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We improve and extend the irrationality results proved by the authors (*Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **4**:3 (2005), 389–437) for dilogarithms of positive rational numbers to results of linear independence over $\mathbb{Q}$ of 1, $\mathrm{Li}_1(x)$ and $\mathrm{Li}_2(x)$ for suitable $x \in \mathbb{Q}$, both for $x > 0$ and for $x < 0$.

1. Introduction

The polylogarithm of order k, arising in Euler's work, is defined by

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad (|x| < 1).$$

Qualitative and quantitative irrationality results for dilogarithms $\mathrm{Li}_2\left(\frac{1}{z}\right)$ with $z \in \mathbb{Z}$, $z \in (-\infty, -5] \cup [7, +\infty)$, were proved in [Hata 1993]. Hata's results were subsequently improved and extended in [Rhin and Viola 2005], and more recently in [Viola and Zudilin 2018] and in [Marcovecchio 2016]. We showed in [Rhin and Viola 2005], henceforth abbreviated [RV05], that $\mathrm{Li}_2\left(\frac{r}{s}\right) \notin \mathbb{Q}$ for all integers r, s with $r \geq 1$ and $s \geq s_1(r) > r$, where $s_1(r)$ can be explicitly computed, and gave new irrationality measures of such $\mathrm{Li}_2\left(\frac{r}{s}\right)$. In particular we proved that $\mathrm{Li}_2\left(\frac{1}{6}\right) \notin \mathbb{Q}$, with an explicit irrationality measure, thus extending Hata's range $[7, +\infty)$ mentioned above to $[6, +\infty)$.

Viola and Zudilin [2018] proved qualitative and quantitative results of linear independence over $\mathbb{Q}$ of the four numbers

$$1, \quad \mathrm{Li}_1\left(\frac{1}{z}\right) = -\log\left(1 - \frac{1}{z}\right) = -\mathrm{Li}_1\left(\frac{1}{1-z}\right), \quad \mathrm{Li}_2\left(\frac{1}{z}\right) \quad \text{and} \quad \mathrm{Li}_2\left(\frac{1}{1-z}\right)$$

for all $z = \frac{s}{r}$ with integers r and s satisfying

$$r \geq 1 \quad \text{and} \quad s \geq s_2(r) > r, \tag{1-1}$$

where again $s_2(r)$ is explicit. Specifically, they proved that, for any $z = \frac{s}{r}$ satisfying (1-1) and for any $\varepsilon > 0$, there exist an effective constant $C(\varepsilon, z) > 0$, and an explicit linear independence measure $\mu(z) > 0$ over $\mathbb{Q}$ such that

$$\left| a_0 + a_1 \mathrm{Li}_1\left(\frac{1}{z}\right) + a_2 \mathrm{Li}_2\left(\frac{1}{z}\right) + a_3 \mathrm{Li}_2\left(\frac{1}{1-z}\right) \right| > C(\varepsilon, z)\, A^{-\mu(z)-\varepsilon} \tag{1-2}$$

for all $(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4 \setminus \{(0, 0, 0, 0)\}$, where $A = \max\{|a_0|, |a_1|, |a_2|, |a_3|\}$. In particular, in [Viola and Zudilin 2018] the authors proved inequalities (1-2) with $z = \frac{s}{r}$ for $r = 1$ and $s \geq 9$, for $r = 2$ and $s \geq 143$, for $r = 3$ and $s \geq 742$, for $r = 4$ and $s \geq 2355$.

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In the present paper we prove lower bounds of type (1-2) for

$$\left|a_0 + a_1 \mathrm{Li}_1\Bigl(\frac{1}{z}\Bigr) + a_2 \mathrm{Li}_2\Bigl(\frac{1}{z}\Bigr)\right|, \tag{1-3}$$

without considering $\mathrm{Li}_2(1/(1-z))$, for all $(a_0, a_1, a_2) \in \mathbb{Z}^3 \setminus \{(0,0,0)\}$ and for $z = \frac{s}{r}$ with integers r, s satisfying $r \geq 1$ and either $s \geq s_+(r) > r$, or $s \leq s_-(r) < 0$, with explicit $s_+(r)$ and $s_-(r)$. We combine some technical results employed in [RV05] and [Viola and Zudilin 2018]. In particular we use the permutation-group method, introduced by the authors in [Rhin and Viola 1996] and later used in a series of papers including [RV05] and [Viola and Zudilin 2018], and the saddle-point method in $\mathbb{C}^2$ in the form used in [Viola and Zudilin 2018]. The latter tool allows us to improve and extend the irrationality measures of $\mathrm{Li}_2\bigl(\frac{1}{z}\bigr)$ for $z = \frac{s}{r} > 1$ obtained in [RV05] to linear independence measures over $\mathbb{Q}$ of 1, $\mathrm{Li}_1\bigl(\frac{1}{z}\bigr)$ and $\mathrm{Li}_2\bigl(\frac{1}{z}\bigr)$, both for $z = \frac{s}{r} > 1$ and for $z = \frac{s}{r} < 0$. Moreover, our method yields improvements upon the linear independence measures of 1, $\mathrm{Li}_1\bigl(\frac{1}{z}\bigr)$ and $\mathrm{Li}_2\bigl(\frac{1}{z}\bigr)$ for $z < 0$ given in [Marcovecchio 2016, Table p. 231], and a new proof of the linear independence measures for $z > 1$ given therein.

We point out that, in contrast to (1-2), the lower bounds for (1-3) obtained in the present paper do not simultaneously involve values of Li_2 at positive and at negative rational numbers. Treating separately the cases $z > 1$ and $z < 0$ in (1-3), as we do in this paper, yields linear independence results for 1, $\mathrm{Li}_1\bigl(\frac{1}{z}\bigr)$ and $\mathrm{Li}_2\bigl(\frac{1}{z}\bigr)$ stronger than those obtained through the construction in [Viola and Zudilin 2018] under the constraint $a_2 = 0$ or $a_3 = 0$ in (1-2). This is not surprising, because the algebraic structure of the group $\langle \boldsymbol{\varphi}, \boldsymbol{\lambda} \rangle$ in (2-21) below generated by the permutations $\boldsymbol{\lambda}$ and $\boldsymbol{\varphi}$ in (2-18)–(2-19), used in the present paper as well as in [RV05] to get the arithmetical correction $\int_\Omega \mathrm{d}\psi(x) + \int_{\Omega'} \mathrm{d}\psi(x)$ in (4-2), is richer than that of the corresponding group $\langle \boldsymbol{\varphi}, \boldsymbol{\nu} \rangle$ required in [Viola and Zudilin 2018] to deal with the full linear form on the left-hand side of (1-2). This is a consequence of the relation $\boldsymbol{\nu}\boldsymbol{\varphi} = \boldsymbol{\varphi}\boldsymbol{\nu}$ satisfied by the permutations $\boldsymbol{\nu}$ and $\boldsymbol{\varphi}$ in [Viola and Zudilin 2018, (3.5)], which implies that $\langle \boldsymbol{\varphi}, \boldsymbol{\nu} \rangle$ is isomorphic to Klein's Vierergruppe of order 4, while the group $\langle \boldsymbol{\varphi}, \boldsymbol{\lambda} \rangle$, with $\boldsymbol{\lambda}\boldsymbol{\varphi} \neq \boldsymbol{\varphi}\boldsymbol{\lambda}$, is larger, being isomorphic to the group $\mathfrak{S}_2 \times \mathfrak{S}_3$ of order $2! \cdot 3! = 12$.

2. Simultaneous approximations to Li_1 and Li_2

Let $z \in \mathbb{R} \setminus [0, 1]$, and let h, j, k, l, m be nonnegative integers such that $l+m-j$, $m+h-k$, $h+j-l$ and $j+k-m$ are also nonnegative. If $z > 1$, as in [RV05, Section 2] we define the following double integrals $I_z^{(\nu)}(h, j, k, l, m)$ $(\nu = 0, 1, 2)$:

$$I_z^{(0)}(h,j,k,l,m) = z^{-l-m} \int_0^1 \int_0^1 \frac{x^j (1-x)^h y^k (1-y)^l}{(x(1-y)+yz)^{j+k-m+1}} \, \mathrm{d}x \, \mathrm{d}y, \tag{2-1}$$

$$I_z^{(1)}(h,j,k,l,m) = z^{-l-m} \int_0^1 \left(\frac{1}{2\pi i} \oint_{|y-x/(x-z)|=\varrho} \frac{x^j (1-x)^h y^k (1-y)^l}{(x(1-y)+yz)^{j+k-m+1}} \, \mathrm{d}y \right) \mathrm{d}x, \tag{2-2}$$

$$I_z^{(2)}(h,j,k,l,m) = \frac{z^{-l-m}}{2\pi i} \oint_{|x-z|=\sigma} \left(\frac{1}{2\pi i} \oint_{|y-x/(x-z)|=\varrho} \frac{x^j (1-x)^h y^k (1-y)^l}{(x(1-y)+yz)^{j+k-m+1}} \, \mathrm{d}y \right) \mathrm{d}x \tag{2-3}$$

for any $\varrho, \sigma > 0$. We also define the linear combination of (2-1) and (2-2) given by

$$I_z(h,j,k,l,m) = I_z^{(0)}(h,j,k,l,m) - (\log z)\, I_z^{(1)}(h,j,k,l,m), \tag{2-4}$$

where $\log z$ is the real value of the logarithm for $z > 1$.

