

ANNALS OF K-THEORY

vol. 5 no. 4 2020

Zero-cycles with modulus and relative K -theory

Rahul Gupta and Amalendu Krishna



A JOURNAL OF THE K-THEORY FOUNDATION

Zero-cycles with modulus and relative K -theory

Rahul Gupta and Amalendu Krishna

Let D be an effective Cartier divisor on a regular quasiprojective scheme X of dimension $d \geq 1$ over a field. For an integer $n \geq 0$, we construct a cycle class map from the higher Chow groups with modulus $\{\mathrm{CH}^{n+d}(X | mD, n)\}_{m \geq 1}$ to the relative K -groups $\{K_n(X, mD)\}_{m \geq 1}$ in the category of pro-abelian groups. We show that this induces a proisomorphism between the additive higher Chow groups of relative 0-cycles and the reduced algebraic K -groups of truncated polynomial rings over a regular semilocal ring which is essentially of finite type over a characteristic zero field.

1. Introduction	757
2. The relative K -theory and cycles with modulus	764
3. The Milnor K -theory	769
4. The cycle class map	776
5. The case of regular curves	783
6. Proof of Theorem 1.1	788
7. Milnor K -theory, 0-cycles and de Rham–Witt complex	791
8. The relative Milnor K -theory	796
9. The cycle class map in characteristic zero	803
10. The cycle class map for semilocal rings	809
11. Appendix: Milnor vs. Quillen K -theory	814
Acknowledgments	816
References	817

1. Introduction

The story of Chow groups with modulus began with the discovery of additive higher 0-cycles by Bloch and Esnault [2003a; 2003b]. Their hope was that these additive 0-cycle groups would serve as a guide in developing a theory of motivic cohomology with modulus which could describe the algebraic K -theory of nonreduced schemes. Recall that Bloch’s original higher Chow groups (equivalently,

MSC2010: primary 14C25; secondary 19E08, 19E15.

Keywords: algebraic cycles with modulus, relative algebraic K -theory, additive higher Chow groups.

Voevodsky’s motivic cohomology) overlook the difference between nonreduced and reduced schemes.

Motivated by the work of Bloch and Esnault, a theory of motivic cohomology with modulus was proposed by Binda and Saito [2019] in the name of “higher Chow groups with modulus” (recalled in Section 2E). The expectation was that one would be able to describe relative algebraic K -theory in terms of these Chow groups. The theory of Chow groups with modulus generalized the theory of additive higher Chow groups defined by Bloch–Esnault and further studied by Rülling [2007], Krishna–Levine [2008] and Park [2009]. It also generalized the theory of 0-cycles with modulus of Kerz–Saito [2016] and the higher Chow groups of Bloch [1986].

Recall that one way to study the algebraic K -theory of a nonreduced (or any singular) scheme is to embed it as a closed subscheme of a smooth scheme and study the resulting relative K -theory. Since there are motivic cohomology groups which can completely describe the algebraic K -theory of a smooth scheme, what one needs is a theory of motivic cohomology to describe the relative K -theory.

The expectation that the higher Chow groups with modulus should be the candidate for the motivic cohomology to describe the relative K -theory has generated a lot of interest in them in past several years. In a recent work, Iwasa and Kai [2019], constructed a theory of Chern classes from the relative K -theory to a variant of the higher Chow groups with modulus. In [Iwasa and Kai 2018], they proved a Riemann–Roch type theorem showing that the relative group K_0 -group of an affine modulus pair is rationally isomorphic to a direct sum of Chow groups with modulus. An integral version of this isomorphism for all modulus pairs in dimension up to two was earlier proven by Binda and Krishna [2018]. They also constructed a cycle class map for relative K_0 -group in all dimensions.

The above results suggest strong connection between cycles with modulus and relative K -theory. However, an explicit construction of cycle class maps in full generality or Atiyah–Hirzebruch-type spectral sequences, which may directly connect Chow groups with modulus to relative algebraic K -theory, remains a challenging problem today.

1A. Main results and consequences. The objective of this paper is to investigate the original question of Bloch and Esnault [2003a] in this subject. Namely, can 0-cycles with modulus explicitly describe relative K -theory in terms of algebraic cycles? We provide an answer to this question in this paper. We prove two results. The first is that there is indeed a direct connection between 0-cycles with modulus and relative K -theory in terms of an explicit cycle class map. The second is that in many cases of interest, these 0-cycles with modulus are strong enough to completely describe the relative algebraic K -theory. More precisely, we prove the following. The terms and notations used in the statements of these results are

explained in the body of the text. In particular, we refer to [Section 2C](#) for relative K -theory and [Section 2E](#) for higher Chow groups with modulus.

Theorem 1.1. *Let X be a regular quasiprojective variety of pure dimension $d \geq 1$ over a field k and let $D \subset X$ be an effective Cartier divisor. Let $n \geq 0$ be an integer. Then there is a cycle class map*

$$\text{cyc}_{X|D}: \{\text{CH}^{n+d}(X | mD, n)\}_m \rightarrow \{K_n(X, mD)\}_m \quad (1.1)$$

between pro-abelian groups. This map is covariant functorial for proper morphisms, and contravariant functorial for flat morphisms of relative dimension zero.

For those interested in the precise variation in the modulus in the definition of $\text{cyc}_{X|D}$, we actually prove that for every pair of integers $m \geq 1$ and $n \geq 0$, there exists a cycle class map $\text{CH}^{n+d}(X | (n+1)mD, n) \rightarrow K_n(X, mD)$ such that going to pro-abelian groups, we get the cycle class map of [Theorem 1.1](#). For a general divisor $D \subset X$, we do not expect that the cycle class map that we construct in [Theorem 1.1](#) will exist without increasing the modulus. However, if we use rational coefficients, then the usage of pro-abelian groups can indeed be avoided, as the following result shows. In this paper, we use this improved version in the proof of [Theorem 1.3](#).

Theorem 1.2. *Let X be a regular quasiprojective variety of pure dimension $d \geq 1$ over a field k and let $D \subset X$ be an effective Cartier divisor. Let $n \geq 0$ be an integer. Then there is a cycle class map*

$$\text{cyc}_{X|D}: \text{CH}^{n+d}(X | D, n)_{\mathbb{Q}} \rightarrow K_n(X, D)_{\mathbb{Q}}. \quad (1.2)$$

This map is covariant functorial for proper morphisms, and contravariant functorial for flat morphisms of relative dimension zero. Furthermore, it coincides with the map (1.1) on the generators of $\text{CH}^{n+d}(X | D, n)$.

We now address as to why the cycle class maps of [Theorems 1.1](#) and [1.2](#) should be nontrivial and what we expect of these maps. Recall that the relative K -theory $K_n(X, mD)$ has Adams operations (e.g., see [\[Levine 1997\]](#) for their construction). From our construction, we expect the map (1.2) to be injective in the pro-setting, with image $\{K_n(X, mD)_{\mathbb{Q}}^{(d+n)}\}_m$. Here, $K_n(X, mD)_{\mathbb{Q}}^{(d+n)}$ is $(d+n)$ -th eigenspace of the Adams operations. When $D = \emptyset$, the cycle class map $\text{cyc}_X := \text{cyc}_{X|\emptyset}$ is not new and it was constructed by Levine [\[1994\]](#) by a different method. He also showed that in this special case, cyc_X is indeed injective with image $K_n(X)_{\mathbb{Q}}^{(d+n)}$.

When $X = \text{Spec}(k)$ and $D = \emptyset$, the cycle class map cyc_X coincides with Totaro's map $\text{CH}^n(k, n) \rightarrow K_n^M(k) \rightarrow K_n(k)$ [\[Totaro 1992\]](#). Totaro showed that the map $\text{CH}^n(k, n) \rightarrow K_n^M(k)$ is an isomorphism and one knows that the canonical map $K_n^M(k)_{\mathbb{Q}} \rightarrow K_n(k)_{\mathbb{Q}}^{(n)}$ is an isomorphism. The remaining part of this paper is devoted to showing that $\text{cyc}_{X|D}$ is in fact an isomorphism with integral coefficients

for the modulus pair $(\mathbb{A}_R^1, \{0\})$, where R is a regular semilocal ring essentially of finite type over a characteristic zero field.

We make some further remarks on the past works on the cycle class map for 0-cycles with modulus. Following Levine's strategy, Binda [2018] showed that there is a cycle class map to relative K -theory provided one makes the following changes: replace the higher Chow group with modulus by a variant of it (which imposes a stronger version of the modulus condition, originally introduced in [Krishna and Park 2012b]), assume that D_{red} is a strict normal crossing divisor, and assume rational coefficients. Theorem 1.1 imposes none of these conditions. If $D \subset X$ is a regular divisor, a cycle class map was defined in [Krishna and Pelaez 2018, Theorem 1.5] using the stable \mathbb{A}^1 -homotopy theory.

We now describe our results about the cycle class map of Theorem 1.1 for the modulus pair $(\mathbb{A}_R^1, \{0\})$. Recall that in case of the higher K -theory of a smooth scheme X , the cycle-class map $\text{CH}^{n+d}(X, n) \rightarrow K_n(X)$ from the 0-cycle group can not be expected to describe all of $K_n(X)$ (even with rational coefficients). However, we show in our next result that the cycle class map of Theorem 1.1 is indeed enough to describe all of the (integral) relative K -theory of nilpotent extensions of smooth schemes, if we work in the category of pro-abelian groups instead of the usual category of abelian groups. This demonstrates a remarkable feature of relative K -theory which is absent in the usual K -theory.

Before we state the precise result, recall that the additive higher Chow groups are special cases of higher Chow groups with modulus. More precisely, for an equidimensional scheme X , the additive higher Chow group $\text{TCH}^p(X, n+1; m)$ is same thing as the higher Chow group with modulus $\text{CH}^p(X \times \mathbb{A}_k^1 \mid X \times (m+1)\{0\}, n)$ for $m, n, p \geq 0$. To understand the reason for the shift in the value of n , we need to recall that the additive higher Chow groups are supposed to compute the relative K -theory of truncated polynomial extensions and one knows that the connecting homomorphism $\partial: K_{n+1}(X[t]/(t^{m+1}), (x)) \rightarrow K_n(X \times \mathbb{A}_k^1, X \times (m+1)\{0\})$ is an isomorphism when X is regular. Under this dichotomy, we shall use the notation cyc_X for $\text{cyc}_{\mathbb{A}_X^1 \mid (X \times \{0\})}$ whenever we use the language of additive higher Chow groups. In particular, for a ring R , we shall write cyc_R for $\text{cyc}_{\mathbb{A}_R^1 \mid \{0\}}$ while using additive higher Chow groups.

Let R now be a regular semilocal ring which is essentially of finite type over a characteristic zero field. Recall that there is a canonical map $K_*^M(R) \rightarrow K_*(R)$ from the Milnor to the Quillen K -theory of R . For $n \geq 1$, the group $\text{TCH}^n(R, n; m)$ is not a 0-cycle group if $\dim(R) \geq 1$. Hence, Theorem 1.1 does not give us a cycle class map for this group. However, using this theorem for fields and various other deductions, we can in fact prove an improved version of Theorem 1.1. Namely, we can avoid the usage of pro-abelian groups for the existence of the cycle class map with integral coefficients.

Theorem 1.3. *Let R be a regular semilocal ring which is essentially of finite type over a characteristic zero field. Let $m \geq 0$ and $n \geq 1$ be two integers. Then the following hold.*

(1) *There exists a cycle class map*

$$\text{cyc}_R^M : \text{TCH}^n(R, n; m) \rightarrow K_n^M(R[x]/(x^{m+1}), (x)).$$

(2) *The composite map*

$$\text{cyc}_R : \text{TCH}^n(R, n; m) \xrightarrow{\text{cyc}_R^M} K_n^M(R[x]/(x^{m+1}), (x)) \rightarrow K_n(R[x]/(x^{m+1}), (x))$$

coincides with the map of [Theorem 1.1](#) when R is a field.

(3) cyc_R^M and cyc_R are natural in R .

(4) cyc_R^M is an isomorphism.

(5) *The map*

$$\text{cyc}_R : \{\text{TCH}^n(R, n; m)\}_m \rightarrow \{K_n(R[x]/(x^{m+1}), (x))\}_m$$

of pro-abelian groups is an isomorphism.

In other words, [Theorem 1.3](#) (5) says that the relative K -theory of truncated polynomial rings can indeed be completely described by the relative 0-cycles over R (the cycles in $\text{TCH}^n(R, n; m)$ have relative dimension zero over R). This shows that the additive Chow groups defined by Bloch–Esnault [[2003a](#)] and [[Rülling 2007](#)] are indeed the relative K -groups, at least in characteristic zero. This was perhaps the main target of the introduction of additive higher Chow groups by Bloch and Esnault.

By the works of several authors (see [[Elbaz-Vincent and Müller-Stach 2002](#)] and [[Kerz 2009](#)] for regular semilocal rings and [[Nesterenko and Suslin 1989](#)] and [[Totaro 1992](#)] for fields), it is now well known that the motivic cohomology of a regular semilocal ring in the equal bidegree (the Milnor range) coincides with its Milnor K -theory. [Theorem 1.3](#) (4) says that this isomorphism also holds for truncated polynomial rings over such rings. This provides a concrete evidence that if one could extend Voevodsky’s theory of motives to the theory of “non- \mathbb{A}^1 -invariant” motives over so-called fat points (infinitesimal extensions of spectra of fields), then the underlying motivic cohomology groups must be the additive higher Chow groups (see [[Krishna and Park 2012a](#)]).

It should be remarked that the objective of [Theorem 1.3](#) is not to compute the relative K -groups. There are already known computations of these by many authors (e.g., see [[Goodwillie 1985](#); [Hesselholt 2008](#)]). Instead, the above result addresses the question whether these relative (Milnor or Quillen) K -groups could be described by additive 0-cycles.

Theorem 1.3 has following consequences. The first corollary below is in fact part of our proof of **Theorem 1.3**.

Corollary 1.4. *Let R be a regular semilocal ring which is essentially of finite type over a characteristic zero field. Let $n \geq 0$ be an integer. Then the canonical map*

$$\{K_n^M(R[x]/(x^m), (x))\}_m \rightarrow \{K_n(R[x]/(x^m), (x))\}_m$$

of pro-abelian groups is an isomorphism. In particular,

$$\{K_n(R[x]/(x^m), (x))^{(p)}\}_m = 0$$

for $p \neq n$.

Let R be any regular semilocal ring containing \mathbb{Q} . Then the Néron–Popescu desingularization theorem says that R is a direct limit of regular semilocal rings $\{R_i\}$, where each R_i is essentially of finite type over \mathbb{Q} (see [Swan 1998, Theorem 1.1]). One knows from [Rülling 2007, Lemma 1.17] that if each

$$\{K_n^M(R_i[x]/(x^m), (x))\}_{n,m \geq 1}$$

is a restricted Witt complex over R_i , then $\{K_n^M(R[x]/(x^m), (x))\}_{n,m \geq 1}$ is a restricted Witt-complex over $\varinjlim_{i \geq 1} R_i = R$ (see [Rülling 2007, Definition 1.14] for the definition of a restricted Witt-complex). On the other hand, it was shown in [Krishna and Park 2016, Theorem 1.2] that each collection $\{\mathrm{TCH}^n(R_i, n; m)\}_{n,m \geq 1}$ is a restricted Witt-complex over R_i . We therefore obtain our next consequence of **Theorem 1.3**.

Corollary 1.5. *Let R be a regular semilocal ring containing \mathbb{Q} . Then the relative Milnor K -theory $\{K_n^M(R[x]/(x^m), (x))\}_{n,m \geq 1}$ is a restricted Witt-complex over R .*

Bloch [1977, Chapter II] had shown (without using the terminology of Witt-complex) that if R is a regular local ring containing a field of characteristic $p > 2$, then the subgroup of the relative Quillen K -theory of truncated polynomial rings over R , generated by Milnor symbols (the symbolic K -theory in the language of Bloch), has the structure of a restricted Witt-complex. The above corollary extends the result of Bloch to characteristic zero.

The last consequence of **Theorem 1.3** is the following. Park and Ünver [2018] proposed a definition of motivic cohomology of truncated polynomial ring $k[x]/(x^m)$ over a field. They showed that these motivic cohomology in the Milnor range coincide with the Milnor K -theory of $k[x]/(x^m)$ when k is a characteristic zero field. **Theorem 1.3** implies that the Milnor range (relative) motivic cohomology of Park–Ünver coincides with the additive higher Chow groups.

1B. Comments and questions. We make a couple of remarks related to the above results.

(1) Since [Theorem 1.1](#) is characteristic-free, one would expect the same to be true for [Theorem 1.3](#) and [Corollary 1.4](#) as well. Our remark is that [Theorem 1.3](#) and [Corollary 1.4](#) are indeed true in all characteristics $\neq 2$. Since the techniques of our proofs in positive characteristics are different from the present paper, they are presented in [\[Gupta and Krishna 2019\]](#).

(2) Our second remark is actually a question. Recall that Chow groups with modulus are supposed to be the motivic cohomology to describe the relative K -theory, just as Bloch's higher Chow groups describe K -theory. Analogous to Bloch's Chow groups, the ones with modulus exist in all bidegrees. However, as we explained earlier, [Theorem 1.3](#) says that the 0-cycles groups with modulus are often enough to describe all of relative K -theory in the setting of pro-abelian groups. One can therefore ask the following.

Question 1.6. Let R be a regular semilocal ring essentially of finite type over a perfect field. Let $n, p \geq 1$ be two integers such that $n \neq p$. Is $\{\mathrm{TCH}^p(R, n; m)\}_m = 0$?

Note that this question is consistent with the second part of [Corollary 1.4](#). Note also that it is already shown in [\[Krishna and Park 2020b\]](#) that the answer to this question is yes when $p > n$. So the open case is when $p < n$. We hope to address this question in a future work. Reader may recall that when $p < n/2$, the additive version of the deeper Beilinson–Soulé vanishing conjecture says that $\mathrm{TCH}^p(R, n; m)$ should vanish for every $m \geq 1$.

1C. An outline of the paper. We end this section with a brief outline of the layout of this text. In [Sections 2](#) and [3](#), we set up our notations, recollect the main objects of study and prove some intermediate results. In [Section 4](#), we define the cycle class map on the group of generators of 0-cycles. Our definition of the cycle class map is *a priori* completely different from the one in [\[Binda 2018; Levine 1994\]](#). The novelty of the new construction is that it is very explicit in nature and, therefore, it becomes possible to check that it factors through the rational equivalence. We also prove in this section that the cycle class map is natural for suitable proper and flat morphisms. One can check that this map does coincide with more abstractly defined maps of [\[Binda 2018; Levine 1994\]](#) on generators. But we do not discuss this in this paper (see however [Section 4D](#) for a sketch of this).

We break the proof of [Theorem 1.1](#) into two steps. In [Section 5](#), we prove it for a very specific type of curves using the results of [Section 2](#). This is the technical part of the proof of [Theorem 1.1](#). It turns out that the general case can be reduced to the above type using the results of [Section 2D](#). This is done in [Section 6](#). The

idea that we have to increase the modulus for factoring the cycle class map through the rational equivalence is already evident in the technical results of [Section 3B](#).

[Sections 7 and 8](#) constitute the heart of the proof of [Theorem 1.3](#). In [Section 7](#), we provide some strong relations between the additive 0-cycles, relative Milnor K -theory and the big de Rham–Witt complex. In particular, we show that it suffices to know the image of certain very specific 0-cycles under the cycle class map in order to show that it factors through the relative Milnor K -theory of a truncated polynomial ring (see [Lemma 7.3](#)). In [Section 8](#), we give an explicit description of the relative Milnor K -theory in terms of the module of Kähler differentials (see [Lemma 8.4](#)). This allows us to establish the isomorphism between the additive higher Chow groups of 0-cycles and the relative Milnor K -theory.

To pass to Quillen K -theory, we prove a vanishing theorem (see [Proposition 9.5](#)) using some results of [[Krishna 2010](#)]. This allows us to show that the additive 0-cycle groups for fields are isomorphic to the relative K -theory in the setting of pro-abelian groups. In [Section 10](#), we extend the results of [Section 9](#) to regular semilocal rings using the main results of [[Krishna and Park 2020a](#)]. The last section is the appendix which contains some auxiliary results on the relation between Milnor and Quillen K -theory of fields. These results are used in the main proofs.

2. The relative K -theory and cycles with modulus

Here we fix our notations and prove some basic results in relative algebraic K -theory. We shall also recall the definition of the higher Chow groups with modulus.

2A. Notations. We shall in general work with schemes over an arbitrary base field k . We shall specify further conditions on k as and when it is required. We let \mathbf{Sch}_k denote the category of separated finite type schemes over k . Recall that $X \in \mathbf{Sch}_k$ is called regular if $\mathcal{O}_{X,x}$ is a regular local ring for all points $x \in X$. We let \mathbf{Sm}_k denote the full subcategory of \mathbf{Sch}_k consisting of regular schemes. For $X, Y \in \mathbf{Sch}_k$, we shall denote the product $X \times_k Y$ simply by $X \times Y$. For any point $x \in X$, we shall let $k(x)$ denote the residue field of x . For a reduced scheme $X \in \mathbf{Sch}_k$, we shall let X^N denote the normalization of X . For $p \geq 0$, we shall denote the set of codimension p points of a scheme X by $X^{(p)}$. For an affine scheme $X \in \mathbf{Sch}_k$, we shall let $k[X]$ denote the coordinate ring of X .

We shall let $\bar{\square}$ denote the projective space $\mathbb{P}_k^1 = \text{Proj}(k[Y_0, Y_1])$ and let $\square = \bar{\square} \setminus \{1\}$. We shall let $\mathbb{A}_k^n = \text{Spec}(k[y_1, \dots, y_n])$ be the open subset of $\bar{\square}^n$, where (y_1, \dots, y_n) denotes the coordinate system of $\bar{\square}^n$ with $y_j = Y_1^j/Y_0^j$. Given a rational map $f: X \dashrightarrow \bar{\square}^n$ in \mathbf{Sch}_k and a point $x \in X$ lying in the domain of definition of f , we shall let $f_i(x) = (y_i \circ f)(x)$, where $y_i: \bar{\square}^n \rightarrow \bar{\square}$ is the i -th projection. For any $1 \leq i \leq n$ and $t \in \bar{\square}(k)$, we let $F_{n,i}^t$ denote the closed subscheme of $\bar{\square}^n$ given by $\{y_i = t\}$. We let $F_n^t = \sum_{i=1}^n F_{n,i}^t$.

By a closed pair (X, D) in \mathbf{Sch}_k , we shall mean a closed immersion $D \hookrightarrow X$ in \mathbf{Sch}_k , where X is reduced and D is an effective Cartier divisor on X . We shall write $X \setminus D$ as X° . We shall say that (X, D) is a modulus pair if $X^\circ \in \mathbf{Sm}_k$. If (X, D) is a closed pair, we shall let $mD \subset X$ be the closed subscheme defined by the sheaf of ideals \mathcal{I}_D^m , where D is defined by the sheaf of ideals \mathcal{I}_D .

All rings in this text will be commutative and Noetherian. For such a ring R and an integer $m \geq 0$, we shall let $R_m = R[t]/(t^{m+1})$ denote the truncated polynomial algebra over R . We shall write $\mathrm{Spec}(R[t_1, \dots, t_n])$ as \mathbb{A}_R^n . The tensor product $M \otimes_{\mathbb{Z}} N$ will be denoted simply as $M \otimes N$. Tensor products over other bases will be explicitly indicated.

2B. The category of pro-objects. By a pro-object in a category \mathcal{C} , we shall mean a sequence of objects $\{A_m\}_{m \geq 0}$ together with a map $\alpha_m^A: A_{m+1} \rightarrow A_m$ for each $m \geq 0$. We shall write this object often as $\{A_m\}$. We let $\mathrm{Pro}(\mathcal{C})$ denote the category of pro-objects in \mathcal{C} with the morphism set given by

$$\mathrm{Hom}_{\mathrm{pro}(\mathcal{C})}(\{A_m\}, \{B_m\}) = \varprojlim_n \varinjlim_m \mathrm{Hom}_{\mathcal{C}}(A_m, B_n). \quad (2.1)$$

In particular, giving a morphism f as above is equivalent to finding a function $\lambda: \mathbb{N} \rightarrow \mathbb{N}$, a map $f_n: A_{\lambda(n)} \rightarrow B_n$ for each $n \geq 0$ such that for each $n' \geq n$, there exists $l \geq \lambda(n), \lambda(n')$ so that the diagram

$$\begin{array}{ccccc} A_l & \longrightarrow & A_{\lambda(n')} & \xrightarrow{f_{n'}} & B_{n'} \\ & \searrow & & & \downarrow \\ & & A_{\lambda(n)} & \xrightarrow{f_n} & B_n \end{array} \quad (2.2)$$

is commutative, where the unmarked arrows are the structure maps of $\{A_m\}$ and $\{B_m\}$. We shall say that f is strict if λ is the identity function. If \mathcal{C} admits all sequential limits, we shall denote the limit of $\{A_m\}$ by $\varprojlim_m A_m \in \mathcal{C}$. If \mathcal{C} is an abelian category, then so is $\mathrm{Pro}(\mathcal{C})$. We refer the reader to [Artin and Mazur 1986, Appendix 4] for further details about $\mathrm{Pro}(\mathcal{C})$.

2C. The relative algebraic K -theory. Given a closed pair (X, D) in \mathbf{Sch}_k , we let $K(X, D)$ be the homotopy fiber of the restriction map between the Thomason–Trobaugh nonconnective algebraic K -theory spectra $K(X) \rightarrow K(D)$. We shall let $K_i(X)$ denote the homotopy groups of $K(X)$ for $i \in \mathbb{Z}$. We similarly define $K_i(X, D)$. We shall let $K^D(X)$ denote the homotopy fiber of the restriction map $K(X) \rightarrow K(X \setminus D)$. Note that $K^D(X)$ does not depend on the subscheme structure of D but $K(X, D)$ does. Note also that if $D' \subset X$ is another closed subscheme such that $D \cap D' = \emptyset$, then $K^D(X)$ is canonically homotopy equivalent to the homotopy fiber $K^D(X, D')$ of the restriction map $K(X, D') \rightarrow K(X \setminus D, D')$.

If (X, D) is a closed pair, we have the canonical restriction map $K(X, (m + 1)D) \rightarrow K(X, mD)$. In particular, this gives rise a pro-spectrum $\{K(X, mD)\}$ and a levelwise homotopy fiber sequence of pro-spectra

$$\{K(X, mD)\} \rightarrow K(X) \rightarrow \{K(mD)\}. \tag{2.3}$$

If $X = \text{Spec}(R)$ is affine and $D = V(I)$, we shall often write $K(X, mD)$ as $K(R, I^m)$ and $K(X)$ as $K(R)$. For a ring R , we shall let $\tilde{K}(R_m)$ denote the reduced K -theory of R_m , defined as the homotopy fiber of the augmentation map $K(R_m) \rightarrow K(R)$. Observe that there exists a canonical decomposition $K(R_m) \cong \tilde{K}(R_m) \times K(R)$.

