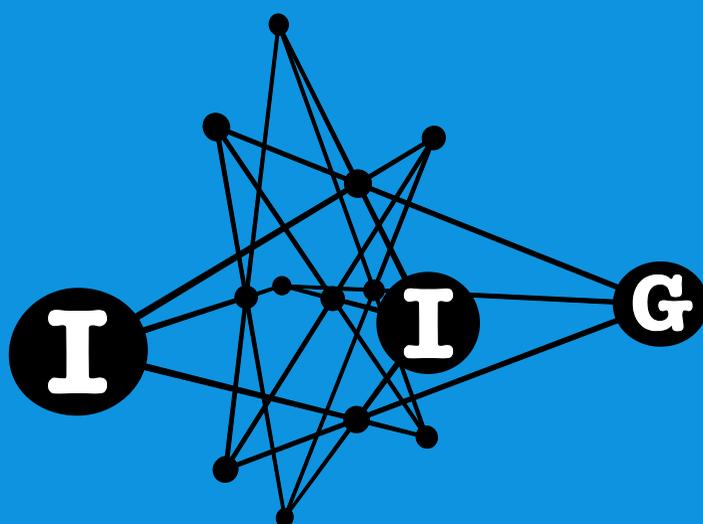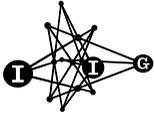# Innovations in Incidence Geometry

Algebraic, Topological and Combinatorial

Tits arrangements on cubic curves

Michael Cuntz and David Geis

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We classify affine rank three Tits arrangements whose roots are contained in the locus of a homogeneous cubic polynomial. We find that there exist irreducible affine Tits arrangements which are not locally spherical.

1. Introduction

Given a real reflection group, its set of reflecting hyperplanes defines a possibly infinite arrangement of hyperplanes with the property that every chamber is an open simplicial cone. Thus geometrically, a reflection group may be viewed as a so-called *simplicial arrangement*.

Of course, very few simplicial arrangements come from reflection groups. Another source, for example, are the *Weyl groupoids* (see [Heckenberger and Welker 2011; Cuntz 2011a]). As Weyl groups are invariants of different types of algebras in Lie theory, Weyl groupoids are invariants of (in a certain sense) more general quantum groups, the so-called Nichols algebras (see for example [Heckenberger 2006]). Not much is known about the infinite dimensional Nichols algebras which produce infinite Weyl groupoids. To go beyond the theory of finite dimensional Nichols algebras, it turns out that one needs an appropriate notion of infinite simplicial arrangement, which is the main contribution of [Cuntz et al. 2017], where these are called *Tits arrangements*. Thus a deep understanding of Tits arrangements would be beneficial for many reasons.

However, even the case of finite simplicial arrangements is poorly understood. Simplicial arrangements are quite rare; it is a highly non-trivial problem to classify simplicial arrangements, even finding further examples is very difficult. Among the known (irreducible) simplicial arrangements of rank three (see [Grünbaum 2009; Cuntz 2012]), one observes that the projective root vectors of almost all of them

are contained in a cubic curve since this holds for the known infinite families of irreducible arrangements $\mathcal{R}(1)$ and $\mathcal{R}(2)$; see [Cuntz 2011b]. Thus one could hope to obtain many, maybe even almost all of the infinite Tits arrangements of rank three by concentrating on those for which the root vectors lie on a cubic curve. Also, there is no infinite family of infinite Tits arrangements known yet, even in the affine case, i.e. when the Tits cone is a half space.

There is yet another reason why cubic curves are interesting in this context: simplicial arrangements of rank three are combinatorially extremal in the sense that they have very few double points; one of the keys for the main result in [Green and Tao 2013] is the fact that arrangements with few double points are close to having the property that their root vectors lie on a cubic curve.

In this paper, we give a classification of *affine rank three Tits arrangements* whose corresponding projective root vectors are contained in the locus of a homogeneous cubic polynomial. Our strategy for the classification builds upon the results obtained in [Cuntz et al. 2017] and on elementary tools from the geometry of the (real) projective plane, like Bézout's theorem and the fact that the conic in $\mathbb{P}^2(\mathbb{R})$ is a selfdual curve.

We find that there are only two classes of irreducible affine Tits arrangements satisfying the above property: namely the arrangement of type $\tilde{A}_2$ whose corresponding projective root vectors are contained in the union of three projective lines, and a new class of arrangements which we call $\tilde{A}_2^0$ (see Figure 1). The projective root vectors of $\tilde{A}_2^0$ are contained in the union of a projective conic σ and a projective line $\mathfrak{l}$ touching σ. It turns out that the arrangement $\tilde{A}_2^0$ is an example of an irreducible affine Tits arrangement which is not locally spherical. More precisely, we have the following main theorem (precise definitions are given in Section 2):

Theorem. *Let the pair $(\mathcal{A}, T)$ be an affine rank three Tits arrangement and assume that the projective root vectors of $\mathcal{A}$ are contained in the locus of a homogeneous cubic polynomial. Then up to projectivities, $\mathcal{A}$ is either a near pencil, an arrangement of type $\tilde{A}_2$, or it is an arrangement of type $\tilde{A}_2^0$.*

This result is established by proving Theorem 2 in Section 3. The necessary definitions and notations are collected in Section 2. In Section 4 we discuss some related open questions.

2. Definitions and notation

We start with the notion of a Tits arrangement in $\mathbb{R}^r$ (see [Cuntz et al. 2017]).

Definition 1. Let $\mathcal{A}$ be a (possibly infinite) set of linear hyperplanes in $V := \mathbb{R}^r$ and let T be an open convex cone in V. We say that $\mathcal{A}$ *is locally finite in T* if for every $x \in T$ there exists a neighborhood $U_x \subset T$ of x, such that the set
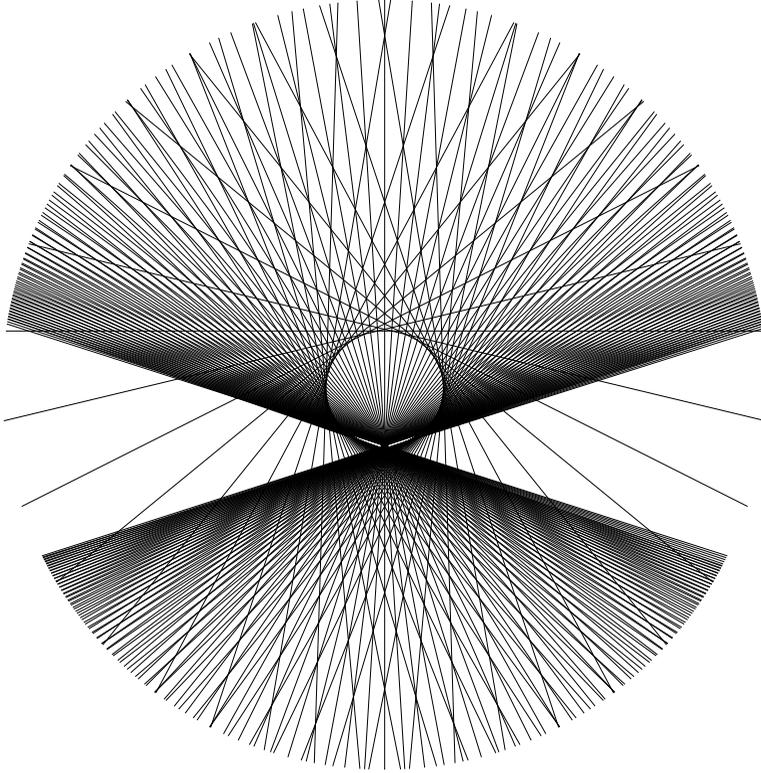
Figure 1. A subset of the arrangement of type $\tilde{A}_2^0$.

$\{H \in \mathcal{A} \mid H \cap U_x \neq \varnothing\}$ is finite. A *hyperplane arrangement (of rank r)* is a pair $(\mathcal{A}, T)$, where T is a convex open cone in V, and $\mathcal{A}$ is a set of linear hyperplanes such that the following holds:

- $H \cap T \neq \varnothing$ for all $H \in \mathcal{A}$.
- $\mathcal{A}$ is locally finite in T.

