ANNALS OF K-THEORY

vol. 5 no. 3 2020

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A JOURNAL OF THE K-THEORY FOUNDATION

ANNALS OF K-THEORY
Vol. 5, No. 3, 2020
https://doi.org/10.2140/akt.2020.5.581

Nisnevich topology with modulus

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In the theory of motives à la Voevodsky, the Nisnevich topology on smooth schemes is used as an important building block. We introduce a Grothendieck topology on proper modulus pairs, which is used to construct a non-homotopy-invariant generalization of motives. We also prove that the topology satisfies similar properties to the Nisnevich topology.

1. Introduction

In the theory of motives à la Voevodsky [2000], the Nisnevich topology on the category of smooth schemes over a field k plays a fundamental role. In this paper, we introduce a Grothendieck topology on proper modulus pairs, which is used to construct a non-homotopy-invariant generalization of motives. We also prove that the topology satisfies similar properties to the Nisnevich topology.

A Nisnevich cover $f : Y \to X$ is an étale cover such that any point $x \in X$ admits a point $y \in Y$ with $f(y) = x$ and $k(y) = k(x)$. Therefore, the Nisnevich topology is finer than the Zariski topology and is coarser than the étale topology. Voevodsky defined the category of effective motives $\mathbf{DM}^{\text{eff}}$ as the derived category of the abelian category of Nisnevich sheaves with transfers $\mathbf{NST}$, modulo $\mathbf{A}^1$-homotopy invariance:

$$\mathbf{DM}^{\text{eff}} := \frac{\mathbf{D}(\mathbf{NST})}{(\mathbf{A}^1\text{-homotopy invariance})}. \tag{1.1.1}$$

This work is supported by RIKEN Special Postdoctoral Researchers (SPDR) Program, by RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program (iTHEMS), and by JSPS KAKENHI Grant (19K23413).
MSC2010: primary 14F20; secondary 14C25, 18F10, 19E15.
Keywords: Nisnevich topology, cd-structure, modulus pairs, motives with modulus.

We briefly recall the definition of **NST**. Let **PST** be the category of additive abelian presheaves on the category of finite correspondences **Cor**. We have a natural functor $\mathbf{Sm} \to \mathbf{Cor}$, where **Sm** denotes the category of smooth schemes over k. Then **NST** is defined to be the full subcategory of **PST** which consists of $F \in \mathbf{PST}$ such that the restriction $F|_{\mathbf{Sm}}$ is a Nisnevich sheaf on **Sm**.

The definition of **NST** is simple, but it is nontrivial that **NST** is an abelian category. It follows from the existence of a left adjoint to the inclusion functor $\mathbf{NST} \to \mathbf{PST}$. A key ingredient of the proof of its existence is the following fact: for any Nisnevich cover $U \to X$, the Čech complex

$$\cdots \to \mathbb{Z}_{\mathrm{tr}}(U \times_X U) \to \mathbb{Z}_{\mathrm{tr}}(U) \to \mathbb{Z}_{\mathrm{tr}}(X) \to 0$$

is exact as a complex of Nisnevich sheaves, where $\mathbb{Z}_{\mathrm{tr}}(-) : \mathbf{Cor} \to \mathbf{PST}$ denotes the Yoneda embedding (see for example [Mazza et al. 2006, Proposition 6.12]). Moreover, the Nisnevich topology is *subcanonical*, i.e., every representable presheaf in **Sm** is a sheaf.

The category of motives $\mathbf{DM}^{\mathrm{eff}}$ has provided vast applications to the study of arithmetic geometry, but on the other hand, it has a fundamental constraint that it cannot capture *non-*$\mathbf{A}^1$*-homotopy-invariant phenomena*, e.g., wild ramification. Indeed, the arithmetic fundamental group $\pi_1(X)$, which captures the information of ramifications, is not $\mathbf{A}^1$-homotopy invariant.

An attempt to develop a theory of motives which captures non-$\mathbf{A}^1$-homotopy-invariant phenomena started in [Kahn et al. 2015]. The strategy is to extend Voevodsky's theory to *modulus pairs*. A *modulus pair* is a pair $M = (\overline{M}, M^\infty)$ of a scheme $\overline{M}$ and an effective Cartier divisor M^∞ on $\overline{M}$ such that the *interior* $M^{\mathrm{o}} := \overline{M} - M^\infty$ is smooth over k. We can define a reasonable notion of morphisms between modulus pairs, and we obtain a category of modulus pairs $\underline{\mathbf{M}}\mathbf{Sm}$. A modulus pair M is *proper* if $\overline{M}$ is proper over k, and we denote by **MSm** the full subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ consisting of proper modulus pairs (see Definition 2.1.1 for details).

These categories embed in categories of "modulus correspondences" $\underline{\mathbf{M}}\mathbf{Cor}$ and **MCor**, just as **Sm** embeds in **Cor** (see Definition 2.3.2). In [Kahn et al. 2015], categories of "modulus sheaves with transfers" $\underline{\mathbf{M}}\mathbf{NST}$ (relative to $\underline{\mathbf{M}}\mathbf{Cor}$) and **MNST** (relative to **MCor**) were introduced, in order to parallel the definition of (1.1.1). However, the proof that these categories are abelian was found to contain a gap. This gap was filled in [Kahn et al. 2019a] for $\underline{\mathbf{M}}\mathbf{NST}$, by showing that its objects are indeed the sheaves with transfers for a suitable Grothendieck topology on $\underline{\mathbf{M}}\mathbf{Sm}$.

In this paper, we construct a Grothendieck topology on **MSm** with nice properties. It will be shown in [Kahn et al. 2019b], using [Kahn and Miyazaki 2019], that the objects of **MNST** are the sheaves (with transfers) for this topology and that this category is abelian. Thus the present paper contains the tools to finish filling the gap of [Kahn et al. 2015]. Moreover, we prove an important exactness result.

Our guide is the following characterization of the Nisnevich topology on **Sm**: the Nisnevich topology is generated by coverings $U \sqcup V \to X$ associated with some commutative square S in **Sm** of the form

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

which satisfies the following properties:

(1) S is a cartesian square,

(2) the horizontal morphisms are open immersions,

(3) the vertical morphisms are étale, and

(4) the morphism $(V - W)_{\mathrm{red}} \to (X - U)_{\mathrm{red}}$ is an isomorphism.

Such squares are called *elementary Nisnevich squares*. Elementary Nisnevich squares form a *cd-structure* on **Sm** in the sense of [Voevodsky 2010]. A remarkable property of the Nisnevich cd-structure is the following fact: a presheaf of sets F on **Sm** is a Nisnevich sheaf if and only if $F(\varnothing) = \{*\}$ and for any elementary Nisnevich square as above, the square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(W) \end{array}$$

is cartesian. This equivalence holds for any cd-structure which is complete and regular; see [Voevodsky 2010, Definitions 2.3, 2.10, Corollary 2.17].

In [Kahn et al. 2019a], a cd-structure on $\underline{\mathbf{M}}\mathbf{Sm}$ is introduced. It is denoted $P_{\underline{\mathrm{MV}}}$, and satisfies properties similar to elementary Nisnevich squares. Its definition will be recalled in Section 4.1. For short, we call the topology on $\underline{\mathbf{M}}\mathbf{Sm}$ associated with $P_{\underline{\mathrm{MV}}}$ the $\underline{\mathrm{MV}}$*-topology*.

Our main result is the following.

Theorem. *The category of proper modulus pairs* **MSm** *admits a cd-structure* P_{MV} *such that the following assertions hold. For short, we call the topology associated with* P_{MV} *the* MV*-topology.*

(1) (Theorems 4.3.1, 4.4.1, 4.4.2) *The cd-structure* P_{MV} *is complete and regular. In particular, a presheaf of sets* F *on* **MSm** *is a sheaf for the* MV*-topology if*

and only if $F(\varnothing) = \{*\}$ *and for any square* $T \in P_{\mathrm{MV}}$ *of the form*

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & M \end{array}$$

the square

$$\begin{array}{ccc} F(M) & \longrightarrow & F(U) \\ \downarrow & & \downarrow \\ F(V) & \longrightarrow & F(W) \end{array}$$

is cartesian.

(2) (Theorem 4.5.1) *The* $\underline{\mathrm{MV}}$*-topology and the* MV*-topology are subcanonical.*

(3) (Corollary 5.2.7) *For any* $M \in \underline{\mathbf{M}}\mathbf{Sm}$, *consider the presheaf* $\mathbb{Z}_{\mathrm{tr}}(M)$ *on* $\underline{\mathbf{M}}\mathbf{Cor}$ *represented by* M, *which is a sheaf for the* $\underline{\mathrm{MV}}$*-topology by [Kahn et al. 2019a, Theorem* 2(2)*]. Then, for any square as above, the following complex of sheaves for the* $\underline{\mathrm{MV}}$*-topology is exact*:

$$0 \to \mathbb{Z}_{\mathrm{tr}}(W) \to \mathbb{Z}_{\mathrm{tr}}(U) \oplus \mathbb{Z}_{\mathrm{tr}}(V) \to \mathbb{Z}_{\mathrm{tr}}(M) \to 0.$$

The organization of the paper is as follows. In Section 2, we recall basic definitions and results on modulus pairs from [Kahn et al. 2019a]. In Section 3, we introduce "the off-diagonal functor", which is a key ingredient to define the cd-structure on the category of proper modulus pairs. In Section 4, we define the cd-structure on the category of proper modulus pairs, and prove that it satisfies completeness and regularity. Finally, in Section 5, we prove the exactness of the Mayer–Vietoris sequences associated with the distinguished squares with respect to the cd-structure.

Notation and convention. Throughout the paper, we fix a base field k. Let **Sm** be the category of separated smooth schemes of finite type over k, and let **Sch** be the category of separated schemes of finite type over k. For any scheme X and for any closed subscheme $F \subset X$, we denote by $\mathbf{Bl}_F(X)$ the blow-up of X along F.

2. Basics on modulus pairs

In this section, we introduce basic notions which we use throughout the paper.

2.1. ***Category of modulus pairs.*** We recall basic definitions on modulus pairs, introduced in [Kahn et al. 2019a]. We also introduce some new notation. In particular, the *canonical model of fiber product* is often useful (see Definition 2.2.2). Though our main interest in this paper is on *proper* modulus pairs, we introduce the general definition of modulus pairs for later use.