Suppose that R is a regular semilocal ring. Let $f(t) \in R[t]$ be a polynomial such that $f(0) \in R^\times$ and let $Z = V(f(t)) \subset \mathbb{A}_R^1$ be the closed subscheme defined by $f(t)$. Since $Z \cap \{0\} = \emptyset$, the composite map $K^Z(\mathbb{A}_R^1) \rightarrow K(\mathbb{A}_R^1) \rightarrow K((m + 1)\{0\})$ is null-homotopic for all $m \geq 0$. Hence, there is a factorization $K^Z(\mathbb{A}_R^1) \rightarrow K(\mathbb{A}_R^1, (m + 1)\{0\}) \rightarrow K(\mathbb{A}_R^1)$. Let $[\mathcal{O}_Z]$ denote the fundamental class of Z in $K_0^Z(\mathbb{A}_R^1)$ (see [Thomason and Trobaugh 1990, Exercise 5.7]). Note that Z may not be reduced or irreducible. Let α_Z denote the image of $[\mathcal{O}_Z]$ under the map $K_0^Z(\mathbb{A}_R^1) \rightarrow K_0(\mathbb{A}_R^1, (m + 1)\{0\})$. Let $\partial_n : \tilde{K}_n(R_m) \rightarrow K_{n-1}(\mathbb{A}_R^1, (m + 1)\{0\})$ denote the connecting homomorphism obtained by considering the long exact homotopy groups sequence associated to (2.3). The homotopy invariance of K -theory on \mathbf{Sm}_k shows that this map is an isomorphism. For $g(t) \in R[t]$, let $\overline{g(t)}$ denote its image in R_m .

Lemma 2.1. *Given $Z = V((f(t)))$ as above, we have*

$$\alpha_Z = \overline{\partial_1((f(0))^{-1}f(t))}.$$

Proof. Since $f(0) \in R^\times$, we note that $Z = \overline{V((f(0))^{-1}f(t))}$. We let $g(t) = (f(0))^{-1}f(t)$ so that $g(0) = 1$ and therefore $\overline{g(t)} \in \tilde{K}_1(R_m)$. We let $\Lambda = \{(a, b) \in R[t] \times R[t] \mid a - b \in (t^{m+1})\}$ be the double of $R[t]$ along the ideal (t^{m+1}) as in [Milnor 1971, Chapter 2]. Let $p_1 : \Lambda \rightarrow R[t]$ be the first projection. Then recall from [Milnor 1971, Chapter 6] that $K_0(R[t], (t^{m+1})) \cong \text{Ker}((p_1)_\# : K_0(\Lambda) \rightarrow K_0(R[t]))$ and [Milnor 1971, Chapter 3] shows that $\partial_1(u) = [M(u)] - [\Lambda] \in K_0(R[t], (t^{m+1}))$, where $M(u)$ is the rank one projective Λ -module given by $M(u) = \{(x, y) \in R[t] \times R[t] \mid u\bar{x} = \bar{y} \text{ in } R_m^\times\}$ for any $u \in R_m^\times$.

Let $u = \overline{g(t)}$ and let $M = M(\overline{g(t)})$. Let $p_2 : M \rightarrow R[t]/(g(t))$ denote the composition of the second projection $M \rightarrow R[t]$ with the surjection $R[t] \twoheadrightarrow R[t]/(g(t))$. It is then easy to see that the sequence

$$0 \rightarrow \Lambda \xrightarrow{\theta} M \xrightarrow{p_2} R[t]/(g(t)) \rightarrow 0$$

is a short exact sequence of Λ -modules if we let $\theta((a, b)) = (a, bg(t)) \in M$. In particular, we get $[\mathcal{O}_Z] = [V((g(t)))] = [M] - [\Lambda] = \partial_1(\overline{g(t)})$. This proves the lemma. \square

2D. The projection formula for relative K -theory. Let (X, D) be a modulus pair in \mathbf{Sch}_k and let S_X be the double of X along D . Recall from [Binda and Krishna 2018, §2.1] that S_X is the pushout $X \sqcup_D X$ of the diagram of schemes $X \leftarrow D \hookrightarrow X$. On each affine open subset $U \subset X$, the double is the spectrum S_U of the ring $\{(a, b) \in \mathcal{O}_U(U) \times \mathcal{O}_U(U) \mid a - b \in \mathcal{I}_D(U)\}$, where $\mathcal{I}_D \subset \mathcal{O}_X$ is the ideal sheaf of D . We have two inclusions $\iota_{\pm}: X \hookrightarrow S_X$ and a projection $p: S_X \rightarrow X$ such that $p \circ \iota_{\pm} = \text{id}_X$. In particular, there is a canonical decomposition $K(S_X) \cong K(S_X, X_-) \times K(X)$. There is an inclusion of modulus pairs $(X, D) \hookrightarrow (S_X, X_-)$, with respect to the embedding $X_+ \hookrightarrow S_X$. This yields the pull-back map $\iota_+^*: K(S_X, X_-) \rightarrow K(X, D)$.

We now let $u: Z \hookrightarrow X$ be a closed immersion such that $Z \cap D = \emptyset$. This gives rise to a closed embedding $Z \hookrightarrow X \xrightarrow{\iota_+} S_X$ such that $Z \cap D = Z \cap X_- = \emptyset$. Since $Z \cap D = \emptyset$, the push-forward map (which exists because $Z \subset X_{\text{reg}}$) $u_*: K(Z) \rightarrow K(X)$ composed with the restriction $K(X) \rightarrow K(D)$ is null-homotopic. Hence, there is a canonical factorization $K(Z) \rightarrow K(X, D) \rightarrow K(X)$ of the push-forward map. We shall denote the map $K(Z) \rightarrow K(X, D)$ also by u_* . It is clearly functorial in (X, D) and Z . Recall also that $K(Z)$ and $K(X, D)$ are module spectra over the ring spectrum $K(X)$ (e.g., see [Thomason and Trobaugh 1990, Chapter 3]). We shall need to know the following result about the map u_* in the proof of Lemma 9.2.

Lemma 2.2. *The push-forward map $u_*: K_*(Z) \rightarrow K_*(X, D)$ is $K_*(X)$ -linear.*

Proof. Since $Z \subset S_X \setminus X_-$, we also have the push-forward map $v_*: K(Z) \rightarrow K(S_X, X_-)$, where we let $v = \iota_+ \circ u$. Suppose we know that

$$u_* = \iota_+^* \circ v_*: K(Z) \xrightarrow{v_*} K(S_X, X_-) \xrightarrow{\iota_+^*} K(X, D)$$

and the lemma holds for v_* . Then for any $\alpha \in K_*(X)$ and $\beta \in K_*(Z)$, we get

$$u_*(u^*(\alpha)\beta) = \iota_+^*(v_*(v^*p^*(\alpha)\beta)) = \iota_+^*(p^*(\alpha)v_*(\beta)) = (p \circ \iota_+)^*(\alpha)u_*(\beta) = \alpha u_*(\beta).$$

We thus have to show the following.

- (1) The lemma holds for the inclusion $Z \hookrightarrow S_X$, and
- (2) $u_* = \iota_+^* \circ v_*$.

To prove (1), we can use that the map $K_*(S_X, X_-) \rightarrow K_*(S_X)$ is a split inclusion (as we saw above). Using this and the fact that $K(S_X, X_-) \rightarrow K(S_X)$ is $K(S_X)$ -linear, it suffices to prove (1) for the composite push-forward map $v_*: K(Z) \rightarrow K(S_X)$. But we already saw above that $K(Z)$ is a module spectrum over $K(S_X)$.

We now prove (2). By the definition of the push-forward maps to the relative K -theory, we have factorizations

$$\begin{CD}
 K(Z) @>>> K^Z(S_X, X_-) @>>> K(S_X, X_-) \\
 @| @VV \iota_+^* V @VV \iota_+^* V \\
 K(Z) @>>> K^Z(X, D) @>>> K(X, D),
 \end{CD} \tag{2.4}$$

such that the square on the right is commutative and the top (resp. bottom) composite arrow is v_* (resp. u_*). Hence, it suffices to show that the left square is commutative.

For showing this, we use the diagram

$$\begin{CD}
 K(Z) @>>> K^Z(S_X, X_-) @>\cong>> K^Z(S_X \setminus X_-) \\
 @| @VV \iota_+^* V @VV \iota_+^* V \\
 K(Z) @>>> K^Z(X, D) @>\cong>> K^Z(X \setminus D),
 \end{CD} \tag{2.5}$$

where the horizontal arrows on the right are the restriction maps induced by the open immersions of modulus pairs. In particular, the square on the right is commutative. The right horizontal arrows are homotopy equivalences by the excision theorem. Hence, it suffices to show that the composite square in (2.5) commutes.

To see this, we note that the composite horizontal arrows in (2.5) have the factorizations:

$$\begin{CD}
 K(Z) @>>> G(Z) @>>> K^Z(S_X \setminus X_-) \\
 @| @| @VV \cong \downarrow \iota_+^* V \\
 K(Z) @>>> G(Z) @>>> K^Z(X \setminus D),
 \end{CD} \tag{2.6}$$

where $G(Z)$ is the K -theory of pseudocoherent complexes on Z [Thomason and Trobaugh 1990, Chapter 3] and $K(Z) \rightarrow G(Z)$ is the canonical map. We are now done because the square on the right in (2.6) clearly commutes. \square

2E. The 0-cycles with modulus. Let k be a field and let (X, D) be an equidimensional closed pair in \mathbf{Sch}_k of dimension $d \geq 1$. We recall the definition of the higher Chow groups with modulus from [Binda and Saito 2019] or [Krishna and Park 2017a]. For any integers $n, p \geq 0$, we let $\underline{z}^p(X | D, n)$ denote the free abelian group on the set of integral closed subschemes of $X \times \square^n$ of codimension p satisfying the following.

- (1) Z intersects $X \times F$ properly for each face $F \subset \square^n$.

- (2) If \bar{Z} is the closure of Z in $X \times \bar{\square}^n$ and $v : \bar{Z}^N \rightarrow X \times \bar{\square}^n$ is the canonical map from the normalization of \bar{Z} , then the inequality (called the modulus condition)

$$v^*(D \times \bar{\square}^n) \leq v^*(X \times F_n^1)$$

holds in the set of Weil divisors on \bar{Z}^N .

An element of the group $\underline{z}^p(X | D, n)$ will be called an admissible cycle. It is known that $\{n \mapsto \underline{z}^p(X | D, n)\}$ is a cubical abelian group (see [Krishna and Levine 2008, §1]). We denote this by $\underline{z}^p(X | D, *)$. We let

$$z^p(X | D, *) = \frac{\underline{z}^p(X | D, *)}{\underline{z}_{\text{degn}}^p(X | D, *)},$$

where $\underline{z}_{\text{degn}}^p(X | D, *)$ is the degenerate part of the cubical abelian group $\underline{z}^p(X | D, *)$. For $n \geq 0$, we let

$$\text{CH}^p(X | D, n) = H_n(z^p(X | D, *))$$

and call them the higher Chow groups with modulus of (X, D) . The direct sum

$$\text{CH}_0(X | D, *) := \bigoplus_{n \geq 0} \text{CH}^{d+n}(X | D, n) = \bigoplus_{n \geq 0} \text{CH}_{-n}(X | D, n) \tag{2.7}$$

is called the *higher Chow group of 0-cycles with modulus*. The subject of this paper is to study the relation between $\text{CH}_0(X | D, *)$ and the relative K -theory $K_*(X, D)$.

We recall for the reader that the groups $\text{CH}^p(X | D, *)$ satisfy the flat pull-back and the proper push-forward properties under certain conditions. We refer the reader to [Binda and Saito 2019] or [Krishna and Park 2017a] for these and other properties of the Chow groups with modulus.

3. The Milnor K -theory

Recall that for a semilocal ring R , the Milnor K -group $K_i^M(R)$ is defined to be the i -th graded piece of the graded Milnor K -theory \mathbb{Z} -algebra $K_*^M(R)$. The latter is defined to be the quotient of the tensor algebra $T_*(R^\times)$ by the two-sided graded ideal generated by homogeneous elements $\{a \otimes (1 - a) \mid a, 1 - a \in R^\times\}$. The image of an element $a_1 \otimes \cdots \otimes a_n \in T_n(R^\times)$ in $K_n^M(R)$ is denoted by the Milnor symbol $\underline{a} = \{a_1, \dots, a_n\}$. If $I \subset R$ is an ideal, the relative Milnor K -theory $K_i^M(R, I)$ is defined to be the kernel of the natural surjection $K_n^M(R) \rightarrow K_n^M(R/I)$. It follows from [Kato and Saito 1986, Lemma 1.3.1] that $K_n^M(R, I)$ is generated by Milnor symbols $\{a_1, \dots, a_n\}$, where $a_i \in \text{Ker}(R^\times \rightarrow (R/I)^\times)$ for some $1 \leq i \leq n$, provided R is a finite product of local rings.

The product structures on the Milnor and Quillen K -theories yield a natural graded ring homomorphism $\psi_R: K_*^M(R) \rightarrow K_*(R)$. If $I \subset R$ is an ideal, we have a natural isomorphism $K_1^M(R, I) \cong \widehat{K}_1(R, I)$, where $\widehat{K}_*(R, I)$ is the group $\text{Ker}(K_*(R) \rightarrow K_*(R/I))$. Using the module structure on $\widehat{K}_*(R, I)$ over $K_*(R)$ and the ring homomorphism $K_*^M(R) \rightarrow K_*(R)$, we obtain a natural graded $K_*^M(R)$ -linear map $\psi_{R|I}: K_*^M(R, I) \rightarrow \widehat{K}_*(R, I)$. The cup product on Milnor K -theory yields maps $K_n^M(R) \otimes K_{n'}^M(R, I) \rightarrow K_{n+n'}^M(R, I)$. In the sequel, we shall loosely denote the image of this map also by $K_n^M(R)K_{n'}^M(R, I)$ (e.g., see [Lemma 3.3](#)).

3A. The improved Milnor K -theory. If R is a semilocal ring whose residue fields are not infinite, then the Milnor K -theory $K_*^M(R)$ does not have good properties. For example, the Gersten conjecture does not hold even if R is a regular local ring containing a field. If R is a finite product of local rings containing a field, Kerz [2010] defined an improved version of Milnor K -theory, which is denoted as $\widehat{K}_*^M(R)$. This is a graded commutative ring and there is natural map of graded commutative rings $\eta^R: K_*^M(R) \rightarrow \widehat{K}_*^M(R)$. For an ideal $I \subset R$, we let $\widehat{K}_*^M(R, I) = \text{Ker}(\widehat{K}_*^M(R) \rightarrow \widehat{K}_*^M(R/I))$. We thus have a natural map $K_*^M(R, I) \rightarrow \widehat{K}_*^M(R, I)$. We state some basic facts about $\widehat{K}_*^M(R)$ in the following result and refer the reader to [[Kerz 2010](#)] for proofs.

Proposition 3.1. *Let R be a finite product of local rings containing a field. Then the map $\eta^R: K_*^M(R) \rightarrow \widehat{K}_*^M(R)$ has the following properties:*

- (1) η^R is surjective.
- (2) η_n^R is an isomorphism for all $n \geq 0$ if R is a field.
- (3) η_n^R is an isomorphism for $n \leq 1$.
- (4) η_n^R is an isomorphism for all n if each residue fields of R are infinite.
- (5) The natural map $K_n^M(R) \rightarrow K_n(R)$ factors through η_n^R .
- (6) The map $\widehat{K}_2^M(R) \rightarrow K_2(R)$ is an isomorphism.
- (7) The Gersten conjecture holds for $\widehat{K}_n^M(R)$.

We now let R be a regular semilocal ring (not necessarily a product of local rings) containing a field. Let F denote the total quotient ring (a product of fields) of R . Recall from [[Kato 1986](#), §1] that there is a (Gersten) complex of abelian groups

$$\begin{aligned}
 K_n^M(F) &\rightarrow \bigoplus_{\text{ht}(\mathfrak{p})=1} K_{n-1}^M(k(\mathfrak{p})) \rightarrow \cdots \\
 &\rightarrow \bigoplus_{\text{ht}(\mathfrak{p})=n-1} K_1^M(k(\mathfrak{p})) \rightarrow \bigoplus_{\text{ht}(\mathfrak{p})=n} K_0^M(k(\mathfrak{p})). \quad (3.1)
 \end{aligned}$$

We let $\widehat{K}_n^M(R)$ denote the kernel of the boundary map

$$\partial : K_n^M(F) \rightarrow \bigoplus_{\text{ht}(\mathfrak{p})=1} K_{n-1}^M(k(\mathfrak{p}))$$

in (3.1). For any $X \in \mathbf{Sch}_k$, the improved Milnor K -theory Zariski sheaf $\widehat{\mathcal{K}}_{n,X}^M$ was defined in [Kerz 2010] whose stalk at a point $x \in X$ is $\widehat{K}_n^M(\mathcal{O}_{X,x})$. As (3.1) gives rise to a resolution of $\widehat{K}_n^M(R_{\mathfrak{p}})$ for every prime ideal $\mathfrak{p} \subset R$ by Proposition 3.1 (7), it follows that $\widehat{K}_n^M(R)$ coincides with the group of global sections of the sheaf $\widehat{\mathcal{K}}_{n,X}^M$ on $X = \text{Spec}(R)$.

Since the composite map $K_n^M(R) \rightarrow K_n^M(F) \rightarrow K_{n-1}^M(k(\mathfrak{p}))$ is well known to be zero for every height one prime ideal $\mathfrak{p} \subset R$, it follows from the definition of $\widehat{K}_n^M(R)$ and the Gersten resolution of Quillen K -theory that there are natural maps

$$K_n^M(R) \rightarrow \widehat{K}_n^M(R) \xrightarrow{\psi_R} K_n(R). \tag{3.2}$$

Suppose now that R is a regular semilocal integral domain of dimension one containing a field and $I \subset R$ is an ideal of height one. Then R/I is a finite product of Artinian local rings. In particular, the improved Milnor K -theory $\widehat{K}_*^M(R/I)$ is defined. We can write $R/I = \prod_{i=1}^r R_{\mathfrak{m}_i}/IR_{\mathfrak{m}_i}$, where $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ are the minimal primes of I . We thus have the canonical maps

$$\widehat{K}_n^M(R) \hookrightarrow \prod_{i=1}^r \widehat{K}_n^M(R_{\mathfrak{m}_i}) \twoheadrightarrow \prod_{i=1}^r \widehat{K}_n^M(R_{\mathfrak{m}_i}/IR_{\mathfrak{m}_i}) \xleftarrow{\cong} \widehat{K}_n^M(R/I), \tag{3.3}$$

where the first arrow is induced from the definition of $\widehat{K}_n^M(R)$ and the Gersten resolutions of the improved Milnor K -theory of the localizations of R . We define the relative improved Milnor K -group $\widehat{K}_n^M(R, I)$ as the kernel of the composite map. Note that this agrees with the relative improved Milnor K -groups defined earlier if R is a product of local rings.

Note that (3.3) also shows that the diagram

$$\begin{array}{ccccc} K_n^M(R) & \longrightarrow & \widehat{K}_n^M(R) & \longrightarrow & K_n(R) \\ \downarrow & & \downarrow & & \downarrow \\ K_n^M(R/I) & \longrightarrow & \widehat{K}_n^M(R/I) & \longrightarrow & K_n(R/I) \end{array} \tag{3.4}$$

commutes. We therefore get the canonical maps of relative K -theories

$$K_*^M(R, I) \rightarrow \widehat{K}_*^M(R, I) \rightarrow \widehat{K}_*(R, I), \tag{3.5}$$

where recall that $\widehat{K}_*(R, I) = \text{Ker}(K_*(R) \rightarrow K_*(R/I))$.

3B. Some results on Milnor- K -theory. We shall need few results on the Milnor K -theory of discrete valuation rings. For a discrete valuation ring R with field of fractions F , we shall let $\text{ord} : F^\times \rightarrow \mathbb{Z}$ denote the valuation map. We begin with the following elementary computation in Milnor K -theory. We shall use the additive notation for the group operation of the Milnor K -theory.

Lemma 3.2. *Let R be a semilocal integral domain with field of fractions F . Let a, b, s, t be nonzero elements of R such that $1 + as, 1 + bt \neq 0$. Then we have the following identity in $K_2^M(F)$.*

$$\{1+as, 1+bt\} = \begin{cases} -\left\{1 + \frac{ab}{1+as}st, -as(1+bt)\right\} & \text{if } 1 + (1+bt)as \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

Proof. Suppose first that $1 + (1 + bt)as = 0$. Then we have

$$\{1 + as, 1 + bt\} = \{1 + as, (-as)^{-1}\} = -\{1 + as, -as\} = 0.$$

Otherwise, we write

$$\begin{aligned} & \left\{1 + \frac{ab}{1+as}st, -as(1+bt)\right\} \\ &= \left\{1 + \frac{ab}{1+as}st, -as\right\} + \left\{1 + \frac{ab}{1+as}st, (1+bt)\right\} \\ &= \left\{1 + \frac{ab}{1+as}st, -as\right\} \\ &\quad + \{1 + (1+bt)as, 1+bt\} - \{1 + as, 1+bt\} \\ &= \{1 + (1+bt)as, -as\} - \{1 + as, -as\} \\ &\quad + \{1 + (1+bt)as, 1+bt\} - \{1 + as, 1+bt\} \\ &= \{1 + (1+bt)as, -(1+bt)as\} \\ &\quad - \{1 + as, -as\} - \{1 + as, 1+bt\} \\ &= \{1 - u, u\} - \{1 - v, v\} - \{1 + as, 1+bt\} \\ &= -\{1 + as, 1+bt\}, \end{aligned} \tag{3.7}$$

where we let $u = -(1 + bt)as$ and $v = -as$. This proves the lemma. □

Lemma 3.3. *Let R be a discrete valuation ring with maximal ideal \mathfrak{m} and field of fractions F . For $m, n \geq 1$, let $K_n^M(F, \mathfrak{m})$ denote the subgroup of $K_n^M(F)$ generated by Milnor symbols $\{y_1, \dots, y_n\} \in K_n^M(F)$ such that $\sum_{i=1}^n \text{ord}(y_i - 1) \geq m$. Then for any $n \geq 0$, we have*

$$K_{n+1}^M(F, \mathfrak{m}) \subseteq (1 + \mathfrak{m}^m)K_n^M(F).$$

Proof. Note that for $n = 0$, we actually have $K_1^M(F, m) = 1 + \mathfrak{m}^m$ and this is obvious from the definition of $K_1^M(F, m)$. We shall prove $n \geq 1$ case by induction on n . We let π denote a uniformizing parameter of R . We can write $y_i = 1 + u_i \pi^{m_i}$ for some $u_i \in R^\times$ and $m_i \in \mathbb{Z}$ for $1 \leq i \leq n$. We first observe that if $m_i \geq m$ for some $i \geq 1$, then $y_i = 1 + u_i \pi^{m_i} \in 1 + \mathfrak{m}^m$ and we are done.

We now assume that $n = 1$. In this case, if some $m_i \leq 0$, then we must have that some $m_j \geq m$ and we are done as above. We can therefore assume that $0 < m_1, m_2 < m$. In this case, Lemma 3.2 says that $\{y_1, y_2\}$ is either zero or it is $- \{1 + u_1 u_2 y_1^{-1} \pi^{m_1+m_2}, -u_1 y_2 \pi^{m_1}\}$. Since $m_1 \geq 0$, we see that $y_1^{-1} \in R^\times$. In particular, $1 + u_1 u_2 y_1^{-1} \pi^{m_1+m_2} \in 1 + \mathfrak{m}^{m_1+m_2}$. We therefore get $\{y_1, y_2\} \in (1 + \mathfrak{m}^m) K_1^M(F)$.

If $n \geq 2$, we have must have $m_i \geq 0$ for some $1 \leq i \leq n$ as $m \geq 0$. Since the permutation of coordinates of a Milnor symbol only changes its sign in the Milnor K -group, we can assume that $m_1 \geq 0$ so that $y_1 \in R^\times$. We can now write $\{y_1, \dots, y_n\} = \{y_1, y_2\} \cdot \{y_3, \dots, y_n\}$. We have seen before that the term $\{y_1, y_2\}$ is either zero or we have

$$\begin{aligned} \{y_1, \dots, y_n\} &= \{1 + u_1 \pi^{m_1}, 1 + u_2 \pi^{m_2}\} \cdot \{y_3, \dots, y_n\} \\ &= - \{1 + u_1 u_2 y_1^{-1} \pi^{m_1+m_2}, -u_1 \pi^{m_1} y_2\} \cdot \{y_3, \dots, y_n\} \\ &= \{-u_1 y_2 \pi^{m_1}\} \cdot \{1 + u_1 u_2 y_1^{-1} \pi^{m_1+m_2}, y_3, \dots, y_n\}. \end{aligned}$$

Since $y_1 \in R^\times$, it follows that $y'_2 := 1 + u_1 u_2 y_1^{-1} \pi^{m_1+m_2} \in 1 + \mathfrak{m}^{m_1+m_2}$. In particular, we see that $\text{ord}(y'_2 - 1) + \sum_{i=3}^n \text{ord}(y_i - 1) \geq \sum_{i=1}^n m_i \geq m$. Hence, the induction hypothesis implies that $\{y'_2, y_3, \dots, y_n\} \in (1 + \mathfrak{m}^m) K_{n-1}^M(F)$. This implies that $\{y_1, \dots, y_n\} = \{-u_1 y_2 \pi^{m_1}\} \cdot \{y'_2, y_3, \dots, y_n\} \in (1 + \mathfrak{m}^m) K_n^M(F)$. \square

Lemma 3.4. *Let R be a discrete valuation ring containing a field. Let \mathfrak{m} and F denote the maximal ideal and the field of fractions of R , respectively. Then the following hold for every integer $n \geq 0$:*

- (1) $(1 + \mathfrak{m}) K_n^M(F) \subseteq \widehat{K}_{n+1}^M(R)$.
- (2) $(1 + \mathfrak{m}^{m+n}) K_n^M(F) \subseteq (1 + \mathfrak{m}^m) \widehat{K}_n^M(R)$ for all $m \geq 1$.

Proof. We shall prove the lemma by induction on n . As the base case $n = 0$ trivially follows, we shall assume that $n \geq 1$. Suppose we show that

$$(1 + \mathfrak{m}) K_1^M(F) \subseteq (1 + \mathfrak{m}) \widehat{K}_1^M(R) \subset \widehat{K}_2^M(R). \tag{3.8}$$

We will then have

$$\begin{aligned} (1 + \mathfrak{m}) K_n^M(F) &\subseteq (1 + \mathfrak{m}) \widehat{K}_1^M(R) K_{n-1}^M(F) \\ &= \widehat{K}_1^M(R) (1 + \mathfrak{m}) K_{n-1}^M(F) \subseteq {}^1 \widehat{K}_1^M(R) \widehat{K}_n^M(R) \subseteq \widehat{K}_{n+1}^M(R), \end{aligned}$$

where \subseteq^1 holds by induction on n . This will prove (1).

