}
\]

\[
x'_0 \in \partial \overline{U}_K \quad \text{and} \quad \sup_{\nu \in \mathbf{n}(x'_0)} \nu \cdot \left( (L + \delta t_0) \frac{x_0 - x'_0}{|x_0 - x'_0|} + \delta i \right) \geq 0. \tag{3-45}
\]

As above, we see that (3-45) cannot hold. We thus conclude that (3-42) and (3-44) are both valid. But, since \(|x_0 - x'_0|\) is bounded away from zero by a quantity not depending on \(\delta'\), and since \(H\) is Lipschitz, we reach a contradiction by selecting \(\delta'\) sufficiently small. \(\square\)

We now point out a convenient way to verify that certain functions satisfy the boundary condition for being a subsolution to (3-7). The condition is a sort of monotonicity property, which we call being “tilted”, and is inspired by Lemma 2.4. Let \(V\) be a subset of \(\mathcal{E}^K\), and \(f : V \to \mathbb{R}\). We say that the function \(f\) is tilted if, for every \(x, y \in V\), we have

\[
y - x \in \overline{U}^*_K \implies f(x) \leq f(y),
\]

where we recall that \(\overline{U}^*_K\) was defined in Lemma 3.3. We may also consider functions \(f\) defined on \(\mathbb{R}_+ \times V\).
(or with $\mathbb{R}_+$ replaced by a subinterval); in this case, we say that the function $f$ is tilted if the function $f(t, \cdot)$ is tilted for every fixed $t \geq 0$. The next lemma provides with a simple characterization of being tilted for Lipschitz functions.

**Lemma 3.5** (characterization of tilted functions). Let $V$ be an open subset of $E^K$, and let $f : V \to \mathbb{R}$ be a Lipschitz function. The function $f$ is tilted if and only if $\nabla f \in \overline{U}_K$ almost everywhere in $V$.

**Proof.** We decompose the proof into two steps.

**Step 1.** We assume that $f$ is tilted, and show that $\nabla f \in \overline{U}_K$ almost everywhere. By Rademacher’s theorem, the function $f$ is differentiable almost everywhere. Let $z \in V$ be a point of differentiability of $f$, and $x \in \overline{U}_K^*$. Since $V$ is open and $f$ is tilted, we have, for every $\varepsilon > 0$ sufficiently small,

$$f(z + \varepsilon x) - f(z) \geq 0.$$

Dividing by $\varepsilon$ and letting $\varepsilon > 0$ tend to zero, we conclude that $x \cdot \nabla f(z) \geq 0$. By Lemma 3.3, this means that $\nabla f(z) \in \overline{U}_K$.

**Step 2.** We assume that $\nabla f \in \overline{U}_K$ almost everywhere, and show that $f$ is tilted. By Fubini’s theorem, the set

$$\{(x, y) \in V^2 : |\{s \in [0, 1] : f \text{ is differentiable at } sy + (1 - s)x\}| = 1\}$$

has full measure (for $I \subseteq [0, 1]$, the notation $|I|$ above denotes its Lebesgue measure). We fix a pair $(x, y)$ in this set. Since the mapping $s \mapsto f(sy + (1 - s)x)$ is Lipschitz, we have

$$f(y) - f(x) = \int_0^1 (y - x) \cdot \nabla f(sy + (1 - s)x) \, ds.$$ 

The result then follows using Lemma 3.3 once more. \qed

Notice that, by Lemmas 2.4 and 3.5, the function

$$(t, q) \mapsto \tilde{F}_N\left(t, \left(\frac{1}{k} \sum_{\ell=1}^k \delta_{q_{a,\ell}}\right)_{a \in \{1, 2\}}\right)$$

is tilted. As announced, the next proposition states that a tilted function automatically satisfies the boundary condition (3-9).

**Proposition 3.6** (boundary condition for subsolution). Let $f \in C([0, T] \times \overline{U}_K)$ be a tilted function, $(t, x) \in (0, T) \times \partial U_K$, and $\phi \in C^\infty((0, T) \times \overline{U}_K)$ be such that $(t, x)$ is a local maximum of $f - \phi$. We have

$$\inf_{v \in \nu(x)} v \cdot \nabla \phi(t, x) \leq 0.$$

**Proof.** We decompose the proof into three steps.

**Step 1.** In this step, we prove a general (and classical) statement concerning the bidual of a closed convex cone. Let $C$ be a closed convex cone, which for simplicity we assume to be in some Euclidean space $E$. 

Let $C'$ be, up to a sign, the cone dual to $C$:

$$C' := \{ y \in E : \text{ for all } x \in C, \ x \cdot y \leq 0 \},$$  \hspace{1cm} (3-46)

and let

$$C'' := \{ x \in E : \text{ for all } y \in C', \ x \cdot y \leq 0 \}. \hspace{1cm} (3-47)$$

In this step, we show that $C'' = C$. Let $f : E \to E$ be such that $f = 0$ on $C$ and $f = +\infty$ otherwise. Its convex dual $f^*$ is such that, for every $y \in E$,

$$f^*(y) = \sup_{x \in E} (x \cdot y - f(x)) = \begin{cases} 0 & \text{ if } y \in C', \\ +\infty & \text{ otherwise.} \end{cases}$$

In the same way, using that $C'$ is a cone, we see that the bidual $f^{**}$ is such that $f^{**} = 0$ on $C''$, and $f^{**} = +\infty$ otherwise. Since $f$ is convex and lower semicontinuous, it is equal to its bidual. This shows that $C'' = C$.

**Step 2.** We now prove another general (and possibly less classical) statement about closed convex cones. Let $E$ be some Euclidean space, and for any $A \subseteq E$, let $\text{int}(A)$ and $\text{conv}(A)$ denote the interior and the convex hull of $A$ respectively. Let $C \subseteq E$ be a closed convex cone, and $C'$ be as in (3-46). Our aim is to show that if the interior of $C$ is not empty, then

$$\text{int}(C) \cap (-C') \neq \emptyset,$$ \hspace{1cm} (3-48)

where we write $-C' := \{ -z : z \in C' \}$. Without loss of generality, we may assume that $C' \neq \{0\}$ (otherwise we have $C = E$, by the result of the previous step, and $0$ belongs to the set on the left side of (3-48)).

We first show that

$$C' \cap (-C') = \{0\}. \hspace{1cm} (3-49)$$

Indeed, if $z \in C' \cap (-C')$, then by the result of the previous step, we must have that $y \cdot z = 0$ for every $y \in C$. Since we assume that the interior of $C$ is not empty, this is only possible if $z = 0$. Let

$$m := \{ z \in C' : |z| = 1 \}.$$

We now show that

$$0 \notin \text{conv}(m). \hspace{1cm} (3-50)$$

Assume instead that $0 \in \text{conv}(m)$. By Carathéodory’s theorem, the point $0$ can then be represented as the barycenter of a finite number of points in $m$. Since $m \subseteq C'$ and $C'$ is convex and contains the origin, this allows us to contradict (3-49). Using Carathéodory’s theorem once more, we can also verify that $\text{conv}(m)$ is compact.

For every $\varepsilon > 0$, we define

$$m_\varepsilon := \{ z \in E : \text{dist}(z, \text{conv}(m)) \leq \varepsilon \},$$

and aim to show that there exists $\varepsilon > 0$ such that

$$C \cap \{ \lambda \nu : \lambda > 0, \ \nu \in m_\varepsilon \} = \emptyset. \hspace{1cm} (3-51)$$
Assume the contrary: for every $\varepsilon > 0$, we could then find $\lambda_\varepsilon > 0$ and $v_\varepsilon \in m_\varepsilon$ such that $\lambda_\varepsilon v_\varepsilon \in \mathcal{C}$. Since $\mathcal{C}$ is a cone, the latter condition means that $v_\varepsilon \in \mathcal{C}$. Since the sets $(m_\varepsilon)_{\varepsilon > 0}$ are compact and nested, and since $\mathcal{C}$ is closed, we can find a limit point $v \in m_0 = \text{conv}(m)$ such that $v \in \mathcal{C}$. We have in particular that $v \in \mathcal{C}'$, but by (3-46), we have $\mathcal{C} \cap \mathcal{C}' = \{0\}$. This implies that $v = 0$. But since $v \in \text{conv}(m)$, this contradicts (3-50).

Notice next that the set $\{\lambda v : \lambda > 0, \ v \in m_\varepsilon\}$ is convex: indeed, for every $\lambda, \lambda' > 0$, $v, v' \in m_\varepsilon$, and $\alpha \in (0, 1)$, we have

$$\alpha \lambda v + (1 - \alpha)\lambda' v' = (\alpha \lambda + (1 - \alpha)\lambda')\left(\frac{\alpha \lambda}{\alpha \lambda + (1 - \alpha)\lambda'} v + \frac{(1 - \alpha)\lambda'}{\alpha \lambda + (1 - \alpha)\lambda'} v'\right),$$

and the quantity between parentheses on the right side belongs to $m_\varepsilon$, since this set is convex. Since $\mathcal{C}$ is also convex, we can find a hyperplane that separates the two disjoint sets appearing in (3-51): there exists $v \in E$, which we may assume to be of unit norm, such that

$$\text{for all } z \in \mathcal{C}, \ z \cdot v \geq 0 \quad \text{and for all } z \in m_\varepsilon, \ v \cdot z \leq 0.$$  

The first property in the previous display yields that $-v \in \mathcal{C}'$. We will now see that $v \in \text{int}(\mathcal{C})$, which will complete the proof of (3-48). For every $v' \in E$ satisfies $|v' - v| \leq \varepsilon$ and $z \in m$, we have

$$v' \cdot z \leq v \cdot z + \varepsilon = v \cdot (z + \varepsilon v) \leq 0,$$

where we used that $z + \varepsilon v \in m_\varepsilon$ in the last inequality. Using the notation in (3-47), this shows that every such $v'$ belongs to $\mathcal{C}''$. By the result of the previous step, we have $\mathcal{C}'' = \mathcal{C}$, and we have thus verified that $v \in \text{int}(\mathcal{C})$.

**Step 3.** We fix $(t, x), \phi$ as in the statement of Proposition 3.6, let

$$\mathcal{C}_0 := \{\lambda(y - x) : \lambda > 0, \ y \in U_K\},$$  

and let $\mathcal{C}$ denote its closure. Since $\mathcal{C}_0$ is open, we have $\text{int}(\mathcal{C}) = \mathcal{C}_0$. For every $\lambda, \lambda' \in (0, \infty)$ and $y, y' \in U_K$, we have

$$\lambda(y - x) + \lambda'(y' - x) = (\lambda + \lambda')\left(\frac{\lambda}{\lambda + \lambda'} y + \frac{\lambda'}{\lambda + \lambda'} y' - x\right) \in \mathcal{C}_0.$$

It follows that $\mathcal{C}_0$ is convex, and thus that $\mathcal{C}$ is a closed convex cone. Let $\mathcal{C}'$ be defined by (3-46), with $E = \mathcal{C}^K$. Since $\mathcal{C}$ has nonempty interior, we can apply the result of the previous step to infer that

$$\mathcal{C}_0 \cap (-\mathcal{C}') \neq \emptyset.$$

Let $v$ denote an element of this set; without loss of generality, we may assume that $v$ is of unit norm. By definition of $\mathcal{C}_0$, there exists $\lambda > 0$ and $y \in U_K$ such that $v = \lambda(y - x)$. Since $-v \in \mathcal{C}'$ and is of unit norm, we also have that $-v \in n(x)$. Since the set $\mathcal{C}$ contains $\overline{U}_K$, we have that $-\mathcal{C}' \subseteq \overline{U}_K^*$, and thus $v \in \overline{U}_K^*$. By convexity of $\overline{U}_K$, for every $\varepsilon \in [0, \lambda^{-1}]$, we have that

$$x + \varepsilon v = (1 - \varepsilon \lambda)x + \varepsilon \lambda y \in \overline{U}_K.$$
By the assumption that \((t, x)\) is a local maximum of \(f - \phi\), for every \(\varepsilon > 0\) sufficiently small, we have

\[ (f - \phi)(t, x + \varepsilon v) \leq (f - \phi)(t, x). \]

Since \(f\) is tilted and \(v \in U_K^*\), we deduce that \(\phi(t, x + \varepsilon v) - \phi(t, x) \geq 0\). Dividing by \(\varepsilon > 0\) and letting it tend to zero, we obtain that

\[ v \cdot \nabla \phi(t, x) \geq 0. \]

Since \(-v \in n(x)\), this is the desired result. \(\square\)

3B. Convergence of finite-dimensional approximations. We now turn to the identification of the solution to (1-7), which we define to be the limit of the solutions to suitable finite-dimensional approximations. From now on, we specialize the results of the previous subsection to the case of

\[ \mathcal{C} := (0, \infty)^2. \]

We aim to approximate each measure in a given pair \((\mu_1, \mu_2) \in (\mathcal{P}(\mathbb{R}_+))^2\) by a measure of the form

\[ \frac{1}{K} \sum_{k=1}^{K} \delta_{x_k, a}, \quad (a \in \{1, 2\}). \quad (3-53) \]

for some (ultimately large) integer \(K\) and some \(x \in \overline{U}_K\), where we set \(x_k = (x_{k,1}, x_{k,2})\), and \(x = (x_1, \ldots, x_K)\). We can clearly map any element of \(\overline{U}_K\) to a pair of probability measures in \((\mathcal{P}(\mathbb{R}_+))^2\) through the mapping defined in (3-53). We can also define a converse operation, from a given pair of measures in \((\mathcal{P}(\mathbb{R}_+))^2\) to an element of \(\overline{U}_K\). Fixing \(K \geq 1\), we set, for every \(\mu = (\mu_1, \mu_2) \in (\mathcal{P}(\mathbb{R}_+))^2\) and \(k \in \{1, \ldots, K\}\),

\[ x_k^{(K)}(\mu) = (x_{k,1}^{(K)}(\mu), x_{k,2}^{(K)}(\mu)) := K \int_{\mathbb{R}_+^2} (F_{\mu_1}^{-1}(u), F_{\mu_2}^{-1}(u)) \, du \in \mathbb{R}_+^2, \quad (3-54) \]

where we recall that the functions \(F_{\mu_a}^{-1}\) were introduced in (1-4). This defines a mapping \(\mu \mapsto x^{(K)}(\mu)\) from \((\mathcal{P}(\mathbb{R}_+))^2\) to \(\overline{U}_K\). Notice that if we map an element \(x\) of \(\overline{U}_K\) to a pair of measures according to (3-53), and then back into an element of \(\overline{U}_K\) through the mapping above, we recover \(x\) (but obviously, some information is lost when we go from a pair of measures to an element of \(\overline{U}_K\) and then back). We also use the notation

\[ \mu^{(K)} = (\mu_1^{(K)}, \mu_2^{(K)}) := \left( \frac{1}{K} \sum_{k=1}^{K} \delta_{x_k^{(K)}(\mu)} \right)_{a \in \{1, 2\}}. \]

The mapping \(\mu \mapsto \mu^{(K)}\) thus takes an element \(\mu\) of \((\mathcal{P}(\mathbb{R}_+))^2\), and returns a pair in \((\mathcal{P}(\mathbb{R}_+))^2\), made of two measures with \(K\) atoms of equal masses (the latter is in some sense the “representative” of \(x^{(K)}(\mu) \in \overline{U}_K\) within the set \((\mathcal{P}(\mathbb{R}_+))^2\)).

The following proposition is the main result of this subsection.
Proposition 3.7 (convergence of finite-dimensional approximations). Let \( \psi \) be the function defined in (2-19), and for each integer \( K \geq 1 \), let \( f^{(K)} : \mathbb{R}_+ \times \bar{U}_K \to \mathbb{R} \) be the viscosity solution to

\[
\begin{aligned}
\frac{\partial_t f^{(K)} - K}{\sum_{k=1}^K \partial_{x_k,1} f^{(K)} \partial_{x_k,2} f^{(K)}} &= 0 & \text{on } (0, \infty) \times \bar{U}_K, \\
\mathbf{n} \cdot \nabla f^{(K)} &= 0 & \text{on } (0, \infty) \times \partial U_K,
\end{aligned}
\]

with initial condition given, for every \( x \in \bar{U}_K \), by

\[
f^{(K)}(0, x) = \psi \left( \frac{1}{K} \sum_{k=1}^K \delta_{x_k,1}, \frac{1}{K} \sum_{k=1}^K \delta_{x_k,2} \right).
\]  

For every \( t \geq 0 \) and \( \mu, \nu \in (\mathcal{P}_2(\mathbb{R}^+))^2 \), the following limit exists and is finite:

\[
f(t, \mu) := \lim_{K \to \infty} f^{(K)}(t, x^{(K)}(\mu)),
\]

where on the right side, we use the notation defined in (3-54). By definition, we interpret this limit as the solution to (1-7). Moreover, there exists a constant \( C < \infty \) such that, for every integer \( K \geq 1 \), \( t \geq 0 \), and \( \mu, \nu \in (\mathcal{P}_2(\mathbb{R}^+))^2 \), we have

\[
|f(t, \mu) - f(t, \nu)| \leq \frac{C}{\sqrt{K}} (t + (\mathbb{E}[X_{\mu_1}^2 + X_{\mu_2}^2])^{1/2}),
\]

as well as

\[
|f(t, \mu) - f(t, \nu)| \leq (\mathbb{E}[|X_{\mu_1} - X_{\nu_1}|^2 + |X_{\mu_2} - X_{\nu_2}|^2])^{1/2}.
\]

Before turning to the proof of Proposition 3.7, we introduce some notation for norms rescaled to be consistent with Wasserstein-type distances on the space of probability measures, according to the correspondences discussed at the beginning of this subsection. For every \( \rho \in [1, \infty] \) and \( x \in \mathcal{E}^K = (\mathbb{R}^2)^K \), we write

\[
|x|_\rho := \left( \frac{1}{K} \sum_{k=1}^K |x_k|^\rho \right)^{\frac{1}{\rho}},
\]

with the usual interpretation as a supremum if \( \rho = \infty \). We also define the norm dual to \( |\cdot|_\rho \) by setting, for \( \tau \in [1, \infty] \) such that \( \frac{1}{\rho} + \frac{1}{\tau} = 1 \) and every \( x \in \mathcal{E}^K = (\mathbb{R}^2)^K \),

\[
|x|_\tau^* := K^{\frac{1}{\tau}} \left( \sum_{k=1}^K |x_k|^{\frac{1}{\tau}} \right)^{\frac{\tau}{\tau}} \left( \frac{1}{K} \sum_{k=1}^K (K|x_k|)^{\frac{1}{\tau}} \right)^{\frac{\tau}{\tau}}.
\]  

We also observe that, by a simple rescaling, the statement of Proposition 3.2 also holds if we replace the displays (3-13)–(3-16), by, respectively,

\[
L := \max(\|\nabla u\|_{2*}, \|\nabla v\|_{2*}, \|\nabla u\|_{L^\infty(0,T \times U_K)}, \|\nabla v\|_{L^\infty(0,T \times U_K)}),
\]

\[
V := \sup \left\{ \frac{|H(p') - H(p)|}{|p' - p|_{2*}} : |p|_{2*}, |p'|_{2*} \leq L + \delta \right\},
\]

\[
M > \sup_{0 \leq t < T, x \in \bar{U}_K} \frac{u(t, x) - v(t, x)}{1 + |x|_2},
\]

\[
(3-62)
\]
and
\[(t, x) \mapsto u(t, x) - v(t, x) - M(|x|_2 + Vt - R)_+.
\]

**Proof of Proposition 3.7.** We decompose the proof into three steps.

**Step 1.** We start with a simple but crucial observation regarding the relationship between the \(f^{(K)}\)'s for different values of \(K\). For all integers \(K, R \geq 1\), we set \(K' := RK\), and for every \(x \in \overline{U}_{K'}\), we define
\[x^{(K,K')} := \left(\frac{1}{R} \sum_{r=1}^{R} x_r, \frac{1}{R} \sum_{r=1}^{R} x_{R+r}, \ldots, \frac{1}{R} \sum_{r=1}^{R} x_{(K-1)R+r}\right) \in \overline{U}_K,
\]
as well as
\[f^{(K,K')}(t, x) := f^{(K)}(t, x^{(K,K')}).
\]
In some sense, the function \(f^{(K,K')}\) is a "lifting" of the function \(f^{(K)}\) to the space \(\overline{U}_{K'}\) (and this lifting is consistent with the identification between measures and elements of \(\overline{U}_K\) discussed at the beginning of this section). Formally, we have for every \(k \in \{1, \ldots, K\}\) and \(r \in \{1, \ldots, R\}\) that
\[
\partial x_{(k-1)R+r} f^{(K,K')}(t, x) = \frac{1}{R} \partial x_k f^{(K)}(t, x^{(K,K')}),
\]
and thus, on a formal level, the function \(f^{(K,K')}\) solves the same equation as \(f^{(K)}\) does, but with a different initial condition. It is not difficult to justify rigorously that \(f^{(K,K')}\) indeed solves this equation in the viscosity sense. In a few words, for instance to verify that \(f^{(K,K')}\) is a subsolution: suppose that \((t, x)\) is a local maximum of \(f^{(K,K')} - \phi\) for some smooth function \(\phi\). Then we can build \(\tilde{\phi} \in C^\infty((0, \infty) \times \overline{U}_K)\) by setting, for every \(x' \in \overline{U}_K\),
\[
\tilde{\phi}(t, x') := \phi(t, x + (x'_1 - x_1^{(K,K')}), \ldots, x'_1 - x_1^{(K,K')}, \ldots, x'_K - x_K^{(K,K')}),
\]
where each coordinate in the inner parenthesis above is repeated \(R\) times. This ensures that \(f^{(K)} - \tilde{\phi}\) has a local maximum at \((t, x^{(K,K')})\). We then use that \(f^{(K)}\) is a subsolution, and the simple relationship between the derivatives of \(\tilde{\phi}\) and those of \(\phi\), to conclude.

**Step 2.** We next leverage on this observation to evaluate the difference between \(f^{(K,K')}\) and \(f^{(K)}\), using Proposition 3.2. Precisely, we will show that there exists a constant \(C < \infty\) such that for every \(t \geq 0\) and \(x \in \overline{U}_{K'},\)
\[
|f^{(K,K')}(t, x) - f^{(K)}(t, x)| \leq \frac{C}{\sqrt{K}}(|x|_2 + t).
\]
(3-63)
By the definition of \(\psi\) in (2-19) and Proposition 2.1, we have that, for every \(x, y \in \overline{U}_K,\)
\[
|f^{(K)}(0, y) - f^{(K)}(0, x)| \leq |x - y|_1.
\]
(3-64)
By Jensen’s inequality, we also have
\[
|f^{(K,K')}(0, y) - f^{(K,K')}(0, x)| \leq |x^{(K,K')} - y^{(K,K')}|_1 \leq |x - y|_1.
\]
(3-65)
Since $|\cdot|_1 \leq |\cdot|_2$, we can appeal to Proposition 3.4 to infer that
\[ \| \nabla f^{(K)} \|_{L^\infty(\mathbb{R}_+ \times U_K)} \leq 1. \quad (3.66) \]
and
\[ \| \nabla f^{(K,K')} \|_{L^\infty(\mathbb{R}_+ \times U_K)} \leq 1. \quad (3.67) \]
In view of the observation preceding this proof, it is thus legitimate to apply Proposition 3.2 with $u$ and $v$ replaced by $f^{(K,K')}$ and $f^{(K')}$, and with the choice of $L = 1$. We also observe that, for every $p, p' \in \mathcal{E}^K = (\mathbb{R}^2)^K$,
\[
\left| K \sum_{k=1}^{K} p_{k,1} p_{k,2} - K \sum_{k=1}^{K} p'_{k,1} p'_{k,2} \right| = \left| K \sum_{k=1}^{K} (p_{k,1} (p_{k,2} - p'_{k,2}) + (p_{k,1} - p'_{k,1}) p'_{k,2}) \right|
\leq \left( K \sum_{k=1}^{K} p'_{k,1} \right)^{\frac{1}{2}} \left( K \sum_{k=1}^{K} (p_{k,2} - p'_{k,2})^2 \right)^{\frac{1}{2}}
+ \left( K \sum_{\ell=1}^{K} (p'_{\ell,2})^2 \right)^{\frac{1}{2}} \left( K \sum_{k=1}^{K} (p_{k,1} - p'_{k,1})^2 \right)^{\frac{1}{2}},
\leq (|p|_2 + |p'|_2) |p - p'|_2.
\]
We can thus for instance choose $V = 3$ when applying Proposition 3.2 to our current setting. For convenience, we also fix $M := 2L + 1 = 3$, which clearly satisfies (3.15) for such choices of $u$ and $v$. We thus deduce that, for every $R \in \mathbb{R}$, the mapping
\[ (t, x) \mapsto f^{(K,K')}(t, x) - f^{(K')}(t, x) - 3(|x|_2 + 3t - R)_+ \quad (3.68) \]
achieves its supremum on $[0] \times \overline{U}_{K'}$. We now derive two different bounds on this supremum, the first one being simple and convenient for large $|x|_2$, the second one covering the case of more moderate values of this quantity. The first bound is a consequence of the estimates (3.66) and (3.67), and of the fact that $f^{(K,K')}(0, 0) = f^{(K')}(0, 0)$: we have
\[ f^{(K,K')}(0, x) - f^{(K')}(0, x) - 3(|x|_2 - R)_+ \leq 3R - |x|_2. \quad (3.69) \]
For the second bound, we first rewrite the supremum of (3.68) over $[0] \times \overline{U}_{K'}$ as
\[
\sup_{x \in \mathcal{E}^{K'}} \left\{ \psi \left( \frac{1}{K'} \sum_{k=1}^{K'} \delta_{x_k} \right) - \psi \left( \frac{1}{K} \sum_{k=1}^{K} \delta_{x_k^{(K,K')}} \right) - 3(|x|_2 - R)_+ \right\}.
\quad (3.70) \]
By (3.64), the difference of $\psi$’s in the supremum above can be bounded by
\[ \frac{1}{K'} \sum_{k=1}^{K} \sum_{r=1}^{R} |x_{(k-1)R+r}^{(K,K')} - x_{k}^{(K,K')}| \leq \frac{1}{K} \sum_{k=1}^{K} \frac{1}{K^2} \sum_{r,r'=1}^{R} |x_{(k-1)R+r} - x_{(k-1)R+r'}|. \quad (3.71) \]
For every $B \in (0, \infty)$, we have
\[
\frac{1}{K'} \sum_{k=1}^{K} |x_k| \mathbb{P}(|x_k| > B) \leq \frac{1}{BK'} \sum_{k=1}^{K} |x_k|^2 = \frac{|x|^2}{B}. \tag{3-72}
\]

On the other hand,
\[
\frac{1}{K} \sum_{k=1}^{K} \frac{2}{R^2} \sum_{1 \leq r' < r \leq R} |x_{(k-1)R+r} - x_{(k-1)R+r'}| \mathbb{P}(|x_{(k-1)R+r'}| < B)
\leq \frac{2}{R^2} \sum_{1 \leq r' < r \leq R} \frac{1}{K} \sum_{k=1}^{K} |x_{(k-1)R+r} - x_{(k-1)R+r'}| \mathbb{P}(|x_{(k-1)R+r'}| < B). \tag{3-73}
\]

By equivalence of norms over $\mathbb{R}^2$, up to a constant factor, we can replace the Euclidean norm in $|x_{(k-1)R+r} - x_{(k-1)R+r'}|$ by the $\ell^1$ norm; and in this case, since $x_{(k-1)R+r'} \leq x_{(k-1)R+r}$, the sum above becomes telescopic. We thus have that, for some absolute constant $C < \infty$,
\[
\frac{2}{R^2} \sum_{1 \leq r' < r \leq R} \frac{1}{K} \sum_{k=1}^{K} |x_{(k-1)R+r} - x_{(k-1)R+r'}| \mathbb{P}(|x_{(k-1)R+r'}| < B) \leq \frac{C B}{K}.
\]

Summarizing, we have shown that, for every $x \in \overline{U}_{K'}$ and $B \in (0, \infty)$,
\[
\left| \psi \left( \frac{1}{K'} \sum_{k=1}^{K'} \delta_{x_k} \right) - \psi \left( \frac{1}{K} \sum_{k=1}^{K} \delta_{x_{(k,K')}} \right) \right| \leq \frac{2|x|^2}{B} + \frac{CB}{K}.
\]

For $B = \sqrt{K} |x|_2$, this becomes, up to a redefinition of $C < \infty$,
\[
\left| \psi \left( \frac{1}{K'} \sum_{k=1}^{K'} \delta_{x_k} \right) - \psi \left( \frac{1}{K} \sum_{k=1}^{K} \delta_{x_{(k,K')}} \right) \right| \leq \frac{C}{\sqrt{K}} |x|_2.
\]

Summarizing, we have thus shown that the quantity inside the supremum in (3-70) is bounded by
\[
\frac{C}{\sqrt{K}} |x|_2 - 3(|x|_2 - R_+).
\]

Notice that the bound in (3-69) is already negative for $|x|_2 \geq 3R$. On the complementary event, the quantity above is clearly bounded by $3CR/\sqrt{K}$. Up to a redefinition of $C < \infty$, we have thus shown that, for every $R > 0$,
\[
\sup_{t \geq 0, x \in \overline{U}_{K'}} \{ f^{(K,K')}(t, x) - f^{(K')}(t, x) - 3(|x|_2 + 3t - R_+) \} \leq \frac{CR}{\sqrt{K}}.
\]

Choosing $R = 3|x|_2 + 3t$ then yields one bound for (3-63). The converse bound is obtained in the same way.

**Step 3.** We complete the proof, by showing that there exists a constant $C$ such that for every $t \geq 0$, $\mu = (\mu_1, \mu_2) \in (P_{\mathbb{R}_+})^2$, and $1 \leq K_1 \leq K_2$, we have
\[
|f^{(K_1)}(t, x^{(K_1)}(\mu)) - f^{(K_2)}(t, x^{(K_2)}(\mu))| \leq \frac{C}{\sqrt{K_1}}((E[X^2_{\mu_1} + X^2_{\mu_2}]^\frac{1}{2} + t). \tag{3-74}
\]
In order to show (3-74), it suffices to verify that, for all integers $K, R \geq 1$, and with $K' := RK$, we have

$$|f^{(K)}(t, x^{(K)}(\mu)) - f^{(K')}(t, x^{(K')}(\mu))| \leq \frac{C}{\sqrt{K}} \left( \E[X_{\mu_1}^2 + X_{\mu_2}^2]^{\frac{1}{2}} + t \right).$$ (3-75)

Indeed, once (3-75) is proved, we can apply it with $(K, K')$ replaced by $(K_1, K_1 K_2)$ and $(K_2, K_1 K_2)$ and obtain (3-74) by the triangle inequality. But (3-75) is almost identical to (3-63): indeed, the latter identity states that the left side of (3-75) is bounded by

$$\frac{C}{\sqrt{K}} (|x^{(K')}(\mu)|_2 + t),$$

and, by Jensen’s inequality,

$$|x^{(K')}(\mu)|_2 = \left( \frac{1}{K'} \sum_{K=1}^{K'} \left| \left( \int_{K=1}^{K} \left( F_{\mu_1}^{-1}(u), F_{\mu_2}^{-1}(u) \right) \, du \right)^{\frac{1}{2}} \leq \left( \E[X_{\mu_1}^2 + X_{\mu_2}^2] \right)^{\frac{1}{2}}. (3-76)$$

Now, it is clear that (3-74) guarantees the existence of the limit in (3-57). It also yields the estimate (3-58), by letting $K_2$ tend to infinity. To show the Lipschitz estimate (3-59), we start from (3-66), which can be rewritten as, for every $t \geq 0$ and $\mu, \nu \in (P_2(\mathbb{R}^+))^2$,

$$|f^{(K)}(t, x^{(K)}(\mu)) - f^{(K)}(t, x^{(K)}(\nu))| \leq |x^{(K)}(\mu) - x^{(K)}(\nu)|_2.$$ (4-1)

As in (3-76), we can then bound the right side above by

$$\left( \E[|X_{\mu_1} - X_{\nu_1}|^2 + |X_{\mu_2} - X_{\nu_2}|^2] \right)^{\frac{1}{2}}.$$

The estimate (3-59) then follows by letting $K$ tend to infinity.

\section{4. The free energy is a supersolution}

The main goal of this section is to show that finite-dimensional approximations of $\tilde{F}_N$ are supersolutions to the finite-dimensional approximations of (1-7), up to a small error. Compared with the previous section, we change the indexing convention and write, for every integer $k \geq 1$,

$$U_k := \{ q = (q_{1,1}, \ldots, q_{1,k}, q_{2,1}, \ldots, q_{2,k}) \in (0, \infty)^{2k} :$$

for all $a \in \{1, 2\}$, for all $\ell \in \{1, \ldots, k - 1\}$, $q_{a, \ell} < q_{a, \ell + 1} \}, (4-1)$$

and we denote the closure of $U_k$ by $\overline{U}_k$.

\textbf{Theorem 4.1} (approximate HJ equation). For each integer $k \geq 1$, $t \geq 0$, and $q \in \overline{U}_k$ indexed as $q = (q_{1,1}, \ldots, q_{1,k}, q_{2,1}, \ldots, q_{2,k})$, denote

$$\tilde{F}_N^{(k)}(t, q) := \tilde{F}_N \left( t, \frac{1}{k} \sum_{\ell=1}^{k} \delta_{q_{1,\ell}}, \frac{1}{k} \sum_{\ell=1}^{k} \delta_{q_{2,\ell}} \right),$$ (4-2)
and let \( f \) be any subsequential limit of \( F_N^{(k)} \) as \( N \) tends to infinity. We have, in the sense of viscosity solutions,

\[
\begin{align*}
\partial_t f - k \sum_{\ell=1}^k \partial_{q_{1,\ell}} f \partial_{q_{2,\ell}} f &\geq -\frac{13}{k} \quad \text{on } (0, \infty) \times U_k, \\
n \cdot \nabla f &\geq 0 \quad \text{on } (0, \infty) \times \partial U_k.
\end{align*}
\]

(4-3)

In the statement above, we understand the notion of subsequential limit in the sense of locally uniform convergence. (By Proposition 2.1, the functions involved are uniformly Lipschitz, and by Lemma 2.6, the initial condition does not depend on \( N \), so the existence of converging subsequences is clear.) Once Theorem 4.1 is proved, we will combine it with the results of the previous section to obtain a proof of Theorem 1.1.

As was announced in Section 2, see in particular (2-23), we need to show that the overlaps \( \sigma_a \cdot \sigma'_a \) are “typically” synchronized with the overlap \( \alpha \land \alpha' \). The argument for achieving this relies on the fact that, possibly after a small perturbation of the energy function, we can ensure that the structure of the Gibbs measure is ultrametric [Panchenko 2013a]. That the ultrametricity can be used to infer synchronization was first observed in [Panchenko 2015]; we revisit the argument in Section 5 below to provide us with a “finitary” version of the statement of synchronization, which is more adapted to the needs of the proof of Theorem 4.1. The small perturbations of the energy function are meant to ensure the validity of the Ghirlanda-Guerra identities. The reader may want to have a brief look at Section 5 to understand better the motivation behind the introduction of such perturbations.

The bird’s eye view proposed above is that typically, the overlaps synchronize. One subtle point is to uncover what “typically” should actually mean. We cannot hope for this synchronization property to hold for every choice of the parameters. Conversely, knowing that synchronization occurs for almost every choice of the parameters is not sufficient. Indeed, this would boil down to considering a limit free energy that satisfies the partial differential equation (3-7) at almost every point. But such a property is unfortunately not sufficient to identify the solution to (3-7) uniquely, and this is the reason why the more involved notion of viscosity solutions is introduced. Roughly speaking, we will be able to show that synchronization occurs at any contact point appearing in the definition of supersolution. More precisely, in the notation of Definition 3.1, at a point where \( f - \phi \) is minimal, we will be able to leverage on the fact that the Hessian of \( f \) must be bounded from below to deduce the validity of the Ghirlanda-Guerra identities, and therefore the synchronization of the overlaps.

We now introduce the small perturbations of the energy function alluded to above. We fix \((\lambda_n)_{n \geq 1}\) an enumeration of the set of rational numbers in [0, 1]. For every integer triple \( h = (h_1, h_2, h_3) \in \mathbb{N}_3^2 \) and \( a \in \{1, 2\} \), we define the random energy \( (H^{a,h}_N (\sigma, \alpha))_{\sigma \in \mathbb{R}^2N, \alpha \in \mathbb{N}^k} \), which is a centered Gaussian field with covariance given, for every \( \sigma, \sigma' \in \mathbb{R}^{2N} \) and \( \alpha, \alpha' \in \mathbb{N}^k \), by

\[
E[H^{a,h}_N (\sigma, \alpha) H^{a,h}_N (\sigma', \alpha')] = N \left( \lambda h_1 \frac{\sigma_a \cdot \sigma'_a}{N} + \lambda h_3 \frac{\alpha \land \alpha'}{k} \right)^{h_3}.
\]

(4-4)

The fact that such a Gaussian random field exists is shown in Lemma A.2 of the Appendix. (It is also seen there that the variables \( \alpha \) can be embedded into a Hilbert space in such a way that \( \alpha \land \alpha' \) becomes the scalar product of the “embedded” variables. Strictly speaking, this observation is required to use
the results of Section 5 with these variables.) We impose the fields \((H_{N}^{a,h})_{a \in \{1,2\}, h \in \mathbb{N}}\) to be independent, and to be independent of the other random variables in the problem. Enlarging the probability space if necessary, we assume that these additional random fields are defined on the probability space with measure \(\mathbb{P}\). Let \(h_+\) be an integer that will be chosen sufficiently large (in terms of \(k\)) in the course of the argument. For convenience, we understand that every element \(x \in \mathbb{R}^{2+2h_+^3}\) is indexed according to

\[ x = (x_1, (x_{1,h})_{h \in \{1,\ldots,h_+\}^3}, x_2, (x_{2,h})_{h \in \{1,\ldots,h_+\}^3}). \]

With this understanding, we set

\[ H_N^{x}(\sigma, \alpha) := N^{-\frac{1}{16}} \sum_{a \in \{1,2\}} (x_a |\sigma_a|^2 + \sum_{h \in \{1,\ldots,h_+\}^3} x_{a,h} H_N^{a,h}(\sigma, \alpha)). \]

The prefactor \(N^{-1/16}\) is meant to ensure that \(H_N^{x}\) will not contribute to the limit free energy; see (4-5) and (4-8). The exponent \(\frac{1}{16}\) is relatively arbitrary; one could replace it by any exponent in the interval \((0, \frac{1}{8})\).

We now define a new free energy that includes the perturbative terms: for every \(t \geq 0\), \(\mu \in (\mathcal{P}(\mathbb{R}_+))^2\) of the form (2-5), and \(x \in \mathbb{R}^{2+2h_+^3}\), we set, with \(H_N^t\) defined in (2-2) and \(H_N^h\) defined in (2-7),

\[ F_N(t, \mu, x) := -\frac{1}{N} \log \int \mathbb{E} \left[ \sum_{\sigma \in h^h} \exp(H_N^t(\sigma) + H_N^h(\sigma, \alpha) + H_N^t(\sigma, \alpha)) \right] P_N(\sigma), \]

as well as

\[ \bar{F}_N(t, \mu, x) := \mathbb{E}[F_N(t, \mu, x)]. \]

In the last two displays, we slightly abuse notation in that we keep denoting the free energy by \(F_N\) (or \(\bar{F}_N\) for its average), although there are now additional variables compared to the quantity defined in (2-10). This abuse of notation does not seem to risk causing much confusion. Indeed, every identity we have seen so far is still valid if \(F_N(t, \mu)\) is replaced by \(F_N(t, \mu, x)\), provided that we redefine the Gibbs measure in (2-11) to include the perturbation terms. Moreover, whenever a risk of confusion arises, we can always write the variables explicitly to dispel it.

We now record a few identities involving the derivatives of \(F_N\) and \(\bar{F}_N\) with respect to this new variable \(x\). We have

\[ \partial_{x_{a,h}} F_N = -N^{-1-\frac{1}{8}} \mathbb{E}(H_N^{a,h}(\sigma, \alpha)) \]

and, by (4-4) and Gaussian integration by parts, see (A-2),

\[ \partial_{x_{a,h}} \bar{F}_N = -N^{-1-\frac{1}{8}} \mathbb{E}(H_N^{a,h}(\sigma, \alpha)) = N^{-\frac{1}{8}} x_{a,h} \mathbb{E}\left( \left( \lambda_{h_1} \sigma_a \cdot \sigma_a' \frac{k}{N} + \lambda_{h_2} \alpha \wedge \alpha' \right)^{h_3} \right) = \left( \lambda_{h_1} |\sigma_a|^2 + \lambda_{h_2} \right)^{h_3}. \]

In particular, recalling that \(\lambda_{h_1}, \lambda_{h_2} \in [0, 1]\), we have

\[ |\partial_{x_{a,h}} \bar{F}_N| \leq 2^{h_3+2} N^{-\frac{1}{8}} |x_{a,h}|. \]

Similarly, for every \(a \in \{1, 2\},\)

\[ \partial_{x_a} F_N = -N^{-1-\frac{1}{8}} \mathbb{E}(|\sigma_a|^2), \]
and in particular,
\[ |\partial_{x_a} F_N| \leq N^{-\frac{1}{p_2}}. \]  
(4-10)

We also have
\[ \partial_{x_a}^2 F_N = -N^{-1-\frac{1}{p_2}} \mathbb{E}[\langle (H_N^{a,h}(\sigma, \alpha))^2 \rangle - \langle H_N^{a,h}(\sigma, \alpha) \rangle^2], \]  
(4-11)
and
\[ \partial_{x_a}^2 F_N = -N^{-1-\frac{1}{p_2}} \mathbb{E}[\langle |\sigma_a|^2 \rangle - \langle |\sigma_a|^2 \rangle^2]. \]  
(4-12)

Before we turn to the proof of Theorem 4.1, we record a useful concentration estimate for the function \( F_N \).

**Proposition 4.2** (concentration of \( F_N \)). Let \( k, h_+ \in \mathbb{N}_+ \) and, for every \( (t, q, x) \in \mathbb{R}_+ \times \overline{U}_k \times \mathbb{R}^{2+h_+^3} \), let
\[ F_N^{(k)}(t, q, x) := F_N \left( t, \frac{1}{k} \sum_{\ell=1}^k \delta_{q_1, \ell}, \frac{1}{k} \sum_{\ell=1}^k \delta_{q_2, \ell}, x \right), \]  
(4-13)
as well as
\[ \bar{F}_N^{(k)}(t, q, x) := \mathbb{E}[F_N^{(k)}(t, q, x)]. \]  
(4-14)

For every \( M < \infty, \ p \in [1, \infty) \) and \( \varepsilon > 0 \), there exists \( C < \infty \) such that for every \( N \geq 1 \),
\[ \mathbb{E} \left[ \sup_{B_M} |F_N^{(k)} - \bar{F}_N^{(k)}|^p \right] \leq CN^{\frac{1}{2}+\varepsilon}, \]
where \( B_M \) denotes the set of \( (t, q, x) \in \mathbb{R}_+ \times \overline{U}_k \times \mathbb{R}^{2+h_+^3} \) for which each coordinate is contained in \([-M, M]\) (in other words, \( B_M \) is the intersection of \( \mathbb{R}_+ \times \overline{U}_k \times \mathbb{R}^{2+h_+^3} \) with the \((3+2k+h_+^3)\)-dimensional \( L^\infty \) ball of radius \( M \)).

**Proof.** By [Panchenko 2013b, Theorem 1.2], there exists \( C < \infty \) such that for every \( (t, q, x) \in B_M \) and \( a \geq 0 \),
\[ \mathbb{P} \left[ |(F_N^{(k)} - \bar{F}_N^{(k)})(t, q, x)|^2 \geq \frac{a}{N} \right] \leq 2 \exp \left( -\frac{a}{C} \right). \]  
(4-15)

In order to conclude, we need some estimate on the modulus of continuity of \( F_N^{(k)} \). We denote
\[ X := 1 + \frac{|(H_N(\sigma))|}{N} + \frac{1}{N} \sum_{a \in \{1, 2\}} \left( \sum_{\ell=0}^k |\langle z_{a(\ell, a) \cdot \sigma_a} \rangle| + \sum_{h \in \{1, \ldots, h_+\}} |\langle H_N^{a,h}(\sigma, \alpha) \rangle| \right). \]

By integration of (2-20), we see that for every \( t, t' \in [0, M], \ q \in \overline{U}_k \), and \( x \in \mathbb{R}^{2+h_+^3} \),
\[ |F_N^{(k)}(t', q, x) - F_N^{(k)}(t, q, x)| \leq C \left( 1 + \frac{|(H_N(\sigma))|}{N} \right) |t' - t|^\frac{1}{2} \]
\[ \leq CX |t' - t|^\frac{1}{2}. \]

Similarly, we can compute, for every \( \ell \in \{1, \ldots, k-1\} \),
\[ \partial_{z_{a,\ell}} F_N^{(k)} = -\frac{1}{N}(2q_a, \ell - 2q_a, \ell-1)^{-\frac{1}{2}} z_{a,\ell-1} \sigma_a - (2q_a, \ell+1 - 2q_a, \ell)^{-\frac{1}{2}} z_{a,\ell+1} \sigma_a, \]
with the understanding that \( q_{a,0} = 0 \) here, and, in the case \( \ell = k \),
\[
\partial_{q_{a,\ell}} F_N^{(k)} = -\frac{1}{N} \langle (2q_{a,k} - 2q_{a,k-1})^{-\frac{1}{2}} z_{a,\ell} \rangle.
\]

By integration, we find that, for every \( q, q' \in \bar{U}_k \), \( t \geq 0 \), and \( x \in \mathbb{R}^{2+h^3}_+ \),
\[
|F_N^{(k)}(t, q', x) - F_N^{(k)}(t, q, x)| \leq CX \sum_{a \in \{1,2\}} \sum_{\ell = 1}^k |q'_{a,\ell} - q_{a,\ell}^\frac{1}{2}|.
\]

Finally, by (4-6) and (4-9), we also have that
\[
|F_N^{(k)}(t, q, x') - F_N^{(k)}(t, q, x)| \leq CN^{-\frac{1}{2}} X |x' - x|.
\]

On the other hand, it follows from (2-21), (2-22), (4-8), and (4-10), that the function \( \bar{F}_N^{(k)} \) is Lipschitz continuous (globally in \( t \) and \( q \), and locally in \( x \)). For every \( \varepsilon \in (0, 1] \), we denote
\[
A_\varepsilon := (\varepsilon \mathbb{Z}^{3+2k+h^3}) \cap B_M.
\]

The previous estimates imply that
\[
\sup_{B_M} |F_N^{(k)} - \bar{F}_N^{(k)}| \leq \sup_{A_\varepsilon} |F_N^{(k)} - \bar{F}_N^{(k)}| + CX \sqrt{\varepsilon},
\]
and therefore, for every \( p \geq 1 \),
\[
\mathbb{E} \left[ \sup_{B_M} |F_N^{(k)} - \bar{F}_N^{(k)}|^p \right] \leq C \mathbb{E} \left[ \sup_{A_\varepsilon} |F_N^{(k)} - \bar{F}_N^{(k)}|^p \right] + C \varepsilon^{\frac{p}{2}} \mathbb{E}[X^p],
\]
with a constant \( C < \infty \) that may depend on \( p \) (in addition to \( k, h^3, \) and \( M \)). We bound the supremum over \( A_\varepsilon \) by the sum over \( A_\varepsilon \) and use (4-15) to get
\[
\mathbb{E} \left[ \sup_{A_\varepsilon} |F_N^{(k)} - \bar{F}_N^{(k)}|^p \right] \leq C |A_\varepsilon| N^{-\frac{p}{2}} = C \varepsilon^{-(3+2k+h^3)} N^{-\frac{p}{2}}.
\]

Using (A-4), we see that, for every \( (t, q, x) \in B_M \),
\[
\mathbb{E} \langle (H_N(\sigma))^2 \rangle \leq CN^{2p},
\]
and similarly, for every \( a \in \{1, 2\} \) and \( \ell \in \{0, \ldots, k\} \),
\[
\mathbb{E} \langle (z_{a,\ell} \cdot \sigma_a)^2 \rangle \leq CN^{2p},
\]
as well as, for every \( h \in \{1, \ldots, h^3\}^3 \),
\[
\mathbb{E} \langle (H_N^{a,h}(\sigma, \alpha))^2 \rangle \leq CN^{2p}.
\]

By Jensen’s inequality, this implies that \( \mathbb{E}[X^p] \leq C \) (in other words, \( \mathbb{E}[X^p] \) is bounded uniformly over \( N \)). We have thus shown that, with \( \alpha := 3 + 2k + h^3 \),
\[
\mathbb{E} \left[ \sup_{B_M} |F_N^{(k)} - \bar{F}_N^{(k)}|^p \right] \leq C \varepsilon^{\frac{p}{2}} N^{-\frac{p}{2}} + C \varepsilon^{\frac{p}{2}}.
\]
Choosing $\varepsilon = N^{-p/(p+2\alpha)}$, we can bound the right side above by $CN^{-p/(2(p+2\alpha))}$. By taking $p$ sufficiently large, we can bring the exponent $\frac{p}{2(p+2\alpha)}$ as close to $\frac{1}{2}$ as desired. By Jensen’s inequality, this proves the claim.