Denote by $\overline{T}$ the topological closure of T with respect to the standard topology of V. If $X \subset \overline{T}$ then the *localization at X (in $\mathcal{A}$)* is defined as

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}.$$

If $X = \{x\}$ we write $\mathcal{A}_x$ instead of $A_{\{x\}}$ and call $(\mathcal{A}_x, T)$ the *parabolic subarrangement at x*. The connected components of $T \setminus \bigcup_{H \in \mathcal{A}} H$ are called *chambers* or *cells*. If K is a chamber then its *walls* are given by the hyperplanes contained in the set

$$W^K := \{H \leq V \mid \dim(H) = r - 1, \langle H \cap \overline{K} \rangle_{\mathbb{R}} = H, H \cap K = \varnothing\}.$$

The arrangement $(\mathcal{A}, T)$ is called *thin* if $W^K \subset \mathcal{A}$ for each chamber K. A *simplicial hyperplane arrangement (of rank r)* is an arrangement $(\mathcal{A}, T)$ such that each chamber K is an open simplicial cone. T is called the *Tits cone* of the arrangement. Finally, a simplicial arrangement is called a *Tits arrangement* if it is also thin.

Remark 1. i) If the pair $(\mathcal{A}, T)$ is a Tits arrangement, we usually omit the reference to T, since it should always be clear from the context.

ii) For a simplicial arrangement $(\mathcal{A}, T)$, the closure of T can be reconstructed from the chambers of $\mathcal{A}$: we have $\overline{T} = \overline{\bigcup_{K \in \mathcal{K}(\mathcal{A})} K}$. In particular, the Tits cone T is determined by $\mathcal{A}$. For details on this, see [Cuntz et al. 2017, Lemma 3.24].

Definition 2. Let the pair $(\mathcal{A}, T)$ be a Tits arrangement and denote the set of chambers by $\mathcal{K}$. Then we have the following thin chamber complex

$$\mathcal{S}(\mathcal{A}, T) := \left\{ \overline{K} \cap \bigcap_{H \in X} H \mid K \in \mathcal{K}, X \subset W^K \right\},$$

whose poset-structure is given by set-wise inclusion. If $U \leq V$ has dimension 1 such that $v := \overline{K} \cap U \in \mathcal{S}(\mathcal{A}, T)$, then v is called a *vertex*. Similarly, if $U' \leq V$ has dimension 2 such that $e := \overline{K} \cap U' \in \mathcal{S}(\mathcal{A}, T)$, then e is called an *edge* or *segment*. A Tits arrangement $(\mathcal{A}, T)$ is called *locally spherical* if all vertices meet T. Finally, if v is a vertex then we define its *weight* to be $w(v) := |\mathcal{A}_v|$.

Remark 2. Let $(\mathcal{A}, T)$ be a Tits arrangement and let v be a vertex. If v meets T, then clearly $w(v) < \infty$ because $\mathcal{A}$ is locally finite in T. However, if v is contained in ∂T, then we necessarily have $w(v) = \infty$ because $\mathcal{A}$ is thin. Altogether, it follows that $w(v) = \infty$ if and only if $v \in \partial T$.

Definition 3. i) Let $(\mathcal{A}, T)$ be a Tits arrangement in $V := \mathbb{R}^r$. If there is a linear form $0 \neq \alpha \in V^*$ such that $T = \alpha^{-1}(\mathbb{R}_{>0})$ is a half-space, then we say that $(\mathcal{A}, T)$ is an *affine Tits arrangement*.

ii) Let $(\mathcal{A}, T)$ be a Tits arrangement in $\mathbb{R}^r$. If $T = \mathbb{R}^r$, then $\mathcal{A}$ is called a *spherical Tits arrangement*.

iii) Let $(\mathcal{A}, T)$ be a Tits arrangement in $\mathbb{R}^3$. Assume that there is $H_0 \in \mathcal{A}$ and a single vertex v such that v is contained in every $H \in \mathcal{A} \setminus \{H_0\}$ while H_0 does not contain v. Then $\mathcal{A}$ is called a *near pencil (arrangement)*. A Tits arrangement $(\mathcal{A}, T)$ in $\mathbb{R}^3$ which is not a near pencil arrangement is said to be *irreducible*.

Remark 3. i) If $(\mathcal{A}, T)$ is affine with corresponding linear form α, the boundary ∂T of the Tits cone is given by the hyperplane $\ker(\alpha)$.

ii) If $(\mathcal{A}, T)$ is spherical, then we have $|\mathcal{A}| < \infty$. Indeed, for such an arrangement we have $0 \in T$ and $0 \in H$ for every $H \in \mathcal{A}$; as by definition $\mathcal{A}$ is locally finite in T, this proves the claim. Moreover, we note that a spherical Tits arrangement in $\mathbb{R}^r$ induces a simplicial cell decomposition of the unit sphere $\mathbb{S}_{r-1}$.

iii) Near pencil arrangements are usually considered trivial.

In the following sections we will be concerned with the case of an affine Tits arrangement $(\mathcal{A}, T)$ of rank three. Then we may view $\mathcal{A}$ as set of projective lines in $\mathbb{P}^2(\mathbb{R})$ and the boundary ∂T of T is again a projective line. Further, in $\mathbb{P}^2(\mathbb{R})$ we have a duality between projective lines and projective points, for which we require the following notation.

Notation. Let $(\mathcal{A}, T)$ be a Tits arrangement of rank three. By abuse of notation we denote the set of projective lines $\{\mathfrak{g} \mid \exists H \in \mathcal{A} : \mathfrak{g} = \pi(H)\}$ by $\mathcal{A}$ as well; here $\pi : \{U \leq \mathbb{R}^3 \mid \dim U \geq 1\} \to \{U \leq \mathbb{P}^2(\mathbb{R})\}$ is the natural projection. If $p \in (\mathbb{P}^2(\mathbb{R}))^*$ then we denote the corresponding dual line by $p^* \subset \mathbb{P}^2(\mathbb{R})$. Likewise, if $\mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$ is a projective line, then its corresponding dual point is denoted by $\mathfrak{l}^* \in \mathbb{P}^2(\mathbb{R})$. Similarly, if $\mathcal{A}$ is a set of projective lines in $\mathbb{P}^2(\mathbb{R})$, we write $\mathcal{A}^* \subset (\mathbb{P}^2(\mathbb{R}))^*$ for the corresponding set of dual projective points (and vice versa). For a set of projective lines $\mathcal{A}$, we call $\mathcal{A}^*$ its corresponding *dual point set*.

Finally, we introduce the following notion of isomorphism of Tits arrangements.

Definition 4. Set $V := \mathbb{R}^3$ and let $(\mathcal{A}, T)$, $(\mathcal{A}', T')$ be two affine Tits arrangements in V. Then these are called *(projectively) isomorphic* if there exists $\phi \in \mathrm{PGL}(V^*)$ such that both $\phi(\mathcal{A}^*) = (\mathcal{A}')^*$ and $\phi(\partial T^*) = (\partial T')^*$.

3. Results and proofs

Now we are ready to prove our main theorem. The main strategy can be summarized as follows: according to the possible factorizations of a homogeneous cubic polynomial P, there are naturally three cases to consider. Namely, P may factor as a product of three linear polynomials, or it may factor as a product of an irreducible quadratic polynomial and a linear polynomial, or P may be irreducible. We examine all three cases and collect all (up to projectivity) affine Tits arrangements $\mathcal{A}$ such that $\mathcal{A}^* \subset V(P)$.

We start with the following lemma which will be used extensively to rule out the possibility of existence of certain Tits arrangements. It basically says that near pencils are the only rank three Tits arrangements containing a segment bounded by two vertices of weight two.

Lemma 1. *Let $\mathcal{A}$ be a Tits arrangement of rank three. Suppose there is a line $\mathfrak{g} \in \mathcal{A}$ containing two vertices v_1, v_2 of weight two such that there is no other vertex contained in the bounded segment between v_1 and v_2 on $\mathfrak{g}$. Then $\mathcal{A}$ is a near pencil.*

Proof. Denote by $\mathfrak{g}_1, \mathfrak{g}_2$ the two lines meeting $\mathfrak{g}$ in v_1 respectively v_2 and set $v := \mathfrak{g}_1 \cap \mathfrak{g}_2$. Since $w(v_1) = w(v_2) = 2$, the vertices v_1, v_2 are in the interior of the Tits cone by Remark 2, and it follows that there are two chambers with vertices

v_1, v_2, v. Every other line $\mathfrak{g}' \in \mathcal{A} \setminus \{\mathfrak{g}\}$ has to avoid these two chambers and hence needs to pass through v. $\qquad\square$

We state some more lemmata, which will turn out to be useful and may be interesting in their own right.

