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Building on work by Kasparov, we study the notion of Spanier–Whitehead K-duality for a discrete group. It is defined as duality in the KK-category between two C^*-algebras which are naturally attached to the group, namely the reduced group C^*-algebra and the crossed product for the group action on the universal example for proper actions. We compare this notion to the Baum–Connes conjecture by constructing duality classes based on two methods: the standard "gamma element" technique, and the more recent approach via cycles with property gamma. As a result of our analysis, we prove Spanier–Whitehead duality for a large class of groups, including Bieberbach's space groups, groups acting on trees, and lattices in Lorentz groups.

Introduction

Alexander duality applies to the homology theory properties of the complement of a subspace inside a sphere in Euclidean space. More precisely, for a finite complex X contained in S^{n+1}, if $\widetilde{H}$ denotes reduced homology or cohomology with coefficients in a given abelian group, there is an isomorphism $\widetilde{H}_i(X) \cong \widetilde{H}^{n-i}(S^{n+1} \setminus X)$, induced by slant product with the pullback of the generator $\mu^*([S^n])$, via the duality map $\mu : X \times (S^{n+1} \setminus X) \to S^n$, $\mu(x, y) = (x-y)/\|x-y\|$.

Ed Spanier and J. H. C. Whitehead generalized this statement and adapted it to the context of stable homotopy theory. Their basic intuition was that sphere complements determine the homology, but not the homotopy type, in general. However, the stable homotopy type can be deduced and provides a first approximation to homotopy type [Spanier and Whitehead 1958]. Thus, the modern statement is phrased in terms of dual objects X, DX in the category of pointed spectra with the smash product as a monoidal structure, and by taking maps to an Eilenberg–Mac Lane spectrum one recovers Alexander duality formally.

The modern version of the duality implies Poincaré duality for compact manifolds and extends in a natural way to generalized cohomology theories such as K-theory. In this setting, a compact spin^c-manifold exhibits Poincaré duality in

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the sense that the K-homology class of the Dirac operator induces by cap product an isomorphism $K^*(M) \to K_{*+n}(M)$, where the shift is given by the dimension [Kasparov 1988].

More generally, the bivariant version of K-theory introduced by Kasparov, which we shall use extensively in this paper, showcases a close relationship to Alexander–Spanier duality; by this we mean that for X, Y finite complexes one has a chain of isomorphisms [Kaminker and Schochet 2019]

$$\mathrm{KK}_*(C(X), C(Y)) \cong \mathrm{KK}_*(\mathbb{C}, C(DX \wedge Y)) \cong K_*(C(DX \wedge Y)) \cong K^*(DX \wedge Y).$$

Having introduced C^*-algebras in this way, as they arise naturally in applications to topology, dynamics, and index theory, and are generally noncommutative, it is natural to seek for generalizations of Spanier–Whitehead duality in the framework of noncommutative geometry.

For a separable, nuclear C^*-algebra A represented on a Hilbert space, the commutant of its projection into the Calkin algebra has some of the properties reminiscent of a Spanier–Whitehead K-dual. This is the *Paschke dual* of A, and satisfies $K_*(P(A)) \cong K^*(A)$. However, in general $P(A)$ is neither separable nor nuclear, the Kasparov product is not defined, so that it seems desirable to explore different routes for the definition of a K-dual.

A. Connes [1994] introduced the appropriate formalism for this question, which shall be described shortly, and in [Connes 1996] he showed the first nontrivial example of a noncommutative Poincaré duality algebra, in the form of the irrational rotation algebra. H. Emerson [2003] proved the same result for the crossed product of a hyperbolic group acting on its Gromov boundary. Examples of pairs of algebras with Spanier–Whitehead duality were also given by Kaminker and Putnam [1997] in the case of Cuntz–Krieger algebras associated to M and its transpose, where M is a square $\{0, 1\}$-valued matrix. Their result is a special case of a more general one, in which the stable and unstable Ruelle algebras of a Smale space are shown to be in duality [Kaminker et al. 2017]. Duality in K-theory also appears in connection with string theory on noncommutative spacetimes [Brodzki et al. 2008; 2009].

In this paper, G is a discrete group which admits a G-compact model $\underline{E}G$ of the classifying space for proper actions [Baum et al. 1994]. We study the question of Spanier–Whitehead duality for the pair of C^*-algebras $C^*_r(G)$ and $C_0(\underline{E}G) \rtimes G$, where the latter is the crossed product for the group action on $\underline{E}G$.

This problem is tightly related to the Baum–Connes conjecture and in particular to the so-called Dirac dual-Dirac method. This goes back to the seminal work of Kasparov [1988, Sections 4 and 6] and is further explored in [Kasparov and Skandalis 1991, Section 6]. In a different direction, the relationship between the assembly map and Fourier–Mukai duality is discussed in [Block 2010].

The idea of an underlying noncommutative duality whenever Dirac and dual-Dirac classes are available is well-known to experts; see for example [Brodzki et al. 2008, Example 2.14; Echterhoff et al. 2008, Theorems 2.9 and 3.1]. In particular work of Emerson and Meyer [2010] shares many ideas with the present paper, while working in the context of equivariant KK-theory and groupoids. See page 472 and Remark 1.18 for more details.

Below are two main results of this paper. More details on statements and terminology are given in the sequel.

Theorem. *Suppose the γ-element exists. Then $C_0(\underline{E}G)\rtimes G$ is a Spanier–Whitehead K-dual of $C^*_r(G)$ (in a canonical way) if and only if G satisfies the strong Baum–Connes conjecture.*

Corollary. *For all a-T-menable groups G which admit a G-compact model of $\underline{E}G$, $C_0(\underline{E}G)\rtimes G$ is a Spanier–Whitehead K-dual of $C^*_r(G)$.*

Noncommutative Spanier–Whitehead duality. Let us see the main notions we will be working with. In what follows the C^*-tensor product is understood to be spatial.

Definition 0.1 (cf. [Brodzki et al. 2008, Section 2.7]). Let A, B be separable C^*-algebras. B is called a *weak Spanier–Whitehead K-dual* of A if there are elements

$$d\in \mathrm{KK}_i(A\otimes B,\mathbb{C}) \quad\text{and}\quad \delta\in \mathrm{KK}_{-i}(\mathbb{C},A\otimes B)$$

such that the induced maps

$$\begin{aligned} d_j &: K_j(A)\to K^{j+i}(B), & d_j(x)&=x\,\widehat{\otimes}_A\, d,\\ \delta_j &: K^j(B)\to K_{j-i}(A), & \delta_j(x)&=\delta\,\widehat{\otimes}_B\, x \end{aligned}$$

are isomorphisms and inverses to each other.

Note that, unlike the case of topological spaces, in the noncommutative context the existence of d, given δ, is an additional requirement.

Some notation: $1_A\in \mathrm{KK}_0(A,A)$ stands for the ring unit, $\sigma: A\otimes B\cong B\otimes A$ denotes the flip isomorphism. Recall as well the homomorphism

$$\tau_B:\mathrm{KK}_*(A,A)\to \mathrm{KK}_*(A\otimes B,A\otimes B),$$

given on cycles as

$$(\phi,H,T)\mapsto(\phi\,\widehat{\otimes}\,1,\,H\,\widehat{\otimes}\,B,\,T\,\widehat{\otimes}\,1),$$

and equally defined via Kasparov product (over the complex numbers) by $\tau_B(x)=x\,\widehat{\otimes}\,1_B=1_B\,\widehat{\otimes}\,x$.

Lemma 0.2 [Emerson 2003, Lemma 9]. *In the setting of Definition 0.1, we have the identities*

$$(\delta_{j+i}\circ d_j)(y)=(-1)^{ij}\,y\,\widehat{\otimes}_A\,\Lambda_A \quad\textit{and}\quad (d_{j-i}\circ\delta_j)(y)=(-1)^{ij}\,\Lambda_B\,\widehat{\otimes}_B\,y,$$

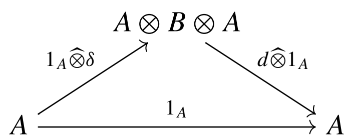
where the elements $\Lambda_A \in \mathrm{KK}_0(A, A)$ *and* $\Lambda_B \in \mathrm{KK}_0(B, B)$ *are defined as*

$$\Lambda_A = \delta \mathbin{\widehat{\otimes}}_B d = (\delta \mathbin{\widehat{\otimes}} 1_A) \mathbin{\widehat{\otimes}}_{A\otimes B\otimes A} (1_A \mathbin{\widehat{\otimes}} \sigma^*(d)),$$
$$\Lambda_B = \delta \mathbin{\widehat{\otimes}}_A d = (\sigma_*(\delta) \mathbin{\widehat{\otimes}} 1_B) \mathbin{\widehat{\otimes}}_{B\otimes A\otimes B} (1_B \mathbin{\widehat{\otimes}} d).$$

Definition 0.3. Let A, B denote C^*-algebras in weak Spanier–Whitehead duality. With notation from Lemma 0.2, if we have $\Lambda_A = 1_A$ and $\Lambda_B = (-1)^i 1_B$, we say that A and B satisfy *Spanier–Whitehead K-duality*.

Note that this definition is symmetric, so that it can equivalently be phrased by saying that B is a Spanier–Whitehead K-dual of A, in alignment with the weak form introduced earlier.

Remark 0.4. In the tensor category (KK, $\otimes$), where objects are C^*-algebras and $\mathrm{Hom}(A, B) = \mathrm{KK}_0(A, B)$, the previous definition (for $i = j = 0$) can be reinterpreted as the statement that A is a dualizable object and B is its dual. In other words the triangle identity



(and its analogue swapping A and B) holds up to the unique isomorphisms coming from braiding and $A \otimes \mathbb{C} \cong A$.

The Spanier–Whitehead K-dual respects tensor products in the following sense: if the dual of A is B and the dual of A' is B', then the dual of $A \otimes B$ is KK-equivalent to $A' \otimes B'$, provided it exists; see [Kaminker and Schochet 2019].

Throughout this paper G denotes a countable discrete group admitting a G-compact model for its universal example for proper actions.

Definition 0.5. *G has (weak) Spanier–Whitehead K-duality* if $C_0(\underline{E}G) \rtimes G$ is a (weak) dual of $C^*_r(G)$.

Remark 0.6. It follows from [Anantharaman-Delaroche 2002, Proposition 2.2] that the action of G on $\underline{E}G$ is amenable. Then by [Anantharaman-Delaroche 2002, Theorem 5.3] the associated full and reduced crossed products are isomorphic. In particular, any covariant pair of representations for $C_0(\underline{E}G)$ and G gives rise to a representation of the reduced crossed product $C_0(\underline{E}G) \rtimes G$, namely the integrated form.

In short, the aim of this paper is identifying an element x belonging to the "representation ring" $\mathrm{KK}^G_0(\mathbb{C}, \mathbb{C})$, and constructing classes d and δ as above in such a way that $\Lambda_{C^*_r(G)}$ and $\Lambda_{C_0(\underline{E}G)\rtimes G}$ are both expressible in terms of x. Then the sought duality is reduced to studying the homotopy class of such an element.

Baum–Connes conjecture: the duality perspective. The Baum–Connes conjecture [Baum et al. 1994] states that the Baum–Connes assembly map

$$\mu^G : \mathrm{KK}^G_*(C_0(\underline{E}G), \mathbb{C}) \to \mathrm{KK}_*(\mathbb{C}, C^*_r(G)) \tag{0.7}$$

is an isomorphism of abelian groups. A generalization "with coefficients" can be introduced by inserting a G-algebra A in the right "slot" of the left-hand side of (0.7), and by considering the corresponding reduced crossed product in the target group:

$$\mu^G_A : \mathrm{KK}^G_*(C_0(\underline{E}G), A) \to \mathrm{KK}(\mathbb{C}, A \rtimes_r G). \tag{0.8}$$

Going back to the case with trivial coefficients (i.e., $A = \mathbb{C}$), since G is a discrete group, the (dual) Green–Julg isomorphism [Blackadar 1998; Kaad and Proietti 2018; Land 2015]

$$\mathrm{KK}^G_*(C_0(\underline{E}G), \mathbb{C}) \cong \mathrm{KK}_*(C_0(\underline{E}G) \rtimes_r G, \mathbb{C})$$

allows us to view the assembly map as a morphism

$$\mathrm{KK}_*(C_0(\underline{E}G) \rtimes G, \mathbb{C}) \to \mathrm{KK}_*(\mathbb{C}, C^*_r(G)). \tag{0.9}$$

We shall see that this map is given by Kasparov product with a certain element

$$\delta \in \mathrm{KK}(\mathbb{C}, C^*_r(G) \otimes C_0(\underline{E}G) \rtimes G)$$

(see Definition 1.1). Thus, the Baum–Connes conjecture for a discrete group G admitting a G-compact model $\underline{E}G$ is equivalent to the assertion that the element δ induces the isomorphism

$$\delta_* : K^*(C_0(\underline{E}G) \rtimes G) \xrightarrow{\cong} K_*(C^*_r(G)).$$

A priori, this isomorphism itself is not enough to conclude that G has weak Spanier–Whitehead K-duality. In this paper, under an assumption (see below), we identify an element

$$d \in \mathrm{KK}(C^*_r(G) \otimes C_0(\underline{E}G) \rtimes G, \mathbb{C})$$

which induces a map

$$d_* : K_*(C^*_r(G)) \to K^*(C_0(\underline{E}G) \rtimes G)$$

which is the inverse of δ_* in favorable circumstances, namely if the Baum–Connes conjecture holds (it is a left inverse in general). Our assumption for constructing such an element d is the existence of the so-called gamma element, or alternatively the (γ)-element for G. Let us briefly review these notions.

