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Shira Gilat and Eliyahu Matzri

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On the norm and multiplication principles for norm varieties

Shira Gilat and Eliyahu Matzri

Let p be a prime, and suppose that F is a field of characteristic zero which is p-special (that is, every finite field extension of F has dimension a power of p). Let $\alpha \in \mathcal{K}_n^M(F)/p$ be a nonzero symbol and X/F a norm variety for α. We show that X has a $\mathcal{K}_m^M$-norm principle for any m, extending the known $\mathcal{K}_1^M$-norm principle. As a corollary we get an improved description of the kernel of multiplication by a symbol. We also give a new proof for the norm principle for division algebras over p-special fields by proving a decomposition theorem for polynomials over F-central division algebras. Finally, for $p = n = m = 2$ we show that the known $\mathcal{K}_1^M$-multiplication principle cannot be extended to a $\mathcal{K}_2^M$-multiplication principle for X.

1. Introduction

Let D be a finite dimensional F-central division algebra. Then D has the reduced norm homomorphism $\mathrm{Nrd} : D \to F$. The norm principle for D states that the image of the reduced norm is an invariant of the class of D in $\mathrm{Br}(F)$, that is, $\mathrm{Nrd}(D) = \mathrm{Nrd}(\mathrm{M}_k(D))$ for any $k \in \mathbb{N}$. Equivalently, $\mathrm{N}_{K/F}(K) \subseteq \mathrm{Nrd}(D)$ for any finite separable field extension K/F splitting D. The multiplication principle states that for any two maximal subfields $K_1, K_2 \subset D$ and elements $k_1 \in K_1, k_2 \in K_2$, there is a third maximal subfield $K_3 \subset D$ and an element $k_3 \in K_3$ such that $\mathrm{Nrd}(k_1)\,\mathrm{Nrd}(k_2) = \mathrm{Nrd}(k_3)$, reflecting the fact that the reduced norm is multiplicative with respect to the multiplication of D.

The above can be rephrased as follows: Let D be a central division algebra over F of index n and let $X = \mathrm{SB}(D)$ be the Severi–Brauer variety of D. Let $A_0(X, \mathcal{K}_1^M)$ be the group of $\mathcal{K}_1^M$-zero cycles on X. It is generated by elements $[x, \lambda]$, where x is a closed point of X and $\lambda \in F(x)$. Let $A_0^n(X, \mathcal{K}_1^M)$ be the subgroup generated

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by elements $[x, \lambda]$, where x is of degree at most n (that is, $[F(x) : F] \le n$). There is a well defined norm homomorphism

$$\mathrm{N} : A_0(X, \mathcal{K}_1^M) \to A_0(\mathrm{spec}(F), \mathcal{K}_1^M) = F^\times,$$

defined on generators by $\mathrm{N}([x, \lambda]) = \mathrm{N}_{F(x)/F}(\lambda)$. The above norm principle can be restated as $\mathrm{N}(A_0(X, \mathcal{K}_1^M)) = \mathrm{N}(A_0^n(X, \mathcal{K}_1^M))$, that is, for any closed point $x \in X$ and $\lambda \in F(x)$ there is a finite number of closed points $x_1, \dots, x_t \in X$ of degree at most n and $\lambda_i \in F(x_i)$ such that $\mathrm{N}_{F(x)/F}(\lambda) = \prod_{i=1}^t \mathrm{N}_{F(x_i)/F}(\lambda_i)$.

The multiplication principle says that for any two closed points $x_1, x_2 \in X$ of degree at most n and $\lambda_1 \in F(x_1)$, $\lambda_2 \in F(x_2)$, there is a third closed point $x_3 \in X$ of degree at most n and $\lambda_3 \in F(x_3)$ such that $\mathrm{N}_{F(x_1)/F}(\lambda_1)\,\mathrm{N}_{F(x_2)/F}(\lambda_2) = \mathrm{N}_{F(x_3)/F}(\lambda_3)$.

This is generalized as follows. Let p be a fixed prime, F a field of characteristic zero which is p-special, and $\alpha \in \mathcal{K}_n^M(F)/p$ a nonzero symbol $\alpha = a_1 \cdots a_n$, where $a_i \in \mathcal{K}_1^M(F)/p$. A crucial part of the proof of the Bloch–Kato conjecture is the fact that symbols have (at least in characteristic zero) p-generic splitting varieties, generalizing Severi–Brauer varieties, introduced by Rost; see [Haesemeyer and Weibel 2009; Rost 2002]. Rost then generalized the norm principle for division algebras (more specifically symbol algebras) to a norm principle for the group of reduced $\mathcal{K}_1^M$-zero cycles:

$$\bar{A}_0(X, \mathcal{K}_1^M) = \mathrm{coker}\big(A_0(X \times X, \mathcal{K}_1^M) \xrightarrow{(\mathrm{pr}_1)_* - (\mathrm{pr}_2)_*} A_0(X, \mathcal{K}_1^M)\big).$$

This principle states that $\mathrm{N}(\bar{A}_0(X, \mathcal{K}_1^M)) = \mathrm{N}(\bar{A}_0^p(X, \mathcal{K}_1^M))$, that is, the image of the norm on $\bar{A}_0(X, \mathcal{K}_1^M)$ is the same as the image of the norm restricted to the subgroup $\bar{A}_0^p(X, \mathcal{K}_1^M)$ generated by elements $[x, \lambda]$, where $x \in X$ is closed of degree at most p.