Lemma 2. *Let $\mathcal{A}$ be an affine Tits arrangement of rank three. Then there is at most one vertex of $\mathcal{A}$ contained in ∂T.*

Proof. By Remark 2, we have $|\mathcal{A}_v| = \infty$ for every vertex $v \in \partial T$. Now suppose there were two different vertices $v, w \in \partial T$. There is a chamber K having v as a vertex. As $\mathcal{A}$ is thin and locally finite, it follows that K has to be contained in the cone C generated by two neighboring lines passing through v. Let $\epsilon > 0$ and consider the ball $B_\epsilon(v)$ centered at v with radius ϵ. Write $C_\epsilon := B_\epsilon(v) \cap C$ and note that there are infinitely many lines passing through w which accumulate at ∂T. From this, it follows that there is a line passing through w and intersecting C_ϵ for every $\epsilon > 0$. Hence, $\mathcal{A}$ contains a line intersecting K, a contradiction. $\qquad\square$

Lemma 3. *Let $\mathcal{A}$ be a Tits arrangement of rank three. Assume that $\mathcal{A}^* \subset V(P)$ for some homogeneous polynomial $P \in \mathbb{R}[x, y, z]$ of degree d. Write $P = Q \cdot \prod_{1 \leq i \leq s} l_i$, where the $l_1, \ldots, l_s \in \mathbb{R}[x, y, z]$ are (not necessarily distinct) linear forms and Q has no linear factors. Then $\mathcal{A}$ determines at most s vertices of weight exceeding d.*

Proof. A vertex v determined by $\mathcal{A}$ is a point in $\mathbb{P}^2(\mathbb{R})$. Therefore, the dual $\mathfrak{l} := v^*$ is a line in $(\mathbb{P}^2(\mathbb{R}))^*$. By Bézout's theorem we know that $|\mathcal{A}^* \cap \mathfrak{l}| \leq |V(P) \cap \mathfrak{l}| \leq d$, unless $\mathfrak{l}$ is a component of $V(P)$. As by assumption $V(P)$ contains at most s linear components, this proves the claim. $\qquad\square$

Lemma 4. *Let $\mathcal{A}$ be a Tits arrangement of rank three. Suppose there is a vertex v of weight two which is surrounded by vertices v_1, v_2, v_3, v_4 of weight three. Then $\mathcal{A}$ is spherical and $|\mathcal{A}| \in \{6, 7\}$.*

Proof. Notice first that all the vertices v, v_1, v_2, v_3, v_4 are in the interior of the Tits cone by Remark 2. We denote the lines intersecting in v by $\mathfrak{l}_1, \mathfrak{l}_2$ and we agree that $v_1, v_3 \in \mathfrak{l}_1$ while $v_2, v_4 \in \mathfrak{l}_2$. It is clear that there are no further vertices lying in the segment between v_1 and v_4 and the same is true for the segments between v_1 and v_2, v_2 and v_3, v_3 and v_4. Denote the line passing through v_i and v_j by $\mathfrak{l}_{i,j}$ and observe that the spherical arrangement $\mathcal{B} \subset \mathcal{A}$ defined by $\mathcal{B} := \{\mathfrak{l}_{i,j} \mid 1 \leq i < j \leq 4\}$ is simplicial. Moreover, there are chambers $K_{i,j}$ of $\mathcal{A}$ which do not contain v but which contain v_i, v_j for $\{i, j\} \in \{\{1, 4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}\}$. As $\mathcal{A}$ is simplicial, these chambers are simplicial cones.

Now suppose there was a line $\mathfrak{g}$ supporting an edge of a chamber $K_{i,j}$ such that $g \in \mathcal{A} \setminus \mathcal{B}$. Then $\mathfrak{g}$ needs to pass through either v_i or v_j. But then the weight of either v_i or v_j needs to be strictly greater than three, contradicting our assumption. This shows that there is precisely one line which one may add to $\mathcal{B}$ in such a way

that the obtained arrangement is simplicial with $w(v_i) = 3$ for $1 \leq i \leq 4$. Namely, the line passing through the points $\mathfrak{l}_{1,2} \cap \mathfrak{l}_{3,4}$ and $\mathfrak{l}_{1,4} \cap \mathfrak{l}_{2,3}$. $\square$

The next proposition and theorem will simplify the proof of Proposition 2.

Proposition 1. *Let $\mathcal{A}$ be a Tits arrangement of rank three. Assume that $\mathcal{A}^*$ is contained in the union of two lines. Then $\mathcal{A}$ is a near pencil.*

Proof. Clearly, we may assume that $|\mathcal{A}| > 4$. So suppose that $\mathcal{A}^* \subset \mathfrak{l}_1 \cup \mathfrak{l}_2$. Then after dualizing the lines $\mathfrak{l}_1$ and $\mathfrak{l}_2$ become two points $v_1, v_2 \in \mathbb{P}^2(\mathbb{R})$ and we have $\mathcal{A}_{v_1} \cup \mathcal{A}_{v_2} = \mathcal{A}$. Without loss of generality we may assume that $w(v_1) = |\mathcal{A}_{v_1}| > 2$. Moreover, as $\mathcal{A}$ is a Tits arrangement there must exist a line $\mathfrak{l} \in \mathcal{A}$ such that $v_1 \notin \mathfrak{l}$. But $\mathfrak{l}$ and every other line of $\mathcal{A}$ which does not pass through v_1 has to pass through v_2. It follows from Lemma 3 that v_2 is the only point on $\mathfrak{l}$ which could potentially be a vertex of weight greater than two. But as $|\mathcal{A}_{v_1}| > 2$, the line $\mathfrak{l}$ contains at least one segment bounded by two vertices of weight two. Hence by Lemma 1 it follows that $\mathcal{A}$ is a near pencil (and a posteriori, there are at most two lines passing through v_2). $\square$

Remark 4. In the situation of Proposition 1 there is a unique $\alpha \in \mathcal{A}^*$ such that $\mathcal{A}^* \setminus \{\alpha\}$ is contained in one of the two lines $\mathfrak{l}_1, \mathfrak{l}_2$ while α lies in the other one.

Theorem 1. *Near pencils are the only Tits arrangements of rank three whose dual point sets lie on a conic.*

Proof. Let $P \in \mathbb{R}[x, y, z]$ be a homogeneous polynomial of degree two and set $\sigma := V(P)$. Suppose that $\mathcal{A}^* \subset \sigma$ for some rank three Tits arrangement $\mathcal{A}$. First, assume that P factors in two distinct linear polynomials. Then by Proposition 1 the only Tits arrangements lying on σ are near pencils. If P factors as a square of a linear polynomial, then every $\alpha \in \mathcal{A}^*$ lies on a single line which means that all lines of $\mathcal{A}$ pass through a single point. Hence $\mathcal{A}$ is not simplicial. Now, finally suppose that P is irreducible. By Lemma 3, the weight of any vertex of $\mathcal{A}$ is bounded by two. But this implies that $\mathcal{A}$ is a near pencil consisting of three lines. $\square$

The next proposition is a first step towards our main theorem. Before stating it, we need to introduce the affine reflection arrangement of type $\tilde{A}_2$.