Similarly, suppose we show for all $m \geq 1$ that

$$(1 + \mathfrak{m}^{m+1})K_1^M(F) \subseteq (1 + \mathfrak{m}^m)\widehat{K}_1^M(R) \subset \widehat{K}_2^M(R). \quad (3.9)$$

Then for any $m \geq 1$ and $n \geq 2$, we will have

$$\begin{aligned} (1 + \mathfrak{m}^{m+n})K_n^M(F) &\subseteq (1 + \mathfrak{m}^{m+n-1})\widehat{K}_1^M(R)K_{n-1}^M(F) \\ &= \widehat{K}_1^M(R)(1 + \mathfrak{m}^{m+n-1})K_{n-1}^M(F) \subseteq^1 \widehat{K}_1^M(R)(1 + \mathfrak{m}^m)\widehat{K}_{n-1}^M(R) \\ &\subseteq (1 + \mathfrak{m}^m)\widehat{K}_n^M(R), \end{aligned}$$

where \subseteq^1 holds by induction on n . This will prove (2). We are therefore left with showing (3.8) and (3.9) in order to prove the lemma.

To prove (3.8), we let π be a uniformizing parameter of R . For $j \in \mathbb{Z}$ and $u, v \in R^\times$, we then have

$$\begin{aligned} \{1 + u\pi, v\pi^j\} &= \{1 + u\pi, v\} + \{1 + u\pi, \pi^j\} \\ &= \{1 + u\pi, v\} + j\{1 + u\pi, \pi\} \\ &= \{1 + u\pi, v\} - j\{1 + u\pi, -u\}, \end{aligned}$$

where the last equality holds because $\{1 + u\pi, -u\pi\} = 0$. It follows that $\{1 + u\pi, v\pi^j\} \in (1 + \mathfrak{m})\widehat{K}_1^M(R)$. If $i \geq 2$, then $\{1 + u\pi^i, v\pi^j\} \in (1 + \mathfrak{m}^{i-1})\widehat{K}_1^M(R) \subseteq (1 + \mathfrak{m})\widehat{K}_1^M(R)$ by (3.9). It remains therefore to prove (3.9).

We now fix $m \geq 1, j \in \mathbb{Z}, a \in R$ and $u \in R^\times$. We consider the element $\{1 + a\pi^{m+1}, u\pi^j\} \in (1 + \mathfrak{m}^{m+1})K_1^M(F)$. We set

$$t = -a\pi^m, \quad v' = (1 + t(-1 - \pi))^{-1} \quad \text{and} \quad v'' = -1 - \pi.$$

With these notations, it is clear that $1 + v't, 1 + v''t \in (1 + \mathfrak{m}^m)$ and $(1 + v't)(1 + v''t) = 1 - \pi t$. In $K_2^M(F)$, we now compute

$$\begin{aligned} \{1 + a\pi^{m+1}, u\pi^j\} &= \{1 + a\pi^{m+1}, u\} + j\{1 + a\pi^{m+1}, \pi\} \\ &= \{1 + a\pi^{m+1}, u\} + j\{1 + (a\pi^m)\pi, \pi\} \\ &= \{1 + a\pi^{m+1}, u\} - j\{1 + (a\pi^m)\pi, -(a\pi^m)\} \\ &= \{1 + a\pi^{m+1}, u\} - j\{1 - \pi t, t\} \\ &= \{1 + a\pi^{m+1}, u\} - j\{(1 + v't)(1 + v''t), t\} \\ &= \{1 + a\pi^{m+1}, u\} - j\{1 + v't, t\} - j\{1 + v''t, t\} \\ &= \{1 + a\pi^{m+1}, u\} + j\{1 + v't, -v'\} + j\{1 + v''t, -v''\} \\ &\in (1 + \mathfrak{m}^m)\widehat{K}_1^M(R). \end{aligned}$$

This proves (3.9) and completes the proof of the lemma. \square

Let us now assume that R is a regular semilocal integral domain of dimension one containing a field. Let $\{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ be the set of maximal ideals of R and let F be the field of fractions of R . If $\underline{m} = (m_1, \dots, m_r)$ is an r -tuple of positive integers, we shall write the relative improved Milnor K -theory $\widehat{K}_*^M(R, \mathfrak{m}_1^{m_1} \cdots \mathfrak{m}_r^{m_r})$ (see (3.3)) as $\widehat{K}_*^M(R, \underline{m})$. For $n \geq 1$, we let

$$K_n^M(F, \underline{m}, \text{sum}) = \bigcap_{i=1}^r (1 + \mathfrak{m}_i^{m_i} R_{\mathfrak{m}_i}) K_{n-1}^M(F).$$

We let $\underline{m} + n = (m_1 + n, \dots, m_r + n)$.

Lemma 3.5. *For every integer $n \geq 1$, we have $K_n^M(F, \underline{m} + n, \text{sum}) \subseteq \widehat{K}_n^M(R, \underline{m})$.*

Proof. If R is local, then Lemma 3.4 says that

$$K_n^M(F, \underline{m} + n, \text{sum}) \subseteq (1 + \mathfrak{m}^n) \widehat{K}_{n-1}^M(R) \subseteq \widehat{K}_n^M(R, \mathfrak{m}^n).$$

In particular, the lemma holds if R is local.

In general, let $y \in K_n^M(F, \underline{m} + n, \text{sum})$. Then it is clear that

$$y \in (1 + \mathfrak{m}^{m_i+n} R_{\mathfrak{m}_i}) K_{n-1}^M(F)$$

for each $1 \leq i \leq r$. It follows by Lemma 3.4 that $y \in (1 + \mathfrak{m}_i^m R_{\mathfrak{m}_i}) \widehat{K}_{n-1}^M(R_{\mathfrak{m}_i}) \subseteq \widehat{K}_n^M(R_{\mathfrak{m}_i})$ for each i . We conclude from (3.1) and Proposition 3.1 that $y \in \widehat{K}_n^M(R)$.

We now consider the diagram

$$\begin{array}{ccc}
 \widehat{K}_n^M(R) & \xrightarrow{\quad\quad\quad} & K_n^M(F) & (3.10) \\
 \downarrow \Delta_R & \searrow \pi & & \downarrow \Delta_F \\
 & & \widehat{K}_n^M(R/\mathfrak{m}_1^{m_1} \cdots \mathfrak{m}_r^{m_r}) & \\
 & & \downarrow \phi & \\
 \bigoplus_{i=1}^r \widehat{K}_n^M(R_{\mathfrak{m}_i}) & \xrightarrow{\quad\quad\quad} & \bigoplus_{i=1}^r K_n^M(F) & \\
 & \searrow \bigoplus_i \pi_i & & \\
 & & \bigoplus_{i=1}^r \widehat{K}_n^M(R_{\mathfrak{m}_i}/\mathfrak{m}_i^{m_i}), &
 \end{array}$$

where ϕ is the last in the sequence of arrows in (3.3).

This diagram is clearly commutative. It follows from the case of local rings shown above that $(\bigoplus_i \pi_i) \circ \Delta_R(y) = 0$. Since ϕ is an isomorphism, we conclude that $\pi(y) = 0$, which is what we wanted to show. \square

Lemma 3.6. *With notations as in Lemma 3.5, we have*

$$K_n^M(F, \underline{m}, \text{sum})_{\mathbb{Q}} \subseteq \widehat{K}_n^M(R, \underline{m})_{\mathbb{Q}}.$$

Proof. The reduction from the semi-local ring R to it being a local ring (dvr) goes through exactly as in the proof of [Lemma 3.5](#) without any change. So the proof of the lemma is eventually reduced to showing the following improved version of [Lemma 3.4 \(2\)](#) for every pair of integers $n \geq 0$ and $m \geq 1$:

$$(1 + \mathfrak{m}^m)K_n^M(F)_{\mathbb{Q}} \subseteq (1 + \mathfrak{m}^m)\widehat{K}_n^M(R)_{\mathbb{Q}}. \quad (3.11)$$

This inclusion is obvious for $n = 0$. To prove this for $n \geq 1$, an easy induction (see [\(3.9\)](#) in the proof of [Lemma 3.4](#)) reduces to the case $n = 1$. We now let π be a uniformizing parameter of R and let $\{1 + a\pi^m, u\pi^j\} \in (1 + \mathfrak{m}^m)K_1^M(F)_{\mathbb{Q}}$, where $u \in R^\times$, $a \in R$ and $j \in \mathbb{Z}$. If $a = 0$, there is nothing to show and so we can write $a = u_0\pi^i$ with $u_0 \in R^\times$ and $i \geq 0$. We then get the following in $K_2^M(F)_{\mathbb{Q}}$:

$$\begin{aligned} \{1 + a\pi^m, u\pi^j\} &= \{1 + u_0\pi^{i+m}, u\pi^j\} \\ &= \{1 + u_0\pi^{i+m}, u\} + j\{1 + u_0\pi^{i+m}, \pi\} \\ &= \{1 + u_0\pi^{i+m}, u\} + \frac{j}{i+m}\{1 + u_0\pi^{i+m}, \pi^{i+m}\} \\ &= \{1 + u_0\pi^{i+m}, u\} - \frac{j}{i+m}\{1 + u_0\pi^{i+m}, -u_0\}. \end{aligned}$$

Since the last term clearly belongs to $(1 + \mathfrak{m}^m)\widehat{K}_1^M(R)_{\mathbb{Q}}$, we conclude the proof of [\(3.11\)](#) and hence of the lemma. \square

4. The cycle class map

In this section, we shall define the cycle class map on the group of 0-cycles with modulus and prove a very special case of [Theorem 1.1](#). The final proof of this theorem will be done by the end of the next section. We fix an arbitrary field k .

4A. The map $\text{cyc}_{X|D}$ on generators. Let (X, D) be a modulus pair in \mathbf{Sch}_k of dimension $d \geq 1$. Let $n \geq 0$ be an integer. We begin by defining the cycle class map $\text{cyc}_{X|D}$ on the group $z^{d+n}(X|D, n)$. Let $Z \in X \times \square^n$ be an admissible closed point. Since Z is a closed point, we have that $Z = \text{Spec}(k(Z))$. We let $p_{\square^n}: X \times \square^n \rightarrow \square^n$ and $p_X: X \times \square^n \rightarrow X$ denote the projection maps. We let $f: Z \rightarrow X$ denote the projection map. It is clear that f is a finite map and its image is a closed point $x \in X$ which does not lie in D . We thus have a factorization $Z \rightarrow \text{Spec}(k(x)) \rightarrow X^o \rightarrow X$ of f . The latter is actually a map of modulus pairs $f: (Z, \emptyset) \rightarrow (X, D)$. Hence, it induces the proper push-forward $f_*: \text{CH}_q(Z, *) \rightarrow \text{CH}_q(X|D, *)$, where $\text{CH}_q(Z, *)$ are Bloch's higher Chow groups of Z [[Bloch 1986](#)].

Now, the closed point $Z \in X \times \square^n$ defines a unique $k(Z)$ -rational point (which we also denote by Z) in \square_Z^n such that the composite projection map $Z \hookrightarrow \square_Z^n \rightarrow Z$ is identity. Furthermore, $[Z] \in z^{d+n}(X|D, n)$ is the image of $[Z] \in z^n(Z, n)$ under

the push-forward map f_* . Since $Z \hookrightarrow \square_Z^n$ does not meet any face of \square^n , it follows that $y_i(Z) \in k(Z)^\times$ for every $1 \leq i \leq n$, where $y_i: \overline{\square}_Z^n \rightarrow \overline{\square}_Z$ is the projection to the i -th factor. In particular, $\{y_1(Z), \dots, y_n(Z)\}$ is a well-defined element of $K_n^M(k(Z))$. We let

$$\text{cyc}_Z^M([Z]) = \{y_1(Z), \dots, y_n(Z)\} \in K_n^M(k(Z)), \tag{4.1}$$

$$\text{cyc}_Z([Z]) = \psi_Z \circ \text{cyc}_Z^M([Z]) \in K_n(Z), \tag{4.2}$$

where recall that $\psi_Z: K_*^M(k(Z)) \rightarrow K_*(k(Z)) = K_*(Z)$ is the canonical map from the Milnor to the Quillen K -theory.

We next recall from Section 2D that as $x = f(Z) \in X^o$ (which is regular), the finite map f defines a map of spectra $f_*: K(Z) \rightarrow K(X, D)$ such that the composite map $K(Z) \rightarrow K(X, D) \rightarrow K(X)$ is the usual push-forward map. The same holds for the inclusion $\iota^x: \text{Spec}(k(x)) \hookrightarrow X$. We let

$$\text{cyc}_{X|D}([Z]) = f_* \circ \text{cyc}_Z([Z]) \in K_n(X, D). \tag{4.3}$$

Extending this linearly, we obtain our cycle class map

$$\text{cyc}_{X|D}: z^{d+n}(X|D, n) \rightarrow K_n(X, D). \tag{4.4}$$

If Z is an admissible closed point as above and $x = f(Z)$, then we have a commutative diagram

$$\begin{array}{ccccc} K_n^M(k(Z)) & \xrightarrow{\psi_Z} & K_n(k(Z)) & & \\ \downarrow N_{Z/x} & & \downarrow T_{k(Z)/k} & \searrow f_* & \\ K_n^M(k(x)) & \xrightarrow{\psi_x} & K_n(k(x)) & \xrightarrow{\iota_*^x} & K_n(X, D), \end{array} \tag{4.5}$$

where $N_{Z/x}$ is the Norm map between the Milnor K -theory of fields [Bass and Tate 1973] (see also [Kerz 2009]) and the right vertical arrow is the transfer (push-forward) map between the Quillen K -theory of fields. The square on the left commutes by Lemma A.3. We can therefore write

$$\begin{aligned} \text{cyc}_{X|D}([Z]) &= f_* \circ \text{cyc}_Z([Z]) \\ &= f_* \circ \psi_Z \circ \text{cyc}_Z^M([Z]) \\ &= f_* \circ \psi_Z(\{y_1(Z), \dots, y_n(Z)\}) \\ &= \iota_*^x \circ \psi_x \circ N_{Z/\iota^x}(\{y_1(Z), \dots, y_n(Z)\}). \end{aligned}$$

It is clear from the definition that for any integer $m \geq 1$, there is a commutative diagram

$$\begin{array}{ccc}
 z^{d+n}(X \mid (m+1)D, n) & \xrightarrow{\text{cyc}_{X \mid D}} & K_n(X, (m+1)D) \\
 \downarrow & & \downarrow \\
 z^{d+n}(X \mid mD, n) & \xrightarrow{\text{cyc}_{X \mid D}} & K_n(X, mD),
 \end{array} \tag{4.6}$$

where the vertical arrows are the canonical restriction maps. We therefore have a strict map of pro-abelian groups

$$\text{cyc}_{X \mid D} : \{z^{d+n}(X \mid mD, n)\}_m \rightarrow \{K_n(X, mD)\}_m. \tag{4.7}$$

We next prove that the map $\text{cyc}_{X \mid D}$ is covariant with respect to proper morphisms of modulus pairs and contravariant for the flat morphisms of modulus pairs which are of relative dimension 0. Note that these are the only general cases where the functoriality of the cycle class map makes sense.

4B. Naturality for flat morphisms. Let (Y, E) and (X, D) be modulus pairs. Let $h : Y \rightarrow X$ be a flat morphisms of relative dimension 0 such that $E = h^*(D)$. Recall from [Krishna and Park 2017a, Proposition 2.12] that we have a pull-back map $h^* : z^{d+n}(X \mid D, n) \rightarrow z^{d+n}(Y \mid E, n)$ such that $h^*([Z]) = [W]$, where $W = (h \times \text{id}_{\square^n})^{-1}(Z)$ and Z a closed point in $X \setminus D \times \square^n$.

Lemma 4.1. *With notations as above, the following diagram commutes:*

$$\begin{array}{ccc}
 z^{d+n}(X \mid D, n) & \xrightarrow{\text{cyc}_{X \mid D}} & K_n(X, D) \\
 \downarrow h^* & & \downarrow h^* \\
 z^{d+n}(Y \mid E, n) & \xrightarrow{\text{cyc}_{Y \mid E}} & K_n(Y, E).
 \end{array} \tag{4.8}$$

Proof. Let Z be a closed point in $z^{d+n}(X \mid D, n)$ and let

$$[W] = \sum_{i=1}^r m_i [W_i] \in z^{d+n}(Y \mid E, n),$$

where W_i are irreducible components of the inverse image scheme W with multiplicities m_i . Let $f^Z : Z \rightarrow X$, $f^W : W \rightarrow Y$ and $f^{W_i} : W_i \rightarrow Y$ denote the respective projections. Let $y(Z) = \text{cyc}_Z^M([Z])$ and $y(W_i) = \text{cyc}_{W_i}^M([W_i])$ as in (4.1). We then have to show that

$$h^* \circ f_*^Z \circ \psi_Z(y(Z)) = \sum_{i=1}^r m_i (f_*^{W_i} \circ \psi_{W_i}(y(W_i))). \tag{4.9}$$

Consider the following diagram:

$$\begin{array}{ccccccccc}
 K(Z) & \longrightarrow & G(Z) & \longrightarrow & K^Z(X \setminus D) & \xleftarrow{\cong} & K^Z(X, D) & \longrightarrow & K(X, D) \\
 \downarrow h^* & & \downarrow h^* & & \downarrow h^* & & \downarrow h^* & & \downarrow h^* \\
 K(W) & \longrightarrow & G(W) & \longrightarrow & K^W(Y \setminus E) & \xleftarrow{\cong} & K^W(Y, E) & \longrightarrow & K(Y, E).
 \end{array} \quad (4.10)$$

Since h is flat, it follows that all the squares in (4.10) commute. Indeed, since the canonical map $K(-) \rightarrow G(-)$ respects flat pull-back, the left-most square in (4.10) commutes. The middle-left square commutes by [Thomason and Trobaugh 1990, Proposition 3.18] and the middle-right square commutes because each map is a pull-back map. Lastly, the right-most square in (4.10) commutes by the definition of the left arrow in the square. As discussed in Section 2D, the composition of the top horizontal arrows is the push forward map f_*^Z and the composition of the top horizontal arrows is the push forward map f_*^W .

It then suffices to show that

$$f_*^W \circ h^* \circ \psi_Z(y(Z)) = \sum_{i=1}^r m_i (f_*^{W_i} \circ \psi_{W_i}(y(W_i))). \quad (4.11)$$

Since $W = (h \times \text{id}_{\square^n})^{-1}(Z)$, we have $y(W_i) = y(Z)$ for each $1 \leq i \leq r$ under the injective map $k(Z)^\times \hookrightarrow k(W_i)^\times$. It then follows that $\psi_{W_i}(y(W_i)) = h_i^* \psi_Z(y(Z)) \in K_n(W_i)$, where $h_i : W_i \rightarrow Z$ is the induced map. Note that $h_i = h \circ g_i$, where $g_i : W_i \hookrightarrow W$ denotes the inclusion of the irreducible component W_i into W . We are therefore reduced to show that

$$f_*^W \circ h^* \circ \psi_Z(y(Z)) = \sum_{i=1}^r m_i (f_*^{W_i} \circ g_i^* \circ h^* \circ \psi_Z(y(Z))). \quad (4.12)$$

We shall actually show that for all $a \in K_n(W)$, we have

$$f_*^W(a) = \sum_{i=1}^r m_i f_*^{W_i} \circ g_i^*(a). \quad (4.13)$$

Observe that the equality (4.12) follows from (4.13) with $a = h^* \circ \psi_Z(y(Z))$. To show (4.13), consider the diagram:

$$\begin{array}{ccccccccc}
 \coprod_{i=1}^r G(W_i) & \xrightarrow{\cong} & G(W_{\text{red}}) & \longrightarrow & K^W(Y \setminus E) & \xleftarrow{\cong} & K^W(Y, E) & \longrightarrow & K(Y, E) \\
 & \searrow (g_i)_* & \downarrow g_* & & \parallel & & \parallel & & \parallel \\
 K(W) & \longrightarrow & G(W) & \longrightarrow & K^W(Y \setminus E) & \xleftarrow{\cong} & K^W(Y, E) & \longrightarrow & K(Y, E).
 \end{array} \quad (4.14)$$

Since $W_{\text{red}} = \coprod_i W_i$, it follows that the push-forward map $\coprod_{i=1}^r G(W_i) \rightarrow G(W_{\text{red}})$ is an isomorphism and the left triangle in (4.14) commutes. Observe that the left-most square commutes because all arrow in the square are (compatible) push-forward maps. As before, the composition of the top horizontal arrows on $G(W_i)$ is the push-forward map $f_*^{W_i}$. Let $b \in G_n(W)$ be the image of $a \in K_n(W)$ under the map $K_n(W) \rightarrow G_n(W)$ induced by the bottom left arrow in (4.14). The equality (4.13) then follows if we show that

$$b = g_* \left(\sum_{i=1}^r m_i g_i^*(b) \right) \in G_n(W). \tag{4.15}$$

The equality (4.15) however follows from the following calculation:

$$\begin{aligned} g_* \left(\sum_{i=1}^r m_i g_i^*(b) \right) &= \sum_{i=1}^r m_i g_{i*} \circ g_i^*(b) \\ &\stackrel{=1}{=} \sum_{i=1}^r m_i g_{i*}(g_i^*(b)[\mathcal{O}_{W_i}]) \\ &\stackrel{=2}{=} b \left(\sum_{i=1}^r m_i g_{i*}([\mathcal{O}_{W_i}]) \right) \\ &\stackrel{=3}{=} b[\mathcal{O}_W] = b, \end{aligned}$$

where $\stackrel{=1}{=}$ follows because for each i , we have $[\mathcal{O}_{W_i}] = 1 \in G_0(W_i)$, $\stackrel{=2}{=}$ follows from the projection formula [Thomason and Trobaugh 1990, Proposition 3.17] for G -theory because $a \in K_n(W)$ and $\stackrel{=3}{=}$ follows as $1 = [\mathcal{O}_W] = \sum_{i=1}^r m_i g_{i*}([\mathcal{O}_{W_i}]) \in G_0(W)$. This completes the proof of the lemma. \square

4C. Naturality for proper morphisms. Let (Y, E) and (X, D) be modulus pairs such that X and Y are regular schemes over k of pure dimension d_X and d_Y . Let $h : Y \rightarrow X$ be a proper morphisms such that $E = h^*(D)$. By [Krishna and Park 2017a, Proposition 2.10], we have a proper push-forward map $h_* : z^{d_Y+n}(Y | E, n) \rightarrow z^{d_X+n}(X | D, n)$ such that $h_*([z]) = [k(z) : k(w)][w]$, where $w = (h \times \text{id}_{\square^n})(z)$ and z is a closed point in $Y \setminus E \times \square^n$. The existence of the push-forward map $h_* : K(Y, E) \rightarrow K(X, D)$ follows from the following lemma.

Lemma 4.2. *The map h induces a proper push-forward map $h_* : K(Y, E) \rightarrow K(X, D)$ of relative K -theory spectra.*

Proof. Since Y is regular, we can assume with out loss of generality that Y is integral. Since X is regular, the map h has finite tor-dimension. By [Thomason and Trobaugh 1990, 3.16.4], we have a push-forward map $f_* : K(Y) \rightarrow K(X)$. Observe that if $E = \emptyset$, then we have a push-forward map $f_* : K(Y) \rightarrow K(X, D)$.

We can therefore assume that $E \neq \emptyset$. To prove the lemma, it suffices to show that Y and D are tor-independent over X . This will in particular imply that $E \rightarrow D$ also has finite tor-dimension. For tor-independence, we note that D is an effective Cartier divisor. Hence, the only possible nontrivial tor term can be $\text{Tor}_{\mathcal{O}_X}^1(\mathcal{O}_Y, \mathcal{O}_D)$. But this is same as the \mathcal{I}_D -torsion subsheaf of \mathcal{O}_Y . Since Y is integral, this torsion subsheaf is nonzero if and only if the ideal \mathcal{I}_E is zero. But this can not happen as E is a proper divisor on Y . This finishes the proof. \square

Remark 4.3. Observe that [Lemma 4.2](#) is true for a general integral scheme (may not be regular) Y over a regular scheme X .

We now prove that the map $\text{cyc}_{X|D}$ in [\(4.4\)](#) commutes with the push-forward map.

Lemma 4.4. *For a cycle $\alpha \in z^{dy+n}(Y|E, n)$, we have $\text{cyc}_{X|D} \circ h_*(\alpha) = h_* \circ \text{cyc}_{Y|E}(\alpha)$.*

Proof. We can assume α is represented by a closed point $z \in z^{dy+n}(Y|E, n)$. We set $w = (h \times \text{id}_{\square^n})(z) \in X \times \square^n$ and $x = p_X(w)$. The compatibility between norm maps in the Milnor K -theory of fields and push-forward maps in the Quillen K -theory (see [Lemma A.3](#)) yields a commutative diagram

$$\begin{array}{ccc}
 K_n^M(k(z)) & \xrightarrow{N_{z/w}} & K_n^M(k(w)) & (4.16) \\
 \downarrow \psi_z & & \downarrow \psi_w & \\
 K_n(k(z)) & \xrightarrow{h_*} & K_n(k(w)) & \\
 \downarrow p_* & & \downarrow p_* & \\
 K_n(Y, E) & \xrightarrow{h_*} & K_n(X, D) &
 \end{array}$$

Using this commutative diagram, we get

$$\begin{aligned}
 \text{cyc}_{X|D} \circ h_*([z]) &= \text{cyc}_{X|D}([k(z) : k(w)][w]) \\
 &= \iota_{w*}(\{y_1(w), \dots, y_n(w)\}^{[k(z):k(w)]}) \\
 &= {}^1 \iota_{w*} \circ \iota_{z/w*} \circ \iota_{z/w}^*(\{y_1(w), \dots, y_n(w)\}) \\
 &= {}^2 \iota_{w*} \circ N_{z/w}(\{y_1(z), \dots, y_n(z)\}) \\
 &= h_* \circ \iota_{z*}(\{y_1(z), \dots, y_n(z)\}) \\
 &= h_* \circ \text{cyc}_{Y|E}([z]).
 \end{aligned}$$

In this set of equalities, recall our notation (preceding [Lemma 5.2](#)) that $y_i(z)$ is the image of z under the i -th projection $\text{Spec}(k(z)) \rightarrow \square_{k(z)}$. In particular, $y_i(z) \in k(z)^\times$ for all $1 \leq i \leq n$. The coordinates $y_i(w) \in k(w)^\times$ have similar meaning.

The map $\iota_{z/w} : \text{Spec}(k(z)) \rightarrow \text{Spec}(k(w))$ is the projection. The equality $=^1$ is a consequence of the projection formula for the Milnor K -theory associated to the resulting inclusion $k(w) \hookrightarrow k(z)$. The equality $=^2$ follows from the fact that $y_i(z) = y_i(w) = \iota_{z/w}^*(y_i(w))$ via the inclusion $k(w) \hookrightarrow k(z)$ for all $1 \leq i \leq n$. This proves the lemma. \square

We end this section with some comments below on $\text{cyc}_{X|D}$ in the nonmodulus case.

4D. Agreement with Levine’s map. We had mentioned in Section 1 that for Bloch’s higher Chow groups of 0-cycles, a cycle class map to the ordinary K -theory of a regular variety was constructed by Levine [1994] with rational coefficients. Binda [2018] constructed such a map in the modulus setting and his map is identical to that of Levine by definition. It is not hard to check that the map $\text{cyc}_{X|D}$ coincides with Levine’s map when $D = \emptyset$ (note that $z^{d+n}(X|D, n) \subseteq z^{d+n}(X, n)$). In particular, it turns out that Levine’s cycle class map for the ordinary 0-cycles exists with integral coefficients. We give a sketch of this agreement and leave the details for the reader.