Proof of Theorem 4.1. We fix the integer $h_+$ sufficiently large that, with the choice of $\varepsilon = k^{-4}$, the statement of Proposition 5.5 holds for some $\delta \geq h_+^{-1}$. Recall that we slightly abuse notation and write $F(k)_N$ both to denote the function in (4-2) and the function in (4-14). We can dispel the confusion by writing $(t, q) \mapsto F(k)_N(t, q)$ for the former and $(t, q, x) \mapsto F(k)_N(t, q, x)$ for the latter. In order to lighten the notation, we drop the superscript $(k)$ and simply write $F_N$ in place of $F(k)_N$ throughout (and similarly for $F(k)_N$). Let $f$ be a subsequential limit of the mapping $(t, q) \mapsto F_N(t, q)$. For convenience, we omit to denote the particular subsequence along which the convergence of $(t, q) \mapsto F_N(t, q)$ to $f$ holds.

Let $(t_\infty, q_\infty) \in (0, \infty) \times \overline{U}_k$ and $\phi \in C^\infty((0, \infty) \times \overline{U}_k)$ be such that $f - \phi$ has a local minimum at $(t_\infty, q_\infty)$. Without loss of generality, we may assume that

$$q_\infty \in \partial \overline{U}_k \implies \max_{\nu \in \partial \nu(t_\infty)} \nu \cdot \nabla \phi(t_\infty, q_\infty) < 0,$$  \hspace{1cm} (4-16)

since in the complementary event, the Neumann boundary condition is satisfied. Under this condition, we will show that

$$\left( \partial_t \phi - k \sum_{\ell=1}^k \partial_{q_\ell} \phi \partial_{q_\ell} \phi \right)(t_\infty, q_\infty) \geq -\frac{13}{k}.$$  \hspace{1cm} (4-17)

Throughout the rest of this proof, we denote by $C < \infty$ a constant whose value may change from one occurrence to another, and is allowed to depend on $k, h_+$ (which itself has already been fixed in terms of $k$), $t_\infty, q_\infty$, and the function $\phi$. We write

$$x_\infty := (1, \ldots, 1) \in \mathbb{R}^{2+2h_+^3}.$$  \hspace{1cm} (4-18)

For every $(t, q) \in (0, \infty) \times \overline{U}_k$, and $x \in \mathbb{R}^{2+2h_+^3}$, we set

$$\tilde{\phi}(t, q, x) := \phi(t, q) - (t-t_\infty)^2 - |q-q_\infty|^2 - |x-x_\infty|^2.$$  \hspace{1cm} (4-19)

The mapping $(t, q, x) \mapsto f(t, q) - \tilde{\phi}(t, q, x)$ has a strict local minimum at $(t_\infty, q_\infty, x_\infty)$. In view of (4-8) and (4-10), the mapping $(t, q, x) \mapsto F_N(t, q, x)$ converges to the mapping $(t, q, x) \mapsto f(t, q)$ locally uniformly. We deduce that there exist $(t_N, q_N, x_N) \in (0, \infty) \times \overline{U}_k \times \mathbb{R}^{2+2h_+^3}$ satisfying

$$\lim_{N \to \infty} (t_N, q_N, x_N) = (t_\infty, q_\infty, x_\infty)$$  \hspace{1cm} (4-20)

and such that, for every $N$ sufficiently large, the function $\tilde{F}_N - \tilde{\phi}$ has a local minimum at $(t_N, q_N, x_N)$; more precisely, for every $N$ sufficiently large,

$$(\tilde{F}_N - \tilde{\phi})(t_N, q_N, x_N) = \inf \{(\tilde{F}_N - \tilde{\phi})(t, q, x) : |t-t_N| + |q-q_N| + |x-x_N| \leq C^{-1}\}. $$  \hspace{1cm} (4-21)
We decompose the rest of the proof into five steps.

**Step 1.** In this step, we show that for every \( N \) sufficiently large and \( |x| \leq C^{-1} \),

\[
-C|x|^2 \leq \bar{F}_N(t_N, q_N, x_N + x) - \bar{F}_N(t_N, q_N, x_N) - x \cdot \nabla_x \bar{F}_N(t_N, q_N, x_N) \leq 0. \tag{4-25}
\]

The second inequality follows from the fact that \( \bar{F}_N \) is a concave function of \( x \) (it is classical to verify that the function \( F_N \) itself is concave in \( x \), since the Hessian of this function is a covariance matrix, up to a minus sign). To show the first inequality in (4-25), we start by writing Taylor’s formula:

\[
\bar{F}_N(t_N, q_N, x_N + x) - \bar{F}_N(t_N, q_N, x_N) \\
= x \cdot \nabla_x \bar{F}_N(t_N, q_N, x_N) + \int_0^1 (1-s)x \cdot \nabla_x^2 \bar{F}_N(t_N, q_N, x_N + sx)x \ ds, \tag{4-26}
\]

where \( \nabla_x^2 \bar{F}_N \) denotes the Hessian of the function \( \bar{F}_N \) in the \( x \) variable. Naturally, the formula above is also valid if we replace \( \bar{F}_N \) by \( \tilde{\phi} \). By (4-21), we have that for every \( |x| \leq C^{-1} \),

\[
\bar{F}_N(t_N, q_N, x_N + x) - \bar{F}_N(t_N, q_N, x_N) \geq \tilde{\phi}(t_N, q_N, x_N + x) - \tilde{\phi}(t_N, q_N, x_N).
\]

Using also (4-24), we obtain that

\[
\int_0^1 (1-s)x \cdot \nabla_x^2 \bar{F}_N(t_N, q_N, x_N + sx)x \ ds \geq \int_0^1 (1-s)x \cdot \nabla_x^2 \tilde{\phi}(t_N, q_N, x_N + sx)x \ ds \geq -C|x|^2.
\]

Combining this with (4-26) yields (4-25).

**Step 2.** We show that, for every \( \varepsilon > 0 \),

\[
\mathbb{E}[|\nabla_x (F_N - \bar{F}_N)(t_N, q_N, x_N)|^2] \leq CN^{-\frac{1}{2} + \varepsilon}, \tag{4-27}
\]

where now we also allow the constant \( C < \infty \) to depend on the choice of \( \varepsilon > 0 \). As observed in the previous step, the function \( F_N \) is concave in the \( x \) variable. We thus have, for every \( x \in \mathbb{R}^{2+2h^3} \),

\[
F_N(t_N, q_N, x_N + x) \leq F_N(t_N, q_N, x_N) + x \cdot \nabla_x F_N(t_N, q_N, x_N).
\]

By (4-25), we also have, for every \( |x| \leq C^{-1} \),

\[
\bar{F}_N(t_N, q_N, x_N + x) \geq \bar{F}_N(t_N, q_N, x_N) + x \cdot \nabla_x \bar{F}_N(t_N, q_N, x_N) - C|x|^2.
\]
For a (deterministic) parameter \( \lambda \in [0, \, C^{-1}] \) to be determined in the course of the argument, we combine the two inequalities above and fix
\[
x = \lambda \frac{\nabla_x (F_N - F_N)(t_N, q_N, x_N)}{\| \nabla_x (F_N - F_N)(t_N, q_N, x_N) \|},
\]
so that \( |x| \leq C^{-1} \), and, for this choice of \( x \),
\[
\lambda \| \nabla_x (F_N - F_N)(t_N, q_N, x_N) \| \leq (F_N - F_N)(t_N, q_N, x_N + x) - (F_N - F_N)(t_N, q_N, x_N) + C\lambda^2.
\]
By Proposition 4.2, we infer that
\[
\lambda^2 \mathbb{E}[\| \nabla_x (F_N - F_N)(t_N, q_N, x_N) \|^2] \leq CN^{-1+2\varepsilon} + C\lambda^4.
\]
Choosing \( \lambda = N^{-1/4+\varepsilon/2} \) yields (4-27).

**Step 3.** We show that the Gibbs measure associated with the choice of parameters \((t_N, q_N, x_N)\) satisfies approximate Ghirlanda–Guerra identities, in the following sense. Recall that we denote by \((\sigma^\ell, \alpha^\ell)_{\ell \geq 1}\) a family of independent copies of \((\sigma, \alpha)\) under \( \langle \cdot \rangle \). For each \( \ell, \ell' \in \mathbb{N}_a \) and \( a \in \{1, 2\} \), we write
\[
R^{\ell, \ell'}_0 := \frac{\sigma^\ell \wedge \alpha^\ell}{k}, \quad R^{\ell, \ell'}_a := \frac{\sigma^\ell_a \cdot \sigma^\ell_a}{N^a}.
\]
and, for each \( n \in \mathbb{N}_a \), we denote by \( R^{\leq n} \) the array
\[
R^{\leq n} := (R^{\ell, \ell'}_a)_{a \in \{0, 1, 2\}, \ell, \ell' \in [1, \ldots, n]}.
\]
In this step, we show that, for every \( \varepsilon > 0 \), \( a \in \{1, 2\} \), \( n, h_1, h_2, h_3 \in \{1, \ldots, h_+\} \), and \( g \in C(\mathbb{R}^{3n^2}) \) satisfying \( \| g \|_{L^\infty} \leq 1 \), we have
\[
\mathbb{E}(f(R^{\leq n})(\lambda h_1 R^{1,n+1}_a + \lambda h_2 R^{0,1,n+1}_a h_3)) - \frac{1}{n} \mathbb{E}(g(R^{\leq n})) \mathbb{E}((\lambda h_1 R^{1,2}_a + \lambda h_2 R^{1,2}_a h_3)) \leq CN^{-1/2+\varepsilon}, \quad (4-28)
\]
where we understand that the Gibbs measure \( \langle \cdot \rangle \) is with the parameters \((t_N, q_N, x_N)\). It follows from (4-25) that
\[
-C \leq \nabla_x^2 F_N(t_N, q_N, x_N) \leq 0.
\]
In particular, by (4-11), for every \( a \in \{1, 2\} \) and \( h = (h_1, h_2, h_3) \in \{1, \ldots, h_+\}^3 \), we have
\[
\mathbb{E}[(\langle H^{a,h}_N(\sigma) \rangle^2 - \langle H^{a,h}_N(\sigma) \rangle)^2] \leq CN^{1+\frac{3}{8}},
\]
and similarly, by (4-12),
\[
\mathbb{E}[(\langle |\sigma_a| \rangle^4 - \langle |\sigma_a| \rangle^2)^2] \leq CN^{1+\frac{3}{8}}.
\]
By (4-6) and (4-27), we also have
\[
\mathbb{E}[(\langle H^{a,h}_N(\sigma) \rangle - \mathbb{E}(H^{a,h}_N(\sigma)))^2] \leq CN^{2+\frac{3}{8}+\varepsilon},
\]
and similarly, by (4-9) and (4-27),
\[ \mathbb{E}[(|\sigma|)^2] \leq CN^{\frac{3}{2} + \frac{1}{8} + \varepsilon}. \]

Since, for any random variable \( X \), we have the variance decomposition
\[ \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}((X - \langle X \rangle)^2) + \mathbb{E}((\langle X \rangle - \mathbb{E}(X))^2), \]
we deduce that
\[ \mathbb{E}((H_{\mathbb{N}}^{a,h}(\sigma) - \mathbb{E}(H_{\mathbb{N}}^{a,h}(\sigma)))^2) \leq CN^{\frac{3}{2} + \frac{1}{8} + \varepsilon}, \quad (4-29) \]
and
\[ \mathbb{E}((|\sigma|)^2 - \mathbb{E}(|\sigma|)^2) \leq CN^{\frac{3}{2} + \frac{1}{8} + \varepsilon}. \quad (4-30) \]

It follows from (4-29) that
\[ \left| \mathbb{E}(g(R^{\leq n})H_{\mathbb{N}}^{a,h}(\sigma^1, \alpha^1)) - \mathbb{E}(g(R^{\leq n}))\mathbb{E}(H_{\mathbb{N}}^{a,h}(\sigma^1, \alpha^1)) \right| \leq CN^{\frac{3}{2} + \frac{1}{8} + \varepsilon}. \]

Recall the expression for \( \mathbb{E}(H_{\mathbb{N}}^{a,h}(\sigma, \alpha)) \) in (4-7). By Gaussian integration by parts, see (A-2), we also have
\[
N^{-1 + \frac{1}{16}} \mathbb{E}(g(R^{\leq n})H_{\mathbb{N}}^{a,h}(\sigma^1, \alpha^1)) = \sum_{\ell=1}^{n} x_{a,h} \mathbb{E} \left( g(R^{\leq n}) \left( \lambda_{h_1} \frac{\sigma_a^1 \cdot \sigma_a^\ell}{N} + \lambda_{h_2} \frac{\alpha^1 \wedge \alpha^\ell}{k} \right)^{h_3} \right)
- n x_{a,h} \mathbb{E} \left( g(R^{\leq n}) \left( \lambda_{h_1} \frac{\sigma_a^1 \cdot \sigma_a^{n+1}}{N} + \lambda_{h_2} \frac{\alpha^1 \wedge \alpha^{n+1}}{k} \right)^{h_3} \right),
\]
where we dropped the dependence on \( N \) and simply wrote \( x_{a,h} \) for the \((a, h)\) coordinate of the vector \( x_N \).

Recall that \( x_N \to x_\infty \) with \( x_\infty \) defined in (4-18), so that for \( N \) sufficiently large, we have \( x_{a,h} \geq \frac{1}{2} \) (this is the point of defining \( x_\infty \) in this way, as opposed to setting \( x_\infty = 0 \)). Matching the term indexed by \( \ell = 1 \) in the sum above with the last term in (4-7), we would like to show that the difference
\[ \mathbb{E} \left( g(R^{\leq n}) \left( \lambda_{h_1} \frac{|\sigma_a^1|^2}{N} + \lambda_{h_2} \right)^{h_3} \right) - \mathbb{E} \left( g(R^{\leq n}) \right) \mathbb{E} \left( \left( \lambda_{h_1} \frac{|\sigma_a^1|^2}{N} + \lambda_{h_2} \right)^{h_3} \right) \]
is small. Using (4-30) and the fact that the mapping \( r \mapsto r^{h_3} \) is Lipschitz over \([0, 2] \), we can bound this difference (in absolute value) by \( CN^{-\frac{1}{4} + \frac{1}{16} + \varepsilon} \). Collecting the terms, we obtain (4-28).

Step 4. We now use the synchronization result of Section 5. Recall from (2-23) that
\[ \left| \partial_t \hat{F}_N - k \sum_{\ell=1}^{k} \partial_{q_{1,\ell}} \hat{F}_N \partial_{q_{2,\ell}} \hat{F}_N \right| \leq \frac{1}{N^2} \sum_{a \in \{1, 2\}} \mathbb{E} \langle (\sigma_a \cdot \sigma_a' - \mathbb{E}(\sigma_a \cdot \sigma_a' | \alpha \wedge \alpha') \rangle^2. \]

By (4-28), Proposition 5.5 (with the quantities \( R_1 \) and \( R_2 \) appearing there being substituted by \( R_a \) and \( R_0 \) in our current notation), and our choice of \( h_+ \), we infer that for every \( N \) sufficiently large,
\[ \left| \partial_t \hat{F}_N - k \sum_{\ell=1}^{k} \partial_{q_{1,\ell}} \hat{F}_N \partial_{q_{2,\ell}} \hat{F}_N \right| (n, q_N, x_N) \leq \frac{13}{k}. \quad (4-31) \]
We will argue in the next step that, for every $N$ sufficiently large,

$$\left(\sum_{\ell=1}^{k} \partial q_{1,\ell} \tilde{F}_N \partial q_{2,\ell} \tilde{F}_N - \sum_{\ell=1}^{k} \partial q_{1,\ell} \tilde{\phi} \partial q_{2,\ell} \tilde{\phi}\right)(t_N, q_N, x_N) \geq 0. \quad (4-32)$$

Temporarily assuming this, and using also (4-22), we thus infer that

$$\left(\partial t \tilde{\phi} - k \sum_{\ell=1}^{k} \partial q_{1,\ell} \tilde{\phi} \partial q_{2,\ell} \tilde{\phi}\right)(t_N, q_N, x_N) \geq -\frac{13}{k}. \quad (4-33)$$

Using (4-20) and the fact that $\tilde{\phi}$ is a smooth function, we deduce that the statement (4-33) also holds at $(t_\infty, q_\infty, x_\infty)$. Recalling also the definition of $\tilde{\phi}$, see (4-19), we conclude that (4-17) holds.

Step 5. There only remains to show that (4-32) holds. If $q_N \in U_k$, then this is immediate, by (4-23). In particular, since $q_N$ converges to $q_\infty$ as $N$ tends to infinity, we know that (4-32) holds for every $N$ sufficiently large whenever $q_\infty \in U_k$. From now on, we assume that $q_\infty \in \partial U_k$. Using that $(t_N, q_N, x_N)$ tends to $(t_\infty, q_\infty, x_\infty)$, the smoothness of the function $\phi$, and the definition of $\tilde{\phi}$ in (4-19), we can infer from (4-16) that, for every $N$ sufficiently large,

$$\max_{v \in n(q_\infty)} v \cdot \nabla \tilde{\phi}(t_N, q_N, x_N) \leq 0. \quad (4-34)$$

We now argue that

$$q_N \in \partial \bar{U}_k \implies \max_{v \in n(q_N)} v \cdot \nabla \tilde{\phi}(t_N, q_N, x_N) \leq 0. \quad (4-35)$$

The set $\bar{U}_k$ can be written as the intersection of $2k$ half-spaces, and the condition that $q_\infty \in \partial U_k$ is equivalent to the statement that $q_\infty$ lies on the boundary of some of those half-spaces, say $D_1, \ldots, D_\ell$. For $N$ sufficiently large, we know that $q_N$ will not be on the boundary of any other half-space than those $D_1, \ldots, D_\ell$. It thus follows that, for $N$ sufficiently large, and whenever $q_N \in \bar{U}_k$, we have that $n(q_N)$ is a subset of $n(q_\infty)$. This yields that (4-35) indeed follows from (4-34).

We now argue that

$$\nabla \tilde{\phi}(t_N, q_N, x_N) \in \bar{U}_k. \quad (4-36)$$

Since the arguments $t_N$ and $x_N$ will be kept fixed, we omit them from the notation. We denote by $(e_{a,\ell})_{a \in \{1,2\}, \ell \in [1,\ldots,k]}$ the canonical basis of $\mathbb{R}^{2k}$, using our indexing convention. We also write $q_N = (q_{a,\ell})_{a \in \{1,2\}, 1 \leq \ell \leq k}$, dropping the dependence on $N$ when writing $q_N$ in coordinates. We fix $a \in \{1,2\}$, and first show that $\partial q_{a,1} \tilde{\phi}(q_N) \geq 0$. If $q_{a,1} = 0$, then this follows from (4-35), since $-e_{a,1} \in n(q_N)$ in this case. Otherwise, since the function $\tilde{F}_N - \tilde{\phi}$ has a local minimum at $q_N$, we have, for every $\varepsilon > 0$ sufficiently small,

$$\tilde{\phi}(q_N - \varepsilon e_{a,1}) - \tilde{\phi}(q_N) \leq \tilde{F}_N(q_N - \varepsilon e_{a,1}) - \tilde{F}_N(q_N).$$

Passing to the limit $\varepsilon \to 0$, and using Lemma 2.4, we get that $\partial q_{a,1} \tilde{\phi}(q_N) \geq 0$. We now fix $\ell \in \{1, \ldots, k-1\}$, and argue that $\partial q_{a,\ell} \tilde{\phi}(q_N) \leq \partial q_{a,\ell+1} \tilde{\phi}(q_N)$. If $q_{a,\ell} = q_{a,\ell+1}$, then this follows from (4-35), since, in this
We denote by \( \nabla \) with the previous display, we infer that the quantity
\[
\tilde{\phi}(q_N + \varepsilon a,\ell - \varepsilon a,\ell+1) - \tilde{\phi}(q_N) \leq F_N(q_N + \varepsilon a,\ell - \varepsilon a,\ell+1) - F_N(q_N).
\]
Passing to the limit, and using Lemma 2.4, we obtain indeed that \( \partial_{q_a,\ell+1} \tilde{\phi}(q_N) \leq \partial_{q_a,\ell+1} \tilde{\phi}(q_N) \).

We are now ready to conclude. For every \( p = (p_{a,\ell})_{a \in \{1,2\}, \ell \in \{1,\ldots,k\}} \in \mathbb{R}^{2k} \), denote
\[
H(p) := \sum_{\ell=1}^{k} p_{1,\ell} p_{2,\ell}.
\]

Our aim is to show that
\[
H(\nabla_q F_N) - H(\nabla_q \tilde{\phi}) \geq 0, \tag{4-37}
\]
where we kept implicit that the gradients are evaluated at \((t_N, q_N, x_N)\). We rewrite the left side of (4-37) in the form
\[
\int_0^1 (\nabla_q F_N - \nabla_q \tilde{\phi}) \cdot \nabla H(s \nabla_q F_N + (1-s) \nabla_q \tilde{\phi}) \, ds. \tag{4-38}
\]
(Evaluation at \((t_N, q_N, x_N)\) is still kept implicit here.) Since \( F_N - \tilde{\phi} \) has a local minimum at \( q_N \), we must have that
\[
\text{for all } y \in \overline{U}_k, \quad (y - q_N) \cdot \nabla_q (F_N - \tilde{\phi})(t_N, q_N, x_N) \geq 0. \tag{4-39}
\]
Recalling the notation \( \overline{U}^*_k \) from (3-17), we infer from (4-39) that
\[
\nabla_q (F_N - \tilde{\phi})(t_N, q_N, x_N) \in \overline{U}^*_k. \tag{4-40}
\]
(Indeed, for every \( z \in \overline{U}_k \), we can choose \( y = q_N + z \in \overline{U}_k \) in (4-39).) By (4-36) and Lemma 2.4, we have that, for every \( s \in [0,1] \),
\[
(s \nabla_q F_N + (1-s) \nabla_q \tilde{\phi})(t_N, q_N, x_N) \in \overline{U}_k.
\]
Finally, notice that, for each \( p = (p_{a,\ell})_{a \in \{1,2\}, \ell \in \{1,\ldots,k\}} \), the vector \( \nabla H(p) \) is obtained by a simple interchange of the index \( a \in \{1,2\} \). In particular, it is clear that \( \nabla H \) maps \( \overline{U}_k \) into itself. Combining this with the previous display, we infer that the quantity \( \nabla H(\cdots) \) appearing in (4-38) belongs to \( \overline{U}_k \). This and (4-40) yield (4-37), as desired. \( \square \)

We can now prove the main theorem of the paper.

**Proof of Theorem 2.7.** The argument consists in combining the results of Proposition 3.7 and Theorem 4.1. We denote by \( (f^{(k)})_{k \geq 1} \) and \( f \) the functions appearing in the statement of Proposition 3.7. By Theorem 4.1 and the comparison principle (3-12), we have, for every integer \( k \geq 1, \ t \geq 0 \) and \( q \in \mathbb{R}^{2k} \),
\[
\lim_{N \to \infty} F_N(t, \frac{1}{k} \sum_{\ell=1}^{k} \delta_{q_1,\ell}, \frac{1}{k} \sum_{\ell=1}^{k} \delta_{q_2,\ell}) + \frac{13t}{k} \geq f^{(k)}(t, q). \tag{4-41}
\]
Let $\mu = (\mu_1, \mu_2) \in (P_2(\mathbb{R}_+))^2$, and, for every integer $k \geq 1$, $a \in \{1, 2\}$, and $\ell \in \{1, \ldots, k\}$, denote

\[ q_{a, \ell}^{(k)} := k \int_{\ell - 1 \over k}^{\ell \over k} F_{\mu_a}^{-1}(u) \, du, \quad \text{and} \quad \mu_a^{(k)} := \sum_{\ell=1}^k \delta_{q_{a, \ell}}. \]

Recall from (3-58) that

\[ |f^{(k)}(t, q^{(k)}) - f(t, \mu)| \leq \frac{C}{\sqrt{k}}(t + (\mathbb{E}[X_1^2 + X_2^2])^{1 \over 2}). \tag{4-42} \]

On the other hand, we have from Proposition 2.1 that

\[ |\bar{F}_N(t, \mu) - \bar{F}_N(t, \mu^{(k)})| \leq \sum_{a=1}^2 \mathbb{E}[|X_{\mu_a} - X_{\mu_a^{(k)}}|] = \sum_{a=1}^2 \mathbb{E}[|F_{\mu_a}^{-1}(U) - k \int_{[\ell \over k]}^{[\ell \over k] + 1 \over k} F_{\mu_a}^{-1}(u) \, du|]. \]

We can bound this term by arguing as in the paragraph starting with (3-71). Indeed, this is the same argument, with the understanding that $K'$ is now infinite. Explicitly,

\[ \mathbb{E} \left[ \left| F_{\mu_a}^{-1}(U) - k \int_{[\ell \over k]}^{[\ell \over k] + 1 \over k} F_{\mu_a}^{-1}(u) \, du \right| \right] = \sum_{\ell=1}^k \int_{\ell - 1 \over k}^{\ell \over k} |F_{\mu_a}^{-1}(r) - k \int_{\ell - 1 \over k}^{\ell \over k} F_{\mu_a}^{-1}(u) \, du| \, dr \]

\[ \leq k \sum_{\ell=1}^k \int_{\ell - 1 \over k}^{\ell \over k} \int_{\ell - 1 \over k}^{\ell \over k} |F_{\mu_a}^{-1}(r) - F_{\mu_a}^{-1}(u)| \, dr \, du. \]

Paralleling (3-72), we write, for a cutoff value $B \in (0, \infty)$ to be determined,

\[ \int_0^1 |F_{\mu_a}^{-1}(u)| \mathbb{1}_{\{F_{\mu_a}^{-1}(u) \geq B\}} \, du \leq \frac{1}{B} \int |F_{\mu_a}^{-1}(u)|^2 \, du = \frac{\mathbb{E}[X_{\mu_a}^2]}{B}, \]

while, as in (3-73),

\[ 2k \sum_{\ell=1}^k \int_{\ell - 1 \over k}^{\ell \over k} \int_{\ell - 1 \over k}^{\ell \over k} |F_{\mu_a}^{-1}(r) - F_{\mu_a}^{-1}(u)| \mathbb{1}_{\{u \leq r, F_{\mu_a}^{-1}(r) \leq B\}} \, du \, dr \]

\[ = 2k \sum_{\ell=1}^k \int_{[0, 1]} \left( F_{\mu_a}^{-1}\left(\frac{\ell - 1}{k} + r\right) - F_{\mu_a}^{-1}\left(\frac{\ell - 1}{k} + u\right) \right) \mathbb{1}_{\{u \leq r, F_{\mu_a}^{-1}(\ell - 1 \over k + r) \leq B\}} \, du \, dr \leq \frac{2B}{k}. \]

Combining the displays above, and choosing $B^2 = k(\mathbb{E}[X_{\mu_1}^2 + X_{\mu_2}^2])$, we arrive at

\[ |\bar{F}_N(t, \mu) - \bar{F}_N(t, \mu^{(k)})| \leq \frac{6}{\sqrt{k}}(\mathbb{E}[X_{\mu_1}^2 + X_{\mu_2}^2])^{1 \over 2}. \tag{4-43} \]

Combining (4-41), (4-42), and (4-43), we deduce that

\[ \liminf_{N \to \infty} \bar{F}_N(t, \mu) \geq f(t, \mu) - \frac{13t}{k} - \frac{C}{\sqrt{k}}(t + (\mathbb{E}[X_{\mu_1}^2 + X_{\mu_2}^2])^{1 \over 2}). \]

Letting the integer $k \geq 1$ tend to infinity, we obtain the desired result. □
5. Synchronization

In this section, we revisit the synchronization result of [Panchenko 2015], see also [Panchenko 2018a; 2018b]. The structure of the reasoning presented here is similar to that in [Panchenko 2015], and emphasizes the fundamental importance of the ultrametric structure of the Gibbs measure. There are a few differences though: one of them is that we state “finitary” versions of the statements; that is, the statements provide approximate criteria that the Gibbs measure may satisfy for large but finite values of $N$ and $k$; the conclusion is then that we have “synchronization up to a small error”. A second difference between the treatment presented here and [Panchenko 2015] is in the phrasing of the synchronization property itself. In [Panchenko 2015], this is stated as the existence of Lipschitz functions that each map the sum of the overlaps of the different species to one of the single-species overlaps. In the present section, we instead choose to phrase the synchronization of different overlaps as the statement that they are monotonically coupled.

As said above, the main powerhouse behind the synchronization result comes from the possibility to enforce the ultrametricity of the Gibbs measure. The fundamental result of [Panchenko 2013a] is that the ultrametricity property is valid as soon as the Ghirlanda–Guerra identities hold; see also the preface to [Panchenko 2013b] for a review of the series of works that preceded this final result. Moreover, as is well-known and was seen again in Section 4, these identities are valid as soon as certain random energy functions become concentrated, a property that one can “build into the measure” by means of a small perturbation of the energy function.

In order to emphasize that the underlying constants in the statements below do not depend on the specific Gibbs measure under consideration, we will state them for rather general measures. We start by stating a finitary version of the statement from [Panchenko 2013a] that Ghirlanda–Guerra identities imply ultrametricity.

**Theorem 5.1 (GG implies ultrametricity [Panchenko 2013a]).** For every $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. Let $G$ be a random probability measure supported on the unit ball of an arbitrary Hilbert space; denote by $\langle \cdot \rangle$ the expectation associated with the measure $G^{\otimes N}$, with canonical random variables $(\sigma^\ell)$ for $\ell \geq 1$, and define, for every $\ell, \ell' \geq 1$, $R^{\ell, \ell'} := \sigma^\ell \cdot \sigma^{\ell'}$, and $R^{\leq n} := (R^{\ell, \ell'})_{1 \leq \ell, \ell' \leq n}$.

Finally, recalling that $\langle \cdot \rangle$ is itself random, denote by $\mathbb{E}$ the expectation with respect to this additional source of randomness. Assume that, for every $n$, $p \in \{1, \ldots, \lfloor \delta^{-1} \rfloor\}$ and $f \in C(\mathbb{R}^{n \times n})$ satisfying $\|f\|_{L^\infty} \leq 1$,

$$\left| \mathbb{E}(f(R^{\leq n})(R^{1,n+1})^p) - \frac{1}{n} \mathbb{E}(f(R^{\leq n}))\mathbb{E}((R^{1,2})^p) - \frac{1}{n} \sum_{\ell=2}^{n} \mathbb{E}(f(R^{\leq n})(R^{1,\ell})^p) \right| \leq \delta. \quad (5-1)$$

Then

$$\mathbb{E}\left(\mathbbm{1}_{\{R^{1,2} \geq \min(R^{1,3}, R^{2,3}) - \varepsilon\}}\right) \geq 1 - \varepsilon. \quad (5-2)$$

**Proof.** We argue by contradiction. Denote by $R := (R^{\ell, \ell'})_{\ell, \ell' \geq 1}$ the entire overlap array, and assume that Theorem 5.1 is false: there exists $\varepsilon > 0$ and, for each $\delta > 0$ no matter how small, a random probability
measure $G$ such that (5-1) holds but (5-2) is violated. Since each entry of $R$ takes values in $[-1, 1]$, up to extraction of a subsequence, we can find a random array $R = (R^\ell,\ell')_{\ell,\ell'\geq 1}$ defined with respect to a certain probability measure $\mathbb{M}$ such that, for each integer $n \geq 1$, the law of the array $R^k := (R^\ell,\ell')_{1 \leq \ell,\ell' \leq n}$ under $\mathbb{M}$ is obtained as the limit law of a subsequence of overlap arrays, each violating (5-2) but satisfying (5-1) for a sequence of values of $\delta$ that tends to zero. In other words, the array $R$ satisfies, for every integer $n$, $p \geq 1$ and $f \in C(\mathbb{R}^{n \times n})$,

$$
\mathbb{M}[f(R^{\leq n})(R^{1,n+1})^p] = \frac{1}{n} \mathbb{M}[f(R^{\leq n})]\mathbb{M}((R^{1.2})^p) + \frac{1}{n} \sum_{\ell=2}^{n} \mathbb{M}[f(R^{\leq n})(R^{1,\ell})^p],
$$

as well as

$$
\mathbb{M}[R^{1.2} \leq \min(R^{1.3}, R^{2.3}) - \varepsilon] \geq \varepsilon.
$$

This was shown to be impossible in [Panchenko 2013a]; see also [Panchenko 2013b, Theorem 2.14]. □

In order to prepare the ground for synchronization statements, we clarify the notion of monotone coupling in the next proposition.

**Proposition 5.2** (monotone coupling). Let $(X, Y)$ be a random vector taking values in $\mathbb{R}^2$, and let $(X', Y')$ be an independent copy of this vector, defined under the probability measure $\mathbb{P}$. The following three statements are equivalent.

1. We have

$$
\mathbb{P}[X < X' \text{ and } Y < Y'] = 0.
$$

2. For every $x, y \in \mathbb{R}$, we have

$$
\mathbb{P}[X \leq x \text{ and } Y \leq y] = \min(\mathbb{P}[X \leq x], \mathbb{P}[Y \leq y]).
$$

3. The law of $(X, Y)$ is

$$
(F^{-1}_X, F^{-1}_Y)(\text{Leb}_{[0,1]}),
$$

that is, the law of $(X, Y)$ is the image of the Lebesgue measure over $[0, 1]$ under the mapping

$$
r \mapsto (F^{-1}_X(r), F^{-1}_Y(r)),
$$

where, for every $r \in [0, 1]$,

$$
F^{-1}_X(r) := \inf\{s \in \mathbb{R} : \mathbb{P}[X \leq s] \geq r\},
$$

and similarly with $X$ replaced by $Y$.

Whenever any of the conditions (1)–(3) appearing in Proposition 5.2 holds, we say that the random variables $X$ and $Y$ are **monotonically coupled**.

**Proof.** We first show that (1) implies (2). The statement (5-4) with the equality sign replaced by "$\leq$" is clear. To show the converse inequality, we argue by contradiction and assume that there exist $x, y \in \mathbb{R}$ such that

$$
\mathbb{P}[X \leq x \text{ and } Y \leq y] < \min(\mathbb{P}[X \leq x], \mathbb{P}[Y \leq y]).
$$
Finally, for every $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. Let $G$ be a random probability measure supported on the Cartesian product of the unit balls of two arbitrary Hilbert spaces; denote by $\langle \cdot \rangle$ the expectation associated with the measure $G^\otimes \mathbb{N}$, with canonical random variables $\langle \sigma_1^\ell \rangle = (\sigma_1^1, \ldots, \sigma_1^\ell)$, and define, for every $a \in \{1, 2\}$ and $\ell, \ell', n \geq 1$,

$$R_{a, \ell, \ell'} := \sigma_a^\ell \cdot \sigma_a^{\ell'}, \quad \text{and} \quad R^{\otimes n} := (R_{a, \ell, \ell'}^\otimes)_{a \in \{1, 2\}, 1 \leq \ell, \ell' \leq n}.$$  

Finally, recalling that $\langle \cdot \rangle$ is itself random, denote by $\mathbb{E}$ the expectation with respect to this additional source of randomness. Assume that, for every $n, h_1, h_2, p \in \{1, \ldots, \lceil \delta^{-1} \rceil\}$ and $f \in C(\mathbb{R}^{2 \times n \times n})$ satisfying $\|f\|_{L^\infty} \leq 1$,

$$\left| \mathbb{E}(f(R^{\otimes n})(\lambda_{h_1} R_1^{1,n+1} + \lambda_{h_2} R_2^{1,n+1})^p) \right| - \frac{1}{n} \mathbb{E}(f(R^{\otimes n)}) \mathbb{E}(\langle \lambda_{h_1} R_1^{1,2} + \lambda_{h_2} R_2^{1,2} \rangle^p) - \frac{1}{n} \sum_{\ell=2}^n \mathbb{E}(f(R^{\otimes n})(\lambda_{h_1} R_1^{1,\ell} + \lambda_{h_2} R_2^{1,\ell})^p) \right| \leq \delta. \quad (5-6)$$

Then, for every $f \in C^\infty(\mathbb{R}^2)$,

$$|\mathbb{E}(f(R_1^{1,2}, R_2^{1,2})) - \mathbb{E}[f(F_1^{-1}(U), F_2^{-1}(U))]| \leq \varepsilon(\|f\|_{L^\infty} + \|\nabla f\|_{L^\infty}), \quad (5-7)$$
where $U$ stands for a uniform random variable over $[0, 1]$, and, for every $a \in \{1, 2\}$ and $r \in [0, 1]$, we write

$$F_a^{-1}(r) := \inf\{s \in \mathbb{R} : \mathbb{P}(\{ |R_a| \leq s \}) \geq r \}. \quad (5-8)$$

The proof of Theorem 5.3 makes use of the following lemma, asserting that if two sequences of random variables converge in law separately, then their monotone coupling converges in law as well.

**Lemma 5.4** (continuity of monotone coupling). Let $(X_n), (Y_n)$ be two sequences of random variables which converge in law to $X$ and $Y$ respectively. Then the associated monotone couplings converge: using the notation in (5-5), and with $U$ a uniform random variable over $[0, 1]$, we have

$$(F_{X_n}^{-1}(U), F_{Y_n}^{-1}(U)) \xrightarrow{\text{law}} (F_X^{-1}(U), F_Y^{-1}(U)). \quad (5-9)$$

**Proof.** Since the law of $F_{X_n}^{-1}(U)$ is that of $X_n$, it is clear that the convergence in (5-9) holds for each coordinate separately. Up to the extraction of a subsequence, we can assume that $(F_{X_n}^{-1}(U), F_{Y_n}^{-1}(U))$ converges in law to some random vector $(A, B)$; we denote by $(A', B')$ an independent copy of this vector. By classical properties of convergence in law and Proposition 5.2, we infer that

$$\mathbb{P}[A < A' \text{ and } B' < B] = 0.$$ 

We conclude using Proposition 5.2 once more. \qed

**Proof of Theorem 5.3.** Step 1. For any two probability measures $\mu, \nu$ on $[-1, 1]^2$, we define

$$\|\mu - \nu\| := \sup\left\{ \int f \, d\mu - \int f \, d\nu : \|f\|_{L^\infty} + \|\nabla f\|_{L^\infty} \leq 1 \right\}.$$ 

In this step, we show that the quantity above, as a function of $(\mu, \nu)$, is continuous for the topology of weak convergence. In other words, if a sequence of probability measures $\mu_n$ over $[-1, 1]^2$ converges weakly to $\mu$, then $\|\mu_n - \nu\|$ converges to $\|\mu - \nu\|$. For every integer $k \geq 1$ and $x \in \mathbb{R}^2$, define

$$P_k(x) := c_k \left( 1 - \frac{|x|^2}{16} \right)^k,$$

where the constant $c_k$ is such that $\int_{[-2, 2]^2} P_k = 1$. Let $f \in C^\infty([-1, 1]^2)$ be such that $\|f\|_{L^\infty} + \|\nabla f\|_{L^\infty} \leq 1$. We may extend $f$ to a Lipschitz function on $\mathbb{R}^2$ such that $\|f\|_{L^\infty} + \|\nabla f\|_{L^\infty} \leq 2$. Denoting the spatial convolution by $*$, we have, for every $x \in [-1, 1]^2$,

$$(f - f * P_k)(x) = \int_{[-2, 2]^2} (f(x) - f(x - y)) P_k(y) \, dy$$

$$= \int_{[-2, 2]^2} \int_0^1 y \cdot \nabla f(x - ty) P_k(y) \, dt \, dy,$$

so

$$\|f - f * P_k\|_{L^\infty} \leq 2 \int_{[-2, 2]^2} |y| P_k(y) \, dy,$$

and the latter quantity tends to 0 as $k$ tends to infinity (uniformly over $f$). On the other hand, $f * P_k$ is
a polynomial of degree at most 2k and, for each fixed k, the coefficients of this polynomial can be bounded in terms of \( \|f\|_{L^\infty} \). In particular, for each fixed k, we have

\[
\sup \left\{ \int f * P_k \, d\mu_n - \int f * P_k \, d\mu : \|f\|_{L^\infty} + \|\nabla f\|_{L^\infty} \leq 1 \right\} \xrightarrow{n \to \infty} 0.
\]

Combining these two facts gives the announced continuity result.

**Step 2.** We need to show that, provided that \( \delta > 0 \) is chosen sufficiently small in terms of \( \varepsilon \), we have

\[
\|\text{Law}(R^{1,2}_1, R^{1,2}_2) - (F^{-1}_1, F^{-1}_2)(\text{Leb}_{[0,1]})\| \leq \varepsilon. \tag{5-10}
\]

In the expression above, we denote by \( \text{Law}(R^{1,2}_1, R^{1,2}_2) \) the law of \( (R^{1,2}_1, R^{1,2}_2) \) under the measure \( \mathbb{E}(\cdot) \). Assuming the contrary, there exist \( \varepsilon > 0 \) and, for \( \delta > 0 \) as small as desired, an overlap distribution satisfying (5-6) but not (5-10). Up to extraction of a subsequence, we can assume that the overlap array converges in law to a limit random overlap \( R \), whose law we denote by \( \mathbb{M} \). By Lemma 5.4 and the result of the previous step, we infer that

\[
\|\text{Law}(R^{1,2}_1, R^{1,2}_2) - (F^{-1}_1, F^{-1}_2)(\text{Leb}_{[0,1]})\| \geq \varepsilon,
\]

where in the expression above, \( F^{-1}_1 \) and \( F^{-1}_2 \) now stand for the inverse cumulative distribution functions of \( R^{1,2}_1 \) and \( R^{1,2}_2 \) respectively (that is, we replace \( \mathbb{E}(\mathbb{I}_{\{R^{1,2}_i \leq s\}}) \) by \( \mathbb{M}[\mathbb{I}_{\{R^{1,2}_i \leq s\}}] \) in (5-8)). In particular, the random variables \( R^{1,2}_1 \) and \( R^{1,2}_2 \) are not monotonically coupled.

**Step 3.** We now show that \( R^{1,2}_1 \) and \( R^{1,2}_2 \) are in fact monotonically coupled, thereby reaching a contradiction. Denote by \( \tilde{R}^{1,2} \) an independent copy of \( R^{1,2} \). (Notice that this is with respect to the “averaged” measure \( \mathbb{M} \), so \( R^{3,4} \) would not qualify as an independent copy of \( R^{1,2} \) in this sense.) By Proposition 5.2, we need to show that

\[
\mathbb{M}[R^{1,2}_1 < \tilde{R}^{1,2}_1 \text{ and } R^{1,2}_2 < \tilde{R}^{1,2}_2] = 0. \tag{5-11}
\]

We first observe that, by the construction of \( R \), we have that for all integers \( n, h_1, h_2, p \geq 1 \) and \( f \in C(\mathbb{R}^{2 \times n \times n}) \),

\[
\mathbb{M}[f(R^{\leq n})(\lambda_{h_1} R^{1,n+1}_1 + \lambda_{h_2} R^{1,n+1}_2)^p] = \frac{1}{n} \mathbb{M}[f(R^{\leq n})] \mathbb{M}[(\lambda_{h_1} R^{1,n}_1 + \lambda_{h_2} R^{1,n}_2)^p] + \frac{1}{n} \sum_{\ell=2}^n \mathbb{M}[f(R^{\leq n})(\lambda_{h_1} R^{1,\ell}_1 + \lambda_{h_2} R^{1,\ell}_2)^p].
\]

Since every continuous function can be uniformly approximated by a polynomial on compact sets, and using the Cramér–Wold theorem, we deduce that conditionally on \( R^{\leq n} \), the law of \( R^{1,n+1} \) is

\[
\frac{1}{n} \text{Law}(R^{1,2}) \quad \text{and} \quad \frac{1}{n} \sum_{\ell=2}^n \delta_{R^{1,\ell}},
\]

where \( \text{Law}(R^{1,2}) \) denotes the law of \( R^{1,2} \) under \( \mathbb{M} \), and \( \delta_{R^{1,\ell}} \) is the Dirac mass at \( R^{1,\ell} \). In particular,

\[
2\mathbb{M}[R^{1,2}_1 < R^{1,3}_1 \text{ and } R^{1,3}_2 < R^{1,2}_2] = \mathbb{M}[R^{1,2}_1 < \tilde{R}^{1,2}_1 \text{ and } \tilde{R}^{1,2}_2 < R^{1,2}_2] + \mathbb{M}[R^{1,2}_1 < R^{1,2}_2 \text{ and } R^{1,2}_2 < R^{1,2}_2] = \mathbb{M}[R^{1,2}_1 < \tilde{R}^{1,2}_1 \text{ and } \tilde{R}^{1,2}_2 < R^{1,2}_2].
\]
The statement (5-11) we aim to show is thus equivalent to
\[ \mathbb{M}[R_1^{1.2} < R_1^{1.3} \text{ and } R_2^{1.3} < R_2^{1.2}] = 0. \] (5-12)

The validity of (5-12) now follows from the fact that \( R_1, R_2, \) and \( R_1 + R_2 \) are ultrametric, which itself is a consequence of Theorem 5.1. Indeed, by ultrametricity, we have

\[ R_1^{1.2} < R_1^{1.3} \Rightarrow R_1^{2.3} = R_1^{1.2}, \]

and

\[ R_2^{1.3} < R_2^{1.2} \Rightarrow R_2^{2.3} = R_2^{1.3}, \]

so that

\[ R_1^{1.2} < R_1^{1.3} \text{ and } R_2^{1.3} < R_2^{1.2} \Rightarrow R_1^{2.3} + R_2^{2.3} < \min(R_1^{1.2} + R_2^{1.2}, R_1^{1.3} + R_2^{1.3}), \]

and the latter statement contradicts the ultrametricity of \( R_1 + R_2 \). This completes the proof of (5-12), and therefore of Theorem 5.3.

As was apparent in (2-23), what we ultimately want to use is not only that two overlaps asymptotically become monotonically coupled, but rather that one of the overlaps can essentially be inferred by observing the other. Even if the two overlaps were perfectly synchronized, this can only be true if the law of the observed overlap is sufficiently “spread out”: in an extreme example, if the observed overlap is essentially of the form

\[ R \quad \text{and} \quad h \in \{1, \ldots, 1 - \delta^{-1}\} \] for some integer \( k \geq 1 \) and parameters \(-1 = q_0 < q_1 < \cdots < q_k \leq 1\). We then have

\[ \mathbb{E}((R_1^{1.2} - \mathbb{E}(R_1^{1.2} | R_2^{1.2}))^2) \leq \frac{12}{k} + \varepsilon k^2 \sup_{\ell \in \{0, \ldots, k-1\}} (q_{\ell+1} - q_\ell)^{-1}. \] (5-13)

**Proof of Proposition 5.5.** By Theorem 5.3, we can choose \( \delta > 0 \) sufficiently small that (5-7) holds for every Lipschitz function \( f : \mathbb{R}^2 \to \mathbb{R} \). We set

\[ \eta := \inf_{\ell \in \{0, \ldots, k-1\}} (q_{\ell+1} - q_\ell), \]

and, for every \( a \in \{1, 2\} \) and \( s \in \mathbb{R} \),

\[ F_a(s) := \mathbb{E}(\mathbb{I}_{[R_a^{1.2} \leq s]}), \]
with $F_a^{-1}$ defined as in $(5-8)$. By assumption, the function $F_2$ is piecewise constant, with discontinuities at $q_1, \ldots, q_k$. Let $\tilde{F}_2$ denote the function which coincides with $F_2$ on the set

$$(\infty, q_0] \cup \{q_0, \ldots, q_k\} \cup [q_k, +\infty),$$

and is affine on each interval $[q_\ell, q_{\ell+1}]$, $\ell \in \{0, \ldots, k-1\}$. The function $\tilde{F}_2$ satisfies $\|\tilde{F}_2\|_{L^\infty} \leq 1$ and $\|\nabla \tilde{F}_2\|_{L^\infty} \leq \eta^{-1}$. Notice that, for every $u \in [0, 1]$,

$$\tilde{F}_2(F_2^{-1}(u)) = F_2(F_2^{-1}(u)) = k^{-1}[ku]. \quad (5-14)$$

We define, for every $x \in \mathbb{R}$,

$$\rho_k(x) := k \max(1, k|x|, 0),$$

and observe that $\int \rho_k = 1$ and that the convolution $F_1^{-1} \ast \rho_k$ is a Lipschitz function, with Lipschitz constant bounded by $k^2$. For every $x, y \in [-1, 1]$, we set

$$f(x, y) = (x - (F_1^{-1} \ast \rho_k)(\tilde{F}_2(y)))^2.$$ 

The Lipschitz constant of this function is bounded by $2\eta^{-1}k^2$, and thus

$$|\mathbb{E}(f(R_1^{1,2}, R_2^{1,2})) - \mathbb{E}[f(F_1^{-1}(U), F_2^{-1}(U))]| \leq 3\varepsilon\eta^{-1}k^2.$$

Using $(5-14)$ and the fact that $F_1^{-1}$ takes values in $[-1, 1]$ and is monotone, we can estimate the second term on the left side above by

$$\mathbb{E}((F_1^{-1}(U) - (F_1^{-1} \ast \rho_k)(k^{-1}[kU]))^2) \leq 2\mathbb{E}[[F_1^{-1}(U) - (F_1^{-1} \ast \rho_k)(k^{-1}[kU])]$$

$$= 2 \sum_{\ell=0}^{k-1} \int_{\frac{\ell+1}{k}}^{\frac{\ell}{k}} |F_1^{-1}(u) - (F_1^{-1} \ast \rho_k)(k^{-1}[ku])| \,du$$

$$\leq 2 \sum_{\ell=0}^{k-1} (F_1^{-1}(\frac{\ell+2}{k}) - F_1^{-1}(\frac{\ell-1}{k}))$$

$$\leq \frac{12}{k}.$$

We have thus shown that

$$\mathbb{E}((R_1^{1,2} - (F_1^{-1} \ast \rho_k)(\tilde{F}_2(R_2^{1,2})))^2) \leq 3\varepsilon\eta^{-1}k^2 + 12k^{-1},$$

and thus in particular, since the conditional expectation is an $L^2$ projection,

$$\mathbb{E}((R_1^{1,2} - \mathbb{E}(R_1^{1,2} | R_2^{1,2}))^2) \leq 3\varepsilon\eta^{-1}k^2 + 12k^{-1}. $$

Up to a redefinition of $\varepsilon$, this is $(5-13)$. \qed
6. The free energy as a saddle-point problem

This final section has a more speculative flavor, and concerns the possibility to rewrite the limit free energy of models such as the one investigated here in the form of a saddle-point problem, a possibility discussed for instance in [Talagrand 2007]. A strong indication in favor of this possibility comes from the study of certain models of statistical inference. The statistical-inference problem most similar to the spin-glass model studied here is probably that of estimating a non-symmetric rank-one matrix. This problem was investigated in [Miolane 2017; Barbier et al. 2017; Reeves 2020; Chen 2020], and it was found there that the free energy could indeed be conveniently represented in the form of a saddle-point problem.