Definition 5. Let $V := \mathbb{R}^3$ with standard basis e_1, e_2, e_3 and denote the corresponding dual basis vectors by $\alpha_1, \alpha_2, \alpha_3 \in V^*$. Consider the matrix

$$C := \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

For $1 \leq i, j \leq 3$ we define reflections $\sigma_i : V^* \to V^*$ via the formulae
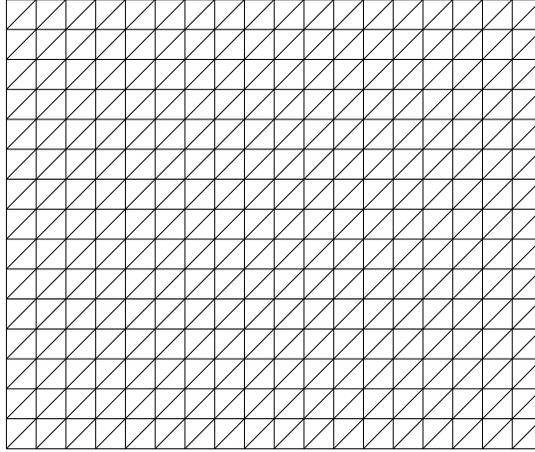
$$\sigma_i(\alpha_j) := \alpha_j - C_{i,j}\alpha_i.$$

Figure 2. A subset of an arrangement of type $\tilde{A}_2$.

The (infinite) subgroup of $\mathrm{GL}(V^*)$ generated by $\sigma_1, \sigma_2, \sigma_3$ is called $\tilde{A}_2$. Let $\mathcal{O}$ be the union of the orbits of $\alpha_1, \alpha_2, \alpha_3$ under $\tilde{A}_2$. Define the arrangement $\mathcal{A} := \{\alpha^{\perp} \mid \alpha \in \mathcal{O}\}$. It is well known that $\mathcal{A}$ defines an affine rank three Tits arrangement. Any arrangement which is projectively isomorphic to $\mathcal{A}$ is then said *to be of type $\tilde{A}_2$.* (See Figure 2 for a visualization of such an arrangement.)

Remark 5. i) The matrices which give rise to affine Tits arrangement in the manner of Definition 5 are called *generalized Cartan matrices of affine type*. A complete classification as well as explicit descriptions of the corresponding root systems can be found for instance in [Kac 1990]. We observe that all arrangements obtainable in this way are locally spherical. Thus, one might believe that this is true for *every* affine Tits arrangement. However, this is not the case (see the proof of Corollary 1).

ii) We observe that for any arrangement $(\mathcal{A}, T)$ of type $\tilde{A}_2$, there are three points $v_1, v_2, v_3 \in \partial T$ such that $\mathcal{A} = \mathcal{A}_{v_1} \cup \mathcal{A}_{v_2} \cup \mathcal{A}_{v_3}$. However, none of these forms a vertex of $\mathcal{A}$. In Figure 2, the points v_1, v_2, v_3 are the intersection points at infinity of classes of mutually parallel lines. Moreover, by the above we see that the dual point set $\mathcal{A}^*$ is contained in the union of three lines (dual to v_1, v_2, v_3 respectively).

Proposition 2. *Let $\mathcal{A}$ be an affine Tits arrangement of rank three. Suppose that $\mathcal{A}^*$ is contained in the union of at most three lines and assume that $\mathcal{A}$ is not a near pencil. Then $\mathcal{A}$ is an arrangement of type $\tilde{A}_2$.*

Proof. Taking into account Theorem 1 it is enough to consider the case where $\mathcal{A}^*$ is contained in the union of exactly three lines: $\mathcal{A}^* \subset \mathfrak{l}_1 \cup \mathfrak{l}_2 \cup \mathfrak{l}_3$. We define $v_1 := \mathfrak{l}_1^*, v_2 := \mathfrak{l}_2^*, v_3 := \mathfrak{l}_3^*$ so that $\mathcal{A} = \mathcal{A}_{v_1} \cup \mathcal{A}_{v_2} \cup \mathcal{A}_{v_3}$. Now we consider three cases:

a) Suppose that there is exactly one $i \in \{1, 2, 3\}$ such that $|\mathcal{A}_{v_i}| = \infty$. We may assume that $i = 1$. Choose a line $\mathfrak{l} \in \mathcal{A}$ such that $\mathfrak{l}$ passes through v_2 but not through v_1. As $|\mathcal{A}_{v_3}| < \infty$, we conclude that $\mathfrak{l}$ contains infinitely many vertices of weight two but only finitely many vertices of weight possibly greater than two. Thus, we find that $\mathfrak{l}$ contains a segment bounded by vertices of weight two. Lemma 1 now shows that $\mathcal{A}$ must be a near pencil, contradicting our assumption.

b) Suppose that there are exactly two indices $i, j \in \{1, 2, 3\}$ such that $|\mathcal{A}_{v_i}| = |\mathcal{A}_{v_j}| = \infty$. We may assume that $\{i, j\} = \{1, 2\}$. Again, we choose a line $\mathfrak{l}$ which passes through v_2 but not through v_1. Because $|\mathcal{A}_{v_3}| < \infty$, we may again conclude that $\mathfrak{l}$ contains infinitely many vertices of weight two but only finitely many vertices of weight possibly greater than two. As in part a), Lemma 1 tells that $\mathcal{A}$ must be a near pencil, which is impossible.

c) Suppose that $|\mathcal{A}_{v_1}| = |\mathcal{A}_{v_2}| = |\mathcal{A}_{v_3}| = \infty$. Then the corresponding points v_1, v_2, v_3 all lie on the line ∂T. In the affine space $\mathbb{E} := \mathbb{P}^2(\mathbb{R}) \setminus \partial T$, the lines through v_1, v_2, v_3 are given by three respective classes of mutually parallel lines. This shows that $\mathcal{A}$ must be of type $\tilde{A}_2$ (compare Figure 2). $\qquad \square$

Remark 6. i) While case a) in the above proof is possible only for near pencil arrangements, we note that there is no Tits arrangement at all satisfying the conditions of case b).

ii) If we drop the condition on $\mathcal{A}$ to be affine, then we find some more possible (spherical) arrangements such that $\mathcal{A}^*$ is contained in the union of three lines: for instance the arrangement of type A(10,3) (as denoted in [Grünbaum 2009]) and some of its subarrangements.

Our next goal is to show that there is no affine Tits arrangement $\mathcal{A}$ such that $\mathcal{A}^*$ is contained in the locus of an irreducible homogeneous cubic polynomial. This is established in the following result, which uses the well known fact that an irreducible singular cubic curve in $\mathbb{P}^2(\mathbb{R})$ has precisely one singular point, which is either an isolated point, a cusp or a double point.

Lemma 5. *There is no affine Tits arrangement of rank three whose dual point set is contained in the locus of an irreducible homogeneous cubic polynomial.*

Proof. Assume that $\mathcal{A}^* \subset C := V(P)$ for some homogeneous irreducible cubic polynomial P. By Lemma 3, we see that $w(v) \leq 3$ for every vertex v. In particular, the arrangement $\mathcal{A}$ cannot be a near pencil. Assume that there exists a vertex v of weight two. Then the above together with Lemma 1 implies that every neighbor of v has weight three. But then by Lemma 4, we conclude that $\mathcal{A}$ is spherical, a contradiction. Thus, every vertex of $\mathcal{A}$ has weight three. In particular, every $\alpha \in \mathcal{A}^*$ is a smooth point of C: every line through a pair of different points $\alpha, \beta \in \mathcal{A}^*$ meets C in a unique third point denoted $\alpha \oplus \beta \in \mathcal{A}^*$. Moreover, as $\mathcal{A}$ is assumed to be

affine, the set $\mathcal{A}^*$ is infinite with unique accumulation point $(\partial T)^*$, in particular $(\partial T)^* \in C$. Now we consider two cases.

Case a): Assume that $(\partial T)^*$ is a smooth point of C. Then we can find $\alpha \in \mathcal{A}^*$ such that the line through α and $(\partial T)^*$ meets C in a third point γ. Now choose a sequence $(\alpha_n)_{n \in \mathbb{N}}$ from $\mathcal{A}^* \setminus \{\alpha\}$ converging towards $(\partial T)^*$. Then by the above, we see that the sequence $(\alpha \oplus \alpha_n)_{n \in \mathbb{N}}$ consists of elements from $\mathcal{A}^*$ and converges towards γ. As by construction $(\partial T)^* \neq \gamma$, this is a contradiction to $\mathcal{A}$ having a unique accumulation point.