The multiplication principle is also generalized and states that the product of two generators of $\bar{A}_0^p(X, \mathcal{K}_1^M)$ is a generator; equivalently, for any two closed points $x_1, x_2 \in X$ of degree at most p and $\lambda_1 \in F(x_1)$, $\lambda_2 \in F(x_2)$, there is a third closed point $x_3 \in X$ of degree at most p and $\lambda_3 \in F(x_3)$ such that

$$\mathrm{N}_{F(x_1)/F}(\lambda_1)\,\mathrm{N}_{F(x_2)/F}(\lambda_2) = \mathrm{N}_{F(x_3)/F}(\lambda_3).$$

Together, the above norm and multiplication principles state the following: Let X be a norm variety for a nonzero symbol $\alpha \in \mathcal{K}_n^M(F)/p$, $x \in X$ any closed point (of arbitrary finite degree) and $\lambda \in F(x)$. Then there is a closed point $y \in X$ of degree at most p and $\gamma \in F(y)$ such that $\mathrm{N}_{F(x)/F}(\lambda) = \mathrm{N}_{F(y)/F}(\gamma)$.

In this work we show that the norm principle can be extended to higher $\mathcal{K}$-cohomology groups, but the multiplication principle does not extend. As an application we recall that using these varieties one can give the following exact sequence describing the kernel of multiplication by a symbol, taken from [Merkurjev and Suslin 2010]; see also [Weibel and Zakharevich 2017] for a similar description.

Theorem 1. *Let F be a field of characteristic prime to p and $\theta \in H^n_{\text{ét}}(F, \mu_p^{\otimes n})$ a symbol, where μ_p denotes the Galois module of all p-th roots of unity. Then for an arbitrary $k \in \mathbb{N}$, there is an exact sequence*

$$\coprod_L H^k_{\text{ét}}(L, \mu_p^{\otimes k}) \xrightarrow{\sum \mathrm{N}_{L/F}} H^k_{\text{ét}}(F, \mu_p^{\otimes k}) \xrightarrow{\cdot\theta} H^{k+n}_{\text{ét}}(F, \mu_p^{\otimes k+n}) \xrightarrow{\prod_E \mathrm{res}_{E/F}} \prod_E H^{k+n}_{\text{ét}}(E, \mu_p^{\otimes k+n}),$$

where the coproduct is taken over all finite splitting field extensions L/F for θ and the product is taken over all splitting field extensions E/F.

As a result of the higher norm principle we can add that the coproduct is taken over all splitting fields L/F such that p^2 does not divide $[L : F]$, as is the case for $p = 2$; see [Orlov et al. 2007] for details.

The work is organized as follows. In Section 3 we prove a generalized norm principle. In Section 4 we give a purely algebraic proof of the main theorem used for the proof of the norm principle in [Haesemeyer and Weibel 2009] for the case of division algebras, resulting in a new proof for the norm principle for division algebras over p-special fields. To this end we prove that if F is p-special (with no restriction on the characteristic) and D is an F-central division algebra, then any polynomial in $D[\lambda]$ of degree less than p splits into linear factors (see Theorem 14). In Section 5 we show that (at least for $p = n = m = 2$) there is no generalized multiplication principle.

2. Background and notations

Let p be a fixed prime, and suppose that F is a field of characteristic zero which is p-special — that is, for any field extension K/F we have that $[K : F]$ is a p-th power, or equivalently (for perfect fields), the absolute Galois group of F is a pro-p group.

Let $a_1, \dots, a_n$ be in $\mathcal{K}^M_1(F)/p \cong F^\times/(F^\times)^p$ and $\alpha = a_1 \cdots a_n$ be a nontrivial symbol in $\mathcal{K}^M_n(F)/p$, where $\mathcal{K}^M_n(F)$ is the n-th Milnor $\mathcal{K}$ group of F. In a work by Rost, it was shown that there exists a "p-generic splitting variety" over F of dimension $p^{n-1} - 1$ for α, namely a smooth, irreducible, projective variety X of dimension $p^{n-1} - 1$, such that for any field extension L/F, α_L vanishes in $K^M_n(L)/p$ if and only if $X(L') \neq \varnothing$, where L'/L is a field extension of dimension prime to p. Such a variety is called a norm variety for α. For a detailed construction of such X we refer the reader to [Haesemeyer and Weibel 2009; Suslin and Joukhovitski 2006].

As an example, in the case $n = 2$, X can be the Severi–Brauer variety of the central simple algebra associated to α by the norm residue map.