The γ-element and the (γ)-element. The following notion of the gamma element originates in [Kasparov 1988].

Definition 0.10 [Tu 2000]. An element x in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ is called a *gamma element* for G if it satisfies the following:

(1) For any finite subgroup $F \subseteq G$, we have

$$\mathrm{Res}_G^F(x) = 1_{\mathbb{C}} \in \mathrm{KK}^F(\mathbb{C}, \mathbb{C}).$$

(2) For some separable, proper G-C^*-algebra P, we have

$$x = \beta \mathbin{\widehat{\otimes}}_P \alpha, \quad \text{where } \alpha \in \mathrm{KK}^G(P, \mathbb{C}),\ \beta \in \mathrm{KK}^G(\mathbb{C}, P).$$

A gamma element for G, if it exists, is a unique idempotent in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ which is characterized by the listed properties. Thus, we call it the gamma element for G and denote it by γ. The existence of the gamma element for G implies that the Baum–Connes assembly map is split-injective for all coefficients A [Tu 2000], and furthermore that the assembly map μ_A^G is surjective if and only if γ acts as the identity on $K_*(A \rtimes_r G)$ via ring homomorphisms

$$\mathrm{KK}^G(\mathbb{C}, \mathbb{C}) \to \mathrm{KK}^G(A, A) \to \mathrm{KK}(A \rtimes_r G, A \rtimes_r G) \to \mathrm{End}(K_*(A \rtimes_r G)). \quad (0.11)$$

The other composition $y = \alpha \mathbin{\widehat{\otimes}} \beta$ is an idempotent in $\mathrm{KK}(P, P)$ which may not be the identity on P in general. Upon replacing P with its "summand" $P_{\mathbb{C}} = yP$, which can be defined as a limit of $P \xrightarrow{y} P \xrightarrow{y} \cdots$ in the category KK^G [Neeman 2001, Proposition 1.6.8], we can arrange α (and β) above to be a weak-equivalence, meaning that $\mathrm{Res}_G^F(\alpha)$ is an isomorphism for any finite subgroup F of G. In this case, the element α in $\mathrm{KK}^G(P_{\mathbb{C}}, \mathbb{C})$ is called the *Dirac element* and can be characterized up to equivalence by the fact that α is a weak-equivalence from a "proper object" $P_{\mathbb{C}}$ to $\mathbb{C}$. Meyer and Nest [2006] showed that the Dirac element always exists for any group G but, in general, it is not known whether $P_{\mathbb{C}}$ can be taken to be a proper C^*-algebra. For most of the known examples, $P_{\mathbb{C}}$ can indeed be assumed to be proper, meaning that we may think $P = P_{\mathbb{C}}$. However, we emphasize that the algebra P appearing in the definition can be quite arbitrary, whereas $P_{\mathbb{C}}$ is a uniquely characterized object.

In [Nishikawa 2019], the first author introduced a notion called the (γ)-element, which can be thought of as an alternative description of the gamma element, bypassing the necessity of a proper algebra P for its definition.

Recall that we assume that G admits a G-compact model for $\underline{E}G$. We use $[-, -]$ to denote the commutator.

Definition 0.12 [Nishikawa 2019, Definition 2.2]. A cycle (H, T) representing an element $[H, T]$ in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ is said to have *property* (γ) if it satisfies the following:

(1) For any finite subgroup $F \subseteq G$, we have

$$\mathrm{Res}_G^F([H, T]) = 1_{\mathbb{C}} \in \mathrm{KK}^F(\mathbb{C}, \mathbb{C}).$$

(2) There is a nondegenerate G-equivariant representation of $C_0(\underline{E}G)$ on H such that

(2a) the function

$$g \mapsto [g \cdot f, T]$$

belongs to $C_0(G, K(H))$: it vanishes at infinity and is compact-operator-valued for any $f \in C_0(\underline{E}G)$;

(2b) for some cutoff function $c \in C_c(\underline{E}G)$ (i.e., c is nonnegative and satisfies $\sum_{g \in G} g(c)^2 = 1$), we have

$$T - \sum_{g \in G} (g \cdot c) T (g \cdot c) \in K(H).$$

An element x in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ is called a (γ)-element for G if it is represented by some cycle with property (γ).

A (γ)-element for G, if it exists, is a unique idempotent in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ which is characterized by the listed properties. Thus, we call it the (γ)-element for G. If there is a gamma element γ for G, there is a cycle with property (γ) representing γ. Thus the two notions, the γ-element and the (γ)-element for G, coincide when γ exists. The existence of the (γ)-element x for G implies that the Baum–Connes assembly map is split-injective for all coefficients A, and furthermore that the assembly map μ_A^G is surjective if and only if x acts as the identity on $K_*(A \rtimes_r G)$ via ring homomorphisms (0.11).

Given the existence of the (γ)-element, [Nishikawa 2019] introduced the so-called (γ)-morphism as a candidate for inverting the assembly map μ^G. This is given by Kasparov product with a certain element

$$\tilde{x} \in \mathrm{KK}^G(C_r^*(G) \otimes C_0(\underline{E}G), \mathbb{C}).$$

The Green–Julg isomorphism allows us to get the corresponding element

$$d \in \mathrm{KK}(C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G, \mathbb{C}).$$

Our proposed strategy aims at realizing weak Spanier–Whitehead duality through elements δ and d respectively corresponding to the assembly map and the (γ)-morphism, which seems to be a natural situation. Furthermore, as a result of Lemma 0.2, the surjectivity and injectivity of the assembly map are controlled by $\Lambda_{C_r^*(G)}$ and $\Lambda_{C_0(\underline{E}G) \rtimes G}$, respectively. This gives yet another interpretation of these two classes.

Equivariant Kasparov duality. In [Emerson and Meyer 2010] the authors study several duality isomorphisms between equivariant bivariant K-theory groups, generalizing Kasparov's first and second Poincaré duality isomorphisms. For many groupoids, both dualities apply to a universal proper G-space, which is the basis for the Dirac dual-Dirac method. In this setting they explain how to describe the Baum–Connes assembly map via localization of categories as in [Meyer and Nest 2006].

The main notion in [Emerson and Meyer 2010] is that of *equivariant Kasparov dual* for a G-space X. It involves an $X \rtimes G$-C^*-algebra P, an element $\alpha \in \mathrm{KK}^G(P, \mathbb{C})$, and an additional class $\Theta \in \mathrm{RKK}^G(X; \mathbb{C}, P)$ (see [Emerson and Meyer 2010, Definition 4.1] for more details). Recall that the category $\mathrm{RKK}^G(X)$ coincides with the range of the pullback functor $p_X^* : \mathrm{KK}^G \to \mathrm{KK}^{X \rtimes G}$ via the collapsing map $p : X \to *$.

The case $X = \underline{E}G$ is particularly relevant for our purposes. The class Θ may be thought as the "inverse" of α up to restriction to finite subgroups. More precisely, if a lifting $\beta \in \mathrm{KK}^G(\mathbb{C}, P)$ of Θ exists, then the axioms of equivariant Kasparov duality guarantee that $\beta \widehat{\otimes}_P \alpha$ is the γ-element and $\alpha \widehat{\otimes}_{\mathbb{C}} \beta = 1_P$. In particular, we have $P = P_{\mathbb{C}}$ and α is a weak equivalence, and hence a Dirac morphism.

Let Z denote the unit space of G and suppose the moment map from $\underline{E}G \to Z$ is proper. Then [Emerson and Meyer 2010, Theorem 5.7] establishes a connection to what we might call "equivariant" Spanier–Whitehead duality. We summarize it below for the convenience of the reader (see also Remark 1.18).

Theorem 0.13. *The triple (P, α, Θ) is a Kasparov dual for X if and only if $C_0(X)$ and P are dual objects in* KK^G (cf. Remark 0.4) *with duality unit and counit induced by Θ and α, respectively.*

Concerning the connection with the Baum–Connes assembly map, we have:

Theorem 0.14 [Emerson and Meyer 2010, Theorem 6.14]. *Suppose $\underline{E}G$ admits a local symmetric Kasparov dual. Then the assembly map μ_A^G is an isomorphism for all proper coefficient algebras A.*

Assuming $\underline{E}G$ to be G-compact, the proof of the previous theorem roughly goes as follows: the second Poincaré duality isomorphism [Emerson and Meyer 2010, Section 6] combined with the Green–Julg isomorphism for proper groupoids [Emerson and Meyer 2009, Theorem 4.2] translate the assembly map μ_A^G into the map $K_*((P \otimes A) \rtimes G) \to K_*(A \rtimes G)$ induced by α. Now it is easy to see from the definition of equivariant Kasparov dual that the element $\tau_A(\alpha) \in \mathrm{KK}^G(P \otimes A, A)$ is invertible when A is a proper C^*-algebra.

Main results. Let us summarize our main results. Recall that G is a countable discrete group with a G-compact model for $\underline{E}G$.

As we have explained in the previous sections, our main strategy for obtaining duality relies on (1) the γ-element, or (2) the (γ)-element. The choice of one over the other does not affect the expression for the unit of Spanier–Whitehead duality; nevertheless, the descriptions of the counit and the elements $\Lambda_{C_r^*(G)}$ and $\Lambda_{C_0(\underline{E}G)\rtimes G}$ depend on the method that we are employing. In practice, the latter elements will be expressible in terms of the γ-element in the first case, and in the terms of the (γ)-element in the second case.

Along this categorization, Theorem A and Corollary B below fall in the first scenario, while Theorem D is an instance of the second. Section 3 contains simple examples of possible applications of duality in K-theory.

Theorem A. *Suppose that the γ-element $\gamma \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ exists and let $P_{\mathbb{C}}$ be the source of the Dirac morphism $\alpha \in \mathrm{KK}^G(P_{\mathbb{C}}, \mathbb{C})$. Then the C^*-algebra $P_{\mathbb{C}} \rtimes G$ is Spanier–Whitehead K-dual to $C_0(\underline{E}G) \rtimes G$.*

A few more comments about this theorem. The source of the Dirac morphism (the "simplicial approximation" in [Meyer and Nest 2006]) can be obtained in a variety of ways: by appealing to the Brown representability theorem, by considering the left adjoint to the embedding functor of projective objects, or by constructing the appropriate homotopy colimit from a projective resolution of $\mathbb{C}$ (here, "projective" is to be understood in a relative sense, i.e., with respect to the homological ideal of weakly contractible objects). Even though $P_{\mathbb{C}}$ may not be a proper algebra in general, its reduced and maximal crossed products are KK-equivalent. This is because $P_{\mathbb{C}}$ belongs to the localizing subcategory of KK^G generated by proper algebras and the reduced and maximal crossed product functors are triangulated functors and commute with countable direct sums; see [Meyer and Nest 2006].

Theorem A provides a fourth characterization of $P_{\mathbb{C}}$, namely as the Spanier–Whitehead K-dual of the classifying space for proper actions. Note that even though our statement is only available after descent — that is, we can only get $P_{\mathbb{C}} \rtimes G$ and not $P_{\mathbb{C}}$ via duality — this is only a minor drawback in the case of discrete groups, for the left-hand side of (0.9) retains the full information of the "topological" K-theory group through the dual Green–Julg isomorphism

$$\mathrm{KK}^G(C_0(\underline{E}G), \mathbb{C}) \cong \mathrm{KK}(C_0(\underline{E}G) \rtimes G, \mathbb{C}).$$

In the situation where, at the KK-theory level, the simplicial approximation is equivalent to the data of G acting on the point, we can replace $P_{\mathbb{C}} \rtimes G$ with $C_r^*(G)$ and obtain Spanier–Whitehead duality for the group as in the next corollary. If the γ-element exists, we define the *strong* Baum–Connes conjecture to be the statement that $j_r^G(\gamma) = 1_{C_r^*(G)}$ in $\mathrm{KK}(C_r^*(G), C_r^*(G))$.

Corollary B. *Suppose the γ-element exists. Then G has Spanier–Whitehead duality if and only if it satisfies the strong Baum–Connes conjecture.*

In light of the result above, we can view the notion of Spanier–Whitehead K-duality for G as a homotopy-theoretic characterization of the strong Baum–Connes conjecture (cf. Remark 3.8).

The main application of the previous corollary is summarized in the result below.

Corollary C. *All a-T-menable groups which admit a G-compact model of $\underline{E}G$ have Spanier–Whitehead K-duality. Examples of a-T-menable groups are the following*:

- *all groups which act properly, affine-isometrically, and cocompactly on a finite-dimensional Euclidean space*,
- *all cocompact lattices of simple Lie groups* $\mathrm{SO}(n,1)$ *or* $\mathrm{SU}(n,1)$,
- *all groups which act cocompactly on a tree.*

Having such an explicit duality should be useful. For example, in principle, it allows us to compute the Lefschetz number of an automorphism of $C^*_r(G)$, or more generally of a morphism f in $\mathrm{KK}(C^*_r(G), C^*_r(G))$; see [Dell'Ambrogio et al. 2014; Emerson 2011].