Clearly the definitions (2-2) and (2-3) make sense also for $z < 0$, whereas the definition (2-1) of $I_z^{(0)}(h, j, k, l, m)$ does not apply if $z < 0$, since in this case the denominator $x(1-y) + yz$ vanishes for (x, y) along a segment of hyperbola inside the unit square $(0, 1) \times (0, 1) \subset \mathbb{R}^2$. In order to define $I_z^{(0)}(h, j, k, l, m)$ for $z < 0$ we use a method introduced in [Viola and Zudilin 2018, Section 2.2]. We apply to $I_z^{(0)}(h, j, k, l, m)$ the change of variables

$$x = \xi, \qquad y = \frac{\eta}{\eta - z}. \tag{2-5}$$

For $z > 1$, (2-5) changes the integration path $[0, 1)$ for y to $[0, -\infty)$ for η. Thus

$$I_z^{(0)}(h, j, k, l, m) = (-1)^{j+k+l+m}\, z^{-j-k} \int_0^1 \xi^j (1-\xi)^h \,\mathrm{d}\xi \int_0^{-\infty} \frac{\eta^k}{(\xi-\eta)^{j+k-m+1}(\eta - z)^{l+m-j+1}}\,\mathrm{d}\eta.$$

Let ζ be any complex number such that $|\zeta| = 1$, $\zeta \neq 1$, and let $\arg\zeta$ denote the argument satisfying $0 < \arg\zeta < 2\pi$. For $\varrho > 0$ let μ_ϱ be the arc $\{|\eta| = \varrho : \arg\eta \text{ from } \arg\zeta \text{ to } \pi\}$. For any ξ, z with $0 < \xi \le 1 < z$, as $\varrho \to +\infty$ we get

$$\left|\int_{\mu_\varrho} \frac{\eta^k}{(\xi-\eta)^{j+k-m+1}(\eta - z)^{l+m-j+1}}\,\mathrm{d}\eta\right| \le \frac{\varrho^k}{(\varrho-1)^{j+k-m+1}(\varrho - z)^{l+m-j+1}} \cdot 2\pi\varrho \ll \varrho^{-l-1} \to 0. \tag{2-6}$$

Therefore, by Cauchy's theorem, for any $z > 1$ and for any $\zeta \in \mathbb{C}$ such that $|\zeta| = 1$, $\zeta \neq 1$, we obtain

$$I_z^{(0)}(h, j, k, l, m) = (-1)^{j+k+l+m}\, z^{-j-k} \int_0^1 \xi^j (1-\xi)^h \,\mathrm{d}\xi \int_0^{\zeta\infty} \frac{\eta^k}{(\xi-\eta)^{j+k-m+1}(\eta - z)^{l+m-j+1}}\,\mathrm{d}\eta, \tag{2-7}$$

where the integration path for η is the half-line $[0, \zeta\infty)$ from 0 to ∞ through ζ. Note that, since $z > 1$ and $\zeta \neq 1$, we have $(\xi - \eta)(\eta - z) \neq 0$ in (2-7).

As in [Viola and Zudilin 2018, Section 2.2], it is easy to prove that the double integral on the right-hand side of (2-7) converges absolutely and uniformly when z varies in any compact region of $\mathbb{C}$ not intersecting the half-line $[0, \zeta\infty)$, and in particular the integrations in ξ and η can be interchanged. Thus

$$I_z^{(0)}(h, j, k, l, m) = (-1)^{j+k+l+m}\, z^{-j-k} \int_0^{\zeta\infty} \left(\int_0^1 \frac{\xi^j (1-\xi)^h \eta^k}{(\xi-\eta)^{j+k-m+1}(\eta - z)^{l+m-j+1}}\,\mathrm{d}\xi\right)\mathrm{d}\eta, \tag{2-8}$$

and $I_z^{(0)}(h, j, k, l, m)$, viewed as a function of the complex variable z, is holomorphic in the cut plane

$$\mathbb{C} \setminus [0, \zeta\infty).$$

Therefore, if in (2-7) or (2-8) we choose $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, $\zeta \neq \pm 1$, by analytic continuation we can move z from the half-line $z > 1$ to the half-line $z < 0$ along a path contained in the lower half-plane $\operatorname{Im} z < 0$ if $\operatorname{Im}\zeta > 0$, or in the upper half-plane $\operatorname{Im} z > 0$ if $\operatorname{Im}\zeta < 0$, because such a path does not cross the cut $[0, \zeta\infty)$.

Since, from (2-7),

$$\overline{I_z^{(0)}(h, j, k, l, m)} = (-1)^{j+k+l+m}\, \bar{z}^{-j-k} \int_0^1 \xi^j (1-\xi)^h \,\mathrm{d}\xi \int_0^{\bar\zeta\infty} \frac{\eta^k}{(\xi-\eta)^{j+k-m+1}(\eta - \bar{z})^{l+m-j+1}}\,\mathrm{d}\eta,$$

for any $z < 0$ we get two values of $I_z^{(0)}(h, j, k, l, m)$, conjugate to each other, one defined by (2-7) or (2-8) for $\operatorname{Im}\zeta > 0$, and the other for $\operatorname{Im}\zeta < 0$. Accordingly, for $z < 0$ we define $I_z(h, j, k, l, m)$ by (2-4), where

$$\log z = \begin{cases} \log|z| - \pi i & \text{if } \operatorname{Im}\zeta > 0 \text{ in } I_z^{(0)}(h,j,k,l,m), \\ \log|z| + \pi i & \text{if } \operatorname{Im}\zeta < 0 \text{ in } I_z^{(0)}(h,j,k,l,m). \end{cases} \tag{2-9}$$

By applying the change of variables (2-5) to the double integrals (2-2) and (2-3) we easily get, similarly to [Viola and Zudilin 2018, (2.11) and (2.12)],

$$I_z^{(1)}(h,j,k,l,m) = (-1)^{j+k+l+m} \frac{z^{-j-k}}{2\pi i} \oint_{\Gamma_{0,1}} \left(\int_0^1 \frac{\xi^j (1-\xi)^h \eta^k}{(\xi-\eta)^{j+k-m+1}(\eta-z)^{l+m-j+1}} \, d\xi \right) d\eta, \tag{2-10}$$

where $\Gamma_{0,1}$ denotes any closed contour for η enclosing the interval $(0, 1)$ but not enclosing z, and, for any $\varrho_1, \varrho_2 > 0$,

$$\begin{aligned} &I_z^{(2)}(h,j,k,l,m) \\ &\quad = (-1)^{j+k+l+m+1} \frac{z^{-j-k}}{2\pi i} \oint_{|\eta-z|=\varrho_1} \left(\frac{1}{2\pi i} \oint_{|\xi-\eta|=\varrho_2} \frac{\xi^j (1-\xi)^h \eta^k}{(\xi-\eta)^{j+k-m+1}(\eta-z)^{l+m-j+1}} \, d\xi \right) d\eta. \end{aligned} \tag{2-11}$$

We showed in [RV05, (2.5) and (2.6)] that the involution $\lambda = \lambda_{x,z} : y \leftrightarrow \tilde{y}$ defined by

$$\tilde{y} = \frac{x(1-y)}{x(1-y) + yz}, \tag{2-12}$$

used as a change of variable for y in the double integrals (2-1), (2-2) and (2-3), transforms the quantities (2-1), (2-2), (2-3), and hence (2-4), according to the permutation $\boldsymbol{\lambda}$ of the exponents defined by

$$\boldsymbol{\lambda} = (j\ \ m)(k\ \ l), \tag{2-13}$$

which acts identically on h. Thus (2-2) and (2-3), and hence (2-10) and (2-11), are invariant under the action of $\boldsymbol{\lambda}$ both for $z > 1$ and for $z < 0$, and the same holds for (2-1) if $z > 1$.