Definition 2. Let X/F be a smooth irreducible projective variety of dimension d and let n be an integer. The group of $\mathcal{K}_m^M$-zero cycles, $A_0(X, \mathcal{K}_m^M)$, is defined as

$$A_0(X, \mathcal{K}_m^M) = \operatorname{coker}\Bigg(\coprod_{\operatorname{codim}(x)=d-1} \mathcal{K}_{m+1}^M(F(x)) \to \coprod_{\operatorname{codim}(x)=d} \mathcal{K}_m^M(F(x)) \Bigg).$$

It is generated by elements $[x, \alpha]$, where x is a closed point of X (marking its index in the coproduct) and $\alpha \in \mathcal{K}_m^M(F(x))$. Also define the subgroup

$$A_0^p(X, \mathcal{K}_m^M) = \big\langle [x, \alpha] \mid x \text{ is closed of degree at most } p \big\rangle.$$

Remark 3. There is a well defined norm map $\mathrm{N} : A_0(X, \mathcal{K}_m^M) \to \mathcal{K}_m^M(F)$ induced by the usual norm on fields: $\mathrm{N}([x, \alpha]) = \operatorname{Cor}_{F(x)/F}(\alpha)$.

As we are going to be interested in norms of elements, we make the following definitions.

Definition 4. For X/F as above define the group

$$\widetilde{A}_0(X, \mathcal{K}_m^M) = A_0(X, \mathcal{K}_m^M)/\operatorname{Ker}(\mathrm{N})$$

and its subgroup,

$$\widetilde{A}_0^p(X, \mathcal{K}_m^M)) = A_0^p(X, \mathcal{K}_m^M))/\operatorname{Ker}(\mathrm{N}).$$

Definition 5. We say that X has a $\mathcal{K}_m^M$-norm principle if $\widetilde{A}_0(X, \mathcal{K}_m^M)) = \widetilde{A}_0^p(X, \mathcal{K}_m^M))$, or equivalently, $\mathrm{N}(A_0(X, \mathcal{K}_m^M)) = \mathrm{N}(A_0^p(X, \mathcal{K}_m^M))$.

Definition 6. We say that X has a $\mathcal{K}_m^M$-multiplication principle if every element in $\widetilde{A}_0^p(X, \mathcal{K}_m^M)$ is a single generator $[x, \beta]$, or equivalently, the norm of every element in $A_0^p(X, \mathcal{K}_m^M)$ can be obtained as the norm of just one generator $[x, \beta]$.

Let $\alpha \in \mathcal{K}_n^M(F)/p$ be a nontrivial symbol. For every m, we have the morphism of multiplication by α:

$$\mathcal{K}_m^M(F)/p \xrightarrow{\cdot\alpha} \mathcal{K}_{n+m}^M(F)/p.$$

Let X be a norm variety for α. Then by Theorem 1, for every m, the kernel $\operatorname{Ker}_m(\alpha)$ of this morphism can be described by the exact sequence

$$\widetilde{A}_0(X, \mathcal{K}_m^M) \xrightarrow{\pi\circ\mathrm{N}} \mathcal{K}_m^M(F)/p \xrightarrow{\cdot\alpha} \mathcal{K}_{n+m}^M(F)/p,$$

where $\pi : \mathcal{K}_m^M(F) \to \mathcal{K}_m^M(F)/p$ is the natural projection. If X has a $\mathcal{K}_m^M$-norm principle we get that the sequence

$$\widetilde{A}_0^p(X, \mathcal{K}_m^M) \xrightarrow{\pi\circ\mathrm{N}} \mathcal{K}_m^M(F) \xrightarrow{\cdot\alpha} \mathcal{K}_{n+m}^M(F)$$

is exact, giving a better description of the kernel of multiplication by α.

Beyond proving the norm principle for X, we would like to give a "nice" generating set for the kernel $\operatorname{Ker}_m(\alpha)$. To this end we make the following definitions.

Definition 7. A basic element of $\mathrm{Ker}(\alpha)$ in $\mathcal{K}_m^M(F)$ is an element of the form $a_1 \cdots a_{m-1} \cdot a_m$, where $a_1, \ldots, a_{m-1} \in \mathcal{K}_1^M(F)/p$ and $a_m \in \mathrm{N}_{L/F}(L^\times)/p$, where L is a splitting field for α of degree (at most) p.

Definition 8. For a symbol α define $\mathrm{BKer}_m(\alpha)$ to be the subgroup of $\mathrm{Ker}_m(\alpha)$ generated by all basic elements of $\mathrm{Ker}_m(\alpha)$.

Remark 9. A description of $\mathrm{Ker}_m(\alpha)$ was given in [Orlov et al. 2007] for the case $p = 2$, where it was proved that $\mathrm{BKer}_m(\alpha) = \mathrm{Ker}_m(\alpha)$ for all m. Also, by the norm and multiplication principles for $\bar{A}_0(X, \mathcal{K}_1^M)$ one has $\mathrm{BKer}_1(\alpha) = \mathrm{Ker}_1(\alpha)$.

We prove that over p-special fields (of characteristic zero), $\mathrm{BKer}_m(\alpha) = \mathrm{Ker}_m(\alpha)$ for a symbol $\alpha \in \mathcal{K}_n^M(F)$ for arbitrary n and m.

3. Norm principle

In this section we prove the higher norm principle for the norm variety of a symbol α. Recall the following well known lemma.