If a cycle with property (γ) is found, then we can deduce the duality in complete analogy with the case of the γ-element (this is how the definition of property (γ) was designed). However, in this case we do not have information on the localization at the weakly contractible objects [Meyer and Nest 2010]. So we get the corresponding statement for Corollary B, but not for Theorem A.

Theorem D. *Suppose there is a (γ)-element $x \in \mathrm{KK}^G(\mathbb{C},\mathbb{C})$ for G. If $j_r^G(x) = 1_{C^*_r(G)} \in \mathrm{KK}^G(C^*_r(G), C^*_r(G))$, then G has Spanier–Whitehead duality.*

1. General framework

Let G be a countable discrete group, and $\underline{E}G$ be a G-compact model of the universal proper G-space. Let A and B be C^*-algebras equipped with a G-action. If the G-action on B is trivial, we recall the dual Green–Julg isomorphism [Blackadar 1998; Kaad and Proietti 2018; Land 2015]

$$\mathrm{GJ} : \mathrm{KK}^G(A, B) \cong \mathrm{KK}(A \rtimes G, B).$$

Given $a \in A$, define $\delta^a_g \in C_c(G, A) \subseteq A \rtimes G$ to be the function

$$\delta^a_g(t) = \begin{cases} a & \text{if } t = g, \\ 0 & \text{if } t \neq g. \end{cases}$$

The *dual coaction* is defined as

$$\Delta : A \rtimes G \to C^*_r(G) \otimes A \rtimes G, \qquad \delta^a_g \mapsto g \otimes \delta^a_g.$$

Let $c \in C_c(\underline{E}G)$ be a cutoff function, and consider the associated projection $p_c \in C_c(G, C_0(\underline{E}G)) \subseteq C_0(\underline{E}G) \rtimes G$ defined by $p_c(g) = cg(c)$. This projection does not depend on c up to homotopy, hence we denote it p_G in the sequel.

Definition 1.1. We define the *canonical duality unit* to be the class

$$\delta = \delta_G = [\Delta(p_G)] \in \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G).$$

The notational dependence on G shall be dropped when clear from the context. In this paper, whenever we say that G has Spanier–Whitehead duality, we implicitly assume that the duality unit is given as above.

Let us recall the definition of Kasparov's descent homomorphism [1988], which plays an important role in this paper. It is denoted j^G below. Suppose (ϕ, H, T) is a Kasparov cycle defining an element of $\mathrm{KK}^G(A, B)$. The G-action on H is denoted $U : G \to \mathrm{End}_{\mathbb{C}}(H)$. The element $j^G([\phi, H, T]) \in \mathrm{KK}(A \rtimes G, B \rtimes G)$ is defined by the cycle $(\tilde{\phi}, H \rtimes G, \widetilde{T})$ given as follows.

The Hilbert C^*-module $H \rtimes G$ is the completion of $C_c(G, H)$ with respect to the norm induced by the $B \rtimes G$-valued inner product

$$\langle \xi | \zeta \rangle(t) = \sum_{g \in G} \beta_{g^{-1}}(\langle \xi(g) | \zeta(gt) \rangle),$$

where $\xi, \zeta \in C_c(G, H)$, $t \in G$, and β denotes the given G-action on B. The right action of $B \rtimes G$ is uniquely determined by the formula

$$(\xi \cdot f)(t) = \sum_{g \in G} \xi(g) \beta_g(f(g^{-1}t)),$$

where $\xi \in C_c(G, H)$, $f \in C_c(G, B)$, and $t \in G$. The representation of $A \rtimes G$ on $H \rtimes G$ is determined by

$$(\tilde{\phi}(f)(\xi))(t) = \sum_{g \in G} \phi(f(g))[U(g)(\xi(g^{-1}t))],$$

where $f \in C_c(G, A)$, $\xi \in C_c(G, H)$, and $t \in G$. Finally the operator $\widetilde{T}$ is defined by $(\widetilde{T}\xi)(t) = T(\xi(t))$ for $\xi \in C_c(G, H)$ and $t \in G$. By using reduced crossed products everywhere, we can similarly define a "reduced version" of the descent homomorphism, denoted j_r^G in the sequel.

Lemma 1.2 [Land 2015, Proposition 4.7]. *Kasparov's descent homomorphism can be factorized as follows*:

$$\begin{array}{ccc}
\mathrm{KK}^G(A, \mathbb{C}) & \xrightarrow{\quad j^G \quad} & \mathrm{KK}(A \rtimes G, C^*(G)) \\
\Big\downarrow{\scriptstyle \mathrm{GJ}} & & \Big\uparrow{\scriptstyle \Delta^*} \\
\mathrm{KK}(A \rtimes G, \mathbb{C}) & \xrightarrow{\tau_{C^*(G)}} & \mathrm{KK}(C^*(G) \otimes A \rtimes G, C^*(G))
\end{array}$$

When the canonical map $A \rtimes G \to A \rtimes_r G$ is an isomorphism (e.g., if G acts properly on A), the version of the previous lemma with *reduced* crossed products also holds. See Remark 0.6.

Lemma 1.3 [Kaad and Proietti 2018, Section 2]. *Let A and B be G-C^*-algebras and suppose the G-action on B is trivial. Consider an element $x \in \mathrm{KK}^G(A, A)$. The following diagram commutes*:

$$\begin{array}{ccc} \mathrm{KK}^G(A, B) & \xrightarrow{\mathrm{GJ}} & \mathrm{KK}(A \rtimes G, B) \\ \Big\downarrow {\scriptstyle x \widehat{\otimes} -} & & \Big\downarrow {\scriptstyle j^G(x) \widehat{\otimes} -} \\ \mathrm{KK}^G(A, B) & \xrightarrow{\mathrm{GJ}} & \mathrm{KK}(A \rtimes G, B) \end{array}$$

It follows from Lemma 1.2 that we have the commutative diagram

$$\begin{array}{ccc} \mathrm{KK}^G(C_0(\underline{E}G), B) & \xrightarrow{\mu_B^G} & \mathrm{KK}(\mathbb{C}, B \otimes C_r^*(G)) \\ {\scriptstyle \mathrm{GJ}} \Big\downarrow {\scriptstyle \cong} & & \Big\downarrow {\scriptstyle =} \\ \mathrm{KK}(C_0(\underline{E}G) \rtimes G, B) & \xrightarrow{\delta \widehat{\otimes}_{C_0(\underline{E}G) \rtimes G} -} & \mathrm{KK}(\mathbb{C}, B \otimes C_r^*(G)) \end{array}$$

Since the definition of the duality counit requires additional information, and depends on the choice of "γ-like" element, the rest of this section gets split in two parts. The torsion-free case is treated in detail at the end of this section.

Argument based on the (γ)-element. Let (H, T) be a G-equivariant Kasparov cycle with property (γ). Let $x = [H, T]$ be the corresponding element in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$. Let

$$\tilde{x} = [H \otimes \ell^2(G), \rho \otimes \pi, (g(T))_{g \in G}] \in \mathrm{KK}^G(C_r^*(G) \otimes C_0(\underline{E}G), \mathbb{C}). \tag{1.4}$$

Here, $\pi : C_0(\underline{E}G) \to B(H)$ is the representation witnessing the conditions for property (γ) of (H, T), ρ stands for the right regular representation, and $C_r^*(G)$ has trivial G-action. By means of the Green–Julg isomorphism, we set

$$d = \mathrm{GJ}(\tilde{x}) \in \mathrm{KK}(C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G, \mathbb{C}).$$

We set $\Lambda_{C_r^*(G)} = \delta \widehat{\otimes}_{C_0(\underline{E}G) \rtimes G}\, d$ and $\Lambda_{C_0(\underline{E}G) \rtimes G} = \delta \widehat{\otimes}_{C_r^*(G)}\, d$. We shall prove

(1) $\Lambda_{C_r^*(G)} = j_r^G(x)$ in $\mathrm{KK}(C_r^*(G), C_r^*(G))$;

(2) $\Lambda_{C_0(\underline{E}G) \rtimes G} = 1_{C_0(\underline{E}G) \rtimes G}$ in $\mathrm{KK}(C_0(\underline{E}G) \rtimes G, C_0(\underline{E}G) \rtimes G)$.

Proposition 1.5. *We have the equality $\Lambda_{C_r^*(G)} = j_r^G(x)$.*

Proof. We claim the Kasparov module

$$[p_G] \widehat{\otimes}_{C_0(\underline{E}G) \rtimes G}\, j_r^G(\tilde{x}) \tag{1.6}$$

is equivalent to $J_r^G(x)$, i.e., there is an isomorphism of Hilbert C^*-modules intertwining the representations and the operators (up to a compact perturbation).

The class in (1.6) is represented by

$$(H \otimes \ell^2(G) \rtimes_r G, (\rho \otimes \pi \rtimes_r 1)(p_G \otimes -), (g(T))_{g\in G} \rtimes_r 1).$$

We have an isomorphism of $C_r^*(G)$-modules

$$H \rtimes_r G \cong (\rho \otimes \pi \rtimes_r 1)(p_G \otimes 1)(H \otimes \ell^2(G) \rtimes_r G) \tag{1.7}$$

given by the assignment

$$\xi \rtimes_r u_g \mapsto \sum_{h\in G} \pi(c)(h\cdot\xi) \otimes \delta_h \rtimes_r u_{hg},$$

where $\xi \in H$, $\delta_h \in \ell^2(G)$, and c is a cutoff function defining p_G. The inverse of the map above is given by the restriction of

$$(\xi)_{h\in G} \rtimes_r u_g \mapsto \sum_{h\in G} h^{-1}\cdot(\pi(c)\xi_h) \otimes \rtimes_r u_{h^{-1}g},$$

where $(\xi)_{h\in G} \in H \otimes \ell^2(G)$. Under the isomorphism in (1.7), the representation $(\rho \otimes \pi \rtimes_r 1)(p_G \otimes -)$ is identified with the left action of $C_r^*(G)$ on $H \rtimes_r G$, and the compressed operator

$$(\rho \otimes \pi \rtimes_r 1)(p_G \otimes 1)((g(T))_{g\in G} \rtimes_r 1)(\rho \otimes \pi \rtimes_r 1)(p_G \otimes 1)$$

is identified with $T' \rtimes_r 1$ on $H \rtimes_r G$, where we define

$$T' = \sum_{g\in G} (g\cdot c)T(g\cdot c).$$

Hence the claim follows by definition of property (γ).

By Lemma 1.2, we have

$$J_r^G(\tilde{x}) = \Delta \otimes_{C_0(\underline{E}G)\rtimes G} \mathrm{GJ}(\tilde{x}).$$

Thus, we have

$$\begin{aligned} J_r^G(x) &= [p_G] \,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, J_r^G(\tilde{x}) \\ &= [p_G] \,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, \Delta \otimes_{C_0(\underline{E}G)\rtimes G} \mathrm{GJ}(\tilde{x}) \\ &= \delta \,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d. \end{aligned}$$

□

Proposition 1.8. *We have the equality* $\Lambda_{C_0(\underline{E}G)\rtimes G} = 1_{C_0(\underline{E}G)\rtimes G}$.

In order to prove the proposition, a few preliminaries are in order. First we generalize the construction in (1.4) to include a coefficient algebra. This is easily done: simply replace $\ell^2(G)$ with the right Hilbert A-module $\ell^2(G, A)$ and define the right regular representation ρ_A^G of $A \rtimes_r G$ (equipped with trivial G-action)

$$a \mapsto (g(a))_{g \in G}, \qquad h \mapsto \rho_h : (a_g)_{g \in G} \mapsto (a_{gh})_{g \in G}$$

for $a \in A, h \in G$. Thus we get a class $\tilde{x}_A$ in $\mathrm{KK}^G(A \rtimes_r G \otimes C_0(\underline{E}G), A)$. We define a group homomorphism

$$\nu_A^G : \mathrm{KK}(\mathbb{C}, A \rtimes_r G) \to \mathrm{KK}^G(C_0(\underline{E}G), A)$$

as the one induced by the class $\tilde{x}_A$ via the index pairing

$$\mathrm{KK}(\mathbb{C}, A \rtimes_r G) \times \mathrm{KK}^G(A \rtimes_r G \otimes C_0(\underline{E}G), A) \to \mathrm{KK}^G(C_0(\underline{E}G), A).$$

This map is referred to as the (γ)-morphism in [Nishikawa 2019]. Note also that $\mathrm{GJ} \circ \nu_{\mathbb{C}}^G$ equals the map d_j from Definition 0.1 (choosing $B = C_0(\underline{E}G) \rtimes G$ as usual). The lemma below is about the naturality property of the assembly map and the (γ)-morphism.

