ANNALS OF K-THEORY

vol. 5 no. 2 2020

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A JOURNAL OF THE K-THEORY FOUNDATION

ANNALS OF K-THEORY
Vol. 5, No. 2, 2020
dx.doi.org/10.2140/akt.2020.5.317

On line bundles in derived algebraic geometry

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We give the first example of a derived scheme X and a line bundle $\mathscr{L}$ on the truncation tX so that $\mathscr{L}$ does not extend to the original derived scheme X. In other words the pullback map $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$, and hence also the pullback map $K^0(X) \to K^0(tX)$, is not surjective. The derived schemes we construct have the further property that while their truncations are projective hypersurfaces, they fail to have any nontrivial line bundles, and therefore they are not quasiprojective.

1. Introduction

Derived algebraic geometry and related methods have recently seen numerous applications in the study of classical schemes, usually by giving either new results about, or new constructions of, various cohomology theories. For example, Kerz et al. [2018] use derived algebraic geometry to prove pro cdh-descent for the algebraic K-theory of Noetherian schemes, and subsequently to settle a conjecture of Weibel on the vanishing of negative K-groups. A more recent example is given in [Annala 2019b], where the algebraic cobordism of general quasiprojective (derived) schemes over a field of characteristic 0 is carried out. Given the position of algebraic cobordism as the universal oriented cohomology theory, and the fact that the cycles in the construction of [loc. cit.] are certain maps from derived schemes, one has ample motivation to study the geometry of general derived k-schemes, and not just, say, specific examples of derived moduli.

It is a well known fact [Kerz et al. 2018] that for an affine derived scheme X, the induced pullback map $K^0(X) \to K^0(tX)$ is an isomorphism. However, due to the form that the descent spectral sequence takes, one would expect the K^0 of a derived scheme to be different from that of its truncation in general. The problem of finding concrete examples of such behavior may be approached via a more computable invariant — the Picard group of X — which is a summand of $K^0(X)$: we have a map $\mathrm{Pic}(X) \to K^0(X)$ that sends a line bundle to its K-theory class, and a one-sided inverse is induced by the perfect determinant map of Schürg, Toën, and Vezzosi [Schürg et al. 2015] as the determinant of a line bundle $\mathscr{L}$, regarded as

MSC2010: 14F05.
Keywords: derived algebraic geometry, deformation theory, Picard group.

a perfect complex, is again $\mathscr{L}$. We can therefore conclude that if the pullback map $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$ fails to be injective (surjective), then so does the map $K^0(X) \to K^0(tX)$.

It is not very hard to find examples of derived schemes X so that the map $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$ is not injective. Some kind of trivial derived enhancement will often have this property: for example, take the derived scheme whose underlying scheme is $\mathbb{P}^2$, and whose structure sheaf is given by the trivial square-zero extension $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)[-1]$. One can compute, either using the descent spectral sequence or the deformation sequences as in Section 2, that the Picard group of X is isomorphic to $\mathbb{Z} \oplus k$.

However, finding an example so that $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$ fails to be surjective is harder, as a trivial extension will not work anymore. It is also a much more interesting question, as many geometric properties of X are governed by the line bundles on X. Consider for example the question of whether or not a derived scheme X is quasiprojective. In [Annala 2019a] it was noted that a derived scheme X is quasiprojective if and only if it has an *ample line bundle*, i.e., a line bundle whose truncation is ample on tX. Hence, the question of whether or not X is quasiprojective can be divided into two parts:

(1) is the truncation tX quasiprojective and

(2) does an ample line bundle on tX extend to X;

and the second question is obviously related to the surjectivity of the map $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$.

The main purpose of this article is finding an example of a derived scheme X such that the pullback map $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$ is not surjective. However, in the examples we construct, $\mathrm{Pic}(X)$ is trivial while the truncation tX is a projective hypersurface, realizing the obstruction (2) to quasiprojectivity. The examples are constructed in Section 3, and, after reducing the problem to one in classical algebraic geometry, verifying the desired properties is an easy computation involving nothing else than just the basic graduate knowledge of algebraic geometry. However, justifying these computations takes a bit more work, and is done in Section 2.