Case b): Assume that $(\partial T)^*$ is a singular point of C. As $(\partial T)^*$ cannot be an isolated point, we may assume that C has either a double point or a cusp at $(\partial T)^*$.

i) Assume first that $(\partial T)^*$ is a cusp of C. By the Weierstrass normal form for cubic polynomials we may assume that $P := y^2 z - x^3$, in particular $(\partial T)^* = (0 : 0 : 1)$. Then in the affine $z = 1$ part of $\mathbb{P}^2(\mathbb{R})$, the curve C consists of two branches C_1, C_2 meeting in $(\partial T)^*$ and given explicitly by

$$C_1 := \{(x : y : 1) \in \mathbb{P}^2(\mathbb{R}) \mid y = x^{3/2}\},$$
$$C_2 := \{(x : y : 1) \in \mathbb{P}^2(\mathbb{R}) \mid y = -x^{3/2}\}.$$

As $(\partial T)^*$ is the unique accumulation point of $\mathcal{A}^*$, there is a point $\alpha_1 \in \mathcal{A}^* \cap (C_1 \cup C_2)$ whose x-coordinate is maximal (when the z-coordinate is normalized to 1). In particular, we have $(0 : 1 : 0) \notin \mathcal{A}^* \cap C$. Without loss of generality, we may assume that $\alpha_1 \in C_2$. Let $\alpha_2 \in C_2 \setminus \{\alpha_1\}$ be the point on C_2 whose x-coordinate is exceeded only by α_1. Similarly, let $\alpha_3 \in C_2 \setminus \{\alpha_1, \alpha_2\}$ be the point on C_2 whose x-coordinate is exceeded only by α_1, α_2. One checks that the point of $\mathcal{A}^* \cap C_1$ with maximal x-coordinate is then given by $\beta := \alpha_1 \oplus \alpha_2$ (remember that $\alpha \oplus \alpha' \in \mathcal{A}^*$ for $\alpha \neq \alpha' \in \mathcal{A}^*$). In particular, if γ denotes the second point of C on the tangent to C at β, then we see that α_3 necessarily has x-coordinate strictly smaller than γ. But then the point $((\alpha_1 \oplus \alpha_2) \oplus \alpha_3) \oplus \alpha_1 \in \mathcal{A}^* \cap C_2$ is different from α_2 and has x-coordinate strictly greater than α_3 but less than α_1, contradicting the choice of α_3.

ii) Now assume that C has a double point at $(\partial T)^*$. We may assume that $C := V(P)$ for $P = y^2 z - x^2(x + z)$. Then in the affine $z = 1$ part of $\mathbb{P}^2(\mathbb{R})$, the curve C is given by the union of the following three sets C_1, C_2, C_3 given explicitly by

$$C_1 := \{(x : y : 1) \in \mathbb{P}^2(\mathbb{R}) \mid x > 0, \ y = x(x + 1)^{1/2}\},$$
$$C_2 := \{(x : y : 1) \in \mathbb{P}^2(\mathbb{R}) \mid x > 0, \ y = -x(x + 1)^{1/2}\},$$
$$C_3 := \{(x : y : 1) \in \mathbb{P}^2(\mathbb{R}) \mid -1 \leq x \leq 0, \ y = \pm x(x + 1)^{1/2}\}.$$

Using the fact that $(\partial T)^* = (0 : 0 : 1)$ is an accumulation point of $\mathcal{A}^*$, we see that both $C_1 \cap \mathcal{A}^*$, $C_2 \cap \mathcal{A}^*$ are infinite. Focusing on these two sets, one can use the same techniques as in i) to obtain another contradiction. $\qquad\square$

Remark 7. If one drops the assumption that A is affine, then there are candidates for (spherical) Tits arrangements $\mathcal{A}$ such that $\mathcal{A}^* \subset V(P)$ for an irreducible cubic polynomial P: namely all spherical arrangements having only vertices of weight two or three. Since these are precisely the arrangements $A(6, 1)$, $A(7, 1)$ and the near pencils with at most four lines, we will not elaborate on this further.

It remains to consider the possibility that $\mathcal{A}^*$ is contained in the locus of a cubic homogeneous polynomial having an irreducible quadratic factor. As preparation, we formulate the following remark and definition which will be useful later.

Definition 6. a) Let σ be an irreducible conic in $\mathbb{P}^2(\mathbb{R})$ and consider a subset $M \subset \sigma$. There exists a projectivity Ψ such that $\Psi(\sigma)$ is given by the polynomial $P := x^2 + y^2 - z^2$ and is thus contained entirely in the affine $z = 1$ part of $\mathbb{P}^2(\mathbb{R})$. We say that $p_1, \ldots, p_k \in M$ are *consecutive with respect to* Ψ, if for any $1 \leq i \leq k-1$ it is true that one of the segments on $\Psi(\sigma)$ bounded by $\Psi(p_i)$, $\Psi(p_{i+1})$ contains no other point of $\Psi(M)$.

b) Consider the map $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ sending $v_1, v_2 \in \mathbb{R}^3$ to their vector product $v_1 \times v_2$. This induces a map $\psi : \left(\mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R}) \right) \setminus \Delta \to (\mathbb{P}^2(\mathbb{R}))^*$, where $\Delta := \{(x, x) \mid x \in \mathbb{P}^2(\mathbb{R})\}$; $(\psi(\langle v_1 \rangle, \langle v_2 \rangle)$ is the line dual to the projective point $\langle v_1 \times v_2 \rangle)$. By a slight abuse of notation, we write $\psi(v_1, v_2) = v_1 \times v_2 \in (\mathbb{P}^2(\mathbb{R}))^*$ for two different projective points $v_1, v_2 \in \mathbb{P}^2(\mathbb{R})$. Observe that for $p, q \in (\mathbb{P}^2(\mathbb{R}))^*$ the vector product $p \times q$ gives the vertex in $\mathbb{P}^2(\mathbb{R})$ obtained as the intersection of the dual lines p^*, q^*. Similarly, if v, v' are two points in $\mathbb{P}^2(\mathbb{R})$, then the vector product $v \times v'$ gives the point in $(\mathbb{P}^2(\mathbb{R}))^*$ which is dual to the line passing through v and v'.

Now we can prove the following statement. (Compare Theorem 3.6 in [Cuntz 2011b], where case c) of the following proposition is examined for spherical Tits arrangements.)

Proposition 3. *Suppose that $\mathcal{A}$ is an affine rank three Tits arrangement and assume that $\mathcal{A}^* \subset \sigma \cup \mathfrak{l}$ for some irreducible conic $\sigma \subset (\mathbb{P}^2(\mathbb{R}))^*$ and an arbitrary line $\mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$. Then the following statements hold*:

  a) $|\mathcal{A}^* \cap \sigma| = \infty$, *unless $\mathcal{A}$ is a near pencil.*

  b) $|\mathcal{A}^* \cap \mathfrak{l}| = \infty$ *and* $(\partial T)^* \in \mathfrak{l}$.

  c) *If* $|\sigma \cap \mathfrak{l}| = 0$ *then $\mathcal{A}$ is a near pencil.*

  d) *If* $|\sigma \cap \mathfrak{l}| = 1$ *then* $\sigma \cap \mathfrak{l} = (\partial T)^*$, *unless $\mathcal{A}$ is a near pencil.*

  e) *If* $|\sigma \cap \mathfrak{l}| = 2$ *then $\mathcal{A}$ is a near pencil.*

Proof. a) Define $\mathcal{B}^* := \mathcal{A}^* \cap \sigma$ and suppose that $|\mathcal{B}^*| < \infty$. Since $\mathcal{A}$ is affine and hence necessarily infinite, the set $L^* := \mathcal{A}^* \cap \mathfrak{l}$ is infinite. We have $\mathcal{A} = \mathcal{B} \cup L$ and it is easy to see that we find a line in $\mathcal{B}$ containing a segment bounded by two vertices of weight two: pick an arbitrary line $\mathfrak{g} \in \mathcal{B} \setminus L$. Then each of the infinitely

many lines in L produces a unique vertex on $\mathfrak{g}$. On the other hand, there are only finitely many lines in $\mathcal{B}$, which could turn some of the above vertices into triple points. By Lemma 1 we conclude that $\mathcal{A}$ is a near pencil.

