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Rational equivalence of cusps

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We prove that two cusps of the same dimension in the Baily–Borel compactification of some classical series of modular varieties are linearly dependent in the rational Chow group of the compactification. This gives a higher dimensional analogue of the Manin–Drinfeld theorem. As a consequence, we obtain a higher dimensional generalization of modular units as higher Chow cycles on the modular variety.

1. Introduction

The classical theorem of Manin [1972] and Drinfeld [1973] asserts that the difference of two cusps is torsion in the Picard group of the modular curve for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. This had stimulated the development of the theory of modular units and cuspidal class groups; see [Kubert and Lang 1981]. The original proof of Manin and Drinfeld used modular symbols and Hecke operators on the cohomology of the modular curve. Later, an interpretation in terms of the mixed Hodge structure of the modular curve minus the cusps was also found [Elkik 1990].

Our purpose in this paper is to prove a generalization of the Manin–Drinfeld theorem for cusps in the Baily–Borel compactification of some higher dimensional classical modular varieties. In higher dimensions, cusps are no longer divisors, but algebraic cycles of various codimension. We wish to clarify their contribution to the Chow group of the Baily–Borel compactification.

The modular varieties of our object of study are of the following three types:

(1) modular varieties of orthogonal type attached to rational quadratic forms of signature $(2, n)$, which have only 0-dimensional and 1-dimensional cusps;

(2) Siegel modular varieties attached to rational symplectic forms; and

(3) modular varieties of unitary type, including the Picard modular varieties, attached to Hermitian forms over imaginary quadratic fields.

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In Cartan's classification of irreducible Hermitian symmetric domains, these correspond to the domains $\mathcal{D}$ of type IV, III, and I, respectively. The Baily–Borel compactification [Baily and Borel 1966] of the modular variety $\Gamma\backslash\mathcal{D}$ for an arithmetic group Γ is obtained by adjoining rational boundary components to $\mathcal{D}$ and then taking the quotient by Γ. Below, by a *cusp* we mean the closure of the image of a rational boundary component in the Baily–Borel compactification.

Our main results are the following.

Theorem 1.1 (orthogonal case). *Let Λ be an integral quadratic lattice of signature $(2, n)$, Γ a congruence subgroup of the orthogonal group $O^+(\Lambda)$, and X_Γ the Baily–Borel compactification of the modular variety defined by Γ. Let Z_1, Z_2 be two cusps of X_Γ of the same dimension, say $k \in \{0, 1\}$. Assume that $n \geq 3$ if $k = 1$. Then we have $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in the rational Chow group $\mathrm{CH}_k(X_\Gamma)_\mathbb{Q} = \mathrm{CH}_k(X_\Gamma) \otimes_\mathbb{Z} \mathbb{Q}$ of X_Γ.*

Theorem 1.2 (symplectic case). *Let Λ be an integral symplectic lattice, Γ a congruence subgroup of the symplectic group $\mathrm{Sp}(\Lambda)$, and X_Γ the Satake–Baily–Borel compactification of the Siegel modular variety defined by Γ. If Z_1, Z_2 are two cusps of X_Γ of the same dimension, say k, then $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_\mathbb{Q}$.*

Theorem 1.3 (unitary case). *Let K be an imaginary quadratic field, Λ a Hermitian lattice over $\mathcal{O}_K$, Γ a congruence subgroup of the unitary group $\mathrm{U}(\Lambda)$, and X_Γ the Baily–Borel compactification of the modular variety defined by Γ. If Z_1, Z_2 are two cusps of X_Γ of the same dimension, say k, then $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_\mathbb{Q}$.*

Note that the equality $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_\mathbb{Q}$ is the same as the equality $N_1[Z_1] = N_2[Z_2]$ in the integral Chow group $\mathrm{CH}_k(X_\Gamma)$ for some natural numbers N_1, N_2. When $k = 0$, we must have $N_1 = N_2$, so $[Z_1] - [Z_2]$ is torsion in $\mathrm{CH}_0(X_\Gamma)$.

In the symplectic case, when Λ has rank ≥ 4, every finite-index subgroup of $\mathrm{Sp}(\Lambda)$ is a congruence subgroup by [Mennicke 1965; Bass et al. 1964]. The case $\mathrm{rk}(\Lambda) = 2$ is just the case of modular curves.

The case $(n, k) = (2, 1)$ in the orthogonal case is indeed an exception. We have self products of modular curves as typical examples of X_Γ in $n = 2$, for which two transversal boundary curves are not homologically equivalent. On the other hand, we should note that some consideration in the case $n = 2$ is necessary for our proof for the case $n \geq 3$.

The proof of Theorems 1.1–1.3 is based on the same simple idea. We connect Z_1 and Z_2 by a chain of submodular varieties or their products, through the interior or the boundary, and use induction on the dimension of modular varieties. This eventually reduces the problem to the Manin–Drinfeld theorem for modular curves. The actual argument requires case-by-case construction depending on the combinatorics of rational boundary components. We need to argue the three cases separately, though the symplectic and the unitary cases are similar. Theorem 1.1 is proved in Section 2, Theorem 1.2 in Section 3, and Theorem 1.3 in Section 4.

In Section 5, as a consequence of these results, we associate an explicit nonzero element of the higher Chow group $\mathrm{CH}_k(\Gamma\backslash\mathcal{D}, 1)_{\mathbb{Q}}$ of the modular variety $\Gamma\backslash\mathcal{D}$ (before compactification) to each pair (Z_1, Z_2) of cusps of maximal dimension k. This gives a higher dimensional analogue of modular units from the viewpoint of algebraic cycles. If the span of all such higher Chow cycles on $\Gamma\backslash\mathcal{D}$ has dimension no less than the number of maximal cusps, we would then obtain a nontrivial subspace of $\mathrm{CH}_k(X_\Gamma, 1)_{\mathbb{Q}}$ for the Baily–Borel compactification X_Γ.

Throughout the paper $\Gamma(N)$ stands for the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N, and $X(N) = \Gamma(N)\backslash\mathbb{H}^*$ the (compactified) modular curve for $\Gamma(N)$. In Section 2 and Section 3, for a free $\mathbb{Z}$-module Λ of finite rank, we denote by $\Lambda^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ its dual $\mathbb{Z}$-module and define $\Lambda_F = \Lambda \otimes_{\mathbb{Z}} F$ for $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. For a $\mathbb{Q}$-vector space V we also write $V^\vee = \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ and $V_F = V \otimes_{\mathbb{Q}} F$ when no confusion is likely to occur.

2. The orthogonal case

In this section we prove Theorem 1.1. We first recall orthogonal modular varieties; see [Scattone 1987; Looijenga 2016]. Let Λ be a free $\mathbb{Z}$-module of rank $2+n$ equipped with a nondegenerate symmetric bilinear form $(\cdot\,,\cdot) : \Lambda \times \Lambda \to \mathbb{Z}$ of signature $(2, n)$. Let

$$Q_\Lambda = \{[\mathbb{C}\omega] \in \mathbb{P}\Lambda_{\mathbb{C}} \mid (\omega, \omega) = 0\}$$

be the isotropic quadric in $\mathbb{P}\Lambda_{\mathbb{C}}$. The open set of Q_Λ defined by the condition $(\omega, \bar{\omega}) > 0$ consists of two connected components, and the Hermitian symmetric domain $\mathcal{D}_\Lambda$ attached to Λ is defined as one of them. This choice is equivalent to the choice of an orientation of a positive definite plane in $\Lambda_{\mathbb{R}}$.