Conventions. Throughout this article, we are going to work over a field k of characteristic 0. Derived schemes over k are ringed spaces (in the ∞-categorical sense), which are locally modeled by spectra of simplicial commutative k-algebras. It is known that under our characteristic assumption, one gets an equivalent theory by replacing simplicial commutative k-algebras by connective commutative k differential graded algebras or by $\mathbb{E}_\infty$-ring spectra over k. As everything in this paper should be assumed to be derived, we will often drop the word "derived" to not burden the exposition; on the contrary, if something is not derived, we will emphasize this by using descriptions such as *classical* or *truncated*. We will denote by $[n]$ the operation of n-fold suspension Σ^n, which in the dg-model corresponds to the homological shift upwards n times. Throughout the article, unless otherwise specified, X will be a derived scheme over k. All derived schemes are assumed to be separated. All rings are assumed to be commutative (in the homotopy theoretic sense), unless otherwise specified.

Given a derived scheme X, its *Picard group* $\mathrm{Pic}(X)$ is defined as the set of equivalence classes of line bundles (locally free sheaves of rank 1) on X, whose group operation is given by the tensor product [Lurie 2018, Definition 2.9.4.1]. A potential alternative definition for the Picard group is as equivalence classes of invertible elements (*invertible sheaves*) of the ∞-category $\mathrm{QCoh}(X)$ of quasicoherent sheaves on X. In [loc. cit.] this is called the *extended Picard group* and denoted by $\mathrm{Pic}^\dagger(X)$. By Corollary 2.9.5.7 therein the difference is not very large: if the underlying topological space of X is connected, then an invertible sheaf is equivalent to $\mathscr{L}[n]$ for some line bundle $\mathscr{L}$ and $n \in \mathbb{Z}$, and therefore $\mathrm{Pic}^\dagger(X)$ is a semidirect product of $\mathrm{Pic}(X)$ and $\mathbb{Z}$.

2. Derived deformation theory of line bundles

An important part of this paper is to have very precise control of the pullback morphism $\mathrm{Pic}(X) \to \mathrm{Pic}(tX)$ for certain types of derived enhancements $tX \hookrightarrow X$. Recall that the Picard group of a derived scheme X can be naturally identified with the first cohomology group $H^1(X; \mathcal{O}_X^\times)$ of the sheaf of units $\mathcal{O}_X^\times$ on X. The sheaf $\mathcal{O}_X^\times$ is defined by the formula

$$\mathcal{O}_X^\times(U) := (\mathcal{O}_X(U))^\times, \tag{2.1}$$

where the right-hand side denotes the space of units of the simplicial k-algebra $\mathcal{O}_X(U)$, i.e., the collection of the components of $\mathcal{O}_X(U)$ corresponding to the units of $\pi_0(\mathcal{O}_X(U))$. Multiplication of A induces on $A^\times$ the structure of a grouplike commutative simplicial monoid, i.e., a spectrum. As the functor $A \to A^\times$ is corepresented by the Laurent polynomials $k[t, t^{-1}]$, units commute with limits, and therefore the above rule truly yields a sheaf.

Suppose now that X is a classical scheme and $\mathcal{F}$ is a connective quasicoherent sheaf on X. Given a derived derivation $d : X \to \mathcal{F}[1]$, we can form the *square-zero extension* X_d of X as the derived enhancement of X whose structure sheaf is given by the top-left corner of the Cartesian diagram

$$\begin{tikzcd} \mathcal{O}_{X_d} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[d, "{(1,d)}"] \\ \mathcal{O}_X \arrow[r, "{(1,0)}"'] & \mathcal{O}_X \oplus \mathcal{F}[1] \end{tikzcd} \tag{2.2}$$

where 0 is the zero derivation (see [Porta and Vezzosi 2015, Definition 1.1] or [Toën and Vezzosi 2008, Definition 1.2.1.7] for the local case). Note that (2.2) remains Cartesian if considered as a square of sheaves of spectra on the underlying topological space X_{top} of X (combine Propositions 2.1.0.3 and 2.2.4.1 of [Lurie 2018]). Moreover, under some assumptions on $\mathcal{F}$ (e.g., the cohomology of $\mathcal{F}$ is concentrated in a single degree), any derived extension $X \hookrightarrow \widetilde{X}$ so that the fiber of the natural map $\mathcal{O}_{\widetilde{X}} \to \mathcal{O}_X$ can be identified with $\mathcal{F}$ (henceforth a *square-zero extension of* X *by* $\mathcal{F}$) arises in the above way from a derived derivation $d : \mathcal{O}_X \to \mathcal{F}[1]$ which is unique up to equivalence [Lurie 2017, Theorem 7.4.1.23] (see also [Porta and Vezzosi 2015, Theorem 3.1] for an easier version that is enough for the purposes of this paper).