Lemma 10 [Gille and Szamuely 2006, p. 195, Corollary 7.2.10]. *Let F be a field of characteristic prime to p which is p-special, and K/F be a field extension of degree p. Then $\mathcal{K}_n^M(K) = \sum \mathcal{K}_{n-1}^M(F)\mathcal{K}_1^M(K)$.*

Also recall the following theorem taken from [Haesemeyer and Weibel 2009] (which is the main ingredient in the proof of the norm principle).

Theorem 11 [Haesemeyer and Weibel 2009, Theorem 9.6]. *Let F be a p-special field of characteristic zero, and E/F a field extension with $[E : F] = p$. Write $E = F[\epsilon]$ with $\epsilon^p \in F$. For a nontrivial symbol $\alpha \in \mathcal{K}_n^M(F)/p$ suppose that $\alpha_E \neq 0$ and that X is a norm variety for α. For $[x, \alpha] \in \bar{A}_0(E)$, where $x \in X_E$ is of degree at most p (over E), there exist points $x_i \in X$ of degree p over F, $t_i \in F$ and $b_i \in F(x_i)$ such that $\mathrm{N}_{E(x)/E}(\alpha) = \prod \mathrm{N}_{E(x_i)/E}(b_i + t_i\epsilon)$.*

We are ready to prove the higher norm principle:

Theorem 12. *Let F be a p special field of characteristic zero, and X a norm variety for a nontrivial symbol $\alpha \in \mathcal{K}_n^M(F)/p$. Then X has a $\mathcal{K}_m^M$-norm principle for any m. Moreover, $\mathrm{Ker}_m(\alpha) = \mathrm{BKer}_m(\alpha)$ for any m.*

Proof. In order to prove the higher norm principle, we have to show that

$$\mathrm{N}(A_0(X, \mathcal{K}_m^M)) = \mathrm{N}(A_0^p(X, \mathcal{K}_m^M)).$$

It is clear that it is enough to show

$$\mathrm{N}(A_0(X, \mathcal{K}_m^M)) \subseteq \mathrm{N}(A_0^p(X, \mathcal{K}_m^M)).$$

Let $[x, \gamma] \in A_0(X, \mathcal{K}_m^M)$, so that $x \in X$ is a closed point and $\gamma \in \mathcal{K}_m^M(F(x))$. Since F is p-special, x is of degree p^t for some $t \geq 1$. If $t = 1$ there is nothing to prove, so we assume $t > 1$. Pick subfields $F \subseteq L \subseteq K \subseteq F(x)$ such that $[F(x) : K] = p$, $[K : L] = p$ and K is not a splitting field of α (if K is a splitting field, then $\mathrm{cor}_{F(x)/F}(\gamma) = \mathrm{cor}_{K/F}(\mathrm{cor}_{F(x)/K}(\gamma))$ and we are done by induction). Write $F(x) = K(x')$ for a closed point $x' \in X_K$. By Lemma 10, we may write $\mathrm{cor}_{F(x)/F}(\gamma) = \mathrm{cor}_{F(x)/F}\big(\sum \gamma_i \cdot b_i\big)$ for some $\gamma_i \in \mathcal{K}^M_{m-1}(K)$, $b_i \in F(x)$. By Theorem 11, there are closed points $x_{ij} \in X_L$ of degree p and $\beta_{i,j} \in L(x_{i,j})$ such that $\mathrm{cor}_{K(x')/K}(b_i) = \sum_j \mathrm{cor}_{K(x_{i,j})/K}(\beta_{i,j})$. Now compute

$$\begin{aligned}
\mathrm{N}([x,\gamma]) &= \mathrm{cor}_{F(x)/F}(\gamma)\\
&= \mathrm{cor}_{F(x)/F}\Big(\sum_i \gamma_i \cdot b_i\Big)\\
&= \mathrm{cor}_{K/F}\Big(\sum_i \gamma_i \cdot \mathrm{cor}_{K(x')/K}(b_i)\Big)\\
&= \mathrm{cor}_{K/F}\Big(\sum_i \gamma_i \cdot \sum_j \mathrm{cor}_{K(x_{i,j})/K}(\beta_{i,j})\Big)\\
&= \mathrm{cor}_{K/F}\Big(\sum_{i,j} \gamma_i \cdot \mathrm{cor}_{K(x_{i,j})/K}(\beta_{i,j})\Big)\\
&= \sum_{i,j} \mathrm{cor}_{K/F}\,\mathrm{cor}_{K(x_{i,j})/K}(\gamma_i \cdot \beta_{i,j})\\
&= \sum_{i,j} \mathrm{cor}_{K(x_{i,j})/F}(\gamma_i \cdot \beta_{i,j})\\
&= \sum_{i,j} \mathrm{cor}_{L(x_{i,j})/F}\,\mathrm{cor}_{K(x_{i,j})/L(x_{i,j})}(\gamma_i \cdot \beta_{i,j})\\
&= \mathrm{N}\Big(\sum_{i,j} [x_{i,j}, \mathrm{cor}_{K(x_{i,j})/L(x_{i,j})}(\gamma_i \cdot \beta_{i,j})]\Big).
\end{aligned}$$