Let $\mathrm{O}(\Lambda)$ be the orthogonal group of Λ, namely the group of isomorphisms $\Lambda \to \Lambda$ preserving the quadratic form. We write $\mathrm{O}^+(\Lambda)$ for the subgroup of $\mathrm{O}(\Lambda)$ preserving the component $\mathcal{D}_\Lambda$. For a natural number N let $\mathrm{O}^+(\Lambda, N) < \mathrm{O}^+(\Lambda)$ be the kernel of the reduction map $\mathrm{O}^+(\Lambda) \to \mathrm{GL}(\Lambda/N\Lambda)$. A subgroup Γ of $\mathrm{O}^+(\Lambda)$ is called a congruence subgroup if it contains $\mathrm{O}^+(\Lambda, N)$ for some level N. A typical example is the kernel of the reduction map $\mathrm{O}^+(\Lambda) \to \mathrm{GL}(\Lambda^\vee/\Lambda)$ for the discriminant group $\Lambda^\vee/\Lambda$.

There are two types of rational boundary components of $\mathcal{D}_\Lambda$: 0-dimensional and 1-dimensional components. The 0-dimensional components correspond to isotropic $\mathbb{Q}$-lines I in $\Lambda_{\mathbb{Q}}$: we take the point $p_I = [I_{\mathbb{C}}] \in Q_\Lambda$, which is in the closure of $\mathcal{D}_\Lambda$, for each such I. The 1-dimensional components correspond to isotropic $\mathbb{Q}$-planes J in $\Lambda_{\mathbb{Q}}$: we take the connected component of $\mathbb{P}J_{\mathbb{C}} - \mathbb{P}J_{\mathbb{R}} \simeq \mathbb{H} \sqcup \mathbb{H}$, say $\mathbb{H}_J$, that is in the closure of $\mathcal{D}_\Lambda$. The union

$$\mathcal{D}_\Lambda^* = \mathcal{D}_\Lambda \sqcup \bigsqcup_{\dim J=2} \mathbb{H}_J \sqcup \bigsqcup_{\dim I=1} p_I$$

is equipped with the Satake topology [Baily and Borel 1966; Borel and Ji 2005]. By [Baily and Borel 1966], the quotient space $X_\Gamma = \Gamma\backslash\mathcal{D}_\Lambda^*$ has the structure of a normal projective variety and contains $\Gamma\backslash\mathcal{D}_\Lambda$ as a Zariski open set.

In Section 2A we prove Theorem 1.1 for 0-dimensional cusps, and in Section 2B for 1-dimensional cusps. Throughout this section U stands for the rank 2 unimodular hyperbolic lattice with Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The symbol $\Lambda_1 \perp \Lambda_2$ stands for the orthogonal direct sum of two quadratic lattices (or spaces) Λ_1, Λ_2, while $\Lambda_1 \oplus \Lambda_2$ just stands for the direct sum of Λ_1, Λ_2 as $\mathbb{Z}$-module (or linear space) and does not necessarily mean that $(\Lambda_1, \Lambda_2) \equiv 0$.

2A. *0-dimensional cusps.* In this subsection we prove Theorem 1.1 for 0-dimensional cusps. Let $I_1 \neq I_2$ be two isotropic lines in $\Lambda_\mathbb{Q}$ and $p_1, p_2 \in X_\Gamma$ the corresponding 0-dimensional cusps. We consider separately the cases where $(I_1, I_2) \equiv 0$ or $(I_1, I_2) \not\equiv 0$. In the former case p_1 and p_2 are joined by a boundary curve, while in the latter case they are joined by a modular curve through the interior of X_Γ.

2A1. *The case* $(I_1, I_2) \equiv 0$. We first assume that $(I_1, I_2) \equiv 0$. The direct sum $J = I_1 \oplus I_2$ is an isotropic plane in $\Lambda_\mathbb{Q}$. Let $\mathbb{H}_J^* = \mathbb{H}_J \sqcup \bigsqcup_{I \subset J} p_I$ and $\Gamma_J \subset \mathrm{SL}(J)$ be the image of the stabilizer of J in Γ. We have a generically injective morphism $f : X_J \to X_\Gamma$ from the modular curve $X_J = \Gamma_J\backslash\mathbb{H}_J^*$ whose image is the 1-dimensional cusp associated to J.

Claim 2.1. *Γ_J is a congruence subgroup of* $\mathrm{SL}(J_\mathbb{Z})$, *where* $J_\mathbb{Z} = J \cap \Lambda$.

Proof. There exists a rank 2 isotropic sublattice $J'_\mathbb{Z}$ in $\Lambda_\mathbb{Q}$ such that $J'_\mathbb{Z} \simeq (J_\mathbb{Z})^\vee$ by the pairing. The lattice $\Lambda_1 = J_\mathbb{Z} \oplus J'_\mathbb{Z}$ is isometric to $U \perp U$. We set $\Lambda_2 = (\Lambda_1)^\perp \cap \Lambda$ and $\Lambda' = \Lambda_1 \perp \Lambda_2$. Recall that Γ contains $\mathrm{O}^+(\Lambda, N)$ for some level N. Since both Λ and Λ' are full lattices in $\Lambda_\mathbb{Q}$, we can find natural numbers N_1, N_2 such that

$$N_1\Lambda' \subset N\Lambda \subset \Lambda \subset N_2^{-1}\Lambda'.$$

If we set $N' = N_1N_2$, this tells us that

$$\mathrm{O}^+(\Lambda', N') \subset \mathrm{O}^+(\Lambda, N) \subset \Gamma \tag{2.2}$$

inside $\mathrm{O}(\Lambda_\mathbb{Q}) = \mathrm{O}(\Lambda'_\mathbb{Q})$. Now we have the embedding

$$\mathrm{SL}_2(\mathbb{Z}) \simeq \mathrm{SL}(J_\mathbb{Z}) \hookrightarrow \mathrm{O}^+(\Lambda'), \qquad \gamma \mapsto (\gamma|_{J_\mathbb{Z}}) \oplus (\gamma^\vee|_{J'_\mathbb{Z}}) \oplus \mathrm{id}_{\Lambda_2},$$

whose image is contained in the stabilizer of J. Since this maps $\Gamma(N')$ into $\mathrm{O}^+(\Lambda', N') \subset \Gamma$, we see that Γ_J contains $\Gamma(N')$. $\square$

Let q_1, q_2 be the cusps of X_J corresponding to I_1, I_2, respectively. By this claim we can apply the Manin–Drinfeld theorem to X_J. Therefore, $[q_1] = [q_2]$ in $\mathrm{CH}_0(X_J)_{\mathbb{Q}}$. Since $f(q_1) = p_1$ and $f(q_2) = p_2$, we obtain

$$[p_1] = f_*[q_1] = f_*[q_2] = [p_2]$$

in $\mathrm{CH}_0(X_\Gamma)_{\mathbb{Q}}$.

2A2. *The case* $(I_1, I_2) \not\equiv 0$. Next we assume that $(I_1, I_2) \not\equiv 0$. In this case $I_1 \oplus I_2$ is isometric to $U_{\mathbb{Q}}$. Its orthogonal complement $(I_1 \oplus I_2)^\perp$ has signature $(1, n-1)$. We choose a vector v of positive norm from $(I_1 \oplus I_2)^\perp$ and put $\Lambda'_{\mathbb{Q}} = I_1 \oplus I_2 \oplus \mathbb{Q}v$. Then $\Lambda'_{\mathbb{Q}}$ has signature (2, 1). Let $\mathcal{D}_{\Lambda'}$ be the Hermitian symmetric domain attached to $\Lambda'_{\mathbb{Q}}$. We have the natural inclusion $\mathcal{D}^*_{\Lambda'} \subset \mathcal{D}^*_{\Lambda}$, which is compatible with the embedding of orthogonal groups

$$\iota : \mathrm{O}^+(\Lambda'_{\mathbb{Q}}) \hookrightarrow \mathrm{O}^+(\Lambda_{\mathbb{Q}}), \qquad \gamma \mapsto \gamma \oplus \mathrm{id}_{(\Lambda'_{\mathbb{Q}})^\perp}.$$

Claim 2.3. *There is a subgroup* $\Gamma' < \mathrm{O}^+(\Lambda'_{\mathbb{Q}})$ *such that* $\iota(\Gamma') \subset \Gamma$ *and* $X' = \Gamma' \backslash \mathcal{D}^*_{\Lambda'}$ *is naturally isomorphic to* $X(N)$ *for some level* N.