Lemma 1.9. *The following diagrams commute for any $f \in \mathrm{KK}^G(A, B)$:*

$$\begin{array}{ccc}
\mathrm{KK}^G(C_0(\underline{E}G), A) & \xrightarrow{\mu_A^G} & \mathrm{KK}(\mathbb{C}, A \rtimes_r G) \\
\Big\downarrow {\scriptstyle -\widehat{\otimes} f} & & \Big\downarrow {\scriptstyle -\widehat{\otimes} J_r^G(f)} \\
\mathrm{KK}^G(C_0(\underline{E}G), B) & \xrightarrow{\mu_A^G} & \mathrm{KK}(\mathbb{C}, B \rtimes_r G)
\end{array}$$

$$\begin{array}{ccc}
\mathrm{KK}(\mathbb{C}, A \rtimes_r G) & \xrightarrow{\nu_A^G} & \mathrm{KK}^G(C_0(\underline{E}G), A) \\
\Big\downarrow {\scriptstyle -\widehat{\otimes} J_r^G(f)} & & \Big\downarrow {\scriptstyle -\widehat{\otimes} f} \\
\mathrm{KK}(\mathbb{C}, B \rtimes_r G) & \xrightarrow{\nu_A^G} & \mathrm{KK}^G(C_0(\underline{E}G), B)
\end{array}$$

Proof. The first diagram commutes by functoriality of descent and associativity of the Kasparov product. By results from [Meyer 2000] any morphism f in $\mathrm{KK}^G(A, B)$ can be written as a composition of $*$-homomorphisms and their inverses in KK. This means it suffices to check the commutativity of the second diagram with respect to $*$-homomorphisms $f : A \to B$. We omit this simple verification. $\square$

Proof of Proposition 1.8. Let $B = C_0(\underline{E}G) \rtimes G$ and regard it as a G-C^*-algebra with the trivial G-action. We have the following diagram:

$$\begin{array}{ccccc}
\mathrm{KK}^G(C_0(\underline{E}G), B) & \xrightarrow{\mu_B^G} & \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes B) & \xrightarrow{\nu_B^G} & \mathrm{KK}^G(C_0(\underline{E}G), B) \\
{\scriptstyle \mathrm{GJ}} \Big\downarrow {\scriptstyle \cong} & & \Big\downarrow {\scriptstyle =} & & {\scriptstyle \mathrm{GJ}} \Big\downarrow {\scriptstyle \cong} \\
\mathrm{KK}(B, B) & \xrightarrow{\delta \widehat{\otimes}_B -} & \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes B) & \xrightarrow{-\widehat{\otimes}_{C_r^*(G)} d} & \mathrm{KK}(B, B)
\end{array}$$

If we prove that the composition on the top is the identity, then it follows that $\Lambda_{C_0(\underline{E}G)\rtimes G} = 1_{C_0(\underline{E}G)\rtimes G}$. Let $D_B : P_B \to B$ be a weak equivalence as in [Meyer and Nest 2006]. Because the diagram

$$\begin{array}{ccccc} \mathrm{KK}^G(C_0(\underline{E}G), P_B) & \xrightarrow{\mu^G_{P_B}} & \mathrm{KK}(\mathbb{C}, P_B \rtimes G) & \xrightarrow{\nu^G_{P_B}} & \mathrm{KK}^G(C_0(\underline{E}G), P_B) \\ \downarrow D_{B*} & & \downarrow J^G_r(D_{B*}) & & \downarrow D_{B*} \\ \mathrm{KK}^G(C_0(\underline{E}G), B) & \xrightarrow{\mu^G_B} & \mathrm{KK}(\mathbb{C}, B \rtimes_r G) & \xrightarrow{\nu^G_B} & \mathrm{KK}^G(C_0(\underline{E}G), B) \end{array}$$

commutes, it suffices to show that $\nu^G_{P_B}$ is a left inverse of the assembly map $\mu^G_{P_B}$. Now, $\mu^G_{P_B}$ is invertible, hence it suffices to show that $\nu^G_{P_B}$ yields a right inverse. A minor generalization of the proof of Proposition 1.5 shows that $\mu^G_{P_B} \circ \nu^G_{P_B}$ coincides with the induced action of $x \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ on $K_*(P_B \rtimes G)$. Recall that x equals the identity when restricted to each finite subgroup $H \subseteq G$, and $P_B \rtimes G$ belongs to the localizing subcategory of KK generated by the $B \rtimes H$'s. Therefore the map $x \mathbin{\widehat{\otimes}} - : K_*(P_B \rtimes G) \to K_*(P_B \rtimes G)$ is the identity by [Meyer and Nest 2006, Theorem 9.3]. □

Remark 1.10. In parallel with Proposition 1.5, one can prove that

$$\Lambda_{C_0(\underline{E}G)\rtimes G} = J^G_r(x \mathbin{\widehat{\otimes}} 1_{C_0(\underline{E}G)}).$$

Again, we set $B = C_0(\underline{E}G) \rtimes G$ and first notice that $\nu^G_B \circ \mu^G_B = x \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)} -$. It is enough to show this when B is replaced by P_B, in which case we can invert the assembly map and write

$$\begin{aligned} (J^G_r(x) \mathbin{\widehat{\otimes}} -) &= (J^G_r(x) \mathbin{\widehat{\otimes}} -) \circ \mu^G_B \circ (\mu^G_B)^{-1}, \\ \mu^G_B \circ \nu^G_B &= \mu^G_B \circ (x \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)} -) \circ (\mu^G_B)^{-1}, \\ \nu^G_B \circ \mu^G_B &= x \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)} -. \end{aligned}$$

To complete the proof one must show that

$$\mathrm{GJ}(x \mathbin{\widehat{\otimes}}_{\mathbb{C}} \mathrm{GJ}^{-1}(1_B)) = J^G_r(x \mathbin{\widehat{\otimes}} 1_{C_0(\underline{E}G)}) \mathbin{\widehat{\otimes}}_B 1_B,$$

but this follows from Lemma 1.3.

We now come to the main result of this subsection.

Theorem 1.11. *Suppose there is a* (γ)*-element* $x \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ *for* G. *If*

$$J^G_r(x) = 1 \in \mathrm{KK}^G(C^*_r(G), C^*_r(G)),$$

then G *has Spanier–Whitehead duality.*

Argument based on the γ-element. Suppose there is a gamma element γ as in Definition 0.10. Following [Guentner et al. 2000, Chapter 15], we define a map s_A for any proper algebra A. This is the G-equivariant $*$-homomorphism

$$s_A : A \rtimes_r G \otimes C_0(\underline{E}G) \to K(A \otimes \ell^2(G)),$$

where $A \rtimes_r G$ is equipped with the trivial G-action, defined as the tensor product of the representation

$$C_0(\underline{E}G) \ni \phi \mapsto (\phi)_{g \in G} \in L(A \otimes \ell^2(G))$$

of $C_0(\underline{E}G)$ on $A \otimes \ell^2(G)$ and the right regular representation

$$A \ni a \mapsto (g(a))_{g \in G} \in L(A \otimes \ell^2(G)), \qquad G \ni g \mapsto 1 \otimes \rho_g$$

of $A \rtimes_r G$ on $A \otimes \ell^2(G)$, where ρ_g is the right translation by g. Here, the G-action on the Hilbert module $A \otimes \ell^2(G)$ is given by the tensor product of the action on A and the left-regular representation. The $*$-homomorphism s_A defines an element

$$s_A \in \mathrm{KK}^G(A \rtimes_r G \otimes C_0(\underline{E}G), A).$$

Proposition 1.12 (see [Guentner et al. 2000, Chapter 15]). *For any proper G-C^*-algebra A, the $*$-homomorphism s_A defines the inverse*

$$s_A : \mathrm{KK}(\mathbb{C}, A \rtimes_r G) \to \mathrm{KK}^G(C_0(\underline{E}G), A)$$

of the assembly map

$$\mu_A^G : \mathrm{KK}^G(C_0(\underline{E}G), A) \to \mathrm{KK}(\mathbb{C}, A \rtimes_r G).$$

Proof. The assembly map is an isomorphism for any proper algebra. Hence, we just show that the composition $\mu_A^G \circ s_A$ is the identity. Take a Kasparov cycle (E, F) for $\mathrm{KK}(\mathbb{C}, A \rtimes_r G)$ where E is a graded $A \rtimes_r G$-module and F is an odd, self-adjoint operator on E satisfying $1 - F^2 \equiv 0$ modulo compact operators.

By Kasparov's stabilization theorem, we can assume E is $A \otimes H \rtimes_r G$ for some graded Hilbert space H with the trivial G-action. The map s_A sends this cycle $(A \otimes H \rtimes_r G, F)$ to the G-equivariant cycle $(A \otimes H \otimes \ell^2(G), \pi, \rho(F))$ for $\mathrm{KK}^G(C_0(\underline{E}G), A)$, where π is a representation of $C_0(\underline{E}G)$ on $A \otimes H \otimes \ell^2(G)$ defined as follows: for ϕ in $C_0(\underline{E}G)$,

$$\pi(\phi)(a_g \otimes v_g \otimes \delta_g) = \phi a_g \otimes v_g \otimes \delta_g$$

and $\rho(F)$ is an operator in $L(A \otimes H \otimes \ell^2(G))$ determined by the map

$$L(A \otimes H \rtimes_r G) = M(A \otimes K(H) \rtimes_r G) \xrightarrow{\rho} M(A \otimes K(H \otimes \ell^2(G))) = L(A \otimes H \otimes \ell^2(G)),$$

which is a natural extension of the right regular representation ρ_A^G of $A \rtimes_r G$ on $A \otimes \ell^2(G)$ described before. Hence, the composition $\mu_A^G \circ s_A$ sends the cycle $(A \otimes H \rtimes_r G, F)$ to the cycle $(p_c(A \otimes H \otimes \ell^2(G) \rtimes_r G), p_c\rho(F) \rtimes_r 1 p_c)$, where we simply denote by p_c the image of a cutoff projection p_c in $C_0(\underline{E}G) \rtimes_G$ by the representation $\pi \rtimes_r 1$.

On the other hand, there is an isomorphism of right Hilbert $A \rtimes_r G$-modules

$$A \otimes H \rtimes_r G \to p_c(A \otimes H \otimes \ell^2(G) \rtimes_r G)$$

given by

$$\xi \rtimes_r u_g \mapsto \sum_{h \in G} c(h(\xi)) \otimes \delta_h \rtimes_r u_{hg} \quad \text{for } \xi \text{ in } A \otimes H.$$

The inverse map is given by

$$(\xi_h)_{h \in G} \rtimes_r u_g \mapsto \sum_{h \in G} h^{-1}(c\xi_h) \rtimes_r u_{h^{-1}g} \quad \text{for } (\xi_h)_{h \in G} \text{ in } A \otimes H \otimes \ell^2(G).$$

Under this isomorphism, the restriction $p_c\rho(F) \rtimes_r 1 p_c$ of $\rho(F) \rtimes_r 1$ on the subspace $p_c(A \otimes H \otimes \ell^2(G) \rtimes_r G)$ of $A \otimes H \otimes \ell^2(G) \rtimes_r G$ corresponds to the operator F on $A \otimes H \rtimes_r G$. In summary, the composition $\mu_A^G \circ s_A$ sends the cycle $(A \otimes H \rtimes_r G, F)$ to itself up to the isomorphism described above. □

For any separable G-C^*-algebra B, we have the commutative diagram

$$\begin{array}{ccc}
\mu_B^G : \mathrm{KK}^G(C_0(EG), B) & \longrightarrow & \mathrm{KK}(\mathbb{C}, B \rtimes_r G) \\
\Big\downarrow {\scriptstyle -\widehat{\otimes}_{\mathbb{C}}\beta} & & \Big\downarrow {\scriptstyle -\widehat{\otimes}_{B \rtimes_r G} j_r^G(\mathrm{id}_B \widehat{\otimes} \beta)} \\
\mu_{B \otimes P}^G : \mathrm{KK}^G(C_0(EG), B \otimes P) & \xrightarrow{\cong} & \mathrm{KK}(\mathbb{C}, (B \otimes P) \rtimes_r G) \\
\Big\downarrow {\scriptstyle -\widehat{\otimes}_P \alpha} & & \Big\downarrow {\scriptstyle -\widehat{\otimes}_{(B \otimes P) \rtimes_r G} j_r^G(\mathrm{id}_B \widehat{\otimes} \alpha)} \\
\mu_B^G : \mathrm{KK}^G(C_0(EG), B) & \longrightarrow & \mathrm{KK}(\mathbb{C}, B \rtimes_r G)
\end{array}$$

where the vertical composition on the left is the identity. With this observation and Proposition 1.12, we see that the element

$$(j_r^G(1_B \widehat{\otimes} \beta)) \widehat{\otimes}_{(B \widehat{\otimes} P) \rtimes_r G} s_{B \otimes P} \widehat{\otimes}_P \alpha \in \mathrm{KK}^G((B \rtimes_r G) \otimes C_0(\underline{E}G), B)$$

induces the left-inverse of the assembly map μ_B^G via Kasparov product. We remark that this is the standard technique for proving the split injectivity of the assembly map in the presence of a γ-element.