Notice that $[L(x_{i,j}) : F] = p^{t-1}$, and clearly $L(x_{i,j})$ splits α, so we are done by induction on t. We proved that $\mathrm{N}(A_0(X, \mathcal{K}^M_m)) = \mathrm{N}(A^p_0(X, \mathcal{K}^M_m))$, so X has a $\mathcal{K}^M_m$-norm principle. The last statement follows from Lemma 10 and the proof thus far. □

Corollary 13. *Let F be a field of characteristic prime to p and $\theta \in H^n_{\text{ét}}(F, \mu_p^{\otimes n})$ a symbol, where μ_p denotes the Galois module of all p-th roots of unity. Then for an arbitrary $k \in \mathbb{N}$, there is an exact sequence*

$$\coprod_L H^k_{\text{ét}}(L, \mu_p^{\otimes k}) \xrightarrow{\sum \mathrm{N}_{L/F}} H^k_{\text{ét}}(F, \mu_p^{\otimes k}) \xrightarrow{\cdot\theta} H^{k+n}_{\text{ét}}(F, \mu_p^{\otimes k+n}) \xrightarrow{\prod_E \mathrm{res}_{E/F}} \prod_E H^{k+n}_{\text{ét}}(E, \mu_p^{\otimes k+n}),$$

where the coproduct is taken over all splitting field extensions L/F for θ of degree not divisible by p^2, and the product is taken over all splitting field extensions E/F.

Proof. This is just applying Theorem 12 to Theorem 1. □

4. Norm principle for division algebras

In this section we give a purely algebraic proof of a variant of Theorem 11 for any division algebra and not just a symbol. When $n = 2$, we have that the m torsion part of the Brauer group ${}_m\mathrm{Br}(F)$ is isomorphic to $\mathcal{K}_2^M(F)/m$. So we may consider symbols as division algebras and their norm varieties as the Severi–Brauer varieties.

Polynomials over division rings. We first recall some known facts concerning polynomials over division algebras. For a more thorough reference we point the reader to [Jacobson 1996, Chapter 1; 1943, Chapter 3; Haile and Rowen 1995]. Let D be an F-central division algebra (i.e., its center is F) and $R = D[\lambda]$ the ring of (left) polynomials over D (where λ is central in R). Let $c \in D$ be a central element. Then there is a well defined substitution homomorphism $\varphi : R \to D$ defined by $\lambda \mapsto c$. In particular, if we can decompose $f(\lambda) = g(\lambda)h(\lambda)$, then $f(c) = g(c)h(c)$.

The polynomial ring $R = D[\lambda]$ is a left (and also right) Euclidean domain; that is, for any $f(\lambda), g(\lambda) \in R$ there are $q(\lambda), r(\lambda) \in R$ such that $f(\lambda) = q(\lambda)g(\lambda) + r(\lambda)$ and $\deg r(\lambda) < g(\lambda)$ or $r(\lambda) = 0$. As a consequence we get that every left (and right) ideal is principal, so R is a (left and right) principle ideal domain (PID). For left ideals of R, $Rf \subseteq Rg$ if and only if $f = hg$ for some $h \in R$. Thus Rf is maximal if and only if f is irreducible, which happens if and only if R/Rf is simple as a (left) module over R.

The two-sided ideals of R are all of the form $Rf = fR$, where $f \in F[\lambda]$. For a left ideal Rf, the maximal two-sided ideal contained in Rf (called the bound of Rf) is the annihilator $I = \mathrm{ann}(R/Rf)$. Note that if Rf is a maximal left ideal (f is irreducible) and $I \neq 0$ then I is a maximal two-sided ideal. Write $I = Rg \neq 0$ (where $g \in F[\lambda]$). Then $R/I \cong D \otimes (F[\lambda]/F[\lambda]g)$, where $F[\lambda]/F[\lambda]g$ is a simple $F[\lambda]$-module of dimension $\deg(g)$ — that is, a field extension of degree $\deg(g)$.

Norm principle for division algebras. For this subsection we assume our base field F is p-special (no restriction on $\mathrm{char}(F)$), and D is any F-central division algebra. We discuss further polynomials over D, and then prove our version of Theorem 11.

Theorem 14. *Let D be an F-central division algebra (which by assumption has index p^t for some t). The only irreducible polynomials over D are of degree a power of p.*

Proof. Let $f \in R = D[\lambda]$ be a polynomial of degree d such that d is not a power of p. Recall from the previous subsection that f is irreducible if and only if $M = R/Rf$ is a simple R-module.

If M is a simple R-module then it is also a simple module over $S = R/\mathrm{ann}(M)$. The ideal $\mathrm{ann}(M)$ is nonzero, since f divides its reduced norm, which is nonzero (see [Haile and Rowen 1995]). This can also be seen by comparing the dimension over F of M and R. As we assume f is irreducible we have that $\mathrm{ann}(M)$ is a maximal two-sided ideal, and since R is a principle ideal domain we can write $\mathrm{ann}(M) = \langle g \rangle$ such that $g \in F[\lambda]$ (see the previous subsection). This implies that $S \cong D \otimes_F E$, where $E = F[\lambda]/\langle g \rangle$ is a field extension of F. As F is p-special we have that $\dim_F(E) = p^s$ for some s, so $\dim_F D \otimes_F E = p^{2t+s}$.