It is easy to see that $I_z^{(0)}(h,j,k,l,m)$ is invariant under the action of the permutation $\boldsymbol{\lambda}$ also in the case $z < 0$, where (2-1) is replaced by the definition (2-7) or (2-8) for $|\zeta| = 1$, $\zeta \neq \pm 1$. Indeed, the change of variables (2-5) transforms the involution (2-12) into the involution $\eta \leftrightarrow \tilde{\eta}$ given by

$$\tilde{\eta} = z\frac{\xi}{\eta}. \tag{2-14}$$

If in (2-7) we use the change of variable (2-14) for η, we get

$$\begin{aligned} &I_z^{(0)}(h,j,k,l,m) \\ &\quad = (-1)^{j+k+l+m} z^{-l-m} \int_0^1 \xi^m (1-\xi)^h \, d\xi \int_0^{-\bar{\zeta}\infty} \frac{\tilde{\eta}^l}{(\xi-\tilde{\eta})^{l+m-j+1}(\tilde{\eta}-z)^{j+k-m+1}} \, d\tilde{\eta}. \end{aligned} \tag{2-15}$$

Since $\operatorname{Im}(-\bar{\zeta}) = \operatorname{Im}\zeta$, by the same argument as in (2-6) we see that the right-hand side of (2-15) equals

$$(-1)^{j+k+l+m} z^{-l-m} \int_0^1 \xi^m (1-\xi)^h \, d\xi \int_0^{\zeta\infty} \frac{\tilde{\eta}^l}{(\xi-\tilde{\eta})^{l+m-j+1}(\tilde{\eta}-z)^{j+k-m+1}} \, d\tilde{\eta}.$$

By (2-7), this is $I_z^{(0)}(h,m,l,k,j)$. Hence for $z < 0$ we get

$$I_z^{(0)}(h,j,k,l,m) = I_z^{(0)}(h,m,l,k,j)$$

both for $\operatorname{Im}\zeta > 0$ and for $\operatorname{Im}\zeta < 0$, as claimed.

Throughout this paper, for any integer $n \geq 1$ we denote by d_n the least common multiple of $1, \ldots, n$, and we set $d_0 = 1$. Also, as in [RV05, (2.9)], we define

$$\begin{aligned} H &= \max\{l+m-j,\ m+h-k,\ h+j-l,\ j+k-m\}, \\ K &= \max\{l+m-j,\ \min\{m+h-k,\ h+j-l\},\ j+k-m\}, \\ \alpha &= \max\{j+k,\ k+l,\ l+m\}, \\ \beta &= \max\{0,\ k+l-h\}, \\ \delta &= \alpha+\beta+h-k-l \end{aligned} \tag{2-16}$$

(note that the integer

$$\delta = \max\{h,\ m+h-k,\ h+j-l,\ j+k,\ k+l,\ l+m\}$$

defined in [RV05, (2.9)] equals $\alpha+\beta+h-k-l$ by virtue of Lemma 2.8 of the same paper). Clearly the integers (2-16) are nonnegative and invariant under the action of the permutation $\boldsymbol{\lambda}$ in (2-13).

We prove the following:

Theorem 2.1. *For any $z \in \mathbb{R} \setminus [0, 1]$, define (2-2), (2-3), (2-7) with $|\zeta| = 1$, $\zeta \neq \pm 1$, (2-9), (2-4) and (2-16). Then*

$$\begin{aligned} d_H d_K z^{\alpha}(z-1)^{\beta} I_z(h, j, k, l, m) &= P(z) - Q(z)\,\mathrm{Li}_2\Big(\frac{1}{z}\Big), \\ d_H d_K z^{\alpha}(z-1)^{\beta} I_z^{(1)}(h, j, k, l, m) &= R(z) - Q(z)\,\mathrm{Li}_1\Big(\frac{1}{z}\Big), \\ d_H d_K z^{\alpha}(z-1)^{\beta} I_z^{(2)}(h, j, k, l, m) &= Q(z), \end{aligned}$$

where

$$P(z), Q(z), R(z) \in \mathbb{Z}[z], \qquad \max\{\deg P(z), \deg Q(z), \deg R(z)\} \leq \delta.$$

Proof. If $z > 1$, the theorem is [RV05, Theorem 2.1]. If $z < 0$, the theorem follows from the case $z > 1$ by analytic continuation, moving z from the half-line $z > 1$ to the half-line $z < 0$ along a path contained in the lower half-plane $\operatorname{Im} z < 0$ if $\operatorname{Im} \zeta > 0$, or in the upper half-plane $\operatorname{Im} z > 0$ if $\operatorname{Im} \zeta < 0$. □

Let

$$\mathcal{S} = \{h,\ j,\ k,\ l,\ m,\ l+m-j,\ m+h-k,\ h+j-l,\ j+k-m\}. \tag{2-17}$$

The action on the set $\mathcal{S}$ of the permutation $\boldsymbol{\lambda}$ defined in (2-13) is

$$\boldsymbol{\lambda} = (j\ \ m)(k\ \ l)(l+m-j\ \ \ j+k-m)(m+h-k\ \ \ h+j-l). \tag{2-18}$$

As in [RV05], we also consider the permutation $\boldsymbol{\varphi}$ whose action on $\mathcal{S}$ is

$$\boldsymbol{\varphi} = (h\ \ m+h-k)(j\ \ j+k-m)(k\ \ m). \tag{2-19}$$

We proved in [RV05, Section 3] that for any $z > 1$ the quotients

$$\frac{I_z^{(\nu)}(h, j, k, l, m)}{h!\,j!\,k!\,l!\,m!} \quad (\nu = 0, 1, 2) \qquad \text{and} \qquad \frac{I_z(h, j, k, l, m)}{h!\,j!\,k!\,l!\,m!} \tag{2-20}$$

are invariant under the action of φ. Hence, by analytic continuation, the same holds in the case $z<0$, where $I_z^{(0)}(h,j,k,l,m)$ and $\log z$ are given by (2-7) and (2-9) with $\zeta\in\mathbb{C}$ such that $|\zeta|=1$, $\zeta\neq\pm1$. Thus the quotients (2-20) are invariant under the action of the whole permutation group

$$\boldsymbol{\Phi}=\langle\boldsymbol{\varphi},\boldsymbol{\lambda}\rangle \tag{2-21}$$

generated by $\boldsymbol{\lambda}$ and $\boldsymbol{\varphi}$, both for $z>1$ and for $z<0$. As we proved in [RV05, Section 3], $\boldsymbol{\Phi}$ is isomorphic to the product $\mathfrak{S}_2\times\mathfrak{S}_3$ of the symmetric groups of orders 2! and 3!, whence

$$|\boldsymbol{\Phi}|=2!\cdot3!=12.$$

Therefore, the transformation formulae in [RV05, p. 416] hold for $I_z^{(\nu)}(h,j,k,l,m)$ $(\nu=0,1,2)$ and for $I_z(h,j,k,l,m)$, in both cases $z>1$ and $z<0$.

The integers α, β and δ in (2-16) are also invariant under the action of the permutation $\boldsymbol{\varphi}$ in (2-19), and hence are invariant under the action of the whole permutation group (2-21), while H and K in (2-16) are not invariant under the action of $\boldsymbol{\varphi}$. Therefore, in place of H and K, we require the integers M and N defined in [RV05, (4.1) and (4.2)], namely

$$M=\max\mathcal{S}$$

and

$$N=\max\{\max{}'(h,\ m+h-k,\ h+j-l),\ j,\ k,\ l,\ m,\ l+m-j,\ j+k-m\},$$

where $\max'$ denotes the second maximum in a finite sequence of real numbers. Clearly M and N are invariant under the action of the permutation group $\boldsymbol{\Phi}$ in (2-21). Also $H\le M$ and $K\le N$, whence Theorem 2.1 holds with M in place of H and N in place of K. Moreover, as in [RV05, p. 417], for any permutation $\boldsymbol{\chi}\in\boldsymbol{\Phi}$, Theorem 2.1 holds with h,j,k,l,m respectively replaced by $\boldsymbol{\chi}(h)$, $\boldsymbol{\chi}(j)$, $\boldsymbol{\chi}(k)$, $\boldsymbol{\chi}(l)$, $\boldsymbol{\chi}(m)$, with polynomials $P_{\boldsymbol{\chi}}(z)$, $Q_{\boldsymbol{\chi}}(z)$, $R_{\boldsymbol{\chi}}(z)\in\mathbb{Z}[z]$ depending on the left coset $\boldsymbol{\chi}\boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}=\langle\boldsymbol{\lambda}\rangle$ is the subgroup of $\boldsymbol{\Phi}$ of order 2 generated by (2-18), and with M, N, α, β, δ all independent of $\boldsymbol{\chi}$.