b) If $\mathcal{A}$ is a near pencil then both statements are easily seen to be true: in this case, all lines except one pass through the point $\mathfrak{l}^*$, thus $|\mathcal{A}^* \cap \mathfrak{l}| = \infty$. Using the fact that $\mathcal{A}$ is locally finite in T, we conclude that $\mathfrak{l}^* \in \partial T$, that is $(\partial T)^* \in \mathfrak{l}$. So we may assume that $\mathcal{A}$ is not a near pencil. We show that the second statement is a consequence of the first. So suppose that $|\mathcal{A}^* \cap \mathfrak{l}| = \infty$ and assume that $(\partial T)^* \notin \mathfrak{l}$. Dualizing we obtain that the point $\mathfrak{l}^*$ does not lie on the line ∂T. Hence $\mathfrak{l}^*$ lies in T and there are infinitely many lines of $\mathcal{A}$ passing through $\mathfrak{l}^*$. But since $\mathcal{A}$ is locally finite in T this is impossible. So it suffices to prove that $|\mathcal{A}^* \cap \mathfrak{l}| = \infty$. We show that $|\mathcal{A}^* \cap \mathfrak{l}| < \infty$ gives a contradiction: fix some $q \in \mathcal{A}^* \cap \sigma$ and for $q \neq p \in \sigma \cap \mathcal{A}^*$ consider the corresponding dual lines p^*, q^* in $\mathbb{P}^2(\mathbb{R})$. Different choices of p will yield different vertices on q^* and part a) implies that there are infinitely many such vertices. If $|\mathcal{A}^* \cap \mathfrak{l}| < \infty$, then only finitely many of these vertices can be turned into triple points. Thus, q^* must contain a segment bounded by two double points, which by Lemma 1 implies that $\mathcal{A}$ is a near pencil. This is the desired contradiction.

For the proof of c) and d) it suffices to note that $\mathcal{A}$ must be a near pencil if $(\partial T)^* \notin \sigma$. Indeed, if this is the case, then we necessarily have $|\mathcal{A}^* \cap \sigma| < \infty$, since points of $\mathcal{A}^*$ may accumulate only in a neighborhood of $(\partial T)^*$ (because $\mathcal{A}$ is locally finite in T). But then part a) implies that $\mathcal{A}$ is a near pencil.

e) After applying a projectivity as in part a) of Definition 6, we may assume that $\sigma = V(P)$ where $P := x^2 + y^2 - z^2$. So σ is contained entirely in the affine $z = 1$ part of $(\mathbb{P}^2(\mathbb{R}))^*$. We write σ' for the conic in $\mathbb{P}^2(\mathbb{R})$ defined by the same polynomial.

Suppose that $\mathcal{A}$ is not a near pencil. As points of $\mathcal{A}^*$ may accumulate only in a neighborhood of $(\partial T)^*$, we have $(\partial T)^* \in \sigma \cap \mathfrak{l}$. Observe that for $p = (a : b : 1) \in \sigma \cap \mathcal{A}^* \subset (\mathbb{P}^2(\mathbb{R}))^*$ the corresponding dual line p^* is the tangent to σ' at the point $(-a : -b : 1) \in \mathbb{P}^2(\mathbb{R})$. In particular, if $(\partial T)^* = (x : y : 1)$, this implies that there is a sequence of tangent lines to σ' converging towards the tangent line at the point $(-x : -y : 1)$, and this tangent line is precisely ∂T. It remains to identify the dual lines q^* corresponding to $q \in \mathfrak{l} \cap \mathcal{A}^*$. We may assume without loss of generality that in the $z = 1$ part of $(\mathbb{P}^2(\mathbb{R}))^*$ the line $\mathfrak{l}$ is given by the equation $y = \lambda$ for some $0 \leq \lambda < 1$. Hence any $q \in \mathfrak{l}$ will have homogeneous coordinates $q = (x_0 : \lambda : 1)$. So if $\lambda > 0$, the equation of the dual line q^* in the $z = 1$ part of $\mathbb{P}^2(\mathbb{R})$ will be $y = -\frac{x_0 \cdot x}{\lambda} - \frac{1}{\lambda}$; if on the other hand $\lambda = 0$, then the equation of q^* will be $x = -\frac{1}{x_0}$. Hence if $\lambda > 0$, then all lines pass through the point $(0 : -\frac{1}{\lambda} : 1)$ which implies that $\mathfrak{l}^* = (0 : -\frac{1}{\lambda} : 1)$; if $\lambda = 0$, then all lines pass through $\mathfrak{l}^* = (0 : 1 : 0)$. This

shows that $\mathfrak{l}^* \notin \sigma'$. Since $(\partial T)^* \in \mathfrak{l}$ we conclude that $\mathfrak{l}^* \in \partial T$. Now we take ∂T as line at infinity. Doing so, we obtain $\mathcal{A}$ as union of tangent lines to a parabola together with infinitely many parallel lines each of which being non-parallel to the symmetry axis of the parabola. But then $\mathcal{A}$ is not simplicial. $\qquad\square$

The following lemma will be the key to proving the main theorem.

Lemma 6. *Let σ be an irreducible conic together with a projectivity Ψ as in part a) of Definition 6. Assume that $\mathfrak{l}$ is a line touching σ and let $\mathcal{A}$ be an irreducible affine rank three Tits arrangement. If $\mathcal{A}^* \subset \sigma \cup \mathfrak{l}$, then $\mathcal{A}$ is determined by specifying four points on σ which are consecutive with respect to Ψ. More precisely, if $p_{-1}, p_0, p_1, p_2, p_3, p_4 \in \mathcal{A}^* \cap \sigma$ are six consecutive points (with respect to Ψ), then we have the following formulae for p_{-1} and p_4 in terms of $p_0, \ldots, p_3$:*

$$p_4 = \left(p_0 \times \left(\mathfrak{l}^* \times (p_1 \times p_3)\right)\right) \times \left(p_1 \times \left(\mathfrak{l}^* \times (p_2 \times p_3)\right)\right), \qquad (1)$$

$$p_{-1} = \left(p_2 \times \left(\mathfrak{l}^* \times (p_0 \times p_1)\right)\right) \times \left(p_3 \times \left(\mathfrak{l}^* \times (p_0 \times p_2)\right)\right). \qquad (2)$$

Moreover, if v is a vertex of weight two of $\mathcal{A}$ with lines $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{A}$ passing through v, then $\mathfrak{l}^* \in \mathfrak{g}_1$ or $\mathfrak{l}^* \in \mathfrak{g}_2$.

Proof. Denote by $L_1, L_2 \subset \mathcal{A}$ the set of lines corresponding to elements in $\mathcal{A}^* \cap \sigma$, $\mathcal{A}^* \cap \mathfrak{l}$ respectively. Observe that every $\mathfrak{h} \in L_2$ passes through the point $\mathfrak{l}^*$ while no line belonging to L_1 passes through $\mathfrak{l}^*$: if $\mathfrak{l}^* \in \mathfrak{g}$ and $\mathfrak{g}^* \in \sigma$ for some $\mathfrak{g}$, then $\mathfrak{g}^* = \mathfrak{l} \cap \sigma = (\partial T)^*$, by part d) of Proposition 3. As $\mathcal{A}$ is thin by definition, we conclude that $\mathfrak{g} \notin \mathcal{A}$.

Note also that every vertex of weight two of $\mathcal{A}$ must lie on a line belonging to L_2. Indeed, assume there was a vertex v of weight two such that $v = \mathfrak{g} \cap \mathfrak{g}'$ for some $\mathfrak{g}, \mathfrak{g}' \in L_1$. As $\mathcal{A}^* \subset \sigma \cup \mathfrak{l}$ and because no line belonging to L_1 passes through $\mathfrak{l}^*$, we may use Lemma 3 to conclude that every neighbor of v has weight bounded by three. But then by Lemma 1 every neighbor of v has weight precisely three, because by assumption $\mathcal{A}$ is not a near pencil. By Lemma 4 we obtain that $\mathcal{A}$ is spherical, a contradiction. In particular, it follows that for every vertex v' obtained as intersection of elements in L_1 there is a line $\mathfrak{h} \in L_2$ passing through v'. Also, every vertex of weight two is a neighbor of $\mathfrak{l}^*$, proving the last claim of the lemma.