If indeed M were a simple module, it would be a simple module of $D \otimes E$, which is either $D \otimes_F E$ (if it is a division algebra) or E^{p^t} (if $D \otimes E \cong \mathrm{M}_{p^t}(E)$) or any other possibility in between. Either way, we get that the dimension of a simple module is a power of p but $\dim_F(M) = d \dim_F(D) = dp^{2t}$ is not a power of p. Hence M is not a simple module, which forces f to be reducible. □

The next corollary is now immediate.

Corollary 15. *For a division algebra D over a p-special field, every polynomial over D of degree less than p splits into linear factors.*

The factorization of polynomials of degree less then p over D enables us to give the following purely algebraic proof of the crucial Theorem 11 (which works for any division algebra, not just symbols).

Corollary 16. *Let D be a division algebra over a p-special field F, and let $F \subset E$ be a field extension of dimension p such that $D_E = D \otimes_F E$ is a division algebra. Then for every $d \in D_E$ there are $d_0, \dots, d_{p-1} \in D$ and $\epsilon \in E$ such that*

$$d = d_0 \prod_{i=1}^{p} (\epsilon - d_i).$$

Proof. Since F is p-special we may write $E = F[\epsilon \mid \epsilon^p \in F]$, and so the extension of D can be written as $D_E = D + D\epsilon + D\epsilon^2 + \cdots + D\epsilon^{p-1}$. Thus, any element $d \in D_E$ is of the form $d = d_0' + d_1'\epsilon + \cdots + d_{p-1}'\epsilon^{p-1}$, where $d_i' \in D$. Looking at the polynomial $f(\lambda) = d_0' + d_1'\lambda + \cdots + d_{p-1}'\lambda^{p-1} \in D[\lambda]$, we have that $f(\epsilon) = d$. By Corollary 15, $f(\lambda)$ splits to linear factors $f(\lambda) = d_0(\lambda - d_1) \cdot (\lambda - d_2) \cdots (\lambda - d_{p-1})$ in $D[\lambda]$, and since ϵ is central we get that $d = f(\epsilon) = d_0(\epsilon - d_1) \cdot (\epsilon - d_2) \cdots (\epsilon - d_{p-1})$. □

Corollary 17. *Suppose F is p-special and D is an F-central division algebra of degree d. Let $E = F[\epsilon \mid \epsilon^p \in F]$ be a field extension of degree p such that D_E is a division algebra. For every element $d \in D_E$, there are maximal subfields $E_i \subseteq D$*

and elements $d_i \in D$ such that

$$\mathrm{N}_{D_E/E}(d) = \mathrm{N}_{E_0 \otimes_F E/E}(d_0) \prod \mathrm{N}_{E_i \otimes_F E/E}(\epsilon - d_i),$$

where $\mathrm{N}_{D_E/E}$ is the reduced norm for D_E.

Proof. The proof follows from the well known fact that for maximal subfields of D the field norm coincides with the reduced norm. In particular, write $d = d_0(\epsilon - d_1) \cdot (\epsilon - d_2) \cdots (\epsilon - d_{p-1})$ as in Corollary 16. Now define E_i to be any maximal subfield of D containing d_i and apply the reduced norm on both sides of the factorization of d to get the required result. □

We use this factorization to get a direct proof of the norm principle for division algebras.

Theorem 18 (norm principle for division algebras). *Let F be a p-special field and D an F-central division algebra of index $d = p^n$. Let E/F be a finite dimensional splitting field for D and let $e \in E$. Then there is a maximal subfield K of D and $k \in K$ such that $\mathrm{N}_{E/F}(e) = \mathrm{N}_{K/F}(k)$.*

Proof. We proceed by induction on the index $\mathrm{ind}(D) = p^n$. The case of $n = 0$ is trivial. We now assume the theorem for $\mathrm{ind}(D) \leq p^k$ and prove it for $\mathrm{ind}(D) = p^{k+1}$. Let E/F be a splitting field for D of degree r, noting that since D is division we have $r \geq \mathrm{ind}(D)$. We proceed by induction on r. The case $r = d$ follows from the fact that in this case E embeds in D as a maximal subfield. We now assume the theorem for $r \leq \mathrm{ind}(D) + s$ and prove it for $r = \mathrm{ind}(D) + s + 1$. As F is p-special we can find a subfield $F \subset E_1 \subset E$ such that E_1 is of degree p over F. Consider $D_{E_1} = D \otimes E_1$. First assume that $D_{E_1} \cong \mathrm{M}_p(D')$ for an E_1-central division algebra D' of index p^k. Then by induction on $\mathrm{ind}(D)$ we get that there is a maximal subfield $T \subset D'$ and $t \in T$ such that $\mathrm{N}_{T/E_1}(t) = \mathrm{N}_{E/E_1}(e)$, implying $\mathrm{N}_{T/F}(t) = \mathrm{N}_{E/F}(e)$. But now, considering T over F, we see that T splits D and $[T : F] = p^{k+1} = \mathrm{ind}(D)$, so T embeds in D as a maximal subfield and we are done.