Let $n \geq 0$ be an integer. Since X is regular, the homotopy invariance implies that the multirelative K -theory exact sequence (see [Levine 1994, §1]) yields an isomorphism

$$\theta_n : K_n(X) \xrightarrow{\cong} K_0(X \times \square^n, \partial \square^n), \tag{4.17}$$

where $K(X \times \square^n, \partial \square^n)$ is the iterated multirelative K -theory of $X \times \square^n$ relative to all codimension one faces.

Let $Z \subset X \times \square^n$ be a closed point. Then Levine’s cycle class map $\text{cyc}_X^L([Z])$ is the image of $1 \in K_0(Z)$ under the composition

$$K_0(Z) \cong K_0^Z(X \times \square^n, \partial \square^n) \rightarrow K_0(X \times \square^n, \partial \square^n) \xleftarrow{\theta_n} K_n(X). \tag{4.18}$$

Let $f : Z \rightarrow X$ denote the projection map (which is finite). We then have a finite map of multiclosed pairs $f : (Z \times \square^n, \partial \square^n) \rightarrow (X \times \square^n, \partial \square^n)$. Since the relative K -theory fiber sequence commutes with finite push-forward for regular schemes (see Lemma 4.2), and since $\text{cyc}_X^L([Z])$ is the image of $[Z] \in K_0(Z \times \square^n, \partial \square^n)$ under the push-forward map f_* , we can assume that $X = \text{Spec}(k)$ and $f : Z \rightarrow \text{Spec}(k)$ is the identity map.

When $n = 0$, the agreement of $\text{cyc}_k([Z])$ and $\text{cyc}_k^L([Z])$ is immediate. When $n \geq 1$ and if we follow our notation of (4.1), then we see that $y_i(Z) = a_i \in k^\times$ for each $1 \leq i \leq n$. In particular, we have $\text{cyc}_k([Z]) = \{a_1, \dots, a_n\} \in K_n(k)$. One therefore has to show that if $z = (a_1, \dots, a_n) \in (k^\times)^n$ is a k -rational point in \square^n , then the class $[k(z)] \in K_0(\square^n, \partial \square^n)$ coincides with $\theta_n(\{a_1, \dots, a_n\})$. But this is an elementary exercise in K -theory using repeated application of relative K -theory exact sequence. For $n = 1$, it already follows from a straightforward generalization

of Lemma 2.1 (where we replace (t^{m+1}) by any ideal of $R[t]$) with identical proof. We leave it to the reader to check the details for $n \geq 2$.

5. The case of regular curves

The goal now is to show that $\text{cyc}_{X|D}$ kills the rational equivalence if we allow the modulus to vary along $\{mD\}_{m \geq 1}$. In this section, we shall prove a very special case of this. The proof of Theorem 1.1 will be reduced to this case in the next section. We consider the following situation. Let $n \geq 0$ be an integer. We let X be a regular connected curve over k and let $D \subset X$ be an effective Cartier divisor. Let $\bar{W} \subset X \times \bar{\square}^{n+1}$ be a closed subscheme such that the following hold:

- (1) The composite map $\bar{W} \xrightarrow{\nu} X \times \bar{\square}^{n+1} \xrightarrow{p_X} X$ is an isomorphism.
- (2) $W = \bar{W} \cap (X \times \square^{n+1})$ is an admissible cycle on $X \times \square^{n+1}$ with modulus $(n+1)D$. That is, $[W] \in z^{n+1}(X | (n+1)D, n+1)$.

Let

$$\partial = \sum_{i=1}^{n+1} (-1)^i (\partial_i^\infty - \partial_i^0) : z^{n+1}(X | E, n+1) \rightarrow z^{n+1}(X | E, n)$$

be the boundary map in the cycle complex with modulus for an effective divisor $E \subset X$. We want to prove the following result in this subsection.

Proposition 5.1. *The class $\text{cyc}_{X|D}(\partial W)$ dies in $K_n(X, D)$ under the cycle class map*

$$z^{n+1}(X | D, n) \xrightarrow{\text{cyc}_{X|D}} K_n(X, D).$$

We shall prove this proposition in several steps. We begin with the following description of the cycle class map on various boundaries of W . Let F denote the function field of X . Let g denote the Milnor symbol $\{g_1, \dots, g_{n+1}\} \in K_{n+1}^M(F)$, where $g_i : \bar{W} \rightarrow \bar{\square}$ is the i -th projection for $1 \leq i \leq n+1$. Note that this symbol is well-defined because no g_i can be identically zero by the admissibility of W . For any closed point $z \in \bar{W}$, let $\text{ord}_z : F^\times \rightarrow \mathbb{Z}$ denote the valuation associated to the discrete valuation ring $\mathcal{O}_{\bar{W},z}$. We let $\partial_z^M : K_{i+1}^M(F) \rightarrow K_i^M(k(z))$ denote a boundary map in the Gersten complex (3.1). We let \mathfrak{m}_z denote the maximal ideal of the local ring $\mathcal{O}_{\bar{W},z}$. A symbol $\{a_1, \dots, \widehat{a}_i, \dots, a_n\}$ will mean the one obtained from $\{a_1, \dots, a_n\}$ by omitting a_i . For any point $z \in X \times \square^{n+1}$, let $y_i : \square_{k(z)}^{n+1} \rightarrow \square_{k(z)}$ denote the projection map to i -th factor. Let $f^z : \text{Spec}(k(z)) \rightarrow X$ denote the projection to X .

Lemma 5.2. *For $1 \leq i \leq n + 1$, we have*

$$\text{cyc}_{X|D}(\partial_i^0 W) = \sum_{z \in \partial_i^0 W} \text{ord}_z(g_i) f_*^z \circ \psi_z(\{y_1(z), \dots, \widehat{y_i(z)}, \dots, y_{n+1}(z)\}),$$

$$\text{cyc}_{X|D}(\partial_i^\infty W) = \sum_{z \in \partial_i^\infty W} \text{ord}_z(1/g_i) f_*^z \circ \psi_z(\{y_1(z), \dots, \widehat{y_i(z)}, \dots, y_{n+1}(z)\}).$$

Proof. We should first observe that the admissibility of W implies that if $z \in \partial_i^t W$ for $t \in \{0, \infty\}$, then we must have $y_j(z) \neq 0$ for all $j \neq i$. In particular, the element $\{y_1(z), \dots, \widehat{y_i(z)}, \dots, y_{n+1}(z)\} \in K_n^M(k(z))$ is well defined. By the definition of $\text{cyc}_{X|D}$, it suffices to show that for $1 \leq i \leq n + 1$, we have

$$\partial_i^0 W = \sum_{z \in \partial_i^0 W} \text{ord}_z(g_i)[z] \quad \text{and} \quad \partial_i^\infty W = \sum_{z \in \partial_i^\infty W} \text{ord}_z(1/g_i)[z]. \tag{5.1}$$

But this is an immediate consequence of the definition of the intersection product of an integral cycle with the faces of $X \times \square^n$. \square

Lemma 5.3. *Suppose that $n \geq 1$ and $\partial_X^M(g) \in \bigoplus_{z \in X^o} K_n^M(k(z))$ under the Milnor boundary map $\partial_X^M : K_{n+1}^M(F) \rightarrow \bigoplus_{z \in X^{(1)}} K_n^M(k(z))$. Then*

$$\sum_{z \in X^o} f_*^z \circ \psi_z \circ \partial_X^M(g) = 0. \tag{5.2}$$

Proof. Suppose first that $D = \emptyset$. In this case, we have a diagram

$$\begin{array}{ccc} K_{n+1}^M(F) & \xrightarrow{\partial_X^M} & \bigoplus_{z \in X^{(1)}} K_n^M(k(z)) \\ \psi_F \downarrow & & \downarrow (\psi_z)_z \\ K_{n+1}(F) & \xrightarrow{\partial_X^Q} & \bigoplus_{z \in X^{(1)}} K_n(k(z)) \xrightarrow{(f_*^z)_z} K_n(X). \end{array} \tag{5.3}$$

The Gersten complex for Milnor K -theory canonically maps to the Gersten complex for the Quillen K -theory by [Lemma A.1](#). In particular, the square in the above diagram is commutative. The bottom row is exact by Quillen’s localization sequence and a limit argument. The lemma follows immediately from this diagram.

We now let $D = m_1x_1 + \dots + m_r x_r$, where x_1, \dots, x_r are distinct closed points of X and m_1, \dots, m_r are positive integers. Let $A = \mathcal{O}_{X,D}$ be the semilocal ring of X at D and let I denote the ideal of D inside $\text{Spec}(A)$. The localization and relativization sequences give us the commutative diagram of homotopy fiber sequences

$$\begin{array}{ccccc}
 \coprod_{z \in X^o} K(k(z)) & \xlongequal{\quad} & \coprod_{z \in X^o} K(k(z)) & & (5.4) \\
 \downarrow & & \downarrow & & \\
 K(X, D) & \longrightarrow & K(X) & \longrightarrow & K(D) \\
 \downarrow & & \downarrow & & \parallel \\
 K(A, I) & \longrightarrow & K(A) & \longrightarrow & K(D).
 \end{array}$$

The associated homotopy groups long exact sequences yield the commutative diagram of exact sequences of abelian groups

$$\begin{array}{ccccc}
 K_{n+1}(A, I) & \xrightarrow{\partial_A^Q} & \bigoplus_{z \in X^o} K_n(k(z)) & \xrightarrow{(f_*^z)_z} & K_n(X, D) & (5.5) \\
 \downarrow & & \parallel & & \downarrow & \\
 K_{n+1}(A) & \xrightarrow{\partial_A^Q} & \bigoplus_{z \in X^o} K_n(k(z)) & \xrightarrow{(f_*^z)_z} & K_n(X) & \\
 u^* \downarrow & & & & & \\
 K_{n+1}(D), & & & & &
 \end{array}$$

where $u : D \hookrightarrow \text{Spec}(A)$ is the inclusion map. It follows from this diagram that there is an exact sequence

$$\widehat{K}_{n+1}(A, I) \xrightarrow{\partial_A^Q} \bigoplus_{z \in X^o} K_n(k(z)) \xrightarrow{(f_*^z)_z} K_n(X, D). \tag{5.6}$$

In order to compare this with the Milnor K -theory, we consider the diagram

$$\begin{array}{ccccc}
 \widehat{K}_{n+1}^M(A) & \xrightarrow{\partial_A^M} & \bigoplus_{z \in X^o} K_n^M(k(z)) & & (5.7) \\
 \downarrow \psi_A & \searrow & \downarrow & \searrow & \\
 & & K_{n+1}^M(F) & \xrightarrow{\partial_X^M} & \bigoplus_{z \in X} K_n^M(k(z)) \\
 & & \downarrow (\psi_z)_z & & \downarrow (\psi_z)_z \\
 K_{n+1}(A) & \xrightarrow{\partial_A^Q} & \bigoplus_{z \in X^o} K_n(k(z)) & & \\
 \downarrow & \searrow \psi_F & \downarrow & \searrow & \\
 & & K_{n+1}(F) & \xrightarrow{\partial_X^Q} & \bigoplus_{z \in X} K_n(k(z)).
 \end{array}$$

The map ψ_A comes from (3.2). By the definition of $\widehat{K}_{n+1}^M(A)$, we know that the composite map

$$\widehat{K}_{n+1}^M(A) \rightarrow K_{n+1}^M(F) \xrightarrow{\partial_z^M} K_n^M(k(z))$$

is zero for all closed points $z \in D$. Hence, the composite

$$\widehat{K}_{n+1}^M(A) \rightarrow K_{n+1}^M(F) \xrightarrow{\partial_X^M} \bigoplus_{z \in X} K_n^M(k(z))$$

factors through the map denoted by ∂_A^M in the above diagram. In particular, the top face of (5.7) commutes. Exactly the same reason shows that the bottom face also commutes. Furthermore, ∂_A^Q is same as the boundary map in the bottom row of (5.5). The left and the right faces clearly commute and so does the front face by Lemma A.1. A diagram chase shows that the back face of (5.7) commutes too.

In order to show (5.2), we consider our final diagram

$$\begin{array}{ccccc}
 & & \widehat{K}_{n+1}^M(A) & & (5.8) \\
 & \nearrow & \downarrow & \searrow \partial_A^M & \\
 \widehat{K}_{n+1}^M(A, I) & \xrightarrow{\partial_A^M} & \bigoplus_{z \in X^o} K_n^M(k(z)) & & \\
 \downarrow \psi_{(A, I)} & & \downarrow \psi_A & & \downarrow (\psi_z)_z \\
 \widehat{K}_{n+1}^M(A, I) & \xrightarrow{\partial_A^Q} & K_{n+1}(A) & \xrightarrow{\partial_A^Q} & \bigoplus_{z \in X^o} K_n(k(z)) \xrightarrow{(f_*^z)_z} K_n(X, D).
 \end{array}$$

The left face of this diagram commutes by (3.5) and we just showed above that the right face commutes. In particular, the bottom face also commutes. Furthermore, the bottom row is same as the exact sequence (5.6). Our assertion will therefore follow if we can show that $g \in \widehat{K}_{n+1}^M(A, I)$ provided $W \in z^{n+1}(X \mid (n+1)D, n+1)$.

Suppose now that $W \in z^{n+1}(X \mid (n+1)D, n+1)$. By the definition of the modulus condition (see Section 2E), it means that $\sum_{j=1}^{n+1} \text{ord}_{x_i}(g_j - 1) \geq (n+1)m_i$ for each $1 \leq i \leq r$. Since $n, m_i \geq 1$, we must have $(n+1)m_i \geq m_i + n$ for each $1 \leq i \leq r$. This implies that $g = \{g_1, \dots, g_{n+1}\} \in K_{n+1}^M(F, m_i + n)$ for every $1 \leq i \leq r$ in the notations of Lemma 3.3. If we now apply Lemma 3.3 with $R = A_{m_i}$, it follows that $g \in (1 + \mathfrak{m}_i^{m_i+n})K_n^M(F)$ for every $1 \leq i \leq r$. In other words, g lies in the intersection $\bigcap_{i=1}^r (1 + \mathfrak{m}_i^{m_i+n})K_n^M(F)$ as an element of $K_{n+1}^M(F)$. It follows from Lemma 3.5 that $g \in \widehat{K}_{n+1}^M(A, \underline{m}) = \widehat{K}_{n+1}^M(A, I)$. \square

Proof of Proposition 5.1. We assume first that $n = 0$. In this case, we will show the stronger assertion that $\text{cyc}_{X \mid D}(\partial W)$ dies in $K_n(X, D)$ if $W \in z^{n+1}(X \mid D, n+1)$. Let A be the semilocal ring and $I \subset A$ the ideal as in the proof of Lemma 5.3. By

(5.4), we have an exact sequence

$$K_1(A, I) \xrightarrow{\partial_A^Q} \bigoplus_{z \in X^o} K_0(k(z)) \xrightarrow{(f_*^z)_z} K_0(X, D) \rightarrow 0. \tag{5.9}$$

Comparing this with the exact sequence

$$K_1(A) \xrightarrow{\partial_A^Q} \bigoplus_{z \in X^o} K_0(k(z)) \xrightarrow{(f_*^z)_z} K_0(X),$$

we see that we can replace $K_1(A, I)$ by $\widehat{K}_1(A, I)$ in (5.9). But then, it is same as the exact sequence

$$(1 + I)^\times \xrightarrow{\partial_A^M} z^1(X | D, 0) \xrightarrow{\text{cyc}_{X|D}} K_0(X, D) \rightarrow 0.$$

Moreover, one knows that $\text{Coker}(\partial_A^M) \cong \text{CH}^1(X | D, 0)$ (e.g, see [Krishna 2015, §2]). We therefore showed that $\text{cyc}_{X|D} : \text{CH}^1(X | D, 0) \rightarrow K_0(X, D)$ is actually an isomorphism.

We now assume for the remaining part of the proof that $n \geq 1$. As before, let $g_i : \overline{W} \rightarrow \square$ denote the projections and let $f = p_X \circ v : \overline{W} \rightarrow X$ be the projection to X . We also recall the element $g = \{g_1, \dots, g_{n+1}\} \in K_{n+1}^M(F)$. Our task is to show that

$$\text{cyc}_{X|D}(\partial W) = \sum_{i=1}^{n+1} (-1)^i \text{cyc}_{X|D}(\partial_i^\infty W - \partial_i^0 W) = 0 \quad \text{in } K_n(X, D) \tag{5.10}$$

if W satisfies the modulus condition for $(n + 1)D$.

Our idea is to compute the cycle class of ∂W in terms of the cycle class of the Milnor boundary $\partial_X^M(g)$. In order to do this, we consider in general a closed point $z \in \overline{W}$. If $z \in \overline{W} \cap F_n^1$, then we must have $g_i(z) = 1$ for some $1 \leq i \leq n + 1$. This means that $g_i - 1 \in \mathfrak{m}_z$. Lemma 3.4 then implies that $\partial_z^M(g) = 0$. If $z \in W \setminus \partial_i^{[0, \infty]} W$ for all $1 \leq i \leq n + 1$, then we must have $g_i \in \mathcal{O}_{\overline{W}, z}^\times$ for all $1 \leq i \leq n + 1$ and hence $\partial_z^M(g) = 0$. If $z \in \partial_i^0 W$ for some $1 \leq i \leq n + 1$, then $z \notin \partial_j^t W$ unless $(t, j) = (0, i)$. This implies that $g_j \in \mathcal{O}_{\overline{W}, z}^\times$ for all $j \neq i$. Furthermore, the image of g_j under the map $\mathcal{O}_{\overline{W}, z}^\times \rightarrow k(z)^\times$ is simply $y_j(z)$. By the definition of the boundary map in the Gersten complex for the Milnor K -theory (e.g., see [Bass and Tate 1973]), we therefore have

$$\partial_z^M(g) = (-1)^i \text{ord}_z(g_i) \{y_1(z), \dots, \widehat{y_i(z)}, \dots, y_{n+1}(z)\} \in K_n^M(k(z)).$$

We have the same expression for $\partial_z^M(g)$ if $z \in \partial_i^\infty W$ for some $1 \leq i \leq n + 1$.

If we identify \overline{W} with X via f so that $W \subseteq X^o = X \setminus D$, it follows from the above computation of $\{\partial_z^M(g) \mid z \in \overline{W}^{(1)}\}$ and the comparison of (5.10) and Lemma 5.2 that the two things hold. Namely,

- (1) The image of g under the Milnor boundary $K_{n+1}^M(F) \xrightarrow{\partial_X^M} \bigoplus_{z \in X} K_n^M(k(z))$ lies in the subgroup $\bigoplus_{z \in \partial W} K_n^M(k(z)) \subset \bigoplus_{z \in X^o} K_n^M(k(z))$.
- (2) The element $\partial_X^M(g)$ maps to $\text{cyc}_{X|D}(\partial W)$ under the composition of maps

$$\bigoplus_{z \in X^o} K_n^M(k(z)) \xrightarrow{(\psi_z)_z} \bigoplus_{z \in X^o} K_n(k(z)) \xrightarrow{(f_*^z)_z} K_n(X, D). \tag{5.11}$$

The proposition is therefore reduced to showing that $\sum_{z \in X^o} f_*^z \circ \psi_z \circ \partial_X^M(g) = 0$ if W lies in $z^{n+1}(X \mid (n+1)D, n+1)$. But this follows at once from Lemma 5.3. \square

6. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1 using the case of regular curves. So let k be any field. Let X be a regular quasiprojective variety of pure dimension $d \geq 1$ over k and let $D \subset X$ be an effective Cartier divisor. We fix an integer $n \geq 0$. In Section 4A, we constructed the cycle class map

$$\text{cyc}_{X|D} : z^{d+n}(X \mid D, n) \rightarrow K_n(X, D).$$

The naturality statements in Theorem 1.1 follow from Lemma 4.4 and Lemma 4.1. To prove Theorem 1.1, it therefore suffices to show the following.

Proposition 6.1. *Let $W \subset z^{d+n}(X \mid (n+1)D, n+1)$ be an integral cycle. Then the image of W under the composition*

$$z^{d+n}(X \mid (n+1)D, n+1) \hookrightarrow z^{d+n}(X \mid D, n+1) \xrightarrow{\partial} z^{d+n}(X \mid D, n) \xrightarrow{\text{cyc}_{X|D}} K_n(X, D)$$

is zero.

We shall prove this proposition in several steps. Let $\overline{W} \subset X \times \overline{\square}^{n+1}$ be the closure of W and let $\nu : \overline{W}^N \rightarrow X \times \overline{\square}^{n+1}$ be the map induced on the normalization of \overline{W} . We begin with a direct proof of one easy case of the proposition as a motivating step.

Lemma 6.2. *Suppose that W lies over a closed point of X . Then the assertion of Proposition 6.1 holds.*

Proof. In this case, the modulus condition implies that such a closed point must lie in X^o . In other words, there is a closed point $x \in X^o$ such that

$$W \in z^n(\text{Spec}(k(x)), n+1) \subset z^{d+n}(X \mid D, n+1).$$

Using the commutative diagram (see the construction of $\text{cyc}_{X|D}$ in Section 4A)

$$\begin{array}{ccccc}
 z^n(\text{Spec}(k(x)), n+1) & \xrightarrow{\partial} & z^n(\text{Spec}(k(x)), n) & \xrightarrow{\text{cyc}_x} & K_n(k(x)) \\
 \downarrow & & \downarrow & & \downarrow \\
 z^{d+n}(X|D, n+1) & \xrightarrow{\partial} & z^{d+n}(X|D, n) & \xrightarrow{\text{cyc}_{X|D}} & K_n(X, D),
 \end{array} \tag{6.1}$$

it suffices to show that $\text{cyc}_x(\partial W) = 0$ in $K_n(k(x))$. We can thus assume that $X = \text{Spec}(k)$ and $D = \emptyset$.

We let F denote the function field of W and let $g = \{g_1, \dots, g_{n+1}\} \in K_{n+1}^M(F)$ denote the Milnor symbol given by the projection maps $g_i : \overline{W} \rightarrow \overline{\square}$. Following the proof of Proposition 5.1, our assertion is equivalent to showing that the composition

$$K_{n+1}^M(F) \xrightarrow{\partial_{\overline{W}}^M} \bigoplus_{z \in \overline{W}^{(1)}} K_n^M(k(z)) \rightarrow \bigoplus_{z \in \overline{W}^{(1)}} K_n(k(z)) \xrightarrow{(f_*^z)_z} K_n(k)$$

kills g . Arguing as in (5.3), it suffices to show that the composite map

$$K_{n+1}(F) \xrightarrow{\partial_{\overline{W}}^Q} \bigoplus_{z \in \overline{W}^{(1)}} K_n(k(z)) \xrightarrow{(f_*^z)_z} K_n(k)$$

kills g . Since this map is same as the composite map

$$K_{n+1}(F) \xrightarrow{\partial_{\overline{W}^N}^Q} \bigoplus_{z \in (\overline{W}^N)^{(1)}} K_n(k(z)) \xrightarrow{(f_*^z)_z} K_n(k),$$

we need to show that g dies under this map. But this is the well-known Weil reciprocity theorem in algebraic K -theory (e.g., see [Weibel 2013, Chapter IV, Theorem 6.12.1]). □

We now proceed to the proof of the general case of Proposition 6.1. In view of Lemma 6.2, we can assume that the projection map $f : \overline{W}^N \rightarrow X$ is finite.

This gives rise to a Cartesian square

$$\begin{array}{ccc}
 \overline{W}^N & & \\
 \downarrow \phi & \searrow & \\
 \overline{W}^N \times \overline{\square}^{n+1} & \longrightarrow & X \times \overline{\square}^{n+1} \\
 \downarrow p' & & \downarrow p_X \\
 \overline{W}^N & \xrightarrow{f} & X.
 \end{array} \tag{6.2}$$

Note that ϕ is a closed immersion. Using the finiteness of f and admissibility of W , it is evident that $W^N = \phi(\overline{W}^N) \cap (\overline{W}^N \times \square^{n+1})$ is an admissible cycle on $\overline{W}^N \times \square^{n+1}$. In other words, it intersects the faces of $\overline{W}^N \times \square^{n+1}$ properly, and satisfies the modulus $(n+1)E$, if we let $E = f^*(D) \subsetneq \overline{W}^N$. Notice that a consequence of the modulus condition for W is that E is a proper Cartier divisor on \overline{W}^N . Since f is finite and $E = f^*(D)$, we have a push-forward map $f_*: z^{n+1}(\overline{W}^N | (n+1)E, *) \rightarrow z^{n+d}(X | (n+1)D, *)$ (see [Krishna and Park 2017a, Proposition 2.10]). Since $(f \times \text{id}_{\square^n})$ takes W^N to W and since $W^N \rightarrow W$ is the normalization map, we see that $f_*([W^N]) = [W] \in z^{n+d}(X | (n+1)D, n+1)$. In particular, we get

$$f_*(\partial W^N) = \partial(f_*([W^N])) = \partial W. \quad (6.3)$$

Proof of Proposition 6.1. In view of Lemma 6.2, we can assume that $f: \overline{W}^N \rightarrow X$ is finite. Using (6.3) and Lemma 4.4, we have that

$$\text{cyc}_{X|D}(\partial W) = \text{cyc}_{X|D} \circ f_*(\partial W^N) = f_* \circ \text{cyc}_{\overline{W}^N|E}(\partial W^N).$$

We can therefore assume that X is a regular curve and $W \in z^{n+1}(X | (n+1)D, n+1)$ is an integral cycle such that the map $p_X: \overline{W} \rightarrow X$ is an isomorphism. We can now apply Proposition 5.1 to finish the proof. \square

Remark 6.3. We remark that throughout the proof of Theorem 1.1, it is only in Lemma 4.2 where we need to assume that X is regular everywhere (see Remark 4.3). One would like to believe that for a proper map of modulus pairs $f: (Y, f^*(D)) \rightarrow (X, D)$, there exists a push-forward map $f_*: K_*(Y, f^*(D)) \rightarrow K(X, D)$. But we do not know how to prove it.

6A. The cycle class map with rational coefficients. If we work with rational coefficients, we can prove the following improved version of Theorem 1.1. This may not be very useful in positive characteristic. However, one expects it to have many consequences in characteristic zero. The reason for this is that the relative algebraic K -groups of nilpotent ideals are known to be \mathbb{Q} -vector spaces in characteristic zero. The proofs of Theorem 1.3 and its corollaries in this paper are crucially based on this improved version.

Theorem 6.4. *Let X be a regular quasiprojective variety of pure dimension $d \geq 1$ over a field k and let $D \subset X$ be an effective Cartier divisor. Let $n \geq 0$ be an integer. Then there is a cycle class map*

$$\text{cyc}_{X|D}: \text{CH}^{n+d}(X | D, n)_{\mathbb{Q}} \rightarrow K_n(X, D)_{\mathbb{Q}}. \quad (6.4)$$

Proof. We shall only indicate where we use rational coefficients in the proof of Theorem 1.1 to achieve this improvement as rest of the proof is just a repetition. Since we work with rational coefficients, we shall ignore the subscript $A_{\mathbb{Q}}$ in an abelian group A in this proof and treat A as a \mathbb{Q} -vector space.