These conditions already suffice to prove the claim. Let $p_0, p_1, p_2, p_3 \in \mathcal{A}^* \cap \sigma$ be four consecutive points (with respect to Ψ). We need to construct the points $p_{-1}, p_4 \in \mathcal{A}^* \cap \sigma$ such that both p_{-1}, p_0, p_1, p_2 and p_1, p_2, p_3, p_4 are consecutive (with respect to Ψ). By symmetry, it suffices to construct p_4. For this, denote the line corresponding to p_i by $\mathfrak{g}_i$ and let $\mathfrak{h}$ be the line passing through the vertices $\mathfrak{l}^*, \mathfrak{g}_1 \cap \mathfrak{g}_3$. Similarly, denote by $\mathfrak{h}'$ the line passing through the vertices $\mathfrak{l}^*, \mathfrak{g}_2 \cap \mathfrak{g}_3$. Then $\mathfrak{g}_4$ is the line passing through the vertices $\mathfrak{g}_0 \cap \mathfrak{h}, \mathfrak{g}_1 \cap \mathfrak{h}'$. From this, one reads off that (1) holds. This completes the proof. $\qquad\square$

Remark 8. Let $P \in \mathbb{R}[x, y, z]$ be a homogeneous cubic polynomial having an irreducible quadratic factor and let $\mathcal{A}$ be a spherical Tits arrangement such that $\mathcal{A}^* \subset V(P)$. If $\mathcal{A}$ is not a near pencil, then one may use Lemma 3 to conclude that there are two possibilities for $\mathcal{A}$: either $\mathcal{A}$ is the arrangement $A(7, 1)$ or $\mathcal{A}$ belongs to the infinite family $\mathcal{R}(1)$ (see [Grünbaum 2009]).

Now we can construct the arrangement of type $\tilde{A}_2^0$ and prove that up to projective isomorphism, it is the only non-trivial affine rank three Tits arrangement whose dual point set is contained in the locus of a cubic polynomial having an irreducible quadratic factor:

Proposition 4. *Up to projectivity, there is only one irreducible affine rank three Tits arrangement $\mathcal{A}$ such that $\mathcal{A}^*$ is contained in the locus of a cubic polynomial P having an irreducible quadratic factor. The arrangement $\mathcal{A}$ may be defined by the following set of dual points*:

$$\mathcal{A}^* = \left\{ \left(k : \tfrac{k(k-1)}{2} : 1 \right), \left(1 : \tfrac{k}{2} : 0 \right) \mid k \in \mathbb{Z} \right\}.$$

Proof. Let $\mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$ be the line corresponding to the linear factor of P and let $\sigma \subset (\mathbb{P}^2(\mathbb{R}))^*$ be the irreducible conic corresponding to the quadratic factor of P. We then have $\mathcal{A}^* \subset \sigma \cup \mathfrak{l} \subset (\mathbb{P}^2(\mathbb{R}))^*$ and by Proposition 3 we may assume that $\mathfrak{l}$ touches σ at the point $(\partial T)^*$.

Let $p_1, p_2, p_3, p_4 \in \mathcal{A}^* \cap \sigma$ be four consecutive points (with respect to some projectivity Ψ). After a change of coordinates we may assume that

$$(\partial T)^* = (0 : 1 : 0), \, p_2 = (1 : 0 : 1),$$
$$p_3 = (2 : 1 : 1), \, p_4 = (3 : 3 : 1).$$

We then have $p_1 = (x : y : z)$ for some $x, y, z \in \mathbb{R}$. Now consider the vertices $v := p_2 \times p_3$, $v' := p_1 \times p_4 \in \mathbb{P}^2(\mathbb{R})$ and let $\mathfrak{g} \subset \mathbb{P}^2(\mathbb{R})$ be the line passing through v and v'. Then the last claim of Lemma 6 implies that $\mathfrak{g} \in \mathcal{A}$ and that $\mathfrak{g}$ passes through the vertex $\mathfrak{l}^*$. As $\mathfrak{l}^* \in \partial T$, we may write $\mathfrak{l}^* = (a : 0 : b)$ for certain $a, b \in \mathbb{R}$. In order to prove the statement we will distinguish four cases.

Case 1. Assume that $x = y = 0$. This implies that $p_1 = (0 : 0 : 1)$. We claim that $\mathfrak{l}^* = (0 : 0 : 1)$. To see this write $\mathfrak{l}^* = (a : 0 : b)$ for some $a, b \in \mathbb{R}$ as above. The fact that $\mathfrak{g}$ passes through $\mathfrak{l}^*$ implies that $a = 0$ and therefore we have $\mathfrak{l}^* = (0 : 0 : 1)$.

Now consider the projectivity $\Phi : (\mathbb{P}^2(\mathbb{R}))^* \to (\mathbb{P}^2(\mathbb{R}))^*$ taking the point p_i to p_{i+1} for $1 \leq i \leq 4$. We obtain

$$\mathcal{A}^* \cap \sigma = \left\{ \Phi^k(p_1) \mid k \in \mathbb{Z} \right\} = \left\{ \left(k : \tfrac{k(k-1)}{2} : 1 \right) \mid k \in \mathbb{Z} \right\},$$

using Lemma 6 and induction. Observe that the lines of $\mathcal{A}$ corresponding to points in $\mathcal{A}^* \cap \mathfrak{l}$ are exactly the lines passing through $\mathfrak{l}^*$ and a vertex of the form $p \times p'$

for p, $p' \in \mathcal{A}^* \cap \sigma$ (see the proof of Lemma 6). We conclude that $\mathcal{A}^* \cap \mathfrak{l} = \left\{ \left(1 : \frac{k}{2} : 0 \right) \mid k \in \mathbb{Z} \right\}$. It is now easy to check that $\mathcal{A}^* = \left\{ \left(k : \frac{k(k-1)}{2} : 1 \right), \left(1 : \frac{k}{2} : 0 \right) \mid k \in \mathbb{Z} \right\}$ defines an irreducible affine Tits arrangement.

Case 2. Assume that $x \neq 0$ and $y = 0$. Then we may assume that $p_1 = (1 : 0 : z)$. Write $\mathfrak{l}^* = (a : 0 : b)$ for $a, b \in \mathbb{R}$. The fact that $\mathfrak{g}$ passes through $\mathfrak{l}^*$ implies that $a \neq 0$. Thus, we may assume that $\mathfrak{l}^* = (1 : 0 : b)$. It follows that $z = \frac{b+4}{3}$ and therefore $p_1 = \left(1 : 0 : \frac{b+4}{3} \right)$. Observe that the five given points $(\partial T)^*$, p_1, p_2, p_3, p_4 on σ determine its equation. Using this together with Lemma 6, the condition $p_5 \in \sigma$ implies that $b \in \{-1, -\frac{3}{2}, -\frac{7}{3}, -3\}$. As p_0, $p_5 \neq p_i$ for $1 \leq i \leq 4$, we conclude that $b \in \{-1, -\frac{3}{2}, -3\}$ is impossible. In the remaining case $b = -\frac{7}{3}$, we observe that the conic σ may be defined by the polynomial $f = -\frac{10}{3} X^2 + 2XY + \frac{28}{3} XZ - \frac{10}{3} YZ - 6Z^2$. By assumption, we know that the line $\mathfrak{l}$ touches σ at the point $(\partial T)^*$. Thus, as $\mathfrak{l}^* = \left(1 : 0 : -\frac{7}{3} \right)$, there exists $0 \neq \lambda \in \mathbb{R}$ such that the following equations are satisfied:

$$ 1 = \lambda \left. \frac{\partial f}{\partial X} \right|_{(\partial T^*)}, \qquad 0 = \lambda \left. \frac{\partial f}{\partial Y} \right|_{(\partial T^*)}, \qquad -\frac{7}{3} = \lambda \left. \frac{\partial f}{\partial Z} \right|_{(\partial T^*)}. $$

The first equation gives $\lambda = \frac{1}{2}$. But then the third equation reads $-\frac{7}{3} = -\frac{5}{3}$. This contradiction shows that Case 2 cannot occur.

Case 3. Assume that $x = 0$ and $y \neq 0$. Then without loss of generality, we may assume that $p_1 = (0 : 1 : z)$. Again, we write $\mathfrak{l}^* = (a : 0 : b)$ for suitable $a, b \in \mathbb{R}$ and as $\mathfrak{g}$ passes through $\mathfrak{l}^*$, we obtain $a \neq 0$. Thus, we may assume that $\mathfrak{l}^* = (1 : 0 : b)$, leading to $z = -\frac{b+3}{3}$. We conclude that $p_1 = (0 : 1 : -\frac{b+3}{3})$. The relation $p_5 \in \sigma$ gives $b \in \{-3, -1\}$. As $p_5 \neq p_i$ for $1 \leq i \leq 4$, we conclude that this is impossible.