Of course, any quantity can be written as a saddle-point problem, so the relevant question is whether there is some natural way for doing so. The point of view provided by Hamilton–Jacobi equations suggests two natural routes for finding variational formulations of the limit free energy. The first one, available only when the nonlinearity in the equation is convex, consists in writing the Hopf–Lax formula for the solution; see for instance [Mourrat 2019]. As was already emphasized, the main feature of the model under consideration here is that the nonlinearity in the equation is not convex (nor concave). The second possible route is based on the fact that, irrespectively of the structure of the nonlinearity, it is also possible to write the solution of a Hamilton–Jacobi equation as a saddle-point problem, provided that the initial condition is concave (or convex), as was suggested also in [Hopf 1965] and then confirmed rigorously using the notion of viscosity solutions in [Bardi and Evans 1984] (see also [Lions and Rochet 1986]).

This second possibility can be applied to good effect in the context of the model of statistical inference studied in [Miolane 2017; Barbier et al. 2017; Reeves 2020; Chen 2020]: as was shown in [Chen 2020], the relevant Hamilton–Jacobi equation is a finite-dimensional version of (1-7), and the initial condition is convex, thereby allowing us to recover the saddle-point formulas obtained in [Miolane 2017; Barbier et al. 2017; Reeves 2020].

However, perhaps surprisingly, this strategy does not seem to work in the context of the model under consideration in this paper, and it is the aim of this section to explore this more precisely.

This point hides an important subtlety, which requires that we introduce more precise language to speak about concavity properties of the initial condition. Indeed, one can endow the set of probability measures with two different geometric structures. Perhaps the more immediate one is to think of it as an affine subspace of the space of signed measures. In this point of view, the natural “straight line” between the measures \( \mu \) and \( \nu \) is given by \( t \mapsto (1 - t)\mu + t\nu \). The second relevant geometric structure on the space of probability measures is that given by optimal transport. In this second point of view, the natural “straight line” between the measures \( \mu \) and \( \nu \) can be seen as the set of laws of the random variables \((1 - t)X_\mu + tX_\nu\), with \( t \) varying in \([0, 1]\), and where the law of \((X_\mu, X_\nu)\) is an optimal coupling between the measures \( \mu \) and \( \nu \) (since we are only concerned with one-dimensional measures here, the coupling given by \((1-5)\) is optimal).

These two points of view give rise to two different notions of convexity, which we will call “affine convexity” and “transport convexity” respectively. (The notion of transport convexity is sometimes also called “displacement convexity”.) The subtlety here is that, at least in the simpler setting of mixed \( p \)-spin
models, the initial condition in (1-7) is affine-concave, as was shown in [Auffinger and Chen 2015]; but, whether for these $p$-spin models or for the bipartite model investigated here, this initial condition is not transport-concave (nor transport-convex). And, since the derivatives in (1-7) are transport-type derivatives, it is the notion of transport concavity (or convexity) that would have been required to guarantee saddle-point formulas by the general mechanism described above.

In the remainder of this section, we examine more precisely what natural attempts at writing saddle-point formulas for the solution to (1-7) may look like, and explain why these attempts fail in general (although we do not exclude the possibility that they be valid for some specific choices of the measures $\pi_1$ and $\pi_2$ in (1-3)).

6A. Attempts based on the Hopf formula. We start by arguing that the initial condition $\psi$ in (1-7) is neither transport-concave nor transport-convex in general. This observation is also valid for models with a single type such as mixed $p$-spin models. The transport concavity (or convexity) of the mapping $\mu \mapsto \psi(\mu)$ would imply in particular that the mapping

$$\chi : \mathbb{R} \rightarrow \mathbb{R},$$

$$h \mapsto \psi((\delta_h, \delta_0))$$

is concave (or convex). Recall from (2-19) that

$$\chi(h) = \psi((\delta_h, \delta_0)) = -\mathbb{E} \log \int \exp((2h)^{\frac{1}{2}} z_1 \sigma_1 - h(\sigma_1)^2) \, d\pi_1(\sigma_1),$$

where here $\sigma_1$ is real-valued, and $z_1$ is a standard one-dimensional Gaussian random variable. Dropping the subscript “1” on $z_1$ and $\sigma_1$ to lighten the notation, and denoting by $\langle \cdot \rangle$ the corresponding Gibbs measure, we have

$$\partial_h \chi = \mathbb{E}\langle \sigma \sigma' \rangle,$$

and

$$\partial_h^2 \chi = \mathbb{E}\langle \sigma \sigma'((2h)^{-\frac{1}{2}} z(\sigma + \sigma') - \sigma^2 - (\sigma')^2) - 2\mathbb{E}\langle \sigma \sigma'((2h)^{-\frac{1}{2}} z\sigma'' - (\sigma'')^2) \rangle\rangle$$

$$= \mathbb{E}\langle \sigma \sigma'((\sigma + \sigma')(\sigma + \sigma' - 2\sigma'') - \sigma^2 - (\sigma')^2) - 2\mathbb{E}\langle \sigma \sigma'(\sigma''(\sigma + \sigma' + \sigma'' - 3\sigma'') - (\sigma'')^2) \rangle\rangle$$

$$= 2\mathbb{E}[\langle \sigma^2 \rangle^2 - 4\langle \sigma^2 \rangle^2\langle \sigma \rangle^2 + 3\langle \sigma \rangle^4]$$

$$= 2\mathbb{E}[\langle \sigma^2 \rangle - \langle \sigma \rangle^2(\langle \sigma^2 \rangle - 3\langle \sigma \rangle^2)].$$

Recall also that when $h = 0$, the Gibbs measure simplifies into being the measure $\pi_1$. It is therefore clear that we can choose the measure $\pi_1$ in such a way that $\partial_h^2 \chi(h = 0)$ has any desired sign: for instance, if $\pi_1$ is the uniform measure on $\{-1, 1\}$, then $\langle \sigma \rangle = 0$ at $h = 0$, so $\partial_h^2 \chi > 0$; but if we choose $\pi_1$ to be the probability measure on $\{-1, 1\}$ such that $\langle \sigma \rangle^2 = \frac{1}{2}$ at $h = 0$, then we have $\langle \sigma^2 \rangle = 1 < 3\langle \sigma \rangle^2 = \frac{3}{2}$, and thus $\partial_h^2 \chi < 0$ at $h = 0$. In both examples, we also have that $\partial_h \chi$ tends to 1 as $h$ tends to infinity. In the case with $\langle \sigma \rangle^2 = \frac{1}{2}$ at $h = 0$, the derivative $\partial_h \chi$ at $h = 0$ is $\frac{1}{2}$ and then decreases, but must then tend to 1. In particular, the function $\partial_h \chi$ is not monotone: that is, the function $\chi$ is neither concave nor convex.
In a possibly confusing twist, for the most studied case in which $\pi_1$ is the uniform measure on $\{-1, 1\}$, one can show that the function $\chi$ is in fact convex. This implies that, at least for the model with a single type, the replica-symmetric solution for this specific choice of measure can in fact be written as a saddle-point problem. But, as is argued here, this is an accident rather than the rule. My understanding is that the solution proposed in [Korenblit and Shender 1985; Fyodorov et al. 1987a; 1987b] is based on this coincidence. As a side note, it is also worth mentioning that the de Ameida–Thouless-type stability criterion employed there is known to be invalid in general, even for models with a single type [Panchenko 2005].

As was recalled in (1-9) (see also [Mourrat 2019]), the limit free energy of mixed $p$-spin models can be expressed in terms of the solution $f_1 = f_1(t, \mu) : \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}_+) \rightarrow \mathbb{R}$ of the equation

$$\begin{cases}
\partial_t f_1 - \int \xi(\partial_\mu f_1) \, d\mu = 0 & \text{on } \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}_+), \\
f_1(0, \cdot) = \psi_1 & \text{on } \mathcal{P}_2(\mathbb{R}_+),
\end{cases}$$

(6-2)

and this solution can be written in variational form using the Hopf–Lax formula: we have

$$f_1(t, \mu) = \sup_{\nu \in \mathcal{P}_2(\mathbb{R}_+)} \left( \psi_1(\nu) - t \mathbb{E}\left[ \xi^*(X_\nu - X_\mu) \right] \right),$$

(6-3)

where $X_\mu, X_\nu$ are defined according to (1-5), and

$$\xi^*(s) := \sup_{r \geq 0} (rs - \xi(r)).$$

We can rewrite this formula as

$$f_1(t, \mu) = \sup_{\nu \in \mathcal{P}_2(\mathbb{R}_+)} \left( \psi_1(\nu) - \mathbb{E}[\sup_{r \geq 0} r(X_\nu - X_\mu) - t\xi(r)] \right)$$

(6-4)

$$= \sup_{\nu \in \mathcal{P}_2(\mathbb{R}_+)} \inf_{f \in L^2([0,1];\mathbb{R}_+)} \left( \psi_1(\nu) - \mathbb{E}[f(U)(X_\nu - X_\mu) - t\xi(f(U))]) \right).$$

(6-5)

One may wonder whether supremum and infimum can be interchanged in the expression above. If this were the case, it would imply in particular that

$$\psi_1(\mu) = \inf_{f \in L^2([0,1];\mathbb{R}_+)} \sup_{\nu \in \mathcal{P}_2(\mathbb{R}_+)} \left( \psi_1(\nu) - \mathbb{E}[f(U)(X_\nu - X_\mu)] \right).$$

But notice that the supremum over $\nu$ above is an affine function of $X_\mu$; taking the infimum, we find that this would imply the transport concavity of $\psi_1$. But we have argued above that this is not so in general. Similarly, replacing $\inf_f \sup_{\nu}$ by $\sup_f \inf_{\nu}$ in the expression above would lead to the conclusion that $\psi_1$ is transport-convex, which has also been excluded in general. Conversely, if $\psi_1$ were actually transport-concave, then interchanging the supremum and the infimum in (6-5) would be valid; and in general, what we find after the interchange is the solution to the same Hamilton–Jacobi equation, but with the initial condition replaced by its transport-concave envelope.

We now come back to the bipartite model investigated in the present paper. The considerations above raise the question of whether the solution to (1-7) can be written as a saddle-point, with respect to the
variables $f = (f_1, f_2) \in (L^2([0, 1]; \mathbb{R}_+))^2$ and $\nu = (\nu_1, \nu_2) \in (P_2(\mathbb{R}_+))^2$, of the functional

$$
\psi(\nu) - E \left[ \sum_{a=1}^{2} f_a(U)(X_{\nu_a} - X_{\mu_a}) - tf_1(U)f_2(U) \right].
$$

But any possible arrangement of inf’s and sup’s leads to a contradiction. If we aim for optimizing first over $f$ and then over $\nu$, in analogy with (6-5), then this amounts to trying to write down a Hopf–Lax formula although the nonlinearity in the equation is neither convex nor concave. The convex (or concave) dual of the mapping $(x, y) \mapsto xy$ is so degenerate that it is easy to rule out this possibility. On the other hand, if we try to optimize first over $\nu$ and then over $f$, then we face the same situation as above: each possibility would imply either that $\psi$ is transport-convex, or that it is transport-concave, and both have been ruled out in general.

6B. A related attempt. A related attempt at generating a candidate variational formula for the limit free energy of the bipartite model is as follows. In the papers [Barra et al. 2015; Panchenko 2015], the authors investigate a large class of models covering in particular the situation in which the definition of $H_N(\sigma)$ in (1-1) is replaced by

$$
N^{-\frac{1}{2}} \sum_{i,j=1}^{N} J_{ij} \sigma_{1,i} \sigma_{2,j} + N^{-\frac{1}{2}} \sum_{a \in \{1,2\}} \sum_{i,j=1}^{N} J_{ij}^{(a)} \sigma_{a,i} \sigma_{a,j},
$$

where $(J_{ij}^{(a)})_{a \in \{1,2\}, 1 \leq i,j \leq N}$ are independent centered Gaussian random variables with a fixed variance, independent of $(J_{ij})$. Assuming that the matrix

$$
A := \begin{pmatrix}
2E[(J_{11}^{(1)})^2] & E[(J_{11})^2] \\
E[(J_{11})^2] & 2E[(J_{11}^{(2)})^2]
\end{pmatrix}
$$

is positive definite, they derive a variational formula for the free energy of the model.

One may wonder whether the formula obtained by ignoring the assumption of positive definiteness of the matrix $A$ necessary for their proofs actually matches the prediction given by the Hamilton–Jacobi equation. We will argue here that this is not so. We have already seen in the previous subsection that writing up a naive Hopf–Lax formula for (1-7) would clearly lead to an invalid prediction. However, the formula given in [Barra et al. 2015; Panchenko 2015], while equivalent to the Hopf–Lax formula in the case when the matrix $A$ is positive definite, is actually different in outlook, and extends to a different expression in the setting when the matrix $A$ is taken to be the matrix of interest to us here, namely

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
$$

(6-6)

Assuming that the matrix $A$ is positive definite, we first explain the derivation of the formula in [Barra et al. 2015; Panchenko 2015] starting from the point of view provided by Hamilton–Jacobi equations. One can check (at least formally, and probably rigorously by combining the arguments of [Barra et al. 2015; Panchenko 2015] with those of [Mourrat and Panchenko 2020]) that the relevant Hamilton–Jacobi
equation for this model is given by
\[
\partial_t f - \frac{1}{2} \int \partial_\mu f \cdot A \partial_\mu f \, d\hat{\mu},
\]
with the same initial condition \(\psi\) defined in (2-19) (that is, \(\psi\) does not depend on \(A\)), and where we used the vector notation \(\partial_\mu f \equiv (\partial_{\mu_1} f, \partial_{\mu_2} f)\). Since we assume that \(A\) is positive definite, we can write down the Hopf–Lax formula
\[
f(t, \mu) = \sup_{\nu \in (\mathcal{P}_2(\mathbb{R}_+))^2} \left( \psi(\nu) - \frac{1}{2t} \mathbb{E}[(X_\nu - X_\mu) \cdot A^{-1}(X_\nu - X_\mu)] \right),
\]
with the notation \(X_\nu = (X_{\nu_1}, X_{\nu_2})\). From here, we could replace \(A\) by the matrix in (6-6), but it is easy to see that with this choice the supremum is infinite. However the formula of [Barra et al. 2015; Panchenko 2015] (for positive definite \(A\)) has an additional restriction on the support of the pair of measures \(\nu\). Under the assumption that \(A\) is positive definite, we can indeed write
\[
f(t, (\delta_0, \delta_0)) = \sup_{\nu \in (\mathcal{P}(\mathbb{R}_+))^2} \left( \psi(\nu) - \frac{1}{2t} \mathbb{E}[X_\nu \cdot A^{-1}X_\nu] \right), \quad (6-7)
\]
where the supremum is taken over every pair of measures \(\nu = (\nu_a)_{a \in \{1, 2\}} \in (\mathcal{P}(\mathbb{R}_+))^2\) with the restriction that, denoting by \(q_a\) the top of the support of \(\nu_a\),
\[
A^{-1} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \leq \begin{pmatrix} 2t \\ 2t \end{pmatrix},
\]
in the sense that the inequality holds component by component. That this additional restriction does not change the value of the supremum in (6-7) in the case when \(A\) is positive definite can be derived from the Lipschitz estimate on \(\psi\) guaranteed by Proposition 2.1, and arguing as in [Mourrat 2019, Step 4]. Blindly replacing the matrix \(A\) by that in (6-6) thus leads to the formula
\[
\sup_{\nu \in \mathcal{P}(\mathbb{R}_+)^2} \left( \psi(\nu) - \frac{1}{t} \mathbb{E}[X_{\nu_1}X_{\nu_2}] \right), \quad (6-8)
\]
One can verify that this formula does not match the solution to (1-7) evaluated at \(\mu = (\delta_0, \delta_0)\). For instance, recalling the notation in (1-3), we may take \(\pi_1\) to be the uniform measure on \([-1, 1]\), \(\pi_2\) to be a non-uniform measure on \([-1, 1]\), and verify that in this case the solution to the equation satisfies \(\partial_t f(0, (\delta_0, \delta_0)) = 0\), but that this property is not satisfied by the expression in (6-8). Assuming that the overlaps are concentrated for small \(t\), the limit free energy should be described by (1-12) in this region, and this would imply that indeed \(\partial_t f(0, (\delta_0, \delta_0)) = 0\).

**Appendix: Gaussian integrals**

**Gaussian integration by parts.** Let \(\mu\) be the law of a \(d\)-dimensional centered Gaussian vector, with covariance matrix \(C \in \mathbb{R}^{d \times d}\). We assume (temporarily) that \(C\) is invertible. In this case, the measure \(\mu\)
has a density with respect to the Lebesgue measure on \( \mathbb{R}^d \), which is proportional to

\[
\exp\left(-\frac{1}{2} x \cdot C^{-1} x\right).
\]

For every bounded and smooth function \( F \in C^\infty(\mathbb{R}^d, \mathbb{R}^d) \), we thus have, by integration by parts,

\[
\int C^{-1} x \cdot F(x) \, d\mu(x) = \int \nabla \cdot F(x) \, d\mu(x),
\]

or equivalently,

\[
\int x \cdot F(x) \, d\mu(x) = \int \nabla \cdot (CF)(x) \, d\mu(x).
\]

This last identity remains valid when \( C \) is not invertible, by approximation. In particular, for every bounded and smooth \( f \in C^\infty(\mathbb{R}^d; \mathbb{R}) \),

\[
\int x_1 f(x) \, d\mu(x) = \sum_{\ell=1}^d \mathbb{E}[x_1 x_\ell] \int \partial_{x_\ell} f(x) \, d\mu(x). \tag{A-1}
\]

One consequence of this observation is the following result. (At least the first part of it is very classical; the last part is certainly also well-known, but I could not find a precise reference).

**Lemma A.1.** Let \( \Sigma \) be a finite set, let \((x_1(\sigma), x_2(\sigma))_{\sigma \in \Sigma}\) be a centered Gaussian random field with respect to the probability measure \( \mathbb{P} \) (with expectation \( \mathbb{E} \)), and let \( P \) be a probability measure on \( \Sigma \). For every \( a, b \in \{1, 2\} \) and \( \sigma, \sigma' \in \Sigma \), we write

\[
C_{ab}(\sigma, \sigma') := \mathbb{E}[x_a(\sigma)x_b(\sigma')].
\]

We denote by \( \langle \cdot \rangle \) the Gibbs measure built from \((x_2(\sigma))\), so that for every \( f : \Sigma \to \mathbb{R} \),

\[
\langle f(\sigma) \rangle := \frac{\int f(\sigma) \exp(x_2(\sigma)) \, d\mathbb{P}(\sigma)}{\int \exp(x_2(\sigma)) \, d\mathbb{P}(\sigma)},
\]

and write \( \sigma', \sigma'' \) for independent copies of the random variable \( \sigma \) under \( \langle \cdot \rangle \). We have

\[
\mathbb{E}(x_1(\sigma)) = \mathbb{E}(C_{12}(\sigma, \sigma) - C_{12}(\sigma, \sigma')) \tag{A-2}
\]

and

\[
\mathbb{E}(x_1^2(\sigma)) = \mathbb{E}(C_{11}(\sigma, \sigma)) + \mathbb{E}[(C_{12}(\sigma, \sigma) - C_{12}(\sigma, \sigma'))(C_{12}(\sigma, \sigma) + C_{12}(\sigma, \sigma') - 2C_{12}(\sigma, \sigma''))]. \tag{A-3}
\]

More generally, we write \((\sigma^\ell)_{\ell \geq 1}\) for a sequence of independent copies of the random variable \( \sigma \) under \( \langle \cdot \rangle \). For every \( p \geq 1 \), there exists a polynomial \( P_p \) (which does not depend on any parameter in the problem) taking as inputs the variables \((C_{ab}(\sigma^k, \sigma^\ell))_{a,b \in \{1, 2\}, k, \ell \geq 1}\) such that

\[
\mathbb{E}(x_1^p(\sigma)) = \mathbb{E}(P_p((C_{ab}(\sigma^k, \sigma^\ell))_{a,b \in \{1, 2\}, k, \ell \geq 1})). \tag{A-4}
\]

Moreover, the polynomial only depends on \((C_{11}(\sigma^k, \sigma^\ell))_{k, \ell \geq 1}\) and \((C_{12}(\sigma^k, \sigma^\ell))_{k, \ell \geq 1}\), and is homogeneous of degree \( p \) provided that we count each occurrence of a variable \( C_{11}(\sigma^k, \sigma^\ell) \) as having degree 2.
Proof. We start by writing
\[ E\langle x_1(\sigma) \rangle = \int E[x_1(\sigma) \frac{\exp(x_2(\sigma))}{\int \exp(x_2(\sigma')) dP(\sigma')} dP(\sigma).] \]

We then apply (A-1) to rewrite the inner expectation as
\[ C_{12}(\sigma, \sigma') \mathbb{E}\left[ \frac{\exp(x_2(\sigma))}{\int \exp(x_2(\sigma')) dP(\sigma')} \right] - \int C_{12}(\sigma, \sigma') \mathbb{E}\left[ \frac{\exp(x_2(\sigma) + x_2(\sigma'))}{(\int \exp(x_2(\sigma'')) dP(\sigma''))^2} \right] dP(\sigma'). \]

Combining the two previous displays leads to (A-2). The argument for (A-3) is similar, except that we now need to compute
\[ \mathbb{E}\left[ (x_1(\sigma))^2 \frac{\exp(x_2(\sigma))}{\int \exp(x_2(\sigma')) dP(\sigma')} \right]. \] (A-5)

In order to apply (A-1) in this case, we split the square of \( x_1(\sigma) \) into two parts, one of them being incorporated into the function “\( f \)” in (A-1). We thus find that the quantity in (A-5) equals
\[ C_{11}(\sigma, \sigma) \mathbb{E}\left[ \frac{\exp(x_2(\sigma))}{\int \exp(x_2(\sigma')) dP(\sigma')} \right] + C_{12}(\sigma, \sigma) \mathbb{E}\left[ \frac{x_1(\sigma) \exp(x_2(\sigma))}{\int \exp(x_2(\sigma')) dP(\sigma')} \right] - \int C_{12}(\sigma, \sigma') \mathbb{E}\left[ \frac{x_1(\sigma) \exp(x_2(\sigma) + x_2(\sigma'))}{(\int \exp(x_2(\sigma'')) dP(\sigma''))^2} \right] dP(\sigma'). \]

This shows that
\[ \mathbb{E}\{(x_1(\sigma))^2\} = \mathbb{E}(C_{11}(\sigma, \sigma)) + \mathbb{E}(x_1(\sigma)(C_{12}(\sigma, \sigma) - C_{12}(\sigma, \sigma'))). \]

For every \( \sigma, \sigma' \in \Sigma \), we define
\[ \tilde{x}_1(\sigma, \sigma') := x_1(\sigma)(C_{12}(\sigma, \sigma) - C_{12}(\sigma, \sigma')). \]

The variables \((\tilde{x}_1(\sigma, \sigma'), x_2(\sigma) + x_2(\sigma'))_{\sigma, \sigma' \in \Sigma}\) form a centered Gaussian field, with
\[ \mathbb{E}[\tilde{x}_1(\sigma, \sigma')(x_2(\sigma'') + x_2(\sigma'''))] = (C_{12}(\sigma, \sigma) - C_{12}(\sigma, \sigma'))(C_{12}(\sigma, \sigma'') + C_{12}(\sigma, \sigma''')). \]

Applying (A-2), we deduce that
\[ \mathbb{E}(x_1(\sigma)(C_{12}(\sigma, \sigma) - C_{12}(\sigma, \sigma'))) = \mathbb{E}(C_{12}(\sigma, \sigma) - C_{12}(\sigma, \sigma'))(C_{12}(\sigma, \sigma) + C_{12}(\sigma, \sigma') - C_{12}(\sigma, \sigma'') - C_{12}(\sigma, \sigma''')), \]

and replacing \( \sigma''' \) by \( \sigma'' \) in the expression above does not change its value. This completes the proof of (A-3). For (A-4), we apply (A-1) again to rewrite
\[ \mathbb{E}\left[ x_1^b(\sigma) \frac{\exp(x_2(\sigma))}{\int \exp(x_2(\sigma')) dP(\sigma')} \right]. \]
as
\[
C_{11}(\sigma, \sigma)(p - 1) \mathbb{E} \left[ \frac{x_1^{p-2}(\sigma) \exp(x_2(\sigma))}{\int \exp(x_2(\sigma')) \, dP(\sigma')} \right] + C_{12}(\sigma, \sigma) \mathbb{E} \left[ \frac{x_1^{p-1}(\sigma) \exp(x_2(\sigma))}{\int \exp(x_2(\sigma')) \, dP(\sigma')} \right] 
- \int C_{12}(\sigma, \sigma') \mathbb{E} \left[ \frac{x_1^{p-1}(\sigma) \exp(x_2(\sigma) + x_2(\sigma'))}{(\int \exp(x_2(\sigma'')) \, dP(\sigma''))^2} \right] \, dP(\sigma').
\]

We can then obtain (A-4) by induction on \( p \).

**Existence of Gaussian process.** The next lemma serves to guarantee that the Gaussian random field introduced in (4-4) indeed exists.

**Lemma A.2.** Let \( p, k, N \geq 1 \) be integers, and \( \lambda_1, \lambda_2 \geq 0 \). There exists a centered Gaussian field \((X(\sigma, \alpha))_{\sigma \in \mathbb{R}^N, \alpha \in \mathbb{N}^k}\) such that, for every \( \sigma, \sigma' \in \mathbb{R}^N \) and \( \alpha, \alpha' \in \mathbb{N}^k \),

\[
\mathbb{E}[X(\sigma, \alpha) X(\sigma', \alpha')] = (\lambda_1 \sigma \cdot \sigma' + \lambda_2 \alpha \land \alpha')^p,
\]

(A-6)

where we recall that the notation \( \alpha \land \alpha' \) was introduced in (2-9).

**Proof.** Recall the definition of the tree \( A \) in (2-6). For each \( n \in \mathbb{N} \), we define the finite approximation

\[
A_n := \{0, \ldots, n\}^0 \cup \cdots \cup \{0, \ldots, n\}^k,
\]

again with the understanding that \( \{0, \ldots, n\}^0 = \{\varnothing\} \), and we denote the set of leaves by \( \mathcal{L}_n := \{0, \ldots, n\}^k \).

By Kolmogorov’s extension theorem, it suffices to construct a Gaussian process \((X_n(\sigma, \alpha))_{\sigma \in \mathbb{R}^N, \alpha \in \mathcal{L}_n}\) such that (A-6) holds for every \( \sigma, \sigma' \in \mathbb{R}^N \) and \( \alpha, \alpha' \in \mathcal{L}_n \).

Let \((f_{\alpha})_{\alpha \in A_n}\) be an orthonormal basis of \( \mathbb{R}^{|A_n|} \), and for each \( \alpha \in \mathcal{L}_n \), let

\[
g_{\alpha} := \sum_{i=1}^{k} f_{\alpha[i]},
\]

so that for every \( \alpha, \alpha' \in \mathcal{L}_n \),

\[
g_{\alpha} \cdot g_{\alpha'} = \alpha \land \alpha'.
\]

(A-7)

Viewing \((\sqrt{\lambda_1} \sigma, \sqrt{\lambda_2} g_{\alpha})\) as a vector in \( \mathbb{R}^N \times \mathbb{R}^{|A_n|} \), we consider the \( p \)-fold tensor product

\[
(\sqrt{\lambda_1} \sigma, \sqrt{\lambda_2} g_{\alpha})^\otimes p \in (\mathbb{R}^N \times \mathbb{R}^{|A_n|})^\otimes p.
\]

Recall that, if we denote by \((e_i)_{i \in \{1, \ldots, N\}}\) an orthonormal basis of \( \mathbb{R}^N \), then an orthonormal basis of the tensor product \((\mathbb{R}^N \times \mathbb{R}^{|A_n|})^\otimes p\) is given by

\[
\mathcal{B} := \{v_1 \otimes \cdots \otimes v_p : v_1, \ldots, v_p \in \{e_i, i \in \{1, \ldots, N\}\} \cup \{f_{\alpha}, \alpha \in A_n\}\}.
\]

We now give ourselves a standard Gaussian vector \( W \) taking values in \( \mathbb{R}^{|\mathcal{B}|} \), and define

\[
X(\sigma, \alpha) := W \cdot (\sqrt{\lambda_1} \sigma, \sqrt{\lambda_2} e_{\alpha})^\otimes p,
\]

where \( e_{\alpha} := \sum_{i=1}^{k} f_{\alpha[i]} \) for every \( \alpha \in A_n \).
so that for every $\sigma, \sigma' \in \mathbb{R}^N$ and $\alpha, \alpha' \in \mathcal{L}_n$,

$$
\mathbb{E}[X(\sigma, \alpha) X(\sigma', \alpha')] = (\sqrt{\lambda_1 \sigma}, \sqrt{\lambda_2 g_{\alpha}})^\otimes p \cdot (\sqrt{\lambda_1 \sigma'}, \sqrt{\lambda_2 g_{\alpha'}})^\otimes p
$$

$$
= \left( (\sqrt{\lambda_1 \sigma}, \sqrt{\lambda_2 g_{\alpha}}) \cdot (\sqrt{\lambda_1 \sigma'}, \sqrt{\lambda_2 g_{\alpha'}}) \right)^p
$$

$$
= (\lambda_1 \sigma \cdot \sigma' + \lambda_2 \alpha \wedge \alpha')^p,
$$

where we used (A-7) in the last step.

\[ \square \]

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CONVERGENCE OF ASYMPTOTIC COSTS FOR RANDOM EUCLIDEAN MATCHING PROBLEMS

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We investigate the average minimum cost of a bipartite matching between two samples of \( n \) independent random points uniformly distributed on a unit cube in \( d \geq 3 \) dimensions, where the matching cost between two points is given by any power \( p \geq 1 \) of their Euclidean distance. As \( n \) grows, we prove convergence, after a suitable renormalization, towards a finite and positive constant. We also consider the analogous problem of optimal transport between \( n \) points and the uniform measure. The proofs combine subadditivity inequalities with a PDE ansatz similar to the one proposed in the context of the matching problem in two dimensions and later extended to obtain upper bounds in higher dimensions.

1. Introduction

The aim of this paper is to extend the results of [Dobrić and Yukich 1995; Boutet de Monvel and Martin 2002; Barthe and Bordenave 2013; Dereich et al. 2013] on the existence of the thermodynamic limit for some random optimal matching problems. Because of their relations to computer science, statistical physics and quantization of measures, optimal matching problems have been the subject of intense research both from the mathematical and physical communities. We refer for instance to [Yukich 1998; Talagrand 2014; Caracciolo et al. 2014] for more details in particular regarding the vast literature.

Probably the simplest and most studied variant of these problems is the bipartite (or Euclidean bipartite) matching on the unit cube in \( d \) dimensions. Given \( p \geq 1 \) and two independent families of i.i.d. random variables \((X_i)_{i \geq 1}\) and \((Y_i)_{i \geq 1}\) with common law the uniform (Lebesgue) measure on \([0, 1]^d\), the problem is to understand the behavior for large \( n \) of

\[
\mathbb{E}\left[\frac{1}{n} \min_{\pi} \sum_{i=1}^{n} |X_i - Y_{\pi(i)}|^p\right],
\]

where the minimum is taken over all permutations \( \pi \) of \( \{1, \ldots, n\} \). It is by now well-known, see [Ajtai et al. 1984; Barthe and Bordenave 2013; Bobkov and Ledoux 2019a; Ledoux 2019], that\(^1\) for \( n \gg 1 \) (see

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\(^1\)The notation \( A \ll 1 \), which we only use in assumptions, means that there exists an \( \varepsilon > 0 \) only depending on the dimension \( d \) and on \( p \geq 1 \), such that if \( A \leq \varepsilon \) then the conclusion holds. Similarly, the notation \( A \lesssim B \), which we use in output statements, means that there exists a global constant \( C > 0 \) depending on the dimension \( d \) and on \( p \geq 1 \) such that \( A \leq CB \). We write \( A \asymp B \) if both \( A \lesssim B \) and \( B \lesssim A \).
[Fournier and Guillin 2015] for some nonasymptotic bounds)

\[ E \left[ \frac{1}{n} \min_{\pi} \sum_{i=1}^{n} |X_i - Y_{\pi(i)}|^p \right] \sim \begin{cases} n^{-\frac{p}{2}} & \text{for } d = 1, \\ \left( \frac{1}{n} \log n \right)^{\frac{p}{2}} & \text{for } d = 2, \\ n^{-\frac{p}{d}} & \text{for } d \geq 3. \end{cases} \quad (1-1) \]

Let us point out that while the case \( d \geq 3, \ p \geq d/2 \), is not explicitly covered in the literature, the proof of [Ledoux 2019] clearly extends to any \( p \neq 2 \) (see also [Bobkov and Ledoux 2019b]). Our main result is the following:

**Theorem 1.1.** For every \( d \geq 3 \) and \( p \geq 1 \), there exists a constant \( f_{bi}^\infty = f_{bi}^\infty(p, d) > 0 \) such that

\[ \lim_{n \to \infty} n^\frac{p}{d} E \left[ \frac{1}{n} \min_{\pi} \sum_{i=1}^{n} |X_i - Y_{\pi(i)}|^p \right] = f_{bi}^\infty. \quad (1-2) \]

This extends earlier results of [Dobrić and Yukich 1995; Boutet de Monvel and Martin 2002; Barthe and Bordenave 2013; Dereich et al. 2013] where the same conclusion was obtained under the more restrictive condition \( p < d/2 \). See also [Talagrand 1992] for bounds on \( f_{bi}^\infty(1, d) \) as \( d \) becomes large. As in the previously quoted papers, our proof is based on a subadditivity argument and makes use of the classical observation that, thanks to the Birkhoff–von Neumann theorem, the bipartite matching problem is actually an optimal transport problem. Indeed, if \( \mu = \sum_{i=1}^{n} \delta_{X_i} \) and \( \lambda = \sum_{i=1}^{n} \delta_{Y_i} \) are the associated empirical measures, then

\[ \min_{\pi} \sum_{i=1}^{n} |X_i - Y_{\pi(i)}|^p = W_p^\mu(\mu, \lambda), \]

where \( W_p \) denotes the Wasserstein distance of order \( p \) (see [Villani 2003; Santambrogio 2015]). However, the papers [Boutet de Monvel and Martin 2002; Barthe and Bordenave 2013; Dereich et al. 2013] rely then upon combinatorial arguments, and in fact their results apply to a larger class of random optimization problems, while [Dobrić and Yukich 1995] strongly uses the dual formulation of optimal transport, which in the case \( p = 1 \) is quite specific, since it becomes a maximization over the set of 1-Lipschitz functions.

The optimal transport point of view allows us to treat the defect in subadditivity as a defect in local distribution of mass rather than a defect in local distribution of points. More precisely, even if \( \mu([0, 1]^d) = \lambda([0, 1]^d) \) it is in general not true that for a given partition of \([0, 1]^d\) in subcubes \( Q_i \), \( \mu(Q_i) = \lambda(Q_i) \). Therefore, in order to use subadditivity, one needs to relax the definition of the matching problem to take into account this disparity. In [Boutet de Monvel and Martin 2002; Barthe and Bordenave 2013; Dereich et al. 2013] this is done by requiring that as many points as possible are matched. Here we allow instead for varying weights. That is, for \( \mu \) and \( \lambda \) containing potentially a different number of points, we consider the problem

\[ E \left[ W_p^\mu \left( \frac{\mu}{\mu([0, 1]^d)}, \frac{\lambda}{\lambda([0, 1]^d)} \right) \right]. \]

The main subadditivity argument for this quantity is contained in Lemma 3.1. In order to estimate the error in subadditivity, we then use in Lemma 3.4 a PDE ansatz similar to the one proposed in the context
of the matching problem in [Caracciolo et al. 2014] and then used in [Ambrosio et al. 2019] (see also [Ledoux 2019; Goldman et al. 2020; Benedetto and Caglioti 2020]) to show that when $d = 2$,

$$\lim_{n \to \infty} \frac{n}{\log n} \mathbb{E} \left[ \frac{1}{n} W_2^2(\mu, \lambda) \right] = \frac{1}{2\pi}.$$  

Notice however that our use of this linearization ansatz is quite different from [Caracciolo et al. 2014; Ambrosio et al. 2019]. Indeed, for us the main contribution to the transportation cost is given by the transportation at the smaller scales and linearization is only used to estimate the higher order error term. On the other hand, in [Caracciolo et al. 2014; Ambrosio et al. 2019] (see also [Ambrosio and Glaudo 2019]), the main contribution to the cost is given by the linearized, i.e., $H^{-1}$, cost. This is somewhat in line with the prediction by [Caracciolo et al. 2014] that for $d \geq 3$, the first order contribution to the Wasserstein cost is not given by the linearized problem while higher order corrections are. In any case, we give in Proposition 5.3 an alternative argument to estimate the error term without relying on the PDE ansatz. There we use instead an elementary comparison argument with one-dimensional transport plans. We included both proofs since we believe that they could prove useful in other contexts.

Let us make a few more comments on the proof of Theorem 1.1. As in [Dobrić and Yukich 1995; Boutet de Monvel and Martin 2002; Barthe and Bordenave 2013], the proof of (1-2) is actually first done on a variant of the problem where the number of points follows a Poisson distribution instead of being deterministic. This is due to the fact that the restriction of a Poisson point process is again a Poisson point process. For this variant of the problem, rather than working on a fixed cube $[0, 1]^d$ with an increasing density of points, we prefer to make a blow-up at a scale $L = n^{1/d}$ and consider in Theorem 5.1, a fixed Poisson point process of intensity one on $\mathbb{R}^d$ but restricted to cubes $Q_L = [0, L]^d$ with $L \gg 1$ (hence the terminology thermodynamic limit). We believe that the subadditivity argument is slightly clearer in these variables (a similar rescaling is actually implicitly used in [Boutet de Monvel and Martin 2002; Barthe and Bordenave 2013]). This setting is somewhat reminiscent of [Huesmann and Sturm 2013], where superadditivity is used to construct an optimal coupling between the Poisson point process and the Lebesgue measure on $\mathbb{R}^d$. In order to pass from the Poisson version of the problem to the deterministic one, we prove a general de-Poissonization result in Proposition 6.1 which can hopefully be useful in other contexts.

Besides the bipartite matching we also treat in Theorems 4.1 and 6.2 the case of the matching to the reference measure. We actually treat this problem first since the proof is a bit simpler. Indeed, while the general scheme of the proof is identical to the bipartite case, the PDE ansatz used in Lemma 3.4 works well for “regular” measures and a more delicate argument is required for the bipartite matching. Notice that by Jensen and triangle inequalities, (1-1) also holds for the matching to the reference measure.

We point out that in [Dobrić and Yukich 1995; Barthe and Bordenave 2013; Dereich et al. 2013], it is more generally proven that if the points $X_i$ and $Y_i$ have a common law $\mu$ supported in $[0, 1]^d$ instead of the Lebesgue measure (for measures with unbounded support a condition on the moments is required), then for $1 \leq p < d/2$ and $d \geq 3$,

$$\limsup_{n \to \infty} n^{\frac{d}{2}} \mathbb{E} \left[ \frac{1}{n} \min_{\pi} \sum_{i=1}^{n} |X_i - Y_{\pi(i)}|^p \right] \leq f_{\infty}^{bi} \int_{[0,1]^d} \left( \frac{d\mu}{dx} \right)^{1-\frac{p}{d}}.$$  


However, when \( p > d/2 \) and without additional assumptions on \( \mu \), the asymptotic rate may be different and thus this inequality may fail; see, e.g., [Fournier and Guillin 2015]. Positive results for specific densities can be obtained nonetheless. For instance, it is proven in [Ledoux and Zhu 2019] that for the standard Gaussian measure \( \mu \) on \( \mathbb{R}^d \), \( d \geq 3 \), the asymptotic bound

\[
 n^p \mathbb{E} \left[ \frac{1}{n} \min_{\pi} \sum_{i=1}^n |X_i - Y_{\pi(i)}|^p \right] \sim 1
\]

holds true also for \( d/2 \leq p < d \).

Finally, we notice that usual results on concentration of measure allow us to improve from convergence of the expectations to strong convergence. However, we are able to cover only the case \( 1 \leq p < d \); see Remark 6.5.

The plan of the paper is as follows. In Section 2, we fix some notation and recall basic moment and concentration bounds for Poisson random variables. In Section 3, we state and prove our two main lemmas, namely, the subadditivity estimate Lemma 3.1 and the error estimate Lemma 3.4. In Section 4, we then prove the existence of the thermodynamic limit for the matching problem of a Poisson point process to the reference measure. The analog result for the bipartite matching between two Poisson point processes is obtained in Section 5. Finally, in Section 6 we pass from the Poissonized problem to the deterministic one and discuss stronger convergence results.

### 2. Notation and preliminary results

We use the notation \( |A| \) for the Lebesgue measure of a Borel set \( A \subseteq \mathbb{R}^d \), and \( \int_A f \) for the Lebesgue integral of a function \( f \) on \( A \). For \( L > 0 \), we let \( Q_L = [0, L]^d \). We denote by \( |p| \) the Euclidean norm of a vector \( p \in \mathbb{R}^N \). For a function \( \phi \), we use the notation \( \nabla \phi \) for the gradient, \( \nabla \cdot \phi \) for the divergence, \( \Delta \phi \) for the Laplacian and \( \nabla^2 \phi \) for the Hessian.

#### 2.1. Optimal transport

In this section we introduce some notation for the Wasserstein distance and recall few simple properties that will be used throughout. Proofs can be found in any of the monographs [Villani 2003; Santambrogio 2015; Peyré and Cuturi 2019] with expositions of theory of optimal transport, from different perspectives.

Given \( p \geq 1 \), a Borel subset \( \Omega \subseteq \mathbb{R}^d \) and two positive Borel measures \( \mu, \lambda \) with \( \mu(\Omega) = \lambda(\Omega) \in (0, \infty) \) and finite \( p \)-th moments, the Wasserstein distance of order \( p \geq 1 \) between \( \mu \) and \( \lambda \) is defined as the quantity

\[
 W_p(\mu, \lambda) = \left( \min_{\pi \in C(\mu, \lambda)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y) \right)^{1/p},
\]

where \( C(\mu, \lambda) \) is the set of couplings between \( \mu \) and \( \lambda \). Moreover, if \( \mu(\Omega) = \lambda(\Omega) = 0 \), we define \( W_p(\mu, \lambda) = 0 \), while if \( \mu(\Omega) \neq \lambda(\Omega) \), we let \( W_p(\mu, \lambda) = \infty \).

Let us recall that since \( W_p \) is a distance, we have the triangle inequality

\[
 W_p(\mu, \nu) \leq W_p(\mu, \lambda) + W_p(\nu, \lambda). \tag{2-1}
\]
We will also use the classical subadditivity inequality
\[
W_p^p \left( \sum_i \mu_i, \sum_i \lambda_i \right) \leq \sum_i W_p^p (\mu_i, \lambda_i),
\] (2-2)
for a finite set of positive measures \( \mu_i, \lambda_i \). This follows from the observation that if \( \pi_i \in C(\mu_i, \lambda_i) \), then \( \sum_i \pi_i \in C(\sum_i \mu_i, \sum_i \lambda_i) \).

**Remark 2.1.** In fact, our results deal with the transportation cost \( W_p^p (\mu, \lambda) \) rather than \( W_p (\mu, \lambda) \). To keep notation simple, we write
\[
W_p^p (\mu, \lambda) = W_p^p (\mu \perp \Omega, \lambda \perp \Omega).
\]
Moreover, if a measure is absolutely continuous with respect to Lebesgue measure, we only write its density. For example,
\[
W_p^p \left( \mu, \mu(\Omega) \frac{\mu(\Omega)}{| \Omega |} \right),
\]
denotes the transportation cost between \( \mu \perp \Omega \) to the uniform measure on \( \Omega \) with total mass \( \mu(\Omega) \).

Occasionally we may write \( W_\Omega (\mu, \lambda) \) instead of \( (W_p^p (\mu, \lambda))^{1/p} \). This may lead to some ambiguity, but it should be clear from the context.

**2.2. Poisson point processes.** As in [Dobrić and Yukich 1995; Boutet de Monvel and Martin 2002; Barthe and Bordenave 2013], we exploit invariance properties of Poisson point processes on \( \mathbb{R}^d \) with uniform intensity in order to obtain simpler subadditivity estimates. We refer, e.g., to [Last and Penrose 2018] for a general introduction to Poisson point processes. Here we only recall that a Poisson point process on \( \mathbb{R}^d \) with intensity one can be defined as a random variable taking values on locally finite atomic measures
\[
\mu = \sum_i \delta_{X_i}
\]
such that, for every \( k \geq 1 \), for any disjoint Borel sets \( A_1, \ldots, A_k \subseteq \mathbb{R}^d \), the random variables \( \mu(A_1), \ldots, \mu(A_k) \) are independent and \( \mu(A_i) \) has a Poisson distribution of parameter \( |A_i| \), for every \( i = 1, \ldots, k \). In particular, if \( A \subseteq \mathbb{R}^d \) is Lebesgue negligible, then \( \mu(A) = 0 \) almost surely.

Existence of a Poisson point process of intensity one is obtained via a superposition argument, noticing that on every bounded subset \( \Omega \subseteq \mathbb{R}^d \) the law of \( \mu \perp \Omega \) can be easily described: conditionally on \( \mu(\Omega) = n \), the measure \( \mu \perp \Omega \) has the same law as the random measure
\[
\sum_{i=1}^n \delta_{X_i},
\]
where \( (X_i)_{i=1}^n \) are independent random variables with uniform law on \( \Omega \). Uniqueness in law can be also obtained, so that translation invariance of Lebesgue measure entails that the process is stationary, i.e., any deterministic translation of the random measure \( \mu \) leaves its law unchanged.

Let us finally recall the classical Cramér–Chernoff concentration bounds for Poisson random variables.
Lemma 2.2. Let $N$ be a Poisson random variable with parameter $n \gg 1$. Then, for every $t > 0$,

$$
P[|N - n| \geq t] \leq 2 \exp\left(-\frac{t^2}{2(t + n)}\right),$$

(2-3)

As a consequence, for every $q \geq 1$,

$$
\mathbb{E}[|N - n|^q] \lesssim n^q.
$$

(2-4)

Proof. For the concentration bound (2-3), see, for instance, [Boucheron et al. 2013]. By the layer-cake representation,

$$
\mathbb{E}[|N - n|^q] \lesssim \int_0^\infty t^{q-1} \exp\left(-\frac{t^2}{2(t + n)}\right) dt
$$

$$
\lesssim \int_0^{\sqrt{n}} t^{q-1} dt + \int_{\sqrt{n}}^n t^{q-1} \exp\left(-\frac{ct^2}{n}\right) dt + \int_n^\infty t^{q-1} \exp(-ct) dt
$$

$$
\lesssim n^q + n^{q-1} \exp(-cn) \lesssim n^q.
$$

□

3. The main lemmas

Our subadditivity argument rests on a general but relatively simple lemma (which we only apply here for rectangles).