As we did before, we need to prove [Proposition 6.1](#) with $W \in z^{d+n}(X | D, n + 1)$. We can again reduce the proof of this proposition to the case when X is a regular curve and $W \in z^{n+1}(X | D, n + 1)$ is an integral cycle such that the map $p_X : \overline{W} \rightarrow X$ is an isomorphism. We thus have to prove [Proposition 5.1](#) with $W \in z^{n+1}(X | D, n + 1)$. In turn, this is reduced to proving [Lemma 5.3](#) when $W \in z^{n+1}(X | D, n + 1)$. However, a close inspection shows that the proof of [Lemma 5.3](#) works in the present case too with no change until its last step where we need to use [Lemma 3.6](#) instead of [Lemma 3.5](#). \square

6B. Chow groups and K -theory with compact support. Let X be a quasiprojective scheme of pure dimension d over a field k and let \overline{X} be a proper compactification of X such that $\overline{X} \setminus X$ is supported on an effective Cartier divisor D . Recall from [[Binda and Saito 2019](#), Lemma 2.9] that $\mathrm{CH}^p(X, n)_c := \varprojlim_{m \geq 1} \mathrm{CH}^p(\overline{X} | mD, n)$ is independent of the choice of \overline{X} and is called the higher Chow group of X with compact support. One can similarly define the algebraic K -theory with compact support by $K_n(X)_c := \varprojlim_{m \geq 1} K_n(\overline{X}, mD)$. It follows from [[Kerz et al. 2018](#), Theorem A] that this is also independent of the choice of \overline{X} . As a consequence of [Theorem 1.1](#), we get

Corollary 6.5. *Let X be a regular quasiprojective scheme of pure dimension d over a field admitting resolution of singularities. Then there exists a cycle class map*

$$\mathrm{cyc}_X : \mathrm{CH}^{n+d}(X, n)_c \rightarrow K_n(X)_c.$$

We remark that even if $\mathrm{CH}^{n+d}(X, n)_c$ is defined without resolution of singularities, this condition is needed in [Corollary 6.5](#) because the usage of [Theorem 1.1](#) requires that X admits regular compactifications (see [Remark 6.3](#)).

7. Milnor K -theory, 0-cycles and de Rham–Witt complex

Our next goal is to show that the cycle class map of [Theorem 1.1](#) completely describes the relative K -theory of truncated polynomial rings in terms of additive 0-cycles in characteristic zero. We shall give a precise formulation of our main result for fields in [Section 9](#) and for semilocal rings in [Section 10](#). In this section, we prove some results on the connection between *a priori* three different objects: the additive 0-cycles, the relative Milnor K -theory and the de Rham–Witt forms. These results will form one of the two keys steps in showing that the cycle class map cyc_k (see [\(7.2\)](#)) factors through the Milnor K -theory in characteristic zero.

7A. The additive 0-cycles. To set up the notations, let k be a field of any characteristic. Let $m \geq 0$ be an integer. Recall that the additive higher Chow groups $\mathrm{TCH}^p(X, *; m)$ of $X \in \mathbf{Sch}_k$ with modulus m are defined so that there are canonical

isomorphisms $\mathrm{Tz}^p(X, n + 1; m) \cong z^p(X \times \mathbb{A}_k^1 \mid X \times (m + 1)\{0\}, n)$ and

$$\mathrm{TCH}^p(X, n + 1; m) \xrightarrow{\cong} \mathrm{CH}^p(X \times \mathbb{A}_k^1 \mid X \times (m + 1)\{0\}, n), \tag{7.1}$$

where the term on the right are the Chow groups with modulus defined in Section 2E. Using a similar isomorphism between the relative K -groups, Theorem 1.1 provides a commutative diagram of pro-abelian groups

$$\begin{CD} \{\mathrm{TCH}^{d+n+1}(X, n + 1; m)\}_m @>\mathrm{cyc}_X>> \{K_{n+1}(X[[t]]/(t^{m+1}), (t))\}_m \\ @V\cong VV @VV\cong V \\ \{\mathrm{CH}^{d+1+n}(X \times \mathbb{A}_k^1 \mid X \times (m + 1)\{0\}, n)\}_m @>\mathrm{cyc}_{\mathbb{A}_k^1 \mid X \times \{0\}}>> \{K_n(X \times \mathbb{A}_k^1, X \times (m + 1)\{0\})\}_m \end{CD} \tag{7.2}$$

for an equidimensional regular scheme X of dimension d and integer $n \geq 0$.

7B. Connection with de Rham–Witt complex. Let k be a field with $\mathrm{char}(k) \neq 2$. Let R be a regular semilocal ring which is essentially of finite type over k . Let $m, n \geq 1$ be two integers. Let $\mathbb{W}_m \Omega_R^*$ be the big de Rham–Witt complex of Hesselholt and Madsen (see [Rülling 2007, §1]). We shall let $\underline{a} = (a_1, \dots, a_m)$ denote a general element of $\mathbb{W}_m(R)$. Recall from [Rülling 2007, Appendix] that there is an isomorphism of abelian groups $\gamma : \mathbb{W}(R) \xrightarrow{\cong} (1 + tR[[t]])^\times$ (with respect to addition in $\mathbb{W}(R)$ and multiplication in $R[[t]]$) such that $\gamma(\underline{a}) = \gamma((a_1, \dots)) = \prod_{i=1}^\infty (1 - a_i t^i)$. This map sends $\mathrm{Ker}(\mathbb{W}(R) \rightarrow \mathbb{W}_m(R))$ isomorphically onto the subgroup $(1 + t^{m+1}k[[t]])^\times$ and hence there is a canonical isomorphism of abelian groups

$$\gamma_m : \mathbb{W}_m(R) \xrightarrow{\cong} \frac{(1 + tR[[t]])^\times}{(1 + t^{m+1}R[[t]])^\times}. \tag{7.3}$$

Under this isomorphism, the Verschiebung map $V_r : \mathbb{W}_m(R) \rightarrow \mathbb{W}_{mr+r-1}(R)$ corresponds to the map on the unit groups induced by the R -algebra homomorphism $R[[t]]/(t^{m+1}) \rightarrow R[[t]]/(t^{r(m+r)})$ under which $t \mapsto t^r$. Recall also that there is a restriction ring homomorphism $\xi_0^R : \mathbb{W}_{m+1}(R) \rightarrow \mathbb{W}_m(R)$ as part of the Witt-complex structure. We shall often use the notation V_r also for the composition

$$\mathbb{W}_m(R) \xrightarrow{V_r} \mathbb{W}_{mr+r-1}(R) \xrightarrow{(\xi_0^R)^{r-1}} \mathbb{W}_{mr}(R).$$

With this interpretation of the Verschiebung map, every element $\underline{a} \in \mathbb{W}_m(R)$ has a unique presentation

$$\underline{a} = \sum_{i=1}^m V_i([a_i]_{\lfloor m/i \rfloor}), \tag{7.4}$$

where for a real number $x \in \mathbb{R}_{\geq 0}$, one writes $\lfloor x \rfloor$ for the greatest integer not bigger than x and $[\]_{\lfloor x \rfloor} : R \rightarrow \mathbb{W}_{\lfloor x \rfloor}(R)$ for the Teichmüller map $[a]_{\lfloor x \rfloor} = (a, 0, \dots, 0)$.

It was shown in [Krishna and Park 2016, Theorem 7.10] that $\{\mathrm{TCH}^*(R, *, m)\}_{m \geq 1}$ is a pro-differential graded algebra which has the structure of a restricted Witt-complex over R in the sense of [Rülling 2007, Definition 1.14]. Using the universal property of $\{\mathbb{W}_m \Omega_R^*\}_{m \geq 1}$ as the universal restricted Witt-complex over R , one gets a functorial morphism of restricted Witt-complexes

$$\tau_{n,m}^R : \mathbb{W}_m \Omega_R^{n-1} \rightarrow \mathrm{TCH}^n(R, n; m). \tag{7.5}$$

It was shown in [Krishna and Park 2020a, Theorem 1.0.2] that this map is an isomorphism. When R is a field, this isomorphism was shown earlier by [Rülling 2007]. We shall use this isomorphism throughout the remaining part of this paper and consequently, will usually make no distinction between the source and target of this map.

7C. Connection with Milnor K -theory. Continuing with the above notations, we have another set of maps

$$\begin{array}{ccc} & \mathrm{TCH}^1(R, 1; m) \otimes \mathrm{CH}^{n-1}(R, n-1) & \\ \tau_{1,m}^R \otimes \nu_{n-1}^R \nearrow & & \searrow \psi_{n,m}^R \\ \mathbb{W}_m(R) \otimes K_{n-1}^M(R) & & \mathrm{TCH}^n(R, n; m). \end{array} \tag{7.6}$$

Here, $\nu_n^R : K_n^M(R) \rightarrow \mathrm{CH}^n(R, n)$ is the semilocal ring analog of the Milnor-Chow homomorphism of Totaro [1992]. It takes a Milnor symbol $\{b_1, \dots, b_{n-1}\}$ to the graph of the function $(b_1, \dots, b_{n-1}) : \mathrm{Spec}(R) \rightarrow \square^{n-1}$. A combination of the main results [Elbaz-Vincent and Müller-Stach 2002; Kerz 2009] implies that this map is an isomorphism. The map $\psi_{n,m}^R$ is given by the action of higher Chow groups on the additive higher Chow groups, shown in [Krishna and Levine 2008]. It takes cycles $\alpha \in \mathrm{TCH}^i(R, n; m)$ and $\beta \in \mathrm{CH}^j(R, n')$ to $\Delta_R^*(\alpha \times \beta) \in \mathrm{TCH}^{i+j}(R, n+n'; m)$, where

$$\Delta_R : \mathrm{Spec}(R) \times \mathbb{A}_k^1 \times \square^{n+n'-1} \rightarrow \mathrm{Spec}(R) \times \mathrm{Spec}(R) \times \mathbb{A}_k^1 \times \square^{n+n'-1}$$

is the diagonal on $\mathrm{Spec}(R)$ and identity on $\mathbb{A}_k^1 \times \square^{n+n'-1}$.

Lemma 7.1. For $\underline{a} = (a_1, \dots, a_m) \in \mathbb{W}_m(R)$ and $\underline{b} = \{b_1, \dots, b_{n-1}\} \in K_{n-1}^M(R)$, one has $\psi_{n,m}^R \circ (\tau_{1,m}^R \otimes \nu_{n-1}^R)(\underline{a} \otimes \underline{b}) = [Z]$, where

$$Z \subset \mathrm{Spec}(R) \times \mathbb{A}_k^1 \times \square^{n-1} \cong \mathbb{A}_R^1 \times_R \square_R^{n-1}$$

is the closed subscheme given by

$$Z = \left\{ (t, y_1, \dots, y_n) \mid \prod_{i=1}^m (1 - a_i t^i) = y_1 - b_1 = \dots = y_{n-1} - b_{n-1} = 0 \right\}. \tag{7.7}$$

Proof. We let $f(t) = \prod_{i=1}^m (1 - a_i t^i)$. In view of the description of $\psi_{n,m}^R$, we only have to show that $\tau_{1,m}^R(\underline{a}) = V(f(t))$. But this is a part of the definition of the restricted Witt-complex structure on $\{\mathrm{TCH}^*(R, *, m)\}_{m \geq 1}$ over R (see [Krishna and Park 2016, Proposition 7.6]). \square

Let $\phi_{n,m}^R: \mathbb{W}_m(R) \otimes K_{n-1}^M(R) \rightarrow \mathbb{W}_m \Omega_R^{n-1}$ be the unique map such that $\tau_{n,m}^R \circ \phi_{n,m}^R = \psi_{n,m}^R \circ (\tau_{1,m}^R \otimes \nu_{n-1}^R)$. The following lemma describes the map $\phi_{n,m}^R$.

Lemma 7.2. *For any $\underline{a} \in \mathbb{W}_m(R)$ and $\underline{b} = \{b_1, \dots, b_{n-1}\} \in K_{n-1}^M(R)$, we have*

$$\phi_{n,m}^R(\underline{a} \otimes \underline{b}) = \underline{a} d \log([b_1]) \wedge \cdots \wedge d \log([b_{n-1}]).$$

Proof. For an ideal $I = (f_1, \dots, f_r) \subset R[t, y_1, \dots, y_{n-1}]$, we let $Z(f_1, \dots, f_r)$ denote the closed subscheme of $\mathrm{Spec}(R[t, y_1, \dots, y_{n-1}])$ defined by I . We let $\underline{a} \in \mathbb{W}_m(R)$ and $\underline{b} = \{b_1, \dots, b_{n-1}\} \in K_{n-1}^M(R)$. We write $b = b_1 \cdots b_{n-1} \in R^\times$. Then we have by (7.4),

$$\begin{aligned} \tau_{n,m}^R(\underline{a} d \log([b_1]) \wedge \cdots \wedge d \log([b_{n-1}])) \\ = \sum_{i=1}^m \tau_{n,m}^R(V_i([a_i]_{[m/i]}) d \log([b_1]) \wedge \cdots \wedge d \log([b_{n-1}])). \end{aligned} \quad (7.8)$$

We now recall that $\tau_{n,m}^R$ is a part of the morphism of restricted Witt-complexes. In particular, we have for each $1 \leq i \leq m$,

$$\begin{aligned} \tau_{n,m}^R(V_i([a_i]_{[m/i]}) d \log([b_1]) \wedge \cdots \wedge d \log([b_{n-1}])) \\ = V_i(Z(1 - a_i t)) \left(\prod_{j=1}^{n-1} Z(1 - b_j^{-1} t) \right) \\ \quad \times (d(Z(1 - b_1 t)) \wedge \cdots \wedge d(Z(1 - b_{n-1} t))) \\ =^1 Z(1 - a_i t^i) \left(\prod_{j=1}^{n-1} Z(1 - b_j^{-1} t) \right) \\ \quad \times (d(Z(1 - b_1 t)) \wedge \cdots \wedge d(Z(1 - b_{n-1} t))), \\ =^2 Z(1 - a_i t^i) Z(1 - b^{-1} t) (d(Z(1 - b_1 t)) \wedge \cdots \wedge d(Z(1 - b_{n-1} t))), \end{aligned} \quad (7.9)$$

where $=^1$ follows from the fact the Verschiebung map on the additive higher Chow groups is induced by the pull-back through the power map $\pi_r: \mathbb{A}_R^1 \rightarrow \mathbb{A}_R^1$, given by $\pi_r(t) = t^r$ (see [Krishna and Park 2016, §6]). The equality $=^2$ follows from the fact that the product in $\mathrm{TCH}^1(R, 1; m)$ is induced by the multiplication map $\mu: \mathbb{A}_R^1 \times_R \mathbb{A}_R^1 \rightarrow \mathbb{A}_R^1$ (see [Krishna and Park 2016, §6]).

Since the differential of the additive higher Chow groups is induced by the anti-diagonal map $(t, \underline{y}) \mapsto (t, t^{-1}, \underline{y})$, we see that $d(Z(1 - b_i t)) = Z(1 - b_i t, y_i - b_i)$.

In particular, we get

$$d(Z(1 - b_1 t)) \wedge \cdots \wedge d(Z(1 - b_{n-1} t)) = Z(1 - bt, y_1 - b_1, \dots, y_{n-1} - b_{n-1}).$$

As $Z(1 - b^{-1} t) \cdot Z(1 - bt) = Z(1 - t) = \tau_{1,m}^R(1)$ is the identity element for the differential graded algebra structure on $\text{TCH}^*(R, *; m)$, we therefore get

$$\begin{aligned} \tau_{n,m}^R(V_i([a_i]_{[m/i]}) d \log([b_1]) \wedge \cdots \wedge d \log([b_{n-1}])) \\ = Z(1 - a_i t^i, y_1 - b_1, \dots, y_{n-1} - b_{n-1}). \end{aligned}$$

Combining this with (7.8), we get

$$\begin{aligned} \tau_{n,m}^R(\underline{ad} \log([b_1]) \wedge \cdots \wedge d \log([b_{n-1}])) \\ = Z\left(\prod_{i=1}^m (1 - a_i t^i), y_1 - b_1, \dots, y_{n-1} - b_{n-1}\right). \end{aligned} \quad (7.10)$$

Since $\tau_{n,m}^R$ is an isomorphism, we now conclude the proof by applying Lemma 7.1. \square

7D. Additive 0-cycles in characteristic zero. We shall assume in this subsection that the base field k has characteristic zero. As above, we let R be a regular semilocal ring which is essentially of finite type over k and $m, n \geq 1$ two integers. For any $\underline{b} = \{b_1, \dots, b_{n-1}\} \in K_{n-1}^M(R)$, we let $b = b_1 \cdots b_{n-1} \in R^\times$. Under our assumption on $\text{char}(k)$, we can prove the following result which is the first key step for showing that the cycle class map cyc_k (see (7.2)) factors through the Milnor K -theory in characteristic zero.

Lemma 7.3. *The map*

$$\psi_{n,m}^R \circ (\tau_{1,m}^R \otimes \nu_{n-1}^R): \mathbb{W}_m(R) \otimes K_{n-1}^M(R) \rightarrow \text{TCH}^n(R, n; m)$$

is surjective. In particular, $\psi_{n,m}^R$ is surjective.

Proof. Let $\phi_{n,m}^R: \mathbb{W}_m(R) \otimes K_{n-1}^M(R) \rightarrow \mathbb{W}_m \Omega_R^{n-1}$ be the unique map such that $\tau_{n,m}^R \circ \phi_{n,m}^R = \psi_{n,m}^R \circ (\tau_{1,m}^R \otimes \nu_{n-1}^R)$. The lemma is then equivalent to showing that $\phi_{n,m}^R$ is surjective.

For $\underline{a} \in \mathbb{W}_m(R)$ and $\underline{b} = \{b_1, \dots, b_{n-1}\} \in K_{n-1}^M(R)$, it follows from Lemma 7.2 that $\phi_{n,m}^R(\underline{a} \otimes \underline{b}) = \underline{ad} \log([b_1]) \wedge \cdots \wedge d \log([b_{n-1}])$. We shall now use that the ground field has characteristic zero. Let $p > m$ be a prime. It then follows from [Rülling 2007, Theorem 1.11, Remark 1.12] that there is a canonical (Ghost) isomorphism

$$\zeta_{r,m}^R: \mathbb{W}_m \Omega_R^r \xrightarrow{\cong} \prod_{i=1}^m \Omega_R^r \quad (7.11)$$

such that

$$\zeta_{r,m}^R(xdy_1 \cdots dy_r) = \left(\frac{1}{j^r} F_j(x) dF_j(y_1) \cdots dF_j(y_r) \right)_{1 \leq j \leq m},$$

where $x, y_i \in \mathbb{W}_m(R)$ and $F_j(y_i)$ means its restriction to $\mathbb{W}_1\Omega_R^r$ via the restriction map of the de Rham–Witt complex. In particular, $\zeta_{0,m}^R(x) = \left(\sum_{d \mid j} dx_d^{j/d} \right)_{1 \leq j \leq m}$ is the classical Ghost map, where $x = (x_1, \dots, x_m) \in \mathbb{W}_m(R)$.

It follows that the following diagram commutes:

$$\begin{CD} \mathbb{W}_m(R) \otimes K_{n-1}^M(R) @>\phi_{n,m}^R>> \mathbb{W}_m\Omega_R^{n-1} \\ @V\zeta_{0,m}^R \otimes \text{id} \cong VV @V\cong VV \zeta_{n-1,m}^R \\ \left(\prod_{i=1}^m R\right) \otimes K_{n-1}^M(R) @>>> \prod_{i=1}^m \Omega_R^{n-1}, \end{CD} \tag{7.12}$$

where the bottom arrow is defined componentwise so that $(a_i)_i \otimes \{b_1, \dots, b_{n-1}\}$ maps to $(1/(i^{n-1})a_i d \log(b_1) \wedge \cdots \wedge d \log(b_{n-1}))_i$. Since we are working with characteristic zero field, it then suffices to show that the map $R \otimes K_{n-1}^M(R) \rightarrow \Omega_R^{n-1}$, given by $a \otimes \underline{b} \mapsto ad \log(b_1) \wedge \cdots \wedge d \log(b_{n-1})$, is surjective. By an iterative procedure, it suffices to prove this surjectivity when $n = 2$.

We now let $a, b \in R$. By Lemma 7.4 below, we can write $b = b_1 + b_2$, where $b_1, b_2 \in R^\times$. We then get $adb = adb_1 + adb_2 = ab_1d \log(b_1) + ab_2d \log(b_2)$. Since Ω_R^1 is generated by the universal derivations of the elements of R as an R -module, we are done. □

Lemma 7.4. *Let R be a semilocal ring which contains an infinite field k . Then every element $a \in R$ can be written as $a = u_1 + u_2$, where $u_1 \in k^\times$ and $u_2 \in R^\times$.*

Proof. Let $M = \{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ denote the set of all maximal ideals of R . Fix $a \in R$. Suppose that there exists $u \in k^\times \subseteq R^\times$ such that $a + u \in R^\times$. Then we are done. Otherwise, every element $u \in k^\times$ has the property that $a + u \in \mathfrak{m}_i$ for some i . Since k is infinite and M is finite, there are two distinct elements $u_1, u_2 \in k^\times$ such that $a + u_1$ and $a + u_2$ both belong to a maximal ideal \mathfrak{m}_j . Then we get $u_1 - u_2 \in \mathfrak{m}_j$. But $u_1 \neq u_2$ in k implies that $u_1 - u_2 \in k^\times$ and this forces $\mathfrak{m}_j = R$, a contradiction. We conclude that there must exist $u \in k^\times$ such that $a + u \in R^\times$. □

8. The relative Milnor K -theory

In this section, we shall prove our second key step (see Lemma 8.4) to show the factorization of the cycle class map through the relative Milnor K -theory and prove the isomorphism of the resulting map.

8A. Recollection of relative Hochschild and cyclic homology. In the next two sections, we shall use Hochschild, André–Quillen and cyclic homology of commutative rings as our tools. We refer to [Loday 1998] for their definitions and some properties that we shall use. While using a specific result from [Loday 1998], we shall mention the exact reference.

Let R be a commutative ring. For an integer $m \geq 0$, recall from Section 2A that the truncated polynomial algebra $R[t]/(t^{m+1})$ is denoted by R_m . Throughout our discussion of truncated polynomial algebras, we shall make no distinction between the variable $t \in R[t]$ and its image in R_m .

If $k \subset R$ is a subring, we shall use the notation $HH_*(R|k)$ for the Hochschild homology of R over k . Similarly, $D_*^{(q)}(R|k)$ for $q \geq 0$ and $HC_*(R|k)$ will denote the André–Quillen and cyclic homology of R over k , respectively. When $k = \mathbb{Z}$, we shall write $HH_*(R|k)$ simply as $HH_*(R)$. Similar notations will be used for $D_*^{(q)}(R|\mathbb{Z})$ and $HC_*(R|\mathbb{Z})$. Note that $HH_*(R) \cong HH_*(R|\mathbb{Q})$, $D_*^{(q)}(R) \cong D_*^{(q)}(R|\mathbb{Q})$ and $HC_*(R) \cong HC_*(R|\mathbb{Q})$ if R contains \mathbb{Q} . We also have $\Omega_{R/\mathbb{Z}}^q \cong \Omega_{R/\mathbb{Q}}^q$ for $q \geq 0$.

Recall that for an ideal $I \subset R$, the relative Hochschild homology $HH_*((R, I)|k)$ is defined as the homology of the complex $\text{Ker}(HH(R|k) \rightarrow HH(R/I|k))$, where $k \subset R$ is a subring and $HH(R|k)$ is the Hochschild complex of R over k . The relative cyclic homology $HC_*((R, I)|k)$ is defined to be the homology of the complex $\text{Ker}(CC(R|k) \rightarrow CC(R/I|k))$, where $CC(R|k)$ is the total cyclic complex of R over k . We refer to [Loday 1998, 1.1.16, 2.1.15] for these definitions. One defines $D_*^{(q)}((R, I)|k)$ similarly. If R is a commutative ring, we shall write $HH_*((R_m, (t))|\mathbb{Z})$, $D_*^{(q)}((R_m, (t))|\mathbb{Z})$ and $HC_*((R_m, (t))|\mathbb{Z})$ simply as $\widehat{HH}_*(R_m)$, $\widehat{D}_*^{(q)}(R_m)$ and $\widehat{HC}_*(R_m)$, respectively. We let $\widetilde{\Omega}_{R_m}^n = \text{Ker}(\Omega_{R_m}^n \rightarrow \Omega_R^n)$. Recall from Section 2C that $\widetilde{K}_*(R_m)$ denotes the relative K -theory $K_*(R_m, (t))$. Suppose now that R contains \mathbb{Q} . For $x \in tR_m$, we shall write $\exp(x) = \sum_{i \geq 0} x^i/i!$ and $\log(1+x) = \sum_{i \geq 1} (-1)^{i-1} x^i/i$. Note that these are finite sums and define homomorphisms

$$tR_m \xrightarrow{\exp} \widetilde{K}_1(R_m) \xrightarrow{\log} tR_m, \quad (8.1)$$

which are inverses to each other.

8B. Relative Milnor K -theory of truncated polynomial rings. Let R be a semilocal ring and let $m \geq 0, n \geq 1$ be two integers. We shall write the relative Milnor K -groups (see Section 3) $K_*^M(R_m, (t))$ as $\widetilde{K}_*^M(R_m)$. Since $\widetilde{K}_*(R_m)$ is same as the kernel of the augmentation map $K_*(R_m) \rightarrow K_*(R)$, we have the canonical map $\psi_{R_m}: \widetilde{K}_*^M(R_m) \rightarrow \widetilde{K}_*(R_m)$.

Let us now assume that R is a semilocal ring containing \mathbb{Q} . Recall that there is a Dennis trace map $\text{tr}_{m,n}^R: K_n(R_m) \rightarrow HH_n(R_m)$ which restricts to the dlog map on the Milnor K -theory (e.g., see [Geller and Weibel 1994, Example 2.1]).

Equivalently, there is a commutative diagram

$$\begin{array}{ccc}
 K_n^M(R_m) & \xrightarrow{d \log} & \Omega_{R_m}^n \\
 \psi_{R_m, n} \downarrow & & \downarrow \epsilon_n \\
 K_n(R_m) & \xrightarrow{\text{tr}_{m, n}^R} & HH_n(R_m),
 \end{array} \tag{8.2}$$

where ϵ_n is the canonical antisymmetrization map from Kähler differentials to Hochschild homology (see [Loday 1998, §1.3.4]).

A very well known result of [Goodwillie 1985] says that the relativization of the Dennis trace map with respect to a nilpotent ideal factors through a trace map $\tilde{K}_n(R_m) \rightarrow \widetilde{HC}_{n-1}(R_m)$ such that its composition with the canonical Connes' periodicity map $B: \widetilde{HC}_{n-1}(R_m) \rightarrow \widetilde{HH}_n(R_m)$ is the relative Dennis trace map. This factorization is easily seen on $\tilde{K}_1(R_m)$ via the chain of maps

$$\tilde{K}_1(R_m) \xrightarrow{\log} tR_m \cong \widetilde{HC}_0(R_m) \xrightarrow{d} \widetilde{HH}_1(R_m) \cong \tilde{\Omega}_{R_m}^1. \tag{8.3}$$

Recall that Connes' periodicity map B on $HC_0(R_m)$ coincides with the differential $d: R_m \rightarrow \Omega_{R_m}^1$ under the isomorphisms $R_m \cong HC_0(R_m)$ and $\Omega_{R_m}^1 \cong HH_1(R_m)$. Goodwillie showed that his factorization $\tilde{K}_n(R_m) \rightarrow \widetilde{HC}_{n-1}(R_m)$ is an isomorphism of \mathbb{Q} -vector spaces. We shall denote this Goodwillie's isomorphism also by $\text{tr}_{m, n}^R$.