For h,j,k,l,m fixed and $n=1,2,3,\dots$, we replace the tuple (h,j,k,l,m) by (hn,jn,kn,ln,mn). Then Theorem 2.1 yields

$$\begin{aligned}
d_{Mn}d_{Nn}z^{\alpha n}(z-1)^{\beta n}I_z(hn,jn,kn,ln,mn)&=P_n(z)-Q_n(z)\mathrm{Li}_2\Big(\frac{1}{z}\Big),\\
d_{Mn}d_{Nn}z^{\alpha n}(z-1)^{\beta n}I_z^{(1)}(hn,jn,kn,ln,mn)&=R_n(z)-Q_n(z)\mathrm{Li}_1\Big(\frac{1}{z}\Big),\\
d_{Mn}d_{Nn}z^{\alpha n}(z-1)^{\beta n}I_z^{(2)}(hn,jn,kn,ln,mn)&=Q_n(z),
\end{aligned}\tag{2-22}$$

with polynomials $P_n(z)$, $Q_n(z)$, $R_n(z)\in\mathbb{Z}[z]$ of degrees not exceeding δn. Similarly, for any permutation $\boldsymbol{\chi}\in\boldsymbol{\Phi}$,

$$\begin{aligned}
d_{Mn}d_{Nn}z^{\alpha n}(z-1)^{\beta n}I_z(\boldsymbol{\chi}(h)n,\boldsymbol{\chi}(j)n,\boldsymbol{\chi}(k)n,\boldsymbol{\chi}(l)n,\boldsymbol{\chi}(m)n)&=P_{\boldsymbol{\chi},n}(z)-Q_{\boldsymbol{\chi},n}(z)\mathrm{Li}_2\Big(\frac{1}{z}\Big),\\
d_{Mn}d_{Nn}z^{\alpha n}(z-1)^{\beta n}I_z^{(1)}(\boldsymbol{\chi}(h)n,\boldsymbol{\chi}(j)n,\boldsymbol{\chi}(k)n,\boldsymbol{\chi}(l)n,\boldsymbol{\chi}(m)n)&=R_{\boldsymbol{\chi},n}(z)-Q_{\boldsymbol{\chi},n}(z)\mathrm{Li}_1\Big(\frac{1}{z}\Big),\\
d_{Mn}d_{Nn}z^{\alpha n}(z-1)^{\beta n}I_z^{(2)}(\boldsymbol{\chi}(h)n,\boldsymbol{\chi}(j)n,\boldsymbol{\chi}(k)n,\boldsymbol{\chi}(l)n,\boldsymbol{\chi}(m)n)&=Q_{\boldsymbol{\chi},n}(z),
\end{aligned}\tag{2-23}$$

with polynomials $P_{\boldsymbol{\chi},n}(z)$, $Q_{\boldsymbol{\chi},n}(z)$, $R_{\boldsymbol{\chi},n}(z)\in\mathbb{Z}[z]$ of degrees not exceeding δn.

By the invariance of the quotients (2-20) under the action of the group $\boldsymbol{\Phi}$, from (2-22) and (2-23) we obtain the identity

$$(\chi(h)n)!\,(\chi(j)n)!\,(\chi(k)n)!\,(\chi(l)n)!\,(\chi(m)n)!\,Q_n(z)=(hn)!\,(jn)!\,(kn)!\,(ln)!\,(mn)!\,Q_{\chi,n}(z), \tag{2-24}$$

whence

$$(\chi(h)n)!\,(\chi(j)n)!\,(\chi(k)n)!\,(\chi(l)n)!\,(\chi(m)n)!\,P_n(z)=(hn)!\,(jn)!\,(kn)!\,(ln)!\,(mn)!\,P_{\chi,n}(z), \tag{2-25}$$
$$(\chi(h)n)!\,(\chi(j)n)!\,(\chi(k)n)!\,(\chi(l)n)!\,(\chi(m)n)!\,R_n(z)=(hn)!\,(jn)!\,(kn)!\,(ln)!\,(mn)!\,R_{\chi,n}(z). \tag{2-26}$$

Let Ω and Ω' be the sets of real numbers $\omega\in[0,1)$ defined in [RV05, p. 420], and let

$$\Delta_n=\prod_{\substack{p>\sqrt{Mn}\\ \{n/p\}\in\Omega}} p,\qquad \Delta_n'=\prod_{\substack{p>\sqrt{Mn}\\ \{n/p\}\in\Omega'}} p\qquad (n=1,2,3,\dots),$$

where p denotes a prime number. By applying to the coefficients of the polynomials in (2-24), (2-25) and (2-26) the discussion in [RV05, p. 418–420], we see that $\Delta_n\Delta_n'$ divides all the coefficients of $P_n(z)$, $Q_n(z)$ and $R_n(z)$. Therefore

$$\begin{aligned}P_n^*(z)&:=(\Delta_n\Delta_n')^{-1}P_n(z)\in\mathbb{Z}[z],\\ Q_n^*(z)&:=(\Delta_n\Delta_n')^{-1}Q_n(z)\in\mathbb{Z}[z],\\ R_n^*(z)&:=(\Delta_n\Delta_n')^{-1}R_n(z)\in\mathbb{Z}[z].\end{aligned}$$

Let

$$D_n=\frac{d_{Mn}d_{Nn}}{\Delta_n\Delta_n'}.$$

Dividing the identities (2-22) by $\Delta_n\Delta_n'$ we get

$$\begin{aligned}D_nz^{\alpha n}(z-1)^{\beta n}I_z(hn,jn,kn,ln,mn)&=P_n^*(z)-Q_n^*(z)\mathrm{Li}_2\Big(\frac{1}{z}\Big),\\ D_nz^{\alpha n}(z-1)^{\beta n}I_z^{(1)}(hn,jn,kn,ln,mn)&=R_n^*(z)-Q_n^*(z)\mathrm{Li}_1\Big(\frac{1}{z}\Big),\\ D_nz^{\alpha n}(z-1)^{\beta n}I_z^{(2)}(hn,jn,kn,ln,mn)&=Q_n^*(z),\end{aligned} \tag{2-27}$$

with

$$P_n^*(z),\,Q_n^*(z),\,R_n^*(z)\in\mathbb{Z}[z],\qquad \max\{\deg P_n^*(z),\deg Q_n^*(z),\deg R_n^*(z)\}\le\delta n. \tag{2-28}$$

By [RV05, (4.13)],

$$\lim_{n\to\infty}\frac{1}{n}\log D_n=M+N-\bigg(\int_\Omega d\psi(x)+\int_{\Omega'}d\psi(x)\bigg), \tag{2-29}$$

where $\psi(x)=\Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Euler gamma-function.

3. The saddle-point method

We shall employ asymptotic formulae, as $n\to\infty$, for the linear forms involving $\mathrm{Li}_1\big(\frac{1}{z}\big)$ and $\mathrm{Li}_2\big(\frac{1}{z}\big)$ and for their common coefficients arising from (2-27). Such asymptotic formulae can be obtained by applying to the integrals in (2-27) the saddle-point method for double complex integrals [Hata 2000, Section 1], in the form used in [Viola and Zudilin 2018].

We henceforth assume the integers belonging to the set $\mathcal{S}$ in (2-17) to be all strictly positive. Moreover, we assume

$$j \le h \quad \text{and} \quad k < m. \tag{3-1}$$

Let, as in [RV05, (5.1)],

$$f_z(x, y) = \frac{x^j(1-x)^h y^k (1-y)^l}{(x(1-y)+yz)^{j+k-m}},$$

and let

$$F_z(\xi, \eta) = \frac{\xi^j(1-\xi)^h \eta^k}{(\xi-\eta)^{j+k-m}(\eta-z)^{l+m-j}}. \tag{3-2}$$

The substitution (2-5) yields

$$(-z)^{-l-m} f_z\left(\xi, \frac{\eta}{\eta-z}\right) = (-z)^{-j-k} F_z(\xi, \eta), \tag{3-3}$$

whence

$$\begin{aligned}
(-z)^{-j-k}\frac{\partial F_z}{\partial \xi} &= (-z)^{-l-m}\frac{\partial f_z}{\partial x}\bigg|_{x=\xi,\ y=\eta/(\eta-z)},\\
(-z)^{-j-k}\frac{\partial F_z}{\partial \eta} &= \frac{(-z)^{-l-m+1}}{(\eta-z)^2}\frac{\partial f_z}{\partial y}\bigg|_{x=\xi,\ y=\eta/(\eta-z)}.
\end{aligned} \tag{3-4}$$