Proof. Let $\Lambda_1 = U \perp \langle 2 \rangle$. Then $\Lambda'_{\mathbb{Q}}$ is isometric to the scaling of $(\Lambda_1)_{\mathbb{Q}}$ by some positive rational number. This gives natural isomorphisms $\mathcal{D}^*_{\Lambda'} \simeq \mathcal{D}^*_{\Lambda_1}$ and $\mathrm{O}^+(\Lambda'_{\mathbb{Q}}) \simeq \mathrm{O}^+((\Lambda_1)_{\mathbb{Q}})$. The group $\mathrm{O}^+((\Lambda_1)_{\mathbb{Q}})$ is related to $\mathrm{SL}_2(\mathbb{Q})$ by the following well-known construction (cf. [Maclachlan and Reid 2003, §2.4]). Let $V \subset M_2(\mathbb{Q})$ be the space of 2×2 matrices with trace 0, equipped with the symmetric form $(A, B) = \mathrm{tr}(AB)$. Then $V \cap M_2(\mathbb{Z})$ is isometric to Λ_1. By conjugation $\mathrm{SL}_2(\mathbb{Q})$ acts on V. This defines a homomorphism

$$\varphi : \mathrm{SL}_2(\mathbb{Q}) \to \mathrm{O}^+(V) = \mathrm{O}^+((\Lambda_1)_{\mathbb{Q}})$$

with $\mathrm{Ker}(\varphi) = \{\pm I\}$. (We have $\mathrm{Im}(\varphi) = \mathrm{SO}^+(V)$, but we do not need this fact.) It is readily checked that $\varphi(\Gamma(N)) \subset \mathrm{O}^+(\Lambda_1, N)$ for every level N. Furthermore, φ is compatible with the Veronese isomorphism

$$\mathbb{H}^* \to \mathcal{D}^*_{\Lambda_1}, \qquad \tau \mapsto e + \tau v_0 - \tau^2 f,$$

where e, f are the standard basis of U and v_0 is a generator of $\langle 2 \rangle$. Now by the same argument as (2.2), there exists a level N such that the embedding ι maps $\mathrm{O}^+(\Lambda_1, N)$ into Γ. This proves our claim. $\square$

Let q_1, q_2 be the cusps of X' corresponding to the isotropic lines I_1, I_2 of $\Lambda'_{\mathbb{Q}}$. By this claim we have a finite morphism $f : X' \to X_\Gamma$ which sends q_1 to p_1 and q_2 to p_2. By the Manin–Drinfeld theorem for X' we have $[q_1] = [q_2]$ in $\mathrm{CH}_0(X')_{\mathbb{Q}}$. Applying f_*, we obtain $[p_1] = [p_2]$ in $\mathrm{CH}_0(X_\Gamma)_{\mathbb{Q}}$. This finishes the proof of Theorem 1.1 for 0-dimensional cusps.

Remark 2.4. If Λ has Witt index 2, $(I_1 \oplus I_2)^\perp$ contains an isotropic line, say I_3. Then we could also apply the result of Section 2A1 to I_1 vs. I_3 and to I_3 vs. I_2, thus obtaining $[p_1] = [p_2]$ via I_3. Together with the case of Section 2A1, this shows that when X_Γ contains at least one 1-dimensional cusp, then any two 0-dimensional cusps can be connected by a chain of 1-dimensional cusps of length ≤ 2, which provides their rational equivalence.

2B. 1-*dimensional cusps.* In this subsection we prove Theorem 1.1 for 1-dimensional cusps.

2B1. *Preliminaries in* $n = 2$. Although the case $n = 2$ is not included in Theorem 1.1 for 1-dimensional cusps, we need to study a specific example in $n = 2$ as preliminaries for the proof for the case $n \geq 3$. We consider the lattice $2U = U \perp U$. Let e_1, f_1 be the standard basis of the first copy of U, and e_2, f_2 be that of the second U. Let $J_1' = \mathbb{Q}e_2 \oplus \mathbb{Q}e_1$ and $J_2' = \mathbb{Q}f_2 \oplus \mathbb{Q}f_1$, which are isotropic planes in $2U_\mathbb{Q}$. We take an arbitrary natural number N and consider the modular surface $S(N) = \mathrm{O}^+(2U, N)\backslash \mathcal{D}_{2U}^*$. Let C_1, C_2 be the boundary curves of $S(N)$ associated to J_1', J_2', respectively.

Lemma 2.5. *We have* $\mathbb{Q}[C_1] = \mathbb{Q}[C_2]$ *in* $\mathrm{CH}_1(S(N))_\mathbb{Q}$.

Proof. Recall that we have the Segre isomorphism

$$\mathbb{H} \times \mathbb{H} \to \mathcal{D}_{2U}, \qquad (\tau_1, \tau_2) \mapsto e_1 - \tau_1\tau_2 f_1 + \tau_1 e_2 + \tau_2 f_2. \tag{2.6}$$

This extends to $\mathbb{H}^* \times \mathbb{H}^* \to \mathcal{D}_{2U}^*$, and maps the boundary components $\mathbb{H} \times (\tau_2 = 0)$, $\mathbb{H} \times (\tau_2 = i\infty)$ of $\mathbb{H}^* \times \mathbb{H}^*$ to the boundary components $\mathbb{H}_{J_1'}$, $\mathbb{H}_{J_2'}$ of $\mathcal{D}_{2U}^*$, respectively.

Let $J_3' = \mathbb{Q}f_2 \oplus \mathbb{Q}e_1$ and $J_4' = \mathbb{Q}e_2 \oplus \mathbb{Q}f_1$. By the pairing we identify $J_2' \simeq (J_1')^\vee$ and $J_4' \simeq (J_3')^\vee$. Then we define an embedding

$$\mathrm{SL}_2(\mathbb{Q}) \times \mathrm{SL}_2(\mathbb{Q}) = \mathrm{SL}(J_1') \times \mathrm{SL}(J_3') \hookrightarrow \mathrm{O}^+(2U_\mathbb{Q})$$

by sending $\gamma_1 \in \mathrm{SL}(J_1')$ to $(\gamma_1|_{J_1'}) \oplus (\gamma_1^\vee|_{J_2'})$ and $\gamma_3 \in \mathrm{SL}(J_3')$ to $(\gamma_3|_{J_3'}) \oplus (\gamma_3^\vee|_{J_4'})$. This embedding of groups is compatible with the isomorphism (2.6) of domains, and it maps $\Gamma(N) \times \Gamma(N)$ into $\mathrm{O}^+(2U, N)$. We thus obtain a finite morphism $f : X(N) \times X(N) \to S(N)$ which maps the boundary curves

$$C_1' = X(N) \times (\tau_2 = 0), \quad C_2' = X(N) \times (\tau_2 = i\infty)$$

of $X(N) \times X(N)$ onto C_1, C_2, respectively. By the Manin–Drinfeld theorem for the second copy of $X(N)$, we have $[C_1'] = [C_2']$ in $\mathrm{CH}_1(X(N) \times X(N))_\mathbb{Q}$. Applying f_*, we obtain the assertion. □