Consider now the induced map $\mathcal{O}_{X_d}^\times \to \mathcal{O}_X^\times$. As units are stable under pullbacks, the square of the previous paragraph yields us a Cartesian square

$$\begin{tikzcd} \mathcal{O}_{X_d}^\times \arrow[r] \arrow[d] & \mathcal{O}_X^\times \arrow[d] \\ \mathcal{O}_X^\times \arrow[r] & (\mathcal{O}_X \oplus \mathcal{F}[1])^\times \end{tikzcd} \tag{2.3}$$

of sheaves of spectra. The lower horizontal map clearly has cofiber equivalent to $\mathcal{F}[1]$, and the induced map $(\mathcal{O}_X \oplus \mathcal{F}[1])^\times \to \mathcal{F}[1]$ is given on the level of simplicial sets by the degreewise formula

$$(a, m) \mapsto \frac{m}{a} \tag{2.4}$$

(as X is discrete, the units are invertible on the nose). We can therefore conclude that we have a distinguished triangle

$$\mathcal{F} \to \mathcal{O}_{X_d}^\times \to \mathcal{O}_X^\times \xrightarrow{\delta_d} \mathcal{F}[1] \tag{2.5}$$

inducing a long exact sequence of cohomology groups. The most important part of this exact sequence, at least for the purposes of this paper, is

$$H^1(X; \mathcal{F}) \to \mathrm{Pic}(X_d) \to \mathrm{Pic}(X) \xrightarrow{\delta_{d*}} H^2(X; \mathcal{F}) \tag{2.6}$$

where the middlemost morphism is the pullback morphism we are interested in. Therefore, if we can understand δ_d well enough, then we can understand the surjectivity of $\mathrm{Pic}(X_d) \to \mathrm{Pic}(X)$. But δ_d has an easy description, given by the following proposition.

Proposition 2.7. *Let* X *be a classical scheme and* $\mathcal{F}$ *a connective quasicoherent sheaf on* X*, and suppose we are given a derivation* d *on* X *taking values in* $\mathcal{F}[1]$*. Then the induced map* $\delta_d : \mathcal{O}_X^\times \to \mathcal{F}[1]$ *of sheaves of simplicial abelian groups as* *in (2.5) is equivalent to the* log derivation $d_{\log}$ *associated to the derivation* d; *i.e., it is defined by the degreewise formula*

$$a \mapsto \frac{d(a)}{a}. \tag{2.8}$$

Remark 2.9. Unless X is smooth, the above formula is deceptive in its simplicity, as the derived derivations of a k-algebra A taking values in a simplicial A-module M are given by degreewise k-derivations $d : \tilde{A} \to M$, where $\tilde{A} \to A$ is a cofibrant replacement [Porta and Vezzosi 2015, Definition 1.1]. Most of the complications go away if X is smooth over k as the cotangent complex is equivalent to the cotangent bundle $\Omega_{X/k}$: one merely takes a fibrant resolution (corresponding to an injective resolution via Dold–Kan) $\widetilde{\mathcal{F}[n]}$ of the quasicoherent sheaf $\mathcal{F}[n]$ and then a derived derivation is the same as a map $\Omega_{X/k} \to \widetilde{\mathcal{F}[n]}$, which in turn is the same as a degreewise derivation $\mathcal{O}_X \to \widetilde{\mathcal{F}[n]}$.

Proof. Taking horizontal cofibers of (2.3) we get an extended diagram

$$\begin{tikzcd}
\mathcal{O}_{X_d}^{\times} \arrow[r] \arrow[d] & \mathcal{O}_X^{\times} \arrow[r, "\delta_d"] \arrow[d, "{(1,d)}"] & \mathcal{F}[1] \arrow[d, "1"] \\
\mathcal{O}_X^{\times} \arrow[r] & (\mathcal{O}_X \oplus \mathcal{F}[1])^{\times} \arrow[r] & \mathcal{F}[1]
\end{tikzcd} \tag{2.10}$$

where the bottom-right horizontal map was identified earlier in (2.4). As the lower composition of the rightmost square is clearly equivalent to $d_{\log}$, the claim follows from the commutativity of the diagram. □

Before drawing the final conclusion of this section, we need to make the following remark.