Owing to (2-8), (2-10) and (2-11), the double integrals occurring in (2-27) can be written as

$$I_z^{(0)}(hn, jn, kn, ln, mn) = \pm z^{-(j+k)n}\int_0^{\zeta\infty}\left(\int_0^1 F_z(\xi,\eta)^n \frac{\mathrm{d}\xi}{\xi-\eta}\right)\frac{\mathrm{d}\eta}{\eta-z}, \tag{3-5}$$

$$I_z^{(1)}(hn, jn, kn, ln, mn) = \pm \frac{z^{-(j+k)n}}{2\pi i}\oint_{\Gamma_{0,1}}\left(\int_0^1 F_z(\xi,\eta)^n \frac{\mathrm{d}\xi}{\xi-\eta}\right)\frac{\mathrm{d}\eta}{\eta-z}, \tag{3-6}$$

$$I_z^{(2)}(hn, jn, kn, ln, mn) = \pm \frac{z^{-(j+k)n}}{2\pi i}\oint_{|\eta-z|=\varrho_1}\left(\frac{1}{2\pi i}\oint_{|\xi-\eta|=\varrho_2} F_z(\xi,\eta)^n \frac{\mathrm{d}\xi}{\xi-\eta}\right)\frac{\mathrm{d}\eta}{\eta-z}. \tag{3-7}$$

Since

$$\begin{aligned}
\frac{1}{F_z}\frac{\partial F_z}{\partial \xi} &= \frac{\partial}{\partial \xi}\log F_z = \frac{j}{\xi} - \frac{h}{1-\xi} - \frac{j+k-m}{\xi-\eta},\\
\frac{1}{F_z}\frac{\partial F_z}{\partial \eta} &= \frac{\partial}{\partial \eta}\log F_z = \frac{k}{\eta} + \frac{j+k-m}{\xi-\eta} - \frac{l+m-j}{\eta-z},
\end{aligned}$$

the saddle-points of $F_z(\xi, \eta)$, i.e., the stationary points of $F_z(\xi, \eta)$ satisfying $F_z(\xi, \eta) \ne 0$, are the solutions of the system

$$\begin{cases}
\dfrac{j}{\xi} - \dfrac{h}{1-\xi} = \dfrac{j+k-m}{\xi-\eta},\\[2mm]
\dfrac{l+m-j}{\eta-z} - \dfrac{k}{\eta} = \dfrac{j+k-m}{\xi-\eta}.
\end{cases} \tag{3-8}$$

As in [Viola and Zudilin 2018, (4.3)], the first equation (3-8) yields

$$\eta = H(\xi) := \xi \, \frac{(m+h-k)\xi + k - m}{(h+j)\xi - j}. \tag{3-9}$$

Subtracting the equations (3-8) and then substituting (3-9), we get the same cubic equation in ξ as in [RV05, (5.4)], namely

$$U(\xi) := \xi((m+h-k)\xi + k - m)((h+j-l)\xi + l - j) - z((h+j)\xi - j)((h+m)\xi - m) = 0. \tag{3-10}$$

Thus, denoting by (ξ_ν, η_ν) $(\nu = 0, 1, 2)$ the saddle-points of $F_z(\xi, \eta)$, we see that ξ_0, ξ_1, ξ_2 are the roots of (3-10), and

$$\eta_\nu = H(\xi_\nu) = \xi_\nu \, \frac{(m+h-k)\xi_\nu + k - m}{(h+j)\xi_\nu - j} \quad (\nu = 0, 1, 2). \tag{3-11}$$

Also, by (3-4),

$$x_\nu = \xi_\nu, \qquad y_\nu = \frac{\eta_\nu}{\eta_\nu - z} \quad (\nu = 0, 1, 2), \tag{3-12}$$

where (x_ν, y_ν) are the saddle-points of $f_z(x, y)$.

We refer to the detailed discussion in [Viola and Zudilin 2018, Section 4] for the application of the saddle-point method in $\mathbb{C}^2$ to the double integrals (3-5), (3-6), (3-7). Let

$$\xi_\pm = \frac{j}{h+j} \pm \frac{\sqrt{hj(m+h-k)(j+k-m)}}{(h+j)(m+h-k)}$$

be the solutions of $\mathrm{d}H/\mathrm{d}\xi = 0$, and let

$$\eta_\pm = \frac{h(j+k-m) + j(m+h-k) \pm 2\sqrt{hj(m+h-k)(j+k-m)}}{(h+j)^2}.$$

The function (3-9) satisfies

$$\eta_+ = H(\xi_+), \qquad \eta_- = H(\xi_-),$$

and maps both the upper and the lower half-circumference of diameter $[\xi_-, \xi_+]$ in the plane of the complex variable ξ onto the real interval $[\eta_-, \eta_+]$. Thus, if we denote by $\mathcal{C}$ and $\mathcal{D}$ the upper and lower half-planes of the complex variable η,

$$\mathcal{C} = \{\operatorname{Im}\eta > 0\}, \qquad \mathcal{D} = \{\operatorname{Im}\eta < 0\},$$

and in the plane of the complex variable ξ we define the regions

$$\begin{aligned}
\mathcal{C}_1 &= \{\operatorname{Im}\xi > 0,\ |\xi - j/(h+j)| > R\},\\
\mathcal{D}_1 &= \{\operatorname{Im}\xi < 0,\ |\xi - j/(h+j)| > R\},\\
\mathcal{C}_2 &= \{\operatorname{Im}\xi < 0,\ |\xi - j/(h+j)| < R\},\\
\mathcal{D}_2 &= \{\operatorname{Im}\xi > 0,\ |\xi - j/(h+j)| < R\},
\end{aligned}$$

where

$$R = \frac{\sqrt{hj(m+h-k)(j+k-m)}}{(h+j)(m+h-k)},$$

we see that (3-9) is a one-to-one mapping of both $\mathcal{C}_1$ and $\mathcal{C}_2$ onto $\mathcal{C}$, and of both $\mathcal{D}_1$ and $\mathcal{D}_2$ onto $\mathcal{D}$.

In Sections 3.1 and 3.2 we shall treat the cases $z > 1$ and $z < 0$, respectively.

3.1. ***The case z > 1.*** As is shown in [RV05, (5.6)] the roots of (3-10) are real, and $\xi_0 < \xi_1 < \xi_2$ using the notation therein. By (3-1) we get

$$0 < \xi_- < \frac{m-k}{m+h-k} < \xi_0 < \frac{j}{h+j} < \xi_+ < 1 < z < \xi_2.$$

Hence, by (3-12) and [RV05, (5.15)],

$$\eta_0 < 0 < \eta_- < \eta_+ < 1 < z < \eta_2 < \xi_2.$$

Let $\xi = \varXi(\eta)$ be a local inverse of (3-9), holomorphic in an open region $\varDelta$ contained in the plane of the complex variable η. We use the notation in [Viola and Zudilin 2018, Sections 4.1 and 4.2], where (ξ_*, η_*) denotes the relevant saddle-point for the double integral considered. Here we apply the saddle-point method to the double contour integral (3-7). We choose

$$\varDelta = \mathcal{C} \cup \mathcal{D} \cup (\eta_+, +\infty), \qquad \varXi : \varDelta \to \mathcal{C}_1 \cup \mathcal{D}_1 \cup (\xi_+, +\infty), \qquad (\xi_*, \eta_*) = (\xi_2, \eta_2).$$

We change the integration path $|\eta - z| = \varrho_1$ in (3-7) to a closed contour $\varGamma \subset \varDelta$ of steepest descent for $|F_z(\varXi(\eta), \eta)|$ enclosing z and passing through 1 and η_2, whence

$$\max_{\eta \in \varGamma} |F_z(\varXi(\eta), \eta)| = |F_z(\xi_2, \eta_2)|,$$

with the maximum attained only at $\eta = \eta_2$. Such a contour $\varGamma$ exists because $\varXi(1) = 1$, whence, by (3-1) and (3-2), $F_z(\varXi(\eta), \eta) \to 0$ as $\eta \to 1$, and $F_z(\varXi(\eta), \eta) \to \infty$ as $\eta \to z$ or $\eta \to \infty$. Furthermore, for any fixed $\eta \in \varGamma$, the function $F_z(\xi, \eta)$ vanishes at $\xi = 1$ and tends to infinity as $\xi \to \eta$ or $\xi \to \infty$. Hence we can change the integration path $|\xi - \eta| = \varrho_2$ to a closed contour δ_η of steepest descent for $|F_z(\xi, \eta)|$, enclosing η and passing through 1 and through the saddle-point $\xi = \varXi(\eta)$. Thus

$$\max_{\xi \in \delta_\eta} |F_z(\xi, \eta)| = |F_z(\varXi(\eta), \eta)|,$$

with the maximum attained only at $\xi = \varXi(\eta)$. Therefore, by Hata's theorem [2000, Section 1],

$$\lim_{n\to\infty} \frac{1}{n} \log |I_z^{(2)}(hn, jn, kn, ln, mn)| = -(j+k)\log z + \log |F_z(\xi_2, \eta_2)|. \tag{3-13}$$