Definition 2.1.1. (1) A *modulus pair* is a pair $M = (\overline{M}, M^{\infty})$ consisting of a scheme $\overline{M} \in \mathbf{Sch}$ (the *ambient space*) and an effective Cartier divisor M^{∞} on $\overline{M}$ (the *modulus divisor*) such that the *interior* $M^{o} := \overline{M} \setminus |M^{\infty}|$ belongs to $\mathbf{Sm}$, where $|M^{\infty}|$ denotes the support of M^{∞}.

Note that M^{o} is a dense open subset of $\overline{M}$. Moreover, we can prove that $\overline{M}$ must be a reduced scheme by using the smoothness of M^{o} and the assumption that M^{∞} is an effective Cartier divisor.

(2) A modulus pair M is called *proper* if the ambient space $\overline{M}$ is proper over k.

(3) An *admissible morphism* $f : M \to N$ of modulus pairs is a morphism between the interiors $f^{o} : M^{o} \to N^{o}$ in $\mathbf{Sm}$ which satisfies *the properness condition*:

– Let Γ be the graph of the rational map $\bar{f} : \overline{M} \dashrightarrow \overline{N}$ which is induced by f^{o}. Then the natural morphism $\Gamma \to \overline{M}$ is proper.

and *the modulus condition*:

– Let Γ^{N} be the normalization of Γ. Then we have the inequality

$$M^{\infty}|_{\Gamma^{N}} \geq N^{\infty}|_{\Gamma^{N}}$$

of effective Cartier divisors on Γ^{N}, where $M^{\infty}|_{\Gamma^{N}}$ and $N^{\infty}|_{\Gamma^{N}}$ denote the pullbacks of M^{∞} and N^{∞} along the natural morphisms $\Gamma^{N} \to \overline{M}$ and $\Gamma^{N} \to \overline{N}$. Note that the pullbacks are defined since the rational map $\bar{f}$ restricts to a morphism f^{o}, and since M^{o} is dense in $\overline{M}$.

If $f : M \to N$ and $g : N \to L$ are admissible morphisms, then the composite $g^{o} \circ f^{o} : M^{o} \to L^{o}$ defines an admissible morphism $M \to N$; see [Kahn et al. 2019a]. If N is proper, then the properness condition above is always satisfied.

(4) We let $\underline{\mathbf{M}}\mathbf{Sm}$ denote the category whose objects are modulus pairs and whose morphisms are admissible morphisms. The full subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ consisting of proper modulus pairs is denoted by $\mathbf{MSm}$.

(5) A morphism $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Sm}$ is called *ambient* if $f^{o} : M^{o} \to N^{o}$ extends to a morphism $\overline{M} \to \overline{N}$ in $\mathbf{Sch}$. Such an extension is unique since $\overline{M}$ is reduced, M^{o} is dense in $\overline{M}$, and $\overline{N}$ is separated. We let $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ (resp. $\mathbf{MSm}^{\mathrm{fin}}$) denote the (nonfull) subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ (resp. $\mathbf{MSm}$) whose objects are modulus pairs (resp. proper modulus pairs) and whose morphisms are ambient morphisms.

(6) A morphism $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Sm}$ is called *minimal* if f is ambient and satisfies $M^{\infty} = \bar{f}^{*}N^{\infty}$.

(7) We let $\underline{\Sigma}_{\mathrm{fin}}$ denote the subcategory of $\underline{\mathbf{M}}\mathbf{Sm}$ whose objects are the same as $\underline{\mathbf{M}}\mathbf{Sm}$ and whose morphisms are those morphisms $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ such that f is minimal, $\bar{f} : \overline{M} \to \overline{N}$ is proper, and $f^{o} : M^{o} \to N^{o}$ is an isomorphism

in **Sm**. Then the canonical functor $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}} \to \underline{\mathbf{M}}\mathbf{Sm}$ induces an equivalence of categories $\underline{\Sigma}_{\mathrm{fin}}^{-1}\,\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}} \xrightarrow{\simeq} \underline{\mathbf{M}}\mathbf{Sm}$ [Kahn et al. 2019a, Proposition 1.9.2].

(8) Let **Sq** be the product category $[0]\times[0]$, where $[0]=\{0\to 1\}$. For any category $\mathcal{C}$, we define $\mathcal{C}^{\mathbf{Sq}}$ to be the category of functors from **Sq** to $\mathcal{C}$. An object T of $\mathcal{C}^{\mathbf{Sq}}$ is given by a commutative diagram

$$\begin{array}{ccc} T(00) & \longrightarrow & T(01) \\ \downarrow & & \downarrow \\ T(10) & \longrightarrow & T(11) \end{array}$$

in $\mathcal{C}$, and a morphism $T_1 \to T_2$ in $\mathcal{C}^{\mathbf{Sq}}$ is given by a set of morphisms $T_1(ij) \to T_2(ij)$, $i, j = 0, 1$, which are compatible with all the edges of the squares.

(9) A morphism $T_1 \to T_2$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ is called *ambient* if for any $i, j = 0, 1$, the morphisms $T_1(ij) \to T_2(ij)$ in $\underline{\mathbf{M}}\mathbf{Sm}$ are ambient. A square $T \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ is called *ambient* if it is contained in $(\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}})^{\mathbf{Sq}} \subset \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$.

The following lemma is often useful.

Lemma 2.1.2. *For any square $T \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$, there exists an ambient square T' which admits an ambient morphism $T' \to T$ which is an isomorphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$.*

Proof. This is just a consequence of a repeated use of the graph trick [Kahn et al. 2019a, Lemma 1.3.6]. Or, the reader can consult the calculus of fractions in [Kahn et al. 2019a, Proposition 1.9.2]. The details are left to the reader. □

2.2. ***Fiber products.*** We discuss fiber products in $\underline{\mathbf{M}}\mathbf{Sm}$ and **MSm**.

Lemma 2.2.1. *Let X be a scheme, and let D_1 and D_2 be effective Cartier divisors on X. Assume that the scheme-theoretic intersection* $\inf(D_1, D_2) := D_1 \times_X D_2$ *is also an effective Cartier divisor on X. Set $X^\infty := D_1 + D_2 - \inf(D_1, D_2)$.*

Then for any morphism $f : Y \to X$ in **Sch** *such that Y is normal and the image of any irreducible component of Y is not contained in $|X^\infty| = |D_1| \cup |D_2|$, we have*

$$f^*X^\infty = \sup(f^*D_1, f^*D_2),$$

where sup *is the supremum of Weil divisors on the normal scheme Y.*

Proof. Since $\inf(D_1, D_2) \times_X Y = \inf(f^*D_1, f^*D_2)$, we are reduced to the case $X = Y$. Moreover, an easy local computation shows that $D_1 - \inf(D_1, D_2)$ and $D_2 - \inf(D_1, D_2)$ do not intersect. The assertion immediately follows from this. See [Kahn et al. 2019a, Lemma 1.10.1, Definition 1.10.2, Remark 1.10.3] for more details. □

Definition 2.2.2. Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, and assume that the fiber product $P^o := M_1^o \times_{N^o} M_2^o$ exists in **Sm**. We define a modulus pair P as follows. Let $\overline{P}_0$ be the scheme-theoretic closure of P^{o} in $\overline{M} \times_{\overline{N}} \overline{M}_2$, and let $\bar{p}_{0,i} : \overline{P}_0 \to \overline{M}_1 \times_{\overline{N}} \overline{M}_2 \xrightarrow{\mathrm{pr}_i} \overline{M}_i$ be the composite of the closed immersion followed by the i-th projection for $i = 1, 2$. Let

$$\overline{P} := \mathbf{Bl}_{(\bar{p}_{0,1}^* M_1^\infty) \times_{\overline{P}_0} (\bar{p}_{0,2}^* M_2^\infty)}(\overline{P}_0) \xrightarrow{\pi_P} \overline{P}_0$$

be the blow-up of $\overline{P}_0$ along the closed subscheme $(\bar{p}_{0,1}^* M_1^\infty) \times_{\overline{P}_0} (\bar{p}_{0,2}^* M_2^\infty)$. Set

$$P^\infty := \pi_P^* \bar{p}_{0,1}^* M_1^\infty + \pi_P^* \bar{p}_{0,2}^* M_2^\infty - E,$$

where $E := \pi_P^{-1}((\bar{p}_{0,1}^* M_1^\infty) \times_{\overline{P}_0} (\bar{p}_{0,2}^* M_2^\infty))$ denotes the exceptional divisor. Then we have $\overline{P} - |P^\infty| = P^{\mathrm{o}} \in \mathbf{Sm}$ by construction, and we obtain a modulus pair $P = (\overline{P}, P^\infty)$.

We call P the *canonical model of fiber product of* f_1 *and* f_2, and we often write

$$M_1 \times_N^{\mathrm{c}} M_2 := P.$$

By construction, we have a commutative diagram

$$\begin{array}{ccc} M_1 \times_N^{\mathrm{c}} M_2 & \xrightarrow{p_2} & M_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow[f_1]{} & N \end{array}$$

in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. Moreover, we have $(M_1 \times_N^{\mathrm{c}} M_2)^{\mathrm{o}} \cong M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$.

Theorem 2.2.3. *Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. Assume that the fiber product $M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ exists in* $\mathbf{Sm}$. *Then the canonical model of fiber product $M_1 \times_N^{\mathrm{c}} M_2$ represents the fiber product $M_1 \times_N M_2$ in* $\underline{\mathbf{M}}\mathbf{Sm}$. *Moreover, if M_1, M_2, N are proper, then $M_1 \times_N^{\mathrm{c}} M_2$ (hence $M_1 \times_N M_2$) is proper.*

Remark 2.2.4. $M_1 \times_N^{\mathrm{c}} M_2$ does not necessarily represent a fiber product in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, and it is not functorial in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. However, under some minimality conditions, they behave nicely in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$.

Proof. We prove that $P := M_1 \times_N^{\mathrm{c}} M_2$ satisfies the universal property of fiber product in $\underline{\mathbf{M}}\mathbf{Sm}$. Let $g_1 : L \to M_1$ and $g_2 : L \to M_2$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}$ which coincide at N. Since $\underline{\mathbf{M}}\mathbf{Sm} \cong \underline{\Sigma}_{\mathrm{fin}}^{-1} \underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, we can find morphisms $L_1 \to L$ in $\underline{\Sigma}_{\mathrm{fin}}$ such that the composite morphisms $L_1 \to L \to M_i$ are ambient for $i = 1, 2$, and such that $\overline{L}_1$ is normal. Since $L_1 \to L$ is an isomorphism in $\underline{\mathbf{M}}\mathbf{Sm}$, we replace L with L_1 and assume that $\overline{L}$ is normal, and that g_1 and g_2 are ambient. Let $p_1 : P \to M_1$ and $p_2 : P \to M_2$ be the ambient morphisms as in Definition 2.2.2.