Lemma 3.1. For every $p \geq 1$, there exists a constant $C > 0$ depending only on $p$ such that the following holds. For every Borel set $\Omega \subset \mathbb{R}^d$, every Borel partition $(\Omega_i)_{i \in \mathbb{N}}$ of $\Omega$, every measure $\mu$ and $\lambda$ on $\Omega$, and every $\varepsilon \in (0, 1)$,

$$
W^p_{\Omega}\left(\mu, \frac{\mu(\Omega)}{\lambda(\Omega)}\lambda\right) \leq (1 + \varepsilon) \sum_i W^p_{\Omega_i}\left(\mu, \frac{\mu(\Omega_i)}{\lambda(\Omega_i)}\lambda\right) + \frac{C}{\varepsilon^{p-1}} W^p_{\Omega}\left(\sum_i \frac{\mu(\Omega_i)}{\lambda(\Omega_i)} \chi_{\Omega_i} \lambda, \frac{\mu(\Omega)}{\lambda(\Omega)}\lambda\right).
$$

(3-1)

Proof. We first use the triangle inequality (2-1) to get

$$
W^p_{\Omega}\left(\mu, \frac{\mu(\Omega)}{\lambda(\Omega)}\lambda\right) \leq \left(W_{\Omega}\left(\mu, \sum_i \frac{\mu(\Omega_i)}{\lambda(\Omega_i)} \chi_{\Omega_i} \lambda\right) + W_{\Omega}\left(\sum_i \frac{\mu(\Omega_i)}{\lambda(\Omega_i)} \chi_{\Omega_i} \lambda, \frac{\mu(\Omega)}{\lambda(\Omega)}\lambda\right)\right)^p.
$$

The proof is then concluded by combining the elementary inequality

$$
(a + b)^p \leq (1 + \varepsilon) a^p + \frac{C}{\varepsilon^{p-1}} b^p
$$

for all $a, b > 0$ and $\varepsilon \in (0, 1)$, (3-2)

with the subadditivity of $W^p_{\Omega}$ (2-2) in the form

$$
W^p_{\Omega}\left(\mu, \sum_i \frac{\mu(\Omega_i)}{\lambda(\Omega_i)} \chi_{\Omega_i} \lambda\right) \leq \sum_i W^p_{\Omega_i}\left(\mu, \frac{\mu(\Omega_i)}{\lambda(\Omega_i)}\lambda\right).
$$

□

Remark 3.2. Alternatively, we could have also stated (3-1) in the slightly more symmetric form:

$$
W^p_{\Omega}\left(\mu, \frac{\lambda}{\mu(\Omega)}\lambda\right) \leq (1 + \varepsilon) \sum_i \frac{\mu(\Omega_i)}{\mu(\Omega)} W^p_{\Omega_i}\left(\frac{\mu}{\mu(\Omega_i)}, \frac{\lambda}{\lambda(\Omega_i)}\right) + \frac{C}{\varepsilon^{p-1}} W^p_{\Omega}\left(\frac{1}{\mu(\Omega)} \sum_i \frac{\mu(\Omega_i)}{\lambda(\Omega_i)} \chi_{\Omega_i} \lambda, \frac{\lambda}{\lambda(\Omega)}\right).
$$
However, the subadditivity argument turns out to be a little bit simpler using (3-1) instead.

Lemma 3.1 shows that in order to estimate the defect in subadditivity, it is enough to bound the local defect of mass distribution. This will be done here through a PDE argument.

**Definition 3.3.** We say that a rectangle \( R = x + \prod_{i=1}^{d} [0, L_i] \) is of moderate aspect ratio if for every \( i, j \), \( L_i/L_j \leq 2 \). A partition \( \mathcal{R} = \{ R_i \} \) of \( R \) is called admissible if for every \( i \), \( R_i \) is a rectangle of moderate aspect ratio and \( 3^{-d} |R| \leq |R_i| \leq |R| \). Notice that in particular \( \# \mathcal{R} \lesssim 1 \) for every admissible partition.

**Lemma 3.4.** Let \( R \) be a rectangle of moderate aspect ratio, \( \mu \) and \( \lambda \) be measures on \( R \) with equal mass, both absolutely continuous with respect to Lebesgue and such that \( \inf_R \lambda > 0 \). Then, for every \( p \geq 1 \),

\[
W_p^R(\mu, \lambda) \lesssim \frac{\text{diam}^p(R)}{(\inf_R \lambda)^{p-1}} \int_R |\mu - \lambda|^p. \tag{3-3}
\]

**Proof.** Let \( \phi \) be a solution of the Poisson equation with Neumann boundary conditions

\[
\Delta \phi = \mu - \lambda \quad \text{in} \quad R \quad \text{and} \quad \nu \cdot \nabla \phi = 0 \quad \text{on} \quad \partial R. \tag{3-4}
\]

We first argue that

\[
W_p^R(\mu, \lambda) \lesssim \frac{1}{(\inf_R \lambda)^{p-1}} \int_R |\nabla \phi|^p. \tag{3-5}
\]

Let us point out that this estimate is well-known and has already been used in the context of the matching problem; see [Ambrosio et al. 2019; Goldman et al. 2020] in the case \( p = 2 \) and [Ledoux 2019, Theorem 2] for general \( p \geq 1 \). Still, we give a proof for the reader’s convenience.

We first argue as in [Goldman et al. 2020, Lemma 2.7], and use triangle inequality (2-1) and the monotonicity of \( W_R \) (2-2) to get

\[
W_R(\mu, \lambda) \leq W_R(\mu, \frac{1}{2}(\mu + \lambda)) + W_R(\frac{1}{2}(\mu + \lambda), \lambda) \leq W_R(\frac{1}{2}\mu, \frac{1}{2}\lambda) + W_R(\frac{1}{2}(\mu + \lambda), \lambda)
\]

\[= 2^{-\frac{1}{p}}(W_R(\mu, \lambda) + W_R(\mu + \lambda, 2\lambda))\]

and thus

\[W_R(\mu, \lambda) \lesssim W_R(\mu + \lambda, 2\lambda).\]

We now recall that by the Benamou–Brenier formula (see [Santambrogio 2015, Theorem 5.28])

\[W_R^p(\mu + \lambda, 2\lambda) = \min_{\rho, j} \left\{ \int_0^1 \int_R \frac{1}{\rho^{p-1}} |j|^p : \partial_t \rho + \nabla \cdot j = 0, \rho_0 = \mu + \lambda, \rho_1 = 2\lambda \right\}.\]

Estimate (3-5) follows using

\[
\rho_t = (1-t)\mu + t\lambda + \lambda \quad \text{and} \quad j = \nabla \phi
\]

as competitor and noticing that for \( t \in [0, 1] \), \( \rho_t \geq \inf_R \lambda \).

We now claim that

\[
\int_R |\nabla \phi|^p \lesssim \text{diam}^p(R) \int_R |\mu - \lambda|^p, \tag{3-6}
\]

which together with (3-5) would conclude the proof of (3-3). Estimate (3-6) is a direct consequence of Poincaré inequality and Calderón–Zygmund estimates for the Laplacian. However, since we did not find
a precise reference for (global) Calderón–Zygmund estimates on rectangles with Neumann boundary conditions, we give here a short proof.

By scaling, we may assume that \( \text{diam}(R) = 1 \). Furthermore, using even reflections along \( \partial R \) we may replace Neumann boundary conditions by periodic ones in (3-4). By Poincaré inequality [Leoni 2017, Proposition 12.29] (notice that thanks to the periodic boundary conditions we now have \( \int_R \nabla \phi = 0 \)),

\[
\int_R |\nabla \phi|^p \lesssim \int_R |\nabla^2 \phi|^p.
\]

By interior Calderón–Zygmund estimates (see for instance [Giaquinta and Martinazzi 2012, Theorem 7.3]), periodicity and the fact that \( R \) has moderate aspect ratio, we get

\[
\int_R |\nabla^2 \phi|^p \lesssim \int_R |\mu - \lambda|^p + \left( \int_R |\nabla^2 \phi|^2 \right)^{\frac{p}{2}}.
\]

By Bochner’s formula and Hölder inequality,

\[
\int_R |\nabla^2 \phi|^2 = \int_R |\mu - \lambda|^2 \lesssim \left( \int_R |\mu - \lambda|^p \right)^{\frac{2}{p}},
\]

which concludes the proof of (3-6).

\( \square \)

**Remark 3.5.** For \( p = 2 \), combining the energy identity

\[
\int_R |\nabla \phi|^2 = \int_R \phi(\lambda - \mu)
\]

with Poincaré inequality, we see that (3-6) (and thus (3-3)) holds for any convex set \( R \). Although the situation for \( p \geq 2 \) is more subtle, see [Fromm 1993, Proposition 2], we believe that (3-6) holds for any rectangle, not necessarily of moderate aspect ratio.

### 4. Matching to the reference measure

In this section, we consider the optimal matching problem between \( \mu \) a Poisson point process on \( \mathbb{R}^d \) with intensity one and the Lebesgue measure. More precisely, for every \( L \geq 1 \) we let

\[
f^\text{ref}_L(L) = \mathbb{E} \left[ \frac{1}{|Q_L|} W^p_{Q_L}(\mu, \kappa) \right],
\]

where \( Q_L = [0, L]^d \) and

\[
\kappa = \frac{\mu(Q_L)}{|Q_L|}
\]

is the generic constant for which this is well defined.

**Theorem 4.1.** For every \( d \geq 3 \) and \( p \geq 1 \), the limit

\[
f^\text{ref}_\infty(L) = \lim_{L \to \infty} f^\text{ref}_L(L)
\]
exists and is strictly positive. Moreover, there exists $C > 0$ depending on $p$ and $d$ such that for $L \geq 1$,

$$f_{\infty}^r \leq f_r^r(L) + \frac{C}{L^{d/2}}. \quad (4-1)$$

The proof follows the argument of [Bouet de Monvel and Martin 2002] (see also [Barthe and Bordenave 2013]) and is mostly based on the following subadditivity estimate.

**Proposition 4.2.** For every $d \geq 3$ and $p \geq 1$, there exists a constant $C > 0$ such that for every $L \geq 1$ and $m \in \mathbb{N}$,

$$f_r^r(mL) \leq f_r^r(L) + \frac{C}{L^{d/2}}. \quad (4-2)$$

**Proof.** We start by pointing out that since $f_r^r(L) \lesssim L^p$, it is not restrictive to assume that $L \gg 1$ in the proof of (4-2).

**Step 1.** (the dyadic case) For the sake of clarity, we start with the simpler case $m = 2^k$ for some $k \geq 1$. We claim that

$$f_r^r(2L) \leq f_r^r(L) + \frac{C}{L^{d/2}}. \quad (4-3)$$

The desired estimate (4-2) would then follow iterating (4-3) and using that $\sum_{k \geq 0} 1/2^{k(d-2)} < \infty$ for $d \geq 3$. In order to prove (4-3), we divide the cube $Q_{2L}$ in $2^d$ subcubes $Q_i = x_i + [Q_L]$ and let $\kappa_i = \mu(Q_i)/|Q_L|$ (and $\kappa = \mu(Q_{2L})/|Q_{2L}|$). Notice that we are considering a partition up to a Lebesgue negligible remainder, which gives no contribution almost surely. By (3-1), for every $\varepsilon \in (0, 1)$,

$$W^p_{Q_{2L}}(\mu, \kappa) \leq (1 + \varepsilon) \sum_i W^p_{Q_i}(\mu, \kappa_i) + \frac{C}{\varepsilon^{p-1}} W^p_{Q_{2L}} \left( \sum_i \kappa_i \chi_{Q_i}, \kappa \right).$$

Dividing by $|Q_{2L}|$, taking expectations and using the fact that $\mathbb{E}[1/|Q_L|W^p_{Q_i}(\mu, \kappa_i)] = f_r^r(L)$ by translation invariance, we get

$$f_r^r(2L) \leq (1 + \varepsilon) \sum_i \frac{|Q_L|}{|Q_{2L}|} f_r^r(L) + \frac{C}{\varepsilon^{p-1}} \mathbb{E} \left[ \frac{1}{|Q_{2L}|} W^p_{Q_{2L}} \left( \sum_i \kappa_i \chi_{Q_i}, \kappa \right) \right]$$

$$= (1 + \varepsilon) f_r^r(L) + \frac{C}{\varepsilon^{p-1}} \mathbb{E} \left[ \frac{1}{|Q_{2L}|} W^p_{Q_{2L}} \left( \sum_i \kappa_i \chi_{Q_i}, \kappa \right) \right].$$

We now estimate $\mathbb{E}[1/|Q_{2L}|W^p_{Q_{2L}}(\sum_i \kappa_i \chi_{Q_i}, \kappa)]$. Using $1/|Q_{2L}|W^p_{Q_{2L}}(\sum_i \kappa_i \chi_{Q_i}, \kappa) \lesssim L^p \kappa$ together with

$$\mathbb{P}[\kappa \leq \frac{1}{2}] \lesssim \exp(-cL^d),$$

which follows from the Cramér–Chernoff bounds (2-3), we may reduce ourselves to the event $\{\kappa \geq \frac{1}{2}\}$. Under this condition, by (3-3), we have

$$\frac{1}{|Q_{2L}|} W^p_{Q_{2L}} \left( \sum_i \kappa_i \chi_{Q_i}, \kappa \right) \lesssim L^p \int_{Q_{2L}} \sum_i |\kappa_i - \kappa|^p \chi_{Q_i} \lesssim L^p \left( \|\kappa - 1\|^p + \sum_i |\kappa_i - 1|^p \right).$$
Recalling that $\mu(Q_i)$ are Poisson random variables of parameter $|Q_i|$ and that $\kappa_i = \mu(Q_i)/|Q_i|$, we get from (2-4),

$$\mathbb{E}[|\kappa - 1|^p] \sim \mathbb{E}[|\kappa_i - 1|^p] \lesssim \frac{1}{L^{d/2}}.$$

Thus,

$$\mathbb{E}\left[\frac{1}{|Q_{2L}|} W^p_{Q_{2L}} \left( \sum_i \kappa_i \chi_{Q_i}, \kappa \right) \right] \lesssim \frac{1}{L^{d/2}} \left( \sum_i \kappa_i \right) \tag{4-4}$$

and we conclude that for $\varepsilon \in (0, 1)$,

$$f_{\text{ref}}(2L) \leq (1 + \varepsilon) f_{\text{ref}}(L) + \frac{C}{L^{\frac{d}{2}}} \left( \sum_i \kappa_i \right) \tag{4-5}$$

Optimizing in $\varepsilon$ by choosing $\varepsilon = L^{-(d-2)/2}$, and using that $f_{\text{ref}}(L)$ is bounded (by (1-1) and (2-3); see, for instance, [Goldman et al. 2018, Proposition 2.7] for details) we conclude the proof of (4-3). Let us point out that we used here boundedness of $f_{\text{ref}}(L)$ for simplicity but that as shown below it can also be obtained as a consequence of our proof.

**Step 2.** (the general case) We now consider the case when $m \in \mathbb{N}$ is not necessarily dyadic. We will partition $Q_{mL}$ into rectangles of almost dyadic size and thus need to deal with slightly more general configurations than dyadic cubes. Let us introduce some notation. Let $R$ be a rectangle with moderate aspect ratio and $R = \{R_i\}$ be an admissible partition of $R$ (recall Definition 3.3). Slightly abusing notation, we define

$$f_{\text{ref}}(R) = \mathbb{E}\left[\frac{1}{|R|} W^p_R(\mu, \kappa)\right]. \tag{4-6}$$

**Step 2.1.** We claim that the following variant of (4-5) holds: for every rectangle $R$ of moderate aspect ratio with $|R| \gg 1$, every admissible partition $\mathcal{R}$ of $R$, and every $\varepsilon \in (0, 1)$, we have

$$f_{\text{ref}}(R) \leq (1 + \varepsilon) \sum_i \frac{|R_i|}{|R|} f_{\text{ref}}(R_i) + \frac{C}{\varepsilon^{p-1}} \left( \sum_i \kappa_i \right) \tag{4-7}$$

Defining $\kappa_i = \mu(R_i)/|R_i|$ and using (3-1) as above, we get

$$f_{\text{ref}}(R) \leq (1 + \varepsilon) \sum_i \frac{|R_i|}{|R|} f_{\text{ref}}(R_i) + \frac{C}{\varepsilon^{p-1}} \mathbb{E}\left[\frac{1}{|R|} W^p_R \left( \sum_i \kappa_i \chi_{R_i}, \kappa \right) \right]. \tag{4-8}$$

The estimate

$$\mathbb{E}\left[\frac{1}{|R|} W^p_R \left( \sum_i \kappa_i \chi_{R_i}, \kappa \right) \right] \lesssim \frac{1}{|R|^{\frac{d(d-2)}{2d}}} \tag{4-8}$$

is then obtained arguing exactly as for (4-4), using first the Crámer–Chernoff bound (2-3) to reduce to the event $\{\kappa \geq \frac{1}{2}\}$ and then (3-3) (recalling that $\text{diam}(R) \sim |R|^{1/2}$ since $R$ has moderate aspect ratio) in combination with (2-4) and the fact that $\#\mathcal{R} \lesssim 1$ since $\mathcal{R}$ is an admissible partition.

**Step 2.2.** Starting from the cube $Q_{mL}$, let us construct a sequence of finer and finer partitions of $Q_{mL}$ by rectangles of moderate aspect ratios and side-length given by integer multiples of $L$. We let $\mathcal{R}_0 = \{Q_{mL}\}$
and define $\mathcal{R}_k$ inductively as follows. Let $R \in \mathcal{R}_k$. Up to translation we may assume that $R = \prod_{i=1}^{d} (0, m_i L)$ for some $m_i \in \mathbb{N}$. We then split each interval $(0, m_i L)$ into $(0, \lfloor m_i/2 \rfloor L) \cup (\lfloor m_i/2 \rfloor L, m_i L)$. It is readily seen that this induces an admissible partition of $R$. Let us point out that when $m_i = 1$ for some $i$, the corresponding interval $(0, \lfloor m_i/2 \rfloor L)$ is empty. This procedure stops after a finite number of steps $K$ once $\mathcal{R}_K = \{Q_L + z_i, z_i \in [0, m_i - 1]^d\}$. It is also readily seen that $2^{K-1} < m \leq 2^K$ and that for every $k \in [0, K]$ and every $R \in \mathcal{R}_k$ we have $|R| \sim (2^{K-k} L)^d$.

Let us prove by a downward induction that there exists $\Lambda > 0$ such that for every $k \in [0, K]$ and every $R \in \mathcal{R}_k$,

$$f^{\text{ref}}(R) \leq f^{\text{ref}}(Q_L) + \Lambda (1 + f^{\text{ref}}(Q_L)) L^{- \frac{d-2}{2}} \sum_{j=K-k}^{K} 2^{-j \frac{d-2}{2}}. \quad (4-9)$$

This is clearly true for $k = K$. Assume that it holds true for $k + 1$. Let $R \in \mathcal{R}_k$. Applying (4-7) with $\varepsilon = (2^{K-k} L)^{-(d-2)/2} \ll 1$, we get

$$f^{\text{ref}}(R) \leq (1 + \varepsilon) \sum_{R_i \in \mathcal{R}_{k+1}, R_i \subset R} \frac{|R_i|}{|R|} f^{\text{ref}}(R_i) + \frac{C}{\varepsilon} \frac{1}{|R|^{\frac{d+2}{2}}}

\begin{equation}
\leq (1 + \varepsilon) \left(f^{\text{ref}}(Q_L) + \Lambda (1 + f^{\text{ref}}(Q_L)) L^{- \frac{d-2}{2}} \sum_{j=K-k+1}^{K} 2^{-j \frac{d-2}{2}}\right) + C(2^{K-k} L)^{- \frac{d-2}{2}}
\end{equation}

$$\leq f^{\text{ref}}(Q_L) + \Lambda (1 + f^{\text{ref}}(Q_L)) L^{- \frac{d-2}{2}} \sum_{j=K-k+1}^{K} 2^{-j \frac{d-2}{2}} \left( \frac{C}{\Lambda} + L^{- \frac{d-2}{2}} \sum_{j=K-k+1}^{K} 2^{-j \frac{d-2}{2}} \right)\].$$

If $L$ is large enough, then $(\sum_{j=K-k+1}^{K} 2^{-j \frac{d-2}{2}}) L^{-(d-2)/2} \leq \frac{1}{2}$. Finally, choosing $\Lambda \geq 2C$ yields (4-9).

Applying (4-9) to $R = Q_{mL}$ and using that $\sum_{j \geq 0} 2^{-j \frac{d-2}{2}} < \infty$, we get

$$f^{\text{ref}}(mL) \leq f^{\text{ref}}(L) + C(1 + f^{\text{ref}}(L)) \frac{1}{L^{\frac{d-2}{2}}}.$$

Since $f^{\text{ref}}(L) \lesssim L^p$, writing that every $L \gg 1$ may be written as $L = mL'$ for some $m \in \mathbb{N}$ and $L' \in [1, 2]$, we conclude that $f^{\text{ref}}(L)$ is bounded and thus (4-2) follows.

\[\square\]

**Remark 4.3.** We point out that as a consequence of the proof of Proposition 4.2 we have for every rectangle $R$ of moderate aspect ratio (recall definition (4-6)),

$$f^{\text{ref}}(R) \lesssim 1. \quad (4-10)$$

We can now prove Theorem 4.1.

**Proof of Theorem 4.1.** The existence of a limit $f^{\text{ref}}_{\infty}$ is obtained from (4-2) arguing exactly as in [Boutet de Monvel and Martin 2002] using the continuity of $L \mapsto f^{\text{ref}}(L)$ (which can be obtained for instance by dominated convergence). The fact that $f^{\text{ref}}(L) \gtrsim 1$ and thus $f^{\text{ref}}_{\infty} > 0$ follows from (1-1) and (2-3). Finally, (4-1) follows from (4-2) by sending $m \to \infty$ for fixed $L$. \[\square\]
Remark 4.4. It may be conjectured from [Caracciolo et al. 2014] that
\[ |f^{\text{ref}}(L) - f_\infty^{\text{ref}}| \lesssim \frac{1}{L^{d-2}}. \]
Let us notice that if we could replace the term \( \varepsilon^{-(p-1)} \) in (4-5) by a constant, then letting \( \varepsilon \to 0 \) we would get the lower bound
\[ f^{\text{ref}}(L) - f_\infty^{\text{ref}} \gtrsim -\frac{1}{L^{\frac{p(d-2)}{2}}}, \]
which, at least for \( p = 2 \), is in line with the conjectured rate. See also Remark 6.4 for rates in the case of a deterministic number of points.

5. Bipartite matching

We now turn to the bipartite matching. For \( \mu \) and \( \lambda \) two independent Poisson point processes of intensity one, we want to study the asymptotic behavior as \( L \to \infty \) of
\[ f^{\text{bi}}(L) = \mathbb{E}\left[ \frac{1}{|Q_L|} W^p_{Q_L}(\mu, \kappa \lambda) \right], \]
with \( \kappa = \mu(Q_L)/\lambda(Q_L) \). Analogously to Theorem 4.1, we have:

Theorem 5.1. For every \( d \geq 3 \) and \( p \geq 1 \), the limit
\[ f^{\text{bi}}_\infty = \lim_{L \to \infty} f^{\text{bi}}(L) \]
exists and is strictly positive. Moreover, there exists \( C > 0 \) depending on \( p \) and \( d \) such that for \( L \geq 1 \),
\[ f^{\text{bi}}_\infty \leq f^{\text{bi}}(L) + \frac{C}{L^{\frac{d-2}{2}}} \cdot \]

The proof of Theorem 5.1 follows the same line of arguments as for Theorem 4.1. We only detail the estimate of the subadditivity defect, i.e., the counterpart of (4-8), since it is more delicate in the bipartite case.

Proposition 5.2. Let \( R \) be a rectangle of moderate aspect ratio with \( |R| \gg 1 \) and \( \mathcal{R} = \{R_i\} \) be an admissible partition of \( R \) (recall Definition 3.3). Defining \( \kappa_i = \mu(R_i)/\lambda(R_i) \) and \( \kappa = \mu(R)/\lambda(R) \), we have for every \( d \geq 3 \) and \( p \geq 1 \),
\[ \mathbb{E}\left[ \frac{1}{|R|} W^p_{R}(\sum_i \kappa_i \chi_{R_i}, \kappa \lambda) \right] \lesssim \frac{1}{|R|^{\frac{d-2}{2p}}} \cdot \]

Proof. As opposed to (4-8), since \( \lambda \) is atomic, we cannot directly use (3-3). The idea is to use as intermediate step the matching between \( \lambda \) and the reference measure. Let us point out that since \( f^{\text{ref}}(R_i) = \mathbb{E}[1/|R_i| W^p_{R_i}(\lambda, \lambda(R_i)/|R_i|)] \) is of order 1, we cannot apply naively the triangle inequality. The main observation is that since \( |\kappa_i - \kappa| \ll 1 \) with overwhelming probability, the amount of mass which actually needs to be transported is very small.

Let \( 1 \gg \theta > 0 \) to be optimized later on. For every \( i \), let
\[ \theta_i := (\kappa - \kappa_i) + \theta \]
and assume that
\[ \theta \geq 2 \max_i |\kappa - \kappa_i| \]  
(5-2)
so that \( \frac{3}{2} \theta \geq \theta_i \geq \frac{1}{2} \theta > 0 \). Notice that thanks to the Cràmer–Chernoff bounds (2-3), (5-2) is satisfied with overwhelming probability as long as \( \theta \gg |R|^{-1/2} \). Using the triangle inequality (2-1) we have
\[
W_R^p \left( \sum_i \kappa_i \chi_{R_i}, \kappa \lambda \right) \leq W_R^p \left( \sum_i \kappa_i \chi_{R_i}, \kappa \lambda \right) + W_R^p \left( \sum_i \chi_{R_i} \left( (\kappa_i - \theta) \lambda + \theta \frac{\lambda(R_i)}{|R_i|} \right) dx \right)
\]
+ \( W_R^p \left( \sum_i \chi_{R_i} \left( (\kappa_i - \theta) \lambda + \theta \frac{\lambda(R_i)}{|R_i|} \right) dx \right) \sum_i \chi_{R_i} \left( (\kappa_i - \theta) \lambda + \theta \frac{\lambda(R_i)}{|R_i|} \right) dx \)
+ \( W_R^p \left( \sum_i \chi_{R_i} \left( (\kappa_i - \theta) \lambda + \theta \frac{\lambda(R_i)}{|R_i|} \right) dx \right), \kappa \lambda \).

Notice that \( \sum_i \theta \lambda(R_i) = \sum_i \theta_i \lambda(R_i) \) by definition of \( \theta_i \) and the fact that \( \kappa \lambda(R) = \mu(R) = \sum_i \kappa_i \lambda(R_i) \) so that the second term is well defined.

We now estimate the three terms separately. The first and third are estimated in a similar way and we thus focus only on the first one. By subadditivity (2-2) of \( W_R^p \), we have
\[
W_R^p \left( \sum_i \kappa_i \chi_{R_i}, \kappa \lambda \right) \leq \theta \sum_i W_R^p \left( \lambda_{i, \lambda} \right).
\]

We turn to the middle term. Again by subadditivity of \( W_R^p \),
\[
W_R^p \left( \sum_i \chi_{R_i} \left( (\kappa_i - \theta) \lambda + \theta \frac{\lambda(R_i)}{|R_i|} \right) \right) \sum_i \chi_{R_i} \left( (\kappa_i - \theta) \lambda + \theta \frac{\lambda(R_i)}{|R_i|} \right) \right) \leq W_R^p \left( \sum_i \chi_{R_i} \theta \frac{\lambda(R_i)}{|R_i|} \right) + \sum_i \chi_{R_i} \theta_i \frac{\lambda(R_i)}{|R_i|} \right).
\]

Using (3-3) in the event \( \{\lambda(R_i)/|R_i| \sim 1\} \) (which has overwhelming probability) and recalling that we assumed \( \frac{2}{3} \theta \geq \theta_i \geq \frac{1}{2} \theta \), we have
\[
W_R^p \left( \sum_i \chi_{R_i} \theta \frac{\lambda(R_i)}{|R_i|} \right) + \sum_i \chi_{R_i} \theta_i \frac{\lambda(R_i)}{|R_i|} \right) \leq \frac{\text{diam}^p(R)}{\theta p-1} \sum_i \kappa_i - \kappa_j |R_i|.
\]

Putting these two estimates together, dividing by \( |R| \) and taking expectations we find
\[
\mathbb{E} \left[ \frac{1}{|R|} W_R^p \left( \sum_i \kappa_i \chi_{R_i}, \kappa \lambda \right) \right] \leq \theta \sum_i f^{\text{ref}}(R_i) + \frac{\text{diam}^p(R)}{\theta p-1} \sum_i \mathbb{E}[|\kappa_i - \kappa_j|^p]
\]
\[ \leq \theta + \frac{1}{\theta p-1} \frac{1}{|R|^{d-2p}}, \]
with the second inequality using (4-10) and (2-4). Optimizing in \( \theta \) by choosing \( \theta = |R|^{-\frac{d-2p}{2p}} \gg |R|^{-1/2} \) (so that (5-2) is satisfied) this yields (5-1). \( \square \)
Comparing (4-4) and (5-1), one may wonder if (5-1) is suboptimal and could be improved. Let us prove that it is not the case, at least if we consider the slightly more regular situation of the matching to a deterministic grid.

**Proposition 5.3.** Let \( \lambda = \sum_{x \in \mathbb{Z}^d} \delta_x \) and for every \( L \gg 1 \), let \((\mu_1, \ldots, \mu_{2d})\) be \( 2d \) independent Poisson random variables of parameter \( L^d \). Writing for \( L \gg 1 \), \( Q_{2L} = \bigcup_{i=1}^{2d} Q_i \) where \( Q_i \) are disjoint cubes of side-length \( L \) and defining \( \kappa_i = \mu_i / \lambda(Q_i) \), \( \kappa = \sum_{i} \mu_i / \lambda(Q_{2L}) \), we have for \( d \geq 3 \) and \( p \geq 1 \),

\[
\mathbb{E} \left[ \frac{1}{|Q_{2L}|} W^p_{Q_{2L}} \left( \sum_i \kappa_i \chi_{Q_i}, \kappa \lambda \right) \right] \sim \frac{1}{L^{\frac{d-2}{2}}}. 
\]

**Proof.** **Step 1.** (the lower bound) We start by proving that

\[
\mathbb{E} \left[ \frac{1}{|Q_{2L}|} W^p_{Q_{2L}} \left( \sum_i \kappa_i \chi_{Q_i}, \kappa \lambda \right) \right] \geq \frac{1}{L^{\frac{d-2}{2}}}. \tag{5-3}
\]

For this we notice that if \( \pi \) is any admissible coupling, then \(|x - y| \geq 1\) for every \((x, y)\) in the support of \( \pi \) with \( x \neq y \) and thus\(^2\)

\[
W^p_{Q_{2L}} \left( \sum_i \kappa_i \chi_{Q_i}, \kappa \lambda \right) \geq W^1_{Q_{2L}} \left( \sum_i \kappa_i \chi_{Q_i}, \kappa \lambda \right).
\]

Let \( \Sigma = \bigcup_i \partial Q_i \cap Q_{2L} \) and \( \varepsilon_i = \text{sign}(\kappa_i - \kappa) \). Using \( \xi(x) = d(x, \Sigma) \sum_i \varepsilon_i \chi_{Q_i}(x) \) as a test function in the dual formulation of \( W^1_{Q_{2L}} \), we obtain

\[
W^1_{Q_{2L}} \left( \sum_i \kappa_i \chi_{Q_i}, \kappa \lambda \right) \geq \sum_i \sum_{x \in \mathbb{Z}^d \cap Q_i} d(X, \Sigma) |\kappa_i - \kappa| \geq L^{d+1} \sum_i |\kappa_i - \kappa|.
\]

Taking expectations we find (5-3).

**Step 2.** (the upper bound) We turn to the corresponding upper bound,

\[
\mathbb{E} \left[ \frac{1}{|Q_{2L}|} W^p_{Q_{2L}} \left( \sum_i \kappa_i \chi_{Q_i}, \kappa \lambda \right) \right] \leq \frac{1}{L^{\frac{d-2}{2}}}. \tag{5-4}
\]

One could argue exactly as for (5-1) but we provide an alternative proof which uses a one-dimensional argument instead of (3-3). Notice that this argument could have also been used to give a different proof of (4-4).

**Step 2.1.** (the one-dimensional estimate) Let \( \lambda = \sum_{x \in \mathbb{Z}} \delta_x \) and for \( \kappa_1, \kappa_2 > 0 \) and \( L \gg 1 \), define \( \kappa = (\kappa_1 \lambda(0, L/2) + \kappa_2 \lambda(L/2, L)) / \lambda(0, L) \). We claim that if \(|\kappa_1 - \kappa_2| \ll L^{-1} \min(\kappa_1, \kappa_2)\) then

\[
W_{(0,L)}^p(\kappa \chi_{(0,L/2)}, \kappa \lambda) \lesssim L^2 |\kappa_1 - \kappa_2| \tag{5-5}
\]

Let us assume without loss of generality that \( \kappa_1 \geq \kappa_2 \). The optimal transport map is essentially symmetric around \( L/2 \) and is given in \((0, L/2)\) by sending a mass \((k+1)\lambda(L/2, L)/\lambda(0, L)(\kappa_1 - \kappa_2)\) from position \( k \)

\(^2\)Recall that \( W^1_{Q_{2L}} \) denotes the 1-Wasserstein distance.
to $k + 1$ (which is admissible since by hypothesis $k|\kappa_1 - \kappa_2| \ll \min(\kappa_1, \kappa_2)$) so that

$$W_{(0, L)}^{p}(\mu, \lambda) \lesssim \sum_{k=0}^{L/2} k|\kappa_1 - \kappa_2| \sim L^2|\kappa_1 - \kappa_2|.$$ 

This proves (5-5).

**Step 2.2.** (proof of (5-4)) The proof is made recursively by layers. In the first step, we pass from $2^d$ cubes to $2^{d-1}$ rectangles of the form $x + (0, L) \times (0, L/2)^{d-1}$. For this we remark that for every cube $Q_i$ there is exactly one cube $Q_j$ such that $Q_j = Q_i + L/2 e_1$ (where $e_i$ is the canonical basis of $\mathbb{R}^d$). Let us focus on $Q_1 = (0, L/2)^d$ and $Q_2 = L/2 e_1 + Q_1$. Let $R = Q_1 \cup Q_2 = (0, L) \times (0, L/2)^{d-1}$. Define

$$\hat{k} = \frac{\kappa_1 \lambda(Q_1) + \kappa_2 \lambda(Q_2)}{\lambda(R)} = \frac{\mu_1 + \mu_2}{\lambda(R)}.$$

We claim that

$$\mathbb{E} \left[ \frac{1}{|R|} W^2_R(\kappa_1 \chi_{Q_1} \lambda + \kappa_2 \chi_{Q_2} \lambda, \hat{k} \lambda) \right] \lesssim \frac{1}{L^{d+2}}.$$

For this we notice that in $R$, the measures $\kappa_1 \chi_{Q_1} \lambda + \kappa_2 \chi_{Q_2} \lambda$ and $\hat{k} \lambda$ are constant in the directions orthogonal to $e_1$ and thus

$$\frac{1}{|R|} W^p_R(\kappa_1 \chi_{Q_1} \lambda + \kappa_2 \chi_{Q_2} \lambda, \hat{k} \lambda) \lesssim \frac{1}{L} W^p_{(0,L)}(\kappa_1 \chi_{(0,L/2)^d} \lambda + \kappa_2 \chi_{(L/2,L)^d} \lambda, \hat{k} \lambda).$$

Since $\kappa_i = \mu_i / (\lambda(Q_i))$, by the Cramér–Chernoff bounds (2-3), we have $|\kappa_i - 1| = O(L^{-d/2})$ with overwhelming probability. Hence, if $d \geq 3$ we may apply (5-5) to get

$$\mathbb{E} \left[ \frac{1}{|R|} W^p_R(\kappa_1 \chi_{Q_1} \lambda + \kappa_2 \chi_{Q_2} \lambda, \hat{k} \lambda) \right] \lesssim L \mathbb{E}[|\kappa_1 - \kappa_2|] \lesssim \frac{1}{L^{d+2}},$$

which proves the claim.

Finally, we will iterate this argument $2^d$ times. At every step $k$ we have $2^{d-k}$ rectangles of the form $x + (0, L)^k \times (0, L/2)^{d-k}$ and each iteration has a cost of the same order (namely, $L^{-(d-2)/2}$). Using triangle inequality this concludes the proof of (5-4).

**Remark 5.4.** Notice that in the proof of (5-5), we locally move a mass of order $L^{-(d-2)/2}$ which corresponds to the optimal choice of $\theta$ in the proof of (5-1).

### 6. De-Poissonization

In this section we discuss how to transfer limit results from matching problems with Poisson point process, i.e., with a random number of points, to those of with a deterministic number of points.

We use the following general result.

**Proposition 6.1.** Let $f : (0, \infty) \times \mathbb{N}^k \to [0, \infty)$, $(L, n) \mapsto f(L|n)$, satisfy the following assumptions:

1. (**p-homogeneity**) $f(L|n) = L^p f(1|n)$, for every $n \in \mathbb{N}^k$, $L > 0$.
2. (**boundedness**) $f(1|n) \leq 1$, for every $n \in \mathbb{N}^k$.
3. (**monotonicity**) $f(1|n) \leq f(1|m)$, for every $m$, $n \in \mathbb{N}^k$ such that $m_i \leq n_i$ for $i = 1, \ldots, k$. 


Define

\[ f(L) = \mathbb{E}[f(L|N_L)]. \]

where \( N_L = (N_{L,i})_{i=1}^k \) are i.i.d. Poisson random variables with parameter \( L^d \). Then,

\[
\lim_{n \to \infty} n^{\frac{2}{d}} f(1(n, \ldots, n)) = \lim_{L \to \infty} f(L) \quad \text{and} \quad \limsup_{n \to \infty} n^{\frac{2}{d}} f(1(n, \ldots, n)) = \limsup_{L \to \infty} f(L).
\]

Proof. Let \( 0 < \delta < L^d \) and introduce the event

\[
A = \{|N_{L,i} - L^d| < \delta \quad \text{for} \quad i = 1, \ldots, k\}.
\]

By independence and Poisson tail bounds (2-3),

\[
\mathbb{P}(A) \geq \left(1 - 2\exp\left(-\frac{1}{2} L^d + \delta\right)\right)^k. \tag{6-1}
\]

We decompose

\[
f(L) = \mathbb{E}[f(L|N_L)|A]\mathbb{P}(A) + \mathbb{E}[f(L|N_L)|A^c]\mathbb{P}(A^c).
\]

If \( A \) holds, we use monotonicity of \( f(L|n) \) to argue that

\[
L^p f(1|a) \leq \mathbb{E}[f(L|N_L)|A] \leq L^p f(1|b),
\]

where \( a = L^d + \delta, \ b = L^d - \delta \). Otherwise, we use (1) and (2) to obtain

\[
0 \leq \mathbb{E}[f(L|N_L)|A^c] \leq L^p.
\]

Combining these inequalities, we have

\[
L^p f(1|a)\mathbb{P}(A) \leq f(L) \leq L^p (f(1|b) + (1 - \mathbb{P}(A))). \tag{6-2}
\]

For any \( n \geq 1 \), let \( L = L(n) \) be such that \( L^d + L^{d/2} \sqrt{2\beta \log L} = n \), for some fixed \( \beta > p \). Then, we have from (6-2) with \( \delta = L^{d/2} \sqrt{2\beta \log L} \),

\[
f(1|n) \leq \frac{f(L)}{L^p\mathbb{P}(A)}.
\]

As \( n \to \infty \), we have

\[
\frac{n^{\frac{2}{d}}}{L^p} = (1 + L^{-\frac{d}{2}} \sqrt{2\beta \log L})^{\frac{2}{d}} = 1 + O\left(\frac{\log n}{n}\right).
\]

Moreover, by (6-1),

\[
\mathbb{P}(A) \geq \left(1 - 2\exp\left(-\frac{1}{2} L^d + \delta\right)\right)^k = 1 - O(L^{-\beta}) = 1 - O(n^{-\frac{d}{2p}}).
\]

It follows that

\[
\limsup_{n \to \infty} n^{\frac{2}{d}} f(1(n, \ldots, n)) \leq \limsup_{L \to \infty} f(L) \quad \text{and} \quad \liminf_{n \to \infty} n^{\frac{2}{d}} f(1(n, \ldots, n)) \leq \liminf_{L \to \infty} f(L).
\]

To obtain the converse inequalities, we choose instead \( L = L(n) \) such that \( L^d - L^{d/2} \sqrt{2\beta \log L} = n \) and argue analogously. \( \square \)
We now apply Proposition 6.1 to matching problems. Let us first consider the case of matching to the reference measure.

**Theorem 6.2.** Let \( d \geq 3 \), \( p \geq 1 \), and \((X_i)_{i \geq 1}\) be i.i.d. uniform random variables on \([0, 1]^d\). Then,

\[
\lim_{n \to \infty} n^p E \left[ W_{\{0,1\}^d}^p \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}, 1 \right) \right] = f^\text{ref}_\infty,
\]

with \( f^\text{ref}_\infty \) as in Theorem 4.1.

**Proof.** Recalling the notation \( Q_1 = [0, 1]^d \), we introduce the function

\[
f(L|n) = L^p E \left[ W_{\{0,1\}^d}^p \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}, 1 \right) \right], \quad \text{if } n \geq 1,
\]

and \( f(L|0) = 0 \). It is clearly bounded and \( p \)-homogeneous in the sense of Proposition 6.1. To show monotonicity, let \( 1 \leq m \leq n \) and use the identity

\[
\frac{1}{n} \sum_{i=1}^{n} \delta_{X_i} = \left( \frac{n}{m} \right)^{-1} \sum_{I \subseteq \{1, \ldots, n\}} \frac{1}{m} \sum_{i \in I} \delta_{X_i}
\]

in combination with the convexity of the transportation cost, to obtain

\[
W_{\{0,1\}^d}^p \left( \sum_{i=1}^{n} \delta_{X_i}, 1 \right) = W_{\{0,1\}^d}^p \left( \left( \frac{n}{m} \right)^{-1} \sum_{I \subseteq \{1, \ldots, n\}} \frac{1}{m} \sum_{i \in I} \delta_{X_i}, 1 \right) \leq \left( \frac{n}{m} \right)^{-1} \sum_{I \subseteq \{1, \ldots, n\}} W_{\{0,1\}^d}^p \left( \frac{1}{m} \sum_{i \in I} \delta_{X_i}, 1 \right).
\]

Taking expectation yields \( f(1|n) \leq f(1|m) \), since for \( I \subseteq \{1, \ldots, n\} \) with \(|I| = m\), \( \{X_i\}_{i \in I} \) have the same law as \( \{X_i\}_{i=1}^{m} \).

Let \( \mu \) be a Poisson point process of intensity one on \( \mathbb{R}^d \) and for \( L > 1 \) let \( N_L = \mu(Q_L) \) be a Poisson random variable of parameter \( L^d \). For \( n \geq 1 \), we notice that

\[
f(L|n) = E \left[ W_{\{0,1\}^d}^p \left( \frac{\mu}{\mu(Q_L)}, \frac{1}{|Q_L|} \right) \left| N_L = n \right. \right] = \frac{L^d}{n} f^\text{ref}_\infty(L|n), \quad (6-3)
\]

where we write, extending the notation from Section 4,

\[
f^\text{ref}_\infty(L|n) = E \left[ \frac{1}{|Q_L|} W_{\{0,1\}^d}^p (\mu, \kappa) \left| N_L = n \right. \right],
\]

with \( \kappa = \mu(Q_L)/|Q_L| \) (and \( f^\text{ref}_\infty(L|0) = 0 \)). Notice that

\[
f^\text{ref}_\infty(L) = E[f^\text{ref}_\infty(L|N_L)].
In order to apply Proposition 6.1 and conclude the proof, we argue that
\[
\lim_{L \to \infty} \mathbb{E}[|f(L|N_L) - f^{\text{ref}}(L|N_L)|] = 0. \tag{6-4}
\]

To this aim, we first bound \( \mathbb{E}[f(L|N_L)] \) uniformly from above as \( L \to \infty \). For this we combine the following two simple facts. First, since \( f(L|n) \leq L^p \), letting \( A_L = \{ N_L \geq |Q_L|/2 \} \), we have
\[
\mathbb{E}[f(L|N_L)|A_L^c] \leq L^p.
\]
Second, since by (2-3) \( \mathbb{P}[A_L] \geq 1 \),
\[
\mathbb{E}[f(L|N_L)|A_L] \overset{(6-3)}{=} \mathbb{E} \left[ \frac{L^d}{N_L} f^{\text{ref}}(L|N_L) \bigg| A_L \right] \overset{\text{fact}}{\leq} \mathbb{E}[f^{\text{ref}}(L|N_L)] = f^{\text{ref}}(L) \lesssim 1,
\]
where in the last inequality we used that \( f^{\text{ref}} \) is bounded as a consequence of Theorem 4.1. Therefore,
\[
\mathbb{E}[f(L|N_L)] = \mathbb{E}[f(L|N_L)|A_L^c] \mathbb{P}[A_L^c] + \mathbb{E}[f(L|N_L)|A_L] \mathbb{P}[A_L] \lesssim L^p \mathbb{P}[A_L^c] + 1 \lesssim L^p \exp(-cL^d) + 1 \lesssim 1.
\]
Using Hölder inequality with \( \frac{1}{q} + \frac{1}{q'} = 1 \), and the fact that \( f(L|N_L) \lesssim L^p \),
\[
\mathbb{E}[|f(L|N_L) - f^{\text{ref}}(L|N_L)|] \overset{(6-3)}{=} \mathbb{E} \left[ \left| \frac{N_L}{|Q_L|} - 1 \right| f(L|N_L) \right] \lesssim L^{-\frac{d}{2}} L^{-\frac{p(q-1)}{q}} \mathbb{E}[f(L|N_L)]^{\frac{1}{q}} \lesssim L^{-\frac{d}{2} + \frac{p(q-1)}{q}}.
\]
Choosing \( q \) close enough to \( 1 \), in particular \( 1 < q < \frac{2p}{2p - d} \) if \( p > \frac{d}{2} \), we get (6-4). \( \square \)

Arguing similarly, we obtain the corresponding result for the bipartite matching on the unit cube, that is Theorem 1.1, which we restate for the reader’s convenience.

**Theorem 6.3.** Let \( d \geq 3 \), \( p \geq 1 \), and \( (X_i)_{i \geq 1}, (Y_i)_{i \geq 1} \), be independent uniform random variables on \([0, 1]^d\). Then,
\[
\lim_{n \to \infty} n^p \mathbb{E} \left[ W^p_{[0,1]^d} \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}, \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i} \right) \right] = f^{\text{bi}}_{\infty},
\]
with \( f^{\text{bi}}_{\infty} \) as in Theorem 5.1.

**Proof.** The proof is very similar to that of Theorem 6.2, but in this case we define the function
\[
f(L|n_1, n_2) = L^p \mathbb{E} \left[ W^p_{[0,1]^d} \left( \frac{1}{n_1} \sum_{i=1}^{n_1} \delta_{X_i}, \frac{1}{n_2} \sum_{i=1}^{n_2} \delta_{Y_i} \right) \right].
\]
for \( n_1, n_2 \geq 1 \), and let \( f(L|n_1, n_2) = 0 \) otherwise. It is clearly bounded and \( p \)-homogeneous. Using the identities
\[
\frac{1}{n_1} \sum_{i=1}^{n_1} \delta_{X_i} = \left( \frac{n_1}{m_1} \right)^{-1} \left( \frac{n_2}{m_2} \right)^{-1} \sum_{I_1 \subseteq \{1, \ldots, n_1\}, |I_1| = m_1} \sum_{|I| = m_1} \frac{1}{m_1} \sum_{i \in I_1} \delta_{X_i},
\]
\[
\frac{1}{n_2} \sum_{i=1}^{n_2} \delta_{Y_i} = \left( \frac{n_1}{m_1} \right)^{-1} \left( \frac{n_2}{m_2} \right)^{-1} \sum_{I_2 \subseteq \{1, \ldots, n_2\}, |I_2| = m_2} \sum_{|I| = m_2} \frac{1}{m_2} \sum_{i \in I_2} \delta_{Y_i},
\]
we obtain monotonicity arguing analogously as in Theorem 6.2. The proof is then concluded as in Theorem 6.2 and we omit the details. \( \square \)

**Remark 6.4** (convergence rates). An inspection of the proof of Proposition 6.1 shows that one can transfer rates of convergence (even only one-sided) from the Poisson case to that of a fixed number of points. In the case of matching to the reference measure this leads to the inequality
\[
f_{\infty}^{\text{ref}} \leq n^p \mathbb{E} \left[ W_{Q_1}^{p} \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}, 1 \right) \right] + C n \frac{2^d}{\delta^2},
\]
while for the bipartite matching we obtain
\[
f_{\infty}^{\text{bi}} \leq n^p \mathbb{E} \left[ W_{Q_1}^{p} \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}, \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i} \right) \right] + C n \frac{2^d}{\delta^2},
\]
for some constant \( C \geq 0 \). Besides being one-sided bounds, these are still far from the conjectured rates in [Caracciolo et al. 2014], which for \( p = 2 \) read
\[
n^2 \mathbb{E} \left[ W_{Q_1}^{2} \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}, \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i} \right) \right] = f_{\infty}^{\text{bi}} + C n \frac{2^d}{\delta^2} + o(n \frac{2^d}{\delta^2}),
\]
with an explicit constant \( C \).

**Remark 6.5** (strong convergence). If \( p < d \), standard concentration of measure arguments allow to obtain strong convergence from convergence of the expected values (see also [Ambrosio et al. 2019] for a similar argument in the case \( p = d = 2 \)). Let us consider for example the case of bipartite matching, and show that, both \( \mathbb{P}\)-a.s. and in \( L^1(\mathbb{P}) \),
\[
\lim_{n \to \infty} n^p \mathbb{E} \left[ W_{Q_1}^{p} \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}, \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i} \right) \right] = f_{\infty}^{\text{bi}}, \tag{6-5}
\]
with \((X_i)_{i \geq 1}, (Y_i)_{i \geq 1}\) independent uniform random variables on \( Q_1 = [0, 1]^d \).