For the integrals (3-5) and (3-6) we can dispense with the saddle-point method, and apply instead the asymptotic formulae [RV05, (5.16) and (5.19)]; i.e., by (3-3) and (3-12),

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log I_z^{(0)}(hn, jn, kn, ln, mn) &= -(l+m)\log z + \log f_z(x_0, y_0)\\ &= -(j+k)\log z + \log |F_z(\xi_0, \eta_0)|\end{aligned} \tag{3-14}$$

and

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \log |I_z^{(1)}(hn, jn, kn, ln, mn)| &\le -(l+m)\log z + \log |f_z(x_1, y_1)|\\ &= -(j+k)\log z + \log |F_z(\xi_1, \eta_1)|.\end{aligned} \tag{3-15}$$

3.2. *The case $z<0$.* Since the polynomial $U(\xi)$ in (3-10) has the leading coefficient

$$(m+h-k)(h+j-l)>0$$

and takes the value $-jmz>0$ at $\xi=0$, the cubic equation (3-10) has a negative root, which we now denote by

$$\xi_2<0.$$

Taking into account that

$$U\left(\frac{j}{h+j}\right)=\frac{h^2jl(j+k-m)}{(h+j)^3}>0,$$

we choose the integers $h,\ j,\ k,\ l,\ m$ such that the remaining roots ξ_0 and ξ_1 of $U(\xi)$ are distinct, and either real with

$$\frac{j}{h+j}<\xi_0<\xi_1<\frac{j}{m+h-k}<\xi_+, \tag{3-16}$$

or complex conjugate,

$$\xi_1=\overline{\xi_0}\quad \text{with } \xi_1\in\mathcal{C}_2,\ \ \xi_0\in\mathcal{D}_2. \tag{3-17}$$

We first apply the saddle-point method to the integral (3-7). From $\xi_2<0$ and (3-11) we get $\xi_2<\eta_2<0$. Again with notation as in [Viola and Zudilin 2018, Sections 4.1 and 4.2], we choose

$$\Delta=\mathcal{C}\cup\mathcal{D}\cup(-\infty,\eta_-),\qquad \Xi:\Delta\to\mathcal{C}_1\cup\mathcal{D}_1\cup(-\infty,\xi_-),\qquad (\xi_*,\eta_*)=(\xi_2,\eta_2).$$

For $\eta\in\Delta$, $\eta\to\infty$, we have $\Xi(\eta)\to\infty$. Thus from (3-2) and (3-9) we see that $F_z(\Xi(\eta),\eta)\to 0$ as $\eta\to 0$, and $F_z(\Xi(\eta),\eta)\to\infty$ as $\eta\to z$ or $\eta\to\infty$. Since $\mathrm{d}F_z(\Xi(\eta),\eta)/\mathrm{d}\eta=0$ at $\eta=\eta_2$, we obtain

$$\xi_2<\eta_2<z<0.$$

Thus there exists a closed contour $\Gamma\subset\Delta$ of steepest descent for $|F_z(\Xi(\eta),\eta)|$ enclosing z and passing through 0 and η_2. Therefore

$$\max_{\eta\in\Gamma}|F_z(\Xi(\eta),\eta)|=|F_z(\xi_2,\eta_2)|,$$

with the maximum attained only at $\eta=\eta_2$. For any fixed $\eta\in\Gamma$ we have $F_z(0,\eta)=0$ and $F_z(\xi,\eta)\to\infty$ as $\xi\to\eta$ or $\xi\to\infty$. Hence we change the path $|\xi-\eta|=\varrho_2$ for ξ to a closed contour δ'_η of steepest descent for $|F_z(\xi,\eta)|$, enclosing η and passing through 0 and $\Xi(\eta)$. Thus with the same discussion as in Section 3.1 we get the analogue of (3-13); i.e.,

$$\lim_{n\to\infty}\frac{1}{n}\log|I_z^{(2)}(hn,jn,kn,ln,mn)|=-(j+k)\log|z|+\log|F_z(\xi_2,\eta_2)|. \tag{3-18}$$

For the application of the saddle-point method to the integrals (3-5) and (3-6) we choose

$$\Delta=\mathbb{C}\setminus[\eta_-,\eta_+],\qquad \Xi:\Delta\to\mathcal{C}_2\cup\mathcal{D}_2\cup(\xi_-,\xi_+),$$

and we distinguish two cases, according to whether (3-16) or (3-17) holds.

<u>First case</u>: $\xi_0,\xi_1\in\mathbb{R}$ satisfy (3-16). By (3-11) and (3-16) we get

$$\eta_+<1<\eta_1<\eta_0.$$

For $\eta \in \Delta$ we now have, by (3-9), $\Xi(\eta) \to j/(h+j)$ as $\eta \to \infty$, and $\Xi(\eta) \to (m-k)/(m+h-k)$ as $\eta \to 0$. Hence, by (3-2), $F_z(\Xi(\eta), \eta) \to \infty$ as $\eta \to z$, and $F_z(\Xi(\eta), \eta) \to 0$ as $\eta \to 0$ or $\eta \to \infty$. Since η_0 and η_1 are, respectively, a local maximum and a local minimum of $|F_z(\Xi(\eta), \eta)|$ for $\eta \in \mathbb{R}$ along the half-line $(\eta_+, +\infty)$, the steepest descent direction for $|F_z(\Xi(\eta), \eta)|$ is the direction of the real line at η_0, and the direction orthogonal to the real line at η_1. Hence, by steepest descent, there exists a path γ from 0 to η_1 in the half-plane $\operatorname{Im} \eta < 0$, with tangent at η_1 orthogonal to the real line, such that the maximum of $|F_z(\Xi(\eta), \eta)|$ on γ is attained only at $\eta = \eta_1$. Thus for the integral (3-6) we may take $\Gamma_{0,1} = \gamma \cup \bar{\gamma}$, whence

$$\max_{\eta \in \Gamma_{0,1}} |F_z(\Xi(\eta), \eta)| = |F_z(\xi_1, \eta_1)|$$

with the maximum attained only at $\eta = \eta_1$; accordingly we choose $(\xi_*, \eta_*) = (\xi_1, \eta_1)$. For any fixed $\eta \in \Gamma_{0,1}$, $\eta \neq 0$, by (3-2) we have $F_z(0, \eta) = F_z(1, \eta) = 0$ and $F_z(\xi, \eta) \to \infty$ as $\xi \to \eta$ or $\xi \to \infty$. Since $\operatorname{Im} \eta$ and $\operatorname{Im} \Xi(\eta)$ have opposite signs, keeping the endpoints $\xi = 0$ and $\xi = 1$ fixed we can continuously deform the integration interval [0, 1] for ξ, without encountering η, to an integration path δ''_η of steepest descent for $|F_z(\xi, \eta)|$, equivalent to [0, 1] by Cauchy's theorem and passing through the saddle-point $\xi = \Xi(\eta)$. Hence

$$\max_{\xi \in \delta''_\eta} |F_z(\xi, \eta)| = |F_z(\Xi(\eta), \eta)| \tag{3-19}$$

with the maximum attained only at $\xi = \Xi(\eta)$. By Hata's theorem,

$$\lim_{n \to \infty} \frac{1}{n} \log |I_z^{(1)}(hn, jn, kn, ln, mn)| = -(j+k) \log |z| + \log |F_z(\xi_1, \eta_1)|. \tag{3-20}$$

For the integral (3-5) we choose $\operatorname{Im} \zeta < 0$. Again by the argument in (2-6), the integration half-line $(0, \zeta\infty)$ for η can be transformed to $\gamma \cup [\eta_1, +\infty)$, where γ is defined as above, without changing the value of (3-5). Clearly

$$\max_{\eta \in \gamma \cup [\eta_1, +\infty)} |F_z(\Xi(\eta), \eta)| = |F_z(\xi_0, \eta_0)|,$$

with the maximum attained only at $\eta = \eta_0$. Thus we take here $(\xi_*, \eta_*) = (\xi_0, \eta_0)$. For any fixed $\eta \in \gamma \cup [\eta_1, +\infty)$ we have $\Xi(\eta) \in \mathcal{D}_2 \cup (\xi_-, \xi_+)$. Hence, as above, the integration interval [0, 1] for ξ in (3-5) can be deformed to a path δ''_η from 0 to 1 passing through $\Xi(\eta)$, equivalent to [0, 1] by Cauchy's theorem and satisfying (3-19). Therefore

$$\lim_{n \to \infty} \frac{1}{n} \log |I_z^{(0)}(hn, jn, kn, ln, mn)| = -(j+k) \log |z| + \log |F_z(\xi_0, \eta_0)|. \tag{3-21}$$