Now, we set d' to be the element in $\mathrm{KK}(C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G, \mathbb{C})$ which corresponds to

$$d'' = (j_r^G(\beta)) \widehat{\otimes}_{P \rtimes_r G} s_P \widehat{\otimes}_P \alpha \in \mathrm{KK}^G(C_r^*(G) \otimes C_0(\underline{E}G), \mathbb{C}).$$

Let

$$\delta = \delta_G \in \mathrm{KK}(\mathbb{C}, C_0(\underline{E}G) \rtimes G \otimes C_r^*(G))$$

as before. We set $\Lambda'_{C_r^*(G)} = \delta \,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d'$, $\Lambda'_{C_0(\underline{E}G)\rtimes G} = \delta \,\widehat{\otimes}_{C_r^*(G)}\, d'$.

Proposition 1.13. *We have*

$$\Lambda'_{C_r^*(G)} = j_r^G(\gamma) \in \mathrm{KK}(C_r^*(G), C_r^*(G))$$

and

$$\Lambda'_{C_0(\underline{E}G)\rtimes G} = 1_{C_0(\underline{E}G)\rtimes G} \in \mathrm{KK}(C_0(\underline{E}G) \rtimes G, C_0(\underline{E}G) \rtimes G).$$

Before giving a proof of Proposition 1.13, let us obtain our main results as its direct consequences:

Theorem 1.14. *If $j_r^G(\gamma) = 1_{C_r^*(G)}$, then G has Spanier–Whitehead duality.*

The previous result has a converse; see Theorem 3.3 for further details.

Theorem 1.15. *If $\mu_{\mathbb{C}}^G$ is an isomorphism, G has weak Spanier–Whitehead duality.*

Theorem 1.16. *In general, if $\gamma \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ exists, then $C_0(\underline{E}G) \rtimes G$ is a Spanier–Whitehead K-dual of $P_{\mathbb{C}} \rtimes G$.*

Proof. Note that $P_{\mathbb{C}} \rtimes G$ is a direct summand (in the category KK) of $C_r^*(G)$ corresponding to the idempotent $j_r^G(\gamma) \in \mathrm{KK}(C_r^*(G), C_r^*(G))$ (see [Neeman 2001, Proposition 1.6.8]). Namely, we have

$$i_{P_{\mathbb{C}}\rtimes G} \in \mathrm{KK}(P_{\mathbb{C}} \rtimes G, C_r^*(G)), \qquad q_{P_{\mathbb{C}}\rtimes G} \in \mathrm{KK}(C_r^*(G), P_{\mathbb{C}} \rtimes G),$$

so that $q_{P_{\mathbb{C}}\rtimes G} \circ i_{P_{\mathbb{C}}\rtimes G} = 1_{P_{\mathbb{C}}\rtimes G}$ and $i_{P_{\mathbb{C}}\rtimes G} \circ q_{P_{\mathbb{C}}\rtimes G} = j_r^G(\gamma)$. We set

$$d_{P_{\mathbb{C}}\rtimes G} = i_{P_{\mathbb{C}}\rtimes G} \,\widehat{\otimes}_{C_r^*(G)}\, d' \in \mathrm{KK}(C_0(\underline{E}G) \rtimes G \otimes P_{\mathbb{C}} \rtimes G, \mathbb{C}),$$
$$\delta_{P_{\mathbb{C}}\rtimes G} = \delta \,\widehat{\otimes}_{C_r^*(G)}\, q_{P_{\mathbb{C}}\rtimes G} \in \mathrm{KK}(\mathbb{C}, C_0(\underline{E}G) \rtimes G \otimes P_{\mathbb{C}} \rtimes G).$$

Then we have

$$\delta_{P_{\mathbb{C}}\rtimes G} \,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d_{P_{\mathbb{C}}\rtimes G} = 1_{P_{\mathbb{C}}\rtimes G}, \qquad \delta_{P_{\mathbb{C}}\rtimes G} \,\widehat{\otimes}_{P_{\mathbb{C}}\rtimes G}\, d_{P_{\mathbb{C}}\rtimes G} = 1_{C_0(\underline{E}G)\rtimes G}.$$

This proves the statement. We only prove the first identity; the other one is proved similarly. For any C^*-algebra D, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{KK}(P_{\mathbb{C}} \rtimes G, D) & \longrightarrow & \mathrm{KK}(C_r^*(G), D) \\
\downarrow {\scriptstyle \delta_{P_{\mathbb{C}}\rtimes G}\widehat{\otimes}_{P_{\mathbb{C}}\rtimes G} -} & & \downarrow {\scriptstyle \delta\widehat{\otimes}_{C_r^*(G)} -} \\
\mathrm{KK}(\mathbb{C}, C_0(\underline{E}G) \rtimes G \otimes D) & \xrightarrow{=} & \mathrm{KK}(\mathbb{C}, C_0(\underline{E}G) \rtimes_r G \otimes D) \\
\downarrow {\scriptstyle -\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d_{P_{\mathbb{C}}\rtimes G}} & & \downarrow {\scriptstyle -\widehat{\otimes}_{C_0(\underline{E}G)\rtimes_r G}\, d'} \\
\mathrm{KK}(P_{\mathbb{C}} \rtimes G, D) & \longrightarrow & \mathrm{KK}(C_r^*(G), D)
\end{array}$$

Here, the top and the bottom horizontal arrows are induced by $i_{P_{\mathbb{C}}\rtimes G}$ and $q_{P_{\mathbb{C}}\rtimes G}$. The right vertical composition is induced by $j_r^G(\gamma)$. It follows that the left vertical composition is the identity. Taking $D = P_{\mathbb{C}} \rtimes G$, we get

$$\delta_{P_{\mathbb{C}}\rtimes G} \,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d_{P_{\mathbb{C}}\rtimes G} = 1_{P_{\mathbb{C}}\rtimes G}. \qquad \square$$

Proof of Proposition 1.13. We directly compute and prove

$$\delta \,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d' = j_r^G(\gamma) \in \mathrm{KK}(C_r^*(G), C_r^*(G)).$$

For simplicity, we prove this for the case when β is represented by a cycle (P, b) for b an essential unitary in $M(P)$, and α by a cycle (H, F), where P is represented on H nondegenerately and F is a G-equivariant essential unitary modulo P. Then d'' is represented by a cycle of the form

$$(H \otimes \ell^2(G), \rho \otimes \pi, N(g(b))_{g\in G} + M(g(F))_{g\in G}),$$

where the G-action on $H \otimes \ell^2(G)$ is the tensor product of the G-action on H and the left regular representation on $\ell^2(G)$, π is a representation of $C_0(\underline{E}G)$ on $H \otimes \ell^2(G)$ given by $\phi \mapsto (\phi)_{g\in G}$, and ρ is a representation of $C_r^*(G)$ on $H \otimes \ell^2(G)$ by the right regular representation $g \mapsto 1 \otimes \rho_g$. Here, M and N are given by the Kasparov technical theorem as usual [Higson 1987; Kasparov 1980; 1995]. If we compute $\delta \otimes_{C_0(\underline{E}G)\rtimes G} d'$, we get the cycle isomorphic to

$$(H \rtimes_r G, \pi_G, T_0 \rtimes_r 1) = j_r^G((H, T_0))$$

where (H, T_0) is a cycle for $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$, π_G is the natural left multiplication by $C_r^*(G)$, and $T_0 = N_0 b + M_0 F_0$. Here F_0 is the average of F: $F_0 = \int_G g(c) F g(c)\, d\mu_G$ and so are N_0 and M_0. The cycle (H, T_0) is (homotopic to) a Kasparov product of α and β. In other words, the element $[H, T_0]$ is the gamma element γ. It follows that

$$\delta \,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d' = j_r^G(\gamma).$$

Now we can prove

$$\delta \,\widehat{\otimes}_{C_r^*(G)}\, d' = 1_{C_0(\underline{E}G)\rtimes G} \in \mathrm{KK}(C_0(\underline{E}G) \rtimes G, C_0(\underline{E}G) \rtimes G)$$

using a simple trick. We have the following diagram for $B = C_0(\underline{E}G) \rtimes G$ with the trivial G action:

$$\begin{array}{ccccc}
\mathrm{KK}^G(C_0(\underline{E}G), B) & \xrightarrow{\mu_B^G} & \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes B) & \xrightarrow{(\mu_B^G)^{-1}} & \mathrm{KK}^G(C_0(\underline{E}G), B) \\
\downarrow \cong & & \downarrow = & & \downarrow \cong \\
\mathrm{KK}(B, B) & \xrightarrow{\delta \widehat{\otimes}_B -} & \mathrm{KK}(\mathbb{C}, C_r^*(G) \otimes B) & \xrightarrow{-\widehat{\otimes}_{C_r^*(G)} d} & \mathrm{KK}(B, B)
\end{array}$$

Here, by $(\mu_B^G)^{-1}$ we simply mean the left inverse of μ_B^G. This shows $\delta \otimes_{C_r^*(G)} d'$ acts as the identity on $\mathrm{KK}(B, B)$, proving the claim. $\square$

Remark 1.17. The previous proof also shows that $d = d'$, as it is intuitive from the fact that the γ-element can be represented by a cycle satisfying property (γ) [Nishikawa 2019].

Remark 1.18. It is natural to use the duality class Θ from page 472 to prove Theorem 1.16. The argument is based on the following diagram, where we set $d' = \mathrm{GJ}(s_P \mathbin{\widehat{\otimes}}_P \alpha)$, and $\mu^G_{P\rtimes G,P}$ is a bivariant assembly map (see Section 3):

$$\begin{array}{ccc}
\mathrm{KK}(\mathbb{C},\, P\rtimes G\otimes C_0(\underline{E}G)\rtimes G) & \xrightarrow{\;-\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d'\;} & \mathrm{KK}(P\rtimes G,\, P\rtimes G) \\
{\scriptstyle \mu^G_{P\otimes C_0(\underline{E}G)\rtimes G}}\Big\uparrow \cong & & \Big\uparrow {\scriptstyle \mu^G_{P\rtimes G,P}} \\
\mathrm{KK}^G(C_0(\underline{E}G),\, P\otimes C_0(\underline{E}G)\rtimes G) & \xrightarrow{\;-\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d'\;} & \mathrm{KK}^G(C_0(\underline{E}G)\otimes P\rtimes G,\, P) \\
{\scriptstyle p^*_{\underline{E}G}}\Big\downarrow \cong & & \cong \Big\downarrow {\scriptstyle p^*_{\underline{E}G}} \\
\mathrm{RKK}^G(\underline{E}G;\, C_0(\underline{E}G),\, P\otimes C_0(\underline{E}G)\rtimes G) & \xrightarrow{\;-\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\, d'\;} & \mathrm{RKK}^G(\underline{E}G;\, C_0(\underline{E}G)\otimes P\rtimes G,\, P)
\end{array}$$

Set $e = \mathrm{GJ}^{-1}(1_{C_0(\underline{E}G)\rtimes G})$ and consider the element $\delta_0 = \Theta \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)} e$ in the bottom left group. Suppressing $p^*_{\underline{E}G}$ from the notation, we compute

$$\begin{aligned}
\delta_0 \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)\rtimes G} d' &= \Theta \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)} e \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)\rtimes G} \mathrm{GJ}(s_P \mathbin{\widehat{\otimes}}_P \alpha) \\
&= (\Theta \mathbin{\widehat{\otimes}}_P \alpha) \mathbin{\widehat{\otimes}}_{\mathbb{C}} (e \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)\rtimes G} \mathrm{GJ}(s_P)) = s_P.
\end{aligned}$$

Now it is routine to check that $\mu^G_{P\rtimes G,P}(s_P) = 1_{P\rtimes G}$. Hence, if we define

$$\delta_{P\rtimes G} \in \mathrm{KK}(\mathbb{C},\, P\rtimes G\otimes C_0(\underline{E}G)\rtimes G)$$

by sending δ_0 through the left vertical isomorphism in the diagram above, we have

$$\delta_{P\rtimes G} \mathbin{\widehat{\otimes}}_{C_0(\underline{E}G)\rtimes G} d' = 1_{P\rtimes G}.$$

The other identity is similarly proved; we skip the details.

Note that this is an improvement over Theorem 1.16, because the existence of Θ is strictly weaker than having a gamma element. A similar argument shows that in general, if $P_{\mathbb{C}}$ is a (categorical) direct summand of some proper algebra, the conclusion of Theorem 1.16 holds, namely $C_0(\underline{E}G)\rtimes G$ is a Spanier–Whitehead K-dual of $P_{\mathbb{C}}\rtimes G$.

The torsion-free case. We treat the torsion-free case separately, partly because it is particularly simple (e.g., condition (1) of Definition 0.10 reduces to a statement in nonequivariant K-theory), and partly because it is among the first cases where the duality classes (i.e., unit and counit) have been identified (albeit in a slightly different language, cf. [Kasparov 1988, Theorems 6.6 and 6.7]).

We assume that G is a countable, discrete, torsion-free group. In this case, because proper actions are automatically free, the space $\underline{E}G$ is identified as the total space EG of the classifying space for principal G-bundles, and our assumption that G admits a G-compact model of $\underline{E}G$ translates into the assumption that G admits a compact model of BG. Denote by [MF] the class

$$[\mathrm{MF}] \in \mathrm{KK}(\mathbb{C}, C^*_r(G) \otimes C(BG))$$

associated to the module of sections of the Miščenko bundle. This is the Hermitian bundle of C^*-algebras obtained from the associated bundle construction

$$EG \times_G C^*_r(G) \to BG,$$

where G acts diagonally, acting on the reduced group C^*-algebra via the left regular representation [Miščenko and Fomenko 1979].