Now assume D_{E_1} is division. Notice that E splits D_{E_1} and is of lesser degree (over E_1). Thus by induction there exist a maximal subfield $T \subset D_{E_1}$ and $t \in T$ such that $\mathrm{N}_{T/E_1}(t) = \mathrm{N}_{E/E_1}(e)$, implying $\mathrm{N}_{T/F}(t) = \mathrm{N}_{E/F}(e)$. Writing $E_1 = F[\epsilon \mid \epsilon^p \in F]$ and using Corollary 17 we get maximal subfields $K_i \subset D$ and elements $d_i \in K_i$ such that

$$\begin{aligned}
\mathrm{N}_{T/F}(t) &= \mathrm{N}_{E_1/F}(\mathrm{N}_{T/E_1}(t)) \\
&= \mathrm{N}_{E_1/F}(\mathrm{N}_{K_0 \otimes_F E_1/E_1}(d_0)) \prod \mathrm{N}_{E_1/F}(\mathrm{N}_{K_i \otimes_F E_1/E_1}(\epsilon - d_i)) \\
&= \mathrm{N}_{K_0 \otimes_F E_1/F}(d_0) \prod \mathrm{N}_{K_i \otimes_F E_1/F}(\epsilon - d_i) \\
&= \mathrm{N}_{K_0/F}(\mathrm{N}_{K_0 \otimes_F E_1/K_0}(d_0)) \prod \mathrm{N}_{K_i/F}(\mathrm{N}_{K_i \otimes_F E_1/K_i}(\epsilon - d_i)).
\end{aligned}$$

Thus $N_{T/F}(t)$ is a product of norms from the maximal subfields $K_i \subseteq D$. Using the fact that the reduced norm is multiplicative and coincides with the field norm for maximal subfields we see that $N_{T/F}(t) = N_{K/F}(d)$, where K is any maximal subfield of D containing d and $d = N_{E/F}(t) \prod N_{K_i \otimes_F E_1/K_i}(\epsilon - d_i)$. □

Remark 19. Using a noncommutative analog of the determinant, the Dieudonné determinant, one can show that over any field the image of the reduced norm of a central simple algebra is an invariant of its class in the Brauer group of F; see [Pierce 1982, 16.5]. The above gives a simple proof of this result for p-special fields.

5. Multiplication principle

In this section we prove that for $p = n = m = 2$ there is no generalized multiplication principle (see Definition 6). We start by quoting the following lemma.

Lemma 20 [Matzri 2019, Lemma 4.1]. *Let F be a p-special field of characteristic prime to p. Let $\alpha \in \mathcal{K}_n^M(F)/p$ be a symbol and $b \in \mathcal{K}_1^M(F)/p$. Then $\alpha \cdot b = 0$ if and only if there exist $s_i \in \mathcal{K}_1^M(F)$, $i = 1, \dots, n$, and a presentation $\alpha = s_1 \cdots s_n$ such that $s_n \cdot b = 0$.*

The proof uses both norm and multiplication principles. Since we already have a generalized norm principle, if there were a generalized multiplication principle we would be able to prove a generalization of Lemma 20.

Lemma 21. *Assume the generalized multiplication principle holds. Let $\alpha \in \mathcal{K}_n^M/p$ and $\beta \in \mathcal{K}_m^M/p$ be symbols. Then, $\alpha \cdot \beta = 0$ if and only if there are presentations $\alpha = a \cdot \alpha'$ and $\beta = b \cdot \beta'$, where $a, b \in \mathcal{K}_1^M(F)$ are such that $a \cdot b = 0$ (that is, b is a norm from the field extension $F[\sqrt[p]{a}]$).*

Proof. The "only if" part is clear. For the other direction, assume that $\alpha \in \mathcal{K}_n^M/p$, $\beta \in cK_m/p$ are symbols such that $\alpha \cdot \beta = 0$. By Theorem 12,

$$\beta = \sum_{[K_i:F]=p} N_{K_i/F}(\beta_i)$$

for some splitting fields K_i and elements $\beta_i \in K_i$. By the generalized multiplication principle this is equal to $\sum_{i=1}^{M} N_{K/F}(\gamma_i \cdot k_i)$, where K is a (single) splitting field for α of degree p, $k_i \in K$ and $\gamma_i \in F$. Thus, as α splits over $K = F[\sqrt[p]{b} : b \in F]$ we can write $\alpha = \alpha' \cdot b$. Now by the projection formula and the description of β we have that $\beta \cdot b = 0$, and so by Lemma 20 we can decompose $\beta = \beta' \cdot a$ such that $a \cdot b = 0$. □

We now show that, at least for $p = n = m = 2$, the generalization of Lemma 20 implied in the last lemma is false. To this end we use the theory of quadratic forms and valuations.