Second case: $\xi_1 \in \mathcal{C}_2$, $\xi_0 = \overline{\xi_1} \in \mathcal{D}_2$. Now $\eta_1 \in \mathcal{C}$, $\eta_0 = \overline{\eta_1} \in \mathcal{D}$. Since $F_z(\Xi(\eta), \eta) \to 0$ as $\eta \to 0$ or $\eta \to \infty$, there exists a path $\gamma' \subset \mathcal{D}$ from 0 to a sufficiently large $\eta' > 0$, passing through the saddle-point η_0, such that

$$\max_{\eta \in \gamma'} |F_z(\Xi(\eta), \eta)| = |F_z(\xi_0, \eta_0)|$$

with the maximum attained only at η_0. For the integral (3-6) we take $\Gamma_{0,1} = \gamma' \cup \overline{\gamma'}$, whence

$$\max_{\eta \in \Gamma_{0,1}} |F_z(\Xi(\eta), \eta)| = |F_z(\xi_0, \eta_0)| = |F_z(\xi_1, \eta_1)|.$$

Thus we must apply the saddle-point method separately for $\eta \in \gamma'$ and $\eta \in \overline{\gamma'}$. We have

$$\left| \oint_{\Gamma_{0,1}} \left(\int_0^1 F_z(\xi, \eta)^n \frac{\mathrm{d}\xi}{\xi - \eta} \right) \frac{\mathrm{d}\eta}{\eta - z} \right| \\ \leq \left| \int_{\gamma'} \left(\int_0^1 F_z(\xi, \eta)^n \frac{\mathrm{d}\xi}{\xi - \eta} \right) \frac{\mathrm{d}\eta}{\eta - z} \right| + \left| \int_{\overline{\gamma'}} \left(\int_0^1 F_z(\xi, \eta)^n \frac{\mathrm{d}\xi}{\xi - \eta} \right) \frac{\mathrm{d}\eta}{\eta - z} \right|. \tag{3-22}$$

For any fixed $\eta \in \gamma'$ or $\eta \in \overline{\gamma'}$, $\eta \neq 0$, we have $F_z(0, \eta) = F_z(1, \eta) = 0$ and $F_z(\xi, \eta) \to \infty$ as $\xi \to \eta$ or $\xi \to \infty$. As above, there exists an integration path δ''_η for ξ from 0 to 1 passing through $\Xi(\eta)$, equivalent to $[0, 1]$ by Cauchy's theorem and satisfying (3-19). By Hata's theorem [2000, (1.9)], the quotient of the absolute values of the integrals over γ' and $\overline{\gamma'}$ in (3-22) tends to a constant as $n \to \infty$, since $|F_z(\xi_0, \eta_0)| = |F_z(\xi_1, \eta_1)|$. It follows that

$$\limsup_{n\to\infty} \frac{1}{n} \log |I_z^{(1)}(hn, jn, kn, ln, mn)| \leq -(j+k) \log |z| + \log |F_z(\xi_1, \eta_1)|. \tag{3-23}$$

In (3-5) we choose $\operatorname{Im} \zeta < 0$, and we change the integration half-line $(0, \zeta\infty)$ for η to a suitable path $\gamma'' \subset \mathcal{D}$ from 0 to ∞, passing through the saddle-point η_0, so that

$$\max_{\eta\in\gamma''} |F_z(\Xi(\eta), \eta)| = |F_z(\xi_0, \eta_0)|$$

with the maximum attained only at η_0. Again, for any $\eta \in \gamma''$ there exists a path δ''_η for ξ from 0 to 1 passing through $\Xi(\eta)$, equivalent to $[0, 1]$ by Cauchy's theorem and satisfying (3-19). Therefore

$$\lim_{n\to\infty} \frac{1}{n} \log |I_z^{(0)}(hn, jn, kn, ln, mn)| = -(j+k) \log |z| + \log |F_z(\xi_0, \eta_0)|. \tag{3-24}$$

4. Linear independence measures

In (2-27) we set $z = \frac{s}{r}$, with integers $r \geq 1$ and s satisfying either $s > r$ or $s < 0$, and we multiply by $r^{\delta n}$. We obtain the following extension of [RV05, (4.12)]:

$$\begin{aligned} D_n s^{\alpha n}(s-r)^{\beta n} r^{(\delta-\alpha-\beta)n} I_{s/r}(hn, jn, kn, ln, mn) &= p_n - q_n \mathrm{Li}_2\Big(\frac{r}{s}\Big), \\ D_n s^{\alpha n}(s-r)^{\beta n} r^{(\delta-\alpha-\beta)n} I^{(1)}_{s/r}(hn, jn, kn, ln, mn) &= p'_n - q_n \mathrm{Li}_1\Big(\frac{r}{s}\Big), \\ D_n s^{\alpha n}(s-r)^{\beta n} r^{(\delta-\alpha-\beta)n} I^{(2)}_{s/r}(hn, jn, kn, ln, mn) &= q_n, \end{aligned} \tag{4-1}$$

with

$$p_n = r^{\delta n} P_n^*\Big(\frac{s}{r}\Big), \qquad q_n = r^{\delta n} Q_n^*\Big(\frac{s}{r}\Big), \qquad p'_n = r^{\delta n} R_n^*\Big(\frac{s}{r}\Big).$$

By (2-28),

$$p_n, p'_n, q_n \in \mathbb{Z}.$$

Let, by (3-3) and (3-12),

$$\begin{aligned} c_\nu &= -\log |f_{s/r}(x_\nu, y_\nu)| = (j+k-l-m) \log \frac{|s|}{r} - \log |F_{s/r}(\xi_\nu, \eta_\nu)| \qquad (\nu = 0, 1), \\ c_2 &= \log |f_{s/r}(x_2, y_2)| = (l+m-j-k) \log \frac{|s|}{r} + \log |F_{s/r}(\xi_2, \eta_2)|, \end{aligned}$$

and

$$c_3 = M + N - \left(\int_{\Omega} \mathrm{d}\psi(x) + \int_{\Omega'} \mathrm{d}\psi(x)\right) + (\alpha - l - m)\log|s| + (m + h - k)\log r + \beta \log|s - r|. \quad (4\text{-}2)$$

Since, by (2-16), $\delta - \alpha - \beta = h - k - l$, applying (2-29), (3-13) and (3-18) in the last equation of (4-1) we obtain

$$\lim_{n\to\infty} \frac{1}{n} \log|q_n| = c_2 + c_3. \quad (4\text{-}3)$$

Similarly, defining

$$r_n^{(1)} = p_n' - q_n \mathrm{Li}_1\Big(\frac{r}{s}\Big), \qquad r_n^{(2)} = p_n - q_n \mathrm{Li}_2\Big(\frac{r}{s}\Big),$$

and using (2-29), (3-14), (3-15), (3-20), (3-21), (3-23) and (3-24) in the first two equations of (4-1), we get

$$\limsup_{n\to\infty} \frac{1}{n} \log|r_n^{(1)}| \le c_3 - c_1 \quad (4\text{-}4)$$

and, by (2-4),

$$\limsup_{n\to\infty} \frac{1}{n} \log|r_n^{(2)}| \le c_3 - \min\{c_0, c_1\}. \quad (4\text{-}5)$$