Proposition 1.19 ([Connes 1994]; for a proof see [Kaad and Proietti 2018]). *The Miščenko module* MF *is the finitely generated projective Hilbert C^*-module described as the completion of $C_c(EG)$ with respect to the norm induced by the $C^*_r(G) \widehat{\otimes} C(BG)$-valued inner product*

$$\langle \xi | \zeta \rangle(t)(x) = \sum_{p(y)=x} \bar{\xi}(y)\zeta(y \cdot t), \tag{1.20}$$

where $\xi, \zeta \in C_c(EG)$, $t \in G$, $x \in BG$, and $p : EG \to BG$ is the quotient map. The right action of $C^*_r(G) \widehat{\otimes} C(BG)$ on M is defined by

$$(\xi \cdot f)(y) = \sum_{g \in G} f(g)(p(y)) \cdot \xi(y \cdot g^{-1}), \tag{1.21}$$

where $\xi \in C_c(EG)$, $f \in C_c(G, C(BG))$, and $y \in EG$.

We have, for any separable C^*-algebra B with trivial G-action [Land 2015; Kaad and Proietti 2018],

$$\begin{array}{ccc} \mathrm{KK}(C(BG), B) & \xrightarrow{[\mathrm{MF}]\widehat{\otimes}_{C(BG)} -} & \mathrm{KK}(\mathbb{C}, C^*_r(G) \otimes B) \\ \downarrow \cong & & \downarrow = \\ \mathrm{KK}^G(C_0(EG), B) & \xrightarrow{\mu^G_B} & \mathrm{KK}(\mathbb{C}, C^*_r(G) \otimes B) \end{array}$$

The vertical isomorphism above is implemented by the strong Morita equivalence between $C(BG)$ and $C_0(EG) \rtimes G$ [Rieffel 1976], whose associated KK-class is denoted $[Y^*]$ below (we use $[Y]$ for the opposite module).

If G admits a compact nonpositively curved manifold as a model for BG, then the element $\underline{d}$ was introduced by Kasparov [1988] as a "dual-Dirac" class

$$\underline{d} \in \mathrm{KK}(C^*_r(G) \widehat{\otimes} C(BG), \mathbb{C}).$$

To be more consistent with the terminology of this paper, $\underline{d}$ should be called the duality counit induced by the γ-element (which exists in this situation). Kasparov went on to show that $\underline{d}$ defines a left inverse for the assembly map.

Hence we see that we are in a situation where Spanier–Whitehead duality comes into play very naturally, with the choice MF = unit and $\underline{d}$ = counit. Note that, while the class $\underline{d}$ requires structural information on the group, the class of the Miščenko bundle relies on very little structure. This is in complete analogy with the canonical unit defined previously.

Proposition 1.22. *The class* MF *coincides with* δ_G *from Definition 1.1 up to* KK-*equivalence. More precisely*, *we have*

$$\delta_G = \mathrm{MF}\,\widehat{\otimes}_{C^*_r(G)\widehat{\otimes}C(BG)}\,\tau_{C^*_r(G)}([Y^*]).$$

Proof. Let us set $Z = \mathrm{GJ}^{-1}([Y]) \in \mathrm{KK}^G(C_0(EG), C(BG))$. It is shown in [Kaad and Proietti 2018] that Z is represented by a G-C^*-correspondence satisfying the isomorphism of Hilbert modules

$$\mathrm{MF} \cong i^*(Y^*)\,\widehat{\otimes}_{C_0(EG)\rtimes G}\,(Z \rtimes_r G)$$

(we are denoting by i the inclusion $\mathbb{C} \hookrightarrow C(BG)$ as constant functions). We want to prove

$$[p_G]\,\widehat{\otimes}_{C_0(EG)\rtimes G}\,[\Delta] = i^*([Y^*])\,\widehat{\otimes}_{C_0(EG)\rtimes G}\,J_r^G(Z)\,\widehat{\otimes}_{C^*_r(G)\widehat{\otimes}C(BG)}\,\tau_{C^*_r(G)}([Y^*]),$$

or equivalently, by Lemma 1.2,

$$[p_G]\,\widehat{\otimes}_{C_0(EG)\rtimes G}\,[\Delta] \\ = i^*([Y^*])\,\widehat{\otimes}_{C_0(EG)\rtimes G}\,\big([\Delta]\,\widehat{\otimes}\,\tau_{C^*_r(G)}(\mathrm{GJ}(Z))\big)\,\widehat{\otimes}_{C^*_r(G)\widehat{\otimes}C(BG)}\,\tau_{C^*_r(G)}([Y^*]).$$

It is well-known that $[p_G] = i^*([Y^*])$ (see for example [Land 2015]), so that by associativity of the Kasparov product we have reduced the problem to showing

$$\tau_{C^*_r(G)}(\mathrm{GJ}(Z))\,\widehat{\otimes}\,\tau_{C^*_r(G)}([Y^*]) = \tau_{C^*_r(G)}(\mathrm{GJ}(Z)\,\widehat{\otimes}_{C(BG)}\,[Y^*]) = 1_{C^*_r(G)\widehat{\otimes}C_0(EG)\rtimes G}.$$

Now $\mathrm{GJ}(Z) = [Y]$ by construction, and hence the proof is complete. □

Now suppose that G is a general torsion-free group, and that a (γ)-element $x = [H, T]$ exists. Inspired by Kasparov's construction, we define the class $\underline{d}$ in $\mathrm{KK}(C^*_r(G) \otimes C(BG), \mathbb{C})$ by setting

$$\underline{d} = [Y]\,\widehat{\otimes}_{C_0(\underline{E}G)\rtimes G}\,d.$$

The element $\underline{d}$ admits a simple description in terms of the cycle (H, T) with property (γ) as follows. The G-equivariant nondegenerate representation π of $C_0(\underline{E}G)$ on H extends to that of the multiplier algebra $C_b(\underline{E}G)$. Together with the representation π_G of G on H, it induces the representation $\pi_G \otimes \pi$ of $C^*_r(G) \otimes C(BG)$ on

H. Here, $C(BG)$ is naturally identified as the subalgebra $C_b(\underline{E}G)$ consisting of G-invariant functions. The representation π_G extends to the representation for $C_r^*(G)$ since π_G is weakly contained in the left regular representation. Indeed, π_G is contained in the (amplified) left regular representation as we have a G-equivariant embedding from H to $H \otimes \ell^2(G)$ given by

$$v \mapsto \sum_h \pi(h(c))v \otimes \delta_h.$$

Proposition 1.23. *The triple* $(H, \pi_G \otimes \pi, T)$ *defines a Kasparov cycle* $[\pi_G \otimes \pi, H, T]$ *for* $\mathrm{KK}(C_r^*(G) \otimes C(BG), \mathbb{C})$. *We have* $[\pi_G \otimes \pi, H, T] = \underline{d}$.

Proof. We need to show that for any G-invariant continuous function ϕ on $\underline{E}G$, the commutator $[T, \phi]$ is compact. By condition (2b) for property (γ), we just need to show that $[T', \phi]$ is compact, where $T' = \sum_{g \in G} g(c)Tg(c)$; c is a cutoff function on $\underline{E}G$. Take any compactly supported function χ on $\underline{E}G$ so that $c\chi = c$.

We have

$$[T', \phi] = \sum_{g \in G} g(c)[T, g(\chi\phi)]g(c) = \sum_{g \in G} g(c)T_g g(c),$$

where $T_g = [T, g(\chi\phi)]$ are compact operators whose norms vanish as g goes to infinity by condition (2a) for property (γ). It follows that $[T', \phi] = \sum_{g \in G} g(c)T_g g(c)$ is compact (see [Nishikawa 2019, Lemmas 2.5 and 2.6]).

We leave to the reader the straightforward check that the element $[H, \pi_G \otimes \pi, T]$ in $\mathrm{KK}(C_r^*(G) \otimes C(BG), \mathbb{C})$ corresponds to d in $\mathrm{KK}(C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G, \mathbb{C})$ by the Morita equivalence between $C(BG)$ and $C_0(EG) \rtimes G$. □

We set

$$\underline{\Delta}_{C_r^*(G)} = [\mathrm{MF}] \,\widehat{\otimes}_{C(BG)}\, \underline{d}, \qquad \underline{\Delta}_{C(BG)} = [\mathrm{MF}] \,\widehat{\otimes}_{C_r^*(G)}\, \underline{d}.$$

The following conclusions are immediate from the discussion above.

Theorem 1.24. *Let G be a torsion-free group and suppose that a* (γ)*-element* $x \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ *exists. We have*

$$\underline{\Delta}_{C_r^*(G)} = J_r^G(x), \qquad \underline{\Delta}_{C(BG)} = 1_{C(BG)}.$$

For example, this is the case when BG is a compact smooth manifold of nonpositive sectional curvature.

2. Examples

In this section we give a few examples and computations to put into context the abstract duality results that have been explained previously. We primarily treat the case of strong Spanier–Whitehead duality, and only briefly discuss the weak case, as it is mostly covered by other results in the literature (see, for example, [Brodzki et al. 2008, Examples 2.14 and 2.17]).

Groups with Spanier–Whitehead K-duality. Let G be a countable discrete group which satisfies (1) and either (2) or (3) of the following:

(1) G admits a G-compact model of $\underline{E}G$;

(2) G admits a γ-element γ such that $j_r^G(\gamma) = 1_{C_r^*(G)}$, or

(3) G admits a (γ)-element x such that $j_r^G(x) = 1_{C_r^*(G)}$.

We recall that the gamma element γ, if it exists, is represented by a cycle with property (γ). Therefore, condition (2) implies (3). Our previous argument shows that such a group G has Spanier–Whitehead K-duality. Thanks to the Higson–Kasparov theorem [2001], we obtain the following:

Theorem 2.1. *All a-T-menable groups which admit a G-compact model of $\underline{E}G$ have Spanier–Whitehead K-duality.*

Examples of such a-T-menable groups include the following:

- all groups which act properly, affine-isometrically, and cocompactly on a finite-dimensional Euclidean space,
- all cocompact lattices of simple Lie groups $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$,
- all groups which act cocompactly on a tree (or more generally on a CAT(0)-cube complex).

For any a-T-menable group G listed above, the gamma element γ can be represented by an explicit cycle with property (γ). Below, we describe an explicit cycle with property (γ) for these groups. As a consequence, we can obtain an explicit cycle d in $\mathrm{KK}(C_r^*(G) \otimes C_0(\underline{E}G) \rtimes G, \mathbb{C})$ which, together with δ, induces the duality between $C_r^*(G)$ and $C_0(\underline{E}G) \rtimes G$.

To begin, we recall from [Kasparov 1988; Valette 2002] that the gamma element exists for any group G which acts properly and isometrically on a simply connected, complete Riemannian manifold M of nonpositive sectional curvature which is bounded from below. In this case, the gamma element for G is represented by an unbounded G-equivariant Kasparov cycle

$$(H_M, D_M),$$

where H_M is the Hilbert space $L^2(M, \Lambda^* T^*_{\mathbb{C}} M)$ of L^2-sections of the complexified exterior algebra bundles on M and where D_M is the self-adjoint operator

$$D_M = d_f + d_f^*$$

on M given by the Witten type perturbation

$$d_f = d + df \wedge$$

of the exterior derivative d; the function f is the squared distance $d^2_M(x_0, x)$ on M for some fixed point x_0 of M. Let

$$F_M = \frac{D_M}{(1 + D_M^2)^{1/2}}$$

be the bounded transform of D_M. The element $[H_M, F_M]$ in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ is the gamma element for G. We now suppose furthermore that the action of G on M is cocompact. In this case, G admits a G-compact model of $\underline{E}G$, namely the manifold M.

Proposition 2.2. *The cycle* (H_M, F_M) *has property* (γ).

Proof. Since $[H_M, F_M]$ is the gamma element for G, it satisfies condition (1) of Definition 0.12. To show that condition (2) holds for $[H_M, F_M]$, we shall apply Theorem 6.1 of [Nishikawa 2019]. We use the natural nondegenerate representation of $C_0(M)$ on H_M by pointwise multiplication. We take the dense subalgebra B of $C_0(M)$ consisting of compactly supported smooth functions. Note that B contains a cutoff function of M. For any function h in B, we have

$$[D_M, g(h)] = [d + d^*, g(h)] = g(c(h)),$$

where $c(h)$ is the Clifford multiplication by the gradient of h which is bounded and compactly supported. We can now use [Nishikawa 2019, Theorem 6.1] to conclude that the bounded transform (H_M, F_M) satisfies condition (2) of property (γ). □

Corollary 2.3. *For all groups G which act properly, affine-isometrically, and cocompactly on a finite-dimensional Euclidean space* $\mathbb{R}^n$, *the G-equivariant cycle* $(H_{\mathbb{R}^n}, F_{\mathbb{R}^n})$ *has property* (γ).