2B2. *The case* $J_1 \cap J_2 = \{0\}$. We go back to the proof of Theorem 1.1. Let Λ have signature $(2, n)$ with $n \geq 3$. Let $J_1 \neq J_2$ be two isotropic planes in $\Lambda_{\mathbb{Q}}$ and $Z_1, Z_2 \subset X_\Gamma$ the corresponding 1-dimensional cusps. We first consider the case where $J_1 \cap J_2 = \{0\}$. In this case the pairing between J_1 and J_2 is perfect because $J_i^\perp / J_i$ is negative definite. The direct sum $\Lambda'_{\mathbb{Q}} = J_1 \oplus J_2$ is isometric to $2U_{\mathbb{Q}}$. We can take an isometry $2U_{\mathbb{Q}} \to \Lambda'_{\mathbb{Q}}$, which maps J'_1, J'_2 to J_1, J_2, respectively. This gives an embedding of orthogonal groups

$$\mathrm{O}^+(2U_{\mathbb{Q}}) \simeq \mathrm{O}^+(\Lambda'_{\mathbb{Q}}) \hookrightarrow \mathrm{O}^+(\Lambda_{\mathbb{Q}}), \qquad \gamma \mapsto \gamma \oplus \mathrm{id}_{(\Lambda'_{\mathbb{Q}})^\perp}, \tag{2.7}$$

which is compatible with the embedding $\mathcal{D}_{2U} \simeq \mathcal{D}_{\Lambda'} \subset \mathcal{D}_\Lambda$ of domains. By the same argument as (2.2), we can find a level N such that the embedding (2.7) maps $\mathrm{O}^+(2U, N)$ into Γ. We thus obtain a finite morphism $f : S(N) \to X_\Gamma$ with $f(C_1) = Z_1$ and $f(C_2) = Z_2$. Sending the equality $\mathbb{Q}[C_1] = \mathbb{Q}[C_2]$ of Lemma 2.5 by f_*, we obtain $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_1(X_\Gamma)_{\mathbb{Q}}$.

2B3. *The case* $J_1 \cap J_2 \neq \{0\}$. We next consider the case where $J_1 \cap J_2 \neq \{0\}$. Let $I = J_1 \cap J_2$ and choose splittings $J_1 = I \oplus I_1$ and $J_2 = I \oplus I_2$. Since $(I_1, I_2) \not\equiv 0$, we have $I_1 \oplus I_2 \simeq U_{\mathbb{Q}}$. Let $\Lambda'_{\mathbb{Q}} = I_1 \oplus I_2$ and $\Lambda''_{\mathbb{Q}} = (\Lambda'_{\mathbb{Q}})^\perp$. Then $\Lambda''_{\mathbb{Q}}$ has signature $(1, n-1)$. Since $n - 1 \geq 2$ and $\Lambda''_{\mathbb{Q}}$ contains at least one isotropic line I, we find that $\Lambda''_{\mathbb{Q}}$ contains infinitely many isotropic lines. We can choose isotropic lines I_3, I_4 in $\Lambda''_{\mathbb{Q}}$ such that I, I_3, I_4 are linearly independent. Put $J_3 = I_4 \oplus I_2$ and $J_4 = I_3 \oplus I_1$. Then J_3, J_4 are isotropic of dimension 2 and we have

$$J_1 \cap J_3 = \{0\}, \quad J_3 \cap J_4 = \{0\}, \quad J_4 \cap J_2 = \{0\}.$$

If $Z_i \subset X_\Gamma$ is the 1-dimensional cusp associated to J_i, we can apply the result of Section 2B2 successively and obtain

$$\mathbb{Q}[Z_1] = \mathbb{Q}[Z_3] = \mathbb{Q}[Z_4] = \mathbb{Q}[Z_2]$$

in $\mathrm{CH}_1(X_\Gamma)_{\mathbb{Q}}$. This finishes the proof of Theorem 1.1 for 1-dimensional cusps.

3. The symplectic case

In this section we prove Theorem 1.2. We first recall Siegel modular varieties (see [Hulek et al. 1993; Looijenga 2016]). Let Λ be a free $\mathbb{Z}$-module of rank $2g$ equipped with a nondegenerate symplectic form $(\cdot, \cdot) : \Lambda \times \Lambda \to \mathbb{Z}$. Let $\mathrm{Sp}(\Lambda)$ be the symplectic group of Λ, namely the group of isomorphisms $\Lambda \to \Lambda$ preserving the symplectic form. For a natural number N we write $\mathrm{Sp}(\Lambda, N)$ for the kernel of the reduction map $\mathrm{Sp}(\Lambda) \to \mathrm{GL}(\Lambda / N\Lambda)$. A subgroup Γ of $\mathrm{Sp}(\Lambda)$ is called a congruence subgroup if it contains $\mathrm{Sp}(\Lambda, N)$ for some level N. When $g \geq 2$, every finite-index subgroup of $\mathrm{Sp}(\Lambda)$ is a congruence subgroup [Mennicke 1965; Bass et al. 1964].

Let

$$\mathrm{LG}_\Lambda = \{[V] \in G(g, \Lambda_{\mathbb{C}}) \mid (\,\cdot\,,\cdot\,)|_V \equiv 0\}$$

be the Lagrangian Grassmannian parametrizing g-dimensional (= maximal) isotropic $\mathbb{C}$-subspaces of $\Lambda_{\mathbb{C}}$. The Hermitian symmetric domain attached to Λ is defined as the open locus $\mathcal{D}_\Lambda \subset \mathrm{LG}_\Lambda$ of those $[V]$ such that the Hermitian form $i(\,\cdot\,,\bar{\cdot}\,)|_V$ on V is positive definite.

Rational boundary components of $\mathcal{D}_\Lambda$ correspond to isotropic $\mathbb{Q}$-subspaces I of $\Lambda_{\mathbb{Q}}$. To each such I we associate the locus $\mathcal{D}_I \subset \mathrm{LG}_\Lambda$ of those $[V]$ which contains I and for which $i(\,\cdot\,,\bar{\cdot}\,)|_V$ is positive semidefinite with kernel $I_{\mathbb{C}}$. If we consider the rational symplectic space $\Lambda'_{\mathbb{Q}} = I^\perp/I$, then $\mathcal{D}_I$ is canonically isomorphic to the Hermitian symmetric domain $\mathcal{D}_{\Lambda'}$ attached to $\Lambda'_{\mathbb{Q}}$ by mapping $[V] \in \mathcal{D}_I$ to $[V/I_{\mathbb{C}}] \in \mathcal{D}_{\Lambda'}$. The union

$$\mathcal{D}^*_\Lambda = \mathcal{D}_\Lambda \sqcup \bigsqcup_{I \subset \Lambda_{\mathbb{Q}}} \mathcal{D}_I$$

is equipped with the Satake topology [Baily and Borel 1966; Borel and Ji 2005; Hulek et al. 1993]. By [Baily and Borel 1966], the quotient space $X_\Gamma = \Gamma\backslash\mathcal{D}^*_\Lambda$ has the structure of a normal projective variety and contains $\Gamma\backslash\mathcal{D}_\Lambda$ as a Zariski open set.

Theorem 1.2 is proved by induction on g. The case $g = 1$ follows from the Manin–Drinfeld theorem. Let $g \geq 2$. Assume that the theorem is proved for every congruence subgroup of $\mathrm{Sp}(\Lambda')$ for every symplectic lattice Λ' of rank $< 2g$. We then prove the theorem for $\Gamma < \mathrm{Sp}(\Lambda)$ with Λ rank $2g$.

Let $I_1 \neq I_2$ be two isotropic $\mathbb{Q}$-subspaces of $\Lambda_{\mathbb{Q}}$ of the same dimension, say g', and $Z_1, Z_2 \subset X_\Gamma$ the corresponding cusps. If we write $g'' = g - g'$, then Z_i has dimension $k = g''(g''+1)/2$. We consider the following three cases separately:

(1) $I_1 \cap I_2 \neq \{0\}$;

(2) the pairing between I_1 and I_2 is perfect;

(3) $I_1 \cap I_2 = \{0\}$ but the pairing between I_1 and I_2 is not perfect.