Remark 2.11. Suppose that $\mathcal{F}$ is a classical quasicoherent sheaf on a classical scheme X and $\mathcal{A}$ is a sheaf of abelian groups on the underlying topological space of X. If $\mathcal{U} = (U_i)_{i \in I}$ is an affine open cover of X, then any Čech cocycle $(\delta_{i_0 \cdots i_n})$ of homomorphisms from $\mathcal{A}$ to $\check{C}^n_{\mathcal{U}}(\mathcal{F})$ represents a morphism $\delta : \mathcal{A} \to \mathcal{F}[n]$, and moreover the induced morphism $\check{H}^j_{\mathcal{U}}(\mathcal{A}) \to \check{H}^j_{\mathcal{U}}(\mathcal{F}) = H^{n+j}(\mathcal{F})$ can be described by the formula

$$(\delta a)_{i_0 \cdots i_{n+j}} = (-1)^n \delta_{i_0 \cdots i_n}(a_{i_n \cdots i_{n+j}}) \tag{2.12}$$

on cocycles.

Indeed, as $\check{C}^*_{\mathcal{U}}(\mathcal{A})$ is weakly equivalent to $\mathcal{A}$, one hopes that there would be only one map (up to homotopy) $\check{C}^*_{\mathcal{U}}(\mathcal{A}) \to \check{C}^{*+n}_{\mathcal{U}}(\mathcal{F})$ that extends the original map $\mathcal{A} \to \check{C}^n_{\mathcal{U}}(\mathcal{F})$. If this were true, then it would suffice to verify that the above formula for δ gives a well defined map of chain complexes $\check{C}^*_{\mathcal{U}}(\mathcal{A}) \to \check{C}^{*+n}_{\mathcal{U}}(\mathcal{F})$. This is indeed true, and the easy algebraic manipulation is left for the reader. Now the fact that the formula given for δ is the right one follows from the fact that the Čech complex maps (quasi-isomorphically) to an injective resolution of $\mathcal{F}[n]$, and that maps to injective resolutions preserve homotopy equivalences.

Proposition 2.13. *Suppose X is a smooth k-scheme and $\mathcal{F}$ is a classical quasi-coherent sheaf on X. Recalling that $\mathrm{Ext}_X^{n+1}(\Omega_{X/k}, \mathcal{F})$ is naturally equivalent to the set of equivalence classes of k-derivations $d : \mathcal{O}_X \to \mathcal{F}[n+1]$, the formula*

$$(d, \mathscr{L}) \mapsto \delta_{d*}(\mathscr{L}) \tag{2.14}$$

where δ_{d*} is defined as in (2.6) determines a biadditive pairing

$$\psi_{\mathcal{F},n} : \mathrm{Ext}_X^{n+1}(\Omega_X, \mathcal{F}) \times \mathrm{Pic}(X) \to H^{n+2}(X; \mathcal{F}) \tag{2.15}$$

that is k-linear in the first argument. Moreover, the pairing has the property that $\psi_{\mathcal{F},n}(d, \mathscr{L})$ vanishes if and only if $\mathscr{L}$ extends to a line bundle on the square-zero extension X_d.

Proof. The fact that $\psi_{\mathcal{F},n}$ detects whether or not a line bundle $\mathscr{L}$ extends to X_d follows directly from the definition, so we are left to show the biadditivity and the k-linearity with respect to the first argument. Proposition 2.7 identifies δ_d with the log derivation $d_{\log}$, and as every map $\Omega_{X/k} \to \mathcal{F}[n+1]$ in the derived category of X is represented by a Čech cocycle $\Omega_{X/k} \to \check{C}^{n+1}_{\mathcal{U}}(\mathcal{F})$, we see that the induced derivation d (and therefore also the induced log derivation $d_{\log}$) is represented by a Čech cocycle of derivations (log derivations).

We can therefore apply Remark 2.11 to our situation, and the formula (2.12) now translates into

$$\psi_{\mathcal{F},n}(d, \alpha)_{k_0,k_1,\dots,k_{n+2}} = d_{k_0,k_1,\dots,k_{n+1};\log}(\alpha_{k_{n+1},k_{n+2}}), \tag{2.16}$$

which is clearly k-linear for d. As log derivations are group homomorphisms, the additivity with respect to the other argument follows as well. □

3. The main results

In this section we are going to give the example. We are also changing our terminology for a more deformation theoretic one: if X is a smooth variety, then a deformation of X over $k \oplus k[i]$ is the same as a square-zero extension of X by the quasicoherent sheaf $\mathcal{O}_X[i]$. This follows for example from [Porta and Vezzosi 2015, Proposition 6.1] (stated and proved only for the $i = 1$ case, but which generalizes easily for $i > 1$ as well) or from a direct globalization of [Lurie 2017, Proposition 7.4.2.5], as equivalence classes of such deformations are identified with $H^{i+1}(X; T_{X/k}) \cong \mathrm{Ext}^{i+1}(\Omega_{X/k}, \mathcal{O}_X)$, $T_{X/k} \cong \Omega_{X/k}^{\vee}$ being the tangent bundle. Hence, the results of the previous section on deformation theory of line bundles can be immediately applied to deformations of X over the higher square-zero extensions $k \oplus k[i]$.