There exists a unique morphism $g^{\mathrm{o}} : L^{\mathrm{o}} \to P^{\mathrm{o}} = M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ in $\mathbf{Sm}$ which is compatible with g_1^{o}, g_2^{o}, p_1^{o}, and p_2^{o}. It suffices to prove that g^{o} defines a morphism

$L \to P$ in $\underline{\mathbf{M}}\mathbf{Sm}$. Let $\Gamma \subset \bar{L} \times \bar{P}$ be the closure of the graph of g^{o}, and let Γ^N be the normalization of Γ. Let $s : \Gamma^N \to \bar{L}$ and $t : \Gamma^N \to \bar{P}$ be the natural projections.

Then, for $i = 1, 2$, we obtain a commutative diagram

$$\begin{array}{ccc} \Gamma^N & \xrightarrow{t} & \bar{P} \\ {\scriptstyle s}\downarrow & \overset{g^{\mathrm{o}}}{\nearrow} & \downarrow{\scriptstyle \bar{p}_i} \\ \bar{L} & \xrightarrow[\bar{g}_i]{} & \overline{M}_i \end{array}$$

where the commutativity follows from the fact that $\bar{p}_i t$ and $\bar{g}_i s$ coincide on the dense open subset $s^{-1}(L^{\mathrm{o}}) \subset \Gamma^N$.

By the construction of P and by Lemma 2.2.1, we have

$$t^* P^\infty = \sup(t^* \bar{p}_1^* M_1^\infty, t^* \bar{p}_2^* M_2^\infty) = \sup(s^* \bar{g}_1^* M_1^\infty, s^* \bar{g}_1^* M_2^\infty),$$

where the second equality follows from the commutativity of the above diagram. Since g_1 and g_2 are ambient and $\bar{L}$ is normal, we have $\bar{g}_i^* M_i^\infty \le L^\infty$. Therefore, we obtain

$$t^* P^\infty \le s^* L^\infty,$$

which shows that g^{o} defines a morphism $g : L \to P$. This proves the first assertion. The last assertion is obvious by construction. □

Corollary 2.2.5. *Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}$. Assume that the fiber product $M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ exists in* **Sm**. *Then there exists a fiber product $M_1 \times_N M_2$ in $\underline{\mathbf{M}}\mathbf{Sm}$. Moreover, if M_1, M_2, and N are proper, then $M_1 \times_N M_2$ is proper.*

Proof. By [Kahn et al. 2019a, Lemma 1.3.6], for each $i = 1, 2$, there exists a morphism $M_i' \to M_i$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ which is invertible in $\underline{\mathbf{M}}\mathbf{Sm}$ and such that the composite $M_i' \to M_i \to N$ is ambient. Theorem 2.2.3 shows that the fiber product $M_1' \times_N M_2'$ exists in $\underline{\mathbf{M}}\mathbf{Sm}$. This also represents a fiber product $M_1 \times_N M_2$, proving the first assertion. The second assertion follows from the construction of the canonical model of fiber product. □

Remark 2.2.6. The inclusion functor $\tau_s : \mathbf{MSm} \to \underline{\mathbf{M}}\mathbf{Sm}$ preserves fiber products by construction.

Given some minimality assumptions, we can say more about the canonical model of fiber product. We do not need this in this paper, but it will be used in the other papers, including [Kahn and Miyazaki 2019].

Proposition 2.2.7. (1) *Let $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, and assume that f_1 is minimal, $M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ is smooth over k and $\overline{M}_1 \times_{\overline{N}} M_2^\infty$ is an effective Cartier divisor on $\overline{M}_1 \times_{\overline{N}} \overline{M}_2$. Then we have*

$$M_1 \times_N^{\mathrm{c}} M_2 = (\overline{M}_1 \times_{\overline{N}} \overline{M}_2, \overline{M}_1 \times_{\overline{N}} M_2^\infty).$$

(2) *Consider the commutative diagram*

$$\begin{array}{ccccc} U_1 & \longrightarrow & V & \longleftarrow & U_2 \\ {\scriptstyle j_1}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j_2} \\ M_1 & \longrightarrow & N & \longleftarrow & M_2 \end{array}$$

in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$, *such that* j_1 *and* j_2 *are minimal, and such that* $M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ *and* $U_1^{\mathrm{o}} \times_{V^{\mathrm{o}}} U_2^{\mathrm{o}}$ *are smooth over k. Then the morphism*

$$j_1 \times j_2 : U_1 \times_V^{\mathrm{c}} U_2 \to M_1 \times_N^{\mathrm{c}} M_2$$

in $\underline{\mathbf{M}}\mathbf{Sm}$, *induced by the universal property of fiber product, belongs to* $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$ *and is minimal.*

(3) *In the situation of* (2), *if* $\bar{j}$, $\bar{j}_1$, $\bar{j}_2$ *are open immersions, if* $U_1 \to V$ *is minimal, and if* $\overline{U}_1 \times_{\overline{V}} \overline{U}_2$ *is normal, then*

$$\overline{j_1 \times j_2} : \overline{U}_1 \times_{\overline{V}} \overline{U}_2 = \overline{U_1 \times_V^{\mathrm{c}} U_2} \to \overline{M_1 \times_N^{\mathrm{c}} M_2}$$

is an open immersion, where the equality follows by (1).

Proof. (1): This follows from the construction of canonical model of fiber product; see also [Kahn et al. 2019a, Corollary 1.10.7].

(2): Let $\overline{P}$ be the closure of $M_1^{\mathrm{o}} \times_{N^{\mathrm{o}}} M_2^{\mathrm{o}}$ in $\overline{M}_1 \times_{\overline{N}} \overline{M}_2$, and $\overline{Q}$ the closure of $U_1^{\mathrm{o}} \times_{V^{\mathrm{o}}} U_2^{\mathrm{o}}$ in $\overline{U}_1 \times_{\overline{V}} \overline{U}_2$. Then the morphisms $\bar{j}_1$ and $\bar{j}_2$ induce a morphism

$$\bar{J} : \overline{Q} \to \overline{P}.$$

Then we obtain the commutative diagrams

$$\begin{array}{ccc} \overline{Q} & \xrightarrow{\bar{J}} & \overline{P} \\ {\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle p_i} \\ \overline{U}_i & \xrightarrow{\bar{j}_i} & \overline{M}_i \end{array}$$

in **Sch** for $i = 1, 2$, where p_i and q_i are the natural i-th projections. Set $F := p_1^* M_1^\infty \times_{\overline{P}} p_2^* M_2^\infty \subset \overline{P}$ and $G := q_1^* U_1^\infty \times_{\overline{Q}} p_2^* U_2^\infty \subset \overline{Q}$. Then the commutativity of the diagrams shows

$$\bar{J}^{-1} F := F \times_{\overline{P}} \overline{Q} = (q_i^* \bar{j}_1^* M_1^\infty) \times_{\overline{Q}} (q_i^* \bar{j}_2^* M_2^\infty) = q_i^* U_1^\infty \times_{\overline{Q}} q_i^* U_2^\infty = G,$$

where the equality in the second line follows from the minimality of j_1 and j_2. Let $\pi_P : \mathbf{Bl}_F(\overline{P}) \to \overline{P}$ and $\pi_Q : \mathbf{Bl}_G(\overline{Q}) \to \overline{Q}$ be the blow-ups. Then, by the universal property of blow-up, $\bar{J}$ lifts to a morphism

$$\bar{J}_1 : \overline{U_1 \times_V^{\mathrm{c}} U_2} = \mathbf{Bl}_G(\overline{Q}) \to \mathbf{Bl}_F(\overline{P}) = \overline{M_1 \times_N^{\mathrm{c}} M_2},$$

which makes the diagram

$$\begin{array}{ccc} \mathbf{Bl}_G(\overline{Q}) & \xrightarrow{\bar{J}_1} & \mathbf{Bl}_F(\overline{P}) \\ {\scriptstyle \pi_Q}\downarrow & & \downarrow{\scriptstyle \pi_P} \\ \overline{Q} & \xrightarrow{\bar{J}} & \overline{P} \end{array}$$

commute. Moreover, letting $F' := \pi_P^{-1}(F)$, $G' := \pi_Q^{-1}(G)$ be the exceptional divisors, the commutativity of the two diagrams as above shows

$$\begin{aligned} \bar{J}_1^*(M_1 \times_N^{\mathrm{c}} M_2)^\infty &= \bar{J}_1^*(\pi_P^* p_1^* M_1^\infty + \pi_P^* p_1^* M_1^\infty - F') \\ &= \pi_Q^* \bar{J}^* p_1^* M_1^\infty + p_Q^* \bar{J}^* \pi_2^* M_2^\infty - G' \\ &= \pi_Q^* q_1^* \bar{j}_1^* M_1^\infty + \pi_Q^* q_2^* \bar{j}_2^* M_2^\infty - G' \\ &= \pi_Q^* q_1^* U_1^\infty + \pi_Q^* q_2^* U_2^\infty - G' \\ &= (U_1 \times_V^{\mathrm{c}} U_2)^\infty, \end{aligned}$$

where the equality in the fourth line follows from the minimality of j_1 and j_2. Therefore, the morphism $\bar{J}_1$ defines a minimal morphism $U_1 \times_V^{\mathrm{c}} U_2 \to M_1 \times_N^{\mathrm{c}} M_2$, as desired.

(3): We take the notation as above. Then $\overline{U}_1 \times_{\overline{V}} \overline{U}_2$ is an open subset of $\overline{P}$. Since $\bar{J}^* F = G$, the minimality of $U_1 \to V$ shows $F \cap \overline{U}_1 \times_{\overline{V}} \overline{U}_2 = \overline{U}_1 \times_{\overline{V}} U_2^\infty$, where the right-hand side is an effective Cartier divisor on $\overline{U}_1 \times_{\overline{V}} \overline{U}_2$. Therefore, the blow-up π_P is an isomorphism over $\overline{U}_1 \times_{\overline{V}} \overline{U}_2$, and the open immersion $\overline{U}_1 \times_{\overline{V}} \overline{U}_2 \to \overline{P}$ uniquely lifts to an open immersion $\overline{U}_1 \times_{\overline{V}} \overline{U}_2 \to \mathbf{Bl}_F(\overline{P})$. □

2.3. ***A remark on elementary correspondences.*** In this subsection, we observe a relationship between cartesian squares and elementary correspondences. First we provide some definitions.