Corollary 2.4. *For all cocompact closed subgroups G of a semisimple Lie group L, the G-equivariant cycle* $(H_{L/K}, F_{L/K})$ *has property* (γ), *where K is a maximal compact subgroup of L.*

Let us look at a few examples.

Poincaré–Langlands duality. In [Niblo et al. 2016] the authors examine the Baum–Connes correspondence for the (extended) affine Weyl group W_a associated to a compact connected semisimple Lie group G. This group can be realized as the group of affine isometries of the Lie algebra $\mathfrak{t}$ of a maximal torus $T \subseteq G$. The structure of W_a is that of a semidirect product $\Gamma \rtimes W$, where Γ is the lattice of translations in $\mathfrak{t}$, and W is the Weyl group of the root system of G.

Ultimately, it is shown that the Baum–Connes conjecture (which holds in this case) is equivalent to T-duality for the aforementioned torus T and the Pontryagin dual $\widehat{\Gamma}$ of the lattice Γ. From the viewpoint of Lie groups, $\widehat{\Gamma}$ equivariantly coincides with the maximal torus $T^\vee$ of the Langlands dual $G^\vee$ of G. In K-theory this is expressed by W-equivariant Spanier–Whitehead duality between the dual tori T and $T^\vee$, which is referred to as "Poincaré–Langlands duality" in [Niblo et al. 2016].

Propositions 1.19–1.23 and Theorem 1.24 can be equivalently applied to get these results, with $C(B\Gamma)$ playing the role of $C(T)$ and $C_r^*(\Gamma)$ playing the role of $C(T^\vee)$ through the Gelfand transform.

The (γ)-element, which belongs to $\mathrm{KK}^{W_a}(\mathbb{C},\mathbb{C})$, in this case can be constructed as explained above with $M=\mathfrak{t}$ and distance function induced by a W-equivariant metric. Equivalently, the bounded transform of the Bott–Dirac operator

$$B_{\mathfrak{t}}=\sum_i(\mathrm{ext}(e_i)+\mathrm{int}(e_1))x_i+(\mathrm{ext}(e_i)-\mathrm{int}(e_i))\,\frac{d}{dx_i}$$

yields a W-equivariant cycle with property (γ), provided that interior multiplication is defined through a W-equivariant metric. The cycle obtained this way is indeed isomorphic to the one obtained through the Witten type perturbation of the de Rham operator, and its KK-class coincides with the classical γ-element which is homotopic to the unit [Higson and Kasparov 2001].

In summary, we obtain equivariant duality classes $\underline{\delta}^W\in\mathrm{KK}^W(\mathbb{C},C(\mathfrak{t}/\Gamma)\otimes C_r^*(\Gamma))$, derived from the Miščenko W-bundle associated to the principal Γ-bundle $\mathfrak{t}\to T$, and $\underline{d}^W\in\mathrm{KK}^W(C(\mathfrak{t}/\Gamma)\otimes C_r^*(\Gamma),\mathbb{C})$, derived from the (γ)-element described above. We can prove

$$\underline{\delta}^W\,\widehat{\otimes}_{C(T)}\,\underline{d}^W=j_r^\Gamma(\gamma),$$

where on the right-hand side we mean "partial" descent with respect to the normal subgroup $\Gamma\subseteq W_a$. As we know, $\gamma=1_{\mathbb{C}}$ in $\mathrm{KK}^{\Gamma\times W}(\mathbb{C},\mathbb{C})$, so that we get

$$\underline{\delta}^W\,\widehat{\otimes}_{C(T)}\,\underline{d}^W=1,\qquad \underline{\delta}^W\,\widehat{\otimes}_{C(T^\vee)}\,\underline{d}^W=1$$

in the equivariant groups $\mathrm{KK}^W(C(T^\vee),C(T^\vee))$, $\mathrm{KK}^W(C(T),C(T))$, respectively.

Lattices in $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$. Let G be a cocompact lattice of a simple Lie group $L=\mathrm{SO}(n,1)$, or $L=\mathrm{SU}(n,1)$. Let K be a maximal compact subgroup of L. Corollary 2.4 shows that the G-equivariant cycle $(H_{L/K},F_{L/K})$ has property (γ). The corresponding element $x=[H_{L/K},F_{L/K}]$ is nothing but the gamma element γ for G, which is shown to be equal to 1_G [Higson and Kasparov 2001; Julg and Kasparov 1995].

Groups acting on trees. Let G be a countable discrete group which acts properly and cocompactly on a locally finite tree Y. The tree Y is the union of the sets Y^0, Y^1 of the vertices and edges of the tree. Without loss of generality, we assume a G-invariant typing on the tree. Namely, we assume a G-invariant decomposition $Y^0=Y^0_0\sqcup Y^0_1$ so that any two adjacent vertices have distinct types. This can be achieved by the barycentric subdivision of the tree. We take E as the geometric realization of the tree. This is a G-compact model of the universal proper G-space. We denote by d the edge path metric on E and hence on Y^0 such that each edge has length 1.

The ℓ^2 space $\ell^2(Y)$ is naturally a graded G-Hilbert space with the even and odd spaces being $\ell^2(Y^0)$, $\ell^2(Y^1)$, respectively. Let $H_{\mathbb{R}}$ be the graded Hilbert space $L^2(\mathbb{R}, \Lambda^*_{\mathbb{C}}(\mathbb{R}))$ as before, but now with the trivial G-action. We construct a Kasparov cycle with the property (γ) on the graded tensor product

$$H_Y = H_{\mathbb{R}} \mathbin{\widehat{\otimes}} \ell^2(Y).$$

Following [Kasparov and Skandalis 1991], we define a nondegenerate representation π of $C_0(E)$ on H_Y, which is diagonal with respect to Y. This is given by a family $(\pi_y)_{y\in Y}$ of representations of $C_0(E)$ on $H_{\mathbb{R}}$ indexed by y in Y. If y is a vertex, we define π_y by sending ϕ in $C_0(E)$ to the multiplication on $H_{\mathbb{R}}$ by the constant $\phi(y)$. If y is an edge with vertices y_0, y_1 of corresponding types, we identify y with the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ via the unique isometry sending y_j to $(-1)^j\frac{1}{2}$. We define π_y by sending ϕ in $C_0(E)$ to the multiplication on $H_{\mathbb{R}}$ by the restriction of ϕ to the edge y extended to the left and right constantly.

Now, like the operator D_M, we define an unbounded, odd, self-adjoint operator D_Y with compact resolvent of index 1, which is almost G-equivariant and has nice compatibility with functions in $C_0(E)$. The bounded transform F_Y of D_Y will give us a desired Kasparov cycle (H_Y, F_Y) with property (γ). For this, we fix a base point y_0 from Y^0. The following construction depends on the choice of y_0. We have the decomposition of H_Y

$$H_Y = H_{\mathbb{R}} \mathbin{\widehat{\otimes}} \mathbb{C}\delta_{y_0} \oplus \bigoplus_{y\in Y^0\setminus\{y_0\}} (H_{\mathbb{R}} \mathbin{\widehat{\otimes}} (\mathbb{C}\delta_y \oplus \mathbb{C}\delta_{e_y})),$$

where for each vertex $y \neq y_0$, e_y is the last edge appearing in the geodesic from y_0 to y and where the symbol δ_* denotes a delta function in $\ell^2(Y)$. Our operator D_Y is block-diagonal with respect to this decomposition. It is given by a family $(D_y)_{y\in Y^0}$ of an unbounded, odd, self-adjoint operators with compact resolvent.

For a vertex $y \in Y^0_j$ of type j, let $B_{\mathbb{R},y}$ be the Bott–Dirac operator on $H_{\mathbb{R}}$ with "origin shifted":

$$B_{\mathbb{R},y} = (\mathrm{ext}(e_1) + \mathrm{int}(e_1))(x - n_y) + (\mathrm{ext}(e_1) - \mathrm{int}(e_1))\,\frac{d}{dx},$$

where $n_y = (-1)^j\left(\frac{1}{2} + d(y, y_0)\right)$. For $y = y_0$, we simply set

$$D_{y_0} = B_{\mathbb{R},y_0} \mathbin{\widehat{\otimes}} 1 \quad \text{on } H_{\mathbb{R}} \mathbin{\widehat{\otimes}} \mathbb{C}\delta_{y_0}.$$

For $y \neq y_0$, we set

$$D_y = B_{\mathbb{R},y} \mathbin{\widehat{\otimes}} 1 + M_{\chi_y} \mathbin{\widehat{\otimes}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{on } H_{\mathbb{R}} \mathbin{\widehat{\otimes}} (\mathbb{C}\delta_y \oplus \mathbb{C}\delta_{e_y}),$$

where M_{χ_y} is the multiplication on $H_{\mathbb{R}}$ by the function χ_y on $\mathbb{R}$ defined as

$$\text{for } y \in Y_0^0, \quad \chi_y(x) = \begin{cases} 0, & x < \frac{1}{2}, \\ \left(x - \frac{1}{2}\right)^2, & \frac{1}{2} \leq x < 1, \\ x - \frac{3}{4}, & 1 \leq x < d(y, y_0), \\ -(x - n_y)^2 + d(y, y_0) - \frac{1}{2}, & d(y, y_0) \leq x < n_y, \\ d(y, y_0) - \frac{1}{2}, & n_y \leq x, \end{cases}$$

$$\text{for } y \in Y_1^0, \quad \chi_y(x) = \begin{cases} d(y, y_0) - \frac{1}{2}, & x < n_y, \\ -(x - n_y)^2 + d(y, y_0) - \frac{1}{2}, & n_y \leq x < -d(y, y_0), \\ -x - \frac{3}{4}, & -d(y, y_0) \leq y < -1, \\ \left(x + \frac{1}{2}\right)^2, & -1 \leq x < -\frac{1}{2}, \\ 0, & -\frac{1}{2} \leq x. \end{cases}$$

Note that for each $y \neq y_0$, D_y is a bounded perturbation of a self-adjoint operator $B_{\mathbb{R},y} \mathbin{\widehat{\otimes}} 1$ with compact resolvent of index 0, and hence so is D_y. All D_y are hence diagonalizable. Therefore, $D_Y = (D_y)_{y \in Y^0}$ is self-adjoint. In order to see that D_Y has compact resolvent, we compute

$$D_y^2 = B_{\mathbb{R},y}^2 \mathbin{\widehat{\otimes}} 1 + M_{\chi_y}^2 \mathbin{\widehat{\otimes}} 1 + \begin{pmatrix} 0 & -M_{\chi_y'} \\ M_{\chi_y'} & 0 \end{pmatrix} \mathbin{\widehat{\otimes}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where χ_y' is the derivative of χ_y. We see that D_y^2 has spectrum far away from 0 as y goes to infinity essentially because the derivatives χ_y' are uniformly bounded in y and because we have

$$(x - n_y)^2 + \chi_y^2 \geq 2\left(\tfrac{1}{2} d(y, y_0) - \tfrac{1}{8}\right)^2$$

everywhere. It follows that D_Y indeed has compact resolvent. Let F_Y be the bounded transform

$$F_Y = \frac{D_Y}{(1 + D_Y^2)^{1/2}}.$$

Proposition 2.5. *A pair (H_Y, F_Y) is a G-equivariant Kasparov cycle with property (γ).*

Proof. Almost G-equivariance follows from

$$D_Y - g(D_Y) = \text{bounded} \quad \text{for } g \in G,$$

which we leave to the reader. To see that $[H_Y, F_Y] = 1_F$ in $R(F)$ for any finite subgroup F of G, we note that the class $[H_Y, F_Y]$ does not depend on the choice of the base point y_0. Hence, we may assume that y_0 is a vertex fixed by the group F. In this case, it is not hard to see that F_Y is an odd, F-equivariant, self-adjoint operator whose graded index is the one-dimensional trivial representation of F spanned by $\xi_0 \widehat{\otimes} \delta_{y_0}$ in $H_{\mathbb{R}} \widehat{\otimes} \mathbb{C}\delta_{y_0}$, where $\xi_0 = e^{-x^2/2}$. This shows $[H_Y, F_Y] = 1_F$. To show that it has condition (2) of property (γ) with respect to the representation π of $C_0(E)$, we shall apply Theorem 6.1 of [Nishikawa 2019] for the dense subalgebra B of $C_0(E)$ consisting of compactly supported functions which are smooth inside each edge and constant near the vertices. Note that B contains a cutoff function of E. First, we can see that for each $y \neq y_0$, the operator $M_{\chi_y} \widehat{\otimes} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ commutes with the representation π. This is due to the vanishing of χ_y for $y \in Y_0^0$ and $y \in Y_1^0$ over $x \le \frac{1}{2}$ and $-\frac{1}{2} \le x$, respectively. For ϕ in B, we compute the commutator $[D_Y, \pi(\phi)]$ as

$$\begin{aligned}
[D_Y, \pi(\phi)] &= [B_{\mathbb{R},y_0} \widehat{\otimes} 1, \pi(\phi)] + \sum_{y \in Y^0 \setminus \{y_0\}} \left[B_{\mathbb{R},y} \widehat{\otimes} 1 + M_{\chi_y} \widehat{\otimes} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \pi(\phi) \right] \\
&= [B_{\mathbb{R},y_0} \widehat{\otimes} 1, \pi(\phi)] + \sum_{y \in Y^0 \setminus \{y_0\}} [B_{\mathbb{R},y} \widehat{\otimes} 1, \pi(\phi)] \\
&= \sum_{y \in Y^0 \setminus \{y_0\}} \left[\begin{pmatrix} 0 & -\frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix}, \pi_{e_y}(\phi) \right] \widehat{\otimes} 1 \\
&= \pi(\phi') \sum_{y \in Y^0 \setminus \{y_0\}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \widehat{\otimes} 1,
\end{aligned}$$

where in the last two, each summand is an operator on $H_{\mathbb{R}} \widehat{\otimes} \mathbb{C}\delta_{e_y}$ and where ϕ' is the derivative of ϕ. Note that each summation is a finite sum since ϕ is compactly supported. We can now use [Nishikawa 2019, Theorem 6.1] to conclude that the bounded transform (H_Y, F_Y) satisfies condition (2) of property (γ). □

Remark 2.6. The construction can be generalized to define a cycle with property (γ) for a group which acts properly and cocompactly on a Euclidean building in the sense of [Kasparov and Skandalis 1991]. In [Brodzki et al. 2019], a different construction is given which provides us a cycle with property (γ) for a group which acts properly and cocompactly on a finite-dimensional CAT(0) cube complex.