Going further, Cathelineau [1990, Theorem 1] showed that the K -group $\tilde{K}_n(R_m)$ and the relative cyclic homology group $\widetilde{HC}_{n-1}(R_m)$ are λ -rings. Furthermore, Goodwillie's map is an isomorphism of λ -rings, thanks to [Cortiñas and Weibel 2009, Theorem 6.5.1]. In particular, it induces an isomorphism between the Adams graded pieces $\text{tr}_{m, n}^R: \tilde{K}_n^{(q)}(R_m) \xrightarrow{\cong} \widetilde{HC}_{n-1}^{(q-1)}(R_m)$ for every $1 \leq q \leq n$. As a corollary of this isomorphism and [Loday 1998, Theorem 4.6.8], we get the following:

Lemma 8.1. *The Dennis trace map induces an isomorphism of \mathbb{Q} -vector spaces*

$$\text{tr}_{m, n}^R: \tilde{K}_n^{(n)}(R_m) \xrightarrow{\cong} \frac{\tilde{\Omega}_{R_m}^{n-1}}{d\tilde{\Omega}_{R_m}^{n-2}}.$$

In order to relate these groups with the Milnor K -theory, we first observe that as the map $K_1^M(R_m) \rightarrow K_1(R_m)$ is an isomorphism, it follows from the properties of γ -filtration associated to the λ -ring structure on K -theory that the canonical map $K_n^M(R_m) \rightarrow K_n(R_m)$ factors through $K_n^M(R_m) \rightarrow F_\gamma^n K_n(R_m)$, where $F_\gamma^\bullet K_n(R_m)$ denotes the γ -filtration. If we consider the induced map on the relative K -groups, it follows that the canonical map $\tilde{K}_n^M(R_m) \rightarrow \tilde{K}_n(R_m)$ factors as

$$\psi_{R_m, n}: \tilde{K}_n^M(R_m) \rightarrow \tilde{K}_n^{(n)}(R_m) = F_\gamma^n \tilde{K}_n(R_m) \hookrightarrow \tilde{K}_n(R_m).$$

It follows from [Stienstra 1985, Theorem 12.3] that $\text{Ker}(\psi_{R_m, n})$ is a torsion group. On the other hand, it follows from [Gorchinskiĭ and Tyurin 2018, Proposition 5.4] that $\tilde{K}_n^M(R_m)$ is a \mathbb{Q} -vector space. If we now apply Soulé's [1985, Théorème 2] computation of $F_\gamma^n \tilde{K}_n(R_m)$, we conclude that the map $\psi_{R_m, n}$ is in fact an isomorphism. We have thus shown the following.

Lemma 8.2. *The maps*

$$\tilde{K}_n^M(R_m) \xrightarrow{\psi_{R_m, n}} \tilde{K}_n^{(n)}(R_m) \xrightarrow{\text{tr}_{m, n}^R} \frac{\tilde{\Omega}_{R_m}^{n-1}}{d\tilde{\Omega}_{R_m}^{n-2}}$$

are all isomorphisms of \mathbb{Q} -vector spaces.

One knows from [Geller and Weibel 1989, 3.4.4] that the map

$$d = B: \tilde{\Omega}_{R_m}^{n-1}/d\tilde{\Omega}_{R_m}^{n-2} \rightarrow \tilde{\Omega}_{R_m}^n$$

is injective. Using the fact that $\text{tr}_{m, n}^R: \tilde{K}_n^M(R_m) \rightarrow \tilde{\Omega}_{R_m}^n$ is multiplicative (e.g., see [Kantorovitz 1999, Property 1.3] or [Loday 1998, 8.4.12]) and it is the usual logarithm on $\tilde{K}_1^M(R_m)$ (see (8.3)), it follows from (8.2) that modulo $d\tilde{\Omega}_{R_m}^{n-2}$, the composite map $\text{tr}_{n, m}^R \circ \psi_{R_m, n}$ is the dlog map:

$$\begin{aligned} d \log(\{1 - tf(t), b_1, \dots, b_{n-1}\}) \\ = \log(1 - tf(t))d \log(b_1) \wedge \dots \wedge d \log(b_{n-1}). \end{aligned} \quad (8.4)$$

8C. More refined structure on $\tilde{K}_n^M(R_m)$. We shall further simplify the presentation of $\tilde{K}_n^M(R_m)$ in the next result. This will be our second key step in factoring the cycle class map through the relative Milnor K -theory and showing that the resulting map is an isomorphism. For this, we assume $m \geq 1$ and look at the diagram

$$\begin{array}{ccc} tR_m \otimes K_{n-1}^M(R) & \longrightarrow & \tilde{K}_n^M(R_m) \\ \text{id} \otimes d \log \downarrow & & \downarrow \text{tr}_{m, n}^R \circ \psi_{R_m, n} \\ tR_m \otimes \Omega_R^{n-1} & & \\ \downarrow & \nearrow \text{dotted arrow} & \\ tR_m \otimes_R \Omega_R^{n-1} & \xrightarrow{\theta_{R_m}^{n-1}} & \tilde{\Omega}_{R_m}^{n-1}/d\tilde{\Omega}_{R_m}^{n-2}, \end{array} \quad (8.5)$$

where the top horizontal arrow is the product

$$a \otimes \{b_1, \dots, b_{n-1}\} \mapsto \{\exp(a), b_1, \dots, b_{n-1}\}.$$

The map $\theta_{R_m}^{n-1}$ in (8.5) is the composition of the canonical map $tR_m \otimes_R \Omega_R^{n-1} \rightarrow \tilde{\Omega}_{R_m}^{n-1}$ (sending $t^i \otimes \omega$ to $t^i \omega$) with the surjection $\tilde{\Omega}_{R_m}^{n-1} \rightarrow \tilde{\Omega}_{R_m}^{n-1}/d\tilde{\Omega}_{R_m}^{n-2}$. Using (8.4),

it is easy to check that (8.5) is commutative. It follows from Lemma 8.2 that $\theta_{R_m}^{n-1}$ factors through a unique map $\tilde{\theta}_{R_m}^n : tR_m \otimes_R \Omega_R^{n-1} \rightarrow \tilde{K}_n^M(R_m)$. We want to show that this map is an isomorphism. Equivalently, $\theta_{R_m}^{n-1}$ is an isomorphism. Since the proof of this is a bit long, we prove that it is surjective and injective in separate lemmas.

Lemma 8.3. *The map $\theta_{R_m}^n$ is surjective for all $n \geq 0$.*

Proof. This is obvious for $n = 0$ and so we assume $n \geq 1$. We now consider the exact sequence

$$R_m \otimes_R \Omega_R^1 \rightarrow \Omega_{R_m}^1 \rightarrow \Omega_{R_m/R}^1 \rightarrow 0.$$

We claim that the first arrow in this sequence is split injective. For this, we consider the map $d' : R_m \rightarrow R_m \otimes_R \Omega_R^1$, given by $d'(\sum_{i=0}^m a_i t^i) = \sum_{i=0}^m t^i \otimes d(a_i)$. The computation

$$\begin{aligned} d' \left(\left(\sum_i a_i t^i \right) \left(\sum_j b_j t^j \right) \right) &= d' \left(\sum_{i,j} a_i b_j t^{i+j} \right) \\ &= \sum_{i,j} t^{i+j} d(a_i b_j) \\ &= \sum_{i,j} b_j t^{i+j} \otimes d(a_i) + \sum_{i,j} a_i t^{i+j} \otimes d(b_j) \\ &= \left(\sum_j b_j t^j \right) \otimes d' \left(\sum_i a_i t^i \right) + \left(\sum_i a_i t^i \right) \otimes d' \left(\sum_j b_j t^j \right) \end{aligned}$$

shows that d' is a \mathbb{Q} -linear derivation on the R_m -module $\Omega_R^1 \otimes_R R_m$. Hence, it induces an R_m -linear map $u : \Omega_{R_m}^1 \rightarrow R_m \otimes_R \Omega_R^1$. Moreover, it is clear that the composite $R_m \otimes_R \Omega_R^1 \rightarrow \Omega_{R_m}^1 \xrightarrow{u} R_m \otimes_R \Omega_R^1$ is identity. This proves the claim. We thus get a direct sum decomposition of R_m -modules $\Omega_{R_m}^1 = (R_m \otimes_R \Omega_R^1) \oplus \Omega_{R_m/R}^1$. As $\Omega_{R_m/R}^n = 0$ for $n \geq 2$, we get $\Omega_{R_m}^n = (R_m \otimes_R \Omega_R^n) \oplus (\Omega_{R_m/R}^1 \otimes_R \Omega_R^{n-1})$ for any $n \geq 1$. This implies that

$$\tilde{\Omega}_{R_m}^n = (tR_m \otimes_R \Omega_R^n) \oplus (\Omega_{R_m/R}^1 \otimes_R \Omega_R^{n-1}). \quad (8.6)$$

The other thing we need to observe is that the exact sequence

$$(t^{m+1})/(t^{2m+2}) \xrightarrow{d} \Omega_{R[t]/R}^1 \otimes_{R[t]} R_m \rightarrow \Omega_{R_m/R}^1 \rightarrow 0$$

implies that $\Omega_{R_m/R}^1 \cong R_m dt / (t^m) dt$. In particular, $\Omega_{R_m/R}^1 \otimes_R \Omega_R^{n-1}$ is generated as an R -module by elements of the form $(\sum_{i=0}^{m-1} a_i t^i dt) \otimes \omega$, where $a_i \in R$.

We now let $\omega \in \Omega_R^{n-1}$. We then get

$$\begin{aligned}
 d\left(\sum_{i=0}^{m-1} \frac{a_i}{i+1} t^{i+1} \otimes \omega\right) &= \sum_{i=0}^{m-1} \frac{a_i}{i+1} t^{i+1} \otimes d\omega + \sum_{i=0}^{m-1} \frac{da_i}{i+1} t^{i+1} \otimes \omega + \sum_{i=0}^{m-1} a_i t^i dt \otimes \omega \\
 &= \sum_{i=0}^{m-1} t^{i+1} \otimes \frac{a_i d\omega + da_i \wedge \omega}{i+1} + \sum_{i=0}^{m-1} a_i t^i dt \otimes \omega. \tag{8.7}
 \end{aligned}$$

It follows from this that the composite map $tR_m \otimes_R \Omega_R^n \hookrightarrow \tilde{\Omega}_{R_m}^n \rightarrow \tilde{\Omega}_{R_m}^n / d\tilde{\Omega}_{R_m}^{n-1}$ is surjective. \square

Lemma 8.4. *For $m, n \geq 1$, the map*

$$\begin{aligned}
 \tilde{\theta}_{R_m}^n : tR_m \otimes_R \Omega_R^{n-1} &\rightarrow \tilde{K}_n^M(R_m); \\
 \tilde{\theta}_{R_m}^n(a \otimes d \log(b_1) \wedge \cdots \wedge d \log(b_{n-1})) &= \{\exp(a), b_1, \dots, b_{n-1}\},
 \end{aligned}$$

is an isomorphism.

Proof. In view of Lemmas 8.2 and 8.3, the assertion that $\tilde{\theta}_{R_m}^n$ is an isomorphism for all $m, n \geq 1$ is equivalent to showing that the map $\theta_{R_m}^n : tR_m \otimes_R \Omega_R^n \rightarrow \tilde{\Omega}_{R_m}^n / d\tilde{\Omega}_{R_m}^{n-1}$ (see (8.5)) is injective for all $n \geq 0$ and $m \geq 1$. We can again assume that $n \geq 1$. We shall prove this by induction on $m \geq 1$. Assume first that $m = 1$. In this case, we want to show that the map $tR \otimes_R \Omega_R^n \rightarrow \tilde{\Omega}_{R_1}^n / d\tilde{\Omega}_{R_1}^{n-1}$ is injective. To show this, we should first observe that every element of $tR \otimes_R \Omega_R^{n-1}$ must be of the form $t \otimes \omega$ and every element of $Rdt \otimes_R \Omega_R^{n-2}$ must be of the form $dt \otimes \omega'$. In this case, we get

$$d(t \otimes \omega + dt \otimes \omega') = dt \wedge \omega + t d\omega - dt \wedge d\omega' = t \otimes d\omega + dt \otimes (\omega - d\omega'). \tag{8.8}$$

If $d(t \otimes \omega + dt \otimes \omega')$ has to lie in $Rt \otimes_R \Omega_R^n$, then we must have $dt \otimes (\omega - d\omega') = 0$. But this implies that $\omega = d\omega'$. Putting this in (8.8), we get $t \otimes d\omega = 0$. This shows that $d\tilde{\Omega}_{R_1}^{n-1} \cap (Rt \otimes_R \Omega_R^n) = 0$. But this is equivalent to the desired injectivity.

To prove the $m \geq 2$ case by induction, we shall need the following

Claim. The restriction map $\text{Ker}(\tilde{\Omega}_{R_{m+1}}^{n-1} \xrightarrow{d} \tilde{\Omega}_{R_{m+1}}^n) \rightarrow \text{Ker}(\tilde{\Omega}_{R_m}^{n-1} \xrightarrow{d} \tilde{\Omega}_{R_m}^n)$ is surjective for every $m \geq 1$.

To prove the claim, recall from (8.6) that

$$\begin{aligned}
 \tilde{\Omega}_{R_m}^{n-1} &\cong (tR_m \otimes_R \Omega_R^{n-1}) \oplus (R_m dt / (t^m) dt \otimes_R \Omega_R^{n-1}) \\
 &\cong \left(\bigoplus_{i=0}^{m-1} Rt^{i+1} \otimes_R \Omega_R^{n-1} \right) \oplus \left(\bigoplus_{i=0}^{m-1} Rt^i dt \otimes_R \Omega_R^{n-2} \right). \tag{8.9}
 \end{aligned}$$

Suppose that $\omega \in \tilde{\Omega}_{R_m}^{n-1}$ is such that $d\omega = 0$. It follows from (8.9) that we can write $\omega = \sum_{i=0}^{m-1} (i+1)^{-1} t^{i+1} \omega_i + \sum_{i=0}^{m-1} t^i dt \wedge \omega'_i$. Therefore, we have

$$\begin{aligned} d\omega &= \sum_{i=0}^{m-1} (i+1)^{-1} t^{i+1} d\omega_i + \sum_{i=0}^{m-1} t^i dt \wedge d\omega_i - \sum_{i=0}^{m-1} t^i dt \wedge d\omega'_i \\ &= \sum_{i=0}^{m-1} (i+1)^{-1} t^{i+1} d\omega_i + \sum_{i=0}^{m-1} t^i dt \wedge (\omega_i - d\omega'_i). \end{aligned} \quad (8.10)$$

Since the left-hand term is zero, it implies from the decomposition (8.9) (for $\tilde{\Omega}_{R_m}^n$) that $d\omega_i = 0 = \omega_i - d\omega'_i$ for $0 \leq i \leq m-1$. If we now let $\omega' = \sum_{i=0}^{m-1} (i+1)^{-1} t^{i+1} \omega'_i \in \tilde{\Omega}_{R_{m+1}}^{n-2}$, we get

$$d\omega' = \sum_{i=0}^{m-1} (i+1)^{-1} t^{i+1} d\omega'_i + \sum_{i=0}^{m-1} t^i dt \wedge \omega'_i = \omega. \quad (8.11)$$

The claim now follows from the surjectivity of the map $\tilde{\Omega}_{R_{m+1}}^{n-2} \rightarrow \tilde{\Omega}_{R_m}^{n-2}$. Indeed, this implies that some $\hat{\omega} \in \tilde{\Omega}_{R_{m+1}}^{n-2}$ maps onto ω' and therefore $d\hat{\omega} \in \tilde{\Omega}_{R_{m+1}}^{n-1}$ maps onto ω . Since $d\hat{\omega}$ is clearly a closed form, we are done.

In order to use the above claim, we let $F_m^n = (Rt^m \otimes_R \Omega_R^n) \oplus (Rt^{m-1} dt \otimes_R \Omega_R^{n-1})$. We then have an exact sequence of R_{m+1} -modules

$$0 \rightarrow F_{m+1}^n \rightarrow \tilde{\Omega}_{R_{m+1}}^n \rightarrow \tilde{\Omega}_{R_m}^n \rightarrow 0. \quad (8.12)$$

Letting $m \geq 1$ and taking the quotient of this short exact sequence by the similar exact sequence for $n-1$ via the differential map, we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & Rt^{m+1} \otimes_R \Omega_R^n & \rightarrow & tR_{m+1} \otimes_R \Omega_R^n & \rightarrow & tR_m \otimes_R \Omega_R^n \rightarrow 0 \\ & & \downarrow & & \downarrow \theta_{R_{m+1}}^n & & \downarrow \theta_{R_m}^n \\ 0 & \rightarrow & F_{m+1}^n / dF_{m+1}^{n-1} & \rightarrow & \tilde{\Omega}_{R_{m+1}}^n / d\tilde{\Omega}_{R_{m+1}}^{n-1} & \rightarrow & \tilde{\Omega}_{R_m}^n / d\tilde{\Omega}_{R_m}^{n-1} \rightarrow 0. \end{array} \quad (8.13)$$

The top row is clearly exact and the above claim precisely says that the bottom sequence is exact. The right vertical arrow is injective by induction. It suffices therefore to show that the map $Rt^{m+1} \otimes_R \Omega_R^n \rightarrow F_{m+1}^n / dF_{m+1}^{n-1}$ is injective. The proof of this is almost identical to that of $m=1$ case. Indeed, it is easy to check that every element of F_{m+1}^{n-1} must be of the form $t^{m+1} \otimes \omega + t^m dt \otimes \omega'$. We therefore get

$$\begin{aligned} d(t^{m+1} \otimes \omega + t^m dt \otimes \omega') &= t^{m+1} d\omega + (m+1)t^m dt \wedge \omega - t^m dt \wedge d\omega' \\ &= t^{m+1} d\omega + t^m dt \wedge ((m+1)\omega - d\omega') \\ &= t^{m+1} d\omega + t^m dt \otimes ((m+1)\omega - d\omega'). \end{aligned} \quad (8.14)$$

If $d(t^{m+1} \otimes \omega + t^m dt \otimes \omega')$ lies in $Rt^{m+1} \otimes_R \Omega_R^n$, then we must have $t^m dt \otimes ((m+1)\omega - d\omega'_i) = 0$. But this implies that $\omega = d((m+1)^{-1}\omega')$. Putting this in (8.14), we get $t^{m+1} \otimes d\omega = 0$. This shows that $dF_{m+1}^{n-1} \cap (Rt^{m+1} \otimes_R \Omega_R^n) = 0$. Equivalently, the left vertical arrow in (8.10) is injective. This proves that $\theta_{R_m}^n$ is injective for all $m \geq 1$ and completes the proof of the lemma. \square

We shall also need the following related result later on.

Lemma 8.5. *Let R be as above. Let $n \geq 0$ and $m \geq 1$ be integers. Then the map $d: tR_m \otimes_R \Omega_R^n \rightarrow \tilde{\Omega}_{R_m}^{n+1}$ is injective.*

Proof. Suppose $\omega = \sum_{i=0}^{m-1} t^{i+1} \otimes \omega_i$ and $d(\omega) = 0$. That is, $\sum_{i=0}^{m-1} t^{i+1} \otimes d\omega_i + \sum_{i=0}^{m-1} (i+1)t^i dt \otimes \omega_i = 0$. But this implies by (8.6) that $\sum_{i=0}^{m-1} (i+1)t^i dt \otimes \omega_i = 0$. Since $\Omega_{R_m/R}^1 \otimes_R \Omega_R^n \cong \Omega_R^n dt \oplus \dots \oplus \Omega_R^n t^{m-1} dt \cong (\Omega_R^n)^m$ as an R -module, we must have $\omega_i = 0$ for each i . In particular, we have $\omega = 0$. \square

9. The cycle class map in characteristic zero

In this section, we shall show that the cycle class map for the additive 0-cycles completely describes the relative K -theory of the truncated polynomial rings over a characteristic zero field in terms of additive 0-cycles. This was perhaps the main target for the introduction of the additive higher Chow groups by Bloch and Esnault [2003a]. We formulate our precise result as follows.

Let k be a characteristic zero field. Let R be a regular semilocal ring which is essentially of finite type over k . Let $m \geq 0, n \geq 1$ be two integers. Recall from Section 8B that the canonical map from the Milnor to the Quillen K -theory induces a map $\psi_{R_m, n}: \tilde{K}_n^M(R_m) \rightarrow \tilde{K}_n(R_m)$. Since this map is clearly compatible with change in $m \geq 0$, we have a strict map of pro-abelian groups

$$\psi_{R, n}: \{\tilde{K}_n^M(R_m)\}_m \rightarrow \{\tilde{K}_n(R_m)\}_m. \tag{9.1}$$

In this section, we shall restrict our attention to the case when R is the base field k itself. We shall prove a general result for regular semilocal rings in the next section. In the case of the field k , every integer $n \geq 1$ has associated to it a diagram of pro-abelian groups:

$$\begin{array}{ccc} & & \{\tilde{K}_n^M(k_m)\}_m \\ & \nearrow & \downarrow \psi_k \\ \{\text{TCH}^n(k, n; m)\}_m & \xrightarrow{\text{cyc}_k} & \{\tilde{K}_n(k_m)\}_m \end{array} \tag{9.2}$$

The following is our main result.

Theorem 9.1. *If k is a field of characteristic zero, then all maps in (9.2) are isomorphisms.*

The proof of this theorem will be done by combining the results of [Section 7](#) and [Section 8](#) with a series of new steps.

9A. Factorization of cyc_k into Milnor K -theory. We follow two step strategy for proving [Theorem 9.1](#). We shall first show that cyc_k factors through the Milnor K -theory and the resulting map is an isomorphism. The second step will be to compare the Milnor and Quillen K -groups in the pro-setting. Apart from showing factorization through the Milnor K -theory, the following result also improves [Theorem 1.1](#) in that it tells us that for additive higher Chow groups of 0-cycles, the cycle class map is a strict morphism of pro-abelian groups (see [Section 2B](#)).

Lemma 9.2. *Let $m \geq 0, n \geq 1$ be two integers. Then the following hold.*

(1) *The map $\text{cyc}_k : \text{Tz}^n(k, n; m) \rightarrow \widetilde{K}_n(k_m)$ descends to a group homomorphism*

$$\text{cyc}_k : \text{TCH}^n(k, n; m) \rightarrow \widetilde{K}_n(k_m).$$

(2) *The map cyc_k has a factorization*

$$\text{TCH}^n(k, n; m) \xrightarrow{\text{cyc}_k^M} \widetilde{K}_n^M(k_m) \xrightarrow{\psi_{k_m, n}} \widetilde{K}_n(k_m).$$

Proof. Since $\text{char}(k) = 0$, we know by the main result of [\[Rülling 2007\]](#) (see also [\[Krishna and Park 2016, Theorem 1.2\]](#)) that each $\text{TCH}^n(k, n; m)$ is a \widetilde{k} -vector space. Similarly, $\widetilde{K}_n(k_m)$ is a \mathbb{Q} -vector space because it is isomorphic to $\widetilde{HC}_{n-1}(k_m)$ by [\[Goodwillie 1985\]](#). The first part of the lemma therefore follows directly from [Theorem 6.4](#).

We shall now prove the second part. Since $\psi_{k_m, n}$ is injective for each $m \geq 0, n \geq 1$ by [Lemma 8.2](#), we only need to show that cyc_k takes a set of generators of the group $\text{TCH}^n(k, n; m)$ to $\widetilde{K}_n^M(k_m)$.

We first assume that $n = 1$. Let $z \in \mathbb{A}_k^1$ be a closed point. We can write $z = \text{Spec}(k[t]/(f(t)))$, where $f(t)$ is an irreducible polynomial. The modulus condition for z implies that $f(0) \in k^\times$. If we let $g(t) = (f(0))^{-1}f(t)$ and let $\overline{g(t)}$ denote the image of $g(t)$ in k_m , then we see that $\overline{g(t)} \in \widetilde{K}_1^M(k_m)$. By the definition of the cycle class map in [\(4.3\)](#), we have that $\text{cyc}_k(z) = [k(z)] \in K_0(\mathbb{A}_k^1, (m+1)\{0\})$. But [Lemma 2.1](#) says that $[k(z)] = \partial(\overline{g(t)})$ under the isomorphism $\widetilde{K}_1^M(k_m) = \widetilde{K}_1(k_m) \xrightarrow{\partial} K_0(k[t], (t^{m+1}))$. So we are done.

Suppose now that $n \geq 2$. In this case, [Lemma 7.3](#) says that $\text{TCH}^n(k, n; m)$ is generated by closed points $z \in \mathbb{A}_k^1 \times \square^{n-1}$ which lie in $\mathbb{A}_k^1 \times \mathbb{G}_m^{n-1} \subset \mathbb{A}_k^n$. Furthermore, $z \in \mathbb{A}_k^n$ is defined by an ideal $I \subset k[t, y_1, \dots, y_{n-1}]$ of the type $I = (f(t), y_1 - b_1, \dots, y_{n-1} - b_{n-1})$, where $b_i \in k^\times$ for each $1 \leq i \leq n-1$. Since z is a closed point, $f(t)$ must be an irreducible polynomial in $k[t]$. Moreover, $f(t)$ defines an element of $\mathbb{W}_m(k) \cong \widetilde{K}_1^M(k_m)$. In particular, $f(0) \in k^\times$.

We next note that the push-forward map $K_*(k(z)) \rightarrow K_*(k(z'))$ is $K_*(k)$ -linear, where $z' = \text{Spec}(k[t]/(f(t)))$ (see [Thomason and Trobaugh 1990, Chapter 3]). The map $K_*(k(z')) \rightarrow K_*(k[t], (t^{m+1}))$ is $K_*(k)$ -linear by Lemma 2.2. It follows that the composition $K_*(k(z)) \rightarrow K_*(k[t], (t^{m+1}))$ is $K_*(k)$ -linear.

It follows therefore from the definition of the cycle class map in (4.3) that under the map $\text{cyc}_k: \text{Tz}^n(k, n; m) \rightarrow K_{n-1}(k[t], (t^{m+1}))$, we have

$$\text{cyc}_k([z]) = \{b_1, \dots, b_{n-1}\} \cdot [k(z')] \in K_{n-1}^M(k) \cdot K_0(k[t], (t^{m+1})) \subseteq K_{n-1}(k[t], (t^{m+1})),$$

where $\{b_1, \dots, b_{n-1}\} \in K_{n-1}^M(k)$. We let $g(t) = (f(0))^{-1}f(t)$ and let $\overline{g(t)}$ be the image of $g(t)$ in k_m via the surjection $k[t] \rightarrow k_m$.