Definition 2.3.1. For any $M_1, M_2 \in \underline{\mathbf{M}}\mathbf{Sm}$, we define $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}$ to be the set of elementary finite correspondences $V : M_1^{\mathrm{o}} \to M_2^{\mathrm{o}}$ which satisfy the following *admissibility conditions*: let $\overline{V}$ be the closure of V in $\overline{M}_1 \times \overline{M}_2$, and let $\overline{V}^N \to \overline{V}$ be the normalization of $\overline{V}$. Let $\mathrm{pr}_i : \overline{V}^N \to \overline{M}_i$ be the i-th projections.

(1) pr_1 is proper.

(2) $\mathrm{pr}_1^* M_1^\infty \geq \mathrm{pr}_2^* M_2^\infty$.

Definition 2.3.2 [Kahn et al. 2019a, Definitions 1.1.1, 1.3.3]. A category $\underline{\mathbf{M}}\mathbf{Cor}$ is defined as follows: the objects are the same as $\underline{\mathbf{M}}\mathbf{Sm}$, and for $M, N \in \underline{\mathbf{M}}\mathbf{Cor}$, the set of morphisms is defined as the free abelian group generated on $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, N)$. Note that $\underline{\mathbf{M}}\mathbf{Cor}(M, N) \subset \mathbf{Cor}(M^{\mathrm{o}}, N^{\mathrm{o}})$ by definition. The composition is given by the composition of finite correspondences. Define $\mathbf{MCor}$ as the full subcategory of $\underline{\mathbf{M}}\mathbf{Cor}$ whose objects are proper modulus pairs.

Proposition 2.3.3. *For any modulus pair M, for any $f : N \to L$ in $\underline{\mathbf{M}}\mathbf{Sm}$, and for any $V \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, N)$, the image*

$$f_+(V) := (\mathrm{Id}_{M^{\mathrm{o}}} \times f^{\mathrm{o}})(V) \subset M^{\mathrm{o}} \times L^{\mathrm{o}}$$

is an irreducible closed subset, and we have $f_+(V) \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, L)$.

Thus, any modulus pair M is associated a covariant functor

$$\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, -) : \underline{\mathbf{M}}\mathbf{Sm} \to \mathbf{Set}.$$

Proof. By [Kahn et al. 2019a, Proposition 1.2.3], the composition of finite correspondences $W := \Gamma_{f^{\mathrm{o}}} \circ V$ belongs to $\underline{\mathbf{M}}\mathbf{Cor}(M, L)$, where $\Gamma_{f^{\mathrm{o}}}$ denotes the graph of $f^{\mathrm{o}} : M^{\mathrm{o}} \to N^{\mathrm{o}}$. By the definition of composition, we can verify that $|W| = f_+(V)$. This implies that $f_+(V)$ is a component of W. Therefore, we have $W \in \underline{\mathbf{M}}\mathbf{Cor}(M, L)$, as desired. □

Proposition 2.3.4. *Let T be a pull-back square in $\underline{\mathbf{M}}\mathbf{Sm}$ of the form*

$$\begin{array}{ccc} T(00) & \xrightarrow{v_T} & T(01) \\ {\scriptstyle q_T}\downarrow & & \downarrow{\scriptstyle p_T} \\ T(10) & \xrightarrow{u_T} & T(11) \end{array} \tag{2.3.5}$$

and let M be a modulus pair. Consider the associated commutative diagram of sets

$$\begin{array}{ccc} \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(00)) & \xrightarrow{v_{T+}} & \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01)) \\ {\scriptstyle q_{T+}}\downarrow & & \downarrow{\scriptstyle p_{T+}} \\ \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(10)) & \xrightarrow{u_{T+}} & \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(11)) \end{array}$$

and set

$$\Pi := \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(10)) \times_{\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(11))} \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01)).$$

Then the induced map $\rho : \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(00)) \to \Pi$ is surjective. Moreover, it is bijective if v_T^{o} is an immersion.

Remark 2.3.6. We can formulate another statement by replacing $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}$ with $\underline{\mathbf{M}}\mathbf{Cor}$ and $(-)_+$ with $(-)_*$, but it is false. Indeed, if α_1 and α_2 are distinct elementary correspondences which have the same image β under p_{T*}, then the image of the (nonelementary) finite correspondence $\alpha := \alpha_1 - \alpha_2$ is zero, which is trivially contained in the image of u_{T*}. But there is no reason why α is contained in the image of v_{T*}.

Proof. The latter statement is clear, since the composite $\mathrm{pr}_2 \circ \rho$ is equal to v_{T+}, which is injective if v_T^{o} is an immersion.

We prove the surjectivity of ρ. Consider any $\alpha_1 \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(10))$ and $\alpha_2 \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01))$, and assume $\beta := u_{T+}(\alpha_1) = p_{T+}(\alpha_2)$. Let ξ_i be the generic point of α_i for $i = 1, 2$.

We need to prove that there exists an element $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(00))$ which maps to α_1 and α_2.

Let

$$\zeta \in (M^{\mathrm{o}} \times T(10)^{\mathrm{o}}) \times_{M^{\mathrm{o}} \times T^{\mathrm{o}}(11)} (M^{\mathrm{o}} \times T(01)^{\mathrm{o}}) \cong M^{\mathrm{o}} \times T(10)^{\mathrm{o}} \times_{T^{\mathrm{o}}(11)} T(01)^{\mathrm{o}} \cong M^{\mathrm{o}} \times T(00)^{\mathrm{o}}$$

be a point which lies over ξ_1 and ξ_2. Let $\gamma := \overline{\{\zeta\}}$ be the closure of ζ in $M^{\mathrm{o}} \times T(00)^{\mathrm{o}}$, endowed with the reduced scheme structure.

Claim 2.3.7. γ is an elementary correspondence from M^{o} to $T(00)^{\mathrm{o}}$.

Proof. We have to prove that γ is finite and surjective over a component of M^{o}. Since $\zeta = (\xi_1, \xi_2) \in \alpha_1 \times_{M^{\mathrm{o}}} \alpha_2$, the scheme γ is naturally a closed subscheme of $\alpha_1 \times_{M^{\mathrm{o}}} \alpha_2$. Moreover, since ζ maps to ξ_i via the projection $\mathrm{pr}_i : \alpha_1 \times_{M^{\mathrm{o}}} \alpha_2 \to \alpha_i$ for each $i = 1, 2$, we obtain dominant maps $\gamma \to \alpha_i$. These maps are finite (hence surjective) since each α_i is finite over M^{o}. Since the natural map $\gamma \to M^{\mathrm{o}}$ factors as $\gamma \to \alpha_1 \to M^{\mathrm{o}}$, and since α_1 is finite and surjective over a component, we obtain the claim. □

Claim 2.3.8. $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(00))$.

Proof. We make a preliminary reduction as follows: since the assertion depends only on the isomorphism class of T in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$, we may assume that T is ambient by Lemma 2.1.2. Moreover, since T is a pull-back diagram, we have $T(00) \cong T(10) \times^{c}_{T(11)} T(01)$, where the right-hand side is the canonical model of fiber product in Definition 2.2.2. Therefore, by replacing $T(00)$ with (the normalization of) $T(10) \times^{\mathrm{c}}_{T(11)} T(01)$ (this preserves the condition that T is ambient by the construction of canonical model), we may assume that $\bar{q}_T^* T(10)^\infty$ and $\bar{v}_T^* T^*(01)$ have a universal supremum in the sense of [Kahn et al. 2019a, Definition 1.10.2] and that $T(00)^\infty = \sup(\bar{q}_T^* T(10)^\infty, \bar{v}_T^* T(01)^\infty)$.

Let $\overline{\gamma}$ be the closure of γ in $\overline{M} \times \overline{T}(00)$. First we check that $\overline{\gamma}$ is proper over $\overline{M}$. Note that the natural map $\overline{\gamma} \to \overline{M}$ factors as $\overline{\gamma} \to \overline{\alpha}_1 \times_{\overline{M}} \overline{\alpha}_2 \to \overline{M}$. The first map is proper since the natural map $\overline{T}(00) \to \overline{T}(10) \times_{\overline{T}(11)} \overline{T}(01)$ is proper by construction of the canonical model of fiber product, and the latter map is proper since the $\overline{\alpha}_i$ are proper over $\overline{M}$ by assumption. This shows that $\overline{\gamma} \to \overline{M}$ is proper, as desired.

Next we check the modulus condition. Let $\overline{\gamma}^N$ be the normalization of $\overline{\gamma}$. Similarly, let $\overline{\alpha}_1$ be the closure of α_1 in $\overline{M} \times \overline{T}(10)$, $\overline{\alpha}_2$ the closure of α_2 in $\overline{M} \times \overline{T}(01)$, and $\overline{\alpha}_i^N$ the normalization of $\overline{\alpha}_i$. By assumption, we have $\alpha_1 \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(10))$

and $\alpha_2 \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01))$, which means

$$M^\infty|_{\overline{\alpha}_1^N} \geq T(10)^\infty|_{\overline{\alpha}_1^N} \quad \text{and} \quad M^\infty|_{\overline{\alpha}_2^N} \geq T(01)^\infty|_{\overline{\alpha}_2^N}.$$

Since $\gamma \to \alpha_i$ are dominant for $i = 1, 2$, we obtain morphisms $\overline{\gamma}^N \to \overline{\alpha}_i^N$ by the universal property of normalization. Therefore, the above inequalities imply

$$M^\infty|_{\overline{\gamma}^N} \geq \bar{q}_T^* T(10)^\infty|_{\overline{\gamma}^N} \quad \text{and} \quad M^\infty|_{\overline{\gamma}^N} \geq \bar{v}_T^* T(01)^\infty|_{\overline{\gamma}^N}.$$

Thus, since $\bar{q}_T^* T(10)^\infty$ and $\bar{v}_T^* T(01)^\infty$ have a universal supremum and $T(00)^\infty = \sup(\bar{q}_T^* T(10)^\infty, \bar{v}_T^* T(01)^\infty)$ by assumption, we obtain

$$\begin{aligned} M^\infty|_{\overline{\gamma}^N} &\geq \sup(\bar{q}_T^* T(10)^\infty|_{\overline{\gamma}^N}, \bar{v}_T^* T(01)^\infty|_{\overline{\gamma}^N}) \\ &= \sup(\bar{q}_T^* T(10)^\infty, \bar{v}_T^* T(01)^\infty)|_{\overline{\gamma}^N} \\ &= T(00)^\infty|_{\overline{\gamma}^N} \end{aligned}$$

by [Kahn et al. 2019a, Remark 1.10.3(3)]. This finishes the proof of the claim. □

By construction, we have $\alpha_1 = q_{T+}(\gamma)$ and $\alpha_2 = v_{T+}(\gamma)$. This finishes the proof of Proposition 2.3.4. □

3. Off-diagonal functor

We introduce the "off-diagonal" functor, which is a key notion used in the definition of the cd-structure on **MSm**.