The case (1) is studied in Section 3A, where Z_1 and Z_2 are joined by a modular variety in the boundary. The case (2) is studied in Section 3B, where Z_1 and Z_2 are joined by a product of two modular varieties (when $g' = 1$) or by a chain of boundary modular varieties (when $g' > 1$). The remaining case (3) is considered in Section 3C, where we combine the results of (1) and (2).

3A. ***The case $I_1 \cap I_2 \neq \{0\}$.*** Assume that $I_1 \cap I_2 \neq \{0\}$. Let $I = I_1 \cap I_2$. In this case $\mathcal{D}_{I_1}, \mathcal{D}_{I_2}$ are in the boundary of $\mathcal{D}_I$. We set $\Lambda'_{\mathbb{Q}} = I^\perp/I$, $I'_1 = I_1/I$, and $I'_2 = I_2/I$. Then I'_1, I'_2 are isotropic subspaces of $\Lambda'_{\mathbb{Q}}$. The isomorphism $\mathcal{D}_I \to \mathcal{D}_{\Lambda'}$ extends to $\mathcal{D}^*_I \to \mathcal{D}^*_{\Lambda'}$ and maps $\mathcal{D}_{I_i}$ to $\mathcal{D}_{I'_i}$. The stabilizer of I in Γ acts on $\Lambda'_{\mathbb{Q}}$ naturally. Let $\Gamma_I < \mathrm{Sp}(\Lambda'_{\mathbb{Q}})$ be its image in $\mathrm{Sp}(\Lambda'_{\mathbb{Q}})$. By a similar argument as Claim 2.1, Γ_I is a congruence subgroup of $\mathrm{Sp}(\Lambda')$ for some lattice $\Lambda' \subset \Lambda'_{\mathbb{Q}}$. If we put $X_I = \Gamma_I \backslash \mathcal{D}^*_{\Lambda'}$, we have a generically injective morphism $f : X_I \to X_\Gamma$ onto the I-cusp.

Let $Z'_1, Z'_2 \subset X_I$ be the cusps of X_I corresponding to $I'_1, I'_2 \subset \Lambda'_{\mathbb{Q}}$, respectively. By the induction hypothesis, we have $\mathbb{Q}[Z'_1] = \mathbb{Q}[Z'_2]$ in $\mathrm{CH}_k(X_I)_{\mathbb{Q}}$. Since $f(Z'_i) = Z_i$, applying f_* gives $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$.

3B. *The case (I_1, I_2) perfect.* Next we consider the case where the pairing between I_1 and I_2 is perfect. We distinguish the cases $g' > 1$ and $g' = 1$ (i.e., top dimensional cusps).

3B1. *The case $g' > 1$.* First let $g' > 1$. We can choose a proper subspace $J_1 \neq \{0\}$ of I_1. We put $J_2 = J_1^\perp \cap I_2$ and $I_3 = J_1 \oplus J_2$. Then I_3 is isotropic of dimension g'. By construction we have $I_1 \cap I_3 \neq \{0\}$ and $I_3 \cap I_2 \neq \{0\}$. Therefore we can apply the result of Section 3A to I_1 vs. I_3 and to I_3 vs. I_2. If Z_3 is the cusp of X_Γ associated to I_3, this gives $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_3] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$.

3B2. *The case $g' = 1$.* Next let $g' = 1$. We set $\Lambda'_{\mathbb{Q}} = I_1 \oplus I_2$, which is a nondegenerate symplectic space of dimension 2. Then $\Lambda''_{\mathbb{Q}} := (\Lambda'_{\mathbb{Q}})^\perp$ is also nondegenerate of dimension $2g-2$ and we have $\Lambda_{\mathbb{Q}} = \Lambda'_{\mathbb{Q}} \perp \Lambda''_{\mathbb{Q}}$. Let $\mathcal{D}_{\Lambda'}, \mathcal{D}_{\Lambda''}$ be the Hermitian symmetric domains attached to $\Lambda'_{\mathbb{Q}}, \Lambda''_{\mathbb{Q}}$, respectively. We have the embedding of domains

$$\mathcal{D}_{\Lambda'} \times \mathcal{D}_{\Lambda''} \hookrightarrow \mathcal{D}_{\Lambda}, \qquad (V', V'') \mapsto V' \oplus V''. \tag{3.1}$$

This is compatible with the embedding of groups

$$\mathrm{Sp}(\Lambda'_{\mathbb{Q}}) \times \mathrm{Sp}(\Lambda''_{\mathbb{Q}}) \hookrightarrow \mathrm{Sp}(\Lambda_{\mathbb{Q}}), \qquad (\gamma', \gamma'') \mapsto \gamma' \oplus \gamma''. \tag{3.2}$$

The isotropic lines I_1, I_2 in $\Lambda'_{\mathbb{Q}}$ correspond to the respective rational boundary points $[(I_1)_{\mathbb{C}}], [(I_2)_{\mathbb{C}}]$ of $\mathcal{D}_{\Lambda'} \simeq \mathbb{H}$. Then (3.1) extends to $\mathcal{D}^*_{\Lambda'} \times \mathcal{D}^*_{\Lambda''} \hookrightarrow \mathcal{D}^*_{\Lambda}$ and maps $[(I_i)_{\mathbb{C}}] \times \mathcal{D}_{\Lambda''}$ to $\mathcal{D}_{I_i}$.

We take some full lattices $\Lambda' \subset \Lambda'_{\mathbb{Q}}$ and $\Lambda'' \subset \Lambda''_{\mathbb{Q}}$. By the same argument as (2.2), we can find a level N such that (3.2) maps $\mathrm{Sp}(\Lambda', N) \times \mathrm{Sp}(\Lambda'', N)$ into Γ. If we put $X' = \mathrm{Sp}(\Lambda', N) \backslash \mathcal{D}^*_{\Lambda'}$ and $X'' = \mathrm{Sp}(\Lambda'', N) \backslash \mathcal{D}^*_{\Lambda''}$, we thus obtain a finite morphism $f : X' \times X'' \to X_\Gamma$. Let p_1, p_2 be the cusps of the modular curve X' corresponding to $I_1, I_2 \subset \Lambda'_{\mathbb{Q}}$, respectively. If we set

$$Z'_i = p_i \times X'' \subset X' \times X'',$$

the above consideration shows that $f(Z'_i) = Z_i$.

We have $[p_1] = [p_2]$ in $\mathrm{CH}_0(X')_{\mathbb{Q}}$ by the Manin–Drinfeld theorem. Taking the pullback by $X' \times X'' \to X'$, we obtain $[Z'_1] = [Z'_2]$ in $\mathrm{CH}_k(X' \times X'')_{\mathbb{Q}}$. Then, taking the pushforward by f, we obtain $\mathbb{Q}[Z_1] = \mathbb{Q}[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$.

3C. *The remaining case.* Finally we consider the remaining case, namely that $I_1 \cap I_2 = \{0\}$ but the pairing between I_1 and I_2 is not perfect. Let $J_1 \subset I_1$ and $J_2 \subset I_2$ be the kernels of the pairing between I_1 and I_2. We choose splittings $I_1 = J_1 \oplus K_1$ and $I_2 = J_2 \oplus K_2$. Then $\dim J_1 = \dim J_2$ and the pairing between K_1 and K_2 is perfect. (We may have $K_i = \{0\}$. This is the case, e.g., when $g' = 1$.) We set $\Lambda'_{\mathbb{Q}} = K_1 \oplus K_2$ and $\Lambda''_{\mathbb{Q}} = (\Lambda'_{\mathbb{Q}})^{\perp}$, which are nondegenerate subspaces of $\Lambda_{\mathbb{Q}}$ with $\Lambda_{\mathbb{Q}} = \Lambda'_{\mathbb{Q}} \perp \Lambda''_{\mathbb{Q}}$. By definition J_1 and J_2 are isotropic subspaces of $\Lambda''_{\mathbb{Q}}$ with $J_1 \cap J_2 = \{0\}$ and $(J_1, J_2) \equiv 0$. We can take another isotropic subspace J_0 of $\Lambda''_{\mathbb{Q}}$ of the same dimension as J_1, J_2 such that the pairings (J_0, J_1) and (J_0, J_2) are perfect. We set $I_3 = J_0 \oplus K_2$ and $I_4 = J_0 \oplus K_1$. Then I_3, I_4 are isotropic subspaces of $\Lambda_{\mathbb{Q}}$ of the same dimension as I_1, I_2. By construction the pairings (I_1, I_3) and (I_2, I_4) are perfect, and we have $I_3 \cap I_4 \neq \{0\}$. Then we can apply the result of Section 3B to I_1 vs. I_3 and to I_2 vs. I_4, and when $K_i \neq \{0\}$ the result of Section 3A to I_3 vs. I_4. (When $K_i = \{0\}$, so that $I_3 = I_4$, the latter process is skipped.) If Z_3, Z_4 are the cusps of X_Γ associated to I_3, I_4, respectively, this shows that