Lemma 22. *Let F be a field of characteristic $\neq 2$, with a valuation $\nu : F \to \Gamma$, where Γ is a well ordered abelian group. Assume that for $a_1, \ldots, a_n \in F^\times$, the images of $\nu(a_1), \ldots, \nu(a_n)$ in $\Gamma/2\Gamma$ are pairwise different. Then the quadratic form $\varphi = \langle a_1, \ldots, a_n\rangle$ is anisotropic.*

Proof. Write $\varphi(v) = \alpha_1^2 a_1 + \cdots + \alpha_n^2 a_n$. Notice that for $i \neq j$ we have that $\nu(\alpha_i^2 a_i) \neq \nu(\alpha_j^2 a_j)$ (for otherwise $\nu(a_i)$ would be equivalent to $\nu(a_j)$ in $\Gamma/2\Gamma$). Thus, $\nu(\varphi(v)) = \nu(\alpha_i^2 a_i)$ for some i such that $\alpha_i \neq 0$. Now,

$$\nu(\alpha_i^2 a_i) = 2\nu(\alpha_i) + \nu(a_i) \neq \infty,$$

so $\varphi(v) \neq 0$ unless $v = 0$. □

Now we can give a counterexample (suggested by Stephen Scully and communicated to us by Stefan Gille)

Proposition 23. *Let $F = \mathbb{Q}(x, y, z)$ and $\alpha = \langle\langle x, y\rangle\rangle$, $\beta = \langle\langle z, -x + yz\rangle\rangle$. Then $\alpha \perp \beta'$ is anisotropic, where β' is the pure subform of β.*

Proof. Take the (x, y, z)-adic valuation of F, with values in the discrete group $\Gamma = \mathbb{Z}^3$ ordered lexicographically from left to right. By the previous lemma, the quadratic forms α, β and $\phi = \alpha \perp \langle -z, x - yz\rangle$ are anisotropic, and so $\alpha \perp \beta' = \phi \perp \langle -xz + yz^2\rangle$ is anisotropic if and only if $-xz + yz^2$ is not a value of ϕ. Assume $\alpha_0^2(-xz+yz^2) = \alpha_1^2 - \alpha_2^2 x - \alpha_3^2 y - \alpha_4^2 z + \alpha_5^2 xy + \alpha_6^2(x - yz)$ for some $\alpha_i \in F$, where $\alpha_0 \neq 0$. Multiplying by a common denominator, we can assume $\alpha_i \in \mathbb{Q}[x, y, z]$. We can rewrite the equation as

$$z(-\alpha_0^2 x + \alpha_4^2 + \alpha_6^2 y) = -\alpha_0^2 yz^2 + \alpha_1^2 - \alpha_2^2 x - \alpha_3^2 y + \alpha_5^2 xy + \alpha_6^2 x.$$

Compering even and odd z-degree, we get that $-\alpha_0^2 x + \alpha_4^2 + \alpha_6^2 y = 0$, which contradicts the fact that α is anisotropic. □

Remark 24. Note that by Springer's theorem, the above example works even if we take prime to 2 closure of F.

Corollary 25. *There is no generalized multiplication principle for the case $p = m = n = 2$.*

Proof. Assume that the generalized multiplication principle holds. By the Milnor conjecture, consider the quadratic Pfister forms $\alpha = \langle\langle x, y\rangle\rangle$ and $\beta = \langle\langle z, -x + yz\rangle\rangle$ in I^2/I^3 over a prime to p closure of $F = \mathbb{Q}(x, y, z)$. We notice that

$$\alpha \cdot \beta = \langle\langle x, y, z, -x + yz\rangle\rangle \cong \langle\langle x, y, -yz, -x + yz\rangle\rangle \cong \langle\langle y, xyz, x - yz, -x + yz\rangle\rangle$$

is hyperbolic. Thus, by Lemma 21 there is a presentation $\beta = \langle\langle b\rangle\rangle \cdot \langle\langle t\rangle\rangle$ such that $\alpha \cdot \langle\langle b\rangle\rangle$ is hyperbolic. Since b is an entry of β if and only if b is a value of the pure subform β', and b is also a value of α, we get that $\alpha \perp \beta'$ is isotropic, in contradiction to Proposition 23. □

We note that even for an odd prime p and $F = \mathbb{Q}(\rho, x, y, z)$, where ρ is a primitive root of unity, we can define $\alpha = (x, y)$ and $\beta = (z, -x + yz)$ in $\mathcal{K}_2^M(F)/p$ and still get that $\alpha \cdot \beta = 0$. We conjecture that this should give a counterexample for the generalized multiplication principle for the case $n = m = 2$, p an odd prime. In order to prove it one would need to show there is no presentation $\alpha = a \cdot b$ such that $b \cdot \beta = 0$. It seems that considering α, β as symbol algebras in the Brauer group of F and using valuation theory one could show such a presentation is not possible, but we could not make it work. For example, if one can show that $\mathrm{Im}(\mathrm{Nrd}_\alpha) \cap \mathrm{Im}(\mathrm{Nrd}_\beta) = F^p$, it would imply the needed condition, but again, we could not do it.

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SHIRA GILAT: shira.gilat@live.biu.ac.il
Department of Mathematics, Bar-Ilan University, Ramat-Gan, Israel

ELIYAHU MATZRI: elimatzri@gmail.com
Department of Mathematics, Bar-Ilan University, Ramat-Gan, Israel

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