Definition 3.1.1. Define $\underline{\mathbf{M}}\mathbf{Et}$ as a category such that

(1) objects are those morphisms $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Sm}$ such that $f^\mathrm{o} : M^\mathrm{o} \to N^\mathrm{o}$ is étale, and

(2) morphisms of $f : M_1 \to N_1$ and $g : M_2 \to N_2$ are those pairs of morphisms $(s : M_1 \to M_2, t : N_1 \to N_2)$ which are compatible with f, g such that s^o and t^o are *open immersions*.

Define **MEt** as the full subcategory of $\underline{\mathbf{M}}\mathbf{Et}$ consisting of those $f : M \to N$ such that $M, N \in \mathbf{MSm}$.

Definition 3.1.2. For modulus pairs M and N, we define the *disjoint union of M and N* by

$$M \sqcup N := (\overline{M} \sqcup \overline{N}, M^\infty \sqcup N^\infty).$$

We have $(M \sqcup N)^\mathrm{o} = M^\mathrm{o} \sqcup N^\mathrm{o}$, and $M \sqcup N$ represents a coproduct of M and N in the category $\underline{\mathbf{M}}\mathbf{Sm}$.

Theorem 3.1.3. *There is a functor*

$$\mathrm{OD} : \underline{\mathbf{M}}\mathbf{Et} \to \underline{\mathbf{M}}\mathbf{Sm}$$

such that for any $f : M \to N$, *there exists a functorial decomposition*

$$M \times_N M \cong M \sqcup \mathrm{OD}(f).$$

Moreover, we have $\mathrm{OD}(f)^{\mathrm{o}} = M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}} \setminus \Delta(M^{\mathrm{o}})$, *where* $\Delta : M^{\mathrm{o}} \to M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}}$ *is the diagonal morphism. In particular, if* f^{o} *is an open immersion, then* $\mathrm{OD}(f)^{\mathrm{o}} = \varnothing$, *and hence* $\mathrm{OD}(f) = \varnothing$. *Moreover, the functor* OD *restricts to a functor*

$$\mathrm{OD} : \mathbf{MEt} \to \mathbf{MSm}.$$

We call these functors the off-diagonal functors.

Proof. First, we prove that for any $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Et}$, there exists a morphism $i : X \to M \times_N M$ such that the induced morphism

$$M \sqcup X \xrightarrow{\Delta \sqcup i} M \times_N M$$

is an isomorphism in $\underline{\mathbf{M}}\mathbf{Sm}$. Take any object $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Et}$. Since f^{o} is étale and separated by the assumption, the diagonal morphism $\Delta : M^{\mathrm{o}} \to M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}}$ is an open and closed immersion. Therefore, we obtain a decomposition into two connected components:

$$M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}} = \Delta(M^{\mathrm{o}}) \sqcup (M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}} - \Delta(M^{\mathrm{o}})).$$

Let P denote the canonical model of fiber product $M \times^{\mathrm{c}}_N M$ as in Definition 2.2.2. Note that $P^{\mathrm{o}} = M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}}$.

Define a closed immersion $\bar{i}_\Delta : \overline{\Delta}(f) \to \overline{P}$ as the scheme-theoretic closure of the open immersion $\Delta(M^{\mathrm{o}}) \to P^{\mathrm{o}} \to \overline{P}$. Set

$$\Delta(f)^\infty := \bar{i}_\Delta^* P^\infty \quad \text{and} \quad \Delta(f) := (\overline{\Delta}(f), \Delta(f)^\infty).$$

Then $\bar{i}_\Delta$ induces a minimal morphism $i_\Delta : \Delta(f) \to P$, and we have $\Delta(f)^{\mathrm{o}} = \Delta(M^{\mathrm{o}})$.

Similarly, define a closed immersion $\bar{i}_{\mathrm{OD}} : \overline{\mathrm{OD}(f)} \to \overline{P}$ as the scheme-theoretic closure of the open immersion $M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}} - \Delta(M^{\mathrm{o}}) \to P^{\mathrm{o}} \to \overline{P}$. Set

$$\mathrm{OD}(f)^\infty := \bar{i}_{\mathrm{OD}}^* P^\infty \quad \text{and} \quad \mathrm{OD}(f) := (\overline{\mathrm{OD}(f)}, \mathrm{OD}(f)^\infty).$$

Then $\bar{i}_{\mathrm{OD}}$ induces a minimal morphism $i_{\mathrm{OD}} : \mathrm{OD}(f) \to P$. Moreover, we have $\mathrm{OD}(f)^{\mathrm{o}} = M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}} - \Delta(M^{\mathrm{o}})$.

The morphisms i_Δ and i_{OD} induce a minimal morphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$:

$$i_\Delta \sqcup i_{\mathrm{OD}} : \Delta(f) \sqcup \mathrm{OD}(f) \to P.$$

By (7) in Definition 2.1.1, this morphism is an isomorphism in $\underline{\mathbf{M}}\mathbf{Sm}$ (not in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$) since $(i_\Delta \sqcup i_{\mathrm{OD}})^{\mathrm{o}} = i_\Delta^{\mathrm{o}} \sqcup i_{\mathrm{OD}}^{\mathrm{o}} : \Delta(f)^{\mathrm{o}} \sqcup \mathrm{OD}(f)^{\mathrm{o}} \to P^{\mathrm{o}} \cong M^{\mathrm{o}} \times_{N^{\mathrm{o}}} M^{\mathrm{o}}$ is an isomorphism in $\mathbf{Sm}$, and since $\bar{i}_\Delta \sqcup \bar{i}_{\mathrm{OD}} : \overline{\Delta(f)} \sqcup \overline{\mathrm{OD}(f)} \to \overline{P}$ is proper by construction.

We claim $\Delta(f) \cong M$. Let $\Delta : M \to P$ ($\cong M \times_N M$) be the diagonal morphism. Then the composite $M \xrightarrow{\Delta} P \cong \Delta(f) \sqcup \mathrm{OD}(f)$ factors through $\Delta(f)$. The inverse morphism is given by $\Delta(f) \to P \xrightarrow{\mathrm{pr}_1} M$, where pr_1 denotes the first projection $P \cong M \times_N M \to M$.

Thus, for any $f : M \to N$ in $\underline{\mathbf{M}}\mathbf{Et}$, we have obtained a decomposition

$$M \times_N M \cong M \sqcup \mathrm{OD}(f).$$

Next we check the functoriality of $\mathrm{OD}(f)$. Let $(f_1 : M_1 \to N_1) \to (f_2 : M_2 \to N_2)$ be a morphism in $\underline{\mathbf{M}}\mathbf{Et}$, i.e., a commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{s} & M_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ N_1 & \xrightarrow{t} & N_2 \end{array}$$

where f_1, f_2, s, and t are morphisms in $\underline{\mathbf{M}}\mathbf{Sm}$ such that f_1^{o} and f_2^{o} are étale and s^{o} and t^{o} are open immersions.

We claim that there exists a unique morphism $\mathrm{OD}(f_1) \to \mathrm{OD}(f_2)$ such that the diagram

$$\begin{array}{ccc} M_1 \times_{N_1} M_1 & \longrightarrow & M_2 \times_{N_2} M_2 \\ {\scriptstyle \cong}\uparrow & & \uparrow{\scriptstyle \cong} \\ M_1 \sqcup \mathrm{OD}(f_1) & \longrightarrow & M_2 \sqcup \mathrm{OD}(f_2) \end{array}$$

commutes. The uniqueness is obvious by the commutativity of the above diagram. For the existence, we need to show that the composite

$$\mathrm{OD}(f_1) \to M_1 \times_{N_1} M_1 \to M_2 \times_{N_2} M_2 \cong M_2 \sqcup \mathrm{OD}(f_2)$$

factors through $\mathrm{OD}(f_2)$. To see this, it suffices to prove that the image of the morphism

$$M_1^{\mathrm{o}} \times_{N_1^{\mathrm{o}}} M_1^{\mathrm{o}} \setminus \Delta(M_1^{\mathrm{o}}) \to M_1^{\mathrm{o}} \times_{N_1^{\mathrm{o}}} M_1^{\mathrm{o}} \xrightarrow{s^{\mathrm{o}} \times s^{\mathrm{o}}} M_2^{\mathrm{o}} \times_{N_2^{\mathrm{o}}} M_2^{\mathrm{o}}$$

lands in $M_2^{\mathrm{o}} \times_{N_2^{\mathrm{o}}} M_2^{\mathrm{o}} \setminus \Delta(M_2^{\mathrm{o}})$, which easily follows from the injectivity of the open immersion s^{o}. This finishes the proof. □

The off-diagonal functor is compatible with base change.

Proposition 3.1.4. *Let $f : M \to N$ be an object of* $\underline{\mathbf{M}}\mathbf{Et}$, *and $N' \to N$ any morphism in* $\underline{\mathbf{M}}\mathbf{Sm}$. *Then the base change $g := f \times_N N'$ belongs to* $\underline{\mathbf{M}}\mathbf{Et}$, *and we have a natural isomorphism* $\mathrm{OD}(g) \cong \mathrm{OD}(f) \times_N N'$.

Proof. The first assertion holds since $g^{\mathrm{o}} = f^{\mathrm{o}} \times_{N^{\mathrm{o}}} N'^{\mathrm{o}}$ is étale as a base change of an étale morphism. We prove the second assertion. Note $(M \times_N M) \times_N N' \cong M' \times_{N'} M'$, where $M' := M \times_N N'$. Consider the following diagram in $\underline{\mathbf{M}}\mathbf{Sm}$:

$$\begin{array}{ccccc} (M \times_N M) \times_N N' & \longleftarrow & (M \sqcup \mathrm{OD}(f)) \times_N N' & \longleftarrow & M' \sqcup (\mathrm{OD}(f) \times_N N') \\ \downarrow & & & & \downarrow h \\ M' \times_{N'} M' & \longleftarrow & & & M' \sqcup \mathrm{OD}(g) \end{array}$$

where all the arrows, except for h, are natural isomorphisms in $\underline{\mathbf{M}}\mathbf{Sm}$, and h is defined to be the composite. By diagram chase, h restricts to the identity map on M' and an isomorphism $\mathrm{OD}(f) \times_N N' \to \mathrm{OD}(g)$. □

4. The cd-structure

In this section, we introduce a cd-structure on **MSm**, and prove its fundamental properties.