For any \( n \geq 1 \), the function
\[
[0, 1]^{d \times 2n} \ni (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto n^p W_{\mathbb{P}} (\mu_X)_i, \mu_Y)_i,
\]
is $2n^{\frac{1}{p} - \min\{\frac{1}{2}, \frac{1}{p}\}}$-Lipschitz with respect to the Euclidean distance, where we write

$$\mu_{(x_i)}_{i=1}^n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}, \quad \text{for } x_i \in [0, 1]^d.$$  

This relies on the triangle inequality (2-1) and the fact that

$$\text{lim}_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \delta_{x_i} = \mu$$

if we endow the set of probability measures on $[0, 1]^d$ with the Wasserstein distance of order $p$. Indeed, for $(x_i)_{i=1}^n, (y_i)_{i=1}^n \in [0, 1]^d \times n$, then

$$W_p(\mu_{(x_i)}_{i=1}^n, \mu_{(y_i)}_{i=1}^n) \leq \left(\frac{1}{n} \sum_{i=1}^n |x_i - y_i|^p\right)^{\frac{1}{p}} \leq n^{-\min\{\frac{1}{2}, \frac{1}{p}\}} \left(\sum_{i=1}^n |x_i - y_i|^2\right)^{\frac{1}{2}}.$$  

Gaussian concentration for the uniform measure on the unit cube [Ledoux 2001, Proposition 2.8] yields that if

$$Z_n = n^{\frac{1}{2}} W_p(\mu_{(x_i)}_{i=1}^n, \mu_{(y_i)}_{i=1}^n),$$

then, for $r > 0$,

$$\mathbb{P}(|Z_n - \mathbb{E}[Z_n]| \geq r) \leq 2 \exp(-cn^{2(\min\{\frac{1}{2}, \frac{1}{p}\} - \frac{1}{2})} r^2),$$

where $c > 0$ is an absolute constant. A standard application of Borel–Cantelli lemma gives that, if $1 \leq p < d$ and $d \geq 3$, then

$$\lim_{n \to \infty} (Z_n - \mathbb{E}[Z_n]) = 0, \quad \mathbb{P}\text{-a.s.} \quad (6-6)$$

Moreover, using the layer-cake formula, we obtain the inequality

$$\mathbb{E}[|Z_n - \mathbb{E}[Z_n]|^p] \leq n^{\frac{p}{2} - \min\{\frac{p}{2}, 1\}}, \quad (6-7)$$

which is infinitesimal as $n \to \infty$. To conclude, it is sufficient to argue that $\lim_{n \to \infty} \mathbb{E}[Z_n] = (f_{\infty}^{\text{bi}})^{1/p}$, since it yields by (6-7) and (6-6) that $\lim_{n \to \infty} Z_n = (f_{\infty}^{\text{bi}})^{1/p}$ in $L^p(\mathbb{P})$ and $\mathbb{P}$-a.s. and hence (6-5). By Theorem 6.2, $\lim_{n \to \infty} \mathbb{E}[Z_n^p] = f_{\infty}^{\text{bi}}$. By Jensen’s inequality,

$$\limsup_{n \to \infty} \mathbb{E}[Z_n]^p \leq \lim_{n \to \infty} \mathbb{E}[Z_n^p] = f_{\infty}^{\text{bi}}.$$

By the elementary inequality (3-2), we have, for any $\varepsilon > 0$,

$$Z_n^p \leq (\mathbb{E}[Z_n] + |Z_n - \mathbb{E}[Z_n]|)^p \leq (1 + \varepsilon)\mathbb{E}[Z_n]^p + \frac{C}{\varepsilon^{p-1}} |Z_n - \mathbb{E}[Z_n]|^p.$$

Taking expectation and letting $n \to \infty$, using (6-7), we obtain

$$f_{\infty}^{\text{bi}} = \lim_{n \to \infty} \mathbb{E}[Z_n^p] \leq (1 + \varepsilon) \liminf_{n \to \infty} \mathbb{E}[Z_n]^p.$$

Letting $\varepsilon \to 0$, we are done.
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GLOBAL RIGIDITY AND EXPONENTIAL MOMENTS FOR SOFT AND HARD EDGE POINT PROCESSES

CHRISTOPHE CHARLIER AND TOM CLAEYS

We establish global rigidity upper bounds for universal determinantal point processes describing edge eigenvalues of random matrices. For this, we first obtain a general result which can be applied to general (not necessarily determinantal) point processes which have a smallest (or largest) point: this allows us to deduce global rigidity upper bounds from the exponential moments of the counting function of the process. Combining our general result with known exponential moment asymptotics for the Airy and Bessel point processes, we improve on the best known upper bounds for the global rigidity of the Airy point process, and we obtain new global rigidity results for the Bessel point process.

Secondly, we obtain exponential moment asymptotics for Wright’s generalized Bessel process and the Meijer-G process, up to and including the constant term. As a direct consequence, we obtain new results for the expectation and variance of the associated counting functions. Furthermore, by combining these asymptotics with our general rigidity theorem, we obtain new global rigidity upper bounds for these point processes.

1. Introduction and statement of results

An important question in recent years in random matrix theory has been to understand how much the ordered eigenvalues of a random matrix can deviate from their typical locations. It has been observed [Johansson 1998; Gustavsson 2005; Arguin et al. 2017; Erdős et al. 2009; 2012; Claeys et al. 2019a] that the individual eigenvalues fluctuate on scales that are only slightly bigger than the microscopic scale. This property is loosely called the rigidity of random matrix eigenvalues. To make this more precise, let us consider the circular unitary ensemble (CUE) which consists of \( n \times n \) unitary Haar distributed matrices. The eigenvalues of such a random matrix lie on the unit circle in the complex plane, and if we denote the eigenangles as \( 0 < \theta_1 \leq \cdots \leq \theta_n \leq 2\pi \), we can expect that \( \theta_j \) will, for typical configurations, lie close to \( \frac{2\pi j}{n} \) because of the rotational invariance of the probability distribution of the eigenvalues. Indeed, it was shown in [Arguin et al. 2017, Theorem 1.5] (see also [Paquette and Zeitouni 2018]) that

\[
\lim_{n \to \infty} \mathbb{P}_{\text{CUE}} \left( (2 - \epsilon) \frac{\log n}{n} < \max_{j=1,\ldots,n} \left| \theta_j - \frac{2\pi j}{n} \right| < (2 + \epsilon) \frac{\log n}{n} \right) = 1
\]

for any \( \epsilon > 0 \). We call this an optimal global rigidity result because the lower and upper bounds of the maximal eigenvalue deviation differ only by a multiplicative factor which can be chosen arbitrarily close to 1. Similar optimal global rigidity results have been obtained in circular \( \beta \)-ensembles [Chhaibi et al.]

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2018; Lambert 2021], in unitary invariant random matrix ensembles [Claeys et al. 2019a], and also for
the sine $\beta$-process [Holcomb and Paquette 2018; Lambert 2021]. In the two-dimensional setting of the
Ginibre ensemble, results of a similar nature were obtained in [Lambert 2020].

One of the important features of random matrix eigenvalues is their universal nature: their
asymptotic behavior on microscopic scales is similar for large classes of random matrix models. For
instance, in many matrix models of Hermitian $n \times n$ matrices, like the GUE, Wigner matrices, and unitary
invariant matrices, the microscopic large $n$ behavior of bulk eigenvalues is described by the sine point
process (see, e.g., [Erdős and Yau 2017]), whereas the microscopic behavior of edge eigenvalues is
described by the Airy point process [Deift and Gioev 2007; Deift et al. 1999; Bourgade et al. 2014;
Forrester 1993; Prähofer and Spohn 2002; Soshnikov 2000; Tracy and Widom 1994a]. For ensembles
of positive-definite Hermitian matrices, the situation is somewhat more complicated. In the Wishart–
Laguerre ensemble and its unitary invariant generalizations, the Bessel point process typically describes
the microscopic behavior of the smallest eigenvalues close to the hard edge 0 [Forrester 1993; Kuijlaars
and Vanlessen 2002; Tracy and Widom 1994b]. However, in a generalization of the Wishart–Laguerre
ensemble known as the Muttalib–Borodin Laguerre ensemble [Borodin 1999; Muttilah 1995], the local
behavior of eigenvalues near the hard edge is described by a different determinantal point process known
as Wright's generalized Bessel point process [Borodin 1999; Kuijlaars and Molag 2019]. Another
generalization of the Bessel process, known as the Meijer-G point process, arises at the hard edge of
Wishart-type products of Ginibre or truncated unitary matrices [Akemann et al. 2013a; 2013b; Kuijlaars
and Zhang 2014; Kuijlaars and Stivigny 2014; Kieburg et al. 2016], and in Cauchy multimatrix ensembles
[Bertola and Bothner 2015; Bertola et al. 2014].

In this paper, we will establish upper bounds for the global rigidity of the Airy point process, the Bessel
point process, and its (determinantal) generalizations arising near the hard edge in Muttalib–Borodin
ensembles and in product random matrix ensembles. We do this by combining asymptotics for exponential
moments of the counting measures of these point processes, which are Fredholm determinants of certain
integral operators, with a global rigidity estimate which can be applied to general point processes which
almost surely have a smallest (or largest) point. In the case of the Airy and Bessel point processes,
asymptotics for the exponential moments are known (see [Bothner and Buckingham 2018; Charlier and
Claeys 2020] for Airy and [Bothner et al. 2019; Charlier 2020] for Bessel) and they allow us to improve
on the best known upper bounds for the Airy point process (see [Zhong 2019] and [Corwin and Ghosal
2020, Theorem 1.6]), and to deduce completely new global rigidity results for the Bessel point process.
As global rigidity estimates give an intuitive idea of how the particles in a point process behave, we
believe that they may lead to a better understanding of these random matrix point processes. Besides that,
global rigidity estimates allow us to control averages of multiplicative statistics; global rigidity estimates
in the Airy point process were for instance used in this way in [Corwin and Ghosal 2020] to obtain
estimates for the lower tail of the Kardar–Parisi–Zhang equation.

Another main contribution of this paper consists in deriving exponential moment asymptotics for
Wright’s generalized Bessel and Meijer-G point processes. We emphasize that we explicitly compute the
multiplicative constant in these asymptotics, which is in general very challenging; see, e.g., [Forrester
2014; Krasovsky 2009] as general references and [Charlier et al. 2019a; Charlier et al. 2019b] in the case of gap probabilities for the processes under consideration in this paper (the exponential moments under consideration here are gap probabilities for thinnings of these processes). As consequences of the exponential moment asymptotics, we obtain asymptotics for the average and variance of the counting functions of these processes, and an upper bound for their global rigidity.

**General rigidity theorem.** Suppose that \( X \) is a locally finite random point process on the real line which has almost surely a smallest particle, and denote the ordered random points in the process by \( x_1 \leq x_2 \leq \cdots \). We write \( N(s) \) for the random variable that counts the number of points \( \leq s \). We will work under the following assumptions, which, as we will see later, are fairly easy to verify in practice.

**Assumptions 1.1.** There exist constants \( C, a > 0, s_0 \in \mathbb{R}, M > \sqrt{2/a} \) and continuous functions \( \mu, \sigma : [s_0, +\infty) \to [0, +\infty) \) such that the following holds:

1. We have
   \[
   \mathbb{E}[e^{-\gamma N(s)}] \leq C e^{-\gamma \mu(s) + \frac{\gamma^2}{2} \sigma^2(s)}
   \]  
   for any \( \gamma \in [-M, M] \) and for any \( s > s_0 \).

2. The functions \( \mu \) and \( \sigma \) are strictly increasing and differentiable on \( (s_0, +\infty) \), and they satisfy
   \[
   \lim_{s \to +\infty} \mu(s) = +\infty \quad \text{and} \quad \lim_{s \to +\infty} \sigma(s) = +\infty.
   \]
   Moreover, \( s \mapsto s \mu'(s) \) is weakly increasing and \( \lim_{s \to +\infty} s \mu'(s)/\sigma^2(s) = +\infty \).

3. The function \( \sigma^2 \circ \mu^{-1} : [\mu(s_0), +\infty) \to [0, +\infty) \) is strictly concave, and
   \[
   (\sigma^2 \circ \mu^{-1})(s) \sim a \log s \quad \text{as} \quad s \to +\infty.
   \]

In the above assumptions, \( C \) and \( s_0 \) are auxiliary constants whose values are unimportant, but on the other hand \( a, \mu, \sigma \) will turn out to encode information about fundamental quantities of the point process under consideration, like the mean and variance of the counting functions.

**Theorem 1.2** (rigidity). Suppose that \( X \) is a locally finite point process on the real line which almost surely has a smallest particle and which is such that Assumptions 1.1 hold. Let us write \( x_k \) for the \( k \)-th smallest particle of the process \( X \), \( k \geq 1 \). Then, there are constants \( c > 0 \) and \( s_0 > 0 \) such that for any small enough \( \epsilon > 0 \) and for all \( s \geq s_0 \),

\[
\mathbb{P}\left( \sup_{k \geq \mu(2s)} \frac{|\mu(x_k) - k|}{\sigma^2(\mu^{-1}(k))} > \sqrt{\frac{2}{a} (1 + \epsilon)} \right) \leq \frac{c \mu(s)^{-\frac{\epsilon}{2}}}{\epsilon}.
\]  

(1-2)

In particular, for any \( \epsilon > 0 \),

\[
\lim_{k_0 \to +\infty} \mathbb{P}\left( \sup_{k \geq k_0} \frac{|\mu(x_k) - k|}{\sigma^2(\mu^{-1}(k))} \leq \sqrt{\frac{2}{a} (1 + \epsilon)} \right) = 1.
\]

**Remark 1.3.** The above result derives an upper bound for the global rigidity via the asymptotics for the first exponential moment of the counting function. Estimates for the first exponential moment however do
not allow us to obtain a sharp lower bound for the global rigidity. For this, one would need more delicate information, like estimates for higher exponential moments, about more complicated averages in the point process, or about convergence of the counting function to a Gaussian multiplicative chaos measure; see, e.g., [Arguin et al. 2017; Berestycki et al. 2018; Claeys et al. 2019a; Lambert et al. 2018]. In point processes arising in random matrix theory for which optimal lower bounds for the global rigidity are available (see, e.g., [Arguin et al. 2017; Chhaibi et al. 2018; Claeys et al. 2019a; Holcomb and Paquette 2018; Paquette and Zeitouni 2018]), it turns out that the upper bounds obtained via the first exponential moment are sharp, and therefore we believe that the upper bound in Theorem 1.2 is, at least for the concrete examples considered related to random matrix theory, close to optimal.

Outline of the proof of Theorem 1.2. We will prove Theorem 1.2 in Section 2 using elementary probabilistic estimates. The most delicate step in the proof consists of establishing a probabilistic bound for the supremum of the normalized counting function of the point process under consideration. For this, we need to use a discretization argument, a union bound, and Markov’s inequality together with the exponential moment asymptotics from Assumptions 1.1. Next, we prove that the bound on the supremum of the normalized counting function implies rigidity of the points, and we quantify the relevant probabilities to obtain Theorem 1.2. This method is similar to that of [Claeys et al. 2019a, Section 4].

Global rigidity for the Airy point process. The Airy point process is a determinantal point process on \( \mathbb{R} \) whose correlation kernel is given by

\[
\kappa^{\text{Ai}}(x, y) = \frac{\text{Ai}(x) \text{Ai}'(y) - \text{Ai}'(x) \text{Ai}(y)}{x - y}, \quad x, y \in \mathbb{R},
\]

where \( \text{Ai} \) denotes the Airy function. This point process describes the largest eigenvalues in a large class of random matrix ensembles, and it has almost surely a largest particle. Upper bounds for the fluctuations of the points have been obtained recently in [Zhong 2019] and [Corwin and Ghosal 2020, Theorem 1.6]. A sharper upper bound can be obtained by combining the exponential moment estimates from [Bothner and Buckingham 2018] with Theorem 1.2.

The Airy point process satisfies Assumptions 1.1 only after considering the opposite points \( x_j = -a_j \), where \( a_1 > a_2 > \cdots \) are the random points in the Airy point process. We write \( N^{\text{Ai}}(s) \) for the number of points \( x_j \) smaller than or equal to \( s \). It was proved in [Bothner and Buckingham 2018] (see also [Bogatskiy et al. 2016; Charlier and Claeys 2020]) that

\[
\mathbb{E}[e^{-2\pi v N^{\text{Ai}}(s)}] = 8^{\nu^2} G(1 + i\nu) G(1 - i\nu) e^{-2\pi v \mu(s) + 2\pi^2 v^2 \sigma^2(s)} (1 + O(s^{-3/2}))
\]
as \( s \to +\infty \) uniformly for \( \nu \) in compact subsets of \( \mathbb{R} \), where \( G \) is Barnes’ \( G \) function, and where

\[
\mu(s) = \frac{2}{3\pi} s^{3/2}, \quad \sigma^2(s) = \frac{3}{4\pi^2} \log s.
\]

It is straightforward to verify from this that the Airy point process satisfies Assumptions 1.1 with

\[
M = 10, \quad \gamma = 2\pi \nu, \quad C = 2 \max_{\nu \in \left[ -\frac{M}{2\pi}, \frac{M}{2\pi} \right]} 8^{\nu^2} G(1 + i\nu) G(1 - i\nu), \quad a = \frac{1}{2\pi^2},
\]

and with \( s_0 \) a sufficiently large constant. Applying Theorem 1.2, we obtain the following result.
Figure 1. Global rigidity for the Airy point process: On the left, the blue dots represent the random points and have coordinates $(k, x_k)$, the blue curves are the upper and lower bounds in (1-5) (with $\epsilon = 0.05$), and the orange curve is parametrized by $(t, \left(\frac{3\pi^2}{2}t\right)^{2/3})$. On the right, the blue dots represent the normalized random points with coordinates $(k, \left(\frac{2}{3\pi}x_k^{3/2} - k\right)/\log k)$. The orange lines indicate the heights $\pm \frac{1}{\pi} \pm \epsilon$ with $\epsilon = 0.05$. We observe the presence of points in the bands between the orange lines, indicating that Theorem 1.4 can be expected to be sharp. The points shown in the figure are not exactly the points in the Airy point process: they are sampled as rescaled extreme eigenvalues of a large GUE matrix, which approximate the points in the Airy point process.

**Theorem 1.4** (rigidity for the Airy process). Let $-x_1 > -x_2 > \cdots$ be the points in the Airy point process. There exists a constant $c > 0$ such that

$$
\mathbb{P}
\left( \sup_{k \geq \mu(s)} \left| \frac{2}{3\pi} x_k^{3/2} - k \right| \log k \geq \frac{\sqrt{1 + \epsilon}}{\pi} \right) \leq c s^{-3\epsilon/4},
$$

as $s \to +\infty$, uniformly for $\epsilon > 0$ small. In particular, for any $\epsilon > 0$, we have

$$
\lim_{k_0 \to \infty} \mathbb{P}
\left( \sup_{k \geq k_0} \left| \frac{2}{3\pi} x_k^{3/2} - k \right| \log k \leq \frac{1}{\pi} + \epsilon \right) = 1.
$$

**Remark 1.5.** This result implies that for any $\epsilon > 0$, the probability that

$$
\left( \frac{3\pi}{2} \right)^{2/3} \left( k - \left( \frac{1}{\pi} + \epsilon \right) \log k \right)^{2/3} \leq x_k \leq \left( \frac{3\pi}{2} \right)^{2/3} \left( k + \left( \frac{1}{\pi} + \epsilon \right) \log k \right)^{2/3}
$$

for all $k \geq k_0$ (1-5) tends to 1 as $k_0 \to +\infty$. Figure 1 illustrates this and supports our belief that Theorem 1.4 is close to optimal (see also Remark 1.3). Let us compare the above with known recent results: from [Corwin and Ghosal 2020, Theorem 1.6], it follows that for any $\epsilon > 0$, the probability that

$$
\left| x_k - \left( \frac{3\pi}{2} \right)^{2/3} k^{2/3} \right| \leq \epsilon k^{2/3}
$$

tends to 1 as $k_0 \to \infty$; in [Zhong 2019], large deviation bounds were obtained which can be used to reduce the right-hand side from $\epsilon k^{2/3}$ to $\log^3 k$, but not to $\log k$ (see, e.g., Proposition 2.2 in [Zhong 2019]).
Global rigidity for the Bessel point process. The Bessel point process is another canonical point process from the theory of random matrices. It models the behavior of the eigenvalues near hard edges in a large class of random matrix ensembles, with the Laguerre–Wishart ensemble as the most prominent example [Forrester 1993]. The Bessel point process is a determinantal point process on $(0, +\infty)$ whose correlation kernel is given by
\[ K_{\alpha}^{Be}(x, y) = \frac{\sqrt{y} J_{\alpha}(\sqrt{x}) J'_{\alpha}(\sqrt{y}) - \sqrt{x} J_{\alpha}(\sqrt{y}) J'_{\alpha}(\sqrt{x})}{2(x - y)}, \quad x, y > 0, \] (1-6)
where $\alpha > -1$ and $J_{\alpha}$ is the Bessel function of the first kind of order $\alpha$. To the best of our knowledge, there are no global rigidity upper bounds available in the literature for the Bessel process, but the corresponding exponential moment asymptotics have been obtained in [Bothner et al. 2019; Charlier 2020], and they allow us to apply Theorem 1.2. Let us write $N^{Be}(s)$ for the number of points $x_j$ smaller than or equal to $s$ in the Bessel process. By [Bothner et al. 2019, equation (1.35)], we have
\[ \mathbb{E}[e^{-2\pi \nu N^{Be}(s)}] = 4\nu^2 e^{\pi \mu(s)} G(1 + i\nu) G(1 - i\nu) e^{-2\pi \nu \mu(s) + 2\pi \nu^2 \sigma^2(s)} (1 + O(s^{-1/2})), \] (1-7)
as $s \to +\infty$ uniformly for $\nu$ in compact subsets of $\mathbb{R}$, with
\[ \mu(s) = \frac{\sqrt{s}}{\pi}, \quad \sigma^2(s) = \frac{1}{4\pi^2 \log s}. \] (1-8)
We verify from (1-7) that the Bessel point process satisfies Assumptions 1.1 with
\[ M = 10, \quad \gamma = 2\pi \nu, \quad C = 2 \max_{\nu \in [-\frac{M}{2\pi}, \frac{M}{2\pi}]} 4\nu^2 e^{\pi \nu \mu(s)} G(1 + i\nu) G(1 - i\nu), \quad a = \frac{1}{2\pi^2}, \]
and with $s_0$ a sufficiently large constant. Applying Theorem 1.2, we obtain the following result.

**Theorem 1.6** (rigidity for the Bessel point process). Let $x_1 < x_2 < \cdots$ be the points in the Bessel point process. There exists a constant $c > 0$ such that
\[ \mathbb{P} \left( \sup_{k \geq \mu(s)} \left| \frac{1}{2\pi^2} x_k^{1/2} - k \right| \log k > \frac{\sqrt{1 + \epsilon}}{\pi} \right) \leq \frac{cs^{-\frac{3}{4}}}{\epsilon}, \]
as $s \to +\infty$, uniformly for $\epsilon > 0$ small. In particular, for any $\epsilon > 0$ we have
\[ \lim_{k_0 \to \infty} \mathbb{P} \left( \sup_{k \geq k_0} \left| \frac{1}{2\pi^2} x_k^{1/2} - k \right| \log k \leq \frac{1}{\pi} + \epsilon \right) = 1. \]

**Remark 1.7.** The above implies that for any $\epsilon > 0$, the probability that
\[ \pi^2 \left( k - \frac{1}{\pi} + \epsilon \right) \log k \right)^2 \leq x_k \leq \pi^2 \left( k + \frac{1}{\pi} + \epsilon \right) \log k \right)^2 \text{ for all } k \geq k_0 \] (1-9)
tends to 1 as $k_0 \to +\infty$. Figure 2 illustrates this estimate.

**Exponential moments and rigidity for Wright’s generalized Bessel process.** Wright’s generalized Bessel process appears as the limiting point process at the hard edge of Mutlib–Borodin ensembles [Borodin
Figure 2. Global rigidity for the Bessel point process: On the left, the blue dots represent the random points and have coordinates \((k, x_k)\), the blue curves are the upper and lower bounds in (1-9) (with \(\epsilon = 0.05\)), and the orange curve is parametrized by \((t, \pi^2 t^2)\). On the right, the blue dots represent the normalized random points with coordinates \((k, (\frac{1}{\pi} x_k^{1/2} - k) / \log k)\). The orange lines indicate the heights \(\pm \frac{1}{\pi} \pm \epsilon\) with \(\epsilon = 0.05\). We observe the presence of points in the bands between the orange lines, indicating that Theorem 1.6 can be expected to be sharp. The points shown in the figure are not exactly the points in the Bessel point process: they are sampled as rescaled extreme eigenvalues of a large Laguerre–Wishart random matrix, which approximate the points in the Bessel point process.

1999; Claeys and Romano 2014; Forrester and Wang 2017; Kuijlaars and Molag 2019; Liu et al. 2016; Zhang 2015; 2017]. It is a determinantal point process on \((0, +\infty)\) which depends on parameters \(\theta > 0\) and \(\alpha > -1\). The associated kernel is given by

\[
\mathbb{K}^{\text{Wr}}(x, y) = \theta (xy)^{\frac{\alpha}{2}} \int_0^1 J_{\alpha+1, \frac{\alpha}{2}}(xt) J_{\alpha+1, \theta}(yt)^{\theta} t^{\alpha} dt, \quad x, y > 0, \tag{1-10}
\]

where \(J_{a, b}\) is Wright’s generalized Bessel function,

\[
J_{a, b}(x) = \sum_{m=0}^{\infty} \frac{(-x)^m}{m! \Gamma(a + bm)}, \quad x > 0.
\]

If \(\theta = 1\), this point process reduces (up to a rescaling) to the Bessel point process:

\[
\mathbb{K}^{\text{Wr}}(x, y)|_{\theta=1} = 4 \mathbb{K}^{\text{Be}}(4x, 4y), \quad x, y > 0. \tag{1-11}
\]

We obtain asymptotics for the exponential moments in this point process.

**Theorem 1.8.** Let \(\nu \in \mathbb{R}\) and let \(N^{\text{Wr}}(s)\) denote the number of points smaller than or equal to \(s\) in Wright’s generalized Bessel process. As \(s \to +\infty\), we have

\[
\mathbb{E}[e^{-2\pi \nu N^{\text{Wr}}(s)}] = C \exp(-2\pi \nu \mu(s) + 2\pi^2 \nu^2 \sigma^2(s) + O(s^{-\frac{\theta}{1+\theta}})), \tag{1-12}
\]
where the functions $\mu$ and $\sigma^2$ are given by

$$
\mu(s) = \frac{1 + \theta}{\pi} \theta^{-\frac{\theta}{1+\theta}} \cos\left(\frac{\pi}{2} \frac{1 - \theta}{1 + \theta}\right) s^{\frac{\theta}{1+\theta}} \quad \text{and} \quad \sigma^2(s) = \frac{\theta}{2\pi^2(1+\theta)} \log s,
$$

and the values of $C$ by

$$
C = \exp\left(\frac{\pi v(1-\theta + 2\alpha)}{1+\theta}\right) \left[4(1+\theta) \theta^{-\frac{\theta}{1+\theta}} \sin^2\left(\frac{\pi \theta}{1+\theta}\right)\right]^{\frac{1}{1+\theta}} G(1+i\nu) G(1-i\nu),
$$

where $G$ is Barnes’ $G$-function. Furthermore, the error term in (1-12) is uniform for $v$ in compact subsets of $\mathbb{R}$.

**Remark 1.9.** By setting $\theta = 1$ in (1-12) and then applying the rescaling $s \mapsto \frac{s}{4}$, we recover the asymptotics (1-7) for the Bessel point process, which is consistent with (1-11).

**Remark 1.10.** It follows from the end of Section 1 in [Borodin 1999] that the left-hand side of (1-12) is invariant under the following changes of the parameters:

$$
s \mapsto s^\theta, \quad \theta \mapsto \frac{1}{\theta}, \quad \text{and} \quad \alpha \mapsto \alpha^* := \frac{1+\alpha}{\theta} - 1.
$$

It follows that the constant $C$ and the functions $\mu$ and $\sigma^2$ must obey the following symmetry relations for any $\theta > 0$ and $\alpha > -1$:

$$
\mu(s, \theta, \alpha) = \mu\left(s^\theta, \frac{1}{\theta}, \alpha^*\right), \quad \sigma^2(s, \theta, \alpha) = \sigma^2\left(s^\theta, \frac{1}{\theta}, \alpha^*\right), \quad C(\theta, \alpha) = C\left(\frac{1}{\theta}, \alpha^*\right),
$$

where we have indicated the dependence of the quantities on $\theta$ and $\alpha$ explicitly. These identities can be verified directly from (1-14) and provide a consistency check of our results.

**Remark 1.11.** It is not entirely obvious that the kernel (1-10) defines a point process. To see this, we note first that the kernel (1-10) arises as the large $n$ limit of the correlation kernel $K_n$ in the Muttilalib–Borodin Laguerre ensemble with $n$ particles (see [Borodin 1999]). Next, from [Lenard 1973] and [Soshnikov 2000, Theorem 1], we know that a kernel defines a point process if and only if it generates locally integrable correlation functions which are symmetric under permutations of variables and satisfy a certain positivity condition. Since $K_n$ must satisfy the symmetry and positivity conditions, and since these conditions are closed under taking limits, we can conclude that (1-10) also defines a point process. The uniqueness of the point process follows from the fact that the process is characterized by its Laplace transform $E e^{-\sum_{k=1}^{\infty} f(x_k)}$ for continuous compactly supported functions $f$, where $x_1, x_2, \ldots$ are the points in the process. For a determinantal point process with a kernel $K$ which is trace-class on any compact, the Laplace transform is characterized by $K$ since

$$
E e^{-\sum_{k=1}^{\infty} f(x_k)} = \det(1 - (1 - e^{-f}) K).
$$

The Fredholm determinant at the right is defined since the trace-norm of $(1 - e^{-f}) K$ is bounded by $\|1 - e^{-f}\|_\infty$ times the trace-norm of $K$ restricted to the support of $f$. Hence the process defined by $K$ is unique. See also, e.g., [Fenzl and Lambert 2020, Remark 11] for a similar situation.

Theorem 1.8 has the following immediate consequence.
Corollary 1.12. As $s \to +\infty$, we have
\[
\mathbb{E}[N^{W_r}(s)] = \mu(s) - \frac{1 - \theta + 2\alpha}{2(1 + \theta)} + O(s^{-1/\nu}), \quad (1-16)
\]
\[
\text{Var}[N^{W_r}(s)] = \sigma^2(s) + \frac{1}{2\pi^2} \log \left[ 4(1 + \theta) \theta^{-\frac{\theta}{1+\theta}} \sin^2 \left( \frac{\pi \theta}{1 + \theta} \right) \right] + \frac{1 + \gamma_E}{2\pi^2} + O(s^{-1/\nu}), \quad (1-17)
\]
where $\gamma_E \approx 0.5772$ is Euler’s constant and the functions $\mu$ and $\sigma^2$ are given by (1-13).

Proof. The asymptotics (1-12) are valid uniformly for $v$ in compact subsets of $\mathbb{R}$. Hence, we obtain (1-16)–(1-17) by first expanding (1-12) as $v \to 0$, and then identifying the power of $v$ using
\[
\mathbb{E}[e^{-2\pi v N^{W_r}(s)}] = 1 - 2\pi v \mathbb{E}[N^{W_r}(s)] + 2\pi^2 v^2 \mathbb{E}[(N^{W_r}(s))^2] + O(v^3), \quad \text{as } v \to 0. \quad \Box
\]

Remark 1.13. Setting $\theta = 1$ in (1-16)–(1-17) and then rescaling $s \mapsto \frac{\gamma}{\theta}$, we recover the asymptotic formula [Bothner et al. 2019, equation (1.35)].

We verify from Theorem 1.8 that Wright’s generalized Bessel process satisfies Assumptions 1.1 with $s_0$ a sufficiently large constant, and
\[
M = 10, \quad \gamma = 2\pi v, \quad C = 2 \max_{v \in [-\frac{M}{s_0}, \frac{M}{s_0}]} C(v), \quad a = \frac{1}{2\pi^2},
\]
where $C = C(v)$ is given by (1-14). We obtain the following global rigidity result by combining Theorem 1.2 with Theorem 1.8.

Theorem 1.14 (rigidity for Wright’s generalized Bessel process). Let $x_1 < x_2 < \cdots$ be the points in Wright’s generalized Bessel point process. There exists a constant $c > 0$ such that
\[
\mathbb{P} \left( \sup_{k \geq \mu(s)} \left| \frac{1 + \theta}{\pi} \theta^{-\frac{\theta}{1+\theta}} \cos \left( \frac{\pi}{2} \frac{1 - \theta}{1 + \theta} \right) x_k^{\frac{\theta}{1+\theta}} - k \right| \log k \right) > \frac{\sqrt{1 + \epsilon}}{\pi} \leq \frac{cs^{-\frac{\theta}{1+\theta}}}{\epsilon},
\]
as $s \to +\infty$, uniformly for $\epsilon > 0$ small. In particular, for any $\epsilon > 0$, we have
\[
\lim_{k_0 \to \infty} \mathbb{P} \left( \sup_{k \geq k_0} \left| \frac{1 + \theta}{\pi} \theta^{-\frac{\theta}{1+\theta}} \cos \left( \frac{\pi}{2} \frac{1 - \theta}{1 + \theta} \right) x_k^{\frac{\theta}{1+\theta}} - k \right| \log k \leq \frac{1}{\pi} + \epsilon \right) = 1.
\]

Remark 1.15. This result implies that for any $\epsilon > 0$, the probability that
\[
\left[ \frac{\pi}{1 + \theta} \cos \left( \frac{\pi}{2} \frac{1 - \theta}{1 + \theta} \right) \left( k - \frac{1}{\pi} \log k \right) \right]^{1 + \theta} \pi \leq x_k \leq \left[ \frac{\pi}{1 + \theta} \cos \left( \frac{\pi}{2} \frac{1 - \theta}{1 + \theta} \right) \left( k + \frac{1}{\pi} \log k \right) \right]^{1 + \theta}
\]
for all $k \geq k_0$, tends to 1 as $k_0 \to +\infty$. We illustrate this numerically in Figure 3 (left) for $\epsilon = 0.05$ and different values of $\theta$.

Exponential moments and rigidity for the Meijer-G process. The Meijer-G process is the limiting point process at the hard edge of Wishart-type products of Ginibre matrices or truncated unitary matrices and
appears also in Cauchy multimatrix models [Bertola and Bothner 2015; Bertola et al. 2014; Kieburg et al. 2016; Kuijlaars and Zhang 2014]. It is a determinantal point process on \((0, +\infty)\) which depends on parameters \(r, q \in \mathbb{N} := \{0, 1, 2, \ldots\}\), \(r > q \geq 0\), \(\nu_1, \ldots, \nu_r \in \mathbb{N}\) and \(\mu_1, \ldots, \mu_q \in \mathbb{N}_{>0}\) such that \(\mu_k > \nu_k\), \(k = 1, \ldots, q\). Its kernel can be expressed in terms of the Meijer-G function:

\[
\mathbb{K}^\text{Me}(x, y) = \int_0^1 G^{1, q}_{q, r+1} \left( \begin{array}{c} -\mu_1, \ldots, -\mu_q \\ 0, -\nu_1, \ldots, -\nu_r \end{array} \right) G^{r, 0}_{q, r+1} \left( \begin{array}{c} \mu_1, \ldots, \mu_q \\ v_1, \ldots, v_r, 0 \end{array} \right) dt, \quad x, y > 0. \tag{1-19}
\]

We obtain exponential moment asymptotics for this point process.

**Theorem 1.16.** Let \(v \in \mathbb{R}\) and let \(N^\text{Me}\) be the counting function of the Meijer-G process, with parameters as above. As \(s \to +\infty\), we have

\[
\mathbb{E}[e^{-2\pi v N^\text{Me}(s)}] = C \exp \left(-2\pi v \mu(s) + 2\pi^2 v^2 \sigma^2(s) + O(s^{-1/v})\right), \tag{1-20}
\]

where the functions \(\mu\) and \(\sigma^2\) are given by

\[
\mu(s) = \frac{1 + r - q}{\pi} \cos \left( \frac{\pi}{2} \frac{r - q - 1}{2 r - q + 1} \right) s^{\frac{1}{r+q}} \quad \text{and} \quad \sigma^2(s) = \frac{1}{2\pi^2(1 + r - q) \log s}, \tag{1-21}
\]

and the values of \(C\) by

\[
C = \exp \left( \frac{2\pi v}{1 + r - q} \left[ \sum_{j=1}^r \nu_j - \sum_{j=1}^q \mu_j \right] \right)^{1/2} G(1 + iv)G(1 - iv),
\]

where \(G\) is Barnes’ \(G\)-function. Furthermore, the error term in (1-12) is uniform for \(v\) in compact subsets of \(\mathbb{R}\).
Remark 1.17. If \( q = 0 \) and if the parameters \( \nu_1, \ldots, \nu_r \) form an arithmetic progression, then the kernel \( \kappa^{Me} \) defines the same point process (up to rescaling) as Wright’s generalized Bessel point process (for a rational \( \theta \)); see [Kuijlaars and Stivigny 2014, Theorem 5.1]. More precisely, if \( r \geq 1 \) is an integer, \( \alpha > -1 \) and
\[
\theta = \frac{1}{r}, \quad \nu_j = \alpha + \frac{j-1}{r}, \quad j = 1, \ldots, r,
\]
then the kernels \( \kappa^{Me} \) and \( \kappa^{Wr} \) are related by
\[
\left( \frac{x}{y} \right)^{\frac{q}{2}} \kappa^{Me}(x, y) = r^r \kappa^{Wr}(r^r x, r^r y).
\]
Therefore, if the parameters satisfy (1-22), we obtain the relations
\[
\mu^{Me}(s) = r^{\nu_1} C^{Wr}, \quad \sigma^{Me}(s)^2 = r^{\nu_1} C^{Wr}, \quad C^{Me} = r^{\nu_1} C^{Wr},
\]
where the quantities with superscript \( Wr \) and \( Me \) are given in Theorems 1.8 and 1.16, respectively. All the identities in (1-24) can be verified by a direct computation; this provides another consistency check of our results.

Remark 1.18. The existence and uniqueness of a point process with correlation kernel (1-19) can be shown in a similar way as outlined in Remark 1.11 for Wright’s generalized Bessel process.

Corollary 1.19. As \( s \to +\infty \), we have
\[
\mathbb{E}[N^{Me}(s)] = \mu(s) - \frac{1}{1+r-q} \left[ \sum_{j=1}^{r} \nu_j - \sum_{j=1}^{q} \mu_j \right] + O(s^{-\frac{1}{1+r-q}}),
\]
\[
\text{Var}[N^{Me}(s)] = \sigma^2(s) + \frac{1}{2\pi^2} \log \left[ 4(1+r-q) \sin^2 \left( \frac{\pi}{1+r-q} \right) \right] + \frac{1+\gamma_E}{2\pi^2} + O(s^{-\frac{1}{1+r-q}}),
\]
where \( \gamma_E \) is Euler’s constant and the functions \( \mu \) and \( \sigma^2 \) are given by (1-21).

Proof. The proof is similar to the proof of Corollary 1.12. \( \square \)

We verify from Theorem 1.16 that the Meijer-G process satisfies Assumptions 1.1 with \( s_0 \) a sufficiently large constant, and
\[
M = 10, \quad \gamma = 2\pi \nu, \quad C = 2 \max_{\nu \in \left[ -\frac{M}{2}, \frac{M}{2} \right]} C(\nu), \quad a = \frac{1}{2\pi^2},
\]
where \( C = C(\nu) \) is given in Theorem 1.16. We obtain the following rigidity result by combining Theorem 1.2 with Theorem 1.16. We emphasize that we do not need the value of the constant \( C \) to prove this result, but that its value might be important in view of other applications.

Theorem 1.20 (rigidity for the Meijer-G process). Let \( x_1 < x_2 < \cdots \) be the points in the Meijer-G point process. There exists a constant \( c > 0 \) such that
\[
\mathbb{P} \left( \sup_{k \geq \mu(s)} \left| \frac{1+r-q}{\pi} \cos \left( \frac{\pi}{2} \frac{r-q-1}{r-q+1} \right) x_k^{\frac{1}{1+r-q}} - k \right| \log k > \frac{\sqrt{1+\epsilon}}{\pi} \right) \leq \frac{c s^{-\frac{\epsilon}{2(1+r-q)}}}{\epsilon},
\]
as \( s \to +\infty \), uniformly for \( \epsilon > 0 \) small. In particular, for all \( \epsilon > 0 \) we have

\[
\lim_{k_0 \to +\infty} \mathbb{P}\left( \sup_{k \geq k_0} \left| \frac{1 + r - q}{\pi} \frac{\cos\left(\frac{\pi}{2} \frac{r - q - 1}{r - q + 1}\right)}{\log k} \frac{1}{s} - k \right| \leq \frac{1}{\pi} + \epsilon \right) = 1.
\]

**Remark 1.21.** The above means that for any \( \epsilon > 0 \), the probability that

\[
\left[ \frac{\pi}{1 + r - q} \frac{1}{\cos\left(\frac{\pi}{2} \frac{r - q - 1}{r - q + 1}\right)} \left( k - \left( \frac{1}{\pi} + \epsilon \right) \log k \right) \right]^{1+r-q} \leq x_k
\]

\[
\leq \left[ \frac{\pi}{1 + r - q} \frac{1}{\cos\left(\frac{\pi}{2} \frac{r - q - 1}{r - q + 1}\right)} \left( k + \left( \frac{1}{\pi} + \epsilon \right) \log k \right) \right]^{1+r-q}
\]

tends to 1 as \( k_0 \to +\infty \). This has been verified numerically for \( q = 0 \) and different values of \( r \) by generating products of \( r \) independent Ginibre matrices [Akemann et al. 2013b; Kuijlaars and Zhang 2014] and is illustrated in Figure 3 (right) for \( \epsilon = 0.05 \).

**Outline of the proofs of Theorems 1.8 and 1.16.** It is well known [Borodin 2011; Johansson 2006; Soshnikov 2000] that the left-hand sides of (1-12) and (1-20) are (for any determinantal point process generated by one of the kernels (1-10) or (1-19), recall Remark 1.11 and Remark 1.18) equal to the Fredholm determinants

\[
det(1 - (1 - t)\mathbb{K}^\text{Wr}_{[0,s]}) \quad \text{and} \quad det(1 - (1 - t)\mathbb{K}^\text{Me}_{[0,s]}), \quad t = e^{-2\pi v}
\]

respectively. The kernels \( \mathbb{K}^\text{Wr} \) and \( \mathbb{K}^\text{Me} \) are known to be integrable in the sense of Its, Izergin, Korepin and Slavnov (IIKS) [Its et al. 1990] only for particular values of the parameters. For example, for \( \theta = p/q \), \( p, q \in \mathbb{N}_{>0}, \mathbb{K}^\text{Wr} \) is integrable of size \( p + q \) [Zhang 2017], and there are associated Riemann–Hilbert (RH) problems of size \( (p + q) \times (p + q) \). We expect the analysis of these RH problems to be rather complicated (except in the simplest case when \( p = q = 1 \)). Furthermore, for irrational values of \( \theta \), \( \mathbb{K}^\text{Wr} \) is not known to be integrable at all. To circumvent this problem, we use the ideas from [Claeys et al. 2019b] to rewrite (1-28) in Section 3 in terms of the determinant of an integrable operator of size 2, and to derive a differential identity in \( s \), i.e., to express, for all values of the parameters, the derivatives

\[
\partial_s \log det(1 - (1 - t)\mathbb{K}^\text{Wr}_{[0,s]}) \quad \text{and} \quad \partial_s \log det(1 - (1 - t)\mathbb{K}^\text{Me}_{[0,s]}),
\]

in terms of the solution, denoted \( Y \), to a 2 \( \times \) 2 RH problem. We then perform, in Section 4, an asymptotic analysis of this RH problem by means of the Deift–Zhou [Deift and Zhou 1993] steepest descent method. The local analysis requires the use of parabolic cylinder functions. By integrating in \( s \) the derivatives (1-29), we obtain

\[
\log det(1 - (1 - t)\mathbb{K}_{[0,s]}) = \log det(1 - (1 - t)\mathbb{K}_{[0,M]}) + \int_M^s \partial_\tilde{s} \log det(1 - (1 - t)\mathbb{K}_{[0,\tilde{s}])}d\tilde{s}, \quad \mathbb{K} = \mathbb{K}^\text{Wr}, \mathbb{K}^\text{Me},
\]

for a certain constant \( M \). By substituting the large \( s \) asymptotics of (1-29) in the integrand of (1-30), we
determine the functions $\mu$ and $\sigma^2$ of Theorems 1.8 and 1.16 in Section 5. However, the quantity

$$\log \det(1 - (1 - t)\mathcal{K}|_{[0,M]})$$

is an unknown constant, so this method does not allow for the evaluation of $C$ (the constants of order 1) of Theorems 1.8 and 1.16. Such constants are notoriously difficult to compute explicitly [Krasovsky 2009], and require the use of other differential identities which are more complicated to analyze. To obtain $C$, we will use a differential identity in $t$, i.e., we will express

$$\partial_t \log \det(1 - (1 - t)\mathcal{K}|_{[0,s]}) = \mathcal{K} = \mathcal{K}^W, \mathcal{K}^M,$$

in terms of $Y$ in Section 3. Large $s$ asymptotics for the derivatives (1-31) appears to be rather complicated to obtain. In particular, it requires the explicit evaluation of certain (regularized) integrals involving parabolic cylinder functions. The key observation is that these integrals do not depend on any other parameters than $t$. Then we evaluate explicitly these complicated integrals by using the known expansion (1-7) from [Bothner et al. 2019; Charlier 2020]. By integrating (1-31) in $t$, we have the identity

$$\log \det(1 - (1 - t)\mathcal{K}|_{[0,s]}) = \log \det(1 - (1 - t)\mathcal{K}|_{[0,s]})_{t=1}$$

$$+ \int_1^t \partial_t \log \det(1 - (1 - \tilde{t})\mathcal{K}|_{[0,s]})d\tilde{t}, \quad \mathcal{K} = \mathcal{K}^W, \mathcal{K}^M,$$

(1-32)

where $t \in (0, 1]$ is fixed. By substituting the large $s$ asymptotics of (1-31) in the integrand of (1-32) and by using the results of [Bothner et al. 2019] in Section 6, we obtain $C$ (and moreover we recover the same functions $\mu, \sigma^2$ as obtained via the differential identity in $s$), since

$$\log \det(1 - (1 - t)\mathcal{K}|_{[0,s]})|_{t=1} = 0.$$

The direct analysis of (1-32) is rather involved. In conclusion, each of the two differential identities has its advantages and disadvantages: the differential identity in $s$ leads to an easier analysis, but does not allow for the evaluation of $C$, while the differential identity in $t$ is significantly more involved but allows us to compute $C$. Another advantage of the differential identity in $s$ is that it allows us, with only limited efforts, to give the optimal estimates $O(s^{-\frac{a}{1+\theta}})$ and $O(s^{-\frac{1}{1+2\theta}})$ for the error terms of Theorems 1.8 and 1.16; with the differential identity in $t$, we are only able to obtain errors of order $O(s^{-\frac{1}{2(1+\theta)}})$ and $O(s^{-\frac{1}{2(1+2\theta)}})$.

2. Proof of Theorem 1.2

In this section, we suppose that $X$ is a locally finite random point process on the real line which has a smallest particle almost surely, with counting function $N(s)$, and which is such that Assumptions 1.1 hold for certain constants $C, a > 0, s_0 \in \mathbb{R}, M > \sqrt{2/a},$ and for certain functions $\mu, \sigma$.

We start by establishing a bound for the tail of the probability distribution of the extremum of the normalized counting function.