Moreover, if $c_0 < c_1$,

$$\lim_{n\to\infty} \frac{1}{n} \log|r_n^{(2)}| = c_3 - c_0, \quad (4\text{-}6)$$

by virtue of (3-14), (3-21) and (3-24). Thus, for a given $z = \frac{s}{r}$, if we choose the integers h, j, k, l, m such that $c_3 < c_0 < c_1$, applying [Viola and Zudilin 2018, Lemma 5.1] with $S = 2$, $\gamma_1 = \mathrm{Li}_1\big(\frac{r}{s}\big)$, $\gamma_2 = \mathrm{Li}_2\big(\frac{r}{s}\big)$ and $\tau = c_0 - c_3$, from (4-4) and (4-6) we deduce the linear independence over $\mathbb{Q}$ of 1, $\mathrm{Li}_1\big(\frac{r}{s}\big)$ and $\mathrm{Li}_2\big(\frac{r}{s}\big)$. Once the linear independence is established, in order to improve the $\mathbb{Q}$-linear independence measure of 1, $\mathrm{Li}_1\big(\frac{r}{s}\big)$ and $\mathrm{Li}_2\big(\frac{r}{s}\big)$ that can be obtained through the above choice of h, j, k, l, m, we can make a new choice of h, j, k, l, m, possibly different from the previous one, with $c_0 \lesseqgtr c_1$ and $c_3 < \min\{c_0, c_1\}$. Then, by Hata's lemma for $S = 2$ generalized by Marcovecchio for any S, i.e., by [Viola and Zudilin 2018, Lemma 5.2] with $S = 2$, $\gamma_1 = \mathrm{Li}_1\big(\frac{r}{s}\big)$, $\gamma_2 = \mathrm{Li}_2\big(\frac{r}{s}\big)$, $d = 0$, $\sigma = c_2 + c_3$ and $\tau = \min\{c_0, c_1\} - c_3$, from (4-3), (4-4) and (4-5) we get for 1, $\mathrm{Li}_1\big(\frac{r}{s}\big)$ and $\mathrm{Li}_2\big(\frac{r}{s}\big)$ the $\mathbb{Q}$-linear independence measure

$$\frac{c_2 + c_3}{c - c_3} \quad \text{with } c = \min\{c_0, c_1\}; \quad (4\text{-}7)$$

i.e., for any $\varepsilon > 0$ and any $(a_0, a_1, a_2) \in \mathbb{Z}^3 \setminus \{(0,0,0)\}$,

$$\left| a_0 + a_1 \mathrm{Li}_1\Big(\frac{r}{s}\Big) + a_2 \mathrm{Li}_2\Big(\frac{r}{s}\Big) \right| > C(\varepsilon) A^{-(c_2+c_3)/(c-c_3)-\varepsilon},$$

where $A = \max\{|a_0|, |a_1|, |a_2|\}$ and where the constant $C(\varepsilon) > 0$ is independent of (a_0, a_1, a_2).

We separately treat the positive and the negative cases, $z > 1$ and $z < 0$. From [RV05, Theorem 5.2] where $c_3 < c_0 < c_1$ is assumed, and from the subsequent discussion in Section 6 of that paper, we obtain the $\mathbb{Q}$-linear independence of 1, $\mathrm{Li}_1\big(\frac{r}{s}\big)$ and $\mathrm{Li}_2\big(\frac{r}{s}\big)$ for all integers r, s with $r \ge 1$ and $s \ge s_+(r) > r$, where $s_+(r)$ is explicit. In [RV05] we found the values $s_+(1) = 6$, $s_+(2) = 51$, $s_+(3) = 173$, $s_+(4) = 423$. Concerning $\mathbb{Q}$-linear independence measures, applying [Viola and Zudilin 2018, Lemma 5.2] we get a new proof of the linear independence measures of 1, $\mathrm{Li}_1\big(\frac{1}{z}\big)$ and $\mathrm{Li}_2\big(\frac{1}{z}\big)$ for $z = \frac{s}{r} > 1$ given in [Marcovecchio 2016, Table p. 231], by choosing $h = \gamma$, $j = \alpha_1$, $k = \beta_1 + \alpha_2 - \alpha_1$, $l = \beta_2 + \alpha_1 - \alpha_2$,

z	h	j	k	l	m	$(c_2+c_3)/(c-c_3)$
$*$ -5	10	8	7	6	10	61.68698...
$*$ -6	10	8	7	6	10	43.51295...
$*$ -7	64	32	37	27	43	32.76353...
$*$ -8	64	32	37	27	43	25.05713...
$*$ -9	64	32	37	27	43	20.87789...
$*$ -10	64	32	37	27	43	18.24158...
$*$ -11	64	32	37	27	43	16.41900...
-12	64	32	37	27	43	15.08001...
$*$ -13	130	65	76	54	86	14.04958...
-14	130	65	76	54	86	13.22908...
-15	130	65	76	54	86	12.58420...
-16	130	65	76	54	86	12.05561...
-17	130	65	76	54	86	11.61188...
-18	130	65	76	54	86	11.23276...
-19	130	65	76	54	86	10.90426...
-20	130	65	76	54	86	10.61629...

Table 1

$m=\alpha_2$ for each z, where $\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma$ are the integers in Marcovecchio's table. Note, however, that in Marcovecchio's notation the quantity μ appearing in the last column of his table equals our linear independence measure (4-7) plus 1; i.e.,

$$\mu=\frac{c_2+c_3}{c-c_3}+1=\frac{c+c_2}{c-c_3}.$$

In the negative case, our method yields new $\mathbb{Q}$-linear independence measures of 1, $\mathrm{Li}_1\bigl(\frac{r}{s}\bigr)$ and $\mathrm{Li}_2\bigl(\frac{r}{s}\bigr)$ for integers $r\geq 1$ and $s\leq s_-(r)<0$ with explicit $s_-(r)$, improving both the results in [Hata 1993, Table 2 p. 386] and in [Marcovecchio 2016, Table p. 231]. For brevity we give numerical results only for $r=1$, and for $-20\leq z=s\leq s_-(1)=-5$. For such values of z it is easy to prove the $\mathbb{Q}$-linear independence of 1, $\mathrm{Li}_1\bigl(\frac{1}{z}\bigr)$ and $\mathrm{Li}_2\bigl(\frac{1}{z}\bigr)$, e.g., with the choice $h=20,\ j=10,\ k=13,\ l=8,\ m=16$, which yields $c_3<c_0<c_1$ for all $z\in\mathbb{Z}$ satisfying $-20\leq z\leq -5$, and hence allows one to apply [Viola and Zudilin 2018, Lemma 5.1]. Then for each z we apply Lemma 5.2 of the same paper with the corresponding values of h, j, k, l, m in Table 1, thus obtaining for 1, $\mathrm{Li}_1\bigl(\frac{1}{z}\bigr)$ and $\mathrm{Li}_2\bigl(\frac{1}{z}\bigr)$ the $\mathbb{Q}$-linear independence measure $(c_2+c_3)/(c-c_3)$ in the last column of the table.

The values of z in Table 1 marked with an asterisk are associated with h, j, k, l, m for which (3-17) holds, i.e., such that the polynomial $U(\xi)$ in (3-10) has complex conjugate roots ξ_0, ξ_1, whence $c=c_0=c_1$. For the remaining values of z, the corresponding h, j, k, l, m yield (3-16) with $c=c_0<c_1$.

As an example we give below all the numerical values for $z=-5$, with $h=10,\ j=8,\ k=7,\ l=6,\ m=10$. We get

$$l+m-j=8,\quad m+h-k=13,\quad h+j-l=12,\quad j+k-m=5,$$

whence

$$M=13,\quad N=12,\quad \alpha=l+m=16,\quad \beta=k+l-h=3,$$

$$\Omega=\left[\tfrac{1}{10},\tfrac{1}{4}\right)\cup\left[\tfrac{3}{10},\tfrac{5}{13}\right)\cup\left[\tfrac{2}{5},\tfrac{5}{12}\right)\cup\left[\tfrac{1}{2},\tfrac{7}{13}\right)\cup\left[\tfrac{4}{7},\tfrac{5}{8}\right)\cup\left[\tfrac{7}{10},\tfrac{10}{13}\right)\cup\left[\tfrac{4}{5},\tfrac{11}{13}\right)\cup\left[\tfrac{6}{7},\tfrac{7}{8}\right)\cup\left[\tfrac{9}{10},\tfrac{12}{13}\right),$$

$$\int_\Omega \mathrm{d}\psi(x)=7.87642\ldots,\qquad \Omega'=\varnothing,$$

$$c_3=M+N-\int_\Omega \mathrm{d}\psi(x)+\beta\log 6=22.49885\ldots.$$

The saddle-points of $F_{-5}(\xi,\eta)$ are

$$(\xi_0,\eta_0)=(0.45905\ldots+i\,0.04354\ldots,\quad 0.96082\ldots-i\,1.38422\ldots),$$
$$(\xi_1,\eta_1)=(\overline{\xi_0},\overline{\eta_0}),$$
$$(\xi_2,\eta_2)=(-12.05913\ldots,\quad -8.56053\ldots).$$

Therefore

$$c=c_0=c_1=23.60655\ldots,\qquad c_2=45.83193\ldots,$$

whence we obtain the $\mathbb{Q}$-linear independence measure of 1, $\mathrm{Li}_1\left(-\frac{1}{5}\right)$, $\mathrm{Li}_2\left(-\frac{1}{5}\right)$ given by

$$\frac{c_2+c_3}{c-c_3}=61.68698\ldots.$$

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