4.1. $\underline{\mathbf{M}}\mathbf{V}$-*squares.* First, let us recall from [Kahn et al. 2019a] the cd-structure on $\underline{\mathbf{M}}\mathbf{Sm}$.

Definition 4.1.1. (1) An $\underline{\mathrm{M}}\mathrm{V}^{\mathrm{fin}}$-*square* is a square $S \in (\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}})^{\mathbf{Sq}}$ such that the morphisms in S are minimal, and such that the resulting square

$$\begin{array}{ccc} \overline{S}(00) & \longrightarrow & \overline{S}(01) \\ \downarrow & & \downarrow \\ \overline{S}(10) & \longrightarrow & \overline{S}(11) \end{array}$$

is an elementary Nisnevich square (on **Sch**).

(2) An $\underline{\mathrm{M}}\mathrm{V}$-*square* is a square $S \in \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ which is isomorphic to the image of an $\underline{\mathrm{M}}\mathrm{V}^{\mathrm{fin}}$-square by the inclusion functor $(\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}})^{\mathbf{Sq}} \to \underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$.

Proposition 4.1.2 [Kahn et al. 2019a, Proposition 3.2.2]. *The* $\underline{\mathrm{M}}\mathrm{V}$-*squares form a complete and regular cd-structure* $P_{\underline{\mathrm{M}}\mathrm{V}}$ *on* $\underline{\mathbf{M}}\mathbf{Sm}$. □

Definition 4.1.3. The topology on $\underline{\mathbf{M}}\mathbf{Sm}$ associated with the cd-structure $P_{\underline{\mathrm{M}}\mathrm{V}}$ is called *the* $\underline{\mathrm{M}}\mathrm{V}$-*topology*.

In the following, we describe OD for $\underline{\mathrm{M}}\mathrm{V}^{\mathrm{fin}}$ and $\underline{\mathrm{M}}\mathrm{V}$-squares.

Lemma 4.1.4. *Let* $f : U \to M$ *be a minimal morphism such that* $\overline{f} : \overline{U} \to \overline{M}$ *is étale. Then we have*

$$\overline{\mathrm{OD}(f)} = \overline{U} \times_{\overline{M}} \overline{U} - \Delta(\overline{U}) \quad \textit{and} \quad \mathrm{OD}(f)^{\infty} = \pi^* M^{\infty} \cap \overline{\mathrm{OD}(f)},$$

where $\Delta : \overline{U} \to \overline{U} \times_{\overline{M}} \overline{U}$ *is the diagonal, and* $\pi : \overline{U} \times_{\overline{M}} \overline{U} \to \overline{M}$ *is the natural morphism.*

Proof. Since $U^{\mathrm{o}} \times_{M^{\mathrm{o}}} U^{\mathrm{o}} - \Delta(U^{\mathrm{o}})$ is dense in $\overline{U} \times_{\overline{M}} \overline{U} - \Delta(\overline{U})$ (as a complement of the divisor $U^{\infty} \times_{\overline{M}} \overline{U} \setminus \Delta(\overline{U})$), and since $U^{\infty} \times_{\overline{M}} \overline{U} = \overline{U} \times_{\overline{M}} U^{\infty} = \pi^* M^{\infty}$, the assertion follows from the construction of $\mathrm{OD}(f)$. □

Proposition 4.1.5. *Let S be an $\underline{\mathbf{MV}}^{\mathrm{fin}}$-square of the form*

$$\begin{array}{ccc} S(00) & \xrightarrow{v_S} & S(01) \\ {\scriptstyle q_S}\downarrow & & \downarrow{\scriptstyle p_S} \\ S(10) & \xrightarrow{u_S} & S(11) \end{array}$$

Then the morphism $\mathrm{OD}(q_S) \to \mathrm{OD}(p_S)$ *is an isomorphism in* $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$.

Proof. Let S be an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square. Then, since $\overline{S}$ is an elementary Nisnevich square, we have a natural isomorphism

$$\overline{S}(00) \times_{\overline{S}(10)} \overline{S}(00) - \Delta_0(\overline{S}(00)) \xrightarrow{\sim} \overline{S}(01) \times_{\overline{S}(11)} \overline{S}(01) - \Delta_1(\overline{S}(01)),$$

where $\Delta_i : \overline{S}(0i) \to \overline{S}(0i) \times_{\overline{S}(1i)} \overline{S}(0i)$ is the diagonal for each $i = 0, 1$. Then, in view of Lemma 4.1.4, the minimality of u_S, p_S, q_S shows that the isomorphism as above induces an isomorphism $\mathrm{OD}(q_S) \to \mathrm{OD}(p_S)$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. ☐

Corollary 4.1.6. *Let S be an $\underline{\mathrm{MV}}$-square. The natural morphism* $\mathrm{OD}(q_S) \to \mathrm{OD}(p_S)$ *is an isomorphism in* $\underline{\mathbf{M}}\mathbf{Sm}$.

Proof. By definition of $\underline{\mathrm{MV}}$-square, there exists an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square S' which is isomorphic to S. Then, noting that there are natural isomorphisms $\mathrm{OD}(q_S) \cong \mathrm{OD}(q_{S'})$ and $\mathrm{OD}(p_S) \cong \mathrm{OD}(p_{S'})$ in $\underline{\mathbf{M}}\mathbf{Sm}$, the assertion follows from Proposition 4.1.5. ☐

4.2. MV-*squares.*

Definition 4.2.1. Let T be an object of $\mathbf{MSm}^{\mathbf{Sq}}$ of the form (2.3.5). Then T is called an MV-*square* if the following conditions hold:

(1) T is a pull-back square in $\mathbf{MSm}$.

(2) There exist an $\underline{\mathrm{MV}}$-square S such that $S(11) \in \mathbf{MSm}$ and a morphism $S \to T$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ such that the induced morphism $S^{\mathrm{o}} \to T^{\mathrm{o}}$ is an isomorphism in $\mathbf{Sm}^{\mathbf{Sq}}$ and $S(11) \to T(11)$ is an isomorphism in $\mathbf{MSm}$. In particular, T^{o} is an elementary Nisnevich square.

(3) $\mathrm{OD}(q_T) \to \mathrm{OD}(p_T)$ is an isomorphism in $\mathbf{MSm}$.

We let P_{MV} be the cd-structure on $\mathbf{MSm}$ consisting of MV-squares. The topology on $\mathbf{MSm}$ associated with the cd-structure P_{MV} is called *the* MV-*topology* for short.

Remark 4.2.2. (1) For any $T \in \mathbf{MSm}^{\mathbf{Sq}}$ with T^{o} an elementary Nisnevich square, the induced morphism $\mathrm{OD}(p_T)^{\mathrm{o}} \to \mathrm{OD}(q_T)^{\mathrm{o}}$ between interiors is an isomorphism in $\mathbf{Sm}$. This follows easily from the definition of elementary Nisnevich squares.

(2) If p_T^o and q_T^o are open immersions, then $\mathrm{OD}(q_T) = \mathrm{OD}(p_T) = \varnothing$. In particular, we have $\mathrm{OD}(q_T) \cong \mathrm{OD}(p_T)$.

Proposition 4.2.3. *Let T be a square in $\mathbf{MSm}^{\mathbf{Sq}}$ which satisfies condition (1), (2), or (3) of Definition 4.2.1. Then, for any morphism $M \to T(11)$ in $\mathbf{MSm}$, the base change square $T_M := T \times_{T(11)} M$ also satisfies (1), (2), or (3), respectively.*

Proof. Since base change of a pull-back diagram is a pull-back diagram, condition (1) is preserved by base change. Proposition 3.1.4 shows that (3) is preserved by the base change.

Finally, we prove that (2) is preserved by base change. Let $S \to T$ be a morphism as in (2), and let $M \to T(11)$ be any morphism in $\mathbf{MSm}$. Then we obtain a morphism $S_M \to T_M$, where $S_M := S \times_{S(11)} M$ and $T_M := T \times_{T(11)} M$. Since $S(11) \cong T(11)$, we obtain $S_M(11) \cong T_M(11)$. Moreover, S_M is an $\underline{\mathrm{MV}}$-square as the base change of an $\underline{\mathrm{MV}}$-square (see [Kahn et al. 2019a, Theorem 4.1.2]), and we have $S_M^o \cong T_M^o$. Therefore, the morphism $S_M \to T_M$ satisfies the requirement in (2). This finishes the proof. □

4.3. *Completeness.*

Theorem 4.3.1. *The cd-structure P_{MV} is complete.*

Proof. By [Voevodsky 2010, Lemma 2.5], it suffices to prove the following:

(1) Any morphism with values in $\varnothing = (\varnothing, \varnothing)$ is an isomorphism.

(2) For any $T \in P_{\mathrm{MV}}$ and any $M \to T(11)$ in $\mathbf{MSm}$, the square $T_M := T \times_{T(11)} M$, which is obtained by base change, belongs to P_{MV}.

But (1) is obvious, and (2) is a direct consequence of Proposition 4.2.3. □

4.4. *Regularity.*

Theorem 4.4.1. *The cd-structure P_{MV} is regular.*

Proof. By [Voevodsky 2010, Lemma 2.11], it suffices to prove that for any $T \in P_{\mathrm{MV}}$, the following assertions hold:

(1) T is a pull-back square in $\mathbf{MSm}$.

(2) $u_T : T(10) \to T(11)$ is a monomorphism.

(3) The fiber products $T(01) \times_{T(11)} T(01)$ and $T(00) \times_{T(10)} T(00)$ exist in $\mathbf{MSm}$, and the derived square

$$\begin{array}{ccc} T(00) & \longrightarrow & T(01) \\ \downarrow{\scriptstyle \Delta_{q_T}} & & \downarrow{\scriptstyle \Delta_{p_T}} \\ T(00) \times_{T(10)} T(00) & \longrightarrow & T(01) \times_{T(11)} T(01) \end{array}$$

which we denote by $d(T)$, belongs to P_{MV}.

The definition of MV-squares gives (1), and (2) holds since $u_T^{\mathrm{o}}: T^{\mathrm{o}}(10) \to T^{\mathrm{o}}(11)$ is an open immersion. We prove (3) by checking the conditions in Definition 4.2.1 for $d(T)$.