$$\mathbb{Q}[Z_1] = \mathbb{Q}[Z_3] = \mathbb{Q}[Z_4] = \mathbb{Q}[Z_2]$$

in $\mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$. This completes the proof of Theorem 1.2.

Remark 3.3. Summing up the argument in the case $g' > 1$, we see that if Z_1 and Z_2 are not top dimensional, we can obtain their rational equivalence through a chain of higher dimensional cusps of length ≤ 5.

4. The unitary case

In this section we prove Theorem 1.3. We first recall modular varieties of unitary type; see [Holzapfel 1998; Looijenga 2016]. Let $K = \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic field with $R = \mathcal{O}_K$ its ring of integers (or more generally an order in K). By a Hermitian lattice over R we mean a finitely generated torsion-free R-module Λ equipped with a nondegenerate Hermitian form $(\cdot\,,\cdot) : \Lambda \times \Lambda \to R$. We let $\Lambda_K = \Lambda \otimes_R K$ and $\Lambda_{\mathbb{C}} = \Lambda \otimes_R \mathbb{C}$, which are Hermitian spaces over K, $\mathbb{C}$, respectively, and in which Λ is naturally embedded. We may assume without loss of generality that the signature (p, q) of Λ satisfies $p \leq q$.

Let $\mathrm{U}(\Lambda)$ be the unitary group of Λ, namely the group of R-linear isomorphisms $\Lambda \to \Lambda$ preserving the Hermitian form. This is the same as K-linear isomorphisms $\Lambda_K \to \Lambda_K$ preserving the lattice Λ and the Hermitian form. We write $\mathrm{SU}(\Lambda)$ for the subgroup of $\mathrm{U}(\Lambda)$ of determinant 1. For a natural number N we write $\mathrm{U}(\Lambda, N)$ for the kernel of the reduction map $\mathrm{U}(\Lambda) \to \mathrm{GL}(\Lambda/N\Lambda)$. A subgroup Γ of $\mathrm{U}(\Lambda)$ is called a congruence subgroup if it contains $\mathrm{U}(\Lambda, N)$ for some level N.

Let $G_\Lambda = G(p, \Lambda_{\mathbb{C}})$ be the Grassmannian parametrizing p-dimensional $\mathbb{C}$-linear subspaces of $\Lambda_{\mathbb{C}}$. The Hermitian symmetric domain $\mathcal{D}_\Lambda$ attached to Λ is defined as the open locus

$$\mathcal{D}_\Lambda = \{[V] \in G_\Lambda \mid (\cdot\,,\cdot)|_V > 0\}$$

of subspaces V to which restriction of the Hermitian form is positive definite. When $p = 0$, this is one point; when $p = 1$, this is a ball in $\mathbb{P}\Lambda_{\mathbb{C}} \simeq \mathbb{P}^q$.

Rational boundary components of $\mathcal{D}_\Lambda$ correspond to isotropic K-subspaces I of Λ_K. For each such I we associate the locus $\mathcal{D}_I \subset G_\Lambda$ of those V which contain I and for which $(\cdot\,,\cdot)|_V$ is positive semidefinite with kernel $I_{\mathbb{C}}$. If we consider $\Lambda'_K = I^\perp/I$, this is a nondegenerate K-Hermitian space of signature $(p-r, q-r)$, where $r = \dim_K I$, and $\mathcal{D}_I$ is naturally isomorphic to the Hermitian symmetric domain $\mathcal{D}_{\Lambda'}$ attached to Λ'_K by sending $[V] \in \mathcal{D}_I$ to $[V/I_{\mathbb{C}}]$. The union

$$\mathcal{D}^*_\Lambda = \mathcal{D}_\Lambda \sqcup \bigsqcup_{I \subset \Lambda_K} \mathcal{D}_I$$

is equipped with the Satake topology [Baily and Borel 1966; Borel and Ji 2005]. By [Baily and Borel 1966], the quotient space $X_\Gamma = \Gamma\backslash\mathcal{D}^*_\Lambda$ has the structure of a normal projective variety and contains $\Gamma\backslash\mathcal{D}_\Lambda$ as a Zariski open set.

The proof of Theorem 1.3 proceeds by induction on q. The case $q = 1$ is the Manin–Drinfeld theorem; we explain this in Section 4A. The inductive argument is done in Section 4B. Since this is similar to the symplectic case, we will be brief in Section 4B.

4A. ***The case $q = 1$.*** Let $q = 1$. Then $r = p = q = 1$, so Λ_K is the (unique) K-Hermitian space of signature $(1, 1)$ containing an isotropic vector, and $\mathcal{D}_\Lambda$ is the unit disc in $\mathbb{P}\Lambda_{\mathbb{C}} \simeq \mathbb{P}^1$. The group $\mathrm{SU}(\Lambda_K)$ is naturally isomorphic to $\mathrm{SL}_2(\mathbb{Q})$, and $\Gamma \cap \mathrm{SU}(\Lambda)$ is mapped to a conjugate of a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ under this isomorphism. This is a classical fact, but since we could not find a suitable reference for the second assertion, we give below a self-contained account for the reader's convenience. Theorem 1.3 in the case $q = 1$ then follows from the Manin–Drinfeld theorem, because we have a natural finite morphism from $X_{\Gamma\cap\mathrm{SU}(\Lambda)}$ to X_Γ.

We embed $K = \mathbb{Q}(\sqrt{-D})$ into the matrix algebra $M_2(\mathbb{Q})$ by sending $\sqrt{-D}$ to $J_D = \begin{pmatrix} 0 & -D \\ 1 & 0 \end{pmatrix}$. Left multiplication by J_D makes $M_2(\mathbb{Q})$ a 2-dimensional K-linear space. We have a K-Hermitian form on $M_2(\mathbb{Q})$ defined by

$$(A, B) = \mathrm{tr}(AB^*) + \sqrt{-D}^{\,-1}\mathrm{tr}(J_D AB^*), \qquad A, B \in M_2(\mathbb{Q}),$$

where for $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we write $B^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. We denote $\Lambda_K = M_2(\mathbb{Q})$ when we want to stress this K-Hermitian structure. Then Λ_K has signature $(1, 1)$ and contains an isotropic vector, e.g., $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Right multiplication by $\mathrm{SL}_2(\mathbb{Q})$ on $M_2(\mathbb{Q})$ is K-linear and preserves this Hermitian form. This defines a homomorphism

$$\mathrm{SL}_2(\mathbb{Q}) \to \mathrm{SU}(\Lambda_K) \tag{4.1}$$

which in fact is an isomorphism; see, e.g., [Shimura 1964, §2].