Groups with weak Spanier–Whitehead K-duality. Let G be a countable discrete group satisfying the conditions (1) and either (2)' or (3)' below:

(1) G admits a G-compact model of $\underline{E}G$;

$(2)'$ G admits a γ-element γ with $j_r^G(\gamma)$ acting as the identity on $K_*(C_r^*(G))$, or

$(3)'$ G admits a (γ)-element x with $j_r^G(x)$ acting as the identity on $K_*(C_r^*(G))$.

Our previous argument shows that such a group G has weak Spanier–Whitehead K-duality. For any word-hyperbolic group, the gamma element is shown to exist and the Baum–Connes conjecture has been verified [Lafforgue 2012; Kasparov and

Skandalis 2003; Mineyev and Yu 2002]. Moreover, any hyperbolic group is known to admit a G-compact model of $\underline{E}G$ [Meintrup and Schick 2002]. Hence, we have the following:

Theorem 2.7. *All word-hyperbolic groups G have weak Spanier–Whitehead K-duality.*

As an example of hyperbolic groups, we can take G to be a cocompact lattices of the simple Lie group $L = \mathrm{Sp}(n, 1)$. As before, the γ-element for G has an explicit representative $(H_{L/K}, F_{L/K})$ with property (γ). We remark that the gamma element $\gamma = [H_{L/K}, F_{L/K}]$ is well-known to be not homotopic to 1_G due to Kazhdan's property (T). Furthermore, Skandalis [1988] showed that $j_r^G(\gamma)$ is not equal to $1_{C_r^*(G)}$. More precisely, what he showed is that $C_r^*(G)$ is not K-nuclear, which in particular implies that it cannot be KK-equivalent to any nuclear C^*-algebra. The same remark that $j_r^G(\gamma) \neq 1_{C_r^*(G)}$ applies to any infinite hyperbolic property (T) group [Higson and Guentner 2004, Theorem 5.2]. In general, when the gamma element γ exists, the equality $j_r^G(\gamma) = 1_{C_r^*(G)}$ implies that $C_r^*(G)$ is KK-equivalent to $P_{\mathbb{C}} \rtimes G$, which satisfies the UCT [Meyer and Nest 2006, Proposition 9.5]; in particular it is K-nuclear. Therefore, if $C_r^*(G)$ is not K-nuclear, we have $j_r^G(\gamma) \neq 1_{C_r^*(G)}$.

3. Some applications

In this section we prove a few results by applying the theory of K-duality developed in the previous pages. Some of the material presented here has been previously treated in the literature via possibly different methods [Dadarlat 2009, Section 3; Emerson and Meyer 2010, Section 5; Kaminker et al. 2017, Section 4.4; Rosenberg and Schochet 1987, Section 7]. Nevertheless, we provide a brief account for completeness, to give a better idea of some applications of our main theorems.

We say a C^*-algebra A is KK-*compact* if the functor sending D to $\mathrm{KK}_*(A, D)$ commutes with filtered colimits. If A is a C^*-algebra with a Spanier–Whitehead K-dual B, then A is KK-compact because $\mathrm{KK}_*(A, D)$ is naturally isomorphic to $\mathrm{KK}_*(\mathbb{C}, D \otimes B)$ and the K-theory functor is continuous.

As explained after Theorem 6.6 of [Meyer and Nest 2006], a C^*-algebra satisfies the *universal coefficient theorem* (UCT) [Blackadar 1998, Section 23] if and only if it belongs to the localizing triangulated subcategory of the KK-category generated by the complex numbers (this category is denoted as $\langle * \rangle$ in [Meyer and Nest 2006]). As in [Dell'Ambrogio et al. 2011], let us denote this subcategory by $\mathcal{T}$. It is known that within this subcategory, an object is dualizable if and only if it is compact:

Proposition 3.1 [Dell'Ambrogio et al. 2011, Proposition 4.1]. *In the subcategory $\mathcal{T} \subseteq \mathrm{KK}$, the full triangulated subcategory $\mathcal{T}_c$ of compact objects coincides with the (closed) symmetric monoidal category $\mathcal{T}_d$ of dualizable objects. Furthermore, both* *these two subcategories are equal to the thick triangulated subcategory generated by the complex numbers.*

Corollary 3.2. *If G has Spanier–Whitehead duality then $C_r^*(G)$ satisfies the UCT.*

Proof. We know that $C_0(\underline{E}G) \rtimes G$ satisfies the UCT [Meyer and Nest 2006, Proposition 9.5]. By assumption, $C_0(\underline{E}G) \rtimes G$ has a Spanier–Whitehead K-dual $C_r^*(G)$. Thus, $C_0(\underline{E}G) \rtimes G$ is KK-compact. By Proposition 3.1, it is dualizable in $\mathcal{T}$. Namely, it has a Spanier–Whitehead K-dual, say A, which satisfies the UCT. On the other hand, it is fairly easy to see that a dual object is unique up to equivalence. Hence, $C_r^*(G)$ is KK-equivalent to A. The claim follows from this. □

The strong Baum–Connes conjecture was introduced in [Meyer and Nest 2006] as the assertion that the canonical Dirac morphism α in $\mathrm{KK}^G(P_{\mathbb{C}}, \mathbb{C})$ induces a KK-equivalence $j_r^G(\alpha)$ from $P_{\mathbb{C}} \rtimes G$ to $C_r^*(G)$. In the presence of the gamma element γ for G, this is equivalent to the assertion that $j_r^G(\gamma) = 1_{C_r^*(G)}$.

Theorem 3.3. *If G has Spanier–Whitehead duality then it satisfies the strong Baum–Connes conjecture. Moreover, if the γ-element exists and G satisfies the strong Baum–Connes conjecture, than G has Spanier–Whitehead duality.*

Proof. Suppose G has Spanier–Whitehead duality. Then we know that the Baum–Connes conjecture holds for G, and so the Dirac morphism α induces an isomorphism $j_r^G(\alpha)_*$ on K-groups from $P_{\mathbb{C}} \rtimes G$ to $C_r^*(G)$. Furthermore, both $P_{\mathbb{C}} \rtimes G$ and $C_r^*(G)$ satisfy the UCT by [Meyer and Nest 2006, Proposition 9.5] and by Corollary 3.2, respectively. It follows that $j_r^G(\alpha)$ is a KK-equivalence [Blackadar 1998, Theorem 23.10.1]. Conversely, if the strong Baum–Connes conjecture holds, we have $j_r^G(\gamma) = 1_{C_r^*(G)}$. Hence, G has Spanier–Whitehead duality by Theorem 1.14. □

As in [Blackadar 1998, Theorem 23.10.5], a C^*-algebra A satisfies the UCT if and only if it is KK-equivalent to a commutative C^*-algebra $C_0(X)$. Furthermore, this X can be taken to be a three-dimensional cell complex [Blackadar 1998, Corollary 23.10.3; Rosenberg and Schochet 1987, Proposition 7.4]. This is because the range of K-theory on such spaces exhausts all countable $\mathbb{Z}/(2)$-graded abelian groups. If $K_*(A)$ is finitely generated, then X can be chosen finite, and a Spanier–Whitehead K-dual exists for such spaces [Emerson and Meyer 2010, Proposition 5.9].

Lemma 3.4. *Suppose A has a Spanier–Whitehead K-dual and satisfies the UCT. Then it has finitely generated K-theory groups.*

Proof. As in the proofs of [Rosenberg and Schochet 1987, Proposition 7.4; Blackadar 1998, Corollary 23.10.3], let $C = C^0 \oplus C^1$ be a commutative C^*-algebra KK-equivalent to A, where C^0 is the mapping cone of a $*$-homomorphism on direct sums of $C_0(\mathbb{R})$, and C^1 is the suspension of such a mapping cone. It is easy to see that C is the inductive limit of subalgebras C_n, where C_n has finitely generated K-theory. Since $\mathrm{KK}_*(A, -)$ is continuous (since A is KK-compact), the equivalence $A \to C$ factors through C_n for some $n \in \mathbb{N}$. Then $K_*(A)$ is finitely generated because it is a quotient of $K_*(C_n)$, which enjoys this property. □

Proposition 3.5. *Suppose G satisfies the Baum–Connes conjecture and the γ-element exists. Then $C_r^*(G)$ has finitely generated K-theory groups.*

Proof. If $\gamma \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C})$ exists, then $P_{\mathbb{C}} \rtimes G$ is dualizable by Theorem 1.16. It is known that $P_{\mathbb{C}} \rtimes G$ satisfies the UCT; see [Meyer and Nest 2006, Proposition 9.5]. Thus, $P_{\mathbb{C}} \rtimes G$ has finitely generated K-groups by Lemma 3.4. Recall that in the localization picture the assembly map appears as

$$K_*(P_{\mathbb{C}} \rtimes G) \to K_*(C_r^*(G)). \tag{3.6}$$

Therefore, if (3.6) is an isomorphism the right-hand side is finitely generated. □

Remark 3.7. More generally, $C_r^*(G)$ has finitely generated K-theory groups if G satisfies the Baum–Connes conjecture and the source $P_{\mathbb{C}}$ of the Dirac morphism is a (categorical) direct summand of a proper algebra. This is because by Remark 1.18, $P_{\mathbb{C}} \rtimes G$ has a Spanier–Whitehead K-dual.

Remark 3.8. By the results in [Dell'Ambrogio et al. 2011], there exists a functor $\mathbb{K}$ from the KK-category to the stable homotopy category satisfying $\pi_n(\mathbb{K}(A)) \cong K_n(A)$. This functor specializes to a full and faithful functor on the subcategory of dualizable objects satisfying the UCT, realizing C^*-algebras as perfect KU-modules (in particular, finite spectra). Hence, the previous results can also be obtained from the well-known fact that homotopy groups are finitely generated in this context.

Define the *n-th dimension-drop algebra* as

$$\mathbb{I}_n = \{ f \in C([0, 1], M_n(\mathbb{C})) \mid f(0) = 0,\ f(1) \in \mathbb{C}1_n \}.$$

We can use this to introduce the mod-n K-theory groups as follows:

$$K_*(B; \mathbb{Z}/(n)) = \mathrm{KK}_*(\mathbb{I}_n, B).$$

It is apparent from this definition that a Baum–Connes conjecture in mod-n K-theory for B would have to introduce coefficients on the left, and we can take this as motivation to find a satisfactory formulation for the full bivariant version of the Baum–Connes conjecture. The approach via localization immediately generalizes to this context, giving us a map

$$\mathrm{KK}_*(A, (P_{\mathbb{C}} \otimes B) \rtimes G) \to \mathrm{KK}_*(A, B \rtimes_r G) \tag{3.9}$$

defined as $y \mapsto y \otimes j_r^G(1_B \widehat{\otimes} \alpha)$, where $\alpha \in \mathrm{KK}(P_{\mathbb{C}}, \mathbb{C})$ is the Dirac morphism, for any (separable) C^*-algebra A and G-C^*-algebra B.

The original definition of the left-hand side (following [Baum et al. 2003] and [Uuye 2011]), what is called the "naive" topological K-group in [Uuye 2011], is given as

$$\varinjlim_{Y \subseteq \underline{E}G} \mathrm{KK}_*^G(C_0(Y, A), B),$$

where the limit ranges as usual over G-invariant G-compact subspaces of $\underline{E}G$. Unlike the simpler case of the conjecture, the definition making use of the naive topological group is *not* equivalent to the definition in (3.9). However, [Uuye 2011] shows that there are natural maps

$$\nu_Y : \mathrm{KK}_*^G(C_0(Y, A), B) \to \mathrm{KK}_*(A, (P_{\mathbb{C}} \otimes B) \rtimes G), \tag{3.10}$$

which make the obvious diagram commute. In addition, if A admits a Spanier–Whitehead K-dual, then (3.10) induces an isomorphism.

Theorem 3.11 [Uuye 2011]. *Suppose A has a Spanier–Whitehead K-dual. Then the comparison map induced by the ν_Y is an isomorphism.*

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