Since $\Delta_{p_T}^{\mathrm{o}}$ and $\Delta_{q_T}^{\mathrm{o}}$ are open immersions, we have $\mathrm{OD}(\Delta_{q_T}) \cong \varnothing \cong \mathrm{OD}(\Delta_{p_T})$ by Theorem 3.1.3. Hence $d(T)$ satisfies (3) in Definition 4.2.1.

Note that $d(T)$ is isomorphic in $\mathbf{MSm}^{\mathbf{Sq}}$ to the diagram

$$\begin{array}{ccc} T(00) & \longrightarrow & T(01) \\ \downarrow & & \downarrow \\ T(00) \sqcup \mathrm{OD}(q_T) & \longrightarrow & T(01) \sqcup \mathrm{OD}(p_T) \end{array}$$

where the vertical maps are the canonical inclusions, and the horizontal maps are induced by v_T. It is easy to see that this diagram is a pull-back diagram, i.e., $d(T)$ satisfies (1) in Definition 4.2.1. Indeed, suppose that we are given a pair of morphisms $f: M \to T(01)$ and $g: M \to T(00) \sqcup \mathrm{OD}(q_T)$ which coincide at $T(01) \sqcup \mathrm{OD}(p_T)$. Then, one sees that $g^{\mathrm{o}}: M^{\mathrm{o}} \to T(00)^{\mathrm{o}} \sqcup \mathrm{OD}(q_T)^{\mathrm{o}}$ factors through $T(00)^{\mathrm{o}}$, which implies that g factors through $T(00)$.

We are reduced to checking Definition 4.2.1(2) for $d(T)$. Consider the following diagram in $\underline{\mathbf{M}}\mathbf{Sm}$:

$$\begin{array}{ccc} (T(00)^{\mathrm{o}}, \varnothing) & \longrightarrow & T(01) \\ \downarrow & & \downarrow \\ (T(00)^{\mathrm{o}}, \varnothing) \sqcup \mathrm{OD}(q_T) & \longrightarrow & T(01) \sqcup \mathrm{OD}(p_T) \end{array}$$

which we denote by $d(T)_0$, where the vertical maps are the canonical inclusions. Then $d(T)_0$ is an $\underline{\mathrm{MV}}$-square since $\mathrm{OD}(q_T) \cong \mathrm{OD}(p_T)$, and there exists a natural morphism $d(T)_0 \to d(T)$. It induces an isomorphism $d(T)_0^{\mathrm{o}} \cong d(T)^{\mathrm{o}}$, and we have $d(T)_0(11) \cong d(T)(11)$. Therefore, $d(T)$ satisfies (2) in Definition 4.2.1. This finishes the proof. □

Theorem 4.4.2. *Let F be a presheaf with values in* **Sets** *on* **MSm**. *Then F is a sheaf with respect to the* MV*-topology if and only if $F(\varnothing) = 0$ and for any $\underline{\mathrm{MV}}$-square $T \in P_{\mathrm{MV}}$, the square*

$$\begin{array}{ccc} F(T(11)) & \longrightarrow & F(T(10)) \\ \downarrow & & \downarrow \\ F(T(01)) & \longrightarrow & F(T(00)) \end{array}$$

is cartesian.

Proof. This follows from [Voevodsky 2010, Corollary 2.17], Theorem 4.3.1, and Theorem 4.4.1. □

4.5. *Subcanonicity.* In this subsection, we prove the following result. Recall that a Grothendieck topology is *subcanonical* if every representable presheaf is a sheaf.

Theorem 4.5.1. *The* $\underline{\mathrm{MV}}$*-topology and the* MV*-topology are subcanonical.*

We need the following elementary observation.

Lemma 4.5.2. *Let P be a complete and regular cd-structure on a category $\mathcal{C}$. Then the topology associated with P is subcanonical if and only if every square $T \in P$ is cocartesian in $\mathcal{C}$.*

Proof. Let $\mathcal{Y}$ denote the Yoneda embedding of $\mathcal{C}$ into the category of presheaves on $\mathcal{C}$. All squares $T \in P$ are cocartesian in $\mathcal{C}$ if and only if for any $T \in P$ and for any $X \in \mathcal{C}$, the square

$$\begin{array}{ccc} \mathcal{Y}(X)(T(11)) & \xrightarrow{u_T^*} & \mathcal{Y}(X)(T(10)) \\ {\scriptstyle p_T^*}\downarrow & & \downarrow{\scriptstyle q_T^*} \\ \mathcal{Y}(X)(T(01)) & \xrightarrow{v_T^*} & \mathcal{Y}(X)(T(00)) \end{array} \tag{4.5.3}$$

is cartesian in $\mathcal{C}$. The latter condition is equivalent to that for any $X \in \mathcal{C}$, the representable presheaf $\mathcal{Y}(X)$ is a sheaf for the topology associated with P by [Voevodsky 2010, Corollary 2.17]. This finishes the proof. □

We also need the following results:

Lemma 4.5.4 [Krishna and Park 2012, Lemma 2.2]. *Let $f : X \to Y$ be a surjective morphism of normal integral schemes, and let D, D' be two Cartier divisors on Y. If $f^*D' \leq f^*D$, then $D' \leq D$.* □

Proposition 4.5.5. (1) *Any* $\underline{\mathrm{MV}}$*-square is cocartesian in* $\underline{\mathbf{M}}\mathbf{Sm}$.

(2) *Any* MV*-square is cocartesian in* $\underline{\mathbf{M}}\mathbf{Sm}$*, and hence in* $\mathbf{MSm}$.

Proof. (1): Let S be an $\underline{\mathrm{MV}}$-square. We may assume that S is an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square since cocartesianness is stable under isomorphisms. Let $S(10) \to M$ and $S(01) \to M$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}$ which coincide after restricting to $S(00)$. Since S^{o} is an elementary Nisnevich square, it is cocartesian in $\mathbf{Sm}$. Therefore, the morphisms $S(10)^{\mathrm{o}} \to M^{\mathrm{o}}$ and $S(01)^{\mathrm{o}} \to M^{\mathrm{o}}$ induce a unique morphism $h^{\mathrm{o}} : S(11)^{\mathrm{o}} \to M^{\mathrm{o}}$. It suffices to check that h^{o} induces a morphism $S(11) \to M$ in $\underline{\mathbf{M}}\mathbf{Sm}$.

Let Γ be the graph of the rational map $\overline{S}(11) \dashrightarrow \overline{M}$, and let $\Gamma^N \to \Gamma$ be the normalization. For any $(ij) \in \mathbf{Sq}$, set

$$S_1(ij) := (\overline{S}(ij) \times_{\overline{S}(11)} \Gamma^N, S^{\infty}(ij) \times_{\overline{S}(11)} \Gamma^N).$$

The minimal morphisms $S_1(ij) \to S_1(kl)$ are induced by $S(ij) \to S(kl)$ for all $(ij) \to (kl)$ in $\mathbf{Sq}$, and they form an $\underline{\mathrm{MV}}^{\mathrm{fin}}$-square S_1. Moreover, $S_1(ij)$ are normal for all $(ij) \in Sq$, and the composites

$$\bar{h}_{ij} : \bar{S}_1(ij) \to \bar{S}(11) \dashrightarrow \overline{M}$$

are morphisms of schemes for all $(ij) \in \mathbf{Sq}$ by construction. Moreover, the morphisms $\bar{S}_1(ij) \to \bar{S}(ij)$ are proper (by the properness of Γ over $\bar{S}(11)$). Therefore, by the minimality of $S_1(ij) \to S(ij)$, the morphism $S_1 \to S$ is an isomorphism in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$.

Claim 4.5.6. $S_1^\infty(11) \geq \bar{h}_{11}^* M^\infty$.

Proof. The admissibility of $S(10) \to M$ and $S(01) \to M$ implies that of $S_1(10) \to M$ and $S_1(01) \to M$. Since $\bar{S}_1(10)$ and $\bar{S}_1(01)$ are normal, we have $S_1(ij)^\infty \geq \bar{h}_{ij}^* M^\infty$ for $(ij) = (10), (01)$. Since $\bar{S}_1(10) \sqcup \bar{S}_1(01) \to \bar{S}_1(11)$ is a surjection between normal schemes and since $S_1(10) \to S_1(11)$ and $S_1(01) \to S_1(11)$ are minimal, Lemma 4.5.4 implies

$$S_1(11)^\infty \geq \bar{h}_{11}^* M^\infty. \qquad \square$$

By Claim 4.5.6, we have a morphism $S_1(11) \to M$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathrm{fin}}$. The composite $S(11) \xleftarrow{\sim} S_1(11) \to M$ gives the desired morphism. The uniqueness of the morphism follows from the fact that the elementary Nisnevich square S^{o} is cocartesian in $\mathbf{Sm}$. This finishes the proof of (1).

(2): Let T be an MV-square. Then condition (2) of Definition 4.2.1 shows that there are an $\underline{\mathrm{M}}\mathrm{V}$-square S and a morphism $S \to T$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ such that $S(11) \cong T(11)$. Let $f : T(10) \to M$ and $g : T(01) \to M$ be morphisms in $\underline{\mathbf{M}}\mathbf{Sm}$ which coincide after restriction to $T(00)$. Then the composites

$$f_S : S(10) \to T(10) \to T(11) \quad \text{and} \quad g_S : S(01) \to T(01) \to T(11)$$

coincide after restriction to $S(00)$. Then f_S and g_S induce a unique morphism $h : T(11) \cong S(11) \to M$ since S is cocartesian in $\underline{\mathbf{M}}\mathbf{Sm}$ by (1). Since $S^{\mathrm{o}} \cong T^{\mathrm{o}}$, we have $h \circ u_T = f$ and $h \circ p_T = g$. This finishes the proof of Proposition 4.5.5. $\square$

Proof of Theorem 4.5.1. This follows from Lemma 4.5.2 and parts (1) and (2) of Proposition 4.5.5. $\square$

5. Mayer–Vietoris sequence

5.1. *Easy Mayer–Vietoris.*

Definition 5.1.1. For any sheaf F on a site $\mathcal{C}$, we denote by $\mathbb{Z}F$ the sheaf associated with the presheaf $\mathcal{C} \ni X \mapsto \mathbb{Z}(F(X))$, where for any set S, we denote by $\mathbb{Z}S$ the free abelian group generated on S.