Case 4. Assume that both $x \neq 0$ and $y \neq 0$. Then we may suppose that $p_1 = (1 : y : z)$. Write $\mathfrak{l}^* = (a : 0 : b)$ for suitable $a, b \in \mathbb{R}$. As before, by considering the line $\mathfrak{g}$, we conclude that $-3za - 3ay - by + 4a + b = 0$. Suppose that $a = 0$. Then without loss of generality $b = 1$ and we have $y = 1$, in particular $p_1 = (1 : 1 : z)$. As $p_5 \in \sigma$, we conclude that $z \in \{\frac{1}{3}, \frac{1}{2}\}$. Again, this is not possible because p_0, $p_5 \neq p_i$ for $1 \leq i \leq 4$.

Hence, we may assume that $a = 1$. In particular, we have $z = \frac{4}{3} - \frac{b(y-1)}{3} - y$ and $p_1 = \left(1 : y : \frac{4}{3} - \frac{b(y-1)}{3} - y \right)$.

Suppose that $b \neq -3$. Using the condition $p_5 \in \sigma$, we compute that y is one of 1, $\frac{-3b^2 - 10b - 7}{2(b+3)}$, $\frac{2b^2 + 5b + 3}{2(b^2 + 3b + 3)}$. As $p_1 \neq p_4$, we can exclude the case $y = 1$.

Likewise, if y is the last of the three numbers, we obtain $p_1 = p_5$, a contradiction. So we must have $y = \frac{-3b^2 - 10b - 7}{2(b+3)}$. This implies that $p_1 = \left(1 : \frac{-3b^2 - 10b - 7}{2(b+3)} : \frac{b^2 + 4b + 5}{2} \right)$. Therefore, the conic σ may be defined by the polynomial

$$ f := (b-1)X^2 + 2XY - (b-7)XZ - 2(b+4)YZ - 6Z^2. $$

To see this, one only has to check that $f(p_i) = 0$ for $1 \leq i \leq 5$. The line $\mathfrak{l}$ touches σ at the point $(\partial T)^* = (0 : 1 : 0)$. Therefore, as $\mathfrak{l}^* = (1 : 0 : b)$, we know that there exists $0 \neq \lambda \in \mathbb{R}$ such that the following equations hold:

$$ 1 = \lambda \left. \frac{\partial f}{\partial X} \right|_{(\partial T^*)}, \quad 0 = \lambda \left. \frac{\partial f}{\partial Y} \right|_{(\partial T^*)}, \quad b = \lambda \left. \frac{\partial f}{\partial Z} \right|_{(\partial T^*)}. $$

The first equation gives $\lambda = \frac{1}{2}$. Thus, the third equation yields $b = -2$ and we obtain $p_1 = \left(1 : \frac{1}{2} : \frac{1}{2}\right) = (2 : 1 : 1) = p_3$, a contradiction.

It remains to consider the case $b = -3$. Then we have $\mathfrak{l}^* = (1 : 0 - 3)$ and $p_1 = \left(1 : y : \frac{1}{3}\right)$. Clearly, we have $y \neq 1$ because $p_1 \neq p_4$. Then Lemma 6 yields $p_5 = (3 : 3 : 1) = p_4$, another contradiction. $\qquad\square$

Corollary 1. *There are irreducible affine rank three Tits arrangements which are not locally spherical.*

Proof. This follows from Proposition 4. The arrangement constructed there is such an example: the vertex $\mathfrak{l}^*$ is incident with infinitely many lines of $\mathcal{A}$. $\qquad\square$

Finally, using Proposition 2, Lemma 5, Proposition 3, and Proposition 4, we obtain the promised main theorem:

Theorem 2. *Let $\mathcal{A}$ be an affine rank three Tits arrangement such that $\mathcal{A}^*$ is contained in the locus of a homogeneous polynomial of degree three. Then up to projectivity, $\mathcal{A}$ is either a near pencil, an arrangement of type $\tilde{A}_2$, or it is an arrangement of type $\tilde{A}_2^0$.*

4. Open questions and related problems

In this section we point out some possibly interesting related problems. First, we ask if there exists an affine rank three Tits arrangement $\mathcal{A}$ (viewed as arrangement of lines in the real projective plane) such that $\mathcal{A}^*$ is contained entirely in the locus of an irreducible homogeneous polynomial.

Problem 1. *Is there an irreducible homogeneous polynomial $P \in \mathbb{R}[x, y, z]$ such that $\mathcal{A}^* \subset V(P)$ for a suitable irreducible affine rank three Tits arrangement $\mathcal{A}$?*

Observe that given a Tits arrangement $\mathcal{A}$ and an irreducible homogeneous polynomial P of degree d such that $\mathcal{A}^* \subset V(P)$, it follows immediately that $\mathcal{A}$ is locally spherical. Indeed, suppose there was a vertex v of $\mathcal{A}$ such that infinitely many lines of $\mathcal{A}$ pass through v. Then after dualizing it follows that infinitely many points of $\mathcal{A}^*$ lie on the line v^*. But since by assumption $\mathcal{A}^* \subset V(P)$, it follows that infinitely many points lie on the intersection $V(P) \cap v^*$. But Bézout's theorem tells that $|V(P) \cap v^*| \leq d \cdot 1 = d < \infty$, because P was assumed to be irreducible and hence v^* cannot be a component of $V(P)$. This contradiction shows that $\mathcal{A}$ must be locally spherical.

This leads to the next problem. Are there other examples of irreducible affine rank three Tits arrangements which are not locally spherical?

Problem 2. *Classify* (*up to projectivities*) *all irreducible affine rank three Tits arrangements* $\mathcal{A}$ *which are not locally spherical.*

Observe that if $\mathcal{A}$ is not locally spherical, then by Lemma 2 there is precisely one vertex v on the boundary of the Tits cone T. In particular, it follows that for every line $\mathfrak{l} \neq v^*$ we have $|\mathcal{A}^* \cap \mathfrak{l}| < \infty$. If in addition we know that $\mathcal{A}^* \subset V(P)$ for some homogeneous polynomial P of degree d, then by Bézout's theorem the last inequality can be strengthened to

$$|\mathcal{A}^* \cap \mathfrak{l}| \leq |V(P) \cap \mathfrak{l}| \leq d$$

for every line $\mathfrak{l} \neq v^*$ which is not a component of $V(P)$.

We close this section by proposing the following final problem which is probably the most difficult:

Problem 3. *Classify* (*up to projectivities*) *all affine rank three Tits arrangements* $\mathcal{A}$ *such that* $\mathcal{A}^* \subset V(P)$ *for some homogeneous polynomial* $P \in \mathbb{R}[x, y, z]$.

A solution to the last problem seems to be an important step towards a classification of all affine rank three Tits arrangements. Indeed, if $\mathcal{A}$ is such an arrangement and if $\mathcal{A} = \bigcup_{i \in I} L_i$ for some finite index set I and sets of mutually parallel lines $L_i, i \in I$, then $\mathcal{A}^*$ is contained in the locus of a polynomial P of degree $|I|$: the polynomial P is a product of linear factors corresponding to the sets $L_i, i \in I$. For example, affine Tits arrangements coming from Nichols algebras of diagonal type are always of this type.

Even if we enlarge $\mathcal{A}$ by finitely many countable subsets of tangent lines to certain conics, we still find a polynomial P' such that the enlarged arrangement is contained in the locus of P'. The polynomial P' may be taken as the product of P together with the irreducible quadratic polynomials defining the (dual) conics in question. This gives the impression that the class of rank three affine Tits arrangements lying on the locus of some polynomial is rather large, as demonstrated by the fact that only usage of at most quadratic polynomials already leads to nontrivial considerations. It may even be conjectured that for every irreducible rank three affine Tits arrangement $\mathcal{B}$ there is a certain polynomial Q such that $\mathcal{B}^* \subset V(Q)$. If this is true, then clearly a solution to Problem 3 amounts to a complete classification of affine rank three Tits arrangements.

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