**Lemma 2.1.** There exist $c > 0$ and $s_0 > 0$ such that for any $\epsilon > 0$ sufficiently small and $s > s_0$,

$$\mathbb{P}\left(\sup_{x>s} \left| \frac{N(x) - \mu(x)}{\sigma^2(x)} \right| > \sqrt{\frac{2}{a}(1+\epsilon)} \right) \leq \frac{c \mu(s)^{-\epsilon}}{2\epsilon}.\quad (2-1)$$
In particular, for any $\epsilon > 0$, 
\[ \lim_{s \to +\infty} \mathbb{P} \left( \sup_{x > s} \left| \frac{N(x) - \mu(x)}{\sigma^2(x)} \right| \leq \sqrt{\frac{2}{a}} (1 + \epsilon) \right) = 1. \]

**Proof.** Let us define $\kappa_k = \mu^{-1}(k)$. We start by noting that for $x \in [\kappa_{k-1}, \kappa_k]$, $k \in \mathbb{N}$, we have by monotonicity of $\mu$ and of the counting function $N$ that 
\[ N(x) - \mu(x) \leq N(\kappa_k) - \mu(\kappa_k) = N(\kappa_k) - \mu(\kappa_k) + 1, \]
and since $\sigma$ is increasing, we also have $\sigma^2(x) \geq \sigma^2(\kappa_{k-1})$. For large enough $s$, it follows that
\[ \sup_{x > s} \frac{N(x) - \mu(x)}{\sigma^2(x)} \leq \sup_{k,k > s} \frac{N(\kappa_k) - \mu(\kappa_k) + 1}{\sigma^2(\kappa_k - 1)}. \]

Hence, by a union bound, for any $\gamma > 0$ we have
\[ \mathbb{P} \left( \sup_{x > s} \frac{N(x) - \mu(x)}{\sigma^2(x)} > \gamma \right) \leq \sum_{k,k > s} \mathbb{P} \left( \frac{N(\kappa_k) - \mu(\kappa_k) + 1}{\sigma^2(\kappa_k - 1)} > \gamma \right) \]
\[ = \sum_{k,k > s} \mathbb{P} \left( e^{\gamma N(\kappa_k)} > e^{\gamma \mu(\kappa_k) - \gamma^2 \sigma^2(\kappa_{k-1})} \right) \leq \sum_{k,k > s} \mathbb{E} \left( e^{\gamma N(\kappa_k)} \right) e^{-\gamma \mu(\kappa_k) + \gamma^2 \sigma^2(\kappa_{k-1})}, \]
where the last inequality is obtained by applying Markov’s inequality on the positive random variable $e^{\gamma N(\kappa_k)}$. Using (1-1) in (2-2), we obtain
\[ \mathbb{P} \left( \sup_{x > s} \frac{N(x) - \mu(x)}{\sigma^2(x)} > \gamma \right) \leq C e^{\gamma} \sum_{k,k > s} e^{-\frac{\gamma^2}{2} \sigma^2(\kappa_k) e^{\gamma^2 (\sigma^2(\kappa_k) - \sigma^2(\kappa_{k-1}))}}. \]

Because $(\sigma^2 \circ \mu^{-1})$ is strictly concave and behaves as $(\sigma^2 \circ \mu^{-1})(k) \sim a \log k$ as $k \to +\infty$, we have that
\[ e^{\gamma^2 (\sigma^2(\kappa_k) - \sigma^2(\kappa_{k-1}))} = e^{\gamma^2 [\sigma^2(\mu^{-1}(k)) - \sigma^2(\mu^{-1}(k-1))]} \]
decreases with $k$ and is uniformly bounded in $k$ by a constant which we denote as $C'$ and which we can choose independently of $\gamma \in [0, M]$. Using also the fact that $\sigma^2$ and $\mu$ are increasing, we obtain
\[ \mathbb{P} \left( \sup_{x > s} \frac{N(x) - \mu(x)}{\sigma^2(x)} > \gamma \right) \leq C C' e^{\gamma} \sum_{k,k > s} e^{-\frac{\gamma^2}{2} \sigma^2(\mu^{-1}(k))} \]
\[ \leq C C' e^{\gamma} \left( e^{-\frac{\gamma^2}{2} \sigma^2(s)} + \int_{\mu(s)}^{\infty} e^{-\frac{\gamma^2}{2} (\sigma^2 \circ \mu^{-1})(x)} \, dx \right). \]

Similarly, we obtain
\[ \mathbb{P} \left( \sup_{x > s} \frac{\mu(x) - N(x)}{\sigma^2(x)} > \gamma \right) \leq \sum_{k,k > s} \mathbb{P} \left( \frac{\mu(\kappa_{k-1}) - N(\kappa_{k-1}) + 1}{\sigma^2(\kappa_{k-1})} > \gamma \right) \]
\[ = \sum_{k,k+1 > s} \mathbb{P} \left( e^{-\gamma N(\kappa_k)} > e^{-\gamma \mu(\kappa_k) - \gamma^2 \sigma^2(\kappa_k)} \right) \leq \sum_{k,k+1 > s} \mathbb{E} \left( e^{-\gamma N(\kappa_k)} \right) e^{\gamma \mu(\kappa_k) + \gamma^2 \sigma^2(\kappa_k)}. \]
Using again (1-1) and the fact that $\sigma$ and $\mu$ are increasing, we then get
\[
P\left(\sup_{x > s} \frac{\mu(x) - N(x)}{\sigma^2(x)} > \gamma\right) \leq C e^{\gamma^2} \sum_{k_{2s+1} > s} e^{-\frac{\gamma^2}{2} \sigma^2(\mu^{-1}(k))} \leq C e^{\gamma^2} \left(2 e^{-\frac{\gamma^2}{2} \sigma^2(\mu^{-1}(s))} + \int_{\mu(s)}^{\infty} e^{-\frac{\gamma^2}{2} \sigma^2(\mu^{-1}(x))} dx\right).
\] (2-4)

By combining (2-3) and (2-4), we obtain
\[
P\left(\sup_{x > s} \left|\frac{N(x) - \mu(x)}{\sigma^2(x)}\right| > \gamma\right) \leq C(C' + 2)e^{\gamma^2} \left(e^{-\frac{\gamma^2}{2} \sigma^2(\mu^{-1}(s)) - 1} + \int_{\mu(s)}^{\infty} e^{-\frac{\gamma^2}{2} \sigma^2(\mu^{-1}(x))} dx\right).
\]

It follows from criteria (2) and (3) of Assumptions 1.1 that the right-hand side converges to 0 as $s \to +\infty$, provided that $\gamma > \sqrt{2/a}$. More precisely, for $\gamma \in (\sqrt{2/a}, M]$ the right-hand side is smaller than
\[
2C(C' + 2)e^{M} \left((\mu(s) - 1)^{-\gamma^2} + \int_{\mu(s)}^{\infty} x^{-\gamma^2} dx\right) \leq 3C(C' + 2)e^{M} \frac{\mu(s)^{1-\gamma^2}}{a^{\gamma^2}}.
\]
for all sufficiently large $s$. In conclusion, for any $\gamma \in (\sqrt{2/a}, M]$, we have
\[
P\left(\sup_{x > s} \left|\frac{N(x) - \mu(x)}{\sigma^2(x)}\right| > \gamma\right) \leq 3C(C' + 2)e^{M} \frac{\mu(s)^{1-\gamma^2}}{a^{\gamma^2}}.
\]

We obtain the claim after setting $c = 6C(C' + 2)e^{M}$ and $\gamma = \sqrt{2/a}(1 + \epsilon)$. \hfill \Box

Next, assuming a bound for the extremum of the normalized counting function, we derive the global rigidity of the points in the process.

**Lemma 2.2.** Let $\epsilon > 0$. For all sufficiently large $s$, if the event
\[
\sup_{x > s} \left|\frac{N(x) - \mu(x)}{\sigma^2(x)}\right| = \sup_{x > s} \left|\frac{N(x) - \mu(x)}{\sigma^2(x)}\right| \leq \sqrt{\frac{2}{a}}(1 + \epsilon)
\] (2-5)
holds true, then we have
\[
\sup_{k \geq \mu(2s)} \left|\frac{\mu(x_k) - k}{\sigma^2(\sigma^2 \circ \mu^{-1})(k)}\right| \leq \sqrt{\frac{2}{a}}(1 + 2\epsilon).
\] (2-6)

**Proof.** We start by proving that
\[
x_k > s, \quad \text{for all } k \geq \mu(2s),
\] (2-7)
for all large enough $s$. Suppose that $x_k \leq s < 2s \leq \kappa_k$, where $\kappa_k = \mu^{-1}(k)$. Then
\[
\mu(2s) \leq \mu(\kappa_k) = k = N(x_k) \leq N(s),
\]
which implies by Assumptions 1.1 that
\[
\frac{N(s) - \mu(s)}{\sigma^2(s)} \geq \frac{\mu(2s) - \mu(s)}{\sigma^2(s)} \geq \frac{\inf_{\xi \leq s \leq 2s} \mu'(\xi) \xi}{\sigma^2(s)} \geq \frac{\inf_{\xi \leq s \leq 2s} \xi \mu'(\xi)}{2 \sigma^2(s)} = \frac{s \mu'(s)}{2 \sigma^2(s)}.
\] (2-8)
Again by Assumptions 1.1, the right-hand side of (2-8) tends to $+\infty$ as $s \to +\infty$, so there is a contradiction with (2-5), provided that $s$ is chosen large enough. We conclude that $x_k > s$ for all $k \geq \mu(2s)$, provided that $s$ is large enough.

We split the proof of (2-6) into two parts. We first prove the following upper bound for $\mu(x_k)$:

$$\mu(x_k) \leq k + \sqrt{\frac{2}{a}(1 + 2\epsilon)(\sigma^2 \circ \mu^{-1})(k)}, \quad \text{for all } k \geq \mu(2s). \tag{2-9}$$

Define $m = m(k)$ as the unique integer such that $\kappa_{k+m} < x_k \leq \kappa_{k+m+1}$. If $m < 0$, then (2-9) is immediately satisfied. Let us now treat the case $m \geq 0$. Since $k \geq \mu(2s)$, we have $x_k > s$ by (2-7). Therefore, we use (2-5) together with $m \geq 0$ to conclude that

$$\sqrt{\frac{2}{a}(1 + \epsilon)} \geq \frac{\mu(x_k) - N(x_k)}{\sigma^2(x_k)} \geq \frac{m}{(\sigma^2 \circ \mu^{-1})(k + m + 1)},$$

and it follows that

$$m \leq \sqrt{\frac{2}{a}(1 + \epsilon)(\sigma^2 \circ \mu^{-1})(k + m + 1)} \leq \sqrt{\frac{2}{a}(1 + \epsilon)((\sigma^2 \circ \mu^{-1})(k) + (m + 1)(\sigma^2 \circ \mu^{-1})'(k))},$$

where we used the concavity of $\sigma^2 \circ \mu^{-1}$ from Assumptions 1.1. This inequality can be rewritten as

$$\left(1 - \sqrt{\frac{2}{a}(1 + \epsilon)(\sigma^2 \circ \mu^{-1})'(k)}\right)m \leq \sqrt{\frac{2}{a}(1 + \epsilon)((\sigma^2 \circ \mu^{-1})(k) + (\sigma^2 \circ \mu^{-1})'(k)).$$

Since $\sigma \circ \mu^{-1}$ is concave, the derivative $(\sigma \circ \mu^{-1})'$ is decreasing, and thus for $k \geq k_0$ we have

$$(\sigma \circ \mu^{-1})(k) = (\sigma \circ \mu^{-1})(k_0) + \int_{k_0}^{k} (\sigma \circ \mu^{-1})'(\tilde{k})d\tilde{k} \geq (\sigma \circ \mu^{-1})(k_0) + (\sigma \circ \mu^{-1})'(k)(k - k_0). \tag{2-10}$$

Since $(\sigma \circ \mu^{-1})(k) \sim a \log(k)$ as $k \to +\infty$, (2-10) yields $(\sigma \circ \mu^{-1})'(k) \to 0$ as $k \to +\infty$. We deduce that, for any fixed $\delta > 0$,\n
$$(1 - \delta)m \leq (1 + \delta)\sqrt{\frac{2}{a}(1 + \epsilon)(\sigma^2 \circ \mu^{-1})(k) - (1 - \delta), \quad \text{for all } k \geq \mu(2s),}$$

provided that $s$ is large enough. We choose $\delta > 0$ sufficiently small such that

$$\frac{1 + \delta}{1 - \delta} \sqrt{\frac{2}{a}(1 + \epsilon)} < \sqrt{\frac{2}{a}(1 + 2\epsilon)}.$$

Therefore, we achieve the inequality

$$m + 1 \leq \sqrt{\frac{2}{a}(1 + 2\epsilon)(\sigma^2 \circ \mu^{-1})(k)}, \quad \text{for all } k \geq \mu(2s),$$

provided that $s$ is large enough. It follows that

$$\mu(x_k) \leq \mu(\kappa_{k+m+1}) = k + m + 1 \leq k + \sqrt{\frac{2}{a}(1 + 2\epsilon)(\sigma^2 \circ \mu^{-1})(k)}.$$


In the second part of the proof, we show the following lower bound for \( \mu(x_k) \):

\[
\sqrt{\frac{2}{a}}(1+\epsilon)(\sigma^2 \circ \mu^{-1})(k) \leq \mu(x_k), \quad \text{for all } k \geq \mu(2s)
\]  

(2-11)

which is even slightly better than what is required to prove (2-6). Suppose that \( \mu(x_k) < k - m \) with \( m > 0 \). By combining (2-7) with (2-5), we have

\[
\sqrt{\frac{2}{a}}(1+\epsilon) \geq \frac{N(x_k) - \mu(x_k)}{\sigma^2(x_k)} > \frac{m}{\sigma^2(x_k)} > \frac{m}{(\sigma^2 \circ \mu^{-1})(k)}, \quad \text{for all } k \geq \mu(2s),
\]

and it follows that \( m < \sqrt{\frac{2}{a}}(1+\epsilon)(\sigma^2 \circ \mu^{-1})(k) \), which proves the lower bound.

\( \Box \)

Furthermore, Lemma 2.2 implies that

\[
\text{Theorem 1.2 follows by a direct application of Bayes' formula, combining (2-12) and (2-13).}
\]

It now suffices to combine the above two results in order to obtain Theorem 1.2.

**Proof of Theorem 1.2.** It follows from Lemma 2.1 that there exists \( c > 0 \) such that for all \( \epsilon > 0 \) sufficiently small and for all \( s \) sufficiently large, we have

\[
\mathbb{P}\left( \sup_{x > s} \frac{|N(x) - \mu(x)|}{\sigma^2(x)} \leq \sqrt{\frac{2}{a}}\left(1 + \frac{\epsilon}{2}\right) \right) \geq 1 - \frac{c\mu(s)^{-\frac{\epsilon}{4}}}{\epsilon}.
\]  

(2-12)

Furthermore, Lemma 2.2 implies that

\[
\mathbb{P}\left( \sup_{k \geq \mu(2s)} |\mu(x_k) - k| \leq \sqrt{\frac{2}{a}}(1+\epsilon) \left| \sup_{x > s} \frac{|N(x) - \mu(x)|}{\sigma^2(x)} \right| \leq \sqrt{\frac{2}{a}}\left(1 + \frac{\epsilon}{2}\right) \right) = 1,
\]  

(2-13)

for all sufficiently large \( s \). Theorem 1.2 follows by a direct application of Bayes’ formula, combining (2-12) and (2-13).

\( \Box \)

### 3. RH problem and differential identities

**Double contour integral representation for the kernels.** For convenience, let us write

\[ \mathcal{K}^{(1)} = \mathcal{K}^{\text{Me}} \quad \text{and} \quad \mathcal{K}^{(2)} = \mathcal{K}^{\text{Wr}}, \]

where \( \mathcal{K}^{\text{Me}} \) and \( \mathcal{K}^{\text{Wr}} \) have been defined in (1-19) and (1-10), respectively. For our analysis, we will use the following double contour representation for these kernels [Claeys et al. 2019b]:

\[
\mathcal{K}^{(j)}(x, y) = \frac{1}{4\pi^2} \int_{\gamma} du \int_{\tilde{\gamma}} dv \frac{F^{(j)}(u)}{F^{(j)}(v)} x^{-u} y^{v-1}, \quad j = 1, 2,
\]

(3-1)

with

\[
F^{(1)}(z) = \frac{\Gamma(z) \prod_{k=1}^{q} \Gamma(1 + \mu_k - z)}{\prod_{k=1}^{r} \Gamma(1 + v_k - z)}, \quad F^{(2)}(z) = \frac{\Gamma(z + \frac{q}{2})}{\Gamma\left(\frac{q+1-z}{2}\right)}.
\]

(3-2)

For \( j = 1 \), we require \( r, q \in \mathbb{N} \), \( r > q \geq 0 \), \( v_1, \ldots, v_r \in \mathbb{N} \) and \( \mu_1, \ldots, \mu_q \in \mathbb{N}_{>0} \) such that \( \mu_k > v_k \), \( k = 1, \ldots, q \). If \( q = 0 \), the product in the numerator is understood as 1. For \( j = 2 \), we require \( \alpha > -1 \) and \( \theta > 0 \). The contours \( \gamma, \tilde{\gamma} \) are both oriented upward, do not intersect each other, and intersect the real
\[ \gamma \rightarrow -\frac{\alpha}{2} - 1 \quad \gamma \rightarrow -\frac{\alpha}{2} - 2 \quad \gamma \rightarrow -\frac{\alpha}{2} + 1 \quad \gamma \rightarrow -\frac{\alpha}{2} + 1 + \theta \]

\[ \quad -\frac{\alpha}{2} + 1 + \frac{1}{2} \alpha \quad -\frac{\alpha}{2} + 1 + \frac{1}{2} \alpha + 1 + 2 \theta \]

**Figure 4.** The contours \( \gamma \) and \( \tilde{\gamma} \).

line on the interval \((0, 1 + \nu_{\text{min}})\) if \( j = 1 \), with \( \nu_{\text{min}} := \min\{\nu_1, \ldots, \nu_{\gamma}\} \), and on the interval \((-\frac{\alpha}{2}, 1 + \frac{\alpha}{2})\) if \( j = 2 \). Furthermore, \( \gamma \) tends to infinity in sectors lying strictly in the left half plane, and \( \tilde{\gamma} \) tends to infinity in sectors lying strictly in the right half plane, see Figure 4.

**Integrable kernels.** As mentioned at the end of Section 1, the kernels \( \mathbb{K}^{(j)}, j = 1, 2 \) are known to be integrable only for particular values of the parameters. With minor modifications of [Claeys et al. 2019b, Propositions 2.1 and 2.2], we obtain the following.

**Proposition 3.1.** Let \( t \in (0, +\infty) \). For \( j = 1, 2 \), we have

\[ \det(1 - (1 - t)\mathbb{K}^{(j)}|_{[0, s]}) = \det(1 - \mathbb{M}^{(j)}_{s, t}), \quad j = 1, 2, \quad (3-3) \]

where \( \mathbb{M}^{(j)}_{s, t} \) is the trace-class integral operator acting on \( L^2(\gamma \cup \tilde{\gamma}) \) with kernel

\[ \mathbb{M}^{(j)}_{s, t}(u, v) = \frac{f(u)^T g(v)}{u - v}, \]

where \( f \) and \( g \) are given by

\[ f(u) = \frac{1}{2\pi i} \left( \frac{\chi_{\gamma}(u)}{s^u \chi_{\tilde{\gamma}}(u)} \right), \quad g(v) = \left( -\frac{\sqrt{1 - t} F^{(j)}(v)^{-1} \chi_{\tilde{\gamma}}(v)}{\sqrt{1 - ts^{-v} F^{(j)}(v)} \chi_{\gamma}(v)} \right), \quad (3-4) \]

and \( \chi_{\gamma} \) and \( \chi_{\tilde{\gamma}} \) are the characteristic functions of \( \gamma \) and \( \tilde{\gamma} \), respectively. The determination of \( \sqrt{1 - t} \) in the definition of \( g \) is unimportant (but the same determination must be chosen for both entries of \( g \)). For definiteness, we require

\[ \sqrt{1 - t} \in [0, 1) \quad \text{if} \; t \in (0, 1], \quad (3-5) \]

\[ \sqrt{1 - t} \in [0, i \infty) \quad \text{if} \; t \in [1, +\infty). \quad (3-6) \]

**Proof.** The proof for \( t = 0 \) (including the fact that \( \mathbb{M}^{(j)}_{s, t} \) is trace-class) can be found in [Claeys et al. 2019b, Propositions 2.1 and 2.2] and relies on a conjugation of \( \mathbb{K}^{(j)}|_{[0, s]} \) with a Mellin transform. The proof for arbitrary values of \( t \in (0, +\infty) \) only requires minor modifications: the quantity \( \mathbb{H}^{(j)}_{s}(v, z) \) of [Claeys...
et al. 2019b, equation (2.10)]

\[
\frac{\sqrt{1-t}}{2\pi i} \int_\gamma \frac{du}{2\pi i} \frac{s^{z-u} F^{(j)}(u)}{F^{(j)}(v)(v-u)(z-u)},
\]

and the kernels \(A^{(j)}\) and \(B^{(j)}\) of [Claeys et al. 2019b, equation (2.19)] need to be modified to

\[
A^{(j)}(u, z) = \frac{\sqrt{1-t} s^{z-u} F^{(j)}(u)}{2\pi i (z-u)}, \quad B^{(j)}(v, u) = \frac{\sqrt{1-t}}{2\pi i F^{(j)}(v)(v-u)},
\]

where the determination of \(\sqrt{1-t}\) is unimportant, as long as it is the same for \(A^{(j)}\) and \(B^{(j)}\).

Using a method developed by Its, Izergin, Korepin, and Slavnov [Its et al. 1990], we will establish differential identities in \(s\) and \(t\) for the logarithm of the Fredholm determinants (3-3) in terms of the following RH problem:

**RH problem for \(Y = Y^{(j)}, \quad j = 1, 2\):**

(a) \(Y : \mathbb{C} \setminus (\gamma \cup \tilde{\gamma}) \rightarrow \mathbb{C}^{2 \times 2}\) is analytic.

(b) \(Y(z)\) has continuous boundary values \(Y_\pm(z)\) as \(z\) approaches the contour \(\gamma \cup \tilde{\gamma}\) from the left (+) and right (−), according to its orientation, and we have the jump relations

\[
Y_+(z) = Y_-(z)J(z), \quad z \in \gamma \cup \tilde{\gamma},
\]

with jump matrix \(J = J^{(j)}\) given by

\[
J(z) = I - 2\pi i f(z)g(z)^T = \begin{cases}
\begin{pmatrix}
1 & -\sqrt{1-t} s^{-z} F^{(j)}(z) \\
0 & 1
\end{pmatrix}, & z \in \gamma, \\
\begin{pmatrix}
1 & \sqrt{1-t} s F^{(j)}(z)^{-1} \\
0 & 1
\end{pmatrix}, & z \in \tilde{\gamma}.
\end{cases}
\] (3-7)

(c) As \(z \rightarrow \infty\), there exists \(Y_1 = Y_1^{(j)}(s, t)\) independent of \(z\) such that

\[
Y(z) = I + \frac{Y_1}{z} + \mathcal{O}(z^{-2}).
\]

**Remark 3.2.** We have some freedom in the choice of \(\gamma\) and \(\tilde{\gamma}\). We choose them symmetric with respect to the real line. This symmetry will be useful later to simplify computations.

**Lemma 3.3.** For \(j = 1, 2\), we have the following differential identities:

\[
\partial_s \log \det(1 - (1-t)\kappa^{(j)}_{[0,x]}) = \frac{Y_{1,11}}{s} = -\frac{Y_{1,22}}{s}, \quad (3-8)
\]

\[
\partial_t \log \det(1 - (1-t)\kappa^{(j)}_{[0,x]}) = \frac{-1}{2(1-t)} \int_{\gamma \cup \tilde{\gamma}} \text{Tr}[Y^{-1}(z)Y'(z)(J(z) - I)] \frac{dz}{2\pi i}. \quad (3-9)
\]

\(^1\)There is a factor \(\frac{1}{2\pi i}\) missing in the expressions for \(H_s^{(j)}(v, z)\) and \(B^{(j)}(v, u)\) of [Claeys et al. 2019b, equations (2.10) and (2.19)].
Remark 3.4. We do not mention whether we take the + or − boundary values of $Y$ in the integrand of (3-9). This is without ambiguity, because
\[
\text{Tr}[Y^{-1}_+(z)Y'_+(z)(J(z) - I)] = \text{Tr}[Y^{-1}_-(z)Y'_-(z)(J(z) - I)] = \text{Tr}[Y^{-1}_+(z)Y'_+(z)(J(z) - I)].
\]

Proof. Both (3-8) and (3-9) are specializations of more general results from [Bertola 2010]. For the proof of (3-8), we refer to [Claeys et al. 2019b, Theorem 1.1], and for the proof of (3-9), we apply [Bertola 2010, Section 5.1] (with $\partial = \partial_t$) to obtain
\[
\partial_t \log \det (1 - (1-t)\kappa^{(j)}_{|[0,s]}) = \int_{\gamma \cup \tilde{\gamma}} \text{Tr}[Y^{-1}_-(z)Y'_-(z)\partial_t J(z)J^{-1}(z)] \frac{dz}{2\pi i}.
\]

From (3-7), it is straightforward to verify that
\[
\partial_t J(z)J(z)^{-1} = \frac{-1}{2(1-t)}(J(z) - I),
\]
which yields (3-9) and finishes the proof. □

4. Steepest descent analysis

In this section, we use the Deift–Zhou [Deift and Zhou 1993] steepest descent method to perform an asymptotic analysis of $Y = Y^{(j)}$, $j = 1, 2$, as $s \to +\infty$ uniformly for $t$ in compact subsets of $(0, +\infty)$. The first transformation $Y \mapsto U$ in Section 4A is a rescaling which is identical to the one from [Claeys et al. 2019b]. The rest of the analysis differs drastically from [Claeys et al. 2019b], and we highlight the main ideas for it here. As common in steepest descent analysis of RH problems, we will need to do a saddle points analysis of a phase function appearing in the jump matrix for $U$ (see Section 4B). In contrast to the approach of [Claeys et al. 2019b], the opening of the lenses is done in two steps $U \mapsto \hat{T} \mapsto T$ presented in Section 4C, and we emphasize that this is somewhat unusual, in that it requires two different factorizations of the jump matrix, each on a different part of the jump contour. The global parametrix $P^{(\infty)}$ of Section 4D approximates $T$ everywhere in the complex plane except near the saddle points $b_1$ and $b_2$. In Sections 4E and 4F, we construct local parametrices $P^{(b_k)}$ in terms of parabolic cylinder functions. The local parametrix $P^{(b_k)}$ is defined in a small disk $D_{b_k}$ centered at $b_k$ and satisfies the same jumps as $T$. The global and local parametrices are completely different than the ones in [Claeys et al. 2019b], because of the different jump matrix factorizations used for opening the lenses. The last step $T \mapsto R$ of the steepest descent analysis is completed in Section 4G. A matrix $R$ is built in terms of $T$, $P^{(\infty)}$, $P^{(b_1)}$, and $P^{(b_2)}$, and we show that it satisfies a small norm RH problem. In particular, $R(z)$ is close to the identity matrix as $s \to +\infty$. We also compute the first two subleading terms of $R$ which are needed for the proofs of Theorems 1.8 and 1.16.
4A. First transformation $Y \mapsto U$. We first rescale the variable of the RH problem for $Y$ in a convenient way. In the same way as in [Claeys et al. 2019b, Section 3.1], we define

$$U(\zeta) = s^\frac{1}{2} Y(is^\rho \zeta + \tau)s^{-\frac{1}{2}} \sigma_3, \quad (4-1)$$

where $\tau = \tau^{(j)}$ and $\rho = \rho^{(j)}$, $j = 1, 2$, are given by

$$\tau^{(1)} = \frac{v_{\min} + 1}{2}, \quad \rho^{(1)} = \frac{1}{r - q + 1}, \quad (4-2)$$

$$\tau^{(2)} = \frac{1}{2}, \quad \rho^{(2)} = \frac{\theta}{\theta + 1} \quad (4-3)$$

and $v_{\min} := \min\{v_1, \ldots, v_r\}$. The matrix $U$ satisfies the following RH problem.

RH problem for $U$:

(a) $U : \mathbb{C} \setminus (\gamma_U \cup \tilde{\gamma}_U) \to \mathbb{C}^{2 \times 2}$ is analytic, where

$$\gamma_U = \{\zeta \in \mathbb{C} : is^\rho \zeta + \tau \in \gamma\} \quad \text{and} \quad \tilde{\gamma}_U = \{\zeta \in \mathbb{C} : is^\rho \zeta + \tau \in \tilde{\gamma}\}. \quad (4-4)$$

The contour $\gamma_U$ (resp. $\tilde{\gamma}_U$) lies in the upper (resp. lower) half plane and is oriented from left to right.

(b) $U$ satisfies the jumps $U_+(\zeta) = U_-(\zeta) J_U(\zeta)$ for $\zeta \in \gamma_U \cup \tilde{\gamma}_U$ with

$$J_U(\zeta) = \begin{cases} \begin{pmatrix} 1 & -\sqrt{1 - t} s^{-is^\rho \zeta} F(is^\rho \zeta + \tau) \\ 0 & 1 \end{pmatrix} & \text{if } \zeta \in \gamma_U, \\ \begin{pmatrix} 1 & 0 \\ \sqrt{1 - t} s^{-is^\rho \zeta} F(is^\rho \zeta + \tau)^{-1} & 1 \end{pmatrix} & \text{if } \zeta \in \tilde{\gamma}_U. \end{cases}$$

(c) As $\zeta \to \infty$, we have

$$U(\zeta) = I + \frac{U_1}{\zeta} + O(\zeta^{-2}),$$

where $U_1 = U_1^{(j)}(s, t)$ is given by

$$U_1 = \frac{1}{is^\rho} s^\frac{1}{2} \sigma_3 Y_1 s^{-\frac{1}{2}} \sigma_3.$$

Remark 4.1. Since $\gamma$ and $\tilde{\gamma}$ are symmetric with respect to the real line, the contours $\gamma_U$ and $\tilde{\gamma}_U$ are symmetric with respect to $i\mathbb{R}$. Furthermore, we note that the function

$$\zeta \mapsto f(\zeta) := s^{-is^\rho \zeta} F(is^\rho \zeta + \tau)$$

satisfies the symmetry relation $f(\zeta) = \overline{f(-\zeta)}$, and thus we also have $J_U(\zeta) = J_U(-\zeta)$ for $\zeta \in \gamma_U \cup \tilde{\gamma}_U$. By uniqueness of the RH solution $U$, we conclude that

$$U(\zeta) = \overline{U(-\zeta)}, \quad \zeta \in \mathbb{C} \setminus (\gamma_U \cup \tilde{\gamma}_U).$$
4B. Saddle point analysis. We choose the branch for log $F(j)$, $j = 1, 2$, such that

$$
\log F^{(1)}(z) = \log \Gamma(z) - \sum_{k=1}^{r} \log \Gamma(1 + v_k - z) + \sum_{k=1}^{q} \log \Gamma(1 + \mu_k - z),
$$

$$
\log F^{(2)}(z) = \log \Gamma \left( z + \frac{\alpha}{2} \right) - \log \Gamma \left( \frac{q}{2} + 1 - z \right),
$$

where $z \mapsto \log \Gamma(z)$ is the log-gamma function, which has a branch cut along $(-\infty, 0]$. Therefore, $z \mapsto \log F^{(1)}(z)$ has a branch cut along $(-\infty, 0] \cup [1 + v_{\min}, +\infty)$, and $z \mapsto \log F^{(2)}(z)$ has a branch cut along $(-\infty, -\frac{q}{2}] \cup [1 + \frac{q}{2}, +\infty)$. Asymptotics for $\log(s^{-is^0\zeta} F(is^0\zeta + \tau))$ as $s \to +\infty$ and simultaneously $s^0\zeta \to \infty$, $|\arg(\zeta) \pm \frac{\pi}{2}| > \epsilon > 0$ were computed in [Claeys et al. 2019b] and are given by

$$
\log(s^{-is^0\zeta} F(is^0\zeta + \tau)) = is^0[c_1\log(i\zeta) + c_2\log(-i\zeta) + c_3\zeta] + c_4 \log(s) + c_5 \log(i\zeta) + c_6 \log(-i\zeta) + c_7 + \frac{c_8}{is^0\zeta} + O\left(\frac{1}{s^{2\rho\zeta^2}}\right),
$$

where the constants $c_i = c_i^{(j)}$, $j = 1, 2$ are given by [Claeys et al. 2019b, equations (3.10)–(3.12)].

The values of $c_7$ and $c_8$ turn out to be unimportant for us. We recall the values of the other constants here. For $j = 1$, we have

$$
c_1 = 1, \quad c_2 = r - q, \quad c_3 = -(r - q + 1),
$$

$$
c_4 = \frac{v_{\min}}{2} - \frac{\sum_{k=1}^{r} v_k - \sum_{k=1}^{q} \mu_k}{r - q + 1}, \quad c_5 = \frac{v_{\min}}{2}, \quad c_6 = (r - q) \frac{v_{\min}}{2} - \sum_{j=1}^{r} v_j + \sum_{k=1}^{q} \mu_k,
$$

and for $j = 2$, we have

$$
c_1 = 1, \quad c_2 = \frac{1}{\theta}, \quad c_3 = -\frac{\theta + 1 + \log \theta}{\theta},
$$

$$
c_4 = \frac{(\theta - 1)(1 + \alpha)}{2(\theta + 1)}, \quad c_5 = \frac{\alpha}{2}, \quad c_6 = -\frac{\theta - \alpha - 1}{2\theta}.
$$

Following [Claeys et al. 2019b], we define $h(\zeta) = h^{(j)}(\zeta)$, $j = 1, 2$, by

$$
h(\zeta) = -c_1\log(i\zeta) - c_2\log(-i\zeta) - c_3\zeta,
$$

where the principal branch is chosen for the logarithms, and $G = G^{(j)}$ is defined via

$$
s^{-is^0\zeta} F(is^0\zeta + \tau) = e^{-is^0h(\zeta)} G(\zeta; s).
$$

We have

$$
\log G(\zeta; s) = c_4 \log s + c_5 \log(i\zeta) + c_6 \log(-i\zeta) + c_7 + \frac{c_8}{is^0\zeta} + O\left(\frac{1}{s^{2\rho\zeta^2}}\right)
$$

as $s \to +\infty$ such that $s^0\zeta \to \infty$, $|\arg(\zeta) \pm \frac{\pi}{2}| > \epsilon > 0$. On the other hand, as $s \to +\infty$ and simultaneously $\zeta \to 0$ such that $s^0\zeta = O(1)$, and such that $is^0\zeta + \tau$ is bounded away from the poles of $F$, we

---

The superscripts $j = 1, 2$ in this paper correspond to the superscripts $j = 2, 3$ in [Claeys et al. 2019b]. Also, there is a typo in [Claeys et al. 2019b, equation (3.12)] for the constant $c_8^{(3)}$. The correct value of $c_8^{(3)}$ can be found in [Charlier et al. 2019a, equation (2.4)].
have $G(\zeta; s) = O(1)$. The jumps for $U$ can be rewritten in terms of $h$ and $G$ as follows:

$$J_U(\zeta) = \begin{cases} 
\frac{1}{\sqrt{-1}} e^{-i\pi h(\zeta)} G(\zeta; s) & \text{if } \zeta \in \gamma_U, \\
\sqrt{-1} e^{i\pi h(\zeta)} G(\zeta; s) & \text{if } \zeta \in \tilde{\gamma}_U. 
\end{cases} (4-8)$$

4B1. Saddle points of $h$. The saddle points of $h$ are the solutions to $h'(\zeta) = 0$. Using (4-5), this equation can be written explicitly as

$$-(c_1 + c_2 + c_3) - c_1 \log(i\zeta) - c_2 \log(-i\zeta) = 0. (4-9)$$

A direct computation shows that this equation admits two solutions $\zeta = b_2$ and $\zeta = b_1$, where

$$b_2 = -b_1 = \exp\left(-\frac{c_1 + c_2 + c_3}{c_1 + c_2}\right) \exp\left(\frac{i\pi c_2 - c_1}{2c_1 + c_2}\right). (4-10)$$

For the Meijer-G point process (i.e., $j = 1$), we have $c_1 + c_2 + c_3 = 0$ and $c_2 > c_1$, and therefore $b_2$ lies on the unit circle in the quadrant $Q_1 := \{\zeta \in \mathbb{C} : \text{Re } \zeta \geq 0, \text{Im } \zeta \geq 0\}$. For Wright’s generalized Bessel process (i.e., $j = 2$), $b_2$ lies on the circle centered at the origin of radius $\exp\left(-\frac{c_1 + c_2 + c_3}{c_1 + c_2}\right)$: $b_2$ is in the quadrant $Q_1$ for $\theta \leq 1$, and in the quadrant $Q_4 := \{\zeta \in \mathbb{C} : \text{Re } \zeta \geq 0, \text{Im } \zeta \leq 0\}$ for $\theta \geq 1$. Let us define

$$\ell := \text{Re}(ih(b_2)) = -(c_1 + c_2) \exp\left(-\frac{c_1 + c_2 + c_3}{c_1 + c_2}\right) \sin\left(\frac{\pi c_2 - c_1}{2c_1 + c_2}\right).$$

We consider the zero set of $\text{Re}(ih) - \ell$:

$$\mathcal{N} = \{\zeta \in \mathbb{C} : \text{Re}(ih(\zeta)) = \ell\},$$

which is visualized in Figure 5.
Lemma 4.2. The set $\mathcal{N}$ consists of five simple curves $\Gamma_j$, $j = 1, \ldots, 5$ and satisfies the symmetry $\mathcal{N} = -\mathcal{N}$. Three of these curves, say $\Gamma_1, \Gamma_2$ and $\Gamma_3$, join $b_2$ with $b_1$. The curve $\Gamma_4$ starts at $b_2$ and leaves the right half plane in the sector $\arg \zeta \in (-\epsilon, \epsilon)$ for any $\epsilon > 0$. The last curve satisfies $\Gamma_5 = -\overline{\Gamma_4}$. In particular, $\mathcal{N}$ divides the complex plane in four regions: two unbounded regions, and two bounded regions. Furthermore, the sign of $\Re(ih(\zeta)) - \ell$ in each of these regions is as shown in Figure 5.

Proof. We divide the proof in four steps.

Claim 1: $\mathcal{N}$ intersects the imaginary axis at three distinct points $y_1 < y_2 < y_3$ such that $y_1 < 0$ and $y_3 > 0$.

To prove this, it suffices to inspect the graph of the function $y \mapsto \Re(ih(iy))$ for $y \in \mathbb{R}$. It is a simple computation to verify that

$$\Re(ih(iy)) = (c_1 + c_2)y \log |y| + c_3y, \quad y \in \mathbb{R}. $$

This function is odd in the variable $y$, is equal to 0 at $y = 0$, tends to $+\infty$ as $y \to +\infty$ and admits a local minimum at $y = y_* := \exp\left(-\frac{c_1 + c_2 + c_3}{c_1 + c_2}\right)$ where it takes the value

$$\Re(ih(iy_*)) = -(c_1 + c_2) \exp\left(-\frac{c_1 + c_2 + c_3}{c_1 + c_2}\right) < \ell. \quad (4.11)$$

Since $y \mapsto \Re(ih(iy))$ is odd, and since

$$\Re(ih(-iy_*)) = (c_1 + c_2) \exp\left(-\frac{c_1 + c_2 + c_3}{c_1 + c_2}\right) > \ell,$$

the equation $\Re(ih(iy)) = \ell$ admits three solutions $y_1, y_2, y_3$ satisfying

$$y_1 < -y_*, \quad y_2 \in (-y_*, y_*), \quad y_3 > y_*.$$

Claim 2: For any $\epsilon \in \left(0, \frac{\pi}{2}\right)$, there exists $\rho_\epsilon > 0$ such that for all $\rho \geq \rho_\epsilon$, $\mathcal{N}$ intersects $\{\rho \, e^{i\phi} : \phi \in (-\epsilon, \epsilon)\}$ at a single point.

This follows from a direct computation using the following expression for $\zeta = \rho \, e^{i\phi}$, $\phi \in (-\epsilon, \epsilon)$:

$$\Re(ih(\zeta)) = \Re(\zeta)\left[(c_1 + c_2)(\tan \phi \log \rho + \phi) + c_3 \tan \phi - \frac{\pi}{2}(c_2 - c_1)\right].$$

Claim 3: There exists no closed curve $\Gamma$ lying entirely in either the left or right half plane such that $\Gamma \subset \mathcal{N}$.

Since $h$ is analytic in $\mathbb{C} \setminus i\mathbb{R}$, $\zeta \mapsto \Re(ih(\zeta))$ is harmonic in $\mathbb{C} \setminus i\mathbb{R}$. Let $\Gamma \subset \mathbb{C} \setminus i\mathbb{R}$ be a closed curve such that $\Gamma \subset \mathcal{N}$. The maximum principle for harmonic functions implies that $\Re(ih(\zeta)) \equiv \ell$ on the interior of $\Gamma$. Since $\Re(ih(\zeta))$ is nonconstant on any open disk, we conclude that there exists no such curve $\Gamma$.

Proof of Lemma 4.2. Since $h'(b_2) = 0$ and $h''(b_2) \neq 0$, there are four curves $\{\Gamma_j\}_{j=1}^4$ emanating from $b_2$ that belong to $\mathcal{N}$. From Claim 3, none of these curves is a closed curve lying entirely in the right half plane. We conclude that these curves must leave the right half plane either on $i\mathbb{R}$ or at $\infty$. From Claim 1 and Claim 2, three curves $\Gamma_j$, $j = 1, 2, 3$ leave the right half plane on $i\mathbb{R}$ at $y_1, y_2$ and $y_3$, respectively, and the last curve $\Gamma_4$ leaves the right half plane at $\infty$ in the sector $\arg \zeta \in (-\epsilon, \epsilon)$ (for any $\epsilon > 0$ fixed).
By \( \text{Re}(ih(-\zeta)) = \text{Re}(ih(\zeta)) \), \( \mathcal{N} \) is symmetric with respect to \( i\mathbb{R} \) and this proves that \( \Gamma_j, \ j = 1, 2, 3 \), join \( b_2 \) and \( b_1 \), and that there exists \( \Gamma_5 \subset \mathcal{N} \) in the left half plane satisfying \( \Gamma_5 = -\overline{\Gamma_4} \). The sign of \( \text{Re}(ih(\zeta)) - \ell \) in the topmost bounded region is negative by (4-11). Since the sign of \( \text{Re}(ih(\zeta)) - \ell \) changes every time a curve \( \gamma_j, j \in \{1, \ldots, 5\} \) is crossed, this determines the sign of \( \text{Re}(ih(\zeta)) - \ell \) in the other regions as well.

\[ \square \]

**4C. Second transformation** \( U \mapsto T \). We will now define \( T \) in terms of \( U \) in two steps, \( U \mapsto \hat{T} \) and \( \hat{T} \mapsto T \). The transformation \( U \mapsto \hat{T} \) is similar to the one from [Claeys et al. 2019b, Section 3.2]. Let us define the union of two line segments \( \Sigma_5 := [b_1, 0] \cup [0, b_2] \), as shown in Figure 6. \( \hat{T} \) consists of analytic continuations of \( U \) in different regions, such that it has jumps on \( \bigcup_{j=1}^{5} \Sigma_j \) instead of \( \gamma_U \cup \tilde{\gamma}_U \), where the contours \( \Sigma_1, \ldots, \Sigma_5 \) are shown in Figure 6. More precisely, denote \( U_I \) for the analytic continuation of the function \( U \) as defined in the region above the contour \( \gamma_U \), \( U_{II} \) for the analytic continuation of \( U \) as defined in the region between \( \gamma_U \) and \( \tilde{\gamma}_U \), and \( U_{III} \) for the analytic continuation of \( U \) as defined in the region below \( \tilde{\gamma}_U \); then with the regions I', II', III' as in Figure 6, we define \( \hat{T} = U_I \) in region I', \( \hat{T} = U_{II} \) in the two regions II', and \( \hat{T} = U_{III} \) in region III'.

\( \hat{T} \) satisfies the same RH conditions as \( U \), except for a modified jump relation on \( \Sigma_5 \).

**RH problem for \( \hat{T} \):**

(a) \( \hat{T} \) is analytic in \( \mathbb{C} \setminus \bigcup_{j=1}^{5} \Sigma_j \), where the contour \( \bigcup_{j=1}^{5} \Sigma_j \) is shown in Figure 6 and is chosen to be symmetric with respect to \( i\mathbb{R} \).

(b) For \( \zeta \in \bigcup_{j=1}^{5} \Sigma_j \), we have \( \hat{T}^+(\zeta) = \hat{T}^-(\zeta) J_{\hat{T}}(\zeta) \), where

\[ J_{\hat{T}}(\zeta) = \begin{cases} 
    \left( \begin{array}{cc}
    0 & \frac{1}{\sqrt{1-i\ell} e^{-i\phi(h(\zeta,s))}} \end{array} \right) & \text{if } \zeta \in \Sigma_1 \cup \Sigma_2, \\
    \left( \begin{array}{cc}
    0 & 1 \\
    \frac{1}{\sqrt{1-i\ell} e^{i\phi(h(\zeta,s))}} & 0 \\
    \end{array} \right) & \text{if } \zeta \in \Sigma_3 \cup \Sigma_4, \\
    \left( \begin{array}{cc}
    1 & -\frac{1}{\sqrt{1-i\ell} e^{-i\phi(h(\zeta,s))}} \\
    \frac{1}{\sqrt{1-i\ell} e^{i\phi(h(\zeta,s))}} & \sqrt{1-i\ell} e^{-i\phi(h(\zeta,s))} \\
    \end{array} \right) & \text{if } \zeta \in \Sigma_5. 
\end{cases} \]
Figure 7. The jump contour for the RH problem for $T$.

(c) As $\zeta \to \infty$, we have

$$\hat{T}(\zeta) = I + \frac{U_1}{\zeta} + O(\zeta^{-2}).$$

As $\zeta \to b_1$ and as $\zeta \to b_2$, we have $\hat{T}(\zeta) = O(1)$.

We note that

$$\left(\frac{1}{\sqrt{1-\ell} e^{i\sigma_3 \ell} G(\xi;\sigma_3)} - \sqrt{1-\ell} e^{-i\sigma_3 \ell} \hat{G}(\zeta; s)\right)$$

$$= \left(\frac{1}{\sqrt{1-\ell} e^{i\sigma_3 \ell} G(\xi;\sigma_3)} - 1\right) \left(\frac{1}{1 - \sqrt{1-\ell} e^{-i\sigma_3 \ell} \hat{G}(\zeta; s)}\right)$$

$$= \left(\frac{1}{\sqrt{1-\ell} e^{i\sigma_3 \ell} G(\xi;\sigma_3)} - 1\right) \left(\frac{1}{1 - \sqrt{1-\ell} e^{-i\sigma_3 \ell} \hat{G}(\zeta; s)}\right).$$

We have used the factorization (4-12) in the transformation $U \mapsto \hat{T}$ to collapse part of the contours on $\Sigma_5$. In the transformation $\hat{T} \mapsto T$, we now use the other factorization (4-13) to open lenses on the other side of $\Sigma_5$.

Let $\Sigma_6$, $\Sigma_7$ be curves as shown in Figure 7, and define $T$ as

$$T(\zeta) = s^{-\frac{c_0}{2} \sigma_3} e^{\frac{c_0}{2} \sigma_3} \hat{T}(\zeta) H(\zeta) e^{-\frac{c_0}{2} \sigma_3} S^{-\frac{c_0}{2} \sigma_3},$$

where

$$H(\zeta) = \begin{cases} 
\left(\frac{1}{\sqrt{1-\ell} e^{i\sigma_3 \ell} G(\xi;\sigma_3)} - 1\right) & \text{if } \xi \in \text{int}(\Sigma_5 \cup \Sigma_6), \\
\left(\frac{1}{1 - \sqrt{1-\ell} e^{-i\sigma_3 \ell} \hat{G}(\zeta; s)}\right) & \text{if } \xi \in \text{int}(\Sigma_5 \cup \Sigma_7), \\
I & \text{otherwise}.
\end{cases}$$
Note \( e^{-is\theta h(\zeta)}G(\zeta; s) \) is analytic (in particular has no poles) in the lower half plane, while \( e^{is\theta h(\zeta)}G(\zeta; s)^{-1} \) is analytic in the upper half plane, so that \( H(\zeta) \) is analytic for \( \zeta \in \mathbb{C} \setminus (\Sigma_5 \cup \Sigma_6 \cup \Sigma_7) \). Note also that \( h(\zeta) \) tends to 0 as \( \zeta \to 0 \). Then \( T \) satisfies the following RH problem:

**RH problem for \( T \):**

(a) \( T : \mathbb{C} \setminus \bigcup_{j=1}^{7} \Sigma_j \to \mathbb{C}^{2 \times 2} \) is analytic. The contour \( \bigcup_{j=1}^{7} \Sigma_j \) is shown in Figure 7 and is chosen to be symmetric with respect to \( i\mathbb{R} \).

(b) It satisfies the jumps \( T_+(\zeta) = T_-(\zeta) J_T(\zeta) \) for \( \zeta \in \bigcup_{j=1}^{7} \Sigma_j \), where

\[
J_T(\zeta) = \begin{cases} 
\begin{pmatrix} 1 & -\sqrt{-1}e^{-s\phi(h(\zeta)-\ell)} \tilde{\eta}(\zeta; s) \\ 0 & 1 \end{pmatrix} & \text{if } \zeta \in \Sigma_1 \cup \Sigma_2, \\
\begin{pmatrix} 1 & \sqrt{-1}e^{s\phi(h(\zeta)-\ell)} \tilde{\eta}(\zeta; s)^{-1} \\ 0 & 1 \end{pmatrix} & \text{if } \zeta \in \Sigma_3 \cup \Sigma_4, \\
\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \text{if } \zeta \in \Sigma_5, \\
\begin{pmatrix} 1 & \sqrt{-1}e^{s\phi(h(\zeta)-\ell)} \tilde{\eta}(\zeta; s)^{-1} \\ 0 & 1 \end{pmatrix} & \text{if } \zeta \in \Sigma_6, \\
\begin{pmatrix} 1 & -\sqrt{-1}e^{s\phi(h(\zeta)-\ell)} \tilde{\eta}(\zeta; s) \\ 0 & 1 \end{pmatrix} & \text{if } \zeta \in \Sigma_7,
\end{cases}
\]  

(4-15)

where

\[
\tilde{\eta}(\zeta; s) = \eta(\zeta; s)s^{-\sigma_4}.
\]  

(4-16)

(c) As \( \zeta \to \infty \), we have

\[
T(\zeta) = I + \frac{T_1}{\zeta} + O(\zeta^{-2}),
\]  

(4-17)

where

\[
T_1 = s^{-\frac{c_4}{2}} e^{\frac{iz\ell}{2}} U_1 e^{-\frac{i\rho h(\zeta)}{2}} \sigma_3 s^{-\frac{\rho h(\zeta)}{2}} \sigma_3 \frac{1}{i\sigma_6} s^{-\frac{\rho h(\zeta)}{2}} e^{\frac{i\rho h(\zeta)}{2}} \sigma_3 Y_1 s^{-\frac{\rho h(\zeta)}{2}} e^{-\frac{i\rho h(\zeta)}{2}} \sigma_3 s^{\frac{\rho h(\zeta)}{2}} \sigma_3.
\]  

(4-18)

As \( \zeta \to b_1 \) and as \( \zeta \to b_2 \), we have \( T(\zeta) = O(1) \).