For any $M \in \underline{\mathbf{M}}\mathbf{Sm}$ (or $\mathbf{MSm}$), we set $\mathbb{Z}(M) := \mathbb{Z}\mathcal{Y}(M)$, where $\mathcal{Y}(M)$ denotes the presheaf of sets represented by M.

Theorem 5.1.2. *Let T be an* MV*-square. Then the complex*

$$0 \to \mathbb{Z}(T(00)) \to \mathbb{Z}(T(10)) \oplus \mathbb{Z}(T(01)) \to \mathbb{Z}(T(11)) \to 0$$

of sheaves on **MSm** *is exact.*

Proof. This follows from [Voevodsky 2010, Lemma 2.18], Theorem 4.4.1, and Theorem 4.5.1. □

5.2. *Mayer–Vietoris with transfers.*

Theorem 5.2.1. *Let $T \in \mathbf{MSm}^{\mathbf{Sq}}$. Assume that T^{o} is an elementary Nisnevich square, and that T satisfies (1) and (3) in Definition 4.2.1. Recall the notation*

$$\begin{array}{ccc} T(00) & \xrightarrow{v_T} & T(01) \\ {\scriptstyle q_T}\downarrow & & \downarrow{\scriptstyle p_T} \\ T(10) & \xrightarrow{u_T} & T(11) \end{array} \tag{5.2.2}$$

from Definition 4.2.1. Then for any $M \in \underline{\mathbf{M}}\mathbf{Sm}$, the complex

$$0 \to \mathbb{Z}_{\mathrm{tr}}(T(00))(M) \xrightarrow{(q_{T*}, v_{T*})} \mathbb{Z}_{\mathrm{tr}}(T(10))(M) \oplus \mathbb{Z}_{\mathrm{tr}}(T(01))(M) \xrightarrow{p_{T*} - u_{T*}} \mathbb{Z}_{\mathrm{tr}}(T(11))(M) \tag{5.2.3}$$

of abelian groups is exact.

Proof. The assertion is equivalent to requiring that the commutative square

$$\begin{array}{ccc} \underline{\mathbf{M}}\mathbf{Cor}(M, T(00)) & \xrightarrow{v_{T*}} & \underline{\mathbf{M}}\mathbf{Cor}(M, T(01)) \\ {\scriptstyle q_{T*}}\downarrow & & \downarrow{\scriptstyle p_{T*}} \\ \underline{\mathbf{M}}\mathbf{Cor}(M, T(10)) & \xrightarrow{u_{T*}} & \underline{\mathbf{M}}\mathbf{Cor}(M, T(11)) \end{array} \tag{5.2.4}$$

be cartesian. Note that the horizontal maps are injective.

The following lemma is key. Recall the notation from Proposition 2.3.3.

Lemma 5.2.5. *Let $\alpha_1, \alpha_2 \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01))$ be elementary correspondences with $\alpha_1 \neq \alpha_2$. Assume that $p_{T+}(\alpha_1) = p_{T+}(\alpha_2)$ holds in $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(11))$. Then α_1 and α_2 belong to the image of v_{T*}.*

Proof. Set $P := T(01) \times_{T(11)} T(01)$, and consider the commutative diagram

$$\begin{array}{ccc} \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, P) & \xrightarrow{\mathrm{pr}_{1+}} & \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01)) \\ \downarrow{\scriptstyle \mathrm{pr}_{2+}} & & \downarrow \\ \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01)) & \longrightarrow & \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(11)) \end{array}$$

in **Set**. By Proposition 2.3.4, there exists an element $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, P)$ such that $\mathrm{pr}_{1+}(\gamma) = \alpha_1$ and $\mathrm{pr}_{2+}(\gamma) = \alpha_2$.

We have a canonical identification

$$\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, P) \cong \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(01)) \sqcup \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, \mathrm{OD}(p_T))$$

induced by $P \cong T(01) \sqcup \mathrm{OD}(p_T)$. Through this identification, we may regard $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, \mathrm{OD}(p_T))$ as a subset of $\underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, P)$.

Claim 5.2.6. $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, \mathrm{OD}(p_T))$.

Proof. Let ξ_1, ξ_2, and ζ be the generic points of α_1, α_2, and γ. Then ζ lies over ξ_1 and ξ_2. Since $\xi_1 \neq \xi_2$ by the assumption that $\alpha_1 \neq \alpha_2$, we have $\zeta \notin M^{\mathrm{o}} \times \Delta(T(01)^{\mathrm{o}})$, where $\Delta(T(01)^{\mathrm{o}})$ denotes the image of $\Delta : T(01)^{\mathrm{o}} \to T(01)^{\mathrm{o}} \times_{T(11)^{\mathrm{o}}} T(01)^{\mathrm{o}}$. This implies that $\zeta \in M^{\mathrm{o}} \times \mathrm{OD}(p_T)^{\mathrm{o}}$. Therefore, we have

$$\gamma \in \mathbf{Cor}(M^{\mathrm{o}}, \mathrm{OD}(p_T)^{\mathrm{o}}) \cap \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, P) = \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, \mathrm{OD}(p_T)). \qquad \square$$

By construction, we have $\alpha_i = \mathrm{pr}_i(\gamma) = |(\mathrm{pr}_i)_*(\gamma)|$, where

$$\mathrm{pr}_i : T(01)^{\mathrm{o}} \times_{T(11)^{\mathrm{o}}} T(01)^{\mathrm{o}} \to T(01)^{\mathrm{o}}, \quad i = 1, 2,$$

are the projections. Thus, in order to prove $\alpha_i \in \underline{\mathbf{M}}\mathbf{Cor}(M, T(00))$ for $i = 1, 2$, it suffices to prove that $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}(M, T(00) \times_{T(10)} T(00))$. Since by the above claim $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}(M, \mathrm{OD}(p_T))$, and since $\mathrm{OD}(q_T) \cong \mathrm{OD}(p_T)$ by condition (3) of Definition 4.2.1, we have $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}(M, \mathrm{OD}(q_T)) \subset \underline{\mathbf{M}}\mathbf{Cor}(M, T(00) \times_{T(10)} T(00))$. This finishes the proof of Lemma 5.2.5. $\square$

Now we are ready to prove that (5.2.4) is cartesian. Let $\alpha \in \underline{\mathbf{M}}\mathbf{Cor}(M, T(01))$ and assume $p_{T*}(\alpha) \in \underline{\mathbf{M}}\mathbf{Cor}(M, T(10))$. Write $\alpha = \sum_{i \in I} m_i \alpha_i$, where I is a finite set, $m_i \in \mathbb{Z} - \{0\}$, and the α_i are elementary correspondences which are distinct from each other. Then we have $\alpha_i \in \underline{\mathbf{M}}\mathbf{Cor}(M, T(01))$ for all $i \in I$. Set

$$J := \{i \in I \mid \exists j \in I - \{i\},\ |p_{T*}(\alpha_i)| = |p_{T*}(\alpha_j)|\}.$$

Then by Lemma 5.2.5, we have $\alpha_i \in \underline{\mathbf{M}}\mathbf{Cor}(M, T(00))$ for all $i \in J$. Let $i \in I - J$, and set $\beta := |p_{T*}(\alpha_i)|$. Then the coefficient of β in $p_{T*}(\alpha)$ is nonzero, and therefore $\beta \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(10))$. By Proposition 2.3.4, there exists a unique element $\gamma \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(00))$ such that $v_{T+}(\gamma) = \alpha_i$ and $q_{T+}(\gamma) = \beta$. Since $T(00)^{\mathrm{o}} \to T(01)^{\mathrm{o}}$ is an open immersion, this implies $\alpha_i = \gamma \in \underline{\mathbf{M}}\mathbf{Cor}^{\mathrm{el}}(M, T(00))$. This finishes the proof of the exactness of (5.2.3). $\square$

Recall from [Kahn et al. 2019a, Theorem 2(2)] that for any $M \in \underline{\mathbf{M}}\mathbf{Sm}$, the presheaf $\mathbb{Z}_{\mathrm{tr}}(M)$ on $\underline{\mathbf{M}}\mathbf{Sm}$ is a sheaf for the $\underline{\mathrm{MV}}$-topology.

Corollary 5.2.7. *Let T be an* MV*-square. Then the complex*

$$0 \to \mathbb{Z}_{\mathrm{tr}}(T(00)) \xrightarrow{(q_{T*}, v_{T*})} \mathbb{Z}_{\mathrm{tr}}(T(10)) \oplus \mathbb{Z}_{\mathrm{tr}}(T(01)) \xrightarrow{p_{T*} - u_{T*}} \mathbb{Z}_{\mathrm{tr}}(T(11)) \to 0$$

of sheaves on $\underline{\mathbf{M}}\mathbf{Sm}$ *for the* $\underline{\mathrm{MV}}$*-topology is exact.*

Proof. By Theorem 5.2.1, it suffices to prove the surjectivity of the last maps of the complexes. Take a morphism $S \to T$ in $\underline{\mathbf{M}}\mathbf{Sm}^{\mathbf{Sq}}$ as in (2) of Definition 4.2.1. Then the map

$$\mathbb{Z}_{\mathrm{tr}}(S(10)) \oplus \mathbb{Z}_{\mathrm{tr}}(S(01)) \to \mathbb{Z}_{\mathrm{tr}}(S(11)) = \mathbb{Z}_{\mathrm{tr}}(T(11))$$

is epi in $\underline{\mathbf{M}}\mathbf{NST}$ by [Kahn et al. 2019a, Theorem 4.5.7]. Since the map factors through

$$\mathbb{Z}_{\mathrm{tr}}(T(10)) \oplus \mathbb{Z}_{\mathrm{tr}}(T(01)),$$

we are done. □

Acknowledgements

The author thanks Shuji Saito deeply for many helpful discussions on the first draft of the paper. It enabled the author to find a simple proof of Theorem 5.2.1. The author's gratitude also goes to Bruno Kahn who encouraged him to find a conceptual formulation of the cd-structure, which led to a considerable improvement of the paper. Finally, the author thanks the referee for correcting some errors in the first version of the paper, and for providing helpful suggestions.

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Received 25 Nov 2019. Revised 31 Mar 2020. Accepted 20 Apr 2020.

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Annals of K-Theory is a journal of the K-Theory Foundation (ktheoryfoundation.org). The K-Theory Foundation acknowledges the precious support of Foundation Compositio Mathematica, whose help has been instrumental in the launch of the Annals of K-Theory.

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Annals of K-Theory (ISSN 2379-1681 electronic, 2379-1683 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

AKT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANNALS OF K-THEORY

2020 vol. 5 no. 3