Let $\Lambda \subset \Lambda_K$ be a full R-lattice. We shall show that for every level N the image of $\mathrm{SU}(\Lambda, N) = \mathrm{SU}(\Lambda_K) \cap \mathrm{U}(\Lambda, N)$ by (4.1) is conjugate to a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Let

$$\mathcal{O} = \{X \in M_2(\mathbb{Q}) \mid \Lambda X \subset \Lambda\}.$$

This is an order in $M_2(\mathbb{Q})$; see [Maclachlan and Reid 2003, §2.2]. Then $\mathrm{SU}(\Lambda) = \mathcal{O}^1$, where for any subset $\mathcal{S}$ of $M_2(\mathbb{Q})$ we write $\mathcal{S}^1 = \mathcal{S} \cap \mathrm{SL}_2(\mathbb{Q})$. Take a maximal order $\mathcal{O}_{\max}$ of $M_2(\mathbb{Q})$ containing $\mathcal{O}$. Since $\mathcal{O}$ is of finite index in $\mathcal{O}_{\max}$, there exists a natural number N_0 such that $N_0\mathcal{O}_{\max} \subset \mathcal{O}$. Therefore,

$$I + NN_0\mathcal{O}_{\max} \subset I + N\mathcal{O} \subset \mathcal{O} \subset \mathcal{O}_{\max}.$$

Since $(I + N\mathcal{O})^1 \subset \mathrm{SU}(\Lambda, N)$, this implies that

$$(I + NN_0\mathcal{O}_{\max})^1 \subset \mathrm{SU}(\Lambda, N) \subset \mathrm{SU}(\Lambda) \subset \mathcal{O}^1_{\max}.$$

Since every maximal order of $M_2(\mathbb{Q})$ is conjugate to $M_2(\mathbb{Z})$, there exists $g \in \mathrm{GL}_2(\mathbb{Q})$ such that

$$\Gamma(NN_0) \subset \mathrm{Ad}_g(\mathrm{SU}(\Lambda, N)) \subset \mathrm{Ad}_g(\mathrm{SU}(\Lambda)) \subset \mathrm{SL}_2(\mathbb{Z}).$$

This proves our claim.

4B. ***Inductive step.*** Let $q \geq 2$. Suppose that Theorem 1.3 is proved for all Hermitian lattices of signature (p', q') with $p' \leq q' < q$. We then prove the theorem for Hermitian lattices of signature (p, q) with $p \leq q$. Since the argument is similar to the symplectic case, we will just indicate the outline. Let $I_1 \neq I_2$ be two isotropic K-subspaces of Λ_K of the same dimension, say r, and $Z_1, Z_2 \subset X_\Gamma$ the associated cusps. We make the following classification:

(1) $I_1 \cap I_2 \neq \{0\}$;

(2) the pairing between I_1 and I_2 is perfect;

(3) $I_1 \cap I_2 = \{0\}$ but the pairing between I_1 and I_2 is not perfect.

(1) This is similar to Section 3A. In this case Z_1 and Z_2 are joined by the cusp associated to $I_1 \cap I_2$, to which we can apply the induction hypothesis.

(2) The case $r = 1$ is similar to Section 3B2. If we set $\Lambda'_K = I_1 \oplus I_2$ and $\Lambda''_K = (\Lambda'_K)^\perp$, these are nondegenerate of signature $(1, 1)$ and $(p-1, q-1)$, respectively. Then Z_1 and Z_2 are joined by the embedding $\mathcal{D}_{\Lambda'} \times \mathcal{D}_{\Lambda''} \hookrightarrow \mathcal{D}_\Lambda$. We can apply the Manin–Drinfeld theorem to $\mathcal{D}_{\Lambda'}$.

The case $r > 1$ is similar to Section 3B1. We can interpolate Z_1 and Z_2 by a third cusp by taking a proper subspace $J_1 \neq \{0\}$ of I_1 and setting $I_3 = J_1 \oplus (J_1^\perp \cap I_2)$. Then we can apply the result of case (1) to I_1 vs. I_3 and to I_3 vs. I_2.

(3) This is similar to Section 3C. We take splittings $I_1 = J_1 \oplus K_1$ and $I_2 = J_2 \oplus K_2$ such that $(J_1, I_2) \equiv 0$, $(J_2, I_1) \equiv 0$ and (K_1, K_2) is perfect. We choose an isotropic subspace J_0 from $(K_1 \oplus K_2)^\perp$ with (J_1, J_0) and (J_2, J_0) perfect, and put $I_3 = J_0 \oplus K_2$ and $I_4 = J_0 \oplus K_1$. Then we apply case (2) to I_1 vs. I_3 and to I_4 vs. I_2, and case (1) to I_3 vs. I_4 when $K_i \neq \{0\}$. This proves Theorem 1.3.

Remark 4.2. As in the symplectic case, we see that when Z_1, Z_2 are not top dimensional, their rational equivalence can be obtained through a chain of higher dimensional cusps of length ≤ 5.

5. Modular units and higher Chow cycles

Let Γ, $\mathcal{D}_\Lambda$, and X_Γ be as in the previous sections. As a consequence of Theorems 1.1–1.3, we can associate to each pair of maximal cusps of X_Γ a nonzero higher Chow cycle of the modular variety $Y_\Gamma = \Gamma \backslash \mathcal{D}_\Lambda$. This gives a higher dimensional analogue of modular units [Kubert and Lang 1981] from the viewpoint of algebraic cycles.

Let $Z_1 \neq Z_2$ be two cusps of X_Γ of the same dimension, say k. By our result, we have $[Z_1] = \alpha[Z_2]$ in $\mathrm{CH}_k(X_\Gamma)_\mathbb{Q}$ for some $\alpha \neq 0 \in \mathbb{Q}$. On the other hand, we can also view Z_1, Z_2 as k-cycles on the boundary $\partial X_\Gamma = X_\Gamma - Y_\Gamma$, which is an equidimensional reduced closed subscheme of X_Γ.

Lemma 5.1. *When the cusps Z_1, Z_2 are not top dimensional, $[Z_1] = \alpha[Z_2]$ holds already in $\mathrm{CH}_k(\partial X_\Gamma)_\mathbb{Q}$.*

Proof. When Z_1, Z_2 are not top dimensional, the proofs of Theorems 1.1–1.3 and Remarks 2.4, 3.3, and 4.2 show that we can connect Z_1 and Z_2 by a chain of higher dimensional cusps. To be more precise, we have (congruence) modular varieties $X_1, \dots, X_N$, their cusps $Z_i^+, Z_i^- \subset X_i$ of dimension k, and a finite morphism $f_i : X_i \to X_\Gamma$ onto a cusp of X_Γ, such that $f_i(Z_i^-) = f_{i+1}(Z_{i+1}^+)$ for each i and $f_1(Z_1^+) = Z_1$, $f_N(Z_N^-) = Z_2$. By induction on dimension, we have $[Z_i^+] = \alpha_i [Z_i^-]$ in $\mathrm{CH}_k(X_i)_\mathbb{Q}$ for some $\alpha_i \in \mathbb{Q}$. Since f_i factors through

$$X_i \to \partial X_\Gamma \subset X_\Gamma,$$

we have

$$[f_i(Z_i^+)] = \alpha_i' [f_i(Z_i^-)]$$

in $\mathrm{CH}_k(\partial X_\Gamma)_\mathbb{Q}$ for some $\alpha_i' \in \mathbb{Q}$. It follows that

$$[Z_1] = \left(\prod_i \alpha_i' \right) [Z_2]$$

in $\mathrm{CH}_k(\partial X_\Gamma)_\mathbb{Q}$. □

Consider the localization exact sequence of higher Chow groups [Bloch 1986; 1994] for the Baily–Borel compactification

$$Y_\Gamma \overset{j}{\hookrightarrow} X_\Gamma \overset{i}{\hookleftarrow} \partial X_\Gamma.$$

The first few terms of this sequence are written as

$$\cdots \to \mathrm{CH}_k(X_\Gamma, 1)_{\mathbb{Q}} \overset{j^*}{\to} \mathrm{CH}_k(Y_\Gamma, 1)_{\mathbb{Q}} \overset{\delta}{\to} \mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}} \overset{i_*}{\to} \mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}} \to \cdots,$$

where δ is the connecting map. By Lemma 5.1, the $\mathbb{Q}$-linear subspace of $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$ generated by the k-dimensional cusps has dimension 1 if k is not the maximal dimension of cusps. On the other hand, when $k = \dim \partial X_\Gamma$, the k-dimensional (= maximal) cusps are irreducible components of ∂X_Γ, so $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$ is freely generated over $\mathbb{Q}$ by those cusps. Let t be the number of maximal cusps of X_Γ. Since the image of $i_* : \mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}} \to \mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$ has dimension 1 by Theorems 1.1–1.3, we find that

$$\dim \mathrm{Im}(\delta) = \dim \mathrm{Ker}(i_*) = t - 1.$$

Let us construct some explicit elements of $\mathrm{CH}_k(Y_\Gamma, 1)_{\mathbb{Q}}$ whose images by δ generate $\mathrm{Im}(\delta) = \mathrm{Ker}(i_*)$.