**Remark 4.3.** We choose the jump contour for \( T \) to be symmetric with respect to \( i\mathbb{R} \) for later use (it will make the analysis simpler). Using this symmetry, we show in a similar way as in Remark 4.1 that \( J_T(\zeta) = J_T(-\zeta) \) for \( \zeta \in \bigcup_{j=1}^{7} \Sigma_j \). By uniqueness of the solution to the RH problem for \( T \), this implies the symmetry

\[
T(\zeta) = T(-\zeta), \quad \zeta \in \mathbb{C} \setminus \bigcup_{j=1}^{7} \Sigma_j.
\]  

(4-19)

By Lemma 4.2, the jumps for \( T \) tend to \( I \) exponentially fast as \( s \to +\infty \) on \( (\bigcup_{j=1}^{7} \Sigma_j) \setminus \Sigma_5 \), and this convergence is uniform outside neighborhoods of \( b_1 \) and \( b_2 \).

For convenience, we use the notation

\[
\tilde{\ell} := \text{Im}(ih(b_2)) = -\text{Im}(ih(b_1)) = (c_1 + c_2) \exp\left(-\frac{c_1 + c_2 + c_3}{c_1 + c_2}\right) \cos\left(\frac{\pi}{2} \frac{c_2 - c_1}{c_1 + c_2}\right).
\]  

(4-20)
4D. Global parametrix. In this subsection we construct the global parametrix \( P^{(\infty)} \). We will show in Section 4G that \( P^{(\infty)} \) approximates \( T \) outside of neighborhoods of \( b_1 \) and \( b_2 \).

**RH problem for \( P^{(\infty)} \):**

(a) \( P^{(\infty)} : \mathbb{C} \setminus \Sigma_5 \to \mathbb{C}^{2 \times 2} \) is analytic.

(b) It satisfies the jumps

\[
P_+^{(\infty)}(\xi) = P_-^{(\infty)}(\xi) \begin{pmatrix} 1 & 0 \\ 0 & \imath \end{pmatrix}, \quad \xi \in \Sigma_5.
\]

(c) As \( \xi \to \infty \), we have

\[
P^{(\infty)}(\xi) = I + \frac{1}{\xi} P^{(\infty)}_1 + \mathcal{O}(\xi^{-2}). \tag{4-21}
\]

(d) As \( \xi \) tends to \( b_1 \) or \( b_2 \), \( P^{(\infty)}(\xi) \) remains bounded.

Conditions (a)–(c) for the RH problem for \( P^{(\infty)} \) are obtained from the RH problem for \( T \) by ignoring the jumps on \( \bigcup_{j=1}^7 \Sigma_j \setminus \Sigma_5 \). Condition (d) has been added to ensure uniqueness of the solution of the RH problem for \( P^{(\infty)} \). This solution can be easily obtained by using Cauchy’s formula and is given by

\[
P^{(\infty)}(\xi) = D(\xi)^{-\sigma_3}, \tag{4-22}
\]

where

\[
D(\xi) = \exp\left( \imath \nu \int_{\Sigma_5} \frac{d\xi}{\xi - \zeta} \right) = \exp\left( \imath \nu \log \left[ \frac{\xi - b_2}{\xi - b_1} \right] \right), \quad \nu := -\frac{1}{2\pi} \log t \in \mathbb{R}, \tag{4-23}
\]

where the branch for the log is taken along \( \Sigma_5 \). The function \( D \) satisfies

\[
D_+(\zeta) = D_-(\zeta) t, \quad \zeta \in \Sigma_5,
\]

\[
D(\xi) = 1 + \frac{D_1}{\xi} + \mathcal{O}(\xi^{-2}), \quad \text{as } \xi \to \infty,
\]

where \( D_1 = -\imath \nu (b_2 - b_1) = -2\imath \nu \text{Re} b_2 \). From (4-21) and (4-22), we obtain

\[
P_1^{(\infty)} = -D_1 \sigma_3. \tag{4-24}
\]

We will also need asymptotics for \( P^{(\infty)}(\xi) \) as \( \xi \to b_2 \). From (4-23), as \( \xi \to b_2 \) we have

\[
D(\xi) = \left( \frac{\xi - b_2}{b_2 - b_1} \right)^{\imath \nu} \left( 1 - \imath \nu \frac{\xi - b_2}{b_2 - b_1} + \mathcal{O}(1) \right),
\]

which implies by (4-22) that

\[
P^{(\infty)}(\xi) = \left( \frac{\xi - b_2}{b_2 - b_1} \right)^{-\imath \nu \sigma_3} \left( I + \imath \nu \frac{\xi - b_2}{b_2 - b_1} \sigma_3 + \mathcal{O}(1) \right), \quad \text{as } \xi \to b_2. \tag{4-25}
\]

It is also straightforward to verify from (4-22) and (4-23) that \( P^{(\infty)} \) satisfies the symmetry relation

\[
P^{(\infty)}(\xi) = \overline{P^{(\infty)}(-\bar{\xi})}, \quad \xi \in \mathbb{C} \setminus \Sigma_5. \tag{4-26}
\]
Local parametrix at $b_2$. We construct the local parametrix $P^{(b_2)}$ in a small disk $D_{b_2}$ around $b_2$ with radius independent of $s$. We require $P^{(b_2)}$ to satisfy the same jumps as $T$ inside $D_{b_2}$, to remain bounded as $\zeta \to b_2$, and to match with $P^{(\infty)}$ on the boundary of $D_{b_2}$, in the sense that

$$P^{(b_2)}(\zeta) = (I + o(1))P^{(\infty)}(\zeta), \quad \text{as } s \to +\infty,$$

uniformly for $\zeta \in \partial D_{b_2}$. The solution can be constructed in terms of the solution $\Phi_{PC}$ to the parabolic cylinder model RH problem presented in the Appendix. This model RH problem depends on a parameter $q$; in our case we need to choose $q = \sqrt{1 - t}$. Let us define

$$f(\zeta) = \sqrt{-2(h(\zeta) - h(b_2))},$$

(4-27)

This is a conformal map from $D_{b_2}$ to a neighborhood of $0$ satisfying $f(b_2) = 0$ and

$$f'(b_2) = \sqrt{\frac{c_1 + c_2}{b_2}} \frac{\sqrt{c_1 + c_2}}{\exp(-\frac{c_1 + c_2}{2(c_1 + c_2)}) \exp(i\frac{\pi c_2}{4c_1 + c_2})} \quad \text{and} \quad f''(b_2) = -\frac{1}{3b_2} f'(b_2).$$

(4-28)

In small neighborhoods of $D_{b_2}$ and $D_{b_1}$, we slightly deform the contour $\bigcup_{j=1}^7 \Sigma_j$ such that it remains symmetric with respect to $i \mathbb{R}$ and such that it satisfies

$$f \left( \bigcup_{j=1}^7 \Sigma_j \cap D_{b_2} \right) \subset \Sigma_{PC}, \quad f(\Sigma_5) \subset (-\infty, 0],$$

(4-29)

where $\Sigma_{PC}$ is shown in Figure 8. The local parametrix is given by

$$P^{(2)}(\zeta; s) = E(\zeta; s)\Phi_{PC}(s^{\frac{\nu}{2}}f(\zeta); \sqrt{1 - t})e^{\frac{s}{2}(i(h(\zeta) - h(b_2)))\sigma_3} \widetilde{G}(\zeta; s)^{-\frac{\sigma_3}{2}},$$

(4-30)

where $E$ is analytic in $D_{b_2}$ and given by

$$E(\zeta; s) = P^{(\infty)}(\zeta)\widetilde{G}(\zeta; s)^{\frac{\sigma_3}{2}} e^{-\frac{\nu}{8}i\overline{\ell}\sigma_3}(s^{\frac{\nu}{2}}f(\zeta))^{i\nu\sigma_3},$$

(4-31)

where $\nu = \nu(t) \in \mathbb{R}$ is given by (4-23) and the branch cut for $(s^{\frac{\nu}{2}}f(\zeta))^{i\nu\sigma_3}$ is taken along $\Sigma_5 \cap D_{b_2}$. Note that $\widetilde{G}(\zeta; s)$ depends on $s$, but by (4-7) and (4-16) it is bounded as $s \to +\infty$ uniformly for $\zeta \in D_{b_2}$ (see also (4-36) below). Since $\nu \in \mathbb{R}$ and $\overline{\ell} \in \mathbb{R}$ (see (4-20)), we thus have $E(\zeta; s) = \mathcal{O}(1)$ as $s \to +\infty$, 
As for later use, we note that this implies
\[
E(\xi; s) = \alpha(s)^{\mathcal{O}}(I + \beta(s)\sigma_3(\xi - b_2) + \mathcal{O}((\xi - b_2)^2)), \quad \xi \to b_2, \tag{4-32}
\]
\[
\alpha(s) = \left. [(b_2 - b_1) f'(b_2)s^\xi]^{1/s} \tilde{G}(b_2; s) \frac{1}{\xi} e^{-is\xi} \right|, \tag{4-33}
\]
\[
\beta(s) = \frac{i\nu}{b_2 - b_1} + \frac{1}{2} (\log \tilde{G})(b_2; s) - \frac{i\nu}{6b_2}. \tag{4-34}
\]
As $s \to +\infty$, for any $N \in \mathbb{N}$, we have
\[
P^{(b_2)}(\xi) P^{(\infty)}(\xi)^{-1} = I + E(\xi; s) \left( \sum_{j=1}^{N} \Phi_{PC,j} \xi_s + O(s^{-2}) \right) E(\xi; s)^{-1} + O(s^\rho) \tag{4-35}
\]
uniformly for $\xi \in \partial D_{b_2}$, where the matrices $\Phi_{PC,1}$ and $\Phi_{PC,2}$ are given by (A-2). In particular, the matrix $\Phi_{PC,1}$ is expressed in terms of the quantities $\beta_{12}$ and $\beta_{21}$ defined in (A-3). Furthermore, the matrices $\Phi_{PC,2k}$ are diagonal for every $k \geq 1$ and the matrices $\Phi_{PC,2k-1}$ are off-diagonal for every $k \geq 1$; see again (A-2).

We need to expand $E(\xi; s)$ as $s \to \infty$. By the expansion (4-7) of $\mathcal{G}$ and the definition (4-16) of $\tilde{\mathcal{G}}$, we get
\[
\log \tilde{\mathcal{G}}(\xi; s) = c_5 \log(i\xi) + c_6 \log(-i\xi) + c_7 + \frac{c_8}{is^\rho \xi} + O(s^{-2}) \quad \text{as } s \to +\infty, \tag{4-36}
\]
uniformly for $\xi \in D_{b_2}$, where the error term can be expanded in a full asymptotic series in integer powers of $s^{-\rho}$. We deduce from this that
\[
\tilde{\mathcal{G}}(b_2; s) = (ib_2)^{c_5}(-ib_2)^{c_6}e^{c_7}\left(1 + \frac{c_8}{is^\rho b_2} + O(s^{-2})\right), \tag{4-37}
\]
\[
(\log \tilde{\mathcal{G}})'(b_2; s) = \frac{c_5 + c_6}{b_2} - \frac{c_8}{is^\rho b_2^2} + O(s^{-2}), \tag{4-38}
\]
as $s \to \infty$, and by (4-31), we can write
\[
E(\xi; s) = \sum_{j=0}^{N} E_j(\xi; s)s^{-\rho j} + O(s^{-(N+1)\rho}), \quad \text{as } s \to +\infty, \tag{4-39}
\]
for any $N \in \mathbb{N}$, uniformly for $\xi \in D_{b_2}$, and where the diagonal matrices $E_j(\xi; s)$ depend on $s$ but are bounded; in particular
\[
E_0(b_2; s) = [(b_2 - b_1) f'(b_2)s^\xi]^{1/s} \tilde{G}(b_2; s)^{\sigma_3} e^{-is\xi}, \tag{4-40}
\]
\[
E_0'(b_2; s) = \beta_0(s) E_0(b_2; s) \sigma_3, \tag{4-41}
\]
\[
\beta_0(s) = \frac{i\nu}{b_2 - b_1} + \frac{c_5 + c_6}{2b_2} - \frac{i\nu}{6b_2}. \tag{4-42}
\]
For later use, we note that this implies
\[
\mathcal{E}(s) := \frac{E_0(b_2; s)_{11}}{E_0(b_2; s)_{11}} = \exp(2i \arg(E_0(b_2; s)_{11}))
\]
\[
= \left[(b_2 - b_1) f'(b_2)s^\xi\right]^{2i\nu} \exp(ic_5 \arg(ib_2) + ic_6 \arg(-ib_2)) e^{-i\xi s^\rho}. \tag{4-43}
\]
Figure 9. The jump contour $\Sigma_R$ in the RH problem for $R$.

4F. Local parametrix at $b_1$. We construct the local parametrix $P^{(b_1)}$ in a small disk $\mathcal{D}_{b_1}$ around $b_1$ in a similar way as we defined $P^{(b_2)}$ in $\mathcal{D}_{b_2}$. More precisely, we require $P^{(b_1)}$ to satisfy the same jumps as $T$ inside $\mathcal{D}_{b_1}$, to remain bounded as $\zeta \to b_1$, and to satisfy the matching condition

$$P^{(b_1)}(\zeta) = (I + o(1))P^{(\infty)}(\zeta), \quad \text{as } s \to +\infty,$$

uniformly for $\zeta \in \partial \mathcal{D}_{b_1}$. It is possible to construct $P^{(b_1)}(\zeta)$ in a similar way as $P^{(b_2)}(\zeta)$ in terms of parabolic cylinder functions. To avoid unnecessary analysis and computations, we choose $\mathcal{D}_{b_1} = -\mathcal{D}_{b_2}$, and we rely on the symmetry $J_T(\zeta) = J_T(-\zeta)$ for $\zeta \in \bigcup_{j=1}^7 \Sigma_j$ (see Remark 4.3) to conclude directly that the function

$$P^{(b_1)}(\zeta) = P^{(b_2)}(-\zeta), \quad \zeta \in \mathcal{D}_{b_1} \setminus \bigcup_{j=1}^7 \Sigma_j$$  \hspace{1cm} (4-44)

satisfies the required conditions for the local parametrix.

4G. Small norm RH problem. In this section we show that, as $s$ becomes large, $P^{(\infty)}(z)$ approximates $T(z)$ for $z \in \mathbb{C} \setminus \bigcup_{j=1}^2 \mathcal{D}_{b_j}$ and $P^{(b_j)}(z)$ approximates $T(z)$ for $z \in \mathcal{D}_{b_j}$, $j = 1, 2$. We define

$$R(\zeta) = \begin{cases} T(\zeta)P^{(\infty)}(\zeta)^{-1} & \text{if } \zeta \in \mathbb{C} \setminus (\mathcal{D}_{b_1} \cup \mathcal{D}_{b_2}), \\ T(\zeta)P^{(b_1)}(\zeta)^{-1} & \text{if } \zeta \in \mathcal{D}_{b_1}, \\ T(\zeta)P^{(b_2)}(\zeta)^{-1} & \text{if } \zeta \in \mathcal{D}_{b_2}. \end{cases}$$  \hspace{1cm} (4-45)

Since $P^{(b_j)}$, $j = 1, 2$, have the exact same jumps as $T$ inside the disks, $R$ is analytic in $\bigcup_{j=1}^2 \mathcal{D}_{b_j} \setminus \{b_j\}$. Furthermore, since $T(z)$ and $P^{(b_j)}(z)^{-1}$ remain bounded as $z \to b_j$, $j = 1, 2$, we conclude that $R(z)$ is also bounded as $z \to b_j$, $j = 1, 2$. Thus the singularities of $R$ at $b_1$ and $b_2$ are removable and $R$ is analytic in the entire open disks. $R$ satisfies the following RH problem.

RH problem for $R$:

(a) $R : \mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2 \times 2}$ is analytic, where

$$\Sigma_R = \partial \mathcal{D}_{b_1} \cup \partial \mathcal{D}_{b_2} \cup \bigcup_{j=1}^7 \Sigma_j \setminus (\mathcal{D}_{b_1} \cup \mathcal{D}_{b_2} \cup \Sigma_5).$$
The contour $\Sigma_R$ is oriented as shown in Figure 9. In particular, we orient the circles $\partial D_{b_1}$ and $\partial D_{b_2}$ in the clockwise direction.

(b) For $\xi \in \Sigma_R$, $R$ satisfies the jumps $R_+(\xi) = R_-(\xi) J_R(\xi)$, where

\[
J_R(\xi) = P^{(b_1)}(\xi) P^{(\infty)}(\xi)^{-1}, \quad \xi \in \partial D_{b_1},
\]

\[
J_R(\xi) = P^{(b_2)}(\xi) P^{(\infty)}(\xi)^{-1}, \quad \xi \in \partial D_{b_2},
\]

\[
J_R(\xi) = P^{(\infty)}(\xi) J_T(\xi) P^{(\infty)}(\xi)^{-1}, \quad \xi \in \Sigma_R \setminus (\partial D_{b_1} \cup \partial D_{b_2}).
\]

(c) $R(\xi)$ remains bounded as $\xi$ tends to the points of self-intersection of $\Sigma_R$.

As $\xi \to \infty$, there exists $R_1 = R_1(s)$ such that

\[
R(\xi) = I + \frac{R_1}{\xi} + O(\xi^{-2}). \tag{4-46}
\]

**Remark 4.4.** The contour $\Sigma_R$ is symmetric with respect to $i \mathbb{R}$. Furthermore, by (4-26) and (4-44), the jumps $J_R$ satisfy the symmetry relation $J_R(\xi) = J_R(-\xi)$ for $\xi \in \Sigma_R$. Hence, by uniqueness of the solution to the RH problem for $R$, we conclude that

\[
R(\xi) = R(-\xi), \quad \xi \in \mathbb{C} \setminus \Sigma_R. \tag{4-47}
\]

From Lemma 4.2 and the fact that $P^{(\infty)}$ is independent of $s$ and uniformly bounded outside $D_{b_1} \cup D_{b_2}$, we have

\[
J_R(\xi) = I + O(e^{-cs|\xi|}), \quad \text{as } s \to +\infty \tag{4-48}
\]

uniformly for $\xi \in \Sigma_R \setminus (\partial D_{b_1} \cup \partial D_{b_2})$, and uniformly for $t$ in compact subsets of $(0, \infty)$. By substituting the expansion (4-39) in (4-35), we infer that, for any $N \in \mathbb{N}$, $J_R$ has an expansion in the form

\[
J_R(\xi) = J_R(\xi; s) = I + \sum_{j=1}^{N} J_R^{(j)}(\xi; s) s^{-\frac{j\rho}{2}} + O(s^{-\frac{(N+1)\rho}{2}}), \quad \text{as } s \to +\infty, \tag{4-49}
\]

where all coefficients $J_R^{(j)}(\xi; s)$ satisfy the symmetry $J_R^{(j)}(\xi; s) = J_R^{(j)}(-\xi; s)$ and are bounded as $s \to +\infty$, uniformly for $\xi \in \partial D_{b_1} \cup \partial D_{b_2}$ and for $t$ in compact subsets of $(0, \infty)$. The first two coefficients $J_R^{(j)}(\xi; s)$, for $j = 1, 2$ are given by

\[
J_R^{(j)}(\xi; s) = E_0(\xi; s) \frac{\Phi_{\text{PC},j}}{f(\xi)^j} E_0(\xi; s)^{-1}, \quad j = 1, 2. \tag{4-50}
\]

The jump relation for $R$ can also be written in the additive form $R_+ = R_- + (J_R - I)$, and together with the asymptotics for $R$, this implies the integral equation

\[
R(\xi) = R(\xi; s) = I + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{R_-(\xi; s)(J_R(\xi; s) - I)}{\xi - \zeta} d\xi. \tag{4-51}
\]

We conclude from (4-48) and (4-49) that $R$ satisfies a small norm RH problem as $s \to +\infty$, and by standard theory [Deift et al. 1999], it follows that $R$ exists for sufficiently large $s$. Moreover, substituting
(4-48) and (4-49) in (4-51) and expanding as \( s \to +\infty \), we obtain

\[
R(\xi; s) = 1 + \sum_{j=1}^{N} \frac{R^{(j)}(\xi; s)}{s^{\frac{j}{2}}} + O(s^{-\frac{(N+1)p}{2}}), \quad \text{as } s \to +\infty,
\]

(4-52)

\[
R'(\xi; s) = \sum_{j=1}^{N} \frac{R^{(j)'(\xi; s)}}{s^{\frac{j}{2}}} + O(s^{-\frac{(N+1)p}{2}}), \quad \text{as } s \to +\infty,
\]

uniformly for \( \xi \in \mathbb{C} \setminus \Sigma_R \), and uniformly for \( t \) in compact subsets of \((0, \infty)\). All the coefficients \( R^{(j)} \)
can in principle be computed iteratively. In particular, a substitution of (4-49) and (4-52) in \( R_+ = R_- J_R \)
yields

\[
R_+^{(1)}(\xi; s) = R^{(1)}(\xi; s) + J_R^{(1)}(\xi; s), \quad R_+^{(2)}(\xi; s) = R^{(2)}(\xi; s) + J_R^{(1)}(\xi; s) J_R^{(1)}(\xi; s) + J_R^{(2)}(\xi; s)
\]

for \( \xi \in \partial D_{b_1} \cup \partial D_{b_2} \). These jumps, together with the asymptotics \( R^{(1)}(\xi; s) = O(\xi^{-1}) \) and \( R^{(2)}(\xi; s) = O(\xi^{-1}) \) as \( \xi \to \infty \), imply that

\[
R^{(1)}(\xi; s) = \frac{1}{2\pi i} \int_{\partial D_{b_1}} \frac{J_R^{(1)}(\xi; s)}{\xi - \xi} d\xi + \frac{1}{2\pi i} \int_{\partial D_{b_2}} \frac{J_R^{(1)}(\xi; s)}{\xi - \xi} d\xi,
\]

(4-53)

and

\[
R^{(2)}(\xi; s) = \frac{1}{2\pi i} \int_{\partial D_{b_1}} \frac{R_{-}^{(1)}(\xi; s) J_R^{(1)}(\xi; s) + J_R^{(2)}(\xi; s)}{\xi - \xi} d\xi
\]

\[
+ \frac{1}{2\pi i} \int_{\partial D_{b_2}} \frac{R_{-}^{(1)}(\xi; s) J_R^{(1)}(\xi; s) + J_R^{(2)}(\xi; s)}{\xi - \xi} d\xi.
\]

(4-54)

where we recall that \( \partial D_{b_1} \) and \( \partial D_{b_2} \) are oriented clockwise.

In the rest of this section, we evaluate \( R^{(1)}(\xi; s) \) and \( R^{(2)}(\xi; s) \) explicitly for \( \xi \in \mathbb{C} \setminus (D_{b_1} \cup D_{b_2}) \), and
we prove that \( R^{(k)}(\xi; s) \) can be chosen diagonal for \( k \) even and off-diagonal for \( k \) odd.

The expression (4-50) for \( J_R^{(1)} \) can be analytically continued from \( \partial D_{b_2} \) to the punctured disk \( D_{b_2} \setminus \{b_2\} \),
and we note that \( J_R^{(1)}(\xi; s) \) has a simple pole at \( \xi = b_2 \). Therefore, for \( \xi \) outside the disks, a residue calculation gives

\[
\frac{1}{2\pi i} \int_{\partial D_{b_2}} \frac{J_R^{(1)}(\xi; s)}{\xi - \xi} d\xi = \frac{A^{(1)}(s)}{\xi - b_2}, \quad \text{with } \quad A^{(1)}(s) = \text{Res}(J_R^{(1)}(\xi; s), \xi = b_2).
\]

(4-55)

To evaluate the first integral that appears at the right-hand side of (4-53), we appeal to the symmetries of
Remark 4.4 to write

\[
\frac{1}{2\pi i} \int_{\partial D_{b_1}} \frac{J_R^{(1)}(\xi)}{\xi - \xi} d\xi = \frac{1}{2\pi i} \int_{\partial D_{b_2}} \frac{J_R^{(1)}(\xi)}{\xi + \xi} d\xi = \frac{-A^{(1)}(s)}{\xi - b_1}.
\]

(4-56)

Therefore, it remains to evaluate \( A^{(1)}(s) = \text{Res}(J_R^{(1)}(\xi; s), \xi = b_2) \). From (4-40), (4-50), and (A-2), we
immediately obtain that
\[
A^{(1)}(s) = \frac{1}{f'(b_2)} E_0(b_2; s) \Phi_{PC,1} E_0(b_2; s)^{-1} = \frac{1}{f'(b_2)} \begin{pmatrix} 0 & \beta_{12} E_0(b_2; s)^2 \\ \beta_{21} E_0(b_2; s)^{-2} & 0 \end{pmatrix},
\]
which is an off-diagonal matrix, and \( E_0(b_2; s) \) has been explicitly evaluated in (4-40). Combining (4-53) with (4-55) and (4-56), we obtain
\[
R^{(1)}(\zeta; s) = \frac{A^{(1)}}{\zeta - b_2} - \frac{A^{(1)}}{\zeta - b_1}, \text{ for } \zeta \in \mathbb{C} \setminus (D_{b_1} \cup D_{b_2}),
\]
where \( A^{(1)} \) is given by (4-57).

For the computation of \( R^{(2)} \), we recall that \( J^{(1)}_R \) and \( J^{(2)}_R \) are given by (4-50), and we note that \( J^{(2)}_R \) can be simplified as
\[
J^{(2)}_R(\zeta) = \frac{1}{f(\zeta)^2} E_0(\zeta; s) \Phi_{PC,2} E_0(\zeta; s)^{-1} = \frac{1}{f(\zeta)^2} \Phi_{PC,2}, \quad \zeta \in \partial D_{b_2},
\]
where we have used that both \( E_0(\zeta; s) \) and \( \Phi_{PC,2} \) are diagonal matrices. We note that \( J^{(2)}_R \) can also be analytically continued from \( \partial D_{b_2} \) to the punctured disk \( D_{b_2} \setminus \{b_2\} \). Let us start by evaluating the integral over \( \partial D_{b_2} \), which appears at the right-hand side of (4-54). For \( \zeta \in \mathbb{C} \setminus (\partial D_{b_1} \cup \partial D_{b_2}) \), since \( R^{(1)}_1 \) is analytic on \( D_{b_1} \cup D_{b_2} \), and since \( J^{(j)}_R \) admits a pole of order \( j \) at \( b_2 \), \( j = 1, 2 \), we have
\[
\frac{1}{2\pi i} \int_{\partial D_{b_2}} \frac{R^{(1)}(\xi; s) J^{(1)}_R(\xi; s) + J^{(2)}_R(\xi; s)}{\xi - \zeta} d\xi = \frac{A^{(2)}(s)}{\zeta - b_2} + \frac{B^{(2)}(s)}{(\zeta - b_2)^2},
\]
\[
A^{(2)}(s) = R^{(1)}(b_2; s) A^{(1)}(s) + \text{Res}(J^{(2)}_R(\xi; s), \xi = b_2),
\]
\[
B^{(2)}(s) = \text{Res}((\xi - b_2) J^{(2)}_R(\xi; s), \xi = b_2).
\]
We again appeal to the symmetry \( \zeta \mapsto -\bar{\zeta} \) of Remark 4.4 to evaluate the integral over \( \partial D_{b_1} \):
\[
\frac{1}{2\pi i} \int_{\partial D_{b_1}} \frac{R^{(1)}(\xi; s) J^{(1)}_R(\xi; s) + J^{(2)}_R(\xi; s)}{\xi - \zeta} d\xi = \frac{1}{2\pi i} \int_{\partial D_{b_2}} \frac{R^{(1)}(\xi; s) J^{(1)}_R(\xi; s) + J^{(2)}_R(\xi; s)}{\xi + \bar{\zeta}} d\xi
\]
\[
= -\frac{A^{(2)}(s)}{\zeta - b_1} + \frac{B^{(2)}(s)}{(\zeta - b_1)^2}.
\]
To evaluate \( A^{(2)} \) and \( B^{(2)} \) explicitly, it remains to compute \( R^{(1)}(b_2; s) \), and the two residues
\[
\text{Res}(J^{(2)}_R(\xi; s), \xi = b_2) \quad \text{and} \quad \text{Res}((\xi - b_2) J^{(2)}_R(\xi; s), \xi = b_2).
\]
It is fairly easy to compute the residues (4-60) from the expression (4-50) for \( J^{(2)}_R(\zeta; s) \). We obtain
\[
B^{(2)}(s) = \text{Res}((\xi - b_2) J^{(2)}_R(\xi; s), \xi = b_2) = \frac{1}{f'(b_2)^3} \begin{pmatrix} \frac{(1+iv)^2}{2} & 0 \\ 0 & \frac{(1-iv)^2}{2} \end{pmatrix},
\]
\[
\text{Res}(J^{(2)}_R(\xi; s), \xi = b_2) = -\frac{f''(b_2)}{2 f'(b_2)^3} \begin{pmatrix} (1+iv) & 0 \\ 0 & (1-iv) \end{pmatrix}.
\]
where \( f'(b_2) \) and \( f''(b_2) \) are given by (4-28). Since
\[
R^{(1)}_-(\xi; s) = R^{(1)}_+(\xi; s) - J^{(1)}_R(\xi; s),
\]
and since \( R^{(1)}_+(\xi; s) \) has already been computed in (4-58), we obtain
\[
R^{(1)}(b_2; s) = \frac{A^{(1)}(s)}{b_2 - b_1} - \text{Res} \left( \frac{J^{(1)}_R(\xi; s)}{\xi - b_2}, \xi = b_2 \right)
\]
\[
= -A^{(1)}(s)\frac{1}{b_2 - b_1} + \frac{1}{f'(b_2)} \begin{pmatrix}
0 & (-\frac{1}{6b_2} - 2\beta_0(s))\beta_{12}E_0(b_2; s)^{21}_{12} \\
(-\frac{1}{6b_2} + 2\beta_0(s))\beta_{21}E_0(b_2; s)^{12}_{21} & 0
\end{pmatrix},
\]
and the constant \( \beta_0(s) \) is given by (4-42). Summarizing, we have
\[
R^{(2)}(\xi; s) = \frac{A^{(2)}(s)}{\xi - b_2} + \frac{B^{(2)}(s)}{(\xi - b_2)^2} + \frac{1}{f'(b_2)} \begin{pmatrix}
0 & (-\frac{1}{6b_2} - 2\beta_0(s))\beta_{12}E_0(b_2; s)^{21}_{12} \\
(-\frac{1}{6b_2} + 2\beta_0(s))\beta_{21}E_0(b_2; s)^{12}_{21} & 0
\end{pmatrix},
\]
for \( \xi \in \mathbb{C} \setminus (\partial D_{b_1} \cup \partial D_{b_2}) \).

Lemma 4.5. For any \( j \geq 1 \), the matrix \( R^{(2j-1)} \) is off-diagonal and the matrix \( R^{(2j)} \) is diagonal.

Proof. By (4-51), the matrices \( R^{(j)} \) can be computed recursively as follows:
\[
R^{(j)}(\xi; s) = \frac{1}{2\pi i} \int_{\partial D_{b_1} \cup \partial D_{b_2}} \sum_{\ell=1}^{j} R^{(j-\ell)}_-(\xi; s) J^{(\ell)}_R(\xi; s) d\xi, \quad j \geq 1.
\]
The result follows by induction, provided that the matrices \( J^{(2j)}_R \) are diagonal and \( J^{(2j-1)}_R \) are off-diagonal. To prove this claim, consider (4-35) and (4-39). These imply that \( J^{(2j)}_R(\xi; s) \) from (4-50) is composed of terms of the form
\[
\frac{1}{f^{j-2k}} E_m \Phi_{PC,j-2k}(E^{-1})_{k-m},
\]
for \( m = 0, \ldots, k \) and \( k = 0, 1, \ldots, \left\lfloor \frac{j-1}{2} \right\rfloor \), and where \( E_m \) and \( (E^{-1})_{k-m} \) are diagonal matrices. All these terms are diagonal if \( j \) is even and off-diagonal if \( j \) is odd. \( \qed \)
5. Proofs of Theorems 1.8 and 1.16: part 1

In this section, we use the analysis of Section 4 to prove part of Theorems 1.8 and 1.16 via the differential identity in $s$

$$\partial_s \log \det (1 - (1-t)K(j)|_{[0,s]}) = \frac{Y_{1,11}}{s},$$

(5-1)

which was derived in (3-8). As mentioned in the introduction, the advantage of this differential identity is that it leads to a significantly simpler analysis than the one carried out in Section 6 and that it allows us to prove the optimal bound $O(s^{-\rho})$ for the error terms of (1-12) and (1-20). The main disadvantage is that it does not allow for the evaluation of the constants $C$ of (1-12) and (1-20). These constants will be obtained in Section 6.

By (4-18), we have

$$T_{1,11} = \frac{1}{\rho} Y_{1,11}.$$

On the other hand, for $\zeta$ outside the lenses and outside the disks, we know from (4-45) that

$$T(\zeta) = R(\zeta) P^{(\infty)}(\zeta),$$

from which we deduce, by (4-17), (4-21), (4-46), and (4-52) that

$$T_1 = T_1(s) = P_1^{(\infty)} + R_1(s) = P_1^{(\infty)} + \sum_{j=1}^{2N+1} R_1^{(j)}(s) s^{-\frac{j}{2}} + O(s^{-(N+1)\rho}), \quad \text{as } s \to +\infty,$$

uniformly for $t$ in compact subsets of $\mathbb{R}$, where $N \in \mathbb{N}$ is arbitrary, and where the coefficients $R_1^{(j)}(s)$, $j \geq 1$, are defined via the expansion

$$R_1^{(j)}(\zeta) = \frac{R_1^{(j)}(s)}{\zeta} + O(\zeta^{-2}), \quad \text{as } \zeta \to \infty.$$

We know from Lemma 4.5 that $R_1^{(2j-1)}$ is off-diagonal for all $j \geq 1$. Thus, using (5-1), we find

$$\partial_s \log \det (1 - (1-t)K(j)|_{[0,s]}) = i s^{\rho-1} \left( P_1^{(\infty)} + \sum_{j=1}^{2N+1} R_1^{(j)}(s) s^{-\frac{j}{2}} + O(s^{-(N+1)\rho}) \right)_{11},$$

(5-2)

as $s \to +\infty$. After integrating (5-2), we obtain

$$\log \det (1 - (1-t)K(j)|_{[0,s]}) = \frac{i}{\rho} P_1^{(\infty)} s^\rho + \int_M \frac{i R_1^{(2j)}(s)}{s} ds + \log C_1 + O(s^{-\rho}),$$

(5-3)

as $s \to +\infty$, where $C_1$ is an unknown constant of integration and $M$ is a sufficiently large constant (i.e., $M$ is independent of $s$). An explicit expression for $P_1^{(\infty)}$ has been computed in (4-24). Then, the leading
We now turn to the computation of the second term of (5-3). Using (4-62) and (4-61), we obtain

\[
\frac{i}{\rho} p_{1,11}^{(\infty)} = - \frac{i D_1}{\rho} = - \frac{2 \nu \text{Re} b_2}{\rho}. \tag{5-4}
\]

We now turn to the computation of the second term of (5-3). Using (4-62) and (4-61), we obtain

\[
i R_{1,11}^{(2)}(s) = i (A_{11}^{(2)}(s) - \overline{A_{11}^{(2)}(s)}) = -2 \text{Im} A_{11}^{(2)}(s) = \frac{\nu^2}{c_1 + c_2} + \frac{1}{|f'(b_2)|^2 \text{Re} b_2} \text{Im}(\beta_{21}\overline{\beta}_{12} e(s)^{-2}).
\]

We recall that \(e(s)\) is given by

\[
e(s) = \left((b_2 - b_1)|f'(b_2)|s^2\right)^{2i\nu} \exp(ic_5 \text{arg}(ib_2) + ic_6 \text{arg}(-ib_2))e^{-i\tilde{\ell} s^\nu}.
\]

In particular, it satisfies \(|e(s)| = 1\) and it oscillates rapidly as \(s \to +\infty\), since \(\tilde{\ell} \neq 0\), see (4-20). Therefore,

\[
\int_{\lambda} e^{-}(s)s^{-1}ds = \tilde{c} \int_{\lambda} e^{2i\tilde{\ell} s^\nu - 2i\nu \log s^\nu} s^{-1}ds = \frac{\tilde{c}}{\rho} \int_{\lambda} e^{2i\tilde{\ell} u - 2i\nu \log u} u^{-1}du = \log C_2 + C_3(s),
\]

where \(\tilde{c}\) and \(C_2\) are constants whose exact values are unimportant for us, and \(C_3(s)\) is bounded by

\[
|C_3(s)| = O\left(\frac{1}{\ell s^\nu}\right) \quad \text{as} \quad s \to +\infty.
\]

We conclude that the integral in (5-3) has the asymptotics

\[
\int_{\lambda} s i R_{1,11}^{(2)}(s) ds = \frac{\nu^2}{c_1 + c_2} \log s + \log(C_2) + O(s^{-\nu}), \quad \text{as} \quad s \to +\infty. \tag{5-5}
\]

Since \(\rho = \frac{1}{c_1 + c_2}\), by combining (5-3), (5-4), and (5-5), we get

\[
\log \det(1 - (1 - t)\mathbb{\xi}^{(j)}|_{[0,s]}) = - \frac{2 \nu \text{Re} b_2}{\rho} s^\nu + \nu^2 \log s^\nu + \log(C) + O(s^{-\nu}), \tag{5-6}
\]

as \(s \to +\infty\), where \(C = C_1 C_2\). The values of \(\rho^{(1)}\) and \(\rho^{(2)}\) are given by (4-2) and (4-3), respectively, and \(\text{Re} b_2^{(j)}\), \(j = 1, 2\), can be evaluated by using (4-10) together with the coefficients \(c_1\), \(c_2\) and \(c_3\) (given above (4-5)). Recalling also that

\[
\log \det(1 - (1 - t)\mathbb{\xi}^{(j)}|_{[0,s]}) = \mathbb{E}[e^{-2\pi \nu Y(s)}],
\]

we have now completed the proofs of Theorems 1.8 and 1.16, up to the determination of the constants \(C = C^{(j)}, \quad j = 1, 2\).

6. Proofs of Theorems 1.8 and 1.16: part 2

In this section, we will compute \(C\) via the differential identity in \(t\)

\[
\partial_t \log \det(1 - (1 - t)\mathbb{\xi}^{(j)}|_{[0,s]}) = \frac{-1}{2(1 - t)} \int_{\gamma \cup \tilde{\gamma}} \text{Tr}[Y^{-1}(z)Y'(z)(J(z) - I)] \frac{dz}{2\pi i}, \tag{6-1}
\]

which was derived in Lemma 3.3.
We divide the proofs as a series of lemmas. First, we use the analysis of Section 4 to expand the right-hand side of (6-1) as \( s \to +\infty \).

**Lemma 6.1.** As \( s \to +\infty \), we have

\[
\partial_t \log \det(1 - (1-t)\mathcal{K}^{(j)}|_{[0,s]}) = I_1 + I_2 + 2 \Re \int_{b_2} + \mathcal{O}(e^{-ct^\rho}), \tag{6-2}
\]

where \( c > 0 \) and

\[
I_1 = -\frac{1}{t} \int_{\Sigma_5} \left( \log(e^{-is^\rho h(\xi)} \mathcal{G}(\xi; s)) \right) \frac{d\xi}{2\pi i} = -\frac{2}{t} \Re \left[ \int_{\mathcal{D}_2} \left( -i s^\rho h'(\xi) + (\log \mathcal{G}(\xi; s))' \right) \frac{d\xi}{2\pi i} \right], \tag{6-3}
\]

\[
I_2 = -\frac{1}{t} \int_{\Sigma_5} \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_3] \frac{d\xi}{2\pi i} = -\frac{2}{t} \Re \left[ \int_{\mathcal{D}_2} \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_3] \frac{d\xi}{2\pi i} \right], \tag{6-4}
\]

\[
I_{b_2} = \frac{1}{2\sqrt{1-t}} \int_{\mathcal{D}_2} e^{-s^\rho (ih(\xi) - \ell)} \mathcal{G}(\xi; s) \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_+ + T^\dagger(\xi)T'(\xi)\sigma_-] \frac{d\xi}{2\pi i}.
\]

with

\[
\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

**Proof.** Using the change of variables \( z = is^\rho \xi + \tau \) in (6-1), we obtain

\[
\partial_t \log \det(1 - (1-t)\mathcal{K}^{(j)}|_{[0,s]}) = \frac{-1}{2(1-t)} \int_{\gamma_U \cup \tilde{\gamma}_U} \text{Tr}[U^{-1}(\xi)U'(\xi)(J_U(\xi) - I)] \frac{d\xi}{2\pi i} = I_\gamma + I_{\tilde{\gamma}}.
\]

\[
I_\gamma = \frac{1}{2\sqrt{1-t}} \int_{\gamma_U} e^{-is^\rho h(\xi)} \mathcal{G}(\xi; s) \text{Tr}[U^{-1}(\xi)U'(\xi)\sigma_+] \frac{d\xi}{2\pi i}, \tag{6-6}
\]

\[
I_{\tilde{\gamma}} = \frac{-1}{2\sqrt{1-t}} \int_{\tilde{\gamma}_U} e^{is^\rho h(\xi)} \mathcal{G}(\xi; s) \text{Tr}[U^{-1}(\xi)U'(\xi)\sigma_-] \frac{d\xi}{2\pi i}, \tag{6-7}
\]

where \( U \) is defined in (4-1), \( \gamma_U \) and \( \tilde{\gamma}_U \) are defined in (4-4), and where we have used (4-8). Note that we do not specify whether we take the + or − boundary values of \( U \) in (6-6) and (6-7), which is without ambiguity, see Remark 3.4. Now, we deform the contours of integration by using the analytic continuation...
of $U$ (denoted $\hat{T}$ and defined in Section 4C). We obtain

$$I_y = \frac{1}{2\sqrt{1-t}} \int_{\Sigma_1 \cup \Sigma_2} e^{-is^\rho h(\xi)} G(\xi; s) \text{Tr}[\hat{T}^{-1}(\xi) \hat{T}'(\xi) \sigma_+] \frac{d\xi}{2\pi i}$$

$$+ \frac{1}{2\sqrt{1-t}} \int_{\Sigma_5} e^{-is^\rho h(\xi)} G(\xi; s) \text{Tr}[\hat{T}_+^{-1}(\xi) \hat{T}_+'(\xi) \sigma_+] \frac{d\xi}{2\pi i}, \quad (6-8)$$

$$I_y = \frac{-1}{2\sqrt{1-t}} \int_{\Sigma_3 \cup \Sigma_4} e^{is^\rho h(\xi)} G(\xi; s)^{-1} \text{Tr}[\hat{T}^{-1}(\xi) \hat{T}'(\xi) \sigma_-] \frac{d\xi}{2\pi i}$$

$$+ \frac{1}{2\sqrt{1-t}} \int_{\Sigma_5} e^{is^\rho h(\xi)} G(\xi; s)^{-1} \text{Tr}[\hat{T}_-^{-1}(\xi) \hat{T}_-'(\xi) \sigma_-] \frac{d\xi}{2\pi i}, \quad (6-9)$$

where the contours $\Sigma_1, \ldots, \Sigma_5$ are shown in Figure 6. Once more, we have not specified the boundary values of $\hat{T}$ for the integrals over $\Sigma_j$, $j = 1, \ldots, 4$ of (6-8) and (6-9); again, this is without ambiguity. Note however that this is not the case for the integrals over $\Sigma_5$. For $\xi \in \Sigma_5$, by (4-14) we have

$$\text{Tr}[\hat{T}_+^{-1} \hat{T}_+'(\xi) \sigma_+] = \text{Tr}[\hat{T}_+^{-1} \hat{P}(\xi) \hat{P}'(\xi) \sigma_+] + \text{Tr} \left[ T_+^{-1} T_+'(\xi) e^{i\rho h(\xi)} H_+^{-1} \sigma_+ H_+ e^{-i\rho h(\xi)} \sigma_+ \right],$$

$$e^{-is^\rho h(\xi)} G(\xi; s) \text{Tr}[\hat{T}_+^{-1} \hat{P}(\xi) \hat{P}'(\xi) \sigma_+] = -\frac{\sqrt{1-t}}{t} (\log(e^{-is^\rho h(\xi)} G(\xi; s)))',$n

$$e^{is^\rho h(\xi)} G(\xi; s)^{-1} \text{Tr}[\hat{T}_+^{-1} \hat{P}(\xi) \hat{P}'(\xi) \sigma_-] = \frac{\sqrt{1-t}}{t} (\log(e^{-is^\rho h(\xi)} G(\xi; s)))'.$$n

Therefore, using also the jumps for $T$ given by (4-15), we obtain

$$\frac{1}{2\sqrt{1-t}} \int_{\Sigma_3} e^{-is^\rho h(\xi)} G(\xi; s) \text{Tr}[\hat{T}_+^{-1}(\xi) \hat{T}_+'(\xi) \sigma_+] \frac{d\xi}{2\pi i}$$

$$= -\frac{1}{2t} \int_{\Sigma_5} (\log(e^{-is^\rho h(\xi)} G(\xi; s))) \frac{d\xi}{2\pi i}$$

$$- \frac{1}{2t} \int_{\Sigma_5} \text{Tr}[T^{-1}(\xi) T'(\xi) \sigma_3] \frac{d\xi}{2\pi i}$$

$$- \frac{\sqrt{1-t}}{2t^2} \int_{\Sigma_5} e^{i\rho h(\xi) - E} G(\xi; s)^{-1} \text{Tr}[T^{-1}(\xi) T'(\xi) \sigma_-] \frac{d\xi}{2\pi i}$$

$$+ \frac{1}{2t^2} \int_{\Sigma_5} e^{-i\rho h(\xi) - E} G(\xi; s) \text{Tr}[T^{-1}(\xi) T'(\xi) \sigma_+] \frac{d\xi}{2\pi i},$$
and

\[
\frac{-1}{2\sqrt{1-t}} \int_{\Sigma_5} e^{is^\rho h(\zeta)} \mathcal{G}(\zeta; s)^{-1} \text{Tr}[\hat{T}^{-1}(\zeta) \hat{T}'(\zeta)\sigma_-] \frac{d\zeta}{2\pi i} \\
= - \frac{1}{2t} \int_{\Sigma_5} (\log(e^{-is^\rho h(\zeta)} \mathcal{G}(\zeta; s)))' \frac{d\zeta}{2\pi i} \\
- \frac{1}{2t} \int_{\Sigma_5} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_3] \frac{d\zeta}{2\pi i} - \frac{1}{2t^2\sqrt{1-t}} \int_{\Sigma_6} e^{s^\rho(b(\zeta)-\ell)} \tilde{\mathcal{G}}(\zeta; s)^{-1} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_-] \frac{d\zeta}{2\pi i} \\
+ \frac{\sqrt{1-t}}{2t^2} \int_{\Sigma_7} e^{-s^\rho(b(\zeta)-\ell)} \tilde{\mathcal{G}}(\zeta; s) \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_+] \frac{d\zeta}{2\pi i}.
\]

Thus, using again (4-14) to rewrite the integrals over \( \Sigma_j, j = 1, 2, 3, 4 \), in terms of \( T \), and collecting the above computations, we rewrite (6-8)–(6-9) as

\[
I_\gamma = \frac{1}{2\sqrt{1-t}} \int_{\Sigma_1 \cup \Sigma_2} e^{-s^\rho(b(\zeta)-\ell)} \tilde{\mathcal{G}}(\zeta; s) \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_+] \frac{d\zeta}{2\pi i}, \\
- \frac{1}{2t} \int_{\Sigma_1} (\log(e^{-is^\rho h(\zeta)} \mathcal{G}(\zeta; s)))' \frac{d\zeta}{2\pi i} \\
- \frac{1}{2t} \int_{\Sigma_1} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_3] \frac{d\zeta}{2\pi i} - \frac{\sqrt{1-t}}{2t^2} \int_{\Sigma_6} e^{s^\rho(b(\zeta)-\ell)} \tilde{\mathcal{G}}(\zeta; s)^{-1} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_-] \frac{d\zeta}{2\pi i} \\
+ \frac{\sqrt{1-t}}{2t^2} \int_{\Sigma_7} e^{-s^\rho(b(\zeta)-\ell)} \tilde{\mathcal{G}}(\zeta; s) \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_+] \frac{d\zeta}{2\pi i}, \quad (6-10)
\]

\[
I_\bar{\gamma} = \frac{-1}{2\sqrt{1-t}} \int_{\Sigma_1 \cup \Sigma_4} e^{s^\rho(b(\zeta)-\ell)} \tilde{\mathcal{G}}(\zeta; s)^{-1} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_-] \frac{d\zeta}{2\pi i}, \\
- \frac{1}{2t} \int_{\Sigma_1} (\log(e^{-is^\rho h(\zeta)} \mathcal{G}(\zeta; s)))' \frac{d\zeta}{2\pi i} \\
- \frac{1}{2t} \int_{\Sigma_1} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_3] \frac{d\zeta}{2\pi i} - \frac{1}{2t^2\sqrt{1-t}} \int_{\Sigma_6} e^{s^\rho(b(\zeta)-\ell)} \tilde{\mathcal{G}}(\zeta; s)^{-1} \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_-] \frac{d\zeta}{2\pi i} \\
+ \frac{\sqrt{1-t}}{2t^2} \int_{\Sigma_7} e^{-s^\rho(b(\zeta)-\ell)} \tilde{\mathcal{G}}(\zeta; s) \text{Tr}[T^{-1}(\zeta)T'(\zeta)\sigma_+] \frac{d\zeta}{2\pi i}. \quad (6-11)
\]