Since $\partial: \widetilde{K}_n(k_m) \xrightarrow{\cong} K_{n-1}(k[t], (t^{m+1}))$ is $K_*(k)$ -linear, we see that

$$\{b_1, \dots, b_{n-1}\} \cdot [k(z')] = \{b_1, \dots, b_{n-1}\} \cdot \partial(\overline{g(t)}) = \partial(\{b_1, \dots, b_{n-1}\} \cdot \overline{g(t)}).$$

We are now done because

$$\{b_1, \dots, b_{n-1}\} \cdot \overline{g(t)} \in K_{n-1}^M(k) \cdot \widetilde{K}_1^M(k_m) \subseteq \widetilde{K}_n^M(k_m).$$

We have thus shown that $\text{cyc}_k([z])$ lies in the image of $\widetilde{K}_n^M(k_m)$ under the map ∂ . \square

9B. The main result for cyc_k^M . Let k be a characteristic zero field as before. We shall now show that the cycle class map cyc_k^M that we obtained in Lemma 9.2 is an isomorphism. Since the map $\tau_{n,m}^k: \mathbb{W}_m \Omega_k^{n-1} \rightarrow \text{TCH}^n(k, n; m)$ is an isomorphism, we shall make no distinction between $\mathbb{W}_m \Omega_k^{n-1}$ and $\text{TCH}^n(k, n; m)$ throughout our discussion of the proof of Theorem 9.1. Furthermore, we shall denote $\text{cyc}_k \circ \tau_{n,m}^k$ also by cyc_k in what follows.

Theorem 9.3. *For every pair of integers $m \geq 0, n \geq 1$, the map*

$$\text{cyc}_k^M: \text{TCH}^n(k, n; m) \rightarrow \widetilde{K}_n^M(k_m)$$

is an isomorphism.

Proof. When $m = 0$, the group on the right of cyc_k is zero by definition and the one on the left is zero by [Krishna and Park 2017b, Theorem 6.3]. We can therefore assume that $m \geq 1$.

We can replace $\text{TCH}^n(k, n; m)$ by $\mathbb{W}_m \Omega_k^{n-1}$. Accordingly, we can identify cyc_k^M with $\text{cyc}_k^M \circ \tau_{n,m}^k$. Let $\eta_m^k: \Omega_k^{n-1} \rightarrow t^m k_m \otimes_k \Omega_k^{n-1}$ denote the (k -linear) map $\eta_m^k(a\omega) = -at^m \otimes \omega$. This is clearly an isomorphism of k -vector spaces.

We shall prove the theorem by induction on $m \geq 1$. Suppose first that $m = 1$. In this case, it follows from Lemmas 7.1 and 7.2 that

$$\begin{aligned}
 \text{cyc}_k^M(ad \log(b_1) \wedge \cdots \wedge d \log(b_{n-1})) & \\
 &= \text{cyc}_k^M(Z(1 - at, y_1 - b_1, \dots, y_{n-1} - b_{n-1})) \\
 &= \{1 - at, b_1, \dots, b_{n-1}\} \\
 &= \tilde{\theta}_{k_1}^n((-at) \otimes d \log(b_1) \wedge \cdots \wedge d \log(b_{n-1})) \\
 &= \tilde{\theta}_{k_1}^n \circ \eta_1^k(ad \log(b_1) \wedge \cdots \wedge d \log(b_{n-1})).
 \end{aligned}$$

It follows from this that $\text{cyc}_k^M = \tilde{\theta}_{k_1}^n \circ \eta_1^k$. We now apply Lemma 8.4 to conclude that cyc_k^M is an isomorphism.

Suppose now that $m \geq 2$. Let F_m^n denote the kernel of the restriction map $\tilde{K}_n^M(k_m) \rightarrow \tilde{K}_n^M(k_{m-1})$. It is easy to see that the isomorphism $\tilde{\theta}_{k_m}^n$ of Lemma 8.4 commutes with the restriction map $k_m \rightarrow k_{m-1}$ for all $m \geq 1$. It follows therefore from Lemma 8.4 and the snake lemma that $\tilde{\theta}_{k_{m+1}}^n$ restricts to an isomorphism

$$\tilde{\theta}_{k_{m+1}}^n : \Omega_k^{n-1} \otimes_k t^{m+1} k_{m+1} \xrightarrow{\cong} F_{m+1}^n; \quad (9.3)$$

$$\tilde{\theta}_{k_{m+1}}^n((d \log(b_1) \wedge \cdots \wedge d \log(b_{n-1})) \otimes at^{m+1}) = \{1 + at^{m+1}, b_1, \dots, b_{n-1}\}.$$

We now consider the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega_k^{n-1} & \xrightarrow{V_{m+1}} & \mathbb{W}_{m+1} \Omega_k^{n-1} & \longrightarrow & \mathbb{W}_m \Omega_k^{n-1} \longrightarrow 0 \\
 & & \tilde{\theta}_{k_{m+1}}^n \circ \eta_{m+1}^k \downarrow & & \downarrow \text{cyc}_k^M & & \downarrow \text{cyc}_k^M \\
 0 & \longrightarrow & F_{m+1}^n & \longrightarrow & \tilde{K}_n^M(k_{m+1}) & \longrightarrow & \tilde{K}_n^M(k_m) \longrightarrow 0,
 \end{array} \quad (9.4)$$

where the horizontal arrows on the right in both rows are the restriction maps. In particular, the square on the right is commutative.

We next note that $V_{m+1}(ad \log(b_1) \wedge \cdots \wedge d \log(b_{n-1})) = V_{m+1}(a) d \log(b_1) \wedge \cdots \wedge d \log(b_{n-1})$ (see [Rülling and Saito 2018, Proposition 4.4]). Indeed, by iteration, it is enough to check this for $n = 2$. Furthermore, we can check it in $\text{TCH}^2(k, 2; m)$. Now, we have

$$\begin{aligned}
 V_{m+1}(ad \log(b_1)) &= V_{m+1}(Z(1 - ab_1^{-1}t)d(Z(1 - b_1t))) \\
 &= V_{m+1}(Z(1 - at, y_1 - b_1)) \\
 &= Z(1 - at^{m+1}, y_1 - b_1),
 \end{aligned} \quad (9.5)$$

where the last two equalities follow from the definitions of differential and Verschiebung on the additive higher Chow groups (see the proof of Lemma 7.2). On

the other hand, we have

$$\begin{aligned}
 V_{m+1}(a)d \log(b_1) &= V_{m+1}(a)b_1^{-1}db_1 \\
 &= (V_{m+1}(Z(1-at))Z(1-b_1^{-1}t)d(Z(1-b_1t))) \\
 &= Z(1-at^{m+1})Z(1-b_1^{-1}t)d(Z(1-b_1t)) \\
 &= Z(1-at^{m+1})Z(1-b_1^{-1}t)Z(1-b_1t, y_1-b_1) \\
 &= Z(1-at^{m+1})Z(1-t, y_1-b_1) \\
 &= Z(1-at^{m+1}, y_1-b_1),
 \end{aligned} \tag{9.6}$$

where the equalities again follow from various definitions; see the proof of [Lemma 7.2](#). A combination of [\(9.5\)](#) and [\(9.6\)](#) proves what we had claimed. Since the map $k \otimes K_{n-1}^M(k) \rightarrow \Omega_k^{n-1}$ (given by $a \otimes b \mapsto ad \log(b_1) \wedge \cdots \wedge d \log(b_{n-1})$) is surjective, it follows from [\(7.10\)](#) and [\(9.3\)](#) that the left square in [\(9.4\)](#) is also commutative.

The top row of [\(9.4\)](#) is well known to be exact in characteristic zero (e.g., see [\[Rüling and Saito 2018, Remark 4.2\]](#)). The bottom row is exact by the definition of F_{m+1}^n and the fact that the map $\widetilde{K}_n^M(k_{m+1}) \rightarrow \widetilde{K}_n^M(k_m)$ is surjective. The left vertical arrow in [\(9.4\)](#) is an isomorphism by [\(9.3\)](#) and the right vertical arrow is an isomorphism by induction. We conclude that the middle vertical arrow is also an isomorphism. This completes the proof of the theorem. \square

Remark 9.4. The reader can check that the proof of [Theorem 9.3](#) continues to work for any regular semilocal ring R containing k as long as cyc_R is defined in a way it is for fields. In more detail, the isomorphism η_m^R makes sense. Lemmas [7.1](#) and [7.2](#) as well as [\(9.3\)](#) are all valid for R . The final step [\(9.4\)](#) also holds for R . This observation will be used in the next section in extending [Theorem 9.1](#) to regular semilocal rings.

9C. The final step. We shall now prove our final step in the proof of [Theorem 9.1](#). Namely, we shall show that the reduced Milnor and Quillen K -theories of the truncated polynomial rings are pro-isomorphic. This will finish the proof of [Theorem 9.1](#). The desired pro-isomorphism is based on the following general result about pro-vanishing of relative cyclic homology.

Let R be a ring containing \mathbb{Q} . Recall from [\[Loday 1998, §4.5, 4.6\]](#) that for integers $m \geq 0$ and $n \geq 1$, there are functorial λ -decompositions $\widetilde{H}H_n(R_m) = \bigoplus_{i=1}^n \widetilde{H}H_n^{(i)}(R_m)$ and $\widetilde{H}C_n(R_m) = \bigoplus_{i=1}^n \widetilde{H}C_n^{(i)}(R_m)$. Moreover, there are isomorphisms

$$\widetilde{H}H_n^{(n)}(R_m) \cong \widetilde{\Omega}_{R_m}^n \quad \text{and} \quad \widetilde{H}C_n^{(n)}(R_m) \cong \frac{\widetilde{\Omega}_{R_m}^n}{d\widetilde{\Omega}_{R_m}^{n-1}} \tag{9.7}$$

such that the canonical map $\widetilde{H}\widetilde{H}_n^{(n)}(R_m) \rightarrow \widetilde{H}\widetilde{C}_n^{(n)}(R_m)$ is the quotient map $\widetilde{\Omega}_{R_m}^n \rightarrow \widetilde{\Omega}_{R_m}^n/d\widetilde{\Omega}_{R_m}^{n-1}$. We also have the functorial isomorphisms

$$\widetilde{H}\widetilde{H}_n^{(i)}(R_m) \cong \widetilde{D}_{n-i}^{(i)}(R_m). \quad (9.8)$$

The key result is the following.

Proposition 9.5. *Let R be a regular semilocal ring which is essentially of finite type over a characteristic zero field. Let $0 < i < n$ be two integers. Then*

$$\{\widetilde{H}\widetilde{C}_n^{(i)}(R_m)\}_m = 0.$$

Proof. First we show that $\{\widetilde{H}\widetilde{H}_n^{(i)}(R_m)\}_m = 0$. By (9.8), this is equivalent to showing that $\{\widetilde{D}_{n-i}^{(i)}(R_m)\}_m = 0$. To show this latter vanishing, we let $A = R[t^2, t^3] \subset R[t]$ be the monomial subalgebra generated by $\{t^2, t^3\}$. We then know that the inclusion $A \hookrightarrow R[t]$ is the normalization homomorphism whose conductor ideal is $I = (t^2, t^3) \subset A$ such that $IR[t] = (t^2)$. Since $R[t]$ is regular, we know that $D_{n-i}^{(i)}(R[t]) = 0$ for $0 < i < n$ by [Loday 1998, Theorem 3.5.6]. We now apply part (ii) of [Krishna 2010, Proposition 5.2] with $A = R[t^2, t^3]$ and $B = R[t]$ to conclude that $\{D_{n-i}^{(i)}(R_{2m-1})\}_m = \{D_{n-i}^{(i)}(R[t]/(t^{2m}))\}_m = 0$. But this is same as saying that $\{D_{n-i}^{(i)}(R_m)\}_m = 0$. In particular, we get $\{\widetilde{D}_{n-i}^{(i)}(R_m)\}_m = 0$.

To prove the result for cyclic homology, we first assume $n \geq 2$ and $i = n - 1$. Connes' periodicity exact sequence (see [Loday 1998, Theorem 4.6.9]) gives an exact sequence

$$\widetilde{H}\widetilde{H}_n^{(n-1)}(R_m) \xrightarrow{I} \widetilde{H}\widetilde{C}_n^{(n-1)}(R_m) \xrightarrow{S} \frac{\widetilde{\Omega}_{R_m}^{n-2}}{d\widetilde{\Omega}_{R_m}^{n-3}} \xrightarrow{B} \widetilde{\Omega}_{R_m}^{n-1} \xrightarrow{I} \frac{\widetilde{\Omega}_{R_m}^{n-1}}{d\widetilde{\Omega}_{R_m}^{n-2}}.$$

We have shown that the first term in this exact sequence vanishes. It suffices therefore to show that the map B is injective. But B is same as the differential map $d: \widetilde{\Omega}_{R_m}^{n-2}/d\widetilde{\Omega}_{R_m}^{n-3} \rightarrow \widetilde{\Omega}_{R_m}^{n-1}$ by [Loday 1998, Corollary 2.3.5]. By Lemma 8.4, we therefore have to show that the map $d: \widetilde{\Omega}_R^{n-2} \otimes_R tR_m \rightarrow \widetilde{\Omega}_{R_m}^{n-1}$ is injective. But this follows from Lemma 8.5.

We prove the general case by induction on $n \geq 2$. The case $n = 2$ is just proven above. When $n \geq 3$, we can assume that $1 \leq i \leq n - 2$ by what we have shown above. We now again use the periodicity exact sequence to get an exact sequence of pro-abelian groups:

$$\{\widetilde{H}\widetilde{H}_n^{(i)}(R_m)\}_m \xrightarrow{I} \{\widetilde{H}\widetilde{C}_n^{(i)}(R_m)\}_m \xrightarrow{S} \{\widetilde{H}\widetilde{C}_{n-2}^{(i-1)}(R_m)\}_m.$$

Since $i \leq n - 2$, we get $i - 1 \leq n - 3$. Hence, the induction hypothesis implies that the last term of this exact sequence is zero. We have shown above that the first term is zero. We conclude that the middle term is zero as well. \square

Corollary 9.6. *Let R be a regular semilocal ring which is essentially of finite type over a characteristic zero field. Let $n \geq 0$ be an integer. Then the canonical map*

$$\psi_{R,n} : \{\tilde{K}_n^M(R_m)\}_m \rightarrow \{\tilde{K}_n(R_m)\}_m$$

is an isomorphism of pro-abelian groups.

Proof. The case $n \leq 2$ is well known (e.g, see [Kerz 2010, Proposition 2]) and in fact an isomorphism at every level $m \geq 0$. We can therefore assume that $n \geq 3$. We have seen in Section 8B that Goodwillie provides an isomorphism of \mathbb{Q} -vector spaces

$$\mathrm{tr}_{m,n}^R : \tilde{K}_n(R_m) \xrightarrow{\cong} \widetilde{HC}_{n-1}(R_m) \cong \bigoplus_{i=1}^{n-1} \widetilde{HC}_{n-1}^{(i)}(R_m).$$

Moreover, Lemmas 8.1 and 8.2 say that the map $\psi_{R_m,n} : \tilde{K}_n^M(R_m) \rightarrow \tilde{K}_n(R_m)$ is injective and $\mathrm{tr}_{m,n}^R$ maps $\tilde{K}_n^M(R_m)$ isomorphically onto $\widetilde{HC}_{n-1}^{(n-1)}(R_m)$. We therefore have to show that $\{\widetilde{HC}_{n-1}^{(i)}(R_m)\}_m = 0$ for $1 \leq i \leq n-2$. But this follows from Proposition 9.5. □

Proof of Theorem 9.1. The proof is a combination of Lemma 9.2, Theorem 9.3 and Corollary 9.6. □

10. The cycle class map for semilocal rings

In this section, we shall define the cycle class map for relative 0-cycles over regular semilocal rings and prove an extension of Theorem 9.1 for such rings. Let k be a characteristic zero field and let R be a regular semilocal ring which is essentially of finite type over k . We shall let F denote the total quotient ring of R . Note that R is a product of regular semilocal integral domains and F is the product of their fields of fractions. Since all our proofs for regular semilocal integral domains directly generalize to finite products of such rings, we shall assume throughout that R is an integral domain. We shall let $\pi : \mathrm{Spec}(F) \rightarrow \mathrm{Spec}(R)$ denote the inclusion of generic point. We shall often write $X = \mathrm{Spec}(R)$ and $\eta = \mathrm{Spec}(F)$. We shall let Σ denote the set of all maximal ideals of R .

10A. The sfs cycles. Let $m \geq 0$ and $n \geq 1$ be two integers. Recall from Section 2E and Section 7A that $\mathrm{TCH}^n(R, n; m)$ is defined as the middle homology of the complex $\mathrm{Tz}^n(R, n+1; m) \xrightarrow{\partial} \mathrm{Tz}^n(R, n; m) \xrightarrow{\partial} \mathrm{Tz}^n(R, n-1; m)$. A cycle in $\mathrm{Tz}^n(R, n; m)$ has relative dimension zero over R . For this reason, $\mathrm{TCH}^n(R, n; m)$ is often called the additive higher Chow group of relative 0-cycles on R . When R is a field, it coincides with the one used in the statement of Theorem 9.1.

Since $\mathrm{TCH}^n(R, n; m)$ does not consist of 0-cycles if $\dim(R) \geq 1$, we can not directly apply Theorem 1.1 to define a cycle class map for $\mathrm{TCH}^n(R, n; m)$. We

have to use a different approach for constructing the cycle class map. We shall use the main results of [Krishna and Park 2020b] and the case of fields to construct a cycle class map in this case. We shall show later in this section that this map is an isomorphism. We shall say that an extension of regular semilocal rings $R_1 \subset R_2$ is simple if there is an irreducible monic polynomial $f \in R_1[t]$ such that $R_2 = R_1[t]/(f(t))$.

Let $Z \subset X \times \mathbb{A}_k^1 \times \square^{n-1}$ be an irreducible admissible relative 0-cycle. Recall from [Krishna and Park 2020a, Definition 2.5.2, Proposition 2.5.3] that Z is called an sfs-cycle if the following hold:

- (1) Z intersects $\Sigma \times \mathbb{A}_k^1 \times F$ properly for all faces $F \subset \square^{n-1}$.
- (2) The projection $Z \rightarrow X$ is finite and surjective.
- (3) Z meets no face of $X \times \mathbb{A}_k^1 \times \square^{n-1}$.
- (4) Z is closed in $X \times \mathbb{A}_k^1 \times \mathbb{A}_k^{n-1} = \text{Spec}(R[t, y_1, \dots, y_{n-1}])$ (by (2) above) and there is a sequence of simple extensions of regular semilocal rings

$$R = R_{-1} \subset R_0 \subset \dots \subset R_{n-1} = k[Z]$$

such that $R_0 = R[t]/(f_0(t))$ and $R_i = R_{i-1}[y_i]/(f_i(y_i))$ for $1 \leq i \leq n-1$.

Note that an sfs-cycle has no boundary by (3) above. We let $\text{Tz}_{\text{sfs}}^n(R, n; m) \subset \text{Tz}^n(R, n; m)$ be the free abelian group on integral sfs-cycles and define

$$\text{TCH}_{\text{sfs}}^n(R, n; m) = \frac{\text{Tz}_{\text{sfs}}^n(R, n; m)}{\partial(\text{Tz}^n(R, n+1; m)) \cap \text{Tz}_{\text{sfs}}^n(R, n; m)}. \quad (10.1)$$

We shall use the following result from [Krishna and Park 2020b, Theorem 1.1].

Proposition 10.1. *The canonical map $\text{TCH}_{\text{sfs}}^n(R, n; m) \rightarrow \text{TCH}^n(R, n; m)$ is an isomorphism.*

10B. The cycle class map. By Proposition 10.1, it suffices to define the cycle class map on $\text{TCH}_{\text{sfs}}^n(R, n; m)$. We can now repeat the construction of Section 4 word by word to get our map. So let $Z \subset X \times \mathbb{A}_k^1 \times \square^{n-1}$ be an irreducible sfs-cycle and let $R' = k[Z]$. Let $f: Z \rightarrow X \times \mathbb{A}_k^1$ be the projection map. Let $g_i: Z \rightarrow \square$ denote the i -th projection. Then the sfs property implies that each g_i defines an element of R'^{\times} , and this in turn gives a unique element $\text{cyc}_{R'}^M([Z]) = \{g_1, \dots, g_{n-1}\} \in K_{n-1}^M(R')$. We let $\text{cyc}_{R'}([Z])$ be its image in $K_{n-1}(R')$ under the map $K_{n-1}^M(R') \rightarrow K_{n-1}(R')$. Since Z does not meet $X \times \{0\}$, we see that the finite map f defines a push-forward map of spectra $f_*: K(R') \rightarrow K(R[t], (t^{m+1}))$. We let $\text{cyc}_R([Z]) = f_*(\text{cyc}_{R'}([Z])) \in K_{n-1}(R[t], (t^{m+1}))$. We extend this definition linearly to get a cycle map $\text{cyc}_R: \text{Tz}_{\text{sfs}}^n(R, n; m) \rightarrow K_{n-1}(R[t], (t^{m+1}))$. We can now prove our first result of this section.

Theorem 10.2. *The assignment $[Z] \mapsto \text{cyc}_R([Z])$ defines a cycle class map*

$$\text{cyc}_R: \text{TCH}^n(R, n; m) \rightarrow K_{n-1}(R[t], (t^{m+1})),$$

which is functorial in R .

Proof. Let F be the fraction field of R . We consider the diagram

$$\begin{CD} \partial^{-1}(\text{TZ}_{\text{sfs}}^n(R, n; m)) @>\partial>> \text{TZ}_{\text{sfs}}^n(R, n; m) @>\text{cyc}_R>> K_{n-1}(R[t], (t^{m+1})) \\ @V\pi^*VV @VV\pi^*V @VV\pi^*V \\ \text{TZ}^n(F, n+1; m) @>\partial>> \text{TZ}^n(F, n; m) @>\text{cyc}_F>> K_{n-1}(F[t], (t^{m+1})). \end{CD} \tag{10.2}$$

Assume first that this diagram is commutative. Then [Theorem 9.3](#) says that $\text{cyc}_F \circ \partial \circ \pi^*$ is zero. Equivalently, $\pi^* \circ \text{cyc}_R \circ \partial = 0$. We will be therefore done if we know that the right vertical map π^* between the relative K -groups is injective. To show this, we can replace these relative K -groups by the relative cyclic homology groups by [\[Goodwillie 1985\]](#). These relative cyclic homology groups in turn can be replaced by the Hochschild homology $HH_*(R)$ and $HH_*(F)$ by [\[Hesselholt 2005, Proposition 8.1\]](#). Since R is regular, we can go further and replace $HH_*(R)$ and $HH_*(F)$ by Ω_R^* and Ω_F^* , respectively, by the famous Hochschild–Kostant–Rosenberg theorem. We therefore have to show that the map $\Omega_R^* \rightarrow \Omega_F^*$ is injective. But this follows from [Lemma 10.3](#).

To show that cyc_R is natural for homomorphisms of regular semilocal rings $R \rightarrow R'$, we first observe that [\(10.2\)](#) shows that cyc_R is functorial for the inclusion $R \hookrightarrow F$. Since the right-most vertical arrow in [\(10.2\)](#) is injective, we can replace R and R' by their fraction fields to check the naturality of cyc_R in general. In this case, the naturality of cyc_R follows from [Theorem 1.1](#). It remains now to show that [\(10.2\)](#) is commutative.

The left square is known to be commutative by the flat pull-back property of additive cycle complex. To show that the right square commutes, let $Z \subset \mathbb{A}_R^1 \times_R \square_R^{n-1}$ be an irreducible sfs-cycle and let $R_{n-1} = k[Z]$. Let R_0 be the coordinate ring of the image of Z in \mathbb{A}_R^1 as in the definition of the sfs-cycles. By definition of sfs-cycles, we have the commutative diagram

$$\begin{CD} R @>>> R_0 @>f_0>> R_{n-1} \\ @V\pi VV @VV\pi_0 V @VV\pi_{n-1} V \\ F @>>> F_0 @>f_0>> F_{n-1}, \end{CD} \tag{10.3}$$

where each term in the bottom row is the quotient field of the corresponding term on the top row. Note also that all horizontal arrows are finite maps of regular

semilocal integral domains. In particular, we have $F_0 = R_0 \otimes_R F$ and $F_{n-1} = R_{n-1} \otimes_R F = R_{n-1} \otimes_{R_0} F_0$.

We let $f: Z \rightarrow \mathbb{A}_R^1$ be the projection and let $p: \text{Spec}(R_0) \hookrightarrow \mathbb{A}_R^1$ be the inclusion. We denote the projection $\text{Spec}(F_{n-1}) \rightarrow \mathbb{A}_F^1$ and inclusion $\text{Spec}(F_0) \hookrightarrow \mathbb{A}_F^1$ also by f and p , respectively. Note that f is a finite map which has factorization $Z \rightarrow \mathbb{A}_R^1 \setminus \{0\} \subset \mathbb{A}_R^1$.

We let $\alpha = \text{cyc}_{R_{n-1}}^M([Z]) = \{g_1, \dots, g_{n-1}\} \in K_{n-1}^M(R_{n-1})$. We then have, by definition, $\text{cyc}_R([Z]) = f_* \circ \psi_{R_{n-1}}(\alpha)$ and $\text{cyc}_F(\pi^*([Z])) = f_* \circ \psi_{F_{n-1}} \circ \pi_{n-1}^*(\alpha)$. Using (10.3), we can write these as $\text{cyc}_R([Z]) = p_* \circ f_{0*} \circ \psi_{R_{n-1}}(\alpha)$ and

$$\text{cyc}_F(\pi^*([Z])) = p_* \circ f_{0*} \circ \psi_{F_{n-1}} \circ \pi_{n-1}^*(\alpha).$$

Since $\pi_{n-1}^* \circ \psi_{R_{n-1}} = \psi_{F_{n-1}} \circ \pi_{n-1}^*$, we only have to show that the diagram

$$\begin{array}{ccccc} K(R_{n-1}) & \xrightarrow{f_{0*}} & K(R_0) & \xrightarrow{p_*} & K(R[t], (t^{m+1})) \\ \pi_{n-1}^* \downarrow & & \downarrow \pi_0^* & & \downarrow \pi^* \\ K(F_{n-1}) & \xrightarrow{f_{0*}} & K(F_0) & \xrightarrow{p_*} & K(F[t], (t^{m+1})) \end{array} \quad (10.4)$$

commutes.

The left square commutes by [Thomason and Trobaugh 1990, Proposition 3.18] since $F_{n-1} = R_{n-1} \otimes_{R_0} F_0$ and f_0 is finite. To see that the right square commutes, note that we can replace $K(R_0)$ by $K^{Z_0}(\mathbb{A}_R^1)$ and $K(F_0)$ by $K^{\eta_0}(\mathbb{A}_F^1)$, where $Z_0 = \text{Spec}(R_0)$ and $\eta_0 = \text{Spec}(F_0)$ (see Section 2C). We can do this because $R[t]$ and R_0 are regular. We are now done because the diagram

$$\begin{array}{ccc} K(\mathbb{A}_R^1, (m+1)\{0\}) & \longrightarrow & K(\mathbb{A}_R^1 \setminus Z_0, (m+1)\{0\}) \\ \downarrow & & \downarrow \\ K(\mathbb{A}_F^1, (m+1)\{0\}) & \longrightarrow & K(\mathbb{A}_F^1 \setminus \eta_0, (m+1)\{0\}) \end{array} \quad (10.5)$$

of restriction maps is commutative and the right square in (10.4) is gotten by taking the homotopy fibers of the two rows of (10.5). We have now shown that both squares in (10.2) are commutative. This also shows that cyc_R is compatible with the inclusion $R \hookrightarrow F$. The proof of the theorem is complete. \square

Throughout the rest of our discussion, we shall identify $\text{TCH}^n(R, n; m)$ with $\mathbb{W}_m \Omega_R^{n-1}$ (by (7.5)) and $K_{n-1}(R[t], (t^{m+1}))$ with $\tilde{K}_n(R_m)$ (via the connecting homomorphism).