Let $Z_1 \neq Z_2$ be two maximal cusps of X_Γ, say of dimension $k = \dim \partial X_\Gamma$. As above, we have $i_*(Z_1 - \alpha Z_2) = 0$ in $\mathrm{CH}_k(X_\Gamma)_{\mathbb{Q}}$ for some $\alpha \in \mathbb{Q}$. We construct an element of $\mathrm{CH}_k(Y_\Gamma, 1)_{\mathbb{Q}}$ whose image by δ is $Z_1 - \alpha Z_2$ in $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$. (Such an element must be nonzero because $Z_1 - \alpha Z_2$ is nonzero in $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$.) Recall from the proof of Theorems 1.1–1.3 that, in a basic case, we have a compactified modular curve $X' = X_{\Gamma'}$, its two cusps $p_1, p_2 \in X'$, a k-dimensional compactified modular variety $X'' = X_{\Gamma''}$, and a finite morphism $f : X' \times X'' \to X_\Gamma$ such that $f(p_i \times X'') = Z_i$. (In the orthogonal case X'' is one point when $k = 0$ and a modular curve when $k = 1$; in the symplectic case X'' is a Siegel modular variety of genus $g - 1$; in the unitary case X'' is associated to a unitary group of signature $(p - 1, q - 1)$.) The general case is a chain of such basic cases. For simplicity we assume that (Z_1, Z_2) is such a basic pair.

By the Manin–Drinfeld theorem for X', there exists a modular function F on X' such that $\mathrm{div}(F) = \beta(p_1 - p_2)$ for some natural number β. Let $Y' \subset X'$ and $Y'' \subset X''$ be the modular varieties before compactification. We can view F as an element of $\mathcal{O}^*(Y') = \mathrm{CH}_0(Y', 1)$. Then $\delta(F) = \beta(p_1 - p_2)$ for the connecting map $\delta : \mathrm{CH}_0(Y', 1) \to \mathrm{CH}_0(\partial X')$. Let $\pi : Y' \times Y'' \to Y'$ be the projection and, by abuse of notation, $f : Y' \times Y'' \to Y_\Gamma$ be the restriction of $f : X' \times X'' \to X_\Gamma$. We can pullback the higher Chow cycle F by the flat morphism π and then take its pushforward by the finite morphism f. The result, $f_* \pi^* F$, is an element of $\mathrm{CH}_k(Y_\Gamma, 1)$.

Proposition 5.2. *We have* $\mathbb{Q}\delta(f_*\pi^*F) = \mathbb{Q}(Z_1 - \alpha Z_2)$ *in* $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$.

Proof. We take a desingularization $\widetilde{X}'' \to X''$ of X'', and let $\widetilde{Y}'' \subset \widetilde{X}''$ be the inverse image of Y''. We have the commutative diagram

$$
\begin{array}{ccccccc}
\mathcal{O}^*(Y') & \xrightarrow[\simeq]{\tilde{\pi}^*} & \mathcal{O}^*(Y' \times \widetilde{X}'') & \xrightarrow{\tilde{j}^*} & \mathcal{O}^*(Y' \times \widetilde{Y}'') & & \\
\| & & \| & & \| & & \\
\mathrm{CH}_0(Y', 1) & \xrightarrow[\simeq]{\tilde{\pi}^*} & \mathrm{CH}_k(Y' \times \widetilde{X}'', 1) & \xrightarrow{\tilde{j}^*} & \mathrm{CH}_k(Y' \times \widetilde{Y}'', 1) & \xrightarrow{\tilde{f}_*} & \mathrm{CH}_k(Y_\Gamma, 1) \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\
\mathrm{CH}_0(\partial X') & \xrightarrow[\simeq]{\tilde{\pi}^*} & \mathrm{CH}_k(\partial X' \times \widetilde{X}'') & \xrightarrow{\tilde{i}_*} & \mathrm{CH}_k(\partial(X' \times \widetilde{X}'')) & \xrightarrow{\tilde{f}_*} & \mathrm{CH}_k(\partial X_\Gamma)
\end{array}
$$

The various δ are the connecting maps of each localization sequence, $\tilde{\pi} : X' \times \widetilde{X}'' \to X'$ the projection, $\partial(X' \times \widetilde{X}'') = X' \times \widetilde{X}'' - Y' \times \widetilde{Y}''$, $\tilde{j} : Y' \times \widetilde{Y}'' \hookrightarrow Y' \times \widetilde{X}''$ the open immersion, $\tilde{i} : \partial X' \times \widetilde{X}'' \hookrightarrow \partial(X' \times \widetilde{X}'')$ the closed embedding, and $\tilde{f} : X' \times \widetilde{X}'' \to X_\Gamma$ the proper morphism induced from f. If we send $\mathbb{Q}F \subset \mathrm{CH}_0(Y', 1)_{\mathbb{Q}}$ through this diagram to $\mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$, the image is $\mathbb{Q}(Z_1 - \alpha Z_2)$. The assertion follows by noticing that $\tilde{f}_* \tilde{j}^* \tilde{\pi}^* = f_* \pi^*$. □

In this way, as a "lift" from the modular unit F, we obtain an explicit nonzero element of $\mathrm{CH}_k(Y_\Gamma, 1)_{\mathbb{Q}}$ whose image by δ is $Z_1 - \alpha Z_2$. If we run (Z_1, Z_2) over all basic pairs of maximal cusps, we obtain a set of nonzero elements of $\mathrm{CH}_k(Y_\Gamma, 1)_{\mathbb{Q}}$ whose image by δ generate $\mathrm{Im}(\delta) = \mathrm{Ker}(i_*)$. In general, by this construction we could obtain more than $t - 1$ higher Chow cycles on Y_Γ. This is because

(1) the choice of $X' \times X'' \to X_\Gamma$ is not necessarily unique for the given pair (Z_1, Z_2), and

(2) the number of basic pairs could be larger than $t - 1$.

The point (1) amounts to the situation that two pairs (I_1, I_2), (I_1', I_2') of isotropic subspaces are not Γ-equivalent as pairs, although I_1 is Γ-equivalent to I_1' and I_2 is Γ-equivalent to I_2', respectively. A typical situation of (2) is that for three cusps Z_1, Z_2, Z_3, all pairs (Z_1, Z_2), (Z_2, Z_3), (Z_3, Z_1) are basic.

If the span $V \subset \mathrm{CH}_k(Y_\Gamma, 1)_{\mathbb{Q}}$ of all higher Chow cycles constructed in this way has dimension $\geq t$, the kernel of $\delta : V \to \mathrm{CH}_k(\partial X_\Gamma)_{\mathbb{Q}}$ would then give rise to a nontrivial subspace of $\mathrm{CH}_k(X_\Gamma, 1)_{\mathbb{Q}}$.

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