We note from (4-22) that \( P(\infty) \) is independent of \( s \). From (4-30) and (4-44), we also note that \( P(b_j)(\zeta), j \in \{1, 2\} \), depends on \( s \) but is bounded as \( s \to +\infty \) uniformly for \( \zeta \in \mathcal{D}_b_j \), and that \( P(b_j)^\gamma(\zeta) = \mathcal{O}(s^\rho) \) as \( s \to +\infty \) uniformly for \( \zeta \in \mathcal{D}_b_j \). Using (4-45) and (4-52), we infer that

\[
T(\zeta) = \mathcal{O}(1), \quad T'(\zeta) = \mathcal{O}(s^\rho), \quad \text{as } s \to +\infty,
\]

uniformly for \( \zeta \in \mathbb{C} \setminus \bigcup_{j=1}^7 \Sigma_j \). Since \( \text{Re}(ih(\zeta) - \ell) > 0 \) for \( \zeta \in \Sigma_1 \cup \Sigma_2 \) and \( \text{Re}(ih(\zeta) - \ell) < 0 \) for \( \zeta \in \Sigma_3 \cup \Sigma_4 \) (see Figures 5 and 6), we have

\[
I_\gamma + I_{\bar{\gamma}} = I_1 + I_2 + I_{b_2} + I_{b_1} + \mathcal{O}(e^{-cs^\rho}), \quad \text{as } s \to +\infty,
\]
where \( I_1, I_2 \) and \( I_{b_2} \) are defined in (6-3), (6-4) and (6-5), respectively, and \( I_{b_1} \) is defined in a similar way to \( I_{b_2} \). Using the symmetry \( \zeta \mapsto -\zeta \) (see in particular (4-19)), we obtain

\[ I_{b_1} = \overline{I_{b_2}}, \]

which finishes the proof.

\[ \square \]

**Lemma 6.2.**

\[ I_1 = \frac{Re b_2}{\pi \rho t} s^\rho + \frac{c_1 c_6 - c_2 c_5}{t(c_1 + c_2)} + \mathcal{O}(s^{-\rho}), \quad as \ s \to +\infty. \]

**Proof.** By the definition (6-3) of \( I_1 \), we have

\[ I_1 = \frac{s^\rho}{\pi t} \text{Im} \int_{[0, b_2]} i h'(\zeta)d\zeta - \frac{1}{\pi t} \text{Im} \int_{[0, b_2]} (\log \tilde{G}(\zeta; s))'d\zeta. \]

For the first integral, we use (4-5), (4-20), and (4-10), to obtain

\[ \text{Im} \int_{[0, b_2]} i h'(\zeta)d\zeta = \text{Im}(i h(b_2)) = \tilde{\ell} = (c_1 + c_2) \exp \left(-\frac{c_1 + c_2 + 3}{c_1 + c_2}\right) \cos \left(\frac{\pi c_2 - c_1}{2 c_1 + c_2}\right) = \frac{Re b_2}{\rho}. \]

For the second integral, we find

\[ \int_{[0, b_2]} (\log \tilde{G}(\zeta; s))'d\zeta = \log \tilde{G}(b_2; s) - \log \tilde{G}(0; s). \]

Using (4-7) and (4-6), as \( s \to +\infty \) we have

\[ \int_{[0, b_2]} (\log \tilde{G}(\zeta; s))'d\zeta = c_4 \log s + c_5 \log(ib_2) + c_6 \log(-ib_2) + c_7 - \log F(\tau) + \mathcal{O}(s^{-\rho}), \]

and thus, by (4-10), we get

\[ \text{Im} \int_{[0, b_2]} (\log \tilde{G}(\zeta; s))'d\zeta = c_5 \arg(ib_2) + c_6 \arg(-ib_2) + \mathcal{O}(s^{-\rho}). \]

We split \( I_{b_2} \) into four parts

\[ I_{b_2} = I_{b_2,1} + I_{b_2,2} + I_{b_2,3} + I_{b_2,4}, \quad (6-12) \]

where \( I_{b_2, j}, \ j = 1, 2, 3, 4, \) are given by

\[
\begin{align*}
I_{b_2,1} &= \frac{1}{2\sqrt{1 - t}} \int_{\Sigma_2 \cap \overline{D}_{b_2}} e^{-s^\rho(i\zeta)} G(\zeta; s) Tr[T^{-1}(\zeta) T' (\zeta) \sigma_+] \frac{d\zeta}{2\pi i}, \\
I_{b_2,2} &= \frac{2 - t}{2t^2\sqrt{1 - t}} \int_{\Sigma_1 \cap \overline{D}_{b_2}} e^{-s^\rho(i\zeta)} G(\zeta; s) Tr[T^{-1}(\zeta) T' (\zeta) \sigma_+] \frac{d\zeta}{2\pi i}, \\
I_{b_2,3} &= \frac{1}{2\sqrt{1 - t}} \int_{\Sigma_3 \cap \overline{D}_{b_2}} e^{s^\rho(i\zeta)} G(\zeta; s)^{-1} Tr[T^{-1}(\zeta) T' (\zeta) \sigma_-] \frac{d\zeta}{2\pi i}, \\
I_{b_2,4} &= \frac{(2 - t)}{2t^2\sqrt{1 - t}} \int_{\Sigma_4 \cap \overline{D}_{b_2}} e^{s^\rho(i\zeta)} G(\zeta; s)^{-1} Tr[T^{-1}(\zeta) T' (\zeta) \sigma_-] \frac{d\zeta}{2\pi i}.
\end{align*}
\]
Lemma 6.3. As \( s \to +\infty \), we have

\[
I_{b_2,1} = \frac{1}{2\sqrt{1-t}} \int_{e^{\pi i/4}[0,+]^{+\infty}} \text{Tr}[\Phi^{-1}_{PC}(z)\Phi'_{PC}(z)\sigma_+ \frac{dz}{2\pi i}] + O(s^{-\frac{1}{2}}),
\]

\[
I_{b_2,2} = -\frac{2-t}{2t^2\sqrt{1-t}} \int_{e^{\pi i/4}(+,0]} \text{Tr}[\Phi^{-1}_{PC}(z)\Phi'_{PC}(z)\sigma_+ \frac{dz}{2\pi i}] + O(s^{-\frac{1}{2}}),
\]

\[
I_{b_2,3} = -\frac{1}{2\sqrt{1-t}} \int_{e^{\pi i/4}[0,+\infty)} \text{Tr}[\Phi^{-1}_{PC}(z)\Phi'_{PC}(z)\sigma_- \frac{dz}{2\pi i}] + O(s^{-\frac{1}{2}}),
\]

\[
I_{b_2,4} = -\frac{(2-t)}{2t^2\sqrt{1-t}} \int_{e^{\pi i/4}(+,0]} \text{Tr}[\Phi^{-1}_{PC}(z)\Phi'_{PC}(z)\sigma_- \frac{dz}{2\pi i}] + O(s^{-\frac{1}{2}}).
\]

Proof. From (4.27) and (4.28), we have

\[
s^\rho(ih(\zeta) - \ell) = s^\rho \left( i\tilde{b} - i\frac{f'(\zeta)}{2} \right) = i\tilde{b}s^\rho - \frac{i\tilde{s}^\rho}{2} f'(b_2)^2 (\zeta - b_2)^2 (1 + O(\zeta - b_2)), \quad \text{as} \quad \zeta \to b_2,
\]

so the main contribution as \( s \to +\infty \) in the integrals for \( I_{b_2,j} \), \( j = 1, \ldots, 4 \) comes from the integrand as

\[
s^\rho(\zeta - b_2) = O(1).
\]

We first obtain an expansion for \( \text{Tr}[T^{-1}T'\sigma_\pm] \) as \( \zeta \longrightarrow b_2 \) and simultaneously \( s \to +\infty \). For \( \zeta \) inside the disk \( D_{b_2} \), by (4.45) we have

\[
T(\zeta) = R(\zeta) P^{(b_2)}(\zeta).
\]

Thus for \( \zeta \in D_{b_2} \), we have

\[
\text{Tr}[T^{-1}T'\sigma_\pm] = \text{Tr}[(P^{(b_2)})^{-1}(P^{(b_2)})'\sigma_\pm] + \text{Tr}[(P^{(b_2)})^{-1}R^{-1}R'P^{(b_2)}\sigma_\pm].
\]

We recall from (4.30) that \( P^{(b_2)} \) is given by

\[
P^{(b_2)}(\zeta) = E(\zeta; s)\Phi_{PC}(s^\frac{1}{2}f(\zeta); t) e^{\frac{i\rho}{\pi}(ih(\zeta) - \ell)\sigma_3} \tilde{G}(\zeta; s)^{-\sigma_3}, \quad \zeta \in D_{b_2} \setminus \bigcup_{j=1}^7 \Sigma_j,
\]

and thus

\[
\text{Tr}[(P^{(b_2)})^{-1}(P^{(b_2)})'\sigma_\pm] = e^{+i\rho(\zeta - b_2)^2} \tilde{G}(\zeta; s)^{\pm1} \left( s^\frac{i}{2} f' Tr[\Phi_{PC}^{-1}\Phi'_{PC}\sigma_\pm] + Tr[\Phi_{PC}^{-1}E^{-1}E'\Phi_{PC}\sigma_\pm] \right),
\]

\[
\text{Tr}[(P^{(b_2)})^{-1}R^{-1}R'P^{(b_2)}\sigma_\pm] = e^{+i\rho(\zeta - b_2)^2} \tilde{G}(\zeta; s)^{\pm1} \text{Tr}[\Phi_{PC}^{-1}E^{-1}R^{-1}R'E\Phi_{PC}\sigma_\pm],
\]

where \( \Phi_{PC} \) and \( \Phi'_{PC} \) are evaluated at \( s^\frac{1}{2} f(\zeta) \) and the other functions are evaluated at \( \zeta \). We also recall from (4.31) that

\[
E(\zeta; s) = E^{(\infty)}(\zeta) \tilde{G}(\zeta; s)^{\sigma_3} e^{-\frac{i\rho}{\pi}i\tilde{b}\sigma_3(\zeta^\frac{i}{2} f(\zeta))^{iv\sigma_3}}
\]

is analytic for \( \zeta \in D_{b_2} \), and thus

\[
E^{\pm1}(\zeta; s) = O(1), \quad E'(\zeta; s) = O(1),
\]

(6.17)
as \( s \to +\infty \) uniformly for \( \zeta \in \mathcal{D}_{b_2} \). By (6-13), (6-15), and (6-16), we have
\[
I_{b_2,1} = \frac{1}{2\sqrt{1-t}} \int_{\Sigma_2 \cap \mathcal{D}_{b_2}} \left( s^{\frac{d}{2}} f(\Phi_{PC}^{-1} \Phi_{PC}' s) + \text{Tr}[\Phi_{PC}^{-1} E^{-1}(\Phi_{PC}' s)] \right) \frac{d\zeta}{2\pi i} \\
+ \frac{1}{2\sqrt{1-t}} \int_{\Sigma_2 \cap \mathcal{D}_{b_2}} \text{Tr}[\Phi_{PC}^{-1} E^{-1} R^{-1} R' E \Phi_{PC}' s] \frac{d\zeta}{2\pi i}.
\]
Let us now perform the change of variables
\[
z = s^{\frac{d}{2}} f'(\zeta),
\]
where we recall that \( f \) is injective on \( \mathcal{D}_{b_2} \). Then we have
\[
e^{-s^{\rho}(i\rho(\zeta) - t)} = e^{-is^{\rho}f(\zeta)} = f^{-1}(s^{-\frac{d}{2}} z), \quad dz = s^{\frac{d}{2}} f'(\zeta) d\zeta.
\]
Since \( \Sigma_2 \cap \mathcal{D}_{b_2} \) is mapped by \( f \) to a subset of \( e^{\frac{d}{2}} \mathbb{R} [0, +\infty) \), see (4-29), this change of variables allows us to rewrite \( I_{b_2,1} \) as
\[
I_{b_2,1} = \frac{1}{2\sqrt{1-t}} \int_{e^{\frac{d}{2}} \mathbb{R} [0, s^{\frac{d}{2}}]} \text{Tr}[\Phi_{PC}^{-1}(z) \Phi_{PC}'(z) \sigma_+] \frac{dz}{2\pi i} \\
+ \frac{1}{2\sqrt{1-t}} \int_{e^{\frac{d}{2}} \mathbb{R} [0, s^{\frac{d}{2}}]} \frac{\text{Tr}[\Phi_{PC}^{-1}(z) E^{-1}(\zeta; s) E'(\zeta; s) \Phi_{PC}'(z) \sigma_+] dz}{s^{\frac{d}{2}} f'(\zeta)} \\
+ \frac{1}{2\sqrt{1-t}} \int_{e^{\frac{d}{2}} \mathbb{R} [0, s^{\frac{d}{2}}]} \frac{\text{Tr}[\Phi_{PC}^{-1}(z) E^{-1}(\zeta; s) R^{-1}(\zeta) R'(\zeta) E(\zeta; s) \Phi_{PC}'(z) \sigma_+] dz}{s^{\frac{d}{2}} f'(\zeta)}
\]
where \( r := |f(r_*)| \) with \( r_* \) defined by \( \mathcal{D}_{b_2} \cap \Sigma_2 = \{r_*\} \). We note from (A-1) that
\[
\Phi_{PC}(z) \sigma_+ \Phi_{PC}'(z) = \mathcal{O}(e^{\frac{d}{2}}) \quad \text{as} \quad z \to \infty,
\]
and we conclude from (4-52) and (6-17) that
\[
\frac{1}{2\sqrt{1-t}} \int_{e^{\frac{d}{2}} \mathbb{R} [0, s^{\frac{d}{2}}]} \text{Tr}[\Phi_{PC}^{-1}(z) E^{-1}(\zeta; s) E'(\zeta; s) \Phi_{PC}'(z) \sigma_+] \frac{dz}{2\pi i} = O(s^{-\frac{d}{2}}),
\]
\[
\frac{1}{2\sqrt{1-t}} \int_{e^{\frac{d}{2}} \mathbb{R} [0, s^{\frac{d}{2}}]} \text{Tr}[\Phi_{PC}^{-1}(z) E^{-1}(\zeta; s) R^{-1}(\zeta) R'(\zeta) E(\zeta; s) \Phi_{PC}'(z) \sigma_+] \frac{dz}{2\pi i} = O(s^{-\rho}),
\]
\[
\frac{1}{2\sqrt{1-t}} \int_{e^{\frac{d}{2}} \mathbb{R} [0, s^{\frac{d}{2}}]} \text{Tr}[\Phi_{PC}^{-1}(z) \Phi_{PC}'(z) \sigma_+] \frac{dz}{2\pi i} = \frac{1}{2\sqrt{1-t}} \int_{e^{\frac{d}{2}} \mathbb{R} [0, +\infty)} \text{Tr}[\Phi_{PC}^{-1}(z) \Phi_{PC}'(z) \sigma_+] \frac{dz}{2\pi i} + O(e^{-cs^\rho})
\]
as \( s \to +\infty \), for a certain \( c > 0 \). This finishes the proof for \( I_{b_2,1} \). The proofs of the expressions for the other integrals are similar.

**Lemma 6.4.** We have
\[
I_{b_2} = I_{b_2} + O(s^{-\frac{d}{2}}), \quad \text{as} \quad s \to +\infty,
\]
where \( I_{b_2} \) depends on \( t \) but is independent of the other parameters. More precisely, for \( j = 1 \) (the Meijer-G process), \( I_{b_2} \) is independent of \( r, q, v_1, \ldots, v_r, \mu_1, \ldots, \mu_q \), and for \( j = 2 \) (Wright’s generalized Bessel process), \( I_{b_2} \) is independent of \( \alpha \) and \( \theta \).
Proof. This follows from (6-12), Lemma 6.3, and the fact that \( \Phi_{PC} \) only depends on \( q = \sqrt{1-t} \). \( \square \)

Let \( b_* := \Sigma_5 \cap \partial \mathcal{D}_{b_2} \). We split \( I_2 \) into two parts:

\[
I_2 = -\frac{2}{t} \Re \left[ \int_{[0,b_2]} \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_3] \frac{d\xi}{2\pi i} \right] = I_{2,1} + I_{2,2},
\]

\[
I_{2,1} = -\frac{2}{t} \Re \left[ \int_{[0,b_*]} \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_3] \frac{d\xi}{2\pi i} \right],
\]

\[
I_{2,2} = -\frac{2}{t} \Re \left[ \int_{\Sigma_5 \cap \mathcal{D}_{b_2}} \text{Tr}[T^{-1}(\xi)T'(\xi)\sigma_3] \frac{d\xi}{2\pi i} \right].
\]

**Lemma 6.5.**

\[
I_{2,1} = \frac{2v}{\pi t} \log \left| \frac{b_* - b_2}{b_* - b_1} \right| + \mathcal{O}(s^{-\frac{\varepsilon}{2}}), \quad \text{as } s \to +\infty.
\]

**Proof.** For \( \xi \in [0, b_*] \subset \mathbb{C} \setminus \mathcal{D}_{b_2} \), by (4-45) we have \( T(\xi) = R(\xi) P^{(\infty)}(\xi) \), and thus

\[
\text{Tr}[T^{-1}T'\sigma_3] = \text{Tr}[(P^{(\infty)})^{-1}(P^{(\infty)})'\sigma_3] + \text{Tr}[(P^{(\infty)})^{-1}R^{-1}R'P^{(\infty)}\sigma_3]
\]

\[
= \text{Tr}[(P^{(\infty)})^{-1}(P^{(\infty)})'\sigma_3] + \mathcal{O}(s^{-\frac{\varepsilon}{2}}), \quad \text{as } s \to +\infty,
\]

uniformly for \( \xi \in [0, b_*] \), where we have used (4-52). We recall that \( P^{(\infty)} \) is given by

\[
P^{(\infty)}(\xi) = D(\xi)^{-\sigma_3}, \quad \text{where } D(\xi) = \exp \left( iv \int \frac{d\xi}{\Sigma_5} \right) = \exp \left( iv \log \left[ \frac{\xi - b_2}{\xi - b_1} \right] \right)
\]

and where the branch of the logarithm is taken along \( \Sigma_5 \). Thus

\[
\text{Tr}[(P^{(\infty)})^{-1}(P^{(\infty)})'\sigma_3] = -2(\log D)',
\]

and we find as \( s \to +\infty \),

\[
I_{2,1} = -\frac{2}{t} \Re \left[ \int_{[0,b_*]} \text{Tr}[(P^{(\infty)})^{-1}(P^{(\infty)})'\sigma_3] \frac{d\xi}{2\pi i} \right] + \mathcal{O}(s^{-\frac{\varepsilon}{2}}),
\]

\[
= \frac{4}{t} \Re \left[ \int_{[0,b_*]} (\log D)'/\frac{d\xi}{2\pi i} \right] + \mathcal{O}(s^{-\frac{\varepsilon}{2}}) = \frac{2v}{\pi t} \log \left| \frac{b_* - b_2}{b_* - b_1} \right| + \mathcal{O}(s^{-\frac{\varepsilon}{2}}). \quad \square
\]

**Lemma 6.6.** As \( s \to +\infty \), we have

\[
I_{2,2} = I^{(1)}_{2,2} + I^{(2)}_{2,2} + I^{(3)}_{2,2} + \mathcal{O}(s^{-\frac{\varepsilon}{2}}),
\]

\[
I^{(1)}_{2,2} = -\frac{2}{t} \Re \left[ \int_{\Sigma_5 \cap \mathcal{D}_{b_2}} (s^0 \Im h'(.\xi) - \log \tilde{\gamma})'(\xi; t) \frac{d\xi}{2\pi i} \right],
\]

\[
I^{(2)}_{2,2} = -\frac{2}{t} \Re \left[ \int_{\Sigma_5 \cap \mathcal{D}_{b_2}} f'(\xi) \text{Tr}\left[ \Phi_{PC}^{-1}(s^0 f_{b_2}(\xi)) \Phi_{PC}(s^0 f_{b_2}(\xi)) \sigma_3 \right] \frac{d\xi}{2\pi i} \right],
\]

\[
I^{(3)}_{2,2} = -\frac{2}{t} \Re \left[ \int_{\Sigma_5 \cap \mathcal{D}_{b_2}} \text{Tr}\left[ \Phi_{PC}^{-1}(s^0 f_{b_2}(\xi)) E^{-1}(\xi; s) E'(\xi; s) \Phi_{PC}(s^0 f_{b_2}(\xi)) \sigma_3 \right] \frac{d\xi}{2\pi i} \right].
\]
Proof. For $\xi \in \Sigma_5 \cap D_{b_2}$, by (4-45) we have $T(\xi) = R(\xi) P^{(b_2)}(\xi)$, and thus

$$\text{Tr}[T^{-1} T' \sigma_3] = \text{Tr}[(P^{(b_2)})^{-1} (P^{(b_2)})' \sigma_3] + \text{Tr}[(P^{(b_2)})^{-1} R^{-1} R' P^{(b_2)} \sigma_3].$$

We recall that $P^{(b_2)}$ is given by

$$P^{(b_2)}(\xi) = E(\xi; s) \Phi_{PC}(z) e^{i\theta (i h(\xi) - \ell_t) \sigma_3} \tilde{G}(\xi; s)^{-\eta_2} \text{ with } z = s^2 f(\xi),$$

and thus

$$\text{Tr}[(P^{(b_2)})^{-1} (P^{(b_2)})' \sigma_3] = (s^2 i h' - (\log \tilde{G}))' + s^2 f' \text{Tr}[\Phi_{PC}^{-1} \Phi_{PC}' \sigma_3] + \text{Tr}[\Phi_{PC}^{-1} E^{-1} E' \Phi_{PC} \sigma_3].$$

$$\text{Tr}[(P^{(b_2)})^{-1} R^{-1} R' P^{(b_2)} \sigma_3] = \text{Tr}[\Phi_{PC}^{-1} E^{-1} R^{-1} E' \Phi_{PC} \sigma_3],$$

where $\Phi_{PC}$ and $\Phi_{PC}'$ are evaluated at $z = s^2 f_{b_2}(\xi)$ and the other functions are evaluated at $\xi$. Thus

$$\int_{\Sigma_5 \cap D_{b_2}} \text{Tr}[T^{-1} T' \sigma_3] \frac{d\xi}{2\pi i}$$

$$= \int_{\Sigma_5 \cap D_{b_2}} (s^2 i h' - (\log \tilde{G}))' \frac{d\xi}{2\pi i} + s^2 f' \text{Tr}[\Phi_{PC}^{-1} \Phi_{PC}' \sigma_3] \frac{d\xi}{2\pi i} + \int_{\Sigma_5 \cap D_{b_2}} \text{Tr}[\Phi_{PC}^{-1} E^{-1} E' \Phi_{PC} \sigma_3] \frac{d\xi}{2\pi i}.$$

From the Appendix, we know that

$$\Phi_{PC}(z) \sigma_3 \Phi_{PC}(z)^{-1} = O(1), \quad (6-19)$$

uniformly for $z \in C$. Therefore, by the cyclic property of the trace, and using also the estimates (4-52) and (6-17), we conclude that

$$\int_{\Sigma_5 \cap D_{b_2}} \text{Tr}[\Phi_{PC}^{-1} E^{-1} R^{-1} E' \Phi_{PC} \sigma_3] \frac{d\xi}{2\pi i} = O(s^{-\frac{\rho}{2}}), \quad \text{as } s \to +\infty. \quad \square$$

Lemma 6.7. As $s \to +\infty$, we have

$$I_{2,2}^{(1)} = \frac{\text{Im}(i h(b_2)) - \text{Im}(i h(b_*))}{\pi t} s^\rho + \frac{c_5 + c_6}{\pi t} \arg \frac{b_2}{b_*} + O(s^{-\rho}).$$

Proof. By definition of $I_{2,2}^{(1)},$ we have

$$I_{2,2}^{(1)} = \frac{s^\rho}{\pi t} \text{Im} \int_{\Sigma_5 \cap D_{b_2}} i h'(\xi)d\xi + \frac{1}{\pi t} \text{Im} \int_{\Sigma_5 \cap D_{b_2}} (\log \tilde{G}(\xi; s))'d\xi,$$

and

$$\text{Im} \int_{\Sigma_5 \cap D_{b_2}} i h'(\xi)d\xi = \text{Im}(i h(b_2)) - \text{Im}(i h(b_*)),$$

$$\int_{\Sigma_5 \cap D_{b_2}} (\log \tilde{G}(\xi; s))'d\xi = \log \tilde{G}(b_2; s) - \log \tilde{G}(b_*; 0). \quad (6-20)$$
The right-hand side of (6-20) can be expanded as \( s \to +\infty \) using (4-7), and we find
\[
\int_{\Sigma_{3} \cap \mathcal{D}_{b_{2}}} (\log \tilde{G}(\zeta; s))' d\zeta = (c_{5} + c_{6}) \log \left( \frac{b_{2}}{b_{*}} \right) + O(s^{-\rho}),
\]
and the result follows.

**Lemma 6.8.** Let \( m \in \mathbb{C} \setminus \mathbb{R}^{-} \). As \( s \to +\infty \), we have
\[
I_{2,2}^{(2)} = -\frac{s^{\rho}}{\pi t} [\text{Im}(ih(b_{*})) - \text{Im}(ih(b_{2}))] - \frac{v_{\rho}}{\pi t} \log s - 2\nu \log r + 2\nu \log |m| + T_{2,2}^{(2)}(m) + O(s^{-\frac{\rho}{2}}),
\]
where \( r = |f(b_{*})| = -f(b_{*}) \) and
\[
T_{2,2}^{(2)}(m) = -\frac{2}{t} \text{Re} \left[ \int_{(-\infty,0]} \left( \text{Tr}[\Phi_{PC}^{-1}(z)\Phi_{PC}'(z)\sigma_{3}] - \left( iz - \frac{2i\nu}{z-m} \right) \right) \frac{dz}{2\pi i} \right].
\]

**Proof.** Using the change of variables \( z = s^{\frac{\rho}{2}} f_{b_{2}}(\xi) \) and denoting \( r = |f_{b_{2}}(b_{*})| = -f_{b_{2}}(b_{*}) \), we rewrite \( I_{2,2}^{(2)} \) as
\[
I_{2,2}^{(2)} = -\frac{2}{t} \text{Re} \left[ \int_{[-\rho/2,0]} \text{Tr}[\Phi_{PC}^{-1}(z)\Phi_{PC}'(z)\sigma_{3}] \frac{dz}{2\pi i} \right].
\]
From the expansion (A-1), we get
\[
\text{Tr}[\Phi_{PC}^{-1}(z)\Phi_{PC}'(z)\sigma_{3}] = iz - \frac{2i\nu}{z} + O(z^{-2}), \quad \text{as } z \to -\infty.
\]
Let \( m \in \mathbb{C} \setminus \mathbb{R}^{-} \). We have
\[
\int_{[-\rho/2,0]} \text{Tr}[\Phi_{PC}^{-1}(z)\Phi_{PC}'(z)\sigma_{3}] \frac{dz}{2\pi i} = \int_{[-\rho/2,0]} \left( iz - \frac{2i\nu}{z-m} \right) \frac{dz}{2\pi i} + \int_{[-\rho/2,0]} \left( \text{Tr}[\Phi_{PC}^{-1}(z)\Phi_{PC}'(z)\sigma_{3}] - \left( iz - \frac{2i\nu}{z-m} \right) \right) \frac{dz}{2\pi i}.
\]
Since
\[
\text{Tr}[\Phi_{PC}^{-1}(z)\Phi_{PC}'(z)\sigma_{3}] - \left( iz - \frac{2i\nu}{z-m} \right) = O(z^{-2}), \quad \text{as } z \to -\infty,
\]
we have
\[
\int_{[-\rho/2,0]} \left( \text{Tr}[\Phi_{PC}^{-1}(z)\Phi_{PC}'(z)\sigma_{3}] - \left( iz - \frac{2i\nu}{z-m} \right) \right) \frac{dz}{2\pi i} + O(s^{-\frac{\rho}{2}}), \quad \text{as } s \to +\infty.
\]
On the other hand, the first integral on the right-hand side of (6-22) can be easily expanded as follows:

\[ t \int_{[-s^2,0]} \left( i z - \frac{2i\nu}{z - m} \right) \frac{dz}{2\pi i} = \frac{s^2 r^2}{4\pi} + \frac{\nu}{\pi} \log \left( \frac{s^2 r}{m} + 1 \right) \]

\[ = -\frac{s^2 r^2}{4\pi} + \frac{\nu r}{2\pi} \log s + \frac{\nu}{\pi} \log \frac{r}{m} + O(s^{-\tilde{e}}), \quad \text{as } s \to +\infty. \]

Therefore, as \( s \to +\infty \) we have

\[ I_{2,2} = -\frac{2}{t} \Re \left[ -\frac{s^2 r^2}{4\pi} + \frac{\nu r}{2\pi} \log s + \frac{\nu}{\pi} \log \frac{r}{m} \right] \]

\[ = -\frac{2}{t} \Re \left[ \int_{(0,\infty)} (\text{Tr}[\Phi_{\text{PC}}^{-1}(z)\Phi_{\text{PC}}'(z)\sigma_3]) \left( i z - \frac{2i\nu}{z - m} \right) \frac{dz}{2\pi i} \right] + O(s^{-\tilde{e}}), \]

and the claim follows by noticing that

\[ r^2 = f(b_2^2) = -2(h(b_2) - h(b_2)) = -2(\text{Im}(ih(b_2)) - \text{Im}(ih(b_2))). \]

**Lemma 6.9.** We have

\[ I_{2,2}^{(3)} = \mathcal{O}(1), \quad \text{as } s \to +\infty, \]

\[ I_{2,2}^{(3)} = \mathcal{O}(b_2 - b_2), \quad \text{as } b_2 \to b_2. \]

**Proof.** This follows from the previous estimates (6-17) and (6-19), and the cyclic property of the trace. \( \square \)

**Lemma 6.10.** As \( s \to +\infty \), we have

\[ I_2 = -\frac{\nu}{\pi t} \log s - \frac{2\nu}{\pi t} \log |(b_2 - b_1)f'(b_2)| + I_{2,2}^{(2)}(1) + O(s^{-\tilde{e}}). \]

**Proof.** By combining Lemmas 6.5, 6.6, 6.7, 6.8 and 6.9, as \( s \to +\infty \) we have

\[ I_2 = I_{2,1} + I_{2,2} = I_{2,1} + I_{2,2}^{(1)} + I_{2,2}^{(2)} + I_{2,2}^{(3)} + O(s^{-\tilde{e}}) \]

\[ = \frac{2\nu}{\pi t} \log \left| \frac{b_2 - b_1}{b_2 - b_1} \right| - \frac{\nu r}{\pi t} \log s + \frac{2\nu}{\pi t} \log |f'(b_2)| + \frac{2\nu}{\pi t} \log |m| + I_{2,2}^{(2)}(m) + I_{2,2}^{(3)} + O(s^{-\tilde{e}}) \]

\[ = -\frac{\nu}{\pi t} \log s + \frac{2\nu}{\pi t} \log \left| \frac{b_2 - b_1}{f(b_2)} \right| + \frac{c_5 + c_6}{\pi t} \log \left( \frac{b_2}{b_2} \right) + \frac{2\nu}{\pi t} \log |m| + I_{2,2}^{(2)}(m) + I_{2,2}^{(3)} + O(s^{-\tilde{e}}). \] (6-23)

The term of order \( \mathcal{O}(1) \) as \( s \to +\infty \) in this expansion is given by

\[ \frac{2\nu}{\pi t} \log \left| \frac{b_2 - b_1}{(b_2 - b_1)f(b_2)} \right| + \frac{c_5 + c_6}{\pi t} \log \left( \frac{b_2}{b_2} \right) + \frac{2\nu}{\pi t} \log |m| + I_{2,2}^{(2)}(m) + I_{2,2}^{(3)}(b_2). \] (6-24)

We simplify this term by noticing that the disks can be chosen arbitrarily small (though independent of \( s \)).
Therefore it is possible to evaluate (6-24) simply by taking the limit \( b_\ast \to b_2 \). As \( b_\ast \to b_2 \), we have
\[
\frac{b_\ast - b_2}{(b_\ast - b_1) f'(b_\ast)} = \frac{1}{(b_2 - b_1) f'(b_2)} + \mathcal{O}(b_\ast - b_2), \quad \arg\left(\frac{b_2}{b_\ast}\right) = \mathcal{O}(b_\ast - b_2), \quad I^{(3)}_{2,2} = \mathcal{O}(b_\ast - b_2),
\]
where we have used Lemma 6.9. Therefore, taking the limit \( b_\ast \to b_2 \) in (6-24) and then substituting in (6-23), we obtain
\[
I_2 = -\frac{\nu \rho}{\pi t} \log s - \frac{2\nu}{\pi t} \log |(b_2 - b_1) f'(b_2)| + \frac{2\nu}{\pi t} \log |m| + \mathcal{I}^{(2)}_{2,2}(m) + \mathcal{O}(s^{-\frac{\nu}{2}}), \quad \text{as } s \to +\infty. \tag{6-25}
\]
We have the freedom to choose \( m \in \mathbb{C} \setminus \mathbb{R}^- \). The claim follows after setting \( m = 1 \) in (6-25). \(\Box\)

**Lemma 6.11.** For \( j = 1, 2 \), we have
\[
\partial_t \log \det (1 - (1 - t)\mathbb{K}^{(j)}_{[0,s]}(t)) = \frac{\Re b_2}{\pi \rho t} s^\rho - \frac{\nu \rho}{\pi t} \log s + \frac{c_1 c_6 - c_2 c_5}{t(c_1 + c_2)} - \frac{2\nu}{\pi t} \log |(b_2 - b_1) f'(b_2)| + \partial_t [\log \mathbb{G}(1 + i\nu) \mathbb{G}(1 - i\nu)] + \mathcal{O}(s^{-\frac{\nu}{2}}), \quad \text{as } s \to +\infty. \tag{6-26}
\]
where \( G \) is Barnes’ \( G \)-function.

**Proof.** It follows from Lemmas 6.2, 6.4 and 6.10 that
\[
\partial_t \log \det (1 - (1 - t)\mathbb{K}^{(j)}_{[0,s]}(t)) = I_1 + I_2 + 2 \Re I_{b_2} + \mathcal{O}(e^{-\rho})
\]
\[
= \frac{\Re b_2}{\pi \rho t} s^\rho - \frac{\nu \rho}{\pi t} \log s + \frac{c_1 c_6 - c_2 c_5}{t(c_1 + c_2)} - \frac{2\nu}{\pi t} \log |(b_2 - b_1) f'(b_2)| + \partial_t [\log \mathbb{G}(1 + i\nu) \mathbb{G}(1 - i\nu)] + \mathcal{O}(s^{-\frac{\nu}{2}}) \tag{6-27}
\]
where
\[
\chi(t) := 2 \Re \mathcal{I}_{b_2} + \mathcal{I}^{(2)}_{2,2}(1).
\]
It is rather difficult to obtain an explicit expression for \( \chi(t) \) from a direct analysis. However, it follows from Lemmas 6.4 and 6.10 that \( \chi(t) \) depends on \( t \) but is independent of the other parameters. More precisely, for \( j = 1 \), \( \chi(t) \) is independent of \( r, q, v_1, \ldots, v_\tau, \mu_1, \ldots, \mu_q \), and for \( j = 2 \), \( \chi(t) \) is independent of \( \alpha \) and \( \theta \). We will take advantage of that by using the known result from [Bothner et al. 2019] for the Bessel point process given by (1-7). If \( j = 1 \), then we set \( r = 1, \ q = 0 \) and \( v_1 = 0 \), and if \( j = 2 \), we set \( \theta = 1 \) and \( \alpha = 0 \). In these cases, \( \Re b_2 = 1, \ \rho = \frac{1}{2}, \ c_1 = c_2 = 1, \ c_5 = c_6 = 0, \ f'(b_2) = \sqrt{2} \) and (6-27) becomes
\[
\partial_t \log \det (1 - (1 - t)\mathbb{K}^{(j)}_{[0,s]}(t)) = \frac{2}{\pi t} \sqrt{s} - \frac{\nu}{\pi t} \log s - \frac{\nu}{\pi t} \log 8 + \chi(t) + \mathcal{O}(s^{-\frac{1}{2}}). \tag{6-28}
\]
On the other hand, the asymptotics (1-7) can be differentiated with respect to \( t \) (this follows from the analysis done in [Bothner et al. 2019] and [Charlier 2020]), and we get as \( s \to +\infty \),
\[
\partial_t \log \det (1 - (1 - t)\mathbb{K}_{\text{Be}}_{[0,s]}(t))
\]
\[
= \partial_t \left(-4\nu \sqrt{s} + \nu^2 \log(8\sqrt{s}) + \log(G(1 + i\nu)G(1 - i\nu)) + \mathcal{O}\left(\frac{1}{\sqrt{s}}\right)\right)
\]
\[
= \frac{2}{\pi t} \sqrt{s} - \frac{\nu}{2\pi t} \log s - \frac{\nu}{\pi t} \log 8 + \partial_t \left(\log(G(1 + i\nu)G(1 - i\nu)) + \mathcal{O}\left(\frac{1}{\sqrt{s}}\right)\right). \tag{6-29}
\]
By (1-11) and (1-23), the left-hand sides of (6-28) and (6-29) are equal, and this yields the relation
\[ \chi(t) = \partial_t (\log(G(1+i\nu)G(1-i\nu))). \]
\[ \square \]

**Lemma 6.12.** As \( s \to +\infty \), we have
\[
\log \det(1 - (1-t)\kappa^{(j)})|_{[0,s]} = -2v \Re b_2 \rho s^\rho + v^2 \rho \log s - 2\pi v \frac{c_1c_6 - c_2c_5}{c_1 + c_2} + 2v^2 \log|b_2 - b_1|f'(b_2) + \log(G(1+i\nu)G(1-i\nu)) + O(s^{-\tau}).
\]

**Proof.** It suffices to integrate (6-26) in \( t \).
\[ \square \]

Thus the constants \( C = C^{(j)}, j = 1, 2 \) of Theorems 1.8 and 1.16 are given by
\[
\log C = -2\pi \frac{c_1c_6 - c_2c_5}{c_1 + c_2} + 2v^2 \log|b_2 - b_1|f'(b_2) + \log(G(1+i\nu)G(1-i\nu)).
\]

This expression can be computed more explicitly by substituting the values for the constants \( c_1, c_2, c_5, c_6 \) given at the beginning of Section 4B, and the values (4-10) and (4-28) for \( b_2, b_1 \), and \( f'(b_2) \).

**Appendix: Parabolic cylinder model RH problem**

Let \( q \in \mathbb{T} = [0, 1) \cup i[0, +\infty) \) and let
\[
\nu := -\frac{1}{2\pi} \log(1 - q^2) \in \mathbb{R}.
\]

Consider the following model RH problem.

**RH problem for \( \Phi_{PC} \):**
(a) \( \Phi_{PC} : \mathbb{C} \setminus \Sigma_{PC} \to \mathbb{C}^{2x2} \) is analytic, where
\[
\Sigma_{PC} = \mathbb{R}^- \cup \bigcup_{j=0}^{3} e^{\frac{2j}{4} + j\frac{2}{4}} \mathbb{R}^+,
\]
as shown in Figure 8.

(b) With the contour \( \Sigma_{PC} \) oriented as in Figure 8, \( \Phi_{PC} \) satisfies the jumps
\[
\Phi_{PC,+}(z) = \Phi_{PC,-}(z) \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}, \quad z \in e^{\frac{2j}{4}} \mathbb{R}^+,
\]
\[
\Phi_{PC,+}(z) = \Phi_{PC,-}(z) \begin{pmatrix} 1 & 0 \\ -\frac{q}{1-q^2} & 1 \end{pmatrix}, \quad z \in e^{-\frac{2j}{4}} \mathbb{R}^+,
\]
\[
\Phi_{PC,+}(z) = \Phi_{PC,-}(z) \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, \quad z \in e^{\frac{3j}{4}} \mathbb{R}^+,
\]
\[
\Phi_{PC,+}(z) = \Phi_{PC,-}(z) \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}, \quad z \in e^{\frac{j}{4}} \mathbb{R}^+,
\]
\[
\Phi_{PC,+}(z) = \Phi_{PC,-}(z) \begin{pmatrix} 1 & 0 \\ 1-q^2 & 0 \end{pmatrix}, \quad z \in \mathbb{R}^-.
\]
As $z \to 0$, we have $\Phi_{PC}(z) = O(1)$.
As $z \to \infty$, $\Phi_{PC}$ admits an asymptotic series of the form

$$\Phi_{PC}(z) \sim \left( 1 + \sum_{k=1}^{\infty} \frac{\Phi_{PC,k}(q)}{z^k} \right) z^{-i\nu \sigma_3} e^{i\frac{\nu^2}{4} \sigma_3},$$

where the principal branch is taken for $z^\pm i\nu$, and where

$$\Phi_{PC,1}(q) = \begin{pmatrix} 0 & \beta_{12}(q) \\ \beta_{21}(q) & 0 \end{pmatrix},$$
$$\Phi_{PC,2}(q) = \begin{pmatrix} 0 & \Phi_{PC,2-1}(q)_{12} \\ (1+i\nu)^{1/2} & 0 \end{pmatrix},$$
$$\Phi_{PC,2k-1}(q) = \begin{pmatrix} 0 & 0 \\ \Phi_{PC,2k-1}(q)_{21} & 0 \end{pmatrix}, \quad k \geq 2,$$
$$\Phi_{PC,2k}(q) = \begin{pmatrix} 0 & \Phi_{PC,2k}(q)_{11} \\ 0 & \Phi_{PC,2k}(q)_{22} \end{pmatrix}, \quad k \geq 2,$$

where

$$\beta_{12}(q) = \frac{e^{\frac{3\pi i}{4}} e^{-\frac{\pi i}{4}} \sqrt{2\pi}}{q \Gamma(i\nu)} \quad \text{and} \quad \beta_{21}(q) = \frac{e^{\frac{3\pi i}{4}} e^{-\frac{\pi i}{4}} \sqrt{2\pi}}{q \Gamma(-i\nu)}.$$

The solution $\Phi_{PC}(z) = \Phi_{PC}(z; q)$ can be expressed in terms of the parabolic cylinder function $D_a(z)$ (see [Olver et al. 2010, Chapter 12] for a definition). RH problems related to parabolic cylinder functions were first studied in [Its 1981], and first used in a steepest descent analysis in [Deift and Zhou 1993]. We also refer to [Fokas et al. 2006, Chapter 9, §4], [Bothner 2017, Section 5.2], and [Lenells 2017, Appendix B] for more recent works using $D_a$ to construct certain model RH problems. The solution to the above RH problem for $q \in [0, 1)$ is the same as in [Lenells 2017]; however for $q \in i(0, +\infty)$ it differs from the one of [Lenells 2017] and, for the convenience of the reader, we construct its explicit solution here.

Lemma A.1. The unique solution to the model RH problem for $\Phi_{PC}$ is given by

$$\Phi_{PC}(z) = \Psi(z) B(z)^{-1},$$

where

$$B(z) = \begin{cases} \begin{pmatrix} 1 - q & 0 \\ 0 & 1 \end{pmatrix}, & \text{arg } z \in (0, \frac{\pi}{4}), \\ \begin{pmatrix} 1 - q & 0 \\ 0 & 1 \end{pmatrix}, & \text{arg } z \in \left(\frac{3\pi}{4}, \pi\right), \\ \begin{pmatrix} 1 - q & 0 \\ 0 & 1 \end{pmatrix}, & \text{arg } z \in (-\pi, -\frac{3\pi}{4}), \\ \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}, & \text{arg } z \in (-\frac{3\pi}{4}, 0), \\ I, & \text{elsewhere}, \end{cases}$$

and

$$\Psi(z) = e^{\int_{\gamma_+} \frac{\Phi_{PC,k}(q)}{z^k} \, dz}.$$
where the error terms are uniform with respect to $a$ in compact subsets and arg $z$ in the given ranges.

and

$$
\Psi(z) = \begin{pmatrix}
\psi_{11}(z) & \frac{-i\frac{a}{\pi} + \frac{z}{2}}{\beta_{12}(q)} \\
\left(i\frac{a}{\pi} + \frac{z}{2}\right)\psi_{11}(z) & \psi_{22}(z)
\end{pmatrix}, \quad q \in \mathbb{T}, \ z \in \mathbb{C} \setminus \mathbb{R},
$$

(A-5)

where the functions $\psi_{11}$ and $\psi_{22}$ are defined by

$$
\psi_{11}(z) = \begin{cases}
e^{\frac{a}{\pi}z}D_{-iv}(e^{-\frac{a}{\pi}z}), & \text{Im} z > 0, \\
e^{-\frac{a}{\pi}z}D_{-iv}(e^{\frac{a}{\pi}z}), & \text{Im} z < 0,
\end{cases}
$$

(A-6a)

and

$$
\psi_{22}(z) = \begin{cases}
e^{-\frac{a}{\pi}z}D_{iv}(e^{-\frac{a}{\pi}z}), & \text{Im} z > 0, \\
e^{\frac{a}{\pi}z}D_{iv}(e^{\frac{a}{\pi}z}), & \text{Im} z < 0.
\end{cases}
$$

(A-6b)

Proof. It is a classical fact (see, e.g., [Olver et al. 2010, Chapter 12]) that $D_a(z)$ is an entire function in both $a$ and $z$, which satisfies the second order ODE in $z$,

$$
D_a''(z) = \left(z^2 - \frac{1}{2} - a\right)D_a(z).
$$

Therefore, we verify that the function $\Psi$ defined by (A-5) satisfies the first order matrix differential equation

$$
\Psi'(z) = \frac{i}{2} \sigma_3 \Psi(z) - i \begin{pmatrix} 0 & \beta_{12} \\ -\beta_{21} & 0 \end{pmatrix} \Psi(z), \quad z \in \mathbb{C} \setminus \mathbb{R}.
$$

Since $\Psi_+$ and $\Psi_-$ satisfy the same linear differential equation, there exists $J_{\Psi}$ independent of $z$ such that $\Psi_+(z) = \Psi_-(z) J_{\Psi}$ for $z \in \mathbb{R}$. Using

$$
D_{iv}(0) = \frac{2^i \sqrt{\pi}}{\Gamma(\frac{1}{2} + iv)}, \quad D_{iv}'(0) = -\frac{2^{1+iv} \sqrt{\pi}}{\Gamma(-\frac{1}{2} + iv)},
$$

we obtain after a computation that

$$
J_{\Psi} = \Psi_-(0)^{-1} \Psi_+(0) = \begin{pmatrix} 1 & -q \\ q & 1 - q^2 \end{pmatrix},
$$

where we have also used (A-3). Using the jumps for $\Psi$, it is easy to verify that $\Phi_{PC}$ defined by (A-4) satisfies the jumps of the RH problem for $\Phi_{PC}$. For each $\delta > 0$, the parabolic cylinder function satisfies the asymptotic formula

$$
D_a(z) = z^a e^{-\frac{a}{2}z^2} \left(1 + \frac{a(a-1)}{2z^2} + O(z^{-4})\right)
$$

$$
-\hat{\tau}(z) \sqrt{\frac{2\pi}{\Gamma(-a)}} e^{\frac{a}{2}z^2} z^{-a-1} \left(1 + \frac{(a+1)(a+2)}{2z^2} + O(z^{-4})\right), \quad z \to \infty, \quad a \in \mathbb{C},
$$

$$
\hat{\tau}(z) = \begin{cases}
0, & \arg z \in [-\frac{3\pi}{4} + \delta, \frac{3\pi}{4} - \delta], \\
e^{i\pi a}, & \arg z \in [\frac{\pi}{4} + \delta, \frac{5\pi}{4} - \delta], \\
e^{-i\pi a}, & \arg z \in [-\frac{5\pi}{4} + \delta, -\frac{\pi}{4} - \delta],
\end{cases}
$$

where the error terms are uniform with respect to $a$ in compact subsets and arg $z$ in the given ranges.
Using this formula and the identity
\[ D'_a(z) = \frac{z}{2} D_a(z) - D_{a+1}(z), \]
the asymptotic equation (A-1) follows from a tedious but straightforward computation. This shows that \( \Phi_{\text{PC}} \) given by A.1 satisfies all the conditions of the RH problem for \( \Phi_{\text{PC}} \). □

**Formula for \( \beta_{12} \beta_{21} \).** Since \( \nu \in \mathbb{R} \), we note from [Olver et al. 2010, formula 5.4.3] that
\[
|\Gamma(i\nu)| = \frac{\sqrt{2\pi}}{\sqrt{\nu(e^{\pi\nu} - e^{-\pi\nu})}} = \frac{\sqrt{2\pi}}{\sqrt{\nu q^2 e^{\pi\nu}}},
\]
from which we deduce the identity
\[
\beta_{12} \beta_{21} = \nu. \quad (A-7)
\]

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**References**


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