Let $X \hookrightarrow \mathbb{P}^n$ be a smooth hypersurface of degree $n+1$ defined as the vanishing locus of a homogeneous polynomial F. Without loss of generality we may assume that X does not contain the point $[1 : 0 : \cdots : 0]$ so that $\mathcal{U} = (U_i|_X)_{i\geq 1}^n$, where $(U_i)_{i\geq 0}^n$ is the standard open cover of $\mathbb{P}^n$, is an affine cover of X. Computing the Čech cohomology groups of the structure sheaf $\mathcal{O}_X$ and the tangent bundle $T_{X/k}$ associated to the above covering, one obtains the following two lemmas. The results are completely elementary, and can be worked out by nothing more than a few pages of diagram chasing. For completeness, however, we give short proofs.

Lemma 3.1. *Suppose $n \geq 3$. Then the cohomology group $H^{n-1}(X; \mathcal{O}_X)$ is isomorphic to k, and it is generated by the Čech cocycle*

$$\frac{\partial_0 F}{x_1 \cdots x_n}. \tag{3.2}$$

Moreover, $H^i(X; \mathcal{O}_X) \cong 0$ for $1 \leq i \leq n-2$.

Proof. All the claims other than that the given cocycle generates are standard. The last remaining claim follows from the fact that $\partial_0 F$ has a term cx_0^n, where $c \neq 0$, and $x_0^n/(x_1 \cdots x_n)$ is not a boundary (unlike all other possibilities). □

Lemma 3.3. *Suppose $n \geq 4$. Then the cohomology group $H^{n-2}(X; T_{X/k})$ is isomorphic to k, and it is generated by the Čech cocycle $d = (d_{12\cdots\hat{i}\cdots n})_{1\leq i\leq n}$, where*

$$d_{12\cdots\hat{i}\cdots n} = (-1)^i \frac{(\partial_0 F)\partial_i - (\partial_i F)\partial_0}{x_1 x_2 \cdots \hat{x}_i \cdots x_n}. \tag{3.4}$$

Remark 3.5. As is customary, we use the hat to denote an index or a term which is left out.

Proof. The fact that $H^{n-2}(X; T_{X/k}) \cong k$ follows immediately from Serre duality once we recall that by hard Lefschetz $H^1(X; \Omega^1_{X/k}) \cong H^1(\mathbb{P}^n; \Omega^1_{\mathbb{P}^n}) \cong k$. Moreover, d is a cocycle in derivations on X: clearly all the chosen derivations send F to 0, so they are derivations on X, and they do satisfy the cocycle condition

$$\begin{aligned}\sum_{i=1}^n (-1)^i d_{12\cdots\hat{i}\cdots n} &= \sum_{i=1}^n \frac{(\partial_0 F)\partial_i - (\partial_i F)\partial_0}{x_1 x_2 \cdots \hat{x}_i \cdots x_n} \\ &= \sum_{i=1}^n \frac{x_i(\partial_0 F)\partial_i - x_i(\partial_i F)\partial_0}{x_1 x_2 \cdots x_n} \\ &= \frac{x_0(\partial_0 F)\partial_0 - x_0(\partial_0 F)\partial_0}{x_1 x_2 \cdots x_n} \\ &= 0 \end{aligned} \tag{3.6}$$

as the Euler form equals 0. Hence, d generates $H^{n-2}(X; T_{X/k})$ if it is nontrivial, but this in fact follows from the calculation (3.7) following the lemma. □

We are now ready to show that the line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)|_X$ does not extend to any nontrivial first-order deformation of X over $k \oplus k[n-3]$. Recall that the transition maps α_{ij} of $\mathcal{O}(-1)$ are defined as $\alpha_{ij} = x_j/x_i$.