10C. Factorization through Milnor K -theory. We shall now show that cyc_R factors through the relative Milnor K -theory. The proof is identical to the case of

fields and we shall only sketch it. We shall reduce the proof to the case of fields using the following result.

Lemma 10.3. *For $n \geq 0$ and $m \geq 1$, the map $\pi^* : \mathbb{W}_m \Omega_R^n \rightarrow \mathbb{W}_m \Omega_F^n$ is injective. In particular, the map $\pi^* : \tilde{K}_n^M(R_m) \rightarrow \tilde{K}_n^M(F_m)$ is injective for all $m \geq 0$.*

Proof. Since $\mathbb{W}_m \Omega_R^n \cong (\Omega_R^n)^m$ (and also for F), we need to show that $\Omega_R^n \rightarrow \Omega_F^n$ is injective to prove the first assertion of the lemma. Since $\Omega_F^n \cong \Omega_R^n \otimes_R F$, it suffices to show that Ω_R^1 is a free R -module. Since R is regular, we have $D_1(R|k) = 0$ and $\Omega_{R/k}^1$ is a free R -module. The Jacobi–Zariski exact sequence (see [Loday 1998, 3.5.5]) therefore tells us that $\Omega_R^1 \cong (\Omega_k^1 \otimes_k R) \oplus \Omega_{R/k}^1$. This proves the first part.

For the second part, there is nothing to prove when $m = 0$. When $m \geq 1$, Lemma 8.4 reduces to showing that the map $\Omega_R^n \otimes_R tR_m \rightarrow \Omega_F^n \otimes_F tF_m$ is injective for all $n \geq 0$. Since $\Omega_F^n \otimes_F tF_m \cong \Omega_F^n \otimes_R tR_m$ and tR_m is a free R -module, the problem is reduced to showing that $\Omega_R^n \rightarrow \Omega_F^n$ is injective. We can now use the first part of the lemma. \square

Our second main result of this section is the following. This generalizes the main results of [Elbaz-Vincent and Müller-Stach 2002; Nesterenko and Suslin 1989; Totaro 1992] to truncated polynomial rings.

Theorem 10.4. *Let R and $m \geq 0, n \geq 1$ be as above. Then the cycle class map cyc_R has a factorization*

$$\text{TCH}^n(R, n; m) \xrightarrow{\text{cyc}_R^M} \tilde{K}_n^M(R_m) \xrightarrow{\psi_{R_{m+1}, n}} \tilde{K}_n(R_m).$$

Furthermore, cyc_R^M is natural in R and is an isomorphism.

Proof. We shall use Proposition 10.1, which allows us to repeat the proof of the field case Lemma 9.2 word by word. When $n = 1$, any sfs irreducible cycle $Z \subset \mathbb{A}_R^1$ is of the form $Z = V(f(t))$, where $f(t)$ is an irreducible polynomial such that $f(0) \in R^\times$. We now repeat the argument of the field case and use Lemma 2.1 to finish the proof. The $n \geq 2$ case follows from Lemma 7.3 and the proof is identical to the case of fields. To prove that cyc_R^M is an isomorphism, we again repeat the case of fields and use Remark 9.4. The naturality of cyc_R^M follows from Theorem 10.2 since $\psi_{R_{m+1}, n}$ is injective. \square

Finally, we are now ready to prove Theorem 1.3 (5). We restate it again for reader's convenience.

Theorem 10.5. *Let R be a regular semilocal ring which is essentially of finite type over a characteristic zero field. Let $n \geq 1$ be an integer. Then the cycle class map*

$$\text{cyc}_R : \{\text{TCH}^n(R, n; m)\}_m \rightarrow \{\tilde{K}_n(R_m)\}_m$$

is an isomorphism of the pro-abelian groups.

Proof. Combine Theorem 10.4 and Corollary 9.6. \square

11. Appendix: Milnor vs. Quillen K -theory

In this section, we collect some results on the compatibility of various maps between Milnor and Quillen K -theories of fields. They are used in the proofs of the main results of this paper. We expect these results to be known to experts but could not find their written proofs in the literature.

Let k be a field and let X be a regular scheme which is essentially of finite type over k . Let $x, y \in X$ be two points in X of codimensions p and $p - 1$, respectively, such that $x \in \overline{\{y\}}$. Let

$$Y = \overline{\{y\}}, \quad F = k(y), \quad A = \mathcal{O}_{Y,x} \quad \text{and} \quad l = k(x).$$

Lemma A.1. *For any $n \geq 1$, the diagram*

$$\begin{CD} K_n^M(F) @>\partial^M>> K_{n-1}^M(l) \\ @VVV @VVV \\ K_n(F) @>\partial^Q>> K_{n-1}(l) \end{CD} \tag{A.1}$$

is commutative.

Proof. Let B denote the normalization of A and let S denote the set of maximal ideals of B . Note that B is semilocal so that S is finite. Since the localization sequence for Quillen K -theory of coherent sheaves is functorial for proper push-forward, we have a commutative diagram

$$\begin{CD} G_n(B) @>>> K_n(F) @>\partial^Q>> \bigoplus_{z \in S} K_{n-1}(k(z)) \\ @VVV @VVV @VV \sum_z T_{k(z)/l} V \\ G_n(A) @>>> K_n(F) @>\partial^Q>> K_{n-1}(l), \end{CD} \tag{A.2}$$

where $T_{k(z)/l}$ is our notation for the finite push-forward $K_*(k(z)) \rightarrow K_*(l)$ and $G_*(-)$ is Quillen K -theory of coherent sheaves functor (for proper morphisms). Note that $K_*(B) \xrightarrow{\cong} G_*(B)$.

On the other hand, the boundary map in Milnor K -theory also has the property that the diagram

$$\begin{CD} K_n^M(F) @>\partial^M>> \bigoplus_{z \in S} K_{n-1}^M(k(z)) \\ @VVV @VV \sum_z N_{k(z)/l} V \\ K_n^M(F) @>\partial^M>> K_{n-1}^M(l) \end{CD} \tag{A.3}$$

commutes, where the right vertical arrow is the sum of the Norm maps in Milnor K -theory of fields (see [Bass and Tate 1973; Kato 1980]). The lemma therefore follows if we prove Lemmas A.2 and A.3 below. \square

Lemma A.2. *Let $z \in S$ be a closed point as above and let R denote the discrete valuation ring of F associated to z . Then the diagram*

$$\begin{CD} K_n^M(F) @>\partial^M>> K_{n-1}^M(k(z)) \\ @VVV @VVV \\ K_n(F) @>\partial^Q>> K_{n-1}(k(z)) \end{CD} \tag{A.4}$$

is commutative for every $n \geq 1$.

Proof. It is well-known and elementary to see (using the Steinberg relations) that $K_*^M(F)$ is generated by $K_1^M(F)$ as an $K_*^M(R)$ -module. Furthermore, ∂^M is $K_*^M(R)$ -linear (see [Bass and Tate 1973, §4, Proposition 4.5]). Since the localization sequence such as the one on the top of (A.2) (with B replaced by R) is $K_*(R)$ -linear, it follows that all arrows in (A.4) are $K_*^M(R)$ -linear. It therefore suffices to prove the lemma for $n = 1$. But in this case, both ∂^M and ∂^Q are simply the valuation map of F corresponding to z . \square

Lemma A.3. *Let $k \hookrightarrow k'$ be a finite extension of fields and $n \geq 0$ an integer. Then we have a commutative diagram*

$$\begin{CD} K_n^M(k') @>N_{k'/k}>> K_n^M(k) \\ @VVV @VVV \\ K_n(k') @>T_{k'/k}>> K_n(k). \end{CD} \tag{A.5}$$

Proof. Assume first that $k \hookrightarrow k'$ is a simple extension so that $k' = k[t]/\mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset k[t]$. Let v_∞ be the valuation of $k(t)$ associated to the point $\infty \in \mathbb{A}_k^1$. Its valuation ring $R_\infty \subset k(t)$ has uniformizing parameter t^{-1} . In this case, we have the following diagram:

$$\begin{CD} 0 @>>> K_{n+1}^M(k) @>>> K_{n+1}^M(k(t)) @>\partial^M=(\partial_v)_{v \neq v_\infty}>> \bigoplus_{v \neq v_\infty} K_n^M(k(v)) @>>> 0 \\ @. @VVV @VVV @VVV @. \\ 0 @>>> K_{n+1}(k) @>>> K_{n+1}(k(t)) @>\partial^Q=(\partial_v)_{v \neq v_\infty}>> \bigoplus_{v \neq v_\infty} K_n(k(v)) @>>> 0. \end{CD} \tag{A.6}$$

The horizontal arrows on the left in both rows are induced by the inclusion $k \subset k(t)$. The top row is Milnor’s exact sequence (see [Weibel 2013, Chapter III, Theorem 7.4]). The bottom row is the localization sequence in Quillen K -theory

(using the isomorphism $K_*(k) \xrightarrow{\cong} K_*(k[t])$) and is known to be exact (see [Weibel 2013, Chapter V, Corollary 6.7.1]). The right square commutes by Lemma A.2.

On the other hand, we have another diagram

$$\begin{array}{ccccc}
 & & K_n^M(k') & & \\
 & & \downarrow & \searrow^{N_{k'/k}} & \\
 K_{n+1}^M(k(t)) & \xrightarrow{\partial^M = (\partial_v)_v} & \bigoplus_{v \neq v_\infty} K_n^M(k(v)) & \xrightarrow{\sum_v N_{k(v)/k}} & K_n^M(k) \\
 \downarrow & & \downarrow & & \downarrow \\
 K_{n+1}(k(t)) & \xrightarrow{\partial^Q = (\partial_v)_v} & \bigoplus_{v \neq v_\infty} K_n(k(v)) & \xrightarrow{\sum_v T_{k(v)/k}} & K_n(k) \\
 & & \uparrow & \nearrow_{T_{k'/k}} & \\
 & & K_n(k') & &
 \end{array} \tag{A.7}$$

By the definition of the norm $N_{k'/k}$ in Milnor K -theory, the composition of the horizontal arrows on the top is the map $(-1)\partial_\infty^M: K_{n+1}^M(k(t)) \rightarrow K_n^M(k)$ (see [Weibel 2013, Chapter III, Definition 7.5]). Similarly, the composite of the horizontal arrows on the bottom is the map $(-1)\partial_\infty^Q: K_{n+1}(k(t)) \rightarrow K_n(k)$ (see [Weibel 2013, Chapter V, 6.12.1]). Note that both of these assertions are another way of stating the Weil reciprocity formulas for the Milnor and Quillen K -theories.

Since the left horizontal arrows in both rows are surjective, we are reduced to showing therefore that the diagram

$$\begin{array}{ccc}
 K_{n+1}^M(k(t)) & \xrightarrow{\partial_\infty^M} & K_n^M(k) \\
 \downarrow & & \downarrow \\
 K_{n+1}(k(t)) & \xrightarrow{\partial_\infty^Q} & K_n(k)
 \end{array} \tag{A.8}$$

commutes. But this follows from Lemma A.2. This proves the lemma for simple extensions.

In general, we can write $k' = k(x_1, \dots, x_r)$. Since the norm maps in Milnor K -theory and the push-forward maps in Quillen K -theory satisfy the transitivity property, and since $k \hookrightarrow k'$ is a composite of simple extensions, the proof of the lemma follows. \square

Acknowledgments

Gupta would like to thank TIFR, Mumbai for invitation in March 2019. This paper was written when Krishna was at Max Planck Institute for Mathematics, Bonn in 2019. He thanks the institute for invitation and support. The authors thank the

referee for reading the manuscript thoroughly and providing valuable suggestions to improve its presentation. They also thank the editors some of whom provided very useful comments and suggestions.

References

- [Artin and Mazur 1986] M. Artin and B. Mazur, *Etale homotopy*, Lecture Notes in Mathematics **100**, Springer, 1986. [MR](#) [Zbl](#)
- [Bass and Tate 1973] H. Bass and J. Tate, “The Milnor ring of a global field”, pp. 349–446 in *Algebraic K-theory, II: “Classical” algebraic K-theory and connections with arithmetic* (Seattle, WA, 1972), edited by H. Bass, Lecture Notes in Math. **342**, 1973. [MR](#) [Zbl](#)
- [Binda 2018] F. Binda, “A cycle class map from Chow groups with modulus to relative K -theory”, *Doc. Math.* **23** (2018), 407–444. [MR](#) [Zbl](#)
- [Binda and Krishna 2018] F. Binda and A. Krishna, “Zero cycles with modulus and zero cycles on singular varieties”, *Compos. Math.* **154**:1 (2018), 120–187. [MR](#) [Zbl](#)
- [Binda and Saito 2019] F. Binda and S. Saito, “Relative cycles with moduli and regulator maps”, *J. Inst. Math. Jussieu* **18**:6 (2019), 1233–1293. [MR](#) [Zbl](#)
- [Bloch 1977] S. Bloch, “Algebraic K -theory and crystalline cohomology”, *Inst. Hautes Études Sci. Publ. Math.* **47** (1977), 187–268. [MR](#) [Zbl](#)
- [Bloch 1986] S. Bloch, “Algebraic cycles and higher K -theory”, *Adv. Math.* **61**:3 (1986), 267–304. [MR](#) [Zbl](#)
- [Bloch and Esnault 2003a] S. Bloch and H. Esnault, “The additive dilogarithm”, *Doc. Math.* extra vol. in Kato’s honor (2003), 131–155. [MR](#) [Zbl](#)
- [Bloch and Esnault 2003b] S. Bloch and H. Esnault, “An additive version of higher Chow groups”, *Ann. Sci. École Norm. Sup. (4)* **36**:3 (2003), 463–477. [MR](#) [Zbl](#)
- [Cathelineau 1990] J.-L. Cathelineau, “ λ -structures in algebraic K -theory and cyclic homology”, *K-Theory* **4**:6 (1990), 591–606. [MR](#) [Zbl](#)
- [Cortiñas and Weibel 2009] G. Cortiñas and C. Weibel, “Relative Chern characters for nilpotent ideals”, pp. 61–82 in *Algebraic topology*, edited by N. A. Baas et al., Abel Symp. **4**, Springer, 2009. [MR](#) [Zbl](#)
- [Elbaz-Vincent and Müller-Stach 2002] P. Elbaz-Vincent and S. Müller-Stach, “Milnor K -theory of rings, higher Chow groups and applications”, *Invent. Math.* **148**:1 (2002), 177–206. [MR](#) [Zbl](#)
- [Geller and Weibel 1989] S. Geller and C. Weibel, “ $K(A, B, I)$ II”, *K-Theory* **2**:6 (1989), 753–760. [MR](#)
- [Geller and Weibel 1994] S. C. Geller and C. A. Weibel, “Hodge decompositions of Loday symbols in K -theory and cyclic homology”, *K-Theory* **8**:6 (1994), 587–632. [MR](#) [Zbl](#)
- [Goodwillie 1985] T. G. Goodwillie, “On the general linear group and Hochschild homology”, *Ann. of Math. (2)* **121**:2 (1985), 383–407. [MR](#) [Zbl](#)
- [Gorchinskiĭ and Tyurin 2018] S. O. Gorchinskiĭ and D. N. Tyurin, “Relative Milnor K -groups and differential forms of split nilpotent extensions”, *Izv. Ross. Akad. Nauk Ser. Mat.* **82**:5 (2018), 23–60. [MR](#) [Zbl](#)
- [Gupta and Krishna 2019] R. Gupta and A. Krishna, “Relative K -theory via 0-cycles in finite characteristic”, 2019. [arXiv](#)
- [Hesselholt 2005] L. Hesselholt, “ K -theory of truncated polynomial algebras”, pp. 71–110 in *Handbook of K -theory*, edited by E. M. Friedlander and D. R. Grayson, Springer, 2005. [MR](#) [Zbl](#)

- [Hesselholt 2008] L. Hesselholt, “The tower of K -theory of truncated polynomial algebras”, *J. Topol.* **1**:1 (2008), 87–114. [MR](#) [Zbl](#)
- [Iwasa and Kai 2018] R. Iwasa and W. Kai, “Isomorphisms up to bounded torsion between relative K_0 -groups and Chow groups with modulus”, 2018. To appear in *J. Inst. Math. Jussieu*. [arXiv](#)
- [Iwasa and Kai 2019] R. Iwasa and W. Kai, “Chern classes with modulus”, *Nagoya Math. J.* **236** (2019), 84–133. [MR](#) [Zbl](#)
- [Kantorovitz 1999] M. R. Kantorovitz, “Adams operations and the Dennis trace map”, *J. Pure Appl. Algebra* **144**:1 (1999), 21–27. [MR](#) [Zbl](#)
- [Kato 1980] K. Kato, “A generalization of local class field theory by using K -groups II”, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27**:3 (1980), 603–683. [MR](#) [Zbl](#)
- [Kato 1986] K. Kato, “Milnor K -theory and the Chow group of zero cycles”, pp. 241–253 in *Applications of algebraic K -theory to algebraic geometry and number theory* (Boulder, CO, 1983), edited by S. J. Bloch et al., *Contemp. Math.* **55**, Amer. Math. Soc., Providence, RI, 1986. [MR](#) [Zbl](#)
- [Kato and Saito 1986] K. Kato and S. Saito, “Global class field theory of arithmetic schemes”, pp. 255–331 in *Applications of algebraic K -theory to algebraic geometry and number theory* (Boulder, CO, 1983), edited by S. J. Bloch et al., *Contemp. Math.* **55**, Amer. Math. Soc., Providence, RI, 1986. [MR](#) [Zbl](#)
- [Kerz 2009] M. Kerz, “The Gersten conjecture for Milnor K -theory”, *Invent. Math.* **175**:1 (2009), 1–33. [MR](#) [Zbl](#)
- [Kerz 2010] M. Kerz, “Milnor K -theory of local rings with finite residue fields”, *J. Algebraic Geom.* **19**:1 (2010), 173–191. [MR](#) [Zbl](#)
- [Kerz and Saito 2016] M. Kerz and S. Saito, “Chow group of 0-cycles with modulus and higher-dimensional class field theory”, *Duke Math. J.* **165**:15 (2016), 2811–2897. [MR](#) [Zbl](#)
- [Kerz et al. 2018] M. Kerz, F. Strunk, and G. Tamme, “Algebraic K -theory and descent for blow-ups”, *Invent. Math.* **211**:2 (2018), 523–577. [MR](#) [Zbl](#)
- [Krishna 2010] A. Krishna, “An Artin–Rees theorem in K -theory and applications to zero cycles”, *J. Algebraic Geom.* **19**:3 (2010), 555–598. [MR](#) [Zbl](#)
- [Krishna 2015] A. Krishna, “On 0-cycles with modulus”, *Algebra Number Theory* **9**:10 (2015), 2397–2415. [MR](#) [Zbl](#)
- [Krishna and Levine 2008] A. Krishna and M. Levine, “Additive higher Chow groups of schemes”, *J. Reine Angew. Math.* **619** (2008), 75–140. [MR](#) [Zbl](#)
- [Krishna and Park 2012a] A. Krishna and J. Park, “Mixed motives over $k[t]/(t^{m+1})$ ”, *J. Inst. Math. Jussieu* **11**:3 (2012), 611–657. [MR](#) [Zbl](#)
- [Krishna and Park 2012b] A. Krishna and J. Park, “Moving lemma for additive higher Chow groups”, *Algebra Number Theory* **6**:2 (2012), 293–326. [MR](#) [Zbl](#)
- [Krishna and Park 2016] A. Krishna and J. Park, “On additive higher Chow groups of affine schemes”, *Doc. Math.* **21** (2016), 49–89. [MR](#) [Zbl](#)
- [Krishna and Park 2017a] A. Krishna and J. Park, “A module structure and a vanishing theorem for cycles with modulus”, *Math. Res. Lett.* **24**:4 (2017), 1147–1176. [MR](#) [Zbl](#)
- [Krishna and Park 2017b] A. Krishna and J. Park, “A moving lemma for cycles with very ample modulus”, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **17**:4 (2017), 1521–1549. [MR](#) [Zbl](#)
- [Krishna and Park 2020a] A. Krishna and J. Park, “de Rham–Witt sheaves via algebraic cycles”, 2020. With an appendix by K. Rülling. [Zbl](#) [arXiv](#)
- [Krishna and Park 2020b] A. Krishna and J. Park, “A moving lemma for relative 0-cycles”, *Algebra Number Theory* **14**:4 (2020), 991–1054. [MR](#) [Zbl](#)

- [Krishna and Pelaez 2018] A. Krishna and P. Pelaez, “Motivic spectral sequence for relative homotopy K -theory”, 2018. To appear in *Ann. Sc. Norm. Super. Pisa Cl. Sci.* [arXiv](#)
- [Levine 1994] M. Levine, “Bloch’s higher Chow groups revisited”, pp. 10, 235–320 in *K-theory* (Strasbourg, 1992), Astérisque **226**, Société Mathématique de France, Paris, 1994. [MR](#) [Zbl](#)
- [Levine 1997] M. Levine, “Lambda-operations, K -theory and motivic cohomology”, pp. 131–184 in *Algebraic K-theory* (Toronto, 1996), edited by V. P. Snaith, Fields Inst. Commun. **16**, Amer. Math. Soc., Providence, RI, 1997. [MR](#) [Zbl](#)
- [Loday 1998] J.-L. Loday, *Cyclic homology*, 2nd ed., Grundlehren der Mathematischen Wissenschaften **301**, Springer, 1998. [MR](#) [Zbl](#)
- [Milnor 1971] J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies **72**, Princeton University Press, 1971. [MR](#) [Zbl](#)
- [Nesterenko and Suslin 1989] Y. P. Nesterenko and A. A. Suslin, “Homology of the general linear group over a local ring, and Milnor’s K -theory”, *Izv. Akad. Nauk SSSR Ser. Mat.* **53**:1 (1989), 121–146. In Russian. [MR](#)
- [Park 2009] J. Park, “Regulators on additive higher Chow groups”, *Amer. J. Math.* **131**:1 (2009), 257–276. [MR](#) [Zbl](#)
- [Park and Ünver 2018] J. Park and S. Ünver, “Motivic cohomology of fat points in Milnor range”, *Doc. Math.* **23** (2018), 759–798. [MR](#) [Zbl](#)
- [Rülling 2007] K. Rülling, “The generalized de Rham–Witt complex over a field is a complex of zero-cycles”, *J. Algebraic Geom.* **16**:1 (2007), 109–169. [MR](#) [Zbl](#)
- [Rülling and Saito 2018] K. Rülling and S. Saito, “Higher Chow groups with modulus and relative Milnor K -theory”, *Trans. Amer. Math. Soc.* **370**:2 (2018), 987–1043. [MR](#) [Zbl](#)
- [Soulé 1985] C. Soulé, “Opérations en K -théorie algébrique”, *Canad. J. Math.* **37**:3 (1985), 488–550. [MR](#) [Zbl](#)
- [Stienstra 1985] J. Stienstra, “Cartier–Dieudonné theory for Chow groups”, *J. Reine Angew. Math.* **355** (1985), 1–66. [MR](#) [Zbl](#)
- [Swan 1998] R. G. Swan, “Néron–Popescu desingularization”, pp. 135–192 in *Algebra and geometry* (Taipei, 1995), edited by M.-c. Kang, Lect. Algebra Geom. **2**, Int. Press, Cambridge, MA, 1998. [MR](#) [Zbl](#)
- [Thomason and Trobaugh 1990] R. W. Thomason and T. Trobaugh, “Higher algebraic K -theory of schemes and of derived categories”, pp. 247–435 in *The Grothendieck Festschrift III*, edited by P. Cartier et al., Progr. Math. **88**, Birkhäuser, Boston, 1990. [MR](#) [Zbl](#)
- [Totaro 1992] B. Totaro, “Milnor K -theory is the simplest part of algebraic K -theory”, *K-Theory* **6**:2 (1992), 177–189. [MR](#) [Zbl](#)
- [Weibel 2013] C. A. Weibel, *The K-book: An introduction to algebraic K-theory*, Graduate Studies in Mathematics **145**, American Mathematical Society, Providence, RI, 2013. [MR](#) [Zbl](#)

Received 8 Jan 2020. Revised 23 Apr 2020. Accepted 11 May 2020.

RAHUL GUPTA: rahul.gupta@mathematik.uni-regensburg.de
Fakultät für Mathematik, Universität Regensburg, Regensburg, Germany

AMALENDU KRISHNA: amal@math.tifr.res.in
School of Mathematics, Tata Institute of Fundamental Research, Mumbai, India

ANNALS OF K-THEORY

msp.org/akt

EDITORIAL BOARD

Joseph Ayoub	Universität Zürich Zürich, Switzerland joseph.ayoub@math.uzh.ch
Paul Balmer	University of California, Los Angeles, USA balmer@math.ucla.edu
Guillermo Cortiñas	Universidad de Buenos Aires and CONICET, Argentina gcorti@dm.uba.ar
Hélène Esnault	Freie Universität Berlin, Germany liveesnault@math.fu-berlin.de
Eric Friedlander	University of Southern California, USA ericmf@usc.edu
Max Karoubi	Institut de Mathématiques de Jussieu – Paris Rive Gauche, France max.karoubi@imj-prg.fr
Moritz Kerz	Universität Regensburg, Germany moritz.kerz@mathematik.uni-regensburg.de
Huaxin Lin	University of Oregon, USA livehlin@uoregon.edu
Alexander Merkurjev	University of California, Los Angeles, USA merkurev@math.ucla.edu
Birgit Richter	Universität Hamburg, Germany birgit.richter@uni-hamburg.de
Jonathan Rosenberg	(Managing Editor) University of Maryland, USA jmr@math.umd.edu
Marco Schlichting	University of Warwick, UK schlichting@warwick.ac.uk
Charles Weibel	(Managing Editor) Rutgers University, USA weibel@math.rutgers.edu
Guoliang Yu	Texas A&M University, USA guoliangyu@math.tamu.edu

PRODUCTION

Silvio Levy (Scientific Editor)
production@msp.org

Annals of K-Theory is a journal of the [K-Theory Foundation](http://ktheoryfoundation.org) (ktheoryfoundation.org). The K-Theory Foundation acknowledges the precious support of [Foundation Compositio Mathematica](http://foundationcompositio.com), whose help has been instrumental in the launch of the Annals of K-Theory.

See inside back cover or msp.org/akt for submission instructions.

The subscription price for 2020 is US \$510/year for the electronic version, and \$575/year (+\$25, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Annals of K-Theory (ISSN 2379-1681 electronic, 2379-1683 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

AKT peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing
<http://msp.org/>

© 2020 Mathematical Sciences Publishers

ANNALS OF K-THEORY

2020

vol. 5

no. 4

- On the Rost divisibility of henselian discrete valuation fields of
cohomological dimension 3 677
YONG HU and ZHENGYAO WU
- On the norm and multiplication principles for norm varieties 709
SHIRA GILAT and ELIYAHU MATZRI
- Excision in equivariant fibred G -theory 721
GUNNAR CARLSSON and BORIS GOLDFARB
- Zero-cycles with modulus and relative K -theory 757
RAHUL GUPTA and AMALENDU KRISHNA