We can now just apply the generating derivation d of Lemma 3.3 to $\mathcal{O}_{\mathbb{P}^n}(-1)|_X$ using the pairing of Proposition 2.13:

$$\begin{aligned}
\psi_{\mathcal{O}_X, n-3}(d, \alpha)_{12\cdots n} &= (d_{\log}\alpha)_{12\cdots n} \\
&\overset{(2.12)}{=} (-1)^{n-2} d_{12\cdots n-1;\log}(\alpha_{n-1,n}) \\
&= (-1)^{n-2} \frac{x_{n-1}}{x_n} (-1)^n \frac{(\partial_0 F)\partial_n - (\partial_n F)\partial_0}{x_1 x_2 \cdots x_{n-1}} \left(\frac{x_n}{x_{n-1}} \right) \\
&= \frac{x_{n-1}}{x_n} \frac{(\partial_0 F) x_{n-1}^{-1}}{x_1 x_2 \cdots x_{n-1}} \\
&= \frac{(\partial_0 F)}{x_1 x_2 \cdots x_n}
\end{aligned} \tag{3.7}$$

and the right-hand side is the generator of Lemma 3.1, and therefore known to be nonzero. We have proven the following:

Theorem 3.8. *Let $X \hookrightarrow \mathbb{P}^n$, $n \geq 4$, be a smooth projective hypersurface of degree $n+1$. Then the line bundle $\mathcal{O}(1)|_X$ does not extend to any nontrivial deformation X' of X over $k \oplus k[n-3]$.*

Proof. Indeed, as we noticed earlier, the obstruction class of a line bundle depends k-linearly on the deformation of X. We have shown that the obstruction is not 0 for the generating deformation, and therefore it will not be 0 for any nonzero multiple of it. □

Remark 3.9. Note that when $n = 3$, the derivation δ given in Lemma 3.3 is still a perfectly valid element of $H^1(X; T_X)$, and applying the log differential to $\mathcal{O}(1)|_X$ as above shows that $\mathcal{O}(1)|_X$ does not extend to the deformation associated to δ. This has a moduli theoretic interpretation. Indeed, it is a well known fact that the moduli space of *polarized K3 surfaces* (K3 surfaces equipped with an ample line bundle) is 19-dimensional. Therefore, the kernel, which can easily be checked to be 19-dimensional, of the map $H^1(X; T_X) \to H^2(X; \mathcal{O}_X)$ given by evaluating at $\mathcal{O}(1)|_X$ should be thought as the tangent space of the space of polarized K3 surfaces, sitting inside the tangent space of the moduli of K3 surfaces.

Assume again that $n \geq 4$. It is known that the Picard group of a smooth hypersurface $X \hookrightarrow \mathbb{P}^n$ is isomorphic to $\mathbb{Z}$ and generated by $\mathcal{O}(-1)|_X$. Hence:

Theorem 3.10. *Let $X \hookrightarrow \mathbb{P}^n$, $n \geq 4$, be a smooth projective hypersurface of degree $n+1$, and let X' be a nontrivial deformation of X over $k \oplus k[n-3]$. Then* $\mathrm{Pic}(X') \cong 0$ *and therefore X' fails to be quasiprojective.*

Proof. We have the deformation sequence

$$\cdots \to H^{n-2}(X;\mathcal{O}_X) \to \mathrm{Pic}(X') \to \mathrm{Pic}(X) \xrightarrow{d_{\log}} H^{n-1}(X;\mathcal{O}_X) \to \cdots. \quad (3.11)$$

The computation (3.7) proves $d_{\log}$ is injective: indeed $d_{\log}$ is a homomorphism, it obtains a nonzero value on the generator $\mathcal{O}(-1)|_X$, and its target $H^{n-1}(X;\mathcal{O}_X) \cong k$ (Lemma 3.1) has no torsion. The claim now follows from the fact that $H^{n-2}(X;\mathcal{O}_X)$ (second claim of Lemma 3.1). □

Acknowledgements

The author would like to thank his advisor Kalle Karu for useful discussions relating the material of this article back to classical algebraic geometry, as well as the anonymous referee, whose comments helped improve the exposition. The author is supported by the Vilho, Yrjö, and Kalle Väisälä Foundation of the Finnish Academy of Science and Letters.

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Received 26 Apr 2019. Revised 15 Oct 2019. Accepted 5 Nov 2019.

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Annals of K-Theory is a journal of the K-Theory Foundation (ktheoryfoundation.org). The K-Theory Foundation acknowledges the precious support of Foundation Compositio Mathematica, whose help has been instrumental in the launch of the Annals of K-Theory.

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ANNALS OF K-THEORY

2020 vol. 5 no. 2